FINITENESS PROPERTIES AND NUMERICAL BEHAVIOR OF LOCAL COHOMOLOGY[1]

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Abstract. This manuscript discusses many problems connected with finiteness properties of local cohomology modules, e.g., the associated primes, the minimal primes, Bass numbers, as well as the behavior of injective dimension, Lyubeznik numbers, and content.

This paper is dedicated to Gennady Lyubeznik on the occasion of his 60th birthday, and celebrates his many contributions to commutative algebra.

1. Introduction

We discuss some of the many open questions about the behavior of local cohomology modules of finitely generated modules over Noetherian rings. Several of these questions were raised by Gennady Lyubeznik, and his work has settled important cases of quite a few of them. Others are related to attempts to prove the direct summand conjecture, which is now a theorem [1, 2, 4], or the theory of quasilength and content, developed in joint papers of mine first with Craig Huneke [22] and later with Wenliang Zhang [25]. The relatively recent results of mine I will mention are joint work either with Jack Jeffries [23], Luis Núñez-Betancourt [24], or Wenliang Zhang [25]. We begin with some background and notation for local cohomology and then give a brief discussion in §3 of thirteen questions that were unresolved at the time of the lecture on which this manuscript is based. Note, however, that the question on Lyubeznik numbers of generic determinantal rings in equal characteristic 0 discussed in that lecture has recently been solved: see [36]. The questions about finiteness of associated primes and closed support are discussed further in §4.

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[1] This paper is an expanded version of material from a lecture given by the author at the conference "Homological and Computational Methods in Commutative Algebra," honoring the sixtieth birthday of Gennady Lyubeznik at the University of Minnesota, Minneapolis on August 9, 2017.

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2. Background and notation

Throughout, unless otherwise specified, all given rings are assumed to be commutative Noetherian rings with 1. If $I \subseteq R$ is an ideal and M is an R-module, the i th local cohomology module of M with support in I, denoted $H^i_I(M)$, may be defined as

$$\varinjlim_t \mathrm{Ext}^i_R(R/I^t,\, M).$$

This only depends on the radical of I. If I is generated, up to radicals, by $f_1,\, \dots,\, f_d$, this is the same as the cohomology of the complex $\mathcal{C}^\bullet(f_1^\infty,\, \dots,\, f_d^\infty; M)$, which we may define as

$$\left(\bigotimes_{j=1}^{d}(0 \to R \to R_{f_i} \to 0)\right) \otimes_R M \qquad \text{or}$$

$$0 \to M \to \bigoplus_j M_{f_j} \to \bigoplus_{j_1 \neq j_2} M_{f_{j_1} f_{j_2}} \to \cdots \to M_{f_1 f_2 \cdots f_d} \to 0.$$

The i th term is the direct sum of the $\binom{d}{i}$ localizations of M at all products $f_{j_1} \cdots f_{j_i}$, where $\{j_1, \dots, j_i\}$ is an i element subset of $\{1, 2, 3, \dots, d\}$.

One also has that if I is generated, up to radicals, by $f_1,\, \dots,\, f_d$, then

$$H^i_I(M) = \varinjlim_t H^i(f_1^t, \dots, f_d^t;\, M),$$

where $H^i(f_1^t, \dots, f_d^t;\, M)$ is the *Koszul cohomology*, which is the same as Koszul homology numbered "backwards." (More precisely, one takes Hom of the homological Koszul complex in to M.) In particular,

$$H^d_I(M) = \varinjlim_t \frac{M}{(f_1^t,\, \dots,\, f_d^t)M},$$

where the successive maps in the direct limit system are induced by multiplication by $f_1 \cdots f_d$ on the numerators.

Note that $H^0_I(M) = \bigcup_t \mathrm{Ann}_M I^t \subseteq M$.

If $\mathfrak{m}$ is a maximal ideal of R and M is finitely generated, the modules $H^i_{\mathfrak{m}}(M)$ have DCC. This case is well understood.

When M is Noetherian, the smallest i such that $H^i_I(M) \neq 0$ is the *depth* of M on I (which may be defined as the length of any maximal regular sequence on M in I).

The *associated primes* of the R-module H (whether H is finitely generated or not) are the primes P of R such that there is an injection $R/P \hookrightarrow H$, i.e., such that some nonzero element of H has annihilator precisely P. This set is denoted $\mathrm{Ass}(H)$. The minimal primes in this set will be called the *minimal* primes of H. H has closed support if and only if it has only finitely many minimal primes. Thus, if $\mathrm{Ass}_R(H)$ is finite, then H has closed support.

The *cohomological dimension*, $\mathrm{cd}_R(I)$, or $\mathrm{cd}(I)$ when R is understood from context, of a proper ideal I of a Noetherian ring R is the integer δ such that $H^\delta_I(R) \neq 0$ while $H^i_I(R) = 0$ for $i > \delta$. If $\delta = \mathrm{cd}(I)$ then for $i > \delta$, $H^i_I(M) = 0$ for every R-module M (whether M is finitely generated or not).

3. Thirteen open questions about local cohomology

Regular rings.

Question 1. *What are the best statements one can make about finiteness conditions on the local cohomology modules $H^i_I(R)$ when R is regular and I is an arbitrary ideal?*

Of course, what is meant here needs clarification. Here are three properties that local cohomology modules $H^i_I(R)$ of regular rings R with support in any ideal I have in many cases:

(i) The set of associated primes is finite!

(ii) The Bass numbers are finite!

(iii) The injective dimension is at most the dimension of the support!

Huneke and Sharp proved that these properties hold in the prime characteristic $p > 0$ case [30]. Lyubeznik [39] proved all three in the cases where the ring R is local or affine in equal characteristic 0, for a much larger class of functors (e.g., iterated local cohomology and images of connecting homomorphisms), using the fact that the local cohomology in these cases has the structure of a holonomic D-module over the ring D of differential operators over R. He introduced the theory of F-finite F-modules [40] to prove corresponding results in characteristic $p > 0$. He also used D-modules and some novel ideas to prove these results for unramfied regular local rings [41].

[5] proves (i) for smooth finitely generated algebras over $\mathbb{Z}$. So far as I know this remains an open question for finitely generated regular $\mathbb{Z}$-algebras,

(i) and (ii) are open questions for ramified regular local rings!

(i) is open for regular rings that contain the rationals, even, so far as I know, for finitely generated algebras over a formal power series ring over a field of characteristic 0.

(iii) is shown to be false in mixed characteristic in [19].

Finiteness of associated primes and closed support.

Question 2. *If M is any finitely generated module over a Noetherian ring R and I is any ideal of R, does $H^i_I(M)$ have closed support (equivalently, finitely many minimal primes) for all $i \geq 0$?*

There are many counterexamples to finiteness of associated primes when the Noetherian ring is not regular. Both positive results and counter-examples are discussed in greater detail in §4, but we mention a few results and questions here.

If T is regular of prime characteristic $p > 0$ and $R = T/fT$ is a hypersurface then $H^i_I(R)$ has closed support for every ideal I of R. This was proved by Luis Núñez-Betancourt and the author in [24] and, independently, by a different method, in [33]. So far as I know, the corresponding question for hypersurfaces in equal characteristic 0 is open, as is the question for local complete intersection rings of codimension 2 or more.

Let R be a ring of prime characteristic $p > 0$, and let $F : R \to R$ via $r \mapsto r^p$ denote the Frobenius endomorphism of R. The ring R is called *F-finite* R is module-finite over $F(R) = \{r^p : r \in R\}$. In this case one can consider $F^e : R \to R$ for every $e \geq 1$: let ${}^eR$ denote the R-module obtained from restriction of scalars under this map. When R is F-finite, every ${}^eR$ is a finitely generated R-module. R is said to have *finite Frobenius representation type* if there is a finite set $\mathcal{M}$ of R-modules such that every ${}^eR$ is a direct sum of copies of elements of $\mathcal{M}$ (allowing arbitrarily many repetitions of any element of $\mathcal{M}$). Rings R of prime characteristic $p > 0$ with finite Frobenius representation type have the property that the set of associated primes of every $H^i_I(R)$ is finite. This is proved in the Gorenstein case in [56] and, in general, in [11], and, independently, in [24].

Let G be a linearly reductive group over a field K. We consider only representations that are given by a regular map $G \to \mathrm{GL}(K, V)$, where V is a finite-dimensional vector space over K, and (possibly infinite) direct sums of such representations. Here, *linearly reductive* means that every representation G is a direct sum of simple representations (ones with no nonzero proper vector subspace that is stable under the action of G). In equal characteristic 0, the reductive linear algebraic groups are linearly reductive, but this is not true in positive characteristic. When a linearly reductive group G acts on a K-vector space V, there is a canonical retraction $V \to V^G$, the *Reynolds operator*, where V^G is the largest subspace of V on which G acts trivially. When G acts on a K-algebra S by K-algebra automorphisms, so that $R = S^G$ is the ring of invariants of the action, the map $S \to S^G$ is a splitting of $R \to S$ as a map of R-modules. This gives many instances, especially in equal characteristic 0, where a ring R is a direct summand, as an R-module, of a regular ring S.

In [45] it is shown that if a Noetherian ring S has the property that $\mathrm{Ass}\big(H^i_I(S)\big)$ is finite for every ideal I of S and $R \subseteq S$ is a direct summand of S as an R-module, then $\mathrm{Ass}\big(H^i_I(R)\big)$ is finite for every ideal I of R. This implies, for example, that if S is a regular affine K-algebra, and G is a linearly reductive linear algebraic group acting on S by K-algebra automorphisms, then $R = S^G$ the property that $\mathrm{Ass}\big(H^i_I(R)\big)$ is finite for every ideal I of R.

Suppose that $1 \leq t \leq r \leq s$ are integers, and let $X = (x_{ij})$ be an $r \times s$ matrix of indeterminates over K. Consider the corresponding generic determinantal ring $R = K[x_{ij} : 1 \leq r, 1 \leq j \leq s]/I_t(X)$, where $I_t(X)$ denotes the ideal generated by the $t \times t$ minors of the matrix X. In all characteristics, this ring can be obtained as a ring of invariants of a general linear group G over K acting on a polynomial ring. In characteristic 0, but not in prime characteristic $p > 0$, G is linearly reductive, and so the result of [45] implies that $\mathrm{Ass}\big(H^i_I(R)\big)$ is finite for all ideals I of R. So far as I know, the corresponding question is open if K has prime characteristic $p > 0$ if $t \geq 4$. When $t = 2$, the ring has finite Frobenius representation type, and using this fact the finiteness of $\mathrm{Ass}\big(H^i_I(R)\big)$ is shown in [24] for all I if $t = 2$ or $t = 3$ (in the latter case, one has finite Frobenius representation type after localizing at any of the variables). We record the remaining open cases as an explicit question.

Question 3. *Is* $\mathrm{Ass}\big(H^i_I(R)\big)$ *finite for every generic determinantal ring over a field?*

So far as I know, the same question is also open if K is replaced by $\mathbb{Z}$. Note as well that some surprising results about local cohomology of rings with support in determinantal ideals obtained by Huneke, Katz, and Marley in [27] and by Lyubeznik, Singh, and Walther in [42] are discussed at the end of §4.

It is shown in [24] that an affirmative answer to Question 3 would follow from an affirmative answer to:

A relative conjecture on finiteness of associated primes.

Question 4. *If* $(R, m) \to S$ *is faithfully flat, and* S/mS *is regular, is the set of associated primes of* $H^i_J(S)$ *that contain* m *finite?*

This is an open question even when S is a polynomial ring or Laurent polynomial ring over R. The author conjectures that this is true: this is a relative form of the conjectures on finitemess of associated primes for regular rings. See [46] and [48] and the references for these papers for proofs of this result in some special cases.

Cohomological dimension.

First, let (A, m, K) be a complete, dimension d, equicharacteristic regular local ring with K algebraically closed, and let I be an ideal with big height$(I) = b$. (Recall that the big height of I =max$\{$height$P)$: P is a minimal prime of $I\}$.) Faltings [12] shows that $\mathrm{cd}(A, I) \le d-[(d-1)/b]$. In general this is the best possible result, by a result of Lyubeznik [38]. However, if there is additional hypothesis on I or A/I, Huneke and Lyubeznik are able to improve this result in several directions. In [28] they show that:

(1) If I is prime, $\mathrm{cd}(A, I) \le d - 1 - [(d - 2)/b]$, and this is best possible for all b, d, with $0 < b < d$.

(2) If A/I is normal, $\mathrm{cd}(A, I) \le d - [d/(b + 1)] - [(d - 1)/(b + 1)]$.

We are naturally led to ask:

Question 5. *When* A *is regular, what is the best possible bound on* $\mathrm{cd}(A, I)$ *given that* A/I *is* S_i *and* R_j*?*

This question is raised explicitly by Huneke in [26].

Question 6. *Is the top local cohomology module of a local domain with support in* I *faithful? That is, if* R *is a local domain and* I *is an ideal of cohomological dimension* δ*, is* $H^\delta_I(R)$ *a faithful* R*-module?*

Laura Lynch [37] has proved this in dimension at most 3. Jack Jeffries and the author have proved this in prime characteristic $p > 0$ when the cohomological dimension of I is equal to the number of generators of I. See [23]. Note that if R is equicharacteristic regular local, it is known that *every nonzero local cohomology module is faithful.* One may localize at a minimal prime of the support, and so reduce to the case where the local cohomology is supported only at the maximal ideal. The results of [30], [39], and [40] imply that it is a nonzero finite direct sum of copies of the injective hull of the residue class field, which is well known to be faithful.

Lyubeznik numbers.

Question 7. *How can one explicitly compute Lyubeznik numbers for families of varieties?*

If (R, m, K) is local equicharacteristic and a quotient of a regular local ring S, the integers $\lambda_{i,j}(R) = \dim_K \mathrm{Ext}^i_S\big(K, H^{n-j}_I(S)\big)$, which are called the *Lyubeznik numbers of* R, are independent of the presentation $R = S/I$. Moreover, we may extend the definition to all equicharacteristic local R by taking $\lambda_{i,j}(R) := \lambda_{i,j}(\widehat{R})$; the completion $\widehat{R}$ is always a homomorphic image of a regular local ring. This integer is the i th Bass number of $H^{n-j}_I(S)$ for the maximal ideal. Among the first to look at relations between Lyubeznik numbers and topological or geometric properties of the underlying spectrum are García-López and Sabbah [13], K.-I. Kawasaki [34] and U. Walther [58]

For example, García-López and Sabbah [13] have shown that in the case where R is the coordinate ring of a complex algebraic variety V and m corresponds to an isolated singularity x, the Lyubeznik numbers are expressible in terms of the singular local cohomology (in the topological sense) of V at x. Blickle and Bondu [7] and later Blickle [6] have related results using étale local cohomology in characteristic p.

We describe briefly some of the existing results on computing local cohomology. In his F-module paper [40] Lyubeznik gives an algorithm to test vanishing of $H^i_I(R)$ for polynomial rings over a finite field, based on iterating the Frobenius until stabilization of a certain sequence of kernels occurs. In characteristic zero, The first algorithm to compute localizations is in T. Oaku's paper [49] and the first to compute local cohomology is in Walther's paper [57], which also discusses how one can compute iterated local cohomology, and, hence, Lyubeznik numbers, in the case of equal characteristic 0. Both are based on D-module theory and yield the structure rather than just the vanishing. While it is possible, in principle, to compute Lyubeznik numbers in specific cases in the equal characteristic case using the D-module algorithms in [49], [50], and [57], this does not enable one to deal with naturally occurring infinite families of varieties.

For example, if R is a generic determinantal ring over a field of characteristic 0 at the origin it was not known, at the time of the talk on which this paper is based, how to obtain explicitly the Lyubeznik numbers for generic determinantal rings R when the minors in the ideal determining R have size $t \geq 3$. This particular problem has now been solved in [36], making strong use of representation theory.

The general problem of how to compute Lyubezik numbers by a method that will be useful for families, even in the case of algebraic varieties over the complex numbers, remains very much open when the singularities are not isolated.

There has also been substantial research on the following question of Lyubeznik: do the Lyubeznik numbers at the vertex of an affine cone over a reduced projective scheme of finite type over a field depend on the specific embedding of the projective scheme or only on the projective scheme itself? Wenliang Zhang proved [59] that this has an affirmative answer in prime characteristic $p > 0$. Reichelt, Saito, and Walther [52] have recently given a negative answer for certain projective schemes in equal characteristic 0: these have more than one irreduicible component. The question is open in equal characteristic 0 if the scheme is a variety, i.e., is irreducible.

The absolute integral closure of a local domain.

The *absolute integral closure* of an integral domain R is the integral closure of R in an algebraic closure of its fraction field. It is unique up to non-unique isomorphism, and is denoted R^+. In [21], Craig Huneke and the author show that if R is an excellent local domain of prime characteristic > 0, then R^+ is a big Cohen-Macaulay algebra over R: every system of parameters in R is a regular sequence in R^+. In [29] it is shown that if (R, m) is a local domain of characteristic p that is a homomorphic image of a Gorenstein ring, then R has a module-finite extension S such that $H^i_m(R) \to H^i_m(S)$ is the zero map for all $i < \dim(R)$. This recovers, with a simpler proof, the main result of [21] when R is a homomorphic image of a Gorenstein ring. This suggests that $H^i_m(R^+)$ might be "small" for $i < \dim(R)$ if R is a complete local domain of mixed characteristic p. In [14] it is shown, in essence, that all roots of p kill $H^2_m(R^+)$ if R is a complete local domain of Krull dimension 3. (In dimensions 1 and 2, the lower local cohomology of R^+ vanishes.) In [15] this result is improved: *all* elements of the maximal ideal of R^+ kill $H^2_m(R^+)$. In [17] it is shown that for i strictly smaller than the dimension of the complete local mixed characteristic domain (R, m) with, say, algebraically closed residue class field, every element of $H^i_m(R^+)$ is killed by arbitrarily small powers $c^{1/N}$ of some nonzero element c of R.

We raise the following question in all dimensions:

Question 8. *If* $(R,\, m)$ *is a complete local domain of mixed characteristic, does the maximal ideal of* R^+ *kill* $H^i_m(R^+)$ *for all* $i < \dim(R)$*?*

Quasilength, content of local cohomology, and robust closure.

If M is a finitely generated R-module killed by a power of an ideal I, the quasilength (see [22]) of M with respect to I, $\mathcal{L}_I(M)$, is the length of a shortest filtration of M by cyclic R-modules that are killed by I, i.e., by modules that are homomorphic images of R/I. When I is a maximal ideal, this is simply length of M.

By the *content* of the local cohomology module $H^d_I(R)$ with respect to $x_1, \dots, x_d$, where $I = (x_1, \dots, x_d)$, we mean, if it exists,

$$\lim_{t_1, \dots, t_d \to \infty} \frac{\mathcal{L}_I\big(R/(x_1^{t_1}, \dots, x_d^{t_d})\big)}{t_1 \cdots t_d}.$$

I do not know any example where one can prove the content is strictly between 0 and 1.

The property of having content 1 for $H^d_{(x_1, \dots, x_d)}(R)$ is equivalent to the condition that the quasilength of $R/(x_1^t, \dots, x_d^t)$ is t^d for all t, but it is also equivalent to the condition that the quasilength of R/I^t is the same as the number of monomials of degree at most $t-1$ in d variables. Thus, the condition for content 1 depends only the ideal generated by the f_i and not on the specific choice of d generators. The same holds for content 0.

When the content is 1, we say that $x_1, \dots, x_d$ is a Q-*sequence*, and we call $H^d_I(R)$ *robust.*

In characteristic p, the content is always 0 or 1 — see [22]. In all characteristics, the content can be 0 even when the local cohomology is not 0: this is shown in [25].

Here are several questions related to content — see [22] for more detail.

Question 9. *C*an the content of a top local cohomology module be strictly between 0 and 1 in equal characteristic 0? If K is a field of characteristic 0, $\underline{x} = x_1, \dots, x_d$ and $\underline{y} = y_1, \dots, y_d$ are $2d$ indeterminates over K, and $G_{d,t} = (x_1 \cdots x_d)^t - \sum_{i=1}^{t} y_i x_i^{t+1}$, what is the content of $H^d_{(\underline{x})}\big(K[\underline{x}, \underline{y}]/(G_{d,t})\big)$?

The latter question is open even when $d = 3$, $t = 2$: it is easy to see that the content is at most 26/27, but it is not known whether it is positive or 0.

Question 10. *W*hat is the content of $H^3_{I_2(x_{ij})}(\mathbb{C}[x_{ijj}]$ for generic 2 by 2 minors of a 2×3 matrix $\mathbb{C}$?

Question 11. *I*f S is not Noetherian and we have that $x_1, \dots, x_d$ regular sequence in S, is the content of $H^d_{(x_1, \dots, x_d)}(S)$ equal to 1 (in other words, is $x_1, \dots, x_d$ a Q-sequence)? This is true if S is Noetherian.

Question 12. *I*s a system of parameters $x_1, \dots, x_d$ of a local ring R always a Q-sequence (this is stronger than direct summand conjecture)?

This is true in equal characteristic. The content of $H^d_{(x_1, \dots, x_d)}(R)$ is known to be positive in mixed characteristic. It is not known that the content must be 1 in mixed characteristic even in dimension 3. Note that an affirmative answer to 11. implies an affirmative answer to 12., since we now know that big Cohen-Macaulay algebras exist even in mixed characteristic [1, 2, 3, 16].

We define an algebra S over a local ring R to be *robust* if $H^d_{(x_1, \dots, x_d)}(S)$ is robust for every system of parameters $x_1, \dots, x_d$ of R. We define f to be in the *immediate robust closure* of $(g_1, \dots, g_n)R$ if $R[z_1, \dots, z_n]/(f - \sum_{i=1}^{n} g_i z_i)$ is robust. This generates an idempotent closure operation, *robust closure*. In characteristic p, this agrees with tight closure for complete local domains. See [25] for the proof of this and other details.

Question 13. *D*oes robust closure give an extension of tight closure to mixed characteristic with all of the desirable properties of tight closure?

In particular, we want to have colon-capturing for systems of parameters, and even better, the Dietz axioms (see [10]), the Rebhuhn-Glanz algebra axiom (see [51]), persistence, and a theory of test elements. For the latter one would want to know, at the very least, that if R is a complete local domain and c is a nonzero element such that R_c is regular, then there is a fixed power c^N of c that multiplies the closure of every ideal I of R into I.

4. The finiteness of the set of associated or minimal primes

As mentioned earlier, it is possible that that $\mathrm{Ass}_R\big(H^i_I(R)\big)$ is finite when R is an arbitrary regular ring and I is any ideal, and it is also possible that for every Noetherian ring R, Noetherian R-module M, and ideal I of R, the set of *minimal* primes of $H^i_I(R)$ is always finite. There are many positive results, but the general case remains open for both questions.

One may also study related questions about finiteness of minimal and associated primes for other related functors, e.g., for iterated local cohomology (see Lyubeznik's papers [39, 40]) and for $\operatorname{Ext}^j_R\big(R/I,\, H^i_I(M)\big)$. For the latter, see the paper of Marley and Vassilev [44].

We first survey some counterexamples to the finiteness of the set of associated primes, and then discuss some positive results and reductions in the problem both for the finiteness of the set of associated primes and for the finiteness of the set of minimal primes.

Negative results on associated primes.

In [54], A. Singh showed that with $R = \mathbb{Z}[U,V,W,X,Y,Z]/(UX+VY+WZ)$ and $I = (X,Y,Z)R$, $H^3_I(R)$ has a nonzero p-torsion element for every integer p. In fact, the class of

$$\lambda_p = \frac{(ux)^p + (yv)^p + (wz)^p}{p}$$

mod $(x^p, y^p, z^p)R$ is nonzero in

$$H^3_I(R) = \varinjlim{}_t\, R/(x^t, y^t, z^t)R,$$

but is killed by p. Thus, $H^3_I(R)$ has infinitely many associated primes in R.

In [31], M. Katzman, showed the following:

Theorem 4.1 (Katzman). *Let K be any field, let $S = K[s,t,u,v,x,y]$, let $f = sx^2v^2-(t+s)xyuv+ty^2u^2$, let $R = S/fS$ and let $T = R_{\mathfrak{m}}$ where $\mathfrak{m} = (s,t,u,v,x,y)$.*

Then $H^2_{(u,v)T}(T)$ has infinitely many associated primes.

The counterexamples in the remainder of this subsection are from the paper [55] of A. Singh and I. Swanson.

Theorem 4.2 (Singh and Swanson). *Let K be an arbitrary field, and consider the five dimensional integral domain $R = K[s,t,u,v,x,y]/(su^2x^2+tuvxy+sv^2y^2)$. Let $\mathfrak{m} = (s,t,u,v,x,y)$. Then the local cohomology module $H^2_{(x,y)}(R_{\mathfrak{m}})$ has infinitely many associated primes. Moreover, with $S = R/(s-1)$, one has that $H^2_{(x,y)}(S)$ has infinitely many associated primes. S is an affine domain of dimension four.*

No example of a *local* ring A with $\dim(A) = 4$ and some $\operatorname{Ass}_A\big(H^i_I(A)\big)$ infinite is known.

Theorem 4.3 (Singh and Swanson). *Let K be an arbitrary field, and consider the hypersurface $S = K[s,t,u,v,w,x,y,z]/(g)$ where*

$$g = (su^2x^2 + sv^2y^2 + tuvxy + tw^2z^2).$$

Then S is a normal domain, $\dim(S) = 7$, and $\operatorname{Ass}_S\big(H^3_{(x,y,z)}(S)\big)$ is infinite. This is preserved if we replace S by $S/(s-1)$ or by the localization $S_{(s,t,u,v,w,x,y,z)}$. If K has characteristic 0, then S has rational singularities, and if K has characteristic $p>0$, then S is F-regular.

There is a similar example in [55] that has dimension 8 with these properties that is also a UFD.

Positive results on finiteness of associated primes and on closed support.

By a result of Khashyarmanesh and Salarian [35] and, independently, of Brodmann and Lashgari Faghani [8], if M is finitely generated then the first non-finitely generated $H^i_I(M)$ has finitely many associated primes. Since $H^0_I(M) \subseteq M$ this shows that $\mathrm{Ass}_R\big(H^i_I(M)\big)$ is finite if $i \leq 1$, and also if $i = \mathrm{depth}_I M$.

If R has Krull dimension d, $I \subseteq R$, and M is Noetherian, Brodmann, Rotthaus, and Sharp [9] observe that $\mathrm{Ass}_R\big(H^d_I(M)\big)$ is finite.

In [43], Marley shows that if R is local, then $\mathrm{Ass}_R\big(H^{d-1}_I(M)\big)$ is finite. Marley also shows that, $H^{d-1}_I(M)$ has *finite support*. Moreover, if R is local, he shows that $\mathrm{Ass}_R\big(H^i_I(M)\big)$ is finite in the following cases:

(1) $\dim(M) \leq 3$.

(2) $\dim(R) = 4$ and R is regular on the punctured spectrum.

(3) $\dim(R) = 5$, R is unramified regular, and M is torsion-free.

In [32] Katzman proved that if R is an $\mathbb{N}$-graded Noetherian ring such that R_0 is the homomorphic image of a domain and the ideal I is generated by n forms of positive degree, then $H^n_I(R)$ has closed support.

Katzman also records an argument attributed to Lyubeznik that $H^n_I(R)$ has closed support when R has positive prime characteristic $p > 0$ and I is generated by n elements.

Rotthaus and Sega [53] show that a top local cohomology module $H^j_I(R)$ has closed support whenever R is a standard graded algebra over a Noetherian ring and I is the irrelevant ideal.

We conclude with some surprising results proved by Huneke, Marley, and Katz in [27]. In this paper it is proved that $H^i_I(M)$ has closed support if I has cohomological dimension at most 2 (i.e., $H^i_I(R) = 0$ if $i \geq 3$) or if R is local of dimension 4. They also show $\mathrm{Ass}_R\big(H^i_I(M)\big)$ is finite if R is a normal, local, excellent domain of dimension at most 4.

Another of the results from this paper is that if R is a Noetherian ring and M a finitely generated R-module, then $H^n_I(M)$ has closed support for all n and all n-generated ideals I if and only if this holds when $n = 3$.

They also show that if R is equicharacteristic 0 then for every n generated ideal I there exists a 2×3 matrix A over R such that $H^n_I(R) = H^3_{I_2(A)}(R)$.

Also note that in [27], Corollary 6.5, it is shown that if R is a commutative Noetherian ring containing $\mathbb{Q}$ of Krull dimension at most 5 and $\mathfrak{a}$ is the ideal generated by the size 2 minors of an arbitrary 2×3 matrix over R, then $H^3_{\mathfrak{a}}(R) = 0$.

We conclude by mentioning two related results proved by Lyubeznik, Singh, and Walter in [42]. First, let $\mathfrak{a}$ be the ideal generated by the size t minors of an $m \times n$ matrix over commutative Noetherian ring R, where $1 \le t \le \min\{m, n\}$, and t differs from at least one of m and n. If $\dim(R)A < mn$, then $H_{\mathfrak{a}}^{mn-t^2+1}(R) = 0$.

Finally, Theorem 1.2 of [42] asserts the following. Let $R = \mathbb{Z}[X] = \mathbb{Z}[x_{ij} : i, j]$ be a polynomial ring, where $X = \big(x_{ij}\big)$ is an $m \times n$ matrix of indeterminates over $\mathbb{Z}$. Let I_t denote the ideal generated by the size t minors of X. Then $H^k_{I_t}(R)$ is torsion-free over $\mathbb{Z}$ for all choices of k, t, and is a $\mathbb{Q}$-vector space whenever k is different from the height of I_t. Moreover, with the standard grading on R ($R_0 = \mathbb{Z}$ and all x_{ij} have degree 1), if $\mathfrak{m}$ denotes the prime ideal generated by the entries of X, one has a degree-preserving isomorphsim $H_{\mathfrak{a}}^{mn-t^2+1}(R) \cong H^{mn}_{\mathfrak{m}}(\mathbb{Q} \otimes_{\mathbb{Z}} R)$.

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