VIGRE Algebra QR Review, November 4, 2002

Summary of Session

The following problems are from the list for November 4 — I am including solutions for all of these problems (not all of them were discussed in detail).

1. For each real number λ, find the rank and signature of the real quadratic form

$$x^2 + y^2 + z^2 + 4xy + 2xz + 2\lambda yz.$$

Solution. The matrix corresponding to this quadratic form is $\begin{pmatrix} 1 & 2 & 1 \\ 2 & 1 & \lambda \\ 1 & \lambda & 1 \end{pmatrix}$ where the off diagonal entries are *half* of the corresponding coefficients. We want to find the cogredience class of this matrix. To preserve the cogredience class, if one performs an elementary row operation one must also perform the corresponding column operation, and conversely. (In a problem like this, using "unbalanced" row and column operations will lose all credit.) Subtract twice the first row from the second and then twice the first column from the second to get $\begin{pmatrix} 1 & 2 & 1 \\ 0 & -3 & \lambda-2 \\ 1 & \lambda & 1 \end{pmatrix}$ and then $\begin{pmatrix} 1 & 0 & 1 \\ 0 & -3 & \lambda-2 \\ 1 & \lambda-2 & 1 \end{pmatrix}$. Subtract the first row from the last and then the first column from the last to get $\begin{pmatrix} 1 & 0 & 1 \\ 0 & -3 & \lambda-2 \\ 0 & \lambda-2 & 0 \end{pmatrix}$ and then $\begin{pmatrix} 1 & 0 & 0 \\ 0 & -3 & \lambda-2 \\ 0 & \lambda-2 & 0 \end{pmatrix}$. Add $(\lambda-2)/3$ times the second row to the third and then $(\lambda-2)/3$ times the second column to the third to get $\begin{pmatrix} 1 & 0 & 0 \\ 0 & -3 & \lambda-2 \\ 0 & 0 & (\lambda-2)^2/3 \end{pmatrix}$ and then $\begin{pmatrix} 1 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & (\lambda-2)^2/3 \end{pmatrix}$. This shows that if $\lambda \neq 2$ the rank is 3 and the signature is $2-1=1$ or $(2,1,0)$, while if $\lambda = 2$ the rank is 2 and the signature is $1-1=0$ or $(1,1,0)$.

2. (a) Up to the equivalence relation that comes from linear changes of basis, how many equivalence classes of quadratic forms are there on a real vector space of dimension n? Your answer should be a function of n, and should be explained.

(b) If x, y, u, v are variables and $a > 0$, b, c, $d > 0$, e, f are real constants such that $b^2 > 4ac$ and $e^2 < 4df$, what are the rank and signature of the quadratic form $ax^2 + bxy + cy^2 + du^2 + euv + fv^2$?

Solution. (a) The equivalence (cogredience) classes correspond to the choices of triples of non-negative integers (p,q,r) with $p+q+r=n$. (These also correspond to monomials of degree n in three variables.) There are many ways of solving the counting problem. One elegant may is to note that each solution can be given by a sequence of p dots, then a slash, a sequence of q dots, then a slash, and, finally, a sequence of r dots. E.g., if $n = 7$, the triple $(2, 1, 4)$ corresponds to $\cdot\ \cdot\ /\ \cdot\ /\ \cdot\ \cdot\ \cdot\ \cdot$ while the triple $(0, 7, 0)$ corresponds to $/\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ /$. The number of strings of length $n + 2$ with n dots and 2 slashes is the number of ways to choose where to place the slashes, i.e., $\binom{n+2}{2}$.

In part (b), because the form is the sum of two quadratic forms in mutually disjoint sets of variables one may diagonalize the pieces in x, y and u, v separately, and take the direct sum. The first form has matrix $\begin{pmatrix} a & b/2 \\ b/2 & c \end{pmatrix}$, which has trace $a+c$ and determinant $(4ac - b^2)/4$. Thus, the sum of the eigenvalues and the product are both positive, and we must have two positive eigenvalues. For the second form we get $\begin{pmatrix} d & e/2 \\ f/2 & c \end{pmatrix}$. Since the determinant is negative, the product of the eigenvalues is negative, and so they are nonzero of opposite sign. Thus, the rank is 4 and the signature is $4 - 1 = 3$ or $(3, 1, 0)$.

3. Consider finite dimensional vector spaces over the real numbers with symmetric bilinear forms. Sylvester's theorem assigns to each form a signature (p, q, r) where p, q, r give the numbers of vectors of positive, negative, and zero norm, resp., in an orthogonal basis. Suppose that the real vector space V has a form with signature (p, q, r), and the its subspace W has signature (p', q', r') with respect to the restriction of this form. Prove or give a counterexample to each of the statements: (1) $p' \le p$, (2) $q' \le q$, and (3) $r' \le r$.

Solution. Since p represents the dimension of the largest subspace on which the form is positive definite and q the dimension of the largest subspace on which the form is negative definite, these cannot increase, and (1), (2) are true. Let $e_1, \ldots, e_p$, $f_1, \ldots, f_q$, and $g_1, \ldots, g_r$ be an orthogonal basis for the form such that the form is positive definite on the span of the e_i, negative definite on the span of the f_j, and such that the g_k are orthogonal to all vectors (including themselves). Let $h = \min\{p, q\}$. Then the form is identically 0 on the span of $e_1 + f_1, \ldots, e_h + f_h, g_1, \ldots, g_k$, and so r' can be as large as $\min\{p, q\} + r$. For example, the form whose matrix is $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ is identically 0 on the line spanned by $e_1 + f_1$, so $r = 0$ increases to $r' = 1$ when we restrict to this line. Note that r' can never be larger than $\min\{p, q\} + r$. E.g., if $r' > p + r$ (the proof for the case where $r' > q + r$ is similar) then the space where the form is identically 0 would meet the space where it is negative definite — the dimension of the intersection[1] would be at least $p + r + 1 + q - n = 1$, a contradiction.

4. Let A be a skew-symmetric real $n \times n$ matrix. Prove that (a) $\det(A + I_n) \ge 1$ and that (b) for any positive definite symmetric $n \times n$ matrix B, $\det(A + B) \ge \det B$. [The examiners gave the following hint: Use that a hermitian matrix (a complex matrix whose complex conjugate is its transpose) has real eigenvalues. The problem can be done in other ways, but some theorem is needed.]

[1]Let W_1, W_2 be subspaces of an n-dimensional space. A basis $\mathcal{B}$ for $W_1 \cap W_2$ can be extended to a basis $\mathcal{B} \cup \mathcal{B}'$ for W_1, where the union is disjoint, and to a basis $\mathcal{B} \cup \mathcal{B}''$ for W_2, where the union is disjoint. Then $\mathcal{B} \cup \mathcal{B}' \cup \mathcal{B}''$ is a basis for $W_1 + W_2$. Thus, $\dim(W_1 + W_2) + \dim(W_1 \cap W_2) = \dim W_1 + \dim W_2$, and $\dim(W_1 \cap W_2) = \dim W_1 + \dim W_2 - \dim(W - 1 + W_2) \ge \dim W_1 + \dim W_2 - n$.

Solution. For the first part, note that it suffices to prove that the characteristic polynomial has pure imaginary eigenvalues, which is done two ways (one using the hint given) in the solution to 8. below. If the nonzero eigenvalues are $\pm c_j i$, $1 \le j \le h$, then the characteristic polynomial $f(x) = \det(xI - A)$ is $\prod_{j=1}^{h}(x^2 + c_j^2)x^{n-2h}$. Setting $x = -1$, we find that $(-1)^n \det(I + A) = \prod_{j=1}^{h}(1 + c_j^2)(-1)^{n-2h}$. Since $(-1)^n$ and $(-1)^{n-2h}$ have the same sign, it follows that $\det(I + A) = \prod_{j=1}^{h}(1 + c_j^2) \ge 1$.

For part (b), note that B is orthogonal conjugate to a real diagonal matrix with positive eigenvalues. This will have a square root that is a real diagonal matrix with positive eigenvalues (just use the positive square roots of the eigenvalues). Thus, reversing the conjugation, we see that there is a real symmetric positive definite matrix S such that $S^2 = B$. Since $B = S^2$, B commutes with any power, positive or negative, of S. Then $S^{-1}AS^{-1}$ is skew-symmetric, and so $\det(S^{-1}AS^{-1} + I) \ge 1$, by part (a). Multiply by $\det B = \det(S^2) = (\det S)^2$. The right hand side becomes $\det B$, while the left hand side is the same as $\det\big(S(S^{-1}AS^{-1} + I)S\big) = \det(SS^{-1}AS^{-1}S + SIS) = \det(A + B)$, and so $\det(A + B) \ge \det B$, as required.

5. Let V be the real vector space of all 2×2 real matrices.

(a) Prove that the operation $(A, B) = \operatorname{trace}(AB)$ is a bilinear form on V. Is it symmetric?

(b) Find a basis for V containing the identity matrix and consisting of matrices mutually orthogonal with respect to this form. Evaluate (A, A) for each element A of your basis.

(c) Find the rank and the signature of this form on V and also on the subspace W of V consisting of all matrices of trace zero.

Solution. Multiplication by A is a linear function of B and multiplication by B is a linear function of A. Thus, the map sending the pair (A, B) to the product AB is bilinear. Since the trace map is a linear map from 2×2 matrices to the field, the given form is bilinear. The form is symmetric because AB and BA have the same trace (and, more generally, the same characteristic polynomial, as noted earlier): this may also be shown by a straightforward computation in this case. Some experimentation leads to the basis I, $I' = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, $J = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, and $J' = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. Since I', J, J' have trace 0, they are orthogonal to I. But $I'J = J'$, $I'J' = I'(I'J) = J$, and $JJ' = -I'$ also have trace 0. It is clear that I and I' are independent and so span all diagonal matrices, while J and J' similarly span all diagonal matrices for the complementary diagonal. Thus, these four form a basis. The traces of $I^2 = I$, $(I')^2 = I$, $J^2 = I$ and $(J')^2 = -I$ are 2, 2, 2, and -2, resp. It follows that the rank is 4 (the form is non-degenerate) and the signature is $3 - 1 = 2$ or $(3, 1, 0)$.

Since I', J, J' all have trace 0, they are an orthogonal basis for the matrices of trace 0. The diagonal entries of the corresponding matrix will now be 2, 2, and -2, and so now the rank is 3 and the signature is $2 - 1 = 1$ or $(2, 1, 0)$.

6. Let $\mathbb{R}$ be the field of real numbers and let n be a positive integer. Let V_n be the vector space of bilinear maps from $\mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^3$. Let W_n be the subspace of V_n consisting of

bilinear maps T that satisfy $T(u,v) = -T(v,u)$ for all choices of u and v in $\mathbb{R}^n$. Determine, with justification, formulas for the dimensions of V and W as $\mathbb{R}$-vector spaces.

Solution. The bilinear map is determined uniquely by its values on the n^2 pairs (e_i, e_j), where $e_1, \ldots, e_n$ is the standard basis for $\mathbb{R}^n$, and every specification of values is allowed. When one adds bilinear maps, corresponding values get added. Thus, the space of bilinear forms is isomorphic with the direct sum (or product) of n^2 copies of $\mathbb{R}^3$, and has dimension $3n^2$. If the form is alternate, then it must vanish on the pairs (e_i, e_i) while if $i < j$ then the value on (e_i, e_j) is the negative of the value on (e_j, e_i), and there is no other restriction. Thus, the space of bilinear forms is isomorphic with a direct sum (or product) of copies of $\mathbb{R}^3$, where the number of copies is $(n^2 - n)/2$, and the dimension is $3(n^2 - n)/2$.

7. Let a be a variable real number. Consider the matrix $M_a = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & a \end{pmatrix}$. Determine, as a varies, the rank and signature of this matrix. For which values of a is it diagonalizable? For which values of a is it positive definite?

Solution. Since the matrix is real symmetric, it is always (orthogonally) diagonalizable, with real eigenvalues. To preserve the cogredience class, if one performs an elementary row operation one must also perform the corresponding column operation, and conversely. (In a problem like this, using "unbalanced" row and column operations will lose all credit.) Subtract the first row from the last and then the first column from the last to get: $\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 1 & a-1 \end{pmatrix}$ and then $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & a-1 \end{pmatrix}$. Now subtract the second row from the last and then the second column from the last to get: $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & a-2 \end{pmatrix}$ and then $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & a-2 \end{pmatrix}$ We have preserved the cogredience class. Thus, if $a > 2$ the rank is 3 and the signature is $3 - 0 = 3$ (or (3,0,0)), if $a = 2$ the rank is 2 and the signature is $2 - 0 = 2$ (or (2,0,1)), while if $a < 2$ the rank is 3 and the signature is $2 - 1 = 1$ (or (2,1,0)). The matrix is positive definite precisely when $a > 2$.

8. Let A be an $n \times n$ skew-symmetric real matrix.

(a) Prove or disprove: If A is a matrix of a non-degenerate alternating bilinear form, then every non-zero eigenvalue of A is purely imaginary.

(b) Prove or disprove: If n is odd, the determinant of A is zero.

Solution. One method for (a) is to use the fact that A is orthogonally conjugate to a direct sum of skew-symmetric blocks of size 2 or 1. For such blocks, the result on the eigenvalues is immediate. Another method is to use the fact that if A is skew-symmetric, then the pure imaginary matrix Ai is Hermitian. Since Ai has real eigenvalues, $A = Ai(-i)$ has pure imaginary eigenvalues. From either point of view, the nonzero eigenvalues occur in conjugate pairs, and so A has even rank. Thus, if n is odd, the determinant is zero.

9. Determine, with proof, for which positive integers n the finite field $\mathrm{GF}(5^n)$ with 5^n elements has the property that every element has a unique cube root.

Solution. 0 always has a unique cube root. The property holds iff the cubing map on $\mathrm{GF}(5^n) - \{0\}$ is surjective, which is equivalent to its being injective. This is the case if and only if the multiplicative group, which has order $5^n - 1$, has no element of order 3, i.e., iff 5^n is not 1 mod 3. Since $5 \equiv -1 \bmod 3$, $5^n = 1 \bmod 3$ iff n is even, and so the answer is all odd positive integers n. Alternatively, the condition is that the equation $x^3 - 1 = 0$ have only the root 1, or that $x^2 + x + 1$ be irreducible over the field, or that the field not contain the splitting field of $x^2 + x + 1$, which is $\mathrm{GF}(5^2)$, i.e. that 5^n not be a power of 5^2, which also yields the answer that n must be odd.