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Tensor Products and Exterior Algebra

Let k be a field and let U, V, and W be vector spaces over k.[1] Recall that a *bilinear* map $B : U \times V \to W$ is a function that is linear in each coordinate separately when the other is held fixed. Thus, for all $u, u' \in U$ and $v, v' \in V$ and all $a, a' \in k$,

$$B(au + a'u', v) = aB(u, v) + a'B(u', v) \quad \text{and} \quad B(u, av + a'v') = aB(u, v) + a'B(u, v').$$

Repeated application shows that $B(\sum_{i=1}^m a_i u_i, \sum_{j=1}^n b_j v_j) = \sum_{i,j} a_i b_j B(u_i, v_j)$. If one has a basis[2] u_i for U and a basis v_j for V then there is a unique bilinear map to W taking specified values w_{ij} on the pairs (u_i, v_j) as i and j run independently through the index sets for the two bases. This is correct even when the bases are infinite.

There is a *universal* bilinear map from $U \times V$. One must construct W. The idea is as follows. For every pair $(u, v) \in U \times V$ introduce one basis element $\theta_{u,v}$ for a vector space[3] W_1 over k: W_1 is the span of these. Consider in W_1 the span W_2 of all the elements $\theta_{au+au',v} - a\theta_{u,v} - a'\theta_{u',v}$ and $\theta_{u,av+a'v'} - a\theta_{u,v} - a'\theta_{u,v'}$. The quotient W_1/W_2 is denoted $U \otimes_k V$ and is called the *tensor product* of U and V over k. The image of $\theta_{u,v}$ in the $U \otimes_k V$ is denoted $u \otimes v$. From the construction the elements $u \otimes v$ span $U \otimes_k V$ over k, but not every element need be of this form. There is a function $B : U \times V \to U \otimes_k V$ that sends (u, v) to $u \otimes v$ for all $u \in U$ and $v \in V$. It is immediate from the choice of elements in W_2 that this map is bilinear. It is a universal bilinear map in the following sense: if $B' : U \times V \to W$ is any bilinear map then there is a unique *linear map* $T : U \otimes_k V \to W$ such that $B' = T \circ B$. In fact, it is clear that the value of T on $u \otimes v$ must be $B'(u, v)$, from which uniqueness follows. Existence is also easy: define a linear map on W_1 by specifying its value on $\theta_{u,v}$ (these are a basis) to be $B'(u, v)$. It is straightforward to check that the specified elements spanning W_2 map to 0, and hence all of W_2 maps to 0. Thus, there is an induced linear map of the quotient W_1/W_2 to W that sends $u \otimes v \to B'(u, v)$ for all $u \in U$ and $v \in V$. Of course, W_1/W_2 *is* $U \otimes_k V$.

Exercises

(1) Let $B' : U \times V \to W$ have the same universal property that $B : U \times V \to U \otimes_k W$ has. Prove that there are unique linear maps $T' : W \to U \otimes_k V$ and $T : U \otimes_k V \to W$ such

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[1] All of our discussion will be valid, with only small changes, for modules over any commutative ring with identity. The changes needed for the case where U and V are modules over a commutative ring with identity will be indicated by footnotes. However, we shall not bother to indicate that the phrase "vector space" should change to the word "module."

[2] For this statement, U and V must be free over k, and the bases must be free bases. There is no restriction on W, however.

[3] Free module

that $T'(u, v) = B(u, v)$ and $T(u, v) = B'(u, v)$ for all $u \in U$ and $v \in V$. Show also that T and T' are inverses. Thus, the universal property determines the tensor product up to canonical isomorphism.

(2) Show that if $\{u_i\}_{i\in I}$ span U and $\{v_j\}_{j\in J}$ span V then the elements $u_i \otimes v_j$ span $U \otimes_k V$. Hence, if U and V are finite-dimensional[4] then so is $U \otimes_k V$.

(3) Show that $U \otimes_k V \cong V \otimes_k U$: in fact there is a unique isomorphism sending every $u \otimes v$ to $v \otimes u$.

(4) Show that $(U \oplus_k U') \otimes_k V \cong (U \otimes_k V) \oplus (U' \otimes_k V)$. (Remember, linear maps from tensor products are easily constructed by giving a bilinear map.) This generalizes by induction to finite direct sums, and it is even straightforward to generalize to infinite direct sums. One has a similar fact when V is a direct sum. Thus, $(\bigoplus_i U_i) \otimes_k (\bigoplus_j V_j) \cong \bigoplus_{i,j} U_i \otimes_k V_j$.

(5) The map $k \otimes_k k \to k$ that sends (a, a') to aa' is a universal bilinear map. Thus, $k \otimes_k k \cong k$.

(6) Using (4) and (5), if U has basis[5] $\{u_i\}_{i\in I}$ and V has basis[6] $\{v_j\}_{j\in J}$ then $U \otimes_k V$ has basis[7] $\{u_i \otimes v_j : i \in I, j \in J\}$. In particular, if U, V are finite-dimensional, then the dimension of $U \otimes_k V$ is the product of the dimensions of U and V.

(7) If $S : U \to U'$ and $T : V \to V'$ are linear, there is a unique linear map, denoted $S \otimes_k T$, from $U \otimes_k V \to U' \otimes_k V'$ that sends $u \otimes v$ to $S(u) \otimes T(v)$. This operation behaves well under composition.

(8) If $N \subseteq U$, then $(U/N) \otimes_k V \cong$ the quotient of $U \otimes_k V$ by the image of $N \otimes_k V$. In the field[8] case, N will have a vector space complement N' in U such that $U = N \oplus_k N'$, and it follows that $U \otimes_k V \cong (N \otimes_k V) \oplus_k (N' \otimes_k V)$. Thus, $N \otimes_k V$ injects into $U \otimes_k V$.

(9) More generally, if $N \subseteq U$ and $Q \in V$ then:

$$(U/N) \otimes_k (V/Q) \cong (U \otimes_k V)/\big(\mathrm{Im}\,(U \otimes_k Q) + \mathrm{Im}\,(N \otimes_k W)\big)$$

(10) Show that $(U_1 \otimes_k U_2) \otimes_k U_3 \cong U_1 \otimes_k (U_2 \otimes_k U_3)$ in a unique way such that $(u_1 \otimes u_2) \otimes u_3$ corresponds to $u_1 \otimes (u_2 \otimes u_3)$. This property is referred to as the *associativity* of tensor product. Thus, one frequently writes $U_1 \otimes_k U_2 \otimes_k U_3$ for either of these, making tacit use of the canonical identification, and, similarly, one may use the notation $u_1 \otimes u_2 \otimes u_3$. This extends easily to n-fold tensor products , and so $U_1 \otimes_k \cdots \otimes_k U_n$ has a meaning that is independent, up to canonical isomorphism, of how one inserts parentheses.

Given n vector spaces $U_1, \ldots, U_n$ by an n-*linear* (or *multilinear*) map $U_1 \times \cdots \times U_n \to W$ we mean a map that is linear in each coordinate when all others are held fixed. Note that there is a map $U_1 \times \cdots \times U_n \to U_1 \otimes_k \cdots \otimes_k U_n$ that sends $(u_1, \ldots, u_n)$ to $u_1 \otimes \cdots \otimes u_n$. This is a universal n-linear map, in the sense that there is a bijection of between n-linear

[4]Finitely generated over k

[5]Free basis

[6]Free basis

[7]Is free and has free basis

[8]When k is not a field, this injectivity often fails. First note that if I is an ideal of k then the image of $I \otimes V$ in V is IV (the span of the elements av for $a \in I$ and $v \in V$). It follows that $(k/I) \otimes_k V \cong V/IV$. Consider what happens with $2\mathbb{Z} \subseteq \mathbb{Z}$ when we tensor with $\mathbb{Z}/2\mathbb{Z}$ over $\mathbb{Z}$. The conclusion of injectivity does hold when U/N is a free k-module (and one can get somewhat more general results along these lines).

maps $U_1 \times \cdots \times U_n \to W$ and linear maps $U_1 \otimes_k \cdots \otimes_k U_n \to W$: we leave the details as an exercise.

An $\mathbb{N}$-graded ring is a ring R which can be written as a direct sum $\bigoplus_{n=0}^{\infty} R_n$ in such a way that for all $m,\ n \in \mathbb{N}$, $R_m R_n \subseteq R_{m+n}$. The elements of R_n are said to be *homogeneous of degree* n.

The tensor product of n copies of V with itself is denoted $V^{\otimes n}$ or $T_k^n(V) = T^n(V)$. By convention, $T^0(V) = k$. Then $T(V) = \oplus_n T^n(V)$ becomes an associative (usually non-commutative) $\mathbb{N}$-graded ring with identity, with k in the center: the multiplication is induced by the obvious bilinear maps $T^m(V) \otimes_k T^n(v) \to T^{m+n}(V)$ (each of these maps is an isomorphism). Note that this *tensor algebra* is generated as a ring over k by $T^1(V)$, which we may identify with V. Of course, $T^n(V)$ is the degree n component. Note that if $L : V \to W$ is a linear map, there is an induced map $T^n(L) : T^n(V) \to T^n(W)$, and $T^n(L' \circ L) = T^n(L') \circ T^n(L)$ when the composition $L' \circ L$ is defined. Together these maps give a degree preserving ring homomorphism $T(V) \to T(W)$.

A multilinear map $V^n \to W$ is called *symmetric* if its value does not change when two entries of an n-tuple are interchanged. (Its value is therefore the same no matter how the entries are permuted.) Let $S^n(V)$ denote the quotient of $V^{\otimes n}$ by the subspace I_n spanned by all differences of n-tuples whose entries are the same, up to permutation. Giving a symmetric n-linear map from V^n to W corresponds bijectively with giving a linear map from $S_k^n(V) = S^n(V)$ to W. $S^n(V)$ is called the n th *symmetric power* of V over k. What is more, $S(V) = \bigoplus_n S^n(V)$ is an $\mathbb{N}$-graded algebra, called the *symmetric algebra* of V over k: in fact, it may be viewed as the quotient of $T^n(V)$ by the ideal generated by commutators. If $\{v_i\}_{i \in I}$ is a basis[9] for V, then the symmetric algebra is isomorphic with the commutative polynomial ring over k in variables corresponding bijectively with the v_i. We may identify V with $S^1(V)$. Note that if $L : V \to W$ is a linear map, for all n there is an induced map $S^n(L) : S^n(v) \to S^n(W)$, and $S^n(L' \circ L) = S^n(L') \circ S^n(L)$ when the composition $L' \circ L$ is defined. Together these maps give a degree preserving ring homomorphism from $S(V) \to S(W)$.

A multilinear map $V^n \to W$ is called *alternate* or *alternating* if its value is 0 whenever two entries of an n-tuple are equal. (This implies that switching two entries negates the value. Making an even permutation of the entries will not change the value, while an odd permutation negates the value.) Let $\bigwedge_k^n(v) = \bigwedge^n(v)$ denote the quotient of $V^{\otimes n}$ by the subspace spanned by all n-tuples two of whose entries are equal. We make the convention that $\bigwedge^0 k \cong k$, and note that we may identify $V \cong \bigwedge^1 V$. Then $\bigwedge V = \bigoplus_n \bigwedge V$ is an associative $\mathbb{N}$-graded algebra with k in the center, with $\bigwedge^n(V)$ as the component in degree n. $\bigwedge(V)$ is called the *exterior algebra* of V over k, and $\bigwedge^n(V)$ is called the n th exterior power of V over k. The multiplication on $\bigwedge(V)$ is often denoted $\wedge$. If the vectors v_i span V, then the elements $v_{i_1} \wedge \cdots \wedge v_{i_n}$ span $\bigwedge^n(V)$. If α has degree m and β has degree n, then $\alpha \wedge \beta = (-1)^{mn} \beta \wedge \alpha$. Thus, the even degree elements are all in the center, while any two odd degree elements anti-commute. If $v_1, \ldots, v_n$ is a basis[10]for V, then the elements

[9]Free basis

[10]Free basis

$v_{i_1} \wedge \cdots v_{i_k}$, $1 \le i_1 < \cdots < i_k \le n$ form a basis[11] for $\bigwedge^k(V)$, and $\bigwedge^k(V)$ has dimension $\binom{n}{k}$. In particular, $\bigwedge^N(V) = 0$ if $N > \dim V$ (or, more generally, if V is spanned by fewer than N elements).

Given a linear map $L : V \to W$, there is an induced map $S^n(L) : \bigwedge^n(v) \to \bigwedge^n(W)$, and $\bigwedge^n(L' \circ L) = \bigwedge^n(L) \circ \bigwedge^n(L')$ when the composition $L' \circ L$ is defined. Together these maps give a ring homomorphism of $\bigwedge(V) \to \bigwedge(W)$ that preserves degrees.

If V has dimension n with basis $v_1, \ldots, v_n$ and $L : V \to V$ has matrix A, then $\bigwedge^n(L) : V \to V$ sends $v_1 \wedge \cdots \wedge v_n$ to $\det(A) v_1 \wedge \cdots \wedge v_n$. (We have that

$$\bigwedge^n(v_1 \wedge \cdots \wedge v_n) = (a_{11}v_1 + \cdots + a_{n1}v_n) \wedge \cdots \wedge (a_{n1}v_1 + \cdots + a_{nn}v_n).$$

Expanding by the generalized distributive law yields n^n terms each of which has the form $a_{i_1,1} \cdots a_{i_n,n} v_{i_1} \wedge \cdots \wedge v_{i_n}$. If two of the i_t are equal, this term is 0. If they are all distinct, the v_{i_t} constitute all the elements $v_1, \ldots, v_n$ in some order: call the corresponding permutation σ. Rearranging the v_j gives $\operatorname{sgn}(\sigma) a_{i_1,1} \cdots a_{i_n,n} v_1 \wedge \cdots \wedge v_n$. The sum of all of the $n!$ surviving terms is $\det(A) v_1 \wedge \cdots \wedge v_n$, using one of the standard definitions of $\det(A)$. The fact that the determinant of a product of two $n \times n$ matrices is the product of the determinants may be deduced from the fact that $\bigwedge^n$ preserves composition.

Problems new this week

Several of these problems deal with earlier topics.

1. Let V and W be finite-dimensional vector spaces over a field k. Construct a canonical isomorphism $\mathrm{End}_k(V) \otimes_k \mathrm{End}_k(W) \cong \mathrm{End}_k(V \otimes_k W)$. You should construct the proposed isomorphism without choosing bases, but you may wish to uses bases in the course of proving that the map you have defined is in fact an isomorphism.

2. Let V be an n-dimensional vector space and let $\Lambda = \bigwedge V = \bigoplus_{r=0}^n \bigwedge^r V$ be its exterior algebra. Show that the elements of Λ of even degree form a commutative subalgebra.

3. Let V be the vector space of 3×1 complex column vectors. Consider the transformation of V defined by the matrix $\begin{pmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 3 & -1 & 0 \end{pmatrix}$ with respect to the standard basis e_1, e_2, e_3 (these are the columns of the identity matrix).

(a) Find the matrix of the induced linear transformation on $\bigwedge^2 V$ with respect to the basis $e_1 \wedge e_2$, $e_1 \wedge e_3$, $e_1 \wedge e_2$.

(b) Let V be a complex vector space of finite dimension n, and let $T : V \to V$ be a linear transformation of V. For a natural number d, $1 \le d \le n$, consider the naturally induced transformation $\bigwedge^d T : \bigwedge^d V \to \bigwedge^d V$. Prove that if T has rank r, then $\bigwedge^d T$ has rank $\binom{r}{d}$, where this binomial coefficient is interpreted as 0 if $d > r$.

4. Let V be an n-dimensional real vector space and f a nonzero element of V. If $h \in V \otimes_{\mathbb{R}} V$, prove that there exist elements $f_1 = f, f_2, \ldots, f_n, g_1, \ldots, g_n \in V$ such that $h = f_1 \otimes g_1 + \cdots + f_n \otimes g_n$.

[11] Free basis

Prove also that there exists $h \in V \otimes_{\mathbb{R}} V$ such that $h \neq f_1 \otimes g_1 + \cdots f_{n-1} \otimes g_{n-1}$ for any choice of $f_i, g_j \in V$.

5. Let V and W be real vector spaces of finite dimension m and n, respectively.

(a) What is the dimension of the vector space $V \otimes_{\mathbb{R}} W$? (No proof needed.)

(b) What is the dimension of the vector space $\bigwedge^k V$ for every integer k? (No proof needed.)

(c) Consider $Y = \bigoplus_{k=0}^{\infty} \bigwedge^k (V \otimes_{\mathbb{R}} W)$ and $Z = \left(\bigoplus_{i=0}^{\infty} \bigwedge^i (V)\right) \otimes_{\mathbb{R}} \left(\bigoplus_{j=0}^{\infty} \bigwedge^j (W)\right)$. (The direct sums are actually finite, i.e., all but finitely many terms are 0.) Find the dimensions of the vector spaces Y and Z, and determine all values of m, n such that Y and Z have the same dimension. (It may be useful that $2^N = (1+1)^N$.)

6. Let $f(x)$ and $g(x)$ be monic polynomials in the variable x over a field K, of degrees m and n, respectively, where m, n are positive integers. Let R be the quotient ring $K[x]/(f(x))$ and let S be the quotient ring $K[x]/(g(x))$.

(a) What is the condition on $f(x)$ for R to be a field? What is the condition on $f(x)$ for R to be an integral domain?

(b) Express the K-vector space dimensions of the following K-vector spaces in terms of m and n (and i, j in part (4)): (1) R; (2) S; (4) $R \otimes_K S$; (4) $\bigwedge^i R \otimes_K \bigwedge^j S$

7. Let F be the splitting field of an equation of degree 6 over the rationals, thought of as a subfield of the complex numbers $\mathbb{C}$.

(a) Must the Galois group of F over $\mathbb{Q}$ be embeddable in S_6? Explain your answer.

(b) Can F have a subfield of degree 25 over $\mathbb{Q}$? Explain.

(c) Let L be the field obtained by adjoining the real root of the equation $x^7 - 2 = 0$ to $\mathbb{Q}$. Describe, with proof, all subfields of L. Can $L \cap F$ be larger than $\mathbb{Q}$? Explain.

(d) Suppose that F has a strictly smaller subfield F_0 that has dimension precisely 360 as a $\mathbb{Q}$-vector space. How many automorphisms of F over $\mathbb{Q}$ fix all elements of F_0? Explain.

8. (a) Let $(V, +)$ and $(W, +)$ be $\mathbb{Q}$-vector spaces and suppose that f is an isomorphism of V with W *as abelian groups*. Prove that f must be an isomorphism of $\mathbb{Q}$-vector spaces.

(b) An abelian group $(G, +)$ is called *divisible* if for every positive integer n, every element of G has the form ng for some $g \in G$. Prove that a finite divisible group must be zero.

9. What are all the possible Jordan forms for a complex 4×4 matrix A satisfying $A^3 = A$? Explain your answer.

10. Consider a polynomial $f(x)$ with rational coefficients. Let G be the Galois group of the splitting field of f over $\mathbb{Q}$. In each case below, discuss whether or not f is solvable by radicals.

(a) G is a simple non-abelian group.

(b) G is abelian.

(c) G is isomorphic to a semi-direct product of two nilpotent groups.

(d) G contains A_5.