

INDECOMPOSABLE CANONICAL MODULES AND CONNECTEDNESS

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1. Introduction

Throughout this paper all rings are commutative, with identity, and Noetherian, unless otherwise specified. In particular, “local ring” always means Noetherian local ring, unless otherwise specified. Our objective is to prove a generalization of Faltings’ connectedness theorem [Fal1, Fal2], which asserts that in a complete local domain (R, m, K) of dimension n , if $I \subseteq m$ is an ideal generated by at most $n - 2$ elements, then the punctured spectrum of R/I is connected. Our result (see Theorems 3.3 and 3.6) draws the same conclusion without the hypothesis that R be a domain: we assume instead that R is complete, equidimensional (i.e., for every minimal prime p of R , $\dim R/p = \dim R$), and that one of the following conditions, which we shall prove are equivalent, holds:

- a) $H_m^n(R)$ (local cohomology with support in m) is indecomposable.
- b) The canonical module ω of R is indecomposable.
- c) The S_2 -ification of R is local.
- d) For every ideal I of height two or more, $\operatorname{Spec} R - V(I)$ is connected.
- e) Given any two distinct minimal primes p, q of R , there is a sequence of minimal primes $p = p_0, \dots, p_i, \dots, p_r = q$ such that for $0 \leq i < r$, the height $(p_i + p_{i+1}) \leq 1$.

§2 details the properties of canonical modules for a not necessarily Cohen-Macaulay ring, as well as the process of S_2 -ification. By and large the results of §2 are known, but in some cases we have not found a convenient reference. We note here only that our definition is that a canonical module ω for (R, m, K) is a finitely generated R -module such that $\operatorname{Hom}_R(\omega, E) \cong H_m^{\dim R}(R)$, where E is an injective hull for K over R . The main results are developed in §3.

2. Canonical modules and S_2 -ification

It will be convenient to have a notation for the ideal that turns out to be the annihilator of the canonical module.

(2.1) Definition. If R is a local ring we shall denote by $\mathbf{j}(R)$ the largest ideal which is a submodule of R of dimension smaller than $\dim R$. Then $\mathbf{j}(R)$ is nonzero if and only if some

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prime P of $\text{Ass } R$ is such that $\dim R/P < \dim R$, and then $\mathbf{j}(R) \supseteq \text{Ann}_R P$. Thus, $\mathbf{j}(R) = (0)$ iff R is equidimensional and unmixed (where *unmixed* means that (0) has no embedded primes). Moreover, $\mathbf{j}(R)$ consists of all elements $r \in R$ such that $\dim R/\text{Ann}_R r < \dim R$.

Throughout this section $E = E_R(K)$ denotes an injective hull of the residue field of the local ring (R, m, K) and $^\vee$ denotes the exact functor $\text{Hom}_R(_, E)$ on R -modules. We begin by summarizing many of the known properties of canonical modules, most of which we shall need in this paper.

(2.2) Remark. Let (R, m, K) be a local ring with $\dim R = d$.

a) If R is complete, then R has a canonical module, and any canonical module is isomorphic with $H_m^d(R)^\vee$.

b) Any two canonical modules for R are (non-canonically) isomorphic.

c) If R is a homomorphic image of a Gorenstein ring, then R has a canonical module. If $R = S/J$, where S is local, then $\text{Ext}_S^h(R, S)$ is a canonical module for R , where $h = \dim S - \dim R$. More generally, if $S \rightarrow R$ is local, R is module-finite over the image of S , S is Cohen-Macaulay with canonical module ω_S , and $h = \dim S - \dim R$, then $\text{Ext}_S^h(R, \omega_S)$ is a canonical module for R . In particular, if R is a module-finite extension of a regular (or Gorenstein) local ring A , then $\text{Hom}_A(R, A)$ is a canonical module for R . (The same holds when R is module-finite over the image of A and the two have the same dimension.)

d) A canonical module for R must be killed by $\mathbf{j}(R)$, and is also a canonical module for $R/\mathbf{j}(R)$, while any canonical module for $R/\mathbf{j}(R)$ is a canonical module for R . Thus, R has a canonical module if and only if $R/\mathbf{j}(R)$ has a canonical module.

For parts e)-i) we let (R, m, K) be a local ring with canonical module ω .

e) The kernel of the map $R \rightarrow \text{Hom}_R(\omega, \omega)$ is $\mathbf{j}(R)$. Thus, ω is faithful if and only if R is equidimensional and unmixed.

f) The module ω and its completion are both S_2 . Moreover, $\text{Hom}_R(\omega, \omega)$ is a commutative semilocal ring module-finite over the image of R and it is S_2 both as an R -module and as a ring in its own right. It may be identified with a subring of the total quotient ring of $R/\mathbf{j}(R)$. Moreover, its m -adic completion is S_2 .

g) For every prime P of R such that $\dim R/P = \dim R$, the ring $(R/P)^\wedge \cong \hat{R}/P\hat{R}$ is equidimensional and unmixed. If $\mathbf{j}(R) = (0)$ then $\mathbf{j}(\hat{R}) = (0)$.

h) $R \rightarrow \text{Hom}_R(\omega, \omega)$ is an isomorphism if and only if R is S_2 and equidimensional (the latter condition follows from S_2 if R is catenary), and also iff $\hat{R}$ is S_2 . Thus, if R has a canonical module and R is equidimensional and S_2 , then $\hat{R}$ is S_2 .

i) If R is equidimensional, then for every prime P of R , ω_P is a canonical module for R_P .

For parts j)-l) we suppose that (R, m, K) is local with $\mathbf{j}(R) = (0)$ and let ω be a canonical module for R . Let $S = \text{Hom}_R(\omega, \omega)$. Let $m_1, \dots, m_s$ denote the maximal ideals of S . Note that ω is an S -module, precisely because $S = \text{Hom}_R(\omega, \omega)$. Then:

j) Every maximal ideal of S has height equal to $\dim R$.

k) When R is complete, so that S is product of local rings S_i , one for every maximal ideal m_i of S , and ω is, correspondingly, a product of modules ω_i over the various S_i , then ω_i is the canonical module for S_i for every i .

1) The module ω is a canonical module for S in the sense that ω_Q is a canonical module for S_Q for every prime ideal Q of S .

Proof. Parts a) - c) are standard and can be found in [HK]. The rest of the results can be found in either [G] or [A1-2]. In particular, see Theorem 3.2 and Corollary 4.3 in [A2].

(2.3) Discussion. Let (R, m, K) be an equidimensional and unmixed local ring, i.e., such that $\mathbf{j}(R) = (0)$. We shall say that a ring S is an S_2 -ification of R if it lies between R and its total quotient ring, is module-finite over R , is S_2 as an R -module, and has the property that for every element $s \in S - R$, the ideal $\mathcal{D}(s)$, defined as $\{r \in R : rs \in R\}$, has height at least two. We are interested in this notion because if $\mathbf{j}(R) = (0)$ and R has a canonical module ω then it has an S_2 -ification, to wit, $\text{Hom}_R(\omega, \omega)$ identified with a subring of the total quotient ring of R . Moreover, whenever R has an S_2 -ification, it is unique. We prove several propositions in this direction.

(2.4) Proposition. *Let (R, m, K) be a local ring with $\mathbf{j}(R) = (0)$ and let T be its total quotient ring. If $f \in T$ let $\mathcal{D}(f) = \{r \in R : rf \in R\}$. Let S be the subring of T consisting of all elements $f \in T$ such that $\text{ht } \mathcal{D}(f) \geq 2$. Then R has an S_2 -ification if and only if S is module-finite over R , in which case S is the unique S_2 -ification of R .*

Proof. It is easy to verify that S is a subring of T containing R , since $\mathcal{D}(r) = R$ for $r \in R$ (and the height is $+\infty$), $\mathcal{D}(s \pm s') \supseteq \mathcal{D}(s) \cap \mathcal{D}(s')$, and $\mathcal{D}(ss') \supseteq \mathcal{D}(s)\mathcal{D}(s')$. Moreover, it is immediate from the way that we defined an S_2 -ification that it must be contained in T . We next observe that if $S_0 \subseteq S \subseteq T$ with S_0 module-finite over R and S_0 is S_2 as an R -module then $S_0 = S$. To see this, suppose that $f \in S - S_0$. Since $\mathcal{D}(f)$ has height at least two (but cannot be R) and S_0 is a faithful R -module, we must have that there is a regular sequence x, y of length two on S_0 in $\mathcal{D}(f)$. Now $xf, yf \in R$, and so we have that $x(yf) - y(xf) = 0$ is a relation on x, y with coefficients in S_0 . It follows that $xf \in xS$, so that $xf = xs$ with $s \in S_0$. But x is a nonzerodivisor in S_0 , hence in R , and so also in T , the total quotient ring of R . Thus, $f = s$, and $f \in S_0$.

Now suppose that S is module-finite over R . We must show that S is S_2 . The depth of S on a height one ideal of R is at least one, since the ideal must contain a nonzerodivisor of R (we have that $\mathbf{j}(R) = (0)$) and this will be a nonzerodivisor in T . Suppose that I has height at least two. Choose elements x, y in I , nonzerodivisors, such that $(x, y)R$ has height two. We claim that x, y form a regular sequence on S (and this will complete the proof). For suppose that we have a relation $xs = ys'$ with elements s and s' of S . In the total quotient ring let $f = s/y = s'/x$. Choose ideals I, I' of R of height at least two such that $Is \subseteq R$ and $I's' \subseteq R$. Then $II'(x, y)$ multiplies f into R (since $I'xf = I's'$ and $Iyf = Is$), and this ideal has height two. Thus, $f \in S$, and so $s' = xf \in xS$. $\square$

(2.5) Remarks. Note that if an R -algebra S' is R -isomorphic to the S_2 -ification S of R , then there is a unique R -isomorphism $S' \cong S$. (For each element f of S' we can choose a nonzerodivisor $r \in R$ such $fr = r' \in R$. Then if $\phi : S' \rightarrow S$ is the isomorphism we must have $\phi(r') = \phi(fr) = r\phi(f)$, which determines $\phi(f)$ uniquely.) Thus, we shall talk about S_2 -ifications which are not literally subrings of the total quotient ring of R : they are always, however, canonically identifiable with such a subring.

Note also that if S is an S_2 -ification of R , then we can choose finitely many generators for S as an R -module, and for each of these generators an ideal of height at least two in R that multiplies S into R . It follows that there is an ideal of height at least two in R that multiplies S into R : intersect the ideals chosen for the individual generators.

(2.6) Proposition. *If (R, m, K) has an S_2 -ification S then for every prime ideal P of R , S_P is an S_2 -ification of R_P .*

Proof. S_P is S_2 over R_P , module-finite, and identifiable with a subring of the total quotient ring. Any element has a unit multiple of the form $s/1$, where s is in S . Then $\mathcal{D}(s/1) \subseteq \mathcal{D}(s)_P \subseteq R_P$ has height at least two. $\square$

(2.7) Proposition. *If (R, m, K) is local, $\mathbf{j}(R) = 0$, and ω is a canonical module for R , then $R \rightarrow \operatorname{Hom}_R(\omega, \omega)$ is an S_2 -ification of R .*

Proof. We know by 2.2 f) that $\operatorname{Hom}_R(\omega, \omega)$ is a module-finite extension of R that may be identified with a subring of the total quotient ring of R . We also know that it is S_2 . Therefore, it will suffice to show that if $s \in \operatorname{Hom}_R(\omega, \omega)$, then $\mathcal{D}(s)$ has height at least two. If not, it will be contained in some height one prime P of R , and P will be in the support of $R/\mathcal{D}(s) \cong (R(\operatorname{id}_\omega) + Rs)/R(\operatorname{id}_\omega) \subseteq \operatorname{Hom}_R(\omega, \omega)/R$, and $\operatorname{Hom}_R(\omega, \omega)/R$ is not supported at any height one prime by 2.2 h) and i). $\square$

3. Connectedness theorems

We first give the statement of the Faltings' connectedness theorem for complete domains (see [Fal1], [Fal2]). In our improvement we will follow the lines of an argument given in [BR]. We will use the following result, the local Hartshorne-Lichtenbaum vanishing theorem (see [Ha], [CS], and [BH]). In its basic form, it asserts that for a complete local domain (R, m, K) of dimension n , if $I \subseteq m$ is not primary to m then $H_I^n(R) = 0$. In a more precise form, it asserts that if (R, m, K) is any complete local ring of dimension n and $I \subseteq m$ is an ideal, then $H_I^n(R) \neq 0$ if and only if there exists a minimal prime ideal p of R such that $\dim R/p = \dim R$ and $I + p$ is primary to m .

(3.1) Theorem (Faltings' connectedness theorem). *Let (R, m, K) be an analytically irreducible local ring of dimension n , and let $\mathfrak{A}$ be an ideal of R generated by at most $n - 2$ elements. Then the punctured spectrum of $\operatorname{Spec} R/\mathfrak{A}$ is connected. (In other words, there do not exist ideals I, J of R such that $\operatorname{Rad} I \cap \operatorname{Rad} J = \operatorname{Rad} \mathfrak{A}$ and $\operatorname{Rad}(I + J) = m$ unless one of the ideals is primary to m and the other has the same radical as $\mathfrak{A}$.)*

(3.2) Remarks. We do not even need the condition that $\mathfrak{A}$ be generated by $n - 2$ or fewer elements: all that is needed is that $H_{\mathfrak{A}}^{n-1}(R) = H_{\mathfrak{A}}^n(R) = 0$, and the second condition is automatic if R is a complete local domain and $\mathfrak{A}$ is not m -primary.

Faltings' original proof was for the equicharacteristic case. In [BR] a much simpler proof was given, whose outline we shall follow here. In [HH2], §6 (see also [HH1]) it is shown that the integral closure R^+ of a complete (or excellent) local domain of positive characteristic p in an algebraic closure of its fraction field is "Cohen-Macaulay" in the sense that every system of parameters in R is a regular sequence in R^+ . The characteristic p case of the

Faltings' connectedness theorem can be deduced from the Cohen-Macaulay property for R^+ , and then the equal characteristic 0 case also follows by the technique of reduction to characteristic p . It was this point of view which led us to suspect that the condition that R is a domain could be weakened.

Our next main objective here is to generalize so that R need not be a domain. We shall state all of our results in the complete case. In each instance, one can achieve the illusion of greater generality by starting with an arbitrary local ring and requiring that its completion satisfy the hypotheses we want.

(3.3) Theorem. *Let (R, m, K) be a complete equidimensional local ring such that, equivalently, $H_m^n(R)$ is an indecomposable R -module or such that ω_R , its canonical module, is indecomposable. (This is automatic if R is a domain, since ω_R is then an ideal of R .) Let $\mathfrak{A}$ be a proper ideal of R generated by $n - 2$ or fewer elements. Then the punctured spectrum of $R/\mathfrak{A}$ is connected.*

Proof. If not let I, J be ideals which give a disconnection, so that $I \cap J$ has the same radical as $\mathfrak{A}$, $I + J$ is primary to m , but neither I nor J is primary to m . The Mayer-Vietoris sequence for local cohomology then yields:

$$\cdots \rightarrow H_{I \cap J}^{n-1}(R) \rightarrow H_{I+J}^n(R) \rightarrow H_I^n(R) \oplus H_J^n(R) \rightarrow H_{I \cap J}^n(R) \rightarrow \cdots$$

and the first and last terms displayed are zero, since $I \cap J$ has the same radical as an ideal with at most $n - 2$ generators. Since $I + J$ is primary to m , this yields an isomorphism:

$$H_m^n(R) \cong H_I^n(R) \oplus H_J^n(R).$$

The fact that $H_m^n(R)$ is indecomposable implies that one of the summands, say $H_J^n(R)$, is zero. But then the local Hartshorne-Lichtenbaum vanishing theorem implies that for every prime $P \in \mathcal{P}$, the set of minimal primes of $R, J + P$ is not primary to m . Let P be one of these minimal primes. The intersection of $I + P$ and $J + P$ is still, up to radicals, $\mathfrak{A} + P$, while the sum is still primary to m . Thus, applying the local Hartshorne-Lichtenbaum vanishing theorem to the domain R/P , we see that $I + P$ must be primary to m for every minimal prime P . But this implies that $\bigcap_{P \in \mathcal{P}} (I + P)$ is primary to m , and up to radicals this is the same as $I + \bigcap_{P \in \mathcal{P}} P$. Since $\bigcap_{P \in \mathcal{P}} P$ is the ideal of nilpotents, we find that I itself is primary to m , a contradiction. $\square$

This result motivates a study of when the canonical module of a complete local equidimensional ring is indecomposable. We begin by associating a graph with such a ring.

(3.4) Definition. Let R be an equidimensional local ring. We denote by Γ_R the (undirected) graph whose vertices are the minimal primes of R , and whose edges are determined by the following rule: if P, Q are distinct minimal primes of R , then $\{P, Q\}$ is an edge of Γ_R if and only if $P + Q$ has height one.

We next observe:

(3.5) Proposition. *Let (R, m, K) be a local ring with canonical module ω . Suppose that $\mathbf{j}(R) = (0)$, and let $S = \text{Hom}_R(\omega, \omega)$, the S_2 -ification of R . Then:*

- a) For every prime ideal P of R and Q of S , if Q lies over P then $\text{ht } Q = \text{ht } P$.*
- b) For every ideal I of R , $\text{height } IS = \text{height } I$.*
- c) Contraction gives a bijection of the minimal primes of S with the minimal primes of R and a bijection of the height one primes of S with the height one primes of R .*

Proof. a) To study the primes of S lying over P , we first replace R , ω , and S , by R_P , ω_P , and S_P . Thus, there is no loss of generality in supposing that P is the maximal ideal of R . But then the result follows from 2.2 j).

b) If P is a prime ideal containing I whose height is the same as that of I , then there is a prime ideal Q of S lying over P . Then Q contains IS , and so $\text{ht } IS \leq \text{ht } Q = \text{ht } P = \text{ht } I$. On the other hand, if Q is a prime ideal containing IS whose height is the same as that of IS , then Q contracts to a prime ideal P containing I , and so $\text{ht } IS = \text{ht } Q = \text{ht } P \geq \text{ht } I$.

c) Any height k prime P of R has at least one prime of S lying over it, and all such primes have height k by part a). Moreover, all height k primes of S lie over height k primes of R . Thus, it suffices to show that when $k = 0, 1$, there is at most one prime of S lying over P . But the primes of S lying over P correspond to the primes of S_P lying over PS_P , and S_P is the S_2 -ification of R_P . When $\dim R_P \leq 1$, R_P is its own S_2 -ification, and the result follows. $\square$

We are now ready to prove a central result:

(3.6) Theorem. *Let (R, m, K) be a complete local equidimensional ring with $\dim R = n$. The following conditions are equivalent:*

- a) $H_m^n(R)$ is indecomposable.*
- b) The canonical module $\omega = \omega_R$ of R is indecomposable.*
- c) The S_2 -ification S of $R/\mathbf{j}(R)$ is local.*
- d) For every ideal J of height at least two, $\text{Spec } R - V(J)$ is connected.*
- e) Γ_R is connected.*

Proof. We shall prove that $a) \Leftrightarrow b) \Leftrightarrow c) \Rightarrow d) \Leftrightarrow e) \Rightarrow c)$. The equivalence of a) and b) is clear. Now assume b). The module ω is also a canonical module for $R/\mathbf{j}(R)$ and, consequently, for the S_2 -ification $\text{Hom}_R(\omega, \omega) = \text{Hom}_{R/\mathbf{j}(R)}(\omega, \omega)$ of $R/\mathbf{j}(R)$ as well. If the S is not local, then ω is a product of nonzero factors corresponding to the various factors rings of S , and this will yield a non-trivial direct sum decomposition of ω over R . On the other hand, if $S \cong \text{Hom}_R(\omega, \omega)$ is local, it contains no idempotents other than 0, 1, and this implies that ω is indecomposable. Thus, $c) \Rightarrow b)$ as well, and we have established the equivalence of the first three conditions.

Now assume that S is local, and that $I, I' \subseteq m$ are such that $I \cap I'$ is nilpotent but $I + I'$ has height at least two. We can replace I, I' by powers and assume that $II' = 0$ but $I + I'$ has height at least two. This situation is preserved when we pass to $R/\mathbf{j}(R)$, for when R is equidimensional $\mathbf{j}(R)$ consists of nilpotents. Thus, we might as well assume that $\mathbf{j}(R) = (0)$. Next, note that $IS + I'S$ has height at least two (since its height is the same as that of $I + I'$, by the preceding proposition). Moreover, if neither I nor I' is primary to m (i.e., if neither has height equal to $\dim R$) then neither IS nor $I'S$ is primary to the maximal ideal of S .

Thus, there is no loss of generality in assuming that R is S_2 . The ideal $I + I'$ will contain a regular sequence $u + u', v + v'$ of length two, where $u, v \in I$ and $u', v' \in I'$. The relation $v(u + u') - u(v + v') = 0$ then shows that $u \in (u + u')R$, while $u' \in (u + u')R$ similarly. This yields $u = (u + u')a$, $u' = (u + u')b$, and so $u + u' = (u + u')(a + b)$. Since $u + u'$ is not a zerodivisor, $1 = a + b$, and it follows that at least one of a, b is a unit. Suppose that a is a unit: the other case is similar. Then $u = (u + u')a$ implies that u is a nonzerodivisor, while $uI' \subseteq II' = (0)$ then implies that $I' = 0$, and so I is primary to m . This completes the proof that $c) \Rightarrow d)$.

We next want to see that $d) \Leftrightarrow e)$. Suppose that one has ideals I, I' such that $I \cap I'$ is nilpotent. Then we can replace I, I' by their radicals while only increasing $I + I'$. Then each of I, I' is a finite intersection of primes. For each minimal prime p of R , $p \supseteq I \cap I'$, and so p must contain either a minimal prime of I or a minimal prime of I' . Thus, p must be either a minimal prime of I or a minimal prime of I' . If we omit all non-minimal primes from the primary decomposition of I (respectively, I') and intersect the others, we get two larger ideals whose intersection is still $\text{Rad}(0)$. Thus, it is possible to give I, I' such that $\text{Rad}(I \cap I') = \text{Rad}(0)$ and $I + I'$ has height two if and only if one can do this with ideals I, I' coming from a partition of the minimal primes of R into two nonempty sets, with I the intersection of the minimal primes in one set and I' the intersection of the minimal primes in the other set. If one set consists of $\{p_1, \dots, p_h\}$ and the other of $\{q_1, \dots, q_k\}$ we shall have $I = \cap_i p_i$, $I' = \cap_j q_j$, and $I + I'$ will then have the same radical as $\cap_{i,j} (p_i + q_j)$, and will have height at least two if and only if every $p_i + q_j$ has height at least two. Thus, $d)$ fails if and only if the minimal primes can be partitioned into two nonempty sets such that no edge of Γ joins a vertex in one set to a vertex in the other, which is precisely the condition for Γ to be disconnected. Thus, $d) \Leftrightarrow e)$.

Finally, we show that $e) \Rightarrow c)$. Suppose that Γ is connected. We want to prove that the S_2 -ification of $R/\mathbf{j}(R)$ is local. The graph associated with $R/\mathbf{j}(R)$ is the same as that associated with R , so that we may assume that $\mathbf{j}(R) = (0)$. If the S_2 -ification S of R has two or more maximal ideals, say $\mathcal{M}_1, \dots, \mathcal{M}_r$, where $r \geq 2$, for each $\mathcal{M}_j$ let $\mathcal{P}_j$ denote the set of minimal primes of S contained in $\mathcal{M}_j$. Then $\mathcal{P}_j$ is evidently non-empty. There is a bijection between the minimal primes of S and those of R , so that for each $\mathcal{P}_j$ there is a corresponding set of minimal primes $\mathcal{Q}_j$ of R . To complete the argument, it will suffice to show that if i, j are different then it is impossible to have an edge joining a vertex in $\mathcal{Q}_i$ to a vertex in $\mathcal{Q}_j$. If there were such an edge, there would be a height one prime P of R containing both a minimal prime in $\mathcal{Q}_i$ and a minimal prime in $\mathcal{Q}_j$. Then $R_P \cong S_P$, and it follows that the unique prime of S lying over P contains both a prime of $\mathcal{P}_i$ and a prime of $\mathcal{P}_j$. Let $\mathcal{M}$ be a maximal ideal of R containing P . Then $\mathcal{M}$ contains both a prime of $\mathcal{P}_i$ and a prime of $\mathcal{P}_j$, which is impossible: S is a finite product of local rings, and each prime ideal of S is therefore contained in a unique maximal ideal of S , forcing $\mathcal{M}_i = \mathcal{M} = \mathcal{M}_j$. $\square$

When R is not complete, it is necessary to study the graph associated with the minimal primes in the completion: the domain property is frequently lost when one completes. However, the characterization in (3.6c) behaves better, as we see in (3.7) below. First note that in the sequel, if (R, m, K) is equidimensional (but possibly has $\mathbf{j}(R) \neq (0)$), by an S_2 -ification of R we mean an S_2 -ification for $R/\mathbf{j}(R)$.

(3.7) Corollary. *Let (R, m, K) be an equidimensional local ring with canonical module ω . Let $n = \dim R$. Then $H_m^n(R)$ is indecomposable if and only if the S_2 -ification of R is local. In particular, if R is S_2 , then $H_m^n(R)$ is indecomposable.*

Proof. Killing $\mathbf{j}(R)$, if it is not zero, does not affect either issue, and so we may assume that $\mathbf{j}(R) = (0)$. The S_2 -ification of $\hat{R}$ is $\text{Hom}_{\hat{R}}(\hat{\omega}, \hat{\omega}) \cong \hat{R} \otimes_R \text{Hom}_R(\omega, \omega)$. Equidimensionality is preserved by completion here, and the issue of whether a semilocal ring is local is not affected by completing with respect to its Jacobson radical. $\square$

The next Proposition is well-known (e.g. see [B, 5.2] which gives a much more general result), but we include a proof as it is fairly short.

(3.8) Proposition. *If (R, m, K) is an excellent local, equidimensional ring then R has an S_2 -ification S , and $\hat{S}$ is the S_2 -ification of $\hat{R}$.*

Proof. We may first kill $\mathbf{j}(R)$, and we henceforth suppose that it is 0. Because R is excellent, this is preserved by completion. Let S be the set of elements of the total quotient ring $\mathcal{T}(R)$ of R that are multiplied into R by an ideal of height two or more. It will suffice to show that S is module-finite over R . Note that $\mathcal{T}(R) \subseteq \mathcal{T}(\hat{R})$, since $\hat{R}$ is flat over R . If S is not module-finite over R we can choose a sequence of elements $\{s_i\}$ in S such that the sequence of R -submodules $\sum_{i=1}^j R s_i \subseteq S$ is strictly increasing with j . Clearly, each $s_i \in \mathcal{T}(\hat{R})$. Since $\hat{R}$ has an S_2 -ification, we can choose j so large that $\sum_{i=1}^j \hat{R} s_i = \sum_{i=1}^{j+1} \hat{R} s_i$. We can choose a nonzerodivisor $a \in R$ such that for $0 \leq i \leq j+1$, $a s_i \in R$. Then $\sum_{i=1}^j \hat{R} a s_i = \sum_{i=1}^{j+1} \hat{R} a s_i$. Since $\hat{R}$ is faithfully flat over R , we find that $\sum_{i=1}^j R a s_i = \sum_{i=1}^{j+1} R a s_i$, and so $a s_{j+1} \in \sum_{i=1}^j R a s_i$. Since a is a nonzerodivisor in R (and hence in S), it follows that $s_{j+1} \in \sum_{i=1}^j R s_i$, a contradiction.

Thus, R has an S_2 -ification, S . When we complete, since the fibers of $R \rightarrow \hat{R}$ are Cohen-Macaulay, we see that $\hat{S}$ is S_2 as an $\hat{R}$ -module. It is clear that it is contained in the total quotient ring of $\hat{R}$: S/R is killed by a nonzerodivisor a in R , and so $\hat{S}/\hat{R}$ is also killed by a . Moreover, if I is an ideal of R of height at least two killing S/R , then $I\hat{R}$ kills $\hat{S}/\hat{R}$. It follows that $\hat{S}$ is the S_2 -ification of $\hat{R}$. $\square$

(3.9) Proposition. *Let (R, m, K) be an excellent equidimensional local ring.*

- a) The S_2 -ification of R is local if and only if the S_2 -ification of $\hat{R}$ is local.*
- b) The S_2 -ification of R is local if and only if the S_2 -ification of R_{red} is local.*
- c) If R is S_2 and $x_1, \dots, x_k$ is a part of a system of parameters, then the S_2 -ification of $R/(x_1, \dots, x_k)R$ is local.*

Proof. a) This is immediate from (3.8).

b) Since R is excellent, $(\hat{R})_{\text{red}} \cong (R_{\text{red}})^\wedge$. Thus, we may assume that R is complete. We may also assume that $\mathbf{j}(R) = 0$. The result then follows from the fact that the graphs associated with R and R_{red} are the same.

c) The issues are unaffected by completing R and killing the nilpotents. It is easy to see that $B = R/(x_1, \dots, x_k)R$ is again equidimensional. If the S_2 -ification is not local then there is a localization of B at a prime of height at least two such that the punctured spectrum is disconnected, and this ring may be viewed as a quotient of a localization R_Q

of R . But R_Q is S_2 and has a canonical module (it is a localization of a complete ring, and so is a homomorphic image of a localization of a complete regular local ring), so that Corollary 3.7 applies. The result now follows from Theorem 3.3. $\square$

(3.10) Remark. Even for complete domains, the fact that the S_2 -ification of the local ring R is local does not imply, in general, that the S_2 -ification of every local ring of R is local. Some primes of R may have more than one prime of the S_2 -ification lying over them.

For example, consider $R = K[x, y, yz, z(z-x), z^2(z-x)] \subseteq S = K[x, y, z]$. This extension is integral, since z satisfies $Z^2 - xZ - z(z-x) = 0$. The element z is multiplied into R by the height two ideal $(y, z(z-x))$. Now $P = (y, yz, z(z-x), z^2(z-x))R \subseteq R$ is a height two prime, but two prime ideals of S lie over it: $(y, z)S$ and $(y, z-x)S$.

If we complete R, S at their homogeneous maximal ideals both rings remain domains. $\hat{S}$ is the S_2 -ification of $\hat{R}$, and is local. However, there are two primes of $\hat{S}$ lying over $\hat{P} = P\hat{R}$, and so the S_2 -ification of $\hat{R}_{\hat{P}}$ is not local.

(3.11) Remark. Suppose that one is trying to give an elementary proof of Faltings' connectedness theorem in the generality we have obtained here, perhaps without using local cohomology. It would suffice to prove that if R is a complete reduced S_2 local ring and x is a single parameter, then the S_2 -ification of R/xR is local. The connectedness theorem can be reduced to the case of parameters, and if one knows the single fact stated above, one can carry through an induction on the number of parameters. However, it is quite possible that the case of a single parameter is no easier than the general case. We next note:

(3.12) Proposition. *If $(R, m, K), (S, n, K)$ are two complete equidimensional local rings with algebraically closed coefficient field K and the S_2 -ifications of R, S are local then so is the S_2 -ification of $T = R \hat{\otimes}_K S$.*

Proof. We may assume that R, S are reduced, and so is T . We also note that when R, S are domains, then T is a domain. It follows that every minimal prime of T has the form $p \hat{\otimes}_K S + R \hat{\otimes}_K q$ where p is a minimal prime of R and q is a minimal prime of S . (Any prime of T will contract to some prime P of R , and also to some prime Q of S . Hence, it must contain $P \hat{\otimes}_K S + R \hat{\otimes}_K Q$, which is the kernel of the map $R \hat{\otimes}_K S \rightarrow R/P \hat{\otimes}_K S/Q$. This immediately shows that the minimal primes are a subset of the ideals $p \hat{\otimes}_K S + R \hat{\otimes}_K q$ for p minimal in R and q minimal in S . Since it is easy to see that these ideals are mutually incomparable, they are all minimal primes.) Let Γ, Γ' be the graphs associated with R, S , respectively. The vertices of the graph associated with $R \hat{\otimes}_K S$ are in bijective correspondence with the set $\Gamma \times \Gamma'$. There is an edge from (p, q) to (p', q') if and only if $\text{ht}((p + p') \hat{\otimes}_K S + R \hat{\otimes}_K (q + q')S) = 1$, which happens iff either $\text{ht}(p + p') = 1$ and $q = q'$ or $p = p'$ and $\text{ht}(q + q') = 1$. But then, in the graph associated with $R \hat{\otimes}_K S$, with its vertices identified with $\Gamma \times \Gamma'$, we have that each subgraph $\Gamma \times \{q\}$ is connected for every $q \in \Gamma'$, and each subgraph $\{p\} \times \Gamma'$ is connected for every $p \in \Gamma$. It follows that $\Gamma \times \Gamma'$ is connected. $\square$

(3.13) Remarks on the graded case. Now suppose that R is a finitely generated $\mathbb{N}$ -graded K -algebra with $R_0 = K$. Let m be the homogeneous maximal ideal of R . If R is a domain, so is its completion (with respect to the homogeneous maximal ideal), since its

completion has a filtration with respect to which the associated graded ring is R , which is a domain. This implies that the homogeneous primes of R remain prime when we complete. The minimal primes (in fact, all associated primes) of R are homogeneous, and so correspond to the minimal primes of the completion. If R is equidimensional one can check whether $H_m^n(R)$ is indecomposable by checking whether the graph associated with the minimal primes of R is connected: It is not necessary to complete, since the completion will have the same graph. One then gets an immediate family of corollaries of the connectedness theorems given here for intersections of projective varieties. One can also apply the technique of reduction to the diagonal to prove theorems: it may be desirable in that case to assume that the field is algebraically closed, so that products of irreducible components remain irreducible.

It is also worth noting that one can give a graded resolution of R over a polynomial ring. Using Ext to compute the canonical module then produces a graded canonical module. The automorphisms of it yield a “global” S_2 -ification S of R . However, while the ring S is a graded module over R , it need not have the property that S_0 is K : when S decomposes, one has non-trivial idempotents in S_0 .

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