

QUASILENGTH, LATENT REGULAR SEQUENCES, AND CONTENT OF LOCAL COHOMOLOGY

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This paper is dedicated to Paul Roberts on the occasion of his sixtieth birthday.

0. INTRODUCTION

We introduce the notion of *quasilen*gth. Let M be a finitely generated module over a ring R and let $\underline{x} = x_1, \dots, x_d$ be a sequence of elements of R . Neither R nor M needs to be Noetherian. Let $I = (\underline{x})R$. Suppose that M is killed by some power of I . The *I-quasilen*gth of M is the least number of factors in a finite filtration of M by cyclic modules each of which is a homomorphic image of R/I .

We use the notion of quasilen

gth to define two nonnegative real numbers $\underline{h}_{\underline{x}}^d(M)$ and $\mathfrak{h}_{\underline{x}}^d(M)$ that are intended heuristically as “measures” of the local cohomology module $H_I^d(M)$. Each may be defined as a lim inf of normalized quasilengths: see §2. The second is actually a limit. In general, one has

$$0 \leq \underline{h}_{\underline{x}}^d(M) \leq \mathfrak{h}_{\underline{x}}^d(M) \leq \nu(M)$$

where $\nu(M)$ denotes the least number of generators of M . Hence, when $M = R$ one has

$$0 \leq \underline{h}_{\underline{x}}^d(R) \leq \mathfrak{h}_{\underline{x}}^d(R) \leq 1$$

If $H_I^d(M) = 0$, then $\underline{h}_{\underline{x}}^d(M) = 0$ (see Proposition 2.2). We do not have an example in which we can prove that $\underline{h}_{\underline{x}}^d(R)$ is strictly between 0 and 1. We can show that $\underline{h}_{\underline{x}}^d(R) = 1$

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if and only if $\mathfrak{h}_{\underline{x}}^d(R) = 1$ (Theorem 3.8(b)). In positive prime characteristic p we can prove that $\underline{\mathfrak{h}}_{\underline{x}}^d(R)$ and $\mathfrak{h}_{\underline{x}}^d(R)$ are equal, and that $\mathfrak{h}_{\underline{x}}^d(R)$ must be either 0 or 1 (Theorem 3.9). Whether these numbers depend on the choice of the sequence \underline{x} of generators or only on M and the radical of the ideal (\underline{x}) is an open question, although we can show that the condition that $\mathfrak{h}_{\underline{x}}^d(R) = 1$ depends only on the ideal I and d , and not on the choice of the d generators x_1, \dots, x_d for I (Theorem 3.8(a)). In any case, we are inclined to view the numbers $\underline{\mathfrak{h}}_{\underline{x}}^d(M)$ and $\mathfrak{h}_{\underline{x}}^d(M)$ as giving quantitative information about the “size” of $H_I^d(M)$.

Note that our basic reference for local cohomology theory is [GrHa].

When d and \underline{x} are understood from context we shall also refer to $\underline{\mathfrak{h}}_{\underline{x}}^d(M)$ as the *h-content* of $H_I^d(M)$. The positivity of $\underline{\mathfrak{h}}_{\underline{x}}^d(R)$ gives a necessary condition for there to exist a map of R to a Noetherian ring such that the x_i map to generators of an ideal of height d . In fact, if R contains a field, it is necessary that $\underline{\mathfrak{h}}_{\underline{x}}^d(R) = 1$. See Theorem 4.1.

We are particularly interested in the case where $M = R$, especially the case where R is a local ring of Krull dimension d and \underline{x} is a system of parameters for R . In §4 we study the conjecture that if x_1, \dots, x_d is a system of parameters for a local ring R , then $\mathfrak{h}_{\underline{x}}^d(R) = \underline{\mathfrak{h}}_{\underline{x}}^d(R) = 1$. This conjecture reduces to the case of a complete local domain. We prove the result (Theorem 4.7) when R is equicharacteristic by reduction to characteristic $p > 0$. We prove in the case of an excellent reduced equidimensional local ring that $\mathfrak{h}_{\underline{x}}^d(R) = \underline{\mathfrak{h}}_{\underline{x}}^d(R)$ without restriction on the characteristic. In mixed characteristic we prove that $\underline{\mathfrak{h}}_{\underline{x}}^d(R) = 1$ if $\dim(R) \leq 2$, and that $\underline{\mathfrak{h}}_{\underline{x}}^d(R) > 0$ always. In fact, if μ is the multiplicity of the system of parameters \underline{x} and λ is the length of $R/(\underline{x})$, we show that $\underline{\mathfrak{h}}_{\underline{x}}^d(R) \geq \mu/\lambda$ (Theorem 4.6). The conjecture that $\underline{\mathfrak{h}}_{\underline{x}}^d(R) = 1$ for every system of parameters of every local ring R implies the direct summand conjecture. In mixed characteristic, we do not know that $\underline{\mathfrak{h}}_{\underline{x}}^d(R) = 1$ even in dimension 3, although the direct summand conjecture [Heit] and the existence of big Cohen-Macaulay algebras [Ho6] are known. We cannot deduce the result from the existence of big Cohen-Macaulay algebras in dimension 3 because we have not been able to prove that if \underline{x} is a regular sequence in a ring that is not necessarily Noetherian, then $\underline{\mathfrak{h}}_{\underline{x}}^d(R) = 1$, although we conjecture this.

In §3 we use quasilength to give conditions that may possibly characterize when a sequence x_1, \dots, x_d of elements of a ring R has the property that there exists an R -algebra S such that x_1, \dots, x_d is a regular sequence on S . We call such a sequence of elements a *latent regular sequence* in R . We also consider sequences such that there exists an R -module M on which the sequence is regular: we refer to these as *latent regular sequences for modules*. We do not know whether every latent regular sequence for modules is a latent regular sequence. These notions are closely related to the notion of a *seed* in [Di1–2]. We also introduce the notion of a *Q-sequence*. We raise the following question: is a Q-sequence the same as a latent regular sequence? See §3, Question 3.6.

One motivation for our study is that these ideas ought to be useful in investigating the existence of big Cohen-Macaulay algebras over local rings, including the mixed char-

acteristic case. Another is that results on \mathfrak{h} -content may well be helpful in studying the direct summand conjecture, and related conjectures, as indicated above. Thus, these notions may be useful in settling the local homological conjectures (for background, we refer the reader to [Du], [EvG1–2], [Heit], [Ho1–5], [PS1–2], and [Ro1–5]). In any case, in studying quasilenngth and content one is immediately led to many questions that are important and appear to be difficult. We conclude this introduction with some examples of such questions.

Question 0.1. Let Λ be either a field K or an unramified discrete valuation domain (V, pV) of mixed characteristic $p > 0$. Let $X_1, \dots, X_d, Y_1, \dots, Y_d$ be indeterminates over Λ . We define

$$f = f_{d,t} = X_1^t \cdots X_d^t - \sum_{j=1}^d Y_j X_j^{t+1}.$$

Let

$$R = R_{d,t} = \Lambda[X_1, \dots, X_d, Y_1, \dots, Y_d]/(f_{d,t}).$$

(In mixed characteristic, one may also consider a variant definition by replacing R by $R/(X_1 - p)$.) We ask whether $\underline{h}_x^d(R) = 0$. The direct summand conjecture follows if one can prove this, which is a weakening of the condition that $H_x^d(R) = 0$. See Remark 4.10 and Example 3.11.

Question 0.2. Here is a second question that appears to be difficult. Consider the minors of an $n \times (n+1)$ matrix of indeterminates over \mathbb{Z} or over a field, where $n \geq 2$. Let I be the ideal they generate. We know that these are not a latent regular sequence (not even a latent regular sequence on modules): see Example 3.1. Can one calculate the \mathfrak{h} -content of $H_I^{n+1}(R)$? In characteristic $p > 0$ it is 0, but over \mathbb{Z} or \mathbb{Q} we do not know the answer even if $n = 2$.

Question 0.3. Finally, suppose that x_1, \dots, x_d is a regular sequence on R . Let $I_t = (x_1^t, \dots, x_d^t)R$ for every $t \geq 1$, and let $I = I_1$. It is easy to see (cf. Proposition 1.2) that for every $t \geq 1$, R/I_t has a filtration with t^d factors each of which is isomorphic with $R/(x_1, \dots, x_d)$. We conjecture that there is no shorter filtration with cyclic factors that are homomorphic images of R/I . This is equivalent to the statement that I -quasilenngth of R/I_t is t^d . This is true if R is Noetherian (see Proposition 1.2(c)), but we have not been able to prove this statement in the general case even if $d = 2$ and $n = 3$!

1. QUASILENGTH

Let R be a ring, M an R -module, and I a finitely generated ideal of R . We define M to have *finite I -quasilenngth* if there is a finite filtration of M in which the factors are cyclic modules killed by I , so that the factors may be viewed as cyclic (R/I) -modules. The *I -quasilenngth* of M is then defined to be the minimum number of factors in such a filtration. If M does not have finite I -quasilenngth, we define its I -quasilenngth to be $+\infty$. We denote the I -quasilenngth of M over R as $\mathcal{L}_I^R(M)$. The ring R and/or the ideal I may be omitted from the terminology and notation if they are clear from context. We denote the least number of generators of M over R as $\nu_R(M)$ or simply $\nu(M)$, and the length of M over R as $\lambda_R(M)$ or simply $\lambda(M)$.

Here are some basic properties of I -quasilenngth.

Proposition 1.1. *Let R be a ring, I a finitely generated ideal of R , and M an R -module.*

- (a) *M has finite I -quasilenngth if and only if M is finitely generated and killed by a power of I . In fact, $\nu(M) \leq \mathcal{L}_I(M)$, and $I^{\mathcal{L}_I(M)}$ kills M .*
- (b) *If M is killed by I , $\mathcal{L}_I(M) = \nu_R(M) = \nu_{R/I}(M)$.*
- (c) *If I is maximal, then $\mathcal{L}_I(M)$ is finite if and only if M is killed by a power of I and has finite length as an R -module, and then $\mathcal{L}_I(M) = \lambda(M)$.*
- (d) *Assume that $0 \rightarrow M' \rightarrow M \rightarrow M'' \rightarrow 0$ is exact. If M' and M'' have finite I -quasilenngth then so does M , and $\mathcal{L}(M) \leq \mathcal{L}(M') + \mathcal{L}(M'')$. If M has finite I -quasilenngth then M'' does as well, and $\mathcal{L}(M'') \leq \mathcal{L}(M)$. If M has finite I -quasilenngth, then M' has finite I -quasilenngth if and only if it is finitely generated.*
- (e) *If M has a finite filtration in which every factor has finite I -quasilenngth then M has finite I -quasilenngth bounded by the sum of the I -quasilenngths of the factors.*
- (f) *If M has finite I -quasilenngth with $I^n M = 0$ and we interpret I^0 as R , then $\mathcal{L}(M) \leq \sum_{j=0}^{n-1} \nu(I^j M / I^{j+1} M)$ and $\mathcal{L}(M) \leq \sum_{j=0}^{n-1} \nu(\text{Ann}_M I^{j+1} / \text{Ann}_M I^j)$.*
- (g) *If S is an R -algebra then $\mathcal{L}_{IS}^S(S \otimes_R M) \leq \mathcal{L}_I^R(M)$.*
- (h) *$\mathcal{L}_I(M) = 0$ if and only if $M = 0$.*
- (i) *If $I = P$ is prime, $\mathcal{L}_P(M)$ is at least the length of M_P as an R_P -module.*

Proof. Given filtrations of M' and M'' , the filtration of M' together with the inverse image of the filtration of M'' in M yields a filtration of M whose factors are the union of

the sets of factors from the filtrations of M' and M'' . This proves the first statement in (d). The second statement follows from the fact that a filtration of length h on M whose factors are cyclic R/I -modules induces a quotient filtration on M'' of the same length whose factors are also cyclic R/I -modules. We postpone the proof of the third statement until we have proved part (a).

Part (e) follows from the first statement in part (d) by an immediate induction on the length of the filtration.

To prove (b), note that if u_1, \dots, u_h generate M , then the submodules $Ru_1 + \dots + Ru_j$ give a filtration of M whose factors are cyclic modules killed by I . Therefore, $\mathcal{L}(M) \leq \nu(M)$.

If M has finite quasilength then, since the factors are all cyclic modules, liftings of the generators of the factors to M generate M . This shows that, in general, $\nu(M) \leq \mathcal{L}(M)$. It follows that $\nu(M) = \mathcal{L}(M)$ when I kills M .

If $0 \rightarrow Q' \rightarrow Q \rightarrow Q'' \rightarrow 0$ is exact, \mathfrak{A} kills Q' , and \mathfrak{B} kills Q'' , then $\mathfrak{A}Q \subseteq Q'$ and so $\mathfrak{A}\mathfrak{B}$ kills Q . It follows that the product of the annihilators of the factors in a finite filtration of M kills M . If $\mathcal{L}_I(M)$ is finite, we therefore have that $I^{\mathcal{L}_I(M)}$ kills M . On the other hand $\nu(M) \leq \mathcal{L}_I(M)$. Part (a) is now proved except for the “if” part. But if M is finitely generated and killed by I^h , then every $I^j M$ is finitely generated (since I and, hence, each I^j is). From part (e),

$$\mathcal{L}_I(M) \leq \sum_{j=0}^h \mathcal{L}_I(I^j M / I^{j+1} M) = \sum_{j=0}^h \nu(I^j M / I^{j+1} M)$$

by part (b), and this completes the proofs of both (a) and (f). The third statement in part (d) also follows, because whatever power of I kills M also kills M' .

Both statements in (f) are immediate from parts (e) and (b). If M is finitely generated and killed by a power of I , then each $I^j M$ is finitely generated, and so these give a filtration of M with finitely generated factors $I^j M / I^{j+1} M$ killed by I . If M is not Noetherian, some of the factors on the left in the second inequality may need infinitely many generators: the inequality is true but uninteresting in this case.

Part (c) is clear, because when $I = m$ is maximal, the only nonzero cyclic (R/I) -module is R/m .

Part (g) is clear because given any finite filtration of M by modules M_j such that every M_j / M_{j-1} is cyclic and killed by I , we may use the images of the $S \otimes_R M_j$ to give a filtration of $S \otimes_R M$ whose factors are cyclic S -modules killed by IS , and its length is the same as the length of the original filtration. Note that (g) is obvious if $\mathcal{L}_I^R(M) = \infty$.

Part (h) is obvious. Part (i) follows from parts (g) and (c) by choosing $S = R_P$. \square

Let Λ be a ring and let $T = \Lambda[X_1, \dots, X_d]$ be a polynomial ring in d variables over Λ . Let J be an ideal of T generated by monomials in X_1, \dots, X_d that contains a power of

every X_i . Then T/J is a finitely generated free module on the monomials in X_1, \dots, X_d not in J , and we refer to its rank as the *co-rank* of J . For example, the co-rank of $(X_1^{t_1}, \dots, X_d^{t_d})$ is $t_1 \cdots t_d$.

Now suppose that R is any ring and $x_1, \dots, x_d \in R$. Suppose that we are given an ideal \mathfrak{A} of R generated by monomials in x_1, \dots, x_d that are given explicitly, in the sense that the d -tuples of exponents are given explicitly, and that a power of every x_i is given as a generator. If Λ is any ring (one may always use \mathbb{Z}) that maps to R , we may form a corresponding ideal J in $\Lambda[X_1, \dots, X_d]$ generated by those $X_1^{h_1} \cdots X_d^{h_d}$ such that $x_1^{h_1} \cdots x_d^{h_d}$ is one of the explicitly given generators of \mathfrak{A} . We refer, somewhat imprecisely, to the co-rank of \mathfrak{A} as the *co-rank* of J . Alternatively, we say that a monomial is *formally* in \mathfrak{A} if it has the form $x_1^{k_1} \cdots x_d^{k_d}$ and there is a given generator $x_1^{h_1} \cdots x_d^{h_d}$ of \mathfrak{A} such that $k_j \geq h_j \geq 0$ for $1 \leq j \leq d$. Then the co-rank of \mathfrak{A} is the number of monomials in x_1, \dots, x_d that are not formally in \mathfrak{A} .

Proposition 1.2. *Let R be a ring, let $I = (x_1, \dots, x_d)$ be an ideal of R , and let M and N be R -modules. Let $\underline{t} = (t_1, \dots, t_d)$ be a d -tuple of positive integers and let $I_{\underline{t}} = (x_1^{t_1}, \dots, x_d^{t_d})$.*

- (a) *If N is finitely generated and killed by $I_{\underline{t}}$, then $\mathcal{L}_I(N) \leq t_1 \cdots t_d \nu(N/IN)$. In fact, N has a filtration by $t_1 \cdots t_d$ modules such that every factor is a homomorphic image of N/IN . In particular, $\mathcal{L}_I(R/I_{\underline{t}}) \leq t_1 \cdots t_d$.*
- (b) *If x_1, \dots, x_d is a regular sequence on M and $N = M/I_{\underline{t}}M$, then N has a filtration by $t_1 \cdots t_d$ modules each of which is isomorphic to $N/IN \cong M/IM$.*
- (c) *If R is ring such that x_1, \dots, x_d is a regular sequence on a Noetherian R -module M , then $\mathcal{L}_I(R/I_{\underline{t}}) = t_1 \cdots t_d$. In particular, if x_1, \dots, x_d is a regular sequence in a Noetherian ring R , then $\mathcal{L}_I(R/I_{\underline{t}}) = t_1 \cdots t_d$.*
- (d) *Let \mathfrak{A} be an ideal generated by a set of monomials in x_1, \dots, x_d containing a power of every x_j , and suppose that the number of monomials in the x_j not formally in \mathfrak{A} is a . Let \mathfrak{B} be another such ideal such that the number of monomials not formally in \mathfrak{B} is b . Suppose that every generator of \mathfrak{B} is formally in \mathfrak{A} . Then $\mathcal{L}_I(\mathfrak{A}M/\mathfrak{B}M) \leq (b - a)\nu(M/IM)$.*

Proof. (a) N has a filtration by $t_1 \cdots t_d$ modules each of which is a homomorphic image of N/IN . To see this, note that N has a filtration

$$N \supseteq x_1 N \supseteq x_1^2 N \supseteq \cdots \supseteq x_1^{t_1-1} N \supseteq x_1^{t_1} N = 0$$

with t_1 factors, each of which is a homomorphic image of $N/x_1 N$, since there is a surjection $N/x_1 N \twoheadrightarrow x_1^j N/x_1^{j+1} N$ induced by multiplication by x_1^j on the numerators. We may use induction on d to complete the proof: each of these factors will have a filtration with $t_2 \cdots t_d$ factors killed by $(x_2, \dots, x_d)R$ as well as x_1 , and each of these factors will

be of a homomorphic image of N/x_1N and therefore of N/IN . The result now follows from parts (b) and (e) of Proposition (1.1).

(b) With x_1 not a zerodivisor, the surjection $M/x_1M \twoheadrightarrow x_1^j M/x_1^{j+1}M$ induced by multiplication by x_1^j is an isomorphism. This yields a filtration of $M/x_1^{t_1}M$ by factors each isomorphic to M/x_1M . The result now follows by induction on d from the fact that x_2, \dots, x_d is a regular sequence on each of these factors.

(c) We know that $\mathcal{L}_I(R) \leq t_1 \cdots t_d$. We obtain a contradiction if $\mathcal{L}_I(R) = h < t_1 \cdots t_d$. This remains true when we replace R by $R/\text{Ann}_R M$ by Proposition 1.1(g), and likewise when we replace R by its localization at a minimal prime in the support of $M/I_{\underline{t}}M$. Hence, there is no loss of generality in assuming that R is a local ring and that $M/I_{\underline{t}}M$ has finite length. The ideals J_i of R that give the filtration of length h (since the factors are cyclic, J_{i+1} is generated over J_i by one element r_{i+1} such that $Ir_{i+1} \subseteq J_i$) may be expanded to M . The result is a filtration of M with h factors, each of which has the form

$$\frac{(J_i + r_{i+1}R)M}{J_i M} \cong \frac{r_{i+1}M}{(J_i M \cap r_{i+1}M)}.$$

We have a surjection of M onto the latter (sending $u \mapsto r_{i+1}u$) that kills IM . Hence, the length of each factor is at most $\lambda(M/IM)$, and it follows that $\lambda(M/I_{\underline{t}}M) \leq h \lambda(M/IM)$. However, $M/I_{\underline{t}}M$ also has a filtration with $t_1 \cdots t_d$ factors each isomorphic with M/IM , and it follows that $\lambda(M/I_{\underline{t}}M) = t_1 \cdots t_d \lambda(M/IM)$, a contradiction.

(d) The ideal \mathfrak{A} is generated over \mathfrak{B} by the set S of monomials in the x_j that are formally in \mathfrak{A} and not formally in \mathfrak{B} . The number of monomials in S is $b - a$, and these can be adjoined successively to \mathfrak{B} to give a sequence of ideals

$$\mathfrak{B} = \mathfrak{B}_0 \subseteq \mathfrak{B}_1 \subseteq \cdots \subseteq \mathfrak{B}_{b-a} = \mathfrak{A}$$

such that each ideal \mathfrak{B}_{i+1} is generated over its predecessor \mathfrak{B}_i by one monomial μ such that, in every instance, $I\mu \subseteq \mathfrak{B}_i$. This yields a sequence

$$\mathfrak{B}M \subseteq \mathfrak{B}_1M \subseteq \cdots \subseteq \mathfrak{A}M$$

such that each of the $b - a$ factors is a homomorphic image of M/IM , and the result follows. \square

Remark on notation. Throughout the rest of this paper, we shall frequently use the notations I and $I_{\underline{t}}$ as in Proposition 1.2 when it is understood what $\underline{x} = x_1, \dots, x_d$ is from context.

Remark 1.3. Quasilenhth is a natural notion but there are difficulties in working with it. One of these is that we do not know, *a priori*, how to choose a filtration of a module which gives the quasilenhth or even gives a result that is close to the quasilenhth. Given a specific

module, every choice of suitable filtration gives an upper bound for the quasilength, but it is very hard to prove lower bounds for the quasilength. In some cases, one can get adequate information from arguments making use of length, but there are many important cases where this method does not give a result that is close to optimal.

Example 1.4. Let K be an infinite field, let $S = K[s, t, u, v]$ be a polynomial ring, and let $R = S/\mathfrak{A}$ where $\mathfrak{A} = (uvs, uvt, v^2s, v^2t)S$, the product of the ideals $(u, v)S$ and $(vs, vt)S$. Let $x_1 = vs$ and $x_2 = vt$. Let $I = (x_1, x_2)R$. Then $I^2 = 0$. We can see that $\mathcal{L}_I(R) = 2$ using the filtration $0 \subseteq vR \subseteq R$. There are several points that we want to make. Both of the inequalities in part (f) of Proposition 1.1 are strict in this case. We have $\nu(R/I) + \nu(I/I^2) = 1 + 2 = 3$. Let $J = (u, v)R$. We also have that $\text{Ann}_R I = J$, and $\nu(R/J) + \nu(J) = 1 + 2 = 3$. Over an infinite field one might use the following strategy to attempt to calculate $\mathcal{L}_I(R)$. Choose generators for the annihilator of I in R , and consider an element in general position in the vector space they span. Let this element generate the first ideal in a filtration. Kill this ideal, and then continue recursively in this way. In the present example, one starts by killing an element of the form $c_1u + c_2v$ where c_1 and c_2 are nonzero scalars. Regardless of how the scalars are chosen, the quotient is isomorphic with $K[s, t, v]/(v^2s, v^2t)$. This still has quasilength 2. Therefore, the proposed strategy does not give the quasilength in the example under consideration: it is necessary to begin the filtration with an ideal generated by an element that is, in some sense, in special position in $\text{Ann}_R I$. It appears to be very difficult to give an algorithm for calculating quasilength even in very simple situations in which the quasilength is known to be small and the ambient ring is finitely generated over a field. See also Remark 2.7 and the last paragraph of Example 3.1.

2. HEURISTIC MEASURES OF LOCAL COHOMOLOGY

Suppose that M is a finitely generated module over the ring R , $\underline{x} = x_1, \dots, x_d$, and $I = (x_1, \dots, x_d)R$. Let $\underline{t} = (t_1, \dots, t_d)$ denote a d -tuple of positive integers. Let $I_{\underline{t}} = (x_1^{t_1}, \dots, x_d^{t_d})R$, and for $k \in \mathbb{N}$ let $\underline{t} + k$ denote the d -tuple $(t_1 + k, \dots, t_d + k)$. We define

$$(I_{\underline{t}}M)^{\text{lim}} = \bigcup_{k=0}^{\infty} ((I_{\underline{t}+k}M) :_M (x_1 \cdots x_d)^k).$$

The notation is somewhat inaccurate, since $(I_{\underline{t}}M)^{\text{lim}}$ depends on knowing M , x_1, \dots, x_d , and \underline{t} , not just on $I_{\underline{t}}M$. However, we believe that what is meant will always be clear from the context. Observe that if we allow d -tuples $\underline{k} = (k_1, \dots, k_d) \in \mathbb{N}^d$, we also have that

$$(I_{\underline{t}}M)^{\text{lim}} = \bigcup_{\underline{k} \in \mathbb{N}^d} ((I_{\underline{t}+\underline{k}}M) :_M x_1^{k_1} \cdots x_d^{k_d}).$$

Note that

$$H_I^d(M) = \varinjlim_{\underline{t}} M/I_{\underline{t}}M,$$

where the maps in the direct limit system are such that the map $M/I_{\underline{t}}M \rightarrow M/I_{\underline{t}+\underline{k}}M$ is induced by multiplication by $x_1^{k_1} \cdots x_d^{k_d}$ on the numerators. It follows that $(I_{\underline{t}}M)^{\text{lim}}$ is the kernel of the composite map $M \rightarrow M/I_{\underline{t}}M \rightarrow H_I^d(M)$, so that

$$H_I^d(M) = \varinjlim_{\underline{t}} M/(I_{\underline{t}}M)^{\text{lim}},$$

and the maps in the direct limit system are now injective.

We write $\underline{t} \geq s$ for $s \in \mathbb{N}$ to mean that every $t_j \geq s$.

We now define

$$\underline{h}_x^d(M) = \lim_{s \rightarrow \infty} \inf \left\{ \frac{\mathcal{L}_I(M/((I_{\underline{t}}M)^{\text{lim}}))}{t_1 \cdots t_d} : \underline{t} \geq s \right\}.$$

By Proposition 1.2(a), every element of every set is at most

$$\mathcal{L}_I(M/I_{\underline{t}})/(t_1 \cdots t_d) \leq \nu(M/I_{\underline{t}}M) \leq \nu(M),$$

and since the sets are decreasing with s , the terms in the limit are nondecreasing. Hence:

Proposition 2.1. *With notation as above, the limit $\underline{h}_x^d(M)$ always exists, and we have that*

$$0 \leq \underline{h}_x^d(M) \leq \nu(M). \quad \square$$

We emphasize that no finiteness hypotheses are needed. We note:

Proposition 2.2. *With notation as above, if $H_I^d(M) = 0$, then $\underline{h}_x^d(M) = 0$.*

Proof. If the local cohomology vanishes, we have that every $M/(I_{\underline{t}}M)^{\text{lim}} = 0$, and so all the quasilengths are 0. \square

We next introduce a variant notion that, for certain purposes, is easier to work with. We shall see that when x_1, \dots, x_d is a system of parameters for an excellent, equidimensional, reduced local ring R and $M = R$, the two notions agree.

Again, let $x_1, \dots, x_d \in R$ be any sequence of elements of the ring R and let M be a finitely generated R -module. We define

$$\underline{h}_x^d(M) = \lim_{s \rightarrow \infty} \inf \left\{ \frac{\mathcal{L}_I(M/(I_{\underline{t}}M))}{t_1 \cdots t_d} : \underline{t} \geq s \right\}.$$

This limit exists by the same reasoning used for $\underline{h}_x^d(M)$, but we can now assert something stronger. We first observe:

Lemma 2.3. *Let $R, x_1, \dots, x_d \in R$ and M be as above and let a d -tuple of integers $\underline{t} = (t_1, \dots, t_d) \geq 1$ be given. Let $I = (x_1, \dots, x_d)R$. Let a real number $\epsilon > 0$ be given. Then there exists an integer $s > 0$ such that for all d -tuples of integers $\underline{T} = (T_1, \dots, T_d) \geq s$,*

$$\frac{\mathcal{L}_I(M/I_{\underline{T}}M)}{T_1 \cdots T_d} \leq \frac{\mathcal{L}_I(M/I_{\underline{t}}M)}{t_1 \cdots t_d} + \epsilon.$$

Proof. Use the division algorithm to write

$$T_j = q_j t_j + r_j, \quad 1 \leq j \leq d,$$

where $q_j, r_j \in \mathbb{N}$ and $0 \leq r_j < t_j$ for all j . Let $\underline{t}' = (q_1 t_1, \dots, q_d t_d)$. Then $M/I_{\underline{T}}M$ has the submodule $I_{\underline{t}'}M/I_{\underline{T}}M$ with quotient $M/I_{\underline{t}'}M$, and so we have

$$(*) \quad \mathcal{L}_I(M/I_{\underline{T}}M) \leq \mathcal{L}_I(M/I_{\underline{t}'}M) + \mathcal{L}_I(I_{\underline{t}'}M/I_{\underline{T}}M).$$

We want to give upper bounds for both terms on the right. We can think of $x_j^{q_j t_j}$ as $(x_j^{t_j})^{q_j}$. It follows from Proposition 1.1(a) that $M/I_{\underline{t}'}M$ has a filtration with $q_1 \cdots q_d$ factors each of which is a homomorphic image of $M/I_{\underline{t}}M$. Hence,

$$\mathcal{L}_I(M/I_{\underline{t}'}M) \leq q_1 \cdots q_d \mathcal{L}_I(M/I_{\underline{t}}M).$$

By Proposition 1.2(d),

$$\mathcal{L}_I(I_{\underline{t}'}M/I_{\underline{T}}M) \leq (T_1 \cdots T_d - (q_1 t_1) \cdots (q_d t_d)) \nu(M/IM).$$

After we divide by $T_1 \cdots T_d$, these two estimates coupled with (*) yield

$$\frac{\mathcal{L}_I(M/I_{\underline{T}}M)}{T_1 \cdots T_d} \leq \frac{q_1 \cdots q_d \mathcal{L}_I(M/I_{\underline{t}}M)}{T_1 \cdots T_d} + \left(1 - \frac{q_1 t_1}{T_1} \cdots \frac{q_d t_d}{T_d}\right) \nu(M/IM).$$

The first summand on the right hand side increases if we replace the denominator by $(q_1 t_1) \cdots (q_d t_d)$. In the second summand, we note that $q_j t_j / T_j$ is the same as $1 - r_j / T_j > 1 - t_j / T_j$. Thus,

$$\frac{\mathcal{L}_I(M/I_{\underline{T}}M)}{T_1 \cdots T_d} \leq \frac{\mathcal{L}_I(M/I_{\underline{t}}M)}{T_1 \cdots T_d} + \left(1 - \left(1 - \frac{t_1}{T_1}\right) \cdots \left(1 - \frac{t_d}{T_d}\right)\right) \nu(M/IM).$$

Since \underline{t} is fixed, it is clear that the second term on the right is eventually $\leq \epsilon$ when the T_j are sufficiently large. \square

This yields:

Theorem 2.4. *With notation as in Proposition 2.3,*

$$\mathfrak{h}_{\underline{x}}^d(M) = \inf \left\{ \frac{\mathcal{L}_I(M/I_{\underline{t}}M)}{t_1 \cdots t_d} : \underline{t} \geq 1 \right\} = \lim_{\underline{t} \rightarrow \infty} \frac{\mathcal{L}_I(M/I_{\underline{t}}M)}{t_1 \cdots t_d}.$$

If $I_t = (x_1^t, \dots, x_d^t)R$ for $t \geq 1$, we also have

$$\mathfrak{h}_{\underline{x}}^d(M) = \lim_{t \rightarrow \infty} \frac{\mathcal{L}_I(M/I_t M)}{t^d}.$$

Proof. Let

$$\eta = \inf \left\{ \frac{\mathcal{L}_I(M/I_{\underline{t}}M)}{t_1 \cdots t_d} : \underline{t} \geq 1 \right\}.$$

Let γ be any element of the set. By the preceding Lemma, $\mathfrak{h}_{\underline{x}}^d(M) \leq \gamma + \epsilon$ for all $\epsilon > 0$, and so $\mathfrak{h}_{\underline{x}}^d(M) \leq \gamma$. It follows that $\mathfrak{h}_{\underline{x}}^d(M) \leq \eta$, while the opposite inequality is obvious. Thus, $\mathfrak{h}_{\underline{x}}^d(M) = \eta$. Let $\epsilon > 0$ be given. Let

$$\gamma_{\underline{t}} = \frac{\mathcal{L}_I(M/I_{\underline{t}}M)}{t_1 \cdots t_d}.$$

Choose a specific d -tuple $\underline{\tau}$ such that $\gamma_{\underline{\tau}} \leq \eta + \epsilon/2$. From the Lemma, there exists s such that for all $\underline{t} \geq s$, $\gamma_{\underline{t}} \leq (\eta + \epsilon/2) + \epsilon/2 = \eta + \epsilon$, and so $\eta \leq \gamma_{\underline{t}} \leq \eta + \epsilon$ for all $\underline{t} \geq s$. Both statements about limits follow. \square

Remark. Let $R \twoheadrightarrow S$ be a surjective homomorphism of rings, let $\underline{x} = x_1, \dots, x_d \in R$, and let $\underline{y} = y_1, \dots, y_d$ be the images of x_1, \dots, x_d in S . Let $J = IS = (y_1, \dots, y_d)S$, let M, N be S -modules, and let ${}_R M$ denote the R -module obtained from M by restriction of scalars. Then it is obvious ${}_R(J_{\underline{t}}M)^{\lim} = (I_{\underline{t}}{}_R M)^{\lim}$, that if N is killed by a power of J then $\mathcal{L}_I^R({}_R N) = \mathcal{L}_J^S(N)$, and we may apply this when $N = M/(J_{\underline{t}}M)^{\lim}$ (respectively, $M/J_{\underline{t}}M$) to conclude that $\mathfrak{h}_{\underline{x}}^d({}_R M) = \mathfrak{h}_{\underline{x}}^d(M)$ and that $\mathfrak{h}_{\underline{x}}^d({}_R M) = \mathfrak{h}_{\underline{y}}^d(M)$.

We next observe that both notions of content can only decrease under base change.

Proposition 2.5. *Let $R \rightarrow S$ be a ring homomorphism, let x_1, \dots, x_d be a sequence of elements of R . Let M be any R -module, and let y_1, \dots, y_d be the images of x_1, \dots, x_d in S . Then $\mathfrak{h}_{\underline{x}}^d(M) \geq \mathfrak{h}_{\underline{y}}^d(S \otimes_R M)$ and $\mathfrak{h}_{\underline{x}}^d(M) \geq \mathfrak{h}_{\underline{y}}^d(M)$. In particular, $\mathfrak{h}_{\underline{x}}^d(R) \geq \mathfrak{h}_{\underline{x}}^d(S)$ and $\mathfrak{h}_{\underline{x}}^d(R) \geq \mathfrak{h}_{\underline{y}}^d(S)$.*

Proof. The statement for $\mathfrak{h}_{\underline{x}}^d(M)$ is immediate from Proposition 1.1(g) and the fact that $(x_1^{t_1}, \dots, x_d^{t_d})R$ expands to $(y_1^{t_1}, \dots, y_d^{t_d})S$. The argument for $\mathfrak{h}_{\underline{x}}^d(M)$ is similar, but needs the fact that the image of

$$S \otimes_R ((x_1^{t_1}, \dots, x_d^{t_d})M)^{\lim}$$

in $S \otimes_R M$ is contained in

$$((y_1^{t_1}, \dots, y_d^{t_d})(S \otimes_R M))^{\lim},$$

which is entirely straightforward to verify. \square

Proposition 2.6. *Let $x_1, \dots, x_d \in R$, let k_1, \dots, k_d be positive integers, let $\underline{y} = x_1^{k_1}, \dots, x_d^{k_d}$, and let M be any R -module. Let $I = (\underline{x})R$ and $J = (\underline{y})R$. Let \underline{t} denote a variable d -tuple of positive integers. Let $I_{\underline{t}}$ denote $(x_1^{t_1}, \dots, x_d^{t_d})R$, and let $J_{\underline{t}}$ denote $(y_1^{t_1}, \dots, y_d^{t_d})R$. We write $\underline{k} \cdot \underline{t}$ for the d -tuple whose i th term is $k_i t_i$.*

- (a) $\mathcal{L}_I(M/I_{\underline{t}}M) \leq k_1 \cdots k_d \mathcal{L}_J(M/J_{\underline{t}}M)$.
- (b) $(J_{\underline{t}}M)^{\lim}$ with respect to \underline{y} is the same as $(I_{\underline{k} \cdot \underline{t}}M)^{\lim}$ with respect to \underline{x} .
- (c) $\mathcal{L}_I(M/(I_{\underline{t}}M)^{\lim}) \leq k_1 \cdots k_d \mathcal{L}_J(M/(J_{\underline{t}}M)^{\lim})$.
- (d) $\mathfrak{h}_{\underline{x}}^d(M) \leq \mathfrak{h}_{\underline{y}}^d(M)$ and $\underline{h}_{\underline{x}}^d(M) \leq \underline{h}_{\underline{y}}^d(M)$.

Proof. Part (b) follows from the fact that $J_{\underline{t}} = I_{\underline{k} \cdot \underline{t}}$, and the usual identification of the local cohomology modules $H_I^d(M)$ and $H_J^d(M)$ and hence of the maps $M \rightarrow H_I^d(M)$ and $M \rightarrow H_J^d(M)$. Parts (a) and (c) follow from the fact that a filtration of the module occurring on the right hand side of the inequality that has h factors that are homomorphic images of R/J can be refined to one that has $k_1 \cdots k_d h$ factors that are homomorphic images of R/I , since R/J has a filtration with $k_1 \cdots k_d$ factors that are homomorphic images of R/I . The statements in (d) follow from (a) and (c) and the definitions of content. \square

Remark 2.7. Remark 1.3 and Example 1.4 emphasized the difficulty in finding an algorithm or procedure that calculates quasilength. We want to point out that in trying to study, for example, $\mathfrak{h}_{\underline{x}}^d(R)$, it would be very useful to have a procedure if it gave a result asymptotic to $\mathcal{L}_I(R/I_t)$ as $t \rightarrow \infty$. This may well be much easier than finding a method that yields precise quasilengths.

Let $R \subseteq S$ be a module-finite extension and let x_1, \dots, x_d be a sequence of elements of R . We shall say that the map $R \subseteq S$ is \underline{x} -split if there is a positive integer h and an R -linear map $S^{\oplus h} \rightarrow R$ whose image contains a power of every x_j , i.e., whose image has the same radical as $(\underline{x})R$. This holds, in particular, if $R \subseteq S$ is split as a map of R -modules, in which case we may take $h = 1$.

Theorem 2.8. *Let $R \subseteq S$ be a module-finite extension such that S is generated as an R -module by r elements. Suppose that this extension is \underline{x} -split, so that there exists an R -linear map $S^{\oplus h} \rightarrow R$ whose image is an ideal containing a power of every x_i . Then $\mathfrak{h}_{\underline{x}}^d(R) \geq \mathfrak{h}_{\underline{x}}^d(S) \geq \frac{1}{rh} \mathfrak{h}_{\underline{x}}^d(R)$, and $\mathfrak{h}_{\underline{x}}^d(R)$ and $\mathfrak{h}_{\underline{x}}^d(S)$ are both 0 or both positive. In particular, if $R \rightarrow S$ splits over R , then $\mathfrak{h}_{\underline{x}}^d(R) \geq \mathfrak{h}_{\underline{x}}^d(S) \geq \frac{1}{r} \mathfrak{h}_{\underline{x}}^d(R)$.*

Proof. Let $I = (\underline{x})R$ and I_t be defined as usual. Suppose that we have an R -linear map $\theta : S^{\oplus h} \rightarrow R$ whose image J contains a power of every x_i . Suppose that we have a filtration of $S/I_t S$ with $L = \mathcal{L}_{I_t S}^S(S/I_t S)$ factors each of which is a homomorphic image

of S/IS . Since S is generated by r elements over R , each factor can be filtered further as an R -module so that one has a filtration by at most r homomorphic images of R/I . This yields a filtration of S/I_tS by R -submodules with at most rL factors each of which is a homomorphic image of R/I . We then obtain a filtration \mathcal{F} of $(S/I_tS)^{\oplus h}$ by R -submodules with at most rhL factors such that every factor is a homomorphic image of R/I .

The map θ induces a map $\bar{\theta} : (S/I_tS)^{\oplus h} \rightarrow R/I_tR$ whose image is $(J + I_t)/I_t$. We may apply $\bar{\theta}$ to the R -modules in \mathcal{F} to obtain a filtration of $(J + I_t)/I_t$ such that the factors consist of rhL homomorphic images of R/I . This yields

$$\mathcal{L}_I(R/I_t) \leq rhL + \mathcal{L}_I^R(R/(J + I_t)) \leq rhL + \mathcal{L}_I^R(R/J) = rh\mathcal{L}_{IS}^S(S/I_tS) + \mathcal{L}_I^R(R/J).$$

We may now divide by t^d and take the limit of both sides at $t \rightarrow \infty$. Since $\mathcal{L}_I^R(R/J)$ is constant, we obtain the inequality on the right in the statement of the theorem. The inequality on the left is the last statement in Proposition 2.5. \square

Remark 2.9. Let $x = x_i$. If $R_x \subseteq S_x$ splits over R_x , we can often obtain an R -linear map $S \rightarrow R$ whose value on 1 is a power of x . This is true (1) if x is a nonzerodivisor in R , or (2) if the kernel of $R \rightarrow R_x$ is killed by x^t for some fixed t , or (3) if S is finitely presented over R . In each of these cases we can use the composite map $S \rightarrow S_x \rightarrow R_x$ (where the map on the right is the splitting) to get a map $f : S \rightarrow R_x$ with $f(1) = 1$. The image of each of the finitely many generators of S will have the form r_j/x^{k_j} for some suitably large k_j . If k is the supremum of the k_j , and x is a nonzerodivisor, the values of $x^k f$ are in R , and its value on 1 is x^k . More generally, let \mathfrak{A} be the kernel of $R \rightarrow R_x$ and suppose that it is killed by x^t . Let $\bar{R} = R/\mathfrak{A}$. Then, as in the case where x is a nonzerodivisor, we get a map $g : S \rightarrow \bar{R}$ whose value on 1 is the image of x^k . Since $\bar{R} \cong x^t R \subseteq R$, we get a map $S \rightarrow R$ whose image on 1 is x^{t+k} . In the case where S is finitely presented, we may use that $(\text{Hom}_R(S, R))_x \cong \text{Hom}_{R_x}(S_x, R_x)$ instead. If, for every i , one of the numbered conditions holds, then we get a map $S^d \rightarrow R$ whose image contains a power of every x_i .

3. LATENT REGULAR SEQUENCES AND Q-SEQUENCES

We recall that x_1, \dots, x_d is a *latent regular sequence* (respectively, a *latent regular sequence for modules*) in R if there exists an R -algebra S (respectively, an R -module M) such that x_1, \dots, x_d is a regular sequence on S (respectively, M). Note that, by definition, for the sequence to be regular, we must have that $S/(x_1, \dots, x_d)S \neq 0$ (respectively, that $M/(x_1, \dots, x_d)M \neq 0$). Of course, a latent regular sequence is also a latent regular sequence for modules.

We note that if such an algebra or module exists, then we may localize at a minimal prime in the support of $S/(x_1, \dots, x_d)S$ (respectively, $M/(x_1, \dots, x_d)M$), so that we

may assume that S is quasilocal (respectively, that M is a module over a local ring of R). When we refer to *completion* in the I -adic topology, we shall always mean the *separated I -adic completion*. When we complete S or M with respect to the I -adic topology, the regular sequence x_1, \dots, x_d becomes a permutable regular sequence on the completion. See [BS], Cor. 1.2, Th. 1.3, and Prop. 1.5. Hence, latent regular sequences (respectively, latent regular sequences on modules) are permutable, i.e., a permutation of such a sequence is again such a sequence.

Moreover, if R is a local ring, x_1, \dots, x_d is a system of parameters, and x_1, \dots, x_d is a regular sequence on M , a module or algebra, then every system of parameters for R is a regular sequence on the I -adic completion of M (which is an algebra if M is an algebra). Again, see [BS]. Thus, the existence of big Cohen-Macaulay algebras over a local ring is equivalent to the statement that some (equivalent, every) system of parameters is a latent regular sequence, and there is a parallel statement for modules. Hence, the study of latent regular sequences is closely related to the study of seeds over a complete local domains in [Di1–2].

Example 3.1. Let $A = (r_{ij})$ be an $n \times (n+1)$, $n \geq 2$, matrix over the ring R , and let $\Delta_1, \dots, \Delta_{n+1}$ be the sequence of $n \times n$ minors of A with alternating signs: specifically, Δ_i is the product of $(-1)^{i-1}$ and the $n \times n$ minor obtained by omitting the i th column. Then $\Delta_1, \dots, \Delta_{n+1}$ is *not* a latent regular sequence on modules. To see this, let I be the ideal that these elements generate, and suppose that they form a regular sequence on M . Let $u \in M - IM$. Each row of the matrix gives a relation on $\Delta_1, \dots, \Delta_{n+1}$. If we multiply by a given element of M , this becomes a relation with coefficients in M . It follows that every for all i, j , $r_{ij}M \subseteq IM$. But then $IM \subseteq I^n M$, since every minor is a homogeneous polynomial of degree n in the r_{ij} . It follows that $Iu \subseteq I^n M$, and the fact that the Δ_i form a regular sequence then implies that $u \in I^{n-1}M$, a contradiction.

Let Λ denote either \mathbb{Z} or a field. Let $A = (X_{ij})$ denote an $n \times (n+1)$ matrix of indeterminates over Λ , and let R be the polynomial ring in the X_{ij} over Λ . Let \underline{x} denote the sequence of $n \times n$ minors of A , and let $d = n+1$. Let $I = (\underline{x})$. We are very interested in the behavior of $H_I^d(R)$. If Λ is a field of characteristic p , we know that $H_I^d(R) = 0$ by a result of Peskine and Szpiro [PS1], Prop. 4.1, and, hence, $\underline{h}_{\underline{x}}^d(R) = 0$. It follows from Theorem 3.9 below that $\underline{h}_{\underline{x}}^d(R) = 0$ as well. If K is of equal characteristic 0, we know that $H_I^d(R) \neq 0$. In this case, $B = K[\Delta_1, \dots, \Delta_{n+1}]$, which is a polynomial ring in all characteristics, is a direct summand as a B -module of R (because it is a ring of invariants of the linearly reductive group $\mathrm{SL}(n, K)$ acting on R), and so $H_{(\underline{x})B}^d(B) \neq 0$ is a direct summand over B of $H_I^d(R)$. In equal characteristic 0 and over \mathbb{Z} we do not know the values of $\underline{h}_{\underline{x}}^d(R)$ and $\underline{h}_{\underline{x}}^d(R)$.

We can say more. The following argument, using results of [Ly], is due to G. Lyubeznik. Assume that K has characteristic 0 and continue the notations of the preceding paragraph. Then $H_I^d(R)$ is a holonomic D -module. After localization at any X_{ij} , the ideal I is generated by the $n-1$ size minors of an $(n-1) \times n$ matrix, and so $H_I^d(R)$ is supported

only at the homogeneous maximal ideal. Since it is a holonomic D-module, it follows from the results of [Ly] that it is a finite direct sum of copies of the injective hull E of R/m , where $m = (x_{ij} : i, j)R$ is the maximal ideal generated by all the variables, and $E \cong H_m^{(n+1)n}(R)$. It follows from the results of [W] that when $n = 2$, $H_I^3(R) \cong H_m^6(R)$. See also Theorem 6.1 and its proof in [HuKM], where this isomorphism is shown to have surprising applications. In general, in equal characteristic 0, $H_I^d(R) \cong H_{\underline{x}'}^{(n+1)n}(R)^{\oplus k_n}$, where $k_n > 0$ is an integer, and \underline{x} is a string formed from the $(n+1)n$ indeterminates x_{ij} . We conjecture that $k_n = 1$ in general, but so far as we know this is an open question except when $n = 2$. Note that from Theorem 4.7, it follows that $\underline{h}_{\underline{x}'}^{(n+1)n}(R) = \mathfrak{h}_{\underline{x}'}^{(n+1)n}(R) = 1$ in equal characteristic 0 and $p > 0$ for all n . However, this does not *a priori* yield any information about the case where the string of elements from the ring consists only of the $n+1$ minors and the exponent is $n+1$, even though the local cohomology module may be the same. Note that in characteristic $p > 0$ in the latter case $\underline{h}_{\underline{x}}^{n+1}(R) = 0$.

Let K be a field and let I be the ideal generated of a 2×3 matrix of indeterminates over K . To underline the difficulty of calculating I -quasilength, we note that we do not know what it is for the quotient of R by ideal I_2 generated by the squares of the 3 minors. By mapping to the polynomial ring $K[y, z]$ so that the matrix becomes $\begin{pmatrix} 1 & 0 & 0 \\ 0 & y & z \end{pmatrix}$, one sees that the quasilength is at least 4, while it is obviously bounded above by 8 (in characteristic 2, it is bounded by 7, because the product of the minors is in I_2). We have not been able to prove more.

Example 3.2. It may be tempting to believe that if $x_1, \dots, x_d \in R$, x_1 is not a zerodivisor in R , but x_2, \dots, x_d is a latent regular sequence in R/x_1R , then x_1, \dots, x_d is a latent regular sequence in R . But this is false. Consider the situation in the preceding paragraph when $n = 2$ and $\Lambda = K$ is a field. We have already seen the $\Delta_1, \Delta_2, \Delta_3$ is not a latent regular sequence on modules. But the images of Δ_2 and Δ_3 do form a latent regular sequence in $R/(\Delta_1)$. Let x, x', y, z, s, t be new indeterminates over K , let $D = xt - x's - 1$, and let $S = (K[x, x', s, t]/(D))[y, z]$. Map $R/(\Delta_1)$ as a K -algebra to S by sending the entries of the matrix X to the corresponding entries of the matrix $\begin{pmatrix} x & ys & zs \\ x' & yt & zt \end{pmatrix}$. Note that the map is well-defined because the second and third columns of the image matrix form a matrix with determinant 0. Under this map, the images of Δ_2 and Δ_3 are $-z$ and y , respectively, which is a regular sequence in S . \square

By an *equational constraint* on $\underline{x} = x_1, \dots, x_d$ and R , we mean a finite family of polynomials F_1, \dots, F_h over \mathbb{Z} with coefficients in \mathbb{Z} in variables $X_1, \dots, X_d, Y_1, \dots, Y_s$ (s may vary). We shall say that \underline{x} and R *satisfy* the constraint if there do *not* exist elements $r_1, \dots, r_s \in R$ such that $F_i(x_1, \dots, x_d, r_1, \dots, r_s) = 0$, $1 \leq i \leq h$. We shall say that a condition \mathcal{C} on \underline{x} and R is *equational* if there is a family of equational constraints such that \underline{x} and R satisfy \mathcal{C} if and only if \underline{x} and R satisfy all of the equational constraints in the family. The following result is already known, except for the terminology of “latent regular sequences.”

Theorem 3.3. *The condition that $x_1, \dots, x_d \in R$ be a latent regular sequence (respectively, a latent regular sequence for modules) is equational.*

We briefly mention the idea of the proof. By a *module* (respectively, *algebra*) *modification* of an R -algebra (respectively, module) M of type k with respect to x_1, \dots, x_d we mean a map $M \rightarrow M'$, where

$$x_{k+1}u_{k+1} = \sum_{i=1}^k x_i u_i$$

with all of $u_1, \dots, u_{k+1} \in M$, is a relation, $0 \leq k \leq d-1$, and M' is either

$$M \rightarrow M[Z_1, \dots, Z_k]/(u_{k+1} - \sum_{i=1}^k x_i Z_i)$$

in the algebra case, where the Z_j are indeterminates, or

$$M \rightarrow (M \oplus Re_1 \oplus \dots \oplus Re_k)/R(u_{k+1} - \sum_{i=1}^k x_i e_i),$$

in the module case, where $Re_1 \oplus \dots \oplus Re_k$ is an R -free module with free basis e_1, \dots, e_k . Then x_1, \dots, x_d is a latent regular sequence (respectively, a latent regular sequence for modules) if and only if for every sequence

$$R \rightarrow M_1 \rightarrow \dots \rightarrow M_r$$

algebra (respectively, module) modifications of types k_1, \dots, k_r , respectively, we have that $1 \in R$ does not map into $(x_1, \dots, x_d)M_r$. The failure of this condition for specific r and k_1, \dots, k_r is easily seen to be equivalent to the failure of an equational constraint on R . For details in the module case we refer the reader to [Ho2] §4, and for the algebra case to (3.31) of [HH5]. \square

Let $x_1, \dots, x_d \in R$. Let the notations $I_{\underline{t}}$ and I_t be as in (2.3) and (2.4). We say that x_1, \dots, x_d form a \mathbb{Q} -sequence if the following equivalent conditions hold:

- (1) For all $\underline{t} = t_1, \dots, t_d$, $\mathcal{L}_I(R/I_{\underline{t}}) = t_1 \cdots t_d$.
- (2) For all $t \geq 1$, $\mathcal{L}_I(R/I_t) = t^d$.
- (3) $\mathfrak{h}_{\underline{x}}^d(R) = 1$.

The equivalence is immediate from Theorem 2.4. We shall show that the condition that x_1, \dots, x_d form a \mathbb{Q} -sequence depends only on d and the ideal $I = (x_1, \dots, x_d)R$. See Theorem 3.8(a). Moreover, Remark 3.7 and Theorem 3.8(b) give additional equivalent conditions for x_1, \dots, x_d to form a \mathbb{Q} -sequence (for example, it is equivalent that $\mathfrak{h}_{\underline{x}}^d(R) = 1$).

If x_1, \dots, x_d is a regular sequence in R , we conjecture that x_1, \dots, x_d is a \mathbb{Q} -sequence, but we cannot prove this even if $d = 2$. We note:

Proposition 3.4. *The condition that x_1, \dots, x_d be a Q-sequence in R is equational.*

Proof. The failure of the condition is equivalent to the existence of t and $h < t^d$ such that R/I_t has a filtration with h cyclic factors such that each quotient is killed by (x_1, \dots, x_d) . This in turn is equivalent to the existence of elements $r_1, \dots, r_h \in R$ with $r_h = 1$ such every $x_i r_j$ is in the ideal generated by x_1^t, \dots, x_d^t and the r_i for $i < j$. We may then take the terms in the filtration to be the ideals J_k/I_t where J_t is generated over I_t by r_1, \dots, r_k . If Z_j is the variable corresponding to r_j , the polynomials we want to vanish are $Z_h - 1$ and

$$X_i Z_j - \sum_{\nu=1}^{j-1} Y_{i,j,\nu} Z_\nu - \sum_{\nu=1}^d V_{i,j,\nu} X_\nu^t, \quad 1 \leq j \leq h, 1 \leq i \leq d,$$

where the $Y_{i,j,\nu}$ and $V_{i,j,\nu}$ are auxiliary variables. \square

We also note:

Remark 3.5. Note that an equational condition on x_1, \dots, x_d holds in R if and only if it holds for all finitely generated subalgebras of R that contain x_1, \dots, x_d . Also note that if we have a direct limit system of rings R^j and for each R^j a sequence of d elements \underline{x}^j such that $\underline{x}^j \mapsto \underline{x}^k$ under $R^j \rightarrow R^k$ for $j \leq k$, then the direct limit of these sequences satisfies the equational condition if and only if all of the \underline{x}^j satisfy it.

Hence, both observations apply to the following three conditions:

- (1) x_1, \dots, x_d is a latent regular sequence.
- (2) x_1, \dots, x_d is a latent regular sequence for modules.
- (3) x_1, \dots, x_d is a Q-sequence.

Question 3.6. Is it the case that a sequence x_1, \dots, x_d in R is a latent regular sequence if and only if it is a Q-sequence? Note that we do not know either direction, since we have not been able to show that if x_1, \dots, x_d is a regular sequence in R , then it is a Q-sequence.

Remark 3.7. If the quasilenngth of R/I_t is t^d , where $I_t = (x_1^t, \dots, x_d^t)R$, then for any ideal J generated by monomials in the elements x_1, \dots, x_d that contains I_t , the I -quasilenngth of R/J is the co-rank h of J (see the discussion in the two paragraphs immediately preceding Proposition 1.2). For if R/J has a filtration with $k < h$ factors that are images of R/I , then R/I_t has a filtration with $k + (t^d - h) < t^d$ factors that are images of R/I , since $\mathcal{L}_I(J/I_t) \leq t^d - h$ by Proposition 1.2(d), and this gives a contradiction. Hence, if x_1, \dots, x_d is a Q-sequence, the I -quasilenngth of R/J for any ideal J generated by monomials in x_1, \dots, x_d that contains a power of every x_i is the same as the co-rank of J , since $J \supseteq I_t$ for all sufficiently large t .

We next observe:

Theorem 3.8. *Let R be any ring and let $x_1, \dots, x_d \in R$.*

(a) *The condition that $\mathfrak{h}_{\underline{x}}^d(R) = 1$ depends only on d and $I = (x_1, \dots, x_d)R$, and not on the specific choice of d generators for I . More specifically, $\mathfrak{h}_{\underline{x}}^d(R) = 1$ if and only if for every integer n , $\mathcal{L}_I(R/I^n)$ is the co-rank $\binom{t+d-1}{d}$ of the ideal I^n .*

(b) *The following conditions are equivalent:*

- (1) $\mathfrak{h}_{\underline{x}}^d(R) = 1$.
- (2) x_1, \dots, x_d is a \mathbb{Q} -sequence.
- (3) For all d -tuples \underline{t} of positive integers, $\mathcal{L}_I(R/I_{\underline{t}}^{\text{lim}}) = t_1 \cdots t_d$.
- (4) $\underline{\mathfrak{h}}_{\underline{x}}^d(R) = 1$.

Proof. (a) By Remark 3.7, to check that the x_1, \dots, x_d are a \mathbb{Q} -sequence it suffices to show for any sequence of monomial ideals J_t cofinal with the ideals I_t that every $\mathcal{L}_I(R/J_t)$ is the same as the co-rank of J_t . In particular, we may use $J_t = I^t$.

(b) It is clear that (3) \Rightarrow (4) \Rightarrow (1) (since $\underline{\mathfrak{h}}_{\underline{x}}^d(R) \leq \mathfrak{h}_{\underline{x}}^d(R) \leq 1$) and we already know that (1) \Leftrightarrow (2). Hence, it suffices to assume (2) and prove (3). Suppose that $R/I_{\underline{t}}^{\text{lim}}$ has a filtration with $h < t_1 \cdots t_d$ factors that are homomorphic images of R/I . This will also be true for R/J , where J is obtained by enlarging $I_{\underline{t}}$ using finitely many elements of $I_{\underline{t}}^{\text{lim}}$. Let y denote $x_1 \cdots x_d$. Then we may assume without loss of generality that J is contained $I_{\underline{t}+k} : y^k$ for a suitable positive integer k . The cyclic submodule C of $R/I_{\underline{t}+k}$ generated by y^k is killed by J , and so has a filtration with h factors that are images of R/I . But $R/(I_{\underline{t}} + y^k R)$ has co-rank $(t_1 + k) \cdots (t_d + k) - t_1 \cdots t_d$ (we may do this calculation in the case where the x_i are indeterminates over some base ring), and so $R/I_{\underline{t}+k}$ has a filtration with $(t_1 + k) \cdots (t_d + k) - t_1 \cdots t_d + h < (t_1 + k) \cdots (t_d + k)$, factors that are homomorphic images of R/I , contradicting the assumption that x_1, \dots, x_d is a \mathbb{Q} -sequence. \square

We are now in a position to prove:

Theorem 3.9 (dichotomy in positive characteristic). *Let $x_1, \dots, x_d \in R$, where R has prime characteristic $p > 0$. Then $\mathfrak{h}_{\underline{x}}^d(R)$ must be either 0 or 1. Thus, either \underline{x} is a \mathbb{Q} -sequence or else $\mathfrak{h}_{\underline{x}}^d(R) = 0$. Moreover, $\underline{\mathfrak{h}}_{\underline{x}}^d(R) = \mathfrak{h}_{\underline{x}}^d(R)$.*

Proof. Let I denote $(x_1, \dots, x_d)R$ and let I_t denote $(x_1^t, \dots, x_d^t)R$ as usual. Note that if Q is any power of p , then $I_t^{[q]} = I_{tq}$. If the elements do not form a \mathbb{Q} -sequence we can choose an integer $t > 0$ such that $\mathcal{L}_I(R/I_t) = h < t^d$. By Remark 3.7, we can replace t by any larger integer, and so we may assume that $t = q = p^e$ is a power of p and that R/I_q has a filtration in which the factors are h homomorphic images of R/I with $h < q^d$.

We prove by induction on n that R/I_{q^n} has a filtration in which the factors are h^n homomorphic images of R/I . The case $n = 1$ is given. At the inductive step, assume that

one has such a filtration for R/I_{q^n} . Let S denote R viewed as an algebra over itself using the e th iterate F^e of the Frobenius endomorphism. By taking images in $S \otimes_R R/I_{q^n}$ we obtain a filtration of $S/I_{q^n}S$ with h^n factors, each of which is a homomorphic image of S/IS . But $S/I_{q^n}S = R/I_{q^{n+1}}$, and $S/IS = R/I_q$. Thus, $R/I_{q^{n+1}}$ has a filtration \mathcal{F} with h^n factors, each of which is a homomorphic image of R/I_q . Since R/I_q has a filtration with h factors each of which is a homomorphic image of R/I , we may refine the filtration \mathcal{F} to a filtration of $R/I_{q^{n+1}}$ with $h \cdot h^n = h^{n+1}$ factors, each of which is a homomorphic image of R/I . This completes the induction.

Since $\frac{h^n}{(q^n)^d} = \left(\frac{h}{q^d}\right)^n$ and $\frac{h}{q^d} < 1$, we have that $\lim_{n \rightarrow \infty} \frac{\mathcal{L}_I(R/I_{q^n})}{(q^n)^d} = 0$.

For the final statement, note that if $\mathfrak{h}_{\underline{x}}^d(R) = 0$ then $\underline{h}_{\underline{x}}^d(R) \leq \mathfrak{h}_{\underline{x}}^d(R)$ must also be zero, while if $\mathfrak{h}_{\underline{x}}^d(R) = 1$, then $\underline{h}_{\underline{x}}^d(R) = 1$ by Theorem 3.8(b). \square

Definition 3.10. Let $x_1, \dots, x_d \in R$ and let $I = (\underline{x})R$. We say that $H_I^d(R)$ is *robust* for \underline{x} if $\underline{h}_{\underline{x}}^d(R) = 1$. We show in §4 that if x_1, \dots, x_d is a system of parameters for an equicharacteristic local ring (R, m) , then $H_I^d(R) = H_m^d(R)$ is robust for \underline{x} (Theorem 4.7).

Example 3.11: Paul Roberts's calculation of local cohomology. Let K be a field of characteristic 0. Let

$$R = K[x_1, x_2, x_3, y_1, y_2, y_3]/(x_1^2x_2^2x_3^2 - y_1x_1^3 - y_2x_2^3 - y_3x_3^3)$$

be the ring $R_{3,2}$ considered in Question 0.1 of the Introduction, or its localization at $(\underline{x}, \underline{y})$, or the completion of that ring. The main result of [Ro6] is that $H_{(\underline{x})}^3(R) \neq 0$. This provides an example of a nonzero local cohomology module that is not robust, since it is clear that $\mathcal{L}_I(R/I_3)$ is at most 7, and so $\mathfrak{h}_{\underline{x}}^3(R) \leq 7/8$. We do not know the values of $\underline{h}_{\underline{x}}^3(R)$ and $h_{\underline{x}}^3(R)$ in this case.

Example 3.12: the case of one element. Let R be any ring. It is quite easy to see that $x \in R$ is a latent regular sequence for modules if and only for all n , $x^n \notin x^{n+1}R$. The necessity of this condition is clear. For sufficiency note that $J = xR + \bigcup_n \text{Ann}_R x^n \neq R$, for if $rx + v = 1$ and $vx^n = 0$ then $x^n = x^n \cdot 1 = x^n(rx + v) = rx^{n+1}$. If we localize at a minimal prime Q of J , then the image of x is not nilpotent, and x is in the maximal ideal of R_Q . We may now kill a minimal prime of R_Q to obtain a quasilocal domain in which the image of x is a nonzero element of the maximal ideal, and, hence, a nonzerodivisor. \square

Example 3.13: the case of two elements. Let R be any ring and $x, y \in R$. Let $I = (x, y)$, and let \overline{R} be the image of R in R_{xy} . Let $\mathcal{T}_{x,y}(R)$ denote the submodule of R_{xy} consisting of all elements u that $I^n u \in \overline{R}$ for some positive integer n . It is easy to verify that $S = \mathcal{T}_{x,y}(R)$ is a subring of R_{xy} . Then x, y is a latent regular sequence for modules if and only if $(x, y)S \neq S$, in which case x, y is a regular sequence on S . See §12 of [Ho5]

(where $\mathcal{T}_{x,y}(R)$ is denoted $\Theta(R; x, y)$). Thus, x, y is a latent regular sequence for modules if and only if it is a latent regular sequence.

Example 3.14. Consider the polynomial ring $R = K[s, t, u, v]$, where K is a field. Let $P = (s, t)R$ and $Q = (u, v)R$. Then $J = PQ = P \cap Q$ is the radical of the ideal $I = (x_1, x_2, x_3)R$, where $x_1 = su$, $x_2 = tv$, and $x_3 = sv - tu$. In fact, $(sv)^2 - x_3sv - x_1x_2 = 0$, from which it follows that sv is integral over I , and, consequently, tu is integral over I as well. It is clear that x_1, x_2, x_3 is *not* a latent regular sequence. To see this, suppose it were regular on an R -algebra S . Then either PS or QS must be a proper ideal of S or else $PQS = S$ and this forces $IS = S$. But, even in the non-Noetherian case, a proper ideal generated by two elements cannot contain a regular sequence of length 3 ([Nor], Theorem 13, p. 150).

We next note that by the Mayer-Vietoris sequence for local cohomology, we have an exact sequence:

$$\cdots \rightarrow H_P^3(R) \oplus H_Q^3(R) \rightarrow H_I^3(R) \rightarrow H_{P+Q}^4(R) \rightarrow H_P^3(R) \oplus H_Q^3(R) \rightarrow \cdots$$

Since $H_P^i(R) = H_Q^i(R) = 0$ for $i > 2$, this gives an isomorphism $H_I^3(R) \cong H_m^4(R)$ where $m = P + Q$ is the maximal ideal of R . Thus, $H_I^3(R) \neq 0$.

Of considerable interest here is that we have not been able to determine whether x_1, x_2, x_3 is a Q-sequence in any characteristic! We can show that $\mathcal{L}_I(R/I_n)$ lies between cn^2 and n^3 where c is a positive constant, but have not been able to get finer information. In positive characteristic, we know that x_1, x_2, x_3 is a Q-sequence if and only if $\mathfrak{h}_{\underline{x}}^3(R) > 0$, which is equivalent to the condition that $\mathcal{L}_I(R/I_t) > c'n^3$ for some positive constant c' and all n , by Theorem 3.9. But we have not been able to prove any such lower bound. Note that if we map $R \rightarrow K[s, t]$ as a $K[s, t]$ -algebra by sending $u \mapsto 1$ and $v \mapsto 1$, \underline{x} specializes to $s, t, s - t$. Since R/I maps to K . This shows that $\mathcal{L}_I(R/I_t)$ is at least the K -vector space dimension of $K[s, t]/(s^n, t^n, (s - t)^n)$. Clearly, this is at least the dimension of $K[s, t]/(s, t)^n$, which is $\binom{n+1}{2}$.

Likewise, we cannot determine whether $\mathcal{L}_J(R/J^n)$ is bounded below by cn^3 . Because $J^2 \subseteq I \subseteq J$, this would yield the corresponding fact for I . We feel that it is striking that it is very hard to calculate quasilenlength even in quite simple examples involving a monomial ideal in a polynomial ring in a small number of variables.

If it turns out that x_1, x_2, x_3 is a Q-sequence, it would show that Q-sequences are not necessarily latent regular sequences. If x_1, x_2, x_3 is *not* a Q-sequence in characteristic p for some $p > 0$, it would provide an example of a nonzero local cohomology module that has content 0. Both possibilities are of interest.

4. THE CASE OF A SYSTEM OF PARAMETERS

Our main objective in this section is to prove the following:

Theorem 4.1. *Let x_1, \dots, x_d be elements of a Noetherian ring R . If the height of the ideal $I = (x_1, \dots, x_d)R$ is d , or if R maps to a Noetherian ring S such that the height of $(x_1, \dots, x_d)S$ is d , then $\underline{h}_x^d(R) > 0$. Moreover, if we also know that R contains a field, then $\underline{h}_x^d(R) = \underline{h}_x^d(R) = 1$.*

By Proposition 2.5, the statement for S is immediate if we can prove the statement for R . Moreover, we may localize at a minimal prime of I of height d , complete, and kill a minimal prime of the resulting complete local ring such that the dimension of the quotient is d . Hence, Theorem 4.1 reduces to the case where x_1, \dots, x_d is a system of parameters in a complete local domain of Krull dimension d .

We first want to prove that in every excellent reduced equidimensional local ring R with system of parameters x_1, \dots, x_d , we have $\underline{h}_x^d(R) = \underline{h}_x^d(R)$. We need some preliminary results.

Lemma 4.2. *Let (R, m, K) be a local ring of dimension $d \geq 1$. Let x_1, \dots, x_d be a system of parameters for R . Let $I = (x_1, \dots, x_d)R$, and for each $\underline{t} = t_1, \dots, t_d$ consisting of positive integers, let $I_{\underline{t}} = (x_1^{t_1}, \dots, x_d^{t_d})R$. Let M be any finitely generated R -module such that $\dim(M) < d$. Then*

(a)

$$\lim_{\underline{t} \rightarrow \infty} \frac{\lambda(M/I_{\underline{t}}M)}{t_1 \cdots t_d} = 0.$$

(b) *Suppose that c is part of a system of parameters for R , i.e., that $\dim(R/cR) = \dim(R) - 1$. Then*

$$\lim_{\underline{t} \rightarrow \infty} \frac{\lambda(\text{Ann}_{R/I_{\underline{t}}}c)}{t_1 \cdots t_d} = 0.$$

Proof. (a) We use induction on $\dim(R)$ and $\dim(M)$. If $\dim(M) = 0$ then the numerator is bounded and the result is clear. This also handles the case where $\dim(R) = 1$. Now suppose that $\dim(R) \geq 2$. If the result holds for all the factors in a finite filtration of M , then it holds for M : this comes down to the case of a filtration of length 2, say $0 \subseteq M_1 \subseteq M_2 = M$. Let $\overline{M} = M/M_1$. The result follows from the exactness of

$$0 \rightarrow M_1/(I_{\underline{t}}M \cap M_1) \rightarrow M/I_{\underline{t}}M \rightarrow \overline{M}/I_{\underline{t}}\overline{M} \rightarrow 0$$

and the fact that we have a surjection $M_1/I_{\underline{t}}M_1 \twoheadrightarrow M_1/(I_{\underline{t}}M \cap M_1)$. Thus, there is no loss of generality in assuming that M is a prime cyclic module. Since we may also assume that it has positive dimension, it follows that at least one x_i , say x_1 , is a nonzerodivisor on M . Let $M' = M/x_1M$. Then $M/I_{\underline{t}}M$ has a filtration by t_1 homomorphic images of $M'/(x_2^{t_2}, \dots, x_d^{t_d})M'$ (whether x_1 is a nonzerodivisor or not), and so

$$\frac{\lambda(M/I_{\underline{t}}M)}{t_1 \cdots t_d} \leq \frac{\lambda(M'/(x_2^{t_2}, \dots, x_d^{t_d})M')}{t_2 \cdots t_d}.$$

The result now follows from the induction hypothesis applied to the ring R/x_1R , the system of parameters consisting of the images of x_2, \dots, x_d in this ring, and the module M' .

(b) Let $S = R/I_{\underline{t}}$ and $J = \text{Ann}_S c$. Then $S/J \cong cS$ and so $\lambda(J) = \lambda(S) - \lambda(cS) = \lambda(S/cS)$. Thus, it suffices to show that

$$\lim_{\underline{t} \rightarrow \infty} \frac{\lambda(R/(I_{\underline{t}} + cR))}{t_1 \cdots t_d} = 0.$$

This is part (a) applied to R/cR . \square

The following result follows at once from Lemma 3.2 on p. 61 of [HH4].

Lemma 4.3. *Let R be an excellent equidimensional reduced local ring and let $c_0 \in R$ be any element such that R_{c_0} is Cohen-Macaulay. Then c_0 has a power c such that for every system of parameters x_1, \dots, x_d for R and for all $k, 0 \leq k \leq d-1$,*

$$c((x_1, \dots, x_k)R :_R x_{k+1}) \subseteq (x_1, \dots, x_k)R$$

and c kills the Koszul homology $H_i(x_1, \dots, x_k; R)$ for all $i \geq 1$. Moreover, such an element c may always be chosen to be part of a system of parameters for R .

Note that the final statement follows because the localization of R at its minimal primes is Cohen-Macaulay, and the Cohen-Macaulay locus is open in an excellent ring, so that there exists an element c_0 not in any minimal prime such that R_{c_0} is a Cohen-Macaulay ring.

Lemma 4.4. *Let R be an excellent equidimensional reduced local ring of Krull dimension d and let c be chosen as in Lemma 4.2. Let x_1, \dots, x_d be any system of parameters, let $\underline{t} = t_1, \dots, t_d$ be positive integers and let $I_{\underline{t}} = (x_1^{t_1}, \dots, x_d^{t_d})$. Then $c^d I_{\underline{t}}^{\text{lim}} \subseteq I_{\underline{t}}$.*

Proof. Suppose that $u \in I_{\underline{t}}^{\text{lim}}$. Then we have that

$$x_1^{k_1} \cdots x_d^{k_d} u \in (x_1^{t_1+k_1}, \dots, x_d^{t_d+k_d})R,$$

where the $k_i \in \mathbb{N}$. Let h be the number of k_i that are positive. It suffices to show that $c^h u \in I_{\underline{t}}$. This reduces at once to the case where there is only one positive value of k , say k_1 (systems of parameters are permutable), for then we obtain that

$$x_2^{k_2} \cdots x_d^{k_d} cu \in (x_1^{t_1}, x_2^{t_2+k_2}, \dots, x_d^{t_d+k_d})R,$$

and the result follows by induction on h . But if

$$x_1^{k_1} u = r_1 x^{t_1+k_1} + \sum_{j=2}^d r_j x_j^{t_j}$$

then

$$x_1^{k_1} (u - r_1 x^{t_1}) \in (x_2^{t_2}, \dots, x_d^{t_d})R$$

and we have that

$$c(u - r_1 x^{t_1}) \in (x_2^{t_2}, \dots, x_d^{t_d})R$$

from which $cu \in I_{\underline{t}}$ follows at once. \square

Note that in characteristic p , instead of c^d as above, we may use a test element for tight closure. In fact, it is the development of tight closure (cf. [HH1–3], [Hu] for background) that inspired this argument.

Theorem 4.5. *Let (R, m, K) be an excellent reduced equidimensional local ring of Krull dimension d and let x_1, \dots, x_d be a system of parameters. Then $\underline{\mathfrak{h}}_{\underline{x}}^d(R) = \mathfrak{h}_{\underline{x}}^d(R)$.*

Proof. It will suffice to show that

$$\lim_{\underline{t} \rightarrow \infty} \frac{\mathcal{L}_I(R/I_{\underline{t}}^{\text{lim}}) - \mathcal{L}_I(R/I_{\underline{t}})}{t_1 \cdots t_d} = 0.$$

From the short exact sequence

$$0 \rightarrow I_{\underline{t}}^{\text{lim}}/I_{\underline{t}} \rightarrow R/I_{\underline{t}} \rightarrow R/I_{\underline{t}}^{\text{lim}} \rightarrow 0$$

we have that

$$\mathcal{L}_I(R/I_{\underline{t}}^{\text{lim}}) \leq \mathcal{L}_I(R/I_{\underline{t}}) \leq \mathcal{L}_I(R/I_{\underline{t}}^{\text{lim}}) + \mathcal{L}_I(I_{\underline{t}}^{\text{lim}}/I_{\underline{t}})$$

and so the difference in the numerator is bounded by

$$\mathcal{L}_I(I_{\underline{t}}^{\text{lim}}/I_{\underline{t}}) \leq \lambda(I_{\underline{t}}^{\text{lim}}/I_{\underline{t}}).$$

Hence, it suffices to show that

$$\lim_{\underline{t} \rightarrow \infty} \frac{\lambda(I_{\underline{t}}^{\text{lim}}/I_{\underline{t}})}{t_1 \cdots t_d} = 0.$$

Choose a parameter c for R as in Lemma 4.2. By Lemma 4.3, $I_{\underline{t}}^{\text{lim}}/I_{\underline{t}} \subseteq \text{Ann}_{R/I_{\underline{t}}} c^d$ for all \underline{t} , and the result now follows from Lemma 4.2(b). \square

Recall that in any local ring (R, m, K) , if $\underline{x} = x_1, \dots, x_d$ is a system of parameters then Lech's Theorem asserts that

$$\lim_{\underline{t} \rightarrow \infty} \frac{\lambda(R/I_{\underline{t}})}{t_1 \cdots t_d} = \mu,$$

where μ is the multiplicity of the system of parameters \underline{x} . See [Le].

Theorems 4.6 and 4.7 below complete the proof of Theorem 4.1.

Theorem 4.6. *Let R be an equidimensional reduced local ring and x_1, \dots, x_d a system of parameters for R . Then $\underline{h}_{\underline{x}}^d(R) = \mathfrak{h}_{\underline{x}}^d(R) \geq \mu/\lambda(R/I)$, where μ is the multiplicity of the system of parameters x_1, \dots, x_d . If $\dim(R) \leq 2$, $\underline{h}_{\underline{x}}^d(R) = \mathfrak{h}_{\underline{x}}^d(R) = 1$.*

Proof. Since $R/I_{\underline{t}}$ has a filtration by $\mathcal{L}_I(R/I_{\underline{t}})$ cyclic modules each of which is a homomorphic image of R/I , we have that $\lambda(R/I_{\underline{t}}) \leq \mathcal{L}_I(R/I_{\underline{t}})\lambda(R/I)$, and so

$$\frac{\mathcal{L}_I(R/I_{\underline{t}})}{t^d} \geq \frac{\lambda(R/I_{\underline{t}})}{t^d} / \lambda(R/I).$$

Taking the limits of both sides as $t \rightarrow \infty$ yields the required result.

For the final statement it suffices to consider the case of a complete local domain, and we may replace this ring by its normalization, which is Cohen-Macaulay. The result now follows from Proposition 1.2(c). \square

We can now show that for any equicharacteristic local ring R and system of parameters x_1, \dots, x_d , $H_{(\underline{x})}^d(R)$ is robust for \underline{x} (see Definition 3.10).

Theorem 4.7. *For an equicharacteristic local ring of dimension d , if x_1, \dots, x_d is a system of parameters, then $\mathfrak{h}_{\underline{x}}^d(R) = \underline{h}_{\underline{x}}^d(R) = 1$. Equivalently, every system of parameters is a Q -sequence.*

Proof. We first consider the case where the ring contains a field of characteristic p . If there is a counterexample, we may map to a counterexample that is a complete local domain. Then $\underline{h}_{\underline{x}}^d(R) = \mathfrak{h}_{\underline{x}}^d(R)$, and it suffices to show that $\mathfrak{h}_{\underline{x}}^d(R) = 1$. By Theorem 4.6, $\mathfrak{h}_{\underline{x}}^d(R) > 0$, and then Theorem 3.9 implies that $\mathfrak{h}_{\underline{x}}^d(R) = 1$.

We give a second proof for the characteristic $p > 0$ case. Again, we complete, and so we may assume that R is a module-finite extension of a complete regular local ring A which has x_1, \dots, x_d are a regular system of parameters. We know that $\mathfrak{h}_{\underline{x}}^d(A) = 1$, since

A is a regular and, hence, Cohen-Macaulay Noetherian ring. Moreover, it is known that $A \rightarrow R$ splits over A (see [Ho1], Theorem 2, p. 31), and, hence, $\underline{h}_x^d(R) > 0$ by Theorem 2.8. Again, $\underline{h}_x^d(R) = 1$ then follows from Theorem 3.9.

To show the result in equal characteristic zero, we make use of the fact that whether x_1, \dots, x_d is a Q-sequence is an equational condition, by Theorem 3.3. But the main result Theorem 5.2 of [Ho2] on reduction to characteristic $p > 0$ then implies the desired conclusion at once. \square

Remark. The argument in the second paragraph of the proof above generalizes as follows. Let $R \subseteq S$ be a module-finite extension of rings of positive prime characteristic p , and let $x_1, \dots, x_d \in R$. Suppose that $R \subseteq S$ is \underline{x} -split, i.e., that there is an R -linear map $S^h \rightarrow R$ whose image contains a power of every x_i , which holds if $R \rightarrow S$ splits as a map of R -modules. Then x_1, \dots, x_d is Q-sequence in R if and only if it is Q-sequence in S . This is immediate from Theorem 2.8 and Theorem 3.9. \square

Remark 4.8. If the direct summand conjecture fails, then for some local ring R we have a system of parameters x_1, \dots, x_d such that $x_1^{t+1} \cdots x_d^{t+1} \in I_t$. This yields a filtration of R/I_{t+1} with fewer than $(t+1)^d$ terms, which shows that $\underline{h}_x^d(R) < 1$ in R . Hence, the conjecture that $\underline{h}_x^d(R) = \underline{h}_x^d(R) = 1$ for every system of parameters of every local ring R implies the direct summand conjecture.

The following completes the proof of Theorem 4.1.

Corollary 4.9. *Let $x_1, \dots, x_d \in R$, where R is a Noetherian ring containing a field. A necessary condition for x_1, \dots, x_d to map to elements generating an ideal of height d in some Noetherian ring S is that $\underline{h}_x^d(R) > 0$. If R contains a field, it is necessary that $\underline{h}_x^d(R) = 1$.*

Proof. We may replace S by its localization at a minimal prime of $(\underline{x})S$ of height d . The result is now immediate from Proposition 2.5 and Theorems 4.6 and 4.7. \square

Remark 4.10. Consider the ring $R = R_{d,t}$ defined in Question 0.1 of the Introduction. (In mixed characteristic p , one may alternatively replace R by $R/(X_1 - p)$.) If one could prove that $\underline{h}_x^d(R) = 0$, then Corollary 4.9 shows that the images of the X_i in R cannot map to a system of parameters in a local ring. This establishes the monomial conjecture and, hence, the direct summand conjecture. (In mixed characteristic, if one uses $R/(X_1 - p)$ it establishes the monomial conjecture in mixed characteristic for systems of parameters containing p , but Theorem 6.1 of [Ho4] implies that this suffices for the general case.) By Corollaries 6.10 and 6.11 of [Ho4], we have that $H_I^d(R) = 0$ in characteristic p , and also if $d = 2$ in all characteristics, so that $\underline{h}_x^d(R) = 0$ in those cases. If $d \geq 3$, we do not know whether $\underline{h}_x^d(R) = 0$ in equal characteristic 0, nor in mixed characteristic p .

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