THE FROBENIUS STRUCTURE
OF LOCAL COHOMOLOGY

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1. INTRODUCTION

All given rings in this paper are commutative, associative with identity, and Noetherian. Throughout, p denotes a positive prime integer. For the most part, we shall be studying local rings, i.e., Noetherian rings with a unique maximal ideal. Likewise our main interest is in rings of positive prime characteristic p. If (R, m) is local of characteristic p, there is a natural action of the Frobenius endomorphism of R on each of its local cohomology modules $H_m^j(R)$. We call an R-submodule N of one of these local cohomology modules F-*stable* if the action of F maps N into itself.

One of our objectives is to understand when a local ring, (R, m), especially a reduced Cohen-Macaulay local ring, has the property that only finitely many R-submodules of its local cohomology modules F-stable. When this occurs we say that R is FH-*finite*. Of course, when the ring is Cohen-Macaulay there is only one non-vanishing local cohomology module, $H_m^d(R)$, where $d = \dim(R)$. The problem of studying the F-stable submodules of $H_m^d(R)$ arises naturally in tight closure theory, taking a point of view pioneered by K. Smith [Sm1,2,3]. E.g., if R is complete, reduced, and Gorenstein, the largest proper F-stable submodule of $H_m^d(R)$ corresponds to the tight closure of 0 (in the finitistic sense: see [HH2], §8), and its annihilator is the test ideal of R. Also see Discussion 2.10 here.

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Among our main results is the fact that if R is F-pure and Gorenstein (or even quasi-Gorenstein) then $H_m^d(R)$ has only finitely F-stable R-submodules. This follows from another of our main results: for any local ring R of prime characteristic $p > 0$, if Frobenius acts injectively on an Artinian R-module M, then the set of annihilators of F-stable submodules of M is a finite set of radical ideals closed under primary decomposition. See §3. The results of this section overlap those of the independent paper [Sh], particularly (1.12), (3.7), and (3.10). Other results related to F-stable submodules of local cohomology may be found in [En1], [En2], and [Ka].

We shall also study the problem of determining conditions under which the local cohomology modules of R have finite length in the category of R-modules with Frobenius action. We say that R has *finite FH-length* in this case. In §4 we introduce the notion of an *anti-nilpotent* Frobenius action on an Artinian module over a local ring. Using results of [Ly] and [Ho4], we show that the local cohomology of a local ring R of characteristic p is anti-nilpotent if and only if the local cohomology of $R[[x]]$ has finite FH-length, in which case the local cohomology of R and every formal power series ring over R is FH-finite.

In §5 we show that if R is the face ring of a finite simplicial complex localized or completed at its homogeneous maximal ideal, then R is FH-finite. See Theorem (5.1).

2. NOTATION AND TERMINOLOGY

Discussion 2.1: Some basics about tight closure. Unless otherwise specified, we shall assume throughout that R is a Noetherian ring of positive prime characteristic p, although this hypothesis is usually repeated in theorems and definitions. R° denotes the complement of the union of the minimal primes of R, and so, if R is reduced, R° is simply the multiplicative system of all nonzerodivisors in R. We shall write $\boldsymbol{F}^e$ (or $\boldsymbol{F}_R^e$ if we need to specify the base ring) for the *Peskine-Szpiro* or *Frobenius* functor from R-modules to R-modules. Note that $\boldsymbol{F}^e$ preserves both freeness and finite generation of modules, and is exact precisely when R is regular (cf. [Her], [Ku1]). If $N \subseteq M$ we write $N^{[q]}$ for the image of $F^e(N)$ in $F^e(M)$, although it depends on the inclusion $N \to M$, not just on N. If $u \in M$ we write u^{p^e} for the image $1 \otimes u$ of u in $F^e(M)$. With this notation, $(u + v)^q = u^q + v^q$ and $(ru)^q = r^q u^q$ for $u, v \in M$ and $r \in R$.

From time to time, we assume some familiarity with basic tight closure theory in prime characteristic $p > 0$. We use the standard notation $N*_M$ for the tight closure of the submodule N in the module M. If M is understood, the subscript is omitted, which is frequently the case when $M = R$ and $N = I$ is an ideal. We refer the reader to [HH1,2,3,6] and [Hu] for background in this area.

In particular, we assume cognizance of certain facts about test elements, including the notion of a *completely stable* test element. We refer the reader to [HH2, §6 and §8], [HH1], [HH3, §6], and [AHH, §2] for more information about test elements and to §3 of [AHH] for a discussion of several basic issues related to the localization problem for tight closure.

Discussion 2.2: Local cohomology and the action of the Frobenius endomorphism. Our basic reference for local cohomology is [GrHa]. Let R be an arbitrary Noetherian ring, let I be an ideal of R and let M be any R-module. The ith *local cohomology module* $H^i_I(M)$ with support in I may be obtained in several ways. It may be defined as $\varinjlim_t \operatorname{Ext}^i_R(R/I^t, M)$: here, any sequence of ideals cofinal with the powers of I may be used instead of the the sequence of powers, $\{I^t\}_t$. Alternatively, we may define $C^\bullet(f; R)$ to be the complex $0 \to C^0 \to C^1 \to 0$ where $C^0 = R$, $C^1 = R_f$ and the map is the canonical map $R \to R_f$, and then if $\underline{f}$ is a sequence of elements $f_1, \ldots, f_n$ we may define $C^\bullet(\underline{f}; R)$ to be the tensor product over R of the n complexes $C^\bullet(f_i; R)$. Finally, let $C^\bullet(\underline{f}; M)$ denote $C^\bullet(\underline{f}; R) \otimes_R M$, which has the form:

$$0 \to M \to \bigoplus_i M_{f_i} \to \bigoplus_{i<j} M_{f_i f_j} \to \cdots \to M_{f_1 \cdots f_n} \to 0.$$

The cohomology of this complex turns out to be $H^\bullet_I(M)$, where $I = (f_1, \ldots, f_n)R$, and actually depends only on the radical of the ideal I.

By the standard theory of local duality (cf. [GrHa, Theorem (6.3)]) when (S, m_S, L) is Gorenstein with $\dim(S) = n$ and M is a finitely generated S-module, $H^i_m(M) \cong \operatorname{Ext}^{n-i}_S(S, M)^\vee$ as functors of M, where $N^\vee = \operatorname{Hom}_R\big(N, E_S(L)\big)$. Here, $E_S(L)$ is an injective hull of L over S. In particular, if (R, m, K) is local of Krull dimension d and is a homomorphic image of a Gorenstein local ring S of dimension n, then $\omega_R = \operatorname{Ext}^{n-d}(R, S)$ whose Matlis dual over S, and, hence, over R as well, is $H^d_m(R)$. We refer to a finitely generated R-module ω_R as a *canonical module* for R if $\omega_R^\vee = H^d_m(R)$. It is unique up to isomorphism, since its completion is dual to $H^d_m(R)$. Our discussion shows that a canonical module exists if R is a homomorphic image of a Gorenstein ring; in particular, ω_R exists

if R is complete. When R is Cohen-Macaulay, one has that $H_m^i(M) \cong \operatorname{Ext}_R^{d-i}(M, \omega_R)$ functorially for all finitely generated R-modules M.

When R is a normal local domain, ω_R is isomorphic with as an R-module with an ideal of pure height one, i.e., with a divisorial ideal.

Finally, suppose that $(R, m, K) \to (S, m_S L)$ is a local homomorphism such that S a module-finite extension of R. Let $\omega = \omega_R$ be a canonical module for R. Then $\operatorname{Hom}_R(S, \omega)$ is a canonical module for S. Here, the rings need not be Cohen-Macaulay, nor domains. To see this, note one can reduce at once to the complete case. We have that

$$H_{m_S}^d(S) \cong H_m^d(S) \cong S \otimes H_m^d(R).$$

Then $E_S(L)$ may be identified with $\operatorname{Hom}_R\big(S, E_R(K)\big)$; moreover. on S-modules, the functors $\operatorname{Hom}_R\big(\_, E_R(K)\big)$ and $\operatorname{Hom}_S(\_, E_S(L))$ are isomorphic. Hence

$$\operatorname{Hom}_S\big(H_{\mathfrak{n}}^d(S), E_S(L)\big) \cong \operatorname{Hom}_R\big(S \otimes_R H_m^d(R), E_R(K)\big).$$

By the adjointness of tensor and Hom, this becomes

$$\operatorname{Hom}_R\big(S, \operatorname{Hom}_R\big(H_m^d(R), E_R(K)\big)\big) \cong \operatorname{Hom}_R(S, \omega),$$

as required.

When $M = R$ we have an action of the Frobenius endomorphism on the complex $C^\bullet(\underline{f}; R)$ induced by the Frobenius endomorphisms of the various rings R_g where g is a product of a subset of $f_1, \ldots, f_n$, and the action on the cohomology is independent of the choice of f_i.

An alternative point of view is that, quite generally, if $M \to M'$ is any map of R-modules then there is an induced map $H_I^i(M) \to H_I^i(M')$. When S is an R-algebra and I an ideal of R we get a map $H_I^i(R) \to H_I^i(S)$ for all i, and $H_I^i(S)$ may be identified with $H_{IS}^i(S)$. In particular, we may take $S = R$ and let the map $R \to S$ be the Frobenius endomorphism. Since $IS = I^{[p]}$ here, this gives a map $H_I^i(R) \to H_{I^{[p]}}^i(R)$. But since $\operatorname{Rad}(I^{[p]}) = \operatorname{Rad}(I)$, $H_{I^{[p]}}^i(R) \cong H_I^i(R)$ canonically. The map $H_I^i(R) \to H_I^i(R)$ so obtained again gives the *action of the Frobenius endomorphism* on $H_I^i(R)$. We shall denote this action by F: note that $F(ru) = r^p F(u)$.

Definition 2.3. When R has prime characteristic $p > 0$, we may construct a non-commutative, associative ring $R\{F\}$ from R which is an R-free left module on the symbols

1, F, F^2, ... , F^e, ... by requiring that $Fr = r^p F$ when $r \in R$. We shall say that an R-module M is an $R\{F\}$-*module* if there is given an action $F : M \to M$ such that for all $r \in R$ and for all $u \in M$, $F(ru) = r^p u$. This is equivalent to the condition that M be an $R\{F\}$-module so as to extend the R-module structure on M. We then call an R-submodule N of M F-*stable* if $F(N) \subseteq N$, which is equivalent to requiring that N be an $R\{F\}$-submodule of N. If M is any $R\{F\}$-module and S is an R-algebra then there is an $S\{F\}$-module structure on $S \otimes_R M$ determined by the condition that $F(s \otimes u) = s^p \otimes F(u)$.

In particular, since we have an $R\{F\}$-module structure on $H^i_I(R)$, we may refer to the F-stable submodules of $H^i_I(R)$.

If R is local of Krull dimension d and x_1, ... , x_d is a system of parameters, then $H^d_m(R)$ may be identified with $\varinjlim_t R/(x_1^t, \ldots, x_d^t)$, where the tth map in the direct limit system

$$R/(x_1^t, \ldots, x_d^t) \to R/(x_1^{t+1}, \ldots, x_d^{t+1})$$

is induced by multiplication by $x_1 \cdots x_d$. If R is Cohen-Macaulay the maps in this direct limit system are injective. When $H^d_m(R)$ is thought of as a direct limit in this way, we write $\langle r; x_1^t, \ldots, x_d^t \rangle$ for the image in $H^d_m(R)$ of the element represented by r in $R/(x_1^t, \ldots, x_d^t)$. The action of the Frobenius endomorphism on the highest local cohomology module in this case may be described as sending

$$\langle r; x_1^t, \ldots, x_d^t \rangle \mapsto \langle r^p; x_1^{pt}, \ldots, x_d^{pt} \rangle.$$

Discussion 2.4: another point of view for F-stable submodules. Let (R, m, K) be local of Krull dimension d, where R has characteristic $p > 0$. Consider an F-stable submodule $N \subseteq H^d_m(R)$. Suppose that R is reduced. We have an isomorphism of (R, m, K) with (S, n, L) where $S = R^{1/p}$ given by $\Phi : R \to R^{1/p}$, where $\Phi(r) = r^{1/p}$. We have a commutative diagram:

$$
\begin{array}{ccc}
R & \xrightarrow{F} & R \\
{\scriptstyle =}\downarrow & & \downarrow{\scriptstyle \Phi} \\
R & \xrightarrow[\iota]{} & R^{1/p}
\end{array},
$$

where $\iota : R \subseteq R^{1/p}$ is the inclusion map. In general, when $\Phi : R \to S$ is any ring isomorphism, for each submodule N of $H^i_I(R)$ there is a corresponding submodule N' of

$H^i_{\Phi(I)}(S)$. In fact, if $\Psi : S \to R$ is Φ^{-1}, and we use ${}^\Psi Q$ to indicate restriction of scalars from R-modules to S-modules, then $H^i_{\Phi(I)}(S)$ is canonically isomorphic with ${}^\Psi\big(H^i_I(R)\big)$ and N' is the image of ${}^\Psi N$ in $H^i_{\Phi(I)}(S)$. Note that ${}^\Psi\_$ is an exact functor.

When $S = R^{1/p}$ and and $I = m$, the modules $H^i_{\Phi(m)}(S)$, $H^i_n(S)$, $H^i_{mS}(S)$, and $H^i_m(S)$ may all be identified: the first three may be identified because $\Phi(m)$ and mS both have radical n, and the last two because if $f_1, \ldots, f_h$ generate m their images $g_1, \ldots, g_h$ in S generate mS and the complexes $C^\bullet(f;\, S)$ and $C^\bullet(g;\, S)$ are isomorphic. The condition that N is F-stable is equivalent to the condition that N maps into N' in $H^i_m(S) \cong H^i_{\Phi(m)}(S)$. A very important observation is this:

$(**)$ With notation as just above, if N is F-stable and $J \subseteq R$ kills N (e.g., if $J = \mathrm{Ann}_R N$) then $\Phi(J)$ kills the image of N in $H^i_m(R^{1/p})$.

The hypothesis that N is F-stable means that N maps into the corresponding submodule N', and N' is clearly killed by $\Phi(J)$.

Definition 2.5. A local ring $(R,\, m)$ of Krull dimension d is FH-*finite* if, for all i, $0 \le i \le d$, only finitely many R-submodules of $H^i_m(R)$ are F-stable. We shall that R has *finite FH-length* if for all i, $H^i_m(R)$ has finite length in the category of $R\{F\}$-modules.

Our main focus in studying the properties of being FH-finite and of having finite FH-length is when the local ring R is Cohen-Macaulay. Of course, in this case there is only one nonzero local cohomology module, $H^d_m(R)$. However, in §5 we show that every face ring has finite FH-length.

Since every $H^i_m(R)$ has DCC even in the category of R-modules, we know that $H^i_m(R)$ has finite length in the category of $R\{F\}$-modules if and only if it has ACC in the category of $R\{F\}$-modules. Of course, it is also equivalent to assert that there is a finite filtration of $H^i_m(R)$ whose factors are simple $R\{F\}$-modules.

Discussion 2.6: purity. Recall that a map of R-modules $N \to N'$ is *pure* if for every R-module M the map $N \otimes_R M \to N' \otimes_R M$ is injective. Of course, this implies that $N \to N'$ is injective, and may be thought of as a weakening of the condition that $0 \to N \to N'$ split, i.e., that N be a direct summand of N'. If N'/N is finitely presented, $N \to N'$ is pure if and only if it is split. For a treatment of the properties of purity, see, for example, [HH5, Lemma (2.1), p. 49].

An R algebra S is called *pure* if $R \to S$ is pure as a map of R-modules, i.e., for every

R-module M, the map $M = R \otimes_R M \to S \otimes_R M$ is injective. A Noetherian ring R of characteristic p is called F-*pure* (respectively, F-*split*) if the Frobenius endomorphism $F : R \to R$ is pure (respectively, split). Evidently, an F-split ring is F-pure and and an F-pure ring is reduced. If R is an F-finite Noetherian ring, F-pure and F-split are equivalent (since the cokernel of $F : R \to R$ is finitely presented as a module over the left hand copy of R), and the two notions are also equivalent when (R, m, K) is complete local, for in this case, $R \to S$ is split iff $R \otimes_R E \to S \otimes_R E$ is injective, where $E = E_R(K)$. An equivalent condition is that the map obtained by applying $\operatorname{Hom}_R(\_, E)$ be surjective, and since $R \cong \operatorname{Hom}_R(E, E)$, by the adjointness of tensor and Hom that map can be identified with the maps $\operatorname{Hom}_R(S, R) \to \operatorname{Hom}_R(R, R) \cong R$.

We say that a local ring R is F-*injective* if F acts injectively on all of the local cohomology modules of R with support in m. This holds if R is F-pure.

When R is reduced, the map $F : R \to R$ may be identified with the algebra inclusion $R \subseteq R^{1/p}$, and so R is F-pure (respectively, F-split) if and only if it is reduced and the map $R \subseteq R^{1/p}$ is pure (respectively, split).

Lemma 2.7. *Let (R, m, K) be a Noetherian local ring of positive prime characteristic p and Krull dimension d.*

(a) *R is FH-finite (respectively, has finite FH-length) if and only if its completion $\widehat{R}$ is FH-finite.*

(b) *Suppose that $(R, m) \to (S, m_S)$ is a local homomorphism of local rings such that mS is primary to the maximal ideal of S, i.e., such that the closed fiber S/mS has Krull dimension 0. Suppose either that $R \to S$ is flat (hence, faithfully flat), split over R, or that S is pure over R. If S is FH-finite, then R is FH-finite, and if S has finite FH-length, then R has finite FH-length. More generally, the poset of F-stable submodules of any local cohomology module $H_m^i(R)$ injects in order-preserving fashion into the poset of F-stable submodules of $H_{m_S}^i(S)$.*

(c) *R is F-injective if and only if $\widehat{R}$ is F-injective. R is F-pure if and only if $\widehat{R}$ is F-pure.*

Proof. Completion does not affect either what the local cohomology modules are nor what the action of Frobenius is. Since each element of a local cohomology module over R is killed by a power of m, these are already $\widehat{R}$-modules. Thus, (a) is obvious. (The Cohen-Macaulay property also does not change: this follows, for example. from the fact

that it equivalent to the vanishing of all local cohomology modules $H_m^i(R)$ for $i < d$.)

Part (b) follows from the fact that the local cohomology modules of S may be obtained by applying $S \otimes_R \_$ to those of R, and that the action of F is then the one discussed in Definition 2.3 for tensor products, i.e., $F(s \otimes u) = s^p \otimes F(u)$. From this one sees that if N is F-stable in $H_m^i(R)$, then $S \otimes_R N$ is F-stable in $S \otimes_R H_m^i(R) \cong H_{m_S}^i(S)$. Thus, we only need to see that if $N \subseteq N'$ are distinct F-stable submodules of $H_m^i(R)$, then the images of $S \otimes N$ and $S \otimes N'$ are distinct in $S \otimes H_m^i(R)$. It suffices to see this when $R \to S$ is pure: the hypothesis of faithful flatness or that $R \to S$ is split over R implies purity. But N'/N injects into $H_m^i(R)/N$, and this in turn injects into $S \otimes_R \left(H_m^i(R)/N \right)$ by purity, so that the image of $u \in N' - N$ is nonzero in

$$ S \otimes_R \left(H_m^i(R)/N \right) \cong H_m^i(S)/\mathrm{Im}\,(S \otimes N). $$

This shows that $1 \otimes u$ is in the image of $S \otimes N'$ in $H_m^i(S)$ but not in the image of $S \otimes N$.

Part (c), in the case of F-injectivity, follows from the fact that it is equivalent to the injectivity of the action of F on the $H_m^i(R)$, and that neither these modules nor the action of F changes when we complete. In the case of F-purity, we prove that if R is F-pure then so is $\widehat{R}$: the other direction is trivial. Consider an ideal I of the completion, and suppose that there is some element u of the completion such that $u \notin I$ but $u^p \in I^{[p]}$. Choose N such that $u \notin I + m^N \widehat{R}$. We see that we may assume that I is primary to the maximal ideal of $\widehat{R}$, which implies that it is the expansion of its contraction J to R. Then we may choose $v \in R$ such that $v - u \in I = J\widehat{R}$. But then $v \notin J$ but $v^p - u^p \in J^{[p]}\widehat{R}$, and since $u^p \in J^p\widehat{R}$ we have that $v^p \in J^{[p]}\widehat{R} \cap R = J^{[p]}$ and so $v \in J$, a contradiction. Thus, ideals of $\widehat{R}$ are contracted with respect to Frobenius, and, consequently, $\widehat{R}$ is reduced. Then $\widehat{R} \to \widehat{R}^{1/p}$ is cyclically pure, by the contractedness of ideals with respect to Frobenius that we just proved, which shows that it is pure: see [Ho2], Theorem (1.7). It follows that $\widehat{R}$ is F-pure. $\square$

Discussion 2.8: the gamma construction. Let K be a field of positive characteristic p with a p-base Λ. Let Γ be a fixed cofinite subset of Λ. For $e \in \mathbb{N}$ we denote by $K_{\Gamma,e}$ the purely inseparable field extension of K that is the result of adjoining p^e th roots of all elements in Γ to K, which is unique up to unique isomorphismover K.

Now suppose that (R, m) is a complete local ring of positive prime characteristic p and that $K \subseteq R$ is a coefficient field, i.e., it maps bijectively onto R/m. Let $x_1, \ldots, x_d$ be a

system of parameters for R, so that R is module-finite over $A = K[[x_1, \ldots, x_d]] \subseteq R$. Let A_Γ denote

$$\bigcup_{e \in \mathbb{N}} K_{\Gamma,e}[[x_1, \ldots, x_d]],$$

which is a regular local ring that is faithfully flat and purely inseparable over A. Moreover, the maximal ideal of A expands to that of A_Γ. We shall let R_Γ denote $A_\Gamma \otimes_A R$, which is module-finite over the regular ring A_Γ and which is faithfully flat and purely inseparable over R. The maximal ideal of R expands to the maximal ideal of R_Γ. The residue class field of R_Γ is K_Γ.

We note that R_Γ depends on the choice of coefficient field K for R, and the choice of Γ, but does not depend on the choice of system of parameters $x_1, \ldots, x_d$. We refer the reader to §6 of [HH3] for more details. It is of great importance that R_Γ is F-finite, i.e., finitely generated as a module over $F(R_\Gamma)$. This implies that it is excellent: see [Ku2].

It is shown in [HH3] that, if R is reduced, then for any sufficiently small choice of the cofinite subset Γ of Λ, R_Γ is reduced. It is also shown in [HH3] that if R is Cohen-Macaulay (respectively, Gorenstein), then R_Γ is Cohen-Macaulay (respectively, Gorenstein).

Lemma 2.9. *Let R be a complete local ring of positive prime characteristic p. Fix a coefficient field K and a p-base Λ for K. Let notation be as in the preceding discussion.*

(a) *Let W be an Artinian R-module with an $R\{F\}$-module structure such that the action of F is injective. Then for any sufficiently small choice of Γ co-finite in Λ, the action of F on $R_\Gamma \otimes_R W$ is also injective.*

(b) *Suppose that F acts injectively on a given local cohomology module of R. Then F acts injectively on the corresponding local cohomology module of R_Γ for all sufficiently small cofinite Γ. In particular, if R is F-injective, then so is R_Γ.*

(c) *Suppose that R is F-pure. Then for any choice of Γ cofinite in the p-base such that R_Γ is reduced, and, hence, for all sufficiently small co-finite Γ, R_Γ is F-pure.*

Proof. For part (a), let V denote the finite-dimensional K-vector space that is the socle of W. Let $W_\Gamma = R_\Gamma \otimes_R W$. Because the maximal ideal m of R expands to the maximal ideal of R_Γ, and R_Γ is R-flat, the socle in W_Γ may be identified with $V_\Gamma = K_\Gamma \otimes V$. If F has a nonzero kernel on W_Γ then that kernel has nonzero intersection with V_Γ, and that intersection will be some K-subspace of V_Γ. Pick Γ such that the dimension of the kernel is minimum. Then the kernel is a nonzero subspace T of V_Γ whose intersection with

$V \subseteq V_\Gamma$ is 0. Choose a basis $v_1, \ldots, v_h$ for V and choose a basis for T as well. Write each basis vector for T as $\sum_{j=1}^{h} a_{ij} v_j$, where the a_{ij} are elements of K_Γ. Thus, the rows of the matrix $\alpha = (a_{ij})$ represent a basis for T. Put the matrix α in reduced row echelon form: the leftmost nonzero entries of the rows are each 1, the columns of these entries are distinct, proceeding from left to right as the index of the row increases, and each column containing the leading 1 of a row has its other entries equal to 0. This matrix is uniquely determined by the subspace T. It has at least one coefficient a not in K (in fact, at least one in every row), since T does not meet V.

Now choose $\Gamma' \subseteq \Gamma$ such that $a \notin K_{\Gamma'}$, which is possible by Lemma (6.12) of [HH3]. Then the intersection of T with $V_{\Gamma'}$ must be smaller than T, or else T will have a K_Γ basis consisting of linear combinations of the v_j with coefficients in $K_{\Gamma'}$, and this will give a matrix β over $K_{\Gamma'}$ with the same row space over K_Γ as before. When we put β in row echelon form, it must agree with α, which forces the conclusion that $a \in K_{\Gamma'}$, a contradiction.

Part (b) follows immediately from part (a).

To prove part (c), consider a choice of Γ sufficiently small that R_Γ is reduced. Let E be the injective hull of K over R. For each power m^t of the maximal ideal of R, we have that $R_\Gamma/(m^t) \cong K_\Gamma \otimes_K R/m^t$. Thus, the injective hull of K_Γ over R_Γ may be identified with $K_\Gamma \otimes_K E$. We are given that the map $E \to E \otimes_R R^{1/p}$ is injective. We want to show that the map

$$E_\Gamma \to E_\Gamma \otimes_{R_\Gamma} (R_\Gamma)^{1/p}$$

is injective. Since the image of a socle generator in E is a socle generator in E_Γ, it is equivalent to show the injectivity of the map $E \to E_\Gamma \otimes_{R_\Gamma} (R_\Gamma)^{1/p}$.

The completion of R_Γ may be thought of as the complete tensor product of K_Γ with R over $K \subseteq R$. However, if one tensors with a module in which every element is killed by a power of the maximal ideal we may substitute the ordinary tensor product for the complete tensor product. Moreover, since R_Γ is reduced, we may identify $(R_\Gamma)^{1/p}$ with $(R^{1/p})_{\Gamma^{1/p}}$: the latter notation means that we are using $K^{1/p}$ as a coefficient field for $R^{1/p}$, that we are using the pth roots $\Lambda^{1/p}$ of the elements of the p-base Λ (chosen for K) as a p-base for $K^{1/p}$, and that we are using the set $\Gamma^{1/p}$ of pth roots of elements of Γ as the co-finite subset of $\Lambda^{1/p}$ in the construction of $R_{\Gamma^{1/p}}^{1/p}$. But $(K^{1/p})_{\Gamma^{1/p}} \cong (K_\Gamma)^{1/p}$. Keeping in mind that every element of E_Γ is killed by a power of the maximal ideal, and that

$E_\Gamma \cong K_\Gamma \otimes_K E$, we have that

$$E_\Gamma \otimes_{R_\Gamma} R_\Gamma^{1/p} \cong (K_\Gamma \otimes_K E) \otimes_{K_\Gamma \otimes_K R} (K_\Gamma^{1/p} \otimes_{K^{1/p}} R^{1/p})$$

and so, writing L for K_Γ, we have that

$$E_\Gamma \otimes_{R_\Gamma} R_\Gamma^{1/p} \cong (L \otimes_K E) \otimes_{L \otimes_K R} (L^{1/p} \otimes_{K^{1/p}} R^{1/p}).$$

Now, if K is any ring, L and R are any K-algebras, S is any $(L \otimes_K R)$-algebra (in our case, $S = L^{1/p} \otimes_{K^{1/p}} R^{1/p}$), and E is any R-module, there is an isomorphism

$$(L \otimes_K E) \otimes_{L \otimes_K R} S \cong E \otimes_R S$$

which maps $(c \otimes u) \otimes s$ to $u \otimes cs$. (The inverse map sends $u \otimes s$ to $(1 \otimes u) \otimes s$. Note that $(c \otimes u) \otimes s = (1 \otimes u) \otimes cs$ in $(L \otimes_K E) \otimes_{L \otimes_K R} S$.)

Applying this fact, we find that $E_\Gamma \otimes_{R_\Gamma} R_\Gamma^{1/p}$ is isomorphic with

$$E \otimes_R (L^{1/p} \otimes_{K^{1/p}} R^{1/p}) \cong E \otimes_R (R^{1/p} \otimes_{K^{1/p}} L^{1/p}) \cong (E \otimes_R R^{1/p}) \otimes_{K^{1/p}} L^{1/p}$$

by the commutativity and associativity of tensor product. But E injects into $E \otimes_R R^{1/p}$ by hypothesis and the latter injects into $(E \otimes_R R^{1/p}) \otimes_{K^{1/p}} L^{1/p}$ simply because $K^{1/p}$ is a field and $L^{1/p}$ is a nonzero free module over it. $\square$

Discussion 2.10: finiteness conditions on local cohomology as an F-module and tight closure. We want to make some connections between F-submodules of local cohomology and tight closure theory. Let R be a reduced local ring of characteristic $p > 0$. Let us call a submodule of $H = H_m^d(R)$ *strongly proper* if it is annihilated by a nonzerodivisor of R. Assume that R has test elements. The *finitistic* tight closure of 0 in a module M is the union of the submodules 0_N^* as N runs through the finitely generated submodules of M. It is not known, in general, whether the tight closure of 0 in an Artinian module over a complete local ring is the same as the finitistic tight closure: a priori, it might be larger. Cf. [LySm1,2], [El], and [St] for results in this direction.

However, if (R, m, K) is an excellent reduced equidimensional local ring with $\dim(R) = d$, the two are the same for $H_m^d(R)$: if u is in the tight closure of 0 and represented by f mod $I_t = (x_1^t, \ldots, x_d^t)$ in $H_m^d(R) = \varinjlim_t R/(x_1^t, \ldots, x_d^t)R$, then there exists $c \in R^\circ$ such that for all $q = p^e \gg 1$ the class of cu^q maps to 0 under the map $R/I_{qt} \to H_m^d(R)$, i.e.,

for some k_q, $cu^q(x_1 \cdots x_d)^{k_q} \in I_{qt+k+q}$ for all $q \gg 1$. But $I_{qt+k_q} :_R (x_1 \cdots x_d)^{k_q} \subseteq I_{qt}^*$ (since R is excellent, reduced, and local, it has a completely stable test element, and this reduces to the complete case, which follows from [HH2], Theorem (7.15)), and so if d is a test element for R we have that $cdu^q \in I_{qt} = I_t^{[q]}$ for all $q \gg 0$, and so the class of u mod I_t is in the tight closure of 0 in R/I_t and hence in the image of R/I_t in $H_m^d(R)$, as required.

Then the finitistic tight closure of 0 in H is an an F-stable strongly proper submodule of $H_m^d(R)$. The reason is that it is immediate from the definition of tight closure that if $u \in 0_N^*$ then $u^q \in 0_{N^{[q]}}^*$, where $q = p^e$ and $N^{[q]}$ denotes an image of $F^e(N)$ in $F^e(H)$ for some ambient module $H \supseteq N$. In particular, $u^q \in 0_{F^e(N)}^*$. Moreover, if c is a test element for the reduced ring R, then $c \in R^\circ$ and so c is a nonzerodivisor, and c kills 0_N^* for every finitely generated R-module N.

Conversely, any strongly proper F-stable submodule $N \subseteq H$ is in the tight closure of 0. If c is a nonzerodivisor that kills N and $u \in N$, then $cu^q = 0$ for all q: when we identify $F^e(H)$ with H, u^q is identified with $F^e(u)$.

What are the strongly proper submodules of $H = H_m^d(R)$? If (R, m, K) is complete with $E = E_R(K)$ the injective hull of the residue class field and canonical module $\omega := \operatorname{Hom}_R(H_m^d(R), E)$, then submodules of H correspond to to the proper homomorphic images of ω: the inclusion $N \subseteq H$ is dual under $\operatorname{Hom}_R(\_, E)$ to a surjection $\omega \twoheadrightarrow \operatorname{Hom}_R(N, E)$. If R is a domain, for every proper N we have that $\omega \twoheadrightarrow \operatorname{Hom}_R(N, E)$ is a proper surjection, and therefore is killed by a nonzerodivisor. Therefore, we have the following results of K. E. Smith [Sm3] (see Proposition (2.5), p. 169 and the remark on p. 170 immediately following the proof of (2.5)). (See also [Sm1], Theorem (3.1.4), where the restricted generality is not needed.)

Proposition 2.11 (K. E. Smith). *If R is a reduced equidimensional excellent local ring of characteristic p, then the tight closure 0^* of 0 in $H_m^d(R)$ (which is the same in the finitistic and ordinary senses) is the largest strongly proper F-stable submodule of $H_m^d(R)$. If R is a complete local domain, it is the largest proper F-stable submodule of $H_m^d(R)$.* $\square$

A Noetherian local ring is called F-*rational* if some (equivalently, every) ideal generated by parameters is tightly closed. An excellent F-rational local ring is a Cohen-Macaulay normal domain. The completion of an excellent F-rational local ring is again F-rational. Cf. [HH3], Proposition (6.27a). From this and the discussion above we have at once (see

[Sm3], Theorem (2.6), p. 170):

Proposition 2.12 (K. E. Smith). *Let R be an excellent Cohen-Macaulay local ring of characteristic p and Krull dimension d. Then R is F-rational if and only if $H_m^d(R)$ is a simple $R\{F\}$-module.*   □

Example 2.13. The ring obtained by killing the size t minors of a matrix of indeterminates in the polynomial ring in those indeterminates is F-rational and, in fact, weakly F-regular, i.e., every ideal is tightly closed. The local ring at the origin is therefore FH-finite: the unique non-vanishing local cohomology module is $R\{F\}$-simple.

Proposition 2.14. *Let R be a Cohen-Macaulay local domain and suppose that there is an m-primary ideal $\mathfrak{A}$ such that $\mathfrak{A}I^* \subseteq I$ for every ideal I of R generated by part of a system of parameters. Then R has finite FH-length.*

Proof. Let d be the dimension of R. By the discussion above, every proper F-stable submodule of $H = H_m^d(R)$ is contained in 0_H^*. But the discussion above shows that 0_H^* is a union of submodules of the form I^*/I where I is a parameter ideal, and so 0_H^* is killed by $\mathfrak{A}$, and has finite length even as an R-module.   □

See Theorem (4.21), which gives a stronger conclusion when the residue class field is perfect and R is F-injective.

Example 2.15. Let $R = K[[X, Y, Z]]/(X^3 + Y^3 + Z^3)$, where K is a field of positive characteristic different from 3. Then R is a Gorenstein domain, and the tight closure of 0 in $H_m^2(R)$ is just the socle, a copy of K: the tight closure of every parameter ideal is known to contain just one additional element, a representative of the generator of the socle modulo the parameter ideal. Evidently R is FH-finite. It is known (see, for example, [HR], Prop. (5.21c), p. 157) that R is F-injective if and only if the characteristic of K is congruent to 1 modulo 3. If the characteristic is congruent to 2 modulo 3, $R[[t]]$ does not have FH-finite length by Theorem (4.15) of §4.

Example 2.16. We construct a complete local F-injective domain of dimension one (hence, it is Cohen-Macaulay) that is not FH-finite. Note that Theorem (4.21) implies that there are no such examples when the residue class field of the ring is perfect.

Let K be an infinite field of characteristic $p > 0$ (it will be necessary that K not be perfect) and let L be a finite algebraic extension field of K such that

(1) $[L : L^p[K]] > 2$ (all one needs is that the dimension of of $L/L^p[K]$ over K is at least 2) and

(2) L does not contain any element of $K^{1/p} - K$ (equivalently, $L^p \cap K = K^p$).

Then the quotient L/K has infinitely many $K\{F\}$-submodules but F acts injectively on it. Moreover, if $R = K + xL[[x]] \subseteq L[[x]]$ then R is a complete local one-dimensional domain that is F-injective but not FH-finite.

The conditions in (1) and (2) above may be satisfied as follows: if k is infinite perfect of characteristic $p > 2$, $K = k(u, v)$, where u and v are indeterminates, and

$$L = K[y]/(y^{2p} + uy^p - v),$$

then (1) and (2) above are satisfied.

Proof. The image of L under F is L^p — this need not be a K-vector space, but $L_1 = L^p[K]$ is a K-vector space containing the image of F. All of the K-vector subspaces of L strictly between L_1 and L are F-stable, and there are infinitely many. The statement that F acts injectively on L/K is exactly the statement that $L^p \cap K = K^p$.

With R as above, the exact sequence

$$0 \to R \to L[[x]] \to L/K \to 0$$

yields a long exact sequence for local cohomology:

$$0 \to L/K \to H_m^1(R) \to H_m^1(L[[x]]) \to 0,$$

Since L/K embeds in $H_m^1(R)$ as an F-stable submodule and m kills it, its R-module structure is given by its K-vector space structure. Moreover, since F is injective on $H_m^1(L[[x]])$, F-injectivity holds for R iff it holds for L/K.

This establishes all assertions except that the given example satisfies (1) and (2). Note that the equation $y^{2p} + uy^p - v$ is irreducible over $k[y, u, v]$ (the quotient is $k[y, u]$). Suppose that L contains an element w of $K^{1/p}$ not in K. Then $[K[w] : K] = p$, and so $[L : K[w]] = 2$. It follows that y satisfies a monic quadratic equation over $K[w]$.

But if we enlarge $K[w]$ to all of $K^{1/p}$ we know the quadratic equation that y satisfies: $y^2 + u^{1/p}y - v^{1/p} = 0$, which is clearly irreducible over $K^{1/p} = k(u^{1/p}, v^{1/p})$. This quadratic

is unique, so we must have $u^{1/p}$, $v^{1/p}$ are both in $K[w]$, a contradiction, since adjoining both produces an extension of K of degree p^2.

It remains to determine $[L : L^p[K]] = [K[y] : K[y^p]]$. Since y^p satisfies an irreducible quadratic equation over K, $[K[y^p] : K] = 2$, and so $[K[y] : K[y^p]] = 2p/2 = p > 2$, by assumption. $\square$

3. ANNIHILATORS OF F-STABLE SUBMODULES AND THE FH-FINITE PROPERTY FOR F-PURE GORENSTEIN LOCAL RINGS

In this section we shall prove that if R is a local ring of positive prime characteristic p and M is an Artinian $R\{F\}$-module such that F acts injectively on M, then the set of annihilator ideals in R of F-stable submodules of M is a finite set of radical ideals closed under primary decomposition. In fact, it consists of a finite set of prime ideals and their intersections. From this we deduce that the an F-pure Gorenstein local ring is FH-finite.

We say that a family of radical ideals of a Noetherian ring is *closed under primary decomposition* if for every ideal I in the family and every minimal prime P of I, the ideal P is also in the family.

The following result is critical to the proofs of these theorems.

Theorem 3.1. *Let M be a Noetherian module over an excellent local ring (R, m). Then there is no family $\{N_\lambda\}_{\lambda \in \Lambda}$ of submodules of M satisfying all four of the conditions below:*

(1) *the family is closed under finite sums*

(2) *the family is closed under finite intersection*

(3) *all of the ideals $\mathrm{Ann}_R(M/N)$ for N in the family are radical, and*

(4) *there exist infinitely many modules in the family such that if N, N' are any two of them, the minimal primes of N are mutually incomparable with the minimal primes of N'.*

Hence, if a family of submodules $\{N_\lambda\}_{\lambda \in \Lambda}$ of M satisfies conditions (1), (2), and (3) above and the set

$$\{\mathrm{Ann}_R(M/N_\lambda) : \lambda \in \Lambda\}$$

is closed under primary decomposition, then this set of annihilators is finite.

Proof. Assume that one has a counterexample. We use both induction on the dimension of R and Noetherian induction on M. Take a counterexample in which the ring has minimum dimension. One can pass to the completion. Radical ideals stay radical, and (4) is preserved (although there may be more minimal primes). The key point is that if P, Q are incomparable primes of R, and $\widehat{R}$ is the completion of R, then $P\widehat{R}$, $Q\widehat{R}$ are radical with no minimal prime in common. A common minimal prime would contain $(P+Q)\widehat{R}$, a contradiction, since the minimal primes of $P\widehat{R}$ lie over P). This, applied together with the fact that the minimal primes of $\widehat{M/N}$ are minimal over $P\widehat{R}$ for some minimal prime P of M/N enables us to pass to the completion.

Take infinitely many N_i as in (4). Let M_0 be maximal among submodules of M contained in infinitely many elements of the N_i. Then the set of modules in the family containing M_0 gives a new counterexample, and we may pass to all quotients by M_0 (M_0 need not be in the family to make this reduction.) Thus, by Noetherian induction on M we may assume that every infinite subset of the N_i has intersection 0.

Consider the set of all primes of R in the support of an M/N_i. If $Q \neq m$ is in the support of infinitely many we get a new counterexample over R_Q. The $(N_i)_Q$ continue to have the property that no two have a minimal prime in common (in particular, they are distinct). Since R had minimum dimension for a counterexample, we can conclude that every Q other than m is in the support of just finitely many M/N_i.

Choose h as large as possible such that there infinitely many primes of height h occurring among the minimal primes of an M/N_i. Then there are only finitely many primes of height $h+1$ or more occurring as a minimal prime of an M/N_i, and, by the preceding paragraph, each one occurs for only finitely many N_i. Delete sufficiently many N_i from the sequence so that the no prime of height bigger than h occurs among the minimal primes of the M/N_i.

Let $D_1(i) = \bigcap_{s=1}^{i} N_s$. By Chevalley's lemma, $D_1(i_1)$ is contained in $m^2 M$ for i_1 sufficiently large: fix such a value of i_1. Let $W_1 = D_1(i_1)$. Let $D_2(j) = \bigcap_{s=i_1+1}^{j} N_s$. Then $D_2(j)$ is contained in $m^2 M$ for sufficiently large j: fix such a value i_2. Recursively, we can choose a strictly increasing sequence of integers $\{i_t\}_t$ with $i_0 = 0$ such that every

$$W_t = \bigcap_{s=i_{t-1}+1}^{i_t} N_s$$

is contained in m^2M. In this way we can construct a sequence W_1, W_2, W_3, ... with the same properties as the N_i but such that all of them are in m^2M. Now $W_1 + W_2 + \cdots + W_t$ stabilizes for $t \gg 0$, since M is Noetherian, and the stable value W is contained in m^2M. There cannot be any prime other than m in the support of M/W, or it will be in the support of M/W_j for all j and this will put it in the support of infinitely many of the original N_j. Hence, the annihilator of M/W is an m-primary ideal, and, by construction, it is contained in m^2 and, therefore, not radical, a contradiction.

It remains only to prove the final statement. If the set of annihilators were infinite it would contain infinitely many prime ideals. Since there are only finitely many possibilities for the height, infinitely many of them would be prime ideals of the same height. The modules in the family having these primes as annihilator satisfy (4), a contradiction. $\square$

By applying Theorem 3.1 to the family of quotients of R by the ideals, we immediately have:

Corollary 3.2. *A family of radical ideals in an excellent local ring closed under sum, intersection, and primary decomposition is finite.* $\square$

Discussion 3.3. For any local ring (R, m, K) we let E denote an injective hull of the residue class field, and we write $\_^{\vee}$ for $\mathrm{Hom}_R(\_, E)$. Note that E is also a choice for $E_{\widehat{R}}(K)$, and that is submodules over R are the same as its submodules over $\widehat{R}$. E is determined up to non-unique isomorphism: the obvious map $\widehat{R} \to \mathrm{Hom}_R(E, E)$ is an isomorphism, and so every automorphism of E is given by multiplication by a unit of $\widehat{R}$.

Now suppose that R is complete. Then $R^{\vee} \cong E$ and $E^{\vee} \cong R$, by Matlis duality. Matlis duality gives an anti-equivalence between modules with ACC and modules with DCC: in both cases, the functor used is a restriction of $\_^{\vee}$, In particular, the natural map $N \to N^{\vee\vee}$ is an isomorphism whenever N has DCC or ACC. Note that there is an order-reversing bijection between ideals I of R and submodules N of E given by $I \mapsto \mathrm{Ann}_E I$ and $N \mapsto \mathrm{Ann}_R N$: this is a consequence of the fact that the inclusion $N \hookrightarrow E$ is dual to a surjection $N^{\vee} \twoheadleftarrow R$ so that $N^{\vee} \cong R/I$ for a unique ideal I of R, and since $I = \mathrm{Ann}_R N^{\vee}$, we have that $I = \mathrm{Ann}_R N$. Note that $N \cong N^{\vee\vee} \cong (R/I)^{\vee} \cong \mathrm{Hom}_R(R/I, E) \cong \mathrm{Ann}_E I$. When R is regular or even if R is Gorenstein, $E \cong H^d_m(R)$.

When R is complete local and W is Artinian, Matlis duality, provides a bijection between the submodules of W and the surjections from $W^{\vee} = M$, and each such surjection is

determined by its kernel N. This gives an order-reversing bijection between the submodules of W and the submodules of M. Specifically, $V \subseteq W$ corresponds to $\mathrm{Ker}\,(W^\vee \twoheadrightarrow V^\vee) = \mathrm{Ker}\,(M \twoheadrightarrow V^\vee)$, and $N \subseteq M$ corresponds to $\mathrm{Ker}\,\big(M^\vee \twoheadrightarrow (M/N)^\vee\big)$. Here, $M^\vee = (W^\vee)^\vee \cong W$ canonically. This bijection converts sums to intersections and intersections to sums: the point is that the sum (intersection) of a family of submodules is the smallest (respectively, largest) submodule containing (respectively, containing) all of them, and the result follows from the fact that the correspondence is an order anti-isomorphism. Since the annihilator of a module kills the annihilator of its dual, Matlis duality preserves annihilators: it is obvious that the annihilator of a module kills its dual, and we have that each of the two modules is the dual of the other. In particular, under the order-reversing bijection between submodules V of W and submodules N of M, we have that $\mathrm{Ann}_R V = \mathrm{Ann}_R(M/N)$.

Discussion 3.4. Let $(R,\, m,\, K) \to (S,\, n,\, L)$ be local, and suppose that mS is n-primary and that L is finite algebraic over K: both these conditions hold if S is module-finite over R. Let $E = E_R(K)$ and $E_S(L)$ denote choices of injective hulls for K over R and for L over S, respectively. The functor $\mathrm{Hom}_R(\_,\, E)$ from S-modules to S-modules may be identified with $\mathrm{Hom}_R(\_ \otimes_S S,\, E) \cong \mathrm{Hom}_S\big(\_,\, \mathrm{Hom}_R(S,\, E)\big)$, which shows that $\mathrm{Hom}_R(S,\, E)$ is injective as an S-module. Every element is killed by a power of the maximal ideal of S, since mS is primary to n, and the value of the functor on $L = S/n$ is $\mathrm{Hom}_R(L,\, E) \cong \mathrm{Hom}_R(L,\, K)$ since the image of L is killed by m. But this is L as an L-module. Thus, $E_S(L) \cong \mathrm{Hom}_R(S,\, E)$, and the functor $\mathrm{Hom}_R(\_, E)$, on S-modules, is isomorphic with the functor $\mathrm{Hom}_S(\_,\, E_S(L))$.

We next observe:

Proposition 3.5. *Let R be a ring of characteristic p and let W be an $R\{F\}$-module.*

  (a) *If F acts injectively on W, the annihilator in R of every F-stable submodule is radical.*

  (b) *If I is the annihilator of an F-stable submodule V of W, then $I :_R f$ is also the annihilator of an F-stable submodule, namely, fV. Hence, if I is radical with minimal primes $P_1,\, \ldots,\, P_k$ then every P_j (and every finite intersection of a subset of the P_j is the annihilator of an F-stable submodule of M.*

Proof. If V is F-stable and $u \in R$ is such that $u^p \in \mathrm{Ann}_R V$, then $F(uV) = u^p F(V) \subseteq u^p V = 0$. Since F is injectively on W, $uV = 0$. This proves part (a). For part (b), note

that fV is F-stable since $F(fV) = f^p F(V) \subseteq f^p V \subseteq fV$ and $u(fV) = 0$ iff $(uf)V = 0$ iff $uf \in \mathrm{Ann}_R V = I$ iff $u \in I :_R f$. For the final statement, choose f in all of the P_j except P_i, and note that $I :_R f = P_i$. More generally, given a subset of the P_j, choose f in all of the minimal primes except those in the specified subset. $\square$

Theorem 3.6. *Let R be a local ring of positive prime characteristic p and let W be an Artinian $R\{F\}$-module. Suppose that F acts injectively on W. Then the*

$$\{\mathrm{Ann}_R V : V \text{ is an F-stable submodule of } W\}$$

is a finite set of radical ideals, and consists of all intersections of the finitely many prime ideals in it.

Proof. By Proposition (3.5), it suffices to prove that family of annihilators is finite. We may replace R by its completion without changing M or the action of F on M. The set of F-stable submodules is unaffected. The annihilator of each such submodule in R is obtained from its annihilator in $\widehat{R}$ by intersection with R. Therefore, it suffices to prove the result when R is complete, and we henceforth assume that R is complete.

As in (3.3) fix an injective hull E of K and let $\_^\vee = \mathrm{Hom}_R(\_, E)$. Matlis duality gives a bijection of submodules of W with submodules of $M = W^\vee$. The F-stable submodules of W are obviously closed under sum and intersection. Therefore, the submodules N of M that correspond to them are also closed under sum and intersection. We refer to these as the *co-stable* submodules of M. The annihilators of the modules M/N, where N runs through the co-stable submodules of M, are the same as the annihilators of the F-stable submodules of W. We may now apply the final statement of Theorem (3.1). $\square$

It is now easy to prove the second main result of this section. Recall the a local ring (R, m, K) of Krull dimension d is *quasi-Gorenstein* if $H_m^d(R)$ is an injective hull of K: equivalently, this means that R is a canonical module for R in the sense that its Matlis dual is $H_m^d(R)$.

Theorem 3.7. *Let (R, m, K) be a local ring of prime characteristic $p > 0$. If R is F-pure and quasi-Gorenstein, then $H_m^d(R)$ has only finitely many F-stable submodules. Hence, if R is F-pure and Gorenstein, the R is FH-finite.*

Proof. There is no loss of generality in replacing R by its completion. We apply Theorem (3.6) to the action of F on $H_m^d(R) = E$. The point is that because $E^\vee = R$, the dual of the

F-stable module V has the form R/I, where I is the annihilator of V, and so V is uniquely determined by its annihilator. Since there are only finitely many possible annihilators, there are only finitely many F-stable submodules of $H^d_m(R)$. $\square$

4. F-PURITY, FINITE LENGTH, AND ANTI-NILPOTENT MODULES

In this section we prove that certain quotients by annihilators are F-split, and we study the family of F-stable submodules of the highest local cohomology both in the F-pure Cohen-Macaulay case, and under less restrictive hypotheses. We do not know an example of an F-injective ring which does not have finite FH-length, but we have not been able to prove that one has finite FH-length even in the F-split Cohen-Macaulay case. We also give various characterizations of when a local ring has finite FH-length.

Theorem 4.1. *Let (R, m, K) be a local ring of prime characteristic $p > 0$ of Krull dimension d. Suppose that R is F-split. Let N be an F-stable submodule of $H^d_m(R)$, and let $J = \operatorname{Ann}_R N$. Then R/J is F-split. In fact, if $T : R^{1/p} \to R$ is any R-linear splitting, then for every such annihilator ideal J, $T\big(\Phi(J)\big) \subseteq J$, and so T induces a splitting $(R/J)^{1/p} \cong R^{1/p}/\Phi(J) \twoheadrightarrow R/J$.*

Proof. Let $H = H^d_m(R)$. When we apply $\_ \otimes_R H$ to $\iota : R \subseteq R^{1/p}$ and to $T : R^{1/p} \to R$, we get maps $\alpha : H \to R^{1/p} \otimes H \cong H^d_m(R^{1/p}) \cong H^d_n(R^{1/p})$, where n is the maximal ideal of $R^{1/p}$, and also a map $\tilde{T} : H^d_m(R^{1/p}) \to H$. Let $u \in h$ and $s \in R^{1/p}$. In fact, we may view α as $\iota \otimes_R \operatorname{id}_H$ and $\tilde{T}$ as $T \otimes_R \operatorname{id}_H$.

Therefore, $\tilde{T}(s\alpha(u)) = (T \otimes_R \operatorname{id}_H)\big(s(1 \otimes u)\big) = (T \otimes_R \operatorname{id}_H)(s \otimes u) = T(s) \otimes u$, and since $T(s) \in R$, this is simply $T(s)u$. To show that $T(J^{1/p}) \subseteq J$, we need to prove that $T(J^{1/p})$ kills N in $H^d_m(R)$. Take $u \in N$ and $j \in J$. Then $T(j^{1/p})u = \tilde{T}(j^{1/p})\alpha(u)$, taking $s = j^{1/p}$. But now, since N is an F-stable submodule of $H^d_m(R)$, α maps N into the corresponding submodule N' of $H^d_m(R^{1/p})$, whose annihilator in $R^{1/p}$ is $\Phi(J)$. We therefore have that $j^{1/p}$ kills $\alpha(N) \subseteq N'$. This is the displayed fact $(**)$ in Discussion (2.4). Therefore, $T(j^{1/p})u = 0$, and $T(j^{1/p}) \in J$. $\square$

We now want to discuss the condition of having finite FH-length.

Proposition 4.2. *Let R be a characteristic p local ring with nilradical J, and let M be an Artinian $R\{F\}$-module. Then M has finite $R\{F\}$-length if and only if JM has finite length as an R-module and M/JM has finite $(R/J)\{F\}$-length.*

Proof. JM has a finite filtration by submodules J^tM, and F acts trivially on each factor. $\square$

Because of Proposition (4.2), we shall mostly limit our discussion of finite FH-length to the case where R is reduced.

Discussion 4.3. Let (R, m, K) be a local ring of characteristic $p > 0$ and let M be an Artinian $R\{F\}$-module. We note that M is also an Artinian $\widehat{R}\{F\}$-module with the same action of F, and we henceforth assume that R is complete in this discussion. We shall also assume that R is reduced. (In the excellent case, completing will not affect whether the ring is reduced.) Fix an injective hull $E = E_R(K)$ for the residue field and let $\_^\vee$ denote the functor $\mathrm{Hom}_R(\_, E)$.

Lemma 4.3. *Let (R, m, K) be a local ring of characteristic p. Then we may construct S local and faithfully flat over R with maximal ideal mS such that S is complete and faithfully flat over R, such that S is Cohen-Macaulay if R is Cohen-Macaulay, such that S is F-injective if R is, and such that S is F-split if R is F-pure. For every i, the poset of F-stable modules of $H_m^i(R)$ injects by a strictly order-preserving map into the poset of F-stable modules of $H_{mS}^i(S) = H_m^i(S)$. Hence, R is FH-finite (respectively, has finite FH-length) if S is FH-finite (respectively, has FH-finite length).*

Proof. By Lemma (2.7), we may first replace R by its completion $\widehat{R}$. We then choose a coefficient field K and a p-base Λ for K, and replace $\widehat{R}$ by $\widehat{R}_\Gamma$ for Γ a sufficiently small cofinite subset of Λ, using Lemma (2.9). Finally, we replace $\widehat{R}_\Gamma$ by its completion. The map on posets is induced by applying $S \otimes_R \_$. $\square$

Discussion 4.4: reductions in the Cohen-Macaulay F-pure case.

Consider the following three hypotheses on a local ring R of prime characteristic $p > 0$.

(1) R is F-injective and Cohen-Macaulay, with perfect residue class field.

(2) R is F-pure.

(3) R is F-pure and Cohen-Macaulay.

In all three cases we do not know, for example, whether the top local cohomology module has only finitely many F-stable submodules. The point we want to make is that Lemma (4.3) permits us to reduce each of these questions to the case where R is complete and F-finite. Moreover, because F-pure then implies F-split, in cases (2) and (3) the hypothesis that R be F-pure may be replaced by the hypothesis that R be F-split.

If $V \subseteq V' \subseteq W$ are F-stable R-submodules of the $R\{F\}$-module W, we refer to V'/V as a *subquotient* of W. We next note the following:

Proposition 4.5. *Let R be a ring of positive prime characteristic p and let W be an $R\{F\}$-module. The following conditions on W are equivalent:*

(a) *If V is an F-stable submodule of W, then F acts injectively on W/V*

(b) *F act injectively on every subquotient of W.*

(c) *The action of F on any subquotient of W is not nilpotent.*

(d) *The action of F on any nonzero subquotient of W is not zero.*

(e) *If $V \subseteq V'$ are F-stable submodules of W such that $F^k(V') = V$ for some $k \geq 1$ then $V' = V$.*

Proof. (a) and (b) are equivalent because a subquotient V'/V is an $R\{F\}$-submodule of W/V. If the action of F on a subquotient $V''/V \subseteq W/V$ is not injective, the kernel has the form V'/V where V' is F-stable. Hence (b) and (d) are equivalent. If F is nilpotent on V''/V it has a nonzero kernel of the form V'/V. This shows that (c) is also equivalent. (e) follows because F^k kills V'/V if and only if $F^k(V') \subseteq V$. $\square$

Definition 4.6. With R and W as in Proposition (4.5) we shall say that W is *anti-nilpotent* if it satisfies the equivalent conditions (a) through (e).

The following is Theorem 4.7 on p. 108 of [Ly]:

Theorem 4.7 (Lyubeznik). *Let R be a local ring of prime characteristic $p > 0$ and let W be an Artinian R-module that has an action F of Frobenius on it. Then W has a finite filtration*

$$0 = L_0 \subseteq N_0 \subseteq L_1 \subseteq N_1 \subseteq \cdots \subseteq L_s \subseteq N_s = M$$

by F-stable submodules such that every N_j/L_j is nilpotent, i.e., killed by a single power of F, while every L_j/N_{j-1} is simple in the category of $R\{F\}$-modules, with a nonzero action of F on it. The integer s and the isomorphism classes of the modules L_j/N_{j-1} are invariants of W. $\square$

Note that the assumption that the action of F on a simple module $L \neq 0$ is nonzero is equivalent to the assertion that the action of F is injective, for if F has a non-trivial kernel it is an $R\{F\}$-submodule and so must be all of L.

We have at once:

Proposition 4.8. *Let the hypothesis be as in (4.7). and let W have a filtration as in (4.5). Then:*

(a) *W has finite length as an $R\{F\}$-module if and only if each of the factors N_j/L_j has finite length in the category of R-modules.*

(b) *W is anti-nilpotent if and only if in some (equivalently, every filtration) as in (4.6), the nilpotent factors are all zero.*

Proof. (a) This comes down to the assertion that if a power of F kills W then W has finite length in the category of $R\{F\}$-modules iff it has finite length in the category of R-modules. But W has a finite filtration with factors $F^j(W)/F^{j+1}(W)$ on which F acts trivially, and the result is obvious when F acts trivially.

It remains to prove (b). If W is anti-nilpotent, then the nilpotent factors in any finite filtration must be 0, since they are subquotients of W. Now suppose that W has a finite filtration by simple $R\{F\}$-modules on which F acts injectively. Suppose that we have $0 \subseteq V \subseteq V' \subseteq W$ such that F acts trivially on V'/V. This filtration and the filtration by simple $R\{F\}$-modules on which F acts injectively have a common refinement in the category of $R\{F\}$-modules. This implies that V'/V has a finite filtration in which all the factors are simple $R\{F\}$-modules on which F acts injectively. Since F must be zero on the smallest nonzero submodule in the filtration, this is a contradiction. $\square$

Corollary 4.9. *Let R be a local ring of positive prime characteristic p. If M is an Artinian R-module that is anti-nilpotent as an $R\{F\}$-module, then so is every submodule, quotient module, and every subquotient of M in the category of $R\{F\}$-modules.*

Proof. It suffices to show this for submodules and quotients. But if N is any $R\{F\}$-submodule, the filtration $0 \subseteq N \subseteq M$ has a common refinement with the filtration of M with factors that are simple $R\{F\}$-modules with non-trivial F-action. $\square$

We also note the following, which is part of [Ly], Theorem 4.2.

Theorem 4.10 (Lyubeznik). *Let $T \to R$ be a surjective map from a complete regular local ring T of prime characteristic $p > 0$ onto a local ring (R, m, K). Then there exists a*

contravariant additive functor $\mathcal{H}_{T,R}$ *from the category of* $R\{F\}$*-modules that are Artinian over* R *to the category of* F_T*-finite modules in the sense of Lyubeznik such that*

(a) $\mathcal{H}_{T,R}$ *is exact.*

(b) $\mathcal{H}_{T,R}(M) = 0$ *if and only if the action of some power of* F *on* M *is zero.*

Theorem 4.11. *Let* R *be a local ring of positive prime characteristic* p. *Let* M *be an Artinian* R*-module that is anti-nilpotent as an* $R\{F\}$*-module. Then* M *has only finitely many* $R\{F\}$*-submodules.*

Proof. We may replace R by its completion and write R as T/J where T is a complete regular local ring of characterisitc p. By (4.10) above, we have a contravariant exact functor $\mathcal{H}_{T,R}$ on $R\{F\}$-modules Artinian over R to F_T-finite modules in the sense of Lyubeznik. This functor is faithfully exact when restricted to anti-nilpotent modules, and all subquotients of M are anti-nilpotent. It follows that if M_1 and M_2 are distinct $R\{F\}$-submodules of M, then N_1 and N_2 are distinct, where $N_i = \mathrm{Ker}\left(\mathcal{H}(M) \to \mathcal{H}(M_i)\right)$ for $i = 1, 2$. By the main result of [Ho4], an F_T-finte module in the sense of Lyubeznik over a regular local ring T has only finitely many F-submodules, from which the desired result now follows at once. $\square$

Discussion 4.12: local cohomology after adjunction of a formal indeterminate. Let R be any ring and M an R-module. Let x be a formal power series indeterminate over R. We shall denote by $M\langle x^{-1}\rangle$ the $R[[x]]$-module $M \otimes_{\mathbb{Z}} (\mathbb{Z}[x, x^{-1}]/\mathbb{Z}[x])$. This is evidently an $R[x]$-module, and since every element is killed by a power of x, it is also a module over $R[[x]]$. Note that if R is an A-algebra, this module may also be described as $M \otimes_A (A[x, x^{-1}]/A[x])$. In particular, $M\langle x^{-1}\rangle \cong M \otimes_R (R[x, x^{-1}]/R[x])$, and if R contains a field K, $M\langle x^{-1}\rangle \cong M \otimes_K K[x, x^{-1}]/K[x]$. We have that $M \cong M \otimes x^{-n}$ for all $n \geq 1$ via the map $u \mapsto u \otimes x^{-n}$, and we write Mx^{-n} for $M \otimes x^{-n}$. As an R-module, $M\langle x^{-1}\rangle \cong \bigoplus_{n=1}^{\infty} Mx^{-n}$, a countable direct sum of copies of M. The action of x kills Mx^{-1} and for $n > 1$ maps $Mx^{-n} \cong Mx^{-(n-1)}$ in the obvious way, sending $ux^{-n} \mapsto ux^{-(n-1)}$. Then $M \to M\langle x^{-1}\rangle$ is a faithfully exact functor from R-modules to $R[[x]]$-modules. If R has prime characteristic $p > 0$ and M is an $R\{F\}$-module, then we may also extend this to an $R[[x]]\{F\}$-module structure on $M\langle x^{-1}\rangle$ by letting F send $ux^{-n} \mapsto F(u)x^{-pn}$. This gives a convenient way of describing what happens to local cohomology when we adjoin a formal power series indeterminate to a local ring R.

Proposition 4.13. *Let R be a Noetherian ring, let I be a finitely generated ideal of R, and let x be a formal power series indeterminate over R. Let J denote the ideal $(I, x)R[[x]]$ of $R[[x]]$.*

   (a) *For every i, $H_J^i(R[[x]]) \cong H_I^i(R)\langle x^{-1}\rangle$. If R has prime characteristic $p > 0$ then the action of Frobenius on $H_J^i(R[[x]])$ agrees with the action on $H_I^i(R)\langle x^{-1}\rangle$ described above.*

   (b) *In particular if (R, m, K) is local and and $J = \mathfrak{n}$, the maximal ideal of $R[[x]]$, then for every i, $H_\mathfrak{n}^i(R[[x]]) \cong H_m^i(R)\langle x^{-1}\rangle$.*

   (c) *If (R, m, K) is local and M is Artinian, then $M\langle x^{-1}\rangle$ is Artinian over $R[[x]]$.*

   (d) *If (R, m, K) is local of prime characteristic $p > 0$ and M is a simple $R\{F\}$-module on which the action of F is not 0, then $M\langle x^{-1}\rangle$ is a simple $R[[x]]\{F\}$-module.*

   (e) *If (R, m, K) is local of prime characteristic $p > 0$ and M is an anti-nilpotent $R\{F\}$-module, then $M\langle x^{-1}\rangle$ is an anti-nilpotent $R[[x]]\{F\}$-module.*

Proof. (a) Let $f_1, \ldots, f_n \in R$ generate I. Then $H_I^i(R)$ is the ith cohomology of the complex $C^\bullet(\underline{f}; R)$, and $H_J^i(R[[x]])$ is the ith cohomology module of the complex $C^\bullet(\underline{f}, x; R[[x]]) \cong C^\bullet(\underline{f}; R) \otimes_R C^\bullet(x; R[[x]])$. The complex $C^\bullet(x; R[[x]])$ is simply

$$0 \to R[[x]] \to R[[x]]_x \to 0$$

and has augmentation $R\langle x^{-1}\rangle$. Since $R[[x]]$, $R[[x]]_x$, and $R\langle x^{-1}\rangle$ are all R-flat, we have that $H_J^i(R[[x]])$ is the ith cohomology module of the mapping cone of the injection of complexes $C^\bullet(\underline{f}; R[[x]]) \hookrightarrow C^\bullet(\underline{f}; R[[x]]_x)$, which may be identified with the cohomology of the quotient complex, and so with the cohomology of $C^\bullet(\underline{f}; R\langle x^{-1}\rangle) \cong C^\bullet(\underline{f}; R)\langle x^{-1}\rangle$. Since $R\langle x^{-1}\rangle$ is R-flat (in fact, R-free) applying $\_ \otimes_R R\langle x^{-1}\rangle$ commutes with formation of cohomology, from which the result follows. Part (b) is immediate from part (a).

To prove (c), note that every element of $M\langle x^{-1}\rangle$ is killed by a power of m and of x, and so by a power of $\mathfrak{n}$. It therefore suffices to see that the annihilator of $\mathfrak{n}$ is a finite-dimensional vector space over K. But the annihilator of x is Mx^{-1}, and the annihilator of m in Mx^{-1} is isomorphic with the annihilator of m in M.

We next prove (d). Since the kernel of the action of F on M is an F-stable R-submodule of M, the fact that M is a simple $R\{F\}$-module implies that F acts injectively on M. Suppose that N is a nonzero $R[[x]]\{F\}$-submodule of M, and that $u_1 x^{-1} + \cdots + u_k x^{-d} \in N$. By multiplying by x^{d-1} we see that $u_k x^{-1} \in N$. Hence, N has nonzero intersection

$N_1 x^{-1}$ with $M x^{-1}$. N_1 is an R-submodule of M. It is also F-stable, since if $u x^{-1} \in N$ then $x^{p-1} F(u x^{-1}) = F(u) x^{-1} \in N \cap M x^{-1}$. Thus, N contains $M x^{-1}$. In every degree h, let $N_h x^{-n} = N \cap M x^{-h}$. Then $N_h \neq 0$, for if $u \in M - \{0\}$ and $q = p^e > h$, $x^{(} q - h) F^e(u x^{-1}) = F^e(u) x^{-h}$, and $F(u) \neq 0$. Moreover, the R-submodule $N_h \subseteq M$ is F-stable, for if $v x^{-h} \in N_h x^{-h}$, then $x^{ph-h} F(v x^{-h}) = F(v) x^{-h} \in N_h x^{-h}$. Thus, $N_h = M$ for all $h \geq 1$, and so $N = M$.

For part (e), if M has a finite filtration by simple $R\{F\}$-modules M_j on which F has nonzero action, then applying $N \mapsto N\langle x^{-1}\rangle$ gives a finite filtration of $M\langle x^{-1}\rangle$ with factors $M_j\langle x^{-1}\rangle$, each of which is a simple $R[[x]]\{F\}$-module by part (d) on which F has nonzero action. $\square$

Theorem 4.14. *Let (R, m, K) be a local ring of prime charjacteristic $p > 0$. Let x be a formal power series indeterminate over R. Let M be an $R\{F\}$-module that is Artinian as an R-module. Then the following are equivalent:*

(1) *M is anti-nilpotent.*

(2) *$M\langle x^{-1}\rangle$ has finite length over $R[[x]]\{F\}$.*

(3) *$M\langle x^{-1}\rangle$ has only finitely many F-stable submodules over $R[[x]]$.*

When these equivalent conditions hold, M has only finitely many F-stable modules over R.

Proof. We show that $(2) \Rightarrow (1) \Rightarrow (3)$. Assume (2). If M is not anti-nilpotent, it has a subquotient $N \neq 0$ on which the action of F is 0. Then $N\langle x^{-1}\rangle$ is a subquotient of $M\langle x^{-1}\rangle$, and so has finite length as an $R[[x]]\{F\}$-module. Since F kills it, it must have finite length as an $R[[x]]$-module. But this is clearly false, since no power of x kills $N\langle x^{-1}\rangle$.

To see that $(1) \Rightarrow (3)$, note that by Proposition (4.13e), the fact that M is anti-nilpotent implies that $M\langle x^{-1}\rangle$ is anti-nilpotent over $R[[x]]$, and the result now follows from Theorem (4.11). The implication $(3) \Rightarrow (2)$ is obvious. $\square$

The next result is an immediate corollary.

Theorem 4.15. *Let (R, m, K) be a local ring of prime characteristic $p > 0$ and let $x = x_1$ and $x_2, \ldots, x_n$ be formal power series indeterminates over R. Let $R_n = R[[x_1, \ldots, x_n]]$, where $R_0 = R$, and let m_n be its maximal ideal. Then the following conditions on R are equivalent:*

(1) *The local cohomology modules $H^i_m(R)$ are anti-nilpotent.*

(2) *The ring $R[[x]]$ has FH-finite length.*

(3) *The ring R_n is FH-finite for every $n \in \mathbb{N}$.*

(4) *The local cohomology modules $H^i_{m_n}(R_n)$ are anti-nilpotent over R_n for all $n \in \mathbb{N}$.*

(5) *The ring R_n has FH-finite length for all $n \in \mathbb{N}$.*

When these conditions hold, R is F-injective.

Proof. We have that $(1) \Rightarrow (4)$ by $(4.13e)$ and a straightforward induction on n. This implies that R_n is FH-finite for all n by Theorem (4.11). Thus, $(4) \Rightarrow (3) \Rightarrow (5) \Rightarrow (2)$, and it suffices to prove that $(2) \Rightarrow (1)$, which is a consequence of Theorem (4.14).

The statement that R is then F-injective is obvious, since F acts injectively on any anti-nilpotent module. $\square$

Corollary 4.16. *Let (R, m, K) be an F-pure Gorenstein local ring prime characteristic $p > 0$ of Krull dimension d. Then $H^d_m(R)$ is anti-nilpotent, and so F acts injectively on every subquotient of $H^d_m(R)$.*

Proof. The hypothesis also holds for $R[[x]]$, and so the result follows from (3.7) and (4.15) $\square$

Corollay 4.17. *Let (R, m, K) be an F-pure Gorenstein local ring prime characteristic $p > 0$ of Krull dimension d. Let J be an ideal of R such that $dim\,(R/J) = d$. Then $H^d_m(R/J)$ is anti-nilpotent, and so F acts injectively on $H^d_m(R/J)$. Hence, if R/J is Cohen-Macaulay, it is F-injective.*

Proof. Since R and R/J have the same dimension, the long exact sequence for local cohomology gives an $R\{F\}$-module surjection $H^d_m(R) \twoheadrightarrow H^d_m(R/J)$, which shows that $H^d_m(R/J)$ is anti-nilpotent as an $R\{F\}$-module, and therefore as an $(R/J)\{F\}$ module as well. $\square$

We shall next need the following result from [Wat]: the F-pure case is attributed there to Srinivas. See Theorem 2.7 of [Wat] and the comment that precedes it.

Theorem 4.18 (Watanabe and Srinivas). *Let $h : (R, m, K) \to (S, \mathfrak{n}, L)$ be a local homomorphism of local normal domains of prime characterisitic $p > 0$ such that S is module-finite over R and the map h is étale in codimension one. If R is strongly F-regular, then so is S. If R is F-pure, then so is S.*

The explicit statement in [Wat] is for the F-regular case, by which the author means the weakly F-regular case. However, the proof given uses the criterion (i) of Proposition (1.4), that the local ring (R, m, K) is weakly F-regular if and only if 0 is tightly closed in $E_R(K)$, which is correct for finitistic tight closure but not for the version of tight closure being used in [Wat]. In fact, condition (i) as used in [Wat] characterizes strong F-regularity in the F-finite case, and we take it as the definition of strong F-regularity here.

On the other hand, there are no problems whatsoever in proving the final statement about F-purity. The action of Frobenius $F_S : E_S(L) \to F_S\big(E_S(L)\big)$ is shown to be the same as the action of Frobenius when $E_S(L)$ is viewed as R-module. Since R is F-pure, the Frobenius action $M \to F_R(M)$ is injective for any R-module.

Corollary 4.19 (Watanabe). *Let (R, m) be a normal local domain of characteristic $p > 0$. Let I be an ideal of pure height one, and suppose that I has finite order $k > 1$ in the divisor class group of R. Choose a generator u of $I^{(k)}$. We let*

$$S = R \oplus It \oplus \cdots \oplus I^{(j)} \oplus \cdots \oplus I^{(k-1)}$$

with $I^{(k)}$ identified with R using the isomorphism $R \cong I^{(k)}$ such that $1 \mapsto u$. (If t is an indeterminate, we can give the following more formal description: form $T = \bigoplus_{j=0}^{\infty} I^{(j)} t^j \subseteq R[t]$, and let $S = T/(ut^k - 1.)$ This ring is module-finite over R, and if k is relatively prime to p, it is étale over R in codimension one.

Hence, if k is relatively prime to p, then S is strongly F-regular if and only if R is strongly F-regular, and S is F-pure if and only if R is F-pure.

Moreover, if $I \cong \omega$ is a canonical module for R, then S is quasi-Gorenstein.

The final statement is expected because, by the discussion of canonical modules for module-finite extensions in (2.2), we have that $\omega_S \cong \mathrm{Hom}_R(S, \omega)$ and $\mathrm{Hom}_R(\omega^{(i)}, \omega) \cong \omega^{-(i-1)} \cong \omega^{(k-(i-1))}$. See [Wa], [TW], and [YW, §3], and [Si] for further details and background on this technique. The result above will enable us to use this "canonical cover trick" to prove the Theorem below by reduction to the quasi-Gorenstein case. A word of caution is in order: even if R is Cohen-Macaulay, examples in [Si] show that the auxiliary ring S described in (4.19) need not be Cohen-Macaulay, and so one is forced to consider the quasi-Gorenstein property. There are examples (see [Si, Theorem (6.1)]) where R is F-rational but S is not Cohen-Macaulay. On the other hand, if R is strongly F-regular, the result of [Wat] shows that S is as well: in particular, S is Cohen-Macaulay in this case.

Theorem 4.20. *Let R be an F-pure normal local domain of Krull dimension d such that R has canonical module $\omega = \omega_R$ of finite order k relatively prime to p in the divisor class group of R. Then $H^d(R)$ is anti-nilpotent, so that R is FH-finite.*

Proof. Since the hypotheses are stable under adjunction of a power series indeterminate, it follows from Theorem (4.15) that it is sufficient to show that R is FH-finite. We may identify ω with a pure height one ideal of I of R. We form the ring S described in (4.19). Then S is F-pure and quasi-Gorenstein, and so $H_m^d(S)$ has only finitely many F-stable submodules by Theorem (3.7). The same holds for $H_m^d(R)$ by Lemma (2.7b), while the lower local cohomology modules of R with support in m vanish. □

The following improves the conclusion of (2.14) with some additional hypotheses.

Theorem 4.21. *Let (R, m, K) be an F-injective Cohen-Macaulay local ring with of prime characteristic $p > 0$ such that K is perfect. Suppose that R has an m-primary ideal $\mathfrak{A}$ such that $\mathfrak{A}I^* \subseteq I$ for every ideal I generated by a system of parameters. Let $d = \dim(R)$. Then $H_m^d(R)$ is anti-nilpotent, so that R and every formal power series ring over R is FH-finite.*

Proof. Let $H = H_m^d(R)$ and let $V = 0_H^*$, which, as in the proof of (2.14), is killed by $\mathfrak{A}$ and has finite length. Then $R/\mathfrak{A}$ is a complete local ring with a perfect residue class field, and contains a unique coefficient field K. This gives V the structure of a K-module, i.e., it is a finite-dimensional K-vector space, and $F : V \to V$ is K-linear if we let the action of K on the second copy of V be such that $c \cdot v = c^p v$ for $c \in K$. Since K is perfect, the dimension of V does not change when we restrict F in this way. Since R is F-injective, the action of F on V is then a vector space isomorphism, and is then also an isomorphism when restricted to subquotients that are $K\{F\}$-modules. It follows that V is anti-nilpotent over $R\{F\}$, and to complete the proof it will suffice to show that F cannot act trivially on the the simple $R\{F\}$-module H/V.

Choose a system of parameters $x_1, \ldots, x_d$ for R. Let $I_t = (x_1^t, \ldots, x_d^t)R$. For any sufficiently large value of t, we may identify V with I^*/I. If F acts trivially on H/V, then for all large t, the image of $1 \in R/I_t \subseteq H$ under F is 0 in H/V, which means that $(x_1 \cdots x_d)^{p-1} \in I_{pt}^*$, and then $\mathfrak{A}(x_1 \cdots x_d)^{p-1} \subseteq I_{pt}$. This implies that

$$\mathfrak{A} \subseteq I_{pt} :_R (x_1 \cdots x_d)^{p-1} = I_{pt} + (x_1 \cdots x_d)^{pt-p}R$$

for all $t \gg 0$, which is clearly false. □

5. FACE RINGS

We give a brief treatment of the decomposition of the local cohomology of face rings over a field with support in the homogeneous maximal ideal. This is discussed in [BH], §5.3, although not in quite sharp enough a form for our needs here, and there are sharp results in substantially greater generality in [BBR]: see Theorem 5.5, p. 218. However, neither result discusses the $R\{F\}$-structure.

Let K be a fixed field of positive characteristic p and let Δ be an abstract finite simplicial complex with vertices $x_1, \ldots, x_n$. Let I_Δ denote the ideal in the polynomial ring $S = K[x_1, \ldots, x_n]$ generated by all monomials in the x_j such that the set of variables occurring in the monomial is not a face of Δ. This ideal is evidently generated by the square-free monomials in the x_j corresponding to minimal subsets of the variables that are not faces of Δ. Let $K[\Delta] = S/I_\Delta$, the *face ring* (or *Stanley-Reisner ring*) of Δ over K. The minimal primes of $K[\Delta]$ correspond to the maximal faces σ of Δ: the quotient by the minimal prime corresponding to σ is a polynomial ring in the variables occurring in σ. The Krull dimension of $K[\Delta]$ is therefore one more than the dimension of the simplicial complex Δ.

If σ is any face of Δ, the link, denoted link(σ), of σ in Δ is the abstract simplicial complex consisting of all faces τ of Δ disjoint from σ such that $\sigma \cup \tau \in \Delta$. The link of the empty face is Δ itself. By a theorem of G. Reisner [Rei], $K[\Delta]$ is Cohen-Macaulay if and only if the reduced simplicial cohomology of every link vanishes except possibly in the top dimension, i.e., in the dimension of the link itself.

Note that the reduced simplicial cohomology $\widetilde{H}^i(\Delta; K)$ of a finite simplicial complex $\Delta \neq \emptyset$ is the same as the simplicial comhomology unless $i = 0$, in which case its dimension as a K-vector space is one smaller. If Δ is an i-simplex, the reduced simplicial cohomology vanishes in all dimensions, unless Δ is empty, in which case we have $\widetilde{H}^{-1}(\emptyset; K) \cong K$: $\widetilde{H}^i(\emptyset; K) = 0$ for all $i \neq -1$. Note also that $\emptyset$ is the only simplicial complex Δ such that $\widetilde{H}^i(\Delta; K) \neq 0$ for a value of $i < 0$.

Let m be the homogeneous maximal ideal of $K[\Delta]$. We shall show that $K[\Delta]$ and its completion are FH-finite in all cases, and in fact, the local cohomology modules are anti-nilpotent. This follows from the following theorem, which also recovers Reisner's result

[Rei] mentioned above in a finer form: it also gives a completely explicit description of all the $H_m^i(K[\Delta])$, including their structure as $R\{F\}$-modules. We write $|\nu|$ for the cardinality of the set ν. If $\nu \in \Delta$, we let

$$K[\nu] = K[\Delta]/(x_i : x_i \notin \nu),$$

which is a $K[\Delta]$-algebra and is also the polynomial ring over K in the variables x_j that are vertices of ν. Then $H_m^i(S_\nu)$ vanishes except when $i = |\nu|$. When $i = \nu$ it is the highest nonvanishing local cohomology of a polynomial ring, and, if the characteristic of K is $p > 0$, it is a simple $R\{F\}$-module on which F acts injectively.

Note that if $p > 0$ is prime, $\kappa = \mathbb{Z}/p\mathbb{Z}$, R and K are rings of characteristic p, and H is an $R\{F\}$-module H, the $K \otimes_\kappa H$ has the structture of a $(K \otimes_\kappa R)\{F\}$-module: the action of F is determined by the rule $F(c \otimes u) = c^p \otimes F(u)$ for all $c \in K$ and $u \in H$. This is well-defined because the action of F restricts to the identity map on $\mathbb{Z}/p\mathbb{Z}$.

Theorem 5.1. *With $R = K[\Delta]$ as above, let κ denote the prime field of K. Let m and μ be the homogeneous maximal ideals of R and $\kappa[\Delta]$, respectively. Then*

$$(*) \quad H_m^i(R) \cong \bigoplus_{\nu \in \Delta} \widetilde{H}^{i-1-|\nu|}(\mathrm{link}(\nu); K) \otimes_\kappa H_\mu^{|\nu|}(\kappa[\nu]).$$

If K has characterisitc $p > 0$, this is also an isomorphism of $R\{F\}$-modules, with the action of F described in the paragraph above. Hence, every $H_m^i(R)$ is a finite direct sum of simple $R\{F\}$-modules on which F acts injectively.

If (R_1, m_1) is either R_m or its completion, then for all i, $H_m^i(R)$ may be identified with $H_{m_1}^i(R_1)$, and $H_{m_1}^i(R_1)$ is a finite direct sum of simple $R_1\{F\}$-modules on which F acts injectively, and so is anti-nilpotent and FH-finite over R_1.

Proof. If σ is a subset of the x_j we denote by $x(\sigma)$ the product of the x_j for $j \in \sigma$. Thus, in $K[\Delta]$, the image of $x(\sigma)$ is nonzero if and only if $\sigma \in \Delta$. Our convention is that $\sigma = \emptyset$ is in Δ, is the unique face of dimension -1, and that $x(\emptyset) = 1$. We write $[\Delta]_i$ for the set of faces of Δ of dimension i. Then $H_m^i(K[\Delta])$ is the ith cohomology module of the complex $C^\bullet$ whose ith term is displayed below:

$$0 \to K[\Delta] \to \cdots \to \bigoplus_{\sigma \in \Delta_{i-1}} K[\Delta]_{x(\sigma)} \to \cdots$$

The initial nonzero term $K[\Delta]$ may be thought of as $K[\Delta]_{x(\emptyset)}$ and the highest nonzero terms occur in degree $\dim(\Delta) + 1$. Let $\theta = (\theta_1, \ldots, \theta_n) \in \mathbb{Z}^n$ by an n-tuple of integers. We want to calculate that θ-graded piece of $H^i_m(K[\Delta])$. This is the same as the ith cohomology of the θ-graded piece of the complex: denote by $C^\bullet[\theta]$ the θ-graded piece of the complex $C^\bullet$. Let $\mathrm{neg}(\theta)$ (respectively, $\mathrm{pos}(\theta)$, respectively, $\mathrm{supp}(\theta)$) denote the set of variables x_i such that θ_i is stictly negative (respectively, strictly positive, respectively, nonzero). Thus, $\mathrm{supp}(\theta)$ is the disjoint union of $\mathrm{neg}(\theta)$ and $\mathrm{pos}(\theta)$.

Let $\nu = \mathrm{neg}(\theta)$ and $\pi = \mathrm{pos}(\theta)$. Then $K[\Delta]_{x(\sigma)}$ has a nonzero component in degree θ if and only if $\nu \subseteq \sigma$ and $\sigma \cup \pi \in \Delta$, and then there is a unique copy of K corresponding to θ in the complex. By deleting the variables occurring in $\nu = \mathrm{neg}(\theta)$ from each face, we find that $C^\bullet[\theta]$ corresponds, with a shift in degree by the cardinality $|\nu|$ of ν, to the complex used to calculate the reduced simplicial cohomology of $\Delta_{\nu,\pi}$, where this is the subcomplex of the link of ν consisting of all simplices τ such that $\tau \cup \pi \in \Delta$. If π is non-empty, $\Delta_{\nu,\pi}$ is a cone on any vertex in π. Hence, the graded component of local cohomology in degree θ can be nonzero only when $\mathrm{pos}(\theta) = \emptyset$ and $\nu = \mathrm{neg}(\theta) \in \Delta$. It now follows that

$$[H^i_m(K[\Delta])]_\theta \cong \bigoplus_\nu \widetilde{H}^{i-|\nu|-1}(\mathrm{link}\, \nu) x^\theta$$

if $\pi = \emptyset$ and $\nu \in \Delta$ is the set of variables corresponding to strictly negative entries in θ, and is zero otherwise.

It follows that we may identify

$$H^i_m(K[\Delta]) \cong \bigoplus_{\nu \in \Delta} \left(\bigoplus_{\mathrm{supp}(w)=\mathrm{neg}(w)=\nu} \widetilde{H}^{i-|\nu|-1}(\mathrm{link}\, \nu) w \right),$$

where w runs through all non-positive monomials such that the set of variables with strictly negative exponents is ν.

We next want to show that if $\nu \in \Delta$, then

$$\bigoplus_{\mathrm{supp}(\theta)=\mathrm{neg}(\theta)=\nu} [H^i_m(K[\Delta]]_\theta \cong \widetilde{H}^{i-|\nu|-1}(\mathrm{link}(\nu);\, K) \otimes_K H^{|\nu|}_m(K[\nu]).$$

The term on the right may also be written as $\widetilde{H}^{i-|\nu|-1}(\mathrm{link}(\nu);\, K) \otimes_\kappa H^{|\nu|}_\mu(\kappa[\nu])$. We also need to check that the actions of F agree.

The action of F on $C^\bullet(\underline{x}; K[\Delta])$ is obtained from the action of F on $C^\bullet(\underline{x}; \kappa[\Delta])$ by applying $K \otimes_\kappa \_$. Thus, we reduce at once to the case where $K = \kappa$, which we assume henceforth.

Let $\gamma \otimes w$ be an element in the cohomology, where $\gamma \in \widetilde{H}^{i-|\nu|-1}(\mathrm{link}(\nu); K)$ and w is a monomial in the variables of ν with all exponents strictly negative. The action of x_i by multiplication is obvious in most cases. If $x_i \notin \nu$, the product is 0. If x_i occurs with an exponent other than -1 in w, one simply gets $\gamma \otimes (x_i w)$. The main non-trivial point is that if x_i occurs with exponent -1 in w, x_i kills $\gamma \otimes w$. To verify this, let $\nu' = \nu - \{x_i\}$. Take a cocycle η that represents γ. After we multiply by x_i, we get an element of $H^{i-|\nu|}\big(\mathrm{link}(\nu')\big)$. Note that each simplex remaining when we delete the variables in ν' involves x_i, and so that the cocycle η' we get from η may be viewed as a cocycle of the complex used to compute the reduced simplicial cohomology of the closed star of x_i in $\mathrm{link}(\nu')$. Since this closed star is a cone, that cohomology is 0. This shows that x_i kills every homogeneous component whose degree has -1 in the ith coordinate.

We have completed the calculation of the structure of the local cohomology as an R-module. On the other hand, given $\nu \in \Delta$, because the field is κ, when F acts on the complex

$$\bigoplus_{\mathrm{supp}(\theta) = \neg(\theta) = \nu} [C^\bullet(\underline{x}; R)]_\theta$$

the value of F acting on an element of the form ηw, where η is a cocycle, if simply ηw^p, and so it follows that

$$F(\gamma \otimes w) = \gamma \otimes w^p.$$

This shows that $R\{F\}$-module structure is preserved by the isomorphism $(*)$. $\qquad\square$

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