[Preliminary Version]

SECOND COEFFICIENTS OF HILBERT-KUNZ FUNCTIONS FOR DOMAINS

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Abstract. Let $(R, \mathfrak{m}, k)$ be an excellent (e.g., F-finite) equidimensional local Noetherian ring of prime characteristic p with $\dim(R) = d$, I an ideal of R such that $\lambda(R/I) < \infty$ and M a finitely generated R-module. We study the existence of $\beta(M) \in \mathbb{R}$ such that $\lambda(M/I^{[q]}M) = e_{HK}(I, M)q^d + \beta(M)q^{d-1} + O(q^{d-2})$. We refer to $\beta(M)$ as the second coefficient of the Hilbert-Kunz function. In particular, we show the existence of such $\beta(M)$ when the defining ideal of the singular locus of R has height at least 2.

0. Introduction

Throughout this paper R is a Noetherian commutative ring of prime characteristic p with $\dim(R) = d$ and I is an arbitrarily given ideal of R such that $\lambda_R(R/I) < \infty$. We write $q = p^n$ where n is a varying non-negative integer. For any q, we denote by $I^{[q]}$ the ideal generated by $\{r^q \mid r \in I\}$.

We use $\lambda_R(-)$ (or $\lambda(-)$ if R is understood) to denote the length of an R-module. Given any finitely generated R-module M, there is the Hilbert-Kunz function $e_n(I, M) = \lambda(M/I^{[q]}M)$, which is considered as a map from $\mathbb{N}$ to $\mathbb{N}$. To simplify notation, we often write $e_n(I, M)$ as $e_n(M)$ if no confusion arises.

Remark 0.1. Let R, I, M be as above. It is enough to understand the Hilbert-Kunz functions over local rings: Indeed, let $V(I) = \{\mathfrak{m} \mid \mathfrak{m} \in \operatorname{Spec}(R), I \subseteq \mathfrak{m}\}$, which is a finite set consists of maximal ideals of R. Then we have $e_n(M) = \lambda(M/I^{[q]}M) = \sum_{\mathfrak{m}\in V(I)} \lambda_{R_\mathfrak{m}}(M_\mathfrak{m}/I^{[q]}M_\mathfrak{m}) = \sum_{\mathfrak{m}\in V(I)} e_n(IR_\mathfrak{m}, M_\mathfrak{m})$.

For this reason, we assume R is local most of the time. By the notation $(R, \mathfrak{m}, k)$, we indicate that R is local with its maximal ideal being $\mathfrak{m}$ and its residue field being $k = R/\mathfrak{m}$.

By a result of [Mo], $e_n(I, M) = \alpha(M)q^d + O(q^{d-1})$ for some $\alpha(M) \in \mathbb{R}$. This $\alpha(M)$ is usually called the Hilbert-Kunz multiplicity of M with respect to I and is denoted by $e_{HK}(I, M)$. (Recall that, given functions $f,\ g : \mathbb{N} \to \mathbb{R}$, we write $f(n) = O(g(n))$ if there exists $C \in \mathbb{R}$ such that $|f(n)| \le |Cg(n)|$ for all $n \in \mathbb{N}$, while we say $f(n) = o(g(n))$ if $\lim_{n\to\infty} f(n)/g(n) = 0$.)

The above result of [Mo] has been pushed further in [HMM].

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Theorem 0.2 ([HMM]). *Let $(R, \mathfrak{m}, k)$ be an excellent local normal ring of prime characteristic p with a perfect residue field and $\dim(R) = d$. Then $e_n(M) = e_{HK}(I, M)q^d + \beta q^{d-1} + O(q^{d-2})$ for some $\beta \in \mathbb{R}$.*

We are going to study the issue more generally. Let $C_1(R)$ be the quotient of the Grothendieck group $G_0(R)$ by its subgroup spanned by $\{[R/P] \in G_0(R) \mid \dim(R/P) < d-1\}$ (see Notation 1.1 (6)). Our result generalizes [HMM] as follows.

Theorem (Corollary 2.5). *Let $(R, \mathfrak{m}, k)$ be an excellent equidimensional reduced local Noetherian ring of prime characteristic p such that the singular locus of R is defined by an ideal of height at least 2. Then there exists a group homomorphism $\beta : C_1(R) \to \mathbb{R}$ such that, for any finitely generated torsion free R-module M, we have*

$$e_n(M) = e_{HK}(I, M)q^d + \beta(c_1(M))q^{d-1} + O(q^{d-2}).$$

In general, for any finitely generated R-module M (not necessarily torsion free), there exists $b(M) \in \mathbb{R}$ such that

(1) $e_n(M) = e_{HK}(I, M)q^d + b(M)q^{d-1} + O(q^{d-2})$.
(2) $\lambda(\mathrm{Tor}_1^R(R/I^{[q]}, M)) = \big(b(M) - \beta(c_1(M))\big)q^{d-1} + O(q^{d-2})$.

In proving the above result, we reduce to the F-finite case by the Γ-construction as in [HH]. Recall that R is defined to be F-finite if R is module-finite over $R^q := \{r^q \mid r \in R\}$ for all q (or equivalently, for $q = p$). If R is F-finite, then R is excellent by [Ku]. In particular, its singular locus is a closed subset $V(J) \subseteq \mathrm{Spec}(R)$ defined by an ideal J.

Observe that the above result fails to hold in the following example, in which R is not a domain.

Example 0.3 ([Mo]). Let $R = k[[X, Y]]/(X^5 - Y^5)$ where k is any field of prime characteristic $p \equiv 2$ or $3 \mod 5$. Then $e_n(R) = 5q + c_n$ with $c_n = -4$ when n is even, while $c_n = -6$ when n is odd.

For any R-module M and for any $n \geq 0$, we can derive an R-module structure on the set M by $r \cdot m := r^{p^n}m$ for any $r \in R$ and $m \in M$. We denote the derived R-module by ${}^nM$. In this terminology, we see that R is F-finite if and only if ${}^1R$ (equivalently, ${}^nR$ for every $n \in \mathbb{N}$) is a finitely generated R-module.

Remark 0.4. If $(R, \mathfrak{m}, k)$ is local and $[k : k^p] = p^a$, then it is easy to see that $e_n(I, {}^eM) = \lambda({}^nM/I^{[q]} \cdot {}^eM) = p^{ea}\lambda(M/I^{[qp^e]}M) = p^{ea}e_{n+e}(I, M)$ for any $n, e \in \mathbb{N}$. If we choose e such that $\sqrt{0}^{[p^e]} = 0$, then ${}^eM$ may be considered as a module over $R/\sqrt{0}$. Thus, to study the behavior of $e_n(M)$ when $n \to \infty$, we may assume R is reduced without loss of generality.

1. Sufficient and necessary conditions for the existence of $\beta(M)$

Notation 1.1. Keep the default assumptions on R, I and d.

(1) Denote $\mathrm{Spec}(R, i) = \{P \in \mathrm{Spec}(R) \mid \dim(R/P) = d - i\}$ for any $0 \leq i \leq d$.
(2) Denote $f(M) = \oplus_{P \in \mathrm{Spec}(R,0)}(R/P)^{\lambda_{R_P}(M_P)}$ for any given finitely generated R-module M.

(3) We say that an F-finite ring R satisfies condition $(*)$ if

$$(*) \qquad \lambda_R(\mathrm{Tor}_1^R(R/I^{[q]},\, {}^e(f(R)))) = O(q^{d-2}) \quad \text{for all sufficiently large } e \in \mathbb{N}.$$

(4) We say that an F-finite local ring $(R, \mathfrak{m}, k)$ satisfies condition $(**)$ if, setting $a = \log_p[k : k^p]$,

$$(**) \qquad \lambda_R(\mathrm{Tor}_1^R(R/I,\, {}^n(f(R)))) = O(q^a q^{d-2}) \quad \text{as } n \to \infty.$$

(5) Denote $W = R \setminus (\cup_{P \in \mathrm{Spec}(R,0)} P)$. We say an R-module M is W-torsion-free if every element of W is a non-zero-divisor on M. Similarly, we say M is W-torsion if $W \cap \mathrm{Ann}_R(M) \neq \emptyset$, which is equivalent to $\dim(M) < d$. Notice that if R is a domain then W-torsion-free (or W-torsion) is the same as torsion-free (or torsion).

(6) Let $G_0(R)$ be the Grothendieck group of R. For any $0 \le i \le d$, we denote by $C_i(R)$ the quotient of $G_0(R)$ by the subgroup spanned by $\{[R/P] \in G_0(R) \mid P \in \cup_{j>i} \mathrm{Spec}(R,j) \text{ i.e., } \dim(R/P) < d - i\}$. Moreover, for any finitely generated R-module M, we denote by $c_i(M)$ the image of $[M]$ in $C_i(R)$. We also denote by $C(R)$ the kernel of the natural map $C_1(R) \to C_0(R)$ and, moreover, we write $c(M) = c_1(M) - c_1(f(M)) \in C(R)$ for any finitely generated R-module M.

(7) Given finitely generated W-torsion R-modules M and N, we write $M \sim N$ if there exists an exact sequence $0 \to K \to M \to N \to C \to 0$ such that $\dim(K \oplus C) \le d - 2$.

Discussion 1.2. (1). Recall that R is called equidimensional if $\min(R) = \mathrm{Spec}(R, 0)$. If R is catenary (e.g., F-finite) and equidimensional, then $\mathrm{Spec}(R, i)$ consists of all prime ideals P such that $\mathrm{height}(P) = i$.

(2). The natural group homomorphism $G_0(R) \to C_0(R)$, which factors through $C_1(R)$, splits. Hence the natural group homomorphism $C_1(R) \to C_0(R)$ also splits.

(3). Consequently, $C_1(R) \cong C(R) \oplus C_0(R)$. And it is easy to see that, for any finitely generated R-module M, $c(M)$ is exactly the projection of $c_1(M)$ to $C(R)$. For any W-torsion R-module T, we see that $c_1(T) = 0$ if and only if $c(T) = 0$.

(4). If R is normal catenary, then $C(R)$ is the class group of R.

(5). $f(M) = f(N)$ if and only if $c_0(M) = c_0(N)$.

(6). Given finitely generated W-torsion R-module M and N, we see that $M \sim N$ if and only if $M_P \cong N_P$ for all $P \in \mathrm{Spec}(R, 1) \cap (\mathrm{Supp}(M) \cup \mathrm{Supp}(N))$.

(7). Suppose $M \sim N$. Say we have exact sequences $0 \to K \to M \to L \to 0$ and $0 \to L \to N \to C \to 0$ such that $\dim(K \oplus C) \le d - 2$. From these two exact sequences we see that

$$\left|\left(e_n(M) - \lambda(\mathrm{Tor}_1^R(R/I^{[q]}, M))\right) - \left(e_n(N) - \lambda(\mathrm{Tor}_1^R(R/I^{[q]}, N))\right)\right| \le O(q^{d-2}) + \lambda(\mathrm{Tor}_2^R(R/I^{[q]}, C)),$$

which relies on the fact that $e_n(T) + \lambda(\mathrm{Tor}_1^R(R/I^{[q]}, T)) = O(q^{\dim(T)})$ for any finitely generated R-module T, which is proved in [HMM, Lemma 1.1]. Assume, moreover, that R satisfies S_2. Then choose an R-regular sequence $\underline{x} = x_1, x_2 \in \mathrm{Ann}(C)$. Since $\mathrm{pd}_R(R/(\underline{x})R) = 2$, we have

$$\lambda(\mathrm{Tor}_2^R(R/I^{[q]}, R/(\underline{x})R)) = \lambda(\mathrm{Tor}_1^R(R/I^{[q]}, R/(\underline{x})R)) - e_n(R/(\underline{x})R),$$

which equal to $O(q^{d-2})$ by [HMM, Lemma 1.1]. Then, as there exists an exact sequence $0 \to D \to (R/(\underline{x})R)^r \to C \to 0$, the long exact sequence forces $\lambda(\mathrm{Tor}_2^R(R/I^{[q]}, C)) = O(q^{d-2})$. Consequently, we have (under the S_2 assumption)

$$e_n(M) - \lambda(\mathrm{Tor}_1^R(R/I^{[q]}, M)) = e_n(N) - \lambda(\mathrm{Tor}_1^R(R/I^{[q]}, N)) + O(q^{d-2}).$$

(8). Suppose $M \sim N$ and $(R, \mathfrak{m}, k)$ is local and F-finite with $[k : k^p] = p^a$. Then we also have that

$$e_n(M) - q^{-a}\lambda(\mathrm{Tor}_1^R(R/I, {}^nM)) = e_n(N) - q^{-a}\lambda(\mathrm{Tor}_1^R(R/I, {}^nN)) + O(q^{d-2}),$$

which relies on the fact that $e_n(T) + q^{-a}\lambda(\mathrm{Tor}_1^R(R/I, {}^nT)) + q^{-a}\lambda(\mathrm{Tor}_2^R(R/I, {}^nT)) = O(q^{\dim(T)})$ for any finitely generated R-module T, which is proved in [Se, Page 278, Theorem].

(9). Suppose R is catenary (e.g., F-finite) and equidimensional. For any finitely generated R-module M, we can write $c_1(M) = \sum_{i=1}^t c_1(R/Q_i)$ with $Q_i \in \mathrm{Spec}(R)$. For each Q_i, choose a prime ideal $P_i \subseteq Q_i$ such that $P_i \in \mathrm{Spec}(R, 0)$. Let $K = \oplus_{i=1}^t Q_i/P_i$. Then $c_1(M) + c_1(K) = \sum_{i=1}^t c_1(R/Q_i) + \sum_{i=1}^t (c_1(R/P_i) - c_1(R/Q_i)) = \sum_{i=1}^t c_1(R/P_i) = c_1(f(M)) + c_1(f(K))$, that is $c_1(M \oplus K) = c_1(f(M \oplus K)) \in C_1(R)$. Notice that K is W-torsion-free.

(10). Suppose R is catenary (e.g., F-finite) and equidimensional and $x \in C_1(R)$, say $x = \sum_{i=1}^r c_1(R/Q_i) - \sum_{i=r+1}^s c_1(R/Q_i)$ with $Q_i \in \mathrm{Spec}(R)$. For each Q_i, choose a prime ideal $P_i \subseteq Q_i$ such that $P_i \in \mathrm{Spec}(R, 0)$. Let $M = (\oplus_{i=1}^r R/P_i) \oplus (\oplus_{i=r+1}^s Q_i/P_i))$ and $N = (\oplus_{i=1}^r Q_i/P_i) \oplus (\oplus_{i=r+1}^s R/P_i))$. It is easy to check that $x = c_1(M) - c_1(N)$ and M, N are both W-torsion-free.

Many of the implications in the next Proposition are implicit in [HMM].

Proposition 1.3. *Let $(R, \mathfrak{m}, k)$ be a reduced F-finite equidimensional Noetherian local ring of prime characteristic p with $\dim(R) = d$. Consider the following statements (with $q = p^n$):*

(1) R satisfies $(*)$ and, moreover, for any finitely generated W-torsion R-module T such that $c_1(T) = c_1(f(T)) = 0$ (i.e., $c(T) = c_1(T) = 0$) and all sufficiently large $e \in \mathbb{N}$, $e_n({}^eT) - \lambda(\mathrm{Tor}_1^R(R/I^{[q]}, {}^eT)) = O(q^{d-2})$.

(2) $e_n(M) - e_n(f(M)) = O(q^{d-2})$ for all finitely generated W-torsion-free R-module M such that $c_1(M) = c_1(f(M))$ (i.e., $c(M) = 0$).

(3) $e_n(M) - e_n(N) = O(q^{d-2})$ for all finitely generated W-torsion-free R-modules M and N such that $c_1(M) = c_1(N)$.

(4) There exists a group homomorphism $\tau : C(R) \to \mathbb{R}$ such that $e_n(M) - e_n(N) = \tau(c_1(M) - c_1(N))q^{d-1} + O(q^{d-2})$ for all finitely generated W-torsion-free R-modules M and N satisfying $c_0(M) = c_0(N)$.

(5) There exists a group homomorphism $\beta : C_1(R) \to \mathbb{R}$ such that

$$e_n(M) = e_{HK}(I, M)q^d + \beta(c_1(M))q^{d-1} + O(q^{d-2})$$

for every finitely generated W-torsion-free R-module M.

(6) For any finitely generated W-torsion-free R-module M and for any $e \in \mathbb{N}$, $\lambda(\mathrm{Tor}_1^R(R/I^{[q]}, {}^eM)) = O(q^{d-2})$.

(7) For any finitely generated W-torsion-free R-module M, there exists e_0 such that $\lambda(\mathrm{Tor}_1^R(R/I^{[q]}, {}^eM)) = O(q^{d-2})$ for all $e_0 \le e \in \mathbb{N}$.

(8) R satisfies $(*)$.

Then $(1) \Rightarrow (2) \Leftrightarrow (3) \Leftrightarrow (4) \Leftrightarrow (5) \Rightarrow (6) \Rightarrow (7) \Rightarrow (8)$*. If, moreover, R satisfies S_2, then* $(8) \Rightarrow (1)$ *and, hence, all the above statements are equivalent.*

Proof. Denote $a = \log_p[k : k^p]$. The assumption implies that W consists of non-zero-divisors of R.

(1) $\Rightarrow$ (2). There exists an exact sequence $0 \to M \to f(M) \to T \to 0$ so that T is W-torsion and $c_1(T) = 0$. Choose $e \gg 0$ such that $\lambda(\mathrm{Tor}_1^R(R/I^{[q]}, {}^e(f(M))) = O(q^{d-2})$ and $\lambda(\mathrm{Tor}_1^R(R/I^{[q]}, {}^eT) - e_n({}^eT) = O(q^{d-2})$ by (1). Then there is a long exact sequence

$$\mathrm{Tor}_1^R\left(\frac{R}{I^{[q]}}, {}^e(f(M))\right) \longrightarrow \mathrm{Tor}_1^R\left(\frac{R}{I^{[q]}}, {}^eT\right) \longrightarrow \frac{{}^eM}{I^{[q]}\cdot {}^eM} \longrightarrow \frac{{}^e(f(M))}{I^{[q]}\cdot {}^e(f(M))} \longrightarrow \frac{{}^eT}{I^{[q]}\cdot {}^eT} \longrightarrow 0.$$

Thus $p^{ea}e_{n+e}(M) - p^{ea}e_{n+e}(f(M)) = \lambda(\mathrm{Tor}_1^R(R/I^{[q]}, {}^eT)) - e_n({}^eT) - O(q^{d-2}) = O(q^{d-2})$, which implies $e_n(M) - e_n(f(M)) = O(q^{d-2})$.

(2) $\Rightarrow$ (3). By Discussion 1.2(9), there exists a finitely generated W-torsion-free R-module K such that $c_1(M \oplus K) = c_1(f(M \oplus K)) \in C_1(R)$. Notice that $c_1(M) = c_1(N)$ implies that $f(M) = f(N)$ and hence $c_1(N \oplus K) = c_1(f(N \oplus K)) \in C_1(R)$. Now the claim follows from (2) applied to $M \oplus K$ and $N \oplus K$. (3) $\Rightarrow$ (2) is trivial.

(3) $\Rightarrow$ (4). As $c_1({}^1M \oplus N^{p^{d-1+a}}) = c_1({}^1N \oplus M^{p^{d-1+a}})$, we apply (3) to ${}^1M \oplus N^{p^{d-1+a}}$ and ${}^1N \oplus M^{p^{d-1+a}}$, which gives that

$$\begin{aligned} e_n({}^1M \oplus N^{p^{d-1+a}}) - e_n({}^1N \oplus M^{p^{d-1+a}}) &= O(q^{d-2}) &&\text{that is}\\ (e_n({}^1M) - e_n({}^1N)) - p^{d-1+a}(e_n(M) - e_n(N)) &= O(q^{d-2}) &&\text{that is}\\ (e_{n+1}(M) - e_{n+1}(N)) - p^{d-1}(e_n(M) - e_n(N)) &= O(q^{d-2}) &&\text{which gives}\\ e_n(M) - e_n(N) &= t(M,N)q^{d-1} + O(q^{d-2}) \end{aligned}$$

for some $t(M,N) \in \mathbb{R}$, in which t is viewed as a map. For every element $x \in C(R)$, we define $\tau(x) = t(M,N)$ provided $x = c_1(M) - c_1(N)$ with M and N W-torsion-free finitely generated over R (cf. Discussion 1.2(10)). To check well-definedness, say $x = c_1(M') - c_1(N')$ with M' and N' W-torsion-free. Then $c_1(M \oplus N') = c_1(M' \oplus N)$, which implies $e_n(M \oplus N') = e_n(M' \oplus N) + O(q^{d-2})$, that is, $e_n(M) - e_n(N) = e_n(M') - e_n(N') + O(q^{d-2})$ by (4), which forces $t(M,N) = t(M',N')$. Now that we have showed that $\tau : C(R) \to \mathbb{R}$ is well-defined, it is straightforward to verify that τ is a group homomorphism.

(4) $\Rightarrow$ (5). As $c_0({}^1M) = c_0(M^{p^{d+a}})$, we apply (4) to ${}^1M$ and $M^{p^{d+a}}$, which gives that (with $\tau(c_1({}^1M) - c_1(M^{p^{d+a}})) = b'(M) = p^a b''(M) \in \mathbb{R}$)

$$\begin{aligned} e_n({}^1M) - e_n(M^{p^{d+a}}) &= b'(M)q^{d-1} + O(q^{d-2}) &&\text{that is}\\ e_n({}^1M) - p^{d+a}e_n(M) &= b'(M)q^{d-1} + O(q^{d-2}) &&\text{that is}\\ e_{n+1}(M) - p^d e_n(M) &= b''(M)q^{d-1} + O(q^{d-2}) &&\text{which gives}\\ e_n(M) &= e_{HK}(I,M)q^d + b(M)q^{d-1} + O(q^{d-2}) &&\text{(cf. [HMM, Theorem 1.11])} \end{aligned}$$

with $b(M) = b''(M)/(p^{d-1} - p^d) = \tau(c_1({}^1M) - c_1(M^{p^{d+a}}))/(p^{d-1+a} - p^{d+a})$, in which b is considered as a map. For every element $x \in C_1(R)$, set $\beta(x) = b(M) - b(N)$ if $x = c_1(M) - c_1(N)$ with M and N finitely generated W-torsion-free R-modules (cf. Discussion 1.2(9)). It is straightforward to check that $\beta : C_1(R) \to \mathbb{R}$ is a well-defined group homomorphism.

(5) $\Rightarrow$ (3). This is trivial as $c_1(M) \mapsto e_{HK}(I, M)$ is well-defined and determines a group homomorphism from $C_1(R)$ to $\mathbb{R}$.

(5) $\Rightarrow$ (6). It suffice to prove $\lambda(\operatorname{Tor}_1^R(R/I^{[q]}, M)) = O(q^{d-2})$ as the assumption of M being W-torsion-free implies ${}^eM$ being W-torsion-free for all $e \in \mathbb{N}$. Choose an exact sequence $0 \to M' \to G \to M \to 0$ such that G is free of finite rank over R. Then G and hence M' are W-torsion-free. Now $\lambda(\operatorname{Tor}_1^R(R/I^{[q]}, M)) = e_n(M') - e_n(G) + e_n(M) = \big(e_{HK}(I, M') - e_{HK}(I, G) + e_{HK}(I, M)\big)q^d + \big(\beta(c_1(M')) - \beta(c_1(G)) + \beta(c_1(M))\big)q^{d-1} + O(q^{d-2}) = O(q^{d-2})$.

(6) $\Rightarrow$ (7). This is obvious.

(7) $\Rightarrow$ (8). This follows immediately as R is W-torsion-free.

(8) $\Rightarrow$ (1) in case R satisfies S_2. Let A be the free abelian group generated by the set of all isomorphic classes $\{[R/Q] \mid Q \in \operatorname{Spec}(R, 1)\}$. Then $C(R)$ is a quotient of A modulo a subgroup generated by $\{\sum_{Q \in \operatorname{Spec}(R,1)} \lambda_{R_Q}((R/(P + xR))_Q)[R/Q] \mid P \in \operatorname{Spec}(R, 0), x \in R \setminus P\}$.

The assumption $c_1(T) = c(T) = 0$ implies that there exist $r \le s$, $P_i \in \operatorname{Spec}(R, 0)$, $x_i \notin P_i$ for $1 \le i \le s$ such that

$$\sum_{Q \in \operatorname{Spec}(R,1)} \lambda_{R_Q}(T_Q)[R/Q] + \sum_{i=1}^{r} \sum_{Q \in \operatorname{Spec}(R,1)} \lambda_{R_Q}((R/(P_i + x_iR))_Q)[R/Q]$$
$$= \sum_{i=r+1}^{s} \sum_{Q \in \operatorname{Spec}(R,1)} \lambda_{R_Q}((R/(P_i + x_iR))_Q)[R/Q]$$

as elements in the (free abelian) group A.

Choose e_0 such that the statement of $(*)$ always holds for $e \ge e_0$ and such that

$$\sqrt{\operatorname{Ann}_R(T \oplus (\oplus_{i=1}^s R/(P_i + x_iR)))}^{[p^{e_0}]} \subseteq \operatorname{Ann}_R(T \oplus (\oplus_{i=1}^s R/(P_i + x_iR))).$$

Then for all $e \ge e_0$, we have ${}^eT \oplus (\oplus_{i=1}^r {}^e(R/(P_i + x_iR)) \sim \oplus_{i=r+1}^s {}^e(R/(P_i + x_iR))$. Therefore, to prove the claim of (1), it suffices to prove that, for any $P \in \operatorname{Spec}(R, 0), x \notin P, e_0 \le e \in \mathbb{N}$, we always have

$$e_n({}^e(R/(P + xR))) - \lambda(\operatorname{Tor}_1^R(R/I^{[q]}, {}^e(R/(P + xR)))) = O(q^{d-2}).$$

Indeed, there is an exact sequence $0 \to {}^e(R/P) \to {}^e(R/P) \to {}^e(R/(P + xR)) \to 0$, which gives a long exact sequence

$$\operatorname{Tor}_1^R\Big(\frac{R}{I^{[q]}}, {}^e(R/P)\Big) \longrightarrow \operatorname{Tor}_1^R\Big(\frac{R}{I^{[q]}}, {}^e(R/(P_i + x_iR))\Big)$$
$$\longrightarrow \frac{{}^e(R/P)}{I^{[q]} \cdot {}^e(R/P)} \longrightarrow \frac{{}^e(R/P)}{I^{[q]} \cdot {}^e(R/P)} \longrightarrow \frac{{}^e(R/(P_i + x_iR))}{I^{[q]} \cdot {}^e(R/(P_i + x_iR))} \longrightarrow 0,$$

which implies $e_n({}^e(R/(P+xR))) - \lambda(\mathrm{Tor}_1^R(R/I^{[q]}, {}^e(R/(P+xR)))) = e_n({}^e(R/P)) - e_n({}^e(R/P)) + O(q^{d-2}) = O(q^{d-2})$.

Now the proof is complete. □

Example 1.4. Suppose $(R, \mathfrak{m}, k)$ is normal. Then statement (2) of Proposition 1.3 is verified in [HMM, Theorem 1.4]. Therefore statements (1) through (8) of Proposition 1.3 all hold.

Proposition 1.5. *Let $(R, \mathfrak{m}, k)$ be a reduced F-finite equidimensional local Noetherian ring of prime characteristic p. Denote $[k : k^p] = p^a$. Consider the following statements (with $q = p^n$):*

(1) R satisfies $(**)$.

(2) R satisfies $(**)$ and, moreover, for any finitely generated W-torsion R-module T such that $c(T) = 0$, $e_n(T) - q^{-a}\lambda(\mathrm{Tor}_1^R(R/I, {}^nT)) = O(q^{d-2})$.

(3) $e_n(M) - e_n(f(M)) = O(q^{d-2})$ for all finitely generated W-torsion-free R-module M such that $c_1(M) = c_1(f(M))$ (i.e., $c(M) = 0$).

(4) $e_n(M) - e_n(N) = O(q^{d-2})$ for all finitely generated W-torsion-free R-modules M and N such that $c_1(M) = c_1(N)$.

(5) There exists a group homomorphism $\tau : C(R) \to \mathbb{R}$ such that $e_n(M) - e_n(N) = \tau(c_1(M) - c_1(N))q^{d-1} + O(q^{d-2})$ for all finitely generated W-torsion-free R-modules M and N satisfying $c_0(M) = c_0(N)$.

(6) There exists a group homomorphism $\beta : C_1(R) \to \mathbb{R}$ such that

$$e_n(M) = e_{HK}(I, M)q^d + \beta(c_1(M))q^{d-1} + O(q^{d-2})$$

for every finitely generated W-torsion-free R-module M.

(7) $q^{-a}\lambda(\mathrm{Tor}_1^R(R/I, {}^nM)) = O(q^{d-2})$ for any finitely generated W-torsion-free R-module M.

Then (7) ⇔ (1) ⇒ (2) ⇒ (3) ⇔ (4) ⇔ (5) ⇔ (6).

Proof. The proof is very similar to (and actually simpler than) the proof of Proposition 1.3.

(1) ⇒ (2). Let A be the free abelian group generated by the set of all isomorphic classes $\{[R/Q] \mid Q \in \mathrm{Spec}(R,1)\}$. Then $C(R)$ is a quotient of A modulo a subgroup generated by $\{\sum_{Q\in\mathrm{Spec}(R,1)} \lambda_{R_Q}((R/(P+xR))_Q)[R/Q] \mid P \in \mathrm{Spec}(R,0), x \in R \setminus P\}$.

The assumption $c_1(T) = c(T) = 0$ implies that there exist $r \le s$, $P_i \in \mathrm{Spec}(R,0)$, $x_i \notin P_i$ for $1 \le i \le s$ such that

$$\sum_{Q\in\mathrm{Spec}(R,1)} \lambda_{R_Q}(T_Q)[R/Q] + \sum_{i=1}^{r} \sum_{Q\in\mathrm{Spec}(R,1)} \lambda_{R_Q}((R/(P_i+x_iR))_Q)[R/Q]$$
$$= \sum_{i=r+1}^{s} \sum_{Q\in\mathrm{Spec}(R,1)} \lambda_{R_Q}((R/(P_i+x_iR))_Q)[R/Q]$$

as elements in the (free abelian) group A.

Choose n_0 such that

$$\sqrt{\mathrm{Ann}_R(T \oplus (\oplus_{i=1}^s R/(P_i+x_iR)))}^{[p^{n_0}]} \subseteq \mathrm{Ann}_R(T \oplus (\oplus_{i=1}^s R/(P_i+x_iR))).$$

Then for all $n \geq n_0$, we have ${}^{n}T \oplus (\oplus_{i=1}^{r} {}^{n}(R/(P_i + x_iR)) \sim \oplus_{i=r+1}^{s} {}^{n}(R/(P_i + x_iR))$. Therefore, to prove the claim of (2), it suffices to prove that, for any $P \in \operatorname{Spec}(R, 0)$ and $x \notin P$, we always have

$$\lambda(R/I \otimes {}^{n}(R/(P + xR))) - \lambda(\operatorname{Tor}_1^R(R/I, {}^{n}(R/(P + xR)))) = O(q^{d-2}q^a).$$

Indeed, there is an exact sequence $0 \to {}^{n}(R/P) \to {}^{n}(R/P) \to {}^{n}(R/(P + xR)) \to 0$, which gives a long exact sequence

$$\operatorname{Tor}_1^R\left(\frac{R}{I}, {}^{n}(R/P)\right) \longrightarrow \operatorname{Tor}_1^R\left(\frac{R}{I}, {}^{n}(R/(P_i + x_iR))\right) \longrightarrow \frac{{}^{n}(R/P)}{I \cdot {}^{n}(R/P)} \longrightarrow \frac{{}^{n}(R/P)}{I \cdot {}^{n}(R/P)} \longrightarrow \frac{{}^{n}(R/(P_i + x_iR))}{I \cdot {}^{n}(R/(P_i + x_iR))} \longrightarrow 0,$$

which implies

$$\begin{aligned}\lambda(R/I \otimes {}^{n}(R/(P + xR))) &- \lambda(\operatorname{Tor}_1^R(R/I, {}^{n}(R/(P + xR))))\\ &= \lambda(R/I \otimes {}^{n}(R/P)) - \lambda(R/I \otimes {}^{n}(R/P)) + O(q^{d-2}q^a) = O(q^{d-2}q^a).\end{aligned}$$

(2) $\Rightarrow$ (3). There exists an exact sequence $0 \to M \to f(M) \to T \to 0$ so that T is W-torsion and $c_1(T) = 0$. Then, as $n \to \infty$, $\lambda(\operatorname{Tor}_1^R(R/I, {}^{n}(f(M))) = O(q^{d-2}q^a)$ and $\lambda(\operatorname{Tor}_1^R(R/I, {}^{n}T) - \lambda(R/I \otimes {}^{n}T) = O(q^{d-2}q^a)$ by (1). Also there is a long exact sequence

$$\operatorname{Tor}_1^R\left(\frac{R}{I}, {}^{n}(f(M))\right) \longrightarrow \operatorname{Tor}_1^R\left(\frac{R}{I}, {}^{n}T\right) \longrightarrow \frac{{}^{n}M}{I \cdot {}^{n}M} \longrightarrow \frac{{}^{n}(f(M))}{I \cdot {}^{n}(f(M))} \longrightarrow \frac{{}^{n}T}{I \cdot {}^{n}T} \longrightarrow 0.$$

Thus $q^a e_n(M) - q^a e_n(f(M)) = \lambda(\operatorname{Tor}_1^R(R/I, {}^{n}T)) - q^a e_n(T) - O(q^{d-2}q^a) = O(q^{d-2}q^a)$, which implies $e_n(M) - e_n(f(M)) = O(q^{d-2})$.

(3) $\Leftrightarrow$ (4) $\Leftrightarrow$ (5) $\Leftrightarrow$ (6). This is proved in Proposition 1.3.

(7) $\Rightarrow$ (1). This follows immediately as R is W-torsion-free.

(1) $\Rightarrow$ (7). By Discussion 1.2(8), there exists a finitely generated W-torsion-free R-module K such that $c_1(M \oplus K) = c_1(f(M \oplus K)) \in C_1(R)$. Thus, as it suffices to prove the claim for $M \oplus K$, we may assume $c_1(M) = c_1(f(M))$ without loss of generality. There exists an exact sequence $0 \to f(M) \to M \to T \to 0$ so that $c_1(T) = 0$ and T is W-torsion. Then, for any $n \in \mathbb{N}$, there is a long exact sequence

$$\operatorname{Tor}_1^R\left(\frac{R}{I}, {}^{n}(f(M))\right) \longrightarrow \operatorname{Tor}_1^R\left(\frac{R}{I}, {}^{n}M\right) \longrightarrow \operatorname{Tor}_1^R\left(\frac{R}{I}, {}^{n}T\right) \longrightarrow \frac{{}^{n}(f(M))}{I \cdot {}^{n}(f(M))} \longrightarrow \frac{{}^{n}M}{I \cdot {}^{n}M} \longrightarrow \frac{{}^{n}T}{I \cdot {}^{n}T} \longrightarrow 0,$$

which gives the desired conclusion

$$\begin{aligned}&\lambda(\operatorname{Tor}_1^R(R/I, {}^{n}M))\\ &= q^a\big(e_n(M) - e_n((f(M))\big) + \big(q^a e_n(T) - \lambda(\operatorname{Tor}_1^R(R/I, {}^{n}T))\big) - O(q^a q^{d-2})\\ &= q^a O(q^{d-2}) + q^a O(q^{d-2}) - O(q^a q^{d-2})\\ &= O(q^a q^{d-2}),\end{aligned}$$

by $(**)$ applied to $f(M)$, (3) applied to M, and by (2) applied to T. □

2. Applications

Theorem 2.1 (See [HMM, Theorem 1.12]). *Let $(R, \mathfrak{m}, k)$ be an F-finite reduced equidimensional Noetherian local ring of prime characteristic p satisfying condition* (5) *of Proposition 1.3 or condition* $(**)$. *Then there exists a group homomorphism $\beta : C_1(R) \to \mathbb{R}$ and, for any finitely generated R-module M, there exists $b(M) \in \mathbb{R}$ such that*

(1) $e_n(M) = e_{HK}(I, M)q^d + b(M)q^{d-1} + O(q^{d-2})$.
(2) $\lambda(\mathrm{Tor}_1^R(R/I^{[q]}, M)) = \big(b(M) - \beta(c_1(M))\big)q^{d-1} + O(q^{d-2})$; *and* $q^{-a}\lambda(\mathrm{Tor}_1^R(R/I, {}^nM)) = \big(b(M) - \beta(c_1(M))\big)q^{d-1} + O(q^{d-2})$ *in case of* $(**)$.

Proof. As condition $(**)$ implies Proposition 1.5(6), which is the same as Proposition 1.3(5), we may simply assume Proposition 1.3(5).

Let $T = \{x \in M \,|\, x/1 = 0 \in W^{-1}M\}$ be the W-torsion submodule of M. Then $M' = M/T$ is W-torsion-free and there is an exact sequence $0 \to T \to M \to M' \to 0$. Observe that $e_{HK}(I, M) = e_{HK}(I, M')$. There also exists an exact sequence $0 \to N \to G \to M \to 0$ such that G is free of finite rank over R. Then G and hence M' are W-torsion-free.

Let $\beta : C_1(R) \to \mathbb{R}$ be as in Proposition 1.3(5). Then apply $R/I^{[q]} \otimes_R$ to $0 \to T \to M \to M' \to 0$ and the same argument as in the proof of [HMM, Theorem 1.12] shows part (1), that is $e_n(M) = e_{HK}(I, M)q^d + b(M)q^{d-1} + O(q^{d-2})$ for some $b(M) \in \mathbb{R}$.

To prove (2), notice that the long exact sequence of Tor gives $\lambda(\mathrm{Tor}_1^R(R/I^{[q]}, M)) = e_n(N) - e_n(G) + e_n(M) = \big(e_{HK}(I, N) - e_{HK}(I, G) + e_{HK}(I, M)\big)q^d + \big(\beta(c_1(N)) - \beta(c_1(G)) + b(M)\big)q^{d-1} + O(q^{d-2}) = \big(b(M) - \beta(c_1(M))\big)q^{d-1} + O(q^{d-2})$. In case of $(**)$, notice that the long exact sequence of Tor also gives $\lambda(\mathrm{Tor}_1^R(R/I, {}^nM)) = q^a e_n(N) - q^a e_n(G) + q^a e_n(M) + O(q^a q^{d-2}) = \big(e_{HK}(I, N) - e_{HK}(I, G) + e_{HK}(I, M)\big)q^a q^d + \big(\beta(c_1(N)) - \beta(c_1(G)) + b(M)\big)q^a q^{d-1} + O(q^a q^{d-2}) = \big(b(M) - \beta(c_1(M))\big)q^a q^{d-1} + O(q^a q^{d-2})$, that is, $q^{-a}\lambda(\mathrm{Tor}_1^R(R/I, {}^nM)) = \big(b(M) - \beta(c_1(M))\big)q^{d-1} + O(q^{d-2})$. □

Corollary 2.2. *Let $(R, \mathfrak{m}, k)$ be an F-finite equidimensional Noetherian local ring of prime characteristic p such that $R/\sqrt{0}$ satisfies condition* (5) *of Proposition 1.3 or condition* $(**)$. *Then, for any finitely generated R-module M, there exists $b(M) \in \mathbb{R}$ such that $e_n(M) = e_{HK}(I, M)q^d + b(M)q^{d-1} + O(q^{d-2})$.*

Proof. There exists e such that $\sqrt{0}^{[p^e]} = 0$. Then ${}^eM$ may be considered as a finitely generated module over $R/\sqrt{0}$. As it suffices to prove the claim for ${}^eM$, we assume R is reduced and satisfies condition (5) of Proposition 1.3 or condition $(**)$ without loss of generality. Now the claim follow immediately from Theorem 2.1. (See Remark 0.4.) □

Theorem 2.3. *Let $(R, \mathfrak{m}, k)$ be an F-finite Noetherian local equidimensional reduced ring of prime characteristic p. Suppose there is an module-finite extension ring R' of R in the total fraction ring of R such that (a) $R'_{\mathfrak{n}}$ satisfies condition* (2) *of Proposition 1.3 or condition* $(**)$ *for every $\mathfrak{n} \in V(IR') \subseteq \mathrm{Spec}(R')$, and (b) $\mathrm{Ann}_R(R'/R)$ has height at least 2. Then there exists a group homomorphism $\beta : C_1(R) \to \mathbb{R}$ such that, for any finitely generated torsion free R-module M, we have*

$$e_n(M) = e_{HK}(I, M)q^d + \beta(c_1(M))q^{d-1} + O(q^{d-2}).$$

In general, for any finitely generated R-module M (not necessarily torsion free), there exists $b(M) \in \mathbb{R}$ such that

(1) $e_n(M) = e_{HK}(I, M)q^d + b(M)q^{d-1} + O(q^{d-2})$.

(2) $\lambda(\mathrm{Tor}_1^R(R/I^{[q]}, M)) = \big(b(M) - \beta(c_1(M))\big)q^{d-1} + O(q^{d-2})$.

Proof. As condition $(**)$ implies Proposition 1.5(3), which is the same as Proposition 1.3(2), we may simply assume Proposition 1.3(2)(3).

Throughout this proof, we will denote $M \otimes_R R'$ by M' and denote the torsion submodule of M' by $T(M')$ for any given R-module M. Thus $M'/T(M')$ is a torsion-free R'-module. As an R'-module, $e_n(IR', M') = \lambda_{R'}(M'/I^{[p^n]}M')$. As an R-module, $e_n(I, M') = \lambda_R(M'/I^{[p^n]}M')$.

For any exact sequence $0 \to M_1 \to M \to M_2 \to 0$ of finitely generated R-modules, there is an induced exact sequence $0 \to K \to M_1' \to M' \to M_2' \to 0$ for some finitely generated R'-module K. As $(R')_P = R_P$ (and hence $K_P = 0$) for any $P \in \mathrm{Spec}(R, 0) \cup \mathrm{Spec}(R, 1)$, we see that $\dim_{R'}(K) = \dim_R(K) < d - 1$. This implies that $c_1(M) \mapsto c_1(M \otimes_R R')$ defines a group homomorphism $C_1(R) \to C_1(R')$.

For any finitely generated torsion-free R-module M, we have an induced long exact sequence $\mathrm{Tor}_1^R(M, R'/R) \to M \to M' \to M \otimes_R R'/R \to 0$, which actually implies an exact sequence $0 \to M \to M' \to M \otimes_R R'/R \to 0$ since M is torsion-free while $\mathrm{Tor}_1^R(M, R'/R)$ is torsion. This implies that $e_n(I, M) - e_n(I, M') = O(q^{d-2})$ by [HMM, Lemma 1.1]. Moreover, for any $P \in \mathrm{Spec}(R, 0) \cup \mathrm{Spec}(R, 1)$, we see that $(M')_P \cong M_P$ is torsion-free, meaning $(T(M'))_P = 0$. Hence $\dim_{R'}(T(M')) = \dim_R(T(M')) < d - 1$. Also, notice that, any $\mathfrak{n} \in V(IR')$, $\dim(R'_\mathfrak{n}) = \dim(R)$ by the dimension formula. Consequently, $c_1(M'_\mathfrak{n}) = c_1(M'_\mathfrak{n}/T(M')_\mathfrak{n}) \in C_1(R'_\mathfrak{n})$ and $e_n(IR'_\mathfrak{n}, M'_\mathfrak{n}) = e_n(IR'_\mathfrak{n}, M'_\mathfrak{n}/T(M')_\mathfrak{n}) + O(q^{d-2})$ for every $\mathfrak{n} \in V(IR')$. It is easy to see that $M'_\mathfrak{n}/T(M')_\mathfrak{n}$ is a torsion-free module over $R'_\mathfrak{n}$.

By Proposition 1.3 and Theorem 2.1, it suffices to show that $e_n(I, M) - e_n(I, N) = O(q^{d-2})$ for all finitely generated torsion-free R-modules M and N provided that $c_1(M) = c_1(N)$. For any such M and N, we have $c_1(M') = c_1(N') \in C_1(R')$ and hence, by the paragraph above, $c_1(M'_\mathfrak{n}/T(M')_\mathfrak{n}) = c_1(N'_\mathfrak{n}/T(N')_\mathfrak{n}) \in C_1(R'_\mathfrak{n})$ for every $\mathfrak{n} \in V(IR')$. By the assumption on $R'_\mathfrak{n}$, we have $e_n(IR'_\mathfrak{n}, M'_\mathfrak{n}/T(M')_\mathfrak{n}) = e_n(IR'_\mathfrak{n}, N'_\mathfrak{n}/T(N')_\mathfrak{n}) + O(q^{d-2})$ for every $\mathfrak{n} \in V(IR')$, which implies $e_n(IR'_\mathfrak{n}, M'_\mathfrak{n}) = e_n(IR'_\mathfrak{n}, N'_\mathfrak{n}) + O(q^{d-2})$ for every $\mathfrak{n} \in V(IR')$ by last paragraph. By Remark 0.1, we get $e_n(IR', M') = e_n(IR', N') + O(q^{d-2})$, which implies the desired result that $e_n(I, M) = e_n(I, N) + O(q^{d-2})$ from what have been shown in the last paragraph. □

As a corollary, we conclude that it suffices to consider the S_2 rings as far as the current issue is concerned. Recall that the S_2-ification of an F-finite local Noetherian reduced ring always exists.

Corollary 2.4. *Let $(R, \mathfrak{m}, k)$ be an F-finite equidimensional local Noetherian reduced ring of prime characteristic p and R' be the S_2-ification of R. Suppose R' satisfies condition $(*)$ or $(**)$ locally at every $\mathfrak{n} \in V(IR')$. Then there exists a group homomorphism $\beta : C_1(R) \to \mathbb{R}$ such that, for any finitely generated torsion free R-module M, we have*

$$e_n(M) = e_{HK}(I, M)q^d + \beta(c_1(M))q^{d-1} + O(q^{d-2}).$$

In general, for any finitely generated R-module M (not necessarily torsion free), there exists $b(M) \in \mathbb{R}$ such that

(1) $e_n(M) = e_{HK}(I,M)q^d + b(M)q^{d-1} + O(q^{d-2})$.
(2) $\lambda(\mathrm{Tor}_1^R(R/I^{[q]}, M)) = \big(b(M) - \beta(c_1(M))\big)q^{d-1} + O(q^{d-2})$.

Proof. Since R' has S_2, Proposition 1.3(2) is satisfied over R'. By construction of R', $\mathrm{Ann}_R(R'/R)$, as an ideal of R, has height at least 2. Now apply Theorem 2.3. □

A special case of the above corollary is the following.

Corollary 2.5. *Let $(R, \mathfrak{m}, k)$ be an excellent equidimensional Noetherian reduced ring of prime characteristic p such that the singular locus of R is defined by an ideal of height at least 2. Then there exists a group homomorphism $\beta : C_1(R) \to \mathbb{R}$ such that, for any finitely generated torsion free R-module M, we have*

$$e_n(M) = e_{HK}(I,M)q^d + \beta(c_1(M))q^{d-1} + O(q^{d-2}).$$

In general, for any finitely generated R-module M (not necessarily torsion free), there exists $b(M) \in \mathbb{R}$ such that

(1) $e_n(M) = e_{HK}(I,M)q^d + b(M)q^{d-1} + O(q^{d-2})$.
(2) $\lambda(\mathrm{Tor}_1^R(R/I^{[q]}, M)) = \big(b(M) - \beta(c_1(M))\big)q^{d-1} + O(q^{d-2})$.

Proof. By the Γ-construction, we may assume R is F-finite without loss of generality. (First, notice that $\widehat{R}$ remains equidimensional and reduced with its singular locus defined by an ideal of height at least 2. Then, by the Γ-construction (see [HH, Section 6]), there exists a faithfully flat local and purely inseparable extension $(\widehat{R}^\Gamma, \mathfrak{m}\widehat{R}^\Gamma)$ of $(\widehat{R}, \mathfrak{m}\widehat{R})$ such that $\widehat{R}^\Gamma$ is an F-finite, reduced and equidimensional local ring. Moreover, by choosing Γ small enough, one can make sure that $\widehat{R}$ and $\widehat{R}^\Gamma$ have the same singular locus under the natural homeomorphism $\mathrm{Spec}(\widehat{R}) \cong \mathrm{Spec}(\widehat{R}^\Gamma)$. Thus, the singular locus of $\widehat{R}^\Gamma$ is defined by an ideal of height at least 2. It is easy to see that there is a well-defined group homomorphism $C_1(R) \to C_1(\widehat{R}^\Gamma)$ induced by $[M] \mapsto [M \otimes_R \widehat{R}^\Gamma]$. Moreover, as $\mathfrak{m}\widehat{R}^\Gamma$ is the maximal ideal of $\widehat{R}^\Gamma$, the Hilbert-Kunz functions $e_n(I,M)$ over R and $e_n(I\widehat{R}^\Gamma, M \otimes_R \widehat{R}^\Gamma)$ over $\widehat{R}^\Gamma$ are the same for any finitely generated R-module M.)

Let R' be the integral closure of R in its total fraction ring. Then $\mathrm{Ann}_R(R'/R)$ is an ideal of R with height at least 2. (Therefore R' is the S_2-ification of R.) By [HMM], R' satisfies Proposition 1.3(2). Now apply Theorem 2.3 or Corollary 2.4. □

Remark 2.6. Let R' be as in the above proof and let $\mathfrak{A} := (R :_R R') = \mathrm{Ann}_R(R'/R)$. Then $\mathfrak{A}M$ is an R-submodule of M and $\dim(M/\mathfrak{A}M) \le \dim(R) - 2$ since $\dim(R/\mathfrak{A}) \le \dim(R) - 2$. But, as $\mathfrak{A}$ is also an ideal of R', $\mathfrak{A}M$ is an R'-module and the result of [HMM] applies. This should give an alternate proof to Corollary 2.5.

Example 2.7. Let $S = k[X_1, X_2, \ldots, X_d]$ where k is a field of characteristic p and $d \ge 2$, and $k \subseteq R \subseteq S$ such that $X_1^{n_1}X_2^{n_2}\cdots X_d^{n_d} \in R$ for all $n_1 + n_2 + \cdots + n_d \gg 0$. Then $\mathrm{height}_R(S/R) = d$ and the above result applies. Notice that R is not normal unless $R = S$.

Similarly, let $S = k[[X_1, X_2, \ldots, X_d]]$ where k is a field of characteristic p and $d \ge 2$, and $k \subseteq R \subseteq S$ such that $X_1^{n_1}X_2^{n_2}\cdots X_d^{n_d}S \subset R$ for all $n_1 + n_2 + \cdots + n_d \gg 0$. Then $\mathrm{height}_R(S/R) = d$ and the above result applies. Notice that R is not normal unless $R = S$.

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