Tight Closure and Base Change

by

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To my
sweetheart
Kevin

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TABLE OF CONTENTS

CHAPTER 1

Introduction

1.1 History

Characteristic p arguments have played an increasingly important role in commutative algebra since 1973, when Peskine and Szpiro used them in their joint thesis to answer questions in characteristic 0 ([PS73]). Hochster generalized this in [Hoc75] with a metatheorem which allows systems of equations in equal characteristic 0 to be reduced to characteristic p.

Further harnessing the power of characteristic p, collaboration between Hochster and Huneke within the last decade has resulted in the notion of the *tight closure* of an ideal. This operation produces a (possibly) larger ideal which is still very close to the original ideal in some sense. Using this new operation, Hochster and Huneke produced an elegant proof of a strengthened form of the Briançon-Skoda theorem. The strengthened version of this theorem asserts that in a Noetherian ring of characteristic p, the integral closure of the n^{th} power of an n-generated ideal is contained in the tight closure of the original ideal. To see that this is a generalization of the "usual" Briançon-Skoda theorem, it suffices to observe that all ideals are tightly closed in regular rings. This observation turns out to be critical for many applications of tight closure.

Another important property is *colon capturing.* If I and J are ideals in R, then we define $I : J = \{r \in R \mid rJ \subseteq I\}$. That is, $I : J$ is the set of elements of R that multiply J into I. If $x_1, \dots, x_n$ are part of a system of parameters in a local ring, then in many circumstances $(x_1, \dots, x_{n-1}) : x_n$ is contained in the tight closure of $(x_1, \dots, x_{n-1})$. One obtains a very similar containment when the ring is Cohen-Macaulay; indeed, tight closure captures the failure of a ring to be Cohen-Macaulay. This is an important concept in the proof that the absolute integral closure of a ring is a balanced big Cohen-Macaulay algebra.

Tight closure has been used to produce new proofs of other seemingly unrelated theorems as well. For example, tight closure can be used to prove the monomial conjecture, the syzygy theorem, and that rings of invariants of linearly reductive groups acting on regular rings are Cohen-Macaulay.

Despite the great success that tight closure has enjoyed so far, there are still many open questions. For instance, it is unknown whether tight closure is a "geometric" operation. If it is, then given a finitely generated algebra over a domain, one would expect an ideal which is tightly closed in the generic fiber to be tightly closed for almost all fibers. This is the main topic for Chapter 6. One would also expect tight closure to commute with completion, and progress in this direction is discussed in Chapter 7.

Other open questions deal with the computability of tight closure. In general, it seems difficult to explicitly compute the tight closure of an arbitrary ideal. Some respite can be had when dealing with monomial ideals, however, as it is at least known that the tight closure of a monomial ideal is again monomial. A similar result is proved in Chapter 5 for binomial ideals.

1.2 Outline of Thesis

In Chapter 2 we present many of the definitions and concepts that will be used throughout the subsequent chapters. This includes some of the standard theorems and properties of tight closure.

Chapter 3 introduces a new method for reducing questions on the tight closure of modules to questions on the tight closure of ideals. The main idea is to pass from a free module to its symmetric algebra and to study tight closure there.

Chapter 4 discusses when information on the tight closure of ideals in the associated graded ring can be lifted to information on tight closure in the ring itself. This is then used to obtain a partial answer to the deformation question.

Chapter 5 shows that the tight closure of a binomial ideal in a ring defined by binomial ideals is again binomial. Other related ideals are also shown to be binomial in this setting, and a proof of Karen E. Smith is presented showing that tight closure commutes with localization in these rings.

Chapter 6 discusses situations where tight closure information on the generic fiber is passed to almost all fibers. Specifically, if an ideal is tightly closed in the generic fiber, or if the generic fiber is strongly F-regular, then we examine when the same holds for almost all fibers.

Chapter 7 presents various reductions that can be made in attempting to show that completion commutes with localization.

CHAPTER 2

Background

In this chapter, we give some basic definitions and concepts that will be used throughout the subsequent chapters. For the remainder of this chapter, R will denote a commutative Noetherian ring of positive prime characteristic p. e will always denote a nonnegative integer, and $q = p^e$ will be a power of the characteristic of R.

2.1 Ideals and Modules

First we discuss many of the definitions and theorems which deal primarily with ideals and modules.

Definition 2.1.1. Given an ideal $I \subseteq R$ and $q = p^e$, $I^{[q]} = \{i^q \mid i \in I\}$.

Since q is a power of the characteristic of R, the binomial theorem asserts that $(x + y)^q = x^q + y^q$. Using this, it is easy to see that $I^{[q]}$ is an ideal, and if I is generated by $g_1, \ldots, g_d$, then $I^{[q]}$ is generated by $g_1^q, \ldots, g_d^q$. This would not be the case if q were not a power of the characteristic.

Definition 2.1.2. Given an ideal $I \subseteq R$, the *Frobenius closure* I^F of I in R is the set of all elements $g \in R$ such that $g^q \in I^{[q]}$ for some $q = p^e$.

Note that if $g^{q_0} \in I^{[q_0]}$, then $g^q \in I^{[q]}$ for all $q \geq q_0$ as well. A closure operation that typically produces a somewhat larger ideal than the Frobenius closure is *tight closure*.

Definition 2.1.3. An element $x \in R$ is said to be in the *tight closure* I^* of an ideal $I \subseteq R$ if there exists c not contained in any minimal prime of R such that $cx^q \in I^{[q]}$ for $q = p^e$ sufficiently large.

For convenience, we will often use R° to denote the complement of the union of the minimal primes of R.

Remark 2.1.4. Due to Theorem 2.1.5 below, many problems in tight closure reduce to the domain case. When R is a domain, the condition that c not be contained in any minimal prime simply means that c is not zero.

Theorem 2.1.5. *Let R be a Noetherian ring of characteristic p with minimal primes $\mathfrak{p}_1, \ldots, \mathfrak{p}_k$. Then an element $x \in R$ is in the tight closure I^* of an ideal $I \subseteq R$ if and only if the image of x in $R/\mathfrak{p}_i$ is in the tight closure of $IR/\mathfrak{p}_i$ (over $R/\mathfrak{p}_i$) for each i.*

Proof. See [HH90, Proposition 6.25] □

Discussion 2.1.6. There are several justifications for terminology "tight closure". One such is the observation that the tight closure of an ideal is contained in the integral closure of the ideal, and is typically much smaller. In particular, in a regular ring, all ideals are tightly closed, while they are not necessarily all integrally closed. See [HH90, Section 5] for a thorough discussion of this.

It is often possible to find an element c that can be used to check tight closure for all ideals. That is, we can often choose c in Definition 2.1.3 so that it does not depend on either the element x or the ideal I. Such elements are called *test elements* and play a crucial role in the theory of tight closure. We formalize this in the following

Definition 2.1.7. An element c not contained in any minimal prime of R is called a *test element* for R if $cx^q \in I^{[q]}$ whenever $x \in I^*$. The ideal $\tau(R)$ generated by the test elements is called the *test ideal* of R.

Remark 2.1.8. The usual definition of the test ideal is $\tau(R) = \cap I : I^*$. [HH90, Proposition 8.23] ensures that these two definitions agree, however.

Remark 2.1.9. The intersection of $\tau(R)$ with the complement of the minimal primes of R precisely gives the test elements. Furthermore, there are often an "abundance" of test elements available in a ring. See Theorem 2.1.10 and Theorem 2.2.6 below, for example.

Theorem 2.1.10. *Let R be an algebra of finite type over an excellent local ring. Suppose that R is reduced, and let c be an element of R° such that R_c is regular (such elements always exist). Then c has a power that is a test element for R.*

Proof. This is a special case of [HH94a, Theorem 6.2]. $\square$

Discussion 2.1.11. The notion of tight closure can be extended to modules via the Peskine-Szpiro functors F^e (see [PS73]). Let S denote R viewed as an R-algebra via the e^{th} Frobenius endomorphism:

$$F^e : R \to S \text{ via } r \mapsto r^q \text{ where } q = p^e.$$

Then $F^e(\_)$ can be identified with the covariant functor $S \otimes_R \_$ from R-modules to S-modules. Since $S = R$, this is really a covariant functor from R-modules to themselves. Note that the R-module structure on $F^e(M)$ is given by $r(s \otimes m) = (rs) \otimes m$, while $s \otimes (rm) = (r^q s) \otimes m$.

There is a canonical map $M \to F^e(M)$ sending m to $1 \otimes m$. If $x \in M$ and $q = p^e$, then we will write x^q for the image of x in $F^e(M)$. If M is a free R-module of rank n and $x = (r_1, \ldots, r_n)$, then with this notation we have $x^q = (r_1^q, \ldots, r_n^q)$. In particular, this notation agrees with our notation for the case $M = R$. If N is a submodule of M, then we define

$$N_M^{[q]} = \ker(F^e(M) \to F^e(M/N)) = \operatorname{im}(F^e(N) \to F^e(M)).$$

This is also the same as the R-span of the elements x^q in $F^e(M)$ where $x \in N$. If M is obvious from the context, we will often drop the subscript and simply write $N^{[q]}$.

Definition 2.1.12. Let $N \subseteq M$ be modules over a ring R of characteristic p. Then $x \in M$ is in the *tight closure* N_M^* of N if there exists an element c not contained in any minimal prime of R such that $cx^q \in N_M^{[q]}$ for $q = p^e$ sufficiently large. If $N_M^* = N$, then we say that N is tightly closed in M.

Remark 2.1.13. Again, if M is clear from the context, then we will often write N^* rather than N_M^*.

Remark 2.1.14. It is easy to see that $x \in M$ is in the tight closure of $N \subseteq M$ if and only if $x + N$ is in the tight closure of 0 in M/N. This allows us to reduce questions about the tight closure of submodules to the case where the submodule is 0. Alternatively, we can apply this same trick in reverse; map a free module G onto M and study the tight closure of the inverse image of N under this map rather than the tight closure of N in M. Thus we can assume either that $N = 0$ or that M is free.

2.2 Rings

In this section we cover some of the concepts which use tight closure, as well as some of the properties of tight closure itself.

Definition 2.2.1. A ring R is *weakly F-regular* if every ideal of R is tightly closed.

Definition 2.2.2. R is *F-regular* if every localization of R is weakly F-regular.

Definition 2.2.3. When R is reduced and $R^{1/p}$ is module-finite over R, then R is said to be *strongly F-regular* if for every c not contained in any minimal prime of R, there exists $q = p^e$ such that the R-linear map $R \to R^{1/q}$ sending 1 to $c^{1/q}$ splits as a map of R-modules.

Remark 2.2.4. When R is reduced and $R^{1/p}$ is module-finite over R, we say that R is *F-finite.*

Theorem 2.2.5. *Let R be a Noetherian domain of positive prime characteristic p.*

(i) If R is F-finite and regular then R is strongly F-regular.

(ii) If R is regular then R is F-regular.

(iii) If R is strongly F-regular then R is F-regular.

(iv) If R is F-regular then R is weakly F-regular.

Proof. See [HH89, Theorem 3.1 (c) and (d)] for (i) and (iii) and [HH90, Theorem 4.6] for (ii). That F-regular implies weakly F-regular is immediate. □

Theorem 2.2.6. *Let R be an F-finite Noetherian domain of characteristic p. Then every nonzero element c of R such that R_c is regular has a power which is a test element.*

Proof. This follows immediatedly from [HH94a, theorem 6.23] □

Theorem 2.2.7. *Let R be an F-finite Noetherian domain of positive prime characteristic p. Let c be any nonzero element of R such that R_c is strongly F-regular (such elements always exist by Theorem 2.2.6). Then R is strongly F-regular if and only if there exists $q = p^e$ such that $Rc^{1/q} \subseteq R$ splits over R.*

Proof. See [HH89, Theorem 3.3a] □

One advantage that strongly F-regular rings have over weakly F-regular rings is that they localize. That is, if R is strongly F-regular and W is a multiplicative set in R, then $W^{-1}R$ is also strongly F-regular. It is still unknown whether this holds for weakly F-regular rings. If it does, of course, then weakly F-regular rings are

F-regular. Some other rings that have this nice localization property are F-rational rings, and their definition depends on a special class of ideals called *parameter ideals*.

Definition 2.2.8. Elements $x_1, \ldots, x_n$ of a local ring R form a *system of parameters* if they generate an ideal of height $\dim R$. Elements $x_1, \ldots, x_j$ of an arbitrary ring are *parameters* if their images in every local ring R_P form part of a system of parameters for every prime P containing $x_1, \ldots, x_j$. An ideal generated by parameters is called a *parameter ideal*.

Definition 2.2.9. R is *F-rational* if every ideal generated by parameters is tightly closed.

Theorem 2.2.10. *Let R be a Noetherian domain of positive prime characteristic p.*

(i) If R is weakly F-regular, then R is F-rational.

(ii) If R is F-finite, F-rational and Gorenstein, then R is strongly F-regular.

Proof. (i) is immediate. For (ii), see [HH89, Theorem 3.1f]. $\square$

CHAPTER 3

Reducing from Modules to Ideals

The definition of tight closure for modules is somewhat more technical than the corresponding definition for ideals. It is useful, therefore, to find ways to reduce from the module case to the ideal case. In this chapter, we discuss a new method of doing just that. The main result is given in Theorem 3.2.2.

3.1 Notation

Let $(H, +)$ be a commutative, additive semigroup with identity 0 (we will be interested in $H = \mathbb{N}$ or $\mathbb{Z}$ here). Then a ring S is H-graded if it has a decomposition $S = \oplus_{t \in H} S_t$ where the S_t are abelian groups, $1 \in S_0$, and $S_r S_t \subseteq S_{r+t}$ for all $r, t \in H$. For the rest of this chapter, $S = R[u_1, \ldots, u_d]$ will be an $\mathbb{N}$-graded R-algebra, where elements of R have degree zero and each indeterminate u_i has degree 1.

If M is a free R-module with free generators $u_1, \ldots, u_d$, then we can embed M in S by identifying M with S_1. If N is a submodule of M with generators $f_1, \ldots, f_t$, then we can think of the f_i as elements of S_1 and define J_N to be the ideal in S generated by $f_1, \ldots, f_t$. S is the *symmetric algebra* on M.

3.2 Reductions

An example of a reduction from modules to ideals is given in the following proposition.

Proposition 3.2.1. *Let R be a weakly F-regular ring of characteristic p. Then every submodule N of a finitely generated R-module M is tightly closed.*

Proof. See [HH90, Proposition 8.7]. □

Weak F-regularity was defined using ideals. So this theorem says that if we want to confirm that all submodules of an R-module M are tightly closed, then it suffices to check that all ideals of R are tightly closed. In the same spirit of reducing questions from studying modules to studying ideals, we have the following theorem.

Theorem 3.2.2. *Let N be a submodule of a finitely generated free R-module M. Let $S = R[u_1, \ldots, u_d]$ be the symmetric algebra on M, and let J be the ideal of S defined by the generators of N. If R and S share a common test element, then $v \in N^*_M$ if and only if $v \in J^*$, where the tight closure of J takes place in S.*

Remark 3.2.3. As discussed in Remark 2.1.14, the assumption that M be a free R-module here is not a difficulty. All questions about the tight closure of submodules can be reduced to the case where M is free.

Proof of Theorem 3.2.2. Call the common test element c. Suppose N is generated as a submodule of M by $f_1, \ldots, f_a$. If $v \in N^*$, then cv^q is in the R-span of $f_1^q, \ldots, f_a^q$ for q large. But then cv^q is in the S-span of these elements, and so $v \in J^*$ as well.

Conversely, suppose that $v \in J^*$ in S. Then $cv^q \in (f_1^q, \dots, f_a^q)S$. Since c is in R, it has degree 0 as an element of S. Furthermore, v and $f_1, \dots, f_a$ are all in S_1. As a consequence, cv^q has degree q as do f_i^q for all i. So when we write

$$cv^q = \sum_{j=1}^{a} s_{jq} f_j^q,$$

only the degree 0 terms of the s_{jq} survive. That is, all the higher order terms must cancel each other out. Hence we may assume that the $s_{jq} \in R$ for all j and q. But then cv^q is in the R-span of $f_1^q, \dots, f_a^q$, and we are done. $\square$

The requirement that R and $S = R[u_1, \dots, u_d]$ share a common test element is a fairly weak condition. This holds, for example, in the following setup.

Theorem 3.2.4. *If a reduced ring R is finitely generated as an algebra over an excellent local ring, then R and $S = R[u_1, \dots, u_d]$ have a common test element.*

Proof. This is a special case of [HH94a, Theorem 6.22] $\square$

CHAPTER 4

Deformation

4.1 The Associated Graded Ring

We would like to be able to say something about the tight closure of ideals in a ring given information about the ideal of leading coefficients in the associated graded ring. First we need some notation.

Definition 4.1.1. Let R be a ring and $\mathcal{I} = \{I_j\}_{j\in\mathbb{N}}$ a collection of ideals of R. Then $\mathcal{I}$ is a *filtration* of R if $R = I_0$, $I_t \supseteq I_{t+1}$ for all t, and $I_sI_t \subseteq I_{s+t}$ for all s and t. If $\cap I_j = 0$, then we will say that $\mathcal{I}$ is a *separating filtration.*

Definition 4.1.2. If $\mathcal{I}$ is a filtration of R, then the *associated graded algebra* $\mathrm{gr}_{\mathcal{I}}R$ of R is $\mathrm{gr}_{\mathcal{I}}R = R/I_1 \oplus I_1/I_2 \oplus \cdots$

Definition 4.1.3. If $\mathcal{I}$ is a filtration of a ring R and $c \in R$ is such that $c \in I_r - I_{r+1}$, then the leading form $[c]$ of c is $c + I_{r+1}$ in $\mathrm{gr}_{\mathcal{I}}R$. If $c \in \bigcap_{r=0}^{\infty} I_r$, then $[c] = 0$.

If I is an ideal of R, then we will denote the ideal of the leading forms of I by $[I]$. It is worth noting that $[I]$ is not always generated by the leading forms of generators of I, as the following example illustrates.

Example 4.1.4. Take R to be $K[x,y]$ and $I = (f,g)$ where $f = xy^2 + x^4$ and $g = x^2y + y^4$. Then $xf - yg = x^5 - y^5$ is in the ideal of leading forms of I, but $x^5 - y^5$ is not contained in the ideal generated by the leading forms of f and g. Using the notation given above, $[x^5 - y^5] \in [I]$, but $[x^5 - y^5] \notin ([f],[g])$.

Lemma 4.1.5. *Let R be a Noetherian ring and $\mathcal{I}$ a separating filtration of R. Set $S = R + I_1t + I_2t^2 + \cdots = R[I_1t, I_2t^2, \ldots]$. Then the following are equivalent:*

(i) S is generated by $R + I_1t + \cdots + I_kt^k$ for some positive integer k. In particular, S is finitely generated.

(ii) There exists a positive integer k such that for all $n > k$,

$$I_n = \sum_{\substack{k_1+\cdots+k_r=n \\ k_i \leq k}} \prod_{j=1}^{r} I_{k_j}$$

(iii) S is Noetherian.

(iv) $\mathrm{gr}_{\mathcal{I}}R$ is Noetherian

Proof. (i) $\implies$ (ii): Suppose that S is generated by $R + I_1t + \cdots + I_kt^k$. Fixing $n > k$, this tells us that $I_nt^n \subseteq (I_1t, \ldots, I_kt^k)$. Equating degrees yields

$$I_n \subseteq \sum_{\substack{k_1+\cdots+k_r=n \\ k_i \leq k}} \prod_{j=1}^{r} I_{k_j} \tag{4.1}$$

The opposite inclusion in (4.1) is immediate, so we obtain equality.

(ii) $\implies$ (iii): For some large value of k, we can write $S = R[I_1t, \ldots, I_kt^k]$. Since R is Noetherian, this makes it clear that S is as well.

(iii) $\implies$ (iv): Since S is Noetherian, so is $S[1/t]$. But

$$\mathrm{gr}_{\mathcal{I}}R \cong \frac{S[1/t]}{(1/t)S[1/t]}$$

and so $\mathrm{gr}_{\mathcal{I}}R$ is Noetherian as well.

(iv) $\implies$ (i): Let J_n be the ideal in S generated by $R + I_1t + \cdots + I_nt^n$ and $\mathfrak{a}_n$ the ideal in $\mathrm{gr}_{\mathcal{I}}R$ generated by $R/I_1 \oplus \cdots \oplus I_n/I_{n+1}$. Since $\mathrm{gr}_{\mathcal{I}}R$ is Noetherian and the $\mathfrak{a}_n$ form an ascending chain of ideals, they must stabilize, say at $\mathfrak{a}_k$. Now if S is not finitely generated, then there exists $n(k) > k$ such that $I_{n(k)}t^{n(k)}$ is not contained in the ideal $J_{n(k)-1}$. Choose $g_k \in I_{n(k)}$ such that $g_kt^{n(k)} \notin J_{n(k)-1}$. Since the filtration is separating, there exists $m(k)$ (greater that $n(k)$) such that $g_k \in I_{m(k)} - I_{m(k)+1}$. Hence $g_k + I_{m(k)+1} \in \mathfrak{a}_{m(k)+1}$. But $g_k + I_{m(k)+1} \notin \mathfrak{a}_k$, contradicting our choice of k. Therefore S must be finitely generated. □

Lemma 4.1.6. *Let x be an element of R, and $J = (u_1, \dots, u_d)$ be an ideal of R. If we have a relation in $\mathrm{gr}_{\mathcal{I}}R$ of the form $[x] = \sum a_i[u_i]$, where $a_i \in \mathrm{gr}_{\mathcal{I}}R$, then we can lift this relation to R and obtain an element $y \in J$ such that $[y] = [x]$.*

Proof. Since $\mathrm{gr}_{\mathcal{I}}R$ is graded and $[x]$ is homogeneous, we may assume that $\sum a_i[u_i]$ is homogeneous as well. Suppose the leading form of x has degree r and the leading form of each u_i is t_i. If a_i is nonzero, then we may assume it is homogeneous of degree $r - t_i$. So in our relation, we can write $[a_i]$ instead of a_i (now we are thinking of a_i as an element of R rather than of $\mathrm{gr}_{\mathcal{I}}R$). Let $y = \sum a_iu_i$ be any lifting of this sum to R. We need only check that $[y] = [x]$.

$[y] = [\sum a_iu_i]$. The leading form of y has degree at least r, since every term a_iu_i has degree at least r. Notice that $[a_i][u_i]$ is either 0 or is $[a_iu_i]$. In writing our sum, we only need write in the nonzero terms, so we may assume that $[a_i][u_i] = [a_iu_i]$ for all i. If there is any cancellation in degree r in this sum, then the same cancellation would occur in the sum $\sum[a_i][u_i]$ in $\mathrm{gr}_{\mathcal{I}}R$, in which case we could have eliminated those terms from the relation to begin with.

So $[y] = \sum [a_i][u_i] = [x]$ and we are done. $\square$

Theorem 4.1.7. *Let R be a Noetherian ring of characteristic p, and $\mathcal{I}$ a separating filtration of R. Let J be an ideal of R such that $[J^{[q]}] = [J]^{[q]}$ for all $q = p^e$ sufficiently large. Let c be a test element for R such that $[c]$ is not contained in any minimal prime of $\mathrm{gr}_{\mathcal{I}} R$. If $[J]$ is tightly closed in $\mathrm{gr}_{\mathcal{I}} R$, then J is tightly closed in R.*

Proof. Let $x \in J^*$. If $x \notin J$ then there exists N minimal such that $x \notin J + I_N$. In R we have $cx^q \in J^{[q]}$, and so $[cx^q] \in [J^{[q]}] = [J]^{[q]}$. In $\mathrm{gr}_{\mathcal{I}} R$, we either have $[c][x]^q = [cx^q]$ or $[c][x]^q = 0$. Either way, $[c][x]^q \in [J]^{[q]}$. Since $[c]$ is not in any minimal prime of $\mathrm{gr}_{\mathcal{I}} R$, this says that $[x] \in [J]^* = [J]$.

When we lift this relation to R using Lemma 4.1.6, we obtain $y \in J$ with $[x] = [y]$. Setting $z = x - y$, we see that $z \in J^*$ and the leading term of z has higher degree than the leading term of x. So $z \in J + I_N$, and since $y \in J$, this says that $x \in J + I_N$ as well. This contradicts our choice of N, and so we conclude that $x \in J + I_N$ for all N, whence $x \in J$. $\square$

Remark 4.1.8. Since $[J^{[q]}] \supseteq [J]^{[q]}$ always holds, we only need to check the other containment for this theorem. It is well known that if J is generated by $u_1, \ldots, u_d$ such that $[u_1], \ldots, [u_d]$ is a regular sequence in $\mathrm{gr}_{\mathcal{I}} R$, then $[J]$ is generated by $[u_1], \ldots, [u_d]$. But $J^{[q]} = (u_1^q, \ldots, u_d^q)$, $[u_i^q] = [u_i]^q$ for all i, and $[u_1]^q, \ldots, [u_d]^q$ is a regular sequence. So $[J^{[q]}]$ is generated by $[u_1]^q, \ldots, [u_d]^q$. That is, $[J^{[q]}] = [J]^{[q]}$.

The requirement that c be a test element whose leading form is not in any minimal prime of $\mathrm{gr}_{\mathcal{I}} R$ is also a fairly weak condition. All test elements will satisfy this when

$\mathrm{gr}_{\mathcal{I}}R$ is a domain, but such elements exist in much greater generality as well, as shown in Proposition 4.1.9 below.

Proposition 4.1.9. *Let $(R, \mathfrak{m})$ be an excellent local domain and $\mathcal{I}$ a filtration of R by $\mathfrak{m}$-primary ideals. If the test ideal τ of R has height at least 1, then there exists a test element c of R such that $[c]$ is not contained in any minimal prime of $\mathrm{gr}_{\mathcal{I}}R$.*

Proof. Let S be the Rees ring of R with respect to $\mathcal{I}$:

$$S = \cdots + \frac{1}{t^2}R + \frac{1}{t}R + R + I_1t + I_2t^2 + \cdots$$

It is clear that

$$\frac{S}{\frac{1}{t}S} \cong \mathrm{gr}_{\mathcal{I}}R.$$

We therefore have a short exact sequence

$$0 \to \frac{1}{t}S \to S \xrightarrow{\phi} \mathrm{gr}_{\mathcal{I}}R \to 0$$

We need to show that $[\tau]$ is not contained in any minimal prime $\mathfrak{p}$ of $\mathrm{gr}_{\mathcal{I}}R$. Both $[\tau]$ and $\mathfrak{p}$ are homogeneous ideals, so $\mathfrak{p} : \tau$ is homogeneous as well. Hence $[\tau] \subseteq \mathfrak{p}$ if and only if the same is true after localizing at the unique homogeneous maximal ideal of $\mathrm{gr}_{\mathcal{I}}R$. We may therefore assume that $\mathrm{gr}_{\mathcal{I}}R$ is local.

If $\mathcal{M}$ is the maximal ideal of S that maps onto $\mathfrak{m}$, then replacing S with its localization at $\mathcal{M}$ preserves the short exact sequence above. Since the map ϕ takes $\mathcal{J} = \cdots + \frac{1}{t}R + \tau + (\tau \cap I_1)t + (\tau \cap I_2)t^2 + \cdots$ to $[\tau]$, $[\tau]$ is not contained in any minimal prime of $\mathrm{gr}_{\mathcal{I}}R$ if and only if $\mathcal{J}$ is not contained in any minimal prime of $\frac{1}{t}S$ in S.

A ring and its associated graded ring have the same dimension, so $\dim\,(R/\tau) = \dim\,\mathrm{gr}_{\bar{\mathcal{I}}}(R/\tau) = \dim\,(\mathrm{gr}_{\mathcal{I}}R/[\tau])$. Since the height of τ is at least 1, $\dim\,(R/\tau) <$

dim R and so dim $(\mathrm{gr}_{\mathcal{I}}R/[\tau]) < \dim \mathrm{gr}_{\mathcal{I}}R$. Thus dim $(S/\mathcal{J}) < \dim (S/\frac{1}{t}S) = \dim S - 1$ (this last equality holds because S is a local catenary domain. So all chains of primes connecting 0 and $\mathcal{M}$ have length dim S. And since $1/t$ is a nonzerodivisor in S, all of its minimal primes have height 1).

If P is a minimal prime of $\frac{1}{t}S$, then since S is a local catenary domain, we have that ht $P + \dim S/P = \dim S$. That is, $\dim S/P = \dim S - 1$. So if $\mathcal{J} \subseteq P$, then $\dim S/\mathcal{J} = \dim S - 1$, contradicting our strict inequality above.

Hence $[\tau]$ is not contained in any minimal prime of $\mathrm{gr}_{\mathcal{I}}R$. □

Corollary 4.1.10. *Let (R, m) be a local ring. Suppose that $R/(f)$ is F-regular and that R contains a test element c such that $[c]$ is not contained in any minimal prime of $\mathrm{gr}_{(f)}R$. If there is a sequence J_t of irreducible m-primary ideals cofinal with the powers of m such that for all t and for all large $q = p^e$ $[J_t^{[q]}] = [J_t]^{[q]}$, then R is weakly F-regular.*

Proof. By [HH90, Theorem 8.8], it suffices to show that each J_t is tightly closed. We know

$$\mathrm{gr}_{(f)}R \cong \frac{R}{(f)}[z]$$

where z is an indeterminate. Hence $R/(f)$ is F-regular, then so is $\mathrm{gr}_{(f)}R$. Therefore $[J_t]$ is tightly closed for all t, whence J_t is as well by Theorem 4.1.7. □

CHAPTER 5

Tight Closure in Rings Defined by Binomial Ideals

This chapter describes how tight closure behaves in rings defined by binomial ideals. This builds on [ES96], which asserts that the associated primes and various other related ideals of binomial ideals are also binomial. In particular, if I is the image of a binomial ideal in $R = S/J$ where J is a binomial ideal, then we will show that I^* and I^F are binomial. Under certain circumstances, $\tau(R)$ and characteristic ideals are shown to be binomial as well. Finally, results of [Smi97] are presented showing that tight closure commutes with localization in certain kinds of rings, including rings defined by binomial ideals.

5.1 The Main Theorems

Throughout this chapter $S = K[x_1, \ldots, x_n]$ will be a polynomial ring in n indeterminates over an algebraically closed field K. Once we fix a monomial order for S, then every polynomial $g \in S$ has a unique monomial of highest order, which we will call the *leading monomial* of g. The coefficient of the leading monomial is called the *leading coefficient* of g, and the *leading term* of g is the product of the leading coefficient and the leading monomial of g.

A *binomial* in S is a polynomial with at most two terms, and a *binomial ideal* is an ideal generated by binomials. Throughout this chapter, $\mathfrak{p} \subset S$ will denote a binomial prime ideal. If J is a binomial ideal in S, $R = S/J$ and I is the image of a binomial ideal in R, then we will simply say that I is a binomial ideal in R.

Theorem 5.1.1. *Let $R = S/J$ where J is a binomial ideal in S. If $I \subseteq R$ is a binomial ideal in R, then I^* is binomial as well.*

Proof. Let N be the nilradical of R. Then N is binomial by [ES96, Theorem 3.1] and $N \subseteq I^*$, so we can kill N without affecting the issue of whether I^* is binomial or not. We may therefore assume that J is radical.

If $\mathfrak{p}_1, \ldots, \mathfrak{p}_a$ are the minimal primes of R, then

$$R' = \prod_{j=1}^{a} R'_j \text{ where } R_j = \frac{S}{\mathfrak{p}_j}.$$

R' is F-regular, since each factor R'_j is F-regular by Lemma 5.2.7 below, and products of F-regular rings are F-regular. Hence $I^* \subseteq (IR')^* \cap R = IR' \cap R \subseteq I^*$, and we obtain equality. So we only need to see that $IR'_j \cap R_j$ is binomial for all j. This is shown in Lemma 5.2.12 below. $\square$

Before showing that I^F is binomial, we make the following

Observation 5.1.2. Let $S \subseteq T$ be rings, and $J \subseteq I$ binomial ideals in T. Then

$$\frac{I}{J} \bigcap \frac{S}{J \cap S} \cong \frac{I \cap S}{J \cap S}$$

Proof. Note that the intersection on the left makes sense, as both I/J and $S/J \cap S$ are contained in T/J. Now if x is an element of the left hand side, then $x \in I \mod J$

and $x \in S \mod J$, whence $x \in I \cap S/J \cap S$. Conversely, it is clear that

$$\frac{I \cap S}{J \cap S} \subseteq \frac{I}{J} \text{ and } \frac{I \cap S}{J \cap S} \subseteq \frac{S}{J \cap S}$$

which finishes the proof. $\square$

Theorem 5.1.3. *Let $R = S/J$ for a radical binomial ideal J. If I is a binomial ideal of R, then I^F is also binomial.*

Proof. Let $\tilde{I}$ be the preimage of I in S. Since $I^F = IR^{1/q} \cap R$ for q sufficiently large, we have

$$I^F = I\left(\frac{S}{J}\right)^{1/q} \cap \frac{S}{J} = \frac{I'}{J^{1/q}} \cap \frac{S}{J} \text{ where } I' = \tilde{I}S^{1/q} + J^{1/q}.$$

Now $J^{1/q} \subseteq I'$ are binomial ideals in $S^{1/q}$ and $J^{1/q} \cap S = J$. So we can use Observation 5.1.2 to obtain

$$I^F = \frac{I'}{J^{1/q}} \cap \frac{S}{J^{1/q} \cap S} = \frac{I' \cap S}{J^{1/q} \cap S} = \frac{I' \cap S}{J}$$

$I' \cap S$ is binomial by Lemma 5.2.8, so I^F is binomial as well. $\square$

5.2 Decomposition Lemmas

In order to complete the proofs of our main theorems, we will need to delve into the structure of binomial ideals a bit. This will lead us to algebras generated by monomials, and this in turn will make it necessary to understand some fairly general semigroup decompositions.

For a semigroup $W \subseteq \mathbb{Z}^n$, define $K[W] = K[\{x^\mu\}_{\mu \in W}]$, the K-algebra generated by monomials with exponents in W. With this notation, $S = K[\mathbb{N}^n]$ and $K[x_1^{\pm 1}, \ldots, x_n^{\pm 1}] = K[\mathbb{Z}^n]$. It is easy to see that $K[V \oplus W] \cong K[V] \otimes_K K[W]$; more generally, if $I, J \subseteq K[\mathbb{Z}^n]$ are ideals such that I makes sense as an ideal of $K[V]$ and J makes sense as an ideal of $K[W]$, then $K[\mathbb{Z}^n]/(I+J) \cong (K[V]/I) \otimes_K (K[W]/J)$.

Our first goal is to show that $R = S/\mathfrak{p}$ is generated by monomials over K. We will do this by establishing that $K[\mathbb{Z}^n]/\mathfrak{p}$ is generated by monomials in $x_1^{\pm 1}, \ldots, x_n^{\pm 1}$, and then realizing R as a subring of this.

Notice that if $x_i \in \mathfrak{p}$ for some i, then we can eliminate x_i from both the top and bottom in R and thus assume that $x_i \notin \mathfrak{p}$ for any i. So without loss of generality, $\mathfrak{p}$ is a prime in $T = K[\mathbb{Z}^n]$ as well. Using [ES96, Theorem 2.1], we find that the exponents occurring in the generators of $\mathfrak{p}$ span a direct summand V of $\mathbb{Z}^n$. We obtain

$$T/\mathfrak{p} = K[V \oplus W]/\mathfrak{p} \cong (K[V]/\mathfrak{p}) \otimes_K K[W] \cong K \otimes_K K[W] \cong K[W]$$

Since these isomorphisms carry monomials to monomials, we conclude that $T/\mathfrak{p}$ is generated by monomials in $x_1^{\pm 1}, \ldots, x_n^{\pm 1}$.

T is just a localization of S, and $\mathfrak{p}T$ is a prime ideal of T. So $\mathfrak{p} = \mathfrak{p}T \cap S$, and we have $\phi : R \hookrightarrow K[W]$ defined by

$$S/\mathfrak{p} \hookrightarrow T/\mathfrak{p} = K[\mathbb{Z}^n]/\mathfrak{p} \xrightarrow{\sim} (K[V]/\mathfrak{p}) \otimes_K K[W] \xrightarrow{\sim} K[W].$$

Hence $S/\mathfrak{p}$ is a subring of $K[W]$. To determine this map explicitly, write $\mathbb{Z}^n = V \oplus W$ and find bases $\{f_1, \ldots, f_d\}$ for V and $\{f_{d+1}, \ldots, f_n\}$ for W. Here each $f_i = (f_{i1}, \ldots, f_{in}) \in \mathbb{Z}^n$. If $\{e_1, \ldots, e_n\}$ is the standard basis for $\mathbb{Z}^n$, then write $e_i = \sum \alpha_{ij} f_j$ for some $\alpha_{ij} \in \mathbb{Z}$. Now the map $\phi : R \hookrightarrow K[W]$ is clear: it sends x_i to the monomial whose exponents are defined by the change-of-basis matrix just described. Consequently, $R \cong K[\phi(x_1), \ldots, \phi(x_n)] = K[M] \subseteq K[W]$.

Although we have now shown that R is generated by monomials over K, some of the exponents may be negative. We will eventually need all of the exponents nonnegative in order to obtain the results used in Theorem 5.1.1. This will be the goal of the next couple of lemmas.

Definition 5.2.1. Given a semigroup $W \subseteq \mathbb{Z}^n$, $\mathbb{Q}(W)$ is the $\mathbb{Q}$-vector space spanned by the generators of W.

Lemma 5.2.2. *If $M \subseteq \mathbb{Z}^n$ is a semigroup, then $M = G \times V$ where G is a group and V is a semigroup in which no element has an inverse.*

Proof. Let G be the unique maximal subgroup of M and set $\mathbb{Q}(G)$ equal to the $\mathbb{Q}$-vector space generated by G. Identify $\mathbb{Q}(G)$ and $\mathbb{Z}^n$ with their images in $\mathbb{Q}^n$, and let $W = \mathbb{Q}(G) \cap \mathbb{Z}^n$. Then $M \subseteq \mathbb{Z}^n = W \oplus L$ for some vector space L. We want to examine $M \cap W$. This certainly contains G, and we can show that the converse also holds; if $m \in M \cap W$, then we need to show that its inverse $-m \in M \cap W$ as well. Given such an m, write $m = \sum q_i g_i$ for $q_i \in \mathbb{Q}$ and $g_i \in G$. Every element of $\mathbb{Q}(G)$ can be multiplied into G by a sufficiently large integer k, so $km \in G$, and hence $-km \in G \subseteq M$ as well. But then $-m = (k-1)m + (-km) \in M$, which means that $m \in G$ after all. So we have $G = M \cap W$.

Now define

$$\phi : M \overset{\iota}{\hookrightarrow} W \oplus L \overset{\pi_2}{\to} L$$

and let $V = \mathrm{im}\phi \subseteq L$. We may assume that $\mathbb{Q}(V)$ spans $\mathbb{Q}(L)$ as a $\mathbb{Q}$-vector space, and we need to show that $M \cong G \times V$. Define

$$\psi : M \to G \times V \text{ via } m \mapsto (\pi_1 \circ \iota(m), \pi_2 \circ \iota(m)).$$

ψ is injective as it is the composition of ι and $\pi_1 \times \pi_2$, which is injective componentwise. To see that it is surjective, first notice that $G \subseteq M$, so if $g \in G$, then

$g \mapsto (g, 0)$. Also, ϕ maps M onto V, so given $v \in V$, there is an $m \in M$ such that $\phi(m) = v$. Then $m \mapsto (\pi_1(m), v)$, whence $m - \pi_1(m) \mapsto (0, v)$ as desired. □

It would be nice to deal only with semigroups in which no element has an inverse. To do this, first define

$$H_i = \{(0, \ldots, 0, x_i, x_{i+1}, \ldots, x_n) \in \mathbb{Z}^n \mid x_i > 0\}$$

and set $H = H_1 \cup \ldots \cup H_n$. Then $\Lambda = G \cap H$ is a semigroup with no inverses, and we can make the following useful

Observation 5.2.3. With G and V as above, if G is generated as a $\mathbb{Z}$-module by $g_1, \ldots, g_d$ and $\Gamma = \prod g_i$, then $K[G \times V] = K[\Lambda \times V]_\Gamma$.

Proof.

$$K[G \times V] \cong K[G] \otimes_K K[V] = K[\Lambda]_\Gamma \otimes K[V] \cong K[\Lambda \times V]_\Gamma$$

□

In Theorem 5.1.1 we reduced to the case where R was normal. We need to determine what that corresponds to in terms of these monomial algebras. We will begin by studying the normalization of semigroups.

Definition 5.2.4. A semigroup M of monomials in $x_1, \ldots, x_n$ is *full* if whenever a, b, and c are monomials such that $ab = c$ and b, $c \in M$ then $a \in M$.

Definition 5.2.5. A semigroup of monomials M is *normal* if M is finitely generated, and whenever a, b, and c are monomials in M such that $ab^n = c^n$ for some positive integer n, then there is a monomial $d \in M$ such that $a = d^n$.

Notice that full semigroups of monomials are normal. If G is a group, then G is full, and hence normal. It is also clear that $(\prod V_i)' = \prod V_i'$. So $M' = (G \times V)' = G \times V'$. Since V' is a semigroup without inverses, we can use Observation 5.2.3 to obtain $K[M'] = K[\Lambda \times V']_\Gamma$, where Γ is the same product as before. Furthermore, Λ is normal: take $e, f, g \in \Lambda$ such that $e + kf = kg$. If $g - f \notin \Lambda$, then $f - g \in \Lambda$, whence $e + (k-1)(f-g) \in \Lambda$. But then $e + (k-1)(f-g) = e + kf - kg + g - f = g - f$ is in Λ as well, contradicting the fact that Λ contains no inverses. Consequently, $\Lambda \times V'$ is a normal semigroup.

$\Lambda \times V$ and $\Lambda \times V'$ are also semigroups in which no element has an inverse. We will show below that we can embed them in $\mathbb{Z}_+^r$ for some r. Since $\Lambda \times V \subseteq \Lambda \times V'$, it will suffice to embed $\Lambda \times V'$ this way. We record the following

Lemma 5.2.6. *Let W be a normal semigroup in which no element has an inverse. Then W embeds as a full semigroup in $\mathbb{Z}^r$ for some r with all entries positive, and this embedding preserves monomials.*

Proof. The idea is to mimic [Hoc72, Proposition 1, (2) $\implies$ (3)]. W is the span over $\mathbb{N}$ of a finite number of vectors, say $v_1, \ldots, v_d$. Identify $\mathbb{Q}(W)$ and $\mathbb{Q}_+(W)$ with their counterparts in $\mathbb{Q}^n$. Then $\mathbb{Q}_+(W)$ is a cone in $\mathbb{Q}^n$, and the convex hull of this cone is defined by a (possibly proper) subset of $v_1, \ldots, v_d$, say $v_1, \ldots, v_s$. Let $W^* = \mathrm{Hom}_{\mathbb{Q}}(\mathbb{Q}(W), \mathbb{Q})$ be the vector space dual of $\mathbb{Q}(W)$, and let

$$U = \{w^* \in W^* : w^*(v_j) \geq 0 \text{ for } 1 \leq j \leq s\}$$

Since U is a finite intersection of half-spaces in W^*, there are finitely many functionals $w_1^*, \ldots, w_r^* \in U$ such that U is the $\mathbb{Q}_+$-span of $w_1^*, \ldots, w_r^*$. It follows that

$$\mathbb{Q}_+(W) = \cap_{1 \leq j \leq r} \{v \in W : w_j^*(v) \geq 0\}.$$

By replacing the w_j^* by suitable positive integer multiples, we can assume that $w_j^*(v_i) \in \mathbb{Z}_+^r$ for all i and j. Let $T : W \to \mathbb{Q}^r$ be defined by setting $T = (w_1^*, \ldots, w_r^*)$. By construction, $T(W) \subset \mathbb{Z}_+^r$. To complete the proof, we need only show that T is one-to-one on W and that $T(W)$ is a full sub-semigroup of $\mathbb{Z}_+^r$. Suppose that $h \neq h'$ in W. From among $v_1, \ldots, v_s$, the vectors defining the hull of $\mathbb{Q}_+(W)$, choose a $\mathbb{Q}$-basis for $\mathbb{Q}(W)$, say $v_1, \ldots, v_t$. Let v_j^* be the linear functional on $\mathbb{Q}(W)$ obtained by projection on the v_j^{th} component (so $v_j^*(v_j) = 1$ and $v_j^*(v_i) = 0$ for $i \neq j$). These linear functionals are in U as they are nonnegative on all of $v_1, \ldots, v_d$. Since $h \neq h'$, there is an i such that $v_i^*(h) \neq v_i^*(h')$. We can write $v_i^* = \sum_j q_j w_j^*$, where the $q_j \in \mathbb{Q}_+$. Since $v_i^*(h) \neq v_i^*(h')$, we can choose j such that $w_j^*(h) \neq w_j^*(h')$, and so $T(h) \neq T(h')$.

It remains to show that $T(W)$ is full. Suppose that $T(h) - T(h') \in \mathbb{Z}_+^r$. Then each of $w_1, \ldots, w_r$ is nonnegative on $h - h'$, so that $h - h' \in \mathbb{Q}_+(W)$. Since $h, h' \in W$ and W is normal, $h - h' \in W$ as required. $\square$

Identifying $\Lambda \times V$ and $\Lambda \times V'$ with their completely-positive embeddings, it is now easy to see that $R' = K[M']$; it suffices to observe that $K[M']$ is normal, as then $K[M] \subseteq K[M'] \subseteq K[M]'$, whence the minimality of $K[M]'$ implies that $K[M]' = K[M']$. To see that $K[M']$ is normal, write $K[M'] \cong K[\Lambda \times V']_\Gamma$. Then $K[\Lambda \times V']$ is normal by [Hoc72, Proposition 1], and localization preserves normality.

Furthermore, [Hoc72, Lemma 1] shows that $K[\Lambda \times V']$ is a direct summand of $K[y_1, \ldots, y_r]$ for indeterminates $y_1, \ldots, y_r$, so $K[M'] = K[\Lambda \times V']_\Gamma$ is a direct summand of $K[y_1, \ldots, y_r]_\Gamma$. This proves the following

Lemma 5.2.7. *If $R = K[M] \subseteq K[y_1, \ldots, y_r]$, then $R' = K[M']$ is F-regular.*

This finishes one of the two lemmas required for Theorem 5.1.1. We next turn our attention to what happens to binomial ideals when they are contracted down to subalgebras.

Lemma 5.2.8. *If M is semigroup of monomials in the indeterminates $y_1, \ldots, y_r$, and $I \subseteq K[y_1, \ldots, y_r]$ is a binomial ideal, then $I \cap K[M]$ is also binomial.*

Before presenting the proof, we will need some definitions.

Definition 5.2.9. A set $\{g_1, \ldots, g_d\} \subseteq I$ is a *Gröbner basis* of I if the leading term of any element of I is divisible by one of the leading terms of $g_1, \ldots, g_d$.

Definition 5.2.10. A *reduced Gröbner basis* for an ideal I of S is a Gröbner basis G for I such that all of the elements of G are monic, and for all $g \in G$, no monomial of g lies in the ideal generated by the leading terms of the remaining elements of G.

Proposition 5.2.11. *Let I be a nonzero ideal of S. Then for a given monomial ordering, I has a unique reduced Gröbner basis.*

Proof. See [CLO92, Proposition 2.7.6] □

Proof of Lemma 5.2.8. This essentially follows from [ES96, Cor 1.7]. Here is an expanded proof.

Let $H \in I \cap K[M] = K[m_1, \ldots, m_d]$ and write $H = \sum f_i m_i$ for $f_i \in K[M]$. Denote the terms of f_i by f_{ij} (so $f_i = \sum_j f_{ij}$). We will show that each $f_{ij} m_i$ is either in I, or else there is an α in K and another term f_{kl} such that $f_{ij}m_i - \alpha f_{kl}m_k$ is in I.

Compute a Gröbner basis G for I. Given f_{ij}, if $f_{ij}m_i \in I$, then we are done. Otherwise, the normal form of $f_{ij}m_i$ with respect to G is again a term, and it must cancel against the normal forms of other terms $f_{kl}m_k$. Hence there must be some k, l and α such that the normal form of $\alpha f_{kl}m_k = f_{ij}m_i$, which means that $f_{ij}m_i - \alpha f_{kl}m_k$ is in I as desired. □

We are now ready to finish the proof of Theorem 5.1.1. All that remains to prove is the following lemma.

Lemma 5.2.12. *$IR' \cap R$ is binomial, where $R = S/\mathfrak{p} \cong K[M]$.*

Proof. Write $R' = K[M'] \cong K[\Lambda \times V']_\Gamma$ as above. If $\Gamma \in I$, then $IR' \cap R = R$ is trivially binomial. So we may assume that $\Gamma \notin I$. Since R' is a direct summand of $K[y_1, \ldots, y_r]$ where the y_i are indeterminates, we get

$$IK[\Lambda \times V'] \cap K[\Lambda \times V] = IK[y_1, \ldots, y_r] \cap K[\Lambda \times V]$$

and this is binomial by Lemma 5.2.8. Localizing at Γ yields

$$IR' \cap R = IK[\Lambda \times V']_\Gamma \cap K[\Lambda \times V]_\Gamma = (IK[\Lambda \times V'] \cap K[\Lambda \times V])_\Gamma,$$

which is binomial. □

5.3 Test and Characteristic Ideals

Before studying the test ideal, we first make the following

Definition 5.3.1. The *Conductor* $\mathfrak{C}$ of R' into R is $\{r \in R \mid rR' \subseteq R\}$. Equivalently, $\mathfrak{C}$ is the largest ideal of R which is also an ideal of R'.

Remark 5.3.2. If R is reduced, then $\mathfrak{C}$ always contains a nonzerodivisor.

Proof of Remark 5.3.2. Write $R = S/\cap_1^t \mathfrak{p}_j$. If $\mathfrak{C}$ does not contain a nonzerodivisor, then $\mathfrak{C} \subseteq \cup \mathfrak{p}_j$, and so by prime avoidance, $\mathfrak{C} \subseteq \mathfrak{p}_i$ some i. Without loss of generality, assume $\mathfrak{C} \subseteq \mathfrak{p}_1$.

Let $b \in \cap_2^t \mathfrak{p}_j - \mathfrak{p}_1$ and let c be a lifting of a nonzero element of the conductor of $S/\mathfrak{p}_1$. Then

$$cpR' = cp\left(\left(\frac{S}{\mathfrak{p}_1}\right)' \times \cdots \left(\frac{S}{\mathfrak{p}_t}\right)'\right) = \left(cp\left(\frac{S}{\mathfrak{p}_1}\right)'\right) \times \{0\} \times \cdots \{0\}$$

and since b is contained in every $\mathfrak{p}_j$ except $\mathfrak{p}_1$, we can rewrite this as

$$cpR' = p\left(\frac{S}{\mathfrak{p}_1}\right) \times p\left(\frac{S}{\mathfrak{p}_2}\right) \times \cdots p\left(\frac{S}{\mathfrak{p}_t}\right)$$

and this is the image of bR in $\prod(S/\mathfrak{p}_j)$. Furthermore, this is nonzero, since otherwise $cp(S/\mathfrak{p}_1)' = 0$, which would force either c or b into $\mathfrak{p}_1$, which is impossible. Therefore $cp \in \mathfrak{C} - \mathfrak{p}_1$, contradicting $\mathfrak{C} \subseteq \mathfrak{p}_1$. Hence $\mathfrak{C}$ is not contained in $\cup \mathfrak{p}_j$ and so $\mathfrak{C}$ contains a nonzerodivisor. $\square$

Now let $u \in \mathfrak{C}$ be a nonzerodivisor and assume $R = S/J$ as before. Then $(uR)^* = uR' \cap R = uR'$ which means that uR' is an ideal of R, call it $\mathfrak{a}$. If $c \in \tau(R)$, then $c\mathfrak{a} = c(uR)^* \subseteq uR$, and so $cuR' \subseteq uR$. But since u is a nonzerodivisor on R (and so on R' as well) we can cancel the u's to get $cR' \subseteq R$, whence $c \in \mathfrak{C}$.

Conversely, if $c \in \mathfrak{C}$, then for $I \subseteq R$, is suffices to show that $cI^* \subseteq I$ for all ideals I of R, as then $c \in \cap I : I^* = \tau(R)$. Since $I^* = IR' \cap R$,

$$cI^* = c(IR' \cap R) \subseteq I(cR') \cap R = IR = I.$$

This argument proves

Lemma 5.3.3. *If $R = S/J$ is reduced, then $\tau(R) = \mathfrak{C}$.*

Now to see that the test ideal is binomial, it suffices to show that the conductor is binomial. Even more is true when $\mathfrak{C}$ contains a monomial nonzerodivisor.

Theorem 5.3.4. *If $R = S/J$ is reduced and $\mathfrak{C}$ contains a monomial nonzerodivisor, then $\mathfrak{C}$ is a monomial ideal.*

Proof. Let $u \in \mathfrak{C}$ be a monomial nonzerodivisor. Then $\mathfrak{C} = R : R' = uR : uR'$, and this will be a monomial ideal if uR' is a monomial ideal in R. Writing $J = \cap \mathfrak{p}_i$ we get

$$R \subseteq \prod \frac{S}{\mathfrak{p}_i} = \prod K[M_i] \subseteq \prod K[M_i'] = R'$$

and for each i, $K[M_i'] = K[\mu_{i1}, \ldots, \mu_{id(i)}]$. Now uR' is generated by the $u\mu_{ij}$, so uR' really is a monomial ideal and we're done. □

If R is a domain, then this is a special case of how characteristic ideals behave. First we need a fact concerning the existence of automorphisms in rings of the form $K[M]$.

Lemma 5.3.5. *If $R = K[\mu_1, \ldots, \mu_d]$, then there is a K-algebra automorphism ϕ of R such that $\phi(\mu_i) = \lambda_i \mu_i$ and $\lambda_i \neq \prod_{i \neq j} \lambda_j^{q_j}$ for any $q_j \in \mathbb{Q}$.*

Proof. Pick $\lambda_1 \in K - \{0\}$. Since K is algebraically closed, it is uncountable, so $K_1 = K - \{\lambda_1^q\}_{q \in \mathbb{Q}}$ is also uncountable. Pick $\lambda_2 \in K_1 - \{0\}$ and set $K_2 = K_1 -$

$\{\lambda_1^{q_1}\lambda_2^{q_2}\}_{q_1,q_2\in\mathbb{Q}}$. Pick $\lambda_3 \in K_2 - \{0\}$ and continue in this manner until $\lambda_1, \ldots, \lambda_d$ have all been chosen.

Now suppose that there is an i such that $\lambda_i = \prod_{i\neq j} \lambda_j^{q_j}$ for some set of $q_j \in \mathbb{Q}$. Let t be the largest index for which q_j is nonzero. Note that $i < t$ by the construction of the λ_j's. Solving for λ_t, obtain

$$\lambda_t = (\lambda_i \prod_{i\neq j} \lambda_j^{-q_j})^{1/q_t} = \lambda_i^{1/q_t} \prod_{i\neq j} \lambda_j^{-q_j/q_t}$$

which contradicts our choice for λ_t. Hence $\lambda_1, \ldots, \lambda_d$ have the desired property. $\square$

Theorem 5.3.6. *If $R = S/\mathfrak{p}$ is a domain and I is a characteristic ideal of R, then I is monomial.*

Proof. Let $v = \sum_{i=1}^{t} \alpha_i v_i$ be an element of I, where each $\alpha_i \in K - \{0\}$ and v_i is a monomial in R. We need to show that each v_i belongs to I. Suppose for now that we can find an automorphism ψ of R such that $\psi(v_i) = \gamma_i v_i$ for some $\gamma_i \in K$ and such that if $i \neq j$ then $\gamma_i \neq \gamma_j$. Then we can use induction to prove that each v_i belongs to I.

If $t = 1$ then the assertion is certainly true. For $t \geq 2$, we have $v, \psi(v) \in I$ whence

$$v - \frac{1}{\gamma_1}\psi(v) = \sum_{i=2}^{t} \alpha_i(1 - \frac{\gamma_i}{\gamma_1})v_i \in I \text{ and each } \alpha_i(1 - \frac{\gamma_i}{\gamma_1}) \neq 0$$

by our choice of γ_i's, so we're done by induction.

To see that such an automorphism ψ exists, write $R = K[\mu_1, \ldots, \mu_d]$ and note that we can find a $\phi \in \mathrm{Aut}(R)$ as described by Lemma 5.3.5. Each v_i is a product of μ_i's, say

$$v_i = \prod_{j=1}^{d} \mu_j^{a_{ij}}.$$

Hence $\phi(v_i) = \gamma_i v_i$ where

$$\gamma_i = \prod_{j=1}^{d} \lambda_j^{a_{ij}}.$$

Set $\psi = \phi$. If $i \neq j$ but $\gamma_i = \gamma_j$, then there is a largest t such that $\lambda_t^{a_{it}} \neq \lambda_t^{a_{jt}}$. Solving for this λ_t, we obtain λ_t as a product of λ_i's with all the i's less than t. This contradicts our construction of ϕ in Lemma 5.3.5, and so we are done. $\square$

5.4 Localization

In this section we show that tight closure commutes with localization in certain kinds of rings, including rings defined by binomial ideals. Throughout this section, R will denote a Noetherian commutative ring of prime characteristic. The results of this section represent unpublished results of Karen E. Smith.

Proposition 5.4.1. *If tight closure commutes with localization in R/P for each minimal prime P of R, then tight closure commutes with localization in R.*

Remark 5.4.2. See [AHH93, Proposition 3.8] for a slightly more general version of this. For completeness, we provide a detailed proof here.

Proof. Let $z/1$ be in $(IR[U^{-1}])^*$. Then modulo each minimal prime P of R disjoint from U, the image of $z/1$ is in $(IR/P[U^{-1}])^* = (IR/P)^*R[U^{-1}]$. For each minimal prime P_i disjoint from U, there is some $u_i \in U$ such that $u_i z \in (IR/P_i)^*$. By setting $u = \prod u_i$, we see that $uz \in (IR/P)^*$ for all minimal primes of R disjoint from U. On the other hand, for each minimal prime Q not disjoint from U, we can find an element in $Q \cap U$. Let v be their product. Now $uvz \in (IR/P)^*$ for every minimal prime of R. This proves that $uvz \in I^*$ in R. We conclude that $z/1$ is in $I^*R[U^{-1}]$. So tight closure commutes with localization for R. $\square$

Proposition 5.4.3. *Let R be a domain and S a finite domain extension of R in which tight closure commutes with localization. Then tight closure commutes with localization in R.*

Proof. Let $z \in R$. If $z/1 \in (IR[U^{-1}])^*$, then the same equations expanded to $S[U^{-1}]$ show that $z/1 \in (IS[U^{-1}])^*$. Because tight closure commutes with localization in S, we have $z/1 \in (IS)^*S[U^{-1}]$. So there exists $u \in U$ such that $uz \in (IS)^*$. Thus $uz \in (IS)^* \cap R \subset I^*$. It follows that $z/1 \in I^*R[U^{-1}]$ and so tight closure commutes with localization in R. □

An immediate corollary is the following.

Corollary 5.4.4. *Let R be a ring with the following property: for each minimal prime P of R, the ring R/P has a finite extension that is F-regular. Then tight closure commutes with localization for R.*

Corollary 5.4.5. *Tight closure commutes with localization in rings of the form S/J where S is a polynomial ring and J is generated by binomials. That is, for all ideals I and all multiplicative systems U, $I^*R[U^{-1}] = (IR[U^{-1}])^*$.*

Proof. The minimal primes of a binomial ideal are binomial by [ES96, Theorem 7.1], and a polynomial ring modulo a prime binomial ideal is isomorphic to a subalgebra of a polynomial ring $K[t_1, \ldots, t_d]$ generated by monomials in the t_i's. The normalization of such an algebra is also a monomial subalgebra, and is the ring of invariants for the action of some torus on $K[t_1, \ldots, t_d]$. Thus its normalization is a direct summand of the regular ring $K[t_1, \ldots, t_d]$ (because tori are linearly reductive), and hence is

F-regular. So the hypothesis of the above corollary are satisfied, and therefore tight closure commutes with localization in such a ring. □

Remark 5.4.6. Proofs of the facts used above can be found earlier in this chapter for the most part. Another general reference for semi-group rings that also contains proofs of these facts is [BH93].

Remark 5.4.7. Although the proof just given is written with tight closure of ideals, essentially the same arguments work for arbitrary finitely generated modules.

CHAPTER 6

Tight Closure and Base Change

Given a domain A and a finitely generated A-algebra R, it is natural to ask how tight closure behaves under various base changes. For instance, if $A \to R$ has a property $\mathcal{P}$ on the generic fiber, then do almost all of the fibers have property $\mathcal{P}$ as well? One would expect an affirmative answer for "geometric" properties. Candidate properties include strongly F-regular, F-regular, weakly F-regular and F-rational.

In characteristic 0, all known examples either have one of these listed properties for almost all fibers, or else they do not have the property on any dense set. For example, Fedder shows in [Fed93, Example 2.9] that

$$\mathbb{Z} \to \frac{\mathbb{Z}[x, y, z]}{x^2 + y^3 + z^5}$$

is F-regular for all fibers over p when $p > 5$. In contrast, [HH94b, Theorem 7.12] ensures that

$$\mathbb{Z} \to \frac{\mathbb{Z}[x, y, z]}{x^2 + y^3 + z^7}$$

is not F-regular for any p, as $a(R) > 0$.

Not all properties behave consistently on almost all fibers, however. For example, one can define a ring R of characteristic p to be F-pure if the Frobenius endomorphism is pure. Then there are rings which are F-pure on a dense set and non F-pure on a dense set as well, such as

$$\mathbb{Z} \to \frac{\mathbb{Z}[x,y,z]}{x^3+y^3+z^3}$$

This is F-pure for $p \equiv 1 \mod 3$ and is not F-pure for $p \equiv 2 \mod 3$. (see [McD96, Lemma 4.3.2] for details).

6.1 Preliminaries

Discussion 6.1.1. Throughout the rest of this chapter, A will denote a Noetherian domain of characteristic p with fraction field L. L' will be an extension field of L, and R_A will be a finitely generated A-algebra. Since we are primarily interested here in which properties pass from the generic fiber to almost all fibers, we will assume that A is infinite; otherwise A is a field. To emphasize this setup, we will often say "Let $A \to R_A$ be as in (6.1.1)."

Given a prime ideal P of A, we write $\kappa = \kappa(P)$ for the field A_P/PA_P. Setting $R_\kappa = \kappa \otimes_A R$, we call the algebra $\kappa \to R_\kappa$ the *fiber* of $A \to R_A$ over P. If $P = (0)$, then κ is the fraction field L of A, and we refer to $L \to L \otimes R_A$ as the *generic fiber* of $A \to R_A$. If P is a maximal ideal of A, then $\kappa(P)$ is isomorphic to A/P and $\kappa \to R_\kappa$ is called a *closed fiber*. Given an element $c_A \in R_A$, define $c_\kappa = 1 \otimes c_A \in \kappa \otimes R_A$. Similarly, given an ideal I_A of R_A, I_κ will denote the image of I_A in R_κ. When a property holds for all fibers over primes not containing a certain element $a \in A^\circ$, then we say the property holds for *almost all* fibers.

Proposition 6.1.2. *Let $A \to R_A$ be as in (6.1.1). Let I and J be ideals of R_A.*

(i) If $I :_{R_A} J = \mathfrak{a}$ then $I_\kappa :_{R_\kappa} J_\kappa = \mathfrak{a}_\kappa$ for almost all κ.

(ii) If $I \cap J = \mathfrak{a}$, then $I_\kappa \cap J_\kappa = \mathfrak{a}_\kappa$ for almost all κ.

Proof. (i) and (ii) follow from [HH, Theorem 2.3.5] parts (g) and (f) respectively. □

Definition 6.1.3. Let notation be as in (6.1.1). Then $c_A \in R_A$ is a *universal test element* for $A \to R_A$ if after A is replaced by a suitable localization at one element of A° the following conditions are satisfied:

(i) $c_A \in R_A^\circ$

(ii) For every homomorphism $A \to \Lambda$ where Λ is a regular domain of positive characteristic, c_Λ is a completely stable test element for R_Λ.

Remark 6.1.4. In particular, if A is a domain of characteristic p and P is a maximal ideal of A, then $\Lambda = \kappa(P)$ is a field (hence a regular domain) of characteristic p. So if c_A is a universal test element, then c_κ is a test element for almost all closed fibers.

Definition 6.1.5. With $A \to R_A$ as in (6.1.1), we say that $A \to R_A$ is generically separable if $L' \otimes_A R_A$ is reduced for all field extensions L' of L. Note that this is equivalent to saying that $L \to R_L$ is separable.

6.2 Fibers

We are now ready to look at one of the potentially "geometric" properties mentioned before. In this section, we will focus on determining when strong F-regularity passes from the generic fiber to almost all fibers.

Definition 6.2.1. $A^\infty = \bigcup A^{1/q}$ and L^∞ is the fraction field of A^∞.

Before tackling the main theorem of this section, we first record the following separability lemma.

Lemma 6.2.2. *Let $L \to E$ be a separable field extension, where L is infinite and E is generated by $y_1, \ldots, y_n$ over L. If* tr.deg $E/L = s$, *then for all choices of α_{ij} in a Zariski open set of L^{sn}, E is a finite separable extension of $F = L(u_1, \ldots, u_s)$ where*

$$u_i = \sum_{j=1}^{n} \alpha_{ij} y_j \quad \textit{for} \quad 1 \leq i \leq s.$$

Proof. $\Omega_{E/L}$ is generated by $dy_1, \ldots, dy_n$ over E. After renumbering the y_i, we may assume that $dy_1, \ldots, dy_s$ are a basis for this E-vector space. There exists an $(n-s)\times s$ matrix B over E such that

$$\begin{pmatrix} dy_{s+1} \\ \vdots \\ dy_n \end{pmatrix} = B \begin{pmatrix} dy_1 \\ \vdots \\ dy_s \end{pmatrix}$$

Setting $u_i = \sum_{j=1}^{n} \alpha_{ij} y_j$ for $1 \leq i \leq s$ where the $\alpha_{ij} \in L$ are as yet undetermined, we see that $du_i = \sum_{j=1}^{n} \alpha_{ij} dy_j$. Hence

$$\begin{pmatrix} du_1 \\ \vdots \\ du_s \end{pmatrix} = C \begin{pmatrix} dy_1 \\ \vdots \\ dy_n \end{pmatrix} = C \begin{pmatrix} \mathrm{id}_{s\times s} \\ B \end{pmatrix} \begin{pmatrix} dy_1 \\ \vdots \\ dy_s \end{pmatrix}$$

where $C = (\alpha_{ij})$ is an $s \times n$ matrix over L.

The du_i span $\Omega_{E/L}$ if and only if

$$\operatorname{rank} C \begin{pmatrix} \mathrm{id}_{s\times s} \\ B \end{pmatrix} \geq s \tag{6.1}$$

and this holds precisely when some $s \times s$ minor of C does not vanish.

Hence (6.1) defines a Zariski open set U in L^{sn}, and it is nonempty as $C = \begin{pmatrix} \mathrm{id}_{s\times s} & 0 \end{pmatrix}$ is contained in U.

Choosing $(\alpha_{ij}) \in U$ and setting $F = L(u_1, \dots, u_s)$, we see that ϕ is an isomorphism in the exact sequence $E \otimes_F \Omega_{F/L} \xrightarrow{\phi} \Omega_{E/L} \to \Omega_{E/F} \to 0$, since the du_i generate $\Omega_{F/L}$. This means that $\Omega_{E/F} = 0$, whence $F \to E$ is separable ([Mat86, Theorem 26.9]).

But $\Omega_{E/F} = 0$ also ensures that tr.deg $E/F = 0$ by using the Mac Lane-Cartier equality in [GD67, Ch.0, §21.7], whence $F \to E$ is finite. $\square$

Lemma 6.2.3. *Let $A \to R_A$ be as in (6.1.1). If $A \to R_A$ is generically separable, then after localizing at a nonzero element of A, $A \to R_A$ factors through a polynomial extension T of A such that $T \to R_A$ is generically separable.*

Proof. Let $y_1, \dots, y_n$ generate R_A over A (and hence E over L). Using Lemma 6.2.2, choose $u_1, \dots, u_s \in R_L$ such that E is finite separable over $L(u_1, \dots, u_s)$. Then R_L is module-finite over $L[u_1, \dots, u_s]$.

After localizing A at one element, the u_i will be elements of R_A. After further localizing A at one element, we can force the equations of integral dependence for the y_i over $L[u_1, \dots, u_s]$ to hold over $T = A[u_1, \dots, u_s]$ as well. This T is the polynomial extension we seek. $\square$

Theorem 6.2.4. *Let $A \to R_A$ be as in (6.1.1) such that R_A is a domain. Assume that $A \to R_A$ is generically separable and that $L^\infty \otimes R_A$ is strongly F-regular. Then after localizing A at one element, R_{κ^∞} is strongly F-regular for all $\kappa = \kappa(\mu)$ with μ maximal.*

Proof. By Lemma 6.2.3, we can localize at a nonzero element of A and obtain a polynomial extension T of A such that the map factors $A \to T \to R_A$ with $T \to R_A$ generically separable.

Now $L^\infty \otimes_A R_A \cong R_A[L^\infty]$ via the natural map sending $l \otimes r$ to lr. Since $R_A[L^\infty]$ is a domain (hence reduced), the map $L^\infty \otimes R_A \hookrightarrow (L^\infty \otimes R_A)^{1/q}$ sending 1 to $c^{1/q}$ is an injection, and it splits because $L^\infty \otimes R_A$ is strongly F-regular.

Let ϕ denote this splitting map. Since $(R_{A^\infty})^{1/q} \subseteq (L^\infty \otimes_A R_A)^{1/q}$, we can restrict ϕ to $(R_{A^\infty})^{1/q}$. $(R_{A^\infty})^{1/q}$ is module-finite over R_{A^∞}, so we can write it as the span over R_{A^∞} of $\theta_1, \ldots, \theta_d$ for some d. For each i we get

$$\phi(\theta_i) = \sum_j \frac{a_{ij}}{b_{ij}} \otimes_A r_{ij} \in L^\infty \otimes_A R_A$$

where $a_{ij}, b_{ij} \in A^\infty$. For each b_{ij}, there exists q_{ij} a power of p such that $b'_{ij} = b_{ij}^{q_{ij}} \in A$. Set $b = \prod_{i,j} b'_{ij}$. After localizing A at b, the image of ϕ will be contained in R_{A^∞}. Furthermore, ϕ will still carry $c^{1/q}$ to 1, and so will be a splitting for the map $R_{A^\infty} \hookrightarrow (R_{A^\infty})^{1/q}$ where $1 \mapsto c^{1/q}$.

Any maximal ideal of A^∞ has the form μ^∞ where μ is a maximal ideal of A. After tensoring, we obtain that the map

$$\frac{A^\infty}{\mu^\infty} \otimes_{A^\infty} R_{A^\infty} \to \frac{A^\infty}{\mu^\infty} \otimes_{A^\infty} (R_{A^\infty})^{1/q} \text{ sending 1 to } 1 \otimes c^{1/q}$$

splits. But

$$\frac{A^\infty}{\mu^\infty} \otimes_{A^\infty} R_{A^\infty} = R_{\kappa^\infty} \text{ and } \frac{A^\infty}{\mu^\infty} \otimes_{A^\infty} (R_{A^\infty})^{1/q} \cong (R_{\kappa^\infty})^{1/q}$$

where this last isomorphism holds because we have

$$\frac{(R_{A^\infty})^{1/q}}{\mu^\infty} \xrightarrow{q^{\text{th}} \text{ power map}} \frac{R_{A^\infty}}{\mu^\infty} \cong R_{\kappa^\infty}$$

Taking q^{th} roots, we get that

$$(R_{\kappa^\infty})^{1/q} \cong \frac{(R_{A^\infty})^{1/q}}{\mu^\infty}$$

as desired.

Since $R_{\kappa^\infty} \to (R_{\kappa^\infty})^{1/q}$ sending 1 to $c^{1/q}$ splits for almost all κ^∞, we are done. □

6.3 Specialization

If tight closure is a geometric property, then when an ideal is tightly closed in the generic fiber, we would expect it to be tightly closed for almost all closed fibers as well. Before discussing this, we recall the following theorem of Hochster and Roberts.

Theorem 6.3.1 (Generic Freeness). *Let A be a Noetherian domain, R a finitely generated A-algebra, and S a finitely generated R-algebra. Let E be a finite S-module, $M \subset E$ an R-submodule which is finite over R, and $N \subset E$ an A-submodule which is finite over A. Set $D = E/(M + N)$. Then there exists a nonzero element $a \in A$ such that D_a is a free A_a-module.*

Proof. See [Mat86, Theorem 24.1]. □

Theorem 6.3.2. *Let $A \to R_A$ be as in (6.1.1) with R_A reduced. Assume that I is is primary to a prime ideal P, where P is maximal with respect to being disjoint from the fraction field L of A. If the image of I is tightly closed in the generic fiber $L \otimes R_A$, then I_κ is tightly closed for almost all fibers.*

Remark 6.3.3. The assumption that the image of I is tightly closed in the generic fiber $L \otimes R_A$ would be true, in particular, if tight closure were known to commute with localization.

Proof of Theorem 6.3.2. [HH, Corollary 2.4.8] guarantees that R_A has a universal test element c in this setup. Using Theorem 6.3.1, we can localize at a nonzero element of A and change notation to force R_A/I to be a free A-module.

Now define $V = \rho^{-1}(\tilde{V})$ where

$$\rho : R_L \to \frac{R_L}{I_L}$$

and $\tilde{V}$ is the socle of R_L/I_L).

For each i, also define $W_i = \{w \in R_L \mid cw^q \in I_L^{[q]} \text{ for all } q \leq p^i\}$. Then

$$V \cap (\bigcap_{i \geq 0} W_i) = I_L, \tag{6.2}$$

for if there exists w in this intersection, then $cw^q \in I_L^{[q]}$ for all q, which implies that $w \in (I_L)^* = I_L$. The other inclusion is immediate.

Now the intersection described in equation (6.2) maps to 0 in R_L/I_L. Since this is a finite-dimensional L-vector space, there is a fixed δ such that we can restrict the intersection to $i \leq \delta$ and still obtain 0 in the vector space. Hence

$$V \cap (\bigcap_{i \leq \delta} W_i) = I_L$$

Setting $q_\delta = p^\delta$, what we have just shown is that

$$(I_L :_{R_L} P_L) \cap (\bigcap_{q \leq q_\delta} (I_L :_{R_L} c^{1/q})) = I_L$$

After localizing A at one element, this holds over R_A:

$$(I : P) \cap \bigcap_{q \leq q_\delta} (I :_{R_A} c^{1/q}) = I$$

Then since this holds for almost all fibers, we obtain that

$$\{w \in \text{ Socle}\left(\frac{R_\kappa}{I_\kappa}\right) \text{ such that } cw^{q_\delta} \in {I_\kappa}^{[q_\delta]}\} = \{0\}$$

for almost all κ.

Hence ${I_\kappa}^* = I_\kappa$ for almost all fibers. $\square$

CHAPTER 7

Completion

If R is an excellent local domain, then completion is a smooth base change. As a consequence, tight closure will commute with completion whenever it commutes with localization (cf [HH94a]). Since the localization problem is still open, however, we discuss current progress in solving the (possibly) more tractable completion problem.

7.1 Preliminaries

Before launching into a discussion of completion, we first define a closely related operation called *plus closure.*

Definition 7.1.1. Let R be a domain. Then the *absolute integral closure* R^+ of R is the integral closure of R in an algebraic closure of its fraction field.

Definition 7.1.2. Given an ideal I of a domain R, the *plus closure* I^+ of I in R is given by $IR^+ \cap R$.

One advantage that plus closure has over tight closure is that it is easy to see that plus closure commutes with localization. So whenever the two closures agree, the localization problem becomes trivial. Given an arbitrary ideal I in a domain R of characteristic p, however, it is currently unknown whether $I^* = I^+$. As the next proposition shows, one containment always holds.

Proposition 7.1.3. *Let I be an ideal in a domain R of characteristic p. Then $I^+ \subseteq I^*$.*

Proof. See [Smi94, Proposition 2.14] for an explicit proof of this, or [HH94b, Theorem 5.22] for a more general theorem. □

If we restrict the class of ideals that we are looking at, then we obtain the reverse inclusion as well.

Theorem 7.1.4. *Let R be a locally excellent Noetherian domain of characteristic $p \neq 0$. If I is any parameter ideal of R, then $I^* = I^+$.*

Proof. See [Smi94, Theorem 5.1] □

7.2 Reductions

Our belief that tight closure commutes with completion is formulated in the following

Conjecture 7.2.1. *Let M be a finitely generated module over a Noetherian ring R, and let $N \subseteq M$ be an R-submodule of M. Then $N^{*\hat{}} = N^{\hat{}*}$.*

If we restrict our attention to the ideal case, then we obtain a suggestive sufficient condition for tight closure to commute with completion.

Proposition 7.2.2. *Let (R, m) be an excellent local ring and suppose that I an ideal of R. If for all t there exists N such that $(I + m^N)^* \subseteq I^* + m^t$, then $(I\hat{R})^* = I^*\hat{R}$.*

Proof. If $u \in (I\hat{R})^* - I^*\hat{R}$, then there exists t such that $u \notin I^*\hat{R} + m^t\hat{R}$. Choosing N as in the hypothesis, we get that $u \notin (I + m^N)^*\hat{R} = ((I + m^N)\hat{R})^*$ (this last equality holds, because localization commutes with tight closure for ideals primary to the maximal ideal). But this is clearly a contradiction, because then $u \in (I\hat{R})^* \subseteq ((I + m^N)\hat{R})^*$. Therefore $(I\hat{R})^* = I^*\hat{R}$. □

This proposition is suggestive because its main assumption is very similar to what we obtain when an ideal is plus closed. More precisely, we have

Theorem 7.2.3. *Let (R, m) be an excellent domain, and suppose that $I = I^+$. If R/I is reduced then $(I + m^N)^+ \subseteq I + m^{N-c}$ for some c. Equivalently,*

$$m^N \frac{R^+}{IR^+} \cap \frac{R}{I} \subseteq m^{N-c} \frac{R}{I}$$

Proof. Let S be a module-finite extension domain of R. Then we have $R \subseteq S \subset R^+$ and if we write $S = \sum R\theta_i$, then

$$\frac{S}{(IS)^+} \subseteq \frac{S}{IS} = \frac{R}{I} \otimes_R S = \sum \frac{R}{I}\theta_i$$

So $R/I \to S/(IS)^+$ is module-finite. Furthermore, this is an injection, as $I \subseteq (IS)^+ \cap R \subseteq I^+ = I$. This yields

$$\frac{R}{I} \overset{\text{m.f.}}{\subseteq} \frac{S}{(IS)^+} \subseteq \frac{R^+}{IR^+}$$

By Artin-Rees, we know that

$$W_S \overset{\text{def}}{=} \frac{S}{(IS)^+}[mz] \cap \frac{R}{I}[z] \text{ is finitely generated as an } \frac{R}{I}[mz]\text{-module.}$$

Then for $W_{R^+} \stackrel{\text{def}}{=} \dfrac{R^+}{IR^+}[mz] \cap \dfrac{R}{I}[z]$, we have $W_{R^+} = \displaystyle\bigcup_{R \to S \text{ a m.f. extn of domains}} W_S$ is integral over $(R/I)[mz]$ (because each W_S is integral over $(R/I)[mz]$).

Let N be the normalization of $(R/I)[mz]$. Then R excellent implies $(R/I)[mz]$ is excellent, and hence $(R/I)[mz] \to N$ is module-finite (this is where we are using that R/I is reduced and that R is excellent). Hence $(R/I)[mz] \to W_{R^+}$ is also module-finite. (The reasoning here is that when we take the field of fractions of $(R/I)[mz]$, we will be inverting the elements of m, and so we will actually get the field of fractions of $(R/I)[z]$. But W_{R^+}, by definition, lies in $(R/I)[z]$. So $W_{R^+} \subseteq N$, which forces W_{R^+} to be module-finite over $(R/I)[mz]$).

$$\text{Hence } W_{R^+} \ \left(\text{which is defined as} \bigoplus_i \left(m^i \frac{R^+}{IR^+} \cap \frac{R}{I}\right) z^i\right)$$

is finitely generated as an $(R/I)[mz]$-module, which means that all generators are of degree $\leq c$ for some c.

Therefore

$$m^{n+c}\frac{R^+}{IR^+} \cap \frac{R}{I} = m^n \left(m^c \frac{R^+}{IR^+} \cap \frac{R}{I}\right) \subset m^n \frac{R}{I} \quad \text{for all } n$$

and so we are done.

$\square$

Although the proof of Conjecture 7.2.1 remains elusive, various reductions are possible. To begin with, we may assume that R is a domain via the following proposition.

Proposition 7.2.4. *If tight closure does not commute with completion, then there exists a domain R and R-modules $N \subseteq M$ such that $N^{*\wedge} \subsetneq N^{\wedge *}$.*

Proof. If N is a submodule of M such that $N^{*\wedge} \subsetneq N^{\wedge *}$, then we can replace M with M/N^*. With this new module, we have $0^*_M = 0$, while $0^*_{\hat{M}} \ni u \neq 0$ for some element $u \in \hat{M}$. Let $\mathfrak{p}_1, \ldots, \mathfrak{p}_k$ be the minimal primes of R. Let W_i be the tight closure of 0 in $M/\mathfrak{p}_i M$ over $R/\mathfrak{p}_i$ pulled back to M. Then $\bigcap W_i = 0$ as tight closure can be checked modulo the minimal primes, and so $(\bigcap W_i)^\wedge = \bigcap \widehat{W}_i = 0$, which means that $u \notin \widehat{W}_i$ for some i, say $i = 1$. Replacing M by M/W_1 and R by $R/\mathfrak{p}_1$, we see that with this new setup, $0^*_M = 0$ and $0^*_{\hat{M}} \neq 0$. □

So this gives a counterexample to Conjecture 7.2.1 where R is a domain. Furthermore, this proposition reduces to the case where the zero submodule is tightly closed in M, but not in $\hat{M}$. This will be the case for the rest of the chapter.

Now that we have reduced to the case of a domain, there are several different directions in which we can try to make further reductions. The first direction that we will consider is reducing to a minimal counterexample. This will yield an interesting intersection property. The other type of reduction that we will consider is an analysis of the associated primes of our module.

Proposition 7.2.5. *Given a Noetherian ring R, if Conjecture 7.2.1 fails, then it fails for an R-module M with $N = 0$ such that tight closure commutes with completion for all nonzero submodules of M.*

Proof. If Conjecture 7.2.1 fails, then $\mathcal{F} = \{N \subseteq M \mid N^{*\wedge} \subsetneq N^{\wedge *}\}$ is a nonempty partially ordered set. Since M is Noetherian, $\mathcal{F}$ contains a maximal element V. V must be tightly closed, as otherwise V^* would be a larger element of $\mathcal{F}$. Replacing M by M/V yields $0^*_M = 0$ and $0^*_{\hat{M}} \neq 0$. Furthermore, given any nonzero submodule N of M, $N^{*\wedge} = N^{\wedge *}$. This has the additional effect of forcing tight closure to commute with completion for all submodules of M/N when $N \neq 0$. $\square$

We will call the module M obtained in Proposition 7.2.5 a *minimal counterexample* to Conjecture 7.2.1, since the conjecture holds for all submodules contained in M. Given such a minimal counterexample, we obtain the following intersection property.

Proposition 7.2.6. *Let M be a minimal counterexample to Conjecture 7.2.1. Then there exists a submodule W of M such that for any finite number of nonzero elements $v_1, \ldots, v_k$ of W, we have*

$$\bigcap_{i=1}^{k} (Rv_i)^* \cap W \neq 0$$

Proof. Since $0^*_{\hat{M}} \neq 0$, we can find $u \in 0^*_{\hat{M}}$ such that $\mathrm{Ann}_{\hat{R}} u = \mathfrak{q} \in \mathrm{Spec}\hat{R}$. Then $\mathfrak{p} = \mathfrak{q} \cap R$ is a prime ideal of R, and we can set $W = \mathrm{Ann}_M \mathfrak{p}$. For any nonzero element $v \in W$, $u \in (Rv)^{*\hat{}}$ or else $M/(Rv)^*$ would yield a new counterexample, contradicting the results of Proposition 7.2.5. So for any finite number of nonzero elements $v_1, \ldots, v_k$ of W,

$$\bigcap_{i=1}^{k} (Rv_i)^* \cap W \neq 0 \text{ and } u \in \bigcap_{i=1}^{k} (Rv_i)^{*\hat{}} \cap \hat{W} = \left(\bigcap_{i=1}^{k} (Rv_i)^* \right)^{\hat{}} \cap \hat{W}.$$

$\square$

The conclusion of this last proposition is actually a little stronger than stated, as we do not obtain just any submodule W for which this finite intersection is nonzero. Rather, W is the annihilator of a prime of R.

As mentioned earlier, instead of reducing to a minimal counterexample as above, we can instead focus on the associated primes of M. Our goal now is to reduce to the case where M has a single associated prime.

Proposition 7.2.7. *If M has no embedded primes, $0 = \cap N_j$ is the unique irreducible primary decomposition of (0), and $N_j^{*\wedge} = N_j^{\wedge *}$ for all j, then $0^*_{\hat{M}} = 0$.*

Proof. First note that if $0^*_{\hat{M}} \neq 0$, we can find $u \in 0^*_{\hat{M}}$ such that $\mathrm{Ann}_{\hat{R}} u = \mathfrak{q} \in \mathrm{Spec}\hat{R}$. Then $\mathfrak{p} = \mathfrak{q} \cap R$ is a prime ideal of R, and we can set $M_0 = \mathrm{Ann}_M \mathfrak{p}$. Now $u \in \mathrm{Ann}_{\hat{M}} \mathfrak{q} \subseteq \mathrm{Ann}_{\hat{M}} \mathfrak{p}\hat{R} = \hat{M}_0$, we get that $0^*_{M_0} \subseteq 0^*_M = 0$, and $0^*_{\hat{M}_0} \ni u \neq 0$. So replacing M with M_0, we may assume that $\mathfrak{p}M = 0$, which means that if $Q \in \mathrm{Ass}_R M$, then $\mathfrak{p} \subseteq Q$.

Now define $V = \ker(M \to M_\mathfrak{p})$, and write $V = \mathrm{Ann}_M a$ for some $a \in R - \mathfrak{p}$. If $x \in V^*$, then there exists $c \in R^0$ such that $cx^q \in V^{[q]}$ for q large. Hence $ca^q x^q \in a^q V^{[q]} = (aV)^{[q]} = 0$ for q large. Then $ax \in 0^* = 0$, and since x annihilates a, we get that $x \in V$. Thus V is tightly closed, and if $u \notin \hat{V}$, then once we replace M by M/V we will have $0^*_M = 0$ and $0^*_{\hat{M}} \neq 0$ since the image of u in this new M remains nonzero. But if $u \in \hat{V} = \mathrm{Ann}_{\hat{M}} a$ then $au = 0$, and hence $a \in \mathrm{Ann}_{\hat{R}} u = \mathfrak{q}$. This forces $a \in \mathfrak{q} \cap R = \mathfrak{p}$, contradicting our choice of $a \in R - \mathfrak{p}$. Hence $u \notin \hat{V}$, and we can replace M by M/V to obtain a new counterexample. With this new M, we still know that $\mathfrak{p}M = 0$, and we now have in addition that all the elements of $R - \mathfrak{p}$ are nonzerodivisors on M. This means that if $Q \in \mathrm{Ass} M$, then $Q \subseteq \mathfrak{p}$. We have now reduced to the case where $\mathrm{Ass} M = \{\mathfrak{p}\}$. Furthermore, since

$$\mathrm{Ass}_{\hat{R}} \hat{M} = \bigcup_{P \in \mathrm{Ass}_R M} \mathrm{Ass}\left(\frac{\hat{R}}{P\hat{R}}\right) = \mathrm{Ass}\left(\frac{\hat{R}}{\mathfrak{p}\hat{R}}\right),$$

if $Q \in \mathrm{Ass}_{\hat{R}} \hat{M}$, then $Q \cap R = \mathfrak{p}$. $\square$

This leads us to the case where $Ass(M) = \{\mathfrak{p}\}$, and we consider next what happens in the event that $\mathfrak{p}M = 0$. We have that M is torsion-free as an $R/\mathfrak{p}$-module, and so $M \hookrightarrow (R/\mathfrak{p})^t$ for some t. But 0 is tightly closed in $R/\mathfrak{p}$, since $\mathfrak{p}$ is prime. Hence 0 is tightly closed in $(R/\mathfrak{p})^t$ as well, as explained in Remark 7.2.9 below. But when R is excellent, R reduced implies that $\hat{R}$ is reduced as well. Therefore $\mathfrak{p}\hat{R}$ is radical, whence 0 is tightly closed in $\hat{R}/\mathfrak{p}\hat{R}$, and so tightly closed in $(\hat{R}/\mathfrak{p}\hat{R})^t$ as well. Since $\hat{M} \hookrightarrow (\hat{R}/\mathfrak{p}\hat{R})^t$, this means that 0 is tightly closed in $\hat{M}$, and so tight closure commutes with completion in this case. This proves the following

Theorem 7.2.8. *If $N \subseteq M$ are finitely generated modules over an excellent local ring R, $\mathfrak{p}M = 0$, and M is torsion-free as an $R/\mathfrak{p}$-module, then $N^{*\hat{}} = N^{\hat{}*}$.*

Remark 7.2.9. If 0 is tightly closed in R, then 0 is tightly closed in R^t as well. For if $u = (u_1, \ldots, u_t) \in 0^*_{R^t}$, then there exists $c \in R^0$ such that $cu^q = 0$ for q large. Hence $cu_i^q = 0$ for large q, which means that $u_i \in 0^*_R = 0$ for all i, and so $u = (0, \ldots, 0)$.

We can also use Theorem 3.2.2 to reduce from the module case to the ideal case.

Proposition 7.2.10. *Let (R, m) be an excellent local domain and M a finitely generated R-module. Suppose that N is a submodule of M such that $N^* = N$, but $\hat{N}^* \neq \hat{N}$. Then there is an ideal J_1 in a localization of a polynomial extension of R such that $J_1^* = J_1$ but $\hat{J_1}^* \neq \hat{J_1}$.*

Proof. By Remark 2.1.14, we may assume that M is a free R-module and write $M = \sum_{j=1}^n Re_j$. Write $N = \mathrm{Span}_R\{f_1, \ldots, f_d\}$ where each $f_i = \sum_{j=1}^n f_{ij}e_j$. Set $S_0 = R[x_1, \ldots, x_n]$, the symmetric algebra of M, and let J be the ideal of S_0 defined by the images $g_i = \sum_{j=1}^n f_{ij}x_j$ of the generators of N. Then $J^* \cap [S_0]_1 = J \cap [S_0]_1$ by Theorem 3.2.2.

Let $S = R[x_1, \dots, x_n]_{\mathfrak{m}}$, where $\mathfrak{m} = (m + (x_1, \dots, x_n))R[x_1, \dots, x_n]$. We may identify $\hat{S}$ with $\hat{R}[[x_1, \dots, x_n]]$, and $\hat{S} \supseteq T = \hat{R}[x_1, \dots, x_n]$.

Take $u \in \hat{N}^* - \hat{N}$. The image of u in $[T]_1$ is in $(JT)^* - JT$ by Theorem 3.2.2, and hence the image v of u in $\hat{S}$ is in $(J\hat{S})^*$. We shall show that $v \notin J\hat{S}$.

Since $J^{[q]}T$ is homogeneous for all q, its associated primes are homogeneous as well. Hence all of the associated primes of $J^{[q]}T$ are contained in the unique maximal homogeneous ideal $\hat{m} + (x_1, \dots, x_n)$ of T. Consequently, $T - (\hat{m} + (x_1, \dots, x_n))$ consists of nonzerodivisors on $J^{[q]}T$ for all q.

It follows that any homogeneous element h of $[S_0]_1$ in $(JS)^*$ is in the image of N in S_0. For if c is a test element of R, then $ch^q \in J^{[q]}S \implies ch^q \in J^{[q]}R[x_1, \dots, x_n] \implies h \in (JR[x_1, \dots, x_n])^* \cap R[x_1, \dots, x_n] \implies h \in \mathrm{im}(N)$ using Theorem 3.2.2 once again.

To conclude that $v \notin J\hat{S}$, it will suffice to show that $J\hat{S} \cap T = JT$. For if this holds and $v \in J\hat{S}$, then the image of u in T is in JT, which we have already seen is false. $\hat{S}$ is faithfully flat over $T_{\hat{m}+(x_1,\dots,x_n)}$. Therefore it is sufficient if $(JT)_{\hat{m}+(x_1,\dots,x_n)} \cap T = JT$. But this holds, as J is homogeneous and all elements of $T - (\hat{m} + (x_1, \dots, x_n))$ are nonzerodivisors on J.

Finally, set $J_1 = (JS)^*$ to obtain the conclusion.

$\square$

7.3 A graded example

Theorem 7.3.1. *Let R be a graded domain generated by 1-forms. Let $m = \oplus_{i>0} R_i$ and suppose that $I = I^*$ is a homogeneous ideal. Then $(I + m^t)^* = I + m^t$ for all t.*

Proof. $(I + m^t)^* \supseteq I + m^t$ is clear. For the other direction, let $I = (f_1, \dots, f_d)$. We may assume that $\deg f_i < t$ for all i, as otherwise $f_i \in m^t$. $I + m^t$ is homogeneous, so $(I + m^t)^*$ is as well, by [HH94b, Theorem 4.2b]. So suppose that $y \in (I + m^t)^*$ with y homogeneous and not in m^t. Then there exists $c \neq 0$ such that $cy^q \in I^{[q]} + (m^t)^{[q]}$ for all sufficiently large q, and we may assume that c is homogeneous as well (see [HH94b, Theorem 4.2e]).

For q large, $\deg(c) + q\deg(y) < qt$. Since $(m^t)^{[q]}$ consists of elements of degree $\geq qt$, $cy^q \in I^{[q]}$ and so $y \in I^* = I$. Hence $(I + m^t)^* \subseteq I + m^t$ and we obtain equality. $\square$

Remark 7.3.2. Using Proposition 7.2.2, we obtain that tight closure commutes with completion as well, if the ideal in question is homogeneous and the ring is a graded domain generated by 1-forms.

CHAPTER 8

Open Questions

There are many related questions which are currently open. The following is a short list organized by chapter.

Chapter 4

(1) If f is a nonzerodivisor and $R/(f)$ is (weakly) F-regular, then is the same true for R?

(2) Under what conditions does $[J^{[q]}] = [J]^{[q]}$?

(3) When does R have a test element c such that $[c]$ is not contained in any minimal prime of $\mathrm{gr}_{\mathcal{I}} R$?

Chapter 5

(4) If $R = S/J$ where S is a polynomial ring and J is binomial, then is τ_R binomial?

Chapter 6

(5) If I is tightly closed in the generic fiber, then is it tightly closed for almost all fibers?

(6) If R_{κ^∞} is strongly F-regular, then is $R_\kappa \subseteq R_{\kappa^\infty}$ strongly F-regular as well?

(7) If the generic fiber is strongly F-rational (see [Vél95]), then are almost all the fibers strongly F-rational as well?

(8) If the generic fiber is F-rational, F-regular, or weakly F-regular, then do almost all the fibers have this property as well?

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ABSTRACT

Tight Closure and Base Change

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We first introduce a very general method of reducing questions about the tight closure of submodules to questions about the tight closure of ideals. Next we examine when the weak F-regularity of the associated graded ring forces the ring itself to be weakly F-regular. This is then used to give a partial answer to the deformation question; under certain circumstances, we show that if $R/(f)$ is F-regular, then so is R.

In Chapter 5 we show that the tight closure and Frobenius closure of binomial ideals are again binomial. Situations where the test ideal and characteristic ideals in binomial rings are binomial are also discussed. A new result of Karen E. Smith is also presented showing that tight closure commutes with localization in rings defined by binomial ideals.

Given a domain A and an A-algebra R_A, we also examine when information on the tight closure of ideals in the generic fiber of R_A passes down to almost all fibers. This is followed by a discussion of how tight closure behaves under completion.