

UNIVERSAL LEX IDEAL APPROXIMATIONS OF EXTENDED HILBERT FUNCTIONS AND HAMILTON NUMBERS

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ABSTRACT. Let $R^{(h)}$ denote the polynomial ring in variables x_1, \dots, x_h over a specified field K . We consider all of these rings simultaneously, and in each use lexicographic (lex) monomial order with $x_1 > \dots > x_h$. Given a fixed homogeneous ideal I in $R^{(h)}$, for each d there is unique lex ideal generated in degree at most d whose Hilbert function agrees with the Hilbert function of I up to degree d . When we consider $IR^{(N)}$ for $N \geq h$, the set $\mathfrak{B}_d(I, N)$ of minimal generators for this lex ideal in degree at most d may change, but $\mathfrak{B}_d(I, N)$ is constant for all $N \gg 0$. We let $\mathfrak{B}_d(I)$ denote the set of generators one obtains for all $N \gg 0$, and we let $b_d = b_d(I)$ be its cardinality. The sequences b_1, \dots, b_d, \dots obtained in this way may grow very fast. Remarkably, even when $I = (x_1^2, x_2^2)$, one obtains a very interesting sequence, 0, 2, 3, 4, 6, 12, 924, 409620, \dots . This sequence is the same as $H_{d-1} + 1$ for $d \geq 2$, where H_d is the d th Hamilton number. The Hamilton numbers were studied by Hamilton and by Hammond and Sylvester because of their occurrence in a counting problem connected with the use of Tschirnhaus transformations in manipulating polynomial equations.

1. INTRODUCTION

There has been a great deal of work recently (see, for example, [1, 2, 3, 5, 7, 8, 9, 15, 16, 22] and their references) on the behavior of invariants of ideals generated in at most a given degree and with at most a given number of generators when the number of variables is not bounded in any way. The problems we study were motivated by a related question described below. Suppose we fix the Hilbert function of a homogeneous ideal I in a polynomial ring over a field K in several variables. An example would be to fix the Hilbert function of the ideal generated by x_1^2, x_2^2 . It is easy to see that the Hilbert functions of all the extensions of I to polynomial rings in more variables, where the number of variables N may be arbitrarily large, are determined by the original Hilbert function. We think of the Hilbert functions of I and its extensions as a model for the behavior of a large class of ideals. In our example, the Hilbert function of the ideal generated by a regular sequence consisting of any two quadratic forms, not necessarily x_1^2, x_2^2 , is obtained once the number of variables is sufficiently large.

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Throughout this paper, the terms “lexicographic” and “lex” are used interchangeably in describing both monomial orders and ideals. In dealing with lexicographic order, we always assume $x_1 > x_2 > \cdots > x_N$. A monomial ideal I will be called *lexicographic* or *lex* (some authors use the term *lexsegment*) if whenever $\mu' > \mu$ are monomials of degree d and $\mu \in I$ then $\mu' \in I$.

Suppose that an ideal I_N in $R^{(N)} := K[x_1, \dots, x_N]$ has the same Hilbert function as $IR^{(N)}$. When one calculates a Gröbner basis, with respect to any monomial order, for I_N , one obtains an initial ideal, a monomial ideal with the same Hilbert function. The length of the calculation depends, of course, on the choice of I , and can grow with N , but an upper bound can be predicted if one has an upper bound for the number of minimal generators of the initial ideal. By a result of Macaulay [20] the lex ideal with the same Hilbert function has the largest number of minimal generators of any homogeneous ideal with the same Betti numbers. This is generalized in [4, 13, 27], where it is shown that the lex ideal with a given Hilbert function has the largest graded Betti numbers of any homogeneous ideal with that Hilbert function. We were therefore led to explore this question: when one considers the extension of I to a polynomial ring in N variables, where $N \gg 0$, how many generators are needed in each degree d for the lex ideal with the same Hilbert function as the extension of I ? The existence of such a lex ideal was shown by Macaulay [20]. It turns out that for any specific degree, the number of minimal generators needed in that degree is constant for all $N \gg 0$. See Discussion 3.6. For other background in this area see [14, 17, 23, 24, 25, 26, 28, 29, 30]. A general treatment of monomial ideals is given in [12].

Our results give bounds on the lengths of Gröbner basis calculations, for any monomial order, for ideals of a certain “shape,” where the shape is described in terms of an ideal in a relatively small number of variables. Fix the field K . We shall say that a finitely generated graded module M_1 over $R^{(h_1)}$ has the same *shape* as a finitely generated graded module M_2 over $R^{(h_2)}$ if M_1 has the same graded Betti numbers as M_2 . If we extend M_1 to a polynomial ring in more variables by a degree-preserving map, the shape is preserved. The shape, together with the number of variables, determines the Hilbert function.

We illustrate this idea by focusing on the case of a regular sequence of quadratic forms of length two, which is an example of a simple shape, coming from the ideal $(x_1^2, x_2^2) \subseteq K[x_1, x_2]$. For all regular sequences of quadratic forms of length 2 generating an ideal I , the modules R/I have the same graded Betti numbers.

Other examples are a regular sequence of length h of forms of specified degrees d_1, \dots, d_h or $t \times t$ minors of an $r \times s$ matrix of forms F_{ij} of degrees d_{ij} such that the t size minors are homogeneous and the height of the ideal is the same as in the generic case: in this case, the shape may vary with the characteristic of the base field (cf. [11]).

In the case of a regular sequence of quadratic forms of length two, suppose that we have such a sequence in the polynomial ring in N variables, and we want to bound the length of a Gröbner basis calculation, with respect to some monomial basis, for the ideal they generate. Such a bound can be determined from the least number of generators of the initial ideal one obtains, and the number of generators is largest for a lex ideal whose Hilbert function is the same as the one obtained from the two quadratic forms, by Macaulay’s result or the results of [4, 13, 27] cited above. Similarly, in general, the numbers of generators of the various lex

ideals with the same Hilbert function, once the ideal is extended to $R^{(N)}$, give information bounding the length of the Gröbner basis calculation for ideals of a given shape as the number of variables N grows.

In the case of our example of a regular sequence consisting of two quadratic forms, our detailed results are given in Theorems 5.1 and 5.2 in §5.

We now make all this precise, and describe in detail the main question that we study. Throughout this paper, let K be a fixed, specified, but arbitrary field K . Let I be a specified ideal in $R^{(h)}$. Observe that if we want to study homogeneous regular sequences F_1, \dots, F_n with respective degrees m_1, \dots, m_n , we may take $I = (x_i^{m_i} : 1 \leq i \leq n)R^{(h)}$. For all $N \geq h$, let $\mathcal{L}_{I,N}$ denote the unique lex ideal with the same Hilbert function as $IR^{(N)}$. In Discussion 3.6, we shall see that the set $\mathfrak{A}_d(I, N)$ (respectively, $\mathfrak{B}_d(I, N)$) of minimal generators of $\mathcal{L}_{I,N}$ that have degree d (respectively, degree at most d) is constant for all $N \gg 0$, and we let $\mathfrak{A}_d(I)$ (respectively, $\mathfrak{B}_d(I)$) denote the stable value. Let $a_d(I)$ and $b_d(I)$ denote the respective cardinalities of the sets $\mathfrak{A}_d(I)$ and $\mathfrak{B}_d(I)$. Clearly, $b_d(I)$ is the sum of the values $a_j(I)$ for $j \leq d$. Our primary objective is the study of the sequences of numbers $a_d(I)$ and $b_d(I)$. One of our main results is a recursive formula for b_{d+1} in the general case: see Theorem 3.7.

When I is generated by a regular sequence consisting of two quadratic forms, for $d \geq 2$ the sequence $b_d(I)$ coincides with the sequence $H_{d-1} + 1$, where H_d is d th Hamilton number, studied by Hamilton and others in a completely different context: see §4. These numbers have double exponential growth (cf. Proposition 4.1(f)).

When a lex ideal is extended to a polynomial ring in more variables, it may fail to be lexicographic. For example, (x_1^2, x_1x_2, x_2^2) is lexicographic in $R^{(2)}$ but not in $R^{(3)}$, because in the larger ring $x_1x_3 > x_2^2$. We shall say that an ideal of $R^{(h)}$ is a *universal lex* ideal if its extension to $R^{(N)}$ is lexicographic for all $N \geq h$. The usage of *universal* is the same as in [12]. This notion will be quite useful in understanding the behavior of $a_d(I)$ and $b_d(I)$. Universal lex ideals are studied in detail in §2.

Notation 1.1. We need two kinds of notation connected with binomial coefficients. We use $\binom{n}{r}$ with its standard meaning for $n \geq r \geq 0$, and it is defined to be zero if

n or r is negative or $r > n$. If $r \geq 0$, we write $\langle \frac{n}{r} \rangle$ for $\frac{1}{r!} \prod_{j=0}^{r-1} (n-j)$ for all $n \in \mathbb{Z}$,

where the product is 1 if $r = 0$. For fixed $r \geq 0$, this is the unique polynomial in n of degree r that agrees with $\binom{n}{r}$ for all $n \geq r$. In fact, it is easy to see that

$$\binom{n}{r} = \langle \frac{n}{r} \rangle \text{ whenever } n \geq 0.$$

Note that $\langle \frac{n}{0} \rangle = 1$ for all $n \in \mathbb{Z}$, that $\langle \frac{n}{r} \rangle = 0$ for $0 \leq n < r$, and that for $n > 0$,

$$\text{(Eq. 1.1)} \quad \langle \frac{-n}{r} \rangle = (-1)^r \binom{n+r-1}{r}.$$

Also note that for all $n \in \mathbb{Z}$ and all $r \geq 1$,

$$\text{(Eq. 1.2)} \quad \langle \frac{n}{r} \rangle = \langle \frac{n-1}{r} \rangle + \langle \frac{n-1}{r-1} \rangle,$$

since this well-known identity for binomial coefficients holds for $n \geq r$ and both sides are polynomials in n . We note that the usual binomial coefficient identities

$$\text{(Eq. 1.3) (i) } \binom{n}{r} = \binom{n-1}{r} + \binom{n-1}{r-1} \quad \text{and} \quad \text{(ii) } \binom{n}{r} - \binom{n-1}{r-1} = \binom{n}{r-1}$$

hold for all $n, r \geq 1$, even if r is large, so that a numerator may exceed a denominator.

We make frequent use of the fact that for $N > 0$, the Hilbert function of $R^{(N)}$ is $t \mapsto \binom{N+t-1}{N-1}$ for all $t \in \mathbb{Z}$. Note that this not correct when $N = 0$: there is one incorrect value, when $t = 0$, since $\binom{-1}{-1}$ is 0, not 1.

2. UNIVERSAL LEX IDEALS

We shall show that universal lexicographic ideals are very constrained.

Notation 2.1. Suppose that we have a universal lexicographic ideal and that it has minimal generators in degrees $d_1 < d_2 < \dots < d_h = d$ where $d_1 \geq 1$, and that the number of minimal generators in degree d_i is $\alpha_i \geq 1$. We shall let Γ denote the set of pairs $\{(d_j, \alpha_j)\}$. There is no constraint on the number of variables, and we may assume that N variables, where $N \gg 0$, are available. This information uniquely determines the generators and the ideal, since in choosing the next minimal generator to use in given degree, it must be the largest monomial of that degree that is not a multiple of any previously chosen monomials. Let $\mathfrak{L}_{\Gamma, N}$ denote this universal lexicographic ideal whenever N is sufficiently large. It will be convenient to define $\beta_0 := 0$ and $\beta_i := \alpha_1 + \alpha_2 + \dots + \alpha_i$, $1 \leq i \leq h$, so that Theorem 2.4 below gives a complete, explicit description of the set of minimal generators once Γ is specified.

Notation 2.2. For this same ideal and for $i \in \mathbb{N}$, we can define a_i and b_i in \mathbb{N} as follows. We let $a_i := \alpha_j$ if i is one of the d_j , and $a_i := 0$ otherwise. We define b_i as the sum of the α_j such that $d_j \leq i$, which is the same as the largest of the β_j for $j \leq i$. Note that b_i is also the sum of the a_j such that $j \leq i$. Thus, a_i is the number of minimal generators of the ideal in degree i , and b_i is the number of minimal generators of the ideal of degree at most i .

Since the quotient of a polynomial ring in M variables by an ideal generated by m variables is polynomial ring in $M - m$ variables, we have the following preliminary result:

Lemma 2.3. *The Hilbert function of the ideal generated by m variables in a polynomial ring in $M > m$ variables over a field is $t \mapsto \binom{M+t-1}{M-1} - \binom{M-m+t-1}{M-m-1}$. \square*

Theorem 2.4. *Consider the universal lexicographic ideal $\mathfrak{L}_{\Gamma, N}$ corresponding to Γ as described in Notation 2.1. Let $\mu_1 := x_1^{d_1-1}$ and, recursively, if $1 \leq j \leq h-1$, $\mu_{j+1} := \mu_j x_{\beta_j+1}^{d_{j+1}-d_j}$. Thus, $\mu_2 = x_1^{d_1-1} x_{\beta_1+1}^{d_2-d_1}$, and $\mu_h = x_1^{d_1-1} x_{\beta_1+1}^{d_2-d_1} \dots x_{\beta_{h-1}+1}^{d_h-d_{h-1}}$. Then, in degree d_j , the minimal generators of $\mathfrak{L}_{\Gamma, N}$ will be*

$$(\dagger_j) \quad \mu_j x_{\beta_{j-1}+1}, \mu_j x_{\beta_{j-1}+2}, \dots, \mu_j x_{\beta_{j-1}+i}, \dots, \mu_j x_{\beta_{j-1}+\alpha_j},$$

where the last term may also be written as $\mu_j x_{\beta_j}$.

We let \mathfrak{B}_Γ denote this set of generators. The number of generators of this ideal, is evidently β_h , which is also the total number of variables occurring in the generators. For any $N \geq \beta_h$, $\mathfrak{L}_{\Gamma, N}$ is the ideal generated by \mathfrak{B}_Γ in $R^{(N)}$.

For any fixed $N \geq \beta_h$, the set of monomials in $\mathfrak{L}_{\Gamma, N}$ that are multiples of a minimal generator of degree d_j but not of a minimal generator of degree d_i for $i < j$ is the same as the set of monomials in

$$\mu_j(x_{\beta_{j-1}+1}, x_{\beta_{j-1}+2}, \dots, x_{\beta_{j-1}+i}, \dots, x_{\beta_{j-1}+\alpha_j})S_j$$

where S_j is the polynomial ring generated over K by all of the consecutive variables $x_{\beta_{j-1}+1}, \dots, x_N$, so that S_j is a polynomial ring in $N - \beta_{j-1}$ variables over K .

Hence, for $N > \beta_h$, the Hilbert function of the ideal $\mathfrak{L}_{\Gamma, N}$ is

$$\begin{aligned} (\dagger) \quad t &\mapsto \sum_{j=1}^h \left(\binom{N - \beta_{j-1} + t - d_j}{N - \beta_{j-1} - 1} - \binom{N - \beta_j + t - d_j}{N - \beta_j - 1} \right) \quad \text{or} \\ (\dagger') \quad t &\mapsto \sum_{j=1}^h \left(\binom{N - \beta_{j-1} + t - d_j}{t - d_j + 1} - \binom{N - \beta_j + t - d_j}{t - d_j + 1} \right). \end{aligned}$$

Before giving the proof, we make a remark and exhibit the first three cases without using the β notation.

Remark 2.5. Requiring that $N > \beta_h$ rather than $N \geq \beta_h$ is only needed because we need to make sure that the formula for the Hilbert function for the term subtracted is correct when $j = h$, $N = \beta_h$, and $t = d_j$. In this case the term subtracted should be 1, not 0.

In degree d_1 we must first have $x_1^{d_1}$, and then $x_1^{d_1-1}x_2, \dots, x_1^{d_1-1}x_{\alpha_1}$ as the minimal generators. In degree d_2 we must have

$$x_1^{d_1-1}x_{\alpha_1+1}^{d_2-d_1+1}, x_1^{d_1-1}x_{\alpha_1+1}^{d_2-d_1}x_{\alpha_1+2}, \dots, x_1^{d_1-1}x_{\alpha_1+1}^{d_2-d_1}x_{\alpha_1+\alpha_2}$$

as the new minimal generators.

In degree d_3 we must have

$$x_1^{d_1-1}x_{\alpha_1+1}^{d_2-d_1}x_{\alpha_1+\alpha_2+1}^{d_3-d_2+1}$$

as the largest new minimal generator and then

$$x_1^{d_1-1}x_{\alpha_1+1}^{d_2-d_1}x_{\alpha_1+\alpha_2+1}^{d_3-d_2}x_{\alpha_1+\alpha_2+2}, \dots, x_1^{d_1-1}x_{\alpha_1+1}^{d_2-d_1}x_{\alpha_1+\alpha_2+1}^{d_3-d_2}x_{\alpha_1+\alpha_2+\alpha_3}$$

as the other new minimal generators.

Proof. We need to prove that for all $N \geq \beta_h$, the universal lexicographic ideal $\mathfrak{L}_{\Gamma, N}$ is the same as the ideal generated by \mathfrak{B}_Γ . We shall prove all of the following statements simultaneously by induction on j .

- (1) Every monomial occurring in the set of generators of $\mathfrak{L}_{\Gamma, N}$ for $N \gg 0$ involves exactly one new variable, the next largest variable not already used.
- (2) All monomials occurring among the generators of $\mathfrak{L}_{\Gamma, N}$ for $N \gg 0$ in degree at least d_j are divisible by μ_j and, if a variable occurs in μ_j , it cannot occur in a monomial generator of total degree $> d_j$ with an exponent strictly larger than it has in μ_j .

- (3) The minimal generators of $\mathfrak{L}_{\Gamma, N}$ for $N \gg 0$ occurring in degree d_j are precisely the monomials displayed in (\dagger_j) , and the new variables,

$$x_{\beta_{j-1}+2}, \dots, x_{\beta_{j-1}+i}, \dots, x_{\beta_{j-1}+\alpha_j},$$

introduced in the monomials after the first do not occur in any other minimal generator of $\mathfrak{L}_{\Gamma, N}$ for $N \gg 0$.

From these statements, we will be able to show easily that \mathfrak{B}_{Γ} generates $\mathfrak{L}_{\Gamma, N}$ for all $N \gg 0$. It is clear that the specified generators are correct for degree d_1 , since the monomials specified are the largest of that degree no matter how many variables $N \gg 0$ there are. Assume the statements above and the specification of generators for $\mathfrak{L}_{\Gamma, N}$, where $N \gg 0$, are correct through degree d_{j-1} for some j with $2 \leq j \leq h$. We need to show the same for the monomials in degree d_j . No monomial in degree higher than d_{j-1} can occur with a bigger exponent on any of the variables in μ_{j-1} than it has in μ_{j-1} : if we increase the exponent on x_{β_i+1} we get a multiple of the largest minimal generator of degree d_i . On the other hand, all monomials chosen in degree d_j or higher degree must be multiples of μ_{j-1} . The reason is that for sufficiently large N , there are arbitrarily many multiples of μ_{j-1} in every degree: anything not a multiple of μ_{j-1} is smaller than these, or else a multiple of a generator of lower degree. No previously used variable can occur in the terms of degree d_j or higher: if that happened, it would produce a multiple of a chosen monomial of lower degree. The smallest monomial we might use comes from multiplying by a power of the next new variable. We cannot use that new variable again.

We now discuss the final statements of the theorem. We comment only on the formula (\dagger) . We have mutually disjoint contributions to the Hilbert function: since μ_j has degree $d_j - 1$, the contribution in degree t from multiples of elements of degree d_j not already in the ideal is the number of monomials of degree $t - (d_j - 1)$ in the ideal

$$(x_{\beta_{j-1}+1}, x_{\beta_{j-1}+2}, \dots, x_{\beta_{j-1}+i}, \dots, x_{\beta_{j-1}+\alpha_j})S_j.$$

Now apply Lemma 2.3 with $M = N - \beta_{j-1}$, $m = \alpha_j$, and t replaced by $t - (d_j - 1)$. Note that $M - m = N - \beta_j$. \square

Recall the definition of the a_i and b_j from Notation 2.2. Note that the formula (\dagger) in Theorem 2.4 Hilbert function is still correct if we insert additional degrees for which the value of the corresponding α_j is 0: each of the extra summands in the formula is the difference of two terms that are equal, since $\beta_j = \beta_{j-1}$. The terms of the original formula are recovered from the terms in the new formula where $b_j \neq b_{j-1}$.

Corollary 2.6. *Given a universal lex ideal \mathfrak{L} generated in degree at most d , with the b_j as above (i.e., b_j is the total number of minimal generators of \mathfrak{L} of degree at most j), we have that, for $N > b_d$, the Hilbert function of \mathfrak{L} is*

$$(\dagger\dagger) \quad t \mapsto \sum_{j=1}^d \left(\binom{N - b_{j-1} + t - j}{N - b_{j-1} - 1} - \binom{N - b_j + t - j}{N - b_j - 1} \right) \quad \text{or}$$

$$(\dagger\dagger') \quad t \mapsto \sum_{j=1}^d \left(\binom{N - b_{j-1} + t - j}{t - j + 1} - \binom{N - b_j + t - j}{t - j + 1} \right).$$

3. THE CASE OF HILBERT FUNCTIONS OF ARBITRARY IDEALS AND MODULES

Discussion 3.1. We fix a field K . Suppose that we are given a homogeneous ideal I in $R^{(h)}$ or a finitely generated \mathbb{Z} -graded module M over $R^{(h)}$. Let H denote the Hilbert function of I or M . The function H uniquely determines the Hilbert function when ideal I or M is extended to $K[x_1, \dots, x_N]$ for any $N \geq h$ (one forms the extension by tensoring either with $R^{(N)}$ over $R^{(h)}$ or, equivalently, with $K[x_{h+1}, \dots, x_N]$ over K): upon tensoring over K with a polynomial ring in $N - h$ variables we get a Hilbert function $H^{(N)}$ whose value on t is given by

$$H^{(N)}(t) = \sum_{0 \leq i \leq t} H(i) \binom{N - h + t - i - 1}{t - i}.$$

The second factor in each of the summands is the number of monomials of degree $t - i$ in $N - h$ variables.

As usual, we define twists $M(s)$ for \mathbb{Z} -graded modules M , where $s \in \mathbb{Z}$, by $[M(s)]_t := [M]_{t+s}$. Using a graded free resolution by finitely generated free $R^{(N)}$ -modules and degree preserving maps (each free module is a finite direct sum of twists $R^{(N)}(s)$, $s \in \mathbb{Z}$ of $R^{(N)}$), we may express the Hilbert function $H^{(N)}$ as a \mathbb{Z} -linear combination of binomial coefficient functions $\binom{N + t + s - 1}{t + s} = \binom{N + t + s - 1}{N - 1}$, and so the the Hilbert function $H^{(N)}$ of $R^{(N)} \otimes_{R^{(h)}} M$ described in the preceding paragraph can be written in the form

$$\sum_{s \in \mathbb{Z}} c_s \binom{N + t + s - 1}{t + s} = \sum_{s \in \mathbb{Z}} c_s \binom{N + t + s - 1}{N - 1}$$

where the $c_s \in \mathbb{Z}$ and all but finitely many of the c_s are 0. The c_s are uniquely determined by I or M : if G_\bullet is a finite free graded resolution of I or M over $R^{(h)}$, then c_s is the difference between the total number of occurrences of $R^{(h)}(s)$ as a summand of G_j with j even and the total number of occurrences of $R^{(h)}(s)$ as a summand of G_j with j odd, and is independent of the choice of the graded resolution G_\bullet . We refer to this function of N and t , which is defined for all $t \in \mathbb{Z}$ and all $N \geq h$, as the *extended Hilbert function* of I or M .

We shall say that a function of $N \geq 1, t \in \mathbb{Z}$ is an *LCBC function* (LCBC stands for “linear combination of binomial coefficients”) if it is a \mathbb{Z} -linear combination of the functions $\binom{N + t + s - 1}{t + s} = \binom{N + t + s - 1}{N - 1}$, where s varies in \mathbb{Z} . We shall say that a polynomial in $\mathbb{Q}[Z, t]$ is an *LCBC polynomial* if it a \mathbb{Z} -linear combination of the corresponding functions $\left\langle \binom{N + t + s - 1}{N - 1} \right\rangle$, which are polynomial in t . Thus, s indexes a \mathbb{Z} -basis for the LCBC functions (respectively, polynomials).

Remark 3.2. Given an LCBC function G with coefficients c_s , if we fix $t = d$, it agrees with a polynomial G_d in N for $N \gg 0$. To get $G_d(N)$ when G has coefficients c_s , in $\sum_{s \in \mathbb{Z}} c_s \binom{N + d + s - 1}{N - 1}$ we may replace $c_s \binom{N + d + s - 1}{N - 1}$ by $c_s \left\langle \binom{N + d + s - 1}{N - 1} \right\rangle$ if $s \geq -d$. However, if $s < -d$, the term $c_s \binom{N + d + s - 1}{N - 1}$

vanishes for all $N \gg 0$, and must be replaced by 0. If $s \geq -d$,

$$c_s \left\langle \begin{matrix} N + d + s - 1 \\ N - 1 \end{matrix} \right\rangle = c_s \left\langle \begin{matrix} N + d + s - 1 \\ d + s \end{matrix} \right\rangle,$$

and the constant term of this polynomial in N is 0 unless $s = -d$, in which case it is c_{-d} . Hence:

Proposition 3.3. *Let G be an extended Hilbert function as in Discussion 3.1 such that*

$$G(N, t) = \sum_{s \in \mathbb{Z}} c_s \binom{N + t + s - 1}{N - 1}.$$

If we fix $t = d$, the polynomial $G_d(N)$ that agrees with $G(N, d)$ for all $N \gg 0$ is

$$\sum_{s \geq -d} c_s \left\langle \begin{matrix} N + d + s - 1 \\ d + s \end{matrix} \right\rangle,$$

and the constant term $G_d(0)$ of $G_d(N)$ is c_{-d} . \square

Theorem 3.4. *The coefficients c_s that occur in the description of an LCBC function are uniquely determined by the values of the function for $N \gg 0$ and $t \gg 0$. Hence, with each LCBC function there is a uniquely associated LCBC polynomial.*

Moreover, if two LCBC functions agree for all $N \gg 0$ in a fixed degree $t = d$, then their difference is constant for all $N \gg 0$ when $t = d + 1$. In fact, suppose one function Q has coefficients c_s and the other Q' has coefficients c'_s . Then they agree for all $N \gg 0$ in degree d if and only if $c_s = c'_s$ for $s \geq -d$, in which case $Q(N, d + 1) - Q'(N, d + 1) = c_{-d-1} - c'_{-d-1}$ for all $N \gg 0$.

Proof. The result of the second paragraph implies the result of the first. We may work with $Q - Q'$ and 0. For $s < -d$ the function with $c_s - c'_s$ as coefficient in $Q - Q'$ has a negative denominator and vanishes. As s takes on the values $-d, -d + 1, -d + 2, \dots, -d + j, \dots$ $Q - Q'$ cannot vanish identically for large N unless all the coefficients $c_s - c'_s$ are 0. But then, when we substitute $d + 1$ for d , the only term that does not vanish is the one where $s = -d - 1$, and one gets the specified constant $c_{-(d+1)} - c'_{-(d+1)}$ as the value. \square

Remark 3.5. The function $\left\langle \begin{matrix} N + \gamma \\ N + \delta \end{matrix} \right\rangle$ with $\gamma \geq \delta$ (where γ, δ may be negative) equals $\left\langle \begin{matrix} N + \gamma \\ \gamma - \delta \end{matrix} \right\rangle$, and the constant term is $\left\langle \begin{matrix} \gamma \\ \gamma - \delta \end{matrix} \right\rangle$.

Discussion 3.6. For every $N \geq h$, there is a unique lex ideal $\mathcal{L}_{I,N}$ in $R^{(N)}$ that has the same Hilbert function as $IR^{(N)}$: see [20] or [23] for an expository version. We shall prove that for a fixed degree d , these lex ideals eventually all have the same set of minimal generators in degree at most d , and are universal lex. We shall let $\mathfrak{B}_{I,d}$ denote the set of generators in degree at most d . We prove this by induction on d . Suppose that we have constructed the ideal up to degree d , and we want to construct it in degree $d + 1$. We have a formula for the Hilbert function of $IR^{(N)}$ and a formula for the Hilbert function of the approximation up through degree d . As functions of N, t these agree in degree d for all $N \gg 0$. By the result above, the difference of their values in degree $d + 1$ is constant for all $N \gg 0$. Denote this constant a_{d+1} . If we take a new universal lex ideal with a_{d+1} new minimal

generators in degree $d + 1$, it will agree with the Hilbert function of $IR^{(N)}$ in all degrees $\leq d + 1$ for all $N \gg 0$. This will lead to a formula for a_{d+1} in terms of b_1, \dots, b_d and the coefficients in the LCBC that gives the Hilbert function of I . Then $b_{d+1} = b_d + a_{d+1}$.

We carry this out in detail. Suppose that I is a homogeneous ideal of $R^{(h)}$ whose extended Hilbert function has integer coefficients $c_s(I)$. Here, s varies in \mathbb{Z} but has only finitely many nonzero values. We develop recursive formulas for $b_d(I)$, which is the least number of generators of the universal lexicographic ideal whose Hilbert function agrees with that of I in degrees up to and including d . Thus $b_d = 0$ if d is strictly less than the degree of any minimal generator of I , and if d_0 is the least degree of a minimal generator of I , b_{d_0} is the number of minimal generators of I of degree d_0 .

We write $a_d(I) = b_d(I) - b_{d-1}(I)$. For the rest of this discussion, we omit I from the notation, and abbreviate $b_d = b_d(I)$ and $a_d = a_d(I)$. Suppose that we know b_j for $j \leq d$. We obtain the recursion as follows. First, we have that a_{d+1} is the difference between the value of the extended Hilbert function of I in degree $d + 1$ and the extended Hilbert function of the universal lex ideal we have already constructed with generators in degree at most d : moreover, we know that the difference is constant, independent of N , by Theorem 3.4. Second, we have a formula ($\dagger\dagger'$) for the extended Hilbert function of the universal lex ideal for $N \gg 0$ by Corollary 2.6. Hence, by Remark 3.2 and Proposition 3.3,

$$a_{d+1} = c_{-(d+1)} - \sum_{j=1}^d \left(\left\langle \begin{array}{c} N - b_{j-1} + (d+1) - j \\ d - j + 2 \end{array} \right\rangle - \left\langle \begin{array}{c} N - b_j + (d+1) - j \\ d - j + 2 \end{array} \right\rangle \right),$$

and the value of this expression is independent of N , since it is a polynomial in N and constant for $N \gg 0$.

By Remark 3.5, we may substitute $N = 0$ to get the constant value

$$a_{d+1} = c_{-(d+1)} - \sum_{j=1}^d \left(\left\langle \begin{array}{c} -b_{j-1} + d - j + 1 \\ d - j + 2 \end{array} \right\rangle - \left\langle \begin{array}{c} -b_j + d - j + 1 \\ d - j + 2 \end{array} \right\rangle \right).$$

For $1 \leq j \leq d - 1$, there are two terms in the summation involving b_j : one of these is the second term occurring for index j and the other is the first term occurring for index $j + 1$. These two terms may be combined using Equation 1.2:

$$(\#) \quad - \left\langle \begin{array}{c} -b_j + d - j + 1 \\ d - j + 2 \end{array} \right\rangle + \left\langle \begin{array}{c} -b_j + d - j \\ d - j + 1 \end{array} \right\rangle = - \left\langle \begin{array}{c} -b_j + d - j \\ d - j + 2 \end{array} \right\rangle.$$

Using Equation 1.1, this becomes

$$-(-1)^{d-j} \left\langle \begin{array}{c} b_j + 1 \\ d - j + 2 \end{array} \right\rangle.$$

This yields

$$a_{d+1} = c_{-d+1} - \left(\left\langle \begin{array}{c} -b_0 + d \\ d + 2 \end{array} \right\rangle - \left(\sum_{j=1}^{d-1} (-1)^{d-j} \left\langle \begin{array}{c} b_j + 1 \\ d - j + 2 \end{array} \right\rangle \right) - \left\langle \begin{array}{c} -b_d + 1 \\ 2 \end{array} \right\rangle \right).$$

By Equation 1.1, $\left\langle \begin{smallmatrix} -b_d + 1 \\ 2 \end{smallmatrix} \right\rangle = \left\langle \begin{smallmatrix} b_d \\ 2 \end{smallmatrix} \right\rangle$. Assuming that I is not the unit ideal, we have that $b_0 = 0$, and so $\left\langle \begin{smallmatrix} -b_0 + d \\ d + 2 \end{smallmatrix} \right\rangle = 0$. Hence:

$$a_{d+1} = c_{-d+1} + \left(\left\langle \begin{smallmatrix} b_d \\ 2 \end{smallmatrix} \right\rangle + \sum_{j=1}^{d-1} (-1)^{d-j} \left\langle \begin{smallmatrix} b_j + 1 \\ d - j + 2 \end{smallmatrix} \right\rangle \right).$$

By adding b_d , we obtain a formula for b_{d+1} .

Theorem 3.7. *Let I be a proper homogeneous ideal in $R^{(h)}$ and abbreviate $c_s = c_s(I)$, which is defined in Discussion 3.1, and $b_j = b_j(I)$. Then:*

$$b_{d+1} = c_{-(d+1)} + b_d + \left\langle \begin{smallmatrix} b_d \\ 2 \end{smallmatrix} \right\rangle + \sum_{j=1}^{d-1} (-1)^{d-j} \left\langle \begin{smallmatrix} b_j + 1 \\ d - j \end{smallmatrix} \right\rangle = c_{-(d+1)} + \sum_{j=1}^d (-1)^{d-j} \left\langle \begin{smallmatrix} b_j + 1 \\ d - j \end{smallmatrix} \right\rangle$$

□

4. HAMILTON NUMBERS

In this section we describe a sequence of integers studied by Hamilton and described in [18], [19], [21] and [31] that originally arose in studying the behavior of generalizations of the Tschirnhaus transformation.¹ For further background we refer the reader to [6] and [10] as well as to the papers already cited. Of course, the Hamilton numbers arise here in a completely different context.

We first give a self-contained treatment of the Hamilton numbers that includes all of the results we need. We then connect this treatment with the one given in [18], which has two small errors in it, one of which is noted in [19].

We begin by defining a sequence ℓ_n recursively for $n \geq 0$ by letting $\ell_0 = 3$ and using the recursion

$$(Eq. 4.1) \quad \ell_{n+1} = 1 + \binom{\ell_n}{2} - \binom{\ell_{n-1}}{3} + \cdots + (-1)^n \binom{\ell_0}{n+2}, \quad n \geq 0.$$

The general term on the right, after the initial 1, is $(-1)^j \binom{\ell_{n-j}}{j+2}$, where $0 \leq j \leq n$. Note that, typically, many of the rightmost terms vanish. We shall soon see that $\ell_n > 0$ for all $n \geq 0$, so that the formula just above is also correct if we replace $\binom{\ell_{n-j}}{j+2}$ by $\left\langle \begin{smallmatrix} \ell_{n-j} \\ j+2 \end{smallmatrix} \right\rangle$, since these agree when the numerator is positive.

We may then define H_n for $n \geq 1$ by the formula $H_n := \ell_{n-1} - 1$. We have:

¹Briefly, H_n is the least degree of an equation from which n consecutive terms after the term of highest degree can be eliminated by a sequence of transformations related to Tschirnhaus transformations without needing to solve an equation of degree $> n$ to determine the transformation.

n	ℓ_n	H_n
0	3	
1	4	2
2	6	3
3	12	5
4	48	11
5	924	47
6	409620	923
7	83763206256	409619
8	3508125906290858798172	83763206255
9	6153473687096578758448522809275077520433168	3508125906290858798171

For the curious, ℓ_{10} is 18932619208894981833333582059033329370801266249535902023330546944758507753065602135844 which is larger than the approximation $1.89 \cdot 10^{85}$.

The growth rate of ℓ_n is double exponential. More precisely:

Proposition 4.1. *We have the following:*

(a) For all $n \in \mathbb{N}$, $\frac{\ell_n^2}{3} \leq \ell_{n+1} \leq \frac{\ell_n^2}{2}$.

(b) If $M > L > k \geq 2$ are integers such that $M \geq \frac{1}{3}L^2$ then $\binom{M}{k} \geq \binom{L}{k+1}$.

(c) For all $n \in \mathbb{N}$, the binomial coefficient terms $\binom{\ell_{n-j}}{j+2}$ on the right hand side of the recursion Eq. 4.1 for ℓ_{n+1} are nonincreasing, $0 \leq j \leq n$.

(d) $\lim_{n \rightarrow \infty} \frac{\ell_{n+1}}{\ell_n^2} = \frac{1}{2}$.

(e) For all $n \geq 1$, $3 \cdot 2^{2^{n-1}} \leq \ell_{n+1}$, and for all $n \geq 0$, $\ell_{n+1} \leq 2 \cdot 2^{2^n}$.

(f) ℓ_n is asymptotic to $2 \cdot 2^{\rho 2^n}$ where

$$\rho = 0.2756687129668628532825852274380553674012976$$

to 43 decimal places. If we let $\rho_n := \frac{\log_2(\ell_{n+1}) - 1}{2^n}$, then for all $n \geq 4$,

$$\rho_n \geq \rho \geq \rho_n - \frac{1}{2^{n-3} \ln 2 \sqrt{\ell_n}}.$$

Proof. We first prove (b). Since $(k+1)!/k! = k+1$ and since, for $0 \leq j < L$, $\frac{M-j}{L-j} \geq \frac{M}{L}$, we have

$$\frac{\binom{M}{k}}{\binom{L}{k+1}} = (k+1) \frac{M}{L} \frac{M-1}{L-1} \cdots \frac{M-j}{L-j} \cdots \frac{M-(k-1)}{L-(k-1)} \frac{1}{L-k} \geq (k+1) \left(\frac{M}{L}\right)^k \frac{1}{L} \geq$$

$$3 \left(\frac{M}{L}\right)^2 \frac{1}{L} = \frac{3M^2}{L^3} \geq \frac{3\left(\frac{L^2}{3}\right)^2}{L^3} = \frac{L}{3} \geq 1.$$

We next prove (a) and (c) simultaneously by induction on n . One may check (a) and (c) explicitly for $0 \leq n \leq 4$ by calculating ℓ_n for $0 \leq n \leq 5$. Assume both (a) and (c) hold for integers up to and including n . In the recursion for ℓ_{n+1} , the binomial coefficients on the right are nonincreasing because the first of the

inequalities in part (a) holds for $M = \ell_{n-j}$ and $L = \ell_{n-j-1}$ until we reach the terms that are 0. The alternating sum of the terms of a nonincreasing sequence of nonnegative real numbers is nonnegative, from which we have at once that

$$(\dagger) \quad 1 + \binom{\ell_n}{2} - \binom{\ell_{n-1}}{3} \leq \ell_{n+1} \leq 1 + \binom{\ell_n}{2}.$$

To complete the inductive proof of (a) and (c), we abbreviate $L := \ell_n$. The second inequality in (\dagger) yields $\ell_{n+1} \leq 1 + \frac{L(L-1)}{2} \leq \frac{L^2}{2}$. The first inequality in (\dagger) together with the fact that $L \geq \frac{\ell_{n-1}^2}{3}$, so that $\ell_{n-1} \leq (3L)^{1/2}$ and $\binom{\ell_{n-1}}{3} \leq \frac{(3L)^{3/2}}{6}$ yields

$$(*) \quad \ell_{n+1} \geq 1 + \frac{L(L-1)}{2} - \frac{3\sqrt{3}L\sqrt{L}}{6},$$

and so we are done if $\frac{L^2}{3} \leq \frac{L^2-L}{2} - \frac{\sqrt{3}L\sqrt{L}}{2}$. If we multiply by $\frac{6}{L\sqrt{L}}$ and collect terms, we see that this is equivalent to $\sqrt{L} \geq \frac{3}{\sqrt{L}} + 3\sqrt{3}$. This is true if $L \geq 48$, and the cases that involve a smaller value of L , where $0 \leq n \leq 4$, are handled by explicit calculation in the base case for the induction. Hence, (a) and (c) are proved.

With $L = \ell_n$, the inequalities

$$1 + \frac{L(L-1)}{2} - \frac{3\sqrt{3}L\sqrt{L}}{6} \leq \ell_{n+1} \leq 1 + \binom{L}{2}$$

were established in $(*)$ and (\dagger) in the preceding paragraph. If we divide by L^2 , we see that $\frac{\ell_{n+1}}{\ell_n^2}$ is trapped between two numbers both of which approach $\frac{1}{2}$ as $n \rightarrow \infty$. This establishes part (d).

Part (e) now follows from the inequalities in part (a) by a straightforward induction.

For part (f), for $n \geq 0$ let $\rho_n := \frac{\log_2(\ell_{n+1}) - 1}{2^n}$, so that $\ell_{n+1} = 2 \cdot 2^{\rho_n 2^n}$. Note that $\rho_0 = 1$, that every $\rho_n \geq 0$, and that $\rho_{n+1} < \rho_n$ since

$$\ell_{n+2} < \frac{1}{2}\ell_{n+1}^2 \Rightarrow \log_2(\ell_{n+2}) < 2\log_2(\ell_{n+1}) - 1.$$

Hence, as $n \rightarrow \infty$, ρ_n converges to $\rho \geq 0$. Also, by $(*)$ above with n increased by one,

$$\ell_{n+2} \geq 1 + \binom{\ell_{n+1}}{2} - \frac{\sqrt{3}}{2}\ell_{n+1}\sqrt{\ell_{n+1}} = 1 + \frac{\ell_{n+1}}{2}(\ell_{n+1} - 1 - \sqrt{3}\sqrt{\ell_{n+1}})$$

and since $\sqrt{3}\sqrt{\ell_{n+1}} \leq 2\sqrt{\ell_{n+1}}$ for all n , we have

$$\ell_{n+2} \geq 1 + \frac{\ell_{n+1}^2}{2} - \frac{\ell_{n+1}}{2} - \ell_{n+1}^{3/2} \geq \frac{\ell_{n+1}^2}{2} - 2\ell_{n+1}^{3/2}, \quad n \geq 3.$$

Hence:

$$(\#) \quad 2 \cdot 2^{\rho_{n+1} 2^{n+1}} \geq \frac{1}{2}(2 \cdot 2^{\rho_n 2^n})^2 - 2(2 \cdot 2^{\rho_n 2^n})^{\frac{3}{2}} =$$

$$2^{\rho_n 2^{n+1}+1} - 2^{3\rho_n 2^{n-1}+5/2} = 2^{\rho_n 2^{n+1}+1} \left(1 - \frac{1}{u_n}\right)$$

where

$$(\ddagger) \quad u_n = \frac{2^{\rho_n 2^{n+1}+1}}{2^{3\rho_n 2^{n-1}+5/2}} = 2^{\rho_n 2^{n-1}-3/2} = \frac{1}{4} \sqrt{\ell_{n+1}}.$$

Note that if $n \geq 4$, $u_n > 2$ and u_n increases with n . Applying \log_2 to the first and last terms in (#), we have

$$1 + \rho_{n+1} 2^{n+1} > \rho_n 2^{n+1} + 1 + \log_2 \left(1 - \frac{1}{u_n}\right) \text{ or } \rho_{n+1} > \rho_n + \frac{1}{2^{n+1}} \log_2 \left(1 - \frac{1}{u_n}\right).$$

Note that for $\log_2(1-x) = \frac{\ln(1-x)}{\ln 2} = \frac{-1}{\ln 2} (x + x^2/2 + \dots)$ is negative for $0 < x < 1$, and $x + x^2/2 + x^3/3 + \dots < x + x^2 + x^3 + \dots = \frac{x}{1-x} < 2x$ if $0 < x < 1/2$. Hence, once $n \geq 4$, so that $u_n > 2$,

$$\rho_{n+1} > \rho_n - \frac{1}{2^{n+1}} \frac{1}{\ln 2} \frac{2}{u_n}.$$

This shows that ρ_{n+1} is trapped between ρ_n and $\rho_n - \epsilon_n$ with $\epsilon_n = \frac{1}{(\ln 2) 2^n u_n}$.

Clearly, $\epsilon_{n+1} < \frac{\epsilon_n}{2}$, and it follows that

$$0 < \rho_n - \rho < \left(1 + \frac{1}{2} + \frac{1}{4} + \dots\right) \epsilon_n = 2\epsilon_n = \frac{1}{2^{n-3} (\ln 2) \sqrt{\ell_{n+1}}},$$

using the formula (\ddagger) for u_n displayed above. Thus, ρ_n converges to ρ very rapidly. Now,

$$\frac{\ell_{n+1}}{2 \cdot 2^{\rho 2^n}} = \frac{2 \cdot 2^{\rho_n 2^n}}{2 \cdot 2^{\rho 2^n}} = 2^{(\rho_n - \rho) 2^n} < 2^{2\epsilon_n 2^n} = 2^{\frac{2}{(\ln 2) u_n}} \rightarrow 1$$

as $n \rightarrow \infty$, which establishes the asymptotic estimate in part (f).

To estimate ρ using ρ_9 , note that since $\ell_{10} > 1.89 \cdot 10^{85}$, we have that

$$\rho_9 - \rho < 2\epsilon_9 < \frac{1}{64 (\ln 2) \sqrt{18.9 \cdot 10^{42}}} < 5.19 \cdot 10^{-45}.$$

Since the 44 th and 45 th digits of ρ_9 after the decimal point are both 3, this error will not affect earlier digits, and ρ_9 will agree with ρ to at least 43 decimal places, which is what is shown in part (f). \square

We shall need:

Proposition 4.2. *The following alternate recursive formulas for ℓ_n hold, as well as the recursive formula shown for H_{n+1} .*

$$(a) \text{ For all } n \geq 1, \quad \ell_{n+1} = \sum_{j=1}^{n+1} (-1)^{j+1} \binom{\ell_{n-j+1} + 1}{j+1}.$$

$$(b) \text{ For all } n \geq 0, \quad \ell_n = 3 + \sum_{j=0}^{n-1} (-1)^j \binom{\ell_{n-j-1} - 1}{j+2}.$$

$$(c) \text{ For all } n \geq 0, \quad H_{n+1} = 2 + \sum_{j=0}^{n-1} (-1)^j \binom{H_{n-j}}{j+2}.$$

Proof. (a) For $n = 1$, this may be verified by direct calculation. Assume $n \geq 2$. Let E_n be the equation displayed as Eq. (4.1).

$$(E_n) \quad \ell_{n+1} = 1 + \binom{\ell_n}{2} - \binom{\ell_{n-1}}{3} + \cdots + (-1)^n \binom{\ell_0}{n+2}, \quad n \geq 0.$$

Since $n \geq 2$ we may write E_{n-1} as a formula for $\binom{\ell_n}{1}$, subtract the equation E_{n-1} from E_n , transpose $\binom{\ell_{n-1}}{1}$ to the right hand side of the equation, and apply Equation 1.3(i) to all pairs of terms with the same numerator and consecutive integer denominators to obtain the required result.

The recurrence in (b) follows by induction on n . For the inductive step, if $n \geq 1$, note that E_{n-1} can be rewritten as

$$\ell_n = \binom{\ell_{n-1} - 1}{2} + \ell_{n-1} + \sum_{j=1}^{n-1} (-1)^j \binom{\ell_{n-1-j}}{j+2}.$$

by substituting $1 + \binom{\ell_{n-1}}{2} = \binom{\ell_{n-1} - 1}{2} + \ell_{n-1}$ on the right side. We replace ℓ_{n-1} , the second term on the right, by the formula from the equation obtained by replacing n by $n - 1$ in (b), which is the induction hypothesis. This yields

$$\ell_n = \binom{\ell_{n-1} - 1}{2} + \left(3 + \sum_{j=0}^{n-2} (-1)^j \binom{\ell_{n-2-j} - 1}{j+2} \right) + \sum_{j=1}^{n-1} (-1)^j \binom{\ell_{n-1-j}}{j+2}.$$

We shift the summation variable in the first sum by 1 so that it varies from 1 to $n - 1$ and combine terms to obtain:

$$\ell_n = 3 + \binom{\ell_{n-1} - 1}{2} + \sum_{j=1}^{n-1} \left((-1)^{j-1} \binom{\ell_{n-1-j} - 1}{j+1} + (-1)^j \binom{\ell_{n-1-j}}{j+2} \right).$$

The required formula for ℓ_n is now obtained by applying Equation 1.3(ii) to the terms in the summation.

One obtains (c) from (b) by replacing each occurrence of ℓ_k by $H_{k+1} + 1$ and subtracting 1 from both sides of the equation. \square

We next give a description of the Hamilton numbers following [18]. We generate an array, shown below, in which the zeroth row consists of 1 followed by an infinite sequence of zeros. We use the notation λ_{ij} , $i, j \geq 0$ for the j th entry of the i th row.

The rows have indentations from the left that are nondecreasing. The indentation

1	0	0	0	0	0	0	0	...
1	1	1	1	1	1	1	1	...
2	3	4	5	6	7	...		
1	5	9	14	20	27	...		
6	15	29	49	76	...			
5	21	50	99	175	...			
4	26	76	175	350	...			
3	30	106	281	631	...			
2	33	139	420	1051	...			
1	35	174	594	1645	...			
36	210	804	2449	...				
35	246	1050	3499	...				
34	281	1331	4830	...				
33	315	1646	6476	...				
32	348	1994	8470	...				
⋮	⋮	⋮	⋮	...				
1	875	23694	3765664	...				
876	24570	401134	...					
⋮	⋮	⋮	⋮	⋮				

of the $i + 1$ th row is the same as for the i th row if the first entry of the i th, call it $\lambda = \lambda_{i,j_i}$, is greater than 1. In this case, the first entry of the $i + 1$ th row is $\lambda - 1$, and for all other entries,

$$(**) \quad \lambda_{i+1,j} = \sum_{t \leq j} \lambda_{i,t}.$$

If the first entry of the i th row is 1, the indentation increases by 1, and all entries of the $i + 1$ th row are given by the formula (**). We refer to all the consecutive rows with the same indentation as a *block*. The blocks in the table above are separated by horizontal lines.

Consider the sequence of entries beginning rows where the indentation has just increased by 1, but including the zeroth row. These entries are shown in boldface in the table above. This is the sequence consisting of the first entries of the initial rows of the blocks. This sequence is $1, 1, 2, 6, 36, \dots$. Call this sequence a_n , $n = 0, 1, 2, 3, 4, \dots$. (This use of the notation a_n , which is consistent with [18], occurs in this paper only in this paragraph and the next: it has a different meaning in the later sections.) Note that for $n \geq 1$ there are a_n rows with indentation n , and that their leading entries are the consecutive integers from a_n to 1 in descending order.

Let $s_{n+1} = a_0 + \dots + a_n$, for $n \geq 0$, where $s_0 = 0$. In [18], H_n is defined as $s_n + 1$ for $n \geq 1$. Thus $H_1 = s_1 + 1 = 2$. We extend the definition so that $H_0 = s_0 + 1 = 1$. A recursive formula for H_{n+1} is then derived in [18] from properties in the description of the table above: this formula is displayed at the top of page 498 of [18]. However, there is an error in this formula, as noted in [19]: the term subtracted on the left should have been 2 rather than 1. The corrected formula leads to a recursion recorded in [21] (by giving formulas in Maple and

Mathematica), namely:

$$H_1 := 2 \quad \text{and} \quad H_{n+1} = 2 + \sum_{i=0}^{n-1} (-1)^i \binom{H_{n-i}}{i+2}, \quad n \geq 1.$$

This agrees with part (c) of Proposition 4.2, which shows that our treatment of the Hamilton numbers yields the same sequence as that in [18]. We note that [18] also treats the numbers ℓ_n , but there is another error, because one has $\ell_n := H_{n+1} + 1$, $n \geq 0$, and in [18], H_n is used instead of H_{n+1} .

Note that our recursion for ℓ_{n+1} is the same as the recursion given on page 498 of [18], except that we have replaced n by $n + 1$.

5. THE CASE OF A REGULAR SEQUENCE OF TWO QUADRATIC FORMS

The next result and the variant that follows describe the behavior obtained from a regular sequence of quadratic forms, making the connection with the Hamilton numbers noted in §1. This result is somewhat surprising, since it means that the number of generators needed for the lex ideal whose Hilbert function agrees with the Hilbert function of an ideal generated by a regular sequence of two quadratic forms has double exponential growth.

Theorem 5.1. *Over any field K , if I is the ideal generated by x_1^2, x_2^2 in $K[x_1, x_2]$, then $b_d(I) = H_{d-2} + 1$ for $d \geq 2$.*

This is clearly correct for $d = 2$. For $d \geq 3$ we may restate this result as:

Theorem 5.2. *Over any field K , with I as above, $b_d(I) = \ell_{d-3}$ for $d \geq 3$.*

Proof. The extended Hilbert function for an ideal I generated by a regular sequence consisting of two quadratic forms, from the Koszul complex resolution

$$0 \rightarrow R^{(N)}(-4) \rightarrow R^{(N)}(-2)^{\oplus 2} \rightarrow I \rightarrow 0$$

is $(N, t) \mapsto 2 \binom{N+t-3}{N-1} - \binom{N+t-5}{N-1}$. That is, $c_{-2} = 2, c_{-4} = -1$ and all other values of c_s are 0. We know that $b_0 = b_1 = 0$ and $b_2 = 2$. The recursion is

$$\begin{aligned} b_{d+1} &= c_{-(d+1)} + b_d + \left\langle \begin{matrix} b_d \\ 2 \end{matrix} \right\rangle + \sum_{j=1}^{d-1} (-1)^{d-j} \left\langle \begin{matrix} b_j + 1 \\ d-j+2 \end{matrix} \right\rangle \\ &= c_{-(d+1)} + \left\langle \begin{matrix} b_d + 1 \\ 2 \end{matrix} \right\rangle + \sum_{j=1}^{d-1} (-1)^{d-j} \left\langle \begin{matrix} b_j + 1 \\ d-j+2 \end{matrix} \right\rangle = c_{-(d+1)} + \sum_{j=1}^d (-1)^{d-j} \left\langle \begin{matrix} b_j + 1 \\ d-j+2 \end{matrix} \right\rangle \end{aligned}$$

Since we have $b_0 = b_1 = 0$ and $b_2 = 2$ we have the following:

$$\text{When } d = 2, \text{ we get } b_3 = 0 + (-1) \left\langle \begin{matrix} 1 \\ 3 \end{matrix} \right\rangle + \left\langle \begin{matrix} 3 \\ 2 \end{matrix} \right\rangle = 3.$$

$$\text{When } d = 3, \text{ we get } b_4 = -1 + \left\langle \begin{matrix} 1 \\ 4 \end{matrix} \right\rangle - \left\langle \begin{matrix} 3 \\ 3 \end{matrix} \right\rangle + \left\langle \begin{matrix} 4 \\ 2 \end{matrix} \right\rangle = -1 + 0 - 1 + 6 = 4.$$

To prove that $b_d = \ell_{d-3}$ for $d \geq 3$ (this is checked above for $d = 3, d = 4$), it will be convenient to extend the definition of ℓ so that $\ell_{-3} = 0, \ell_{-2} = 0$, and $\ell_{-1} = 2$. Then we want to show by induction that $b_d = \ell_{d-3}$ for all $d \geq 0$. From

this definition and the calculations above we have that $b_d = \ell_{d-3}$ for $0 \leq d \leq 4$. For $d \geq 4$ we have that $c_{-(d+1)} = 0$ and so the recursion for b_{d+1} becomes

$$b_{d+1} = \sum_{j=1}^d (-1)^{d-j} \left\langle \begin{matrix} b_j + 1 \\ d - j + 2 \end{matrix} \right\rangle, \quad d \geq 4,$$

and the induction hypothesis yields that

$$b_{d+1} = \sum_{j=1}^d (-1)^{d-j} \left\langle \begin{matrix} \ell_{j-3} + 1 \\ d - j + 2 \end{matrix} \right\rangle, \quad d \geq 4,$$

To complete the proof, it suffices to show that the right hand side is $\ell_{d+1-3} = \ell_{d-2}$. By Proposition 4.2(a) we have that for $n \geq 1$,

$$\ell_{n+1} = \sum_{i=1}^{n+1} (-1)^{i+1} \binom{\ell_{n-i+1} + 1}{i+1}.$$

The required result follows by letting $n = d - 3$ and $j = d + 1 - i$ (note the last sentence of the first paragraph of Notation 1.1). \square

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