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IDEALS GENERATED BY QUADRATIC POLYNOMIALS

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Throughout this talk let R denote a polynomial ring over an arbitrary field K: say $R = K[x_1, \ldots, x_N]$. We will denote the projective dimension of the R-module M over R by $\mathrm{pd}(M)$. The following conjecture was posed by M. Stillman.

Conjecture 1. *There is an upper bound, independent of N, on pd(R/I) where I is any ideal of R generated by n homogeneous polynomials of given degrees $d_1, \ldots, d_n$.*

One motivation for proving this conjecture comes from its equivalence to the following open question (the proof of equivalence due to Caviglia):

Conjecture 2. *There is a bound on the Castelnuovo-Mumford regularity of an ideal in a polynomial ring that depends only on the number of its minimal generators and the degrees of those generators.*

Conjecture 1 clearly holds when $n \leq 2$ or when all of the $d_i = 1$, but even the simplest next case of $n = 3$ and $d_1 = d_2 = d_3 = 2$ requires non-trivial arguments. D. Eisenbud and C. Huneke proved that if I is generated by three quadratic forms, then $\mathrm{pd}(R/J) \leq 4$.
B. Engheta showed the existence of a bound on the projective dimension for three cubic forms; he has shown that $\mathrm{pd}(R/I) \leq 36$ (but in all known examples of three cubics $\mathrm{pd}(R/I) \leq 5$).We prove Conjecture 1 for the case when all of the d_i are at most 2 and n is arbitrary.

Since a base change of the field K to a larger field does not affect the projective dimension, we often pass to the case where K is infinite.

We use recursion on h to define a function $B(m, n, h)$, $m, n, h \geq 0$ with $h \leq n$, as follows. Let
$B(m, n, 0) = m + mn = m(n + 1)$.
If $h \geq 1$, let $B(m, n, h) = (m + h)(n^3 + n^2 + n + 1) + h(n + 1) + B\big((m + h)n^2, n, h - 1\big)$.
Let $C(s)$ be the largest value of $B(m, n, h) + h$ for $m + n = s$ and $0 \leq h \leq n-1$. Let $C_0(s)$ be the largest value of $B(0, n, h) + m + h$ for $m + n = s$ and $0 \leq h \leq n - 1$. Both $C_0(s)$ and $C(s)$ are asymptotic to $2s^{2s}$.

Main Theorem 1. *Let* $R = K[x_1, \ldots, x_N]$ *be a polynomial ring over a field* K*. Let* $F_1, \ldots, F_{m+n}$ *be forms of degree at most 2 generating ideal* I*, and suppose that the linear* F_i *span a* K*-vector space of dimension* m*. Let* h *be the height of the ideal generated by the images of those* F_j *of degree 2 in the ring obtained from* R *by killing the ideal generated by all of the* F_i *that are linear. Suppose that* K *is infinite. After a linear change of variables, there are at most* $b \le B(m, n, h)$ *variables* $\underline{y} = y_1, \ldots, y_b$ *and at most* $c \le h$ *quadratic forms* $\underline{G} = G_1, \ldots, G_c$ *in* I *such that*

(1) The elements $y_1, \ldots, y_b, G_1, \ldots, G_c$ *form a regular sequence in* R

(2) The F_i *are in the polynomial ring* $K[\underline{y}, \underline{G}]$.

Hence, if K *is infinite,* $F_1, \ldots, F_{m+n}$ *are in the* K*-subalgebra of* R *generated by a regular sequence of length at most* $B(m, n, h) + h$ *whose terms are linear and quadratic forms. It also follows that, if* K *is infinite, the elements* $F_1, \ldots, F_{m+n}$ *are contained in the* K*-subalgebra of* R *generated by a regular sequence of at most* $C(m + n)$ *forms of degree at most 2. Consequently, for every field* K*, the projective dimension of* R/I *is at most* $B(0, n, h) + m + h$*, and is also at most* $C_0(m + n)$.

Note that the last statement follows from the first because the projective dimension drops by m when we kill m variables in I and pass to the polynomial ring in the remaining variables, replacing the original F_j by the images of the quadratic F_j in the smaller polynomial ring.

Note also that if $h = n$ then there is a much better result, since in that case $F_1, \ldots, F_{m+n}$ is already a regular sequence.

Given polynomials of degree at most 2 that need not be forms, each is the sum of at most one quadratic form, one linear form, and a scalar that is already in K. Hence:

Main Theorem 2. *Let* $R = K[x_1, \ldots, x_N]$ *be a polynomial ring over a field* K*. Let* $F_1, \ldots, F_s$ *be polynomials of degree* ≤ 2*, and let* $I = (F_1, \ldots, F_s)$*. If* K *is infinite,* $F_1, \ldots, F_s$ *are in the* K*-subalgebra of* R *generated by a regular sequence of at most* $C(2s)$ *forms of degree* ≤ 2*. Consequently, for any field* K*, the projective dimension of* R/I *is at most* $C(2s)$*.* □

Finally, observe that these theorems hold even if we allow the ambient polynomial ring to have a set of variables of arbitrary infinite cardinality, because any given finite set of polynomials will only involve finitely many of the variables.

To prove these theorems, we will need some lemmas.

Lemma 1. *Let I be a homogeneous ideal in the polynomial ring R. If we take the image of I after killing some of the variables, its height does not increase.*

Lemma 2. *If $F_{r+1}, \ldots, F_{r+s}$ is a regular sequence modulo a set of variables containing the variables occurring in $F_1, \ldots, F_r$, then they form a regular sequence modulo the ideal $(F_1, \ldots, F_r)$.*

Proof. The variables killed generate a ring A, and we may form the quotient $B = A/(F_1, \ldots, F_r)A$. The polynomial ring C in the rest of the variables over B is flat over B. It suffices to prove the statement after replacing B, C by their localizations at their homogeneous maximal ideals.

But then the result follows from H. Matsumura, Commutative Algebra, Ch. 8, (20.F): If the image of a sequence in the closed fiber of a flat local extension is regular, so is the original sequence. □

A regular sequence of forms in a polynomial ring R over a field K can be extended to a homogeneous system of parameters $\underline{F} = F_1, \ldots, F_n$. Then R is module-finite and free over the polynomial ring $K[\underline{F}]$. Thus:

Lemma 3. *Let $F_1, \ldots, F_t$ be a regular sequence of forms in a polynomial ring R over a field K. Then R is free, hence, faithfully flat over $A = K[F_1, \ldots, F_t]$. Hence, for any ideal J of R whose generators lie in A, $pd(R/J) \le t$.* □

Lemma 4. *Let K be any field, let $\underline{y} = y_1, \ldots, y_r$, and $\underline{z} = z_1, \ldots, z_s$ be $r+s$ indeterminates over K, let $\alpha_1, \ldots, \alpha_n \in K[\underline{y}]$, and let $\beta_1, \ldots, \beta_n \in K[\underline{z}]$. Then we can map the polynomial ring $K[T_1, \ldots, T_n]$ onto $K[\alpha_1, \ldots, \alpha_n]$ as a K-algebra so that each $T_i \mapsto \alpha_i$. Call the kernel P. We can also map $K[T_1, \ldots, T_n]$ onto $K[\beta_1, \ldots, \beta_n]$ as a K-algebra so that each $T_i \mapsto \beta_i$. Suppose this kernel is also P. Finally, we can map $K[T_1, \ldots, T_n]$ onto $K[\alpha_1 + \beta_1, \ldots, \alpha_n + \beta_n] \subseteq K[\underline{y}, \underline{z}]$ as a K-algebra so that for every i, $T_i \mapsto \alpha_i + \beta_i$. Suppose the the kernel of this map contains P. Assume also that P is homogeneous (automatic if the α_i or the β_i are homogeneous of the same degree over K). Then P is generated by linear forms over K.*

The point is that the sum of two different generic solutions for the equations in the homogeneous prime is again a solution, which makes the corresponding algebraic set closed under addition and, hence, a vector space.

The basic step: putting the F_i and variables in standard form.

We are going to use the construction of putting the F_i and the variables in standard form described just below iteratively.

We say that the F_i and the variables x_j are in *standard form* if:

(1) $F_{n+i} = x_i$ for $1 \leq i \leq m$ and the other F_i are quadratic or 0.

(2) None of the F_i for $1 \leq i \leq n$ has a monomial term in $K[x_1, \ldots, x_m]$.

(3) $F_1, \ldots F_h, x_1, \ldots, x_m, x_{m+h+1}, \ldots, x_N$ is a regular sequence.

(4) There is a nonnegative integer $r \leq (m+h)n$ such that when the F_i are written as polynomials in the variables $x_1, \ldots, x_{h+m}$ with coefficients in the ring $K[x_{h+m+1}, \ldots, x_N]$, the coefficients occurring for the variables $x_1, \ldots, x_{h+m}$ are in the K-span of the variables $x_{h+m+1}, \ldots, x_{h+m+r}$.

(5) There is an integer $s \in \mathbb{N}$ with $s \leq (m+h)n^2$ such that when the F_i are written as polynomials in the variables $x_1, \ldots, x_{h+m+r}$ with coefficients in the ring $K[x_{h+m+r+1}, \ldots, x_N]$, the coefficients of $x_{h+m+1}, \ldots, x_{h+m+r}$ are in the K-span of t he variables $x_{h+m+r+1}, \ldots, x_{h+m+r+s}$.

(6) We call $x_{m+1}, \ldots, x_{m+h}$ *front variables*, and use the alternative notation $\underline{u} = u_1, \ldots, u_h$ for them. Let f_i be the image of F_i under the K-homomorphism $\pi : R \to K[u_1, \ldots, u_h]$ that kills all the variables except the front variables while fixing the front variables (note that π kills F_i for $i > n$). Then there is an integer d, $h \leq d \leq n$, such that $f_1, \ldots, f_d$ are linearly independent over K while $f_i = 0$ for $i > d$.

We call $f_1, \ldots, f_n$ the *front polynomials*.

Under the conditions (1) — (6) above we call $x_1, \ldots, x_m$ *leading variables*. We call $x_{h+m+1}, \ldots, x_{h+m+r}$ the *primary coefficient variables* and use the alternative notation $\underline{v} = v_1, \ldots, v_r$ for them. We shall refer to $x_{h+m+r+1}, \ldots, x_{h+m+r+s}$ as the *secondary coefficient variables* and use the alternative notation $\underline{w} = w_1, \ldots, w_s$ for them. We refer to $x_{h+m+r+1}, \ldots, x_N$ as the *tail variables*. We use the alternative notation $\underline{z} = z_1, \ldots, z_{N-(h+m+r)}$ for the tail variables.

The tail variables include the secondary coefficient variables $\underline{w}$.

Let τ denote the K-homomorphism from R to the polynomial ring in the tail variables that fixes the tail variables and kills the others. We write $g_i = \tau(F_i)$. Again, τ kills F_i for $i > n$. We call $g_1, \ldots, g_n$ the *tail polynomials*.

We can map the polynomial ring $K[T_1, \ldots, T_n]$ onto $K[f_1, \ldots, f_n]$ by $T_i \mapsto f_i$. The kernel is a prime ideal P in $K[T_1, \ldots, T_n]$, which we call the ideal of *front relations*.

We want to see that these conditions can be achieved. Note, however, that standard form is far from unique.

Achieving standard form.

We want to show that the sequence can be put in standard form without changing the ring it generates. If the F_i are linearly dependent over K, one or more F_i may be replaced by 0, and we assume that any 0 elements occur in the final spots among the first n terms of the sequence. Hence, we may assume that the linear forms that occur are linearly independent. We may assume that the F_i are numbered so that any linear forms occur as a final segment, and we may assume these forms are also an initial segment of the variables. Thus, we have $x_i = F_{n+i}$, $1 \leq i \leq m$.

We may now subtract from each F_i, $1 \leq i \leq n$, the sum of all terms that occur involving $x_1, \dots, x_m$.
Since K is infinite, after replacing $F_1, \dots, F_h$ by suitably general linear combinations of the quadratic forms $\underline{F} = F_1, \dots, F_n$, we may assume that the images of $F_1, \dots, F_h$ form a maximal regular sequence in $(\underline{F})\overline{R}$, where $\overline{R} = R/(x_1, \dots, x_m)R$. Note that for $i \leq n$, every F_i will be quadratic or else 0. We have that $F_1, \dots, F_h, x_1, \dots, x_m$ is a regular sequence, since regular sequences of forms are permutable. Since K is infinite we may extend $F_1, \dots, F_h, x_1, \dots, x_m$ to a homogeneous system of parameters using linear forms, which we may assume are the variables $x_{m+h+1}, \dots, x_N$.

That is, $F_1, \ldots, F_h, x_1, \ldots, x_m, x_{h+m+1}, \ldots, x_N$ form a homogeneous system of parameters for $K[x_1, \ldots, x_N]$. The images of $F_1, \ldots, F_h$ when we kill the variables in this regular sequence are precisely the elements $f_1, \ldots, f_h \in K[u_1, \ldots, u_h]$. Hence, the quadratic forms $f_1, \ldots, f_h$ are a homogeneous system of parameters for $K[u_1, \ldots, u_h]$. In particular, they are linearly independent over K. The remaining F_i may be permuted so that $f_1, \ldots, f_d$ is a K-vector space basis for the K-span of the f_i. For $i > d$ we may subtract a K-linear combination of $F_1, \ldots, F_d$ from F_i to arrange that f_i be zero.

When $F_1, \ldots, F_n$ are written as polynomials in the leading and front variables $x_1, \ldots, x_m, u_1, \ldots, u_h$ (with coefficients in the polynomial ring in the remaining variables) there are most $m + h$ linear terms: the coefficients are linear forms in $x_{m+h+1}, \ldots, x_N$. All these coefficients from all of the F_i span a K-vector space of dimension $r \leq n(m + h)$. We make a linear change of the variables $x_{m+h+1}, \ldots, x_N$ so that the variables $x_{m+h+1}, \ldots, x_{m+h+r}$ span this space. These are the primary coefficient variables, which are also denoted $v_1, \ldots, v_r$.

Next, when $F_1, \ldots, F_n$ are written as polynomials in the leading, front, and primary coefficient variables $x_1, \ldots, x_m, u_1, \ldots, u_h, v_1, \ldots, v_r$ (with coefficients in the polynomial ring in the remaining variables) there are most $n(m+h)$ terms that are linear involving one of the v_j: the coefficients are linear forms in $x_{m+h+r+1}, \ldots, x_N$. All these coefficients from all of the F_i span a K-vector space of dimension $s \leq n^2(m+h)$. We make a linear change of the variables $x_{m+h+1}, \ldots, x_N$ so that the variables $x_{m+h+r+1}, \ldots, x_{m+h+r+s}$ span this space. These are the secondary coefficient variables, which we also denote $w_1, \ldots, w_s$.

Finally, we refer to $x_{m+h+r+1}, \ldots, x_N$ as the *tail variables*. This set is the complement in $\{x_1, \ldots, x_N\}$ of the union of leading, front, and primary coefficient variables. We emphasize again that the tail variables include the secondary coefficient variables.

The conditions for standard form are now satisfied. Once we have standard form, it is unaffected by permuting $F_{h+1}, \ldots, F_d$, or replacing them with linear combinations that have the same K-span.

Crucial point: if this procedure is carried through a second time using the forms consisting of all leading, front, primary and secondary coefficient variables (these are the new leading variables) and the tails, thus producing a second set of tails, then one of two things happens:

(1) The second set of tails generates an ideal of height at most $h - 1$ modulo the ideal generated by the new leading variables. (It may also happen that the set of original tails generates an ideal of height at most $h-1$ modulo the ideal generated by the original leading variables, which is an easier case.)

(2) The leading, front, primary and secondary coefficient variables and the nonzero elements in first set of tails form a regular sequence.

Either condition yields an estimate of what is needed for $B(m, n, h)$. The first condition requires a larger value and leads to the recursive definition given earlier. It remains to explain why these are the only possibilities.

Key behavior of standard form and the proof of the main theorems

Placing $F_1, \ldots, F_{m+n}$ and the variables in standard form puts surprising constraints on the forms. In particular, parts (c), and (d) of the Key Lemma below play a central role in the proof of the main results.

Key Lemma. *Let $F_1, \ldots, F_{m+n}$ consist of quadratic and linear forms in $R = K[x_1, \ldots, x_N]$ which, together with the variables $x_1, \ldots, x_N$ are assumed to be in standard form. Let all notation and terminology, including m, n, h, r, s, d, f_i, g_i, u_i, v_i, w_i, P, π, and τ, be as above. Let $I = (F_1, \ldots, F_{m+n})R$. Then:*

(a) $f_1, \ldots, f_h$ *are a homogenous system of parameters for* $K[u_1, \ldots, u_h]$*, and since* $K[f_1, \ldots, f_h] \subseteq K[f_1, \ldots, f_d] = K[f_1, \ldots, f_n] \subseteq K[u_1, \ldots, u_h]$, *Krull dim* $K[f_1, \ldots, f_n] = h$, *and ht* $P = n - h$.

(b) *Every* F_i *uniquely has the form* $f_i + e_i + g_i$ *where* $e_i \in K[\underline{x}, \underline{u}, \underline{v}, \underline{w}] = B$ *with* $\underline{x} = x_1, \ldots, x_m$, *and is also in the ideal* $(\underline{x}, \underline{v})B$. *Hence,* $K[F_1, \ldots, F_{m+n}] \subseteq K[g_1, \ldots, g_n, \underline{x}, \underline{u}, \underline{v}, \underline{w}]$.

(c) *If* $H \in P$, *the* $H(g_1, \ldots, g_n) = 0$. *That is, the tail polynomials satisfy the front relations!*

(d) *If* $i > d$, *then* $g_i = 0$, *i.e.,* $F_i = e_i \in (\underline{x}, \underline{v}, \underline{w})R$ *and* $F_i \in K[\underline{x}, \underline{u}, \underline{v}, \underline{w}]$.

Proof. We already know (a). For (b), e_i is clearly the sum of all terms in F_i not involving only front variables nor only tail variables. From the condition (2) of the definition of standard form there are no terms involving only leading variables. The terms in e_i that involve some x_i or u_i in degree 1 have coefficients in the span of the v_i by the definition of primary coefficient variables. That is, $e_i = \sum_{j=1}^{m} L_j x_j + \sum_{k=1}^{h} L'_k u_k + e'_i$ where the L_j, L'_j are in the K-span of $v_1, \ldots, v_r$ and the terms in e'_i are quadratic in the v_t or linear in the v_t, and those that are linear in the v_t have coefficients in the K-span of $w_1, \ldots, w_s$ by the definition of secondary coefficient variables.

To prove (c), note that P is homogeneous, and we may assume H is homogeneous of degree k. Since $-f_i \equiv e_i + g_i \mod I$, we have $0 = (-1)^k H(f_1, \ldots, f_n) = H(-f_1, \ldots, -f_n) \equiv H(e_1 + g_1, \ldots, e_n + g_n) \mod I$. Therefore $H(e_1+g_1, \ldots, e_n+g_n) \in I$. If $H(g_1, \ldots, g_n) \neq 0$, we get a contradiction by showing that $F_1, \ldots, F_h, H(e_1 + g_1, \ldots, e_n + g_n)$ is a regular sequence of length $h+1$ in $R/(x_1, \ldots, x_m)$. To this end, it suffices to show that we have a regular sequence modulo $(\underline{x}, \underline{v})$. Killing the v_j kills the e_j by (b) above, and so it suffices to show that $f_1+g_1, \ldots, f_h+g_h, H(g_1, \ldots, g_n)$ is a regular sequence in the polynomial ring in the front and tail variables.

This is immediate from Lemma 2: since regular sequences of forms are permutable, it suffices to show $H(g_1, \ldots, g_n)$, $f_1+g_1, \ldots, f_h+g_h$ is a regular sequence. Since the first element is nonzero, it is a nonzerodivisor involving only the tail variables, and mod the tail variables the remaining terms become the regular sequence $f_1, \ldots, f_h$.

Finally, (d) follows from (c) and the fact that $f_i = \pi(F_i) = 0$ for $i > d$: we may take $H = T_i$, and this gives that $g_i = 0$. The remaining statements now follow from (b) above. $\square$

Proof of Main Theorem 1. If $h = 0$, then $F_1, \ldots, F_n$ are expressible as sums of multiples by linear forms of $x_1, \ldots, x_m$. All the multipliers together span a vector space of dimension at most mn, and so all of $F_1, \ldots, F_n$ can be expressed in terms of $m + mn$ variables, which include the variables $x_1, \ldots, x_m$.

We now assume $h \geq 1$ and use induction on h. We may assume standard form. We consider two cases, (1) and (2). Case (2) has subcases (2a) and (2b). Note that Case (1) is, in a sense, subsumed in Case (2a). Let h' be the height of the ideal generated by the tail polynomials modulo the secondary coefficient variables. Of course, $h' \leq h$.

Case 1: $h' \le h - 1$. By the induction hypothesis, we may work in $K[\underline{z}]$ with the sequence of tail polynomials and secondary coefficient variables $\underline{w}$ to select at most $k \le B\big(n^2(m+h), n, h-1\big)$ tail variables $z_{j_1}, \ldots, z_{j_k}$, such that $K[g_1, \ldots, g_n, w_1, \ldots, w_s]$ is contained in the K-algebra generated by these z_{j_ν} and a sequence of at most $h - 1$ distinct quadratic forms that, together with the z_{j_ν}, form a regular sequence. When we include the $\underline{x}$, $\underline{u}$, $\underline{v}$ (which are disjoint from the tail variables), we still have a regular sequence, and by part (b) of the Key Lemma, the algebra these generate contains $K[F_1, \ldots, F_{m+n}]$.

In this case the constraint placed on $B(m,n,h)$ is that it be at least

$$m + h + n(m+h) + B\big(n^2(m+h), n, h-1\big).$$

Case 2: $h' = h$. Let $S = K[\underline{z}]$ (tail variables). variables. We work with sequence $g_1, \ldots, g_n, w_1, \ldots, w_s$ in S. We may now put these in standard form. We need not change $g_1, \ldots, g_h$: these already form a regular sequence of length h mod the secondary coefficient variables. (So that an initial segment of the new front polynomials will be independent over K while the rest are 0, we may adjust $F_{h+1}, \ldots, F_d$, replacing them by K-linear combinations with the same span.)

The reason is that there is a homomorphism of K-algebras $K[f_1, \ldots, f_n] \to K[g_1, \ldots, g_n]$ sending $f_i \mapsto g_i$ for all i. This is well-defined precisely because of part (d) of the Key Lemma: the g_i satisfy all the relations in P on the f_i. It is clearly surjective. it follows that $K[g_1, \ldots, g_n]$ is module-finite over $K[g_1, \ldots, g_h]$. But $K[g_1, \ldots, g_n]$ maps onto the new ring of front polynomials, which will have Krull dimension h. Thus, $K[g_1, \ldots, g_h]$ must have dimension h, and no less. This implies that $g_1, \ldots, g_h$ is a regular sequence, even mod the secondary coefficient variables.

Thus, the prime ideal of relations on $g_1, \ldots, g_n$ must be exactly P. The same holds for the new ideal of front relations on the front polynomials of the g_i. We let $\alpha_1, \ldots, \alpha_n$ denote the new front polynomials let $\beta_1, \ldots, \beta_n$ denote the new tail polynomials. Let h'' denote the height of the ideal generated by the new tail polynomials in the ring generated over K by the new tail variables modulo the ideal generated by the new secondary coefficient variables.

Subcase (2a): $h'' \leq h-1$**.** This is very much like Case (1). We have that $K[F_1, \ldots, F_{m+n}]$ is contained in the polynomial ring generated over K by the original leading, front, primary coefficient variables, secondary coefficient variables, and the original tail polynomials. But $K[w_1, \ldots, w_s, g_1, \ldots, g_n]$ is contained in turn in the K algebra generated by the secondary coefficient variables (which are also the new leading variables), the new front, new primary coefficient , and new secondary coefficient variables, and the new tails. We count variables as follows:

Type of variable or term	Cardinality
leading	m
front	h
primary extr.	$(m+h)n$
secondary extr. = new leading	$(m+h)n^2$
new front	h
new primary coeff.	$\big((m+h)n^2+h\big)n$
new secondary coeff. & new tails	$B\big((m+h)n^2, n, h-1\big)$ by the induction hyp.

In this case the constraint placed on $B(m, n, h)$ is that it be at least

$$(m+h)(n^3+n^2+n+1)+h(n+1)+B\big((m+h)n^2, n, h-1\big).$$

Since this is larger than what was needed in Case (1), we no longer need to consider Case (1). This is the formula used in the recursive definition of $B(m, n, h)$.

Subcase (2b): $h'' = h$. Now we show that $d = h$. First note that the relations on $g_1, \ldots, g_n$ are given exactly by P, and because $h' = h$ we know the relations on $\alpha_1, \ldots, \alpha_n$ are given exactly by P as well. But the elements $\beta_1, \ldots, \beta_n$ are homomorphic images of the elements $g_1, \ldots, g_n$, and so satisfy P, and so are $\alpha_1 + \beta_1, \ldots, \alpha_n + \beta_n$, which must also satisfy P.

Moreover, since $h'' = h$, the algebra generated by the elements $\beta_1, \ldots, \beta_n$ has dimension h, and P gives the ideal of all relations on the elements $\beta_1, \ldots, \beta_n$. By Lemma 4, P is generated by linear forms. Since standard form was set up so that $f_1, \ldots, f_d$ are linearly independent but the f_i for $i > h$ are integral over $K[f_1, \ldots, f_h]$, we must have that $f_i = 0$ for $i > h$, i.e., $d = h$, as claimed. Hence, the only nonzero tail polynomials are $g_1, \ldots, g_h$, which form a regular sequence modulo $x_1, \ldots, x_m$, $\underline{u}$, $\underline{v}$, $\underline{w}$. Then $K[F_1, \ldots, F_{m+n}] \subseteq K[g_1, \ldots, g_h, \underline{x}, \underline{u}, \underline{v}, \underline{w}]$, and the only constraint on $B(m, n, h)$ is that it be at least $(m + h)(1 + n + n^2)$. Thus, the formula obtained in Subcase (2a) will cover all cases. $\square$

The question in higher degree

Question: Is there $C(n,d) \in \mathbb{Z}_+$ such that given n forms $F_1, \ldots, F_n$ of degree at most d in a polynomial ring R over an infinite field K, there exists a regular sequence of forms $G_1, \ldots, G_k \in R$ of degree at most d with $k \le C(n,d)$ such that $F_1, \ldots, F_n \in K[G_1, \ldots, G_k]$? Our main result is the case $d = 2$. An affirmative answer would yield results analogous to our main theorems for arbitrary d: $C(n,d)$ would bound the projective dimension of $R/(F_1, \ldots, F_n)$ over any field, and $C(dn,d)$ would give corresponding results when the F_i are polynomials of degree at most d, not required to be homogeneous.