Cohen-Macaulay Varieties, Geometric Complexes, and Combinatorics

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This paper is dedicated to Richard Stanley on the occasion of his 70th birthday

ABSTRACT. We describe the original reasons for studying Stanley-Reisner rings, including Stanley's idea for proving the upper bound conjecture, but also including their application to proving that certain unions of Cohen-Macaulay varieties are Cohen-Macaulay. There are connections with the Cohen-Macaulay property for determinantal ideals, for Grassmannians, and for toric varieties. We also sketch three different proofs of Reisner's theorem, each with a novel feature.

1. Introduction

It is a great pleasure for me to have the chance to discuss Richard Stanley's enormous, groundbreaking contributions to algebraic combinatorics, which arose in part from his application of the Cohen-Macaulay property for face rings of simplicial complexes, also known as Stanley-Reisner rings, to proving the upper bound conjecture. My conversations with Richard at approximately the time of this work had a great influence on me.

Reisner, who was my second Ph.D. student, and I were studying Stanley-Reisner rings at the same time that Richard was, quite independently, and for different reasons. Richard realized that old results of F. S. Macaulay [**14**] on the behavior of Hilbert functions of $\mathbb{N}$-graded Cohen-Macaulay rings would imply the upper bound conjecture for triangulations of a sphere provided that one knew that a certain ring associated to the triangulation was Cohen-Macaulay. He wrote to me, asking whether triangulations of spheres yielded Cohen-Macaulay rings. Gerald Reisner had very recently proved the main result of his thesis at that time: Reisner's result implied a positive answer to Richard's question. I became one of the first people to know that the upper bound conjecture for spheres had been proved, even though I was not directly involved in proving the result. My understanding is that Richard heard about Reisner's result before he received my letter telling him about it.

Another paper [5] in this volume discusses Stanley's work [19], [20], [21], [22], [23] on this problem and subsequent research that it has inspired. We avoid duplicating the topics of that paper as much as possible. In what follows we explain reasons independent of Stanley's work for studying the face rings associated with simplicial complexes. This interest grew out of work of the author on proving that certain classes of rings were Cohen-Macaulay, including the rings defined by generic determinantal ideals (joint with J. A. Eagon) [10], the homogeneous coordinate rings of standard Grassmann varieties and their Schubert subvarieties [7], and normal subrings of polynomial rings generated by monomials [6], i.e., normal toric rings. Not unexpectedly, all of these classes of rings have strong interactions with algebraic combinatorics, and in a multitude of ways. In the sequel, we discuss this perspective as well as techniques related to Reisner's proof of his main theorem that continue to be of great interest. We also discuss specific developments, some quite recent, that make it seem likely that the interactions between commutative algebra and algebraic combinatorics will be permanently ongoing.

One thing that I want to point out here is that the fact that Stanley and Reisner were studying face rings of simplicial complexes independently at roughly the same time was not a pure coincidence. The upper bound conjecture for cyclic polytopes was proved by McMullen [16] using the shellability of the polytopal boundary complex of a convex polytope, which had recently been proved by Brugesser and Mani [2]. My first proof that normal toric varieties are Cohen-Macaulay made use of the same result from [2]. In the course of that proof, one is led, inductively, to consider a finite union of normal toric varieties, and one needs to prove that this union is Cohen-Macaulay. These varieties fit together, in a sense that will be made precise in the sequel, like the facets of the boundary complex of a convex polytope. The shellability of that complex is crucial in the proof of the Cohen-Macaulay property for that union. I guessed at the time that there should be a characterization of when such unions are Cohen-Macaulay in terms of the topological properties of a topological space built from a simplicial or polytopal complex that models the way the union is put together. This program was begun by Reisner [18], and reached fruition in the main result of Thompson in [24].

Richard was aware of McMullen's work and of the paper [6] while thinking about the upper bound conjecture for spheres, which helps to explain the apparent coincidence.

2. Proving the Cohen-Macaulay property for a union of varieties that resembles a simplicial complex

Let $V_1, \ldots, V_n$ be, for simplicity, algebraic varieties in affine space $\mathbb{A}^n_K$ over an algebraically closed field K. In fact, one can generalize the situation easily to closed subschemes of an arbitrary ambient Noetherian scheme: since the issues that will concern us are local, we need only consider the affine case. One is frequently in the situation of knowing that the V_i and their intersections (several at a time) are Cohen-Macaulay of specified dimensions, and one would like to conclude that the union of the V_i is also Cohen-Macaulay. If the V_i are mutually incomparable, they will correspond to the irreducible components of the union, and in order for the union to be Cohen-Macaulay, the V_i will need to be of the same dimension. Further conditions are needed to guarantee the Cohen-Macaulay property for the union. For example, if there are just two V_i, the dimension of the intersection

$V_1 \cap V_2$ must be exactly one smaller than $d = \dim(V_1) = \dim(V_2)$. This case is treated in Proposition 4.1 (stated there in terms of defining ideals). One can formulate a precise result in terms of the geometry of a simplicial complex (or a more general geometric object, in which, for example, the faces are convex polytopes) associated with the V_i. The simplicial complex tracks how the dimensions of the intersections of the V_i behave as one intersects them. Dimensions are shifted by a constant in this tracking. The varieties are thought of as corresponding to the maximal faces or *facets* of the simplicial complex. For example, the case of two varieties of the same dimension d with an intersection of dimension $d-1$ corresponds to a simplicial complex that consists of two vertices: the vertices are the facets. They have dimension 0 while their intersection, which is empty, has dimension -1. Dimensions are shifted by $-d$ in passing from V_1, V_2, $V_1 \cap V_2$ to the simplicial complex. One would like, in general, to be able to say that if the simplicial complex has sufficiently good topological properties, then the union of V_i will be Cohen-Macaulay when all their intersections are.

There is, in fact, a theorem like this, made explicit in [**24**], which we discuss in §7. The condition on the simplicial complex is that it be *Cohen-Macaulay* over the field K, which turns out to be a purely topological property of its geometric realization, expressible in terms of the vanishing of certain cohomology with coefficients in K. However, one needs an additional condition on the varieties to make this work. If one works with their defining ideals, the additional condition is that intersection distributes over sums in certain cases. If one thinks in terms of the theory of schemes, this is saying that scheme-theoretic union distributes over scheme-theoretic intersection in certain needed instances.

We make all this precise in §5, which is easier to do if we make our statements in terms of defining ideals instead of varieties. In fact, the ideals need not be prime, and this allows much greater generality. However, before doing this, we discuss the notion of a Cohen-Macaulay ring, and we consider some examples that triggered the line of thought discussed above.

3. The notion of depth, and Cohen-Macaulay rings

In this section we collect some facts from commutative algebra that we shall need in the sequel. If R is a ring, $x_1, \ldots, x_n \in R$, and M is an R-module, $x_1, \ldots, x_n$ is called a *possibly improper regular sequence* of length n on M if x_1 is a nonzerodivisor on M (i.e., multiplication by x_1 is an injective map $M \to M$), x_2 is a nonzerodivisor on $M/x_1 M$, and so forth, i.e, for all i, $0 \le i \le n-1$, x_{i+1} is a nonzerodivisor on $M/(x_1, \ldots, x_i)M$. The term *weak regular sequence* on M is also used. It is convenient to make the convention that the empty sequence is a possibly improper regular sequence of length 0 on M. If $(x_1, \ldots, x_n)M \ne M$, we say that $x_1, \ldots, x_n$ is a *regular sequence* of length n on M.

For simplicity, we restrict attention now to the case where R is a Noetherian ring and M is a finitely generated R-module. For a more detailed treatment and proofs, not given here, of certain statements below, we refer the reader to [**3**] and [**15**]. If I is an ideal of R and M is an R-module, we define the *depth* of M on I (also referred to as the *grade* of I on M), denoted $\mathrm{depth}_I M$, as follows. If $IM = M$, $\mathrm{depth}_I M = +\infty$. If $IM \ne M$, then $\mathrm{depth}_I M$ is the length of any maximal regular sequence in M contained in I. By a basic theorem, any two such maximal regular

sequences have the same length: see §1.1 and §1.2 of [**3**] for a proof of this fact and of Proposition 3.1 just below.

We note two facts about depth.

PROPOSITION 3.1. If M is a finitely generated R-module over a Noetherian ring R, then if $IM = M$ we have that $\mathrm{Ext}_R^j(R/I, M) = 0$ for all j, while if $IM \neq M$, the smallest integer j such that $\mathrm{Ext}_R^j(R/I, M) \neq 0$ is the depth of M on I.

From the above result and the long exact sequence for Ext we have:

PROPOSITION 3.2. If $0 \to A \to B \to C \to 0$ is an exact sequence of finitely generated modules over a Noetherian ring R, $I \subseteq R$ is an ideal, and $\mathrm{depth}_I B > \mathrm{depth}_I C$, then $depth_I A = \mathrm{depth}_I C + 1$.

In the sequel a *local* ring is a Noetherian ring R with a unique maximal ideal $\mathfrak{m}$. We say that $(R, \mathfrak{m}, K)$ is local to mean that R is local with maximal ideal $\mathfrak{m}$ and residue class field $K := R/\mathfrak{m}$. If $(R, \mathfrak{m}, K)$ is local of Krull dimension d, then d is the least number of generators of an ideal whose radical is $\mathfrak{m}$. A sequence of elements $x_1, \ldots, x_d \in \mathfrak{m}$ in a local Noetherian ring $(R, \mathfrak{m}, K)$ is called a *system of parameters* if the radical of $(x_1, \ldots, x_d)R$ is $\mathfrak{m}$ or, equivalently, $(x_1, \ldots, x_d)R$ is $\mathfrak{m}$-primary.

A local ring R is called *Cohen-Macaulay* if some (equivalently, every) system of parameters is a regular sequence. An equivalent condition is that $\mathrm{depth}_\mathfrak{m} R = d = \dim(R)$. A Noetherian ring R is called *Cohen-Macaulay* if all of its local rings at maximal (equivalenty, at prime) ideals are Cohen-Macaulay. Cohen-Macaulay rings R are also characterized by the property that if $I = (f_1, \ldots, f_n)R$ is a proper ideal of height n, then every associated prime of I is a minimal prime of I (and then, necessarily, of height n). In a Cohen-Macaulay local ring, a sequence of elements is a regular sequence if and only if it is part of a system of parameters. Moreover, if $(R, \mathfrak{m}, K)$ is local and $f_1, \ldots, f_h$ is a regular sequence, then R is Cohen-Macaulay if and only if $R/(f_1, \ldots, f_h)R$ is Cohen-Macaulay. It is also the case that $(R, \mathfrak{m}, K)$ is Cohen-Macaulay if and only if its $\mathfrak{m}$-adic completion $\widehat{R}$ is Cohen-Macaulay.

We also note that a Cohen Macaulay ring $(R, \mathfrak{m}, K)$ of Krull dimension d is called *Gorenstein* if $\mathrm{Ext}_R^d(R, K) \cong K$. There are many other characterizations.

Polynomial and formal power series rings in finitely many variables over a Cohen-Macaulay ring are again Cohen-Macaulay, and the property is preserved by localization or by taking a quotient by an ideal generated by a regular sequence. The following facts are also very useful:

THEOREM 3.3. *Let R be a Cohen-Macaulay ring, and let $P \subseteq Q$ be two prime ideals of R. Then any two* saturated *chains of primes (meaning that there is no prime properly between any two consecutive primes in the chain) joining P to Q have the same length.*

A ring that satisfies the conclusion of the theorem is called *catenary*. Thus, Cohen-Macaulay rings are catenary. See §2.1 of [**3**].

THEOREM 3.4. *Let R be a Cohen-Macaulay local ring, and let P be any minimal prime of R. Then $dim(R/P) = dim(R)$.*

3.1. The graded case. In discussing a graded ring R over a field K, we always mean that R is a finitely generated $\mathbb{N}$-graded K-algebra such that $R_0 = K$. We do not require that R be generated by forms of degree 1. We shall always use

$\mathfrak{m}$ for the homogeneous maximal ideal $\bigoplus_{n=1}^{\infty} R_n$ in R. Such a ring R is always a finitely generated module over a subring A generated by algebraically independent forms. That is, A is a graded subring, and isomorphic to a polynomial ring over A. If R is generated by forms of degree 1 and the base field K is infinite, then A can also be taken to be generated by forms of degree 1.

In this graded case, R is Cohen-Macaulay if and only if $R_{\mathfrak{m}}$ is Cohen-Macaulay, and R is Cohen-Macaulay if and only if R is a free module over the polynomial subring A for one (equivalently, every) choice of A.

In the graded case, if $f_1, \ldots, f_k$ is a regular sequence of forms of positive degree in the homogeneous maximal ideal of the $\mathbb{N}$-graded ring R, then R is Cohen-Macaulay if and only if $R/(f_1, \ldots, f_k)R$ is Cohen-Macaulay. This situation is entirely similar to the local case.

We also note that in the graded case 3.1, R is Gorenstein if and only if $R_{\mathfrak{m}}$ is Gorenstein.

Throughout the rest of this paper, the graded case, as described above, is the main case, and the reader should always think of this case.

3.2. Pure dimension. We shall say that a Noetherian ring is of *pure dimension* d if for every maximal ideal m and and minimal prime P contained in m the dimension of R_m/P is d.

A domain finitely generated over a field or over $\mathbb{Z}$ has pure dimension. A Cohen-Macaulay ring that is local, or graded as in 3.1, also has pure dimension. The local case is Theorem 3.4.

4. Determinantal ideals, Grassmannians, and Schubert varieties

In this and the following section, we suggest that the reader think about the graded case with the conventions of 3.1 in reading the discussion. However, to achieve natural generality, we frequently assume that certain Noetherian rings R arising have *pure dimension*, as discussed in 3.2.

In [**10**], it is shown that the ideal $I_t(X)$ generated by the size t minors of an $r \times s$ matrix $X = (x_{ij})$ of indeterminates over a field K is prime in the polynomial ring $K[X] := K[x_{ij} : i, j]$ generated by the indeterminates over K, and that $K[X]/I_t(X)$ is Cohen-Macaulay. The idea of the proof is as follows. One considers, for the purpose of induction, a much large class of ideals. These arise as follows. One takes a sequence of submatrices $X_0, \ldots, X_t = X$ of X each of which consists of an initial segment (starting on the left) of the columns of X. Each X_i is a submatrix of X_{i+1}, that is, the horizontal sizes s_i are strictly ascending, and $s_i \geq i - 1$. One considers all the ideals $I_1(X_1) + I_2(X_2) + \cdots + I_t(X_t) + J_h + A_{h,k}$, where J_h is generated by the variables of h consecutive top rows of X (i.e., by all the x_{ij} for $1 \leq i \leq h$ and $1 \leq j \leq s$) and $A_{h,k}$ is generated by the first k variables in the row indexed by $h + 1$ (i.e., by the $x_{h+1,j}$, $1 \leq j \leq k$). If the horizontal size of X_i is $i - 1$, then $I_i(X_i) = (0)$. The idea of the proof is to show that all of these ideals are radical, to determine which are prime, and to show that each of the ideals that is not prime is the intersection of two larger primes in the family. The very large size of this family makes it possible to do this by induction. The primes occur when h is one of the s_i. It is then proved that the primes give Cohen-Macaulay quotients using the following easy but critically important fact:

PROPOSITION 4.1. Let I_1, I_2 be ideals of the Noetherian ring R such that R/I_1 and R/I_2 are Cohen-Macaulay of pure dimension d, while $R/(I_1 + I_2)$ is Cohen-Macaulay of pure dimension $d - 1$. Then $R/(I_1 \cap I_2)$ is Cohen-Macaulay of pure dimension d.

PROOF. It suffices to prove the result after localizing at a maximal ideal of R, so that we may assume that the ring is local. One has the short exact sequence

$$0 \to R/(I_1 \cap I_2) \to R/I_1 \oplus R/I_2 \to R/(I_1 + I_2) \to 0,$$

and the result now follows from Proposition 3.2.   □

The plan of the proof that the prime ideals in the family, including $I_t(X)$, have Cohen-Macaulay quotients is simply this. One uses induction, assuming the result for larger ideals in the family. If I is one of the prime ideals in the family that one wants to prove gives a Cohen-Macaulay quotient, one considers the first variable x not already in I in the first row that is not already contained in I. Then R/I is Cohen-Macaulay if and only if $R/(I + xR)$ is Cohen-Macaulay: since I is prime, x is not a zerodivisor in R/I. If $I + xR$ is a prime in the family of ideals under consideration, the proof is complete, by induction. If not, one can show that $I + xR$ is the intersection of two larger prime ideals in the family: $I + xR = I_1 \cap I_2$, where $I_1 + I_2$ is also a prime ideal in the family. Moreover, $\dim\bigl(R/(I + xR)\bigr) = \dim(R/I_1) = \dim(R/I_2)$, and $\dim\bigl(R/(I_1 + I_2)\bigr) = \dim(R) - 1$. The result now follows from Proposition 4.1.   □

Geometrically, one has the union of two varieties of the same dimension whose intersection has dimension one less, as in the discussion at the beginning of §2. This single idea enables one to prove that a great many geometrically important varieties have the Cohen-Macaulay property!

The application to Grassmann and Schubert varieties is closely related. The following result is used in [**7**] to prove that the homogeneous coordinate rings of standard Grassmann varieties and their Schubert subvarieties are Cohen-Macaulay. Here, the geometric objects (apply Spec) fit together like the faces of an $(n - 1)$-simplex (which give a triangulation of an $(n - 2)$-sphere).

The following result is a variant of Proposition 1.4 of [**7**].

PROPOSITION 4.2. Let $I_1, \ldots, I_n$ be ideals of a Noetherian ring R. Assume that for the least family of ideals closed under union and intersection that contains the ideals $I_1, \ldots, I_n$, intersection distributes over sum. Suppose that all of the rings R/I_j are Cohen-Macaulay of pure dimension d, and that if J is the sum of any k distinct I_j, then R/J is Cohen-Macaulay of pure dimension $d - k + 1$. Then the intersection of any k of I_j gives an ideal I such that R/I is Cohen-Macaulay of pure dimension d.

PROOF. We use induction on n. The induction hypothesis applies to $I_1 + I_n, \ldots, I_{n-1} + I_n$. Hence, the intersection, which is $(I_1 \cap \cdots \cap I_{n-1}) + I_n$, yields a Cohen-Macaulay quotient of dimension $d - 1$, while $I_1 \cap \cdots \cap I_{n-1}$ (using the induction hypothesis again) and I_n both yield Cohen-Macaulay quotients of dimemension d. We may now apply Proposition 4.1.   □

5. Toric varieties, constructability, and shellability

We want to discuss the proof in [**6**] that a normal subring R of a polynomial ring $K[x_1, \ldots, x_n]$ over a field that is generated by monomials is Cohen-Macaulay.

The result is also valid when one adjoins inverses for the variables, but for simplicity we stick with the case of an ordinary polynomial ring.

In order to convey one of the key ideas in the proof, we need to talk about polytopal complexes. The main example that we have in mind in this section is the boundary complex of convex polytope in Euclidean space (a convex polytope is simply the convex hull of a finite set of points).

More generally, we may have a finite set Π of such convex polytopes that includes all of the faces of any polytope occurring, and such that any two of the convex polytopes in the complex meet in a polytope which is a face of each of them. The subcomplexes of Π are the polytopal complexes such that the set of convex polytopes is a subset of Π.

Let $L(\Pi)$ denote the set of finite unions of faces occurring in Π. This is a distributive lattice under $\cap$ and $\cup$. By a Π-family of ideals $\mathcal{I}$ in the ring R we mean a family of ideals closed under sum and intersection together with an order-reversing bijection $\alpha : \mathcal{I} \to L(P)$ such that for all $I, I' \in \mathcal{I}$ we have

(1) $\alpha(I + I') = \alpha(I) \cup \alpha(I')$ and $\alpha(I \cap I') = \alpha(I) \cup \alpha(I')$ and

(2) $\dim(R/I) - \dim(\alpha(I))$ is a constant c, independent of $I \in \mathcal{I}$.

Note that $\mathcal{I}$ must be a distributive lattice under $+$ and $\cap$ when this happens. A question of great interest is this: suppose that one has a polytopal complex Π and a Π-family of ideals $\mathcal{I}$ in R such that for all of the ideals J corresponding to faces, R/J is Cohen-Macaulay of pure dimension (the pure dimension condition is automatic in the graded case: see 3.1 and 3.2). When can one conclude that the smallest ideal J_0 in $\mathcal{I}$, which is the same as the intersection of the ideals corresponding to facets, is such that R/J_0 is also Cohen-Macaulay? With this terminology, Proposition 4.2 may be thought of as asserting that one has this result when the polytope is the boundary complex of a simplex.

In proving that normal rings generated by monomials are Cohen-Macaulay, the idea of the crucial step is this. After embedding the ring generated by monomials in a polynomial ring in a sufficiently good way, one can see that a principal ideal in the ring R generated by a certain monomial μ has a primary decomposition in which the ideals $I_1, \ldots, I_s$ in the primary decomposition correspond to the facets of the boundary of a convex polytope, and that the smallest family of ideals $\mathcal{I}$ containing them is a Π-family of homogeneous ideals, where Π is the boundary complex of a convex polytope. Moreover, one knows by induction that the rings R/J, where J is any face, are Cohen-Macaulay. In this case, the smallest ideal $J_0 = I_1 \cap \cdots \cap I_s$ corresponds to the entire boundary complex. One wants to conclude that $R/\mu R$, and, consequently, R, is Cohen-Macaulay, so that one would like to have a generalization of Proposition 4.2 that is sufficient to complete the proof in this situation.

In fact, such a result can again be proved by repeated application of Proposition 4.1. To see this, we need to discuss the notion of constructibility of polytopal (and, as a special case, simplicial) complexes. The definition is recursive. A complex consisting of just one polytope and all its faces is constructible. A complex is also constructible if it is the union of two constructible subcomplexes of the same dimension d whose intersection is constructible of dimension $d - 1$.

If Π is constructible, it follows by a straightforward induction that when the quotients of R by the ideals corresponding to faces are constructible, so is the quotient R/J_0, where J_0 is the smallest ideal in $\mathcal{I}$.

Therefore the Cohen-Macaulay property for normal rings generated by monomials follows if one knows that the boundary complex of a convex polytope is constructible. A stronger result was proved by Brugesser and Mani [**2**]: the boundary complex is shellable. This is a notion defined recursively in a manner similar to the way that the notion of constructible is defined, but all of the complexes that occur in the recursion are required to be shellable and to be, topologically, balls or spheres. The result of [**2**] is thus used in [**6**] to complete the proof that normal rings generated by monomials are Cohen-Macaulay.

This shellability result was used by McMullen [**16**] in the proof of the upper bound conjecture for the case of triangulations of spheres that can be obtained from the boundary complex of a convex polytope. This suggests the connection between the Cohen-Macaulay property and the upper bound conjecture for spheres.

6. Reisner's theorem

Throughout this section K is a field, $K[x_1, \ldots, x_n]$ is a polynomial ring, Δ is a simplicial complex with vertices $x_1, \ldots, x_n$ (i.e., a family of subsets of $\{x_1, \ldots, x_n\}$ containing the subsets $\{x_i\}$ for $1 \le i \le n$ and closed under taking subsets). The maximal elements of Δ are called *facets*. I_Δ is the ideal of $R = K[x_1, \ldots, x_n]$ generated by monomials in the x_j whose support is not in Δ, where the support of the monomial μ is the set of x_j that divide μ. Every ideal of R generated by square-free monomials has the form I_Δ for a unique choice of Δ. We denote by $H^i(\Delta, A)$ the simplicial cohomology of Δ with coefficients in A (this agrees with, say, singular cohomology of the geometric realization $|\Delta|$ of Δ), and $\widetilde{H}^i(\Delta, A)$ the *reduced simplicial cohomology* of Δ with coefficients in A. (This is different from $H^i(\Delta, A)$ only when $i = 0$. If $i = 0$ and $A = K$ is a field, $\widetilde{H}^0(\Delta, K)$ is a vector space of dimension one smaller than $H^0(\Delta, K)$, and so vanishes when Δ is connected.)

The closed star of a vertex x of Δ is the subcomplex of Δ consisting of all simplices of Δ that contain x. It is a cone with vertex x over the *link* of x: the link is the subcomplex consisting of all $\sigma \in \Delta$ such that $x \notin \sigma$ and $\sigma \cup \{x\} \in \Delta$. One can iterate taking links of vertices. If one iterates, first taking the link of x_0, then the link of x_1 within the link of x_0, and so forth, through x_k, the resulting subcomplex can be described in terms of simplex $\tau := \{x_0, \ldots, x_k\} \in \Delta$ as the set of $\sigma \in \Delta$ disjoint from τ such that $\sigma \cup \tau \in \Delta$. This is called the *link* of τ in Δ. Reisner's theorem [**18**] states:

THEOREM 6.1. *The ring $R_\Delta := K[X]/I_\Delta$ is Cohen-Macaulay over the field K if and only if when Λ is Δ or any link of any simplex in Δ, one has that $\widetilde{H}^i(\Lambda, K) = 0$ for $0 \le i < dim(\Lambda)$.*

When Δ satisfies either of the conditions that this theorem asserts are equivalent, it is called *Cohen-Macaulay* over K. This is actually a topological condition on the geometric realization $|\Delta|$ of Δ: that is clearly true for the vanishing conditions on the reduced cohomology of Δ itself. But, as is shown in [**17**], the condition on the vanishing of the cohomology of the link of τ can be rephrased as a condition on the local cohomology (in the topological sense) of Δ at a point relatively interior to τ. (The local cohomology at y may be thought of as the cohomology of the

space obtained by collapsing the complement of an small open neighborhood of y to a point, and taking the direct limit as the open neighborhood becomes smaller and smaller. At a point y relatively interior to $|\tau|$, there are arbitrarily small open neighborhoods where the space obtained when one collapses the complement of the neighborhood to a point is the suspension of $|\tau|$.) Hence, the needed vanishing of cohomology of links can be replaced by a suitable statement about vanishing of local cohomology at points of $|\Delta|$.

We comment briefly on some proofs of Reisner's theorem. Reisner used reduction to characteristic $p > 0$ to handle all cases. One first shows that the Cohen-Macaulay conditions on the localizations of R_Δ at the variables x_i (one adjoins an inverse for x_i) are equivalent to the those for the links of the various x_i in Δ. One therefore needs to think about what characterizes the Cohen-Macaulay property for a locally Cohen-Macaulay graded ring. It turns out that one simply needs vanishing of certain positive degree cohomology of the twists $\mathcal{O}_X(t)$ of the structure sheaf of $X := \mathrm{Proj}(R_\Delta)$ as t varies in the integers. These always vanish for large t. In the locally Cohen-Macaulay case they also vanish for sufficiently large negative t. The rings R_Δ are F-split: one has the Frobenius map $F : R \to R$, so that one has $F(R) \subseteq R$, and this splits as a map of $F(R)$-modules. See [**9**] for a treatment. This condition implies that $\mathcal{O}(t)$ injects into $\mathcal{O}(p^h t)$ for all h, and this gives the required vanishing of cohomology for all t except 0. Then one completes the argument by showing that the needed vanishing of cohomology of $\mathcal{O}_X$ is equivalent to the vanishing of $\widetilde{H}^i(\Delta, K)$ for $i < \dim(\Delta)$.

One can prove the theorem in a more purely algebraic way by using (algebraic) local cohomology modules to measure depth. One then studies the multigraded pieces of the local cohomology $H^i_{\mathfrak{m}}(R_\Delta)$ where $\mathfrak{m} = (x_1, \ldots, x_n)R_\Delta$. See, for example, [**3**].

A third proof of the sufficiency of Reisner's condition is discussed briefly in the next section.

We note that much other information about R/I_Δ can be obtained from the study of Δ: cf. [**8**], for example.

7. Thompson's theorem

The following is a slight variant (with the same proof) of the main result of [**24**], which establishes the result one expects on unions of Cohen-Macaulay varieties without a constructibility hypothesis.

THEOREM 7.1. *Let Δ be a simplicial complex that is Cohen-Macaulay over a field K, i.e., which satisfies Reisner's criterion. Suppose $\mathcal{I}$ is a Δ-family of ideals in a Noetherian K-algebra R, and that for every face of Δ, if I is the corresponding ideal in $\mathcal{I}$, then R/I is Cohen-Macaulay of pure dimension. Let I_0 be the least ideal in $\mathcal{I}$, which corresponds to Δ (it is the intersection of the ideals corresponding to maximal faces). Then R/I_0 is Cohen-Macaulay.*

This result implies the sufficiency of Reisner's criterion for R_Δ to be Cohen-Macaulay: in Reisner's case, the quotients by the ideals corresponding to faces are polynomial rings.

8. Hodge algebras, F-split rings, and F-regular rings

The theory of Hodge algebras (quotients by generic determinantal ideals and homogeneous coordinate rings of Grassmannians are included among these), developed in [**4**], and for which there is an expository account in [**3**], permits one to deform algebras arising in algebraic geometry to Stanley-Reisner rings, and then to deduce the Cohen-Macaulay property from the topological properties of a simplicial complex associated to a poset (the simplices correspond to totally ordered subsets), and, consequently, from combinatorial properties of the poset.

Checking whether a ring is F-split can sometimes be achieved by this method (when $K[\Delta]$ is Gorenstein), and proving that a ring is F-split is often a step towards proving F-regularity, a much stronger condition controlling the singularities associated with the ring. Cf. [**11**] and [**12**].

There is now considerable evidence that one can expect numerous interactions between the studies of F-split rings, F-regular rings, rings that are of particular combinatorial interest such as homogeneous coordinate rings of Richardson varieties ([**13**] gives a recent example) and cluster algebras ([**1**] gives another recent example), and Stanley-Reisner rings. The applications of Stanley-Reisner rings will be of ongoing importance in commutative algebra and algebraic geometry.

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