On the Varieties of Pairs of Matrices whose Product is Symmetric

by

Charles Christopher Mueller

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Doctoral Committee:

Professor Melvin Hochster, Chair
Professor Karen E. Smith
Professor J. Tobias Stafford
Associate Professor James P. Tappenden
Assistant Professor Neil M. Epstein

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TABLE OF CONTENTS

CHAPTER 1

Introduction

1.1 Background and History

1.1.1 Ideals defined by matrix equations: a brief history

The study of ideals generated by "generic" matrix equations has been, and continues to be, an important direction of research in both Algebra and Algebraic Geometry. Perhaps most well-known, and earliest studied, are the determinantal ideals and their associated rings and varieties of the same name. In 1916, Macaulay [28] proved that the ideal generated by maximal minors of a matrix of indeterminate entries are unmixed. These results were extended to arbitrary sized minors by Eagon [11] in his 1961 Ph.D. thesis. The following year, Eagon and Northcott [12] proved the perfection of determinantal ideals by constructing a finite free resolution and, in 1971, Hochster and Eagon [22] developed the method of principal radical systems to show that determinantal rings are Cohen-Macaulay normal domains. Bruns [3] computed the divisor class group and, with Herzog [4], calculated the **a**-invariant. In 1994, Hochster and Huneke [25] proved that determinantal rings have the property that all ideals are tightly closed in characteristic $p > 0$.

Geometrically, determinantal rings are related to the homogeneous coordinate rings of certain Schubert subvarieties of Grassmannians. Their geometric significance was realized at least as far back as the 1930's. T. Room [31], in the preface of his book "The geometry of determinantal loci," stated "...it appears that practically all the loci about the projective properties of which anything is known either are included in the class of determinantal loci, or are closely connected with it."

A generalization of determinantal rings and varieties called ladder determinantal rings and varieties was introduced in 1988 by Abhyankar [1] in his studies of singularities of Schubert varieties of flag manifolds. Through a series of papers ladder determinantal rings were shown to be a domain [30], Cohen-Macaulay [20], and normal [9]. Conca computed the divisor class group [7] and, together with Herzog [10], using tight closure techniques and a result of K.E. Smith [32], showed that ladder determinantal rings have rational singularities in characteristic 0.

Not all rings and varieties defined by matrix equations are defined in terms of determinants, nor have all the rings and varieties defined by matrix equations been as quick in yielding results as (ladder) determinantal rings. For example, the commuting variety, defined by pairs of commuting matrices of indeterminates X, Y such that $XY = YX$, was shown to be irreducible in 1955 by Motzkin and Taussky [29] and by Gerstenhaber [17], through different means in 1961. Yet it remains open as to whether the commuting variety is reduced or Cohen-Macaulay.

Our thesis investigates the rings and varieties defined by pairs of matrices of indeterminates whose product is symmetric, that is $XY = (XY)^{tr}$, and relies on facts already established for certain specific determinantal rings. Many of the methods and techniques used in this thesis were originally developed to study determinantal rings and ladder determinantal rings. Applying these techniques, we prove that these rings are Cohen-Macaulay and normal. Furthermore, we establish they have rational singularities in characteristic 0 and we conjecture that all of these rings are F-regular in characteristic $p > 0$.

1.1.2 Singularities

Over the complex numbers varieties are smooth when they are locally analytically isomorphic with an open set in $\mathbb{C}^n$. The varieties we study are typically not smooth, but we establish several important properties for the singularities. For example, we show that they are Cohen-Macaulay and normal (which implies the singular locus has codimension 2).

In characteristic 0, Hironaka [21] proved that one can always find a smooth variety that maps properly and birationally onto a given variety by a process called "blowing-up." The original variety has rational singularities if it is, roughly speaking, "cohomologically indistinguishable" from a smooth variety that one gets via blowing-up. We prove that the varieties we study have rational singularities in characteristic 0.

Singularities of a variety in characteristic 0 can often be studied by reduction to characteristic $p > 0$ methods, more specifically, by considering the action of Frobenius on the coordinate ring in characteristic $p > 0$.

1.1.3 Tight Closure

Although tight closure is primarily a notion for rings of characteristic p, it has strong connections with the study of the singularities of algebraic varieties over fields of characteristic zero. The tight closure of an ideal I is a possibly larger ideal, denoted by I^*, which is always contained in the integral closure of I and is frequently much smaller. Hochster and Huneke observed that for large classes of rings, the geometric notion of rational singularities is analogous to a certain property which could be formulated in terms of tight closure, namely the property that parameter ideals are tightly closed. These rings were eventually named F-rational rings by

Fedder and Watanabe [15]. K.E. Smith [32] proved that, in characteristic $p > 0$, F-rational rings are pseudo-rational in the sense of Lipman and Tessier [27] and, in characteristic 0, that F-rational type implies rational singularities. The converse, that rational singularities in characteristic 0 implies F-rational type, is a theorem of Hara [18]. Making use of Smith's result, Conca and Herzog [10] showed that ladder determinantal varieties have rational singularities.

The theory of tight closure also draws attention to rings in which all ideals are tightly closed, called weakly F-regular rings. Rings such that all localizations are weakly F-regular are called F-regular. These properties turn out to be significant. Hochster-Roberts [26] proved that rings of invariants of linear reductive groups acting on regular rings are Cohen-Macaulay and, in characteristic $p > 0$, the closely related result that direct summands of regular rings are Cohen-Macaulay. These results were forerunners of tight closure theory. Using tight closure techniques, the Hochster-Roberts theorem can actually be proved for the much larger class of weakly F-regular rings [23]. This, in turn, is the characteristic p analogue of Boutot's theorem [2] which states that, over an algebraically closed field of characteristic 0, direct summands of polynomial rings with rational singularities, have rational singularities.

1.2 Notations, Conventions and Known Results

By an $\mathbb{N}$-graded ring, R, we shall always mean a ring $R = \oplus_{n\geq 0} R_n$ finitely generated over a field $R_0 = K$. We shall denote by m the homogeneous (irrelevant) maximal ideal of R. For a graded R-module M, we shall denote by $[M]_i$ the i^{th} graded piece of M.

By a *system of parameters* for a $\mathbb{N}$-graded ring R, we shall mean a sequence of homogeneous elements of R whose images form a system of parameters for R_m.

We say that $I = (x_1, \ldots, x_n)$, an ideal of R, is a *parameter ideal* if the images of $x_1, \ldots, x_n$ form part of a system of parameters in R_P, for every prime ideal P containing I.

Let R be a commutative Noetherian ring of characteristic $p > 0$, let I be an ideal of R and let R^0 denote the complement of the union of the minimal prime ideals of R. The *tight closure* I^* of I is the set of elements $z \in R$ for which there exists $c \in R^0$ such that $cz^{p^e} \in I^{[p^e]}$ for all $e >> 0$, where $I^{[p^e]}$ denotes the ideal generated by all elements $a^{p^e}, a \in I$. If $I = I^*$, we say I is *tightly closed.*

A ring R, of characteristic $p > 0$, is *weakly F-regular* if every ideal of R is tightly closed and is *F-regular* if every localization is weakly F-regular. R is called *F-rational* if every parameter ideal is tightly closed.

We now summarize some known results:

Theorem 1.1. *Let R be a Noetherian ring of characteristic $p > 0$.*

a) R is regular $\Rightarrow$ R is F-regular $\Rightarrow$ R is weakly F-regular $\Rightarrow$ R is F-rational $\Rightarrow$ R is normal and Cohen-Macaulay (provided R is a homomorphic image of a Cohen-Macaulay ring). ([24], 3.4, 4.2)

b) For Gorenstein rings, F-rationality and F-regularity are equivalent ([24], 4.7).

c) Suppose that R is, in addition, $\mathbb{N}$-graded over a perfect field with graded maximal ideal m. Then R is F-rational if and only if R_m is F-rational ([25], 1.4).

1.3 Outline of Main Results

Our main thesis results center on the algebraic varieties defined by pairs of matrices whose product is symmetric. More formally, let $A = (a_{ij})$ and $B = (b_{ij})$ be matrices of algebraically independent indeterminates, over a field K, of size $n \times s$, $s \times n$, respectively. For any matrix X, let $I_t(X)$ denote the ideal generated by the t-sized minors of X. Let $K[A,B]$ be the polynomial ring in the $2ns$ entries of A and B. Throughout this thesis we denote $I^{n,s} = I_1(AB - (AB)^{tr})$, the ideal of $K[A,B]$ generated by the entries of the $n \times n$ skew-symmetric matrix $AB - (AB)^{tr}$. Let $V^{n,s} = V(I^{n,s})$ be the algebraic set in $\mathbb{A}_K^{2ns}$ corresponding to $I^{n,s}$ and let $R^{n,s} = K[A,B]/I^{n,s}$, which will be shown to be the coordinate ring associated to the variety $V^{n,s}$ (this would be clear if we knew that $R^{n,s}$ is reduced; we shall prove that it is, in fact, a domain.)

Our main results are that, for all $n, s \geq 1$, $R^{n,s}$ is a Cohen-Macaulay (5.2), normal domain (5.11) and $V^{n,s}$ has rational singularities in characteristic 0 7.9.

In chapter 2 we establish that $R^{n,s}$ is a normal Cohen-Macaulay domain in several cases, which provide "base" cases for proofs by induction. We also show that $V^{n,s}$ is an irreducible algebraic set for all $n, s \geq 1$. Thus we reduce the problem of showing that $R^{n,s}$ is a domain to showing that $I^{n,s}$ is a radical ideal.

An algebraic set X is often specified as the points $V(I)$ where the elements of the ideal I vanish. It is typically difficult to prove that I contains all polynomials that vanish on X (that I is radical). Our key results on Cohen-Macaulayness and normality are proved by establishing facts like this for a much larger class of auxiliary ideals. In a number of instances it requires great effort even to show that the sets arising are irreducible.

The focus of chapter 3 is to establish the irreducibility of the corresponding algebraic sets of certain auxiliary ideals. We use these facts, together with the method of principal radical systems, to show, in chapter 4, that $I^{n,s}$ is a radical ideal for all $n, s \geq 1$. It then follows that $R^{n,s}$ is a domain for all $n, s \geq 1$. Using the family of auxiliary ideals, the main result of chapter 5 is that $R^{n,s}$ is a Cohen-Macaulay normal domain for all $n, s \geq 1$. We also compute the divisor class group of $R^{n,s}$.

Hara and Watanabe in [19] give a criterion for F-rationality for Cohen-Macaulay rings which are $\mathbb{N}$-graded algebras finitely generated over $R_0 = K$, a field of characteristic $p > 0$, which leads us to study the Hilbert function and related invariants, such as the **a**-invariant. The **a**-invariant only depends on the Hilbert polynomial of $R^{n,s}$. Since these rings are Cohen-Macaulay, we can recover this polynomial from the Hilbert polynomial of the ring modulo a homogeneous system of parameters. In chapter 6, we explicitly construct a linear homogeneous system of parameters for $R^{n,s}$ for all $n, s \geq 1$ and are able to show that the **a**-invariant is negative for all $n, s \geq 1$. We compute its value for $n \leq s+1$ and conjecture its value for $n > s+1$.

At this point, proving that $R^{n,s}$ is F-rational has been reduced to showing that $R^{n,s}$ is F-injective. Using Gröbner basis techniques and a result Conca and Herzog [10] we further reduce F-rationality to showing that, for a certain monomial order, $K[A, B]/\mathrm{in}(I^{n,s})$ is Cohen-Macaulay and $\mathrm{in}(I^{n,s})$ is generated by square-free monomials, which we prove in the $n \leq s+1$ case. A result of K.E. Smith [32] implies that $V^{n,s}$ has rational singularities in characteristic 0 for $n \leq s+1$. Using these results and a criterion due to Kempf [16], we establish that $V^{n,s}$ has rational singularities in characteristic 0 for all n, s.

CHAPTER 2

Varieties and Complete Intersections

2.1 Pairs of Matrices whose Product is Symmetric

The focus of this chapter is to establish that $R^{n,s}$ is a Cohen-Macaulay normal domain in several cases. These facts will serve as "base cases" when proving that $R^{n,s}$ is a Cohen-Macaulay normal domain for all $n, s \geq 1$. We first establish that $V^{n,s}$ is an irreducible algebraic set for all $n, s \geq 1$, and then show that $R^{n,s}$ is a complete intersection for $n \leq s+1$ and is a unique factorization domain for $n \leq s$.

Remark 2.1. Note that if L denotes the algebraic closure of K then, since L is faithfully flat over K, we have that $R^{n,s} \subseteq L \otimes_K R^{n,s}$. Thus, we may assume K is algebraically closed when showing $R^{n,s}$ is a domain or reduced. What is more, we may assume K is algebraically closed when proving that $R^{n,s}$ is Cohen-Macaulay, since the Cohen-Macaulay property for finitely generated K-algebras is stable under change of field. Unless otherwise stated, we will assume that K is algebraically closed henceforth.

2.1.1 The Base Cases

When $n = 1$ and $s \geq 1$ the product of the matrices is a 1×1 (symmetric) matrix and $I^{n,s}$ is the zero ideal.

$$\begin{pmatrix} a_{11} & \cdots & a_{1s} \end{pmatrix} \begin{pmatrix} b_{11} \\ \vdots \\ b_{s1} \end{pmatrix} = \begin{pmatrix} a_{11}b_{11} + \ldots + a_{1s}b_{s1} \end{pmatrix}$$

Thus $R^{1,s}$ is isomorphic to $K[A, B]$, a regular domain of dimension $2s$.

When $n > 1$ and $s = 1$, our matrices have the form:

$$\begin{pmatrix} a_{11} \\ \vdots \\ a_{n1} \end{pmatrix}, \begin{pmatrix} b_{11} & \cdots & b_{1n} \end{pmatrix}$$

$I^{n,s}$ is generated by the $\binom{n}{2}$ upper (equivalently, lower) triangular entries of the alternating $n \times n$ matrix, $AB - (AB)^{tr}$. These entries are given by the polynomials: for $1 \leq i < j \leq n$

$$a_{i1}b_{1j} - a_{j1}b_{1i}$$

We note that, in more general cases, the $\binom{n}{2}$ polynomials generating $I^{n,s}$ may be thought of as coming from the difference of dot products: for $1 \leq i < j \leq n$,

$$\mathtt{R}_i^A \cdot \mathtt{C}_j^B - \mathtt{R}_j^A \cdot \mathtt{C}_i^B$$

where $\mathtt{R}_i^A$ denotes the i^{th} row of A and $\mathtt{C}_j^B$ denotes the j^{th} column of B.

The defining polynomials for $I^{n,s}$ (in the $s = 1$ case) are the same as the 2×2 minors of the $2 \times s$ matrix, C, whose rows are A^{tr} and B. It follows that $R^{n,1}$ is isomorphic to $K[C]/I_2(C)$, a determinantal ring, which is known [22] to be a normal Cohen-Macaulay domain of dimension $n + 1$.

2.2 Group Actions

We have a right action by K-algebra automorphisms on $K[A, B]$ by $GL_n(K) \times GL_s(K)$ given by:

$$(\gamma, \delta) : (A, B) \mapsto (\gamma A \delta^{-1}, \delta B \gamma^{tr})$$

where the notation means that the entries of A are mapped to the entries of $\gamma A \delta^{-1}$ and the entries of B are mapped to the entries of $\delta B \gamma^{tr}$. This is a right action on $K[A, B]$ and $I^{n,s}$ is stable since $AB - (AB)^{tr}$ maps to $\gamma(AB - (AB)^{tr})\gamma^{tr}$. Thus induces a right action on $R^{n,s}$.

The induced action on $V^{n,s}$ sends

$$(\alpha, \beta) \mapsto (\gamma \alpha \delta^{-1}, \delta \beta \gamma^{tr})$$

and is a left action on $V^{n,s}$.

This action preserves the pair $(\mathrm{rank}(A), \mathrm{rank}(B))$. One can do row operations on A, with corresponding column operations on B, and conversely.

Note that we also have an action by $\mathbb{Z}_2 = \{\bar{0}, \bar{1}\}$ on $R^{n,s}$ and $V^{n,s}$ given by:

$$\bar{1} : (A, B) \mapsto (B^{tr}, A^{tr})$$

$$\bar{1} : (\alpha, \beta) \mapsto (\beta^{tr}, \alpha^{tr})$$

2.3 Localization Properties

We want to study what happens when we localize $R^{n,s}$ at any maximal minor of A or B.

Proposition 2.2. *Let Δ be any maximal minor of A or B and let M denote the submatrix associated to Δ, i.e., $det(M) = \Delta$. Then $R^{n,s}_{\Delta}$ is a regular domain of dimension $2ns - \binom{n}{2}$ for $n \leq s$ and of dimension $ns + \binom{s+1}{2}$ for $n \geq s$.*

If $R^{n,s} \subseteq S$, we can let $(\gamma, \delta) \in \mathrm{GL}_n(S) \times \mathrm{GL}_s(S)$ act on the pair (A, B) by

$$(\gamma, \delta) : (A, B) \mapsto (\gamma A \delta^{-1}, \delta B \gamma^{tr}) = (A', B')$$

producing a new pair of matrices over S.

$$I_1(AB - (AB)^{tr})S = I_1(A'B' - (A'B')^{tr})S$$

because $A'B' - (A'B')^{tr} = \gamma(AB - (AB)^{tr})\gamma^{tr}$ and $AB - (AB)^{tr} = \gamma^{-1}(A'B' - (A'B'))^{tr}(\gamma^{tr})^{-1}$.

Proof: Let $m = \min\{n, s\}$. By the $\mathbb{Z}_2$-action we may assume, without loss of generality, Δ is a minor of A. We consider two cases: $n \geq s$ and $n < s$. In either case, by permuting rows and columns, we may further assume that M is the $m \times m$ submatrix of A in the upper left-hand corner.

Case 1: $n \geq s$

A has the form:

$$\begin{pmatrix} M \\ \hline A_0 \end{pmatrix}$$

When $n = s$ we take A_0 to be the empty set.

$(M^{-1}, I) \in GL_n(R^{n,s}_\Delta) \times GL_s(R^{n,s}_\Delta)$ acts on

$$(A, B) = (\begin{pmatrix} M \\ \hline A_0 \end{pmatrix}, \left(\begin{array}{c|c} B_0 & B_1 \end{array} \right))$$

to give

$$(A', B') = (\begin{pmatrix} \mathrm{I} \\ \hline M^{-1}A_0 \end{pmatrix}, \left(\begin{array}{c|c} B_0(M^{-1})^{tr} & B_1(M^{-1})^{tr} \end{array} \right))$$

Then entries of $A_0' = M^{-1}A_0$, $B_0' = B_0(M^{-1})^{tr}$, $B_1' = B_1(M^{-1})^{tr}$ and M along with $1/\Delta$ generate $K[A, B][1/\Delta]$ and so the entries of A_0', B_0', B_1' and M are all algebraically independent over K and $K[A, B]_\Delta = K[A_0', B_0', B_1', M]_\Delta$.

The symmetry condition implies that $R^{n,s}_\Delta = K[A,B]_\Delta/I^{n,s}$ is isomorphic to

$$K[A_0', B_0', B_1', M]_\Delta/(I_1(B_0' - B_0'^{tr}) + I_1(B_1' - (A_1'B_0')^{tr}) + I_1(A_1'B_1' - (A_1'B_1')^{tr}))$$

After substituting $(A_1'B_0')^{tr}$ for B_1', the relation $A_1'B_1' = (A_1'B_1')^{tr}$ is trivially satisfied using the relation $B_0' = B_0'^{tr}$. Thus $R^{n,s}_\Delta$ is isomorphic to

$$K[M, A_0, B_0][1/\Delta]/(I_1(B_0 - B_0^{tr}))$$

which is the localization of a polynomial ring, hence a regular domain. We note that the dimension of $R^{n,s}_\Delta = ns + \binom{s+1}{2}$ when $n \geq s$.

Case 2: $n \leq s$

A has the form:

$$\begin{pmatrix} M | A_0 \end{pmatrix}$$

When $n = s$ we take A_0 to be the empty set.

Let

$$(\mathrm{I}\ ,\ \begin{array}{c c} & \begin{array}{cc} n & s\text{-}n \end{array} \\ \begin{array}{c} n \\ s\text{-}n \end{array} & \begin{pmatrix} M & A_0 \\ 0 & \mathrm{I} \end{pmatrix} \end{array}) \in GL_n(R^{n,s}_\Delta) \times GL_s(R^{n,s}_\Delta)$$

act on

$$(A, B) = (\begin{pmatrix} M & \big| & A_0 \end{pmatrix}, \begin{pmatrix} B_0 \\ \hline B_1 \end{pmatrix})$$

Note that δ is given, but

$$\delta^{-1} = \begin{array}{c c} & \begin{array}{cc} n & s\text{-}n \end{array} \\ \begin{array}{c} n \\ s\text{-}n \end{array} & \begin{pmatrix} M^{-1} & -M^{-1}A_0 \\ 0 & \mathrm{I} \end{pmatrix} \end{array}$$

acts (on the right) on A and gives

$$(A', B') = (\left(\begin{array}{c|c} \mathrm{I} & 0 \end{array}\right), \left(\begin{array}{c} MB_0 + A_0B_1 \\ \hline B_1 \end{array}\right))$$

Then entries of $A'_0 = A_0$, $B'_0 = MB_0 + A_0B_1$, $B'_1 = B_1$ and M along with $1/\Delta$ generate $K[A, B][1/\Delta]$ and so the entries of A'_0, B'_0, B'_1 and M are all algebraically independent over K and $K[A, B]_\Delta = K[A'_0, B'_0, B'_1, M]_\Delta$.

The symmetry condition implies that $R^{n,s}_\Delta = K[A, B]_\Delta / I^{n,s}$ is isomorphic to

$$K[A'_0, B'_0, B'_1, M]_\Delta / (I_1(B'_1 - B'^{tr}_1))$$

a localization of a polynomial ring, hence a regular domain. We note that the dimension of $R^{n,s}_\Delta = 2ns - \binom{n}{2}$ when $n \leq s$. $\square$

We also want to remark on what happens when we localize $R^{n,s}$ at an entry of A or B. By permuting variables we may assume, without loss of generality, that the entry is a_{11}. The ring $R^{n,s}[1/a_{11}]$ is isomorphic to the localization of a polynomial ring in $2s+n-1$ indeterminates over $R^{n-1,s-1}$. In fact, following the above argument, it is easy to see that

Remark 2.3. $R^{n,s}[1/a_{11}] \cong R^{n-1,s-1}[a_{11}, \ldots, a_{1s}, a_{21}, \ldots, a_{n1}, b_{11}, \ldots, b_{s1}][1/a_{11}]$

2.4 On Irreducibility

Theorem 2.4. *$V^{n,s}$ is an irreducible algebraic set for all $n, s \geq 1$.*

$V^{n,s}$ is irreducible if and only if the radical of $I^{n,s}$ has a unique minimal prime. Let Δ denote a maximal minor of A or B. There is a one-to-one correspondence between primes of $R^{n,s}_\Delta$ and primes of $R^{n,s}$ not containing Δ. Let $P_\Delta (= \bigcup_t \mathrm{Ann}_{R^{n,s}} \Delta^t)$ be the unique minimal prime of $R^{n,s}$ not containing Δ. To complete the proof of 2.4, it suffices to show that $R^{n,s}$ has a unique minimal prime ideal. First we show:

Proposition 2.5. *For any maximal minor, Δ', of A or B, $P_\Delta = P_{\Delta'}$.*

This amounts to showing that no maximal minor of A or B is mapped to zero in $R^{n,s}_\Delta$.

*P*roof: For $n \geq s$: we may again assume A and B have the following forms

$$\left(\begin{array}{c} \mathrm{I} \\ \hline A_1 \end{array}\right), \left(\begin{array}{c|c} B_0 & B_0 A_1^{tr} \end{array}\right)$$

where B_0 is a symmetric $s \times s$ matrix and I is the $s \times s$ identity matrix.

If Δ' is a minor of A which consists of h rows of I, then Δ' is (up to sign) an $(s-h)$-sized minor of A_1 which does not vanish modulo $I_1(B_0 - B_0^{tr})$.

Suppose Δ' is a minor of B. We can factor B so that it has the form

$$B_0 \left(\begin{array}{c|c} \mathrm{I} & A_1^{tr} \end{array}\right)$$

Specializing B_0 to the identity matrix, we have that Δ' does not vanish in $R^{n,s}_\Delta$.

For $s \geq n$: We assume our matrices have the form

$$\left(\begin{array}{c|c} \mathrm{I} & 0 \end{array}\right), \left(\begin{array}{c} B_0 \\ \hline B_1 \end{array}\right)$$

We only need to see that the n-sized minor of B_0 is not mapped to zero. But specializing B_0 to the identity matrix gives a minor which does not vanish in $R^{n,s}_\Delta$.$\square$

So, for any maximal minor, Δ, of A or B, there exists a unique minimal prime, P_Δ, of $R^{n,s}$ not containing Δ, and all choices of Δ give the same minimal prime of $R^{n,s}$, say P. Thus, P is killed by a power of the ideal generated by the maximal minors of both A and B, $I_m(A) + I_m(B)$ with $m =\min\{n, s\}$, and

Remark 2.6. If there is another minimal prime of $R^{n,s}$, it must contain $I_m(A)+I_m(B)$.

Proposition 2.7. *The algebraic set corresponding to P contains $V^{n,s}$ (i.e., $V^{n,s} = V(P)$).*

Proof: Suppose not. Suppose that $V^{n,s} = V(P) \cup Y$, where Y is some algebraic set. As a subspace of $2ns$ affine space over K, we may regard

$$V^{n,s} = \{(\alpha, \beta) \in M_{n\times s}(K) \times M_{s\times n}(K) : \alpha\beta = (\alpha\beta)^{tr})\}$$

Pick $(\alpha_0, \beta_0) \in V^{n,s} - V(P)$ and let $X_{\beta_0} = \{(\alpha, \beta) \in V^{n,s} : \ \beta = \beta_0\}$. So

$$X_{\beta_0} \cap V^{n,s} = (X_{\beta_0} \cap V(P)) \cup (X_{\beta_0} \cap Y)$$

But X_{β_0} is irreducible, so $X_{\beta_0} = X_{\beta_0} \cap Y$, thus $X_{\beta_0} \subseteq Y$. But on Y all maximal minors of A and B vanish. So $X_{\beta_0} \subseteq V^{n,s} \cap V(I_m(A) + I_m(B))$. But this is false. Given a fixed β_0 with all maximal minors vanishing there always exists a matrix α_0 with some maximal minor not vanishing. Indeed, by letting $GL_n(K) \times GL_s(K)$ act on $X_{\beta_0} \mapsto X_{\delta\beta_0\gamma^T}$, we may assume β_0 has the form

$$\left(\begin{array}{c|c} \mathrm{I} & 0 \\ \hline 0 & 0 \end{array}\right)$$

where I is the identity matrix of size less than m $(= \min\{n, s\})$. For any such β_0, choose α_0 to be the block matrix formed by the identity matrix of size m in the upper left-hand corner and zeros elsewhere. $\square$

We have in fact shown

Remark 2.8. $R^{n,s}$ has a unique minimal prime and the dimension of $R^{n,s}$ is $ns + \binom{s+1}{2}$ for $n \geq s$ and is $2ns - \binom{n}{2}$ for $n \leq s$.

By remark 2.6, we note

Remark 2.9. For all $n, s \geq 1$, the singular locus of $R^{n,s}$ contains the ideal generated by the maximal minors of A and the maximal minors of B.

2.5 Complete Intersections

Definition 2.10. Suppose R is the coordinate ring of an affine variety over an algebraically closed field K. Then R has the form $R = S/I$ where S is a polynomial ring over K, and R is called a complete intersection if I is generated by the least possible number of elements, namely codim(V)= height(I). Then V is the intersection of codim(V) hypersurfaces, and I is generated by an S-sequence.

Theorem 2.11. *For $n \leq s$, $R^{n,s}$ is a complete intersection and a unique factorization domain.*

Proposition 2.12. *For $n \leq s+1$, $R^{n,s}$ is a complete intersection.*

*P*roof: The dimension of $K[A,B]$ is $2ns$ and $I^{n,s}$ can be generated by $\binom{n}{2}$ elements. For $n \leq s$, the dimension of $R^{n,s}$ is $2ns - \binom{n}{2}$. To prove the proposition, it suffices to show that height of $I^{n,s}$ is greater than or equal to $\binom{n}{2}$, or equivalently that $(2ns) - (2ns - \binom{n}{2})$ is less than or equal to $\binom{n}{2}$.

When $n = s+1$, the dimension of $R^{n,s}$ is $3s(s+1)/2$ and we only need to see that $\binom{s+1}{2} \geq (2s(s+1) - 3s(s+1)/2)$. □

Proposition 2.13. *For $n \leq s$, $R^{n,s}$ is a domain.*

*P*roof: Following Theorem 2.10 in [6], we precede by induction on n. The case $n = 1$ is handled in 2.1.1. So assume $n > 1$. Since $\mathrm{depth}_{I_1(A)+I_1(B)}K[A,B] = 2ns > \mathrm{depth}_{I^{n,s}}K[A,B] = \binom{n}{2}$, $I_1(A) + I_1(B)$ is not contained in any associated prime of $I^{n,s}$. By the induction hypothesis and 2.3, $R^{n,s}[1/a_{11}]$ is a Cohen-Macaulay domain so the exists a unique minimal prime, P, of $R^{n,s}$ not containing a_{11}. By 2.8, there exists only one minimal prime, thus a_{11} is a nonzerodivisor for $R^{n,s}$ and $R^{n,s}$ is a domain. □

Proposition 2.14. *For $n \leq s$, $R^{n,s}$ is a normal domain.*

For this argument, we do not assume K to be algebraically closed.

*P*roof: Following Theorem 2.11 in [6] In order to show normality we apply criteria based on Serre's conditions. By 2.12 and 2.13, we know $R^{n,s}$ is a Cohen-Macaulay domain. Thus normality is reduced to showing that $R^{n,s}_P$ is a regular local ring for all prime ideals of height at most one. We proceed by induction on the size of the matrices.

The statements are obvious if $n = 1$. Let $n > 1$. Consider a prime ideal P in $R^{n,s}$ such that $\text{depth} R^{n,s}_P \leq 1$. Then $\text{depth}_{\tilde{P}} K[A, B] \leq 1$. Because of

$$\text{depth}_{I_1(A)+I_1(B)} K[A, B] = 2ns > \binom{n}{2} + 1$$

there is an indeterminate A_{ij} which has residue class a_{ij} not contained in $\tilde{P}$. Clearly we may assume $a_{ij} = a_{11}$. Then by 2.3 and the inductive hypothesis $R^{n,s}[1/a_{11}]$ is normal. Consequently, $R^{n,s}_P$ is a normal domain. □

We note that the above holds more generally when K is a normal domain.

Lemma 2.15. *Any entry of A or B is a prime element of $R^{n,s}$.*

*P*roof: By the group action, it suffices to show that a_{11} is a prime element of $R^{n,s}$. Let Δ be a maximal minor of B in the lower right-hand corner. Then $\{a_{11}, \Delta\}$ form a permutable regular sequence for $R^{n,s}$. Hence Δ is a nonzerodivisor for $R^{n,s}/a_{11}R^{n,s}$. Therefore,

$$R^{n,s}/a_{11}R^{n,s} \subseteq (R^{n,s}/a_{11}R^{n,s})_\Delta = R^{n,s}_\Delta/a_{11}R^{n,s}_\Delta$$

So it suffices to show that the image of a_{11} in $R^{n,s}_\Delta$ is prime. Up to multiplication by a unit, the image is an entry in A which is prime in $R^{n,s}_\Delta$. Thus any entry in A or B is prime in $R^{n,s}$. □

Lemma 2.16. *Let S be Noetherian domain and x a prime element of S. If S_x is factorial, then S is factorial.*

*P*roof: See Lemma 2.2.18 in [5]. □

Corollary 2.17. *$R^{n,s}$ is a unique factorization domain for $n \leq s$.*

*P*roof: Apply 2.3, 2.15, and 2.16. □

CHAPTER 3

Preliminary Results on Auxiliary Ideals

3.1 Irreducible Families of Algebraic Sets

The next two chapters are devoted to showing that $R^{n,s}$ is a Cohen-Macaulay normal domain for all $n, s \geq 1$. Since we have already established the claim for $1 \leq n \leq s$ [Theorem 2.11] and $s = 1$ [Section 2.1.1], we will focus on the cases, $n > s > 1$. We use the method of principal radical systems developed by Hochster and Eagon [22]. The method requires that we establish facts like these for a much larger family of auxiliary ideals. We use this method to prove

Theorem 3.1. *Let K be a field, let n and s be positive integers and let A and B be matrices of indeterminates over K of size $n \times s$ and $s \times n$, respectively. Then $I^{n,s} = I_1(AB - (AB)^{tr})$ is a prime ideal, i.e., $R^{n,s} = K[A, B]/I^{n,s}$ is a domain.*

The idea is to include $I^{n,s}$ in a large family of ideals. Typically these ideals are radical rather than prime. The result is proved by reverse induction, in that the largest ideal(s) in the family are shown to be radical first. The family has the property that for each ideal I in the family there is an ideal of the $I + xR$ in the family which, by the induction hypothesis, is known to be radical.

By previous remarks [2.1], we may assume K is algebraically closed. We assume this throughout this chapter.

We first focus on showing that several of the ideals in our families have radicals that are prime. Thus, once we show that they are radical, it will follow that they are prime. We think of points in $\mathbb{A}_K^{2ns}$ as corresponding to pairs of matrices of sizes $n \times s$ and $s \times n$ over K.

Our approach will be to kill entries in A and B, one at a time, in a carefully chosen order, in such a way that, eventually, we decrease n or s or both. To this end, we define several auxiliary ideals in the next section.

3.2 Notation

Let J_i denote the ideal in $K[A, B]$ generated by the first i entries of the first row of A. Let J'_i denote the ideal generated by the last i entries of the first column of B. If we make the convention that J_0 and J'_0 are the zero ideal, then, for $0 \leq i \leq s$, we have

$$J_i = (a_{11}, \ldots, a_{1i})$$
$$J'_i = (b_{s1}, \ldots, b_{s-i+1,1})$$

Let $A|_j$ denote the $n \times j$ submatrix formed by the first j columns of A. We make the natural convention that $I_{s+1}(A|_{s+1})$ is the zero ideal, thus $I_j(A|_j)$ is defined for $1 \leq j \leq s+1$.

For $n \geq s$, $0 \leq i \leq s$ and $1 \leq j \leq s+1$, we define the following ideals (in $K[A, B]$) and closed algebraic sets (in $\mathbb{A}_K^{2ns}$):

$$F_{i,j}^{n,s} = I^{n,s} + J_i + I_j(A|_j)$$

$$G_{i,j}^{n,s} = I^{n,s} + J_s + J'_i + I_j(A|_j)$$

Therefore:

$$V(F_{i,j}^{n,s}) = V(I^{n,s}) \cap V(J_i) \cap V(I_j(A|_j))$$

$$V(G^{n,s}_{i,j}) = V(I^{n,s}) \cap V(J_s) \cap V(J'_i) \cap V(I_j(A|_j))$$

Remark 3.2. For all $n \geq s$ and $1 \leq j \leq s+1$, $F^{n,s}_{s,j} = G^{n,s}_{0,j}$ and for $n > s$ and $1 \leq j \leq s+1$, $G^{n,s}_{s,j} = F^{n-1,s}_{0,j}$.

Our first step is to show certain ideals in these families have radicals which are prime ideals (equivalently, the corresponding algebraic set is irreducible).

Theorem 3.3. *For $n \geq s$, $0 \leq i \leq s$, $1 \leq j \leq s+1$ and $i \neq j-1$,*

$$V(F^{n,s}_{i,j}) \text{ is an irreducible algebraic set.}$$

For $n \geq s$, $1 \leq i \leq s$, $1 \leq j \leq s+1$ and $i+j \leq s$,

$$V(G^{n,s}_{i,j}) \text{ is an irreducible algebraic set.}$$

Theorem 3.3 will be proved in various cases, which we separate into propositions. Before we do this, we need to define some notation and make a general reduction.

3.3 Matrix Notations

A and B are matrices of indeterminates. Capital letters with subscripts and/or primes like A_0, B', C_1, etc., are used for block submatrices after some indeterminates in A, B have been specialized to 0.

When (A, B) is specialized to a point of $\mathbb{A}^{2ns}_K$, we use (α, β) to represent that point. Greek letters with subscripts and/or primes like $\alpha', \beta_0, \varepsilon_1$, etc. are used for block submatrices after the entries of (A, B) have been specialized to elements in the field.

3.4 Connected Algebraic Groups

Let V be a closed algebraic set and let W be a subset of V. Let $v \in V$. If H is a connected and, hence, irreducible linear algebraic group acting on both V and

W and v is in the closure of W, then, for every $h \in H$, hv is in the closure of W. Therefore it suffices to show that one element of the orbit of V is in the closure of W.

Remark 3.4. In the sequel, each stabilizer that we consider is a finite product of general linear groups and affine spaces and, therefore, connected.

3.5 Reductions

Working modulo $I_1(A|_1)$, our pairs of matrices have the block form:

$$\left(\begin{array}{cc} & \begin{array}{cc} 1 & s\text{-}1 \end{array} \\ n & \begin{pmatrix} 0 & A' \end{pmatrix} \end{array}, \begin{array}{cc} & n \\ \begin{array}{c} 1 \\ s\text{-}1 \end{array} & \begin{pmatrix} B_0 \\ B' \end{pmatrix} \end{array} \right)$$

and $AB = A'B'$. Therefore $K[A, B]/(I^{n,s} + I_1(A|_1))$ is a polynomial ring over $K[A', B']/I^{n,s-1}$, which, when clear from context, we tacitly denote by $R^{n,s-1}$. In this way, it is clear that $R^{n,s}/I_1(A|_1)$ is isomorphic to $R^{n,s-1}[B_0]$.

For $0 \leq k \leq s$ and $0 \leq l \leq n$, let $\mathcal{J}_{k,l}$ denote the ideal generated by the first k columns of A, the first l rows of A and the first l columns of B. We include these ideals in our families. If $l = n$, the ideal $\mathcal{J}_{k,n}$ is generated by the entries of A and B, hence $\mathcal{J}_{k,n}$ is a maximal (prime) ideal. So there is no loss of generality in assuming that $l < n$.

Modulo $\mathcal{J}_{k,l}$, A and B have the block form:

$$\begin{array}{cc} & \begin{array}{cc} k & s\text{-}k \end{array} \\ \begin{array}{c} l \\ n\text{-}l \end{array} & \begin{pmatrix} 0 & 0 \\ 0 & A' \end{pmatrix} \end{array}, \quad \begin{array}{cc} & \begin{array}{cc} l & n\text{-}l \end{array} \\ \begin{array}{c} k \\ s\text{-}k \end{array} & \begin{pmatrix} 0 & B_0 \\ 0 & B' \end{pmatrix} \end{array}$$

and

$$AB = \begin{pmatrix} 0 & 0 \\ & \\ 0 & A'B' \end{pmatrix}$$

Therefore $R^{n,s}/\mathcal{J}_{k,l}$ is isomorphic to $R^{n-l,s-k}[B_0]$.

Remark 3.5. In our inductive proofs of ring-theoretic properties that are stable under polynomial extensions and for which we know the result for smaller sized matrices, there is no loss of generality in assuming that $k = l = 0$ and that $j > 1$.

3.6 Irreducible Cases

Since the property that a ring has a unique minimal prime ideal is stable under polynomial extension and since this is know for smaller sized matrices, by remark 3.5 we may assume $j > 1$ when proving $K[A, B]/F_{i,j}^{n,s}$ or $K[A, B]/G_{i,j}^{n,s}$ has a unique minimal prime.

Let $V(I_{k+1}(\alpha|_j))$ denote the set of points $(\alpha, \beta) \in \mathbb{A}_K^{2ns}$ such that the rank of the first j columns of α is less than or equal to k.

Proposition 3.6. *For $n \geq s$, $0 \leq k \leq j$ and $1 < j \leq s$,*

$$V^{n,s} \cap V(I_{k+1}(A|_j)) \text{ is an irreducible algebraic set.}$$

Proof: First note that when $k = 0$, $V^{n,s} \cap V(I_1(A|_j))$ is the same as $V^{n,s} \cap V(\mathcal{J}_{j,0})$, which is irreducible. So we assume $k > 0$. We may further assume $k < s$. Fix k, with $0 < k < s$. The subgroup, H, of $GL_n(K) \times GL_s(K)$ that stabilizes $V^{n,s} \cap V(I_{k+1}(A|_j))$ consists of pairs of matrices of the form:

$$GL_n(K) \times \begin{array}{c c} & \begin{array}{cc} j & s\text{-}j \end{array} \\ \begin{array}{c} j \\ s\text{-}j \end{array} & \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \end{array}$$

Let U be the subset of $V^{n,s} \cap V(I_{k+1}(A|_j))$ where the rank of α is maximal ($\mathrm{rank}(\alpha) = s - j + k$.) Any point, (α, β), of U has an element in its orbit of the form:

$$\begin{array}{c c} & \begin{array}{cc} j-k & s\text{-}j\text{+}k \end{array} \\ \begin{array}{c} 1 \\ s\text{-}j\text{+}k \\ n\text{-}s\text{+}j\text{-}k\text{-}1 \end{array} & \begin{pmatrix} 0 & 0 \\ 0 & 1_{s-j+k} \\ 0 & 0 \end{pmatrix} \end{array}, \quad \begin{array}{c c} & \begin{array}{ccc} 1 & s\text{-}j\text{+}k & n\text{-}s\text{+}j\text{-}k\text{-}1 \end{array} \\ \begin{array}{c} j\text{-}k \\ s\text{-}j\text{+}k \end{array} & \begin{pmatrix} \beta_0 & \varsigma_0 & \omega_0 \\ 0 & \varsigma_1 & 0 \end{pmatrix} \end{array}$$

with ς_1 symmetric. We tacitly use the fact that $k < s$ in order to get the first row of α to be zero. In doing this, we will also show $V(F^{n,s}_{s,j}) = (V^{n,s} \cap V(J_s) \cap V(I_j(A|_j)))$ is irreducible for $1 < j \leq s$.

Let F denote the irreducible family of pairs of matrices with the above form. By remark 3.4, H is connected (irreducible) and $H \times F$ surjects onto U, thus U is irreducible.

Lemma 3.7. *The closure of U, $\overline{U}$, in $V^{n,s} \cap V(I_{k+1}(A|_j))$ contains the pairs of matrices $(\alpha, \beta) \in V^{n,s} \cap V(I_{k+1}(A|_j))$ such that* $\mathrm{rank}(\alpha|_j) = k$

Lemma 3.8. $\overline{U} \supset \{(\alpha, \beta) \in V^{n,s} \cap V(I_{k+1}(A|_j)) \mid \mathrm{rank}(\alpha|_j) \leq k\}$

*P*roof of Lemma 3.7: We proceed by reverse induction on $r = I_s(\alpha)$ and we assume α's with higher rank are in $\overline{U}$. In general, any point in $(V^{n,s} \cap V(I_{k+1}(A|_j)))$

with $\mathrm{rank}(\alpha|_j) = k$ has an element in its orbit of the form:

$$\begin{array}{cc} & \begin{array}{ccc} j\text{-}k & r & s\text{-}r\text{+}k\text{-}j \end{array} \\ \begin{array}{c} 1 \\ r \\ n\text{-}r\text{-}1 \end{array} & \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1_r & 0 \\ 0 & 0 & 0 \end{pmatrix} \end{array} \begin{array}{cc} & \begin{array}{ccc} 1 & r & n\text{-}r\text{-}1 \end{array} \\ \begin{array}{c} j\text{-}k \\ r \\ s\text{-}r\text{+}k\text{-}j \end{array} & \begin{pmatrix} \beta_0 & \varsigma_0 & \omega_0 \\ 0 & \varsigma_1 & 0 \\ \beta_2 & \varsigma_2 & \omega_2 \end{pmatrix} \end{array}$$

where ς_1 is symmetric and r is the rank of α.

In order to show this family of pairs of matrices is contained in $\overline{U}$, it is enough to show that a dense subset is in $\overline{U}$. So consider the nonempty open, hence dense, subset defined by the non-vanishing of the $(1,1)$ entry of the ω_2 block matrix (we aim to clear the $j-k+r+1$ row of β: this leaves α's form invariant). Up to the group action, (α,β) have the form:

$$\begin{array}{cc} & \begin{array}{cccc} j\text{-}k & r & 1 & s\text{-}r\text{+}k\text{-}j\text{-}1 \end{array} \\ \begin{array}{c} 1 \\ r \\ 1 \\ n\text{-}r\text{-}2 \end{array} & \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1_r & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \end{array} \begin{array}{cc} & \begin{array}{cccc} 1 & r & 1 & n\text{-}r\text{-}2 \end{array} \\ \begin{array}{c} j\text{-}k \\ r \\ 1 \\ s\text{-}r\text{+}k\text{-}j\text{-}1 \end{array} & \begin{pmatrix} \beta_0 & \varsigma_0 & \omega_0 & \varepsilon_0 \\ 0 & \varsigma_1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \beta_3 & \varsigma_3 & \omega_3 & \varepsilon_3 \end{pmatrix} \end{array}$$

where ς_1 is symmetric.

By the induction hypothesis, the pairs of matrices of the form:

$$\begin{array}{cc} & \begin{array}{cccc} j\text{-}k & r & 1 & s\text{-}r\text{+}k\text{-}j\text{-}1 \end{array} \\ \begin{array}{c} 1 \\ r \\ 1 \\ n\text{-}r\text{-}2 \end{array} & \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1_r & 0 & 0 \\ 0 & 0 & t & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \end{array} \begin{array}{cc} & \begin{array}{cccc} 1 & r & 1 & n\text{-}r\text{-}2 \end{array} \\ \begin{array}{c} j\text{-}k \\ r \\ 1 \\ s\text{-}r\text{+}k\text{-}j\text{-}1 \end{array} & \begin{pmatrix} \beta_0 & \varsigma_0 & \omega_0 & \varepsilon_0 \\ 0 & \varsigma_1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \beta_3 & \varsigma_3 & \omega_3 & \varepsilon_3 \end{pmatrix} \end{array}$$

where $t \neq 0$ and ς_1 is symmetric are in $\overline{U}$, and so the closure (including $t = 0$) is contained in $\overline{U}$. This proves lemma 3.7. $\square$

The proof of lemma 3.8 is similar but we use reverse induction on k. Assume $\alpha|_j$ with higher rank are in $\overline{U}$.

In general, any point (α, β) in $V^{n,s} \cap V(I_{k+1}(A|_j))$ with $\text{rank}(\alpha|_j) < k$ has an element in its orbit of the form:

$$\begin{array}{cc} & \begin{array}{ccc} \scriptstyle j\text{-}k+1 & \scriptstyle r & \scriptstyle s\text{-}r+k\text{-}j\text{-}1 \end{array} \\ \begin{array}{c} \scriptstyle 1 \\ \scriptstyle r \\ \scriptstyle n\text{-}r\text{-}1 \end{array} & \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1_r & 0 \\ 0 & 0 & 0 \end{pmatrix} \end{array} \begin{array}{cc} & \begin{array}{ccc} \scriptstyle 1 & \scriptstyle r & \scriptstyle n\text{-}r\text{-}1 \end{array} \\ \begin{array}{c} \scriptstyle j\text{-}k+1 \\ \scriptstyle r \\ \scriptstyle s\text{-}r+k\text{-}j\text{-}1 \end{array} & \begin{pmatrix} \beta_0 & \varsigma_0 & \omega_0 \\ 0 & \varsigma_1 & 0 \\ \beta_2 & \varsigma_2 & \omega_2 \end{pmatrix} \end{array}$$

where ς_1 is symmetric and r is the rank of α.

In order to show this family of pairs of matrices is contained in $\overline{U}$, it is enough to show that a dense subset is in $\overline{U}$. So consider the nonempty open, hence dense, subset defined by the non-vanishing of the $(j-k+1, 1)$ entry of the ς_0 block matrix. Up to the group action, (α, β) have the form:

$$\begin{array}{cc} & \begin{array}{cccc} \scriptstyle j\text{-}k & \scriptstyle 1 & \scriptstyle r & \scriptstyle s\text{-}r+k\text{-}j\text{-}1 \end{array} \\ \begin{array}{c} \scriptstyle 1 \\ \scriptstyle 1 \\ \scriptstyle r \\ \scriptstyle n\text{-}r\text{-}2 \end{array} & \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1_r & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \end{array} \begin{array}{cc} & \begin{array}{cccc} \scriptstyle 1 & \scriptstyle 1 & \scriptstyle r & \scriptstyle n\text{-}r\text{-}2 \end{array} \\ \begin{array}{c} \scriptstyle j\text{-}k \\ \scriptstyle 1 \\ \scriptstyle r \\ \scriptstyle s\text{-}r+k\text{-}j\text{-}1 \end{array} & \begin{pmatrix} \beta_0 & \varsigma_0 & \omega_0 & \varepsilon_0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \omega_2 & 0 \\ \beta_3 & \varsigma_3 & \omega_3 & \varepsilon_3 \end{pmatrix} \end{array}$$

where ω_2 is symmetric.

By the induction hypothesis, the pairs of matrices of the form:

$$\begin{array}{cc} & \begin{array}{cccc} \scriptstyle j\text{-}k & \scriptstyle 1 & \scriptstyle r & \scriptstyle s\text{-}r+k\text{-}j\text{-}1 \end{array} \\ \begin{array}{c} \scriptstyle 1 \\ \scriptstyle 1 \\ \scriptstyle r \\ \scriptstyle n\text{-}r\text{-}2 \end{array} & \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & t & 0 & 0 \\ 0 & 0 & 1_r & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \end{array} \begin{array}{cc} & \begin{array}{cccc} \scriptstyle 1 & \scriptstyle 1 & \scriptstyle r & \scriptstyle n\text{-}r\text{-}2 \end{array} \\ \begin{array}{c} \scriptstyle j\text{-}k \\ \scriptstyle 1 \\ \scriptstyle r \\ \scriptstyle s\text{-}r+k\text{-}j\text{-}1 \end{array} & \begin{pmatrix} \beta_0 & \varsigma_0 & \omega_0 & \varepsilon_0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \omega_2 & 0 \\ \beta_3 & \varsigma_3 & \omega_3 & \varepsilon_3 \end{pmatrix} \end{array}$$

where $t \neq 0$ and ω_2 is symmetric are in $\overline{U}$, so the closure (including $t = 0$) is contained in $\overline{U}$. This proves lemma 3.8. $\square$

When $k = j - 1$ we note

Corollary 3.9. *For $n \geq s$ and $1 < j \leq s$,*

$$V(F^{n,s}_{0,j}) = (V^{n,s} \cap V(I_j(A|_j))) \text{ is an irreducible algebraic set.}$$

We also note as a corollary that this proof shows that

Corollary 3.10. *For $n \geq s$ and $1 < j \leq s$,*

$$V(G^{n,s}_{0,j}) = (V^{n,s} \cap V(J_s) \cap V(I_j(A|_j))) \text{ is an irreducible algebraic set.}$$

Proposition 3.11. *For $n \geq s$ and $0 \leq i < s$,*

$$V(F^{n,s}_{i,s+1}) = V^{n,s} \cap V(J_i) \text{ is an irreducible algebraic set.}$$

*P*roof: Fix i. We may assume $i > 0$. For each $i < k \leq s$, let $V^{n,s}_{a_{1k}}$ be the dense open subset of $V^{n,s}$ such that the a_{1k} entry of α is not zero. Regard $V^{n,s} \cap V(J_i)$ as the closed subset of $V^{n,s}$ such that the first i entries of the first row of α are zero.

Consider the map

$$V^{n,s}_{a_{1k}} \xrightarrow{\varphi_k} V^{n,s} \cap V(J_i)$$

given by $(\alpha, \beta) \mapsto (\alpha\delta^{-1}, \delta\beta)$ where δ^{-1} is given by adding the $1 \times s$ row vector

$$(\frac{-a_{11}}{a_{1k}}, \ldots, \frac{-a_{1i}}{a_{1k}}, 0, \ldots, 0)$$

to the k^{th} row of the $s \times s$ identity matrix. In other words, this map "clears" the first i entries of the first row of α.

For each k, $V^{n,s}_{a_{1k}}$ is irreducible, hence the image of φ_k is irreducible. Let W_k denote the closure of the image of φ_k in $V^{n,s} \cap V(J_i)$. We note that W_k contains all pairs (α, β) in $V^{n,s} \cap V(J_i)$ such that the a_{1k} entry of α is not zero

Remark 3.12. Since the product of any matrix with the zero matrix is always symmetric, to show a subset is not empty we take α to be of the form needed and β to be the zero matrix. For instance, in the above example $V^{n,s}_{a_{1k}}$ is nonempty; take α with a 1 in the $(1, k)$ entry and zeros elsewhere and β to be the zero matrix.

Lemma 3.13. *For all k, l with $i < k, l \leq s$, $W_k = W_l$, i.e., they all give the same irreducible component of $V^{n,s} \cap V(J_i)$ which we denote by W.*

Consider the set, $Y \subset V^{n,s} \cap V(J_i)$ of pairs of matrices (α, β) with $a_{1k} \neq 0$ and $a_{1l} \neq 0$. This is an open subset of both W_k and W_l with non-trivial intersection. (Take B to be the zero matrix.) Thus $\overline{Y} = W_k = W_l$.

Definition 3.14. Let Y be an algebraic set and $\{U_k\}$ be a finite family of nonempty irreducible subsets of Y. If for all k, k', $U_k, U_{k'}$ intersect (pairwise) in a nonempty open subset then the closure of U_k in Y is the same as the closure of $U_{k'}$ in Y. When this occurs, we say that the U_k, "give the same irreducible component."

Lemma 3.15. $W = V^{n,s} \cap V(J_i)$

Whether a pair (α, β) is in W depends on its orbit under the group action. The subgroup, H, of $GL_n \times GL_s$ which stabilizes $V^{n,s} \cap V(J_i)$ consists of pairs (γ, δ) of the following block form:

$$\begin{array}{cc} & \begin{array}{cc} 1 & n\text{-}1 \end{array} \\ \begin{array}{c} 1 \\ n\text{-}1 \end{array} & \begin{pmatrix} * & 0 \\ * & * \end{pmatrix} \end{array} \times \begin{array}{cc} & \begin{array}{cc} i & s\text{-}i \end{array} \\ \begin{array}{c} i \\ s\text{-}i \end{array} & \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \end{array}$$

We know W contains all pairs such that α has a nonzero entry in the first row. It remains to show that W contains $V^{n,s} \cap V(J_s)$. But by a previous remark, $V^{n,s} \cap V(J_s)$ has two irreducible components. So it is enough to show that a dense subset of each component is contained in W.

Lemma 3.16. *The open subset, U, of $V^{n,s} \cap V(J_s) \cap V(J'_s)$ where α has maximal rank (rank$(\alpha) = s$) is contained in W.*

Every pair in U has an element in its orbit of the form

$$\begin{array}{cc} & s \\ \begin{array}{c} 1 \\ s \\ n\text{-}s\text{-}1 \end{array} & \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \end{array} \quad \begin{array}{cc} & \begin{array}{ccc} 1 & s & n\text{-}s\text{-}1 \end{array} \\ s & \begin{pmatrix} 0 & \beta' & 0 \end{pmatrix} \end{array}$$

with β' symmetric.

Consider the irreducible family in $V^{n,s} \cap V(J_i)$

$$\begin{array}{cc} & \begin{array}{cc} s\text{-}1 & 1 \end{array} \\ \begin{array}{c} 1 \\ s\text{-}1 \\ 1 \\ n\text{-}s\text{-}1 \end{array} & \begin{pmatrix} 0 & t \\ 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix} \end{array} \quad \begin{array}{cc} & \begin{array}{cccc} 1 & s\text{-}1 & 1 & n\text{-}s\text{-}1 \end{array} \\ \begin{array}{c} s\text{-}1 \\ 1 \end{array} & \begin{pmatrix} t\omega_0 & \varsigma_0 & \omega_0 & 0 \\ t\omega_1 & \omega_0^{tr} & \omega_1 & 0 \end{pmatrix} \end{array}$$

with ς_0 symmetric.

For $t \neq 0$ this family is contained in W so its closure (including $t = 0$) is contained in W. Thus $W \supseteq V^{n,s} \cap V(J_s) \cap V(J'_s)$.

Lemma 3.17. *The open subset, U, of $V^{n,s} \cap V(J_s) \cap V(I_s(A|_s))$ where α has maximal rank (rank$(\alpha) = s - 1$) is contained in W.*

Up to the group action, every pair in U has an element in its orbit of the form

$$\begin{array}{cc} & \begin{array}{cc} 1 & s\text{-}1 \end{array} \\ \begin{array}{c} 1 \\ s\text{-}1 \\ n\text{-}s \end{array} & \begin{pmatrix} 0 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix} \end{array} \quad \begin{array}{cc} & \begin{array}{ccc} 1 & s\text{-}1 & n\text{-}s \end{array} \\ \begin{array}{c} 1 \\ s\text{-}1 \end{array} & \begin{pmatrix} \beta_0 & \varsigma_0 & \omega_0 \\ 0 & \varsigma_1 & 0 \end{pmatrix} \end{array}$$

where ς_1 is symmetric.

Consider the irreducible family in $V^{n,s} \cap V(J_i)$ parameterized by $t \in \mathbb{A}^1_k$,

$$\begin{array}{cc} & \begin{array}{ccc} 1 & s\text{-}2 & 1 \end{array} \\ \begin{array}{c} 1 \\ s\text{-}2 \\ 1 \\ n\text{-}s \end{array} & \begin{pmatrix} 0 & 0 & t \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \end{array} \begin{array}{cc} & \begin{array}{cccc} 1 & s\text{-}2 & 1 & n\text{-}s \end{array} \\ \begin{array}{c} 1 \\ s\text{-}2 \\ 1 \end{array} & \begin{pmatrix} \beta_0 & \varsigma_0 & \omega_0 & \varepsilon_0 \\ t\omega_1 & \varsigma_1 & \omega_1 & 0 \\ t\omega_2 & \omega_1^{tr} & \omega_2 & 0 \end{pmatrix} \end{array}$$

where ς_1 is symmetric. For $t \neq 0$ this family is contained in W, so its closure is contained in W (including $t = 0$). Thus $W \supset V^{n,s} \cap V(J_s)$

Proposition 3.18. *For $n \geq s$, $1 \leq i \leq s$ and $1 \leq j \leq s$ with $i \neq j - 1$,*

$$V(F^{n,s}_{i,j}) = V^{n,s} \cap V(J_i) \cap V(I_j(A|_j)) \text{ is an irreducible algebraic set.}$$

*P*roof: We've show when $i = s$ and when $j = 1$, so assume $i < s$ and $j > 1$. Let $(V^{n,s} \cap V(I_j(A|_j)))_{a_{1k}}$ denote the nonempty open (irreducible) subset of $V^{n,s} \cap V(I_j(A|_j))$ defined by the non-vanishing of the $(1,k)$ entry of α. The group action on $V^{n,s} \cap V(I_j(A|_j))$ (as well as the fact that $a_{1k} \neq 0$) allows us to first clear the k^{th} column of α (except the first entry), then clear the remaining entries in the first row of α. Note that this action can drop the rank of α by at most one, so the pairs obtained from performing such actions remain in our set. As such we have, for each k such that $i < k \leq s$, a map

$$(V^{n,s} \cap V(I_j(A|_j)))_{a_{1k}} \xrightarrow{\phi_k} V(F^{n,s}_{i,j})$$

If we denote by W_k the closure of the image of ϕ_k we have that all such k give the same irreducible component, denoted W.

Lemma 3.19. $W \supset V(F^{n,s}_{i,j})$

Remains to show that W contains pairs where the first row of α is zero. That is, we need to show $V^{n,s} \cap V(J_s) \cap V(I_j(A|_j))$ is contained in W. Since this is a variety, it is enough to show that the open subset defined by α's of maximal rank are in W.

The maximum rank for any α in $V^{n,s} \cap V(J_s) \cap V(I_j(A|_j))$ is $s-1$ since $\alpha|_j$ can have a maximum rank of $j-1$ and $n > s$. So up to the group action we need to see that pairs of matrices in $V(F^{n,s}_{i,j})$ of the form

$$\begin{array}{c c} & \begin{array}{cc} 1 & s\text{-}1 \end{array} \\ \begin{array}{c} 1 \\ s\text{-}1 \\ n\text{-}s \end{array} & \begin{pmatrix} 0 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix} \end{array}, \quad \begin{array}{c c} & \begin{array}{ccc} 1 & s\text{-}1 & n\text{-}s \end{array} \\ \begin{array}{c} 1 \\ s\text{-}1 \end{array} & \begin{pmatrix} \beta_0 & \varsigma_0 & \omega_0 \\ 0 & \varsigma_1 & 0 \end{pmatrix} \end{array}$$

with ς_1 symmetric, are in W.

Consider the irreducible family in $V(F^{n,s}_{i,j})$ parameterized by $t \in \mathbb{A}^1_K$,

$$\begin{array}{c c} & \begin{array}{ccc} 1 & s\text{-}2 & 1 \end{array} \\ \begin{array}{c} 1 \\ s\text{-}2 \\ 1 \\ n\text{-}s \end{array} & \begin{pmatrix} 0 & 0 & t \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \end{array} \quad \begin{array}{c c} & \begin{array}{cccc} 1 & s\text{-}2 & 1 & n\text{-}s \end{array} \\ \begin{array}{c} 1 \\ s\text{-}2 \\ 1 \end{array} & \begin{pmatrix} \beta_0 & \varsigma_0 & \omega_0 & \varepsilon_0 \\ t\omega_1 & \varsigma_1 & \omega_1 & 0 \\ t\omega_2 & \omega_1^T & \omega_2 & 0 \end{pmatrix} \end{array}$$

When t is nonzero, this family is in W, so its closure is contained in W. Thus $V(F^{n,s}_{i,j})$ is irreducible.

Proposition 3.20. *For $n \geq s$, $1 \leq i < s-1$ and $j = s - i$,*

$V(G^{n,s}_{i,s-i}) = V^{n,s} \cap V(J_s) \cap V(J'_i) \cap V(I_{s-i}(A|_{s-i}))$ *is an irreducible algebraic set.*

Proof: The subgroup, H, of $GL_n \times GL_s$ which stabilizes $V(G^{n,s}_{i,s-i})$ consists of pairs of matrices of the form

$$\begin{array}{cc} & \begin{array}{cc} 1 & n\text{-}1 \end{array} \\ \begin{array}{c} 1 \\ n\text{-}1 \end{array} & \begin{pmatrix} * & 0 \\ * & * \end{pmatrix} \end{array} \times \begin{array}{cc} & \begin{array}{cc} s\text{-}i & i \end{array} \\ \begin{array}{c} s\text{-}i \\ i \end{array} & \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \end{array}$$

and a typical pair in $V(G^{n,s}_{i,s-i})$ has the form

$$\begin{array}{cc} & \begin{array}{cc} s\text{-}i & i \end{array} \\ \begin{array}{c} 1 \\ n\text{-}1 \end{array} & \begin{pmatrix} 0 & 0 \\ \alpha_0 & \alpha_1 \end{pmatrix} \end{array}, \begin{array}{cc} & \begin{array}{cc} 1 & n\text{-}1 \end{array} \\ \begin{array}{c} s\text{-}i \\ i \end{array} & \begin{pmatrix} \beta_0 & \varsigma_0 \\ 0 & \varsigma_1 \end{pmatrix} \end{array}$$

with the rank of $\alpha_0 < s-i$.

For $1 \leq k \leq s-i$, if $b_{k1} \neq 0$ then, the symmetry and rank conditions imply that the k^{th} column of α is zero. Let U_k be the open subset of $V(G^{n,s}_{i,s-i})$ defined by the non-vanishing of the b_{k1} entry of β. Up to the group action, we may assume that the b_{11} entry is nonzero. Denote by F the family of matrices in $V(G^{n,s}_{i,s-i})$ of the form:

$$\begin{array}{cc} & \begin{array}{cc} 1 & s\text{-}1 \end{array} \\ \begin{array}{c} 1 \\ n\text{-}1 \end{array} & \begin{pmatrix} 0 & 0 \\ 0 & \alpha' \end{pmatrix} \end{array}, \begin{array}{cc} & \begin{array}{cc} 1 & n\text{-}1 \end{array} \\ \begin{array}{c} 1 \\ s\text{-}1 \end{array} & \begin{pmatrix} 1 & 0 \\ 0 & \varsigma' \end{pmatrix} \end{array}$$

F is irreducible by the induction hypothesis and $H \times F$ surjects onto U_k for all k. Thus U_k is irreducible and all such U_k give the same irreducible component, $W = \overline{U_k} = \overline{U_{k'}}$

So it remains to show that $W \supseteq V(G^{n,s}_{s,s-i})$. Consider the irreducible family, F', in $V(G^{n,s}_{i,s-i})$ consisting of pairs of the form

$$\begin{array}{cc} & \begin{array}{cc} 1 & s\text{-}1 \end{array} \\ \begin{array}{c} 1 \\ n\text{-}1 \end{array} & \begin{pmatrix} 0 & 0 \\ 0 & \alpha_1 \end{pmatrix} \end{array}, \begin{array}{cc} & \begin{array}{cc} 1 & n\text{-}1 \end{array} \\ \begin{array}{c} 1 \\ s\text{-}1 \end{array} & \begin{pmatrix} t & \varsigma_0 \\ 0 & \varsigma_1 \end{pmatrix} \end{array}$$

When $t \neq 0$, $H \times F'$ is contained in W, so its closure (including $t = 0$) is contained in W. Thus W contains pairs in $V(G^{n,s}_{i,s-i})$ of the form

$$\begin{array}{cc} & \begin{array}{cc} 1 & s\text{-}1 \end{array} \\ \begin{array}{c} 1 \\ n\text{-}1 \end{array} & \begin{pmatrix} 0 & 0 \\ 0 & \alpha_1 \end{pmatrix} \end{array}, \quad \begin{array}{cc} & \begin{array}{cc} 1 & n\text{-}1 \end{array} \\ \begin{array}{c} 1 \\ s\text{-}1 \end{array} & \begin{pmatrix} 0 & \varsigma_0 \\ 0 & \varsigma_1 \end{pmatrix} \end{array}$$

In order to show that $W \supseteq V(G^{n,s}_{s,s-i})$ its enough to show that W contains a dense subset. Consider the dense open subset on which α_1 has maximal rank. The pair has an element in its orbit of the form

$$\begin{array}{cc} & \begin{array}{cc} 1 & s\text{-}1 \end{array} \\ \begin{array}{c} 1 \\ s\text{-}1 \\ n\text{-}s \end{array} & \begin{pmatrix} 0 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix} \end{array}, \quad \begin{array}{cc} & \begin{array}{ccc} 1 & s\text{-}1 & n\text{-}s \end{array} \\ \begin{array}{c} 1 \\ s\text{-}1 \end{array} & \begin{pmatrix} 0 & \varsigma_0 & \omega_0 \\ 0 & \varsigma_1 & 0 \end{pmatrix} \end{array}$$

with ς_1 symmetric. But these are in W. $\square$

Proposition 3.21. *For $n \geq s$, $1 < i < s$, $1 < j < s$ and $i + j < s$,*

$$V(G^{n,s}_{i,j}) = V^{n,s} \cap V(J_s) \cap V(J'_i) \cap V(I_j(A|_j)) \text{ is an irreducible algebraic set.}$$

Proof: For $j < k \leq s - i$, let U_k denote the open (irreducible) subset of $V(G^{n,s}_{0,j})$ defined by the non-vanishing of the $(k, 1)$ entry of β. We have a map $U_k \to V(G^{n,s}_{i,j})$. Let W_k the closure of the image of ϕ_k. We have that all such k give the same irreducible component, denoted W.

So it remains to show $V(G^{n,s}_{s-j,j})$ is contained in W. This is an irreducible set, so it is enough to show a dense subset is contained in W. Consider the open subset such that β has maximal rank and $b_{11} \neq 0$. Because of the group action, it is enough to show that pairs of the form

$$\begin{array}{cc} & \begin{array}{cc} 1 & s\text{-}1 \end{array} \\ \begin{array}{c} 1 \\ s\text{-}1 \\ n\text{-}s \end{array} & \begin{pmatrix} 0 & 0 \\ 0 & \alpha' \\ 0 & 0 \end{pmatrix} \end{array}, \quad \begin{array}{cc} & \begin{array}{ccc} 1 & s\text{-}1 & n\text{-}s \end{array} \\ \begin{array}{c} 1 \\ s\text{-}1 \end{array} & \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \end{array}$$

with α' symmetric, are in W.

Consider the family of matrices of the form

$$\begin{array}{cc} & \begin{array}{cc} 1 & s\text{-}1 \end{array} \\ \begin{array}{c} 1 \\ n\text{-}1 \end{array} & \begin{pmatrix} 0 & 0 \\ -t\mathsf{C}_j^{\alpha'} & \alpha' \end{pmatrix} \end{array} \begin{array}{cc} & \begin{array}{cc} 1 & n\text{-}1 \end{array} \\ \begin{array}{c} 1 \\ s\text{-}1 \end{array} & \begin{pmatrix} 1 & 0 \\ te_j & 1 \end{pmatrix} \end{array}$$

where α' is symmetric, e_j is the j^{th} elementary column vector and $\mathsf{C}_j^{\alpha'}$ denotes the j^{th} column of the matrix α'.

When $t \neq 0$, this family is in W so its closure is contained in W. The $W = V(G_{i,j}^{n,s})$. $\square$

This completes the proof of Theorem 3.3. $\square$

CHAPTER 4

Principal Radical Systems and the Domain Property

We have already established [2.4] that $R^{n,s}$ has a unique minimal prime ideal for all $n, s \geq 1$. To prove that $R^{n,s}$ is a domain is equivalent to proving that $I^{n,s}$ is radical. For this we use the method of principal radical systems mentioned earlier. We will need the following elementary lemmas (see [22]):

Lemma 4.1. *Let S be a Noetherian ring that is either local or $\mathbb{N}$-graded, and let $x \in S$ be in the maximal ideal or be a form of positive degree. Suppose that N is the nilradical of S, that N is prime, that $x \notin N$ and that S/xS is reduced. Then $N = 0$.*

*P*roof: Suppose that $u \in N$. Since S/xS is reduced, we must have that $u = xv$ for some $v \in S$. Since $xv \in N$, $x \notin N$, and N is prime, we must have that $v \in N$. Therefore $N = xN$. By Nakayama's lemma for local or graded rings, $N = 0$. □

Corollary 4.2. *Let S be a Noetherian ring that is either local or $\mathbb{N}$-graded, and let $x \in S$ be in the maximal ideal or be a form of positive degree. Suppose that I is a (homogeneous in the graded case) proper ideal of S with radical P, where P is prime, that $x \notin P$, and that $P + xS$ is radical. Then $I = P$, i.e., I is prime.* □

Lemma 4.3. *Let S be Noetherian, let I be an ideal of S, let J be the radical of I, and suppose that $J \subseteq P$ where P is prime. Suppose that $I + xS$ is radical where $x \notin P$ and that $xP \subseteq I$. Then $I = J$, i.e., I is radical.*

Proof: Suppose that $u \in J$. Then $u \in I + xS$, say $u = i + xs$, where $i \in I$ and $s \in S$. Then $xs = u - i \in J \subseteq P$, and so $s \in P$. Since $xP \subseteq I$, we have that $xs \in I$ and so $u = i + xs \subseteq I$. $\square$

In our case, Lemma 4.1 will be used to show that the ideals in our families whose corresponding algebraic sets are irreducible are radical. Typically we have that $V(I)$ is irreducible and that $(I + x)$ is radical and we only need to show that $x \notin \mathrm{Rad}(I)$. This is typically done via a "specialization" argument (i.e., see proof of 4.9).

We typically use Lemma 4.3 to show that the ideals in our families whose corresponding algebraic sets are reducible are radical. As it turns out, these ideals have exactly two minimal primes and we say that these ideals "bifurcate." Geometrically, this means that the corresponding algebraic set is the union of two irreducible algebraic sets. In these instances, we systematically use the following notation:

$$\begin{tikzcd}
 & I \cap J \arrow[dl] \arrow[dr] & \subset \; ((I \cap J) + x) \\
I \arrow[dr] & & J \arrow[dl] \\
 & I + J &
\end{tikzcd}$$

The induction hypothesis implies the result for larger ideals, so we always have that $((I \cap J) + x)$ is radical and, by Theorem 3.3, we will always have that the radical of J, the radical of I and the radical of $I + J$ are all prime ideals. To apply Lemma 4.1 we will frequently need to show that $xJ \subseteq (I \cap J)$. For this we use Cramer's Rule:

Proposition 4.4. (Cramer's Rule:) *Let S be a commutative ring and X an $n \times n$ matrix with entries in S. Write X in terms of its columns $X = (\mathsf{C}_1 \cdots \mathsf{C}_n)$.*

Let $y_1, \ldots, y_n \in R$ be such that $y_1\mathsf{C}_1 + \cdots + y_n\mathsf{C}_n = Z$ for some column matrix Z. Then for every i we have

$$y_i \cdot \det(X) = \det(\mathsf{C}_1 \cdots \mathsf{Z} \cdots \mathsf{C}_n)$$

where Z is in the i^{th} position.

With the notation established in Section 3.2, we prove

Theorem 4.5. *For $n \geq s$, $0 \leq i \leq s$ and $1 \leq j \leq s+1$, the families of ideals*

$$F_{i,j}^{n,s} = I^{n,s} + J_i + I_j(A|_j)$$

$$G_{i,j}^{n,s} = I^{n,s} + J_s + J_i' + I_j(A|_j)$$

are all radical.

We postpone the proof momentarily to discuss two corollaries and an example.

Corollary 4.6. *For all $n, s \geq 1$, $R^{n,s}$ is a domain.*

And by combining Theorem 4.5 and Theorem 3.3 we have

Corollary 4.7. *For $n \geq s$, $0 \leq i \leq s$, $1 \leq j \leq s+1$ with $i \neq j-1$,*

$$K[A,B]/F_{i,j}^{n,s} = K[A,B]/(I^{n,s} + J_i + I_j(A|_j))$$

is a domain.

For $n \geq s$, $1 \leq i \leq s$, $1 \leq j \leq s+1$ and $i + j \leq s$,

$$K[A,B]/G_{i,j}^{n,s} = K[A,B]/(I^{n,s} + J_s + J_i' + I_j(A|_j))$$

is a domain.

Example 4.8. As mentioned earlier [2.1.1], $R^{n,1}$ is a domain. For heuristic reasons, we verify that $R^{2,1}$ is a domain using the method of principal radical systems. The issue at hand is whether $I^{2,1} = (a_{11}b_{21} - a_{21}b_{11})$ is radical. Suppose this were not obvious and consider the ideal $I^{2,1} + J_1 = (a_{11}, a_{21}b_{11})$. Both $I^{2,1} + J_1 + I_1(A|_1) = (a_{11}, a_{21})$ and $I^{2,1} + J_1 + J_1' = (a_{11}, b_{11})$ are prime ideals containing $I^{2,1} + J_1$. Applying Lemma 4.3 we have that if $b_{11}(I^{2,1} + J_1 + I_1(A|_1)) \subseteq (I^{2,1} + J_1)$ then $I^{2,1} + J_1$ is radical. But this is obvious: $b_{11}(a_{11}, a_{21}) \subseteq (a_{11}, b_{11})$. To show that $I^{2,1}$ is radical, by Lemma 4.1, it is enough to show that a_{11} is not in the Rad $(I^{2,1})$. Specializing a_{11} to 1 and all other entries of A and B to zero gives a point where $I^{2,1}$ vanishes but a_{11} does not. Hence, $I^{2,1}$ is radical.

*P*roof of Theorem 4.5: We shall prove the claim in cases, which we separate into propositions. We note that, by remark 3.5 and Theorem 2.11, we may assume the results if either n or s (or both) decrease and we may assume $j > 1$. So it remains to show that the families of ideals $F^{n,s}_{i,j} = I^{n,s} + J_i + I_j(A|_j)$, $G^{n,s}_{i,j} = I^{n,s} + J_s + J_i' + I_j(A|_j)$ are radical for $n \geq s$, $0 \leq i \leq s$ and $1 < j \leq s + 1$.

Proposition 4.9. *For $n \geq s$, $0 < i \leq s$ and $1 < j \leq s$,*

$$G^{n,s}_{i,j} = I^{n,s} + J_s + J_i' + I_j(A|_j) \text{ is radical.}$$

Note: $I_{s+1}(A|_{s+1})$ is the zero ideal thus the case $G^{n,s}_{i,s+1}$ is handled in Proposition 4.11 and the case $G^{n,s}_{0,j}$ is handled by Proposition 4.12 (using the families $F^{n,s}_{s,j}$. We also want to note that $I_j(A|_j) \supset I_{j+1}(A|_{j+1})$.

*P*roof: The result is known by the induction hypothesis and Proposition 4.12 when $i = s$, so fix i such that $0 < i < s$ and assume the result is known for larger i values.

For $1 < j \leq s - i$, Rad$(G^{n,s}_{i,j})$ is prime Theorem 3.3 and by induction hypothesis $G^{n,s}_{i+1,j} = (G^{n,s}_{i,j} + b_{s-i,1})$ is radical. By Lemma 4.1, it suffices to show that $b_{s-i,1}$ is not in $G^{n,s}_{i,j}$. But if we specialize $b_{s-i,1}$ to 1 and all other entries of A and B to zero, we have a point where all the generators of $G^{n,s}_{i,j}$ vanish but $b_{s-i,1}$ does not.

When $j > s-i$, we have a bifurcation:

$$\begin{array}{ccc} & I^{n,s} + J_s + J'_i + I_j(A|_j) \ \subset \ I^{n,s} + J_s + J'_{i+1} + I_j(A|_j) & \\ \swarrow & & \searrow \\ I^{n,s} + J_s + J'_s + I_j(A|_j) & & I^{n,s} + J_s + J'_i + I_{s-i}(A|_{s-i}) \\ \searrow & & \swarrow \\ & I^{n,s} + J_s + J'_s + I_{s-i}(A|_{s-i}) & \end{array}$$

Which may be written as:

$$\begin{array}{ccc} & G^{n,s}_{i,j} \ \subseteq \ (G^{n,s}_{i,j} + b_{s-i,1}) & \\ \swarrow & & \searrow \\ G^{n,s}_{s,j} & & G^{n,s}_{i,s-i} \\ \searrow & & \swarrow \\ & G^{n,s}_{s,s-i} & \end{array}$$

$G^{n,s}_{i,s-i}$ is prime by Theorem 3.3 and $G^{n,s}_{i+1,j} = (G^{n,s}_{i,j} + b_{s-i,1})$ is radical by the induction hypothesis, by Lemma 4.1 it suffices to show that $b_{s-i,1}\ I_{s-i}(A|_{s-i}) \subseteq I_j(A|_j)$ modulo $I^{n,s} + J_s + J'_i$.

Let Δ be any $(s-i)$-sized minor of $A|_{s-i}$ and let M be the matrix corresponding to Δ (i.e., - det $(M) = \Delta$). Write M in terms of its columns, C^M_1. The symmetry condition implies $b_{1,1}\mathsf{C}^M_1 + \cdots + b_{s-i,1}\mathsf{C}^M_{s-i} = [0]$. So by Cramer's rule we have:

Remark 4.10. $b_{k,1}$ annihilates $I_{s-i}(A|_{s-i})$ modulo $I^{n,s} + J_s + J'_i$ for $1 \le k \le s-i$.

This proves our claim. $\square$

Proposition 4.11. *For $n \ge s$ and $0 < i \le s$,*

$$F^{n,s}_{i,s+1} = I^{n,s} + J_s + J'_i \ \textit{is radical.}$$

*P*roof: When $i = s$ the result is known, so assume $1 \le i < s$. For all such i we have a bifurcation:

$$\begin{array}{ccc} & I^{n,s} + J_s + J'_i \ \subseteq \ I^{n,s} + J_s + J'_{i+1} & \\ & \swarrow \qquad \searrow & \\ I^{n,s} + J_s + J'_s & & I^{n,s} + J_s + J'_i + I_{s-i}(A|_{s-i}) \\ & \searrow \qquad \swarrow & \\ & I^{n,s} + J_s + J'_s + I_{s-i}(A|_{s-i}) & \end{array}$$

By Lemma 4.3 and Cramer's rule (see Remark 4.10) we are done. □

Proposition 4.12. *For $n \ge s$, $1 \le i \le s$ and $1 < j \le s$,*

$$F^{n,s}_{i,j} = I^{n,s} + J_i + I_j(A|_j) \text{ is radical.}$$

We prove this in cases:

Lemma 4.13. *For $1 < j \le s$, $F^{n,s}_{s,j}$ is radical.*

*P*roof: $\mathrm{Rad}(F^{n,s}_{s,j})$ is prime and $G^{n,s}_{1,j}$ is radical, so it is enough to show that b_{s1} is not in $F^{n,s}_{s,j}$. But specializing b_{s1} to 1 and all other entries of A and B to zero, we have a point where all the generators of $F^{n,s}_{s,j}$ vanish but b_{s1} does not. So assume $i < s$ and assume the result for larger i.

Lemma 4.14. *For $1 < j \le i < s$, $F^{n,s}_{i,j}$ is radical.*

*P*roof: We already know that $\mathrm{Rad}(F^{n,s}_{i,j})$ is prime and the $F^{n,s}_{i+1,j}$ is radical by the induction hypothesis. So it remains to show that $a_{1,i+1}$ is not in $F^{n,s}_{i,j}$. But specializing $a_{1,i+1}$ to 1 and all other entries to zero, we have a point where all the generators of $F^{n,s}_{i,j}$ vanish but $a_{1,i+1}$ does not. □

Lemma 4.15. *For $1 \le i < s$ and $i = j - 1$, $F^{n,s}_{i,j}$ is radical.*

*P*roof: When this happens, there is a bifurcation:

$$\begin{tikzcd}
 & F^{n,s}_{j-1,j} \subseteq F^{n,s}_{j,j} \arrow[dl] \arrow[dr] & \\
F^{n,s}_{j-1,j-1} \arrow[dr] & & F^{n,s}_{j,j} \arrow[dl] \\
 & F^{n,s}_{j,j-1} &
\end{tikzcd}$$

$F^{n,s}_{j-1,j-1}$ and $F^{n,s}_{j,j}$ are prime, so by Lemma 4.3 we need to show that $a_{1,j}\ I_{j-1}(A|_{j-1}) \subseteq I_j(A|_j)$ modulo $I^{n,s} + J_i$. This is done by Cramer's rule 4.4.

Let Δ be any $(j-1)$-sized minor $A|_{j-1}$ and let M be the matrix corresponding to Δ (i.e., - det $(M) = \Delta$). If $\tilde{M}$ denotes the $j \times j$ minor of A whose first row is $a_{11}, a_{12}, \ldots, a_{1,j}$ and lower left hand $(j-1)$-sized block is M (the remaining entries are forced.) Then by expanding $\det(\tilde{M})$ along the first row shows that $a_{1,j}\ I_{j-1}(A|_{j-1}) \subseteq I_j(A|_j)$ modulo $I^{n,s} + J_i$. $\square$

Lemma 4.16. *For $1 \le i < j-1$, $F^{n,s}_{i,j}$ is radical.*

*P*roof: In this case, we know $\mathrm{Rad}(F^{n,s}_{i,j})$ is prime and $F^{n,s}_{i+1,j}$ is radical. Specializing $a_{1,i+1}$ to 1 and all other entries to zero, we have a point where all the generators of $F^{n,s}_{i,j}$ vanish but $a_{1,i+1}$ does not. $\square$

Proposition 4.17. *For $n \ge s$ and $1 < j \le s+1$,*

$$F^{n,s}_{0,j} = I^{n,s} + I_j(A|_j) \text{ is radical.}$$

*P*roof: The result is known when $j = s+1$, so assume that $1 < j \le s$. In this case, we know $\mathrm{Rad}(I^{n,s} + I_j(A|_j))$ is prime and $I^{n,s} + J_1 + I_j(A|_j)$ is radical. Specializing a_{11} to 1 and all other entries to zero, we have a point where all the generators of $I^{n,s} + I_j(A|_j)$ vanish but a_{11} does not. $\square$

Proposition 4.18. *For $n \geq s$ and $1 \leq i \leq s$,*

$$F^{n,s}_{i,s+1} = I^{n,s} + J_i \textit{ is radical.}$$

*P*roof: When $i = s$ we have a bifurcation:

$$\begin{array}{ccc} & I^{n,s} + J_s \ \subseteq \ I^{n,s} + J_s + J'_1 & \\ & \swarrow \quad \searrow & \\ I^{n,s} + J_s + J'_s & & I^{n,s} + J_s + I_s(A|_s) \\ & \searrow \quad \swarrow & \\ & I^{n,s} + J_s + J'_s + I_s(A|_s) & \end{array}$$

It is enough to show that b_{s1} annihilates $I_s(A|_s)$. But this was shown earlier (see 4.10). So assume the result for larger i values.

For $1 \leq i < s$, $\mathrm{Rad}(I^{n,s} + J_i)$ is prime and $I^{n,s} + J_{i+1}$ is radical. Specializing $a_{1,i+1}$ to 1 and others to zero, we have a point where all the generators of $I^{n,s} + J_i$ vanish but $a_{1,i+1}$ does not. $\square$

This completes the proof of Theorem 4.5. $\square$

CHAPTER 5

Cohen-Macaulayness and Normality

In this chapter we prove that $R^{n,s}$ is Cohen-Macaulay and normal for all $n, s \geq 1$. We refer the reader to Section 3.2 for notational definiteness. We establish the Cohen-Macaulay property using reverse induction on the size of the matrices to prove:

Theorem 5.1. *For $n \geq s$ and $0 \leq i \leq s-1$,*

$$K[A,B]/F^{n,s}_{i,s+1}$$

$$K[A,B]/G^{n,s}_{i,s-i}$$

$$K[A,B]/G^{n,s}_{i+1,s-i}$$

are Cohen-Macaulay rings.

Corollary 5.2. *For all $n, s \geq 1$, $R^{n,s}$ is a Cohen-Macaulay domain.*

First, we will need to calculate the dimension of several quotient rings of our families of ideals.

5.1 Dimension Calculations

We will need the following Theorem, due to Conca [8].

Theorem 5.3. *Let K be a field and Z be an $m \times n$ ($m \leq n$) matrix of indeterminates in which an $s \times s$ submatrix is symmetric. Then the determinantal ring of t-minors* *vanishing,* $K[Z]/I_t(Z)$*, associated to* Z *is a Cohen-Macaulay domain of dimension* $(m+n+1-t)(t-1)-\binom{s}{2}$ *when* $t \leq s$*.*

*P*roof: See [8]. $\square$

Proposition 5.4. *For* $n \geq s$ *and* $1 \leq j \leq s$*,*

the dimension of $K[A,B]/F^{n,s}_{0,j} = K[A,B]/(I^{n,s} + I_j(A|_j))$ *is* $ns + \binom{s}{2} + j - 1$*.*

*P*roof: When $j = 1$ the dimension is $ns + \binom{s}{2}$ by comments preceding Remark 3.5. For $j > 1$, $K[A,B]/(I^{n,s} + I_j(A|_j))$ is a domain so it is enough to calculate the dimension of $(V^{n,s} \cap V(I_j(A|_j)))$ on an open set.

We take the open set defined by β's of maximal rank. Up to the group action, pairs have the form

$$\begin{array}{cc} & \begin{array}{cc} \scriptstyle j & \scriptstyle s\text{-}j \end{array} \\ \begin{array}{c} \scriptstyle j \\ \scriptstyle s\text{-}j \\ \scriptstyle n\text{-}s \end{array} & \begin{pmatrix} \alpha_0 & \alpha_1 \\ \alpha_1^{tr} & \varsigma_1 \\ 0 & 0 \end{pmatrix} \end{array}, \quad \begin{array}{cc} & \begin{array}{ccc} \scriptstyle j & \scriptstyle s\text{-}j & \scriptstyle n\text{-}s \end{array} \\ \begin{array}{c} \scriptstyle j \\ \scriptstyle s\text{-}j \end{array} & \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \end{array}$$

with α_0 and ς_1 symmetric and the j-minors of the first j columns of α vanishing. Let α' denote the $s \times j$ matrix formed by the first j columns and first s rows of α. In block form $\alpha' = \begin{pmatrix} \alpha_0 \\ \hline \alpha_1^{tr} \end{pmatrix}$. The ideal generated by the j-minors of the first j columns of α is the same as $I_j(\alpha')$. So the dimension of $K[A,B]/(I^{n,s} + I_j(A|_j))$ is $ns + \binom{s-j+1}{2}$ (the free entries of β and the free entries of ς_1) plus the dimension of $K[A_0, A_1]/(I_1(A_0 - A_0^{tr}) + I_j(A'))$.

First we note that the ideal generated by the j-minors of α' is the same as the ideal generated by the j-minors of $(\alpha')^{tr}$. By Theorem 5.3 the dimension of $K[A_0, A_1]/(I_1(A_0 - A_0^{tr}) + I_j(A'))$ equals $(s+1)(j-1) - \binom{j}{2}$. Thus the dimension of

$K[A,B]/(I^{n,s} + I_j(A|_j))$ is

$$ns + \binom{s-j+1}{2} + (s+1)(j-1) - \binom{j}{2} = ns + \binom{s}{2} + j - 1.\square$$

Proposition 5.5. *For $n \geq s$, $1 \leq i < s$ and $j = s - i$,*

the dimension of $K[A,B]/G^{n,s}_{i,s-i}$ equals $ns + \binom{s}{2} - i$.

*P*roof: We note that $i + j = s$, hence $G^{n,s}_{i,s-i}$ is a prime ideal by Corollary 4.7. As such we compute the dimension of $K[A,B]/G^{n,s}_{i,s-i}$ on the open set of $V(G^{n,s}_{i,s-i})$ where the β's take maximal rank and b_{11} is nonzero. Up to the group action pairs have the form

$$\begin{array}{c} \\ \scriptstyle 1 \\ \scriptstyle s\text{-}i\text{-}1 \\ \scriptstyle i \\ \\ \end{array}\begin{array}{c} \begin{array}{ccc} \scriptstyle 1 & \scriptstyle s\text{-}i\text{-}1 & \scriptstyle i \end{array} \\ \begin{pmatrix} 0 & 0 & 0 \\ 0 & \alpha_0 & \alpha_1 \\ 0 & \alpha_1^{tr} & \alpha_2 \\ 0 & 0 & 0 \end{pmatrix} \end{array}, \quad \begin{array}{c} \\ \scriptstyle 1 \\ \scriptstyle s\text{-}i\text{-}1 \\ \scriptstyle i \end{array}\begin{array}{c} \begin{array}{cccc} \scriptstyle 1 & \scriptstyle s\text{-}i\text{-}1 & \scriptstyle i & \end{array} \\ \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} \end{array}$$

with α_0 and α_2 symmetric.

The dimension of this open set is the number of free entries in β, $ns - i$, plus the number of free entries of the symmetric $s - 1$ sized matrix. $\square$

We note that by Proposition 5.4 the dimension of $K[A,B]/G^{n,s}_{s,j}$ equals

$$ns - s + \binom{s}{2} + j - 1$$

.

5.2 Cohen-Macaulay Property

We have already established that $R^{n,s}$ is a Cohen-Macaulay for $n \leq s+1$ (2.11) and for $s = 1$ (2.1.1). We first prove:

Theorem 5.6. *For $1 \leq j \leq s$, $K[A,B]/F_{0,j}^{s,s}$ and $K[A,B]/F_{0,j}^{s+1,s}$ are Cohen-Macaulay domains.*

*P*roof: By Remark 3.5 we may assume $j > 1$. It is well-known (see [22]) that the determinantal ring $K[A|_j]/I_j(A|_j)$ is a Cohen-Macaulay, normal domain of dimension $(n+1)(j-1)$. If we let A' denotes the $n \times (s-j)$ matrix formed by the last $s-j$ columns of A and let

$$R_t(A|_j)[A',B] = \frac{K[A|_j]}{I_j(A|_j)}[A',B]$$

denote the polynomial extension of $K[A|_j]/I_j(A|_j)$ in the entries of A' and B, we have that $R_t(A|_j)$ is a Cohen-Macaulay domain of dimension $(n+1)(j-1)+ns+n(s-j) = 2ns-n+j-1$, when $n \geq s$. But in these cases, $K[A,B]/F_{0,j}^{n,s}$ is a domain (4.7) of dimension $ns + \binom{s}{2} + j - 1$ (5.4). When $n = s$ or $n = s+1$, the dimension drops by $\binom{n}{2}$ and $I^{n,s}$ is generated by $\binom{n}{2}$ elements, thus $I^{n,s}$ is generated by a homogeneous system of parameters in a Cohen-Macaulay ring, thus a regular sequence. $\square$

To complete the proof of the Cohen-Macaulay property, we recall some well-known results:

Lemma 5.7. *Let R be an $\mathbb{N}$-graded algebra finitely generated over $R_0 = K$. If x is a nonzerodivisor for R and R/xR is Cohen-Macaulay then R is Cohen-Macaulay.*

Lemma 5.8. *Consider the short exact sequence:*

$$0 \to S/(I \cap J) \to S/I \oplus S/J \to S/(I+J) \to 0$$

If S/I and S/J are Cohen-Macaulay rings of dimension m and if $S/(I+J)$ is Cohen-Macaulay of dimension $m+1$, then $S/(I \cap J)$ is Cohen-Macaulay of dimension m.

*P*roof of Theorem 5.1: Fix $n \geq s$. We assume the result holds for smaller sized matrices. By Remark 3.5 we may further assume that $j > 1$. For $1 \leq$

$i \le s-1$, $K[A,B]/F^{n,s}_{i-1,s+1}$ is a domain and a_{1i} is a nonzerodivisor. Therefore, by Lemma 5.7, we reduce to showing that $K[A,B]/(I^{n,s}+J_s)$ is Cohen-Macaulay. But $K[A,B]/(I^{n,s}+J_s)$ is not a domain. In fact, any minimal prime of J_s (in $R^{n,s}$) contains all of the $s \times s$ minors of A $(= I_s(A|_s))$ or all of the entries of the first column of B $(= J'_s)$. Let $P = I^{n,s}+J_s+I_s(A|_s)$ and $Q = I^{n,s}+J_s+J'_s$. It follows that $V(I^{n,s}+J_s) = V(P) \cup V(Q)$. Since all of these ideals are radical, we have that $I^{n,s}+J_s = P \cap Q$. $\square$

Before going further, we prove:

Proposition 5.9. *$K[A,B]/G^{n,s}_{i,s-i}$ is Cohen-Macaulay for $1 \le i < s$, which implies that $K[A,B]/G^{n,s}_{i,s-i+1}$ is Cohen-Macaulay of the same dimension.*

*P*roof: First note that in these families we have $i+j=s$ and $i+j=s+1$, respectively. When $i=s-1$, $K[A,B]/G^{n,s}_{s-1,1}$ is isomorphic to a polynomial ring in n indeterminates (corresponding to the first row of B) over

$$K[A',B']/I_1(A'B'-(A'B')^{tr})$$

$(A', B'$ are size $(n-1,s-1))$which, by the induction hypothesis, is Cohen-Macaulay. $G^{n,s}_{s-1,2}$ bifurcates as:

$$\begin{tikzcd}[column sep=small] & G^{n,s}_{s-1,2} \arrow[dl] \arrow[dr] & \\ G^{n,s}_{s,2} \arrow[dr] & & G^{n,s}_{s-1,1} \arrow[dl] \\ & G^{n,s}_{s,1} & \end{tikzcd}$$

$G^{n,s}_{s,2}, G^{n,s}_{s-1,1}, G^{n,s}_{s,1}$ are all Cohen-Macaulay and the dimension of the first two is one more than the third. By Lemma 5.8, $G^{n,s}_{s-1,2}$ is Cohen-Macaulay of dimension

$$ns + \binom{s}{2} - s + 1$$

Now assume the result for larger i.

For any $1 \leq i < s-1$, $G^{n,s}_{i,s-i+1}$ bifurcates as:

$$
\begin{tikzcd}
 & G^{n,s}_{i,s-i+1} \arrow[dl] \arrow[dr] & & \\
G^{n,s}_{s,s-i+1} \arrow[dr] & & G^{n,s}_{i,s-i} \arrow[dl] \arrow[r, phantom, "\subseteq"] & G^{n,s}_{i+1,s-i} \\
 & G^{n,s}_{s,s-i} & &
\end{tikzcd}
$$

$G^{n,s}_{s,s-i+1}, G^{n,s}_{s,s-i}, G^{n,s}_{i+1,s-i}$ are all Cohen-Macaulay. Since $G^{n,s}_{i,s-i}$ is prime, $b_{s-i,1}$ is a nonzerodivisor on $K[A,B]/G^{n,s}_{i,s-i}$. The induction hypothesis yields $G^{n,s}_{i+1,s-i}$ is Cohen-Macaulay, so by Lemma 5.7 we have $G^{n,s}_{i,s-i}$ is Cohen-Macaulay. And by Lemma 5.8 and dimension considerations, we have that $K[A,B]/G^{n,s}_{i,s-i+1}$ is Cohen-Macaulay of the same dimension.

Returning to our proof that $R^{n,s}$ is Cohen-Macaulay, we had reduced the question to showing that $K[A,B]/(I^{n,s}+J_s)$ is Cohen-Macaulay. Since $I^{n,s}+J_s$ bifurcates, by Lemma 5.8 its enough if $K[A,B]/(I^{n,s}+J_s+J'_s)$ and $K[A,B]/(I^{n,s}+J_s+I_s(A|_s))$ are Cohen-Macaulay of the same dimension and $K[A,B]/G^{n,s}_{s,s}$ is Cohen-Macaulay of dimension one less. By the induction hypothesis and previous results we have that $K[A,B]/(I^{n,s}+J_s+J'_s)$ and $K[A,B]/G^{n,s}_{s,s}$ are Cohen-Macaulay of the correct dimension. So we've reduced to showing that $K[A,B]/(I^{n,s}+J_s+I_s(A|_s))$ is Cohen-Macaulay of dimension $ns - \binom{s}{2}$. Since this ring is a domain, b_{s1} is a nonzerodivisor and it is enough to show $K[A,B]/G^{n,s}_{1,s}$ is Cohen-Macaulay of dimension $ns + \binom{s}{2} - 1$. So by Lemma 5.7, we're done. $R^{n,s}$ is a Cohen-Macaulay domain of dimension $ns + \binom{s+1}{2}$.

5.3 Normality

Lemma 5.10. *Let S be a Noetherian ring and x an S-regular element. If S_x is a normal domain and S/xS is reduced, then S is normal.*

Theorem 5.11. *For all $n, s \geq 1$, $R^{n,s}$ is a normal domain for all $n, s > 1$.*

We know the result when either $n \leq s$ or $s = 1$, so fix $n > s$ and assume the result holds if either n or s (or both) decrease.

$K[A,B]/(I^{n,s} + J_1)$ is a domain, thus a_{11} is a $R^{n,s}$-regular. So it suffices to show that the localization of $R^{n,s}$ at a_{11} is a normal domain. After localizing a_{11} and performing elementary row and column operation we may assume our pairs have the form:

$$\begin{array}{cc} & \begin{array}{cc} 1 & s\text{-}1 \end{array} \\ \begin{array}{c} 1 \\ n\text{-}1 \end{array} & \begin{pmatrix} 1 & 0 \\ 0 & A' \end{pmatrix} \end{array}, \quad \begin{array}{cc} & \begin{array}{cc} 1 & n\text{-}1 \end{array} \\ \begin{array}{c} 1 \\ s\text{-}1 \end{array} & \begin{pmatrix} B_0 & (A'B_1)^{tr} \\ B_1 & C' \end{pmatrix} \end{array}$$

with $A'C'$ symmetric.

$R^{n,s}$ localized at a_{11} is isomorphic to a polynomial ring over

$$K[A', C'][1/a_{11}]/I_1(A'C' - (A'C')^{tr})$$

which is normal by the induction hypothesis.

Remark 5.12. By previous remarks (2.9) we have that the singular locus contains the ideal $I_s(A) + I_s(B)$. We have shown that $R^{n,s}$ is factorial for $n \leq s$. To show normality for $n > s$ it suffices to show that the singular locus has depth greater than or equal to 2. Since $R^{n,s}$ is Cohen-Macaulay, it is enough to show that the height of $I_s(A) + I_s(B)$ is greater than or equal to 2.

For $n \geq s$, $K[A,B]/(I^{n,s} + I_s(A))$ is a domain thus $\Delta \in I_s(A)$ is $R^{n,s}$-regular. Since $\Delta' \in I_S(B)$ is a nonzerodivisor on $K[A,B]/(I^{n,s} + I_s(A))$ the height of the ideal generated by Δ, Δ' is two and the singular locus contains this ideal. This gives an alternate proof that $R^{n,s}$ is normal for $n \geq s$.

It is well-known that the divisor class group of a unique factorial ring is trivial. Thus, for $n \leq s$, the divisor class group of $R^{n,s}$ is 0. We note the following results (see [5], p. 315):

Let R be a Noetherian normal domain

Gauss' lemma The divisor class group of R is isomorphic to the divisor class group of $R[t]$ (where t is an indeterminate.)

Nagata's theorem If $S \subset R$ is multiplicatively closed and S is generated by prime elements then the divisor class group of R is isomorphic to the divisor class group of $S^{-1}R$.

Consider the case when $n \geq s+1$, by 2.3 and 2.15, we may repeatedly apply Gauss' lemma and Nagata's theorem and eventually have that the divisor class group of $R^{n,s}$ is isomorphic to the divisor class group of $R^{n-s+1,1}$. It is well-known (see, for instance, [6]) that the divisor class group of the determinantal rings defined by the vanishing of the 2×2 minors of a $2 \times n$ matrix is isomorphic to $\mathbb{Z}$. So we have

Corollary 5.13. *The divisor class group of $R^{n,s}$ is trivial for $n \leq s$ and is isomorphic to $\mathbb{Z}$ for $n > s$.*

CHAPTER 6

Linear Homogeneous System of Parameters and the a-invariant

In this chapter we explicitly construct a linear system of parameters for $R^{n,s}$ for all $n, s \geq 1$ and, using Gröbner bases techniques and deformation theory, we show that the **a**-invariant of $R^{n,s}$ is negative for all $n, s \geq 1$.

6.1 Definitions

Throughout this section we fix $S = K[X_1, \ldots, X_n]/I$, a polynomial ring over a field K, and $I \subset (X_1, \ldots, X_n)$. We denote by x_i the residue class of X_i for $i = 1, \ldots, n$ and set $m = (x_1, \ldots, x_n)$. Moreover, for the remainder of this thesis, we fix the order of the indeterminates to be $X_1 > \ldots > X_n$ and we fix the monomial order to be revlex, as defined below.

Definition 6.1. The reverse lexicographic order, denoted by the subscript $_{\text{revlex}}$, is a total ordering of monomials defined as follows: if $\alpha = (a_1, \ldots, a_n)$ and $\beta = (b_1, \ldots, b_n)$, then $x^\alpha >_{\text{revlex}} x^\beta$ means that $\deg(x^\alpha) > \deg\,(x^\beta)$ or that $\deg(x^\alpha) = \deg(x^\beta)$ and there exists an integer j with $1 \leq j \leq n$ such that $a_i = b_i$ for $i > j$ while $a_j < b_j$.

We denote by $\text{in}(f)$ the initial monomial (with respect to revlex) of a polynomial $f \in S$, and by $\text{in}(I)$ the ideal generated by all $\text{in}(f)$, $f \in I$. The reader is referred to ([13], Chapter 15) for a more detailed discussion of Gröbner bases.

If I is, in addition, a homogeneous ideal then $S = K \oplus S_1 \oplus S_2 \oplus \ldots$ is graded and the Hilbert function is defined to be $H_S(i) = \dim_K[S]_i$, while the Hilbert-Poincaré series is defined to be $P_S[t] = \sum_{i=0}^{\infty} \dim_K[S]_i t^i$. For I homogeneous we have $H_S(i) = H_{\mathrm{in}(S)}(i)$ for all i and $\dim(S)$ $=\dim(S/\mathrm{in}(I))$ (see [13]).

If y is a nonzerodivisor in S and has degree l, the short exact sequence of graded modules

$$0 \to S(-l) \xrightarrow{y} S \to S/yS \to 0$$

shows that $P_{S/yS}(t) = (1 - t^l)P_S(t)$. Hence, if $y_1, \ldots, y_d$ is a linear homogeneous system of parameters and if S is, in addition, Cohen-Macaulay, then we have that

$$P_{S/(y_1,\ldots,y_n)S}(t) = (1-t)^d P_S(t)$$

We are interested in showing that the **a**-invariant of $R^{n,s}$ is negative. We make use of the fact that the **a**-invariant of S is the same as the degree of the Hilbert-Poincaré series viewed as a rational function in t. Obviously we have that the **a**-invariant of S equals the **a**-invariant of $S/\mathrm{in}(I)$. But we also have that, in the Cohen-Macaulay case, if $F_1, \ldots, F_d$ is a homogeneous system of parameters for S then the **a**-invariant of S equal the **a**-invariant of $S/(F_1, \ldots, F_d)S - \sum_{i=1}^{d} \deg(F_i)$

Remark 6.2. If S is a finitely generated $\mathbb{N}$-graded Cohen-Macaulay algebra over a field, then the **a**-invariant of S is negative if and only if the **a**-invariant of S modulo a linear homogeneous system of parameters is less than the Krull dimension of S.

6.2 Construction of New Matrices

Let $K[x_i : 1 \leq i \leq 2ns]$ be the polynomial ring over a field in the $2ns$ indeterminates. We construct two matrices (of indeterminates) X, Y of sizes $n \times s$, $s \times n$, respectively. X will be composed of only odd subscripted values and Y of even subscripted values as follows: For $1 \leq k \leq n-1$ the $(n-k)^{th}$ row of X contains $\min\{k, s\}$ new odd subscripted variables on the left, but with subscripts increasing from right to left. For $1 \leq k \leq n-1$ the k^{th} diagonal (counting from the upper right corner) of Y contains $\min\{k, s\}$ new even subscripted variables with subscripts increasing as one moves to the left and up.

Position in X	Subscripted Value	Position in Y	Subscripted Value
$(n-1, 1)$	1	$(1, n)$	2
$(n-2, 2)$	3	$(2, n)$	4
$(n-2, 1)$	5	$(1, n-1)$	6
$(n-3, 3)$	7	$(3, n)$	8
$(n-3, 2)$	9	$(2, n-1)$	10
$(n-3, 1)$	11	$(1, n-2)$	12
$\vdots$	$\vdots$	$\vdots$	$\vdots$
$(1, 1)$	$2d-1$	$(1, 2)$	$2d$

with $d = ns - \binom{s+1}{2}$ for $n \geq s$ or $d = \binom{n}{2}$ for $n \leq s$. The remaining entries are assigned the remaining (largest) subscripted values, but their order is not important for our purposes. We will denote them by *.

Example 6.3. When $n = 6$ and $s = 4$, our matrices have the form:

$$X = \begin{pmatrix} x_{27} & x_{25} & x_{23} & x_{21} \\ x_{19} & x_{17} & x_{15} & x_{13} \\ x_{11} & x_9 & x_7 & * \\ x_5 & x_3 & * & * \\ x_1 & * & * & * \\ * & * & * & * \end{pmatrix}, Y = \begin{pmatrix} * & x_{28} & x_{20} & x_{12} & x_6 & x_2 \\ * & * & x_{26} & x_{18} & x_{10} & x_4 \\ * & * & * & x_{24} & x_{16} & x_8 \\ * & * & * & * & x_{22} & x_{14} \end{pmatrix}$$

When $n = 4$ and $s = 6$, our matrices have the form:

$$X = \begin{pmatrix} x_{11} & x_9 & x_7 & * & * & * \\ x_5 & x_3 & * & * & * & * \\ x_1 & * & * & * & * & * \\ * & * & * & * & * & * \end{pmatrix}, Y = \begin{pmatrix} * & x_{12} & x_6 & x_2 \\ * & * & x_{10} & x_4 \\ * & * & * & x_8 \\ * & * & * & * \\ * & * & * & * \\ * & * & * & * \end{pmatrix}$$

We are interested in the initial ideal, with respect to revlex, of

$$J_{n,s} = I_1(XY - (XY)^{tr}).$$

The $\binom{n}{2}$ polynomials generating $J_{n,s}$ may be thought of as coming from the differences of dot products:

$$\mathtt{R}_i^X \cdot \mathtt{C}_j^Y - \mathtt{R}_j^X \cdot \mathtt{C}_i^Y \quad 1 \leq i < j \leq n$$

Our construction of X, Y has the nice property that, for $1 \leq i < j \leq n$,

$$\text{in}(\mathtt{R}_i^X \cdot \mathtt{C}_j^Y - \mathtt{R}_j^X \cdot \mathtt{C}_i^Y) = \text{in}(\mathtt{R}_i^X \cdot \mathtt{C}_j^Y).$$

Furthermore, by virtue of our choice of ordering, we have that the initial ideal of $J_{n,s}$ contains $x_i x_{i+1}$ for $1 \leq i \leq 2d - 1$, i odd, and where $d = ns - \binom{s+1}{2}$ when $n \geq s$

and $d = \binom{n}{2}$ when $n \leq s$. Note that the two formulas for d agree for $n = s+1$. This is not surprising since, when $n \leq s+1$, $K[X,Y]/J^{n,s}$ is a complete intersection domain (see remark 6.5) and, in these cases, we have that the initial monomials of the generators of $J_{n,s}$, the $x_i x_{i+1}$'s, are mutually distinct and form a regular sequence on $K[X,Y]$. Thus,

Remark 6.4. For $n \leq s+1$ and for i odd, the $x_i x_{i+1}$ actually generate $\mathrm{in}(J_{n,s})$.

In our examples above, after taking the product, it is easy to see that $\mathrm{in}(J_{6,4})$ contains $\{x_1x_2, x_3x_4, x_5x_6, x_7x_8, x_9x_{10}, x_{11}x_{12}, x_{13}x_{14}, \ldots, x_{27}x_{28}\}$, but also contains $x_{21}x_{14}$ and $\mathrm{in}(J_{4,6})$ contains $\{x_1x_2, x_3x_4, x_5x_6, x_7x_8, x_9x_{10}, x_{11}x_{12}\}$.

Remark 6.5. Since there is no danger of confusion, we denote by

$$K[X,Y] = K[x_i : 1 \leq i \leq 2ns].$$

By mapping the (i,j) entry of A, B to the (i,j) entry in X, Y, respectively, we have an obvious isomorphism

$$R^{n,s} \cong K[X,Y]/J_{n,s} := S^{n,s}$$

6.3 Linear Homogeneous System of Parameters

Let Q be the ideal of $K[x_i : 1 \leq i \leq 2ns]$ generated by x_i for $2d+1 \leq i \leq 2ns$ (that is, the *'d entries of both X and Y) as well as $x_i - x_{i+1}$ for $1 \leq i \leq 2d-1$, i odd.

Theorem 6.6. *With the notation above, Q is a linear homogeneous system of parameters for $S^{n,s}$.*

We must show that the number of generators of Q equals the dimension of $S^{n,s}$ and that the dimension of $S^{n,s}/QS^{n,s}$ is zero (or equivalently that $S^{n,s}/QS^{n,s}$ is generated over K by nilpotent elements.) We will return to the proof after establishing:

Proposition 6.7. *The number of generators of Q equals the dimension of $S^{n,s}$.*

*P*roof: By remark 6.5, the dimension of $S^{n,s}$ equals the dimension of $R^{n,s}$. Identifying every x_i in X with either 0 or x_{i+1}, as prescribed above, gives ns generators. As for the matrix Y, we only have the additional *'d entries, which get identified to zero. There are $ns - d$ such entries. So Q is generated by $2ns - d$ linear polynomials. For $n \leq s$,

$$2ns - d = 2ns - \binom{n}{2} = \dim(S^{n,s})$$

and for $n \geq s$,

$$2ns - d = 2ns - (ns - \binom{s+1}{2}) = ns + \binom{s+1}{2} = \dim(S^{n,s}) \ \square$$

*P*roof of Theorem 6.6: We will show that the

$$\dim(K[X,Y]/\mathrm{in}(J_{n,s} + Q)) = 0.$$

This approach will allow us to bound the **a**-invariant of $S^{n,s}$.

The initial ideal of $J_{n,s}$ contains $x_i x_{i+1}$ for $1 \leq i \leq 2d-1$, i odd. It is not hard to see that $\mathrm{in}(J_{n,s} + Q) \supseteq (x_i^2 : 1 \leq i \leq 2d-1, i \text{ odd})$ and, hence, that the dimension of $K[X,Y]/(\mathrm{in}(J_{n,s} + Q))$ is zero. So Q is a linear homogeneous system of parameters for $S^{n,s}$. $\square$

Remark 6.8. For $n \leq s+1$, we have that $\mathrm{in}(J_{n,s}+Q) = \mathrm{in}(J_{n,s})+Q = (x_1^2, x_3^2, \ldots, x^2_{2\binom{n}{2}-1})$. Clearly, the dimension of $K[X,Y]/\mathrm{in}(J_{n,s} + Q)$ is zero, but we also note that the Hilbert function in degree i is given by $\binom{\binom{n}{2}}{i}$ which is well-known, since

$$K[X,Y]/\mathrm{in}(J_{n,s} + Q)$$

is an Artinian complete intersection in $\binom{n}{2}$ indeterminates. By comments preceding remark 6.2, the **a**-invariant of $S^{n,s}$ equals the **a**-invariant of $K[X,Y]/\mathrm{in}(J_{n,s} + Q)$

minus $\dim(S^{n,s})$. Thus, for $n \leq s+1$, the **a**-invariant of $S^{n,s}$ equals

$$\binom{n}{2} - (2ns - \binom{n}{2}) = -2(ns - \binom{n}{2}).$$

For $n > s+1$, the **a**-invariant of $K[X,Y]/\mathrm{in}(J_{n,s}+Q)$ is bounded by the **a**-invariant of $K[X]/(x_1^2,\ldots,x_d^2)$ which is $d = ns - \binom{s+1}{2}$. So, for $n > s+1$, to show the **a**-invariant of $S^{n,s}$ is negative, by 6.2, it is enough to show

$$d = ns - \binom{s+1}{2} < ns + \binom{s+1}{2} = \dim(R^{n,s}).$$

By remark 6.5, we note as a corollary:

Corollary 6.9. *For $n, s \geq 1$, the* **a***-invariant of $R^{n,s}$ is negative.*

We conjecture that, for $n \geq s+1$, the **a**-invariant of $R^{n,s}/QR^{n,s} = \binom{s+1}{2}$ and, hence, the **a**-invariant of $R^{n,s} = -ns$. We note that this holds for $n = s+1$ as well as the determinantal case, $s = 1, n > 1$.

Example 6.10. Let us again consider the case when $s = 1$ and $n > 1$. By previous remarks [2.1.1], $R^{n,1} = K[A,B]/I^{n,1}$ is isomorphic to $K[C]/I_2(C)$:

$$A = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \\ \vdots \\ a_n \end{pmatrix}, B = \begin{pmatrix} b_1 & b_2 & b_3 & b_4 & \ldots & b_n \end{pmatrix}, C = \begin{pmatrix} a_1 & a_2 & a_3 & a_4 & \ldots & a_n \\ b_1 & b_2 & b_3 & b_4 & \ldots & b_n \end{pmatrix}$$

With the notation above, it is known that $(a_n, b_1, a_i - b_{i+1} : 1 \leq i \leq n-1)$ is a system of parameters for $K[C]/I_2(C)$. We note that our construction of a system of parameters agrees with the determinantal case. It is also known that the **a**-invariant of $K[C]/I_2(C)$ is $-n$ [4], which coincides with our above conjecture in the $s = 1$ case.

CHAPTER 7

Rational Singularities in Characteristic 0

The purpose of this chapter is to prove that $V^{n,s}$ has rational singularities in characteristic 0 for all n, s. We first use a criterion for the F-rationality of Cohen-Macaulay rings in the graded case due to Hara and Watanabe to establish that $R^{n,s}$ is F-rational for all n, s if and only if it is F-injective for all n, s. By a result of Conca and Herzog, it suffices to show that $K[A, B]/\mathrm{in}(I^{n,s})$ is F-injective and Cohen-Macaulay for a suitable monomial order on $K[A, B]$. We exhibit a certain specific order on the indeterminates and show, in several cases, that $\mathrm{in}(I^{n,s})$ is generated by square-free monomials and $K[A, B]/\mathrm{in}(I^{n,s})$ is Cohen-Macaulay, hence $R^{n,s}$ is F-rational in these cases. In particular we show that $R^{n,s}$ is F-regular for $n \leq s + 1$, from which it follows, by a result of K.E. Smith that $V^{n,s}$ has rational singularities in characteristic 0 for $n \leq s + 1$. Using the fact that $V^{s,s}$ has rational singularities and a criterion due to Kempf, we obtain that $V^{n,s}$ has rational singularities in characteristic 0 for all n, s.

7.1 F-rationality

Singularities of a variety in characteristic 0 can often be studied by reduction to characteristic $p > 0$ methods. More specifically, by considering the action of Frobenius on the coordinate ring in characteristic $p > 0$.

A result of Hara and Watanabe [19] states,

Theorem 7.1. *Let R be a Cohen-Macaulay, $\mathbb{N}$-graded algebra finitely generated over $R_0 = K$, a perfect field of characteristic $p > 0$. Then R is F-rational if and only if the following three conditions hold: 1) R is F-injective, 2) there exist homogenous elements u_j of positive degree such that every R_{u_j} is F-rational and the u_j generate an ideal primary to the homogenous maximal ideal of R, and 3) the* **a***-invariant of R is negative.*

In our situation, Property 3 follows by reverse induction on the size of the matrices. Indeed, if one localizes $R^{n,s}$ at any $a_{i,j}$ or $b_{i,j}$, the situation is the same as when one localizes at $a_{1,1}$. By 2.3, one gets a polynomial ring, localized at one variable, over a ring of the form $R^{n-1,s-1}$. By 2.1.1, it follows by induction on n or s that this ring is F-rational.

We have already shown 6.9 that the **a**-invariant of $R^{n,s}$ is negative for $n, s \geq 1$. Thus, proving that $R^{n,s}$ is F-rational for all n, s is equivalent to showing that $R^{n,s}$ is F-injective for all n, s.

7.2 Deformation of F-injectivity

A theorem of Conca and Herzog [10] states,

Theorem 7.2. *Let $S = K[X_1, \ldots, X_n]$ be a polynomial ring over a field of characteristic $p > 0$, let I be contained in the homogenous maximal ideal, m, of S. Let $<$ be a monomial ordering and let $in(I)$ denote the initial ideal with respect to $<$.*
If $S/in(I)$ is Cohen-Macaulay and F-injective then S/I is Cohen-Macaulay and F-injective.

The approach to showing that $R^{n,s}$ is F-injective is twofold: first demonstrate an order of the indeterminates and show that the initial ideal of $I^{n,s}$ is generated by square-free monomials. For $n \leq s+1$, this was done in 6.4 and implies the F-purity of $K[A,B]/\mathrm{in}(I^{n,s})$ by Fedder [14] which, in turn implies F-injectivity [26].

Secondly, we need to show that $K[A,B]/\mathrm{in}(I^{n,s})$ is Cohen-Macaulay for a suitable monomial order. For $n \leq s+1$, it was shown in 6.8 that, using revlex, $K[A,B]/\mathrm{in}(I^{n,s})$ is a complete intersection, hence:

Corollary 7.3. *For $n \leq s+1$, $R^{n,s}$ is F-rational in characteristic $p > 0$.*

Which immediately implies, by a result of K.E. Smith [32]:

Corollary 7.4. *For $n \leq s+1$, $V^{n,s}$ has rational singularities in characteristic 0.*

Moreover, since complete intersections are Gorenstein, by a result of Hochster and Huneke [24], we have

Corollary 7.5. *For $n \leq s+1$, $R^{n,s}$ is F-regular in characteristic $p > 0$.*

We conjecture that the rings $R^{n,s}$ are F-regular for all $n, s \geq 1$.

7.3 Rational Singularities in Characteristic 0

The following is a well-known consequence of Kempf's criterion for rational singularities in ([KKMS], p. 50), which asserts that if a Cohen-Macaulay variety X over a field of characteristic 0 has a desingularization $\pi : Y \to X$, then X has rational singularities if and only if the direct image of the canonical sheaf on Y is the canonical sheaf on X.

Lemma 7.6. *Let X be a reduced and irreducible Cohen-Macaulay variety over a field of characteristic 0. Suppose that $\pi : Y \to X$ is a proper birational surjection, and that $U' \subseteq Y$, $U \subseteq X$ are open sets such that the restriction of π to U' gives an isomorphism of U' with U. Suppose that Y has rational singularities, and that $Y - U'$ has codimension at least 2 in Y. Then X has rational singularities.*

*P*roof: Let Y' be a desingularization of Y, which will also be a desingularization of X. The direct image of the canonical sheaf on Y' is the canonical sheaf on Y, since Y has rational singularities. Therefore it suffices to show that the direct image of the canonical sheaf on Y is the canonical sheaf on X. We may restrict attention to an open affine set in X and its inverse image in Y. Therefore, we may assume that X is affine. Let s be a section of the canonical sheaf on X. We may restrict s to U. This gives a section s' of the canonical sheaf on Y restricted to U', and it suffices to show that it extends to all of Y. The problem is local on Y: the extension, if it exists, will be unique, since the canonical sheaf is torsion-free. The defining sheaf of ideals I of $Y - U'$ has height at least two on each open affine, and therefore depth at least two. Thus, the first local cohomology of the canonical sheaf, which is Cohen-Macaulay, with support in I, vanishes, and it follows that the section extends. $\square$

In our cases, X will be $V^{n,s}$ where $n > s + 1$. We think of points in $\mathbb{A}_K^{2ns}$, which we will denote by $\mathbb{A}_K^{n,s} \times \mathbb{A}_K^{s,n}$, as corresponding to pairs of variable matrices, (α, β), of respective sizes $n \times s, s \times n$ over a field K. Given a pair of matrices, $(\alpha, \beta) \in \mathbb{A}_K^{n,s} \times \mathbb{A}_K^{s,n}$, we denote by $[\alpha|\beta^{tr}]$ the $n \times 2s$ matrix whose first s columns are the columns of the matrix α and whose last s column are the columns of the matrix β^{tr}. Let Grass(s, n) denote the Grassmann variety of s-dimensional vector subspaces of affine n-space over K.

Lemma 7.7. *For $(\alpha, \beta) \in V^{n,s}$ and for $n \geq s$, rank$([\alpha|\beta^{tr}]) \leq s$.*

*P*roof: By 2.4, $V^{n,s}$ is irreducible. So it suffices to prove that the $(s+1)$-sized minors of $[\alpha|\beta^{tr}]$ vanish on a Zariski open, dense set. Therefore, we may assume that $\alpha\beta$, α, and β all have maximal rank $(= s)$. But then the column space of α must be the same as the column space of $\alpha\beta$ and the transposed row space of β must be the same as the row space of $(\alpha\beta)^{tr}$. Since $\alpha\beta = (\alpha\beta)^{tr}$, we have that the column space of α equals the column space of β^{tr}. $\square$

Corresponding to the notation in Lemma 7.6, we define $Y = Y^{n,s}$ to be the closed algebraic subset of $\mathbb{A}_K^{n,s} \times \mathbb{A}_K^{s,n} \times \mathrm{Grass}(s,n)$ given by triples (α, β, H) where $(\alpha, \beta) \in V^{n,s}$, $H \subseteq \mathrm{Grass}(s,n)$ and $\mathrm{Im}([\alpha|\beta^{tr}]) \subseteq H$. We note that given $(\alpha, \beta) \in V^{n,s}$ there exists $H \subseteq \mathrm{Grass}(s,n)$ such that H contains the column space of $[\alpha|\beta^{tr}]$. Furthermore, H is unique if $\mathrm{rank}([\alpha|\beta^{tr}]) = s$.

The projection

$$\mathbb{A}_K^{n,s} \times \mathbb{A}_K^{s,n} \times \mathrm{Grass}(s,n) \to \mathbb{A}_K^{n,s} \times \mathbb{A}_K^{s,n}$$

is a projective, hence proper, morphism since $\mathrm{Grass}(s,n)$ is a projective variety. Composing and restricting the range we have a surjective, projective morphism

$$Y \xrightarrow{\pi} V^{n,s}$$

which is an isomorphism on the open, dense subset of $V^{n,s}$ where $\mathrm{rank}([\alpha|\beta^{tr}]) = s$ (since H is uniquely determined in this case).

To show $Y^{n,s}$ has rational singularities, we give a finite open cover of $\mathrm{Grass}(n,s)$ by open sets $U_i \cong \mathbb{A}^{s(n-s)}$ such that the inverse image, W_i, of U_i in $Y^{n,s}$ has the property that $W_i \cong V^{s,s} \times U_i$. The U_i are the sets defined by the non-vanishing of a Plücker coordinate. Without loss of generality, we may assume that, if the $n \times s$ matrix whose column space represents H is γ, the top $s \times s$ minor of γ does not vanish.

Assuming the top minor does not vanish, we get a unique basis for H such that the top $s \times s$ submatrix of γ is the identity. This gives an isomorphism of $U_i \cong \mathbb{A}^{s(n-s)}$

by letting H correspond to this $(n-s)\times s$ matrix, γ_0, formed from the last $(n-s)$ rows of γ.

$$\begin{array}{ccccc} Y & \hookrightarrow & W_i & \cong & V^{s,s}\times \mathbb{A}^{s(n-s)} \\ \downarrow & & \downarrow & & \downarrow \\ \mathrm{Grass}(n,s) & \hookrightarrow & U_i & \cong & \mathbb{A}^{s(n-s)} \end{array}$$

Once we have a basis for H, and so a unique representation matrix γ, we get an isomorphism of the fiber over H

$$\{(\alpha,\beta)\in V^{n,s}\mid \mathrm{Im}([\alpha|\beta^{tr}])\subseteq H\}\times H$$

with $V^{s,s}$.

There exist unique $s\times s$ matrices α',β' such that $\alpha=\gamma\alpha'$, $\beta=\beta'\gamma^{tr}$ and $\alpha'\beta'=(\alpha'\beta')^{tr}$. There is a one-to-one correspondence that sends (α,β,H) to (α',β',γ).

We have established in 7.4 that $V^{s,s}$ has rational singularities. Thus $V^{s,s}\times\mathbb{A}_K^{(n-s)s}$ has rational singularities and, hence, so does $Y^{n,s}$. It remains to show that the dimension of the variety

$$\{(\alpha,\beta,H)\subset V^{n,s}\times \mathrm{Grass}(s,n)\mid \mathrm{rank}(\mathrm{Im}([\alpha|\beta^{tr}]))<s\}$$

is less than or equal to $\dim(V^{n,s})-2=ns+\binom{s+1}{2}-2$.

The dimension of $\mathrm{Grass}(s,n)$ is $s(n-s)$ and, by symmetry, all the fibers have the same dimension, so it remains to show that the dimension of a fiber of π is less than or equal to $s^2+\binom{s+1}{2}-2$. The dimension of $V^{s,s}$ is $s^2+\binom{s+1}{2}$ [2.8], so it suffices to show that the height of the ideal $I_s(A)+I_s(B)$ is greater than or equal to 2. But this was shown in Remark 5.12.

By Lemma 7.6,

Theorem 7.8. *$V^{n,s}$ has rational singularities in characteristic 0 for $n>s+1$*

Corollary 7.9. *$V^{n,s}$ has rational singularities in characteristic 0 for all n, s.*

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ABSTRACT

On the Varieties of Pairs of Matrices whose Product is Symmetric

by

Charles Christopher Mueller

Chair: Melvin Hochster

We study the varieties and their coordinate rings of pairs of matrices of indeterminates whose product is symmetric. Hochster and Eagon developed the method of Principal Radical Systems to show that determinantal rings are Cohen-Macaulay normal domain. We use this method to show our rings are Cohen-Macaulay normal domains.

We explicitly construct a linear homogeneous system of parameters for our rings and show that the **a**-invariant of our rings is negative. Thus, we reduce the question of F-rationality to that of F-injectivity by a criterion due to Hara and Watanabe. Using Gröbner bases techniques and deformation theory, along with a theorem of Conca and Herzog we establish, in several cases, that our rings are F-injective, hence F-rational in these cases.

A result of K.E. Smith implies that, in these cases, the corresponding varieties have rational singularities in characteristic 0. Using these results and a criterion

due to Kempf, we establish that the varieties of pairs of matrices whose product is symmetric have rational singularities in characteristic 0.