VIGRE Algebra QR Review, November 25, 2002

Problems new this week

These problems deal with a mixture of topics.

1. Prove that an $n \times n$ matrix A over the real number numbers $\mathbb{R}$ has rank $\leq r$, where $1 \leq r \leq n$ if and only if A is the sum of r or fewer $n \times n$ matrices of rank at most 1.

2. Let B_1 and B_2 be real symmetric bilinear forms on finite-dimensional real vector spaces V_1 and V_2 respectively. Show that there is a bilinear form B on $V_1 \otimes_{\mathbb{R}} V_2$ such that $B(u_1 \otimes u_2, v_1 \otimes v_2) = B_1(u_1, v_1)B_2(u_2, v_2)$. Show that B is symmetric. Express the rank of B in terms of the ranks of B_1 and B_2, and the signature (p, q, n) of B in terms of the signatures (p_1, q_1, n_1) and (p_2, q_2, n_2) of B_1 and B_2.

3. Let G be the set of $n \times n$ matrices over the complex numbers obtained from the $n \times n$ identity matrix by a permutation of columns.

(a) Prove that G is a subgroup of $GL(n)$ isomorphic to S_n.

Let G act on the set M_n of $n \times n$ complex matrices by left multiplication.

(b) Let $n = 3$. What is the cardinality of the orbit of the 3×3 matrix $A = (a_{ij})$ such that $a_{11} = a_{21} = a_{32} = a_{33} = 1$ and all other entries are 0? What is the cardinality of the stabilizer of A in G?

(c) Give a necessary and sufficient condition for the orbit of a matrix B under G to have cardinality $n!$. Is this condition always satisfied when B is invertible? Can it ever be satisfied if one of the rows of the matrix is 0?

4. Find a primitive element (field generator) for $\mathbb{Q}(\sqrt{2}, \sqrt[3]{5})$. Show that this field is not a Galois extension of $\mathbb{Q}$, and find its Galois closure.

5. Fix positive integers $r \leq n$ and a complex polynomial $f(x)$ of degree r. Prove that there are, up to similarity, only finitely many complex $n \times n$ matrices that have minimal polynomial f.

Give an example of a finite-dimensional complex vector space V with infinitely many linear transformations having the same minimal polynomial.

6. Find the Galois group of the splitting field of the polynomial $x^5 - 7x^4 + 3$ over the rational numbers $\mathbb{Q}$.

7. (a) Prove that every group of order $77 = 7 \cdot 11$ is abelian.

(b) Let G be a group of order $231 = 3 \cdot 7 \cdot 11$. Prove that any subgroup H of G of order 11 is contained in the center of G.

8. Let $R = k[x]$ be the polynomial ring in one variable over a field k. Recall that a proper ideal I of R is *prime* if whenever $f, g \in R$ and $fg \in I$, either $f \in I$ or $g \in I$.

(a) Show that the set of nonzero prime ideals of R is in bijective correspondence with the set of irreducible monic polynomials in R.

(b) Find the prime ideals that contain both $x^5 - x^3$ and $x^4 - 1$. Describe these prime ideals by giving generators for each.

9. Let G and H be non-abelian simple groups. Prove that every non-trivial normal subgroup N of $G \times H$ must contain G or contain H.

10. Let k be a field and let $k[x]$ be the polynomial algebra in one variable over k. Let D be the k-subalgebra of the k-algebra $\mathrm{End}_k k[x]$ of all k-vector space endomorphisms of $k[x]$ generated by the "derivative operator" d (the k-linear map defined by $d(x^n) = nx^{n-1}$ for all non-negative integers n) and the operator $\mathbf{x}$ (the linear operator defined by $\mathbf{x}(f) = xf$ for all polynomials f). Note that there is a natural D-module structure on $k[x]$. (Recall that a module is *simple* if it has no proper non-trivial submodules.)

(a) Is D a commutative ring? Prove it or give a counterexample.

(b) Prove that if k has characteristic zero, then $k[x]$ is a simple D-module.

(c) Is $k[x]$ a simple D-module if the characteristic of k is non-zero? Prove it or give a counterexample.