Nonnegativity of Intersection Multiplicities in Ramified Regular Local Rings following Gabber/De Jong/Berthelot

What follows is an exposition of Gabber's proof that intersection multiplicities in the sense of Serre are nonnegative, which uses a result of de Jong on the existence of alterations, a weak form of resolution of singularities. The proof is aimed at the only open case, ramified regular local rings (of mixed characteristic). The argument also shows that intersection multiplicities vanish when they are supposed to. Strict positivity remains an open question, so far as I know. The argument presented here is based on an exposition of Berthelot, *P. Berthelot, Altérations de variétes algébriques [d'après A. J. de Jong], Séminaire BOURBAKI, 48ème année, n^O 815, pp. 815-01 – 815-39.* The work of de Jong may be found in *A. J. de Jong, Smoothness, semi-stability, and alterations, Preprint 916, Univ. Utrecht (1995), to appear Publ. Math. I.H.E.S.* and in *A. J. de Jong, Families of curves and alterations, preprint (1996).* References to the work of Serre are all to his classic lecture notes, *J.-P. Serre, Algèbre locale · Multiplicités, Springer-Verlag Lecture Notes in Math. No. 11, Séconde Edition, 1965.*

I have provided more detail than Berthelot concerning a number of points in the proof (and less in some instances). I have avoided the use of derived categories, although some familiarity with spectral sequences is essential. Also, the argument given for the independence of multiplicity from the choice of a section when intersecting with a constant section of a trivial vector bundle over a projective scheme is different from that given by Berthelot, and the use of Artin approximation follows an argument of S. P. Dutta.

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1. The main case

By using Artin approximation and making a faithfully flat extension of a coefficient ring (which in this case will be a complete discrete valuation ring in which the residual characteristic generates the maximal ideal), one can reduce to studying the problem when (A, m, K) is a regular local ring such that A has characteristic 0 but the characteristic p of $A/m = K$ is positive and $p \in m^2$, and such that K is algebraically closed and A is essentially of finite type over a complete discrete valuation ring. Details are given in an Appendix.

Let $X = \operatorname{Spec} A$. We are studying $Y = \operatorname{Spec} A/P$ and $Z = \operatorname{Spec} A/Q$ where P, Q are prime ideals of A such the only point of $Y \cap Z$ is the closed point s of X, i.e., such that $P+Q$ is primary to m. We shall write also that $S_1 = \operatorname{Spec}(A/m)$, so that s is the unique point of S_1. By the early results of Serre, these hypotheses imply that $\dim Y + \dim Z \leq \dim X$. We are trying to show by induction on $\dim Y + \dim Z$ that $\chi^A(Y, Z) \geq 0$, with equality if $\dim Y + \dim Z < \dim X$, where $\chi^A(Y, Z) = \sum_i (-1)^i \ell(\operatorname{Tor}_i^A(A/P, A/Q))$, and where ℓ indicates the length of a finite length A-module.

2. The use of an alteration — shifting to projective space

One chooses, using the results of de Jong, a surjective projective morphism $Z' \to Z$, where Z' is regular and such that the restriction to some non-empty open affine of Z is a finite morphism. Let μ be the degree of the extension of function fields. We may view Z' as closed in a projective space $\mathbb{P} = \mathbb{P}_A = \mathbb{P}_A^N$ over A, and let Y' be the full inverse image of Y in $\mathbb{P}$.

From the induction hypothesis, the projection formula, and a spectral sequence argument one can show (and we give further details below) that

$$\chi^{\mathbb{P}}(Y', Z') = \mu \chi^A(Y, Z),$$

where, by definition,

$$\chi^{\mathbb{P}}(Y', Z') = \sum_{i,\, j} (-1)^{i+j} \ell(H^i(\mathbb{P}, \mathcal{T}or_j^{\mathcal{O}_{\mathbb{P}}}(\mathcal{O}_{Y'}, \mathcal{O}_{Z'}))).$$

Here, $\mathcal{T}or$ indicates sheaf Tor. All but finitely many of the sheaf Tor's vanish since $\mathbb{P}$ is regular. Since these sheaf Tor's are supported only on $Y' \cap Z'$, which maps to s under the map $\mathbb{P} \to X$, the Grothendieck cohomology of the sheaf Tor's is supported only at s and so has finite length.

One may see that the displayed equality holds in an elementary way as follows. First choose a finite free resolution of A/P over A in which the modules occurring are finitely generated free A-modules, and let $G_\bullet$ denote the resolution, but with A/P replaced by 0.

The pullback of this complex to $\mathbb{P}$ gives a resolution of $\mathcal{O}_{Y'}$ by free $\mathcal{O}_{\mathbb{P}}$-modules. Now we may apply $\otimes_{\mathcal{O}_{\mathbb{P}}}\mathcal{O}_{Z'}$ to obtain a complex $\mathcal{F}_\bullet$ that may be thought of as $G_\bullet \otimes_A \mathcal{O}_{Z'}$: its homology gives the sheaf Tor's of $\mathcal{O}_{Y'}$ with $\mathcal{O}_{Z'}$ over $\mathcal{O}_{\mathbb{P}}$. Then one has that

$$\chi^{\mathbb{P}}(Y', Z') = \sum_{i,\, j}(-1)^{i+j}\ell(H^i(\mathbb{P}, \mathcal{T}or_j^{\mathcal{O}_{\mathbb{P}}}(\mathcal{O}_{Y'}, \mathcal{O}_{Z'}))) = \sum_{i,\, j}(-1)^{i+j}\ell(H^i(\mathbb{P}, H_j(\mathcal{F}_\bullet))),$$

which, by a spectral sequence argument, is the same as

$$\sum_{i,\, j}(-1)^{i+j}\ell(H_j(H^i(\mathbb{P}, \mathcal{F}_\bullet))).$$

(One may form an injective Cartan-Eilenberg resolution of the finite complex $\mathcal{F}_\bullet$ in the category of sheaves of abelian groups. We refer the reader to Chapter XVII, Section 1. of *H. Cartan and S. Eilenberg, Homological Algebra, Oxford Univ. Press, 1956* for details, but we do recall that, in particular, the Cartan-Eilenberg resolution is a double complex such that the i th column gives an injective resolution of $\mathcal{F}_i$ (but with $\mathcal{F}_i$ replaced by 0), while the homology of the rows consists of injectives and the homology of the rows taken at the i th spot gives an injective resolution of $H_i(\mathcal{F}_\bullet)$. One gets a new double complex by taking the global sections of every injective occurring. (For the rows, it turns out that the construction of the Cartan-Eilenberg complex is such that taking global sections and taking homology of rows commute.) The two spectral sequences for iterated (co)homology associated with this double complex of global sections give the required comparison of Euler characteristics.)

But the complex

$$H^i(\mathbb{P}, \mathcal{F}_\bullet) = H^i(\mathbb{P}, G_\bullet \otimes_A \mathcal{O}_{Z'}) \cong G_\bullet \otimes_A H^i(\mathbb{P}, \mathcal{O}_{Z'}),$$

so that

$$H_j(H^i(\mathbb{P}, \mathcal{F}_\bullet)) \cong \mathrm{Tor}_j^A(A/P, H^i(\mathbb{P}, \mathcal{O}_{Z'}))$$

and so

$$\chi^{\mathbb{P}}(Y', Z') = \sum_{i,\, j}(-1)^{i+j}\ell(\mathrm{Tor}_j^A(A/P, H^i(\mathbb{P}, \mathcal{O}_{Z'}))),$$

and the right hand side can be rewritten as

$$\sum_{i}(-1)^i\chi^A(A/P, H^i(\mathbb{P}, \mathcal{O}_{Z'})).$$

Because the calculation of $H^i(\mathbb{P}, \mathcal{O}_{Z'})$ commutes with localization on A, we know that for $i \geq 1$ each $H^i(\mathbb{P}, \mathcal{O}_{Z'})$, viewed as an A-module, has support strictly smaller than $V(Q)$, and so all the $\chi^A(A/P, H^i(\mathbb{P}, \mathcal{O}_{Z'}))$ vanish for $i \geq 1$ by the induction hypothesis. Moreover, when $i = 0$ the same fact concerning localization shows that $H^0(\mathbb{P}, \mathcal{O}_{Z'})$ has a prime filtration in which μ of the factors are copies of A/Q and the remaining factors are prime cyclic A/Q-modules of smaller dimension than A/Q. The induction hypothesis shows that the factors of smaller dimension contribute 0 to the Euler characteristic, and the desired statement follows.

3. Passing to the normal bundle — a spectral sequence of Serre

Consider a finitely generated module M over a Noetherian ring B and suppose that we have elements $x_1, \ldots, x_r$ generating an ideal I of B. We can filter the Koszul complex $K_\bullet = K_\bullet(x_1, \ldots, x_r; M)$ so that the tth filtered piece of K_i is $I^{t-i}K_i$, where $I^{t-i} = B$ when $t \leq i$. Note that this makes sense because the entries of the matrices defining the maps in the Koszul complex are in I. Let X_i denote the image of x_i in I/I^2, the first graded piece of $\mathrm{gr}_I B$. The spectral sequence associated to this filtration has E_1 term $H_\bullet(X_1, \ldots, X_r; \mathrm{gr}_I M)$ and converges to (an associated graded of) $H_\bullet(x_1, \ldots, x_r; M)$. If $x_1, \ldots, x_r$ is a regular sequence, then $C = \mathrm{gr}_I B$ is a polynomial ring in the X_i over B/I. See Serre, Ch. IV, Section 3. In particular, $X_1, \ldots, X_r$ is also a regular sequence and both $H_\bullet(x_1, \ldots, x_r; M)$ and $H_\bullet(X_1, \ldots, X_r; \mathrm{gr}_I M)$ can be reinterpreted as Tor's. Thus, there is a spectral sequence whose E_1 term is $\mathrm{Tor}_\bullet^C(B/I, \mathrm{gr}_I M)$ which converges to (an associated graded of) $\mathrm{Tor}_\bullet^B(B/I, M)$, where B/I is viewed as the cyclic C-module obtained by killing the ideal spanned by all elements of positive degree in C. When B is local, this spectral sequence turns out to be independent of the choice of minimal generators for I (still assuming, of course, that I is generated by a regular sequence of length r). One chooses an invertible matrix that takes one set of generators to the other. This matrix induces an isomorphism both of the original Koszul complexes (automatically compatible with the filtrations, which are staggered I-adic) and also of the Koszul complexes of associated gradeds. Once one identifies the Koszul complexes with the corresponding Tor's, one has a canonical spectral sequence over the local ring, independent of the choice of generators for the complete intersection. This set-up consequently globalizes: B may be replaced by a Noetherian scheme, M by a coherent sheaf on B, and I by a sheaf of ideals that is generated locally by a regular sequence of length r. We can define the spectral sequence locally on the open affines of a cover: they need to be sufficiently small that the ideal is a complete intersection on each of them. We then need to check that they "glue" correctly on overlaps of sets in the cover. But if we consider any point in the overlap, they do agree in a small neighborhood of the point, since we may check this over the local ring of the point, where the spectral sequence is canonical, i.e., independent of the choice of generators for the complete intersection. We can now apply this to our situation.

Let $\mathcal{J}$ denote the sheaf of ideals on $\mathbb{P}$ that defines Z'. Let E be the (total space of) the normal bundle to Z' in $\mathbb{P}$, i.e., $E = \mathcal{S}\mathrm{pec}\, \mathrm{gr}_{\mathcal{J}} \mathcal{O}_{\mathbb{P}}$, where $\mathcal{S}\mathrm{pec}$ indicates sheaf Spec. Then the globalized Serre spectral sequence yields the fact that

$$\chi^{\mathbb{P}}(\mathcal{O}_{Y'}, \mathcal{O}_{Z'}) = \chi^{E}(\mathcal{G}, \mathcal{O}_{Z'}),$$

where $\mathcal{G} = \mathcal{S}\mathrm{pec}\, \mathrm{gr}_{\mathcal{J}} \mathcal{O}_{Y'}$ is viewed as a sheaf on E and $\mathcal{O}_{Z'}$ on the right indicates the copy of Z' which corresponds to the zero section of the bundle E. Here, χ^E is defined as a double alternating sum of Grothendieck cohomology of sheaf Tor's once again — but now, the reason that there are only finitely many sheaf Tor's is that the zero section of a vector bundle is locally a complete intersection in (the total space of) the bundle. (If one has a short exact sequence of coherent sheaves with finite length Grothendieck cohomology, say $0 \to \mathcal{F} \to \mathcal{G} \to \mathcal{H} \to 0$, then, by the long exact sequence for Grothendieck cohomology, $\chi(\mathcal{G}) = \chi(\mathcal{F}) + \chi(\mathcal{H})$, where $\chi(\mathcal{F})$ denotes the alternating sum of the lengths of the Grothendieck cohomology of $\mathcal{F}$. Now suppose that one has a finite complex $\mathcal{F}_\bullet$ of coherent sheaves such that every $\mathcal{F}_i$ and every $H_i(\mathcal{F}_\bullet)$ has finite length Grothendieck cohomology. Then $\sum_i (-1)^i \chi(\mathcal{F}_i) = \sum_i (-1)^i \chi(H_i(\mathcal{F}_\bullet))$, since χ is additive. This enables us to use spectral sequences to compare alternating sums of values of χ. Note also that if we use the defintion of χ to eliminate it from the notation, we get the kind of double alternating sum that we have been discussing.)

4. Passing to the closed fiber

There is a power of m that kills $\mathcal{G}$ and one may consider instead of $\mathcal{G}$ the various factors G that occur in the m-adic filtration. When working with one of these G, one may replace E by its fiber E_s over the closed point and Z' by Z'_s. We now think of Z'_s as the zero section of the bundle E_s. We now want to show that $\chi^{E_s}(G, \mathcal{O}_{Z'_s}) \geq 0$, with equality when the dimension of the support of G is less than r, the rank of E_s (and of E). (It turns out that $\dim X - (\dim Y + \dim Z) = r - \dim \mathcal{G}$, and the dimension of $\mathcal{G}$ is the supremum of the dimensions of the supports of the various factors G.) The result for $\mathcal{G}$ then follows from the additivity of χ^E.

5. A vector bundle argument

We shall show in the next two sections that we can map a trivial bundle over $W = Z'_s$ onto E_s, and this is a point we want to examine in considerable detail. Once we have done this, we may prove nonnegativity by replacing E_s by this trivial bundle over Z'_s, and G by its pullback. One may check locally that the sheaf Tor's don't change. Since the dimension of the support of G and the dimension of the fiber have increased by the same amount, we have not disturbed whether a given inequality or equality holds between them. Note that we will now be working with a new value for r.

Think of the trivial bundle as $\mathbb{A} \times W$, where $\mathbb{A}$ is an affine space of some dimension which we shall take to be the new value of r. Then there is a section $\lambda \times W = W(\lambda)$ through each point of λ of $\mathbb{A}$, say with defining sheaf $\mathcal{O}_{W(\lambda)}$. The next point is that $\chi^{\mathbb{A}\times W}(G, \mathcal{O}_{W(\lambda)})$ is independent of the choice of λ. Let $X_1, \ldots, X_r$ denote coordinate functions in $\mathbb{A}$, so that we may think of the structure sheaf of $\mathbb{A} \times W$ as $\mathcal{O}_W[X_1, \ldots, X_r]$, so that if U is an open affine in W the ring of global sections on U is $\mathcal{O}_W(U)[X_1, \ldots, X_r]$. Then the complex $K_\bullet(X_1 - \lambda_1, \ldots, X_r - \lambda_r; G)$ has its homology the sheaf Tor's that we need to use to calculate $\chi^{\mathbb{A}\times W}(G, \mathcal{O}_{W(\lambda)})$. By the same spectral sequence argument used in the second section (depending on a Cartan-Eilenberg resolution) we can see that $\chi^{\mathbb{A}\times W}(G, \mathcal{O}_{W(\lambda)})$ may be thought of as

$$\sum_{i,j} (-1)^{i+j} \ell(H_j(X_1 - \lambda_1, \ldots, X_r - \lambda_r; H^i(\mathbb{A} \times W, G))),$$

where each $N = N_i = H^i(\mathbb{A} \times W, G)$ may also be thought of as an i th higher direct image of G under the projection map $\mathbb{A} \times W \to \mathbb{A}$. Since this is a projective morphism, each N is a finitely generated module over $K[X_1, \dots, X_r]$. The fact that $\chi^{\mathbb{A}\times W}(G, \mathcal{O}_{W(\lambda)})$ is independent of λ now follows from the observation that, for any such module N,

$$\sum_j (-1)^j \ell(H_j(X_1 - \lambda_1, \dots, X_r - \lambda_r; N))$$

does not depend on λ: it is, in fact, the torsion-free rank of N, since, as a function of N, it vanishes on modules of dimension smaller than r, has the value 1 when $N = K[X_1, \dots, X_r]$, and is additive (these remarks follow from Serre: in fact, this number is the intersection multiplicity of N with $K[X_1, \dots, X_r]/(X_1 - \lambda_1, \dots, X_r - \lambda_r)$ at the point λ).

We now study $\chi^{\mathbb{A}\times W}(G, \mathcal{O}_{W(\lambda)})$ from a different point of view: we choose λ carefully and think of the iterated (co)homology in the other order. Specifically, if $\dim G < r$ its projection onto $\mathbb{A}$ is not all of $\mathbb{A}$, and we can choose λ so that the the support of G does not meet $W(\lambda)$ and so the sheaf Tor's vanish. If $\dim G = r$ (i.e., the dimension of its support is r) we might still be able to choose a point not in the projection of the support of G on $\mathbb{A}$ (actually we would prefer to show this does not happen for some one of the factors G, which would then establish strict positivity when the sum of the dimensions of Y and Z is the dimension of the ambient space), but if the projection map is onto, since (the support of) G has dimension r and so does $\mathbb{A}^r$, there must still be a point λ with a finite inverse image in the support of G. Since the intersection of the support of G with $W(\lambda)$ is then finite for this choice of λ, the higher Grothendieck cohomology all vanishes, while the sheaf Tor's can be computed as a direct sum of ordinary Tor's over local rings at the finitely many points in the support. Each of these ordinary Tor's can be viewed as Koszul homology because the section is defined locally by a regular sequence. The Euler characteristics of Koszul homology are then positive, by Serre's results.

Thus, the main point that we have not addressed is why one can map a trivial bundle onto E_s: this is done in the next two sections.

6. The injectivity of a map induced by d

We shall see that mapping a trivial bundle onto E_s comes down to showing the injectivity of a map induced by d after tensoring with a residue field. What follows is a local description of the map, and a more complete explanation of what is happening.

Recall that (A, m, K) is a ramified regular local ring with K algebraically closed, so that A/m^2 has, canonically, the structure of a K-algebra: the characteristic is now p and there is a unique coefficient field. (The argument can be made to work in the equicharacteristic case, but, on the face of it, it does not work for unramified mixed characteristic regular local rings. However, one can get around this simply by adjoining a square root of p to the ring. One can replace modules over the original ring by their tensor products with the new ring, and the Tor's get tensored with the new larger regular ring. The issues are unaffected, since dimensions don't change, and modules of finite length stay of finite length when one tensors, but the length is multiplied by the length of the closed fiber of the extension of regular rings. Thus, one can reduce to the ramified case.)

Let $A[u_1, \ldots, u_N] = A[u]$ correspond to one of the open affine pieces U of projective space over A. Let the intersection of Z' with U be defined by $I \subseteq A[u]$. Let s be the closed point of $\operatorname{Spec} A$, so that the intersection of Z'_s with U is defined by $I + mA[u]$.

The restriction of the map d to U is constructed in the discussion that follows. Since $B = (A/m^2)[u]$ is a K-algebra, there is a well-defined K-linear derivation $d : B \to \Omega_{B/K}$. By composition with $A[u] \to B$ one gets a $\mathbb{Z}$-linear map $A[u] \to \Omega_{B/K}$ which is a derivation. For any ideal J of $A[u]$ we may restrict this map to J and then we get a composite map $J \to \Omega_{B/K} \to (B/J) \otimes_B \Omega_{B/K}$. From the fact that d is a derivation one sees that this map kills J^2 and that the induced map $J/J^2 \to (B/J) \otimes_B \Omega_{B/K}$ is (B/J)-linear. We shall continue to denote the map $J/J^2 \to (B/J) \otimes_B \Omega_{B/K}$ by the letter d. Note that if $J \subseteq J'$ we have a composite map

$$J/J^2 \to (B/J) \otimes_B \Omega_{B/K} \to (B/J') \otimes_B \Omega_{B/K}$$

which we shall also denote by d, although if we need to be precise we may use the notation $d_{J,J'}$. It is immediate from the fact that all versions of d are induced by the usual universal derivation $d : B \to \Omega_{B/K}$ that if $J \subseteq J' \subseteq J''$ and one considers

$$J/J^2 \to J'/J'^2 \to (B/J'') \otimes_B \Omega_{B/K},$$

where the first map is induced by $J \subseteq J'$ and the second map is $d_{J',J''}$ then the composite is $d_{J,J''}$. Of course, since we have a surjection $A[u] \to B$, the various tensor products over B can be written over $A[u]$ instead.

In particular, by taking $J = I$ and $J' = I + mA[u]$ we obtain $d : I/I^2 \to C \otimes_{A[u]} \Omega_{B/K}$, where $C = A[u]/(I + mA[u])$. Here, $I + mA[u]$ defines the intersection of Z'_s with U. This is a "piece" of the map we are interested in, but restricted to the open affine U. Now we want to show that this map is an injection once we tensor with the residue field of a closed point of Z'_s that lies in U. We think of this closed point as a point of $\operatorname{Spec} A[u]$, so that it corresponds to a maximal ideal M of $A[u]$ containing $I + mA[u]$. Taking $J' = J'' = M$, we see that the map

$$(A[u]/M) \otimes_{A[u]} (I/I^2) \to (A[u]/M) \otimes_{A[u]} (C \otimes_{A[u]} \Omega_{B/K})$$

or

$$K \otimes_{A[u]} (I/I^2) \to K \otimes_{A[u]} \Omega_{B/K}$$

factors

$$K \otimes_{A[u]} (I/I^2) \to M/M^2 \to K \otimes_{A[u]} \Omega_{B/K},$$

where K on the left in these tensor products should be thought of as $A[u]/M$. We may localize at M before killing M, and then, since $A[u]_M$ is regular and $A[u]_M/I_M$ is regular (since Z' is), I_M is generated by part of a regular system of parameters for $A[u]_M$, and this shows that the first map is injective. The second is injective by an easy calculation: the main point is that the Zariski cotangent space M/M^2 for $A[u]$ for any maximal ideal M containing m is isomorphic in an obvious way with the Zariski cotangent space at the corresponding point of $B = (A/m^2)[u]$, since $m^2 \subseteq M^2$, and for the K-algebra B, we may identify the Zariski cotangent space with $(A[u]/M) \otimes_{A[u]} \Omega_{B/K}$. Thus, it is essential to work with $(A/m^2)[u]$ in this part of the argument: one cannot simply work with $(A/m)[u]$ instead.

7. Mapping a trivial bundle onto the fiber of the normal bundle

Recall that a sheaf is generated by global sections if one can map a direct sum of copies of the structure sheaf onto it (this direct sum can be taken to be finite in the case of a coherent sheaf). To show that we can map a trivial bundle onto the total space of the closed fiber of the normal bundle, we must show that, if $\mathcal{H}om$ indicates sheaf Hom and $\mathcal{I}$ defines Z' in $\mathbb{P} = \mathbb{P}^N_A$, then the sheaf

$$\mathcal{H}om_{\mathcal{O}_{Z'_s}}(\mathcal{O}_{Z'_s} \otimes_{\mathcal{O}_{Z'}} \mathcal{I}/\mathcal{I}^2,\, \mathcal{O}_{Z'_s})$$

is generated by global sections, and this sheaf may be identified with

$$\mathcal{H}om_{\mathcal{O}_{Z'}}(\mathcal{I}/\mathcal{I}^2, \mathcal{O}_{Z'_s}).$$

(When no base is specified for sheaf Hom it may be taken over $\mathcal{O}_{\mathbb{P}}$, although frequently one may use the structure sheaf of a smaller subscheme whose defining sheaf of ideals annihilates both sheaves.)

We proceed as follows. First, recall that $S_1 = \operatorname{Spec} K$ and let $X_1 = \operatorname{Spec}(A/m^2)$. Since $p \in m^2$ and K is algebraically closed and, in particular, is perfect, the ring A/m^2 has characteristic p and has a unique coefficient field $K \subseteq A/m^2$, so that X_1 is a scheme over K in a canonical fashion. In fact for indeterminates $T_1, \ldots, T_n$ over K, $A/m^2 \cong K[T_1, \ldots, T_n]/I^2$, where $I = (T_1, \ldots, T_n)$. Recall that $\mathbb{P} = \mathbb{P}^N_A$. Note that $P_1 = X_1 \otimes_A \mathbb{P}$ has the structure of a scheme over K, since X_1 does, and that it may also be identified with $X_1 \times_{S_1} \mathbb{P}^N_K$. Also, $\mathbb{P}^N_K$ may be thought of as $\mathbb{P}_s$. If we let $\Omega^1_{P_1} = \Omega^1_{P_1/S_1}$, this product decomposition induces a direct sum decomposition

$$\Omega^1_{P_1} \cong \Omega^1_{P_1/X_1} \oplus \pi^*\Omega^1_{X_1/S_1}$$

where π is the product projection $P_1 \to X_1$. One can describe this direct sum decomposition over an open affine $U = \operatorname{Spec} A[u_1, \ldots, u_N]$ in algebraic terms as follows:

$$\Omega^1_{P_1}(U) \cong \Omega^1_{(A/m^2)[u]/K} \cong (\oplus_{i=1}^N (A/m^2)du_i) \oplus (A/m^2[u]) \otimes_{A/m^2} \Omega^1_{(A/m^2)/K}$$

(in fact, for any two K-algebras D, D', we have that there is an isomorphism $\Omega^1_{D \otimes_K D'/K} \cong D \otimes_K \Omega^1_{D'/K} \oplus D' \otimes_K \Omega^1_{D/K}$); here, take $D = A/m^2$ and $D' = K[u_1, \ldots, u_N]$).

It follows that

$$\mathcal{H}om_{\mathcal{O}_{\mathbb{P}}}(\Omega^1_{P_1}, \mathcal{O}_{\mathbb{P}_s}) \cong \mathcal{H}om_{\mathcal{O}_{\mathbb{P}}}(\Omega^1_{P_1/X_1}, \mathcal{O}_{\mathbb{P}_s}) \oplus \mathcal{H}om\,(\pi^*\Omega^1_{X_1/S_1}, \mathcal{O}_{\mathbb{P}_s}).$$

The first summand on the right may be identified with the tangent sheaf to $\mathbb{P}_s \cong \mathbb{P}^N_K$, and so is generated by global sections. The second factor is the same as

$$\pi^*(\mathcal{H}om\,(\Omega^1_{(A/m^2)/K}, K)).$$

On an open affine U, this corresponds to the fact that

$$\mathrm{Hom}_{K[u]}(K[u] \otimes_K \Omega^1_{(A/m^2)/K}, K[u]) \cong K[u] \otimes_K \mathrm{Hom}_K(\Omega^1_{(A/m^2)/K}, K).$$

It follows that the second summand is also generated by global sections, since the pullback of a sheaf generated by global sections is generated by global sections, and on the affine X_1 every coherent sheaf is generated by global sections.

Thus, $\mathcal{H}om_{\mathcal{O}_{\mathbb{P}}}(\Omega^1_{P_1}, \mathcal{O}_{\mathbb{P}_s})$ is generated by global sections on $\mathcal{O}_{P_s}$. We have a surjection $\mathcal{O}_{\mathbb{P}_s} \to \mathcal{O}_{Z'_s}$ and both are killed by m. Hence, the induced map obtained by applying $\mathcal{H}om\,(\Omega^1_{P_1}, \_)$ is onto, since it may be identified with the result of applying $\mathcal{H}om_{\mathcal{O}_{\mathbb{P}_s}}(\Omega^1_{P_1} \otimes K, \_)$, and the sheaf $\Omega^1_{P_1} \otimes K$ is locally free on $\mathbb{P}_s$. It follows that $\mathcal{H}om_{\mathcal{O}_{\mathbb{P}}}(\Omega^1_{P_1}, \mathcal{O}_{Z'_s})$ is generated by global sections over $\mathbb{P}_s$ and, hence over Z'_s. To complete the proof, it will therefore suffice to show that there is a surjection $\mathcal{H}om_{\mathcal{O}_{\mathbb{P}}}(\Omega^1_{P_1}, \mathcal{O}_{Z'_s}) \twoheadrightarrow \mathcal{H}om_{\mathcal{O}_{Z'}}(\mathcal{I}/\mathcal{I}^2, \mathcal{O}_{Z'_s})$, or, equivalently, $\mathcal{H}om_{\mathcal{O}_{\mathbb{P}}}(\Omega^1_{P_1} \otimes \mathcal{O}_{Z'_s}, \mathcal{O}_{Z'_s}) \twoheadrightarrow \mathcal{H}om_{\mathcal{O}_{Z'}}(\mathcal{I}/\mathcal{I}^2, \mathcal{O}_{Z'_s})$.

To this end, first note that there is a map $d : \mathcal{I}/\mathcal{I}^2 \to \Omega^1_{P_1} \otimes \mathcal{O}_{Z'_s}$. The local description of this map over an open affine U in $\mathbb{P}$ has been discussed in detail in the preceding section. Moreover, it was shown in the preceding section that if we localize at a point of Z' and tensor with the residue field *this map is injective.* The same holds if we consider instead the induced map $\mathcal{I}/\mathcal{I}^2 \otimes \mathcal{O}_{Z'_s} \to \Omega^1_{P_1} \otimes \mathcal{O}_{Z'_s}$. Both sheaves are locally free (recall that $\mathcal{I}$ is locally a complete intersection, and also that m kills $\mathcal{O}_{Z'_s}$ and that $\Omega^1_{P_1} \otimes K$ is locally free on $\mathbb{P}_s \cong \mathbb{P}^N_K$). It follows that the dual map of sheaves obtained by applying $\mathcal{H}om\,(\_, \mathcal{O}_{Z'_s})$ is surjective: it suffices to check this after localizing at a point of Z', and by Nakayama's lemma, it then suffices to prove surjectivity after tensoring with the residue field. But the maps so obtained are vector space duals to the maps that were shown to be injective in the preceding section. □

Appendix: The use of Artin approximation

The treatment below is different from the one in Berthelot's article: in particular spectral sequences are not used. The argument we give follows the lines of one given by S. P. Dutta, *A theorem on smoothness — Bass-Quillen, Chow groups and intersection multiplicity of Serre*, preprint.

In studying the question of whether intersection multiplicities of finitely generated modules over a regular local ring are positive or nonnegative or zero, one may always replace the ring and the modules by their completions. We indicate here how we may use (M. Artin's original version) of the Artin approximation theorem to reduce to the case of a local ring of an affine algebra over a complete discrete valuation ring.

Recall that Artin's theorem asserts that if one has a local ring essentially of finite type over an excellent (e.g., complete) discrete valuation ring then given a finite system of polynomial equations over the ring with a solution in its completion, then one may find a solution in the Henselization congruent to the original solution modulo any given power of the maximal ideal of the completion. This is referred to as "approximating" the original solution.

The complete ramified regular local ring is well known to have the form

$$V[[x_1, \dots , x_n]]/(p - f)$$

where (V, pV) is a complete discrete valuation ring of mixed characteristic p and f is in the square of the maximal ideal m_T of $T = V[[x_1, \dots , x_n]]$. Note that V will have a faithfully flat extension (W, pW) which is also a complete discrete valuation ring such that p generates the maximal ideal, but such that W/pW is algebraically closed. Thus, we may replace the original regular local ring A by $B = W[[x_1, \dots , x_n]]/(p - f)$. Then B is faithfully flat over A and the maximal ideal of A expands to the maximal ideal of B. We may replace the original modules by their tensor products with B without changing their dimensions or the Serre multiplicity.

Thus, there is no loss of generality in assuming that $A = T/(p - f)$ where $T = V[[x_1, \dots , x_n]]$, V is a complete discrete valuation ring with maximal ideal pV, the residue field K of V (which is also the residue field of A) is algebraically closed, and $f \in m - T^2$. Suppose that one has a counterexample: we think of it as being given by a pair of finitely generated modules M, N over T such that M and N are killed by $p - f$. We think of these modules as the cokernels of finite matrices (α_{ij}), (β_{hk}) over T.

We want to use Artin approximation to replace this example by an example with the same properties constructed over the Henselization S of the ring $V[x_1, \dots , x_n]_{m_0}$ where $m_0 = (p, x_1, \dots , x_n)$. The Henselization is a direct limit of local rings essentially of finite type over $V[x_1, \dots , x_n]_{m_0}$, and it follows easily that the counterexample descends to a local ring essentially of finite type over V.

To descend to the Henselization we want to think of f and the α_{ij}, β_{hk} as solutions in T of a finite system of polynomial equations over S. There will also be additional auxiliary elements involved in these equations and their solution, i.e., the system of polynomial equations will involve many variables besides those that correspond to f and and the α_{ij}, β_{hk}. The idea is to construct a large family of equations satisfied by f, α_{ij}, β_{hk}, and some auxiliary elements such that when we take a new solution in S, congruent to the original solution modulo a certain high power of m_T, we can use this solution to get a counterexample over S. The trick is to express everything that we need to know about the original example equationally. We shall write Z for the variable in the equations corresponding to f, and U_{ij}, V_{hk} for the variables corresponding to the α_{ij}, β_{hk}.

Since, in particular, we may keep the solution the same modulo m_T^2, there is no difficulty in guaranteeing that the "new" choice of f, call it g, will be in m_S^2. The question is, how do we keep track of all the other things we need to preserve "equationally." We explain how to do this in a sequence of remarks below. In the sequel, we shall use the expression "keep track of" to mean "keep track of equationally."

Once we have explained how to keep track of everything equationally we are done: The new counterexample will consist of a pair of modules over $S/(p-g)$. The new modules will be the cokernels of two matrices: the entries of the defining matrices of M, N are replaced by the values of the U_{ij}, V_{hk} in S in the new solution provided by using Artin approximation, and the cokernels of the resulting matrices give the S-modules that we want to get a counterexample over S. (They will be killed by $p-g$: see (3) below.)

(1) We can keep track of the fact that a sequence of $n+1$ elements is part of a system of parameters by writing down equations expressing a power of each of $p, x_1, \ldots, x_n$ as a linear combination (the coefficients from the ring that are used will eventually be auxiliary unknowns) of elements in the sequence. (We also need to keep the elements in the maximal ideal, but we shall assume here and throughout that we are always keeping elements the same modulo at least m_T^2.)

(2) We can keep track of the fact that a sequence of elements is part of a system of parameters (since the ring is regular, this is equivalent to being a regular sequence) by extending the sequence to a full system of parameters and keeping track of the full system.

(3) We can keep track of the fact that $p-f$ (or another element, which may involve variables) kills a module by writing down equations that express the product of $p-f$ with each relevant standard basis vector as a linear combination of columns of the matrix defining the module.

(4) We may keep track of a finite free resolution of a finitely generated module by keeping track of all the matrices occurring. The condition that we have a complex is certainly equational in the entries. The condition that the minors of a given size vanish if their size is greater than the rank is equational. The determinantal ranks of these matrices are thus preserved if we approximate modulo a high enough power of m_T (this enables us to keep nonzero minors whose size is the rank nonzero). The Buchsbaum-Eisenbud acyclicity criterion guarantees the acyclicity of the new complex if we keep the depths of the ideals of minors sufficiently big. But this simply requires certain linear combinations of minors to be regular sequences, and we may apply (2).

(5) We may keep track of the fact that two modules with possibly different presentations are isomorphic by lifting the map between them and its inverse to maps of free modules. We may similarly keep track of the fact that the composition of two given maps of finitely

generated modules is a third given map. Note also that when we have presentations for two modules we get a presentation for their tensor product.

(6) We may keep track of the fact that we have a short exact sequence of modules as follows. We construct finite free resolutions of the rightmost and leftmost module in the sequence and then use these, in the usual way, to construct a resolution of the middle module. In this way we get a short exact sequence of finite free complexes such that the columns give resolutions of the original three modules while the rows are split exact. One can keep track of the acyclicity of the columns as in (4), while it is easy to keep track of the fact that a short exact sequence of free modules is split exact.

(7) We can keep track of the length and, in fact, the isomorphism class of a finite length module over T by using the fact that it has a finite presentation over S.

(8) We can keep track of a finite complex of finitely generated modules and its homology by breaking it up into short exact sequences that carry the information and keeping track of all of them.

(9) We can keep track of the fact that a module M has dimension at least h by embedding a cyclic submodule of M as a nonzero submodule of one obtained by killing part of a regular sequence of length h. (One can find a cyclic submodule of the form T/P with the correct dimension, and P will be a minimal prime of an ideal generated by a regular sequence.) One can keep track of the fact that a finitely generated module M has dimension at most h by exhibiting a sequence of elements $y_1, \dots, y_h$ of the maximal ideal such that $M/(y_1, \dots, y_h)M$ has finite length. Hence, one can keep track of the dimension of a finitely generated module.

With these remarks, we see that we can keep track of a counterexample, since we can keep track of the dimensions of M and N, of the fact that both are killed by $p - f$, of the fact that their tensor product is killed by a power of the maximal ideal, of a finite resolution of M by finitely generated free modules over $T/(p - f)$, and of the finite length homology of the complex obtained by tensoring that resolution with N (and, hence, of the intersection multiplicity) while descending to S.