Length functions determined by killing powers of several ideals in a local ring

by

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CHAPTER I

Introduction

1.1 Motivation

Intuition suggests that the parabola $y = x^2$ intersects the line $y = 0$ more closely than does the line $y = x$. This intuition can be made more concrete by associating with each isolated point of intersection of two plane curves a number called the *intersection multiplicity* of the curves at that point. The intersection multiplicity at a point P is, roughly speaking, equal to the number of points of intersection you would expect to get near P if you perturbed the intersecting varieties slightly. For example, if ϵ is small (but nonzero), then $y = x^2$ and $y = 0 + \epsilon$ intersect in 2 points nearby the origin, so the intersection multiplicity of these two curves is 2. On the other hand, $y = x$ and $y = \epsilon$ only intersect once, so the intersection multiplicity is 1 in this case.

There is also a more algebraic definition.

Definition 1. Let $f, g \in \mathbb{C}[x, y]$ be polynomials defining plane curves in $\mathrm{A}^2(\mathbb{C})$. Suppose the origin is an isolated point of the intersection of these curves. Define the *intersection multiplicity of f and g at the origin* to be

$$\chi(f, g) = \text{length}\left(\frac{\mathbb{C}[x, y]_{(x,y)}}{(f, g)}\right).$$

Here $\text{length}(\mathbb{C}[x, y]_{(x,y)})/(f, g)$ is the vector space dimension of $\mathbb{C}[x, y]_{(x,y)}$ over $\mathbb{C}$.

More generally, an arbitrary module M over a ring R is said to have finite length if the length l of a chain of distinct submodules $0 = M_0 \subsetneq \cdots \subsetneq M_l = M$ is bounded; if M has finite length, the length of M is the maximum value of l over all such chains of submodules.

For example, if $f = y - x^2$ and $g = y$, then

$$\frac{\mathbb{C}[x,y]_{(x,y)}}{(f,g)} \cong \frac{\mathbb{C}[x,y]}{(y-x^2,y)} \cong \frac{\mathbb{C}[x]}{(x^2)},$$

and the elements 1 and x form a basis for this ring as a $\mathbb{C}$-vector space, so the intersection multiplicity of $y = x^2$ and $y = 0$ is, once again, 2.

More generally, it should be possible to define an intersection multiplicity for a pair of varieties in $\mathrm{A}^n(\mathbb{C})$ meeting in an isolated point. Let R be the ring of local functions at the point of intersection, let m be the unique maximal ideal of R, and let I and J be the ideals of R corresponding to the two varieties. The requirement that the point of intersection be an isolated point forces the ideal $I + J$ to be m-primary, which is equivalent to requiring that m be the only prime ideal containing $I + J$, that some power of m be contained in $I + J$, or that $R/(I + J)$ have finite length. Since $R/(I + J)$ has finite length, it would seem natural to define the intersection multiplicity to be this length; unfortunately, this turns out not to be a satisfactory definition.

The definition that works is due to Serre ([25]), and requires that we define functors Tor_k^R which give an R-module $\mathrm{Tor}_k^R(M,N)$ for any two R-modules M and N. In particular, the functor Tor_0^R is just the tensor product, so that $\mathrm{Tor}_0^R(M,N) \cong M \otimes_R N$. The intersection multiplicity of the varieties defined by the ideals I and J is defined to be the alternating sum

$$\chi\left(\frac{R}{I}, \frac{R}{J}\right) = \sum_{k \geq 0} (-1)^k \operatorname{length}\left(\mathrm{Tor}_k^R\left(\frac{R}{I}, \frac{R}{J}\right)\right).$$

The fact that $I + J$ is m-primary assures that these lengths are finite, and the fact that R is a localization of a polynomial ring implies that the Tor_k's are eventually zero. Therefore this sum is well-defined. It turns out that all of the Tor_k's for positive k are zero in the case where I and J both define plane curves, so in that case the intersection multiplicity is equal to the first term of the alternating sum, which is the length of

$$\mathrm{Tor}_0^R\left(\frac{R}{I}, \frac{R}{J}\right) \cong \frac{R}{I} \otimes_R \frac{R}{J} \cong \frac{R}{I+J}.$$

This is just Definition 1.

As is the case with plane curves, this definition of the intersection multiplicity tells how many points of intersection should be expected nearby the given point when the two varieties are perturbed slightly. However, Serre's definition also makes sense, more generally, when the ring of local functions on $\mathrm{A}^n(\mathbb{C})$ is replaced by an arbitrary regular local ring, and when R/I and R/J are replaced by finitely generated R-modules M and N. In such a situation, define the intersection multiplicity of M and N to be

$$\chi(M, N) = \sum_{k \geq 0} (-1)^k \operatorname{length}(\mathrm{Tor}_k^R(M, N)).$$

Let d and e be the Krull dimensions of M and N and let n be the dimension of R. Thinking of d and e as corresponding to the dimensions of two intersecting varieties, and n as corresponding to the dimension of the ambient space, geometric intuition would suggest the following conjectures, originally due to Serre.

i. If $\operatorname{length}(M \otimes_R N)$ is finite, then $d + e \leq n$; if the varieties corresponding to M and N are too "large", then we would expect their intersection to be larger than an isolated point.

ii. If $d + e < n$, then $\chi(M, N) = 0$; the varieties corresponding to M and N are too "small", and we would expect them to cease intersecting completely if they were perturbed slightly.

iii. If $d + e = n$, then $\chi(M, N) > 0$. In this case we'd expect the corresponding varieties to continue to intersect after being moved slightly.

Serre proved the first statement, but proved the other two only for equicharacteristic regular local rings ([25]). The second has been established for arbitrary regular local rings by P. Roberts ([20]) and, independently, by H. Gillet and C. Soulé ([8]). The third statement is still open in this generality, although O. Gabber has shown at least that $\chi(M, N) \geq 0$ (see [23] or [1]).

Serre's definition of the intersection multiplicity can also be generalized to the case where more than two varieties intersect; to do so requires defining the functors Tor_k for more than two modules. We present the definition in this generality below. This definition of intersection multiplicity proves again to be equivalent, in the case of complex varieties, to a definition that depends on counting the number of points of intersection after the varieties are moved slightly.

In this thesis we investigate functions

$$f_k = \mathrm{length}(\mathrm{Tor}_k(M_1/I_1^{a_1}M_1, \ldots, M_n/I_n^{a_n}M_n)),$$

where R is an arbitrary local ring (not necessarily regular); M_i is a finitely generated R-module and I_i an ideal of R, for $1 \leq i \leq n$; and $I_1 + \cdots + I_n + \mathrm{Ann}\, M_1 + \cdots + \mathrm{Ann}\, M_n$ is primary to the maximal ideal of R.

Special cases of these functions have already been found to be interesting in their own right; see, for example, the discussion below of Hilbert-Kunz functions. It is also hoped that a better understanding of these functions may lead to insights into the theory of intersection multiplicities; for example, it seems reasonable to attempt to calculate the intersection multiplicity of varieties defined by ideals I_1 and I_2 using only properties of the function

$$f_0(a_1, a_2) = \text{length}(R/(I_1^{a_1} + I_2^{a_2})),$$

although this has not yet proved possible. We are, however, able to establish certain properties of the functions f_k in some generality. In particular, they are shown to have rational generating functions in the following cases:

i. $n = 2$, $M_1 = M_2 = R$, $k \geq 2$;

ii. $n = 2$, $M_1 = M_2 = R$, k is arbitrary, R is regular, and

$$\bigoplus_{a_1, a_2 \geq 0} (I_1^{a_1} \cap I_2^{a_2}) t_1^{a_1} t_2^{a_2} \subset R[t_1, t_2]$$

is a finitely generated R-algebra;

iii. $M_1, \ldots, M_n = R$ is a polynomial or power series ring, $I_1, \ldots, I_n$ are principal monomial ideals, and $k = 0$; more generally, a weaker result is proved in the case when $I_1, \ldots, I_n$ are monomial but not necessarily principal.

1.2 Basic Definitions

All rings are commutative, and local rings are in addition assumed to be Noetherian.

Definition 2. Given n modules $M_1, \ldots, M_n$ over a ring R, define the R-module $\text{Tor}_k^R(M_1, \ldots, M_n)$ by

i. choosing a projective resolution

$$\cdots \to P_{i,3} \to P_{i,2} \to P_{i,1} \to P_{i,0} \to M_i \to 0$$

for each M_i,

ii. tensoring together all of these projective resolutions (with the modules M_i removed) and taking the total complex,

$$\cdots \to \bigoplus_{\substack{(i_1,\dots,i_n)\\ \sum_j i_j=2}} P_{1,i_1}\otimes\cdots\otimes P_{n,i_n} \to \bigoplus_{\substack{(i_1,\dots,i_n)\\ \sum_j i_j=1}} P_{1,i_1}\otimes\cdots\otimes P_{n,i_n} \to P_{1,0}\otimes\cdots\otimes P_{n,0}\to 0,$$

and

iii. taking the kth homology of this complex.

This defines a functor, covariant in each of the n variables, which specializes for $n = 2$ to the usual Tor. It shares most of the basic properties with the usual 2-variable Tor; for example,

i. A short exact sequence in any of the variables gives the usual long exact sequence.

ii. Any element of R that kills one of M_i also kills $\mathrm{Tor}_k(M_1, \dots, M_n)$.

iii. $\mathrm{Tor}_0^R(M_1, \dots, M_n) = M_1 \otimes_R \cdots \otimes_R M_n$.

iv. If all but one of the M_i is flat, then $\mathrm{Tor}_k(M_1, \dots, M_n) = 0$ for all $k > 0$.

v. If the M_i are all finitely generated over R, so is $\mathrm{Tor}_i(M_1, \dots, M_n)$.

Definition 3. Let R be a Noetherian ring, n a positive integer, $M_1, \dots, M_n$ finitely generated R-modules, and $I_1, \dots I_n$ ideals of R. Let $J = I_1 + \cdots + I_n + \mathrm{Ann}_R(M_1) + \cdots + \mathrm{Ann}_R(M_n)$, and assume that R/J has finite length. For k a non-negative integer, define

$$f_k(a_1, \dots, a_n) = \mathrm{length}(\mathrm{Tor}_k(M_1/I_1^{a_1}M_1, \dots, M_n/I_n^{a_n}M_n)).$$

Note that the modules $\mathrm{Tor}_k(M_1/I_1^{a_1}M_1, \dots, M_n/I_n^{a_n}M_n))$ do have finite length, since they are finitely generated modules that are killed by the ideal $J' = I_1^{a_1} + \cdots +$

$I_n^{a_n} + \operatorname{Ann} M_1 + \cdots + \operatorname{Ann} M_n$, and hence are finitely generated modules over R/J', which has finite length. (This is because J' has the same radical as the ideal J above, and R/J is assumed to be finite length.)

In this thesis, the ring R will always be local, and the modules $M_1, \ldots, M_n$ will almost always be equal to R.

1.3 Examples

i. If $n = 1$, then f_0 is the usual Hilbert function, and is eventually a polynomial of degree $d = \dim R$.

ii. In [3] and [4], W. Brown studies f_0 in the case $n = 2$ and $M = R$; he calls this function the Hilbert function of I_1 and I_2. In [3], he gives sufficient conditions for f_0 to be eventually polynomial (in other words, for there to exist (b_1, b_2) such that $f_0(a_1, a_2)$ agrees with a polynomial for all $(a_1, a_2) \geq (b_1, b_2)$). In [4], he considers the case where f_0 is eventually a polynomial in a_1, a_2, and $\min(a_1, a_2)$. Both papers also provide several examples of such functions.

iii. If $M_1 = \cdots = M_N = M$, $I_1, \ldots, I_n$ are all principal ideals, and R has characteristic $p > 0$, then the function $\mathrm{HK}_{M,I} : a \mapsto f_0(p^a, \ldots, p^a)$ is called the Hilbert-Kunz function on M of the ideal $I = I_1 + \cdots + I_n$. The more standard definition is in terms of bracket powers,

$$I^{[p^e]} = (\{ i^{p^e} | i \in I \}).$$

With this notation, $\mathrm{HK}_{M,I}(a) = \operatorname{length}(M/I^{[p^e]}M)$. Because R has characteristic p, $I^{[p^e]} = I_1^{p^e} + \cdots + I_n^{p^e}$, which shows that $\mathrm{HK}_{M,I}$ is dependent only on I and not on the choice of $I_1, \ldots, I_n$.

Monsky has shown ([16]) that this function has the form

$$e_{\mathrm{HK}}(M, I)p^{da} + O(p^{a(d-1)})$$

where $d = \dim M$ and $e_{\mathrm{HK}}(M, I)$ is a positive real constant. The constant $e_{\mathrm{HK}}(R, m)$ may be written $e_{\mathrm{HK}}(R)$, and $\mathrm{HK}_{R,m}$ may be written HK_R, and called just the Hilbert-Kunz function of R.

It is not even known in general whether $e_{HK}(R)$ is rational. However, many results have been proved in particular cases; see [16], [5], [17], [11], [19], [18], [6], and [24].

The Hilbert-Kunz function also has an important connection with the theory of tight closure. If $I \subset J$ are m-primary ideals of R, then $I^* = J^*$ if and only if $e_{HK}(R, I) = e_{HK}(R, J)$. This statement is true when R is a complete local domain, and in somewhat more generality. (See [14], thm. 8-17, or see the more thorough discussion in [13])

iv. In [7], M. Contessa considers the generalization of the previous situation to the case of arbitrary k. She is able to use results about the higher Tors to determine the form of Hilbert-Kunz functions of modules over regular rings of dimension at most two. In this thesis, we similarly find that results for $k > 0$, in addition to being interesting in their own right, have applications to the case $k = 0$.

CHAPTER II

Quasipolynomial Functions and $\mathbb{N}^n$-graded modules

Given the functions $f_k : \mathbb{N}^n \to \mathbb{N}$ defined previously, there are corresponding generating functions

$$F_k(x_1, \ldots, x_n) = \sum_{\alpha=(a_1,\ldots,a_n)\in\mathbb{N}^n} f_k(\alpha)x^\alpha.$$

Later we will prove that, under certain conditions, these generating functions are rational. In fact, we will prove that the functions f_k are *quasipolynomial*, a somewhat stronger condition. The first section of this chapter discusses the basic properties of quasipolynomial functions.

Given a $\mathbb{N}^n$-graded module M having finite length in each graded piece, there is a natural notion of a Hilbert function $\mathbb{N}^n \to \mathbb{N}$ given by

$$(a_1, \ldots, a_n) \mapsto \text{length}(M_{(a_1,\ldots,a_n)}).$$

For $n = 1$ this is the usual Hilbert function. The second part of this chapter is devoted to showing that certain $\mathbb{N}^n$-graded algebraic objects (including, for example, finitely generated modules over finitely generated algebras over fields) have Hilbert functions which are quasipolynomial functions. This fact will be used in proofs of the results of the following chapter.

2.1 Quasipolynomial Functions

Most of the results of this section are known, though not perhaps in the form we need them; see, for example, [15].

Most of the functions considered here are defined on $\mathbb{N}^n$. However, it will be convenient to identify such functions with functions defined on all of $\mathbb{Z}^n$ that are zero for elements of $\mathbb{Z}^n$ not in $\mathbb{N}^n$.

Definition 4. Given linearly independent vectors $\alpha_1, \dots, \alpha_e \in \mathbb{N}^n$, say that a function $f : \mathbb{N}^n \to \mathbb{Z}$ is *periodic with respect to* $\alpha_1, \dots, \alpha_e$ if, for any i, the function $\alpha \mapsto f(\alpha + \gamma) - f(\alpha)$ is identically zero on $\mathbb{N}^n$, for γ any element of $\mathbb{N}\alpha_1 + \cdots + \mathbb{N}\alpha_e$.

Definition 5. Given $\beta \in \mathbb{N}^n$ and linearly independent vectors $\alpha_1, \dots, \alpha_e \in \mathbb{N}^n$, the *cone with vertex* β *generated by* $\alpha_1, \dots, \alpha_e$ is

$$C_{\beta,\alpha_1,\dots,\alpha_e} = \beta + \mathbb{R}_{\geq 0}\alpha_1 + \cdots + \mathbb{R}_{\geq 0}\alpha_e \subset \mathbb{R}^n_{\geq 0}.$$

Definition 6. Let $\alpha_1, \dots, \alpha_e \in \mathbb{N}^n$ be linearly independent vectors in $\mathbb{R}^n$, and let β be another element of $\mathbb{N}^n$. Let $C = C_{\beta,\alpha_1,\dots,\alpha_e}$ be the cone with vertex β generated by $\alpha_1, \dots, \alpha_e$. Call $f : \mathbb{N}^n \to \mathbb{Z}$ *simple quasipolynomial of polynomial degree* d *on* $\beta, \alpha_1, \dots, \alpha_e$ if $f(x) = 0$ for $x \notin C$ and

$$f(x) = \sum_{\alpha \in \mathbb{N}^n} c_\alpha(x) x^\alpha$$

for $x \in C$, where c_α is periodic with respect to $\alpha_1, \dots, \alpha_e$, is identically zero for all $\alpha = (a_1, \dots, a_n)$ such that $\sum a_i > d$, and is not identically zero for at least one α satisfying $\sum a_i = d$.

Note that polynomials of degree d are examples of simple quasipolynomials of diagonal degree d. We use the term "polynomial degree" and not just "degree" to distinguish it from another notion of degree.

Definition 7. The *cumulative degree* of a simple quasipolynomial of polynomial degree d on $\beta, \alpha_1, \ldots, \alpha_e$ is $d + e$.

The cumulative degree will turn out to be the more useful number for our purposes. It will also be convenient to allow simple quasipolynomial functions of cumulative degree 0, defined as follows.

Definition 8. A function $f : \mathbb{N}^n \to \mathbb{Z}$ is *simple quasipolynomial of cumulative degree* 0 if it is nonzero on at most a single element of $\mathbb{N}^n$.

Lemma 9. *Let $F(x_1, \ldots, x_n)$ be the generating function of $f : \mathbb{N}^n \to \mathbb{Z}$, fix $\beta \in \mathbb{N}^n$, and fix a linearly independent set of vectors $\alpha_1, \ldots, \alpha_e \in \mathbb{N}^n$. Suppose*

$$F(x_1, \ldots, x_n) = \frac{1}{\prod_{j=1}^{e}(1 - x^{\alpha_j})^{d_j}}$$

with $\sum_{j=1}^{e} d_j \leq d$. Then f is the simple quasipolynomial function of cumulative degree d given by

$$f(a_1, \ldots, a_n) = \begin{cases} \prod_j \binom{d_j - 1 + m_j}{m_j} & \textit{if } (a_1, \ldots, a_n) = \sum_j m_j \alpha_j, \\ 0 & \textit{otherwise.} \end{cases}$$

Proof. Given $\alpha \in \mathbb{N}^n$, there is at most one way of writing α as an $\mathbb{N}$-linear combination of $\alpha_1, \ldots, \alpha_e$. This means that the coefficient of x^α in F is nonzero if and only if α can be written as the $\mathbb{N}$-linear combination of $\alpha_1, \ldots, \alpha_e$. If α is such an exponent, write $\alpha = \sum m_j \alpha_j$. Then the coefficient of x^α in F is the product of the coefficients of $x^{m_j \alpha_j}$ in $1/(1 - x^{\alpha_j})$, and it is easy to verify that these coefficients are just the binomial coefficients given in the description of f above. ∎

Definition 10. Given a polynomial $P(x_1, \ldots, x_n)$, define the support of P, $\operatorname{Supp} P$, to be the set of all $\alpha \in \mathbb{N}^n$ such that the coefficient of x^α in P is nonzero.

Definition 11. Given linearly independent vectors $\alpha_1, \ldots, \alpha_e \in \mathbb{N}^n$ and another vector $\beta \in \mathbb{N}^n$, let

$$\Pi_{\beta,\alpha_1,\ldots,\alpha_e} = \left\{ \alpha \in \mathbb{N}^n \,\middle|\, \alpha = \beta + \sum_i m_i \alpha_i,\ 0 \le m_i < 1 \right\}.$$

Given any $\alpha \in C_{\beta,\alpha_1,\ldots,\alpha_e} \cap \mathbb{N}^n$, there is a unique representative of α modulo $\alpha_1, \ldots, \alpha_e$ in $\Pi_{\beta,\alpha_1,\ldots,\alpha_e}$; in the following, we use $\overline{\alpha}$ to denote that unique representative.

Lemma 12. *Let $F(x_1, \ldots, x_n)$ be the generating function of $f : \mathbb{N}^n \to \mathbb{Z}$, fix $\beta \in \mathbb{N}^n$, and fix a linearly independent set of vectors $\alpha_1, \ldots, \alpha_e \in \mathbb{N}^n$. Suppose*

$$F(x_1, \ldots, x_n) = \frac{P}{\prod_{j=1}^{e}(1 - x^{\alpha_j})^{d_j}}$$

with $\sum_{j=1}^{e} d_j \le d$ and $\operatorname{Supp} P \subset \Pi_{\beta,\alpha-1,\ldots,\alpha_e}$. Write $P = \sum_\alpha c_\alpha x^\alpha$. Then f is the simple quasipolynomial function of cumulative degree d given by

$$f(\alpha) = \begin{cases} \prod_j c_{\overline{\alpha}} \binom{d_j - 1 + m_j}{m_j} & \text{if } \alpha = \overline{\alpha} + \sum_j m_j \alpha_j. \end{cases}$$

Proof. The generating function F can be written as the sum of the fractions

$$\frac{c_\alpha x^\alpha}{\prod_{j=1}^{e}(1 - x^{\alpha_j})^{d_j}}.$$

The result then follows immediately from the previous lemma. ☐

Lemma 13. *Fix $\beta \in \mathbb{N}^n$, and fix a linearly independent set of vectors $\alpha_1, \ldots, \alpha_e \in \mathbb{N}^n$. Then functions of the form*

$$F(x_1, \ldots, x_n) = \frac{P}{\prod_{j=1}^{e}(1 - x^{\alpha_j})^{d_j}}$$

with $\operatorname{Supp} P \subset \Pi = \Pi_{\beta,\alpha_1,\ldots,\alpha_e}$, and $\sum_{j=1}^{e} d_j \le d$, form a basis for the generating functions of the simple quasipolynomial functions of cumulative degree at most d on $\beta, \alpha_1, \ldots, \alpha_e$.

Proof. This is an immediate consequence of the previous lemma and of the fact that the functions

$$(a_1, \dots, a_e) \mapsto \prod_j \binom{d_j - 1 + a_j}{a_j},$$

with $\sum(d_j - 1)$ at most d, form a basis for the polynomials of degree at most d. $\square$

Theorem 14. *Let f_1 be simple quasipolynomial on β, α_1, $\dots$, α_e of degree d. Pick $\beta' \in \mathbb{N}^n$ and linearly independent α'_1, $\dots$, $\alpha'_{e'}$ such that*

$$C' = C_{\beta', \alpha'_1, \dots, \alpha'_{e'}} \subset C = C_{\beta, \alpha_1, \dots, \alpha_e}.$$

Define a new function f' such that $f'(\alpha) = \alpha$ for $\alpha \in C'$ and $f'(\alpha) = 0$ for $\alpha \notin C'$. Then there exist positive integers m_i such that f' is simple quasipolynomial of degree at most d on β', $m_1\alpha'_1$, $\dots$, $m_{e'}\alpha'_{e'}$.

Proof. For each α'_i, choose m_i such that $m_i\alpha'_i \in \mathbb{N}\alpha_1 + \cdots \mathbb{N}\alpha_e$. Since f is simple quasipolynomial, we know

$$f(x) = \sum_{\alpha \in \mathbb{N}^n} c_\alpha(x) x^\alpha$$

for $x \in C$, where c_α is periodic with respect to $\alpha_1, \dots, \alpha_e$, is identically zero for all $\alpha = (a_1, \dots, a_n)$ such that $\sum a_i > d$. Since the c_α's are periodic with respect to $\alpha_1, \dots, \alpha_e$, they are also periodic with respect to $m_1\alpha'_1, \dots, m_{e'}\alpha_{e'}$. Therefore f' is also simple quasipolynomial. $\square$

Theorem 15. *If f_1 and f_2 are both simple quasipolynomial functions on β, α, $\dots$, α_e, of cumulative degree d_1 and d_2 respectively, then $f_1 + f_2$ is also simple quasipolynomial on β, α_1, $\dots$, α_e, of degree at most* $\max(d_1, d_2)$.

Proof. The proof is immediate from the definition of a simple quasipolynomial function. $\square$

Theorem 16. *If* $f : \mathbb{N}^n \to \mathbb{N}$ *is polynomial of degree* d*, then*

$$f^*(a) = \sum_{a_1+\cdots+a_n=a} f(a_1, \ldots, a_n)$$

is polynomial of degree $d+n-1$*.*

Proof. The set of generating functions of polynomials of degree at most d has a basis consisting of functions of the form

$$(x_1, \ldots, x_n) \mapsto \frac{1}{\prod_{i=1}^n (1-x_i)^{d_i}}$$

where $d_i \geq 1$ and $\sum_i (d_i - 1) \leq d$. Given a function f with generating function F, the generating function of f^* is $F(x, \ldots, x)$. Under this transformation the basis element above becomes

$$\frac{1}{(1-x)^{\sum d_i}},$$

which is clearly the generating function of a polynomial of degree at most $d+n-1$.

Thus f^* is a polynomial of degree at most $d+n-1$; it only remains to prove that f^* has at least degree $d+n-1$. To do this, it suffices to show that f_d^* has degree $d+n-1$, where f_d is the degree d part of f. We will prove this by bounding f_d^* below by a polynomial of degree $d+n-1$.

If $f_d(a_1, \ldots, a_n) < 0$ for some $(a_1, \ldots, a_n) \in \mathbb{N}^n$, then the polynomial function

$$a \mapsto f(aa_1, \ldots, aa_n) = f_d(aa_1, \ldots, aa_n) + \text{lower order terms}$$

has negative leading term, and must eventually take on negative values. This is contrary to our hypotheses, so such $(a_1, \ldots, a_n)$ must not exist, and $f_d : \mathbb{N}^n \to \mathbb{N}$ must take on only nonnegative values. Furthermore, we can choose $(a_1, \ldots, a_n) \in \mathbb{R}^n_{\geq 0}$ such that $\sum a_i = 1$ and $f_d(a_1, \ldots, a_n) > 0$; if f_d were zero for every such $(a_1, \ldots, a_n)$ then it would be identically zero. In fact, the subset of

$$A = \left\{ (a_1, \ldots, a_n) \in \mathbb{R}^n_{\geq 0} \,\middle|\, \sum_i a_i = 1 \right\}$$

on which f_d is zero is a proper algebraic subset of A. Therefore there is a closed $(n-1)$-dimensional disk $\Delta \subset A$ of radius δ around the chosen point $(a_1, \dots, a_n)$ such that $f_d(\beta) \geq c$ for c a fixed, positive real number and β any element of Δ.

Let a be any positive integer, and let α be any element of Δ. Then $f_d(a\alpha) = a^d f_d(\alpha) \geq ca^d$. Therefore f_d is bounded below by a^d on the set $a\Delta = \{a\alpha | \alpha \in \Delta\}$. By the same reasoning as that used in Lemma 50, the number of points in $\mathbb{N}^n \cap a^d$ is greater than or equal to the volume of $(a-b)\Delta$ for a fixed constant $b \in \mathbb{R}$. This volume is a polynomial $V(a)$ of degree $n-1$ in a. Therefore

$$\begin{aligned} f_d^*(a) &= \sum_{\sum a_i = a} f_d(a_1, \dots, a_n) \\ &\geq \sum_{(a_1,\dots,a_n)\in\mathbb{N}^n\cap(a-b)\Delta} f_d(a_1, \dots, a_n) \\ &\geq \sum_{(a_1,\dots,a_n)\in\mathbb{N}^n\cap(a-b)\Delta} a^{d-1}c \\ &\geq V(a)a^d c \end{aligned}$$

Therefore $f_d^*(a)$ is bounded below by the product $V(a)a^dc$, which is a polynomial of degree $d+n-1$. $\square$

Theorem 17. *Given $f : \mathbb{N}^n \to \mathbb{N}$ simple quasipolynomial of cumulative degree d, define a new function $f^* : \mathbb{N} \to \mathbb{N}$ by*

$$f^*(a) = \sum_{a_1+\cdots+a_n=a} f(a_1, \dots, a_n).$$

Then f^* is simple quasipolynomial of cumulative degree d.

Proof. Write F for the generating function associated with f, and F^* for the generating function associated with f^*. Then $F^* = F(x, \dots, x)$. By Theorem 13, $F(x, \dots, x)$ is clearly the generating function of a simple quasipolynomial function if

$F(x, \ldots, x)$ is. It is also clear that f has degree at most d. The fact that the degree of f is not less than d follows from the same fact for polynomials. $\square$

Definition 18. A function $f : \mathbb{N}^n \to \mathbb{Z}$ is *quasipolynomial of degree* d if it can be written as the sum of simple quasipolynomial functions of cumulative degree at most d, and if it cannot be written as the sum of simple quasipolynomial functions of cumulative degree less than d.

Sometimes we will say that a function $f : \mathbb{N}^n \to \mathbb{Z}$ is *quasipolynomial on* $\alpha_1, \ldots, \alpha_e \in \mathbb{N}^n$; by this we mean that f can be written as the sum of simple quasipolynomial functions on subsets of $\alpha_1, \ldots, \alpha_e$.

It might be more accurate to call such functions eventually quasipolynomial; this would make our usage agree, for example, with the definition given in 4.4 of [26].

It should be obvious from the definition that simple quasipolynomial functions of cumulative degree d are also quasipolynomial functions of degree d.

Theorem 19. *If $f_1 : \mathbb{N}^n \to \mathbb{Z}$ and $f_2 : \mathbb{N}^n \to \mathbb{Z}$ are quasipolynomial, of degrees d_1 and d_2 respectively, then $f_1 + f_2$ is quasipolynomial of degree at most* $\max(d_1, d_2)$. *Also, if $d_1 \neq d_2$, then the degree of $f_1 + f_2$ is exactly* $\max(d_1, d_2)$.

Proof. Immediate from the definition. $\square$

The following theorems provide a few examples of quasipolynomial functions.

Lemma 20. *If $f(\alpha)$ is zero for all but finitely many values of α, then f is quasipolynomial of degree* 0.

Proof. The proof follows immediately from the definition of a simple quasipolynomial of degree 0. $\square$

Theorem 21. *A function $f : \mathbb{N} \to \mathbb{Z}$ is quasipolynomial on* 1 *(in other words, is the sum of simple quasipolynomial functions on $\{\beta, 1\}$, for various β's) of degree d iff it is eventually polynomial of degree $d-1$.*

Proof. A simple quasipolynomial function $f(\alpha)$ on $\beta, 1$ is a function that is zero for $\alpha < \beta$, and that agrees for $\alpha \geq \beta$ with a polynomial with coefficients that are periodic with respect to 1 and therefore are constant. Such a function is clearly eventually polynomial, as is the sum of such functions.

For the converse, let f' be the polynomial function that f eventually agrees with. A polynomial of degree $d-1$ is simple quasipolynomial of degree d. The difference between f and f' is then nonzero for only finitely many values of $\mathbb{N}$, and is accounted for by the previous lemma. □

Theorem 22. *If a function $f : \mathbb{N} \to \mathbb{Z}$ is quasipolynomial on $e_1, \ldots, e_n$ of degree d, where e_i is the vector in $\mathbb{N}^n$ that is one in the ith position but zero elsewhere, then there exists $\alpha' \in \mathbb{N}^n$ such that $f(\alpha)$ agrees with a polynomial of degree $d-n$ for all $\alpha \geq \alpha'$. (Here we take a polynomial of negative degree to be identically zero.)*

Proof. A simple quasipolynomial function on a proper subset of $e_1, \ldots, e_n$ is easily seen to be eventually zero, whereas a simple quasipolynomial function on $e_1, \ldots, e_n$ agrees eventually with a polynomial with constant coefficients, as in the previous proof. The sum of such functions is, therefore, eventually a polynomial, and the statement about the degree follows from the definition of cumulative degree. □

The converse to this theorem is false; if f is eventually polynomial but has non-polynomial behavior on, for example, a finite set of lines extending parallel to one of the coordinate axes, then f is not quasipolynomial.

Theorem 23. *The product of two quasipolynomial functions in distinct sets of variables is quasipolynomial of degree equal to the sum of the degrees of the two quasipolynomial functions.*

Proof. Let $f(x_1, \dots, x_m)$ and $g(y_1, \dots, y_n)$ be the two functions. Let $f = f_1 + \cdots f_r$, with each f_i simple quasipolynomial, and with f_1 having degree equal to the degree of f. Choose $g_1, \dots, g_s$ similarly. Then

$$fg = \sum_{i,j} f_i g_j,$$

so it is clear that fg is quasipolynomial if the product of two simple quasipolynomial functions is simple quasipolynomial. If, in addition, such products have degrees equal to the sums of the degrees of their factors, then the statement about degrees will also be proved, since $f_1 g_1$ would then have the correct degree, with all the other products having equal or lesser degrees. Thus we may assume without loss of generality that f and g are simple quasipolynomial. Write $F(x_1, \dots, x_m)$ and $G(y_1, \dots, y_n)$ for the generating functions of f and g, respectively. The generating function of fg is just FG and, by theorem 13, FG is the generating function of a simple quasipolynomial function of the correct degree. $\square$

Quasipolynomial functions have a very simple characterization in terms of generating functions. Some preliminary lemmas will be required before the proof of this fact.

Lemma 24. *Let $F(x_1, \dots, x_n)$ be the generating function of $f : \mathbb{N}^n \to \mathbb{Z}$. Assume*

$$F(x_1, \dots, x_n) = \frac{P(x_1, \dots, x_n)}{\prod_{j=1}^{e} (1 - x^{\alpha_j})^{d_j}}$$

with $\alpha_1, \dots, \alpha_d \in \mathbb{N}^n$ vectors that are linearly independent in $\mathbb{R}^n$ and P a polynomial. Then f is quasipolynomial of cumulative degree at most $\sum_{j=1}^{e} d_j$.

Proof. The function F is the sum of fractions of the form

$$\frac{x^\gamma}{\prod_{j=1}^{e}(1-x^{\alpha_j})^{d_j}}$$

Each of these is the generating function of a simple quasipolynomial function of degree d, by Theorem 13. $\square$

Lemma 25. *Let $F(x)$ be a rational function of the form*

$$F(x) = \frac{P(x)}{\prod_{j=1}^{e}(1-x^{\alpha_j})^{d_j}}.$$

If the α_j's are linearly dependent, then $F(x)$ can be rewritten in the form

$$F(x) = \sum_i \frac{P_i(x)}{\prod_{j=1}^{e_i}(1-x^{\alpha_{i,j}})^{d_{i,j}}},$$

such that each e_i is strictly less than e.

Proof. Choose a linear dependence relation among the α_j's. By clearing denominators and renumbering the α_j's if necessary, this relation can be written

$$m_1\alpha_1 + \cdots + m_k\alpha_k = m_{k+1}\alpha_{k+1} + \cdots + m_e\alpha_e$$

where each m_k is a non-negative integer. Write γ for the element of $\mathbb{N}^n$ that both sides of this equation are equal to. Let $d = 1+\sum_{i=1}^{k}(d_i-1)$, and let $d' = 1+\sum_{i=k+1}^{e}(d_i-1)$. Since

$$(1-x^{m\alpha_i}) = (1-x^{\alpha_i})(1+x^{\alpha_1}+x^{2\alpha_1}+\cdots+x^{(m-1)\alpha_1}),$$

F can be rewritten in the form

$$\frac{P'(x)}{\prod_{j=1}^{e}(1-x^{m_j\alpha_j})^{d_j}},$$

with P' a new polynomial. Next write

$$F(x) = \frac{P'(x)(1-x^\gamma)^{d+d'}}{(1-x^\gamma)^{d+d'}\prod_{j=1}^{e}(1-x^{m_j\alpha_j})^{d_j}}, \tag{2.1}$$

and note that

$$
\begin{aligned}
(1 - x^{\gamma}) = (1 - x^{\sum_{i=1}^{k}(m_i\alpha_i)}) &= (1 - x^{m_1\alpha_1})x^{\sum_{i=2}^{k}(m_i\alpha_i)} \\
&+ (1 - x^{m_2\alpha_2})x^{\sum_{i=3}^{k}(m_i\alpha_i)} \\
&+ \cdots \\
&+ (1 - x^{m_k\alpha_k}),
\end{aligned}
$$

which shows that $(1 - x^{\gamma})$ is contained in the ideal generated by $(1 - x^{m_1})$, $\ldots$, $(1 - x^{m_k})$. Similarly, $(1 - x^{\gamma})$ is also contained in the ideal generated by $(1 - x^{m_{k+1}})$, $\ldots$, $(1 - x^{m_e})$. This fact, together with our choice of d and d', allows us to rewrite $(1 - x^{\gamma})^{d+d'}$ as the sum of terms each of which is contains one of both $(1 - x^{m_i})^{d_i}$ for $1 \leq i \leq k$ and $(1 - x^{m_i})^{d_i}$ for $k < i \leq e$ as factors. Finally we are able to write F in the desired form by expanding the numerator of Equation 2.1, breaking up the fraction into a sum of fractions, and canceling these factors. □

Theorem 26. *Let $F(x_1, \ldots, x_n)$ be the generating function of $f : \mathbb{N}^n \to \mathbb{Z}$. Then f is quasipolynomial iff F can be written in the form*

$$F(x) = \frac{P(x)}{\prod_{j=1}^{e}(1 - x^{\alpha_j})^{m_j}}.$$

Proof. If F is in the given form, then we can use repeated applications of the previous lemma to rewrite F as the sum of terms each of which is in the form specified in Lemma 24. Thus f is the sum of quasipolynomial functions and must itself be quasipolynomial.

Conversely, if f is quasipolynomial, then f can be written as the sum of simple quasipolynomial functions. The form of the generating function of each of these simple quasipolynomial functions is given by Theorem 13, and the sum of such functions can clearly be written in the form required. □

Theorem 27. *Let $f : \mathbb{N}^n \to \mathbb{N}$ be quasipolynomial of degree d. Then*

$$f^*(a) = \sum_{a_1+\cdots+a_n=a} f(a_1, \ldots, a_n)$$

is also quasipolynomial of degree d.

Proof. If f is quasipolynomial, and $F(x_1, \ldots, x_n)$ is its generating function, then the generating function of f^* is $F(x, \ldots, x)$. The fact that f is quasipolynomial then follows immediately from Theorem 26. It is also clear from this theorem that the degree of f^* is at most d. The only remaining difficulty is to show that the degree cannot be less than d.

There must exist β, α_1, $\ldots$, α_e such that f restricted to $C_{\beta,\alpha_1,\ldots,\alpha_e}$ is simple quasipolynomial of cumulative degree d. If not, then we could cover $\mathbb{N}^n$ by cones in such a way that on each cone f was simple quasipolynomial (using Theorem 14) and had degree less than d. This would express f as the sum of simple quasipolynomial functions of cumulative degree less than d, contradicting the fact that f has degree d.

Let f' be the restriction of f to $C_{\beta,\alpha_1,\ldots,\alpha_e}$. Then $f' \leq f$, so $f'^* \leq f^*$. By Theorem 17, f'^* has degree d; it follows immediately from the definition of simple quasipolynomial that f^* must have degree at least d. □

Theorem 28. *If $f, g : \mathbb{N}^n \to \mathbb{N}$ are quasipolynomial and $f \leq g$, then* $\deg f \leq \deg g$.

Proof. The previous theorem allows us to reduce to the case $n = 1$. A quasipolynomial function of degree 1 is eventually equal to a polynomial with periodic coefficients, and the result is trivial for such functions. □

Theorem 29. *Let $f : \mathbb{N}^n \to \mathbb{Z}$, $g : \mathbb{N}^n \to \mathbb{Z}$, and $\beta \in \mathbb{N}^n$ be such that $f(\alpha + \beta) - f(\alpha) = g(\alpha)$. If g is quasipolynomial, then so is f. In addition, if $f : \mathbb{N}^n \to \mathbb{N}$, so all the values of f are nonnegative, then the cumulative degree of f is one greater than that of g.*

Proof. Let F and G be the generating functions of f and g, respectively. The relation $f(\alpha + \beta) - f(\alpha) = g(\alpha)$ is equivalent to the relation $F = G/(1 - x^\beta)$. It is clear, then, that F is the generating function of a quasipolynomial function if G is. It remains to show that the degree increases by one when the values of f are known to be nonnegative.

Let b be the sum of the coordinates of the vector β. Then

$$F^* = \frac{G^*}{1 - x^b},$$

where F^* and G^* are the generating functions associated with, respectively, f^* and g^*. Thus by the previous theorem we can assume without loss of generality that $n = 1$.

It is not hard to see that in the case $n = 1$ a quasipolynomial function g is agrees eventually with a polynomial with periodic coefficients. Such a function is eventually bounded below by a function with generating function of the form

$$\frac{c}{(1 - x^N)^d},$$

where c a positive real number and d is the degree of g. Therefore f is bounded below by a function with generating function

$$\frac{c}{(1 - x^N)^d(1 - x^b)},$$

and it follows that f has degree $d + 1$. $\square$

2.2 $\mathbb{N}^n$-graded algebras.

We now apply the results of the previous section to the problem of determining the Hilbert functions of certain $\mathbb{N}^n$-graded algebraic objects.

Theorem 30. *Let S be an $\mathbb{N}^n$-graded algebra such that S_0 is Artin and S is finitely generated as an algebra over S_0. Let M be a finitely generated, $\mathbb{N}^n$-graded S-module of dimension d. Then the Hilbert function $f(\alpha) = \mathrm{length}(M_\alpha)$ is quasipolynomial of degree d.*

Proof. The proof will be by induction on d. If $d = 0$ then M has finite length, and $\mathrm{length}(M_\alpha)$ is zero for all but finitely many α, so f is quasipolynomial of degree 0.

Now assume that $d > 0$ and that the theorem is proved for all smaller d. Because the sum of quasipolynomial functions is quasipolynomial, we can take a primary filtration of M and assume without loss of generality that M is P-primary for some $P \in \mathrm{Spec}(S)$.

We can map an $\mathbb{N}^n$-graded polynomial ring onto S and work over the polynomial ring instead of over S. So we may as well assume S is polynomial; say $S = S_0[s_1, \ldots, s_e]$, with $\deg(s_i) = \alpha_i \in \mathbb{N}^n$.

Since $d = \dim M > 0$, there must be some s_i not contained in P; otherwise M would have finite length. That s_i must be a non-zerodivisor on M. This means that the sequence

$$0 \to M \xrightarrow{s_i} M \to \frac{M}{s_i M} \to 0$$

is exact, so

$$\begin{aligned} \mathrm{length}(M_\alpha) - \mathrm{length}(M_{\alpha-\alpha_i}) &= \mathrm{length}\left(\left(\frac{M}{s_i M}\right)_\alpha\right) \\ f(\alpha) - f(\alpha - \alpha_i) &= f_{M/s_i M}(\alpha). \end{aligned}$$

Since s_i is a non-zerodivisor on M, M/s_iM has dimension $d-1$. Therefore, by the induction assumption, M/s_iM is quasipolynomial of degree $d-2$, and f_M is quasipolynomial of degree $d-1$. □

It should be clear from the proof that if $\alpha_1, \ldots, \alpha_e$ are generators of S, then the generating function of the Hilbert function can be written as a rational function with denominator $\prod_i (1 - x^{\alpha_i})$.

A few examples might be helpful.

i. If M is a finitely generated module over an $\mathbb{N}$-graded algebra S that is generated by elements of degree 1, then the Hilbert function f_M is quasipolynomial on $\{1\}$, and hence is eventually polynomial, as expected.

ii. If M is a finitely generated module over an $\mathbb{N}^2$-graded algebra S, and if every algebra-generator of S over S_0 has degree $(0,1)$ or $(1,0)$, then S is quasipolynomial of degree $d = \dim S$ on $(0,1)$ and $(1,0)$. By Theorem 22, there exists a β such that, for all $\alpha \geq \beta$, $f_S(\alpha)$ is equal to a polynomial of degree at most $d-2$. Thus we obtain Theorem 2 of [2], that Hilbert functions of bigraded modules are eventually polynomial of degree at most $(\dim M) - 2$. More results about these functions can be found in [27].

iii. More generally, let M be a finitely generated module over a $\mathbb{N}^n$-graded algebra S, and assume every algebra-generator of S over S_0 has degree e_i, where e_i is the vector in $\mathbb{N}^n$ that is one in the ith position but zero elsewhere. Again, there is a β such that $f_M(\alpha)$ agrees with a polynomial of degree at most $d-n$ for all $\alpha \geq \beta$. (See [12], where it is also shown that all the highest degree monomials of this polynomial have nonnegative coefficients.)

iv. If $M = S = k[s_1, \ldots, s_e]$, and if the degree of s_i is α_i, then the generating function of the Hilbert function f_S is

$$\frac{1}{\prod_i (1 - x^{s_1})}$$

and f_S is quasipolynomial of degree d. It is not hard to see that any quasipolynomial function of the form

$$\frac{P(x)}{\prod_i (1 - x^{\alpha_i})}$$

such that $P(x)$ has nonnegative coefficients can be written as the sum of translations of such functions. Direct sums of modules have Hilbert functions that are the sums of the Hilbert functions of the individual modules. In this way, any quasipolynomial function of this form can be realized as the Hilbert function of some finitely generated module over such an S. In fact, this identifies exactly the set of all functions that arise as the Hilbert functions of $\mathbb{N}^n$-graded modules.

Theorem 31. *Let S be an $\mathbb{N}^n$-graded finite algebra over a local ring R, and let T be an $\mathbb{N}^n$-graded, finitely generated, S-algebra. Assume that $S_0 = T_0 = R$. Let N and M be finitely-generated $\mathbb{N}^n$-graded algebras over T and S, respectively, with $N \supset M$, and suppose that for every element of N there is a power of m_R multiplying that element into M. Then $f_{N/M} : \alpha \to \operatorname{length}((N/M)_\alpha)$ is quasipolynomial of degree at most $\dim N = \dim(T/(\operatorname{Ann}_T N))$.*

Proof. We can filter N/M by N/TM and TM/M. The quotient N/TM is a T-module, not just an S-module. Also, N/TM is a finitely generated T-module, and since every element of N is multiplied into M (and hence into TM) by a power of m, it follows that there exists an n such that m^n kills every element of N/TM. Therefore N/TM can be thought of as a module over T/m^nT. This module now satisfies the conditions of the previous theorems, so the Hilbert function of N/TM is quasipolynomial. Since $\operatorname{Ann}_T N/TM \supset \operatorname{Ann} T$, this function is in addition quasipolynomial of degree at most $\dim N$.

It remains only to show that TM/M has a quasipolynomial Hilbert function of degree at most $\dim N$. So assume $N = TM$. Suppose that T is generated over S by r generators, so $T = S[t_1, \dots, t_r]$. Suppose that r is at least 2, and that the theorem has already been proved for smaller values of r. Apply the theorem with $S[t_1]$ replacing S and $S[t_1]M$ replacing M, and with T and N as before. It is clear that the hypotheses of the theorem are still satisfied, and T is generated over $S[t_1]M$ by $r-1$ elements, so by assumption the Hilbert function of $N/S[t_1]M$ is quasipolynomial. Similarly, apply the theorem with S and M as before, but T replaced by $S[t_1]$ and N by $S[t_1]M$, and the conclusion is that $S[t_1]M/M$ also has a quasipolynomial Hilbert function. The Hilbert function of N/M is the sum of the Hilbert functions of $N/S[t_1]M$ and $S[t_1]M/M$, and sum of two quasipolynomial functions is quasipolynomial, so the result follows by induction if only we can deal with the case $r = 1$.

So assume $N = S[t]M$ (where $t = t_1$). Construct the S-module

$$N' = \bigoplus_{i=0}^{\infty} \frac{t^{i+1}M + t^i M + \cdots + tM + M}{t^i M + t^{i-1}M + \cdots + tM + M}.$$

This module has the same Hilbert series as N/M; to see this, note that the quotient modules that we're summing are just successive quotients of modules in the filtration

$$\frac{M}{M} \subset \frac{tM + M}{M} \subset \frac{t^2M + tM + M}{M} \subset \cdots,$$

and note that

$$\bigcup_{i=0}^{\infty} \frac{t^i M + t^{i-1}M + \cdots + M}{M} = \frac{N}{M},$$

thanks to our assumption that $N = TM = S[t]M$.

Note also that $t(t^iM + \cdots + M) \subset (t^{i+1}M + \cdots + M)$, so multiplication by t induces a well-defined map

$$\frac{t^iM + \cdots + M}{t^{i-1}M + \cdots + M} \to \frac{t^{i+1}M + \cdots M}{t^iM + \cdots + M},$$

and this in turn induces a well-defined map $N' \to N'$. Thus multiplication by t makes sense on N', making N' a T-module as well as an S-module.

In addition, N' is generated over T by any set of generators for M, and in particular is finitely generated because M is. Any element of N' is killed by a power of m_R because N' is a direct sum of subquotients of N/M, which has the property that any element is killed by a power of m_R. Since N' is a T-module, it must also be the case that any element of N' is killed by a power of m_RT. Because N' is finitely generated, it follows that N' is killed by some fixed power of m_RT, hence is a finitely generated module over $T/(m_RT)^j$ for some j. The fact that N', and hence N/M, has a quasipolynomial Hilbert function, follows now from theorem 6. Note also that $\operatorname{Ann}_T N' \supset \operatorname{Ann}_T N$, so $\dim N' \leq \dim N$, and N/M has a quasipolynomial Hilbert function of degree at most $\dim N$. $\square$

CHAPTER III

Results about $\mathrm{Tor}_k(R/I_1^{a_1}, R/I_2^{a_2})$

This chapter applies the results of the previous chapter to prove theorems that describe the functions

$$f_k(a_1, \ldots, a_n) = \mathrm{length}(\mathrm{Tor}_k(M_1/I_1^{a_1}M_1, \ldots, M_n/I_n^{a_n}M_n))$$

in the case $n \geq 2$, $M_1 = \cdots = M_n = R$.

3.1 $\mathrm{Tor}_k(R/I_1^{a_1}, R/I_2^{a_2})$ for $k \geq 2$

Let I be any ideal of a ring R. The short exact sequence $0 \to I \to R \to R/I \to 0$ gives a long exact sequence

$$\begin{array}{cccccccc} \cdots & \to & \mathrm{Tor}_k(I, M) & \to & 0 & \to & \mathrm{Tor}_k(R/I, M) & \to \\ & \to & \mathrm{Tor}_{k-1}(I, M) & \to & 0 & \to & \mathrm{Tor}_{k-1}(R/I, M) & \to & \cdots \\ \cdots & \to & \mathrm{Tor}_0(I, M) & \to & M & \to & M/IM & \to & 0, \end{array} \tag{3.1}$$

which yields an isomorphism $\mathrm{Tor}_k(R/I, M) \cong \mathrm{Tor}_{k-1}(I, M)$ for $i \geq 2$ and an injection $\mathrm{Tor}_1(R/I, M) \to \mathrm{Tor}_0(I, M)$.

If I_1 and I_2 are two ideals of R, then $\mathrm{Tor}_k(R/I_1, R/I_2) \cong \mathrm{Tor}_{k-1}(I_1, I_2)$ for $i \geq 3$, by two applications of the above isomorphisms. Since direct sums commute with Tor, there is also an isomorphism

$$\bigoplus_{a_1, a_2 \geq 0} \mathrm{Tor}_k^R(R/I_1^{a_1}, R/I_2^{a_2}) \cong \mathrm{Tor}_k^R(\bigoplus_{a_1 \geq 0} I_1^{a_1}, \bigoplus_{a_2 \geq 0} I_2^{a_2})$$

for $k \geq 3$. Let t_1 and t_2 be indeterminates; then we can replace $\bigoplus_{a_i \geq 0} I_i^{a_i}$ by the algebra $R[I_i t] = \bigoplus_{a_i \geq 0} I_i^{a_i} t^{a_i}$. This allows us to impose the structure of an algebra on $\bigoplus_{1_1, a_2 \geq 0} \operatorname{Tor}_k^R(R/I_1^{a_1}, R/I_2^{a_2})$.

Lemma 32. *Let I_1 and I_2 be ideals of a ring R. Let $S_i = R[I_i t_i] = \bigoplus_{a_i \geq 0} I_i^{a_i} t_i^{a_i}$. Then the R-modules*

$$\bigoplus_{a_1, a_2 \geq 0} \operatorname{Tor}_k^R(R/I_1^{a_1}, R/I_2^{a_2})$$

can be given the structure of $\mathbb{N}^2$-graded, finitely generated $S_1 \otimes_R S_2$-modules, for $k \geq 2$.

Proof. For $k > 2$, we know that this direct sum is isomorphic to $\operatorname{Tor}_{k-2}^R(S_1, S_2)$. Let $r_{j,1}, \ldots, r_{j,m_i}$ generate I_i, and let $T_1 = R[x_{1,1}, \ldots, x_{1,m_1}]$ and $T_2 = R[x_{2,1}, \ldots, x_{2,m_2}]$ be polynomial rings, and map T_i onto S_i by mapping $x_{i,j}$ onto $r_{i,j}$. This makes S_i into a T_i-module. Choose a resolution of S_i by finitely generated projective T_i-modules,

$$\cdots \to P_{i,1} \to P_{i,0} \to S_i.$$

Since T_i is a free R-module, this resolution is also a resolution for S_i over R. Therefore $\operatorname{Tor}_k^R(S_1, S_2)$ is the homology of the total complex

$$\cdots \to P_{1,1} \otimes_R P_{2,0} \oplus P_{1,0} \otimes_R P_{2,1} \to P_{1,0} \otimes_R P_{1,0} \to 0.$$

The R-modules in this complex are easily seen to also be finitely generated $T_1 \otimes_R T_2$-modules, and the maps $T_1 \otimes T_2$-maps. Therefore the homology modules $\operatorname{Tor}_k^R(S_1, S_2)$ are also finitely generated modules over $T_1 \otimes T_2$. In addition, note that these modules must be killed by the kernels of the maps $T_i \to S_i$; thus, each $\operatorname{Tor}_k^R(S_1, S_2)$ is also a finitely generated $S_1 \otimes S_2$-module.

For $k = 2$, apply direct sums to the end of the long exact sequence of Equation

3.1 above to get

$$0 \to \bigoplus_{a_1,a_2\geq 0} \mathrm{Tor}_2^R(R/I_1^{a_1}, R/I_2^{a_2}) \to R[I_1t_1] \otimes_R R[I_2t_2] \to R[I_1t_1] \otimes_R R[t_2].$$

The map $R[I_1t_2] \otimes_R R[I_2t_2] \to R - I_1t_1 \otimes_R R[t_2]$ is clearly a map of $S_1 \otimes_R S_2$-modules, so the kernel is an $S_1 \otimes_R S_2$-module. □

Theorem 33. *Let I_1 and I_2 be ideals of a local ring (R, m) such that $I_1 + I_2$ is m-primary. Then for $k \geq 2$ the function*

$$f_k(a_1, a_2) = \mathrm{length}(\mathrm{Tor}_k(R/I_1^{a_1}, R/I_2^{a_2}))$$

is quasipolynomial of degree at most $2d$.

Proof. Fix $k \geq 2$. Then $\bigoplus_{a_1,a_2} \mathrm{Tor}_k(R/I_1^{a_1}, R/I_2^{a_2})$ is isomorphic to (or, in the case of $k = 2$, is at least a submodule of) $\mathrm{Tor}_{k-2}(S_1, S_2)$. Since $I + J$ is m-primary, $I_1^{a_1} + I_2^{a_2}$ is also m-primary, so some power of m is contained in $I_1^{a_1} + I_2^{a_2}$ and some power of m kills $\mathrm{Tor}_k(R/I_1^{a_1}, R/I_2^{a_2})$. Therefore every graded piece of $\mathrm{Tor}_{k-2}(S_1, S_2)$ is killed by a power of m. By the previous lemma, $\mathrm{Tor}_{k-2}(S_1, S_2)$ is a finitely generated $S_1 \otimes S_2$-module. It follows that every element of $\mathrm{Tor}_{k-2}(S_1, S_2)$ is also killed by a power of $m(S_1 \otimes S_2)$ and, in fact, that a fixed power $m^N(S_1 \otimes S_2)$ kills every element of the module. Therefore $\mathrm{Tor}_{k-2}(S_1, S_2)$ is actually a module over $(R/m^N) \otimes_R (S_1 \otimes S_2)$.

We now know that the module $\bigoplus_{a_1,a_2} \mathrm{Tor}_k(R/I_1^{a_1}, R/I_2^{a_2})$ is isomorphic, by an isomorphism that preserves the grading, to a finitely generated, bigraded module over $(R/m^N) \otimes_R (S_1 \otimes S_2)$. Therefore f_k is the Hilbert function of a finitely generated bigraded module over the bigraded ring $(R/m^N) \otimes_R (S_1 \otimes S_2)$. Theorem 30, from the previous chapter, says that this Hilbert function is quasipolynomial of degree equal to the dimension of the ring. So it remains only to calculate the dimension of $(R/m^N) \otimes_R (S_1 \otimes S_2)$. The ideal $m(S_1 \otimes S_2)$ is nilpotent, so the ring $(R/m) \otimes_R (S_1 \otimes S_2)$

has the same dimension, and

$$(R/m) \otimes_R (S_1 \otimes S_2) \cong ((R/m) \otimes_R S_1) \otimes_{R/m} ((R/m) \otimes_R S_2).$$

The dimension of $(R/m) \otimes_R S_i = (R/m) \otimes_R R[I_i t]$ is the analytic spread of I_i, which is at most d. The claimed result follows. □

The modules whose Hilbert functions we calculate above have generators all of whose multidegrees are either $(0,1)$ or $(0,1)$. By remarks in the previous chapter, such functions are eventually polynomial of degree at most 2 less than the dimension of the module. Therefore

Corollary 34. *Let I_1 and I_2 be ideals of a local ring (R, m) such that $I_1 + I_2$ is m-primary. Then there exists a $(b_1, b_2) \in \mathbb{N}^2$ such that, for $k \geq 2$ and $(a_1, a_2) \geq (b_1, b_2)$, the function*

$$f_k(a_1, a_2) = \text{length}(\text{Tor}_k(R/I_1^{a_1}, R/I_2^{a_2}))$$

is polynomial of degree at most $2d - 2$.

Note that it is possible that the Tor modules could have dimension less than the sums of the analytic spreads of the two ideals; for example, if I and J are generated by disjoint parts of a system of parameters, then the higher Tor's are all zero. However, I know of no examples showing that nonzero polynomials of smaller degree may occur.

3.2 $\text{Tor}_1(R/I_1^{a_1}, R/I_2^{a_2})$

The next goal is to establish some results about the functions f_1 and f_0. As far as we know, the following may be true.

Conjecture 35. *Let I_1 and I_2 be ideals of a local ring (R, m) such that $I_1 + I_2$ is m-primary. Then the function*

$$f_1(a_1, a_2) = \text{length}(\text{Tor}_1(R/I_1^{a_1}, R/I_2^{a_2}))$$

is quasipolynomial of degree at most $\dim R + 2$.

However, all we know now is this:

Theorem 36. *Let* I_1 *and* I_2 *be ideals of a local ring* (R, m) *such that* $I_1 + I_2$ *is* m*-primary, and such that the* R*-algebra*

$$T = \bigoplus_{a_1,a_2} (I_1^{a_1} \cap I_2^{a_2}) x_1^{a_1} x_2^{a_2} \subset R[x_1, x_2]$$

is finitely generated over R. *Then the function*

$$f(a_1, a_2) = \text{length}(\text{Tor}_1(R/I_1^{a_1}, R/I_2^{a_2}))$$

is quasipolynomial of degree at most $\dim R + 2$.

Proof. Let T be the doubly-graded R-algebra described above. By assumption it is finitely generated. Let S be the doubly-graded R-algebra

$$S = R[I_1 x_1, I_2 x_2] = \bigoplus_{a_1,a_2} I_1^{a_1} I_2^{a_2} x_1^{a_1} x_2^{a_2}.$$

Since $S \subset T$, T is also an S-algebra. Now observe that

$$T/S = \bigoplus_{a_1,a_2} \frac{I_1^{a_1} \cap I_2^{a_2}}{I_1^{a_1} I_2^{a_2}} x_1^{a_1} x_2^{a_2}$$

and recall that $Tor_1^R(R/I_1^{a_1}, R/I_2^{a_2}) = (I_1^{a_1} \cap I_2^{a_2})/I^r J^s$. Apply Theorem 31, with $N = T$, $M = S$, and the conclusion follows; all that remains is to calculate the upper bound on the dimension of R.

Let $M \subset T$ be the ideal generated by mR and by $(I^a \cap J^b) s^a t^b$, for all a, b satisfying $a + b \geq 1$. Clearly M is maximal, with $T/M \cong R/m_R$. Also, $\dim T = \text{height}\, M$. Let P be minimal a minimal prime such that $\dim T/P = \dim T$, and let $p = P \cap R$. Apply the dimension formula to get

$$\begin{aligned} \dim T + 0 = \text{height}\, M + \text{tr. deg}_{R/m_R} T/MT &\leq \text{height}(m_R(R/p)) + \text{tr. deg}_{R/p} T \\ &\leq \dim R + 2 \end{aligned}$$

□

Definition 37. Given a pair (I_1, I_2) of ideals of the local ring (R, m), call the algebra $\bigoplus_{a_1,a_2}(I_1^{a_1} \cap I_2^{a_2})x_1^{a_1}x_2^{a_2}$ the *intersection algebra* of I_1 and I_2. If this algebra is finite, we will say that I_1 and I_2 have *finite intersection algebra.*

The previous theorem can now be restated as follows.

Corollary 38. *If I_1 and I_2, ideals of the local ring (R, m), have finite intersection algebra, then the function*

$$f(a_1, a_2) = \text{length}(\text{Tor}_1(R/I_1^{a_1}, R/I_2^{a_2}))$$

is quasipolynomial of degree at most $\dim R + 2$.

3.3 $\text{Tor}_0(R/I_1^{a_1}, R/I_2^{a_2})$

Throughout this section, (R, m) is a regular local ring of dimension d.

Let I_1 and I_2 be ideals of R such that $I_1 + I_2$ is m-primary, and let $f_\chi(a_1, a_2)$ be the Euler characteristic

$$f_\chi(a_1, a_2) = \chi(R/I_1^{a_1}, R/I_2^{a_2}) = \sum_k (-1)^k \,\text{length}(\text{Tor}_k^R(R/I_1^{a_1}, R/I_2^{a_2})).$$

This sum is well-defined because the Tor's are all zero for $k > \dim R$, and because the condition that $I_1 + I_2$ be m-primary forces all of the Tor_k's to have finite length. The function χ is biadditive (so, for example, $\chi(M \oplus N, P) = \chi(M, P) + \chi(N, P)$) because each of the Tor_k's are. Also, $\chi(M, N)$ is zero whenever M and N are such that $\dim(M) + \dim(N) < d$ (see [8] or [20]). And if $\text{length}(M \otimes N) < \infty$, as is the case when $M = R/I^{a_1}$ and $N = R/I^{a_2}$, then $\dim(M) + \dim(N) \le d$ (see [25]).

Theorem 39. *For large a_1 and a_2, $f_\chi(a_1, a_2)$ is either identically zero, or is a eventually a polynomial of degree d.*

Proof. Fix a_1 and a_2 and let $P_{u,1}, \dots, P_{u,s_u}$ be the primes associated to $I_u^{a_u}$. For $u = 1$ and $u = 2$, choose filtrations of $R/I_u^{a_u}$ by prime cyclic modules, and let $p_{u,i}$ be the number of times that $R/P_{u,i}$ occurs in the corresponding filtration. Then by biaddativity,

$$\chi(R/I_1^{a_1}, R/I_2^{a_2}) = \sum_{i \le s_1, j \le s_2} p_{1,i} p_{2,j} \chi(R/P_{1,i}, R/P_{2,j}).$$

Assume that the $P_{u,i}$'s are ordered so that $P_{u,1}, \dots, P_{u,s'_u}$ have the least height of any of the $P_{u,i}$'s, and let e_u be this minimal height. Since each $P_{u,i}$ contains $I_u^{a_u}$, we know that $P_{1,i} + P_{2,j}$ is m-primary for any i and j. Therefore $e_1 + e_2 \ge d$. This means that if $i > s'_1$ or $j > s'_2$ then $\text{height}(P_{1,i}) + \text{height}(P_{2,j}) > e_1 + e_2 \ge d$, so $\chi(R/P_{1,i}, R/P_{2,j}) = 0$. Therefore

$$\chi(\frac{R}{I_1^{a_1}}, \frac{R}{I_2^{a_2}}) = \sum_{i \le s'_1, j \le s'_2} p_{1,i} p_{2,j} \chi(\frac{R}{P_{1,i}}, \frac{R}{P_{2,j}}).$$

Note that $p_{u,i}$ is independent of the choice of filtration; in fact, $p_{u,i}$ is just the length of $R_{P_{u,i}}/I_u^{a_u} R_{P_{u,i}}$. Also, the ideals $P_{u,1}, \dots, P_{u,s_u}$ are also the associated primes of minimal height for any power of I_1. So, for any a_1 and a_2,

$$\begin{aligned} f_\chi(a_1, a_2) &= \chi(\frac{R}{I_1^{a_1}}, \frac{R}{I_2^{a_2}}) \\ &= \sum_{i \le s'_1, j \le s'_2} \text{length}(\frac{R_{P_{1,i}}}{I_1^{a_1} R_{P_{1,i}}}) \, \text{length}(\frac{R_{P_{2,i}}}{I_2^{a_2} R_{P_{2,i}}}) \chi(\frac{R}{P_{1,i}}, \frac{R}{Q_{2,j}}). \end{aligned}$$

Since I_u is primary to the maximal ideal of $R_{P_{u,i}}$, the functions

$$a_u \mapsto \text{length}(\frac{R_{P_{u,i}}}{I_u^{a_u} R_{P_{u,i}}})$$

are just Hilbert functions, and are eventually polynomial of degree e_u. Therefore, the function $f_\chi(a_1, a_2)$ is polynomial of degree d for sufficiently large a and b, if $e + f = d$, and is identically zero if $e + f > d$, since in that case all of the $\chi(R/P_{1,i}, R/P_{2,j})$'s are zero. $\square$

Actually from the proof and the results of the previous chapter this function is quasipolynomial of degree at most $d+2$, and is the sum of periodic-polynomials on subsets of $\{(1,0),(0,1)\}$. Here is the result in more detail.

Theorem 40. *$f_\chi(a_1,a_2)$ is either identically zero, or quasipolynomial of degree $d+2$.*

Proof. By the proof of the previous theorem, f_χ is the sum of terms of the form $cf(x)g(y)$, where $f(x)$ and $g(y)$ are eventually polynomial and c is a non-negative integer. By Theorem 19, the sum of quasipolynomial functions is quasipolynomial of degree equal to the maximum of the degrees of the summands. So it suffices to show that each such summand is either identically zero or is quasipolynomial of degree $d+2$. By Theorem 21, $f(x)$ and $g(y)$ are each quasipolynomial of degree $e+1$ and $f+1$ (where e and f are as in the proof of the previous theorem). So, Theorem 23, $cf(x)g(y)$ is quasipolynomial of degree $e+f+2=d+2$, as long as c is nonzero. □

Theorem 41. *If I_1 and I_2 have finite intersection algebra, then the function*

$$f_0(a_1,a_2)=\text{length}\left(\frac{R}{I_1^{a_1}+I_2^{a_2}}\right)$$

is quasipolynomial of degree $\dim R+2$.

Proof. By the previous theorems, we already know that f_i is quasipolynomial for $i>0$, and we know that the alternating sum $\chi=f_0-f_1+f_2+\cdots$ is quasipolynomial. Since $f_0=\chi+f_1-f_2+\cdots$, and since the f_i's are eventually zero, this expresses f_0 as a finite sum of quasipolynomial functions. Thus f_0 is quasipolynomial. It remains to calculate the degree.

Both I_1 and I_2 are contained in m, so $I_1^{a_1}+I_2^{a_2}\subset m^{\min(a_1,a_2)}$. Therefore f_0 is bounded below by, for example, the function given by

$$f(a_1, a_2) = \begin{cases} \text{length}(R/m^{(a_1+a_2)/3}) & \text{if } 1/2 \leq a_1/a_2 \leq 2 \\ 0 & \text{otherwise} \end{cases}$$

Since $\text{length}(R/m^{a_1})$ is eventually polynomial of degree d, this function is quasipolynomial of degree $d+2$. So f_0 is bounded below by a quasipolynomial function of degree $d+2$.

Since $I_1 + I_2$ is m-primary, there exists N such that $m^N \subset I_1 + I_2$. It follows that $m^{N(a_1+a_2-1)} \subset I_1^{a_1} + I_2^{a_2}$, so

$$f_0(a_1, a_2) \leq \text{length}(R/m^{N(a_1+a_2)}).$$

Therefore $f_0(a_1, a_2)$ can also be bounded above by a polynomial of degree d, and hence by a quasipolynomial function of degree $d+2$.

The result now follows from Theorem 28. $\square$

CHAPTER IV

Monomial Ideals

This chapter focuses on the cases where $k = 0$, $M_1 = \cdots = M_n = R = k[\![x_1, \ldots, x_d]\!]$, and the ideals $I_1, \ldots, I_n$ are monomial ideals. When $n = 2$ the results of the previous chapter apply, because the pair I_1 and I_2 have finite intersection algebra whenever I_1 and I_2 are both monomial ideals. The first section of this chapter presents a proof of this fact. The rest of the chapter is devoted to a different approach which gives more useful information even for arbitrary n.

4.1 The case $n = 2$

Let $A = k[\![x_1, \ldots, x_d]\!]$, with $x_1, \ldots, x_d$ indeterminates. Let B be a sub-k-alebra of A generated as an algebra by monomials in $x_1, \ldots, x_d$. Then B is also a sub-k-module of A, and is generated over k as a module by monomials. The set of exponents $\alpha \in \mathbb{N}^d$ of monomials x^α generating B as a module over k form a submonoid M of the additive monoid $\mathbb{N}^n$. It is easy to see that M is a finitely generated monoid iff B is a finitely generated R-algebra.

Now consider the situation at hand. Let $I_1, I_2 \subset R = k[\![x_1, \ldots, x_d]\!]$ be ideals generated by monomials. Let T be the intersection algebra of I_1 and I_2. Consider the two algebras

$$\begin{aligned} T_1 &= R[I_1 t_1, t_2] &\subset R[t_1, t_2] \\ T_2 &= R[t_1, I_2 t_2] &\subset R[t_1, t_2]. \end{aligned}$$

If $x^{\beta_1}, \dots, x^{\beta_m}$ generate I, then T_1 is generated over R by $t_2, x^{\beta_1}t_1, \dots, x^{\beta_m}t_1$. The situation for T_2 is the same, so both T_1 and T_2 are finitely generated R-algebras. The intersection of T_1 and T_2 is T. Let M be the submonoid of $\mathbb{N}^d$ consisting of those vectors $(a_1, \dots, a_d, b_1, b_2)$ which occur as the exponents of monomials $x_1^{a_1} \cdots x_d^{a_d} t_1^{b_1} t_2^{b_2}$ contained in T. Define monomials M_1 and M_2 that correspond in the same way to T_1 and T_2. Then M_1 and M_2 are finitely generated, and $M = M_1 \cap M_2$. Therefore we need only show that

Theorem 42. *If M_1 and M_2 are finitely generated submonoids of $\mathbb{N}^m$, then so is $M = M_1 \cap M_2$.*

Proof. This is a direct result of a theorem from integer programming (see, e.g., section I.3 of), which states that if $M \subset \mathbb{N}^n$ is the set of solutions to a finite set of $\mathbb{Z}$-linear equations, then M is a finitely generated monoid.

Let $\alpha_{i,1}, \dots, \alpha_{i,e_i} \in \mathbb{N}^m$ generate M_i. Define a new monoid $M' \subset \mathbb{N}^{m+e_1+e_2}$ as the set of all $(a_1, \dots, a_m, b_{1,1}, \dots, b_{1,e_1}, b_{2,1}, \dots, b_{2,e_2})$ satisfying the $2m$ equations

$$\begin{aligned} (a_1, \dots, a_m) &= e_{1,1}\alpha_{1,1} + \cdots + e_{1,e_1}\alpha_{1,e_1} \\ (a_1, \dots, a_m) &= e_{2,1}\alpha_{2,1} + \cdots + e_{2,e_2}\alpha_{2,e_2} \end{aligned}$$

Then M' is finitely generated, and it is easy to see that $M = M_1 \cap M_2$ is the image of M' under the map which projects $\mathbb{N}^{m+e_1+e_2}$ onto the first m coordinates. Therefore M is also finitely generated. $\square$

Therefore we have

Theorem 43. *If $I_1, I_2 \subset k[\![x_1, \dots, x_d]\!]$ are monomial ideals, then the length functions f_k are quasipolynomial for all $k \geq 0$.*

Proof. Now that we know I_1 and I_2 have finite intersection algebra, this is a consequence of the theorems of the previous chapter. □

4.2 The case $n \geq 2$

Throughout this section, $I_1, \dots, I_n$ will be ideals of $R = k[\![x_1, \dots, x_d]\!]$ which are generated by monomials.

I take for granted the basic definitions and theorems from the theory of convex polytopes as in, for example, the first three chapters of [10].

The monomials of R correspond to elements of $(\mathbb{Z}_{\geq 0})^d \subset \mathbb{R}^d$, under a map "log" which takes a monomial x^α to the integer vector α determined by its exponents. Thus $\log x^\alpha = \alpha$. Similarly, monomial ideals correspond to subsets of $\mathbb{R}^d$; write $\log I$ for the discrete subset of $\mathbb{R}^d$ corresponding to the exponents of the monomials contained in I.

If S is a subset of $\mathbb{R}^d$, aS will denote the subset $\{as | s \in S\}$. Note that $a \log I$ is not equal to $\log I^a$.

Let K_i be the convex hull of $\log I_i$. Write $P = \mathbb{R}^d_{\geq 0} = \left\{(r_1, \dots, r_d) \in \mathbb{R}^d \middle| r_i \geq 0\right\}$, let A_i be the closure of $P \backslash K_i$. I will show that $f_0(a_1, \dots, a_n)$ is closely approximated, when a_1, a_2, $\dots$, a_n are all sufficiently large, by the d-dimensional volume of the region $\bigcap_i a_i A_i$. Along the way, I will show that the volume of $\bigcap_i a_i A_i$ is piecewise a degree d homogeneous polynomial in the a_i's. The conclusion will be that $f(a_1, \dots, a_n)$ is approximated by a forms of degree d in $a_1, \dots, a_n$, for large $a_1, \dots, a_n$.

First I'll prove the theorem about the volume of $\bigcap_i a_i A_i$. But before doing this it would be useful to know what kind of object $\bigcap_i a_i A_i$ is.

Lemma 44. *If I is an ideal generated by finitely many monomials, then $K = \operatorname{conv}(\log I)$ is the intersection of finitely many closed half-spaces.*

Proof. Let $x^{\alpha_1}, \dots, x^{\alpha_k}$ generate an ideal I. Then $\log I$ is the union of the sets $\alpha_i + \mathbb{N}^d$, for $1 \le i \le k$. The convex hull in $\mathbb{R}^d$ of one of these sets is $\alpha_i + P$, so the convex hull K of I in $\mathbb{R}^d$ is the convex hull of the sets $\alpha_i + P$. Now, embed $\mathbb{R}^d$ in $\mathrm{P}^d(\mathbb{R})$ by the map

$$(r_1, \dots, r_d) \mapsto (r_1 : \cdots : r_d : 1),$$

and let $\overline{K}$ and $\overline{P}$ be the closures of K and P in this projective space. Let A be another copy of $\mathbb{R}^d$ in $\mathrm{P}^d(\mathbb{R})$ containing $\overline{K}$. Then $\alpha_i + \overline{P}$ is the convex hull in A of the point α_i and the points at infinity

$$(1 : 0 : \cdots : 0), (0 : 1 : 0 : \cdots : 0), \dots, (0 : \cdots : 0 : 1 : 0).$$

It follows that $\overline{K}$ is the convex hull in A of the points $\{\alpha_i\}$ corresponding to the generators of I along with the above points at infinity. Since $\overline{K}$ is the convex hull of a finite number of points, $\overline{K}$ can be written as the intersection of finitely many half-spaces of A. Half-spaces in A correspond to half-spaces in $\mathbb{R}^d$, so the original set K can also be written as the intersection of finitely many half-spaces. ☐

Thus each of our sets K_i can be written as the intersection of finitely many half-spaces.

Let $H^-_{i,1}, \dots, H^-_{i,k_i}$ be the closed half-spaces whose intersection is K_i, and let $H^+_{i,i}, \dots, H^+_{i,k_i}$ be the complementary closed half-spaces.

Note that $A_i = \bigcup_j H^+_{i,j}$. We can also write

$$A_i = \bigcup_{\substack{S \subset \{1, \dots, k_i\} \\ S \neq \emptyset}} A_{i,S}$$

where

$$A_{i,S} = \bigcap_{j \in S} H^+_{i,j} \cap \bigcap_{j \notin S} H^-_{i,j}.$$

The advantage of this decomposition is that we have written A_i as the union of sets with disjoint interiors: if S' and S'' are distinct non-empty subsets of $\{1..k_i\}$, then there must be some j contained in one but not the other; thus $A_{i,S'} \cap A_{i,S''}$ is contained in the hyperplane $H^+_{i,j} \cap H^-_{i,j}$.

Let T be the set of all pairs $(i,j) \in \mathbb{Z}^2$ with $1 \leq i \leq n$ and $1 \leq j \leq k_j$. If $S \subset T$, write $S_{i'}$ for the set of all $(i,j) \in S$ with $i = i'$. Then

$$\bigcap_i a_i A_i = \bigcup_{\substack{S \subset T \\ S_i \neq \emptyset \text{ for any } i}} \Big(\bigcap_{(i,j) \in S} H^+_{i,j} \cap \bigcap_{(i,j) \notin S} H^-_{i,j} \Big).$$

Proving that this union of intersections is actually equal to $\bigcap_i a_i A_i$ is routine. Once again, the sets that we are taking the union of have disjoint interiors. Also, each of these sets is contained in $\bigcap_i a_i A_i$, hence is bounded, and each is the intersection of finitely many half-spaces, hence is a polytope.

So we have established that $\bigcap_i a_i A_i$ is the union of polytopes with disjoint interiors. To prove that the volume of $\bigcap_i a_i A_i$ is a piecewise polynomial in the a_i, it therefore suffices to prove the following theorem. Note that the $H_{i,j}$'s in the statement below do not correspond exactly with the similarly-named half-spaces above.

Theorem 45. *Let $H^+_{1,1}, \ldots, H^+_{1,k_1}, \ldots, H^+_{n,1}, \ldots, H^+_{n,k_n}$ be half-spaces, and let*

$$Q_{a_1,\ldots,a_n} = \bigcap_{i,j} a_i H_{i,j}$$

be a family of polytopes indexed by $a_1, \ldots, a_n$. Then the volume of $Q_{a_1,\ldots,a_n}$ is a continuous function of $a_1, \ldots, a_n$. In addition, there exist a finite number of hyperplanes in $\mathbb{R}^n_{\geq 0}$ such that, for $a_1, \ldots, a_n$ not contained in any of the hyperplanes, the volume of the polytope $Q_{a_1,\ldots,a_n}$ is given by a degree d form in $a_1, \ldots, a_n$.

Proof. I believe this fact is actually well known, but have been unable to find a complete reference in the literature. Therefore I will present a sketch of the proof.

If $(a_1, \ldots, a_n)$ are fixed, then volume of the polytope $Q_{a_1,\ldots,a_n}$ can be calculated by triangulating $Q_{a_1,\ldots,a_n}$ and then calculating the volume of each simplex in the triangulation. It is also helpful to choose the triangulation so that each vertex of a simplex in the triangulation is also a vertex of the polytope. The existence of such a triangulation can be proved by induction on the dimension of the polytope: first triangulate the boundary of Q (which has lower dimension), then pick a fixed vertex of Q, and construct simplices each of which is a "pyramid" whose vertex is the chosen vertex and whose base is one of the simplices in the triangulation of the boundary.

If $(a_1, \ldots, a_n)$ are now allowed to vary, such a triangulation can be made to vary along with the polytope as long as the combinatorial type of the polytope does not change. The vertices of $Q_{a_1,\ldots,a_n}$ are the intersections of hyperplanes $a_i H_{i,j}$, and it can be shown that locally the location of one of these vertices is determined by solving a system of linear equations that depends linearly on the a_i and that the coordinates of the vertices are therefore locally linear functions of $(a_1, \ldots, a_n)$. The volume of a simplex can be written as a degree d form in the coordinates of its vertices. This means the volume of the simplices in the triangulation of the polytope, and hence the volume of the polytope itself, can be written as a degree d form in $(a_1, \ldots, a_n)$.

This only works for $(a_1, \ldots, a_n)$ constrained to a region where the combinatorial type of the polytope does not change. But it can be shown that the values of $(a_1, \ldots, a_n)$ for which the combinatorial type of $Q_{a_1,\ldots,a_n}$ changes are solutions to one of finitely many systems of linear equations, and hence are contained in the union of a finite set of hyperplanes. $\square$

From the description given in the theorem, it is clear that this volume is actually a quasipolynomial function of $(a_1, \dots, a_n)$. The next goal is to show that f_0 is well approximated by this volume for large enough $(a_1, \dots, a_n)$. However, without even doing this we've already gained some information in the case where the ideals I_i are all principal.

Corollary 46. *If $I_1, \dots, I_n$ are each generated by a single monomial, then f_0 has the form given in the above theorem, hence is quasipolynomial of degree $d+2$.*

Proof. Let $I_i = (x^{\alpha_i})$. In this case A_i is just $P \backslash (\alpha_i + P)$, and the volume of $\cap_i A_I$ is easily seen to be exactly equal to the number of lattice points contained in $\cap_i A_i$, and to the length of $R/(I_1^{a_1} + \cdots + I_n^{a_n})$. The result follows immediately from the previous theorem. □

Of course, this result is exactly what we would expect given the results of the previous chapter. The following corollary is a slight generalization of a result from [6].

Corollary 47. *Let I and J be monomial ideals in $R = k[\![x_1, \dots, x_d]\!]$, where k has characteristic p, and let $S = R/I$. Assume $I + J$ is m_R-primary. Then the Hilbert-Kunz function (see chapter 1), $\mathrm{HK}_{S,J}(e)$, is eventually a polynomial of degree* $\dim S$ *in p^e.*

Proof. Suppose $I = (x^{\alpha_1}, \dots, x^{\alpha_b})$, and $J = (x^{\alpha_{b+1}}, \dots, x^{\alpha_n})$. Let $I_i = (\alpha_i)$. Apply the previous corollary to $I_1, \dots, I_n$, and note that

$$\mathrm{HK}_{S,I}(e) = f_0(\underbrace{1, \dots, 1}_{b}, \underbrace{p^e, \dots, p^e}_{n-b}).$$

Since f_0 is piecewise polynomial, the point $(1, \dots, 1, e, \dots, e)$ must eventually end up in one of the "pieces."

The statement about the degree follows from Monsky's result about the form of Hilbert-Kunz functions [16]. ☐

As noted in [6], the assumption about the characteristic of p is actually unnecessary if we generalize the definition of the Hilbert-Kunz function in a natural way.

The next lemmas will be used to bound $f(a_1, \ldots, a_n)$ using the volume of $\cap a_i A_i$.

Lemma 48. *If H is a half-space containing $\log I$, then aH contains $\log I^a$.*

Proof. We can write $H = \left\{x \in \mathbb{R}^d \middle| \phi(x) \geq c\right\}$ for some linear function ϕ and real number c, and aH can then be described as the set of all $x \in \mathbb{R}^d$ such that $\phi(x) \geq ac$. Let $\alpha \in \log I^m$. Then $\alpha = \alpha_1 + \cdots + \alpha_a + \beta$, with $\alpha_i \in \log I$ and $\beta \in \log R$. Then

$$\begin{aligned} \phi(\alpha) &= \phi(\alpha_1 + \cdots + \alpha_a + \beta) \\ &= \phi(\alpha_1) + \cdots + \phi(\alpha_a) + \phi(\beta) \\ &\geq c + \cdots + c + \phi(\beta) \\ &\geq ac. \end{aligned}$$

For the last step we used the fact that $\phi(\beta) \geq 0$ for any $\beta \in \log R$; establishing this fact will complete the proof. So let β be an arbitrary element of $\log R$, and let α be an arbitrary element of $\log I$. Then $\alpha + m\beta \in \log I$ for any $m \in \mathbb{Z}^+$, so $\phi(\alpha + m\beta) = \phi(\alpha) + m\phi(\beta) \geq c$ for any m. From this it follows that $\phi(\beta) \geq 0$. ☐

It follows from the previous lemma that $a_i A_i$ contains $\log(I_i^{a_i})$. There is also an inclusion the other way.

Lemma 49. *If I is generated by h monomials, and if K is the convex hull of $\log(I)$, then $\log(I^a) \supset \mathbb{Z}^d \cap (a+h)K$.*

Proof. Let $x^{\alpha_1}, \ldots, x^{\alpha_h}$ generate I, and let β be any element of $(a+h)K$. Then β can be written as $a+h$ times an element of K. Since K is convex, this element can be written in the form $\sum_i b_i\alpha_i$, where the b_i's are non-negative real numbers such that $\sum_i b_i \geq 1$. So $\beta = \sum_i (a+h)b_i\alpha_i$. Write c_i for the greatest integer less than or equal to $(a+h)b_i$. Each c_i is non-negative, and

$$\sum_{i=1}^{h} c_i \geq \sum_{i=1}^{h}((a+h)b_i - 1) = (a+h)(\sum_{i=1}^{h} b_i) - h \geq (a+h)\cdot 1 - h = a$$

Therefore

$$x^{\sum_i c_i\alpha_i} = \prod_i (x^{\alpha_i})^{c_i}$$

is an element of I^a, and since each coordinate of the vector $\sum_i (a+h)b_i\alpha_i$ is at least as large as the corresponding coordinate of $\sum_i c_i\alpha_i$,

$$\beta = x^{\sum_i (a+h)b_i\alpha_i}$$

must also be contained in I^a. $\square$

Write $\mathrm{vol}(A)$ for the d-dimensional volume of $A \subset \mathbb{R}^d$, and write $\#(A)$ for the number of points in $A \cap \mathbb{Z}^d$.

Lemma 50. *Let A be a subset of $\mathbb{R}^d$ such that both A and its boundary ∂A have finite volume. Then*

$$\mathrm{vol}(A) - \mathrm{vol}(\partial A)\sqrt{2} \quad \leq \quad \#(A) \quad \leq \quad \mathrm{vol}(A) + \mathrm{vol}(\partial A)\sqrt{2}$$

Proof. Given $\alpha \in \mathbb{Z}^d$, we can form a half-open unit cube $\alpha + [0,1)^d$. This has volume 1 and contains exactly the one integer point α. Form the set A_0 made up exactly of those cubes $\alpha + [0,1)^d$ which are completely contained within A. Note that A_0 is contained in A, and that the volume of A_0 is exactly equal to the number of integer points contained in A_0. Similarly, form a set A^0 that is the union of those cubes $\alpha + [0,1)^d$ which have nonzero intersection with A. Then A^0 contains A, and again has the nice property that the number of integer points in A^0 is equal to the volume of A^0.

Note also that every point of $A \backslash A_0$ lies within $\sqrt{2}$ units of the boundary of S, and, similarly, that every point of $A^0 \backslash A$ is within $\sqrt{2}$ units of the boundary of A. The stated formula follows immediately. $\square$

Theorem 51. *Fix* $(b_1, \ldots, b_n) \in \mathbb{N}^n$, $a > 0$. *Then*

$$\lim_{m \to \infty} \frac{\operatorname{length}\left(\dfrac{R}{I_1^{mb_1} + \cdots + I_n^{mb_n}}\right)}{\operatorname{vol}(\bigcap_i mb_i A_i)} = 1.$$

Proof. Note that $\log(\sum_i I_i^{a_i}) = \bigcap_i \log(I^{a_i})$. Therefore $\operatorname{length}(R/(\sum_i I_i^{a_i}))$ is just the number of points contained in $\mathbb{N}^d$ but not in any $\log(I_i^{a_i})$. Let I be generated by h monomials. Then by Lemmas 48 and 49,

$$(\bigcap_i mb_i A_i) \cap \mathbb{Z}^d \subset \mathbb{N}^d \backslash \log(\sum_i I_i^{mb_i}) \subset (\bigcap_i (mb_i + h) A_i) \cap \mathbb{Z}^d.$$

Therefore

$$\#(\bigcap_i mb_i A_i) \leq \operatorname{length}\left(\frac{R}{\sum_i I_i^{mb_i}}\right) \leq \#(\bigcap_i (mb_i + h) A_i).$$

Let b be the $d-1$-volume of the boundary of $\bigcap_i b_i A_i$, making mb the $d-1$-volume of the boundary of $\bigcap_i mb_i A_i$. Combine the inequalities above with the inequalities given by lemma 50:

$$\begin{aligned}
\operatorname{vol}(\bigcap_i mb_iA_i) - mb\sqrt{2} \\
&\le \#(\bigcap_i mb_iA_i) \\
&\le \text{length}\left(\frac{R}{\sum_i I_i^{a_i}}\right) \\
&\le \#(\bigcap_i (mb_i + h)A_i) \\
&\le \operatorname{vol}(\bigcap_i mb_iA_i) + (m+h)b\sqrt{2}.
\end{aligned}$$

Divide through by $\operatorname{vol}(\bigcap_i mb_iA_i)$ to get

$$\frac{\operatorname{vol}(\bigcap_i mb_iA_i) - mb\sqrt{2}}{\operatorname{vol}(\bigcap_i mb_iA_i)} \le \frac{\text{length}\left(\frac{R}{\sum_i I_i^{b_i}}\right)}{\operatorname{vol}(\bigcap_i mb_iA_i)} \le \frac{\operatorname{vol}(\bigcap(mb_i + h)A_i) + mb\sqrt{2}}{\operatorname{vol}(\bigcap_i mb_iA_i)}.$$

Recalling that $\operatorname{vol}(\bigcap_i mb_iA_i)$ is a degree d form in m and n, it is easy to see that the left- and right- hand sides of this inequality approach 1 as m and n approach infinity along the line $n = ma$. The desired result follows. $\square$

This theorem, together with Theorem 45 and the discussion proving it, shows, roughly speaking, that f_0 agrees asymptotically with a piecewise degree d polynomial when $I_1, \ldots, I_n$ are monomial ideals.

4.3 Examples

In general, the functions f_0 seem to be quite difficult to compute, but in the monomial case, at least. there are lots of very tractable examples. It is worth examing a few, if only to verify that the piecewise behavior really does occur and that f_0 is not always eventually a polynomial.

i. Let $R = \mathbb{Q}[\![x_1, \dots, x_d]\!]$, $I_1 = \cdots = I_n = (x_1, \dots, x_d)$. Then

$$f_0(a_1, \dots, a_n) = \binom{n + \min(a_1, \dots, a_n) - 1}{n}.$$

If we divide $\mathbb{R}^n_{\geq 0}$ into the n subsets

$$A_i = \{(a_1, \dots, a_n) \in \mathbb{N}^n \,|\, a_i \leq a_j \text{ for all } j \neq i\},$$

each of which is the cone over a $(d-1)$-dimensional simplex, then f_0 is polynomial of degree d on each A_i.

ii. Fix a positive integer c and take $R = \mathbb{Q}[\![x_1, x_2]\!]$, $I_1 = (x_1, x_2)^c \cap (x_1)$, and $I_2 = (x_2^c)$. Then it is not hard to calculate that

$$f_0(a_1, a_2) = \begin{cases} \frac{(c-1)^2}{2} a_1^2 + c a_1 a_2 + \frac{c-1}{2} a_1 & \text{for } a_1 \leq \frac{c}{c-1} a_2 \\ c^2 a_1 a_2 - \frac{c^2}{2} a_2^2 + \frac{c}{2} a_1 & \text{for } a_1 \geq \frac{c}{c-1} a_2. \end{cases}$$

APPENDICES

APPENDIX A

Open Questions

i. Is it possible to calculate the intersection multiplicity $\chi(I_1, I_2)$ knowing only the behavior of the function f_0? Or do there exist pairs of ideals (I_1, I_2) and (J_1, J_2) that have different intersection multiplicities ($\chi(I_1, I_2) \neq \chi(J_1, J_2)$) but such that the functions f_0 are identical?

ii. Are the f_i always quasipolynomial when $n = 2$? The results of this thesis are compatible with the following conjecture:

Conjecture 52. *For (R, m) of dimension d and $I_1, I_2 \subset R$ such that $I_1 + I_2$ is m-primary,*

$$f_k(a_1, a_2) = \text{length}(\text{Tor}_k(R/I_1^{a_1}, R/I_2^{a_2}))$$

is quasipolynomial of degree $d+n$, if $k = 0$, is quasipolynomial of degree at most $d+2$, if $k = 1$, and is quasipolynomial of degree $2d$, if $k \geq 2$.

In fact, from this dissertation it might appear that a more general conjecture would be true for arbitrary n; however, there are counterexamples among the Hilbert-Kunz functions. If

$$R = \frac{(\mathbb{Z}/5\mathbb{Z})[\![x_1, x_2, x_3, x_4]\!]}{(x_1^4 + x_2^4 + x_3^4 + x_4^4)},$$

then

$$\mathrm{HK}_R(a) = \frac{168}{61}(5^a)^3 - \frac{107}{61}(3^a),$$

which is not quasipolynomial in 5^a. (See [11] for details.)

For counterexamples to the more restricted conjecture above, ideals I_1, I_2 not having finite intersection algebra would be particularly interesting to examine. Such examples can be constructed from examples existing in the literature of non-finitely generated symbolic power algebras. (See [22], [21], or [9]; the latter paper gives a family of examples based on monomial curves which may be easier to do calculations with.)

In particular, given an ideal $P \subset R$ such that the algebra

$$R \oplus P^{(1)} \oplus P^{(2)} \oplus \cdots$$

is not finitely generated, it is possible to find a $f \in R$ such that the ideals (f) and P do not have finite intersection algebra. To do this, choose f so that $(P^a : f^a) = P^{(a)}$, for all a.

Lemma 53. *If R regular and $P \subset R$ prime, then there exists $f \in R$ such that $(P^a : f^a) = P^{(a)}$ for all a.*

Proof. Since R is a regular local ring, so is R_P. The associated graded ring of a regular local ring is a polynomial ring, so, in particular, $\mathrm{gr}_{PR_P}(R_P) = (R - P)^{-1}\mathrm{gr}_P(R)$ is a domain. Let I be the kernel of the natural map $\mathrm{gr}_P(R) \to (R - P)^{-1}\mathrm{gr}_P(R)$. Then $(R - P)^{-1}I = 0$, and, since I is finitely generated, there exists $f \in R - P$ such that $fI = 0$. (We are implicitly identifying elements in R with their images in $\mathrm{gr}_P(R)$ via the natural map $R \to R/P \subset \mathrm{gr}_P(R)$). This means that $fg \in P^{n+1}$ for any element g of P^n which can be multiplied into

P^{n+1} by an element of $R - P$. More generally, any element $g \in P^n$ which can be multiplied into P^{n+m} by an element of $R - P$ satisfies $f^m g \in P^{n+m}$. The result follows. $\square$

Lemma 54. *If f and P are chosen as suggested above, then the intersection algebra $\bigoplus_{a_1,a_2\geq 0}(P^{a_1} \cap (f^{a_2}))x_1^{a_1}x_2^{a_2}$ is not finitely generated.*

Proof. We will prove the contrapositive. Assume that the intersection algebra is finitely generated. Note that $(P^{a_1} : f^{a_2})f^{a_2} = P^{a_1} \cap (f^{a_2})$, so the algebra can be rewritten as $\bigoplus(P^{a_1} : f^{a_2})x_1^{a_1}(x_2 f)^{a_2}$. Map this to the symbolic power algebra $\bigoplus_a P^{(a)}x^a$ by mapping tf to 1. This is an ungraded ring homomorphism. So the symbolic power is finitely generated, since it can be written as a homomorphic image of the intersection algebra, which we assumed to be finitely generated. $\square$

iii. For the cases where we proved results f_k was quasipolynomial for $k \geq 1$, we only found an upper bound on the degree of f_k. Is this upper bound actually an equality when f_k is not identically zero? If not, is it the best possible upper bound, or is there a smaller one?

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ABSTRACT

Length functions determined by killing powers of several ideals in a local ring

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Given a local ring R and n ideals whose sum is primary to the maximal ideal of R, one may define a function which takes an n-tuple of exponents to the length of the quotient of R by sum of the ideals raised to the respective exponents. This quotient can also be obtained by taking the tensor product of the quotients of R by the various powers of the ideals. This thesis studies these functions as well as the functions obtained by replacing the tensor product by a higher Tor. These functions are shown to have rational generating functions under certain conditions.