PARAMETER-LIKE SEQUENCES AND EXTENSIONS OF TIGHT CLOSURE

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0. INTRODUCTION

We introduce here a notion of closure for ideals (*parameter tight closure*) in arbitrary Noetherian rings, including rings of mixed characteristic, that we hope will have properties parallel to those enjoyed by tight closure in characteristic p. We are not able to prove that the definition proposed here has all the properties that tight closure does: this remains an open question. But we can show that it agrees with tight closure in prime characteristic $p > 0$, that it is, in general, contained in the solid closure introduced in [Ho8] and [Ho9] (which is known to be "too large"), and it appears *very* likely that in many cases it is smaller than solid closure: cf. Discussion (3.6) and Theorem (3.7). We also show that, quite generally, including in mixed characteristic, it captures elements of a domain which are in the expansion of an ideal to an integral extension (sse Theorem (2.5)), and that, in equal characteristic 0, it has so-called "colon-capturing" properties analogous to those of tight closure (see Theorem (3.2)). The case of complete local domains suffices for applications, and so, for simplicity, we often restrict to that case in the sequel.

In fact, closure operations of the kind we have in mind are determined by their behavior over complete local domains. We describe briefly how the definition goes in that case. The underlying idea is to define a notion of "parameter-like" sequence in an algebra, not necessarily Noetherian, over a complete local domain R. The definition is made in terms of

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annihilation properties of certain local cohomology modules. See (2.2) for details. It is, in fact, the case that a system of parameters of a complete local domain R is parameter-like. We next define the notion of a "parameter-preserving" algebra S over the complete local domain R: an R-algebra S is *parameter-preserving* if and only if every system of parameters in R is parameter-like in S. We then define the *parameter tight closure* $I^\natural$ of $I \subseteq R$ to be the smallest ideal J of R containing I such that for every parameter preserving-algebra S over R, the contraction of JS to R is J.

If R is a complete local domain, it turns out that a parameter-preserving algebra is solid, which makes the new closure *a priori* contained in the solid closure defined in [Ho9]. In §3 we show that the solid algebra Roberts uses in [Ro6] to prove that the solid closure of an ideal in a regular ring of equal characteristic zero can be strictly larger than the ideal is not parameter-preserving. Thus, it appears to be possible that the parameter tight closure of any ideal in a regular ring is equal to the ideal, although we cannot prove this. Moreover, we do not know whether parameter tight closure agrees with any of the equal characteristic zero notions of tight closure introduced in [HH10]: but we can show that it contains the largest of them, the big equational tight closure defined there. See Theorem (3.3).

We shall show that this parameter tight closure has many of the properties that we want a tight closure theory to have. However, we do not know whether every ideal of a regular ring is parameter tightly closed, neither in equal characteristic zero nor in mixed characteristic. If the mixed characteristic case could be established, the direct summand conjecture would follow.

Of course, we were led to study the notion of parameter-like sequences because of the possibility of settling many long-standing open questions in mixed characteristic (cf. [Ho1,7], [PS1,2], [Ro1-5,7], [Ho2,3,5,6], [Du1,2], [DHM], [EvG], and [Rang]) via the construction of a suitable analogue of tight closure theory. However, we feel that parameter-like sequences are worthy of study in their own right even without the potential for this application: their behavior appears to be subtle even in finitely generated algebras over a complete local domain.

Other notions, defined quite differently, that generalize tight closure have been explored in [Heit1–3] and [HoV]: these are related to ideas from [Sm] and [HH5]. In particular, in a tremendous breakthrough, [Heit3] resolves the direct summand conjecture in dimension 3. In [Ho11] a main result of [Heit3] is used to prove the existence (in a weakly functorial sense) of big Cohen-Macaulay algebras in dimension 3, even in mixed characteristic.

1. TIGHT CLOSURE IN POSITIVE CHARACTERISTIC

For the theory of tight closure in characteristic p we refer the reader to [Ho8], [HH1–4, 6–12], [Hu], [Sm], and [Bru]. The equal characteristic zero theory is described in [Ho9] and [Ho10], and developed in complete detail in [HH10].

For the moment we shall be working over a Noetherian ring R of positive prime characteristic p. In this situation we shall always let e denote an element of the nonnegative integers $\mathbb{N}$, and write q as an abbreviation for p^e. Thus "for all sufficient large q" means "for all sufficiently large integers q of the form p^e," and so forth.

Recall that a module M over a local ring (R, m, K) is a *balanced big Cohen-Macaulay module* (cf. [Sh]) for R if $M \neq mM$ and every system of parameters for R is a regular sequence on M. If M is also an R-algebra it is called a *big Cohen-Macaulay algebra* for R (i.e., in the context of algebras we shall always assume "balanced" but we omit the word).

We also recall that if R is a domain then an R-module M is called *solid* if there exists a nonzero R-linear map from M to R, i.e., $\mathrm{Hom}_R(M, R) \neq 0$. If (R, M, K) is a complete local domain of dimension d then it turns out that M is solid if and only if $H^d_m(M) \neq 0$. An R-algebra is called *solid* if it is solid when considered as an R-module.

In order to explain, in part, why we are led to consider the notion of *parameter tight closure* we first consider four characterizations of tight closure in the characteristic $p > 0$ case. For simplicity, we consider only the case of ideals, and when it simplifies matters, we assume that the Noetherian ring R of prime characteristic $p > 0$ is a complete local domain. The first characterization given below is actually the definition of tight closure in positive characteristic. The characterizations (2)and (3) below are consequences of Theorems (11.1) and (8.6b), respectively, of [Ho9]. The characterization (4) is a consequence of

Theorem (8.17) of [HH4] and the results of [Mo] on Hilbert-Kunz multiplicities. See also the discussion in [Ho8], p. 179.

(1.1) Characterizations of of tight closure. Let R be a Noetherian ring of prime characteristic $p > 0$. Let $u \in R$ and let $I \subseteq R$ be an ideal. Let I^* denote the tight closure of I.

(1) (Definition) $u \in I^*$ precisely if there exists c not in any minimal prime of R such that $cu^q \in I^{[q]}$ for all nonnegative integers e, where $I^{[q]}$ is the ideal generated by all q *th* powers of elements of I. (When R is a domain, the condition on c is simply that it not be 0.)

(2) Let R be a complete local domain. $u \in I^*$ if and only if $u \in IS \cap R$ for some big Cohen-Macaulay R-algebra S.

(3) Let R be a complete local domain. $u \in I^*$ if and only if $u \in IS \cap R$ for some solid R-algebra S.

(4) Let R be a complete local domain. Assume also that I is m-primary. With $J = I + uR$, we have that $u \in I^*$ if and only if $\lim_{e\to\infty} \dfrac{\ell(R/J^{[p^e]})}{\ell(R/I^{[p^e]})} = 1$. (Here, "$\ell$" indicates length.)

We present these characterizations because every characterization of tight closure in prime characteristic $p > 0$ gives a potential method for generalizing the theory to mixed characteristic. We want to discuss briefly the difficulties that arise from using these characterizations to help motivate the definitions of the next section.

We first note that an analogue of (1) can be defined in equal characteristic zero by reduction to characteristic p. This idea gives a very good extension of tight closure theory to the equal characteristic zero case (cf. [Ho8], [Ho10], [HH10]), but this definition does not seem to lead to any highly useful notion in mixed characteristic.

Condition (2) might lead to a notion that is a good notion in all characteristics, but at this time this idea does little good in mixed characteristic, because big Cohen-Macaulay algebras are not known to exist in mixed characteristic.

Condition (3) leads to a notion that is explored in the author's paper [Ho9], but an example [Ro6] of Paul Roberts shows that solid closure is too big in equal characteristic zero (ideals in regular rings of dimension 3 are not always solidly closed). The situation in mixed characteristic is unresolved, but there it is difficult to prove anything and Roberts' example is discouraging. Solid closure does give some information, but not enough to settle, for example, the direct summand conjecture. For further information about this and related conjectures, we refer the the reader to [Ho1,7], [PS1,2], [Ro1-5,7], [Ho2,3,5,6], [Du1,2], [DHM], [EvG], and [Rang].

In connection with all of these conditions, we note that if R is essentially of finite type over a field or even over an excellent local ring, and has prime characteristic $p > 0$, then $u \in I^*$ if and only if the image of u is in $(ID)^*$ (working over D) for every complete local domain D to which R maps.[1] Thus, under mild conditions on the ring, tight closure theory in prime characteristic $p > 0$ is determined by its behavior for complete local domains.

Condition (4) merits some further comment. Note that in a complete local domain (R, m), the tight closure of I is the intersection of the tight closures of the m-primary ideals containing I, and so tight closure is determined by its behavior on m-primary ideals. The intriguing condition (4) can be rephrased slightly as follows: for m-primary ideals $I \subseteq J$ in a complete local domain (R, m) of prime characteristic p, $J \in I^*$ if and only if I and J have the same Hilbert-Kunz multiplicity. (It is known that, with $d = \dim R$, $\ell(R/I^{[q]}) = \gamma(q^d) + O(q^{d-1})$, where γ, the Hilbert-Kunz multiplicity, is a positive real constant (conjectured, but not known, to be rational) and the term $O(q^{d-1})$ is bounded in absolute value by a constant times q^{d-1}.) Cf. [Mo] for the basic theory, and see [HaMo] for some surprising examples. This exciting tie-in between tight closure and Hilbert-Kunz multiplicities has not, so far, led to any possible extensions of tight closure theory to mixed characteristic.

We want to come back to the conditions (2) and (3). Evidently, if one has a class of R-algebras contained in the solid R-algebras and containing the big Cohen-Macaulay R-algebras, one can use it to define a notion of closure that will agree with tight closure in prime characteristic $p > 0$ and may give a good notion in equal characteristic 0 and in mixed characteristic. In the next section we define a class of algebras, the *parameter-*

[1] One may use [HH7], Prop. (6.23) and Thm. (6.24) to show "only if". To prove "if" one may use that the rings considered have completely stable test elements. One can reduce to looking at the completions of their local rings and then the quotients of those by minimal primes by [HH4], Prop. (6.25).

preserving algebras, and prove that it lies between the class of big Cohen-Macaulay algebras and the class of solid algebras: it is not obvious that parameter-preserving algebras are solid, but that is the content of Theorem (2.7). We shall show in the sequel that this class of algebras gives a notion with many of the properties we want, and, so far as we know, it may have all of the properties that we want. In §3 we show that the algebra that Roberts uses in [Ro6] to show that solid closure is too big (in the sense that not every ideal of a regular ring is solidly closed) is not parameter-preserving. Thus, there is no known "obstruction" to prevent this notion from being a good one in equal characteristic 0 and in mixed characteristic. But whether it has all the properties one would like remains an open question.

2. PARAMETER-LIKE SEQUENCES AND PARAMETER-PRESERVING ALGEBRAS

As discussed earlier, we want to explore here the possibility of defining a closure operation that is provably useful in all characteristics along the following lines: we first define a property, *parameter-preservation*, of algebras that is stronger than being solid but weaker than being a big Cohen-Macaulay algebra. We then define u to be *immediately in the parameter tight closure* of I if $u \in IS \cap R$ for some algebra S having the specified property. We then take the *parameter tight closure* $I^\natural$ of I to be the smallest ideal of R containing I that is closed under immediate parameter tight closure. We can do something similar for modules. The detailed definition is given in (2.2).

Although we are primarily interested in complete local domains, it will be convenient to allow complete local rings of pure dimension as well: recall that R has *pure dimension* d if (0) has no embedded prime ideals, and for every minimal prime P of R, the dimension of R/P is d. This is equivalent to the statement that every nonzero submodule of R has dimension d. Likewise, we say that an R-module has *pure dimension* d if it and all of its nonzero submodules have dimension d.

Much of the sequel depends on the facts (b) and (c) about local cohomology in the following:

(2.1) Lemma. *Let R be a complete local ring of pure dimension d. Then:*

(a) *Given any system of parameters $x_1, \dots, x_d$ for R, R is module-finite over a complete local ring $A \subseteq R$ such that $x_1, \dots, x_d \in A$ and A is either regular (A can always be chosen to be regular in the equal characteristic case), or A is a hypersurface in a complete regular local ring.*

(b) *For $i \neq d$, the module $H^i_m(R)$ is annihilated by an ideal of height at least 2 in R. (The unit ideal has height $+\infty$, and so the condition is satisfied if the local cohomology module vanishes.)*

(c) *Let M be any R-module, not necessarily Noetherian. Let $\mathfrak{B} \subseteq R$ denote the annihilator of $H^d_m(M)$. Then if $H^d_m(M) \neq 0$, $\dim R/\mathfrak{B} = d$.*

Proof. (a) This is quite standard if R contains a field: it has a coefficient field K and one may choose $D = K[[x_1, \dots, x_d]]$. In the mixed characteristic case, choose a coefficient ring B for R. This means that $B \subseteq R$ with $m_B \subseteq m = m_R$, that $B/m_B \to R/m$ is an isomorphism, and that for some mixed characteristic discrete valuation ring V with residual characteristic p such that $m_V = pV$, either $B = V$ or $B = V/(p^t)$. Choose a map ϕ of $V[[X]] = V[[X_1, \dots, X_d]]$ to R so that the X_i map to the x_i, the given system of parameters for R. Then image A of $V[[X]]$ in R has pure dimension d, since $R \supseteq A$ is module-finite, and so $\operatorname{Ker}\phi \subseteq V[[X]]$ is a pure height one ideal of the unique factorization domain $V[[X]]$. But then $\operatorname{Ker}\phi = (f)$ is principal, and R is module-finite over $A \cong V[[X]]/(f)$ as required.

(b) Choose A as in part (a). Then, by local duality over the Gorenstein ring A, the Matlis dual of $H^i_m(R)$ is $\operatorname{Ext}^{d-i}_A(R, A)$, $0 \leq i \leq d-1$, and so it suffices to see that the Ext has an annihilator of height two or more in A: this ideal will expand to an ideal of height two or more in R. Therefore it suffices to see that for every height one prime Q of A, $\operatorname{Ext}^j_{A_Q}(R_Q, A_Q) = 0$ for $j = d - i \neq 0$. But since A_Q is a one-dimensional Gorenstein ring, the Matlis dual of the localized Ext is $H^{1-j}_{QA_Q}(R_Q)$, which is 0 if $j > 1$, or $j < 0$, clearly, and vanishes when $j = 1$ because R_Q is of pure dimensional one over A_Q, and this implies that it is Cohen-Macaulay.

(c) If $\dim R/\mathfrak{B} < d$, then $\mathfrak{B}$ contains an element x_1 that is part of a system of parameters. Hence, we can choose $A \subseteq R$ as in part (a) such that $x_1 \in A$. Since R is module-finite over A, the maximal ideal of A expands to an m-primary ideal of R. Thus, if we think of M is an A-module, the local cohomology does not change, and we still have that $H^d_{m_A}(M) \neq 0$ but that this module is killed by the parameter x_1. We may therefore assume that $R = A$ is a hypersurface. Let $E = E_R(K) \cong H^d_m(R)$. Then $H^d_m(M) \cong M \otimes_R E$ is nonzero and killed by x_1. But $\mathrm{Hom}_R(\_, E)$ is faithfully exact, and so we get that $\mathrm{Hom}_R(H^d_m(M), E) \cong \mathrm{Hom}_R(M \otimes_R E, E)$ is nonzero and killed by x_1, and by the adjointness of tensor and Hom this may be identified with $\mathrm{Hom}_R\bigl(M, \mathrm{Hom}_R(E, E)\bigr) \cong \mathrm{Hom}_R(M, R)$, since R is complete. Since the Hom is nonzero and killed by x_1, there exists a nonzero map $M \to R$ that is killed by x_1. But this means that the image of the map is an ideal of R killed by x_1, and x_1 is not a zerodivisor in R. This is a contradiction. $\square$

If R is a local ring and J is any ideal of R, we define J^{unmx} to be the intersection of the primary components of J corresponding to minimal primes Q of J such that $\dim R/Q = \dim R/J$. (We have restricted this definition to the local case to avoid difficulties that arise from rings having maximal ideals with differing heights.) Note that if R is any local ring of dimension d, then $R/(0)^{\mathrm{unmx}}$ has pure dimension d. In fact, $(0)^{\mathrm{unmx}}$ is the largest ideal I of R such that the dimension of I as an R-module is smaller than d: it consists of all elements of $u \in R$ such that $\dim Ru < d$.

(2.2) Definitions: parameter-like sequences, parameter-preserving algebras, and parameter tight closure. Although we are primarily interested in complete local domains, it will be convenient to allow complete local rings of pure dimension d as well in certain definitions. Thus, let R be a complete local ring of pure dimension d and let S be an R-algebra. Let $x_1, \ldots, x_d$ be a system of parameters for R. Let $T_0 = \mathcal{T}_0(S)$ be the quotient of S by the ideal of all elements that have an annihilator of positive height in R, and, recursively, if $T_i = \mathcal{T}_i(S)$ has been defined for $i < d$ let T_{i+1} be the quotient of $T_i/(x_{i+1}T_i)$ by the ideal of all elements u such $\dim Ru < d-(i+1)$. (Note that $Ru \in T_{i+1}$

is killed by $(x_1, \dots, x_{i+1})$ and that $\dim R/(x_1, \dots, x_{i+1}) = d - (i+1)$.) If we need to make explicit the dependence of $\mathcal{T}_i(S)$ on the choices of R and $\underline{x} = x_1, \dots, x_d$, we shall write $\mathcal{T}_i^R(\underline{x}; S)$, but we shall usually omit either or both of R, $\underline{x}$.

Call a system of parameters $x_1, \dots, x_d$ *parameter-like* in S if $T_d \neq 0$, and for all i, $0 \leq i \leq d-1$, the height of the annihilator $\mathfrak{A}_i$ of $H_m^{d-1-i}(T_i)$ in R is at least $i+2$. We note again that we are making the usual convention that the height of the unit ideal is $+\infty$, and so the condition is satisfied whenever $H_m^{d-1-i}(T_i)$ vanishes. Since R was assumed to have pure dimension, it is equivalent to assert that for every i, $0 \leq i \leq d-1$, either $H^{d-1-i}(T_i) = 0$, or else $\dim R/\mathfrak{A}_i \leq d - i - 2$.

Call S a *parameter-preserving* R-algebra if every system of parameters $x_1, \dots, x_d$ of R is parameter-like in S.

Given $N \subseteq M$, finitely generated R-modules, define $u \in M$ to be in the *immediate parameter tight closure* of N in M if there exists a parameter-preserving R-algebra S such that $1 \otimes u$ is in the image of $S \otimes_R N$ in $S \otimes_R M$ (if $M = R$ and $N = I$ is an ideal, this just says that the image of u in S is in IS). Define the *first parameter tight closure of N in M* to be the submodule of M generated by the elements in the immediate tight closure of N. The first parameter tight closure will be a submodule of M containing N. Iterating this process, we obtain an ascending chain of submodules of M that must stabilize. We define the stable submodule in this chain to be the *parameter tight closure* of N in M, and denote it $N_M^\natural$ or simply $N^\natural$. When N is an ideal of R, M is understood to be R unless otherwise specified.

Alternatively, $N^\natural$ is the smallest submodule of M containing N that has the property that for any element $u \in M$ and any parameter preserving-algebra S over R, if $1 \otimes u$ is in the image of $S \otimes_R N^\natural$ in $S \otimes_R M$, then $u \in N^\natural$.

The definition of parameter-like is rather technical. The results that follow will help explain why it was chosen. The key points that will be established are:

(1) A system of parameters in a complete local domain is parameter-like, and module-finite extensions of complete local domains are parameter-preserving. (Cf. (2.3).)
(2) A big Cohen-Macaulay algebra over R is parameter-preserving. (Cf. (2.6).)
(3) A parameter-preserving algebra is solid. (Cf. (2.7).)

(4) The algebra that is used in [Ro6] to show that not every ideal in a regular ring is solidly closed in equal characteristic 0 is *not* parameter-preserving. (Cf. (3.6) and (3.7).)

These conditions imply many good properties for this closure operation, the majority of which are discussed in §3. We now proceed to the proofs.

(2.3) Theorem. *If R is a complete local domain then every sequence of elements that a system of parameters for R is parameter-like in R and in every module-finite extension domain of S of R. Hence, every module-finite extension domain of R is parameter-preserving, including, of course, R itself.*

Proof. Fix part of a system of parameters in R: it will also be a system of parameters for S. One sees by induction on i that T_i is a local ring module-finite over $R/(x_1, \dots, x_i)$, that T_i has pure dimension $d - i$, using the remark following Fact (2.1), and that it is a quotient of S by a proper ideal. Thus, all the T_i are nonzero. The fact that the annihilators of the local cohomology modules are as stated now follows from Lemma (2.1b). □

We next observe:

(2.4) Lemma. *Let (R, m) be a complete local ring of pure dimension d, and let S be an R-algebra. Let $\underline{x} = x_1, \dots, x_d$ be a system of parameters in R. Let $R_i = \mathcal{T}_i^R(\underline{x}; R)$ (cf. Definition (2.2)), and $T_i = \mathcal{T}_i^R(\underline{x}; S)$. Then:*

(a) *$R_0 = R$ and for every $i \le d$, R_i is a homomorphic image of $R/(x_1, \dots, x_i)R$ that has pure dimension $d - i$. Moreover, S_i is an R_i module. Let y_j be the image of x_j in R_i for $i + 1 \le j \le d$. Then for $i \le j \le d$, $\mathcal{T}_{j-i}^{R_i}(y_{i+1}, \dots, y_d; S_i) = S_j$. Thus, $x_1, \dots, x_d$ is parameter-like in S if and only if $H^{d-1}(T_0)$ has an annihilator of height at least two in R, and the images of $x_2, \dots, x_d$ are parameter-like in T_1 over R_1.*

(b) *If S' is flat over S, then for $1 \le i \le d$, $\mathcal{T}_i(S') \cong S' \otimes_S \mathcal{T}_i(S)$, and, for all j, $H_m^j(\mathcal{T}_i(S')) \cong S' \otimes_S H_m^j(\mathcal{T}_i(S))$. Thus, if $x_1, \dots, x_d$ is parameter-like in S, then it is parameter-like in S' if and only if $S' \otimes_S T_d \neq 0$. In particular, if S' is faithfully flat over S, and $x_1, \dots, x_d$ is parameter-like in S then it is parameter-like in S'. Likewise, if $S' = W^{-1}S$, where W is a multiplicative system in S, then for all i, $0 \le i \le d$, for* *all j, we have that $\mathcal{T}_i(W^{-1}S) \cong W^{-1}T_i$, that $H^j_m\big(T_i(W^{-1}S)\big) \cong W^{-1}H^j_m(T_i)$, and, if $x_1, \ldots, x_d$ is parameter-like in S, then it is parameter-like in $W^{-1}S$ if and only if $W^{-1}T_d \neq 0$.*

(c) *If $x_1, \ldots, x_d$ is parameter-like in S and Q is any prime in the support of T_d, S_Q is parameter-preserving. Thus, in testing immediate parameter tight closure it suffices to consider quasilocal R-algebras (S, Q) over R such that m maps into Q.*

Proof. (a) $R_0 = R$ since R is assumed to have pure dimension, and it is clear that R_i is a homomorphic image of $R/(x_1, \ldots, x_i)$, and that T_i is a module over $R/(x_1, \ldots, x_i)$. The statement that R_i has pure dimension $d - i$ is immediate by induction on i.

Let $u \in R/(x_1, \ldots, x_i)$ be such that Ru has dimension $< d - i$ in $R/(x_1, \ldots, x_i)$. If $v \in T_i$ then the cyclic module Ruv is a homomorphic image of Ru, and so also has dimension $< d - i$. It follows that uv is killed in T_i, and so u kills T_i. Thus, T_i is an R_i-module. Once we know this, we have at once from the definitions that for $i \leq j \leq d$, $\mathcal{T}^{R_i}_{j-i}(y_{i+1}, \ldots, y_d;\, S_i) = S_j$, and the final statement in part (a) is then clear.

For part (b), first note that this holds when $i = 0$. The ideal of S' that we must kill to form $\mathcal{T}_0(S')$ is the union of the annihilators in S' of the positive height ideals of R. For any such ideal I, the annihilator of I in S' is the expansion of its annihilator from S, and so the union is the expansion of the union of the annihilators in S. We may then proceed by induction on i. Killing x_{i+1} times the algebra commutes with tensoring with S' over S, and the next step is like the formation of T_0, but working with $S_i/x_{i+1}S_i$ and R_{i+1} instead of S and R. The statement that local cohomology commutes with tensoring with S' over S is obvious from the Cech complex method of defining local cohomology, and the final statement follows at once.

Part (c) is implied at once by part (b). □

(2.5) Theorem. *If R is a complete local domain, I is an ideal of R, and S is a module-finite extension of R then $IS \cap R \subseteq I^\natural$. Hence, if $I = I^\natural$ for every ideal of R, then R is a direct summand of every module-finite extension.*

Proof. We can replace S by a quotient by a minimal prime of S disjoint from the domain R, and then the first statement is immediate from (2.3) and the definition of parameter tight closure. The second statement then follows from the main result of [Ho4]. □

(2.6) Theorem. *If S is a balanced big Cohen-Macaulay algebra for R then S is parameter-preserving.*

Proof. Note that $S_0 = S$, since any height one (or more) ideal of R will contain an element that is part of a system of parameters, and so the annihilator is 0. By a trivial induction on i, we have $T_i = S/(x_1, \dots, x_i)S$, $1 \le i \le d$, which is a big Cohen-Macaulay algebra over R_i. The Cohen-Macaulay condition on S_i implies that it has depth $d - i$ on the maximal ideal of m, and so all the $H^j_m(S_i) = 0$ for $j < d-i$, and, in particular, for $j = d-1-i$. □

(2.7) Theorem. *If R is a complete local domain and S is a parameter-preserving R-algebra, then S is a solid R-algebra. In fact, if $\dim R = d$, $x_1, \dots, x_d$ is a system of parameters for R, and the T_i are as in Definition (2.2), then we have that for all i, $0 \le i \le d$, $H^{d-i}_m(T_i) \ne 0$. (In particular, this holds when $i = 0$, which yields the fact that S itself is solid.)*

Proof. We use reverse induction on i to show that all the $H^{d-i}_m(T_i) \ne 0$, $i = d, d-1, \dots, 0$. When $i = d$ we have that T_d is a nonzero module killed by $x_1, \dots, x_d$ and, hence, by a power of m. Thus, $H^0_m(T_d) \ne 0$. Now suppose that we have shown that a certain $H^{d-i}_m(T_i) \ne 0$, $1 \le i \le d$. We must show that $H^{d+1-i}_m(T_{i-1}) \ne 0$. Now $x = x_i$ is a nonzerodivisor on T_{i-1} by the construction for T_{i-1}, and so we have a short exact sequence

$$(*) \qquad 0 \to T_{i-1} \xrightarrow{x} T_{i-1} \to T_{i-1}/xT_{i-1} \to 0.$$

Also, we have a short exact sequence $0 \to I \to T_{i-1}/xT_{i-1} \to T_i \to 0$ where I is an ideal of T_{i-1}/xT_{i-1} consisting of elements that are killed by an ideal of positive height in R_i. This means that every finitely generated R_i-submodule of I has dimension $< d - i$ as an R_i-module. We can conclude that $H^{d-i}_m(I) = H^{d-i+1}_m(I) = 0$, and so, from the long exact sequence for local cohomology, $H^{d-i}(T_{i-1}/xT_{i-1}) \cong H^{d-i}(T_i) \ne 0$, by the induction hypothesis.

On the other hand, the short exact sequence $(*)$ displayed above yields a long exact sequence of local cohomology modules part of which is

$$H^{d-i}_m(T_{i-1}) \xrightarrow{x} H^{d-i}_m(T_{i-1}) \to H^{d-i}_m(T_{i-1}/xT_{i-1}) \to H^{d-i+1}_m(T_{i-1})$$

We assume that the last term is 0, and get a contradiction. If the last term is zero, we have a surjection:

$$H_m^{d-i}(T_{i-1})/xH_m^{d-i}(T_{i-1}) \twoheadrightarrow H_m^{d-i}(T_{i-1}/xT_{i-1})$$

Since $H_m^{d-i}(T_{i-1}/xT_{i-1}) \neq 0$, we know that $H_m^{d-i}(T_{i-1}) \neq 0$. By Definition (2.2), since $x_1, \ldots, x_d$ is parameter-like in S, if $\mathfrak{A}$ is the annihilator of $H_m^{d-i}(T_{i-1})$ in R, we have that $\dim R/\mathfrak{A} \leq d-(i-1)-2 = d-1-i$. But $\mathfrak{A}$ annihilates $H_m^{d-i}(T_{i-1}/xT_{i-1})$ as well, and so if $\mathfrak{B}$ is the annihilator of $H_m^{d-i}(T_{i-1}/xT_{i-1})$ we have that $\dim(R/\mathfrak{B}) \leq d-1-i$. If we think of T_{i-1}/xT_{i-1} as a module over R_i (which has pure dimension $d-i$) we see that we have a contradiction, by Lemma (2.1c). □

3. THE NEW CLOSURE OPERATION

(3.1) Theorem. *For a complete local domain R of prime characteristic $p > 0$, parameter tight closure is the same as the tight closure.*

Proof. Let $N \subseteq M$ be finitely generated R-modules. To show that $N^\natural \subseteq M^*$, it suffices to show that if $u \in M$ is in the immediate parameter tight closure of N in M, then $u \in M^*$. This is immediate from the Theorem (8.6) of [Ho9]: since any parameter-preserving algebra is solid, by Theorem (2.7), one has that u is in the solid closure of N in M, and then by it is in N^*, by [Ho9, Thm. (8.6)].

The converse follows from Theorem (11.1) of [Ho9]: if u is in N^*, then there exists a big Cohen-Macaulay R-algebra S such that $1 \otimes u$ is in the image of $S \otimes_R N$ in $S \otimes_R M$, and S is parameter-preserving by Theorem (2.6). □

(3.2) Theorem. *Let R be a complete local domain of equal characteristic, or a complete local domain of mixed characteristic and dimension at most three.*

(a) *(Colon capturing property) Let $x_1, \ldots, x_d$ be a system of parameters for R. Then for $1 \leq i \leq d-1$, if $I = (x_1, \ldots, x_i)R$, then $I : x_{i+1} \subseteq I^\natural$.*

(b) *(Analogue of phantom acyclicity) Let $G_\bullet$ denote a finite complex of finitely generated free modules over R that satisfies the standard conditions on rank and height.*[2] *Then for each $i \geq 1$, the module of cycles $Z_i \in G_i$ is in the parameter tight closure in G_i of the module of boundaries B_i.*

Proof. Of course, in the prime characteristic $p > 0$ case both parts follow from the fact that parameter tight closure agrees with tight closure, for which the statements in this theorem are standard.

However, the proof that we give for equal characteristic 0 also handles the positive characteristic case. By the main results of [HH5] (for characteristic $p > 0$), [HH9] (for equal characteristic 0), and [Ho11] (for mixed characteristic and dimension at most 3), R has a big Cohen-Macaulay algebra S. In S, $x_1, \dots, x_d$ is a regular sequences and so $I :_R x_{i_1} \subseteq I :_S x_{i+1} \subseteq IS$. Part (b) follows similarly, because when we apply $S \otimes_R \_$, the complex $S \otimes_R G_\bullet$ becomes acyclic over S. □

(3.3) Theorem. *Let $N \subseteq M$ be finitely generated modules over a complete local domain R of equal characteristic 0. Then $N^\natural \supseteq N^{*\mathrm{EQ}}$, the big equational tight closure of N in the sense of [HH10].*

Proof. Theorem (11.4) of [Ho9] shows that for any element u of $N^{*\mathrm{EQ}}$ there is a big Cohen-Macaulay algebra S for R such $1 \otimes u$ is in the image of $S \otimes_R N$ in $S \otimes_R M$. □

(3.4) Theorem. *Let R be a complete local domain of dimension at most two. Let $N \subseteq M$ be finitely generated R-modules. Then $N^\natural$ is the same as the solid closure of N in M, and $u \in N^\natural$ if and only if there is a big Cohen-Macaulay algebra S for R such that $1 \otimes u$ is in the image of $S \otimes_R N$ in $S \otimes_R M$.*

Proof. By Proposition (12.3) and Theorem (12.5) of [Ho9], in the dimension two case, an algebra over R is solid if and only if it can be mapped further to a big Cohen-Macaulay algebra. The parameter tight closure is always contained in the solid closure because parameter-preserving algebras are solid. In dimension two, the converse holds because any

[2]This means that the sum of the determinantal ranks of the maps to and from G_i is the rank of G_i, and that the ideal generated by the rank size minors of a matrix of the map $G_i \to G_{i-1}$ has height $\geq i$.

solid algebra can be mapped further to a big Cohen-Macaulay algebra, and big Cohen-Macaulay algebras are parameter-preserving. □

(3.5) Corollary. *Over a complete regular local ring of dimension at most two, every submodule of every finitely generated module is parameter tightly closed.*

Proof. It suffices to check that the immediate parameter tight closure of a submodule is equal to the submodule. But an element is in it if and only if it gets into the expanded submodule after tensoring with a big Cohen-Macaulay algebra, by Theorem (3.4). But a big Cohen-Macaulay algebra over a regular ring is faithfully flat over the regular ring (cf. the parenthetical argument in 6.7 on p. 77 of [HH5]). □

(3.6) Discussion. Let $R = K[[x_1, x_2, x_3]]$, where K is the field of rational numbers or any other field of characteristic 0, and let $S = R[y_1, y_2, y_3]/(F)$, where $F = x_1^2x_2^2x_3^2 - \sum_{j=1}^{3} y_jx_j^3$. Then S is solid by a result of Paul Roberts [Ro6]: this shows that the ideal $(x_1^3,\, x_2^3,\, x_3^3)$ is not solidly closed in $K[[x_1,\, x_2,\, x_3]]$. As an indication that parameter tight closure is likely to behave better than solid closure in equal characteristic 0, we want to prove that x_1, x_2, x_3 is not a parameter-like sequence in S (which is an example of what is called a *forcing algebra* in [Ho9]). The following result handles a much larger class of forcing algebras, showing that none of them is parameter-preserving. We restrict attention to dimension ≥ 3, since we already know that every ideal is parameter tightly closed in complete regular domains of dimension at most 2.

(3.7) Theorem. *Let (V, x_1V) be a complete discrete valuation ring with residue class field K (which may or may not be of equal characteristic), and let $R = V[[x_2,\, \ldots\,,\, x_d]]$, $d \geq 3$, so that R is a complete regular local domain of dimension d with regular system of parameters $x_1,\, \ldots\,, x_d$. Let $S = R[y_1,\, \ldots\,, y_d]/(F)$ where $y_1,\, \ldots\,, y_d$ are indeterminates over R and*

$$F = (x_1 \cdots x_d)^{t-1} - \sum_{j=1}^{d} y_i x_i^t$$

for some fixed integer $t \geq 1$. Then S is not parameter-preserving over R. Specifically, $x_1,\, \ldots\,, x_d$ is not parameter-like in S: in fact, $T_{d-2} = S/(x_1,\, \ldots\,, x_{d-2})S$ is such that $H^1_m(T_{d-2})$ is not killed by an ideal of height two or more in $R_{d-2} = K[[x_{d-1}, x_d]]$.

Proof. Killing an initial segment of $x_1, \ldots, x_{d-2}$ in S produces a domain, from which it follows that $T_i = S/(x_1, \ldots, x_i)S$ for $0 \le i \le d-2$, and

$$T_{d-2} \cong K[[x_{d-1}, x_d]][y_1, \ldots, y_d]/(y_{d-1}x_{d-1}^t + y_d x_d^t),$$

which is a polynomial ring in $y_1, \ldots, y_{d-2}$ over $B = K[[x_{d-1}, x_d]][y_{d-1}, y_d]/(G)$, where $G = y_{d-1}x_{d-1}^t + y_d x_d^t$. The definition of parameter-like for $x_1, \ldots, x_d$ requires that $H^1_m(T_{d-2})$ be 0 or else be killed by a height two ideal of $R_{d-2} = K[[x_{d-1}, x_d]]$. Since $H^1_m(T_{d-2})$ may be identified as the polynomials in $y_1, \ldots, y_{d-2}$ over $H^1_m(B)$, it suffices to see that this fails for $H^1_m(B)$. Let $u = x_{d-1}^t$ and $v = -x_d^t$, so that $B = K[[x_{d-1}, x_d]][y_{d-1}, y_d]/(y_{d-1}u - y_d v)$. Since $K[[x_{d-1}, x_d]][y_{d-1}, y_d]$ is a finitely-generated free module over $K[[u, v,]][y_{d-1}, y_d]$, we have that B is module-finite and free over $C = K[[u, v,]][y_{d-1}, y_d]/(y_{d-1}u - y_d v)$. Then

$$H^1_m(B) \cong H^1_{(x_{d-1}, x_d)}(B) \cong H^1_{(u,v)}(B) \cong B \otimes_C H^1_{(u,v)}(C),$$

and so it will certainly suffice to show that $H^1_{(u,v)}(C)$ is a faithful C-module: if it were annihilated by an ideal of R_{d-2} primary to the maximal ideal, it would be annihilated by an ideal of C primary to the maximal ideal.

Let $A = K[[u, v]]$, and let z be an indeterminate over A. Then the A-algebra surjection $A[[u, v]][y_{d-1}, y_d] \to A[uz, vz]$ sending y_{d-1} to vz and y_d to uz is easily seen to have $(y_{d-1}u - y_d v)$ as its kernel, so that

$$C \cong A[uz, vz] = A \oplus (u, v)Az \oplus (u, v)^2 Az^2 \oplus \cdots,$$

the Rees ring, where the direct sum is over A. Let $Q = (u, v)A$, the maximal ideal of A. Thus,

$$H^1_{(u,v)}(C) \cong \bigoplus_{j=0}^{\infty} H^1_Q(Q^j).$$

From the short exact sequence $0 \to Q^j \to A \to A/Q^j \to 0$ and the corresponding long exact sequence for $H^\bullet_Q(\_)$, we have an exact sequence

$$\cdots \to H^0_Q(A) \to H^0_Q(A/Q^j) \to H^1_Q(Q^j) \to H^1_Q(A) \to \cdots$$

Since A has depth 2 on Q, $H^0_Q(A) = H^1_Q(A) = 0$, and so $H^1_Q(Q^j) \cong H^0_Q(A/Q^j) = A/Q^j$. Thus, $H^1_{(u,v)}(C) \cong \bigoplus_{j=0}^\infty A/Q^j$, so that the annihilator is $\subseteq \bigcap_j Q^j = (0)$, as required. □

4. A GALOIS CONJECTURE

In [Rang] ideas involving the interaction of group cohomology for Galois groups and local cohomology, as well techniques from number theory, are used to prove certain cases of the direct summand conjecture. The question that we mention here is related to the ideas of [Rang] but a bit different, and can be presented in a reasonably elementary way. An affirmative answer would be sufficient to prove the direct summand conjecture. The conjecture is true both in equal characteristic $p > 0$ and in equal characteristic 0, although the reasons why it is true in those two cases are completely different.

Let V be a complete discrete valuation ring, which may be either equal characteristic or mixed characteristic. In the mixed characteristic case assume that the residual characteristic p is the generator of the maximal ideal. In either case, denote the generator of the maximal ideal by $x = x_1$. Let $A = V[[x_2, \ldots, x_d]]$ be a formal power series ring over V. If D is any domain we denote by D^+ an *absolute integral closure* of D, i.e., the integral closure of D an algebraic closure of its fraction field (cf. [Ar]). D^+ is unique up to non-unique isomorphism. Let $\mathcal{F}$ denote the fraction field of A, and then the fraction field of A^+ is an algebraic closure of $\mathcal{F}$, which we denote $\overline{\mathcal{F}}$, although the notation $\mathcal{F}^+$ would also be appropriate. We shall write G for the group of $\mathcal{F}$-automorphisms of $\overline{\mathcal{F}}$, which also acts on A^+. Note that ${A^+}^G = A$ when $\mathcal{F}$ has characteristic zero, which includes the case where A has equal characteristic zero and the case where A has mixed characteristic.

We shall write E for $H^d_m(A)$, the highest (in fact, the only) nonzero local cohomology module of A with support in $m = m_A$, since it is also an injective hull $E_A(K)$ for the residue field $K = A/m$ of A over A. We write $M^\vee$ for $\mathrm{Hom}_A(M, E)$. If (C, n, L) is any complete local ring, we shall call a C-module W *small* if $E_C(L)$, the injective hull of $L = C/n$ over C, cannot be injected into W. Note that if $E_C(L)$ is a submodule of W, then it is actually a direct summand of W, since $E_C(L)$ is an injective module. The condition that a module be small is not a strong restriction.

The result of [Ho7, Thm. (6.1)] implies that in order to prove the direct summand conjecture, it suffices to show that the modules $H^d_m(A^+)$ are not zero. Now $x = x_1$ is a regular parameter in A, and we have a short exact sequence $0 \to A^+ \xrightarrow{x} A^+ \to A^+/xA^+ \to 0$. If we contradict the direct summand conjecture and assume that $H^d_m(A^+) = 0$, part of the corresponding long exact sequence for local cohomology gives:

$$\cdots \to H^{d-1}_m(A^+) \xrightarrow{x} H^{d-1}_m(A^+) \to H^{d-1}_m(A^+/xA^+) \to 0.$$

This implies an isomorphism $H^{d-1}_m(A^+/xA^+) \cong H^{d-1}_m(A^+)/xH^{d-1}_m(A^+)$. The regular ring A/xA injects into A^+/xA^+ (because A is normal, the principal ideal xA is contracted from A^+). If A provides a counterexample to the direct summand conjecture of smallest dimension (or if A has mixed characteristic, provides a counterexample, and $x = p$), then A/xA is a direct summand of A^+/xA^+ as an (A/xA)-module, and it follows that $H^{d-1}_m(A/xA)$ injects into $H^{d-1}_m(A^+/xA^+)$. Evidently, since G acts on A^+, since m is contained in the ring of invariants of this action, and since x is an invariant, G acts on $H^{d-1}_m(A^+)/xH^{d-1}_m(A^+)$, and it is clear that $H^{d-1}_m(A/xA)$ injects into $\big(H^{d-1}_m(A^+)/xH^{d-1}_m(A^+)\big)^G \subseteq H^{d-1}_m(A^+)/xH^{d-1}_m(A^+)$.

We therefore will have a contradiction that establishes the direct summand conjecture if we can prove the following:

(4.1) Galois Conjecture. *Let (A, m, K) be a complete regular local ring of dimension d with fraction field $\mathcal{F}$, let G be the automorphism group of the algebraic closure $\overline{\mathcal{F}}$ over $\mathcal{F}$, and let x be a regular parameter in A. Then $\big(H^{d-1}_m(A^+)/xH^{d-1}_m(A^+)\big)^G$ is a small (A/xA)-module.*

(4.2) Theorem. *The Galois Conjecture (4.1) holds if $\dim A \le 2$ or if A contains a field. In fact, in all of these cases $\big(H^{d-1}_m(A^+)/xH^{d-1}_m(A^+)\big)^G = 0$.*

Proof. The explanation when A contains a field is quite different depending on whether the field has characteristic 0 or positive characteristic. In the first case, it turns out that ^{G} is an exact functor here, so that what we have is $\big(H^{d-1}_m({A^+}^G)/xH^{d-1}_m({A^+}^G)\big)$, and since ${A^+}^G = A$, this is $H^{d-1}_m(A)/xH^{d-1}_m(A)$, and $H^{d-1}_m(A) = 0$. In the positive characteristic case we know from the main result of [HH5] that A^+ is a big Cohen-Macaulay algebra, so that $H^{d-1}_m(A^+) = 0$, and the result follows again. The same argument shows that the conjecture is true when A has dimension at most two. $\square$

From the discussion above, we have the following:

(4.3) Theorem. *If the Galois Conjecture is true whenever A is a formal power series ring $V[[x_2, \ldots, x_d]]$ over a complete discrete valuation domain (V, pV, K) of mixed characteristic and residual characteristic $p > 0$, then the direct summand conjecture is true.* □

5. QUESTIONS

Of course, many open questions remain. We mention some of the most important among these.

Question 1. *In a complete regular local ring containing the rationals, is every ideal parameter tightly closed?*

Question 2. *In an arbitrary complete regular ring, is every ideal parameter tightly closed?*

An affirmative answer to this question would yield the direct summand conjecture in the general case.

Question 3. *Over a complete local domain of equal characteristic 0, does parameter tight closure agree with big equational tight closure?*

Of course, an affirmative answer to Question 3 would yield an affirmative answer for Question 1, since it is known that every ideal of an equicharacteristic zero regular ring is tightly closed if one uses big equational tight closure as the operation.

Note that Theorem (3.3) shows that the parameter tight closure contains the big equational tight closure: it is the converse that is problematic.

Question 4. *Do colon-capturing and an analogue of phantom acyclicity hold for parameter tight closure in mixed characteristic local domains?*

Question 5. *Can one characterize parameter-preserving finitely generated algebras over a complete local domain or parameter-preserving complete local extensions of a complete local domain in a simpler way?*

Even when S is restricted in this way, the problem does not seem easy. Evidently, questions about parameter tight closure are abundant.

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