THE NOTION OF TIGHT CLOSURE IN EQUAL CHARACTERISTIC ZERO[1]

BY MELVIN HOCHSTER[2]

1. Introduction Our objective is to explain how the results of tight closure theory in characteristic p can be extended to equal characteristic zero. The results described in this paper and not otherwise attributed are joint work of the author and Craig Huneke.[3] The detailed treatment of the theory we sketch here is given in [HH6]. One does in fact get, over an arbitrary Noetherian ring containing the rational numbers, $\mathbb{Q}$, a closure operation defined on submodules of finitely generated modules. The operation has the same kind of persistence properties as in the characteristic p case. For regular rings, every ideal (and every submodule of every finitely generated module) is tightly closed. The tight closure of an ideal is contained in the integral closure and is usually much smaller. One has the same kind of colon-capturing properties as in characteristic p, and, more generally, one has an analogous phantom homology theory. In consequence, one has a theory that yields equal characteristic 0 versions of what has been done in characteristic p: one gets a very short proof that direct summands of regular rings are Cohen-Macaulay (and more: they are F-regular), improved versions of the so-called "local homological conjectures" (these conjectures are now theorems, for the most part), and a tight closure version of the Briançon-Skoda theorem.

In [HH6] several notions of tight closure in equal characteristic 0 are developed. There is one for each field of equal characteristic 0, valid for algebras over that field, and a notion, *big equational tight closure*, that yields a larger tight closure than any of the theories linked to a specific field. However, here, for simplicity, we shall focus on the notion associated to the field of rational numbers, also called *equational* tight closure in [HH6], and denoted there by either ${}^{*\mathrm{eq}}$ or ${}^{*\mathbb{Q}}$. Since we shall be dealing with only one notion we denote it $*$, and we omit the adjective "equational." This is the smallest of our "tight closure" notions. It is not known whether the various characteristic zero notions of tight closure actually yield differing results. It is possible that they all agree, whenever they are defined. We should note that some of the definitions here are simpler than those in [HH6], although equivalent in the special case that we are studying.

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We shall use [HH1] and [Hu] as our basic references for characteristic p tight closure theory.

We conclude this section by recalling some of the alternative characterizations of tight closure available in characteristic p. Before immersing ourselves in the technique of reduction to characteristic p, we want to stress the point that a variant of one of these other characterizations may give a good theory in equal characteristic zero or even, possibly, in mixed characteristic. Having a different approach to the equal characteristic zero case would undoubtedly be very valuable.

Recall that for good rings of characteristic p, and, in particular, for rings possessing a completely stable test element, one may test tight closure by passing to the complete local domains of R, i.e., to the rings obtained by localizing R at a prime, completing, and killing a minimal prime. It turns out that $u \in I^*$ if and only if the image of u is in $(IB)^*$ for every complete local domain B of R. Thus, the central problem is to characterize tight closure in a complete local domain (R, m, K).

Part (a) of the Theorem (1.1) below is phrased in terms of Hilbert-Kunz functions: when I is primary to m, $\ell(R/I^{[p^e]})$ as a function of e is called the *Hilbert-Kunz function* (cf. [Ku], [Mo], [HaMo]). (Here, "ℓ" denotes length.) By a result of [Mo], if $\dim R = d$ there is a positive real number β_I such that for all $q = p^e$, $\ell(R/I^{[q]}) = \beta_I\, q^d + O(q^{d-1})$. If $\dim R = 1$ then β_I is a positive integer. In general, β_I is conjectured to be rational, but this is an open question even in dimension two! The behavior of the Hilbert-Kunz functions is quite surprising. For example, if $R = (\mathbb{Z}/5\mathbb{Z})[[x_1, x_2, x_3, x_4]]/(G)$ where $G = \sum_{i=1}^4 x_i^4$, then $\ell(R/m^{[5^e]}) = \frac{168}{61}(5^e)^3 - \frac{107}{61}(3^e)$. Cf. [HaMo].

Recall that the *absolute integral closure* of the domain R (cf. [Ar2]), which we denote R^+, is the integral closure of R in an algebraic closure of its fraction field, and is unique up to non-unique isomorphism.

When R is a domain, an R-module M is called *solid* if it possesses a nonzero R-module map to R. When (R, m, K) is a complete local domain of dimension d, this is equivalent to the condition that the local cohomology module $H_m^d(M) \neq 0$. Cf. [Ho3].

Theorem (1.1). *Let (R, m, K) be a complete local domain of characteristic p. Let $d =$ $\dim R$. Let I be an ideal of R and let $u \in R$. Each of the following statements (some of which include a supplementary hypothesis), gives a characterization of when u is in I^*.*

(a) *Suppose that I is m-primary. Let $J = I + Ru$. Then $u \in I^*$ if and only if $\beta_J = \beta_I$. More generally, if $I \subseteq J \subseteq m$ are m-primary then $J \subseteq I^*$ if and only if $\beta_J = \beta_I$. (This result characterizes tight closure in complete local domains of characteristic p, since an element is in the tight closure of $I \subseteq m$ if and only if it is in the tight closure of all m-primary ideals containing I.)*

(b) *Fix a discrete $\mathbb{Z}$-valuation nonnegative on R and positive on m and extend it to a valuation v of R^+ to $\mathbb{Q}$. Then $u \in I^*$ if and only if there exist elements $\theta \in R^+ - \{0\}$ with $v(\theta)$ arbitrarily small such that $\theta u \in IR^+$.*

(c) *(K. E. Smith* [Sm1].*) Assume that I is generated by part of a system of parameters. Then $u \in I^*$ if and only if $u \in IR^+$.*

(d) *$u \in I^*$ if and only if there exists a big Cohen-Macaulay algebra S for R such that $u \in IS$.*

(e) *$u \in I^*$ if and only if there exists a solid R-algebra S such that $u \in IS$ (see the discussion just prior to the statement of this theorem).*

Proof. By Theorem (8.17) of [HH1], when $I \subseteq J$ are m-primary ideals we have that $J \subseteq I^*$ if and only if $\lim_{e\to\infty} \frac{1}{q^d}\ell(J^{[q]}/I^{[q]}) = 0$, which, by Monsky's result, is equivalent to the condition that $\beta_I = \beta_J$, yielding (a). For (b) see [HH2]. Part (c) is proved in [Sm1]. The results of (d) and (e) are obtained in [Ho3] from the perspective of "solid closure." □

Since R^+ is a big Cohen-Macaulay algebra in the characteristic p case (cf. [HH3]) and since big Cohen-Macaulay algebras are solid, conditions (c), (d), and (e) are closely related. Whether tight closure in locally excellent domains of characteristic p is, in general, simply the contracted expansion from R^+ is an important open question. It reduces to the case of complete local domains. The answer is not known even in dimension 2.

It is known, however, that R^+ is never a big Cohen-Macaulay algebra in equal characteristic 0 if $\dim R \geq 3$. Moreover, the main result of [Ro5] shows that "solid closure," the notion defined by attempting to extend the characterization of tight closure given in part (e), does not behave well in equal characteristic 0: ideals of regular rings of dimension 3 fail to be solidly closed in general. But (d) might give a good notion in equal characteristic 0: see §8. See also [Ho4] and [HH5].

2. Comparison of fibers and a simple example It will turn out that if K is any field of characteristic 0 then in the ring $R = K[X, Y, Z]/(X^3 + Y^3 + Z^3) = K[x, y, z]$, the element z^2 is in the tight closure of $I = (x, y)R$, although it is not in I. Before giving the explanation, we want to recall that if $A \to R$ is a ring homomorphism, P is any prime ideal of A, and $\kappa(P) = A_P/PA_P$, then the *fiber* of $A \to R$ over P is defined as the $\kappa(P)$-algebra $\kappa(P) \otimes_A R$. If P is a maximal ideal μ of A then $\kappa(\mu) = A/\mu$ and the fiber is simply $R/\mu R$: fibers over maximal ideals are referred to as *closed* fibers. On the other hand if A is a domain and $P = (0)$, then $\kappa(P)$ is the fraction field of A, and the fiber over (0) may be identified with $(A - 0)^{-1}R$, and is referred to as the *generic* fiber.

When R is finitely generated over a Noetherian domain A of characteristic zero and one is studying some good property $\mathcal{P}$ of the fibers, especially a geometric property, it is often true that the generic fiber has property $\mathcal{P}$ if and only if there is a dense open subset U of the maximal spectrum of A such that the closed fibers over the points of U have property $\mathcal{P}$: we say that *almost all* closed fibers have $\mathcal{P}$ in this case. It is true that the generic fiber has $\mathcal{P}$ if and only if almost all closed fibers have $\mathcal{P}$ for the following properties:

(1) smoothness
(2) normality
(3) being reduced
(4) the Cohen-Macaulay property
(5) the Gorenstein property
(6) equidimensionality

Let $x_1, \ldots, x_n \in R$, let $u \in R$ and $I = (x_1, \ldots, x_n)R$. It is also true that (the image of) u is nonzero (or not in the expansion of I) in the generic fiber if and only if this is true for almost all closed fibers, and that if $x_1, \ldots, x_n$ generate a proper ideal of height

n in the generic fiber then the images of the x's generate a proper ideal of height n in almost all closed fibers, which will be reduced and equidimensional, but not necessarily domains. (Consider $\mathbb{Z}[x,y]/(x^2+y^2)$: whether the fiber modulo p is a domain depends on the residue of p modulo 4. This kind of behavior can be remedied by adjoining a square root i of -1 to $\mathbb{Z}$, after which one has non-domain behavior for all fibers over $Z[i]$.) There is a detailed and essentially self-contained exposition of results on comparison of fibers in [HH6], §(2.3).

The idea behind our tight closure theory in equal characteristic zero is to formulate the definition so that comparison of fibers "works" *a priori*, by virtue of the definition.

In the example that we are studying, it should turn out that z^2 is in the tight closure of $(x,y)R$ because, roughly speaking, when one "drops down" or "descends" to the finitely generated $\mathbb{Z}$-subalgebra of R, call it $R_{\mathbb{Z}}$, generated by x, y and z over $\mathbb{Z}$ (here, $R_{\mathbb{Z}} \cong \mathbb{Z}[X,Y,Z]/(X^3+Y^3+Z^3)$), then for almost every closed fiber of $\mathbb{Z} \to R_{\mathbb{Z}}$, i.e., modulo all but finitely many positive prime integers p, the image of z^2 modulo p is in the tight closure of $IR_{\mathbb{Z}}/pR_{\mathbb{Z}}$ in $R_{\mathbb{Z}}/pR_{\mathbb{Z}}$. (Fix $p \neq 3$. The rest of this parenthetical remark is carried through entirely modulo p, but we continue to write x, y and z for the images of the variables. It is enough to see that $x(z^{2q}) \in I^{[q]} = (x^q, y^q)$ for all $q = p^e$. Let $\kappa = \mathbb{Z}/p\mathbb{Z}$ and let the subscript $_\kappa$ indicate the result of tensoring with κ. We can write $2q = 3h + r$ where h is a nonnegative integer and $0 \le r \le 2$. Note that 1, z, z^2 is a free basis for R_κ over $\kappa[x,y] = T$, so that every element has a unique representation as $\tau_0 + \tau_1 z + \tau_2 z^2$ with the $\tau_i \in T$. Then $x(z^2)^q$ becomes $x(z^3)^h z^r = x(x^3+y^3)^h z^r$, and this will be in (x^q, y^q) provided that for all choices of nonnegative i, j with $i + j = h$, we have that at least one of the exponents in $x^{3i+1}y^{3j}$ is at least q. But if both exponents are at most $q-1$, we have that $3i + 1 + 3j \le 2q - 2$ or $3h + 1 \le 2q - 2 = 3h + r - 2$, contradicting $r \le 2$.)

We leave it as an exercise for the reader to verify that z is not in the tight closure of $(x,y)R$ and that z^2 is not in the integral closure of $(x,y)R$. Also, show that the ideal generated by x and y is tightly closed in the ring $(\mathbb{Z}/p\mathbb{Z})[X,Y,Z]/(X^2+Y^3+Z^5)$ precisely for those primes p such that $p \ge 7$.

3. A general definition for tight closure Let S be a locally excellent Noetherian ring containing the rationals. Keeping the example of the preceding section in mind, we want to give the general definition of when an element u of S is in the tight closure of an ideal J of S.

Definition (3.1). The element u is in the *tight closure* J^* of J if there exists a finitely generated $\mathbb{Z}$-subalgebra $R_{\mathbb{Z}}$ of S containing u such that, with $I_{\mathbb{Z}} = J \cap R_{\mathbb{Z}}$, one has that, for all but at most finitely many closed fibers R_κ of $\mathbb{Z} \to R_{\mathbb{Z}}$, the image u_κ of u in R_κ is in the (characteristic p) tight closure of the image of I_κ in R_κ.

We should make several remarks here. Once one has a choice of subalgebra $R_{\mathbb{Z}}$ that enables one to see that u is in the tight closure of J, every larger choice works as well: this is a consequence of persistence properties for tight closure in characteristic p. Thus, in the above definition, instead of saying that "there exists a finitely generated $\mathbb{Z}$-subalgebra ..." we could have said "for every sufficiently large finitely generated $\mathbb{Z}$-subalgebra ...." For simplicity we have given the definition only for ideals. But there is no difficulty extending the definition to the case of modules. As in the positive characteristic case, the definitions are set up so that an element u of the Noetherian module M is in the tight closure of $N \subseteq M$ if and only if the image of u in M/N is in the tight closure of 0 in M/N. If we map a finitely generated free R-module $G = R^h$ onto M and replace N by its inverse image in G and u by an element of G that maps to u, then we reduce the problem of defining tight closure to the case where the ambient module is $G = R^h$. The definition in this case is essentially the same as in the case of ideals.*

It is also worth noting in dealing with finitely generated $\mathbb{Z}$-algebras in this context that, after localizing at one nonzero integer a, one may assume that all of the modules, algebras, etc. that one is considering are free over the localized base $A = \mathbb{Z}_a$. This is a special case of the lemma of generic freeness. Even when one has a cokernel of a map of $\mathbb{Z}$-algebras, it is possible to localize and assume that cokernel is free. This means that for almost all closed fibers, an injection of finitely generated $\mathbb{Z}$-algebras stays injective when one passes to the fiber (i.e., works modulo the corresponding prime integer p). The following strong form of generic freeness, developed in [HR], Lemma (8.1), p. 146, suffices for our needs here.

Lemma (3.2) (generic freeness). *Let A be a Noetherian domain, let R be a finitely generated A-algebra, let S be a finitely generated R-algebra, let W be a finitely generated S-module, let M be a finitely generated R-submodule of W and let N be a finitely generated A-submodule of W. Let $V = W/(M + N)$. Then there exists an element $a \in A - \{0\}$ such that V_a is free over A_a.* $\square$

The very important special case where $R = S$ and M, N are both 0, has been known much longer (this case may be found in [Mat], §22.) In most cases where one uses generic freeness, the flatness of the module, algebra, etc. under consideration would suffice.

Note that, for example, since one may localize to make $R_{\mathbb{Z}}/I_{\mathbb{Z}}$ flat, it follows that $I_\kappa \to R_\kappa$ is an injection for almost all fibers.

4. Basic properties We shall use $\langle \_ \rangle$ to indicate the image of a module under a map: which map should be obvious from the context. The most frequent use of this notation is when $N \subseteq M$ are S-modules, S' is an S-algebra, and $\langle S' \otimes_S N \rangle$ denotes the image of $S' \otimes_S N$ in $S' \otimes_S M$.

*For the intrepid reader we comment briefly on the other notions of tight closure in the ideal case. If one is considering algebras over a field K of characteristic 0, in the definition of $*^K$ one replaces $R_{\mathbb{Z}}$ by an affine $\mathbb{Z}$-algebra $R_A \subseteq R$, where A is a suitably large affine $\mathbb{Z}$-subalgebra of K, and then one considers what happens for almost all closed fibers of $A \to R_A$. The largest notion, *big equational tight closure*, denoted $*^{\mathrm{EQ}}$, is governed by what happens in rings of the form C_Q that have local maps to the complete local domains of R, where C is an affine $\mathbb{Z}$-algebra and Q is a prime ideal meeting $\mathbb{Z}$ in (0). For every prime integer $p > 0$ let $B(p)$ be the localization of C/pC at the multiplicative system of all nonzerodivisors on $C/(Q + pC)$. For a ring of the form C_Q, an element $u \in C_Q$, and an ideal $I \subseteq C_Q$, one defines u to be in $I^{*\mathrm{EQ}}$ if for almost all prime integers $p > 0$ the image of u in $B(p)$ is in the characteristic p tight closure of $IB(p)$ in $B(p)$. If R is an L-algebra for some field L and $K \subseteq L$ is a subfield then $I^{*K} \subseteq I^{*L}$. I^*, which is $I^{*\mathbb{Q}}$, is contained in $I^{*\mathrm{EQ}}$, and if R is a K-algebra then I^{*K} is between them. Modules are treated in an entirely similar way.

The following result states many of the basic properties of tight closure in equal characteristic zero. The proof is given in §3.2 of [HH6]. The idea is generally to pass to consideration of one (or more) finitely generated $\mathbb{Z}$-algebras, and then deduce the result from a corresponding result in characteristic p by considering what happens for almost all fibers.

Theorem (4.1) (basic properties of tight closure in characteristic zero). *Let S be a locally excellent Noetherian algebra containing $\mathbb{Q}$. Let $N', N \subseteq M$ be finitely generated S-modules. Let $u \in M$ and let v be the image of u in M/N. Let I be an ideal of S. Unless otherwise indicated, * indicates tight closure in M.*

(a) *N^* is a submodule of M containing N.*
(b) *$u \in N^*{}_M$ if and only if $v \in 0^*{}_{M/N}$.*
(c) *If $N \subseteq N' \subseteq M$ then $N^*{}_M \subseteq N'^*_M$ and $N^*{}_{N'} \subseteq N^*{}_M$.*
(d) *$(N^*)^* = N^*$.*
(e) *$(N \cap N')^* \subseteq N^* \cap N'^*$.*
(f) *$(N + N')^* = (N^* + N'^*)^*$.*
(g) *$(IN)^*{}_M = \big((I^*{}_R)N^*{}_M\big)^*_M$.*
(h) *$(N :_M I)^*{}_M \subseteq N^* :_M I$ (respectively, $(N :_S N')^* \subseteq N^* :_S N'$). Hence, if $N = N^*$ then $(N :_M I)^* = N :_M I$ (respectively, $(N :_S N')^* = N :_S N'$).*
(i) *If $N_i \subseteq M_i$ are finitely many finitely generated S-modules and we identify $N = \oplus_i N_i$ with its image in $M = \oplus_i M_i$ then the obvious injection $\oplus_i N_i^*{}_{M_i} \hookrightarrow M$ maps $\oplus_i N_i^*{}_{M_i}$ isomorphically onto N^*_M.*
(j) *(Persistence of tight closure) Let $S \to S'$ be a ring homomorphism. Let $u \in N^*{}_M$. Then $1 \otimes u \in \langle S' \otimes_S N\rangle^*{}_{S' \otimes_S M}$ over S'.*
(k) *(Persistence of tight closure: second version). Let $S \to S'$ be as in (j), let $u \in N^*_M$, and let $V \subseteq W$ be finitely generated S'-modules. Suppose also that there is an S-homomorphism $\alpha\colon M \to W$ such that $\alpha(N) \subseteq V$. Then $\alpha(u) \in V^*_W$.*
(l) *(Irrelevance of nilpotents) If J is the nilradical of S, then $J \subseteq (0)^*$, and so $J \subseteq I^*$ for all ideals I of S. Consequently, $JM \subseteq N^*$. Moreover, if $\overline{N}$ denotes the image of N in M/JM, then N^* is the inverse image in M of the tight closure $\overline{N}^*{}_{M/JM}$, which may be computed either over S or over S_{red} $(= S/J)$.*
(m) *Let $\mathfrak{p}^{(1)}, \dots, \mathfrak{p}^{(s)}$ be the minimal primes of S and let $S^{(i)} = R/\mathfrak{p}^{(i)}$. Let $M^{(i)} = S^{(i)} \otimes_S M$ and let $N^{(i)}$ be the image of $S^{(i)} \otimes_S N$ in $M^{(i)}$. Let $u^{(i)}$ be the image of u in $M^{(i)}$. Then $u \in N^*$ if and only if $u^{(i)} \in (N^{(i)})^*$ in $M^{(i)}$ over $S^{(i)}$, $1 \le i \le s$.*
(n) *If $R = \Pi_{i=1}^h R_i$ is a finite product and $M = \Pi_i M_i$ and $N = \Pi_i N_i$ are the corresponding product decompositions of M, N, respectively, then $u = (u_1, \dots, u_h) \in M$ is in $N^*{}_M$ over R if and only if for all i, $1 \le i \le h$, $u_i \in N_i{}^*{}_{M_i}$.* □

The next result, which is essentially Theorem (3.4.1) of [HH6], shows that the notion of characteristic zero tight closure we are using can be tested after passing to the complete local domains of R. Some discussion of the ideas in the proof is given following the statement of the theorem.

Theorem (4.2). *Let S be a locally excellent Noetherian algebra containing $\mathbb{Q}$, and let $N \subseteq M$ be finitely generated S-modules. Then the following three conditions on an element $u \in M$ are equivalent:*

(1) $u \in N^*{}_M$.
(2) *For every maximal ideal* m *of* S, *if* B *is the quotient of the completion of* S_m *by a minimal prime, then the image of* u *in* $B \otimes_S M$ *is in the tight closure of* $\langle B \otimes_S N \rangle$ *working over* B.
(3) *For every prime ideal* P *of* S, *if* B *is the quotient of the completion of* S_P *by a minimal prime, then the image of* u *in* $B \otimes_S M$ *is in the tight closure of* $\langle B \otimes_S N \rangle$ *working over* B. □

The proof of this theorem depends on the fact that the Henselization of an excellent local ring of equal characteristic 0 is an approximation ring, i.e., that whenever a finite system of polynomial equations over the ring has a solution in the completion it has a solution in the Henselization and, hence, in a suitable étale extension of the original ring. This result, which generalizes [Ar1], can be deduced from general Néron desingularization, which asserts that an arbitrary regular (meaning flat with geometrically regular fibers) homomorphism of Noetherian rings is a filtered inductive limit of smooth homomorphisms (i.e., smooth of finite type). This means that if $R \to S$ is regular and factors $R \to R_1 \to S$ with R_1 of finite type over R then $R_1 \to S$ factors $R_1 \to S_1 \to S$ where S_1 is smooth over R. General Néron desingularization has a somewhat complicated history and there has been some disagreement concerning the correctness of what has been published. We refer the reader to [Po1, 2], [Og], [And], and [Sp] for further discussion. The special cases that we need here have been established independently in [Rot] and [ArR]. In particular, a self-contained argument that excellent Henselian rings of equal characteristic zero are approximation rings is given in [Rot].

To prove the most interesting implication in Theorem (4.2), namely, that (3) implies (1), one uses the approximation result for the localized completion at every maximal ideal, and then a kind of compactness argument to show that the element is in the tight closure after descending to a faithfully flat étale extension of a finitely generated (over $\mathbb{Q}$) subalgebra of the original ring. Characteristic p results then imply that the faithfully flat étale extension is unnecessary.

Remark. The definition of tight closure given in [HH6] for the case where the ring is not necessarily locally excellent forces the conclusion of Theorem (4.2) to hold: e.g., in the ideal case an element u of the ring is defined to be in the tight closure of I if it is in the tight closure (in the sense that we have already defined) of IB for every complete local domain B of R.

5. Descent from the complete case and the Artin-Rotthaus theorem There is a general strategy for proving results about tight closure: one first passes to the complete local case (often, the case of a complete local domain). In the second step one passes to the study of the finitely generated $\mathbb{Q}$- (and then $\mathbb{Z}$-) subalgebras of that complete local domain. The third step is to pass to closed fibers, using results comparing what happens at the generic fiber with what happens for almost all closed fibers.

To prove results on capturing colons and phantom homology, which we shall treat in the next two sections, one needs, at the second step, to pass to an affine $\mathbb{Z}$-subalgebra of a complete local ring while preserving heights. The following result, which is essentially Theorem (3.5.1) of [HH6], permits one to do this:

Theorem (5.1). *Let K be a field of characteristic zero and let (S, m, L) be a complete local ring that is a K-algebra. Assume that S is equidimensional and unmixed.*

Suppose that R_0 is a subring of S that is finitely generated as a K-algebra, and that we are also given finitely many sequences of elements $\{z_t^{(i)}\}$ in R_0, each of which is part of a system of parameters for S.

Then there is a finitely generated K-algebra R such that the homomorphism $R_0 \hookrightarrow S$ factors $R_0 \hookrightarrow R \to S$ and such that the following conditions are satisfied:

(1) *R is biequidimensional.*
(2) *The image of each sequence $\{z_t^{(i)}\}_t$ in R consists of parameters in the following sense: each subsequence consisting of, say, ν elements generates an ideal that has height ν modulo every minimal prime.*
(3) *If $\mathfrak{m}$ is the contraction of m to R, then $\dim R_{\mathfrak{m}} - \text{depth}\, R_{\mathfrak{m}} = \dim S - \text{depth}\, S$. In particular, $R_{\mathfrak{m}}$ is Cohen-Macaulay iff S is Cohen-Macaulay.*
(4) *If S is reduced (respectively, a domain) then so is R.* □

(N.B. In general, $\dim R_{\mathfrak{m}}$ is substantially bigger than $\dim S$.)

Of course, K may be $\mathbb{Q}$. Having passed to a finitely generated $\mathbb{Q}$-subalgebra, it is then easy to descend further to a finitely generated $\mathbb{Z}$-subalgebra.

We shall not say very much about the proof of Theorem (5.1), except that one thinks of the ring S as module-finite over a complete regular local ring, and reduces the problem of descending from S to that of descending from the complete regular local ring. The proof is then completed by using the following result, which is another special case of general Néron desingularization. A self-contained proof is given by Artin and Rotthaus in [ArR].

Theorem (5.2). *Let K be a field (or an excellent DVR). Let $T = K[[x_1, \dots, x_n]]$ be the formal power series ring in n variables over K. Then every K-algebra homomorphism of a finitely generated K-algebra R to T factors $R \to S \to T$, where the maps are K-algebra homomorphisms and S is smooth (of finite type) over $K[x_1, \dots, x_n]$. Thus, S is regular and $x_1, \dots, x_n$ is a regular sequence in S.*

6. Further properties of tight closure We are now ready to establish a number of important properties of characteristic 0 tight closure that parallel the properties of tight closure in characteristic p.

Theorem (6.1). *Let S be a regular Noetherian ring containing $\mathbb{Q}$ and let $N \subseteq M$ be finitely generated S-modules. Then $N_M^* = N$.*

Sketch of the proof. One reduces to the case where (S, m, L) is a complete local domain. If $u \in N^* - N$ one can preserve this while replacing N by a submodule maximal with respect to being disjoint from u. Passing to M/N, we may assume that $N = 0$, that M is an essential extension of L, and the the socle generator u is in the tight closure of 0. Then the injective hull of M is the same as the injective hull of the residue field, and so if $x_1, \dots, x_n$ is a regular system of parameters for S, we see that we may assume that M embeds in $S/(x_1^t, \dots, x_n^t)S$ for t sufficiently large. Hence we may take u to be the image of $(x_1 \cdots x_n)^{t-1}$, since this element generates the socle. Thus, it will suffice to show that $u = (x_1 \cdots x_n)^{t-1}$ is not in the tight closure of $(x_1^t, \dots, x_n^t)S$ in S.

By Theorem (5.2) if there is an affine $\mathbb{Q}$-algebra containing elements X_i that map to the x_i and such that $X_1^{t-1} \cdots X_n^{t-1}$ is in the tight closure of $(X_1^t, \dots, X_n^t)$, there is also one that is smooth over $\mathbb{Q}$ and in which the X's form a regular sequence. One can descend to a $\mathbb{Z}_a$-subalgebra, where a is a nonzero integer. One now gets a contradiction from the fact that the closed fibers are all regular rings. □

Let $N \subseteq M$ be finitely generated modules over a Noetherian ring S. We shall say that $u \in M$ is in the *regular closure* ${N^{\mathrm{reg}}}_M$ of N in M if for every regular ring T to which S maps, $u_T \in \langle N_T \rangle$ (in M_T). This is slightly different from the notion considered in [HH1] and [HH4], where it was required that S° map into T° (where the superscript $^\circ$ indicates the set of elements of the ring not in any minimal prime). This regular closure is *a priori* smaller than the one considered in [HH1] and [HH4] (although we do not know an example where it is actually strictly smaller). This makes the following Corollary slightly stronger than if it were stated for the notion of [HH1] and [HH4].

Corollary (6.2). *Let S be a Noetherian ring containing $\mathbb{Q}$. Let $N \subseteq M$ be finitely generated S-modules. Then $N_M^* \subseteq {N^{\mathrm{reg}}}_M$.*

Proof. Let $u \in N_M^*$ and suppose that S maps to a regular Noetherian ring T. By the persistence of tight closure, after tensoring with T one has that the image of u is in ${\langle T \otimes_S N \rangle^*}_{T \otimes_S M} = \langle T \otimes_S N \rangle$ (since T is regular). Thus, $u \in {N^{\mathrm{reg}}}_M$. □

Corollary (6.3). *Let S be a Noetherian ring containing $\mathbb{Q}$. Let I be any ideal of S. Then $I^* \subseteq \overline{I}$, the integral closure of I. Hence, all radical ideals of S and, in particular, all prime ideals of S are tightly closed.*

Proof. An element is in $\overline{I}$ if and only if it is in IV for all maps of R to discrete valuation rings V, which shows that $I^{\mathrm{reg}} \subseteq \overline{I}$, and we may apply (6.2) □

We refer to [HH1] and [Hu] for the background of the Briançon-Skoda theorem.

Theorem (6.4) (tight closure Briançon-Skoda theorem). *Let S be a Noetherian ring of equal characteristic zero and let I be an ideal of S generated by at most n elements. Then for every $k \in \mathbb{N}, \overline{I^{n+k}} \subseteq (I^{k+1})^*$. In particular, $\overline{I^n} \subseteq I^*$.*

Proof. Fix generators of I, say $I = (u_1, \dots, u_n)$. It is clear that if an element z is in $\overline{(u_1, \dots, u_n)^{n+k}}$ then this remains true when S is replaced by a suitable affine $\mathbb{Z}$-subalgebra containing z and $u_1, \dots, u_n$. The equation that demonstrates integral dependence here will continue to do so when we pass to closed fibers. The result is now immediate from the definition of tight closure and the fact that the tight closure Briançon-Skoda theorem holds for all the closed fibers. The final statement is the case where $k = 0$. □

Corollary (6.5). *Let S be a Noetherian ring containing $\mathbb{Q}$ and let I be a principal ideal of S. Then $I^* = \overline{I}$.*

Proof. $I^* \subseteq \overline{I}$ by Corollary (6.3) and the other inclusion follows from the generalized Briançon-Skoda theorem (6.4) in the case where $n = 1$ and $k = 0$. $\square$

The following very important result is representative of a large class of results that assert that manipulations of ideals generated by monomials in a fixed system of parameters yield the same result as if the parameters formed a regular sequence (or were indeterminates), *up to tight closure.*

Theorem (6.6) (tight closure captures colons). *Let S be a locally excellent Noetherian ring containing $\mathbb{Q}$. Let $x_1, \dots, x_n$ generate an ideal of height at least n modulo every minimal prime of S. Then $(x_1, \dots, x_{n-1})^* :_S x_nS = (x_1, \dots, x_{n-1})^*$.*

Sketch of the proof. Suppose that $x_n u \in (x_1, \dots, x_{n-1})^*$. We must show that $u \in (x_1, \dots, x_{n-1})^*$. The hypothesis is preserved modulo minimal primes, and also by localization. Thus, we may assume that the ring is an excellent local domain. We may then complete, and we are in the reduced, equidimensional case: the height condition will still hold modulo every minimal prime, and so we may assume that S is a complete local domain. The case where one of the x_i is not in the maximal ideal is easy, and so we may assume that the x_i are part of a system of parameters.

Since $x_n u \in (x_1, \dots, x_{n-1})^*$ we know that there is an affine $\mathbb{Q}$-subalgebra R of S containing $x_1, \dots, x_{n-1}$, x_n, and u such that $x_n u \in \big((x_1, \dots, x_{n-1})R\big)^*$. By Theorem (5.1), we can give a $\mathbb{Q}$-algebra factorization $R \to R_1 \to S$ of $R \to S$ such that R_1 is a domain finitely generated over $\mathbb{Q}$ and such that the images of $x_1, \dots, x_n$ generate a proper ideal of height n in R_1. We can then pass to an affine $\mathbb{Z}$-subalgebra with the same properties. Almost all closed fibers will be reduced, equidimensional, and have the property that the images of the x_i generate an ideal of height n, with the image of ux_n still in the characteristic p tight closure of the ideal generated by the images of $x_1, \dots, x_{n-1}$, and the result now follows from colon-capturing in characteristic p. $\square$

We now define a Noetherian ring containing $\mathbb{Q}$ to be *weakly F-regular* if every ideal is tightly closed and *F-rational* if it is a product of domains in which every ideal of height n generated by n elements is tightly closed. (These notions are considered in greater generality in [HH6].) Thus, weakly F-regular implies F-rational. Exactly as in the characteristic p case, over a weakly F-regular ring the tight closure of every submodule of a finitely generated module is again tightly closed. The proof of the following important result is quite similar to the characteristic p case, and is omitted.

Corollary (6.7). *An F-rational ring is normal and, if it is locally excellent, then it is Cohen-Macaulay. In particular, a weakly F-regular ring is normal and, if it is locally excellent, then it is Cohen-Macaulay.* $\square$

(6.8) Definition and discussion: purity. We say that a map of R-modules $N \to M$ is *pure* if $W \otimes_R N \to W \otimes_R M$ is injective for every R-module W. Since W may be equal to R, this implies that $N \to M$ is injective. If M/N is finitely presented, then $N \hookrightarrow M$

is pure if and only if it splits. It follows that if R is Noetherian, $N \hookrightarrow M$ is pure if and only if $N \to M'$ splits for all $M' \subseteq M$ containing the image of N such that M'/N is finitely generated. Cf. [HR], §6. We shall very often be interested in the condition that a ring homomorphism $R \to S$ be pure (over R), in which case we shall say that R is a *pure subring* of S. We are particularly interested in this condition when R is a Noetherian ring. The condition that a ring homomorphism $R \to S$ be pure implies that every ideal of R is contracted from S.

We now prove a considerable strengthening of the main result of [HR] on the Cohen-Macaulay property for rings of invariants of linearly reductive groups G acting on regular rings: the key point is that in the situations described in [HR], S is regular and the fixed ring S^G is a pure subring of S. The situation just below is therefore much more general.

Theorem (6.9) (generalized Hochster-Roberts theorem). *Every pure subring of an equicharacteristic regular ring is a Cohen-Macaulay ring (and normal: in fact, the completion of each of its local rings is normal).*

Proof. As in the proof for the characteristic p case (see, for example, §7 of [HR]) one can reduce to the case where R is complete. The result now follows from (6.7) and (6.10) below: the proof of (6.10) is left as an exercise. □

Proposition (6.10). *If S is weakly F-regular and R is pure in S, then R is weakly F-regular.* □

Boutot has shown [Bou] that if an affine algebra S over a field of characteristic 0 has rational singularities, and R is pure in S, then R has rational singularities. There is a great deal of evidence that the property of having rational singularities is connected with F-rationality. For example, in [Sm2] it is shown that if $R_{\mathbb{Z}}$ has the property that almost all fibers are F-rational (R is then said to have *F-rational type*; there is a similar definition for *weakly F-regular type*), then $K \otimes_{\mathbb{Z}} R$ has rational singularities for any field K of characteristic 0. It may be possible to characterize rational singularities along these lines.

7. Phantom homology and homological theorems There has been a great deal of work done on a family of problems that used to be known as the "local homological conjectures." Many are now theorems: cf. [PS1,2], [Ho1,2], [Ro1–4], and [Du]. In this section we consider some very general tight closure results that recover many of these results, often in a greatly strengthened form.

Throughout this section let R be a locally excellent Noetherian ring which, for simplicity, we assume is reduced and locally equidimensional. (The results stated here are valid more generally: one wants the conditions we impose to hold modulo every minimal prime of R). Let $G_\bullet$ denote a finite free complex over R, say $0 \to R^{b_d} \to \cdots \to R^{b_0} \to 0$, let α_i denote a matrix of the i th map, $1 \le i \le d$, let r_i be the determinantal rank of α_i, and let I_i be the ideal generated by the size r_i minors of α_i ($\alpha_{d+1} = 0$ and $r_{d+1} = 0$). Recall that $G_\bullet$ satisfies *the standard conditions on rank and height* if for $1 \le i \le d$, $b_i = r_{i+1} + r_i$ and $\operatorname{ht} I_i \ge i$. The complex $G_\bullet$ is called *phantom acyclic* if the cycles in G_i are contained in the tight closure of the boundaries in G_i for $i \geq 1$. We refer to [HH1], [HH4], and [Hu] for more detail. We then get the following analogues of characteristic p results: detailed proofs are given in [HH6], although we shall make some comments on the proofs below.

Theorem (7.1) (phantom acyclicity criterion). *With R and $G_\bullet$ as above, if $G_\bullet$ satisfies the standard conditions on rank and height then $G_\bullet$ is phantom acyclic.*

Sketch of the proof. The argument is very much like the proof of Theorem (6.6). One reduces to the case where the ring is a complete local domain, and then descends to an affine $\mathbb{Q}$-algebra while preserving height conditions using Theorem (5.1). From there it is easy to pass to an affine $\mathbb{Z}$-subalgebra, and then one uses the fact that the hypothesis holds for almost all closed fibers and that the theorem holds in characteristic p to complete the proof. □

This yields at once a very powerful result:

Theorem (7.2) (vanishing theorem for maps of Tor). *Let R be an equicharacteristic regular ring, let S a module-finite extension of R that is torsion-free as an R-module (e.g., a domain), and let $S \to T$ be any homomorphism to a regular ring. Then for every finitely generated R-module M, the map $Tor_i^R(M, S) \to Tor_i^R(M, T)$ is 0 for all $i \geq 1$.*

Sketch of the proof. One reduces to the case where T is complete local regular and then to the case where R is complete and regular as well. A minimal free resolution $G_\bullet$ of M satisfies the standard conditions by the usual acyclicity criterion, and this is preserved when when we pass to $S \otimes_R G_\bullet$, which, by (7.1) is then phantom acyclic: any given cycle in degree $i > 0$ (representing a typical element of $\mathrm{Tor}_i^R(M, S)$) is in the tight closure of the boundaries. This is preserved when we tensor further and pass to $T \otimes_R G_\bullet$. Since T is regular, it is weakly F-regular, and the result follows. □

The mixed characteristic version of (7.2) remains open. If it is true, it implies that regular rings are direct summands of their module-finite extensions and that pure subrings of regular rings are Cohen-Macaulay, both of which are open questions in mixed characteristic. These issues are explored in [HH5] §4, where (7.2) is proved by in equal characteristic by a different method. Finally, we mention the following analogue of a characteristic p result from [HH4] (6.5–6).

Theorem (7.3) (phantom intersection theorem). *Let R and $G_\bullet$ be as above, with $G_\bullet$ of length d, and suppose that the standard conditions on rank and height hold for $G_\bullet$. Let $z \in M = H_0(G_\bullet)$ be any element whose annihilator in R has height $> d$, the length of $G_\bullet$. Then $z \in 0^*_M$. In consequence:*
(1) *if (R, m, K) is local, z cannot be a minimal generator of M.*
(2) *the image of z is 0 in $H_0(S \otimes_R G_\bullet)$ for any regular (or weakly F-regular) ring S to which R maps.*

Sketch of the proof. In proving that $z \in 0^*_M$ one first reduces to the case where R is a complete local domain, and then uses (5.1) to descend first to a $\mathbb{Q}$-algebra and then to a $\mathbb{Z}$-algebra while preserving the rank and height conditions. The result then follows from the characteristic p version by passage to closed fibers. Part (1) then follows because mM

is tightly closed in M (M/mM is a direct sum of copies of R/m, and m is tightly closed in R), while part (2) is obvious. □

Theorem (7.3) is a strengthening of the "improved" new intersection theorem (discussed, for example, in [Ho2]).

8. Questions There are many open questions concerning the behavior of tight closure: we shall touch briefly on only a few of them here, excluding those that depend primarily on further progress in characteristic p. One obvious question is whether the various notions of tight closure ($^{*\mathbb{Q}}$, ^{*K}, and $^{*\mathrm{EQ}}$: see the footnote on p. 5) agree when they are defined.

We next ask whether an element of a complete equicharacteristic local domain R is in the tight closure of an ideal I if and only if it is in IS for some big Cohen-Macaulay algebra S for R. This is correct in characteristic p, and the "only if" part is correct in equal characteristic 0, by Theorem (11.4) of [Ho3] (in fact, this part holds for $^{*\mathrm{EQ}}$). See also Theorem (1.1) here.

For affine $\mathbb{Q}$-algebras we ask whether being weakly F-regular and having weakly F-regular type are the same, and whether being F-rational is the same as having F-rational type. It is an open question for affine $\mathbb{Q}$-algebras whether F-rational type is equivalent to having rational singularities, and there is an analogous question (for which we have not made the necessary definitions) for affine algebras over an arbitrary field. Cf. [Sm2].

In all the known examples where an ideal I of an affine $\mathbb{Q}$-algebra is tightly closed, it actually turns out that one can find $I_{\mathbb{Z}} \subseteq R_{\mathbb{Z}}$ such that I_κ is tightly closed in R_κ for almost all fibers. We do not know whether this is always the case. We also do not know whether the image of an element can be in the tight closure of I_κ for a dense (in the case of $\mathbb{Z}$ this simply means infinite) set of closed fibers without being in the tight closure for almost all closed fibers.

Last, but certainly not least, is the problem of whether there is a theory with similar properties to tight closure that is valid in mixed characteristic: good enough, for example, to prove the vanishing conjecture for maps of Tor (itself, an important open question).

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Department of Mathematics
University of Michigan
Ann Arbor, MI 48109–1003
USA

E-mail:
hochster@math.lsa.umich.edu