BIG COHEN-MACAULAY ALGEBRAS IN DIMENSION THREE VIA HEITMANN'S THEOREM

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1. INTRODUCTION

Recently, in a tremendous breakthrough, R. Heitmann [Heit3] has shown that the direct summand conjecture holds for regular rings of dimension at most three. This was previously known in all dimensions for rings containing a field: what is new is that the result is valid even in mixed characteristic. Here, we shall use the key result in Heitmann's paper to prove the existence of balanced big Cohen-Macaulay algebras in dimension three in a weakly functorial form. For the background of the direct summand conjecture and related homological conjectures and splitting questions we refer the reader to [Ho1–7], [PS1,2], [Ro1–6], [Du1,2], [Rang], [DHM], and [EvG].

Recall that B is called a *balanced big Cohen-Macaulay algebra* for the local ring (R, m) if $mB \neq B$ and every system of parameters for R is a regular sequence on B. Cf. [Sh], [HH5], and [HH9]. Using one of Heitmann's theorems together with ideas from [HH9] and [Ho9], we prove the stronger result that every local ring of dimension at most three has a balanced big Cohen-Macaulay algebra. In fact, we prove these exist in a weakly functorial form. In the equal characteristic case, this result follows from the main results of [HH5] and [HH9].

Our main result, in the simplest case, may be stated as follows.

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Theorem 1.1. *Let R be a local ring of equal characteristic or a local ring of dimension at most 3. Then R has a balanced big Cohen-Macaulay algebra. Let $\phi : R \to S$ be a local homomorphism of complete local domains. If R and S have equal characteristic, or if R and S have characteristic 0 and residual characteristic $p > 0$ and both R and S have dimension at most 3, then there is a commutative diagram*

$$\begin{array}{ccc} B & \longrightarrow & C \\ \uparrow & & \uparrow \\ R & \xrightarrow{\phi} & S \end{array}$$

where B and C are balanced big Cohen-Macaulay algebras for R and S respectively.

Since the result is already known in equal characteristic, we henceforth focus on the mixed characteristic case.

2. HEITMANN'S THEOREM

We restate for purposes of reference one of the main results of [Heit3] (where the result is proved for a more general class of 3-dimensional rings R). We note that there is also an exposition of Heitmann's theorem in [Ro7].

Theorem 2.1 (Heitmann). *Let R be a complete local domain of mixed characteristic $p > 0$, i.e., the residue field of R has prime characteristic p while the fraction field of R has characteristic 0. Suppose that the dimension of R is 3. Let p, x, y be a system of parameters for R and N a positive integer such that p^N, x, y kill the Koszul homology modules $H_1(p^m, x^m, y^m; R)$ for every positive integer m. Also assume that there is an element $\theta \in R$ such that $\theta^{p-1} = p$. Suppose that $p^N u \in (x, y)R$. Then for every positive rational number ϵ, there is a module-finite extension domain T of R with $p^\epsilon u \in (x, y)T$.*

Discussion 2.2. Let R be a ring, and let I be a finitely generated ideal of R, with generators $u_1, \ldots, u_h$. We shall usually take the point of view that identifies the local cohomology $H^\bullet_I(T)$, where T is an R-algebra, with a direct limit, over t, of the Koszul cohomology $H^\bullet(u_1^t, \ldots, u_h^t; R)$. (For the very basic facts about local cohomology used here, including its standard identification with a direct limit of Koszul cohomology, we refer the reader to [GH]. In particular, see Theorem 2.3 on p. 20. There is also an exposition of

relevant material in [HR], §5, pp. 132–5.) It is worth noting for those not familiar with local cohomology that if J is another finitely generated ideal with the same radical as I, then $H^\bullet_J(R)$ and $H^\bullet_I(R)$ may be identified canonically. In particular, in dealing with a complete local domain (R, m), if I is any ideal generated by a system of parameters and j is any integer, we may identify $H^j_I(R)$ with $H^j_m(R)$, and, likewise, we may identify $H^j_I(R^+)$ and $H^j_m(R^+)$ (R^+ is discussed immediately following item (b) of the sequel). We next want to make a detailed examination of what happens for ideals with three generators.

Let u, v, w be three elements of R,and let $I = (u, v, w)R$. Suppose that u, v is a regular sequence in R. We want to make some observations about the Koszul (co)homology modules $H_1(u, v, w; R) \cong H^2(u, v, w; R)$. This module has the form M/N where $M = \{(r_1, r_2, r_3) \in R^3 \,|\, r_1u+r_2v+r_3w \ = 0\}$ and N is the R-span of $(0, -w, v)$, $(w, 0, -u)$ and $(-v, u, 0)$. Projection on the third coordinate maps M onto $\big((u, v)R :_R w\big)$ while mapping N onto (u, v), and so yields a map $\pi : H_1(u, v, w; R) \twoheadrightarrow \big((u, v) :_R w\big)/(u, v)R$. We have this surjection regardless of whether u, v is a regular sequence or not, and, likewise, we have surjections

$$\pi_t : H_1(u^t, v^t, w^t; R) \twoheadrightarrow \big((u^t, v^t) :_R w^t\big)/(u^t, v^t)R$$

for every $t > 0$.

But in the regular sequence case we can say more:

(a) *If u, v is a regular sequence, the maps*

$$\pi_t : H_1(u^t, v^t, w^t; R) \twoheadrightarrow \big((u^t, v^t) :_R w^t\big)/(u^t, v^t)R$$

are isomorphisms for all t.

Quite generally, $H^2_I(R) \cong \varinjlim_t H^2(u^t, v^t, w^t; R)$ where the maps send an element represented by (r_1, r_2, r_3) to (vwr_1, uwr_2, uvr_3) (cf. Thm. 2.3 of [GH]). When u, v is a regular sequence, we may identify $H^2(u^t, v^t, w^t; R) \cong \big((u^t, v^t) :_R w^t\big)/(u^t, v^t)$, and the map to the next term in the direct limit system sends the element represented by r to uvr. When u, v is a regular sequence, this map is injective: it suffices to see that if $uvr \in (u^{t+1}, v^{t+1})$ then $r \in (u^t, v^t)$, which follows very easily from the fact that u, v is a regular sequence. Thus:

(b) *If u, v is a regular sequence in R, then $H^2_I(R)$ may be viewed as an increasing union of the modules*

$$H^2(u^t, v^t, w^t; R) \cong H_1(u^t, v^t, w^t; R) \cong \big((u^t, v^t) :_R w^t\big)/(u^t, v^t)R.$$

In particular, each $H_1(u^t, v^t, w^t; R)$ *injects into* $H^2_I(R)$.

We recall (cf. [Ar]) that the integral closure R^+ of a domain R in an algebraic closure of its fraction field is called an *absolute integral closure* of R. It is unique up to non-unique isomorphism. If R is a complete local domain then R^+ is a directed union of module-finite extensions of R, each of which is a complete local domain, and so is a quasi-local domain (we reserve the term *local* for Noetherian quasi-local rings). Since each ring in the directed union can be replaced by its normalization, we may view R^+ as a directed union of complete normal local domains, each of which is module-finite over R. If R is a complete local domain and $S \subseteq S'$ are module-finite domain extensions, any system of parameters in S is also a system of parameters in S', and we shall also refer to it as a system of parameters of R^+. Equivalently, if $d = \dim R$ $(= \dim R^+)$, then $x_1, \ldots, x_d$ is a system of parameters for R^+ if and only if every element of the maximal ideal of R^+ is nilpotent modulo $(x_1, \ldots, x_d)R^+$.

Note that when R is a complete local domain of mixed characteristic p, if $\epsilon = a/b$ is a positive rational number, so that $a, b > 0$ are positive integers, then R^+ contains roots of $Z^b = p^a$, which we denote, somewhat ambiguously, as p^ϵ. They are well-defined up to multiplication by units of R^+, however. For any fixed choice of ϵ, R^+ is a directed union of normal complete local domains containing all choices of p^ϵ. The statement that a certain family of p^ϵ kills an R^+-module is independent of how we choose the specific values of the various p^ϵ.

The following theorem is an easy consequence of Heitmann's theorem. We note that Heitmann's theorem, in a rough heuristic sense, asserts that the situation for "colon-capturing" in mixed characteristic is a sort of combination of the positive characteristic situation (where R^+ is a balanced big Cohen-Macaulay algebra for complete local domains, and somewhat more generally — see [HH5]) and a situation reminiscent of positive characteristic tight closure theory, but with p itself, somewhat surprisingly, playing the role of a test element. For more about tight closure we refer the reader to [Bru], [Ho10], [HH1-4, 6-12], [Hu], [Sm], and for discussion of other closure operations to [Heit1-3], [Ho8-9], [HoV], [Ho11] and [Bre].

Theorem 2.3. *Let* R *be a complete local domain of dimension at most* 3 *and mixed characteristic* $p > 0$. *If* R *has dimension at most two then* R^+ *is a balanced big Cohen-*

Macaulay algebra for R, and every system of parameters in R^+ is a regular sequence. If $\dim R = 3$, then for every system of parameters x, y, z of R^+ and for every rational number $\epsilon > 0$, if $zu \in (x, y)R^+$, then $p^\epsilon u \in (x, y)R^+$. In particular, for all $\epsilon > 0$, p^ϵ kills $H^2_m(R^+)$, while the lower local cohomology modules are 0.

Proof. Because R^+ is a directed limit of normal rings containing a given system of parameters, any two elements of a system of parameters in R^+ form a regular sequence. Thus, from here on we may assume that $\dim R = 3$.

We first note that the final statement about the local cohomology of R^+ is equivalent to the assertion that for all positive rational ϵ and for every system of parameters x, y, z in R^+, p^ϵ kills $\big((x, y)R^+ :_{R^+} z\big)$, i.e., whenever $uz \in (x, y)R^+$, for $u \in R^+$, then $p^\epsilon u \in (x, y)R^+$. This follows from the discussion in (2.2): for any system of parameters x, y, z, if $I = (x, y, z)R^+$, then $H^2_m(R^+) \cong H^2_I(R^+)$ is the increasing union of submodules each of which is isomorphic to $\big((x^t, y^t)R^+ :_{R^+} z^t\big)/(x^t, y^t)R^+$. Thus, every module of the form $\big((x, y)R^+ :_{R^+} z\big)/(x, y)R^+$ injects into $H^2_m(R^+)$, and $H^2_m(R^+)$ is a union of such submodules.

Now suppose that we have a given element of $H^2_m(R^+)$ and we want to prove that it is killed by p^ϵ. Because R^+ is the directed union of the module-finite extensions R_1 of R, we may assume that the given element is in the image of the map $H^2_m(R_1) \to H^2_m(R^+)$ for a sufficiently large choice of R_1. Moreover, we may replace R_1 by a larger module-finite extension, and therefore there is no loss of generality in assuming that we have $\theta \in R_1$ such that $\theta^{p-1} = p$, and that R_1 is normal. Since R^+ is the directed union of rings with these properties, it will suffice to show that for such choice of R_1 (normal and containing θ as above), the image of $H^2_m(R_1)$ in $H^2_m(R^+)$ is killed by p^ϵ for all $\epsilon > 0$. To simplify notation we drop the subscript and write R instead of R_1: R is fixed in the rest of the argument. Since R is normal, R is Cohen-Macaulay when localized at any prime except the maximal ideal, and so $H^2_m(R)$ has finite length.[1] Thus, we may choose a system of parameters p^N, x, y that kills $H^2_m(R)$, and it follows that these kill $H_1(u, v, w; R)$ for every system of parameters $u, v, w \in R$ by (2.2b). Thus, the hypotheses of Theorem (2.1) are satisfied, and they remain satisfied when we replace p^N, x, and y by positive integer powers. Let $J = (p^N, x, y)$. We view $H^2_m(R) = H^2_J(R)$ as the directed union

[1]By local duality [GH], Thm. 6.3, p. 85, if $R = S/Q$ with S regular local, $\dim S = d$, $H^2_m(R)$ is the Matlis dual of $\mathrm{Ext}^{d-2}_S(S, R)$: since R_P is Cohen-Macaulay for $P \neq m$, this is supported only at m.

of the modules $\big((x^m, y^m) :_R p^{Nm}\big)/(x^m, y^m)R$. Suppose that $w \in \big((x^m, y^m) :_R p^{Nm}\big)$, i.e., $p^{Nm}w \in (x^m, y^m)R$. Then we may apply Theorem (2.1) to p^{Nm}, x^m, y^m, and so $p^\epsilon w \in (x^m, y^m)R^+$ for all positive rational numbers ϵ, which shows that p^ϵ kills the image of the element of $H^2_m(R)$ represented by w in $H^2_m(R^+)$. This completes the proof. □

We conclude this section with the following observation, which we shall need in §5.

Fact 2.4. *If (S, n) is a complete local Noetherin domain, then S^+ is n-adically separated.*

To, see this, suppose that v were a nonzero element in every power of n. Choose a prime Q ideal of S not containing v such that $\dim S/Q = 1$. Then we may replace S by S/Q and S^+ by its quotient by a prime lying over Q. Therefore, we may assume without loss of generality that S has dimension one, and we may replace it by a larger normal ring that contains v. Then S is a DVR, and its ideals, which are principal, are contracted from S^+. Thus, v is in $n^N S^+$ iff it is in n^N, and the result follows. □

3. MODIFICATIONS AND BIG COHEN-MACAULAY ALGEBRAS

Definition 3.1: algebra modifications. Let T be an algebra over a local ring (R, m). Suppose that k is an integer, $0 \leq k < \dim R$, that $x_1, \ldots, x_{k+1} \in m$ is part of a system of parameters for R, and also suppose that $t_1, \ldots, t_{k+1}$ are elements of T satisfying $x_{k+1}t_{k+1} = \sum_{i=1}^k x_i t_i$. Let $X_1, \ldots, X_k$ be indeterminates over T. Then the T-algebra $T' = T[X_1, \ldots, X_k]/(F)$, where $F = \big(t_{k+1} - \sum_{i=1}^k x_i X_i\big)$, is called an *algebra modification* of T with respect to R. By a *sequence of algebra modifications* of T over R we mean a sequence of R-algebras $T_0 = T, T_1, \ldots, T_r$ such that, for $0 \leq j < r$, T_{j+1} is an algebra modification of T_j with respect R. At every stage the choices of k and of $x_1, \ldots, x_{k+1}$ may change.

Such a sequence of algebra modifications over R is called *bad* if the image of $1 \in T$ is in mT_r. It is shown in §3 of [HH9] that T can be mapped to a balanced big Cohen-Macaulay algebra for R if and only if T does not possess a bad sequence of modifications. (One then constructs the balanced big Cohen-Macaulay algebra as a rather large direct limit of various rings T_r constructed from finite sequences of modifications.)

Now suppose that we have a local map of local rings $(R, m) \to (S, n)$ and that we start with an R-algebra T. We may take a sequence of modifications of T over R: call the last term T_r. We may then take a sequence of algebra modifications of $S \otimes T_r$ over S: call the last term U_s. Evidently, in this situation, we have a commutative diagram:

$$\begin{array}{ccc} T_r & \longrightarrow & U_s \\ \uparrow & & \uparrow \\ R & \longrightarrow & S \end{array}$$

We shall refer to the sequence $T = T_0, T_1, \ldots, T_r, U_0 = S \otimes_R T_r, U_1, \ldots, U_s$ as a *double sequence of algebra modifications* over $R \to S$. We shall say that this sequence is *bad* if the image of $1 \in T$ in U_s is in nU_s. Again, by the results of §3 of [HH9], there exists a commutative diagram:

$$\begin{array}{ccc} B & \longrightarrow & C \\ \uparrow & & \uparrow \\ R & \longrightarrow & S \end{array}$$

such that $R \to B$ factors $R \to T \to B$, B is a balanced big Cohen-Macaulay algebra for R, and C is a balanced big Cohen-Macaulay algebra for S if and only if there is no bad double sequence of algebra modifications of T over $R \to S$.

The idea of the proof of the "if" part as follows: one constructs B as a direct limit of finite sequences of modifications of T over R, and then one constructs C as a direct limit of finite sequences of modifications of $S \otimes_R B$ over S. The resulting algebra can be seen to be a direct limit of algebras U_s coming from double sequences of algebra modifications of T.

In particular, there is a commutative diagram

$$\begin{array}{ccc} B & \longrightarrow & C \\ \uparrow & & \uparrow \\ R & \longrightarrow & S \end{array}$$

where B is a balanced big Cohen-Macaulay algebra over R and C is a balanced big Cohen-Macaulay algebra over S if and only if there is no bad double sequence of algebra modifications over $R \to S$.

4. PARTIAL MODIFICATIONS

We want to refine the obstruction to constructing commutative diagrams:

$$\begin{array}{ccc} B & \longrightarrow & C \\ \uparrow & & \uparrow \\ R & \longrightarrow & S \end{array}$$

in which B and C are balanced big Cohen-Macaulay algebras over R and S respectively so as to make it even more "finite" in nature, utilizing an idea that is implicit in the proof of Theorem 11.1 of [Ho9], pp. 153–7. Let $\mathbb{N}$ denote the set of nonnegative integers. If M is any R-module and $X_1, \ldots, X_k$ are indeterminates over R, we let $M[X] = M[X_1, \ldots, X_k]$ denote $M \otimes_R R[X_1, \ldots, X_k]$. We write $M[X]_{\leq N}$ for the submodule spanned by the elements $uX_1^{a_1} \cdots X_k^{a_k}$ such that $u \in M$ and the $a_j \in \mathbb{N}$ with $\sum_j a_j \leq N$. In particular, we may use this notation when $M = R$.

4.1 Partial algebra modifications. We begin with an R-module M. We define a *partial algebra modification of* M to be a map $M \to M'$ where M' is an R-module obtained as follows: for some integer $k \geq 0$ and $x_1, \ldots, x_{k+1}$ that are part of a system of parameters for R and relation $x_{k+1}u_{k+1} = \sum_{j=1}^{k} x_i u_i$, where the $u_i \in M$, choose indeterminates $X_1, \ldots, X_k$ and an integer $N \geq 1$, let $F = \left(u_{k+1} - \sum_{i=1}^{k} x_i X_i\right)$, and let

$$M' = M[X_1, \ldots, X_k]_{\leq N}/FR[X_1, \ldots, X_k]_{\leq N-1},$$

which makes sense because F has degree 1 in the X_j. We shall refer to the integer N as the *degree bound* of the partial algebra modification. Note that if T is an R-algebra and one takes the direct limit over N of the T' for fixed k, $x_1, \ldots, x_{k+1}$, $X_1, \ldots, X_k$, and F, one obtains an algebra modification of T. We can define a sequence of partial algebra modifications of an R-module T as in §3, and, when T is an R-algebra, we call the sequence *bad* precisely if the image of $1 \in T$ in M_r is in mM_r.

Given a local map of local rings $R \to S$ we can now define a *double sequence of partial algebra modifications* of an R-module M with respect to $R \to S$ as follows: we first form

a sequence of partial algebra modifications of M over R, say $M = M_0, M_1, \ldots, M_r$, and then a sequence of partial algebra modifications $N_0 = S \otimes_R M_r$, $N_1, \ldots, N_s$ of N_0 over S. When M is an R-algebra T, the sequence is defined to be *bad* precisely if the image of $1 \in T$ in N_s is in nN_s.

It is quite easy to see that if T is any R-algebra then any S-algebra obtained from T by a double sequence of algebra modifications over $R \to S$ is a direct limit of modules that arise from a double sequence of partial algebra modifications over $R \to S$. In consequence, we have:

Theorem 4.2. *Let $(R, m) \to (S, n)$ be a local homomorphism of local rings, and let T be an R-algebra. Then there exists a commutative diagram:*

$$\begin{array}{ccc} B & \longrightarrow & C \\ \uparrow & & \uparrow \\ R & \longrightarrow & S \end{array}$$

such that $R \to B$ factors $R \to T \to B$, B is a big Cohen-Macaulay algebra for R, and C is a balanced big Cohen-Macaulay algebra for S if and only if there is no bad double sequence of partial algebra modifications of T over $R \to S$.

5. WEAKLY FUNCTORIAL BIG COHEN-MACAULAY ALGEBRAS

Before proving one of our main results, we note the following lemma.

Lemma 5.1. *Let R be a local ring and suppose that M is an R-module, that T is an R-algebra, that c is an element of T that is not a zerodivisor, while $\alpha : M \to T_c$ is an R-linear map. Let $M \to M'$ be a partial algebra modification of M with respect to part of a system of parameters $x_1, \ldots, x_{k+1}$ for R, with degree bound D. Suppose that for every relation $t_{k+1}x_{k+1} = \sum_{j=1}^{k} t_j x_j$ with coefficients in T, we have that $ct_{k+1} \in (x_1, \ldots, x_k)T$. Finally, suppose that $\alpha(M) \subseteq c^{-N}T$.*

Then there is an R-linear map $\beta : M' \to T_c$ with image contained in $c^{-(ND+D+N)}T$ such that the diagram

$$\begin{array}{ccc} T_c & \xrightarrow{=} & T_c \\ {\scriptstyle\alpha}\uparrow & & \uparrow{\scriptstyle\beta} \\ M & \longrightarrow & M' \end{array}$$

commutes.

Proof. We have a relation $x_{k+1}u_{k+1} = \sum_{j=1}^{k} x_i u_i$ with the $u_j \in M$ such that

$$M' = M[X_1, \ldots, X_k]_{\leq N}/FR[X_1, \ldots, X_K]_{\leq N-1},$$

with $F = u_{k+1} - \sum_{j=1}^{k} x_j X_j$. Applying α, we have that, with $v_j = \alpha(u_j)$, there is a relation $x_{k+1}v_{k+1} = \sum_{i=1}^{k} x_i v_i$, where the $v_i \in c^{-N}T$, and then $x_{k+1}(c^N v_{k+1}) = \sum_{i=1}^{k} x_i(c^N v_i)$, and so $c(c^N v_{k+1}) \in (x_1, \ldots, x_k)T$. It follows that $v_{k+1} = \sum_{j=1}^{k} x_j(w_j/c^{N+1})$, where the w_j are in T. Extend $R \to T_c$ to a map $\theta\colon R[X_1, \ldots, X_k] \to T_c$ by letting X_j map to w_j/c^{N+1}, $1 \leq j \leq k$. We obtain β by first defining a map $\gamma\colon M[X_1, \ldots, X_k] \to T_c$ that takes $uX_1^{a_1} \cdots X_k^{a_k}$, where $u \in M$, to

$$\alpha(u)(w_1/c^{N+1})^{a_1} \cdots (w_k/c^{N+1})^{a_k}.$$

In fact, γ is the composition of $\alpha \otimes_R \theta\colon M[X_1, \ldots, X_k] \to T_c \otimes_R T_c$ with the map $T_c \otimes_R T_c \to T_c$ that sends $t \otimes t'$ to tt'. Then γ is $R[X_1, \ldots, X_k]$-linear, and so it kills $FR[X_1, \ldots, X_k]$: in particular, it kills $FR[X_1, \ldots, X_k]_{\leq D-1}$. We obtain β by restricting γ. On $M[X_1, \ldots, X_k]_{\leq D}$, we have that for each term the sum of the a_i is at most D, and since $\alpha(u)$ involves, at worst, c^{-N}, the values of β involve, at worst, $c^{(N+1)D+N} = c^{ND+D+N}$ in the denominator, as required. □

Theorem 5.2. *Let $R \to S$ be a local homomorphism of complete local domains of mixed characteristic and dimension at most 3. Then there is a commutative diagram:*

$$\begin{array}{ccc} B & \longrightarrow & C \\ \uparrow & & \uparrow \\ R & \longrightarrow & S \end{array}$$

where B is a balanced big Cohen-Macaulay algebra over R and C is a balanced big Cohen-Macaulay algebra over S.

Proof. If not there is a bad double sequence of partial algebra modifications

$$R \to M_1 \to \cdots \to M_r \to S \otimes_R M_r \to N_1 \to \cdots \to N_s$$

with the image of $1 \in R$ in N_s actually in nN_s. Let $D > 0$ be an integer at least as large as any of the degree bounds for the partial algebra modifications in this sequence. Let

$D_0 = 0$ and define a sequence D_j recursively by $D_{j+1} = (D_j + 1)D + D_j$ so that $D_1 = D$, $D_2 = D^2 + 2D$, and so forth. Clearly, D_j is a positive integer for $j \geq 1$.

We note that, as explained in [HH9], Proposition (1.2), there is an R-algebra homomorphism $R^+ \to S^+$. A choice of such a homomorphism will induce a map $R_p^+ \to S_p^+$, where the subscript $_p$ indicates adjunction of $1/p$.

Fix any rational number $\epsilon > 0$ and a value of p^ϵ in R^+. If h is an integer (it may be negative) by $p^{h\epsilon}$ we mean $(p^\epsilon)^h$ (when h is negative the value is taken in R_p^+).

We claim that there is a commutative diagram:

$$\begin{array}{ccccccccccccccc}
R_p^+ & \stackrel{=}{\longrightarrow} & R_p^+ & \stackrel{=}{\longrightarrow} & \cdots & \stackrel{=}{\longrightarrow} & R_p^+ & \longrightarrow & S_p^+ & \stackrel{=}{\longrightarrow} & S_p^+ & \stackrel{=}{\longrightarrow} & \cdots & \stackrel{=}{\longrightarrow} & S_p^+ \\
\uparrow & & \uparrow & & & & \uparrow & & \uparrow & & \uparrow & & & & \uparrow \\
R & \longrightarrow & M_1 & \longrightarrow & \cdots & \longrightarrow & M_r & \longrightarrow & S \otimes_R M_r & \longrightarrow & N_1 & \longrightarrow & \cdots & \longrightarrow & N_s
\end{array}$$

in which

(1) the leftmost vertical arrow α_0 is the inclusion $R \subseteq R_p^+$,

(2) the vertical arrows α_i from the M_i, where $R = M_0$, are R-linear,

(3) the vertical arrows β_i from the N_i, where $N_0 = S \otimes_R M_r$, are S-linear,

(4) the image of each M_i in R_p^+ is inside the cyclic module $p^{-D_i\epsilon}R^+$, $0 \leq i \leq r$, and

(5) the image of each N_j is inside the cyclic module $p^{-D_{r+j}\epsilon}S^+$, $0 \leq j \leq s$.

We have already specified α_0. Suppose that the α_j, $j \leq i < r$ have already been constructed so that the squares to the left of the arrow α_i commute. The existence of α_{i+1} is then immediate from Theorem (2.3) and Lemma (5.1).

The composite map $M_r \to R_p^+ \to S_p^+$ induces a map $S \otimes M_r \to S_p^+$ simply because S_p^+ is an S-algebra, and the power of p needed to clear denominators in the image is the same as for the map $M_r \to R_p^+$. Thus, we have β_0, as required. The remaining β_i are constructed recursively, exactly as in the argument for the existence of the α_i: we are applying Theorem (2.3) to S^+ instead of to R^+, and Lemma (5.1) over S instead of over R.

Let $E = D_{r+s}$. Tracing the image of $1 \in R$ around this commutative diagram in two different ways, we find that for every rational $\epsilon > 0$, we have that, in S^+, $1 \in np^{-E\epsilon}S^+$, i.e., that $p^{E\epsilon} \in nS^+$ for arbitrarily small $\epsilon > 0$. Taking $\epsilon = 1/N$, $N > 0$, we find that $p^E \in n^N S^+$ for all $N > 0$, which contradicts Fact (2.4). $\square$

We next note the following refinement of Theorem 5.2: the maps to B and C in Theorem 5.2 can be chosen so as to factor through R^+ and S^+. Note that when R is a complete

local domain, we may define a balanced big Cohen-Macaulay algebra over R^+ to be an R^+-algebra B such that every system of parameters of R^+ is a regular sequence on B, and such that, if $\mathcal{M}$ is the maximal ideal of R^+, we have that $\mathcal{M}B \neq B$. Evidently, an R^+-algebra B is a balanced big Cohen-Macaulay algebra for R^+ if and only if it is a balanced big Cohen-Macaulay algebra for every module-finite extension of R within R^+.

Corollary 5.3. *Let $R \to S$ be a local homomorphism of complete local domains of mixed characteristic and dimension at most 3. Choose a local map $R^+ \to S^+$ so that the diagram*

$$\begin{array}{ccc} R^+ & \longrightarrow & S^+ \\ \uparrow & & \uparrow \\ R & \longrightarrow & S \end{array}$$

commutes. Then there is a commutative diagram:

$$\begin{array}{ccc} B & \longrightarrow & C \\ \uparrow & & \uparrow \\ R^+ & \longrightarrow & S^+ \end{array}$$

where B is a balanced big Cohen-Macaulay algebra over R^+ and C is a balanced big Cohen-Macaulay algebra over S^+.

Proof. The issue is whether there is a bad double sequence of algebra modifications over $R^+ \to S^+$. Because the map $R^+ \to S^+$ is a direct limit of maps $R' \to S'$, where R' is a module-finite extension of R within R^+ and S' is a module-finite extension of S within S^+ containing the image of R', if there is such a bad double sequence of algebra modifications, there will exist such a bad double sequence over $R' \to S'$ for suitably large choices of R' and S'. But this contradicts Theorem 5.2 (with R' and S' playing the roles of R and S). $\square$

Corollary 5.4. *Suppose that $R \to S$ is a local map of local rings of dimension 3, and that there is a minimal prime Q of $\widehat{S}$ lying over P in $\widehat{R}$ such that $\dim(S/Q) = \dim S$, and $\dim \widehat{R}/P = \dim R$. Suppose also that $\widehat{S}/Q$ is mixed characteristic (which implies that $\widehat{R}/P$ is mixed characteristic as well). Then there is a commutative diagram*

$$\begin{array}{ccc} B & \longrightarrow & C \\ \uparrow & & \uparrow \\ R & \longrightarrow & S \end{array}$$

in which B and C are balanced big Cohen-Macaulay algebras over R and S respectively.

Proof. Put together the diagram

$$\begin{array}{ccc} B & \longrightarrow & C \\ \uparrow & & \uparrow \\ \widehat{R}/P & \longrightarrow & \widehat{S}/Q \end{array}$$

guaranteed by the preceding theorem with the diagram

$$\begin{array}{ccc} \widehat{R}/P & \longrightarrow & \widehat{S}/Q \\ \uparrow & & \uparrow \\ R & \longrightarrow & S \end{array}$$

The result now follows because the dimension conditions guarantee that every system of parameters for R (respectively, S) is a system of parameters in $\widehat{R}/P$ (respectively, $\widehat{S}/Q$). $\square$

6. VANISHING OF MAPS OF TOR

The result obtained in the preceding section yields a special case of the vanishing conjecture for maps of Tor discussed at length in §4 of [HH9] and also in [Rang]. For simplicity, we have stated the result when T is local.

6.1 Theorem. *Let $A \to R \to S$ be maps of Noetherian rings such that $A \to S$ is local homomorphism of mixed characteristic regular local rings, R is a module-finite and A-torsion-free extension of A, and A, S have dimension at most 3. Let M be any A-module. Then the map $\operatorname{Tor}_i^A(M, R) \to \operatorname{Tor}_i^A(M, S)$ vanishes for all $i \geq 1$ and for every A-module M.*

Proof. We may complete S and likewise replace A by its completion and M, R by their tensor products with $\widehat{A}$ over A. Thus, we may assume that A, S are complete. Second, let R_1 be the quotient of R by a minimal prime disjoint from $A-\{0\}$ contained in $\operatorname{Ker}(R \to S)$. Because of the factorization $\operatorname{Tor}_i^A(M, R) \to \operatorname{Tor}_i^A(M, R_1) \to \operatorname{Tor}_i^A(M, S)$ it suffices to see that that the second map is 0, and so we may replace R by R_1, and assume that R is a module-finite and, hence, local extension domain of A and that $R \to S$ is a local map

of complete local domains of mixed characteristic and dimension at most 3. By Theorem (5.2), there is a commutative diagram:

$$\begin{array}{ccc} B & \longrightarrow & C \\ \uparrow & & \uparrow \\ R & \longrightarrow & S \end{array}$$

where B, C are balanced big Cohen-Macaulay algebras over R, S respectively. Since R is module-finite over A, B is a balanced big Cohen-Macaulay algebra over A, and, since A is regular, therefore A-flat (cf. [Ho3], Lemma 5.5 or [HH5], §6.7, p. 77), while, similarly, C is faithfully flat over S. Thus, $\operatorname{Tor}_i^A(M, R) \to \operatorname{Tor}_i^A(M, B) \to \operatorname{Tor}_i^A(M, C)$ is 0, since the module in the middle is 0 (B is A-flat), and so the composite $\operatorname{Tor}_i^A(M, R) \to \operatorname{Tor}_i^A(M, S) \to \operatorname{Tor}_i^A(M, C)$ is 0. But the map on the right is injective because C is faithfully flat over S, and so $\operatorname{Tor}_i^A(M, R) \to \operatorname{Tor}_i^A(M, S)$ is 0. □

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