SMALL SUBALGEBRAS OF POLYNOMIAL RINGS AND STILLMAN'S CONJECTURE

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Abstract. Let n, d, η be positive integers. We show that in a polynomial ring R in N variables over an algebraically closed field K of arbitrary characteristic, any K-subalgebra of R generated over K by at most n forms of degree at most d is contained in a K-subalgebra of R generated by $B \leq {}^\eta\mathcal{B}(n, d)$ forms $G_1, \ldots, G_B$ of degree $\leq d$, where ${}^\eta\mathcal{B}(n, d)$ does not depend on N or K, such that these forms are a regular sequence and such that for any ideal J generated by forms that are in the K-span of $G_1, \ldots, G_B$, the ring R/J satisfies the Serre condition R_η. These results imply a conjecture of M. Stillman asserting that the projective dimension of an n-generator ideal of R whose generators are forms of degree $\leq d$ is bounded independent of N. We also show that there is a primary decomposition of the ideal such that all numerical invariants of the decomposition (e.g., the number of primary components and the degrees and numbers of generators of all of the prime and primary ideals occurring) are bounded independent of N.

1. Introduction

Throughout this paper, let R denote a polynomial ring over an arbitrary field K: say $R = K[x_1, \ldots, x_N]$. We prove Stillman's conjecture, which asserts that given a specified number n of forms of specified positive degrees, say at most d, there is a bound for the projective dimension of the ideal I the forms generate that depends on n and d but not on the number N of variables. The conjecture is recorded in [22] and previous work related to it may be found in [1], where the problem is solved for quadrics, and in [3, 7, 10, 11, 12, 17, 18, 20, 21], where bounds are given for small numbers of quadrics and cubics and examples are given, and the degree restriction is shown to be needed, based on much earlier work in [4, 5, 19]. We prove Stillman's conjecture in a greatly strengthened form, as well as many other results, e.g., Theorems A, B, C, D, E, and F below. In fact, we prove that the forms are in a polynomial K-subalgebra generated by a regular sequence with at most $\mathcal{B}(n, d)$ elements, where $\mathcal{B}(n, d)$ does not depend on K or N: we refer to this smaller polynomial ring informally as a "small" subalgebra.

In [2] a number of bounds for degrees 2, 3, and 4 are computed. The methods of [2], particularly for the case $d = 4$, differ substantially from those used here: they yield smaller bounds, although in the case $d = 4$ there are restrictions on the characteristic. Some of the results of [2] are discussed in §4.

Date: February 20, 2017.

2000 *Mathematics Subject Classification.* Primary 13D05, 13F20.

Key words and phrases. polynomial ring, ideal, form, projective dimension, regular sequence, R_η.

[1]The second author was partially supported by grants from the National Science Foundation (DMS–0901145 and DMS–1401384).

For the purpose of proving Stillman's conjecture one can pass to the case where the field is algebraically closed, and we shall assume that K is algebraically closed, unless otherwise stated, throughout the rest of this paper.

We use $\mathbb{N}$ to denote the nonnegative integers and $\mathbb{Z}_+$ the positive integers. We define a nonzero homogeneous polynomial F of positive degree in R to have a k-*collapse* for $k \in \mathbb{N}$, if F is in an ideal generated by k elements of strictly smaller positive degree, and we define F to have *strength* k if it has a $k+1$-collapse but no k-collapse. Nonzero linear forms have strength $+\infty$, and a form has strength at least 1 if and only if it is irreducible. One of the main themes here is that F has a "small" collapse if and only if the singular locus of F has "small" codimension. "Only if" is evident: when $F = \sum_{i=1}^{k} G_i H_i$, the partial derivatives of F are in the $2k$-generated ideal $(G_1, \dots, G_i, H_1, \dots, H_i : 1 \le i \le k)R$. "If" is quite difficult: a precise statement is made in the first paragraph of Theorem A below.

We use V to denote a finite-dimensional graded vector subspace of R spanned by forms of positive degree. If the d is an upper bound for the degree of any element of V, we may write $V = V_1 \oplus \dots \oplus V_i \oplus \dots \oplus V_d$, where V_i denotes the i th graded piece, and we shall say V has *dimension sequence* $(\delta_1, \dots, \delta_d)$ where $\delta_i := \dim_K(V_i)$. This sequence carries the same information as the Hilbert function of V. We regard two such dimension sequences as the same if they become the same after shortening by omitting the rightmost string of consecutive 0 entries.

If F is a form in $K[x_1, \dots, x_N]$, we denote by $\mathcal{D}F$ the K-vector space spanned by the partial derivatives $\partial F/\partial x_i$, $1 \le i \le N$. When the characteristic does not divide d, the degree of F, we have that $F \in (\mathcal{D}F)$, the ideal generated by $\mathcal{D}F$, since Euler's formula asserts that

$$\deg(F)F = \sum_{i=1}^{N} x_i(\partial F/\partial x_i).$$

We shall say that a sequence of elements generating a proper ideal of a ring S is a *prime* sequence (respectively, an R_η-sequence, where $\eta \in \mathbb{Z}_+$), if the quotient of S by the ideal generated by any initial segment is a domain (respectively, satisfies R_η). A prime sequence in S is always a regular sequence. If $S = R$ is a polynomial ring, every R_η-sequence of forms of positive degree for $\eta \ge 1$ is a prime sequence (in fact, the quotients are normal domains), and, hence, a regular sequence.

We call an integer valued function on $\mathbb{N}^h$ *ascending* if it is nondecreasing in each input when the others are held fixed. In all our constructions of functions, it is easy to make them ascending: replace the function $\mathcal{B}$ by the one whose value on $(b_1, \dots, b_h)$ is $\max\{\mathcal{B}(a_1, \dots, a_h) : 0 \le a_i \le b_i \text{ for } 1 \le i \le h\}$. A d-tuple of integer-valued functions on $\mathbb{N}^h$ will be called *ascending* if all of its entries are ascending functions.

The main results are stated below. The proofs are given in §3, after some preliminary results are established in §2.

Theorem A. *There exists an integer ${}^{\eta}A(d) \ge d-1 \ge 0$, ascending as a function of η, $d \in \mathbb{Z}_+$, such that if $R = K[x_1, \dots, x_N]$ is the polynomial ring in N variables over an algebraically closed field K and $F \in R$ is a form of degree $d \ge 1$ of strength at least ${}^{\eta}A(d)$, then the codimension of the singular locus in R/FR is at least $\eta+1$, so that R/FR satisfies the Serre condition* R_η.

Moreover, there are ascending functions $\underline{A} = (A_1, \dots, A_d)$ and, for every integer $\eta \ge 1$, ${}^{\eta}\underline{A} = ({}^{\eta}A_1, \dots, {}^{\eta}A_d)$ from dimension sequences $\delta = (\delta_1, \dots, \delta_d) \in \mathbb{N}^d$ to $\mathbb{N}^d$

with the following property: If V denotes a graded K-vector subspace of R of vector space dimension n with dimension sequence $(\delta_1, \ldots, \delta_d)$, such that for $1 \le i \le d$, the strength of every nonzero element of V_i is at least $A_i(\delta)$ (respectively, ${}^{\eta}A_i(\delta)$), then every sequence of K-linearly independent forms in V is a regular sequence (respectively, is an R_η-sequence).

In fact, if we have the functions ${}^{\eta}A(i)$ described in the first paragraph for $1 \le i \le d$, we may take ${}^{\eta}A_i(\delta) = {}^{\eta}A(i) + 3(n-1)$.

Remark 1.1. The condition that the singular locus of R/FR have codimension at least $\eta+1$ in R/FR, i.e., that R/FR satisfy the Serre condition R_η, is equivalent to the condition that the ideal $FR+(\mathcal{D}F)R$ have height $\eta+2$ in R. (If the characteristic is 0 or does not divide $\deg(F)$, F is in the ideal $(\mathcal{D}F)R$.)

By taking a supremum over values of the ${}^{\eta}A_i$ over all dimension sequences with at most d entries such that sum of the entries is at most n we have at once the result mentioned in the abstract:

Corollary A. *There is an ascending function ${}^{\eta}\mathcal{A}(n,d)$, independent of K and N, such that for all polynomial rings $R = K[x_1, \ldots, x_N]$ over an algebraically closed field K and all ideals I generated by a graded vector space V whose nonzero homogeneous elements have positive degree at most d, if no homogeneous element $V - \{0\}$ is in an ideal generated by ${}^{\eta}\mathcal{A}(n,d)$ forms of strictly lower degree, then R/I satisfies R_η.*

We use this result to prove:

Theorem B (existence of small subalgebras). *There is an ascending function B from dimension sequences $\delta = (n_1, \ldots, n_d)$ to $\mathbb{N}_+$ with the following property. If K is an algebraically closed field and V is a finite-dimensional $\mathbb{N}_+$-graded K-vector subspace of a polynomial ring R over K with dimension sequence δ, then $K[V]$ is contained in a K-subalgebra of R generated by a regular sequence $G_1, \ldots, G_s$ of forms of degree at most d, where $s \le B(\delta)$.*

Moreover, for every $\eta \ge 1$ there is such a function ${}^{\eta}B$ such that, in addition, every sequence consisting of linearly independent homogeneous linear combinations of the elements in $G_1, \ldots, G_s$ is an R_η-sequence.

For example, this theorem implies for $\eta \ge 3$ that all the quotients of R by ideals generated by homogeneous linear combinations of the elements in $G_1, \ldots, G_s$ are unique factorization domains: this follows at once from a theorem of Grothendieck, conjectured by Samuel, for which there is an elementary exposition in [6].

By taking a supremum over all dimension sequences with at most d entries such that the sum of the entries is at most n, we have at once:

Corollary B. *There is an ascending function ${}^{\eta}\mathcal{B}(n,d)$, independent of K and N, such that for all polynomial rings $R = K[x_1, \ldots, x_N]$ over an algebraically closed field K and all graded vector subspaces V of R of dimension at most n whose homogeneous elements have positive degree at most d, the elements of V are contained in a subring $K[G_1, \ldots, G_B]$, where $B \le {}^{\eta}\mathcal{B}(n,d)$ and $G_1, \ldots, G_B$ is an R_η-sequence of forms of degree at most d.*

Theorem B. easily implies a strong form of M. Stillman's Conjecture:

Theorem C. *There is an ascending function C from $\mathbb{N}_+ \times \mathbb{N}_+ \times \mathbb{N} \to \mathbb{N}_+$ with the following property. If R is a polynomial ring over an arbitrary field K and M is a* *module that is the cokernel of an $m \times n$ matrix whose entries have degree at most d, then the projective dimension of M is bounded by $C(m, n, d)$.*

Theorem B yields many other bounds:

Theorem D. *Let K be an algebraically closed field and let $R = K[x_1, \ldots, x_N]$ be the polynomial ring in N variables over K. Let $m, n, d \in \mathbb{Z}_+$, let $\mathcal{M}$ be an $m \times n$ matrix over R whose entries have degree at most d, let M be the column space of $\mathcal{M}$.*

(a) There exists an ascending function $P(m, n, d)$ independent of N and K that bounds the length of a finite free resolution of M, the ranks of the free modules occurring, and the degrees of all of the entries of all of the matrices occurring. Hence, $P(m, n, d)$ bounds sets of generators for the modules of syzygies associated with the resolution. In the graded case, $P(m, n, d)$ bounds the twists of R that occur as summands in a minimal free resolution of M.

(b) There exists an ascending function $E(m, n, d)$ independent of N and K that bounds the number of primary components in an irredundant primary decomposition of M in R^m, the number of and the degrees of the generators of every prime ideal occurring, and the number of generators and the degrees of the entries of the generators for every module in the decomposition. $E(m, n, d)$ can also be taken to bound the exponent on every associated prime ideal P needed to annihilate the corresponding P-coprimary component of M mod M (in the ideal case, the exponent a needed so that P^a is contained in the corresponding primary ideal of the decomposition).

(c) There exists an ascending function $D(k, d)$ independent of N and K that bounds the minimum number of generators of any minimal prime of an ideal generated by a regular sequence consisting of k or fewer d-forms.

Remark 1.2. Part (c) is obvious from part (b), since we may take $D(k, d) = E(1, k, d)$. However, the function $D(k, d)$ plays a special role in the proofs, and may have a much smaller bound.

Free resolutions are not unique, but the specified bounds work for at least one free resolution. Similarly, primary decompositions are not unique, but the specified bounds work for at least one irredundant primary decomposition of M in R^m. Of course, when $m = 1$ we are obtaining such a bound for the primary decomposition of an ideal with n generators when the degrees of the generators are at most d.

We shall refer to the largest degree of any entry of a nonzero element v of the free module R^m over the polynomial ring R as the *degree* of v. We shall say that a set of generators for a submodule of R^m is *bounded* by n, d if it has at most n elements of degree at most d. If $n = d$, we say that the set of generators is bounded by n.

Theorem E. *There exist ascending positive integer-valued functions $\Theta(m, n, r, d)$, $\Lambda(m, n, d, h)$, and $\Gamma(m, n, d)$ of the nonnegative integers $h \geq 2$, m, n, r, d taking values in $\mathbb{Z}_+$ with the following properties. Let $R = K[x_1, \ldots, x_N]$ be a polynomial ring over an algebraically closed field K. Let $G := R^m$. Let M, Q and $M_1, \ldots, M_h$ be submodules of G. Let I be an ideal of R. Suppose that all of $M, Q, M_1, \ldots, M_h$, and I have sets of generators bounded by n, d.*

(a) Given an $m \times r$ matrix over R with entries of degree at most d, thought of as a map from $R^r \to R^m$, and a set of generators for a submodule M of R^m bounded

by n, d, there is a set of generators for $Ker(R^r \to R^m \twoheadrightarrow R^m/M)$ bounded by $\Theta(m, r, n, d)$.

(b) *There exists a set of generators for $M_1 \cap \cdots \cap M_h$ bounded by $\Lambda(m, n, d, h)$.*

(c) *There exist sets of generators for $M :_R Q$, $M :_G I$, and $M :_R I^\infty$ bounded by $\Gamma(m, n, d)$.*

Remark 1.3. Given a map of finitely presented R-modules, we may always think of it as induced by a map of free modules that map onto these R-modules, so that it may be described as the map $R^r/M' \to R^m/M$ determined by the $m \times r$ matrix of a map of the free numerators. The kernel of this map is generated by the images of the generators of the kernel of the map to $R^r \to R^m/M$. Thus, part (a) of Theorem 1 enables one to bound a set of generators for the kernel of a map of finitely presented modules when we have information bounding the sizes and degrees of the presentations and of the matrix of the map of free modules.

Remark 1.4. It is difficult to make a comprehensive statement of all the related results that follow from the main theorems: the following is an example.

Corollary E. *Let $R = K[x_1, \ldots, x_N]$ be a polynomial ring over an algebraically closed field. There exist bounds for the number of generators of the ideal generated by the leading forms of the elements in an ideal generated by n elements of degree at most d that depend on n and d but not on N or K.*

Proof. Let the ideal be $(f_1, \ldots, f_n)R$. Let $F_1, \ldots, F_n$ be the result of homogenizing the f_i with respect to an new variable $x = x_{N+1}$. Then $F_1, \ldots, F_n$ also have degree at most d, and the required ideal is the image of $(F_1, \ldots, F_N) :_{R[x]} x^\infty \bmod x$. □

Theorem F. *There is an ascending function $\Phi(h, d)$ such that, independent of the algebraically closed field K or the integer N, if a form of degree d in the polynomial ring $K[x_1, \ldots, x_N]$ has strength at least $\Phi(h, d)$, then $\mathcal{D}F$ is not contained in an ideal generated by h forms of degree at most $d - 1$.*

Of course, this is obvious from Euler's formula if p does not divide d: in that case we may take $\Phi(d, h) = h$, since F is in the ideal $(\mathcal{D}F)R$. We handle the case where p is a positive prime that may divide d inductively, by using the fact that we know Corollary B for integers less than d. See Proposition 2.6.

2. Preliminary results

Theorem 2.1. *Let K be an algebraically closed field, let $R = K[x_1, \ldots, x_N]$ be a polynomial ring. Let $V = V_1 \oplus \cdots \oplus V_d$, where V_i is spanned by forms of degree i, and suppose that V has finite dimension n. Assume that a homogeneous basis for V is a regular sequence in R. Let X be defined by the vanishing of all the elements of V. Let S be the family of all subsets of V consisting of nonzero forms with mutually distinct degrees, so that the number of elements in any member of S is at most the number of nonzero V_i. For $\sigma \in S$, let C_σ be the codimension of the singular locus of $V(\sigma)$ in $\mathbb{A}^N_K$. Then the codimension in $\mathbb{A}^N_K$ of the singular locus of X is at least $(\min_{\sigma \in S} C_\sigma) - (n - 1)$.*

Proof. We need to study the codimension of the set where the Jacobian has rank at most $n - 1$. We first consider an irreducible component Z of the singular locus where the Jacobian has rank 0, i.e., where it vanishes identically. Let λ_0 be the set of all i such that $V_i \neq 0$. If we form σ by choosing one form G_i of each degree

$i \in \lambda_0$, then Z is in the singular locus of the scheme $Y = V(G_i : i \in \lambda_0)$ defined by the vanishing of these G_i (evidently, the Jacobian of this smaller set of polynomials is still identically 0 on Z), which shows that the dimension of the singular locus of Y is at least as large as the dimension of Z, and hence C_σ is a lower bound for the codimension of Z.

Consider an irreducible component Z of the singular locus, and suppose that on a nonempty open subset U_1 of Z, the Jacobian matrix has rank r, $1 \leq r \leq n-1$. We can choose an $r \times r$ minor μ of the Jacobian matrix that does not vanish on dense open subset U of U_1, and it will suffice to bound below the codimension of U in $\mathbb{A}^N$. Choose $r+1$ rows of the Jacobian matrix that contain the r rows corresponding to the nonvanishing minor. Renumber the F_i so that these rows correspond to $F_1, \ldots, F_{r+1}$. We have a map $U \to \mathbb{P}^r$ that assigns to each point $u \in U$ the non-trivial relation on the rows of the Jacobian matrix J_0 of $F_1, \ldots, F_{r+1}$ when it is evaluated at u: since the J_0 has rank exactly r at u, this relation is unique up to multiplication by a nonzero scalar. In fact, it is given by the $r \times r$ minors of the r columns determined by the nonvanishing minor μ. Since the image of the map $U \to \mathbb{P}^r$ has dimension at most r, the dimension of U is bounded by the sum of r and the dimension of a typical fiber Y of the map. Note that $r \leq n-1$, and the codimension of U in $\mathbb{A}^N$ is bounded below by $C - r$, where C is the codimension of a typical fiber of the map $U \to \mathbb{P}^r$. Consider the fiber over the point $u = [a_1 : \cdots : a_{r+1}] \in \mathbb{P}^r$. Because the a_i give a relation on the rows of the Jacobian matrix corresponding to $F_1, \ldots, F_{r+1}$, it follows that all of the partial derivatives of $F = \sum_{i=1}^{r+1} a_i F_i$ vanish on U. We can break this sum up as a sum of nonzero forms of mutually distinct degrees, say $F = G_{i_1} + \cdots + G_{i_h}$ where $1 \leq i_1 \leq \cdots \leq i_h \leq d$ are the degrees. But then the sum of the rows of the Jacobian matrix for $Z_0 = V(G_{i_1}, \ldots, G_{i_h})$ vanishes on U, and so U is contained in the singular locus of Z_0. The codimension in $\mathbb{A}^N_K$ of the singular locus of Z_0 is bounded below by C_σ with $\sigma = \{G_{i_1}, \ldots, G_{i_h}\}$. Thus the codimension of U in $\mathbb{A}^N_K$ is bounded below by $C_\sigma - r$, where $r \leq n-1$. This yields the stated result. □

Remark 2.2. Note that in a polynomial ring, the height of a homogeneous ideal I does not increase when we kill some of the variables. (Let P be a minimal prime of I whose height is the same as that of I, and let Q be the prime generated by the variables we are killing. The result holds because we may localize at a minimal prime of $P+Q$, and we may apply the result of [24], Théorème 1, part (2), p. V–13, which implies that $\mathrm{height}(P+Q) \leq \mathrm{height}(P) + \mathrm{height}(Q)$.)

Remark 2.3. In the theorem just below, the hypothesis that the degrees associated with the various rows be distinct is crucial: without it, the rows could all be taken to be the same. Having the degrees be all different somehow makes the matrix more like a generic matrix, i.e., a matrix of indeterminates, for which results like the one below have long been known: cf. [9], [14].

Theorem 2.4. *Let K be a field, let R be a polynomial ring over K, and let M be an $h \times N$ matrix such that for $1 \leq i \leq h$, the i th row consists of forms of degree $d_i \geq 0$ and the d_i are mutually distinct integers. Suppose that for $1 \leq i \leq h$, the height of the ideal generated by the entries of the i th row is at least b. (If the row consists of scalars, this is to be interpreted as requiring that it be nonzero.) Then the ideal generated by the maximal minors of the matrix has height at least $b-h+1$.*

Proof. We may enlarge the field to be algebraically closed without loss of generality. We may assume without loss of generality that $d_1 \leq \ldots \leq d_h$. We use induction on h: the case where $h = 1$ is immediate. (If one has a single nonzero row of scalars, the height of the ideal generated by the maximal minors is $+\infty$.) We therefore assume $h \geq 2$ and that the result holds for smaller h. Next, we reduce to the case where the number of variables in R is b, and every non-scalar row generates an ideal primary to the homogeneous maximal ideal. Suppose that the number of variables is greater than b. For each i, choose a subset of the span of the i th row generating an ideal J_i of height b. Choose a linear form that is not in any of the minimal primes of any of the J_i. We may kill this form, and the hypotheses are preserved: the height of the ideal generated by the maximal minors does not increase by Remark 2.2. We may continue in this way until the number of variables is b.

Let P be a minimal prime ideal of the ideal generated by the maximal minors of M. To complete the proof, it will suffice to show that the dimension of the ring R/P is at most $h-1$.

Let $\overline{M}$ denote the image of the matrix M over R/P. It is possible that all of the maximal minors of the matrix formed by a proper subset consisting of $h_0 < h$ of the rows of $\overline{M}$ vanish in R/P. But then the height of the ideal generated by the maximal minors of these rows is at least $b-h_0+1$ by the induction hypothesis, and this shows that the dimension of R/P is at most h_0-1. Hence, we may assume that there is no linear dependence relation on any proper subset of the rows of the image of $\overline{M}$, while the rank of the image $\overline{M}$ is $h-1$. This implies that there are unique elements of the fraction field of R/P, call them $u_1, \ldots, u_{h-1}$, such that $\rho_h = \sum_{i=1}^{h-1} u_i \rho_i$, where ρ_i is the image of the i th row of M. More specifically, since the first $h-1$ rows of $\overline{M}$ are linearly independent over $\mathrm{frac}(R/P)$, we may choose $h-1$ columns forming an $h \times (h-1)$ submatrix $\overline{M}_0$ of $\overline{M}$ such that the $h-1$ size minor Δ of the first $h-1$ rows is not 0. The nonzero relation, unique up to multiplication by a nonzero scalar in $\mathrm{frac}(R/P)$, on the rows of the submatrix $\overline{M}_0$ is given by the vector whose entries are its $h-1$ size minors, which are homogeneous elements of R/P. This must give the relation on the rows of $\overline{M}$. Thus, every u_j can be written as a fraction with denominator Δ whose numerator is one of the other minors of $\overline{M}_0$. Let S be the ring $(R/P)[u_1, \ldots, u_{h-1}]$. Note that u_i has degree $d_h - d_i > 0$, so that S is a finitely generated $\mathbb{N}$-graded algebra over K with $S_0 = K$ generated over K by the images of the x_i and by the u_i. The Krull dimension of S is the same as that of R, since the fraction field has not changed, and that is the same as the height of the maximal ideal of S. But $S/(u_1, \ldots, u_{h-1})S$ is zero-dimensional, since the vanishing of the u_i implies the vanishing of all entries of ρ_h, and these generate an ideal primary to the maximal ideal of $K[x_1, \ldots, x_b]$. It follows that the Krull dimension of S is at most $h-1$, and, hence, the same holds for R/P, as required. □

Theorem 2.5. *Let K be an algebraically closed field, and let V be an n-dimensional graded K-vector space of the polynomial ring $R = K[x_1, \ldots, x_N]$ consisting of forms of degree between 1 and d, so that $V = V_1 \oplus \cdots \oplus V_d$. Assume that a basis for V consisting of forms is a regular sequence in R. Let h denote the number of integers i such that $V_i \neq 0$, so that $h \leq \min\{d, n\}$. Suppose that for every nonzero homogeneous element F of V, the height of the ideal $(\mathcal{D}F)$ in R is at least*

$\eta+h+2n-1$. *Then the codimension of the singular locus of $R/(V)$ in $R(V)$ is at least $\eta+1$.*

Proof. Consider a set σ of homogeneous elements of V of distinct degrees: it has at most h elements. The Jacobian matrix of the elements of σ has at most h rows, and the degrees associated with the rows are distinct. By hypothesis, each row generates an ideal of height $\eta+h+2n-1$ in R. By Theorem 2.4, the height of the ideal of maximal minors is at least $\eta+2n-1+1$. Hence, the codimension C_σ of the singular locus of $V(\sigma)$ in $\mathbb{A}^N_K$ is at least $\eta+2n$. Hence, by Theorem 2.1, the codimension of the singular locus of $R/(V)$ in $\mathbb{A}^N_K$ is at least $\eta+n+1$. When we work mod (V) it can drop, at worst, to $\eta+1$. □

The following result shows that Corollary B in degree $d-1$ implies Theorem F in degree d.

Proposition 2.6. *Suppose that we have a function ${}^3\mathcal{B}(n,d-1)$ for a fixed value of d all n, as in the statement of Corollary B. Then Theorem F holds with $\Phi(h,d)={}^3\mathcal{B}(h,d-1)+1$.*

Proof. Suppose that a form F of degree d in $K[x_1,\,\dots,\,x_N]$ has strength at least ${}^3\mathcal{B}(h,d-1)+1$ but that $\mathcal{D}F$ is contained in the ideal generated by h forms of degree $d-1$ or less. By Corollary B these forms are contained in a subring $K[G_1,\,\dots,\,G_B]$ where $B\le {}^3\mathcal{B}(h,d-1)$ and $G_1,\,\dots,\,G_B$ form an R_3-sequence. Then $\mathcal{D}F$ is also contained in the ideal generated by $G_1,\,\dots,\,G_B$. Since $R/(G_1,\,\dots,\,G_B)$ is a complete intersection that is R_3, it is a UFD. F must be irreducible in this quotient, or else we obtain a homogeneous equation $F=F_1F_2+\sum_{i=1}^B G_iH_i$. This implies that F has a $(B+1)$-collapse, contradicting the hypothesis. Therefore, $G_1,\,\dots,\,G_B,\,F$ is a prime sequence. This implies that the maximal minors of the Jacobian matrix have positive height mod $(G_1,\,\dots,\,G_B,\,F)R$. Hence the row of the Jacobian matrix corresponding to F, whose K-span is $\mathcal{D}F$, cannot be 0 mod $(G_1,\,\dots,\,G_B)$. □

Extension of prime ideals. Recall that a ring homomorphism $R\to S$ is *intersection flat* if it is flat and for every family $\mathcal{I}$ of ideals $\mathcal{I}$ of R, $\bigcap_{I\in\mathcal{I}}(IS)=(\bigcap_{I\in\mathcal{I}}I)S$. Flatness implies this condition when $\mathcal{I}$ is a finite family. If S is free over R, then S is intersection flat. See [15], p. 41. In the situation where $G_1,\,\dots,\,G_B$ is part of a homogeneous system of parameters for the polynomial ring $K[x_1,\,\dots,\,x_N]$, if $G_1,\,\dots,\,G_N$ is a full homogeneous system of parameters we know that we have free extensions $K[G_1,\,\dots,\,G_B]\to K[G_1,\,\dots,\,G_N]$, $K[G_1,\,\dots,\,G_N]\to K[x_1,\,\dots,\,x_N]$ (this is module-finite and free, since the target ring is Cohen-Macaulay), and, if $K\subseteq L$ is an ring extension, $K[x_1,\,\dots,\,x_N]\to L[x_1,\,\dots,\,x_N]$ (since L is free over K). Hence, $K[G_1,\,\dots,\,G_B]\to L[x_1,\,\dots,\,x_N]$ is free and, consequently, intersection flat.

Recall also that R is a *Hilbert ring* if every prime ideal is an intersection of maximal ideals.

We first observe the following:

Theorem 2.7. *Let R be a Hilbert ring, and let S be an R-algebra that is intersection flat. Suppose that for every maximal ideal m of R, S/mS is a domain. Then for every prime ideal P of R, S/PS is a domain.*

Proof. We use induction on the dimension of R/P. The case of dimension 0 is the hypothesis. Now assume that $\dim(R/P)=d>0$. Let $F,G\in S$ be such that

$FG \in PS$. By the induction hypothesis, for every prime $Q \supseteq P$ of R such the height of Q/P is one in R/P, S/QS is a domain. Hence, $F \in QS$ or $G \in QS$. Since R is a Hilbert ring, P is an intersection of maximal ideals m, all of which contain such a Q. Hence, P is the intersection of all such Q, and the family of such Q is infinite. Thus, either F or G, say F, is in Q_iS for infinitely many choices $Q_1, \dots, Q_i \dots$ of the prime Q. Hence, $F \in \bigcap_{i=1}^{\infty} Q_iS = (\bigcap_{i=1}^{\infty} Q_i)S$, because $R \to S$ is intersection flat. But $\bigcap_{i=1}^{\infty} Q_i = P$, since $f \notin P$ cannot have the property that $f + P$ has infinitely many minimal primes in R/P. Hence, $F \in PS$, as required. □

Second, we observe:

Proposition 2.8. *Let R be an $\mathbb{N}$-graded domain and let $F_1, \dots, F_n$ be a regular sequence of forms that generate a prime ideal P. Let $f_1, \dots, f_n$ be elements of R whose leading forms are the elements $F_1, \dots, F_n$. Then $f_1, \dots, f_n$ generate a prime ideal Q.*

Proof. Let $L(g)$ denote the leading form of $g \in R$.

Suppose $gh = \sum_{i=1}^{n} r_i f_i$ with $g, h \notin Q$ and choose this example with the degree of gh minimum, and also so that the largest degree δ of any of the $L(r_i f_i) = L(r_i)F_i$ is minimum. If $\delta > \deg(gh)$, we have $\sum_{i \in S} L(r_i)F_i = 0$, where i runs through the set of indices such that the degree of $r_i f_i$ is δ. Then the vector of $L(r_i)$ is a graded linear combination of Koszul relations on the F_i, $\sum_{ij} h_{ij}(F_j e_i - F_i e_j)$. We can replace each F_i by f_i in this expression to obtain a relation on the f_i, $\sum_i u_i f_i = 0$. Then $gh = \sum_{i=1}^{n} (r_i - u_i) f_i$ has a smaller value for δ on the right hand side. Hence, we may assume that $\delta = \deg(gh)$. But then $L(g)L(h) \in P$, and one of them, say $L(g)$, is in P. We may alter g by subtracting a linear combination of the f_i so as to cancel its leading form and so obtain $g'h \in Q$, contradicting the minimality of the degree of gh. □

Corollary 2.9. *Let K be an algebraically closed field, and let $R = K[g_1, \dots, g_B]$ denote a polynomial ring over K. Suppose $K[g_1, \dots, g_B] \subseteq L[x_1, \dots, x_N] = S$, a polynomial ring over a field L such that the inclusion is graded and $g_1, \dots, g_B$ is a prime sequence in S. Then for every prime ideal P of R, PS is prime.*

Proof. By Proposition 2.8 above, for any $c_1, \dots, c_n \in K$, $g_1 - c_1, \dots, g_n - c_n$ is prime in S. The result is now immediate from Theorem 2.7. □

Corollary 2.9 can also be deduced from [13], Théorème 12.1(viii).

We also note the following fact, which is immediate from [24], Proposition 15, p. IV–25.

Proposition 2.10. *Let R to S be flat extension of Noetherian rings, and let M be a P-coprimary R-module, i.e., the set of associated primes of M is $\{P\}$. Suppose that PS is prime. Then $S \otimes M$ is PS-coprimary.* □

Bounding all data for calculations with ideals or modules when the number of variables is known. The results of this subsection are expected, and likely can be deduced by nonstandard methods as in [8] or possibly even from [23], and they are closely related in both content and methods to those of [13], (9.8). However, what we need is not precisely given in any of those papers, and we give a brief treatment here that contains what we need for both this and subsequent papers.

Let $R = K[x_1, \dots, x_B]$ be a polynomial ring over an algebraically closed field. Consider an $m \times n$ matrix $\mathcal{M}$ with entries in R such that the degrees of the entries are at most a given integer d. Let $M \subseteq R^m$ be the column space of $\mathcal{M}$. In this section we show that bounds for the data of a primary decomposition of M in R^m (respectively, of a finite free resolution of M) can be given in terms of B, m, n, d, where the *data* of the decomposition include the number of associated primes, the number of generators of each, and the number of generators and the degrees of the generators of all the modules in the primary decomposition. As will be evident from the proof, one can keep track of more numerical characteristics. By the *data* of a finite free resolution, we mean the length, the ranks of the free modules modules occurring, and the degrees of the entries of the matrices. The bounds are independent of the choice of K. We also obtain bounds for the operations occurring in Theorem 1 when the number of variables is bounded.

The results of this section are very different from other bounds obtained elsewhere in the paper, because they are allowed to depend on B, the number of variables in the polynomial ring. We shall apply them in situations where we have a bound on B that is independent of K and N.

Theorem 2.11. *Let $h \geq 2$, B, m, n, r, and d vary in the nonnegative integers. Then there exist ascending functions $\mathcal{T}(B, m, r, n, d)$, $\mathcal{G}(B, m, n, d)$, $\mathcal{L}(B, m, n, d, h)$, $\mathcal{E}(B, m, n, d)$, and $\mathcal{P}(B, m, d)$ with values in $\mathbb{Z}_+$ with the properties described below. Let K be an algebraically closed field and let $R = K[x_1, \ldots, x_B]$ be the polynomial ring in B variables over K. Let $m, n, d \in \mathbb{N}$, let $\mathcal{M}$ be an $m \times n$ matrix over R whose entries have degree at most d, and let M be the column space of $\mathcal{M}$.*

(a) *Given an $r \times m$ matrix over R with entries of degree at most d, thought of as a map from $R^r \to R^m$, and a set of generators for a submodule M of R^m bounded by n, d, there is a set of generators for $Ker(R^r \to R^m \twoheadrightarrow R^m/M)$ bounded by $\mathcal{T}(B, m, r, n, d)$.*

(b) *There exist sets of generators for $M \cap N$, $M :_R N$, $M :_G I$, and $M :_R I^\infty$ bounded by $\mathcal{G}(B, m, n, d)$.*

(c) *There exists a set of generators for $M_1 \cap \cdots \cap M_h$ bounded by $\mathcal{L}(B, m, n, d, h)$.*

(d) *There exists an ascending function $\mathcal{E}(B, m, n, d)$ that bounds the number of primary components in an irredundant primary decomposition of M in R^m, the number of and the degrees of the generators of every prime and primary ideal occurring, and the number of generators and the degrees of the entries of the generators for every module in the decomposition.*

(e) *There exists an ascending function $\mathcal{P}(B, m, n, d)$ that bounds, independent of K, the length a free resolution of M, the ranks of the free modules occurring, and the degrees of all of the entries of all of the matrices occurring. In the graded case, $\mathcal{P}(B, m, n, d)$ bounds the twists of R that occur as summands in a minimal free resolution of M.*

Discussion 2.12. We shall prove this result by first considering the case where all the entries of the matrices occurring and all entries of generators of the modules and ideals occurring are replaced by generic polynomials of degree at most d with distinct indeterminate coefficients u_t. Let A denote the polynomial ring over $\mathbb{Z}$ in all the variable coefficients. Then we can cover $\mathrm{Spec}(A)$ by a finite number of locally closed affines $\mathrm{Spec}(A_s)$ for each of which there is a generic calculation of the kernel, intersection, colon, primary decomposition of the module, or a generic finite free resolution (for these, each A_s is replaced by a module-finite extension domain A'_s). These generic calculations specialize to give all ones needed when we replace the variables u_t by elements of an algebraically closed field K: when we make that replacement, we obtain a map $A \to K$ whose kernel $P \in \mathrm{Spec}(A)$ lies in one of the $\mathrm{Spec}(A_i)$ for $i \geq 1$, and the required calculation over K is obtained by extending the map $A_i \to K$ to a map $A'_i \to K$, and then tensoring over A'_i with K. The details are given below.

To accomplish this, we first do the generic calculation or primary decomposition or free resolution over an open affine $\mathrm{Spec}(A_1)$ in $\mathrm{Spec}(A)$. The complementary closed set is a union of closed irreducibles. We can then iterate the procedure with each of these irreducible closed sets. In fact, we can carry out this construction when A is an arbitrary Noetherian domain, and the result will follow readily once we have carried through the first step, i.e., once we have shown that we can find an open affine A_1 and a module-finite extension A'_1 where there is a generic calculation, or primary decomposition, or free resolution. It then follows by Noetherian induction that for each irreducible component of the complement of A_1, one already has a finite cover by locally closed affines as described. The result just below constructs A_1 for an arbitrary Noetherian domain A.

Discussion 2.13. Let A be a Noetherian domain and let $R_A = A[x_1, \ldots, x_B]$. Let $\mathcal{Q}_A$ (respectively, $\mathcal{M}_A$) be a $r \times m$ (respectively, $m \times n$) matrix over R_A and let M_A be the column space of $\mathcal{M}_A$. Let I_A be an ideal of R_A, and let N, $M_{1,A}, \ldots, , M_{h,A}$ be submodules of $G_A := R_A^m$. Let W_A be the kernel of the composite map

$$R_A^r \to R_A^m \twoheadrightarrow R_A^m/M_A,$$

where the map on the left has matrix $\mathcal{Q}_A$. For any A-algebra S, let R_S, M_S, G_S, etc. denote the tensor products over A of R_A, $\mathcal{M}_A$ with S. In the case of $\mathcal{M}_A$ or $\mathcal{N}_A$, $\mathcal{M}_S$ or $\mathcal{N}_S$ is the result of replacing each entry of the matrix considered by its image in S. Let $\mathcal{F}$ denote an algebraic closure of the fraction field of A. In the case ideals I_A or submodules of M'_A of G_A, the change in subscript from A to S indicates that I_A or M_A is to be replace by its image in R_A or G_A.

A well-known form of generic flatness (perhaps more accurately, generic freeness) asserts that if W_A is a finitely generated S_A-module, where S_A is a finitely generated algebra over a Noetherian domain A, one can localize at one element of $a \in A - \{0\}$ so that $(W_A)_a$ is A_a-free. It is also true that if T_A is a finitely generated S_A-algebra, that W_A is a finitely generated T_A-module, and Q_A is a finitely generated S_A-submodule of W_A, one may localize at one element of A so that $(W_A/Q_A)_a$ is A_a-free: see Lemma 8.1 of [16]. Of course, in apply this we may take S_a to be R_A.

Note that for $I_A \subseteq R_A$ or $M'_A \subseteq G_A$ there are two possible meanings for I_S and M'_S: one is $I_A \otimes_A S$ (respectively, $M'_A \otimes_A S$) and the other is its image in R_S (respectively, G'_S). By the theorem on generic freeness, using the image will be the same as the result of tensoring with S if we first replace A by a suitable localization at one element of $A - \{0\}$, which we will be free to do in this section, and we shall assume that A has been replaced by such a localization for which the two agree.

Let $\mathcal{A}$ denote the family of extension rings of A within $\mathcal{F}$ obtained by adjoining finitely many integral elements to A and then localizing at one nonzero element of A. Note that $\mathcal{F}$ is the directed union of the rings in $\mathcal{A}$.

For each of $W_{\mathcal{F}}$ (W_A is defined above as a certain kernel), $M_{!,\mathcal{F}} \cap \cdots \cap M_{h,\mathcal{F}}$, $M_{\mathcal{F}} :_{R_{\mathcal{F}}} Q_{\mathcal{F}}$, $M_{\mathcal{F}} :_{G_{\mathcal{F}}} I_{\mathcal{F}}$, $M_{\mathcal{F}} :_{G_{\mathcal{F}}} I_{\mathcal{F}}^{\infty}$, a chosen irredundant primary decomposition of $M_{\mathcal{F}}$ in $G_{\mathcal{F}}$, and a chosen finite free resolution of $\mathcal{M}$ over $R_{\mathcal{F}}$, one can choose $A' \in \mathcal{A}$, module-finite over $A_1 = A_a$, such that the kernel, intersection, colon, primary decomposition or finite free resolution is defined over A'. It may not have the same property over A', but that can be restored after localizing at one nonzero element of A.

Proposition 2.14. *Let notation and hypotheses be as in Discussion 2.13 just above. After localizing at one more nonzero element of A, we have a calculation of the kernel, intersection, or colon, or a primary decomposition or finite free resolution over $A' \in \mathcal{A}$ which is preserved by arbitrary base change to an algebraically closed field K. Since every map $A_1 \to K$, where K is an algebraically closed field, extends to a map $A' \to K$, the kernel, intersection, colon, primary decomposition and finite free resolution over K arise from the one over A' by specialization, i.e., by base change from A' to K.*

Proof. We shall be applying generic freeness repeatedly with A' replacing A. Since every nonzero element of A' has a nonzero multiple in A, we may assume in these applications that we are localizing at an element of $A - \{0\}$. We may localize at one element of A and achieve a finite number of instances of freeness over A'.

First note that after localizing at one element of $A - \{0\}$, we can preserve the exactness of a finite number of short exact sequences of finitely generated R_A-modules upon tensoring with *any* A'-algebra L. We may also preserve the inclusions in a finite filtration of a finitely generated $R_{A'}$-module, as well as the injectivity of an A'-algebra map $S_{A'} \to T_{A'}$ of finitely generated A'-algebras upon tensoring with an arbitrary A'-algebra L over A'. We may preserve intersections of two (hence, finitely many) submodules $M_{A'}$, $N'_{A'}$ of a finitely generated $R_{A'}$ module $W_{A'}$, because we may preserve the inclusions of $M_{A'}$ and $N_{A'}$ in $W_{A''}$ as well as the exactness of the sequence

$$0 \to M_{A'} \cap N'_{A'} \to M_{A'} \oplus N_{A'} \to M_{A'} + N_{A'} \to 0$$

under arbitrary base change by localizing at one element of $A - \{0\}$ so that all the modules involved become A'-free.

With these remarks it is obvious that we can preserve the exactness of

$$0 \to W'_A \to R^r_{A'} \to R^m_{A'}/M_A \to 0$$

under any base change $A' \to L$. We already know that we can preserve finite intersections of submodules. If $u_1, \ldots, u_t$ generate N_A, we have an exact sequence

$$0 \to M_{A'} : R_{A'}N_{A'} \to R_{A'} \to (G_{A'}/M_{A'})^{\oplus t}$$

were the image of $r \in R_A$ is the vector whose t entries are the images of the elements ru_j in $G_{A'}/M_{A'}$. Likewise, if $f_1, \ldots, f_t$ generate $I_{A'}$, we have an exact sequence

$$0 \to M_{A'} :_{G_{A'}} I_A \to G_{A'} \to (G_A/M_A)^{\oplus t}$$

where the image of $u \in G_{A'}$ is the image of the vector $(f_1u, \ldots, f_tu)$.

We now consider primary decomposition. We can preserve that an element of a module (hence, the module itself) is nonzero, if that is true after tensoring with $\mathcal{F}$. Call the element u_A and the module Q_A. We may localize so that all the terms of $0 \to R_Au_A \to Q_A \to W_A \to 0$ become A-free, and R_Au_A is then a nonzero free A-module. Then R_Lu_L is nonzero for every nonzero A-algebra L, and injects into Q_L. This enables to keep modules distinct, and to keep ideals distinct.

If a primary decomposition of $M_{A'}$ in G_A is irredundant, this can be preserved: we can localize sufficiently that intersection commutes with base change for all finite sets of primary components, and we can keep every intersection that omits a component distinct from $M_{A'}$. Likewise, we can keep all the primes that occur distinct. We need an additional argument to show that the primes remain primes and that components remain primary.

If $P_{A'}$ is such that $P_{\mathcal{F}}$ is prime, we can localize at one element of $A - \{0\}$ and guarantee that P_L is prime for every map of A' to an algebraically closed field L. In fact, it suffices to preserve that $D_{A'} = R_{A'}/P_{A'}$ is a domain for a finitely generated A'-algebra $D_{A'}$, given that $D_{\mathcal{F}}$ is a domain. After localizing at one nonzero element of A, we have that $D_{A'}$ is module-finite over a polynomial ring over A'. After enlarging A' and $D_{A'}$ by adjoining finitely many p^e th roots of elements of A' and of the variables, we may assume that the $D_{A'}$ is contained in a domain $D'_{A'}$ obtained by making a separable extension of the fraction field of a polynomial ring over A' and adjoining finitely many integral elements in that separable extension. By the theorem on the primitive element for separable field extensions, $D'_{A'}$ has the same fraction field as $A'[x_1, \ldots, x_h][\theta]$ where θ satisfies a monic irreducible separable polynomial H'_A over $A'[x_1, \ldots, x_h]$. Now we an choose $G_{A'} \in A'[x_1, \ldots, x_h]$ such that $D'_{A'} \subseteq (A'[x_1, \ldots, x_h]_{G_{A'}})[\theta]$. By inverting an element of $A - \{0\}$ we may assume that $G_{A'}$ is monic. To complete the proof, it suffices to show that we can, after enlarging A', keep the minimal polynomial $H_{A'}$ of θ (which we may also assume is monic) irreducible no matter what field we tensor with. This can be done using Hilbert's Nullstellensatz. For each positive degree s strictly smaller than the degree of $H_{A'}$, we can write down a potential factorization of $H_{A'}$, namely $H_{A'} = H'_{A'}H''_{A'}$, where $H'_{A'}$ has degree s, and we use indeterminates for all the coefficients of $H'_{A'}$ and $H''_{A'}$. Equating corresponding coefficients yields a system of polynomial equations in the unknown coefficients Z_j. We know these equations have no solution in the algebraically closed field $\mathcal{F}$. Hence, the polynomials we are setting equal to 0 generate the unit ideal in $\mathcal{F}[Z_j : j]$. They will therefore still generate the unit ideal in $A'[Z_j : j]$ for a suitably large choice of A'.

We can preserve that a submodule is $P_{A'}$-coprimary: filter the module by torsion-free modules over $R_{A'}/P_{A'}$, and embed each in a free $(R_{A'}/P_{A'})$-module. The filtration and the embedding will be preserved by arbitrary base change after localization at a suitable element of $A - \{0\}$.

Thus, we can choose a primary decomposition that over A' that is preserved by base change to any algebraically closed field.

To preserve $M_{A'} : I_{A'}^{\infty}$ under base change, we note that this is the same as the intersection of those primary components of $M_{A'}$ that the corresponding prime does not contain $I_{A'}$.

That one can preserve a finite free resolution is clear: its exactness is equivalent to the exactness of finitely many short exact sequences. □

We use Proposition 2.13 above to construct the open affine A_1. As mentioned earlier, we now obtain a cover by locally closed open affines as required by Noetherian induction applied to the irreducible components of the complement of $\mathrm{Spec}(A_1)$ in $\mathrm{Spec}(A)$.

Proof of Theorem 2.11. By applying this procedure to A as defined in Discussion 2.12, we obtain finitely many kernels, colons, intersections, primary decompositions or finite free resolutions that give rise to all others needed over any algebraically closed field by specialization. The existence of the bounds stated in Theorem 2.11 is immediate. □

Remark 2.15. It is clear from the argument that we can bound much more if we choose to, by taking a finer stratification. For example, we can bound all the data associated with finite free resolutions of the ideals and/or modules in the primary decomposition, and the same is true for finitely many other ideals and/or modules formed from them by iterated intersection, colon, product, and sum.

3. The proof of the main theorems A, B, C, D, E and F

We shall prove that if Theorems A, B, C, D, E, and F hold for positive integers strictly less than d then they hold also for degree d. We note that all of the theorems are obvious if $d = 1$.

To prove the first statement in Theorem A, let $D := D(k-1, d-1)$, which bounds the number of generators of a minimal prime of an ideal generated by a regular sequence of $k-1$ or fewer forms of degree $d-1$, and which exists by the induction hypothesis. Let Φ be as in Proposition 2.6, which also exists by the induction hypothesis, since one has a function ${}^3\mathcal{B}(n, d-1)$ as in Corollary B. If the strength of the d-form F is at least $\Phi(D, d)$, but the height of $(\mathcal{D}F)R$ is at most $k-1$, we can choose $k-1$ or fewer polynomials in $\mathcal{D}F$ that form a maximal regular sequence, and we can choose an associated (equivalently, minimal) prime of the ideal they generate that contains $\mathcal{D}F$. The number of generators is bounded by $D = D(k-1, d-1)$. Hence, using only those generators of degree at most $d-1$, we obtain that $\mathcal{D}F$ is contained in an ideal J generated by at most D forms of degree $\leq d-1$. By Theorem F, this contradicts the strength assumption on F. In characteristic 0 or if $p > d$, we could simply have assumed that F has strength D.

The statement in the second paragraph of Theorem A follows from the final statement. We use induction on n. The result is clear if $n = 1$. We may assume $n > 1$ and that any $n-1$ or fewer linearly independent homogeneous elements in V form an R_η-sequence. None of the elements in the basis is in the ideal generated by the others: if it were, we would get a graded relation on the basis elements in which one of the coefficients is 1: say it is the coefficient of an element of degree i. Then a nonzero linear combination of elements of degree i has an k-collapse for $k \leq i-1$, a contradiction, since we are assuming ${}^\eta A(i) \geq i-1$. Since the quotient by $n-1$ or fewer elements in the basis satisfies R_η, and so is a domain, we may assume that a basis for V consisting of forms is a regular sequence.

From the property of the ${}^\eta A(i)$ stated in the first paragraph of Theorem A, each row of the Jacobian matrix of a basis for V with respect to $x_1, \ldots, x_N$ generates an ideal of height $\eta + 3n - 3 + 1 + 1 = \eta + 3n - 1$ in the polynomial ring. Since $n \geq h$, where h is the number of nonzero V_i, by Theorem 2.5, the height of the defining ideal of the singular locus of $R/(V)$ in $R/(V)$ is at least $\eta + 1$, so that $R/(V)$ satisfies R_η.

We next show that Theorem A in degree at most d implies Theorem B in degree at most d. Linearly order the dimension sequences $\delta = (\delta_1, \ldots, \delta_d)$ so that $\delta < \delta'$ precisely if $\delta_i < \delta'_i$ for the largest value of i for which the two are different. This is a well-ordering. Assume that ${}^\eta B$ is defined for all values all predecessors of δ. If the vector space is ${}^\eta A(\delta)$-strong, it satisfies R_η and we are done. If not, for some i an element of V_i has an ${}^\eta A_i(\delta)$-collapse, and we can express the element in terms of at most $2 \cdot {}^\eta A_i(\delta)$ forms of lower degree. This enables us to form a new vector space in which δ_j remains the same for $j > i$, and the δ_i decreases by 1, and the earlier δ_i increase by a total of $2 \cdot {}^\eta A_i(\delta)$. If we let δ' run through all dimension sequences, with this property, which precede δ in the well-ordering we may take ${}^{\eta}B(\delta) = \max_{\delta'}\{{}^{\eta}B(\delta')\}$. This completes the proof of Theorem B.

Theorem C is immediate, because if d bounds the degrees of the entries of the matrix, then mnd bounds the number of non-scalar homogeneous components of all entries, and $C(m, n, d) := \max\{B(\delta) : \sum_{t=1}^{d} \delta_t = mnd\}$ bounds the projective dimension of the cokernel.

Theorems D and E follow immediately from the existence of ${}^{\eta}B$ and Theorem 2.11 of the preceding section, while as already noted, Theorem F in degree d follows from Theorem B in degree $d-1$ by Proposition 2.6. □

4. Results in low degree

This section contains *statements only* for some results in degree ≤ 4: see §2 of [2]. The methods of that paper avoid the use of the results on primary decomposition developed in Theorem 2.11, which relies on Proposition 2.13. This permits calculation of values for ${}^{\eta}A$ in degree at most 4 and for $B(n_1, n_2)$ in degree 2.

The following theorems gives a value of ${}^{\eta}A$ and for $B(0,\, n)$ for the degree 2 case in all characteristics. In these theorems, R is a polynomial ring over an algebraically closed field K.

Theorem 4.1. *Let V be a vector space of quadratic forms in R of dimension n over K. If every element of $V - \{0\}$ is $(n-1)$-strong, every sequence of linearly independent elements of V is a regular sequence. If $\eta \geq 1$ and every element of $V - \{0\}$ is $(n - 1 + \lceil \frac{\eta}{2} \rceil)$-strong, then the quotient by the ideal generated by any elements of V satisfies the Serre condition* R_{η}.

Theorem 4.2. *A vector space of quadrics in the polynomials ring R that has dimension n is contained in a polynomial subring generated by a regular sequence consisting of at most $2^{n+1}(n-2) + 4$ linear and quadratic forms. Hence, the projective dimension of R/I, where I is the ideal generated by these forms, is at most $2^{n+1}(n-2) + 4$.*

Theorem 4.3. *Let $b = 2(n_2 + n_3) + \eta + 1$ if $n_2 \neq 0$, and $2(n_2 + n_3) + \eta$ if $n_2 = 0$. Let $\mathfrak{R}(b) = (2b+1)(b-1)$ if the characteristic is not 2 or 3, $\mathfrak{R}(b) = 2(2b+1)(b-1)$ if $\mathrm{char}(K)$ is 2, and $\mathfrak{R}(b) = 2b^2 - b$ if $\mathrm{char}(K)$ is 3. Then we may take*

$$ {}^{\eta}\underline{A}(n_1, n_2, n_3) = \Big(0, \lceil \frac{b}{2} \rceil + n_1, \mathfrak{R}(b) + n_1\Big). $$

Remark 4.4. While bounds independent of the characteristic exist for these low degree cases, the results tend to indicate that better bounds hold if the characteristic is assumed to be 0 or $> d$.

Unfortunately, the above result for degree 3 leads to values for B that are n-fold exponential. In degree 4, the computational results we have, even for the functions A, are technical and discouraging. The formula for A involves an auxiliary function that is exponential. However, it still seems possible that there is a better result that is only quartic in the number of forms.

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