FINITE TOR DIMENSION AND FAILURE OF COHERENCE IN ABSOLUTE INTEGRAL CLOSURES

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This paper is dedicated to the memory of Hideyuki Matsumura.

Abstract. It is shown both in characteristic $p > 0$ and in mixed characteristic $p > 0$ that if R is a perfect ring in the first case or R/pR is perfect in the second case, then, under some additional conditions, the radical of a finitely generated ideal has finite Tor dimension, and bounds are obtained. Let R^+ denote the integral closure of the domain R in an algebraic closure of its fraction field. The results are applied to show that R^+ is not coherent when R is Noetherian of dimension at 3, and, under additional restrictions, when the dimension is 2. Motivation for this question connected with tight closure theory is discussed.

§1 Introduction

Throughout this paper all rings are commutative, associative, with identity, and all modules are unital. Following [Ar2], if R is an integral domain, we refer to an integral closure of R in an algebraic closure of its field of fractions as an *absolute integral closure* for R, or even as *the* absolute integral closure of R, and denote it R^+. Evidently, it is unique up to non-unique isomorphism, since this is true for algebraic closures of fields.

The advent of the recent theory of tight closure, for which we give [HH1] as a basic reference, and its intimate connection with rings of the form R^+, for which we refer the reader to [HH2], [HH4], and [Sm], has created tremendous motivation for studying these rings. Even if one is only interested in the behavior of Noetherian rings, it is now clear that the behavior of the rings R^+ when R is Noetherian provides a wealth of information. Moreover, the properties of these rings turn out to be utterly surprising.

In the next section we review some of the properties of absolute integral closures and their connections with tight closure theory and the existence of big Cohen-Macaulay algebras. We also explain in some detail why the question of whether these rings are coherent or not is very natural from the point of view of tight closure theory.

In the third section we prove some surprising results on finiteness of Tor dimension of certain ideals in these and some related rings. Some of what is proved is foreshadowed in [Ho2].

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In the fourth section we use these results to prove that if R is a complete regular local domain of characteristic p, R^+ is not coherent in general if its Krull dimension is at least two.

§2 Absolute integral closures, tight closure, and big Cohen-Macaulay algebras

A domain is called *absolutely integrally closed* if it has no proper domain extension that is integral over it. It is easy to see that R is absolutely integrally closed if and only if every monic polynomial over R has a factorization over R with monic linear factors. By virtue of this characterization, the property is retained if one kills a prime ideal. Another characterization is that R is absolutely integrally closed if and only if its fraction field is algebraically closed and R is normal, i.e., integrally closed in its fraction field. The property of being absolutely integrally closed is prserved by localization at an arbitary mutliplicative system.

Every domain R has an integral extension domain R^+ that is absolutely integrally closed: simply choose an algebraic closure L of the fraction field K of R, and let R^+ be the integral closure of R in L. It follows that R^+ is unique, as an R-algebra, up to (non-unique) R-isomorphism, since the algebraic closure of a field is unique up to (non-unique) isomorphism. In fact, it is easy to see that if P is prime in R and Q is a prime of R^+ lying over P, then $(R/P)^+ \cong R^+/Q$, and that if W is a multiplicative system in R then $(W^{-1}R)^+ \cong (W^{-1})(R^+)$.

R^+ has some surprising properties: the sum of two (or any family of prime ideals) is either all of R^+ or prime, and the sum of two (or any family) of primary ideals is primary to the sum of the corresponding primes. Cf. [Ar2], [HH2].

If R is a complete or Henselian local domain, then R^+ is quasilocal, i.e., has a unique maximal ideal.

We note the following, which is the main result of [HH2].

Theorem 2.1. *Let R be an excellent domain of characteristic p such that R is local (or semilocal, and with all maximal ideals of the same height). Then every system of parameters for R is a regular sequence in R^+.*

Here, when R is semilocal with all maximal ideals of height n, by a *system of parameters* we mean a sequence $x_1, \ldots, x_n$ of length n in the Jacobson radical of R such that $R/(x_1, \ldots, x_n)R$ is zero dimensional.

It is worth noting that the result corresponding to Theorem 2.1 is false in equal characteristic zero in dimension 3 or higher.

Thus, although R^+ is not a Noetherian ring, it has quite astonishing properties that make it, in some ways, better behaved than a Noetherian ring. Our interest in properties of R^+ related to coherence is motivated, in part, by the following important open question.

Question 2.2. Suppose that R is a locally excellent Noetherian domain and let I be an ideal of R. Is the tight closure I^* of I the same as $IR^+ \cap R$?

It is known that $IR^+ \cap R \subseteq R^*$ quite generally (confer [HH3]) and it is shown that Question 2.2 has an affirmative answer for ideals generated by parameters in [Sm], and, using this, for ideals of finite phantom projective dimension in [Ab2] (see also [Ab1] and [AHH] for the basic theory of modules of finite phantom projective dimension). There is a question analogous to (2.2) for modules, and there is a result in [Ab2] for modules, but, for simplicity, we are restricting attention to the ideal case here.

Proposition 2.3. *Let R be a Noetherian domain of characteristic p and let I be an ideal of R. Let u be an element of R. If $(I + (u))R^+$ is finitely related and u is in the tight closure of I, then u is in $IR^+ \cap R$. In particular, if R is locally excellent and $R/(I + uR)$ has finite phantom projective dimension and $u \in I^*$ then $u \in IR^+ \cap I$.*

Proof. Since R is Noetherian, I is finitely generated, and the fact that $(I + (u))R^+$ is finitely related implies that the ideal $IR^+ :_{R^+} u$ is a finitely generated ideal, call it J, of R^+. Since u is in the tight closure I, we can choose a nonzero element c of R such that $cu^q \in I^{[q]}$ for all $q = p^e \gg 0$, which shows that $c^{1/q}u \in IR^{1/q} \subseteq IR^+$, so that $c^{1/q} \in IR^+ :_{R^+} u$ for all large q and, hence, for all $q = p^e$. It follows that the finitely generated ideal J contains all the elements $c^{1/q}$. But this is impossible if J is not the unit ideal. To see this, choose a module-finite extension domain S of R containing generators of J, and let m be a maximal ideal of S containing J. Then there is a Noetherian valuation domain V containing S such that the maximal ideal tV of V lies over m. It follows that all the elements $c^{1/q}$ are in tV^+. But the valuation (with values in the integers) associated with V extends to a valuation v of V^+ to the rational numbers, and for sufficiently large q, $v(c^{1/q}) = \frac{1}{q}v(c)$ will be smaller than $v(t)$, a contradiction.

The final statement is now immediate: the finite phantom resolution for the quotient $R/(I+(u))$ becomes acyclic when one tensors with R^+, from which it follows that $R/(I+(u))R^+$ has a finite free resolution over R^+, and this implies at once that $(I + (u))R^+$ is finitely related. $\square$

Thus, if R^+ were coherent, one would have at once that $I^* = IR^+ \cap R$ for every ideal I of R, which in turn would imply that tight closure localizes well. We shall soon see that coherence itself fails for R^+. It is still possible that some weaker property than coherence, but still strong enough to make some form of the argument given in the proof of (2.3) go through, does hold. Ironically, our proof of the failure of coherence in R^+ makes use of the fact that very strong vanishing theorems for Tor hold for R^+: these are discussed in the next section.

§3 Tor dimension of radical ideals in perfect reduced rings

We recall that an R-module M has finite *Tor dimension* at most d if, equivalently:

(1) For all modules N, $\mathrm{Tor}_i^R(M, N) = 0$ for all $i > d$.

(2) M has a left resolution by flat modules of length at most d.
(3) For any projective resolution $G_\bullet$ of M, the image of $G_d \to G_{d-1}$ is flat (i.e., any d th module of syzygies is flat).

By convention, $\operatorname{Tor\,dim} 0 = -1$. Note that $\operatorname{Tor\,dim} M = 0$ iff M is a nonzero flat module and $\operatorname{Tor\,dim} M = 1$ if and only if M is not flat but there are flat modules $G_1 \subseteq G_0$ such that $M \cong G_0/G_1$.

For the rest of this section fix a positive prime integer p. We shall use the letter "q" to represent p^e, where e is a nonnegative integer. In this section all the rings R considered are either perfect of prime characteristic p (meaning that the Frobenius endomorphism is an automorphism) or else have the property that $R/\operatorname{Rad}(pR)$ is perfect. More precisely, we shall always assume that R has one of the following two properties:

(1) R is a ring of characteristic p such that the Frobenius endomorphism $F : R \to R$ is an automorphism (note that this implies that R is reduced), or
(2) R is a reduced ring, p is a nonzerodivisor in R, $\operatorname{Rad} pR$ is a direct limit of principal ideals generated by roots of p, and $R/\operatorname{Rad} pR$ is perfect.

Note that rings satisfying these conditions are almost never Noetherian. Condition (2) implies that $\operatorname{Rad} pR$ is flat as an R-module, since every root of p will also be a nonzerodivisor, and a direct limit of flat modules (in this case, free modules) is flat.

Note as well that any reduced ring R of characteristic p has a purely inseparable (in the sense that every element has some p^e th power in R) extension ring R^∞ that is perfect (and this extension is unique, up to unique R-isomorphism): R^∞ may be constructed as the direct limit of the system

$$R \xrightarrow{F} R \xrightarrow{F} R \xrightarrow{F} \cdots \xrightarrow{F} R \xrightarrow{F} R \xrightarrow{F} \cdots ,$$

where F is the Frobenius endomorphism. This limit system of rings isomorphic with R and injective maps may be thought of instead as

$$R \subseteq R^{1/p} \subseteq R^{1/p^2} \subseteq \cdots \subseteq R^{1/p^e} \subseteq R^{1/p^{e+1}} \cdots .$$

Also note that if R is any domain containing $\mathbb{Z}$, then R^+ satisfies (2), as does any normal integral extension S of R closed under taking p th roots. The point is that $\operatorname{Rad} pR$ will be the increasing union $\bigcup_e p_e S$, where the elements p_e are constructed recursively: $p_0 = p$ and, for all e, p_{e+1} is a p th root of p_e. (If $u \in \operatorname{Rad} pR$ then $u^n \in pR$ for all $n \gg 0$, and so we may choose $n = q = p^e$ such that $u^q = ps = (p_e)^q s$ with $s \in S$. Then $(u/p_e)^q \in S$ and so u/p_e is a fraction integral over S and, hence, in S, so that $u \in p_e S$. Thus, $\operatorname{Rad} pS = \bigcup_e p_e S$.

Theorem 3.1. *Suppose that $I \subseteq R$ is an ideal with $I = \operatorname{Rad}(f_1, \ldots, f_d)R$ and R is as in (1) or (2) above, and also suppose that $x_1 = p$ in case (2). Then* $\operatorname{Tor\,dim} R/I \le d$. *Moreover, in case (2), if p is not a zerodivisor on the R-module M then* $\operatorname{Tor}_i^R(R/I, M) = 0$ *if $i > d - 1$.*

We postpone the proof until after we have discussed some preliminary material.

Remark 3.2. Let $\{R_\lambda\}_{\lambda\in\Lambda}$ be a direct limit system of rings with direct limit R. For each $\lambda \in \Lambda$ let M_λ and N_λ by R_λ-modules, and suppose that the $\{M_\lambda\}_{\lambda\in\Lambda}$ is a direct limit system, where each map $M_\lambda \to M_\mu$ with $\lambda \le \mu$ is R_λ-linear. Assume that $\{N_\lambda\}_{\lambda\in\Lambda}$ is a direct limit system in the same sense. Then the respective direct limits M and N are R-modules, and for all i we may identify $\mathrm{Tor}_i^R(M,\, N)$ with the direct limit of the modules $\mathrm{Tor}_i^{R_\lambda}(M_\lambda,\, N_\lambda)$. See [CE, Chapter VI, exercise 17].

We recall some notation and facts from [Ho2]. When R is perfect of characteristic p, an ideal J is radical if and only if $J = F^{-1}(J)$, where F is the Frobenius endomorphism. It follows that the sum of two (or any number of) radical ideals is radical, and that a finite product of radical ideals is radical (and is the same as their intersection). The radical of the principal ideal xR is the same as the ideal $\bigcup_e x^{1/p^e}R$, and we sometimes denote this ideal by $(x^\infty)R$. Let $J_1, \ldots, J_n$ be ideals in a ring S. For each i let $K_\bullet(J_i; S)$ be the complex $0 \to J_i \to S \to 0$. If $\mathbf{J} = J_1, \ldots, J_n$ then

$$K_\bullet(\mathbf{J}; S) = \bigotimes_{i=1}^{n} K_\bullet(J_i; S).$$

We have the following proposition from [Ho2]:

Proposition 3.3. *If T is a perfect algebra of char $p > 0$ and $\mathbf{J} = J_1, \ldots, J_n$ is a sequence of ideals such that $J_i = F^{-1}(J_i)$, at most one of which is not flat, then $K_\bullet(\mathbf{J}; T)$ is acyclic. In particular, if $x_1, \ldots, x_n$ are nonzerodivisors in T then $K_\bullet((x_1^\infty), \ldots, (x_n^\infty); T)$ is a flat resolution of $T/((x_1^\infty), \ldots, (x_n^\infty))$.*

Proof of 3.1. We first consider the situation of (1), where R is perfect. Since every R-module is a direct limit of finitely presented R-modules, it suffices to show for every finite matrix μ of elements of R that, with $M = \mathrm{Coker}\,\mu$, $\mathrm{Tor}_i(R/I, M) = 0$ for $i > d$. Let $K = \mathbb{Z}/p\mathbb{Z}$ and let S vary through rings of the form $S_0^\infty \subseteq R$, where S_0 is a finitely generated subalgebra of R containing the $f's$ and the entries of μ. For every such S, let μ_S denote the map of free S-modules represented by μ. Let $M_S = \mathrm{Coker}\,\mu_S$. Let $I_S = \mathrm{Rad}(x_1, \ldots, x_d)S$. Then by (3.2) we may view $\mathrm{Tor}^i_R(R/I,\, M)$ as the direct limit of the modules $\mathrm{Tor}^i_S(S/I_S,\, M_S)$, and so it suffices to consider the case where R is replaced by S. Since Tor dim may be calculated locally, we may assume that R has the form T^∞ where (T, m) is a Noetherian local ring containing the f's. In particular, we have that T, and, hence, R, has only finitely many minimal primes. Suppose that $I = \mathrm{Rad}(f_1, \ldots, f_d)$.

Assume first that $\mathrm{ht}\, I > 0$. We want to show that we may choose d generators for $(f_1, \ldots, f_d)T$ avoiding the minimal primes of T (since the set of zerodivisors of a reduced ring is the union of the minimal primes). Since $m(f_1, \ldots, f_d) \not\subseteq \cup\{P : P \text{ minimal in } T\}$ we have by [Kap], Theorem 124 that for each i, there exists $g_i \in m(f_1, \ldots, f_d)T$ such that $f_i + g_i$ is not in any minimal prime of T. Replacing f_i by $f_i + g_i$ we may assume that each f_i is a nonzerodivisor in T and, hence, in R. Then $K_\bullet((f_1^\infty), \ldots, (f_d^\infty); R)$ is a flat resolution of $R/\left((f_1^\infty) + \cdots + (f_d^\infty)\right) = R/I$, by Proposition 3.3.

Suppose now that $\mathrm{ht}(f_1, \ldots, f_d)T = 0$. If $d = 0$ there is nothing to prove, so assume that $d \ge 1$. Then $(f_1, \ldots, f_d)T$ is not (0), and so cannot be contained in all the minimal primes of T. Let $g \notin I$ be an element in the intersection of all the minimal primes of T not containing $(f_1, \dots, f_d)T$. Let $J = (g^\infty)$ in R.

Then $I + J$ is a radical ideal of positive height and $I \cap J = 0$. Thus we have a short exact sequence

$$0 \to R/(I \cap J) = R \to R/I \oplus R/J \to R/(I + J) \to 0.$$

It follows that the Tor dimension of $R/I \oplus R/J$ is at most the supremum of the Tor dimensions of R and $R/(I + J)$, and so Tor dim $R/I \leq$ Tor dim $R/(I + J)$. Thus, it will suffice to show that Tor dim $R/(I + J) \leq d$. Since $I + J$ has positive height, it will be enough to show that it is the radical of a d generator ideal. But, writing $f = f_1$, we have that $I + J = \mathrm{Rad}(g + f, f_2, \cdots, f_d)R$, because, since $fg = 0$, we have that $f^2 = f(g + f)$ and $g^2 = g(g + f)$.

We now consider the case where R satisfies (2). If $d = 0$ there is nothing to prove, so we assume that $d \geq 1$ and $f_1 = p$. Let $N = \mathrm{syz}\, M$ be the kernel of a map $Q \twoheadrightarrow M$, where Q is free. Then $\mathrm{Tor}_i^R(R/I, M) \cong \mathrm{Tor}_{i-1}^R(R/I, N)$ for $i > d$ since $d \geq 1$, and so it suffices to prove the last statement in the theorem, concerning the case where p is not a zerodivisor on M. In this case, consider a free resolution

$$\cdots \to G_n \to \cdots \to G_0 \to M \to 0$$

for M. Since p, and, hence, any root of p is a nonzerodivisor on all of these modules, if u denotes any root of p we have that the subcomplex

$$\cdots \to uG_n \to \cdots \to uG_0 \to uM \to 0$$

is exact. Let $J = \mathrm{Rad}\, pR$. Taking a directed union, we obtain that the subcomplex

$$\cdots \to JG_n \to \cdots \to JG_0 \to JM \to 0$$

is exact, and hence that

$$\cdots \to G_n/JG_n \to \cdots \to G_0/JG_0 \to M/JM \to 0$$

is exact. This yields a free resolution of M/JM over the ring R/JR. If we drop the term M/JM and use the resulting free complex to compute $\mathrm{Tor}_i^{R/J}(R/I, M/JM)$, we see that this module is isomorphic to $\mathrm{Tor}_i^R(R/I, M)$. Now, the first module R/I in the first Tor may be thought of as $(R/J)/(I/J)$. Since R/J is a perfect ring, and since I/J is the radical of the ideal generated by the $d - 1$ elements that are the images of $f_2, \dots, f_d$ in this ring, the characteristic p form of (3.1) shows that $\mathrm{Tor}_i^{R/J}(R/I, M/JM) = 0$ for $i > d - 1$, as required. $\square$

Corollary 3.4. *Let R be a perfect ring. Then the ideal (x^∞) is flat for every $x \in R$.*

Proof. By Theorem 3.1, Tor $\dim R/(x^\infty) \le 1$, so (x^∞) is flat by condition (3) of the definition of Tor dimension. □

In fact, one may prove Corollary 3.4 directly by using the equational condition for flatness of a module M (relations on M come from relation already in R) and reducing immediately to the case that $R = S^\infty$ where S is Noetherian. Theorem 3.1 (in case (1)) then follows as a direct application of Theorem 3.3.

Theorem 3.5. *Let R be a Noetherian domain, let Q be a maximal ideal in $S = R^+$, and let $m = Q \cap R$. Let $d = \dim R_m$. Then if S/Q has characteristic $p > 0$,* Tor $\dim_S S/Q \le d$*. If S has characteristic p, then* Tor $\dim_S S/Q = d$*, while if S has characteristic 0 while S/Q has characteristic p (i.e., the mixed characteristic case) then* Tor $\dim_S S/Q \ge d-1$*. Furthermore* Tor $\dim_S S/Q = d$ *in all mixed characteristic cases if and only if the direct summand conjecture holds in mixed characteristic.*

Proof. Choose a system of parameters $x_1, \dots, x_d$ in the maximal ideal of R_m. In mixed characteristic p, we may additionally suppose that $x_1 = p$. The calculation of Tor $\dim_S S/Q$ is local on the maximal ideals of S, and localizing at any maximal ideal other than Q makes S/Q vanish, so that Tor $\dim_S S/Q =$ Tor $\dim_{S_Q} S/Q$ (note that $S/Q \cong S_Q/QS_Q$). But QS_Q will be the radical of $(x_1, \dots, x_d)S_Q$, and so either condition (1) or condition (2) needed to apply Theorem 3.1 will hold, so that Theorem 3.1 yields that Tor $\dim_{S_Q} S/Q \le d$. It remains to see that this Tor dimension is at least d in the equal characteristic p case and is at least $d-1$ in mixed characteristic, and is equal to d in mixed characteristic if and only if the direct summand conjecture holds.

Now suppose that there is an S_Q-module M and a regular sequence $y_1, \dots, y_k$ in QS_Q on M of length k. Suppose also that $QS_QM \ne M$. Then if the highest nonvanishing $\mathrm{Tor}_i^{S_Q}(S/Q, M/(y_1, \dots, y_k)M)$ occurs when $i = h$ (not all can vanish, for our hypothesis implies that the Tor does not vanish when i = 0), then the highest nonvanishing $\mathrm{Tor}_i^{S_Q}(S/Q, M)$ occurs when i = h+k. (By induction on k, this reduces at once to the case where $k = 1$. Let $y = y_1$. The exact sequence $0 \to M \xrightarrow{y} M \to M/yM \to 0$ yields a long exact sequence for Tor in which every third map is given by multiplication by y and so is 0, i.e., one has short exact sequences $0 \to \mathrm{Tor}_j^{S_Q}(S/Q, M) \to \mathrm{Tor}_j^{S_Q}(S/Q, M/yM) \to \mathrm{Tor}_{j-1}^{S_Q}(S/Q, M) \to 0$ for all j, and the stated result now follows easily).

Next note that in mixed characteristic there is an S_Q-module M such that $x_2, \dots, x_d$ is a regular sequence on M, while if S has characteristic p we can choose M such that $x_1, \dots, x_d$ is a regular sequence on M. Morover, in both cases, $QM \ne M$. In the characteristic p case one may use the fact that S_Q is a direct limit of local rings of characteristic p in which $x_1, \dots, x_d$ is a system of parameters. The construction of big Cohen-Macaulay modules by modification will work for the direct limit, for if some finite sequence of modifications of the direct limit were "bad" in the sense of [Ho1], this would also be true for one of the algebras in the direct limit system. In the mixed characteristic case one applies the same construction to the ring obtained from S_Q by killing a minimal prime of p.

This remark, coupled with the observation in the preceding paragraph, shows that the Tor dimension of S/Q must be at least $d-1$ in mixed characteristic and must be at least d in characteristic p.

We now do a finer analysis which makes the connection with the direct summand conjecture in mixed characteristic. The technique we use also gives a second proof that the Tor dimension is at least d in characteristic p. Thus, we shall continue to consider both cases.

We first note that if there is a module M such that $\mathrm{Tor}_d^{S_Q}(S/Q, M) \neq 0$ then, by a direct limit argument there is also such a module that is finitely generated. Since M will then have a finite filtration in which the factors are cyclic modules, there must be such a module which is cyclic. Moreover, since S_Q/J is the direct limit of the modules S_Q/J_0 where J_0 runs through the finitely generated subideals of J, it follows that if there exists M such that $\mathrm{Tor}_d^{S_Q}(S/Q, M) \neq 0$ then there exists such an M of the form S_Q/J where J is finitely generated.

To proceed further, we want to construct an explicit flat resolution of S/Q that can be used to calculate the d th Tor. This has already been done in [Ho2], although in slightly different generality. Again, assume that $x_1, \dots, x_d$ is a system of parameters and that $x_1 = p$ in the mixed characteristic case. If z is any nonzero element of S_Q, let z_n denote a p^n th root of z for all $n \geq 1$. These may be chosen so that $z_n^p = z_{n-1}$ for all $n \geq 1$, where $z_0 = z$. As earlier, let (z^∞) denote the union of the principal ideals generated by the z_n in the domain S_Q. This is a flat ideal of S_Q, since it is a direct union of principal ideals, each of which is a free module over S_Q. This union is independent of how the roots are chosen, since the ring contains all roots of unity. Note that if y, z are two nonzero elements of the ring then $(y^\infty) \otimes_{S_Q} (z^\infty)$ may be naturally identified with $(y^\infty)(z^\infty)$ and this ideal is the same as $((yz)^\infty)$. Then the total complex of the tensor product of the complexes $0 \to (x_i^\infty) \to S_Q \to 0$ gives a flat complex and this complex is acyclic.[1] It now follows that the Tor dimension of S/Q is d if and only if there exists a finitely generated ideal I of S_Q such that the last non-trivial map in the resolution of S/Q has a kernel after we apply $\otimes_{S_Q} S_Q/I$. To check this, it is convenient to think of the

[1]One may use the arguments of [Ho2], but we give a brief sketch of a simpler argument. We prove the result by induction on the number of x_i (but require that $x_1 = p$ in the mixed characteristic case). By the induction hypothesis, tensoring the first $k-1$ of the short complexes together gives a resolution of $J = \sum_{t=1}^{k-1}(x_i^\infty)$, which is a radical ideal (this may be checked modulo (x_1^∞) in the mixed characteristic case, and then in either mixed characteristic or characteristic p follows from the fact that in characteristic p, the p th root of a sum is the sum of the p th roots). We need only show that when we tensor the flat resolution of S_Q/J that we have by induction with the short resolution $0 \to (x_k^\infty) \to S_Q \to 0$, the resulting total complex is acyclic. But its homology consists of the modules $\mathrm{Tor}_t^{S_Q}(S_Q/J, S_Q/(x_k^\infty))$. Since the second entry has a flat resolution of length 1, only the Tor for t=1 is a problem, and $\mathrm{Tor}_1^{S_Q}(S_Q/J, S_Q/(x_k^\infty)) = J \cap (x_k^\infty)/J(x_k^\infty)$. Consider any element u in the intersection of the two ideals. Since u has a p th root v in the ring and both ideals are radical, v is in each of the ideals. Then $u = v^{p-1}v$ is in the product of the ideals, since we may think of one factor as being in J and the other in (x_k^∞).

complex in a slightly different way. We may think of the complex $0 \to (x_i^\infty) \to S_Q \to 0$ as the directed union of the complexes $0 \to (x_{in}) \to S_Q \to 0$, and the latter may be identified with the Koszul complex $0 \to S_Q \to S_Q \to 0$, where the map may be identified with multiplication by x_{in}. Then in the direct limit system, the map on the left copy of S_Q in the n th complex is given by multiplication by $x_{i,n+1}^{p-1}$. Tensoring these together, we see that the resolution of S/Q over S_Q is given by the direct limit of the (homological) Koszul complexes $K_\bullet(x_{1n}, \dots, x_{dn}; S_Q)$. After tensoring with S_Q/I we have that the kernel at the d th spot is the annihilator of $I_n = (x_{1n}, \dots, x_{dn})$ in S_Q/I, and so the condition for the Tor dimension of S/Q to be d is that there exist a finitely generated ideal I of S_Q and an element u in the n th annihilator that is not killed by mapping forward. But then we can give an example with the same properties in which S_Q/I is replaced by S_Q/I_n: we can map S_Q/I_n to S_Q/I so that the image of the class of 1 is u. The class of 1 cannot map to 0 as we map forward in the direct limit system, or the same will be true for u. As we map 1 forward its image is represent by the product of the x_{in} to a power of the form s/q, where q is a power of p, say p^e, and $0 \le s < q$ is an integer. Let $z_i = x_{i,n+e}$. The condition wanted is that $(z_1 \cdots z_d)^s \notin (z_1^q, \dots, z_d^q)$ in S_Q with $s < q$. If the monomial conjecture (or direct summand conjecture: they are equivalent) holds then this is the case, since S_Q is a direct limit of Noetherian local domains in which $z_1, \dots, z_d$ is a system of parameters, and the Tor dimension will be d. This gives a second proof in characteristic p, where the monomial conjecture is known to hold. On the other hand, if the Tor dimension is always d then the direct summand conjecture holds: one only needs the case where R is a formal power series ring over a discrete valuation ring (in which case $S = R^+$ is already quasilocal). The fact that the conditions holds for suitable parameters implies that the local cohomology of S with support in the maximal ideal does not vanish, and this implies the direct summand conjecture by the results of [Ho2]. □

As demonstrated in Theorem 3.5, the Tor dimension of R^+ when R is a local ring of mixed characteristic is intimately connected to the homological conjectures, however, we wish to point out that R^+ is rarely a *balanced* big Cohen-Macaulay algebra in this case.

Proposition 3.6. *Let R be a complete domain having mixed characteristic. If* $\dim R \ge 4$ *then R^+ is* not *a balanced big Cohen-Macaulay algebra for R.*

Proof. Since R is a complete domain, R is module finite over the regular ring $A = V[[x_2, ..., x_d]]$ where V is a complete valuation domain with maximal ideal pV. Hence $R^+ = A^+$. If S is a normal module finite extension algebra of A and $Q \in \operatorname{Spec} S$ lies over $(x_2, \dots, x_d)A$ then the ring S_Q is a normal ring containing the rationals, and hence splits out of any module finite extension via the trace map. Thus any bad relation on $x_2, \dots, x_d$ in S_Q remains a bad relation in $(A^+)_Q = (A_Q)^+ = (S_Q)^+$. If $d \ge 4$ then $\dim S_Q \ge 3$ and so an S_Q can always be obtained which is not Cohen-Macaulay.

Proposition 3.6 leaves open the question of whether or not a system of parameters which includes p can be a regular sequence on R^+. Another interesting question to answer is whether or not $R^+/\operatorname{Rad}(pR)$ is a big Cohen-Macaulay algebra of some sort when R is a (complete) local domain of mixed characteristic.

Theorem 3.5 gives no information when the ring involved is equal characteristic 0. We ask the following question:

Question 3.7. If R is a complete local domain containing $\mathbb{Q}$ then is the Tor dimension of Tor $\dim R^+/m_{R^+}$ equal to $\dim R$?

Lemma 3.8. *Let (R, m) be a quasi-local ring of dimension d such that every d-element, m-primary ideal is a regular sequence. If every finitely generated m-primary ideal is contained in a d-generated ideal then* Tor $\dim R/m \leq d$.

Proof. We have that R/m is the direct limit of the rings R/I, where I runs through the finitely generated m-primary ideals. Computing Tor commutes with direct limits and if $I \subseteq (x_1, \dots, x_d)$ we get $\mathrm{Tor}_{d+1}(R/I, \_) \to \mathrm{Tor}_{d+1}(R/(x_1, \dots, x_d), \_) = 0$. Hence we have that $\mathrm{Tor}_{d+1}(R/m, \_) = 0$. □

Corollary 3.9. *Let (R, m) be a complete domain of dimension two containing $\mathbb{Q}$. Then if every finitely generated m-primary ideal of R^+ is contained in a two-generated ideal of R^+ we have* Tor $\dim R^+/m_{R^+} = 2$.

Proof. R^+ is a direct limit of two-dimensional normal and, hence, Cohen-Macaulay rings. Thus, every pair of elements of R^+ generating a height 2 ideal is an R^+ sequence. Now we may apply Lemma 3.8. □

We do not know if the conditions of Corollary 3.9 are satisfied. By an easy induction it suffices to show that any three-generated m_{R^+}-primary ideal (x, y, z) is contained in a two-generated m_{R^+}-primary ideal (u, v). Under these conditions $R^+ = k[[u, v]]$ and we can assume that $(x, y, z) = (u, v, z)$. Then z satsifies a monic polynomial with coefficients in $k[[u, v]]$.

Lemma 3.10. *If z satisfies a polynomial of the form $X^n - f(u, v)$ then (u, v, z) is contained in a two-generated ideal of R^+.*

Proof. We may assume that $z^n = f(u, v)$ where $f(u, v) \in k[[u, v]]$. If f has a factor g which is a regular parameter in $k[[u^{1/h}, v]]$ for some h then $(u, v, z) \subseteq (g^{1/n}, g')R^+$ where $k[[u^{1/h}, v]] = k[[g, g']]$. We now show that we can force this to happen (possibly after making a module finite extension). Without loss of generality, we may assume f is irreducible in $k[[u, v]]$. Then $k[[u, v,]]/(f)$ is a complete one dimensional domain and has normalization $k'[[t]]$ where k' is a finite algebraic extension of k. Then u maps to γt^h for some unit $\gamma \in k'[[t]]$. γ has an h^{th} root (after a finite extension) since $char(k') = 0$, so after a change of variables we may assume that u maps to t^h. Thus the map extends to $k'[[u^{1/h}, v]]$ and is onto. Thus the kernel of the map has the form $(v - G(u^{1/h}))$ where $G(t)$ is the image of v in $k'[[t]]$. Since f maps to 0, f is a multiple of $v - G(u^{1/h})$, which is a regular parameter in $k'[[u^{1/h}, v]]$. □

Remark 3.11. We do not know how to get the desired result when the element z satisfies a general equation of degree 3 or higher. For instance if z satisfies $z^3 + x^3 + y^3 + xyz = 0$ in $k[[x, y]]^+$ we do not know if $(x, y, z)R^+$ is contained in a two-generated ideal.

§4. Failure of coherence in absolute integral closures in characteristic p

In this section we show that the absolute integral closure of any Noetherian domain of characteristic p and dimension $d \geq 3$ is *not* coherent. We will do this by examining 2-generated height one ideals in the divisor class group of extensions of the ring. First we note the implication of coherence for R^+ on projective dimensions.

Proposition 4.1. *Let S be any perfect integral extension of a Noetherian ring. If S is coherent then $\operatorname{pd}_S S/I < \infty$ for any finitely generated ideal $I \subseteq S$. If $\dim S \leq d$ then $\operatorname{pd}_S S/I \leq d$.*

Proof. Suppose that S is integral over R. Since finite projective dimension can be checked locally we need only show that $\operatorname{pd}_{S_n} S_n/IS_n < \infty$ for every maximal ideal $n \in \operatorname{Spec} S$. Let $P = n \cap R$ and let $d = \dim R_P$. Then $\dim S_n = d$ since $R \subseteq S$ is integral. If $x_1, \ldots, x_d$ is a s.o.p. for R_P then $nS = \operatorname{Rad}(x_1, \ldots, x_d)S_n$. Since S is coherent so is S_n ([Gl, Theorem 3.4.2]) and S_n/IS_n has an S_n-resolution by finitely generated S_n modules ([Gl, Corollary 3.5.2]). Since S_n is quasi-local there is a minimal resolution of S_n/IS_n and the i^{th} betti number is given by the vector space dimension over S_n/nS_n of $\operatorname{Tor}_i^{S_n}(S_n/IS_n, S_n/nS_n)$. S_n is perfect, so by Theorem 3.1, $\operatorname{Tor}\dim S_n/nS_n \leq d$, hence $\operatorname{pd} S_n/IS_n \leq d$. $\square$

Corollary 4.2. *Let R be any Noetherian domain of finite Krull dimension. If R^+ is coherent then R^+ has finite finitistic projective dimension $d =$ Krull $\dim R$.*

We need the following result in [Va]:

Lemma 4.3. [Va, Theorem 5.21 and proof] *Let (R, m) be a quasi-local coherent domain. If $\operatorname{pd} R/(a, b) < \infty$ then $\operatorname{pd} R/(a, b) \leq 2$.*

Suppose that R is a normal Noetherian domain. Then the divisor class group $\operatorname{Cl}(R)$ is an abelian group measuring how far R is from being a UFD. Whenever $R \to S$ is an injective module-finite map of normal domains then there are maps $\operatorname{Cl}(R) \to \operatorname{Cl}(S)$ and $\operatorname{Cl}(S) \to \operatorname{Cl}(R)$ such that the composite map is multiplication by $[S : R]$ on $\operatorname{Cl}(R)$ (see [B]). In particular, if some element u of $\operatorname{Cl}(R)$ is mapped to $0 \in \operatorname{Cl}(S)$ then u is a torsion element of $\operatorname{Cl}(R)$.

Proposition 4.4. *Let (R, m) be an excellent normal Noetherian domain of characteristic p and let $(a, b)R$ be a height one ideal. If R^+ is coherent then there is an element $s \in R^+$, not in any minimal prime of $(a, b)R^+$, for which $[(a, b)R_s] \in \operatorname{Cl}(R_s)$ is a torsion element. If R is Henselian then we may take $s = 1$.*

Proof. By Proposition 4.1 and Lemma 4.3, $\mathrm{pd}_{R^+} R^+/(a,b) \leq 2$. Either $(a,b)R^+$ is principal, or projective of rank 1, or has a resolution

$$0 \to Q \to (R^+)^2 \xrightarrow{[a \quad b]} R^+ \to R^+/(a,b) \to 0$$

where Q is projective of rank 1. If $(a,b)R^+ = \alpha R^+$ then $[(a,b)\overline{R[\alpha]}] = 0$. If $(a,b)R^+$ is projective then it becomes free after localizing at some element $s \in R^+$, i.e., $(a,b)R^+_s$ is principal so we are done. Assume now that the third condition holds. Choose $s \in R^+$ not in any minimal prime of $(a,b)R^+$ such that $Q_s \cong R^+_s$. In this case we get a resolution

$$0 \to R^+_s \xrightarrow{\begin{bmatrix} u \\ v \end{bmatrix}} (R^+_s)^2 \xrightarrow{[a \quad b]} R^+_s \to R^+_s/(a,b) \to 0$$

where $\mathrm{ht}(u,v)R^+_s = 2$ and hence u,v is an R^+_s-regular sequence (since R^+ is a directed union of normal rings). Thus a and b have a GCD in R^+_s. Letting T be the normalization of $R_s[\alpha,u,v]$ (where $a = \alpha u$, $b = -\alpha v$) we get $[(a,b)T_s] = [\alpha T_s] = 0$ in $\mathrm{Cl}(T_s)$. Thus by the above remark, $[(a,b)R_s]$ is a torsion element.

If R is Henselian then R^+ is quasi-local, in which case all projectives are free, so there is no need to localize. □

Theorem 4.5. *Let (R,m) be a complete local domain of characteristic p having dimension $d \geq 3$. Then R^+ is not coherent.*

Proof. Let S be a regular domain contained in R over which R is module finite. Then $S^+ = R^+$ so we may assume that R is regular. Let u,v,w be regular parameters for R. Let $T = R[z] = R[Z]/(Z^2 - Zw - uv)$. Then T is a local normal domain and $(z,u)T$ is a height one prime ideal of T which has infinite order in $\mathrm{Cl}(T)$. Complete rings are Henselian, therefore $R^+ = T^+$ is not coherent by Proposition 4.4. □

We can now show that R^+ is not coherent for any domain of dimension three or higher.

Theorem 4.6. *Let R be any Noetherian domain of characteristic p. If $\dim R \geq 3$ then R^+ is not coherent.*

Proof. Coherence is stable under localization under any multiplicative set, so if m is any maximal ideal of height at least 3, then $(R-m)^{-1}R^+ = (R_m)^+$ is coherent. Thus we can assume that R is local. Let Q be any minimal prime of $\hat{R}$ such that $\dim \hat{R} = \dim \hat{R}/Q$. Letting $S = \hat{R}/Q$ we have an injection $R \to S$. In this case we can consider R^+ to be a subring of S^+. Let $n = R^+ \cap m_{S^+}$. If R^+ is coherent then so is R^+_n. By Theorem 4.5, S^+ is not coherent, and by the proof of Theorem 4.5, we know that the ideal $(z,u)S^+$ is not of finite projective dimension over S^+, where z satisfies $z^2 = zv - wu$ and u,v,w is part of a system of parameters in S. But we can pick u,v,w to live in R, and then $z \in R^+$. By Proposition 4.1 and Lemma 4.3, if R^+ is coherent then $(z,u)R^+_n$ has a finite free resolution of length at most 2, which will become an S^+ resolution upon tensoring with S^+ (this follows from the generalized version of the Buchsbaum–Eisenbud acyclicity criterion since heights of ideals cannot go down in the map $R_n^+ \to S^+$). Therefore R^+ is not coherent. $\square$

We turn now to the case where $\dim R = 2$. We first record the following facts, for which we are indebted to Dale Cutkosky.

Theorem 4.7. *Let k be the algebraic closure of a finite field and suppose that (R, m) is a complete normal domain of dimension 2 with $R/m = k$. Then $\mathrm{Cl}(R)$ is a torsion group.*

Proposition 4.8. *Let k be a field of positive transcendence degree over $\mathbb{Z}/(p)$. Then there exists a finite extension K of k, and $F \in K[[x, y, z]]$ such that $R = K[[x, y, z]]/F$ is a normal domain of dimension 2 and $\mathrm{Cl}(R)$ contains elements of infinite order.*

Theorem 4.9. *Let (R, m) be a complete local domain containing a field of positive transcendence degree over $\mathbb{Z}/(p)$. Then R^+ is not coherent.*

Proof. Let k be a coefficient field for R. Then by the Cohen structure theorem, $R^+ = A^+$, where $A = k[[u, v]]$ is regular. Since k has positive transcendence degree over $\mathbb{Z}/(p)$, R may be assumed to be of the form given by Proposition 4.8 (knowing that K is a *finite* algebraic extension of k gives that R is module finite over A). Thus we may assume that there are elements of infinite order in $\mathrm{Cl}(R)$. Let $I \subseteq R$ be an ideal such that $[IR]$ has infinite order in $\mathrm{Cl}(R)$. If R^+ is coherent then $\mathrm{pd}_{R+} R^+/IR^+ \leq 2$ by Proposition 4.1. IR^+ cannot be principal (since $[IR]$ has infinite order) so $\mathrm{pd}_{R_+} R^+/IR^+ = 2$. But then there is a module finite normal extension $S \supseteq R$ such that $\mathrm{pd}_S S/IS = 2$. In this case $IS = \alpha JS$ where $\mathrm{ht}(JS) \geq 2$ by the Hilbert–Burch theorem, which in turn implies that $[IR]$ is torsion, contradicting the fact that $[IR]$ has infinite order. Thus R^+ is not coherent. $\square$

Remark 4.10. Let k be the algebraic closure of $\mathbb{Z}/(p)$ and let (R, m) be a complete normal domain with $R/m = k$. Then Theorem 4.7 shows that $\mathrm{Cl}(A)$ is torsion. In this case every 2 elements in R^+ do have a GCD, so every two-generated ideal in R^+ has finite projective dimension. To see this let a, $b \in R^+$. We can assume a, $b \in R$ and R is normal. Then $[(a, b)R]$ is torsion in $\mathrm{Cl}(R)$, so for some n, $(a, b)^n = \gamma J$ where $\mathrm{ht}(J) \geq 2$. But then $(a^n, b^n) = \gamma(c, d)$ where $\mathrm{ht}(c, d) \geq 2$, hence $\gamma^{1/n}$ is $\mathrm{GCD}(a, b)$ in $\mathrm{Cl}(R)$. $\square$

Remark 4.11. The method of proof used in this section to show R^+ is not coherent in many situations does not address the projective dimesion of R^+/IR^+ when $\mathrm{ht}(I) > 1$. This is of independent interest, and may be the only way to show R^+ lacks coherence in situation of Remark 4.9.

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