Multiplicities and Chern classes in local algebra, by Paul C. Roberts, Cambridge Tracts in Mathematics 133, Cambridge University Press, Cambridge, England, 1998.

This book contains an exposition of several topics from commutative algebra and algebraic geometry that straddle the border between the two subjects. We first describe some of the ideas from intersection theory that are needed: we give here a geometric description along the lines of [F] (cf. also [FM], [BFM]). We then survey some questions in local algebra that, despite first appearances, turn out to be intimately related to intersection theory. Finally, we discuss specifics about what is done in the book under review. It is worth pointing out right away that several of the subtle problems in local algebra that Roberts settled (handling the last and most difficult case, where the ring does not contain a field) can be settled using the Frobenius endomorphism in positive characteristic, and either by analytic methods or by reduction to positive characteristic if the ring contains a field of characteristic zero. However, none of these methods seems to work when the ring does not contain a field.

One of the basic ideas of intersection theory is to define a graded abelian group, the *Chow group*, $A_*(X)$ of a variety (or, more generally, of a scheme) X, spanned by equivalence classes of subvarieties (the equivalence is an algebraic analogue of homotopy), and then have certain "intersection operations" acting on these classes producing an answer well-defined up to equivalence. The Chern classes, Chern characters (here one needs to allow rational coefficients) and other intersection operators that arise may be thought of simply as operators on all Chow groups enjoying certain kinds of functorial behavior. Where do these intersection operations come from? Sometimes one is simply intersecting with a subvariety or more general closed set — the reader should be warned, however, that in order to get well-defined operations it may be necessary to impose conditions on the closed set. However, one hopes, if the ambient variety is smooth, to be able to intersect with any closed subvariety. More caution is needed here. Intersecting a subvariety V with itself in this theory does not give anything like the same subvariety back. Since one is working with equivalence classes one should think instead of replacing one of the copies of V by something equivalent that meets the original V in a "better" (roughly speaking, more general) way: the intersection will, typically, be smaller than V.

In classical versions of intersection theory practitioners worried a great deal about proving "moving lemmas" that allowed one to show that there really did exist a variety equivalent to one specified but in sufficiently "good" position relative to another. It turns out that such issues can be avoided, and a wonderful treatment along different lines is given in [F].

A very important point is that certain subvarieties of codimension one are closely related to line bundles, and that this enables one to replace the operation of intersecting with subvarieties of codimension one by the operation of restricting a line bundle. By pursuing this idea one can eliminate moving lemmas and even get a theory that works in contexts where there is no base field, such as algebraic schemes over a discrete valuation ring or some other regular ring. Moreover, one can associate intersection operators called *Chern classes* with arbitrary vector bundles (equivalently, locally free sheaves), not just line bundles: this is analogous to what is done in topology.

The Chow group $A_*(X) = \bigoplus_k A_k(X)$ is simple to define formally: to get $A_k(X)$ one starts in degree k with the free abelian group generated by the subvarieties of dimension[1] k, and then one kills certain *divisors*: for each subvariety of dimension $k+1$, say W, and each nonzero element f of the field of rational functions on W, one kills the divisor, $\operatorname{div} f$, of f, which is defined to be $\sum_V (\operatorname{ord}_V f)\,[V]$ as V runs through the finitely many k-dimensional subvarieties of W for which f has nonzero order.[2]

Proper morphisms in algebraic geometry are an analogue of the notion from complex analysis (inverse images of compact sets are compact), but are defined via the property of taking closed sets to closed sets even after a base change. Proper morphisms, including closed immersions, induce a functorial pushforward on Chow groups that preserves degree.[3] Flat morphisms with a well-defined relative dimension r, including open immersions and structural maps of bundles, induce a contravariant flat pullback[4] on Chow groups that shifts degrees upward by r. The intersection operators that one constructs have certain standard compatibilities with these pullbacks and pushforwards, including a so-called *projection formula*: we say a bit more about this below.

The intersection operators determined by line bundles play a central role, and we want to describe these next. The main point is that, given a line bundle L and a subvariety V of dimension k, one gets a class in $A_{k-1}(V)$ simply by restricting the line bundle to V. This idea enables one to define $c_1(L)$, the first Chern class of L. Why does a line bundle on V determine a class α in $A_{k-1}(V)$?

To make the connection, we first think about going in the other direction. Suppose one has an effective Cartier divisor D on V, which simply means a closed subvariety of codimension one defined locally by a single equation, i.e., by the vanishing of a single regular function. For a sufficiently fine open cover, on each open set one can choose a function defining D. On overlaps of pairs of open sets in this cover one has two choices, and there is a unit that multiplies one choice to the other. These units give the transition data to define a line bundle. It is natural to associate the class of the effective Cartier divisor (counting its components with suitable multiplicities) with this line bundle, and this turns out to be independent of choices, up to rational equivalence.

To describe how one may associate a class $\alpha \in A_{k-1}(V)$ with a line bundle L that may not come from an effective Cartier divisor one may make use of the process of blowing up.[5]

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[1]In the classical case of varieties over a field Krull dimension is used, but a somewhat different notion of dimension is used in the more general case of schemes of finite type over a regular base.

[2]To determine the order, one assigns a one-dimensional local ring $A = \mathcal{O}_{V,W}$ to V: this may be thought of as the regular functions on an open affine that meets V, localized at the prime that defines its intersection with V. The order of an element f of $A - \{0\}$ is the length of A/fA: this function extends to a group homomorphism from the nonzero elements of the fraction field to the integers.

[3]If $f : X \to Y$ is proper and $V \subseteq X$ is a variety than $f_*([V]) = d[W]$ with $W = f(V)$ when $\dim W = \dim V$, where d is the degree of the function field $R(V)$ over $R(W)$; if $\dim W < \dim V$ then $f_*([V]) = 0$.

[4]If $f : X \to Y$ is flat, and $V \subseteq Y$ is a variety, $f^*([V])$ is $[f^{-1}(V)] := \sum_{Z \in \mathcal{C}} \operatorname{length}(\mathcal{O}_{Z, f^{-1}(V)})[Z]$, where $\mathcal{C}$ is the set of irreducible components of $f^{-1}(V)$ (for each Z, $\dim Z = \dim V + r$).

[5]*Blowing up* a closed subscheme Z of X is a very general device that produces a map $f : Y \to X$ that is a proper morphism and an isomorphism of $Y - f^{-1}(Z) \to X - Z$. If X is a variety, the map $Y \to X$ is both proper and birational. A key point is that $f^{-1}(Z)$ is an effective Cartier divisor on Y. If one blows up a (closed, rational) point in $\mathbb{A}^n$ its inverse image in the blow-up Y is a copy of $\mathbb{P}^{n-1}$: roughly speaking, in Y the point has been replaced by the set of lines in $\mathbb{A}^n$ through it.

For a suitable blow-up $f: Y \to V$, any line bundle has a pullback of the form $L_1 \otimes L_2^{-1}$, where the L_i are line bundles corresponding to effective Cartier divisors. The L_i are associated with classes $\beta_i \in A_{k-1}(Y)$ as described earlier, and the class α that we seek will be $f_*(\beta_1) - f_*(\beta_2)$.

We can now describe Chern classes of arbitrary vector bundles E on X. One can choose a proper flat morphism $\pi: Y \to X$ such that $\pi^* E$ has a filtration by line bundles. The i th Chern class $c_i(E)$ of E can be viewed as being determined[6] by the i th elementary symmetric function of the first Chern classes of these line bundles. $c_i(E)$ is an operator that lowers degrees by i on $A_*(X)$. Actions of various Chern classes and characters are often indicated with the symbol $\cap$. An example of the kind of projection formula mentioned earlier is that if $X \to Y$ is proper, E is a vector bundle on Y, and $\alpha \in A_*(X)$, then the $f_*(c_i(f^*E) \cap \alpha) = c_i(E) \cap f_*(\alpha)$.

Once one has defined Chern classes of vector bundles one can show that if E is the total space of a vector bundle over a base X, then $A_*(X) \cong A_*(E)$ via flat pullback: the map shifts degrees by the rank of E. The inverse of this map can be thought of as a *Gysin* homomorphism that intersects arbitrary classes in $A_*(E)$ with X, embedded as a closed set in E via the zero section of the vector bundle.

When $X \to Y$ is a regular embedding of codimension d (so that X is determined, locally, by the vanishing of an ideal generated by a regular sequence of length d — the case where $d = 1$ is that of an effective Cartier divisor) one wants, roughly speaking, to intersect arbitrary classes in $A_*(Y)$ with X and get classes in $A_*(X)$: the graded degree should drop by d. Gysin homomorphisms of this sort can be defined by systematic exploitation of the idea of deformation to the normal cone. One finds a flat family over a line almost all of whose fibers are simply the original map $X \to Y$, but at an isolated point the fiber is the inclusion of X in the normal bundle to X in Y, a vector bundle over X. The problem of constructing the general Gysin homomorphism can then be reduced to the vector bundle case described above.

Deformations of this sort are easy to describe algebraically in the affine case. Suppose that Y corresponds to $\operatorname{Spec} R$ and that X is defined by an ideal I. The algebra $R[I/T, T] \subseteq R[T, 1/T]$, where T is an indeterminate, is flat over $K[T]$. If one specializes T to a nonzero scalar in K the fiber is R, while if one specializes T to zero the fiber is the associated graded ring $\operatorname{gr}_I R = R/I \oplus I/I^2 \oplus I^2/I^3 \oplus \cdots$. The technique used in [F] is a globalization of this construction. When I is locally generated by a regular sequence, $\operatorname{Spec}(\operatorname{gr}_I R)$ is a vector bundle over $\operatorname{Spec} R/I$.

When X is a smooth variety of dimension n over a field K, the diagonal map of $\delta: X \to X \times X$ is a regular embedding, and the problem of intersecting the classes associated with two subvarieties V and W within X may be replaced by the problem of intersecting $V \times W$ with $\Delta = \delta(X)$, the diagonal in $X \times X$. For smooth X, if one lets $A^*(X)$ be defined by taking $A^i(X) = A_{n-i}(X)$ for every i, then the intersection product gives the structure of a graded commutative associative ring to $A^*(X)$, the Chow ring.

There is an important further point to be made here: in many instances, one can get

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[6]One can choose π, a composition of structural morphisms of projectivized vector bundles, such that $\pi_*: A_*(Y) \to A_*(X)$ is onto (and $\pi^*: A_*(X) \to A_*(Y)$ is a split monomorphism). One has $c_i(E) \cap \alpha = \pi_*(c_i(\pi^*E) \cap \beta)$ where $\pi_*(\beta) = \alpha$. The formal definition is different from this characterization.

intersections to be defined in a rather small ambient closed subscheme, say Z, of X. The class in $A_*(Z)$ pushes forward to the class in $A_*(X)$. This idea is pursued vigorously in [F], leading to "refined" Gysin homomorphisms and intersection operators of many sorts. For example, when one intersects two pure-dimensional subschemes V and W of a smooth variety X, the intersection can actually be obtained as a class in $A_*(V \cap W)$. In particular, if V and W meet in an isolated point, one can replace X, V, W by their intersections with an affine open neighborhood of this point, and so assume that the intersection, set-theoretically, is P. If V and W meet properly, which in this case means that the sum of their dimensions is the dimension of X, then one can assign an intersection multiplicity to this point P in $V \cap W$: the intersection is given by a class α in $A_0(P)$, which turns out to be $\mathbb{Z}[P]$, and the intersection multiplicity is the coefficient of $[P]$ in α.

One can also define Chern characters: one allows rational coefficients, and then the Chern character of a vector bundle that has a filtration by line bundles is the sum of the formal exponentials of first Chern classes associated with the line bundles. Although one is substituting the operators on $A_*(X)$ in a formal power series, the higher order terms may be dropped, since they will act by shifting degrees down by more than the dimension of X, and so are zero. The formula can be rewritten in terms of the Chern classes of the vector bundles and makes sense even when there is no filtration by line bundles. One can also define the Todd class $\mathrm{td}\, E$ of a vector bundle E similarly: if one has a filtration by line bundles, it is the product of the formal power series for $x/(1-e^{-x})$ evaluated on the first Chern classes of the line bundles in the filtration, and the definition can be extended to the general case as for Chern characters. Again, one needs rational coefficients here. When X is smooth over a field K, its Todd class is the Todd class of its tangent bundle.

In this context one can prove Riemann-Roch theorems and Riemann-Roch formulas of a very general sort. These can be rather technical to state: one may be starting with a complex of vector bundles and a map of singular varieties. But if one has a proper morphism of smooth varieties, and just one vector bundle, these results correspond to the Grothendieck Riemann-Roch theorem. The theorem can be thought of as remedying the failure of proper pushforward on vector bundles (defined, thinking K-theoretically, as the alternating sum of all the direct images) to commute with the Chern character map. The cure is to multiply by the Todd class of the tangent bundle, and then one gets an operator that commutes with proper pushforward.

An important part of the theory presented in [F] is that there is a theory of *localized Chern characters* in the following sense: given a bounded complex of vector bundles $E_\bullet$ on Y that is exact off a closed set X, if $\alpha \in A_*(Y)_{\mathbb{Q}}$ one can assign a class $\mathrm{ch}_X^Y(E_\bullet) \cap \alpha \in A_*(X)_{\mathbb{Q}}$ that pushes forward to $\sum_i (-1)^i \mathrm{ch}(E_i) \cap \alpha$ in $A_*(Y)_{\mathbb{Q}}$ and has all the naturality properties that one could possibly hope for: the definition, based on the so-called "graph construction," is extremely complicated.[7]

[7]However, the localized Chern character may be characterized axiomatically by (i) compatibility with proper morphisms and open immersions, (ii) additivity on short exact sequences of bounded complexes that are exact off X, (iii) the fact that for vector bundles E' on Y, $\mathrm{ch}_X^Y(E' \otimes E_\bullet) = \mathrm{ch}(E'|_X)\mathrm{ch}_X^Y(E_\bullet)$, together with, finally, (iv) the fact that if D is an effective Cartier divisor on a variety Y and $E_\bullet$ is the length one complex $0 \to \mathcal{O}_Y(-D) \subseteq \mathcal{O}_Y \to 0$ (where $\mathcal{O}_Y(-D)$ is simply the sheaf of ideals in $\mathcal{O}_Y$ that defines D, so that $H_1(E_\bullet) = 0$ and $H_0(E_\bullet) = \mathcal{O}_D$), then $\mathrm{ch}_D^Y(E_\bullet) \cap [Y] = (\mathrm{td}\,(N))^{-1} \cap [D]$, where N is the normal bundle to D in Y, which is the same as $\mathcal{O}_Y(D)|_D$; $\mathcal{O}_Y(D) = \mathcal{O}_Y(-D)^{-1}$.

The localized Chern characters lead to Riemann-Roch theorems and formulas for singular X. Suppose, for simplicity, that X is locally closed in a projective scheme over $S = \operatorname{Spec} K$, i.e., X is *quasi-projective* over S. Let $K_0(X)$ be the Grothendieck group of coherent sheaves on X. One can define a map $\tau_X : K_0(X) \to A_*(X)_{\mathbb{Q}}$ as follows. Let $\mathcal{F}$ be a coherent sheaf on X. Embed X as a closed subscheme in some M smooth over S, and resolve the pushforward sheaf of $\mathcal{F}$ to M by a finite complex $E_\bullet$ of locally free sheaves on M. Let T be the tangent bundle of M over S restricted to X. The value of τ_X on $[\mathcal{F}]$ is defined to be $\operatorname{td}(T) \cdot \operatorname{ch}_X^M(E_\bullet) \cap [M] \in A_*(X)_{\mathbb{Q}}$, which is independent of the choice of M and the resolution $E_\bullet$. The compatibility of the map τ_X with proper morphisms together with its other functorial properties yields a very good Riemann-Roch theorem.

One can also develop an important *Riemann-Roch formula* as follows. Let $E_\bullet$ be a complex of locally free sheaves on X that is exact off a closed $Z \subseteq X$ and $\mathcal{F}$ a coherent sheaf on X. Then the homology sheaves $H_i(E_\bullet \otimes_{\mathcal{O}_X} \mathcal{F})$ are supported on Z, and so define classes $[H_i(E_\bullet \otimes \mathcal{F})]$ in $K_0(Z)$. The Riemann-Roch formula asserts that, in $A_*(Z)_{\mathbb{Q}}$, $\sum_i (-1)^i \tau_Z\big([H_i(E_\bullet \otimes \mathcal{F})]\big) = \operatorname{ch}_Z^X(E_\bullet) \cap \tau_X(\mathcal{F})$.

Roberts used a version of this formula to solve several important problems in local algebra. For this, the theory described above needs to be generalized: one allows K to be a suitable regular local ring instead of a field. In order to build the theory in this generality (or even greater generality: [F] develops the theory for schemes of finite type over any base scheme that is regular) one needs to use a different notion of dimension from the usual Krull dimension — a notion that has better stability properties under passing to dense open sets. The notion that is used does agree with Krull dimension for finitely generated algebras over a field.[8]

We now want to change tacks and discuss some conjectures and theorems in local algebra many of which may seem to have nothing whatsoever to do with intersection theory. But they *are* all connected with intersection theory, and for those that are known in all cases, deep results in intersection theory have been the key to their proof. The statements discussed below, except (14), are all known to be true for rings containing a field. Some, thanks to Roberts, are known in complete generality. But many remain conjectures in the case of local rings of mixed characteristic (formal power series in n variables over the p-adic integers is an example of one such ring), where the fraction field of the ring has characteristic zero but the residue field has prime characteristic $p > 0$ because the integer p is in the maximal ideal. The discussion that follows their statements will make the situation clear. Many, but not all, are discussed in Roberts' book. The ones that are known in all cases were proved by Roberts in mixed characteristic using, in part, the intersection theory presented in [F] that was described above. However, (14) is not known even for rings containing a field, even in positive characteristic.

[8] To be precise, if the regular base scheme is S and V is an integral scheme of finite type over S such that the closure of its image in S is T, one defines $\dim_S V$ as the sum of the transcendence degree of the extension of function fields from that of T to that of V minus the codimension of T in S. E.g., if $S = \operatorname{Spec} A$, $T = \operatorname{Spec} A/P$, and $V = \operatorname{Spec} B$, so that $S \leftarrow T \leftarrow V$ corresponds to $A \to A/P \subseteq B$, this dimension is the transcendence degree of the fraction field of B over that of A/P minus the dimension of the local ring A_P. This theory is set up so that $\dim_S S = 0$. Note that if $S = \operatorname{Spec} A$ where A is a regular local ring that is not a field, f is a nonzero element of the maximal ideal of A, and $V = \operatorname{Spec} A_f$, then $\dim V = \dim S - 1$ (using Krull dimension) but $\dim_S V = \dim_S S$.

In all of the following R is a local ring, i.e., a Noetherian ring with a unique maximal ideal, unless otherwise specified, and M (and N) are finitely generated R-modules.

(1) **The zerodivisor theorem.** *If $M \neq 0$ has finite projective dimension (i.e., M has a finite projective resolution: the projective dimension is the length of the shortest such) and $r \in R$ is not a zerodivisor on M, then r is not a zerodivisor on R.* (Cf. [Aus], [PS1].)

(2) **Bass' question.** *If $M \neq 0$ and has a finite injective resolution, then R is a Cohen-Macaulay ring.* (Cf. [Bass].)

(3) **The intersection theorem.** *If $M \otimes_R N \neq 0$ is has finite length, then the Krull dimension of N (i.e., of R modulo the annihilator of N) is at most the projective dimension of M.* (Cf. [PS1].)

(4) **The new intersection theorem.** *Let $0 \to G_n \to \cdots \to G_0 \to 0$ denote a finite complex of free modules over R such that $\bigoplus_i H_i(G_\bullet)$ has finite length but is not 0. Then the Krull dimension of $R \leq n$.* (Cf. [PS2], [Ro1].)

(5) **The improved new intersection conjecture.** *Let $0 \to G_n \to \cdots \to G_0 \to 0$ denote a finite complex of free modules over R such that $H_i(G_\bullet)$ has finite length for $i > 0$ and $H_0(G_\bullet)$ has a minimal generator that is killed by a power of the maximal ideal. Then $\dim R \leq n$.* (Cf. [Ho3].)

(6) **The direct summand conjecture.** *If $R \subseteq S$ is a module-finite ring extension with R a regular ring (R need not be local, but the problem reduces at once to the local case), then R is a direct summand of S as an R-module.* (Cf. [Ho1–3].)

(7) **The canonical element conjecture.** *Let $x_1, \ldots, x_d$ be a system of parameters for R, let $G_\bullet$ denote a projective resolution of the residue field of R with $G_0 = R$, and let $K_\bullet$ denote the Koszul complex of R with respect to $x_1, \ldots, x_d$. Lift the identity map $R = K_0 \to G_0 = R$ to a map of complexes. Then, no matter what the choice of system of parameters or lifting, the last map from $R = K_d \to G_d$ is not 0.* (Cf. [Ho3], [Du2].)

(8) **Existence of balanced big Cohen-Macaulay modules conjecture.** *There is a (not necessarily finitely generated) module W over R such that $m_R W \neq W$ and every system of parameters for R is a regular sequence on W.* (Cf. [Ho2].)

(9) **Cohen-Macaulayness of direct summands conjecture.** *If R is a direct summand of a regular ring S as an R-module, then R is Cohen-Macaulay (R need not be local, but the result reduces at once to the case where R is local).* (Cf. [HR], [Bou], [HH1].)

(10) **The vanishing conjecture for maps of Tor.** *Let $A \subseteq R \to S$ be homomorphisms where R is not necessarily local (one can reduce to that case, however), with A, S regular and R module-finite over A. Let W be any A-module. Then the map $Tor_i^A(W, R) \to Tor_i^A(W, S)$ is zero for all $i \geq 1$.* (Cf. [HH1], [HH3], [Rang].)

(11) **The strong direct summand conjecture.** *Let $A \subseteq R$ be a map of complete local domains, and let Q be a height one prime ideal of R lying over xA, where A and A/xA are regular. Then xA is a direct summand of Q as an A-module.* (Cf. [Rang].)

(12) **Existence of weakly functorial big Cohen-Macaulay algebras conjecture.** *Let $R \to S$ be a local homomorphism of complete local domains. Then there exists an*

R-algebra $\mathcal{B}_R$ that is a balanced big Cohen-Macaulay algebra for R, an S-algebra $\mathcal{B}_S$ that is a balanced big Cohen-Macaulay algebra for S, and a homomorphism $\mathcal{B}_R \to \mathcal{B}_S$ such that

$$\begin{array}{ccc} \mathcal{B}_R & \longrightarrow & \mathcal{B}_S \\ \uparrow & & \uparrow \\ R & \longrightarrow & S \end{array}$$

commutes. (Cf. [HH3].)

(13) **Serre's conjecture on multiplicities.** *Suppose that R is regular of dimension d, and that $M \otimes_R N$ has finite length. Then $\chi(M, N)$, defined as the alternating sum of the lengths of the modules $\mathrm{Tor}_i^R(M, N)$, is 0 if $\dim M + \dim N < d$ and positive if the sum is equal to d (the sum cannot exceed d: that was proved by Serre).* (Cf. [Se].)

(14) **Small Cohen-Macaulay modules conjecture.** *If R is complete, then there exists a finitely generated R-module $M \neq 0$ such that some (equivalently, every) system of parameters for R is a regular sequence on M.* (Cf. [Ho2].)

(1) was conjectured in [Aus], while (2) was raised as a question in [Bass]. These statements may seem unrelated, both to each other and to intersection theory, but in [PS1] both were shown to follow from (3), and (3) was proved in characteristic $p > 0$ and in certain cases for rings containing a field of characteristic 0 by reduction to characteristic p. The general case of (3) reduces to the case where $N = R/I$ is cyclic, and if M is cyclic as well, one is studying $R/(I + J) \cong M \otimes_R N$, and $\mathrm{Spec}\, R/(I + J)$ is the scheme-theoretic intersection of the closed subschemes defined in $\mathrm{Spec}\, R$ by I and J. (4) is a generalization of (3) that was proved by Peskine and Szpiro and, independently, by Roberts (cf. [PS2], [Ro1]). In [Ho2], (8) was proved in equal characteristic and used to prove (1) − (4) in equal characteristic. (5), (6), and (7) are known to be equivalent in the open case (mixed characteristic), and to follow from (8) in general. (Cf. [Ho1-3], [Du2].) Interest in (5) was sparked by [EvG] in their proof of another homological conjecture, the syzygy theorem, in equal characteristic. Roberts gave an analytic proof of (5) in characteristic 0 in [Ro].

Tight closure theory has given a new perspective on the local homological conjectures and has led both to new proofs, sharper theorems, and further conjectures. We refer the reader to [HH1] and [Hu] for more detail. Closely related is the existence, in a weakly functorial sense (cf. (12)), of big Cohen-Macaulay algebras in characteristic $p > 0$ first proved in [HH2] and extended to equal characteristic zero in [HH3], where many applications of the existence of weakly functorial big Cohen-Macaulay algebras are explored. The very powerful vanishing conjecture for maps of Tor given in (10) can be proved in equal characteristic using either tight closure theory or the existence of these algebras.

In very recent work [Rang] it is shown that (11) and (10) are equivalent. Thus, (11), which is a new conjecture, may be viewed in one way as a rather new theorem in the equal characteristic case, while in mixed characteristic, where both (11) and (10) are open, it provides a very down-to-earth approach for attacking (10).

Serre's conjecture on multiplicities has been an open question since the publication of [Se], and has been a strong impetus for research in local algebra. It has been long known that the existence of finitely generated Cohen-Macaulay modules (14) would suffice to prove positivity in the case where intersection multiplicities are supposed to be positive, but there has been little progress on the existence of finitely generated Cohen-Macaulay modules: they are not known to exist in dimension 3, even in equal characteristic. Roberts [Ro3] and Gillet-Soulé [GS], independently, were the first to prove the vanishing part of the conjecture in full generality in mixed characteristic. Roberts used the ideas of [F], and the proof is given in the work under review.

The following diagram summarizes these implications:

$$\begin{array}{ccccccccccc}
 & & 13 & & & & & & & & \\
 & \nearrow & & & & & & & & & \\
14 & & & & & & & & & & 1 \\
 & \searrow & & & & & & & & \nearrow & \\
 & & 8 & \to & 7,6,5 & \to & 4 & \to & 3 & & \\
 & \nearrow & & \nearrow & & & & & & \searrow & \\
12 & \to & 11,10 & & & & & & & & 2 \\
 & & & \searrow & & & & & & & \\
 & & & & 9 & & & & & &
\end{array}$$

where statements separated by commas (7,6,5 in one case and 11,10 in the other) are equivalent. Notice that this graph is connected: these questions are all related and form part of a very large picture.

Summary of status: (1) – (4) are known in all cases. (5) – (13) are known for rings containing a field but not in mixed characteristic (but in the case of (13) only the positivity is an open question). (14) is not known except in dimension at most two, even if the ring contains a field, regardless of characteristic.[9]

The new intersection theorem (4) (and hence all of (1) – (4)) was proved by Paul Roberts in [Ro4] (see also [Ro5]) and the proof is given in his book: it depends on the Riemann-Roch formula as well as a subtle result on the behavior of lengths in characteristic $p > 0$.

Roberts' book gives an exposition of that part of the material from [F], as well as topics in commutative and homological algebra, needed to settle (1)–(4) and also the vanishing statement in (13) (i.e., the case where the sum of the dimensions of M and N is smaller than d).[10] Roberts' treatment is very much an algebraic one: for example, a

[9]Some conjectures discussed in [PS1] have turned out to be false: [Heit] gives a counterexample to the rigidity conjecture. [DHM] gives an example where the intersection of multiplicity (defined using Tor) of a pair of modules, one of finite length and of finite projective dimension (but the other not of finite projective dimension) is negative. Two other questions raised in [PS1] remain very much open. Cf. [PS1, Th. 0.10]. Suppose that R is local, and that M, N are nonzero finitely generated modules with M of finite projective dimension. One question asks whether the length of a maximal regular sequence in the annihilator of M is equal to $\dim R - \dim M$. The other is this: suppose, moreover, that $M \otimes_R N$ has finite length. Is $\dim M + \dim N \leq \dim R$? Affirmative answers to these two questions together imply (3) on the list above (which is now known, [Ro4-5]). An affirmative answer to the second question would strengthen the parenthetical comment for the regular case made in (13). But these questions are open even in equal characteristic.

[10]The work of de Jong on alterations [DeJ] has been used by Gabber, as explained in [Ber], to give a new proof of the vanishing part of (13), and to prove non-negativity of $\chi(M, N)$ when the dimensions of the modules add to d: positivity in that case remains, in general, an open question.

sheaf on $\operatorname{Proj} A$, where A is a graded ring, is defined as an equivalence class of graded modules. The book is remarkably self-contained. In an astonishingly brief space, the author develops a huge amount of material and gets to proofs of some very hard, very deep theorems. Unavoidably, in making this material available to algebraists with a minimum of machinery from algebraic geometry, some geometric insight is lost. The use of graded and multigraded algebras replaces geometry. Many of the main results of [F] are proved in the context where the author needs them, including a local Riemann-Roch formula, the proof of which makes use of a splitting principle for complexes.

The first part of the book develops the basic theory of the Chow group and prerequisites from homological algebra and the theory of local Noetherian rings that are needed in the later chapters. Among the less standard topics are discussions of dualizing complexes, the uses of the Frobenius endomorphism and of Hilbert-Kunz and Dutta multiplicities, and there is a very brief discussion of tight closure theory and reduction of problems from equal characteristic zero to positive characteristic p. The second part of the book contains a highly algebraic treatment of projective schemes, Chern classes, Grassmannians, and of local Chern characters. Versions of the Riemann-Roch theorem and the local Riemann-Roch formula are proved. The final chapter includes several applications, including Roberts' proofs of the vanishing part of Serre's conjecture and his proof of the new intersection theorem, both in mixed characteristic. There are exercise sets at the ends of most chapters.

This book will be of enormous value to algebraists who want to gain an understanding of the powerful techniques presented in [F], and how they can be applied to local algebra. It will likewise be of great service to algebraic geometers who want to gain some understanding of the subtler parts of local algebra. I hope that all of its readers will go on to a fuller exploration of both the ideas of intersection theory and the ideas from local algebra that are introduced in Roberts' book.

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