VIGRE Algebra QR Review, September 16, 2002
Summary of Session

A first installment of notes was distributed. The following topics were reviewed (this is just a sketch: see the notes for details about each):

(1) Right and left actions of groups on sets.

(2) The elements of the orbit of s are in one-to-one correspondence with the cosets of the stabilizer of s; hence, cardinalities of orbits divide the order of the group in the case of a finite group

(3) A group acts on itself by conjugation. Conjugacy classes are orbits, and so their cardinalities divide the order of the group in the case of a finite group. An element of a group is in the center if and only if its conjugacy class contains just one element. A subgroup is normal if and only if its conjugacy class contains just one element.

(3) Sylow theorems: in part, if $|G| = p^k m$ where p is prime, p, k, m are positive integers, and p does not divide m, then G has subgroups of order p^k (the *p-Sylow subgroups*) all of which are conjugate, and the number of them divides m and is $\equiv 1 \bmod p$. If p divides $|G|$, then G has elements of order p.

(4) Nilpotent and solvable groups were discussed. A solvable group has a finite subnormal series in which all factor groups arising from consecutive terms are abelian. A nilpotent group G has a normal series in which for each pair of terms $G_i \subseteq G_{i+1}$, the image of G_{i+1}/G_i in G/G_i is in the center of G/G_i. Thus, abelian $\Rightarrow$ nilpotent $\Rightarrow$ solvable. The property of being nilpotent or solvable passes to subgroups and quotient groups. However, if N is a normal subgroup of G and G/N and N are both nilpotent, G need **not** be nilpotent. S_3 has trivial center and is therefore not nilpotent, although it has a normal cyclic subgroup of order 3 with quotient isomorphic to $\mathbb{Z}_2$.

(5) All finite p-groups are nilpotent, and the only finite groups that are nilpotent are finite products of p-groups (p varying).

(6) Semi-direct products were discussed. G is the semi-direct product of a normal subgroup N by a subgroup H if H maps isomorphically onto G/N. In this case, every element of G can be written uniquely in the form nh with $n \in N$ and $h \in H$. The group structure of G is then uniquely determined if one knows how H acts on N by conjugation, i.e., if one knows for every $h \in H$ the automorphism ϕ_h of N that sends n to hnh^{-1}. The reason is that $(nh)(n_1h_1) = nhn_1(h^{-1}h)h_1 = \big(n\phi_h(n_1)\big)\big(hh_1\big)$. Conversely, given any two groups N, H and any homomorphism ϕ of H into the group automorphisms $\mathrm{Aut}\,(N)$ of N (we write ϕ_h for $\phi(h)$) one can put a group structure on $N \times H$ by the rule $(n, h) * (n_1, h_1) = (n\phi_h(n_1), hh_1)$. This group will contain isomorphic copies of N as a normal subgroup and H as a subgroup, and the action of the copy of H on the copy of N by conjugation will correspond to ϕ. Details are given in the notes.

Solutions to some problems were presented, as follows.

1. Let H be a subgroup of G of index 3. Prove that either H is normal, or that H has a subgroup N of index 2 in H such that N is normal in G.

Solution. G acts on the left cosets of H by left multiplication (one may also use the right action on right cosets). Here, $g \in G$ maps g_1H to $(gg_1)H$. This action is transitive and, in particular, therefore non-trivial. Its kernel N is easily seen to be contained in H (which is the stabilizer of the left coset $eH = H$). But this gives a non-trivial homomorphism $G \to S_3$, and so G/N embeds in S_3. Now $[G : N] = [G : H][H : N] = 3 \cdot [H : N]$, and $[G : N]$ must be a divisor of $|S_3| = 6$ that is divisible by 3, i.e., 3 or 6. Therefore $[H : N]$ must be 1 or 2.

2. Say that a group G has the TFF property if there is a normal subgroup N which is torsion-free and has finite index. Prove that if G has the TFF property then so does any subgroup of G.

Solution. Here, *torsion-free* means that there are no elements of finite order except e. The map $G \to G/N$ may be restricted to H. The kernel in H is obviously $H \cap N$, so that $H \cap N$ is normal in H and $H/(H \cap N)$ embeds in G/N, which is finite. This shows that $H \cap N$ has finite index in H. It is obvious that a subgroup of a torsion-free group is torsion-free, and so $H \cap N$ is torsion-free and gives the required subgroup of H.

6. Find all groups of order 21, up to isomorphism.

Solution. The number of 7-Sylow subgroups must be $\equiv 1 \bmod 7$ and divides 3. Therefore there is only one, call it N, and it must be normal. There are elements of order 3, and one of these will generate a subgroup H of order 3. H cannot meet N, and so H injects into G/N, which has cardinality $21/7 = 3$. Thus, H maps isomorphically onto G/N, and we have that G is the semi-direct product of N by H. Therefore we can construct all groups of order 21 by finding all possible maps from $\mathbb{Z}_3$ into the automorphisms of $\mathbb{Z}_7$. Because any homomorphism from a cyclic group is determined by where a generator (in this case, $1 \in \mathbb{Z}_7$) maps, it is easy to see that every automorphism of $\mathbb{Z}_7$ coincides with multiplication by a nonzero element of this field. The generator of $\mathbb{Z}_3$ must therefore map to such an automorphism, but there is one more restriction: that automorphism must have order dividing 3. The trivial automorphism gives the direct product $\mathbb{Z}_7 \times Z_3 \cong \mathbb{Z}_{21}$. There are two more automorphisms that actually have order 3: these are multiplication by 2 and multiplication by 4 (mod 7). Each of these gives a non-abelian group of order 21, and we have found that there are at most 3 such groups. However, there is one point remaining: do multiplication by 2 and multiplication by 4 really give different non-abelian groups of order 21, or are they the same? This was left for people to think about.

There was some discussion of this problem:

3. Let G be a finite group with with a normal subgroup H such that both G/H' and H are nilpotent. Prove that G is nilpotent. Here H' denotes the derived (or commutator) subgroup of H.

Note that H' is generated by all elements of the form $h_1 h_2 h_1^{-1} h_2^{-1}$ with $h_1, h_2 \in H$, and is the smallest normal subgroup of H such that H/H' is abelian.

Here is a related, variant problem — perhaps solving it will help with 3.

$3'$. Suppose that G is a group, that H is a normal subgroup, that G/H' is nilpotent, that H/H' is in the center of G/H', and that H' is in the center of H. Prove that G is nilpotent. In fact, prove that H' is in the center of G (and then you should see that the nilpotence of G follows right away).