VIGRE Algebra QR Review, September 23, 2002

Note that there is a room change: this session will meet in 3096 East Hall, which has more comfortable chairs. The main theme for this week will still be group theory. Below are the problems that remain from last week, along with 5 new problems. There is just a bit left to do on Problem 6 from the old list.

— Mel

Problems from last week

3. Let G be a finite group with with a normal subgroup H such that both G/H' and H are nilpotent. Prove that G is nilpotent.[1] Here H' denotes the derived (or commutator) subgroup of H.

4. Find all finite groups G with at most 3 conjugacy classes.

5. Let G be a group with $|G| = 30$.

(i) Prove that either G has a normal Sylow 3-subgroup or a normal Sylow 5-subgroup.

(ii) Prove that G is isomorphic with a semidirect product of $\mathbb{Z}_{15}$ by $\mathbb{Z}_2$. (This is equivalent to showing that G has a normal subgroup that is cyclic of order 15 and a subgroup that is cyclic of order 2 meeting it only in the identity element.)

6. Find all groups of order 21, up to isomorphism. (We found one abelian group, and two non-abelian semi-direct products of $\mathbb{Z}_7$ by $\mathbb{Z}_3$: we did not determine whether the latter two are isomorphic.)

7. Prove that a finite simple group of even order is generated by elements of order 2.

8. Let G be a group with center Z. Suppose that G/Z is cyclic. Prove that there exists an element x of G such that every element of G can be written in the form $x^n z$ for some integer n and some $z \in Z$. From this, prove that G is abelian.

9. Let I and J be ideals of a commutative ring with identity R such that $I + J = R$. Prove that for any two positive integers m, n, $I^m + J^n = R$. (The ideal I^m is generated by all products of m elements of I.)

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[1] Don't be discouraged by this problem. It should never have been put on the Algebra QR. No one will get it on a timed exam, and that includes most of the algebra faculty. The simplest solution that I know uses the fact that the subgroup generated by two normal nilpotent subgroups of a group is nilpotent, the fact that a finite nilpotent group is the direct product of its Sylow subgroups, and a fact very similar to problem $3'$ from an earlier handout:

$3''$. If P is a subgroup of G acting on a normal subgroup $H \subseteq G$ with center Z such that the corresponding action of P on $H/Z \subseteq G/Z$ is trivial, then the action of P on H' is trivial.

Problems new this week

1. Determine the group with generators v_1, v_2 subject to the relations $v_1^2 = v_2^2 = 1$ and $(v_1v_2)^3 = 1$.

2. Let $x_1, \ldots, x_n$ be variables over the complex numbers $\mathbb{C}$. Let θ be a primitive nth root of unity in $\mathbb{C}$, and let G be the cyclic group of order n that it generates. Let G act on the field $F = \mathbb{C}(x_1, \ldots, x_n)$ in such a way that θ maps x_i to $\theta^i x_i$, $1 \leq i \leq n$. Determine a finite set of generators for the fixed field F^G.

3. Let G be a group and suppose that G acts (on the left) transitively on a set Ω. For each $\alpha \in \Omega$ let G_α be the stabilizer of α in G.

(a) Prove that a subgroup $H \subseteq G$ acts transitively on Ω if and only if $H \cap xG_\alpha \neq \emptyset$ for all $x \in G$.

(b) Assume that G is finite, that $|\Omega| = p^a$, where p is a positive prime integer and $a \geq 1$, and let P be a Sylow p-subgroup of G. Prove that:

(1) P acts transitively on Ω;

(2) $G = G_\alpha P = PG_\alpha$; and

(3) $G_\alpha \cap P$ is a Sylow p-subgroup of G_α.

4. Let p be a positive prime integer. Prove that every group of order p^n, where n is a positive integer, is nilpotent, and that every group of order p^2 is abelian. (These are standard theorems.)

5. (a) Let x and y be elements of a (non-commutative) ring R such that x and y are both nilpotent and $yx = cxy$, where c is in the center of R. Is $x+y$ necessarily nilpotent? Prove your answer.

(b) Give an example of a ring R with identity and nilpotent elements x, y in R such that $x + y$ is invertible.