DIAMOND CLOSURE

by Melvin Hochster and Juan D. Vélez

Section 1.

INTRODUCTION

Throughout, all rings are commutative, associative, with identity, ring homomorphisms are assumed to preserve the identity, and modules are assumed to be unital.

Our objective is to generalize results of tight closure theory for ideals in Noetherian rings of characteristic $p > 0$ to Noetherian rings that may not contain a field. We refer the reader to [HH1–12], [Hu], [Ho2-3], [AHH] and [Bru] for information about tight closure theory and related topics. The theory that is obtained from the notion of diamond closure presented here is not everything that one would hope for. For example, at this point it does not appear to lead to a solution of the direct summand conjecture (cf. [Ho1]). Its main failing is that not every ideal of a regular ring of mixed characteristic is closed in this sense.

However, it does offer better control of what is contained in a colon ideal formed from a system of parameters in a mixed characteristic local ring than what can be proved for some alternatives, such as solid closure (cf. [Ho2]). In general, it is smaller than the intersection of the integral closure of the ideal I with the inverse image in the ring R of the tight closure of IR/pR in R/pR. Cf. Proposition (2.9).

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The notion of diamond closure is based on defining certain "powers" of ideals $I^{\langle n\rangle}$ (a fixed choice of prime $p > 0$ is assumed). It then turns out that $u \in I^{\diamondsuit}$, which is our notation for the diamond closure of I (we use $I^{\diamondsuit p}$ if it is necessary to indicate p) means that for a suitably restricted choice of $c \in R$, $cu^n \in I^{\langle n\rangle}$ for all $n \gg 0$. The details are given in §2. In §3 several results are proved that show that diamond closure has several of the good properties of tight closure. E.g., if R is a complete local normal domain of mixed characteristic p and S is a module-finite extension of R, then $IS \cap R \subseteq I^{\diamondsuit}$, and if $x_1, \ldots, x_n$ is a system of parameters for R containing p then for any choice of $x_{k+1} \neq p$, with $I = (x_1, \ldots, x_k)R$ one has $I :_R x_{k+1} \subseteq I^{\diamondsuit}$. This last is an analogue of the so-called "colon-capturing" property for tight closure. More general forms of this result are given in §3. In the cases where it applies, this result greatly strengthens the known result that places the colon ideal inside the integral closure of I. (See, for example, Theorem (1.2) of [EHU] and the preceding discussion.)

The diamond closure of an ideal is typically much smaller than the integral closure: it is contained in both the integral closure, and in the inverse image of the tight closure working modulo (p).

In a very brief fourth section a connection is made, for ideals I in finitely generated $\mathbb{Z}$-algebras R, of $\bigcap_p I^{\diamondsuit p}$ with the tight closure in equal characteristic zero (in the sense of [HH12]) of $\mathbb{Q} \otimes_{\mathbb{Z}} I$ in $\mathbb{Q} \otimes_{\mathbb{Z}} R$.

Section 2.

THE DEFINITION OF DIAMOND CLOSURE FOR IDEALS

Throughout, p denotes a fixed positive prime integer. We are very interested in closure operations for local domains such that p is in the maximal ideal. However, for the moment, we place no restriction on the Noetherian ring R.

For every such p we shall define a closure $I^{\diamondsuit}$ for ideals I in R which we refer to as the *diamond closure* of I in R. If p is not clear from context, the term *diamond p-closure* may be used.

In case R has characteristic p, so that $p = 0$ in R, this notion is identical with tight closure, under very mild conditions on R. When R is a local domain with positive residual characteristic it is understood that, unless otherwise specified, p is to be taken to be the characteristic of the residue field. Hence, we shall not usually indicate p in the notation, although the notion is only defined once some positive prime integer has been fixed. If we do need to specify which prime integer is being used we shall write $I^{\diamond p}$ for the diamond closure.

We shall always use q, q', q'', q_i, etc. to indicate powers of p with a nonnegative integer as the exponent: thus, 1 is included.

Throughout the rest of this paper, unless otherwise specified, all given rings are assumed to be Noetherian.

(2.1) Definition. If $\underline{u} = u_1, \dots, u_d$ is a sequence of elements of R and $n = qr$ where q is a power of p and p does not divide r, we denote by $\langle \underline{u} \rangle^{\langle n \rangle}$ the ideal of R generated by all elements of the form $q_1(u_1^{a_1} \cdots u_d^{a_d})^{q_2}$ where $q = q_1 q_2$ is a factorization of q into powers of p and the a_i are nonnegative integers whose sum is $q_1 r$. If $(\underline{u})_{q_2}$ denotes the ideal generated by the q_2 th powers of the u_i then we may also describe this ideal as

$$\sum_{q_1 q_2 = q} q_1((\underline{u})_{q_2})^{q_1 r}.$$

Another description is this: $\langle \underline{u} \rangle^{\langle n \rangle}$ is generated by those elements $q_1 U$ such that $q_1 | n$, U is a monomial in the u_j of degree n, and the exponent a_j on every u_j is such that $q_2 | a_j$.

(2.2) Theorem-Definition. *Let $\underline{u} = u_1, \dots, u_d$ be a sequence of elements of a ring R and let n be a positive integer. Then $\langle \underline{u} \rangle^{\langle n \rangle}$ depends only on the ideal I generated by $u_1, \dots, u_d$ and not on the choice of generators. We denote this ideal $I^{\langle n \rangle}$.*

Moreover:

(a) $I^{\langle n \rangle} \subseteq I^n$.

(b) *If $I \subseteq J$ then $I^{\langle n \rangle} \subseteq J^{\langle n \rangle}$.*

Before giving the proof we note the following fact about the behavior of binomial coefficients:

(2.3) Fact. *If $q = q'q''$ are powers of p, a is a positive integer, b is a positive integer not divisible by p, and $aq \geq bq'$ then $\binom{aq}{bq'}$ is divisible by q''.*

Proof. We have $\binom{aq}{bq'} = \frac{aq}{bq'}k$ with $k = \binom{aq-1}{bq'-1}$, an integer, and $\frac{aq}{bq'}k = \frac{q''ak}{b}$. Since p does not divide b, the result follows. □

Proof of (2.2). We first observe that given two different sequences of generators for the same ideal it suffices to compare the result from each with the result obtained from the concatenation of the two sequences or something even larger, since it is clear that enlarging the sequence can only make the result bigger. From this it easily follows that it suffices to show that the result obtained from a given sequence does not change when we enlarge it either with a multiple of one of its terms (this is trivial, since the various $(\underline{u})_{q_2}$ do not change), or else enlarge it by including the sum of two of its terms. Thus, we only need to show that it does not change when we enlarge a sequence with the sum of two of its terms.

To see this, since the order of the u_i clearly does not matter, we may assume that the additional term is $u_1 + u_2$. Let $\underline{v}$ denote the enlarged sequence. A typical generator of $\langle \underline{v} \rangle^{\langle n \rangle}$ will have the form

$$q_1(u_1 + u_2)^{a_0 q_2} u_1^{a_1 q_2} u_2^{a_2 q_2} h^{q_2}$$

where h is a monomial in $u_3, \cdots, u_n$ of degree D and $a_0 + a_1 + a_2 + D = q_1 r$. When we expand $(u_1 + u_2)^{a_0 q_2}$ by the binomial theorem and multiply out a typical term is

$$q_1 \binom{a_0 q_2}{b} u_1^{b + a_1 q_2} u_2^{c + a_2 q_2} h^{q_2}$$

where b, c are nonnegative integers such that $b + c = a_0 q_2$. If b and c are both divisible by q_2 it is obvious that this term is in $\langle \underline{u} \rangle^{\langle n \rangle}$. If not, we may assume that the highest power of p dividing both b and c is $q_3 < q_2$, and since $b + c$ is divisible by q_2 we have that q_3 is the highest power dividing b and also the highest power dividing c. But in this case $\binom{a_0 q_2}{b}$ is divisible by q_2/q_3, by (2.3). It follows that the given term is in $q_0 (\underline{u})_{q_3}^{q_0 r}$ where $q_0 = q_1(q_2/q_3) = q/q_3$.

Finally, the inclusion $I^{\langle n \rangle} \subseteq I^n$ is immediate from the definition, while the last statement (b) follows from the fact that we may include a given finite set of generators of I in a finite set of generators for J. □

Let I_t denote the ideal generated by all t th powers of elements of I (these are different from the ideals $(\underline{u})_t$ defined previously in Definition (2.1) by taking t th powers of specific generators).

Proposition (2.4). *Let R be a Noetherian ring. If n is a positive integer let $n = qr$ where $q = p^e$, $e \in \mathbb{N}$ and p does not divide r.*

(a) *If R has prime characteristic $p > 0$, then $I^{\langle q\rangle} = I^{[q]}$, and, more generally, $I^{\langle n\rangle} = (I^{[q]})^r$. On the other hand, if p is invertible in R then $I^{\langle n\rangle} = I^n$.*

(b) *In any ring R,*

$$I^{\langle n\rangle} = \sum_{q_1 q_2 = q} q_1 (I_{q_2})^{q_1 r}.$$

(c) *In particular, for all n, $I_n \subseteq I^{\langle n\rangle}$.*

(d) *Let $u_1, \ldots, u_k \in I$. Let $a_1, \ldots, a_k$ be be nonnegative integers whose sum is n. If p^s is the highest power of p dividing the multinomial coefficient $\binom{n}{a_1, \ldots, a_k}$, then each of the monomials $p^s u_1^{a_1} \cdots u_k^{a_k} \in I^{\langle n\rangle}$, and, hence, each of the terms $\binom{n}{a_1, \ldots, a_k} u_1^{a_1} \cdots u_k^{a_k}$ occurring in the multinomial expansion of $(u_1 + \cdots + u_k)^n$ is in $I^{\langle n\rangle}$. It follows that for any ring homomorphism $R \to S$, $(IS)_n \subseteq I^{\langle n\rangle} S$.*

Proof. The second statement in (a) implies the first. The second statement is immediate from the fact that the sum in Definition (2.1) has only one nonzero term in it, corresponding to the choices $q_1 = 1$ and $q_2 = q$. The final statement follows from part (b) below and the fact that with $q_1 = q$, $q_2 = 1$, we have a term in the sum given in part (b) of the form $qI^{qr} = qI^n$, and when p is invertible this is I^n.

For part (b), note that I_{q_2} will be generated by finitely many q_2 powers of elements of I, and these may be included in a set of generators. Thus, we may choose a finite set of generators for I so large that I_{q_2} is the ideal generated by their q_2 powers for all $q_2 \leq q$. We may then use these generators in Definition (2.1), and since $I_{q_2} = (\underline{u})_{q_2}$ (for these specific generators, with notation as in (2.1)) the result follows from the formula displayed in (2.1).

For (c), observe that if $u \in I$ and $n = qr$ then $u^n = 1(u^q)^r \in 1(I_q)^r$.

It remains to prove (d). Let q_2 be the greatest power of p that divides all of the integers a_i. Since we can include the u_i in a set of generators for I, the result follows if we can show that $\binom{n}{a_1, \ldots, a_k}$ is divisible by q/q'. At least one of the a_i is divisible by q' and no higher power of p. For definiteness, we may assume that it is a_1, by renumbering. Now $\binom{n}{a_1, \ldots, a_k} = \binom{n}{a_1}\binom{n-a_1}{a_2, \ldots, a_k}$, and so the result follows from the fact that q/q' divides $\binom{n}{a_1}$, by Fact (2.3). $\square$

The following result enables one to give a variant notion of tight closure in the positive prime characteristic case that agrees with the usual notion whenever one has test elements, i.e., in all good cases. The reason for proving this fact is that it helps motivate the notion of diamond closure, and shows that it agrees with tight closure, under mild hypotheses, if one is in positive prime characteristic.

Before stating the result, we recall that $c \in R^\circ$ is a test element for a Noetherian ring R of characteristic p if for every ideal I of R and element $u \in R$, $u \in I^*$ if and only if $cu^q \in I^{[q]}$ for all $q = p^e$. We also recall the every reduced ring that is essentially of finite type over an excellent local ring has a test element (in fact any $c \in R^\circ$ such that R_c is regular has a power that is a test element: cf. [HH9], §6). It is also worth noting that tight closure can be tested by testing modulo nilpotents, or by testing modulo each of the minimal primes. Thus, in building a theory for a closure operation of this kind, the domain case is the main case.

(2.5) Proposition. *Let R be a Noetherian ring of positive prime characteristic p. Let $u \in R$ and let I be an ideal of R.*

(a) *If there is an element c of R not in any minimal prime of R such that $cu^n \in I^{\langle n \rangle}$ for all sufficient large integers n then $u \in I^*$, the tight closure of I.*

(b) *Conversely, if R has a test element $c \in R^\circ$ and $u \in I^*$, then $cu^n \in I^{\langle n \rangle}$ for all n.*

Proof. (a) The condition is clearly sufficient for u to be in the tight closure, taking only values of n of the form $q = p^e$.

(b) For the converse, note that if u is in I^* then u^q is in the tight closure of $I^{[q]}$ for all q, and so u^{qr} is in the tight closure of $(I^{[q]})^r$ for all q and all r. Since c is a test element, $cu^{qr} \in I^{\langle n \rangle}$ for all large $n = qr$. □

(2.6) Definition. Now we define $I^\diamond$ to be the set of elements $u \in R$ such that there exists c in R not in any minimal prime of pR and not in any minimal prime of R such that $cu^n \in I^{\langle n \rangle}$ for all $n \gg 0$. The remarks of the preceding paragraph make the analogy with tight closure clear: in positive prime characteristic, for rings that have a test element, this will give the tight closure. It turns out that $I^\diamond \subseteq \overline{I}$, and that $I^\diamond$ is contained in the inverse image in R of the tight closure of $I(R/pR)$ in R/pR, as we shall see shortly. But note that it is not even clear that $I^\diamond$ is an ideal without some argument! We verify this next:

(2.7) Proposition. *Let R be a Noetherian ring, let I be an ideal of R, and let $u \in R$. Then*

$I^{\langle n\rangle}$ is an ideal of R. Moreover, the following three conditions on $u \in R$ are equivalent:

(a) $u \in I^{\diamond}$.

(b) *There is an element c not in any minimal prime of pR or R such that for all q and for all positive integers r not divisible by p,*

$$cu^{qr} \in \sum_{q_1 q_2 = q} q_1 (I_{q_2})^{q_1 r}.$$

(c) *There is an element c not in any minimal prime of pR or R such that for all q and for all positive integers r (whether divisible by p nor not),*

$$cu^{qr} \in \sum_{q_1 q_2 = q} q_1 (I_{q_2})^{q_1 r}.$$

Proof. We first show the equivalence of conditions (a), (b) and (c), and then we prove that $I^{\diamond}$ is an ideal. The condition in (b) is simply a restatement of the definition of the diamond closure of I, while it is clear that (c) implies (b). Thus, it suffices to see that (c) implies (b). Suppose that $r = q's$ where s is not divisible by p. Then we may think of qr as $(qq')s$ and so

$$cu^{qr} \in \sum_{q_1 q_2 = qq'} q_1 (I_{q_2})^{q_1 s}.$$

If q_1 divides q we can write $q_2 = q_0 q'$ where $q_1 q_0 = q$ and observe that $(I_{q_0 q'})^{q_1 s} \subseteq (I_{q_0})^{q' q_1 s} = (I_{q_0})^{q_1 r}$. If q divides q_1 the term is contained in $qI^{q_2 q_1 s} = qI^{qq's} = qI^{qr}$. Thus, (a), (b), and (c) are equivalent.

It remains to show that $I^{\diamond}$ is an ideal. It is clearly closed under taking multiples. It suffices to show that if $x \in I^{\diamond}$ and $y \in I^{\diamond}$ then $x + y \in I^{\diamond}$. Choose c not in any minimal prime of pR or R such that $cx^n \in I^{\langle n\rangle}$ for $n \gg 0$ and d not in any minimal prime of pR or R such that $dy^n \in I^{\langle n\rangle}$ for $n \gg 0$. Then a typical term in $cd(x+y)^n$ has the form

$$cd\frac{(qr)!}{(aq_0)!(bq_0)!}x^{aq_0}y^{bq_0}$$

where $aq_0 + bq_0 = n = qr$ (so that $a + b = (q/q_0)r$), and either $q_0 = q$ or $q_0 < q$ and p does not divide a or b. In the first case we have that the element is a multiple of

$$(cx^{aq})(dy^{bq}) \in \Big(\sum_{q_1 q_2 = q} q_1 (I_{q_2})^{q_1 a}\Big)\Big(\sum_{q_3 q_4 = q} q_3 (I_{q_4})^{q_3 b}\Big).$$

In the second case the binomial coefficient is divisible by q/q_0, and we wind up with a multiple of

$$(q/q_0)(cx^{aq})(dy^{bq}) \in \frac{q}{q_0}\Big(\sum_{q_1q_2=q_0} q_1(I_{q_2})^{q_1a}\Big)\Big(\sum_{q_3q_4=q_0} q_3(I_{q_4})^{q_3b}\Big).$$

This is the same formula as in the first case, since $q/q_0 = 1$ in that case. A typical term after we multiply out is $J = (q/q_0)q_1q_3(I_{q_2})^{q_1a}(I_{q_4})^{q_3b}$. Let q' be the greater of q_1, q_3 and q'' the lesser of q_2, q_4. Then $q'q'' = q_0$ and J is contained in $(q/q_0)q'(I_{q''})^d$ where $d = (q_2q_1a+q_4q_3b)/q'' = (q_0a+q_0b)/q'' = qr/q''$, and so J is contained in $(qq'/q_0)(I_{q''})^{qr/q''}$. Since $q'q'' = q_0$, the first coefficient may be rewritten as q/q'' and the result follows. □

(2.8) Theorem. *Let R be a Noetherian ring, and I, J ideals of R.*

(a) *If R has characteristic p, then $I^\diamond \subseteq I^*$, with equality if R has a test element.*

(b) *If p is invertible in R, then $I^\diamond = \overline{I}$.*

(c) *$I \subseteq I^\diamond$. Moreover, if $I \subseteq J$ then $I^\diamond \subseteq J^\diamond$.*

(d) *$(I^\diamond)^\diamond = I^\diamond$.*

(e) *If $R \to S$ is a homomorphism of Noetherian rings such that every minimal prime of S or of pS lies over a prime in R that is either minimal or a minimal prime of pR, and $I \subseteq R$ is an ideal of R, then $I^\diamond$ maps into $(IS)^\diamond$.*

Proof. (a) This is essentially Proposition (2.5).

(b) This follows from the fact that $I^{\langle n\rangle} = I^n$. Note that since pR has no minimal primes, the restriction on the multiplier c is simply that it not be in any minimal prime of R.

(c) The first statement is trivial. The second is immediate from the fact that $I^{\langle n\rangle} \subseteq J^{\langle n\rangle}$ for all n, which is clear from the original definition if we include generators of I among the generators of J.

(d) It suffices to show that when we enlarge I by including a single additional generator $u \in I^\diamond$, we have that $(I+uR)^\diamond = I^\diamond$, for then we may insert the generators of $I^\diamond$ one at a time with changing the closure. Thus, it suffices to show that if $u \in I^\diamond$ and $v \in (I+Ru)^\diamond$ then $v \in I^\diamond$. Choose c not in any minimal prime of R or pR such that $cu^n \in I^{\langle n\rangle}$ for all n, and c' not in any minimal prime of R or pR such that $c'v^n \in (I+Ru)^{\langle n\rangle}$ for all n. We shall show that $(cc')v^n \in I^{\langle n\rangle}$ for all n. Let $n = qr$, as usual, where p does not divide r. Choose a set of generators $u_1, \dots, u_d$ for I so large that $I_{q'}$ is generated by the elements $u_j^{q'}$ for all q' dividing q. Note that $c'v^n$ is a sum of terms $q_1 U u^a$ where q_1 divides q and U is a monomial in $u_1, \dots, u_d$ such that each exponent on any u_j is divisible by $q_2 = q/q_1$, where a is divisible by q_2, and such that the degree of Uu^a in the u_j and u is n. Let $a = q_2 b$. When we multiply by c we may replace cu^a by a sum of terms q_3U' where q_3 divides q_2, U' has total degree $q_2 b = a$ in the u_j and the exponent on every u_j is divisible by q_2/q_3 (cf. (2.7c)). But then, when we multiply out, all term in the sum for $cc'v^n$ have the form $(q_1q_3)(UU')$ where UU' is a monomial of total degree n in the u_j and the degree of any u_j in UU' is at least $q_2/q_3 = n/(q_1q_3)$, as required.

Part (e) is obvious. □

(2.9) Theorem. *Let S be a Noetherian ring and let I be an ideal of S.*

(a) *Let $R = S/pS$, and let $f : S \to R$ be the quotient surjection. Let $I_1 = IR$. Then $I^\diamond \subseteq f^{-1}(I_1^*)$, where I_1^* is the tight closure of I_1 in R. Moreover, $I^\diamond \subseteq \overline{I}$ as well.*

(b) *If I is not contained in any minimal prime of (0) and I is not contained in any minimal prime of p, then $p\overline{I} \subseteq I^\diamond$.*

Proof. (a) The final statement is a consequence of the fact that if $u \in I^\diamond$ then there exists $c \in R^\circ$ such that $cu^n \subseteq I^{\langle n\rangle}$ for $n \gg 0$, and $I^{\langle n\rangle} \subseteq I^n$. The statement that $u \in I^\diamond$ maps into I_1^* in R/pR follows from the fact that we have $cu^q \in I^{\langle q\rangle}$ for all $q \gg 0$. Since c is not in any minimal prime of p, its image in R/pR is not in any minimal prime of the ring, and it is immediate from the definition that $I^{\langle q\rangle}(R/pR) = I_1^{[q]}$.

(b) Let $J = \overline{I}$. Then there exists a positive integer k such that $J^n I^k = I^{n+k}$ for all n. Choose $c_1 \in I$ not in any minimal prime of R or pR. Let $c = c_1^k$. Then $cJ^n \subseteq I^{n+k} \subseteq I^n$ for all $n \in \mathbb{N}$. Thus, if $v \in J$, $c(pv)qr = p^n v^n \in qI^n = q(I_1)^n \subseteq I^{\langle n\rangle}$ for all n. It remains to see that $p\overline{I} \subseteq I^\diamond$.

(2.10) Examples. (a) Let R be a ring such that R/pR is regular. Then $I = I^\diamond$ for every ideal containing p, by (2.9a), since in R/pR every ideal is tightly closed. The same is likewise true if R/pR is weakly F-regular.

(b) Let V be a DVR with maximal ideal pV, where p is a positive prime integer, and let $R = V[[x,y]]$. Then $(x^2,y^2)^\diamond \subseteq (x^2,xy,y^2) \cap (x^2,y^2,p) = (x^2,y^2,pxy)$. On the other hand, by part (b), $pxy \in (x^2, y^2)^\diamondsuit$. This, $(x^2, y^2)^\diamondsuit = (x^2, y^2, pxy)$. Thus, ideals in regular rings are not closed in general.

(c) Let V a complete DVR whose maximal ideal is generated by a prime integer $p \neq 3$, and let $R = V[[x, y, z]]/(f)$ where $f = x^3 + y^3 + z^3 - p$. Note that R is regular! It is still not trivial to understand how diamond closure behaves in this ring. If we kill pR then the tight closure of (x^t, y^t) in the quotient is generated by $(xy)^{t-1}z^2$, $t \geq 1$. It follows that $(x^t, y^t)^\diamondsuit$ in R is contained in $(p, x^t, y^t, (xy)^{t-1}z^2)R$. But x, y is also part of a regular system of parameters, so that $(x^t, y^t)^\diamondsuit \subseteq \overline{(x^t, y^t)R} = (x, y)^t R$. Note that if $t = 1$, $(x, y)R$ is prime and so it its own diamond closure. In particular, the diamond closure need not map onto the tight closure of $(x, y)R/pR$ in R/pR.

If $t = 2$ or 3 then $p = z^3 \bmod (x^t, y^t)$, and we can conclude that the diamond closure is contained in $(z^3, x^t, y^t, (xy)^{t-1}z^2)$ as well as in $(x, y)^t$. Since x, y, z is a regular sequence in R, it is easy to see that this intersection is $(x^t, y^t, z^3(x, y)^t, (xy)^{t-1}z^2)$. We emphasize that $t = 2$ or 3 here. Now, mod (x^t, y^t), $z^3 \equiv p$, and so $z^3(x, y)^t$ is in $I^\diamondsuit$. Since x, y, and z all multiply $(xy)^{t-1}z^2$ into $(x^t, y^t, z^3(x, y)^t)$, we see that $I^\diamondsuit$ must be either $J = (x^t, y^t, z^3(x, y)^t)$ or $J + (xy)^{t-1}z^2R$. The issue can now be resolved by testing whether $(xy)^{t-1}z^2 \in (x^t, y^t)^\diamondsuit$: since pR is prime in R, this is the same as asking whether $\bigcap_n (x^t, y^t)^{\langle n \rangle} :_R ((xy)^{t-1}z^2))^n$ contains an element not in pR. We leave this as an exercise for the reader. The point here is that even in rather simple cases in regular rings, it is not so clear what $I^\diamondsuit$ is.

Section 3.

Some good properties of diamond closure

In this section we show that diamond closure does have several of the good properties that one would hope for in an analogue of tight closure.

Let R be a Noetherian domain such that p is not a unit in R. We shall say the the Noetherian domain R is *p-normal* if the localization of R at each minimal prime of pR is normal. This condition is automatic if $p = 0$, for then the localization is a field. If $p \neq 0$ the condition implies that each localization of R at a minimal prime of P is a DVR. If R is p-normal then the localization of R at the union of the minimal primes of pR is a semilocal PID (or a field). We shall need to assume that R is p-normal for some purposes. This condition is not too restrictive. Of course, it holds whenever R is normal.

In particular, we have the following: for those familiar with the notion of "plus closure" (cf. [Sm]), this result asserts that diamond closure captures plus closure — this is analogous to results for tight closure.

(3.1) Theorem (capturing plus closure). *Let R be a Noetherian domain that is p-normal in the sense described just above. Let I be an ideal of R, and let S be an integral extension ring of R. Then $IS \cap R \subseteq I^{\diamond}$.*

Proof. Let $I = (u_1, \dots, u_h)R$, and suppose that $v \in IS \cap R$, so that $v = \sum_{j=1}^{h} s_i u_i$ with the $s_i \in S$. Then we may replace S by $R[s_1, \dots, s_h]$ and therefore assume that S is module-finite over R. We may also choose a minimal prime P of S disjoint from $R - \{0\}$ and so replace S by S/P. Thus, it suffices to consider the case where S is a domain that is a module-finite extension of R.

Let W be the complement of the union of the minimal primes of pR in R. The p-normality of R implies that the localization R_W is either a PID or a field, and so S_W is free over R_W with 1 as part of a free basis, and, hence, $R_W \to S_W$ splits over R_W. By restricting the splitting to S and multiplying by an element c of W to clear denominators, we obtain an R-linear map $\phi\colon S \to R$ such that $\phi(1) = c \in W$.

Now with $r \in (u_1, \dots, u_h)S$ for every integer n we have that $r^n \in I^{\langle n \rangle}S$, by Proposition (2.4d). Applying ϕ to both sides shows that $cr^n \in I^{\langle n \rangle}$ for all n, as required. □

We can also prove the following result which is an analogue of the "colon-capturing" property of tight closure.

(3.2) Theorem (colon-capturing). *Let R be an A-torsion-free module-finite extension ring of a p-normal domain A, and let $x_1, \dots, x_d$ be a permutable regular sequence in A. Assume that one of the elements, say x_{i_0} is either a power of p or a root of p. Let $x \neq x_{i_0}$ be an element of the sequence. Let I be any ideal of R generated by monomials in the x_j other than x. Then Then $I :_R x \subseteq I^{\diamond}$. In particular, if $k+1 \neq i_0$, and $I = (x_1, \dots, x_k)R$, then $I :_R x_{k+1} \subseteq I^{\diamond}$.*

Proof. Since $A \to R$ is module-finite and R is a torsion-free A-module, when we localize R at $W = A - pA$ it becomes a free module over A_W, which is a PID. It follows that we can choose an element $c \in A - pA$ and a free A-submodule G of R such that $cR \subseteq G$.

By rewriting the monomials with some exponents possibly increased we may assume that x_{i_0} is p, rather than a power of p. This, we may assume without loss of generality that p is a power of x_{i_0}, possibly the first power.

Now suppose that $vx = \sum_{i=1}^{k} \mu_i r_i$, where the μ_i are the monomials in the x_j (other than x) that generate I. Raising both sides to the n th power, we obtain that $v^n x^n$ is a certain R-linear combination of monomials in the x_j with certain multinomial coefficients appearing. This is still true if we replace each multinomial coefficient that appears by the highest power of p that divides it, and it is also true that we can write each such power of p that occurs as a power of x_{i_0} instead. Call the ideal of A generated by these modified monomials J_0. Then $v^n x^n \in J_0 R$ where J_0 is generated by monomials of A in the x_j other than x. Thus, $x^n(cv^n) \in J_0 G$, where $cv^n \in G$. Since G is A-free, we have that the x_i form a permutable regular sequence on G. But then $J_0 G :_G x = J_0 G$ (cf. [EH]), and we have that $cv^n \in J_0 G \subseteq J_0 R$. But the generators of J_0 are in $I^{\langle n \rangle}$ by Proposition (2.4d). Thus, $cv^n \in I^{\langle n \rangle}$ for all n, as required. □

We have the following corollary:

(3.3) Theorem (colon-capturing). *Let R be a complete local ring of pure dimension d (i.e., there are no embedded primes, and all minimal primes have dimension d) of mixed characteristic p and let $x_1, \ldots, x_d$ be part of a system of parameters for R. Assume that a root or power of p occurs among the x_j, say as x_{i_0}. Let $x \neq x_{i_0}$ be an element of the system of parameters. Let I be any ideal of R generated by monomials in the x_j other than x. Then Then $I :_R x \subseteq I^{\diamond}$. In particular, if $k+1 \neq i_0$, and $I = (x_1, \ldots, x_k)R$, then $I :_R x_{k+1} \subseteq I^{\diamond}$.*

Proof. Let V be coefficient ring for R, i.e., a complete DVR subring of R with p as the generator of the maximal ideal such that the inclusion $V \to R$ is local and induces an isomorphism of residue class fields (cf. [C] or [G1, 0_{III}, 10.3] and [G2, §19]). Let y denote the parameters with x_{i_0} omited and let $A = V[[y]][x_{i_0}]$ (where the adjunction of the x_{i_0} is only need in case it is a root of p). Then R is module-finite over A, which is regular, and the condition that it have pure dimension implies that it is torsion-free as an A-module. The x_i form a permutable regular sequence in A, and the result is now immediate from Theorem (3.2). □

(3.4) Theorem (Briançon-Skoda theorem). *Let I be an ideal of a Noetherian ring R*

with d generators and suppose that I is not contained in any minimal prime of either R or pR. Then for every integer $m \geq 0$, $\overline{I^{d+m}} \subseteq (I^{m+1})^{\diamondsuit}$.

Proof. Let $I = (u_1, \dots, u_d)R$. Let $J = \overline{I^{d+m}}$. Then, exactly as in the proof of (2.9b), we can choose $c \in J$ not in any minimal prime of R or pR such that $cJ^n \subseteq (I^{d+m})^n$, and, as is shown in the proof of Theorem (5.4) in [HH4], $I^{dn+mn} \subseteq (u_1^n, \dots, u_d^n)^{m+1} \subseteq (I^{m+1})^{\langle n \rangle}$, since each of the obvious generators of $(u_1^n, \dots, u_d^n)^{m+1}$ is the n th power of a monomial of degree $m+1$ in the u_j. □

(3.5) Remark. While the Briançon-Skoda theorem is known for regular rings even in mixed characteristic (cf. [LT], [LS]), the result above may provide sharper information than is otherwise available in rings of mixed characteristic that are not regular. The same is true of Corollary (4.2) in the next section.

Section 4.

A connection with tight closure in equal characteristic 0

It is natural, in a finitely generated $\mathbb{Z}$-algebra R, to consider the intersection of the ideals $I^{\diamondsuit p}$ as p varies through all positive primes of $\mathbb{Z}$. We shall use $I^{\diamondsuit \infty}$ to denote this intersection We shall use * in this section to indicate tight closure in the sense of [HH12] for affine $\mathbb{Q}$-algebras. (This operation is also referred to as *equational tight closure* in [HH12]).

(4.1) Proposition. *Let R be a finitely generated $\mathbb{Z}$-algebra, I an ideal of R, and let $\_\,_{\mathbb{Q}}$ denote the result of tensoring over $\mathbb{Z}$ with the rational numbers $\mathbb{Q}$. If $u \in I^{\diamondsuit \infty}$ then the image of u in $R_{\mathbb{Q}}$ is in $(I_{\mathbb{Q}})^*$.*

Proof. This is obvious from (2.9a) and the definition of * in [HH12], since we have that the image of u is in $\big(I(R/pR)\big)^*$ for all primes p. □

The following result is an immediate consequence of Theorem (3.4) and the definition of $I^{\diamondsuit \infty}$.

(4.2) Corollary (Briançon-Skoda theorem). *Let I be a d generated ideal of a finitely generated $\mathbb{Z}$-algebra R that is not contained in any minimal prime of R nor of pR for any positive prime integer $p \in \mathbb{Z}$. Then for every positive integer m, $\overline{I^{d+m}} \subseteq (I^{m+1})^{\diamond\infty}$.* □

Bibliography

[AHH] I. M. Aberbach, M. Hochster, and C. Huneke, *Localization of tight closure and modules of finite phantom projective dimension*, J. Reine Angew. Math. (Crelle's Journal) **434** (1993), 67–114.

[Bru] W. Bruns, *Tight closure*, Bull. Amer. Math. Soc. **33** (1996), 447–458.

[C] I. S. Cohen, *On the structure and ideal theory of complete local rings*, Trans. Amer. Math. Soc. **59** (1946), 54–106.

[EH] J. A. Eagon and M. Hochster, *R-sequences and indeterminates*, Quart. J. Math. Oxford (2) **25** (1974), 61–71.

[EHU] D. Eisenbud, C. Hunke, and B. Ulrich, *Order ideals and a generalized Krull height theorem*, preprint.

[G1] A. Grothendieck, *Éléments de la géométrie algébrique III, Étude cohomolgoique des faisceaux cohérents (première partie)*, Publ. Math. de l'I.H.E.S. **11**, 1961.

[G2] ———, *Éléments de la géométrie algébrique IV, Étude locale des schémas et des morphismes de schémas (première partie)*, Publ. Math. de l'I.H.E.S. **20**, 1964.

[Ho1] M. Hochster, *Contracted ideals from integral extensions of regular rings*, Nagoya Math. J. **51** (1973), 25–43.

[Ho2] ———, *Solid closure*, in Commutative Algebra: Syzygies, Multiplicities and Birational Algebra, Contemp. Math. **159**, Amer. Math. Soc., Providence, R. I., 1994, pp. 103–172.

[Ho3] ———, *Tight closure in equal characteristic, big Cohen-Macaulay algebras, and solid closure*, in Commutative Algebra: Syzygies, Multiplicities and Birational Algebra, Contemp. Math. **159**, Amer. Math. Soc., Providence, R. I., 1994, pp. 173–196.

[HH1] M. Hochster and C. Huneke, *Tightly closed ideals*, Bull. Amer. Math. Soc. **18** (1988), 45–48.

[HH2] ———, *Tight closure*, in Commutative Algebra, Math. Sci. Research Inst. Publ. **15**, Springer-Verlag, New York-Berlin-Heidelberg, 1989, pp. 305–324.

[HH3] ———, *Tight closure and strong F-regularity*, Mémoires de la Société Mathématique de France, numéro **38** (1989), 119–133.

[HH4] ———, *Tight closure, invariant theory, and the Briançon-Skoda theorem*, J. Amer. Math. Soc. **3** (1990), 31–116.

[HH5] ———, *Absolute integral closures are big Cohen-Macaulay algebras in characteristic p*, Bull. Amer. Math. Soc. (New Series) **24** (1991), 137–143.

[HH6] ———, *Tight closure and elements of small order in integral extensions*, J. of Pure and Appl. Algebra **71** (1991), 233–247.

[HH7] ———, *Infinite integral extensions and big Cohen-Macaulay algebras*, Annals of Math. **135** (1992), 53–89.

[HH8] ———, *Phantom homology*, Memoirs Amer. Math. Soc., Vol. **103**, No. 490 (1993), 1–91.

[HH9] ———, *F-regularity, test elements, and smooth base change*, Trans. Amer. Math. Soc. **346** (1994), 1–62.

[HH10] ———, *Tight closure of parameter ideals and splitting in module-finite extensions*, J. of Algebraic Geometry **3** (1994), 599–670.

[HH11] ———, *Applications of the existence of big Cohen-Macaulay algebra*, Advances in Math. **111** (1995), 45–117.

[HH12] ———, *Tight closure in equal characteristic 0*, preprint (205 pp.).

[Hu] C. Huneke, *Tight Closure and Its Applications*, Proc. of the CBMS Conference held at Fargo, North Dakota, July, 1995, C.B.M.S. Regional Conference Series, Amer. Math. Soc., Providence, R.I., 1996.

[LS] J. Lipman and A. Sathaye, *Jacobian ideals and a theorem of Briançon-Skoda*, Michigan Math. J. **28** (1981), 199–222.

[LT] J. Lipman and B. Teissier, *Pseudo-rational local rings and a theorem of Briançon-Skoda about integral closures of ideals*, Michigan Math. J. **28** (1981), 97–116.

[Sm] ———, *Tight closure of parameter ideals*, Inventiones Math. **115** (1994), 41–60.

Department of Mathematics
University of Michigan
Ann Arbor, MI 48109–1109
USA

Universidad Nacional de Colombia
A.A.3840
Medellín
Colombia

E-mail:
hochster@math.lsa.umich.edu
jdvelez@perseus.unalmed.edu.co