SOLID CLOSURE

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In memory of my father, Lothar Hochster,
April 20, 1906 – February 17, 1991

CONTENTS

1. INTRODUCTION

Throughout this paper, unless otherwise specified, all rings are assumed to be commutative, associative, with identity, and all modules are assumed to be unital. In [HH4], the author, jointly with Craig Huneke, introduced the notion of tight closure for submodules of finitely generated modules over certain Noetherian rings. The definition is first made in characteristic p using the action of the Frobenius endomorphism. A notion for finitely generated algebras over a field of characteristic zero is then obtained by reduction to characteristic p. No satisfactory notion in mixed characteristic has yet been proposed. Tight closure theory has produced a host of new results and improvements of old results. We refer to the introductions of [HH1-4, 6, 8-10] for more detail, and to these papers as well

1991 Mathematics Subject Classification. Primary 13E05, 13B99, 13A35, 13H05, 13H10, 13D25.
The author was supported in part by a grant from the National Science Foundation.
This paper is in final form and no version of it will be submitted for publication elsewhere.

as [Ab1-2, AHH, FeW, HH11, Hu1-2, Gla, Sm1-3, Vel] and [Wil] for the full development of tight closure theory. The applications include invariant theory (for background, cf. [B], [Hr1-2], [Ke]), the Briançon-Skoda theorem (for background, cf. [BrS, LS, LT, Sk]) and the local homological conjectures (for background, cf. [Du, EvG1-3, Ho1-3, 5-7, PS1-2, Ro1-5, S]).

The existence of a "sufficiently good" parallel theory in mixed characteristic would settle many long standing conjectures.

Our objective in this paper is to introduce a new closure operation, solid closure, defined *a priori* in a characteristic-free manner. The author originally hoped that this theory might play, in mixed characteristic, a role analogous to that of tight closure theory in equal characteristic. Solid closure does turn out to agree, in characteristic p, with the notion of tight closure if the ring is well-behaved (for example, if the ring is finitely generated over an excellent local ring, or if $R^p \subseteq R$ is module-finite (cf. [Ku2]), or, more generally, if the ring has a completely stable weak test element in the sense of §6 of [HH4]; see §8). However, an example of Paul Roberts [Ro6] proves that, in equal characteristic zero, solid closure is "too big." Cf. (7.22-4). Roberts' example shows, for example, that in the ring $K[[x, y, z]]$ or $K[x, y, z]$, where K is a field of characteristic zero and x, y, z are formal indeterminates, the element $x^2y^2z^2$ is in the solid closure of the ideal (x^3, y^3, z^3). Thus, ideals of regular rings of dimension three need not be solidly closed in equal characteristic zero, which is quite different from the situation in positive characteristic.

However, it is still not clear whether every ideal is solidly closed in a regular ring of mixed characteristic. Oddly, if (V, pV) is a discrete valuation ring of mixed characteristic p then $x^2y^2z^2$ is not in the solid closure of (x^3, y^3, z^3) in $V[[x, y, z]]$ (but we do not know whether the ideal is solidly closed), while (x^3, y^3, z^3) is solidly closed in $\mathbb{Z}[x, y, z]$. We do not know whether (p^2, x^2, y^2) is solidly closed in $V[[x, y]]$ (nor in $V[x, y]$). Roberts' result is discouraging, but in some ways it makes the study of solid closure even more intriguing. The reader is referred to §7 and §13 for further discussion of the issues raised here. We note one more point: as a consequence of these remarks, solid closure does not commute with localization, because the solid closure of (x^3, y^3, z^3) in $\mathbb{Q}[x, y, z]$ is not the expansion of the solid closure of (x^3, y^3, z^3) in $\mathbb{Z}[x, y, z]$.

Regardless of whether every ideal turns out to be solidly closed in regular rings of mixed characteristic (that would imply the direct summand conjecture in general), solid closure has a number of aspects that make it worthwhile to study. It gives a novel perspective on tight closure in characteristic p that leads to new results: e.g., Theorem (5.9), viewed as a result about tight closure (this is done explicitly in Corollary (8.8)), is new. Although ideals of regular rings are not solidly closed in general, the solid closure of an ideal is contained in the integral closure and is usually much smaller, even in equal characteristic zero and in mixed characteristic. For example, if V is the ring of p-adic integers, $\mathbf{x} = x_1, \dots, x_n$, and $R = V[[\mathbf{x}]]$, every ideal of R containing p is solidly closed, while the integral closure of the ideal $(p, x_1^k, \dots, x_n^k)$ is $pR + (\mathbf{x})^k R$. Moreover, the solid closure point of view leads to a characterization of tight closure in complete local domains of characteristic p in terms of contracted expansions from a suitable balanced big Cohen-Macaulay algebra: see Theorem (11.1).

Insofar as possible, this manuscript is self-contained. We shall not need to make much use of the theory of tight closure except in §8 and §11, where we compare the two theories, and in those sections we can easily spell out just what is needed.

We next give a brief description of the notion of solid closure for the case of ideals. First note that when we refer to a "local ring (R, m, K)" we mean a Noetherian ring R with a unique maximal ideal m and residue field $K = R/m$.

(1.1) Definitions. *If R is a domain, we shall say that an R-module M is solid if $Hom_R(M, R) \neq 0$. We shall say that an R-algebra S is solid if it is solid as an R-module.*

(1.2) Definition. *Let $I \subseteq R$ be an ideal of a Noetherian ring R and let $x \in R$. If R is a complete local domain we say that x is in the solid closure $I^\bigstar$ of I if there exists a solid R-algebra S such that $x \in IS$. (Note: the notation $I^\blacksquare$ was used for the solid closure of I in an earlier version of this manuscript.) More generally, $x \in I^\bigstar$ if for every complete local domain B arising as the quotient by a minimal prime of the completion of R_m for some maximal ideal m of R, the image of x in B is in $(IB)^\bigstar$.*

There is a similar notion for submodules of a finitely generated R-module: see (5.1).

(1.3) *Remarks.* A module-finite extension S of a Noetherian domain R is always a solid R-algebra. However, in more general situations, even when S and R are both finitely generated over a base field K, it is a subtle and difficult problem to decide whether S is a solid R-algebra for specific choices of R and S. The problem is that the elements of $\mathrm{Hom}_R(S, R)$ correspond to the solutions of an infinite system of linear equations over R (although only countably infinite in the main case, where S is finitely generated as an R-algebra).

We view the condition that S be solid as an R-algebra as a nondegeneracy condition. The reason for the choice of the term is that if R is normal and S is a solid R-algebra then S does not contain any elements of the fraction field of R that are not in R: see Proposition (2.9).

The manuscript is structured as follows. In §2 we develop the basic properties of solid modules and algebras. In §3 we do likewise for formally solid modules and algebras over a Noetherian ring: this notion is convenient in studying solid closure, since the latter is defined in terms of passage to various complete local domains associated with R. In §4 we introduce the notion of a generic forcing algebra for a triple (M, N, u) where $N \subseteq M$ are modules and $u \in M$; these map to the other algebras S such that $1 \otimes u$ is "forced" into $\mathrm{Im}\,(S \otimes N \to S \otimes M)$. This enables us to characterize when u is in the solid closure ${N^\bigstar}_M$ of N in M as follows: $u \in {N^\bigstar}_M$ iff the triple (M, N, u) has a formally solid generic forcing algebra. These ideas and a number of others are explored in §5, where the basic properties of solid closure are proved. Many of the results are parallel to those of [HH4] for tight closure, but the proofs are different. In some instances we have referred to [HH4] for "isolated" arguments that can be read without reference to other parts of [HH4].

In §6 we study minimal solid algebras (they have no proper solid quotients), and exhibit better behavior than other solid algebras in a number of useful ways. In §7 we study rings in which every ideal is solidly closed, producing a theory parallel to the theory of weakly F-regular rings initiated in [HH4]. In §8 we show that solid closure agrees with tight closure for sufficiently good rings of characteristic p, including algebras essentially of finite type over an excellent local ring. In §11 we show more: over a complete local domain R, an element is in the tight closure of an ideal if and only if it is in its contracted expansion from a big Cohen-Macaulay algebra, and a big Cohen-Macaulay algebra is solid. This result also enables us to see that a certain equal characteristic zero notion of tight closure is contained in the solid closure. Moreover, the arguments give a new proof of the existence of big Cohen-Macaulay algebras in characteristic p which is simpler than the argument of [HH7]. However, it does not yield the weakly functorial behavior which is a very important consequence of the results of [HH7]: see [HH12].

In §9 a rather elementary criterion for when an element is in a solid closure over a complete local domain is developed in terms of the behavior of multiples of certain formal power series.

In §10 we introduce a theory of shadow homology, which is parallel to the theory of phantom homology initiated in [HH4] and pursued in [Ab1], [HH8], and [AHH]. We are able to obtain an equicharacteristic analogue of the phantom acyclicity criteria developed in [HH4] and [HH9], but the proof makes use of the existence of big Cohen-Macaulay algebras in equal characteristic, which one has either from [HH7] or from the arguments of §11 here (see also [HH5] and [Hu2]). Note that the result of [HH7] asserts that the integral closure of a complete (or excellent) local domain of characteristic p in an algebraic closure of its fraction field is a big Cohen-Macaulay algebra for the ring, which implies the existence of big Cohen-Macaulay algebras for all local rings of equal characteristic.

In §12, we consider solid closure over rings of dimension two. We prove, in particular, that over a complete local domain of dimension two, an element is in the solid closure of an ideal if and only if it is in the contracted expansion of the ideal from a big Cohen-Macaulay algebra. The point is that in dimension two any solid algebra can be mapped to a big Cohen-Macaulay algebra. But this is false in dimension three in equal characteristic zero.

In §13 we make some further remarks concerning the behavior of solid closure in regular rings. We discuss what is known about which ideals are, or are not, solidly closed. Connections with the Briançon-Skoda theorem are discussed.

The title of the final section is self-explanatory.

To maximize its accessibility the theory of solid closure is presented here as independently as possible of tight closure theory. Nonetheless, it is, most definitely, an offshoot of tight closure theory, and it would not exist without the enormous contributions of Craig Huneke to the development of tight closure.

2. Solid modules and solid algebras

The definitions of solid module and solid algebra were given in (1.1). Although we are mainly interested in the case where R is a Noetherian ring, for the moment we shall not impose any finiteness conditions. The following results give some basic properties of solid modules and algebras. Although most of these results are elementary, they are very important, particularly Corollaries (2.3) and (2.4) and Theorem (2.11). While R is often Noetherian in the applications, it is rare for a solid R-module M to be finitely generated as an R-module: the main case is where M is a finitely generated R-algebra.

(2.1) Proposition. *Let R be a domain.*

(a) *If M, N are solid R-modules (or algebras) then $M \otimes_R N$ is a solid R-module (or algebra). (The same then applies to a nonempty finite family of solid R-modules or solid R-algebras.)*

(b) *If $M \twoheadrightarrow N$ is a surjection of R-modules and N is a solid R-module then M is a solid R-module.*

(c) *If S is an R-algebra and some S-module M is a solid R-module then S is a solid R-algebra.*

(d) *If S is a solid R-algebra then there exists an R-module homomorphism $\alpha\colon S \to R$ such that $\alpha(1) \neq 0$.*

(e) *If M is a solid R-module and S is an extension domain of R then $S \otimes_R M$ is a solid S-module (in particular, we may take S to be any localization of R). In fact, if $\alpha : M \to R$ has image $J \neq 0$, then $id_S \otimes_R \alpha : S \otimes_R M \to S$ has image $JS \neq 0$.*

(f) *If T is a solid R-algebra then every R-algebra S that has an R-homomorphism to T is a solid R-algebra. In particular, T and a polynomial ring (in an arbitrary number of variables over T) are solid or not alike.*

(g) *If M is a solid R-module (or R-algebra) and N is the submodule (or ideal) of M consisting of all elements that are killed by an element of $R - \{0\}$, then M/N is a solid R-module (or algebra). In fact, any R-homomorphism $M \to R$ kills N and so factors $M \twoheadrightarrow M/N \to R$.*

(h) *If M is a solid R-module (or algebra) and $\{I_\lambda\}_\lambda$ is a family of ideals of R such that $\bigcap_\lambda I_\lambda = (0)$, then $M/(\bigcap_\lambda I_\lambda M)$ is a solid R-module (or algebra). In fact, any R-homomorphism $M \to R$ kills $\bigcap_\lambda I_\lambda M$ and so factors $M \twoheadrightarrow M/(\bigcap_\lambda I_\lambda M) \to R$.*

(i) *A finitely presented R-module is solid if and only if it is faithful. In particular, a finitely generated module over a Noetherian domain R is solid if and only if it is faithful.*

(j) *A direct sum of R-modules is solid if and only if at least one of the summands is solid.*

(k) *If M is an R-module such that $N \otimes_R M$ is solid for some choice of R-module N then M is solid.*

(l) *If M is a solid R-module with a finite filtration $M = M_h \supseteq M_{h-1} \supseteq \cdots \supseteq M_0$ then at least one of the factors M_{i+1}/M_i, for $0 \le i \le h-1$, is solid.*

(m) *Let S be a solid R-algebra. If J is an ideal of S such that $J^h = 0$, then S/J is a solid R-algebra. In particular, if S is Noetherian then S_{red} is a solid R-algebra.*

Proof. (a) If $\alpha\colon M \to R$ and $\beta\colon N \to R$ are nonzero maps with images I, J (nonzero ideals of R) then there is a bilinear map $M \times N \to R$ sending (u, v) to $\alpha(u)\beta(v)$, and the corresponding map $\gamma\colon M \otimes_R N \to R$ has nonzero image IJ.

(b) It is clear that if $N \to R$ is nonzero then the composite map $M \twoheadrightarrow N \to R$ is nonzero (it has the same image).

(c) Fix $u \in M$ that has nonzero image under an R-homomorphism $\alpha\colon M \to R$ and define $\beta\colon S \to R$ by $\beta(s) = \alpha(su)$.

(d) Fix a nonzero R-homomorphism $\beta\colon S \to R$ and suppose that $\beta(s) \neq 0$ for some fixed element $s \in S$. Then $\gamma\colon S \to R$ defined by $\gamma(t) = \beta(st)$ has the required property.

(e) This is obvious.

(f) The first statement follows from (c), taking $M = T$ (or from (d), choosing $\alpha: T \to R$ with $\alpha(1) \neq 0$ and composing with the structural R-homomorphism $S \to T$). The second statement follows because each of T and the polynomial ring over T has a T-algebra map to the other (one is the obvious inclusion; for the other, kill the variables), and these T-algebra maps are necessarily R-algebra maps as well.

(g) This is obvious.

(h) This follows from the observation that, since $I_\lambda M$ will map into I_λ for all λ, $\bigcap_\lambda I_\lambda M$ will map into $\bigcap_\lambda I_\lambda = (0)$.

(i) It is clear from (g) that a solid module must be faithful. Now suppose that M is finitely presented and faithful. Let $W = R - \{0\}$. Then $W^{-1}M \neq 0$, and so can be mapped onto the field $F = W^{-1}R$. Since M is finitely presented, $W^{-1}\mathrm{Hom}_R(M, R) \cong \mathrm{Hom}_F(W^{-1}M, F)$, and so $\mathrm{Hom}_R(M, R) \neq 0$.

(j) This is obvious.

(k) Map a (necessarily nonzero) free R-module G onto N. Then $G \otimes_R M$ maps onto $N \otimes_R M$ and so is solid by (b). Since $G \otimes_R M$ is a direct sum of copies of M, M must be solid by (j).

(l) If one has that $0 \to N \to M \to M/N \to 0$ is exact then a map $M \to R$ is either nonzero on N, so that N is solid, or factors through M/N, so that M/N is solid. This handles the case $h = 2$ (the case $h = 1$ is trivial), while the general case follows easily by induction on h.

(m) S has a finite filtration $S \supseteq J \supseteq J^2 \supseteq \cdots J^i \supseteq \cdots \supseteq J^h = (0)$. Hence, by part (l), J^i/J^{i+1} is solid for some i. Since this is an (S/J)-module, S/J is solid by part (c). □

(2.2) Proposition. *Let $R \subseteq S$ be domains. Let M be an S-module viewed also as an R-module via restriction of scalars.*

(a) *If S is embeddable, as an R-module, in a product of copies of R (this is true, for example, if S is embeddable in a free R-module) and M is solid as an S-module then M is solid as an R-module.*

(b) *If $\mathrm{Hom}_R(S, R)$ is embeddable, as an S-module, in a product of copies of S (this is true, for example, if $\mathrm{Hom}_R(S, R)$ is embeddable in a free S-module), and M is solid as an R-module, then M is solid as an S-module.*

Proof. (a) If $S \subseteq P$, where P is a product of copies of R, and $\alpha: M \to S$ is nonzero, then some element of $\alpha(M) \subseteq P$ has a nonzero entry in one coordinate: let $P \to R$ be the product projection corresponding to that coordinate. Then the composite map $M \to S \hookrightarrow P \to R$ is R-linear and nonzero.

(b) Suppose that $\alpha: M \to R$ is nonzero and R-linear. Define $\beta: M \to \mathrm{Hom}_R(S, R)$ to be the S-linear map whose value on $u \in M$ is the homomorphism θ_u defined by $\theta_u(s) = \alpha(su)$. Note that β is nonzero, for if $\alpha(u) \neq 0$ then $\theta_u(1) = \alpha(u) \neq 0$. Suppose that $\mathrm{Hom}_R(S, R) \subseteq Q$, a product of copies of S. Then $\beta(M) \subseteq Q$, and we may choose a product projection $Q \to S$ that is nonzero on some element of $\beta(M)$ (since $\beta(M) \neq 0$). The composite map $M \xrightarrow{\beta} \mathrm{Hom}_R(S, R) \hookrightarrow Q \to S$ yields a nonzero S-linear homomorphism $M \to S$, as required. □

The following immediate corollary is very important in the applications:

(2.3) Corollary (independence of ring for module-finite extensions). *Let $R \subseteq S$ be a module-finite extension of domains, where R is Noetherian. Let M be an S-module. Then M is solid as an S-module if and only if M is solid as an R-module.*

Proof. Since S (respectively, $\mathrm{Hom}_R(S, R)$) is finitely generated and torsion-free as an R-module (respectively, S-module) it is embeddable in a free R-module (respectively, S-module). Thus, both parts of Proposition (2.2) apply. □

This also yields:

(2.4) Corollary (local cohomology criterion). *Let (R, m, K) be a complete local domain of Krull dimension d. An R-module M is solid if and only if $H^d_m(M) \neq 0$.*

Proof. We can represent R as a module-finite extension of a complete regular local ring (A, q, K). Then M is solid as an R-module if and only if it is solid as an A-module. Moreover, since qR is primary to m, $H^d_q(M) = H^d_m(M)$. Thus, there is no loss of generality in assuming that R is regular. In this case $E = H^d_m(R)$ is an injective hull for the residue field of R and $\mathrm{Hom}_R(\_\,, E)$ is faithfully exact. Then $H^d_m(M) \cong M \otimes_R H^d_m(R) = M \otimes_R E$ is nonzero if and only if $\mathrm{Hom}_R(M \otimes_R E, E) \neq 0$. By the adjointness of $\otimes_R$ and Hom_R the latter may be identified with $\mathrm{Hom}_R(M, \mathrm{Hom}_R(E, E)) \cong \mathrm{Hom}_R(M, R)$ (since R is complete, we have that $\mathrm{Hom}_R(E, E) \cong R$, by Matlis duality). Thus, $H^d_m(M) \neq 0$ if and only if its Matlis dual, $\mathrm{Hom}_R(M, R)$, is not zero. □

(2.5) *Remarks on local cohomology.* We use [GrHa] as a general reference for local cohomology.

(a) Let m be an ideal of a Noetherian ring R and let $\mathbf{x} = x_1, \dots, x_d$ be a sequence of elements of R such that m and $I = (\mathbf{x})R$ have the same radical. Then the functors $H^j_I(\_)$ and $H^j_m(\_)$ may be identified. (E.g., if (R, m, K) is local we may choose $\mathbf{x}$ to be a system of parameters for R.) Let $x = x_1 \cdots x_d$ and for each i, $1 \leq i \leq d$, let $y_i = x_1 \cdots x_{i-1} x_{i+1} \cdots x_d$, so that $x_i y_i = x$ for $1 \leq i \leq d$. Then $H^d_m(R) \cong H^d_I(R)$ may be identified with $R_x / (\sum_i \mathrm{Im}\, R_{y_i})$ and if M is any R-module we have that $H^d_m(M) \cong H^d_m(R) \otimes_R M \cong M_x / (\sum_i \mathrm{Im}\, M_{y_i})$. (We have written "Im" for precision: the maps $M_{y_i} \to M_x$ may not be injective if one or more of the x_i is a nonzerodivisor.)

(b) With notation as in (a), let $\mathbf{x}^t$ denote $x_1^t, \dots, x_d^t$. Note that, alternatively, we may view $H^d_m(R)$ as $\varinjlim_t R/(\mathbf{x}^t)$ where the maps between consecutive terms in the direct limit system are induced by multiplication by $x = x_1 \cdots x_d$ on the copies of R in the numerators, or as $(R_{x_1}/\mathrm{Im}\, R) \otimes_R \cdots \otimes_R (R_{x_d}/\mathrm{Im}\, R)$. From this last characterization it is easy to see that if m has the same radical as $(x_1, \dots, x_d)R$, n has the same radical as $(z_1, \dots, z_e)R$ and q has the same radical as $m + n$, then $H^{d+e}_q(R) \cong H^d_m(R) \otimes_R H^e_n(R)$.

(c) As a consequence of the discussion in (a), if R is complete local ring with system of parameters $\mathbf{x}$, then M is solid if and only if $M_x \neq \sum_i \mathrm{Im}\, M_{y_i}$.

(d) If S is an R-algebra then S_x and $\sum_i \mathrm{Im}\, S_{y_i} \subseteq S_x$ are both S-modules, and S_x is generated as an S-module by the elements $1/x^t$. Let z_i denote the image of y_i in S_x. Then $H^d_m(S) \neq 0$ if and only if for some t (equivalently, for all large t) $1/x^t \notin \bigcup_N (\sum_i S \cdot (1/z_i)^N)$ in S_x. This holds if and only if there exists t such that (equivalently, for all sufficiently large t) $x^{N-t} \notin (x_1^N, \dots, x_d^N)S$ for all $N \in \mathbb{N}$ (if the x's are zerodivisors one can still multiply by a sufficiently high power of x to get a valid equation with denominators cleared). Said slightly differently:

(2.6) Observation. *Let R be a Noetherian ring, let $x_1, \dots, x_d \in R$, let m be an ideal of R having the same radical as $(x_1, \dots, x_d)R$, and let S be an R-algebra. (E.g., we may have that (R, m, K) is a local ring of dimension d and that $x_1, \dots, x_d$ is a system of parameters.) Then $H^d_m(S) \neq 0$ if and only if there exists an integer t such that whenever $(x_1 \cdots x_d)^k \in (x_1^N, \dots, x_d^N)S$ for $N, k \in \mathbb{N}$ then $N \leq k + t$.* $\square$

(2.7) *Remark.* Let $(R, m, K) \to (S, n, L)$ be a local homomorphism of local rings and let $\mathbf{x} = x_1, \dots, x_d$ be a system of parameters for R. Note that the images of the x's in S are part of a system of parameters for S if and only if killing them drops the dimension of S by $d = \dim R$. Since the radicals of $(\mathbf{x})S$ and mS are the same, an equivalent condition is that (#) $\dim S = \dim R + \dim S/mS$, and this condition is evidently independent of the system of parameters. It suffices if ht $mS \geq \dim R$ (in which case ht $mS = \dim R$, since mS is generated up to radicals by $\dim R$ elements), and the condition that ht $mS = \dim R$ is equivalent to (#) if S is equidimensional and catenary (for then ht $J + \dim S/J = \dim S$ for every ideal J of S).

Corollary (2.4) and the above discussion imply:

(2.8) Corollary. *Let (R, m, K) be a complete local domain, let (S, n, L) be a local ring, and let $(R, m, K) \to (S, n, L)$ be a local homomorphism such that*

$$dim\, S = dim\, R + dim\, S/mS.$$

Let M be any S-module such that $H^{\dim S}_n(M) \neq 0$; if S is a complete local domain this simply means that M is solid as an S-module. (The condition $H^{\dim S}_n(M) \neq 0$ holds for any finitely generated S-module with $dim\, M = dim\, S$.) Then M is a solid R-module.

Proof. Let $d = \dim R$. Choose a system of parameters $x_1, \dots, x_d$ for R. The hypothesis implies that $x_1, \dots, x_d$ can be extended to a system of parameters $x_1, \dots, x_d, z_1, \dots, z_e$ for S, where $\dim S = d + e$. (For finitely generated modules over a local ring, the highest nonvanishing local cohomology module with support in the maximal ideal occurs at the dimension of the module.) In any case, assume that $H^{d+e}_n(M) \neq 0$. This module may be identified with $H^{d+e}_n(S) \otimes_S M$. Let I be the ideal generated by the x's in R and let J be the ideal generated by the z's in S. By (2.5c) we have that $H^{d+e}_n(S) \cong H^d_{IS}(S) \otimes_S H^e_J(S)$ and so $H^{d+e}_n(M) \cong \big(H^d_{IS}(S) \otimes_S H^e_J(S)\big) \otimes_S M \cong \big(H^d_I(R) \otimes_R S\big) \otimes_S H^e_J(S) \otimes_S M \cong H^d_I(R) \otimes_R \big(S \otimes_S M \otimes_S H^e_J(S) \cong H^d_m(R) \otimes_R \big(M \otimes_S H^e_J(S)\big) \cong \big(H^d_m(R) \otimes_R M\big) \otimes_S H^e_J(S) \cong H^d_m(M) \otimes_S H^e_J(S)$. Since this module is not zero, we must have that $H^d_m(M) \neq 0$, and then Corollary (2.4) shows that M is solid over R. $\square$

The following result (that solid extensions tend not to adjoin fractions) is part of the reason for the use of the word "solid."

(2.9) Proposition. *Let R be a Noetherian domain, let $a, b \in R$ with $b \neq 0$ and let S be a solid R-algebra. Suppose that $a \in bS$. Then the element a/b of the fraction field of R is integral over R. Hence, if R is normal, $a/b \in R$.*

In consequence, if S is a domain solid over the Noetherian domain R and $\theta \in S$ is algebraic over R, then θ is integral over R.

Proof. We begin with the proof of the statement in the first paragraph. We may assume that $a \neq 0$. Since the normalization of R in its fraction field F is an intersection of discrete valuation rings contained in F, if a/b is not integral over R we may choose a discrete valuation ring V with $R \subseteq V \subseteq F$ and $a/b \notin V$. We may replace R, S by $V, V \otimes_R S$ by Proposition (2.1e). Thus, we may assume without loss of generality that $R = V$ is a discrete valuation ring and that $a/b \notin V$. By Proposition (2.1g) we may replace S by a quotient that is torsion-free over V. Then $c = b/a$ is in the maximal ideal of V, and we have that $a \in bS = acS$. Since a is not a zerodivisor in S, it follows that $1 \in cS$, i.e., that c is a unit in S, and so $S = c^n S$ for every positive integer n. But then, by Proposition (2.1h), $S = \bigcap_n c^n S$ must map to zero in V, since $\bigcap_n c^n V = (0)$. This is a contradiction.

Now consider the situation in the second paragraph. Since θ is algebraic over R it satisfies an equation $\sum_{i=0}^n b_i \theta^i = 0$ for some positive integer n with the $b_i \in R$ and $b = b_n \neq 0$. It follows that $b\theta$ is integral over R and so $R[b\theta]$ is module-finite over R. Then S is also solid over $R[b\theta]$ by Theorem 2.3. Since $b\theta \in bS$ it follows from the statement in the first paragraph that the fraction $b\theta/b = \theta$ is integral over $R[b\theta]$, and since this ring is module-finite over R, θ is integral over R. □

(2.10) *Discussion.* In Proposition (2.1e) it was noted that, quite trivially, if M is a solid R-module and S is an extension domain of R then $S \otimes_R M$ is a solid S-module. Somewhat surprisingly, when R is a Noetherian domain this remains true without the hypothesis that the map of domains $R \to S$ be injective. Before giving the proof, we note the following:

(2.11) Lemma. *Let J be a nonzero ideal of a Noetherian domain R and let $P_1, \dots, P_k$ be height one prime ideals of R.*

(a) *If each of the rings R_P is a discrete valuation ring for $P \in \{P_1, \dots, P_n\}$ then there is an R-homomorphism $\theta: J \to R$ such that θ takes on a value not in any of the P_i, i.e., $\theta(J) \nsubseteq \bigcup_i P_i$.*

(b) *There is a module-finite extension R' of R within its fraction field and an R'-linear map $JR' \to R'$ that takes on a value outside any of the primes of R' that lie over one of the P_i: in fact, it takes on a value in R outside any of the P_i.*

Proof. Let $W = \bigcup_i P_i$.

(a) $W^{-1}R$ is a semilocal Dedekind domain and, hence, a PID. It follows that there is an isomorphism $\phi: W^{-1}J \to W^{-1}R$ as modules over $T = W^{-1}R$. Since $\mathrm{Hom}_T(W^{-1}J, T) \cong W^{-1}\mathrm{Hom}_R(J, R)$, ϕ has the form $w^{-1}\theta$ for some $w \in W$ and $\theta \in \mathrm{Hom}_R(J, R)$. Since $\mathrm{Im}\,\phi$ contains $1 \in W^{-1}R$, we have that θ takes on the value $w \in W$.

(b) The normalization of a one-dimensional semilocal Noetherian domain D is also a one-dimensional semilocal Noetherian domain, although it need not, in general, be module-finite over D. (Cf. [N], Theorem 33.2, which guarantees that the integral closure is Noetherian. The fact that there are only finitely many maximal ideals follows easily from the fact that there only finitely many maximal ideals in D.) Such a normalization is therefore a PID. Let S be the normalization of R. Then $W^{-1}S$ is the normalization of $W^{-1}R$, which is a one-dimensional semilocal ring. Thus, $W^{-1}S$ is a PID, and we can choose a $W^{-1}S$-linear map $\phi: JW^{-1}S \to W^{-1}S$ that is an isomorphism. Suppose that

u/w' maps to 1 under this homomorphism, where $u \in JS$ and $w' \in W$. Then u maps to w'. Choose a finite set of generators j_ν of J as an R-module and choose $w \in W$ such that the elements $\phi(j_\nu)$ are all of the form s_ν/w with every $s_\nu \in S$. Choose a module-finite extension R' of R contained in S such that $u \in JR'$ and such that the s_ν are in R'. The restriction of $w\phi$ to JR' has image contained in R' (if we have several elements $a_\nu \in R'$ then $w\phi$, which is $W^{-1}S$-linear, maps $\sum_\nu a_\nu j_\nu$ to $\sum_\nu a_\nu w\phi(j_\nu) = \sum_\nu a_\nu s_\nu$, which is in R'.) Thus, the restriction of $w\phi$ to JR' yields an R'-linear map of JR' to R' whose value on u is $ww' \in W$. □

We are now ready to prove a very important result:

(2.12) Theorem (persistence of solidity). *Let R be a Noetherian domain and let M be a solid R-module. Then for any homomorphism $R \to S$, where S is a domain, $S \otimes_R M$ is a solid S-module.*

Proof. Let $P = \mathrm{Ker}\,(R \to S)$ and let $P = P_h \supset \cdots \supset P_0 = (0)$ be a saturated chain of primes in R descending from P. The map $R \to S$ factors $R \to R/P_1 \to \cdots \to R/P_h \hookrightarrow S$. By the associativity of $\otimes$, in order to prove the result for the composite map it suffices to prove it for each map in the chain. The problem of proving that $S \otimes_R M$ is solid over S thereby reduces to proving the result in two cases: one is the case of an inclusion, and the other is the case where the domain S is obtained from the domain R by killing a height one prime. Since we have already done the case of an inclusion, we may assume that $S = R/P$, where P is a height one prime of S.

Fix a nonzero homomorphism $\alpha\colon M \to R$ with image $J \neq (0)$. By Lemma (2.11) we can choose a module-finite extension domain R' of R within the fraction field of R and an R'-linear map $\theta\colon JR' \to R'$ that takes on a value in $R - P$. Let Q be a prime ideal of R' lying over P. Then $M' = R' \otimes_R M$ is solid over R', by Proposition (2.1e), and, in fact, $\mathrm{id}_{R'} \otimes_R \alpha$ gives an R'-linear map α' of $R' \otimes_R M$ to R' with image JR'. It follows that the composite map $\beta = \theta\alpha'$ from M' to R' takes on a value in $R - P$: this value will be an element of $R' - Q$. Thus, if we tensor with $S' = R'/Q$ we obtain a nonzero (R'/Q)-module homomorphism $M'/QM' \to R'/Q$, so that M'/QM' is solid over R'/Q.

Now, R'/Q is a module-finite extension of R/P, and so it follows from Corollary (2.3) that M'/QM' is solid over R/P. But then

$$M'/QM' \cong (R'/Q) \otimes_R M \cong (R'/Q) \otimes_{R/P} \big((R/P) \otimes_R M\big)$$

is solid over R/P, and it follows from Proposition (2.1k) that $(R/P) \otimes_R M$ is solid over R/P, as required. □

3. Formally solid modules and algebras

We begin with two definitions.

(3.1) Definition. *Let m be a maximal ideal of R. We shall refer to the R-algebra obtained by completing the local ring R with respect to its maximal ideal and then killing a minimal prime as a complete local domain of R.*

(3.2) Definition. *Let R be a Noetherian ring. We shall say that an R-module M is formally solid if for every complete local domain B of R, $B \otimes_R M$ is solid over B.*

From this definition we have at once:

(3.3) Proposition. *Let R be a Noetherian ring and let M be an R-module.*
(a) *M is formally solid if and only if for every maximal ideal m of R, M_m is formally solid over R_m.*
(b) *If R is local, M is formally solid over R if and only if $\widehat{R} \otimes_R M$ is formally solid over $\widehat{R}$.*
(c) *If R is complete local, M is formally solid if and only if for every minimal prime $\mathfrak{p}$ of R, $(R/\mathfrak{p}) \otimes_R M$ is formally solid over $R/\mathfrak{p}$.*
(d) *If R is complete local, M is formally solid over R if and only if M is solid over R.* □

We also note:

(3.4) Proposition. *Let R be a Noetherian ring.*
(a) *If M, N are formally solid R-modules (or algebras) then $M \otimes_R N$ is a formally solid R-module (or algebra). (The same then applies to a nonempty finite family of solid R-modules or solid R-algebras.)*
(b) *If $M \twoheadrightarrow N$ is a surjection of R-modules and N is a formally solid R-module then M is a formally solid R-module.*
(c) *If S is an R-algebra and some S-module M is a formally solid R-module then S is a formally solid R-algebra.*
(d) *If an R-algebra has S has an R-algebra map to a formally solid R-algebra T, then S is formally solid. In particular, T and a polynomial ring (in an arbitrary number of variables) over T are formally solid or not alike.*
(e) *M is formally solid over R if and only if $M_{red} = R_{red} \otimes_R M$ is formally solid over R_{red} (if and only if M_{red} is formally solid over R).*
(f) *A finitely generated R-module M is formally solid if and only if $M_{red} = R_{red} \otimes_R M$ is faithful over R_{red}.*
(g) *If M is an R-module such that $N \otimes_R M$ is formally solid for some choice of R-module N then M is formally solid.*

Proof. (a) This follows from the fact that $B \otimes_R (M \otimes_R N) \cong (B \otimes_R M) \otimes_B (B \otimes_R N)$ and part (a) of Proposition (2.1).

(b) This follows from the right exactness of $\otimes$ and part (b) of Proposition (2.1).

(c) Let B be a complete local domain of R and apply part (c) of Proposition (2.1) to $B \otimes_R M$ viewed as a module over $B \otimes_R S$.

(d) This is immediate from part (c).

(e) This is immediate from the fact that the complete local domains of R are the same as the complete local domains of R_{red}, viewed as R-algebras.

(f) By part (e), we may assume that R is reduced. Note that M is faithful if and only if R can be embedded in a finite direct sum of copies of M. Thus, faithfulness is preserved by flat base change, and is preserved when we localize and complete. Hence, if M is faithful, it remains faithful when we pass to the completed localization of R at a maximal ideal. If $\mathfrak{p}$ is a minimal prime of R, then $(R/\mathfrak{p}) \otimes_R M$ is still supported at $\mathfrak{p}$, and so $(R/\mathfrak{p}) \otimes_R M$

is faithful over $R/\mathfrak{p}$, and hence, solid over $R/\mathfrak{p}$. This shows that if M_{red} is faithful over R_{red}, then M is formally solid.

Now suppose that R is reduced but that M is not faithful. Then M is killed by some element x that is not nilpotent. Localize R at a maximal ideal m containing $\bigcup_t \mathrm{Ann}_R x^t$. Then the image of x is still not nilpotent in R_m, and kills M_m. Likewise, the image of x in the completion C of R is not nilpotent, and so we may kill a minimal prime $\mathfrak{p}$ of C to obtain a complete local domain B of R such that the image of x is nonzero in B.

(g) For every complete local domain B of R, $B \otimes_R (N \otimes_R M) \cong (B \otimes_R N) \otimes_B (B \otimes_R M)$ is solid over B, and so $B \otimes_R M$ is solid over B by Proposition (2.1k). $\square$

In (3.7) we shall prove that the property of being formally solid is preserved by arbitrary Noetherian base change. We first note:

(3.5) Lemma. *Let M be a formally solid module over a complete local ring R and let S be a Noetherian R-algebra. Then $S \otimes_R M$ is formally solid over S.*

Proof. It suffices to prove that $C \otimes_R M$ is solid over C after a further base change to a complete local domain C. Therefore, we may assume without loss of generality that $S = C$ is a complete local domain. Choose a minimal prime $\mathfrak{p}$ of R contained in $\mathrm{Ker}\,(R \to C)$. Then $(R/\mathfrak{p}) \otimes_R M$ is solid over $R/\mathfrak{p}$ by the hypothesis on M, and we may apply Theorem (2.12) to the base change $R/\mathfrak{p} \to C$. $\square$

We shall also need:

(3.6) Lemma. *Let $(R, m, K) \to (S, n, L)$ be a local homomorphism of local rings and suppose that at least one of the following conditions holds:*
(a) *R, S are complete local domains and ht $mS \geq$ ht m ($=$ dim R).*
(b) *R, S are complete local rings and for every minimal prime $\mathfrak{p}$ of R there is a prime ideal $\mathfrak{q}$ of S lying over $\mathfrak{p}$ such that ht $m(S/\mathfrak{q}) \geq$ ht $(m/\mathfrak{p})$ ($=$ dim $R/\mathfrak{p}$).*
(c) *S is faithfully flat over R.*

Then every S-module W that is formally solid as an S-module is formally solid as an R-module.

Moreover, if M is an R-module such that $S \otimes_R M$ is formally solid as S-module then M is formally solid as an R-module.

Proof. The second conclusion follows from the first conclusion by Proposition (3.4g).

(a) By (2.8) (see also Remark (2.7)) and the height condition on S, we have that W is solid over R.

(b) Fix a minimal prime $\mathfrak{p}$ of R and fix a prime ideal $\mathfrak{q}$ of S such that $\mathfrak{q}$ lies over $\mathfrak{p}$ and ht $m(S/\mathfrak{q}) \geq$ ht $m/\mathfrak{p}$. Since W is formally solid over S, by (3.3e) it is solid over S. By Theorem (2.12), $(S/\mathfrak{q}) \otimes_S W \cong S/\mathfrak{q} \otimes_{R/\mathfrak{p}} (R/\mathfrak{p} \otimes_R W)$ is solid over $S/\mathfrak{q}$, and so part (a) applied to $R/\mathfrak{p} \to S/\mathfrak{q}$ shows that $R/\mathfrak{p} \otimes_R W$ is solid over $R/\mathfrak{p}$. Since this holds for every minimal prime $\mathfrak{p}$ of R, we see that (b) is implied by (a) (which is, of course, a special case of (b)).

(c) The map $\widehat{R} \to \widehat{S}$ is also faithfully flat and, in consequence, satisfies the condition in (b). To see this, note that if $\mathfrak{p}$ is a minimal prime of R then $S/\mathfrak{p}S$ is faithfully flat over $R/\mathfrak{p}$, and we can choose a minimal prime $\mathfrak{q}$ of $\mathfrak{p}S$ (which will be a minimal prime of S, since $S_\mathfrak{q}$ is nilpotent modulo $\mathfrak{p}S_\mathfrak{q}$ and $\mathfrak{p}R_\mathfrak{p}$ is nilpotent) such that $\dim S/\mathfrak{q} = \dim S/\mathfrak{p}S$.

Now, $\dim S/\mathfrak{p}S = \dim R/\mathfrak{p} + \dim\big((S/\mathfrak{p}S)/m(S/\mathfrak{p}S)\big)$ ($= \dim R/\mathfrak{p} + \dim S/mS$), by the flatness of $S/\mathfrak{p}S$ over $R/\mathfrak{p}$ (cf. [Mat] (13.B) Theorem 19 (2)). Thus,

$$\dim S/\mathfrak{q} = \dim R/\mathfrak{p} + \dim(S/mS) \geq \dim R/\mathfrak{p} + \dim\big((S/\mathfrak{q})/m(S/\mathfrak{q})\big),$$

so that the dimension of $S/\mathfrak{q}$ falls by at least $\dim R/\mathfrak{p}$ when one kills $m(S/\mathfrak{q})$, and this shows that ht $m(S/\mathfrak{q}) \geq \dim R/\mathfrak{p}$, as required.

Now, the fact the W is formally solid over S implies that $\widehat{S} \otimes_R W \cong \widehat{S} \otimes_{\widehat{R}} (\widehat{R} \otimes_R W)$ is formally solid over $\widehat{S}$, and so, by part (b), over $\widehat{R}$. This implies that $\widehat{R} \otimes_R W$ is formally solid over $\widehat{R}$, and so W is formally solid over R, as required. □

Part (c) of this result is globalized in Theorem (3.8) below.

Notice that in part (b) there may be a suitable choice of $\mathfrak{q}$ although there is no *minimal* prime of S that gives a suitable choice of $\mathfrak{q}$. For example, let $R = K[[x, y]]$ with K a field and let $S = R[[u, v]]/(xv - yu)$. Then $R \subseteq S$ and has the unique minimal prime $\mathfrak{p} = (0)$, and ht $mS = 1$, so that we may not choose $\mathfrak{q} = (0)$ (the only minimal prime of S), but we may take $\mathfrak{q} = (u, v)S$.

We can now prove the following result of basic importance:

(3.7) Theorem (persistence of formal solidity). *If M is a formally solid module over a Noetherian ring R and S is a Noetherian R-algebra then $S \otimes_R M$ is formally solid over S.*

Proof. It suffices to prove this when S is replaced by a complete local domain to which it maps. We may therefore assume that S is a complete local domain. Let P be the contraction of the maximal ideal of S to R, let m be a maximal ideal of R containing P, and let T be the completion of the ring R_m. Let Q be a prime ideal of T lying over P. Then the map $R \to S$ factors $R \to R_m \to R_P \to (R_P)^\frown \to S$ while the faithfully flat map $R_P \to T_Q$ induces a faithfully flat map $(R_P)^\frown \to (T_Q)^\frown$, and the map $R \to (R_P)^\frown \to (T_Q)^\frown$ also factors $R \to R_m \to T \to (T_Q)^\frown$. The fact that M is formally solid over R is preserved when we make a base change from R to T by Proposition (3.3a,b), and since T is complete we may then make a further base change to $(T_Q)^\frown$ by Lemma (3.5). By Lemma (3.6), since $(T_Q)^\frown$ is faithfully flat over $(R_P)^\frown$, we have that $(R_P)^\frown \otimes_R M$ is formally solid over $(R_P)^\frown$. But we may then make a base change from $(R_P)^\frown$ to S by a second application of Lemma (3.5), since $(R_P)^\frown$ is complete. □

(3.8) Theorem. *If the Noetherian ring S is faithfully flat over a Noetherian ring R and W is a formally solid S-module then W is a formally solid R-module.*

Hence, if M is an R-module such that $S \otimes_R M$ is formally solid over S, then M is formally solid over R.

Proof. As in the proof of (3.6), the second conclusion follows from the first.

Let m be a maximal ideal of R and let Q be a prime ideal of S lying over m. Then $R_m \to S_Q$ is faithfully flat. By Theorem (3.7), W_Q is formally solid over S_Q. By Theorem (3.6), W_Q is formally solid over R_m. Since $W_Q \cong S_Q \otimes_{R_m} W_m$ it follows from Proposition (3.4g) that W_m is formally solid over R as well. Since this holds for every maximal ideal m of R, the result follows from Proposition (3.3a). □

We also note:

(3.9) Proposition. *If R is Noetherian, a faithfully flat R-module M is formally solid.*

Proof. The hypothesis is preserved when we pass from R to a complete local domain of R. Thus, we may assume that (R, m, K) is a complete local domain with $\dim R = d$. But then $H^d_m(M) \cong H^d_m(R) \otimes_R M \neq 0$. $\square$

4. Generic forcing algebras

(4.1) Definition. *Let R be a Noetherian ring, let $N \subseteq M$ be finitely generated R-modules, and let $u \in M$. We shall say that an R-algebra S is a forcing algebra for the triple (M, N, u) if the image of u in $S \otimes_R M$ is in $\mathrm{Im}(S \otimes_R N \to S \otimes_R M)$.*

It is trivial to verify that:

(4.2) Observation. *The R-algebra S is a forcing algebra for the triple (M, N, u) if and only if it is a forcing algebra for the triple $(M/N, 0, \overline{u})$, where $\overline{u}$ denotes the image of u in M/N. If T is a forcing algebra for (M, N, u) and there is a homomorphism $T \to S$ then S is also a forcing algebra for (M, N, u).* $\square$

(4.3) *Discussion and definition.* Let $R^k \xrightarrow{A} R^h \to M/N \to 0$ be a finite presentation of M/N, where $A = (r_{ij})$ is an $h \times k$ matrix over R that gives the map and if we choose a vector $(r_1, \dots, r_h) \in R^h$ that represents $\overline{u}$ in $R^h/\mathrm{Im}\, R^k \cong M/N$, then S will be a forcing algebra for the triple (M, N, u) if and only if the image of $\overline{u}$ in $S \otimes_R (M/N)$ is 0, which is equivalent to the statement that the image of $(r_1, \dots, r_h)$ is zero in $S \otimes_R (R^h/\mathrm{Im}\, R^k)$. If we write $(r_1, \dots, r_h)$ instead as a column vector ρ this is equivalent to the existence of a column vector σ in S^k such that $A\sigma$ is the image of ρ.

We can therefore construct a forcing algebra for (M, N, u) by adjoining indeterminates $X_1, \dots, X_h$ to R and then killing the ideal generated by the entries of the matrix $AX - \rho$ in the polynomial ring $R[X_1, \dots, X_h]$, where X is the column vector whose entries are the X_i. We shall refer to this algebra as the *generic forcing algebra* for (M, N, u) for the data A, ρ. We shall also say that it is a *generic forcing algebra* for (M, N, u). Let T be the generic forcing algebra for the data A, ρ. Given any other forcing algebra S for (M, N, u), we can map T to S by sending the X_i to the entries of the vector σ whose existence was observed above.

Thus, if one fixes a generic forcing algebra for (M, N, u), all other forcing algebras may be viewed as, simply, the algebras to which it maps. This explains the terminology.

If $I = (a_1, \dots, a_k)R$ and $u = r \in R$ then there is a generic forcing algebra for (R, I, u) of the particularly simple form

$$R[X_1, \dots, X_h]/(\sum_{i=1}^{k} a_i X_i - r).$$

Here, we are using the data $A = (\, a_1 \quad \dots \quad a_k \,)$ and $\rho = (\, r \,)$.

With this terminology, we observe the following:

(4.4) Proposition. *Let R be a Noetherian ring. Let $N \subseteq M$ be finitely generated R-modules and let $u \in M$. Let T be a generic forcing algebra for (M, N, u) for the data A, ρ.*

(a) *If S is any R-algebra, let A', ρ' denote the images of A, ρ under the homomorphism $R \to S$ applied to every entry. Then $S \otimes_R T$ is a generic forcing algebra for the triple $(S \otimes_R M, S \otimes_R N, 1 \otimes u)$ for the data A', ρ'.*

(b) *T maps as an R-algebra to every other forcing algebra for the triple (M, N, u).* $\square$

(4.5) *Discussion.* It is natural to ask how the generic forcing algebras for (M, N, u) over R vary with choice of data A, ρ. Roughly speaking, they are uniquely determined as R-algebras up to "adjunction of indeterminates": see Proposition (4.6) below for a precise statement. We make the following observations:

(a) If one changes only ρ the algebra does not change, up to R-isomorphism. For ρ will change by adding a vector of the form $A\beta$, where β has entries in R, and the matrix $AX - (\rho + A\beta)$ becomes $AX' - \rho$ if we let $X' = X - \beta$, i.e., if we make a translation of coordinates over R.

(b) Suppose that we change the presentation of M/N by using the same generators for M but a possibly different set of generators for the relations. We may understand the effect on the generic forcing algebra by comparing each of the algebras with the algebra obtained by using the union of the two sets of relations. This enables us to consider only what happens when one enlarges the set of relations by adjoining some additional "redundant" relations. This is equivalent to giving the matrix A some additional columns, each of which is a linear combination of the columns already present.

For simplicity, we may consider the case where there is just one column, which we may suppose has the form $A\delta$ for some column vector δ over R. Then there is one new variable, X_{h+1}, and if X is the column vector whose entries are $X_1, \ldots, X_h$, then to define the new generic forcing algebra we must kill the entries of $AX + A\delta X_{h+1} - \rho = AX' - \rho$ if we let $X' = X + X_{h+1}\delta$. We see that by using coordinates X', X_{h+1} (this is a linear change of coordinates) we have changed the R-isomorphism class of the generic forcing algebra by adjoining an indeterminate.

Thus, if S, T are two generic forcing algebras for (M, N, u) that arise from data that differs only by altering the relations chosen in the presentation of M/N, then there are indeterminates $\mathbf{Y} = Y_1, \ldots, Y_\mu$ and $\mathbf{Z} = Z_1, \ldots, Z_\nu$ such that $S[\mathbf{Y}] \cong T[\mathbf{Z}]$ over R.

(c) Suppose that we change the generators of M. By comparing what happens for each of two sets of generators with what happens when their union is used, we see that it suffices to understand what happens when a set of generators is enlarged, and it is enough to consider the case where we insert a single new redundant generator. Indeed, we may assume that we have chosen generators $u_1, \ldots, u_h$ for M and that the new generator is $-(a_1u_1 + \cdots + a_hu_h)$. Then we get a new presentation in which the matrix, in block form, is $\begin{pmatrix} A & \alpha \\ 0 & 1 \end{pmatrix}$, where α is a column vector whose entries are the a's, the 0 is a row vector of length k, and the element 1 represents a size one block with the single entry 1. We may then take the replacement of ρ to be $\begin{pmatrix} \rho \\ 0 \end{pmatrix}$. If we write our vector of indeterminates as

$X' = \begin{pmatrix} X \\ X_{h+1} \end{pmatrix}$, where X retains its former meaning, then

$$\begin{pmatrix} A & \alpha \\ 0 & 1 \end{pmatrix}\begin{pmatrix} X \\ X_{h+1} \end{pmatrix} - \begin{pmatrix} \rho \\ 0 \end{pmatrix} = \begin{pmatrix} AX + \alpha X_{h+1} - \rho \\ X_{h+1} \end{pmatrix}.$$

Killing the entries of this matrix produces the same generic forcing algebra as before, up to R-isomorphism.

We can summarize the content of this discussion as follows:

(4.6) Proposition. *If R is a Noetherian ring, $N \subseteq M$ are finitely generated R-modules, $u \in M$, and S, T are two generic forcing algebras for (M, N, u) for possibly different data, then there are finite sets of indeterminates $\mathbf{Y}$, $\mathbf{Z}$ such that $S[\mathbf{Y}] = T[\mathbf{Z}]$ as R-algebras.* □

5. Solid closure

We first want to extend Definition (1.2) to the case of submodules of finitely generated modules.

(5.1) Definition. *Let R be a Noetherian ring, let $N \subseteq M$ be finitely generated R-modules, and let $u \in M$. If R is a complete local domain we say that u is in the solid closure $N^\star$ (or $N^\star{}_M$) of N in M over R if there is a solid R-algebra S such that the image $1 \otimes u$ of u in $S \otimes_R M$ is in Im$(S \otimes_R N \to S \otimes_R M)$. In other words, $u \in N^\star$ if and only if (M, N, u) has a forcing algebra S that is solid as an R-algebra.*

In the general case, we say that x is in the solid closure $N^\star$ (or $N^\star{}_M$) of N in M over R if for every complete local domain B of R, the image of x in $B \otimes_R M$ is in the solid closure of Im$(B \otimes_R N \to B \otimes_R M)$ in $B \otimes_R M$ over B. In other words, every complete local domain B of R has a solid B-algebra S such that the image of x in $S \otimes_R M$ is in Im$(S \otimes_R N \to S \otimes_R M)$.

(5.2) *Discussion and notations.* Let $\overline{x}$ denote the image of x in M/N. For any given R-algebra S, the image of x in $S \otimes_R M$ is in Im$(S \otimes_R N \to S \otimes_R M)$ if and only if the image of $\overline{x}$ in $S \otimes_R (M/N)$ is 0. It follows that x is in $N^\star{}_M$ if and only if $\overline{x}$ is in $0^\star{}_{M/N}$. We may also map a finitely generated free module G onto M, let H be the inverse image of N in G and let y be an element of G that maps to x under the surjection $G \twoheadrightarrow M$. Then $x \in N^\star{}_M$ if and only if $y \in H^\star{}_G$.

It follows that issues about solid closures of submodules may be reduced either to the case where the submodule is zero, or to the case where the ambient module is free.

When G is free we denote by HS the module Im$(S \otimes_R H \to S \otimes_R G)$. This is analogous to the notation used when $G = R$ and H is an ideal of R. When the base ring, say R, is understood we shall often use the notation M_S for $S \otimes_R M$, and the notation $\langle N_S \rangle$ to denote Im$(N_S \to M_S)$. Of course, $\langle N_S \rangle$ depends not only on N and S but also on the map $N \to M$, the base ring R, and the R-algebra structure of S. Nonetheless, this notation is convenient, and it will only be used when the ambiguity is not a problem.

Solid closure may be described alternatively in terms of formal solidity for generic forcing algebras.

(5.3) Proposition. *Let R be a Noetherian ring, let $N \subseteq M$ be finitely generated R-modules, and let $u \in M$. Then the following conditions are equivalent:*

(a) $u \in N^{\bigstar}{}_M$.
(b) *For every complete local domain B of R, there is a solid forcing algebra for the triple $(B \otimes_R M, B \otimes_R N, 1 \otimes u)$ over B.*
(c) *For every complete local domain B of R, there is a finitely generated solid forcing algebra for $(B \otimes_R M, B \otimes_R N, 1 \otimes u)$ over B.*
(d) *For every complete local domain B of R, some (equivalently, every) generic forcing algebra for $(B \otimes_R M, B \otimes_R N, 1 \otimes u)$ over B is solid.*
(e) *Some forcing algebra for (M, N, u) over R is formally solid.*
(f) *Some (equivalently, every) generic forcing algebra for (M, N, u) over R is formally solid.*

Proof. (a) and (b) are equivalent by virtue of the definition of solid closure. The weak form of (d) ("some") implies (c) and (c) evidently implies (b). On the other hand, (b) implies the strong form of (d) ("every"), since if T is a forcing algebra over B then for every generic forcing algebra S, S maps as a B-algebra to T and we may apply Proposition (2.1f). This shows that (a), (b), (c) and both forms of (d) are equivalent. Assume these conditions. Let S_R denote a generic forcing algebra for (M, N, u) over R. Then for every complete local domain B of R, $S_B = B \otimes_R S_R$ is a generic forcing algebra for $(B \otimes_R M, B \otimes_R N, 1 \otimes u)$ over B. It is then clear that if some choice of S_R is formally solid then the weak form of (d) holds, while if the strong form of (d) holds then every choice of S_R must be formally solid. Thus, both the strong and weak forms of (f) are equivalent to (d). Since (f) $\Rightarrow$ (e) $\Rightarrow$ (d) in its weak form (similarly), all these conditions are equivalent. $\square$

In the next two propositions we explore the basic properties of solid closure without using the notion of a generic forcing algebra nor the notion of formal solidity. There is little cost in doing so. However, these notions provide a very easy proof for Theorem (5.6), given the stability theorem for formally solid modules that we have already established.

(5.4) Proposition. *Let R be a Noetherian ring, let M be a finitely generated R-module, let N, N', and P denote submodules of M, and let I denote an ideal of R. If there is no subscript on a solid closure, the subscript is understood to be M.*

(a) *$N^{\bigstar}{}_M$ is a submodule of M containing N.*
(b) *If $P \subseteq N \subseteq M$ then $P^{\bigstar}{}_M \subseteq N^{\bigstar}{}_M$ and $P^{\bigstar}{}_N \subseteq P^{\bigstar}{}_M$.*
(c) *If $N_1 \subseteq M_1, \ldots, N_h \subseteq M_h$ are finitely many inclusions of finitely generated modules over R then for every complete local domain B of R there is a solid B-algebra S such that the image of ${N_i}^{\bigstar}{}_{M_i}$ in $S \otimes_R M_i = (M_i)_S$ is contained $\langle (N_i)_S \rangle$ for $1 \le i \le h$.*
(d) $(N^{\bigstar})^{\bigstar} = N^{\bigstar}$.
(e) $(N \cap N')^{\bigstar} \subseteq N^{\bigstar} \cap N'^{\bigstar}$.
(f) $(N + N')^{\bigstar} = (N^{\bigstar} + N'^{\bigstar})^{\bigstar}$.
(g) $(IN)^{\bigstar} = (I^{\bigstar} N^{\bigstar})^{\bigstar}$.
(h) *An arbitrary intersection of solidly closed submodules of M is solidly closed.*
(i) *The solid closure of the ideal (0) in R is the nilradical J of R. The solid closure $N^{\bigstar}$ of N in M contains JM.*

(j) *Let* $R_{red} = R/J$ *with* J *as in* (i) *just above and let* $P_{red} = R_{red} \otimes_R P$ *for every* R*-module* P*. Then the solid closure of* N *in* M *over* R *is the same as the inverse image in* M *of the solid closure of* $Im(N_{red} \to M_{red})$ *in* M_{red}*, and the latter may be calculated either considering these modules over* R_{red} *or considering them over* R*.*

(k) *An element* $u \in M$ *is in* $N^{\bigstar}{}_M$ *if and only if for every minimal prime* $\mathfrak{p}$ *of* R*, the image of* u *in* $M/\mathfrak{p}M$ *is in the solid closure of* $Im(N/\mathfrak{p}N \to M/\mathfrak{p}M)$ *in* $M/\mathfrak{p}M$*, calculated over* $R/\mathfrak{p}R$*.*

(l) $(N:_M I)^{\bigstar}{}_M \subseteq N^{\bigstar}{}_M:_M I$ *and* $(N:_R P)^{\bigstar}{}_R \subseteq N^{\bigstar}{}_M:_R P$*. Moreover,* $N^{\bigstar}{}_M:_M I$ *and* $N^{\bigstar}{}_M:_R P$ *are solidly closed in* M *and* R*, respectively.*

(m) *If* $M_1, \ldots, M_h$ *are finitely generated* R*-modules,* $N_i \subseteq M_i$ *for* $1 \le i \le h$*, and* P_i *denotes the solid closure of* N_i *in* M_i*, then the solid closure of* $N_1 \oplus \cdots \oplus N_h$ *in* $M_1 \oplus \cdots \oplus M_h$ *is* $P_1 \oplus \cdots \oplus P_h$*.*

Proof. (a) Suppose that $u_1, u_2 \in N^{\bigstar}$ and that r is an element of R. We must show that ru_1 and $u_1 + u_2$ are in $N^{\bigstar}$. Let B be a complete local domain of R. We must show that there is a solid B-algebra S such that the images of ru_1 and $u_1 + u_2$ in M_S are in $\langle N_S \rangle$. There exist solid B-algebras S_1, S_2 such that the image of u_i is in $\langle N_{S_i} \rangle$ for $i = 1, 2$. This remains true when S_i is mapped further. Since both S_1 and S_2 map as B-algebras to $S = S_1 \otimes_B S_2$, it suffices to observe that $S_1 \otimes_B S_2$ is also a solid B-algebra, by Proposition (2.1a). (It is obvious that $N^{\bigstar} \supseteq N$.)

(b) This is immediate from the definition.

(c) Choose finitely many generators u_{ij} of N_i for every i. Given B, for every i, j there is a solid B-algebra S_{ij} such that the image of u_{ij} in $S_{ij} \otimes_R M$ is in $\langle (N_i)_{S_{ij}} \rangle$. Let S be the tensor product over B of the finitely many solid B-algebras S_{ij}, which is solid by Proposition (2.1a).

(d) By part (a), $N^{\bigstar} \subseteq (N^{\bigstar})^{\bigstar}$, and so it suffices to show that if $u \in (N^{\bigstar})^{\bigstar}$ then $u \in N^{\bigstar}$. Let B be any complete local domain of R. We must show that there is a solid B-algebra S such that the image of u in M_S is in $\langle N_S \rangle$. By part (c) we know that we can choose a solid B-algebra S_1 such that the image of $N^{\bigstar}$ in M_{S_1} is contained in $\langle N_{S_1} \rangle$. We also know that we can choose a solid B-algebra S_2 such that image of u in M_{S_2} is in $\langle N_{S_2} \rangle$. It follows easily that with $S = S_1 \otimes_B S_2$ the image of u is in contained in $\langle N_S \rangle$ (the fact that S_1 maps to S shows that this image contains the image of $N^{\bigstar}$, and the fact that S_2 maps to S then shows that it contains u).

(e) This is a consequence of the first part of (b).

(f), (g) By (a), $N + N' \subseteq N^{\bigstar} + N'^{\bigstar}$ (respectively, $IN \subseteq I^{\bigstar} N^{\bigstar}$), so that $\subseteq$ follows from (b). Let $P = N + N'$ (respectively, IN) and let $Q = N^{\bigstar} + N'^{\bigstar}$ (respectively, $I^{\bigstar} N^{\bigstar}$). Let B be a complete local domain of R. We must show that there is a solid B-algebra S such that the image of $Q^{\bigstar}$ in M_S is contained in $\langle P_S \rangle$. But by part (c) we can choose S such that the images of $Q^{\bigstar}$, $N^{\bigstar}$, and $N'^{\bigstar}$ (respectively, $Q^{\bigstar}$, $N^{\bigstar}$, and $I^{\bigstar}$) are contained respectively in $\langle Q_S \rangle$, $\langle N_S \rangle$, and $\langle N'_S \rangle$ (respectively, in $\langle Q_S \rangle$, $\langle N_S \rangle$, and IS). It follows that the image of $Q^{\bigstar}$ is contained in the sum of the images of N_S and N'_S (respectively, the product of IS with the image of N_S), and this is the same as the image of P_S.

(h) This follows from the first part of (b).

(i) Since any nilpotent of R is killed by a map to a complete local domain of R, it is clear that $(0)^\bigstar \supseteq J$. But if f is an element of R that is not nilpotent then there is a maximal ideal m of R that contains $\mathrm{Ker}\,(R \to R_f)$, and the image of f in R_m is also not nilpotent. Since R_m injects into its completion C, the image of f in C is not nilpotent, and so there is a minimal prime ideal $\mathfrak{p}$ of C that does not contain f. It follows that the image of f in $B = C/\mathfrak{p}$ (a complete local domain of R) is not zero, and so $f \notin (0)^\bigstar$, as required.

The second statement follows from the fact that the image of JM is zero in M_B for any complete local domain B of R.

(j) The complete local domains B of R are obviously in bijective correspondence with the complete local domains of R_{red}, and the nilradical of R is killed in mapping to any complete local domain B of R. For each solid algebra S over a complete local domain B of R,

$$S \otimes_R (M/N) \cong S \otimes_{R_{red}} (R_{red} \otimes_R (M/N)) \cong S \otimes_{R_{red}} (M_{red}/\mathrm{Im}\, N_{red}).$$

The result is immediate from these remarks.

(k) Let $\mathfrak{p}_1, \ldots, \mathfrak{p}_h$ be the minimal primes of R. For every localization R_m of R, the minimal primes of R_m are given by the expansions of those $\mathfrak{p}_i$ contained in m. Every minimal prime $\mathfrak{q}$ of the completion B of R_m lies over a unique minimal prime of $\mathfrak{p}R_m$ of R_m (where $\mathfrak{p}$ is a unique minimal prime of R) since $R_\mathfrak{p} \to B_\mathfrak{q}$ will be faithfully flat. Conversely, if $\mathfrak{p}$ is a minimal prime of R contained in m and $\mathfrak{q}$ is a minimal prime of $\mathfrak{p}B$ then $\mathfrak{q}$ is a minimal prime of B (again consider the map $R_\mathfrak{p} \to B_\mathfrak{q}$; the elements of $\mathfrak{p}R$ are nilpotent and $\mathfrak{q}B_\mathfrak{q}$ is nilpotent modulo $\mathfrak{p}B_\mathfrak{q}$). It follows that every complete local domain B of R corresponds to a complete local domain of some $R/\mathfrak{p}$ for a (unique) choice of minimal prime $\mathfrak{p}$ of R, and every complete local domain of any $R/\mathfrak{p}$ is a complete local domain of R as well. The stated result is then immediate.

(l) The second statement implies the first, since $N^\bigstar{:}_M I$ (respectively, $N^\bigstar{:}_R P$) is solidly closed and contains $N{:}_M I$ (resepctively, $N{:}_R P$). Let u be an element of $(N^\bigstar{:}_M I)^\bigstar$ (respectively, $(N^\bigstar{:}_R P)^\bigstar$). Let B be a complete local domain of R and let S be a solid B-algebra such that u is in $\langle (N^\bigstar{:}_M I)_S \rangle$ (respectively, such that u is in $\langle (N^\bigstar{:}_R P)_S \rangle$). It follows that the image of Iu (respectively, uP) is in $\langle (N^\bigstar)_S \rangle$. After replacing S by a possibly larger solid B-algebra, we may assume that this image in turn is contained in $\langle N_S \rangle$. Thus, for every complete local domain B of R there is a solid B-algebra S such that the image of Iu (respectively, uP) is contained in $\langle N_S \rangle$. It follows that Iu (respectively, $uP) \subseteq N^\bigstar$, and so $u \in N^\bigstar{:}_M I$ (respectively, $N^\bigstar{:}_R P$).

(m) It is an immediate consequence of the definition that if

$$u = (u_1, \ldots, u_h) \in (N_1 \oplus \cdots \oplus N_h)^\bigstar$$

then u_i is in $N_i^\bigstar$ for all i. The fact that if $u \in \bigoplus_i N_i^\bigstar$ then $u \in (\bigoplus_i N_i)^\bigstar$ is also immediate if one uses part (c) and observes that for every complete local domain B of R, there is a single solid B-algebra S such that $u_i \in \mathrm{Im}\,(S \otimes_R N_i \to S \otimes_R M_i)$ for $1 \le i \le h$. □

(5.5) Proposition. *Let $N \subseteq M$ be finitely generated modules over a Noetherian ring R. Let $u \in M$.*

(a) *u is in the solid closure of N in M over R if and only if for every maximal ideal m of R, $u/1$ is in the solid closure of N_m in M_m over R_m.*

(b) *If R is local, then u is in the solid closure of N in M over R if and only if its image in $\widehat{M}$ is in the solid closure of $\widehat{N}$ over $\widehat{R}$.*
(c) *Suppose that u is multiplied into N by a power of a maximal ideal m of R. Then u is in the solid closure of N in M over R if and only if $u/1$ is in the solid closure of N_m in M_m over R_m.*
(d) *If M/N is supported only at one maximal ideal m of R (so that the natural map $M/N \to (M/N)_m \cong M_m/N_m$ is an isomorphism), then the solid closure of N_m in M_m over R_m is the expansion of the solid closure of N in M over R.*
(e) *Suppose that $R = \prod_{i=1}^h R_i$ is a finite product of rings. (Then every R-module M has a canonical decomposition $M \cong \prod_{i=1}^h M_i$ where M_i is an R_i-module, and if $N \subseteq M$ the decomposition $N \cong \prod_{i=1}^h N_i$ is such that for all i, $N_i \subseteq M_i$.) In this situation an element $u = (u_1, \dots, u_h)$ of a finitely generated R-module $M = \prod_{i=1}^h M_i$ is in the solid closure of $N = \prod_{i=1}^h N_i \subseteq M$ over R if and only if for all i, $1 \le i \le h$, u_i is in the solid closure of N_i in M_i over R_i.*

Proof. (a) Every complete local domain of R_m is a complete local domain of R, and every complete local domain of R is a complete local domain of R_m for some m.

(b) The complete local domains of $\widehat{R}$ are the same as the complete local domains of R.

(c) For any maximal ideal $q \neq m$, the image of u in M_q is in the image of N_q, and so the statement follows at once from part (a).

(d) By Discussion (3.3) we may assume that $N = 0$, and so $M = M/N$ is supported only at m and $M \cong M_m$. The result is then immediate from part (c).

(e) This is immediate from the fact (a) that one may test for membership in a solid closure locally on the maximal ideals of R. □

(5.6) Theorem (persistence of solid closure). *Let R be a Noetherian ring. Let $N \subseteq M$ be finitely generated R-modules and suppose that $u \in M$ is in ${N^\bigstar}_M$. Then for every ring homomorphism $h\colon R \to S$, where S is Noetherian, we have that the image $1 \otimes u$ of u in M_S is in the solid closure of $\langle N_S \rangle$ in M_S over S.*

In particular, if $u \in R$ is in ${I^\bigstar}_R$ for an ideal I of R then $h(u) \in {(IS)^\bigstar}_S$ over S.

Proof. The statement of the second paragraph is immediate from the assertion in the first paragraph. To prove the former, let T denote a generic forcing algebra for (M, N, u) over R. By the equivalence of (a) and (f) in Proposition (5.3), T is formally solid over R. By the persistence theorem for formally solid modules, Theorem (3.7), $S \otimes_R T$ is formally solid over S, and it is a generic forcing algebra for $(S \otimes_R M, S \otimes_R N, 1 \otimes u)$ over S, by Proposition (4.4a). Thus, a second application of the equivalence of (a) and (f) in Proposition (5.3) yields the desired conclusion. □

A very simple but extremely useful consequence of the above is:

(5.7) Corollary. *Let $R \to S$ be any homomorphism of Noetherian rings and let J be a solidly closed ideal of S. Then the contraction I of J to R is solidly closed in R.*

Proof. Let $u \in {I^\bigstar}_R$. Then Theorem (5.6) implies that the image of u in S is in ${(IS)^\bigstar}_S$ and, hence, in ${J^\bigstar}_S$, since I maps into J. But ${J^\bigstar}_S = J$, so that u is in the contraction of J to R. □

(5.8) Corollary. *Let R be a Noetherian ring, let $N \subseteq M$ be finitely generated R-modules, and let $u \in M$. Then $u \in N^\bigstar{}_M$ if and only if for every homomorphism of R to a complete local domain B, the image of u in M_B is in the solid closure over B of $\langle N_B \rangle$ in M_B.*

Proof. The "if" part is clear from the definition of solid closure, while the "only if" part is immediate from Theorem (5.6). □

(5.9) Theorem. *Let $R \to S$ be a ring homomorphism of Noetherian rings, let $N \subseteq M$ be finitely generated R-modules, and let $u \in M$. Suppose that the image of u in M_S is in the solid closure of $\langle N_S \rangle$ in M_S over S. Suppose also that at least one of the following three conditions holds:*

(a) *S is faithfully flat over R.*

(b) *$R \to S$ is a local homomorphism of complete local rings and for every minimal prime $\mathfrak{p}$ of R there is a prime ideal $\mathfrak{q}$ of S lying over $\mathfrak{p}$ such that $ht\, m(S/\mathfrak{q}) \geq ht\, m/\mathfrak{p}$.*

(c) *For every maximal ideal m of R and minimal prime $\mathfrak{p}$ of $(R_m)\widehat{\ }$ there is a prime ideal Q of S lying over m and a prime ideal $\mathfrak{q}$ of $(S_Q)\widehat{\ }$ lying over $\mathfrak{p}$ such that $ht\, m\big((S_Q)\widehat{\ }/\mathfrak{q}\big) \geq dim\big((R_m)\widehat{\ }/\mathfrak{p}\big)$.*

Then u is in the solid closure of N in M over R.

Proof. By the equivalence of (a) and (f) in Proposition (5.3), we know that there is a formally solid S-algebra T that is a forcing algebra for $(S \otimes_R M, S \otimes_R N, 1 \otimes u)$, which trivially implies that T is a forcing algebra for (M, N, u) over R. By applying Theorem (3.8) in case (a) and Lemma (3.6b) in case (b), we see that T is formally solid over R.

To prove (c), note that to show that $u \in N^\bigstar{}_M$ it suffices to prove this for $1 \otimes u$ in M_B with $\langle N_B \rangle$ replacing N for every ring B of the form $(R_m)\widehat{\ }/\mathfrak{p}$. By Theorem (5.6), if $C = (S_Q)\widehat{\ }/\mathfrak{q}$ then $1 \otimes u$ is in the solid closure of $\mathrm{Im}\,(C \otimes_S N)$ in $C \otimes_S M$ over C. Since $B \to C$ satisfies the hypothesis of (b), the result is immediate from part (b). □

Proposition (2.9) coupled with Theorem (5.6) yields:

(5.10) Theorem. *Let R be a Noetherian ring and let I be an ideal of R. Then $I^\bigstar$ is contained in the integral closure $\overline{I}$ of I.*

Moreover, if I is a principal ideal then $I^\bigstar = \overline{I}$.

Proof. Let $u \in I^\bigstar$ and suppose that $u \notin \overline{I}$. Then there is a ring homomorphism from R to a discrete valuation ring V such that the image a of u in V is not in IV. By Theorem (5.6), a is in $(IV)^\bigstar{}_V$ over V. We therefore have a counterexample in the discrete valuation ring V. Since IV is principal, say $IV = bV$ with $b \in V$, and since V is normal, the fact that $a \in (bV)^\bigstar$ implies, by Proposition (2.9), that $a/b \in V$, i.e. that $a \in bV$, a contradiction. This completes the proof of the statement in the first paragraph.

To prove the statement in the second paragraph we may assume that $I = bR$. It will suffice to show that if $a \in \overline{I}$ then a is in $I^\bigstar$; we have already established the other inclusion in general. To accomplish this it suffices to consider what happens after we replace R by a complete local domain of R. The expansion of I is still generated by the image of b and the image of a will still be in $\overline{I}$. Thus, we may assume without loss of generality that R is a complete local domain. If $b = 0$ then $(0) = I = \overline{I} = I^\bigstar$ and we are done. If $b \neq 0$ then an equation of integral dependence for a on bR shows that a/b is integral over R. But then $S = R[a/b]$ is a solid R-algebra, and $a \in bS$, so that $a \in I^\bigstar$, as required. □

The following result is a strong parallel, for solid closure, of Proposition (8.18) of [HH4], which is a corresponding result for tight closure. It shows, in particular, that over a normal Noetherian ring the solid closure of a torsion-free module embedded in a projective module is independent of the embedding.

(5.11) Proposition. *Let R be a reduced Noetherian ring and let M, N, F, G be finitely generated R-modules.*

(a) *If M/N is torsion-free, then N is solidly closed in M. More generally, $N^{\bigstar}{}_M$ may be identified with a submodule of $N' = \mathrm{Ker}\big(M \to (R^\circ)^{-1}(M/N)\big)$, where R° denotes the set of elements of R not in any minimal prime of R. If N is torsion-free, then $N' \subseteq (R^\circ)^{-1}N$.*

(b) *If $N \subseteq G \subseteq F$, where G is projective and F is any module, then $N^{\bigstar}{}_F \cap G = N^{\bigstar}{}_G$. Hence, if G is solidly closed in F, then $N^{\bigstar}{}_F = N^{\bigstar}{}_G$.*

(c) *If R is normal, and $G \subseteq F$ with G projective and F torsion-free, then G is solidly closed. If an arbitrary module N has embeddings in two possibly distinct finitely generated projective modules F and G, then $N^{\bigstar}{}_F \cong N^{\bigstar}{}_G$ canonically.*

Proof. (a) The argument is identical with that used for part (a) of Proposition (8.18) of [HH4], and is omitted.

(b) Suppose that there is an element x in $N^{\bigstar}{}_F$ but not in $N^{\bigstar}{}_G$. Then this remains true after replacing R by a suitable localization and killing a submodule of F maximal with respect to being disjoint from the image of G. We may therefore suppose that (R, m, K) is local, that G is free, and that F is an essential extension of G. It follows that F is torsion-free, and consequently we may replace F by an essential extension that is free. Thus, we may assume that $\beta: G \to F$ is a map of free modules with matrix B, and the injectivity of β implies that $b = \det(B)$ is a nonzerodivisor in R. This remains true when we complete, which does not affect any relevant issue. Finally, we may replace the complete ring R by its quotient by a minimal prime $\mathfrak{p}$: we replace G, F by their tensor products with $R/\mathfrak{p}$ and N by the image of $(R/\mathfrak{p}) \otimes_R N$ in $(R/\mathfrak{p}) \otimes_R F$ (or its image in $(R/\mathfrak{p}) \otimes_R G$; since $b \notin \mathfrak{p}$, the induced map $(R/\mathfrak{p}) \otimes_R G \to (R/\mathfrak{p}) \otimes_R F$ remains injective). By the definition of solid closure, $\mathfrak{p}$ may be chosen so that the image of x is not in the solid closure of (the new) N in (the new) G. As remarked above, the image of b will be nonzero, and so the induced map $G \xrightarrow{B} F$ will still be injective. Thus, there is no loss of generality in assuming that R is a complete local domain and that $G \to F$ is an injection of free modules of equal rank. Choose data A, ρ for the triple (G, N, x). Then BA, $B\rho$ give data for the triple (F, N, x). To complete the argument, it will suffice to show that if the generic forcing algebra T for the data BA, $B\rho$ for the triple (F, N, x) is solid over R, then so is the generic forcing algebra S for the data A, ρ for the triple (G, N, x) over R.

But T is obtained from $R[X]$ by killing the entries of the matrix $BAX - B\rho = B(AX - \rho)$, while S is obtained from $R[X]$ by killing the entries of the matrix $AX - \rho$. Thus, there is an obvious surjection $T \twoheadrightarrow S$. We claim that the kernel of this map is an R-torsion module. The point is that we may multiply $B(AX - \rho)$ by the classical adjoint $\mathrm{adj}\, B$ of B, and this produces $b(AX - \rho)$, which shows that b kills $\mathrm{Ker}\,(T \to S)$. By Proposition (2.1g), T is solid over R if and only if S is solid over R.

(c) The argument is the same as for part (c) of Proposition (8.18) of [HH4] (the argument requires knowing that over one-dimensional regular rings, every submodule of every finitely generated module is solidly closed: this is proved in (7.13) and (7.17) below). □

6. Minimal solid algebras

Many of the problems that we shall encounter concerning the properties of solid algebras can be reduced to the case of what we shall call *minimal* solid algebras. The minimal solid algebras, which are defined just below, are better behaved than ordinary solid algebras in a number of ways. We shall illustrate this by showing that the minimal solid algebras finitely generated over a complete one-dimensional local domain are simply the module-finite extension domains. (This cannot, however, be true in higher dimensions: see Example (6.6).)

(6.1) *Definition and discussion.* Let R be a domain. We shall say that a solid R-algebra S is a *minimal* solid R-algebra if S has no proper homomorphic image that is a solid R-algebra. Thus, every solid Noetherian R-algebra maps onto a minimal solid R-algebra. Likewise, if R is Noetherian, then every finitely generated solid R-algebra maps onto a minimal finitely generated solid R-algebra. The solid Noetherian R-algebras can be described as the Noetherian R-algebras that map onto a minimal solid R-algebra. When a solid R-algebra S is not Noetherian, it is not clear whether the family of ideals I such that S/I is solid will have a maximal element.

Of major interest is the case where R is Noetherian and S is finitely generated over R.

(6.2) Proposition. *Let R be a domain and let S be a minimal solid R-algebra.*

(a) *If M is a solid S-module then M is a solid R-module.*

(b) *If S is Noetherian then S is a domain. (Thus, S is an extension domain of R.)*

(c) *If $S \subseteq T$ is a ring extension such that every nonzero element of T has a nonzero T-multiple in S, (e.g., if T is a domain and the extension of fraction fields $S_{(0)} \subseteq T_{(0)}$ is algebraic), and T is a solid R-algebra then T is a minimal solid R-algebra.*

(d) *If S is a domain then every extension domain T that is finitely presented as an S-module is a minimal solid R-algebra.*

(e) *If R is a Noetherian domain and R' is a module-finite extension domain of R then an R'-algebra S is minimal solid over R' if and only if it is minimal solid over R.*

Proof. (a) Choose a nonzero S-module map $M \to S$ and call the image I, so that I is a nonzero ideal of S. There exists a nonzero R-module map $S \to R$. If this map kills I then it factors $S \twoheadrightarrow S/I \to R$, which implies that S/I is a solid R-algebra, a contradiction. If it does not kill I then the composite map $M \to S \to R$ is nonzero.

(b) S has a finite filtration as an S-module with factors that are of the form S/P, where P is a varying prime ideal of S. By Proposition (2.1), part (1), at least one of these factors is a solid R-module. Since no proper homomorphic image of S is a solid R-module, P must be (0), so that S is a domain.

(c) Suppose that T has a proper homomorphic image T/J, $J \neq 0$, that admits a nonzero R-module map to R. By Proposition (2.1d) we may assume that this map is nonzero on the identity element of T/J. When we restrict the map to S, we get a nonzero R-module map $S \to R$ that kills $J \cap S$ and so induces a nonzero R-module map $S/(J \cap S) \to R$. The hypothesis implies that $J \cap S \neq (0)$, which contradicts the minimality of S.

(d) T is solid over S by Proposition (2.1i) and so is solid over R by (a) above. It is then minimal by (c) above.

(e) By Theorem (2.3), S is solid over R if and only if it is solid over R'. But it is also true that each given proper homomorphic image S/J of S is solid over R if and only if it is solid over R', by the same result. □

We next want to characterize the minimal solid finitely generated algebras over a one-dimensional local ring. We first observe:

(6.3). *Proposition. Let (R, m, K) be a complete one-dimensional local domain, let $x \in m - \{0\}$, and let M be a torsion-free R-module. Then M is solid if and only if $M \neq xM$.*

Thus, if M is a torsion-free R-algebra , M is solid if and only if the image of x in R is not a unit.

Proof. The single element x is a system of parameters for R. By the local cohomology criterion (2.4), a necessary and sufficient condition for M to be solid is that $M_x/\mathrm{Im}\, M$ not be zero. If M is torsion-free the condition that u/x^h be in the image of M in M_x for all $u \in M$ and all h is that $M = x^h M$ for all h. This conditon when $h = 1$ is that $M = xM$, which in turn implies that $M = x^h M$ for all h. □

(6.4) Theorem. *Let (R, m, K) be a complete local domain of dimension one. A finitely generated R-algebra S is a minimal solid R-algebra if and only if S is a module-finite extension domain of R.*

Proof. S is a domain, and x is not invertible in S. It follows that we may choose a maximal ideal $\mathfrak{q}$ of S containing x. If S has dimension two or more then it has infinitely many primes of height one. Not all of these can contain x, since those that contain x will be minimal primes of xS. If P is a height one prime not containing x such that $P \subseteq \mathfrak{q}$ then S/P is torsion-free as an R-module (since $x \notin P$) and x is not invertible in S/P. Thus, S/P is also solid, contradicting the minimality of S. It follows that S has a maximal ideal containing x (and, hence, m) of height at most one. Let τ be the transcendence degree of the fraction field of S over R and let δ be the transcendence degree of $S/\mathfrak{q} = S_\mathfrak{q}/\mathfrak{q}S_\mathfrak{q}$ over $K = R/m$. Since q is maximal, $\delta = 0$. By [Mat] (14.C) Theorem 23 (the dimension formula), we have that $\mathrm{ht}\ \mathfrak{q} = \mathrm{ht}\ P + t - \delta$, so that $\mathrm{ht}\ \mathfrak{q} = 1 + \tau$. Since $\mathrm{ht}\ \mathfrak{q} \leq 1$, we must have that $\mathrm{ht}\ \mathfrak{q} = 1$ and $\tau = 0$. Thus, S is algebraic over R. By Theorem (2.9), since S is solid over R, S must be integral over R and, hence, module-finite over R. □

(6.5) Corollary. *Let R be a complete local domain of dimension one. Then a finitely generated R-algebra S is solid if and only if S has a homomorphic image that is a module-finite extension of R.*

Proof. If S is solid it can be mapped onto a minimal solid R-algebra, and the result is then immediate from Theorem (6.4). On the other hand, an algebra that can be mapped onto a module-finite extension of R is solid by Proposition (2.1), parts (i) and (f). □

(6.6) *Example.* It cannot be true in the equal characteristic case that a minimal solid finitely generated algebra over a complete local domain of dimension 3 is necessarily module-finite over the domain, even when the domain is regular. To see this, choose T to be a complete normal domain of dimension 3 containing a field of characteristic zero such that T is not Cohen-Macaulay. Such domains exist even when the coefficient field is algebraically closed. Represent T as module-finite over a complete regular local ring R (necessarily a formal power series ring in three variables over a field). Let x, y, z be the variables in R. Then, since T is not Cohen-Macaulay there is a relation $uz = vx + wy$ with $u, v, w \in T$ and $u \notin (x, y)T$. By Theorem (10.11) below, u is in the solid closure of $(x, y)T$ in T, and so the generic forcing algebra $T[Z, Z']/(u - xZ - yZ')$ is a finitely generated solid T-algebra. Thus, it can be mapped onto a minimal solid T-algebra S, which will also be a minimal solid R-algebra. We claim that S cannot be module-finite over T (which is equivalent to being module-finite over R). Suppose that it were module-finite. Let d be the degree of the extension of fraction fields, $[G : F]$, with $G = S_{(0)}$ and $F = T_{(0)}$. Then, since T is normal, $(1/d)\,\mathrm{Trace}_{G/F}$ gives a T-module retraction of S to T. Since $u \in (x, y)S$, it would follow (applying the retraction) that $u \in (x, y)T$, a contradiction. □

7. S-REGULAR RINGS

In parallel with tight closure theory, we define a Noetherian ring R to be *weakly S-regular* if every ideal of R is solidly closed, and *S-regular* if $W^{-1}R$ is weakly S-regular for every multiplicative system W in R. (Solid closure does not commute with localization; we do not know whether weakly S-regular implies S-regular.) In characteristic p, and in dimension at most two, regular rings are S-regular. We shall see later that for well-behaved rings of characteristic p, the notion of weak S-regularity coincides with that of weak F-regularity. Cf. Corollary (8.9). On the other hand, by a result of Paul Roberts [Ro6] (cf. (7.22) and (7.23)), an S-regular ring of equal characteristic zero is forced to have a rather small dimension: see Corollary (7.24) below. The situation for regular local rings of mixed characteristic remains mysterious.

Quite generally, S-regular rings are normal, and a weakly S-regular ring containing a field is Cohen-Macaulay. We do not know whether an S-regular ring is Cohen-Macaulay in mixed characteristic.

We first observe:

(7.1) Proposition. *Let R be a Noetherian ring.*

(a) *R is weakly S-regular if and only if every ideal of R primary to a maximal ideal is solidly closed.*

(b) *R is weakly S-regular if and only if R_m is weakly S-regular for every maximal ideal m of R.*

(c) *If (R, m, K) is local, then R is weakly S-regular if and only if $\widehat{R}$ is weakly S-regular.*

(d) *If (R, m, K) is local, a sufficient condition for it to be S-regular is that there exist a sequence of solidly closed irreducible m-primary ideals $\{I_t\}_t$ cofinal with the powers of m. In fact, if such a sequence exists, then for all finitely generated R-modules $N \subseteq M$, N is solidly closed in M.*

Proof. (a) This is a consequence of the fact that every proper ideal of a Noetherian ring is an intersection of a (usually infinite) family of ideals primary to maximal ideals.

(b) The ideals q primary to m in R are in bijective with correspondence with the ideals primary to mR_m in R_m via expansion and contraction, and q is solidly closed in R if and only if qR_m is solidly closed in R_m, by Proposition (5.5d).

(c) The ideals q primary to m in R are in bijective correspondence with the ideals primary to $m\widehat{R} = \widehat{m}$ in $\widehat{R}$ via expansion and contraction, and q is solidly closed in R if and only if $q\widehat{R} = \widehat{q}$ is solidly closed in $\widehat{R}$, by Proposition (5.5b).

(d) Since $N = \bigcap_t (N + m^t M)$ it suffices to do the case where M/N has finite length. By Discussion (5.2) we may assume that $N = 0$ and that M has finite length. Choose I so that it kills M and view M as a module over R/I_t, which is self-injective (i.e., 0-dimensional Gorenstein), since I is m-primary and irreducible. Then R/I_t is the only indecomposable injective module over R/I_t, and the injective hull of M over R/I_t will be a finite direct sum of copies of R/I_t. Thus, $M \subseteq (R/_t)^h$ for some nonnegative integer h. Since I_t is solidly closed in R, 0 is solidly closed in R/I_t, and so 0 is solidly closed in $(R/I_t)^h$ by Proposition (5.4m). But then 0 is solidly closed in M by the second part of Proposition (5.4b). □

(7.3) Definition. *We shall say that elements $x_1, \dots, x_d$ in a Noetherian ring R are parameters if for every prime ideal $P \supseteq (x_1, \dots, x_d)R$, the images $x_1/1, \dots, x_d/1$ of these elements in R form part of a system of parameters for R_P. In particular, an element x not in the union of the minimal primes of R is called a parameter.*

(7.4) Definition. *We shall say that a Noetherian ring R is S-rational if every ideal generated by parameters is solidly closed.*

(7.5) *Remarks.* The terminology is parallel to that used for tight closure: a Noetherian ring R for which tight closure is defined such that every ideal generated by parameters is tightly closed is called *F-rational*. See [HH9] and [FeW] for further discussion. The reason for this terminology is that, in equal characteristic 0, the F-rational rings may coincide with the rings R such that $\operatorname{Spec} R$ has rational singularities. Cf. [Sm1,3].

(7.6) Proposition. *Let R be a Noetherian ring. Suppose either that:*

(a) (0) *is solidly closed (this is equivalent to the statement that R is reduced) and that every principal ideal generated by a parameter is solidly closed or*

(b) *every ideal generated by a parameter is solidly closed and the zero-dimensional connected components of Spec R, if any, are reduced.*

Then R is normal.

Proof. Since (b) ⇒ (a) it will suffice to prove that R is normal assuming (b). Suppose that $R = R_1 \times \cdots \times R_h$, where each $\operatorname{Spec} R_i$ is connected. The parameters in R correspond to h-tuples $(x_1, \dots, x_h)$ such that every x_j is a parameter in R_j. If x_i is a parameter in R_i then

$$y = (1, \dots, 1, x_i, 1, \dots, 1)$$

is a parameter in R. From the fact that yR is solidly closed in R, it follows that x_iR_i is solidly closed in R_i, by Proposition (5.5e). Thus, the condition that every parameter generate a solidly closed ideal is inherited by the rings occurring as factors in the product decomposition of R, and so we may assume without loss of generality that $\operatorname{Spec} R$ is connected.

If $\dim R = 0$, then, by hypothesis, R is reduced and so R is a field. If $\dim R \geq 1$ then every principal ideal generated by a parameter in R is integrally closed, by Theorem (5.10). But this implies that R is normal: see Lemma (5.9) of [HH4]. $\square$

(7.7) Corollary. *If R is S-regular (or S-rational) then R is normal.*

Proof. If R is S-rational then the condition in Proposition (7.6a) holds. $\square$

In order to show that the property of being S-rational passes to local rings at maximal ideals we first note:

(7.8) Lemma. *Let m be a maximal ideal of Noetherian ring R and let I be an ideal of R_m generated by a system of parameters of length d (thus, $\dim R_m = d$). Then there is an ideal J of R generated by d parameters such that $JR_m = I$.*

Proof. We use induction on d. If $d = 0$ then $I = (0)$ in R_m and we may take $J = (0) \subseteq R$. Assume that $d \geq 1$. We claim that there is an element $x \in R$, not in any minimal prime of R, such that $x/1$ is part of a minimal set of generators for I. To see this, let I_0 and I_1 denote the contractions of I and mI to R and let $\mathfrak{p}_1, \dots, \mathfrak{p}_h$ denote the minimal primes of R. We wish to choose x in $I_0 - (I_1 \cup \bigcup_\nu \mathfrak{p}_\nu)$. Since all but at most one of the ideals I_1, $\mathfrak{p}_\nu$ are prime, we can do this unless $I_0 \subseteq I_1$ or $I \subseteq \mathfrak{p} = \mathfrak{p}_\nu$ for some minimal prime of $\mathfrak{p}$ of R. The former is impossible because I, mI are equal to the expansions of their contractions to R, which would imply $I \subseteq mI$ (and so $I = (0)$) if $I_0 \subseteq I_1$. On the other hand, I_0 is the contraction of an R_m-primary ideal of R_m and so is primary to m. If $I_0 \subseteq \mathfrak{p}$ then $m \subseteq \mathfrak{p}$ and so $m = \mathfrak{p}$ and $\dim R_m = 0$.

Thus, we can choose x as specified. We may now apply the induction hypothesis to the ring R/xR, the maximal ideal m/xR, and the ideal IR_m/xR_m. Since $x/1$ is part of a minimal set of generators for IR_m, it is part of a system of parameters generating I, and the ideal IR_m/xR_m will be generated by $d-1$ parameters. By the induction hypothesis we can choose parameters $\overline{y}_1, \dots, \overline{y}_{d-1}$ in R/xR (where the $y_i \in R$ and the bar indicates images modulo xR) such that the images of the $\overline{y}_i$ in R_m/xR_m generate IR_m/xR_m. The fact that x is not in any minimal prime of R then implies that $x, y_1, \dots, y_{d-1}$ are parameters in R, and these elements evidently satisfy the required condition. $\square$

(7.9) Proposition. *Let R be a Noetherian ring.*

(a) *R is S-rational if and only if R_m is S-rational for every maximal ideal m of R.*
(b) *If (R, m, K) is local, then R is S-rational if and only if every ideal generated by a (full) system of parameters is solidly closed.*
(c) *If (R, m, K) is local, then R is S-rational if and only if $\widehat{R}$ is S-rational.*
(d) *If (R, m, K) is Cohen-Macaulay local and $x_1, \dots, x_d$ is one system of parameters, then R is S-rational if and only if $(x_1^t, \dots, x_d^t)$ is solidly closed for infinitely many values of t.*
(e) *If R is Gorenstein, then R is weakly S-regular if and only if it is S-rational.*

Proof. (b) Let $x_1, \dots, x_k$ be part of a system of parameters and extend it to a full system of parameters $x_1, \dots, x_d$ for some $d \geq k$. Then

$$(x_1, \dots, x_k)R = \bigcap_t (x_1, \dots, x_k, x_{k+1}^t, \dots, x_d^t)R,$$

and each of the ideals in the intersection is generated by a full system of parameters and, hence, solidly closed.

(a) To prove "if" assume that every R_m is S-rational and let I be an ideal of R generated by parameters. Suppose that $u \in I^{\star}{}_R - I$. Then this is preserved upon localization at some maximal ideal m of R, and we must have $I \subseteq m$. But then IR_m is generated by part of a system of parameters in R_m, a contradiction.

To prove "only if", note that by part (b) it suffices to show that if I is generated by a system of parameters in R_m then I is solidly closed. If not, we can choose $u \in R$ such that $u/1 \in I^{\star} - I$. Choose J, an ideal generated by parameters in R, such that $JR_m = I$. Then $J = J^{\star}$, since R is S-rational. Let J' be the contraction of JR_m to R, which is the same as the set of elements multiplied into J by an element of $R - m$. Thus, $J' = J :_R w$ for a single element $w \in R - m$. Then J' is an m-primary ideal of R and is solidly closed by part (l) of Proposition (5.4). Since $u/1 \notin I = JR_m$, we have that $u \notin J'$. But then the fact that $u \notin J'^{\star} = J'$ is preserved by localization at m, by Proposition (5.5c), and so u is not in the solid closure of $J'R_m = I$, a contradiction.

(c) By part (b), we need only consider ideals I generated by full systems of parameters in R (and $\widehat{R}$), and there is a bijection between such ideals in R and such ideals in $\widehat{R}$ given by expansion (or completion) and contraction. The issue of whether such an ideal is solidly closed is unaffected by completion, since $R/I \cong \widehat{R}/I\widehat{R}$ and we may apply Proposition (5.5b).

(d) If $\mathbf{y} = y_1, \dots, y_d$ is any full system of parameters then the local cohomology module $H^d_m(R) = H^d_{(\mathbf{y})R}(R)$ is the direct limit of modules $R/(y_1^N, \dots, y_d^N)R$, and the maps in the system are injective. Thus, $R/(\mathbf{y})R$ embeds in $H^d_m(R)$, which is also the increasing union of submodules isomorphic with $R/(x_1^t, \dots, x_d^t)R$. It follows that $R/(\mathbf{y})R$ embeds in $R/(x_1^t, \dots, x_d^t)R$ for any sufficiently large value of t. If $(x_1^t, \dots, x_d^t)R$ is solidly closed in R, then 0 is solidly closed in $R/(x_1^t, \dots, x_d^t)R$, which implies that 0 is solidly closed in $R/(y_1, \dots, y_d)R$, and so $(y_1, \dots, y_d)R$ is solidly closed in R.

(e) Since both properties are local on the maximal ideals of R it suffices to prove the case where (R, m, K) is local of dimension d. Let $x_1, \dots, x_d$ be a system of parameters for R and let $I = (x_1^t, \dots, x_d^t)R$. Then the sequence $\{I_t\}_t$ consists of m-primary irreducible ideals and is cofinal with the powers of m. By the F-rationality of R, the ideals I are solidly closed, and by Proposition (7.1d) this implies that R is weakly F-regular. □

(7.10) *Remark.* In §10 we shall see that every S-rational ring containing a field is Cohen-Macaulay. We do not know whether this is true in mixed characteristic.

(7.11) Corollary. *Every local ring of a weakly S-regular ring or of an S-rational ring is analytically normal.*

Proof. The properties of weak S-regularity and S-rationality are preserved by localization at a maximal ideal and by completion. Thus, the result is clear for maximal ideals. But if P is a prime ideal of R, m is a maximal ideal of R containing P and Q is a prime ideal of the completion T of R that lies over Q, then $(R_P)\widehat{\ } \to (T_Q)\widehat{\ }$ is faithfully flat. Since T is normal and complete, T_Q is excellent and normal, and so $(T_Q)\widehat{\ }$ is normal. It follows that $(R_P)\widehat{\ }$ is normal as well. □

The normality of weakly S-regular rings enables us to prove a converse to Proposition (7.1d).

(7.12) Proposition. *A local ring (R, m, K) is weakly S-regular if and only if there exists a sequence of solidly closed irreducible m-primary ideals $\{I_t\}_t$ cofinal with the powers of m.*

Proof. The sufficiency of the condition was established in Proposition (7.1d). On the other hand, if (R, m, K) is weakly S-regular then it is normal and so *approximately Gorenstein* in the sense of [Ho4] (see also Discussion (8.6) on p.75 of [HH4] for a summary), which means precisely that there exists a sequence of m-primary irreducible ideals $\{I_t\}_t$ cofinal with the powers of m. These ideals will all be solidly closed, since R is S-regular. □

(7.13) Theorem. *If R is weakly S-regular and $N \subseteq M$ are finitely generated R-modules then N is solidly closed in M.*

Proof. Suppose, to the contrary, that $u \in N^{\bigstar}{}_M - N$. Then this will be preserved upon localization at a suitable maximal ideal m of R. Hence, there is no loss of generality in assuming that R is local. But then the result follows from Proposition (7.12) and Proposition (7.1d). □

(7.14) *Discussion.* We recall that a map of R-modules $N \to M$ is called *pure* if for every R-module Q, $Q \otimes_R N \to Q \otimes_R M$ is injective. In particular, $N \to M$ itself must be an injection. This is a weakening of the condition that the map embed N as a direct summand of M over R. This weaker condition has the advantage that it is stable under taking direct limits. See §6 of [HR1], §5 (a) of [HR2], and [Ho4]. If M/N is finitely presented over R then $N \hookrightarrow M$ is pure if and only if it splits (see, for example, Corollary 5.2 on p. 142 of [HR2]). When R is Noetherian, an injection $N \hookrightarrow M$ is pure if and only if $N \hookrightarrow M_0$ splits for every submodule M_0 of M containing the image of N such that $M/\operatorname{Im} N$ is finitely generated. If R is a complete local ring and $R \to M$ is pure, then it splits without any finiteness condition on M: see the second paragraph of the proof of Corollary (6.24), p. 59, of [HH4].

The purity of $R \hookrightarrow M$ implies that for every ideal I of R, $IM \cap R = I$, since applying $(R/I) \otimes_R \_$ yields an injection.

(7.15) Proposition. *Suppose that $A \subseteq R$ are Noetherian rings and that the injection $A \hookrightarrow R$ is pure, or even that every ideal of A is contracted from R. Then if R is weakly S-regular, so is A. In particular, if A is a direct summand of R as an A-module, and R is weakly S-regular then so is A.*

Proof. If $I \subseteq A$, then I is the contraction of $IR \subseteq R$, and so I is solidly closed in A by Corollary (5.7). □

Although the following result is easy, it is rather important:

(7.16) Proposition. *Let R be a Noetherian ring.*

(a) *If R is a Noetherian ring and S is a module-finite extension (or, more generally, a formally solid extension), then $IS \cap R \subseteq I^{\bigstar}$ for every ideal I of R.*

(b) *If R is weakly S-regular, then R is pure in every formally solid extension algebra.*

(c) *If R is weakly S-regular, then R is a direct summand, as a module over itself, of every module-finite extension ring.*
(d) *If R is weakly S-regular and complete local, then R is a direct summand, as a module over itself, of every solid R-algebra.*

Proof. Part (a) is obvious, and (b) follows from (a) together with the fact that weakly S-regular rings are normal, hence, approximately Gorenstein, and so the contractedness of ideals implies purity. Part (c) is immediate from (b) and the fact that pure implies split when the cokernel is finitely presented. Part (d) is immediate from (b) and the fact that when a complete local ring is a pure submodule of a module the map splits: cf. the second paragraph of the proof of (6.24) in [HH4]. □

We next note:

(7.17) Proposition. *A Noetherian ring of dimension at most one is weakly S-regular if and only if it is regular, i.e., if and only if it is a finite product of Dedekind domains, in which case it is S-regular.*

Proof. Since a weakly S-regular ring is normal, and a normal Noetherian ring of dimension at most one is regular, "only if" is clear. Thus, we only need to show that a regular Noetherian ring R of dimension at most one is S-regular. It will suffice to show that every local ring of R is weakly S-regular. But the local rings of R are discrete valuation rings or fields. In such a ring every ideal is principal. Since the solid closure of a principal ideal is equal to the integral closure and since every principal ideal in a normal ring is integrally closed, it follows that in a field or discrete valuation ring every ideal is solidly closed. □

We shall see later that not every regular local ring is S-regular: this is false in equal characteristic zero in dimension bigger than or equal to 3, by a result of Paul Roberts [Ro6]. But it is true in characteristic p and in dimension at most two that regular rings are S-regular, and it is an open question for regular local rings of mixed characteristic. To explore this question further, we first observe:

(7.18) Proposition. *Let (R, m, K) be an analytically irreducible local Gorenstein domain of dimension d with system of parameters $x_1, \dots, x_d$, let $I_t = (x_1^{t+1}, \dots, x_d^{t+1})R$, for $t \geq 0$, let $I = I_0$, let $x = x_1 \cdots x_d$, and let u denote an element of R whose image in R/I generates the socle.*

Let $Y_1, \dots, Y_d$ denote indeterminates over R and let $G_t = x^t u - \sum_{i=1}^d x_i^{t+1} Y_i$. Then R is weakly S-regular if and only if for all $t \geq 0$ (equivalently, for infinitely many values of $t \geq 0$) $H_m^d\big(R[Y]/(G_t)\big) = 0$.

Note here that $H_m^d\big(R[Y]/(G_t)\big) = H_I^d\big(R[Y]/(G_t)\big)$.

Proof. If R is weakly S-regular then, evidently, all the ideals I_t are solidly closed. On the other hand, these ideals form a sequence of m-primary ideals cofinal with the powers of the maximal ideal, and so R is weakly S-regular if and only if infinitely many of the I_t are solidly closed, by Proposition (7.12). Since the image of $x^t u$ generates the socle in R/I_t for every $t \geq 1$, any ideal strictly larger than I_t must contain $x^t u$. Thus, I_t is solidly closed if and only if $x^t u$ is not in its solid closure, i.e., if and only if a generic forcing algebra for $(R, I_t, x^t u)$ is not formally solid. $R[Y]/(G_t)$ is evidently a generic forcing algebra for

$(R, I_t, x^t u)$. Thus, the condition for it not to be formally solid is that $\widehat{R}\otimes_R R[Y]/(G_t)$ not be solid. By Corollary (2.4), this is equivalent to the conditon that $H^d_m\big(\widehat{R}\otimes_R R[Y]/(G_t)\big) = 0$, and this module may be identified with $\widehat{R} \otimes_R H^d_m\big(R[Y]/(G_t)\big) \cong H^d_m\big(R[Y]/(G_t)\big)$, since every element of $H^d_m(N)$ is killed by a power of m for every R-module N. □

In particular, the criterion of this result is valid when R is a regular local ring and the x's are a system of parameters and, in that case, the x's may be chosen to be a minimal set of generators of the maximal ideal of R, in which case we may take $u = 1$. In this very important special case we therefore have:

(7.19) Corollary. *Let (R, m, K) be a regular local ring of dimension d and suppose that $x_1, \ldots, x_d$ is a regular system of parameters, i.e., that $m = (x_1, \ldots, x_d)R$. Let $Y_1, \ldots, Y_d$ denote indeterminates over R and let G_t denote the polynomial $x_1^t \cdots x_d^t - \sum_{i=1}^d x_i^{t+1} Y_i$. Then R is weakly S-regular if and only if for all $t \geq 0$ (equivalently, for infinitely many values of $t \geq 0$) $H^d_m\big(R[Y]/(G_t)\big) = 0$.* □

(7.20) Theorem. *Let R be a regular Noetherian ring.*
(a) *If R has positive prime characteristic p then R is S-regular.*
(b) *If $\dim R \leq 2$ then R is S-regular.*

Proof. In both cases it suffices to show that the local rings of R are weakly S-regular, and these may be replaced by their completions. Thus, we need only show that a complete regular local ring satisfying one of the two hypotheses is weakly S-regular.

(a) Let $x_1, \ldots, x_d$ be a minimal set of generators for the maximal ideal of R. We use the notation of Proposition (7.18) and Corollary (7.19) here. We may take $u = 1$. Consider $R[Y]/(G_t)$. In this ring S we have that $x^t \in I_{t+1}S$. Taking q^{th} powers where $q = p^e$ yields that $x^{qt} \in I_{qt+q}S$. Since $(qt + q) - qt = q$ is unbounded, Observation (2.6) shows that $H^d_I(S) = 0$.

(b) Since we have already handled the case where $\dim R \leq 1$ we may assume that R is a complete regular local domain of dimension 2, with regular system of parameters x_1, x_2, and take $u = 1$. Let $T = \mathbb{Z}[X_1, X_2, Y_1, Y_2]/(G_t)$ where G_t denotes the polynomial $(X_1X_2)^t - \sum_{i=1}^2 Y_i X_i^{t+1}$. We can map T to $S = R[Y]/(G_t)$ by sending X_i to x_i and Y_i to Y_i for $i = 1, 2$. Then $H^2_I(S) \cong S \otimes_T H^2_{(X_1,X_2)T}(T)$.

But $H^2_{(X_1,X_2)T}(T)$ is shown to vanish in §6 of [Ho7], pp. 545-547. The groups $H_{2,t,c}$ studied there are the graded pieces (with respect to a certain $\mathbb{Z}^2$-grading: $c \in \mathbb{Z}^2$ here) of $H^2_{(X_1,X_2)T}(T)$. See Corollary (6.11) of [Ho7]. □

(7.21) *Remarks.* The result of part (a) is implicit in [Ho1] as well as in [Ho7], while the result of part (b) is implicit in [Ho7].

(7.22) *Discussion: some multigraded local cohomology modules.* Let d and t be positive integers, let $X_1, \ldots, X_d$, $Y_1, \ldots, Y_d$ be indeterminates over $\mathbb{Z}$, and let

$$G = (X_1 \cdots X_d)^t - \sum_{i+1}^{d} Y_i X_i^{t+1}.$$

In [Ho7], pp.544-552, there is a study of the d^{th} local cohomology modules $H^d_I(R_{d,t})$ of the rings

$$R_{d,t} = \mathbb{Z}[X_1, \dots, X_d, Y_1, \dots, Y_d]/(G_{d,t}),$$

with support in $I = (X_1, \dots, X_d)R$. More specifically, a $\mathbb{Z}$-grading is introduced such that the degree of X_i is the i^{th} row of a size d identity matrix and the degree of Y has -1 in the i^{th} spot and t's elsewhere. The c^{th} graded piece $H_{d,t,c}$ is calculated, in a certain sense, as an abelian group. These groups are shown to be divisible. When $d = 2$ they are also shown to be finitely generated, and, hence, 0. The question is raised as to whether these groups must always vanish. This has recently been answered negatively by Paul Roberts.

(7.23) *Discussion: Roberts' calculation.* We continue the notation of (7.22). In these terms, Roberts' main result in [Ro6] is that if $d = 3$ then $\mathbb{Q} \otimes_{\mathbb{Z}} H_{3,2,c} \neq 0$ when $c = (-2, -2, -2)$. This is the same as the degree $(-2, -2, -2)$ graded piece of $H^3_I(\mathbb{Q} \otimes_{\mathbb{Z}} R_{3,2})$ with $I = (x_1, x_2, x_3)$. In particular, $H^3_I(\mathbb{Q} \otimes_{\mathbb{Z}} R_{3,2}) \neq 0$. As an immediate consequence we have:

(7.24) Corollary. *Let f, g, h be any three elements in a Noetherian ring S of equal characteristic zero. Then $f^2g^2h^2$ is in $(f^3, g^3, h^3)^\bigstar$. In particular, if x_1, x_2, x_3 are part of a system of parameters for an equicharacteristic zero regular local ring of dimension three (or any equicharacteristic zero local ring of dimension at least three) then $x_1^2x_2^2x_3^2$ is in $(x_1^3, x_2^3, x_3^3)^\bigstar$.*

In consequence, a weakly S-regular ring (or an S-rational ring) containing the rationals has dimension at most two.

Proof. To prove the first statement note that it suffices to do so after passing to a complete local domain of R. Thus, we may assume that R is a complete local domain. If any of f, g, or h is a unit the conclusion is clear. Thus, we may assume that all three are in the maximal ideal of the ring. Then there is a local homomorphism of $A = \mathbb{Q}[[X_1, X_2, X_3]]$ to R carrying X_1, X_2, X_3 to f, g, h respectively, and by the persistence of solid closure, Theorem (5.6), it suffices to prove the result when the ring is $\mathbb{Q}[[X_1, X_2, X_3]]$ and f, g, h are X_1, X_2, X_3. The relevant generic forcing algebra is then $B = A[Y_1, Y_2, Y_3]/(G)$ where

$$G = X_1^2X_2^2X_3^2 - \sum_{i=1}^{3} Y_iX_i^3,$$

and so it will suffice to prove that $H^3_J(B) \neq 0$ with $J = (X_1, X_2, X_3)B$. But $H^3_J(B)$ is $\varinjlim_t B/(X_1^t, X_2^t, X_3^t)B$ (where the maps between consecutive modules are induced by multiplication by $X_1X_2X_3$ acting on the copies of B in the numerators). Let $R = Q[X_1, X_2, X_3, Y_1, Y_2, Y_3]/(G)$. Then $R/(X_1^t, X_2^t, X_3^t)R \cong B/(X_1^t, X_2^t, X_3^t)B$, and it follows that $H^3_J(B) \cong H^3_I(R)$ with $I = (X_1, X_2, X_3)R$. But this R is precisely the ring $\mathbb{Q} \otimes_{\mathbb{Z}} R_{3,2}$ discussed in (7.23), and the result is now immediate from Paul Roberts' result in [Ro6].

The final statement follows at once: if one had an S-regular ring of dimension at least three containing $\mathbb{Q}$, it would have a local ring (at some maximal ideal) of dimension at least three, and that local ring would still be S-regular, by Proposition (7.1b). Choose x_1, x_2, x_3 in the local ring so that they are part of a system of parameters. Then $x_1^2x_2^2x_3^2 \in$

$(x_1^3, x_2^3, x_3^3)^\bigstar = (x_1^3, x_2^3, x_3^3)$, and this contradicts the monomial conjecture, which is known in equal characteristic. (The proof for the S-rational case is the same, using Proposition (7.9a).) $\square$

(7.25) *Remarks.* Despite this result, if $R = \mathbb{Z}[\mathbf{x}]$ or $S = Z[[\mathbf{x}]]$ with $\mathbf{x} = x_1, \dots, x_n$, any ideal I generated by monomials in the x's is solidly closed either in R or in S. In fact, over R, N is solidly closed in M, where M is a finitely generated R-module, if M/N is torsion-free over $\mathbb{Z}$: see Theorem (13.1b). The result for $I \subseteq S$ follows from Theorem (13.1a), since $I = \bigcap \{I + pS : p \text{ prime in } \mathbb{Z}\}$. E.g., (x_1^3, x_2^3, x_3^3) is solidly closed in $\mathbb{Z}[x_1, x_2, x_3]$ and in $\mathbb{Z}[[x_1, x_2, x_3]]$.

The reader is referred to §13 for further discussion.

8. Comparison with tight closure in characteristic p

Our main objective in this section is to show that in many good cases in characteristic p, including the case of algebras essentially of finite type over an excellent local ring, the notion of tight closure defined in [HH4] coincides with the notion of solid closure defined here. The key point is that both can be tested after passing to a complete local domain of R. One of the main results is Theorem (8.6), which shows that the tight closure is always contained in the solid closure and that the two notions agree when the ring has a completely stable weak test element (this includes the case where the ring is essentially of finite type over an excellent local ring).

In §11 we shall prove that for a complete local domain of characteristic p, an element is in the tight closure of a submodule of a finitely generated module M if and only if it is in the expansion after tensoring with a big Cohen-Macaulay algebra. Since big Cohen-Macaulay algebras over a complete local domain are always solid, this is a refinement of the result discussed in the preceding paragraph. In §11 we shall also use this result on big Cohen-Macaulay algebras to show that the solid closure contains the tight closure (defined via reduction to characteristic p) for rings containing $\mathbb{Q}$, insofar as tight closure can be defined.

In the theory of tight closure one needs a technical device, namely, the theory of completely stable test elements (described in Definition (8.3c) below), to control behavior as one passes to the completion of a local ring.

The definitions, discussion, and results given in (8.1) through (8.5) below sketch rapidly all that we need concerning the theory of tight closure from [HH4], [HH3], and [HH9].

(8.1) *Discussion and definitions.* We recall that if R is a ring of positive prime characteristic p, and F^e, for $e \in \mathbb{N}$, denotes the e^{th} iteration of the Frobenius endomorphism of R (so that $F^e(r) = r^{p^e}$), we can define a right exact functor $\mathbf{F}^e$ (or, more precisely, $\mathbf{F}^e_R$) from R-modules to R-modules that preserves finite generation by applying the functor $S \otimes_R \_$, where S denotes R viewed as an R-algebra via the structural homomorphism F^e. Quite generally, for any R-algebra S, $S \otimes_R \_$ is a right exact functor from R-modules to S-modules that preserves finite generation, whose value on R is S, and such that images of R-free modules are S-free. If g denotes the structural homomorphism $R \to S$, then this functor sends $M = \operatorname{Coker}(R^h \xrightarrow{A} R^k)$, where A is given by the matrix (r_{ij}), to the cokernel of the map of free S-modules given by the matrix $g(A) = \big(g(r_{ij})\big)$.

In the present instance, rather confusingly, $S = R$, so that the functor sends R-modules to R-modules. Note that $\mathbf{F}^e(\mathrm{Coker}\,(r_{ij})) \cong \mathrm{Coker}\,(r_{ij}^q)$, where $q = p^e$.

The functors $\mathbf{F}^e$ are known as the *Peskine-Szpiro* or *Frobenius* functors. In the general situation of a base change from R-modules to S-modules, where S is an R-algebra , there is a map $M \to S \otimes_R M$ as R-modules. We shall write the image of $u \in M$ in $\mathbf{F}^e(M)$ as u^{p^e}. Because $S = R$ in our case, $S \otimes_R M$ has two R-module structures. We shall always work with the one obtained by viewing it as an S-module and then "remembering" that $S = R$. With this convention, the map $u \mapsto u^{p^e}$ is linear in the sense that $(ru)^{p^e} = r^{p^e}u^{p^e}$. When $N \subseteq M$ we shall denote the image of $\mathbf{F}^e(N)$ in $\mathbf{F}^e(M)$ as $N^{[p^e]}$. This notation is somewhat imprecise, since one needs to know what M and the embedding $N \subseteq M$ are. When $M = R$ and $N = I$ is an ideal of R, $I^{[p^e]}$ denotes the ideal of R generated by the $(p^e)^{th}$ powers of the elements of I.

(8.2) *Notation.* We shall denote by R° the multiplicative system consisting of all elements of R not in any minimal prime ideal of R. If R is a domain, $R^\circ = R - \{0\}$.

(8.3) Definitions. *Let R be a Noetherian ring of positive prime characteristic p.*

(a) *If $N \subseteq M$ are finitely generated R-modules, we say that $u \in M$ is in the tight closure ${N^*}_M$ (or simply N^* if M is clear from the context) of N in M over R if there exists an element $c \in R^\circ$ such that $cu^{p^e} \in N^{[p^e]}$ in $\mathbf{F}^e(M)$ for all sufficient large nonnegative integers e.*

(b) *An element $c \in R^\circ$ is called a q'-weak test element, where q' is a power of p, if for every pair of finitely generated R-modules $N \subseteq M$ and every element $u \in M$, u is in ${N^*}_M$ if and only if $cu^{p^e} \in N^{[p^e]}$ in $\mathbf{F}^e(M)$ for all $p^e \geq q'$. If $q' = 1$, then c is called a test element.*

(c) *An element $c \in R^\circ$ is called a completely stable q'-weak test element if its image in the completion of every local ring of R is a q'-weak test element for that local ring. If $q' = 1$ then c is called a completely stable test element.*

(8.4) *Remarks.* Tight closure and the notion of test element are introduced in [HH4], but see also [HH1-3] and [Hu1]. The theory is further pursued in [HH6] and [HH8-11]. The following result on the existence of completely stable test elements, which follows from Theorems (5.10) and (6.20) of [HH9] will suffice for our purpose here:

(8.5) Theorem. *Let R be a Noetherian ring of characteristic p and suppose either*

(i) *that R is an algebra essentially of finite type over an excellent local ring or*

(ii) *that R is module finite over $R^p = F(R)$, i.e., that the Frobenius endomorphism is a finite morphism.*

Suppose that $c \in R_{red}$ is such that $(R_{red})_c$ is regular (such elements c always exist). Then c has a power that is a completely stable q'-weak test element for some q'. If R is reduced then c has a power that is a test element.

Thus, a ring satisfying (i) *or* (ii) *has a completely stable weak test element.* $\square$

(8.6) Theorem. *Let $N \subseteq M$ be finitely generated modules over a Noetherian ring R of characteristic p.*

(a) ${N^*}_M \subseteq {N^\bigstar}_M$.

(b) *If R has a completely stable weak test element, then ${N^*}_M = {N^\bigstar}_M$.*

Proof. Suppose that $u \in {N^*}_M$. This is preserved upon completion and also upon killing a minimal prime of the completion. Thus, (a) will follow if one can show tight closure and solid closure agree for complete local domains. (b) will also follow from the complete local domain case if one can show that if a given element $u \in M$ is not in ${N^*}_M$, then one can choose a complete local domain B of R such that u_B is not in ${N_B}^*{}_{M_B}$ over B. But the fact that one has a completely stable q'-weak test element c shows that $cu^{p^e} \notin N^{[p^e]}$ in $\mathbf{F}^e(M)$ for some $p^e \geq q'$, and this continues to be the case after localizing at a suitable maximal ideal and then after completion at that maximal ideal. We can then preserve that the image of u is not in the tight closure after killing a suitable minimal prime of the complete local ring so obtained by (6.25) of [HH4] generalized to the case of modules (the argument is valid without essential change).

Thus, both parts of the theorem follow if we can establish that for a complete local domain (R, m, K), one has ${N^*}_M = {N^\bigstar}_M$. The fact that ${N^\bigstar}_M \subseteq {N^*}_M$ is implicit in Theorem (5.22) of [HH10]. Let $u \in M$ and suppose that there is a solid R-algebra S such that $1 \otimes u \in \langle N_S \rangle$. We may choose an R-module map $\theta\colon S \to R$ such that $\theta(1) = c \in R - \{0\}$. If we apply $\mathbf{F}^e_S$ we find that $1 \otimes u$ is in

$$\operatorname{Im}\big(S \otimes_R \mathbf{F}^e_R(N) \to S \otimes_R \mathbf{F}^e_R(M)\big) = \operatorname{Im}\big(S \otimes_R N^{[p^e]} \to S \otimes_R \mathbf{F}^e_R(M)\big).$$

For every R-module W the map $\theta\colon S \to R$ induces a map $S \otimes_R W \to W$ sending $s \otimes w$ to $\theta(s)w$. If $W_0 \subseteq W$ the diagram:

$$\begin{array}{ccc} S \otimes_R W_0 & \longrightarrow & S \otimes_R W \\ \downarrow & & \downarrow \\ W_0 & \longrightarrow & W \end{array}$$

commutes. Applying this with $W = \mathbf{F}^e(M)$ and $W_0 = N^{[p^e]}$ we find that the image of $1 \otimes u^{p^e}$ is cu^{p^e} and is in $N^{[p^e]}$. Since this holds for all e, we have that $u \in {N^*}_M$.

Now suppose that R is a complete local domain of dimension d and that $u \in {N^*}_M$ for some $u \in M$. Let S be the generic forcing algebra for (M, N, u) for, say, the data A, ρ, so that $S = R[X]/J$, where J is generated by the entries of $AX - \rho$. To complete the argument, it will suffice to show that $H^d_m(S) \neq 0$.

Let $x_1, \dots, x_d$ be a system of parameters for R. Let $x = x_1 \cdots x_d$. We shall prove that $x^t \notin (x_1^{t+1}, \dots, x_d^{t+1})S$ for all t. By Observation (2.6) this condition is sufficient to imply that $H^d_m(S) \neq 0$.

Suppose to the contrary that $x^t \in (x_1^{t+1}, \dots, x_d^{t+1})S$ for some fixed t. Then we can lift this to $R[X]$: we have that

$$(\#) \qquad x^t = \sum_{i=1}^{d} x_i^{t+1} F_i(X) + G, \qquad \text{where } G \in J.$$

We shall obtain a contradiction. To this end, let b be a positive integer that is an upper bound for all the numbers $\deg F_i$.

The fact that u is in the tight closure of N in M implies that ρ is in the tight closure of the column space of A. We therefore have that for all $e \gg 0$, $c\rho^{p^e} = F^e(A)y_e$, where $F^e(A)$ denotes the matrix obtained from A by raising every entry to the p^e power and where y_e is a vector over R. Taking $(p^e)^{th}$ roots, if we let $c_e = c^{1/p^e}$, then $c_e\rho = Az_e$ for every $e \gg 0$, where z_e is a vector over R^{1/p^e} depending on e. Consider the equations (#) over the domain $(R^{1/p^e})_{c_e}$, and substitute $X = c_e^{-1}z_e$. This substitution makes the entries of $AX - \rho$ vanish, and since these generate J, it makes G vanish. This yields

$$x^t = \sum_{i=1}^{d} X_i^{t+1} F_i(c_e^{-1}z_e).$$

If we multiply both sides by c_e^b we clear denominators on the right, and we obtain

$$c^{b/p^e}x^t \in (x_1^{t+1}, \dots, x_d^{t+1})R^{1/p^e}$$

for all $e \gg 0$. Raising both sides to the $(p^e)^{th}$ power yields

$$c^b(x^t)^{p^e} \in \big((x_1^{t+1}, \dots, x_d^{t+1})R\big)^{[p^e]}$$

for all $e \gg 0$. Let K be a coefficient field for R and let $A = K[[x_1, \dots, x_d]]$, a regular ring over which R is module-finite. Replace c^b by a nonzero multiple c' in A. Then for all $e \gg 0$ we have that

$$c'(x^t)^{p^e} \in \big((x_1^{t+1})^{p^e}, \dots, (x_n^{t+1})^{p^e}\big)R \,\cap\, A,$$

and by the main result of [Ho1] (or by Theorem (7.20a) and Proposition (7.16c) here, which recover that result) this is

$$\big((x_1^{t+1})^{p^e}, \dots, (x_n^{t+1})^{p^e}\big)A.$$

(One may also use the main result of §7 of [HH4] concerning operations on ideals generated by monomials in parameters to get a contradiction here.) This shows that x^t is in the tight closure of the ideal $(x_1^{t+1}, \dots, x_d^{t+1})A$, a contradiction, since every ideal of a regular ring of characteristic p is tightly closed by Theorem (4.4) of [HH4]. □

(8.7) *Remark.* If one defines the *formal tight closure* of a submodule N of a finitely generated R-module M over a Noetherian ring R of positive prime characteristic to consist of all elements $u \in M$ such that for every complete local domain B of R, the image $1 \otimes u$ of u in $B \otimes_R M$ is in the tight closure, over B, of $\langle N_S \rangle$, then it is readily apparent from (8.6) that the formal tight closure of N in M is the same as $N^{\star}{}_M$.

(8.8) Corollary. *Let $R \to S$ be a homomorphism of Noetherian rings of characteristic p such that R has a completely stable weak test element. Suppose also that*

(*) for every maximal ideal m of R and minimal prime $\mathfrak{p}$ of $(R_m)^\frown$ there is a prime ideal Q of S lying over m and a prime ideal $\mathfrak{q}$ of $(S_Q)^\frown$ lying over $\mathfrak{p}$ such that ht $m\big((S_Q)^\frown/\mathfrak{q}\big) \geq$ dim $\big((R_m)^\frown/\mathfrak{p}\big)$.

(In particular, this condition holds whenever S is faithfully flat over R.)

Suppose as well that the image of u in $S \otimes_R M$ is in the tight closure of the $\langle N_S \rangle$ in $S \otimes_R M$ over S. Then u is in the tight closure of N in M over R.

Proof. Since the image of u is in the tight closure $\langle N_S \rangle$ in $S \otimes_R M$, by Theorem (8.6a) it is in the solid closure. We may then apply (5.9c) to conclude that $u \in N^{\bigstar}{}_M$ over R, and by Theorem (8.6b) it follows that $u \in N^*{}_M$ over R. □

The result above is an enormous improvement upon Theorem (5.31) of [HH10], in two ways. First, the height condition (*) above is very much weaker than the condition that $R \to S$ be "formally height preserving" in the sense of [HH10]. The latter condition, in essence, restricts Q to be a minimal prime of mS, so that m expands to an ideal primary to the maximal ideal of $(S_Q)^\frown$, and then requires that $\dim (S_Q)^\frown/\mathfrak{p}(S_Q)^\frown \geq \dim (R_m)^\frown/\mathfrak{p}(R_m)^\frown$. (Cf. (5.30a) of [HH10].) To simplify notation, let $B = (R_m)^\frown/\mathfrak{p}(R_m)^\frown$ and let $C = (S_Q)^\frown/\mathfrak{p}(S_Q)^\frown$. Thus, $B \to C$ is a local homomorphism of complete local rings such that the maximal ideal of B expands to the maximal ideal of C. The condition that $\dim C \geq \dim B$ implies that $\dim C = \dim B$ and that $B \to C$ is injective (otherwise, the image of B has dimension d smaller than that of B, and the maximal ideal of B will be the radical of an ideal generated by d elements). Thus, while no prime $\mathfrak{q}$ of C is mentioned in the definition, any minimal prime $\mathfrak{q}$ of C such that $\dim C/\mathfrak{q} = \dim C$ must lie over (0) in B. It follows that $R \to S$ is formally height preserving if and only if for every maximal ideal m of R and minimal prime $\mathfrak{p}$ of $(R_m)^\frown$, there are a minimal prime Q of mS and a minimal prime $\mathfrak{q}$ of $(S_Q)^\frown$ lying over $\mathfrak{p}$ in $(R_m)^\frown$ such that ht $m(S_Q)^\frown/\mathfrak{q} \geq \dim (R_m)^\frown/\mathfrak{p}$. (This is equivalent to $\dim (S_Q)^\frown/\mathfrak{q} = \dim (R_m)^\frown/\mathfrak{p}$.)

It is now clear that condition (*) is much weaker than "formally height preserving". (Consider the example given after the proof of Lemma (3.6): mS is a prime of height one in S.)

The second point is that, even in situations where $R \to S$ is formally height preserving (even, in fact, when $R \to S$ is faithfully flat), Corollary (8.8) above has an advantage, because one does not need to worry about the technical hypotheses (i) or (ii) that are needed for Theorem (5.31) of [HH10].

We also have the following immediately from Theorem (8.6):

(8.9) Corollary. *Let R be a Noetherian ring of characteristic p.*

(a) *If R is weakly S-regular (respectively, S-regular, respectively, S-rational) then R is weakly F-regular (respectively, F-regular, respectively, F-rational).*

(b) *If R has a completely stable weak test element then R is weakly F-regular (respectively, F-regular, respectively, F-rational) if and only if R is weakly S-regular (respectively, S-regular, respectively, S-rational).* □

(8.10) Corollary. *Let R be a Noetherian ring of characteristic p such that either:*

(i) *R is weakly F-regular and has a completely stable weak test element or*

(ii) *The completion of every local ring of R at a maximal ideal is weakly F-regular.*

(In particular, (ii) *is satisfied if R is regular.) Let S be a formally solid R-algebra. Then $R \to S$ is pure as a map of R-modules. In particular, if R is a regular domain and S is a solid R-algebra, then $R \to S$ is pure as a map of R-modules.*

Proof. Since (i) implies (ii), it suffices to prove the result when (ii) holds. $R \to S$ is pure if and only if $R_m \to S_m$ is pure for every maximal ideal m of R, and the issue of whether $R_m \to S_m$ is pure is unaffected by applying $(R_m)\widehat{\ }\otimes_{R_m} \_$. It follows that we may assume that R is complete local, weakly F-regular, and that $R \to S$ is solid. But then R is S-regular and normal. Since R is normal, it is approximately Gorenstein in the sense of [Ho4] (cf. also Discussion (8.6) on p. 75 of [HH4] and the proof of (7.12) here), and so to prove that $R \to S$ is pure it suffices to prove that every ideal of R is contracted from S. But this is immediate from the definition of solid closure and the fact that every ideal of R is solidly closed. □

(8.11) *Remark.* The result of P. Roberts [Ro6] shows that, in equal characteristic zero, if R is regular and $R \to S$ is solid it is not the case that R is necessarily pure in S. For example, let K be a field of characteristic zero, let $R = K[[x_1, x_2, x_3]]$, and let $S = R[y_1, y_2, y_3]/(g)$ where $g = x_1^2x_2^2x_3^2 - \sum_{i=1}^3 y_i x_i^3$. By Roberts' result $R \to S$ is solid, but $(x_1^3, x_2^3, x_3^3)S \cap R$ contains $x_1^2x_2^2x_3^2$, so that $(x_1^3, x_2^3, x_3^3)R$ is not contracted from S.

We do not know what the situation is if, for example, R is a regular local ring of mixed characteristic.

9. A formal power series criterion

Our objective is to give a remarkably elementary characterization of when an element of a complete local domain is in the solid closure of an ideal I. The main result is Theorem (9.3) below. The idea can be generalized to the case of modules: see Remark (9.4c). We need some preliminary definitions and discussion and a lemma.

(9.1) *Definitions and discussion.* Let $\mathbf{X} = X_1, \ldots, X_n$ be indeterminates over a ring R and let $\mathbf{y} = y_1, \ldots, y_n$ be analytic indeterminates over R. Then there is an R-module isomorphism between the R-module homomorphisms of $R[\mathbf{X}]$ to R and the formal power series ring $R[[\mathbf{y}]]$ that sends the homomorphism $\phi\colon R[\mathbf{X}] \to R$ to the power series $\sum_\nu \phi(\mathbf{X}^\nu)\mathbf{y}^\nu$, where $\nu = (\nu_1, \ldots, \nu_n)$ is a multi-index running through $\mathbb{N}^n$, $\mathbf{X}^\nu$ denotes $X_1^{\nu_1} \cdots X_n^{\nu_n}$, and similarly for $\mathbf{y}^\nu$.

Call a power series $\sum_\nu r_\nu \mathbf{y}^\nu$ *special* if for every ν with all entries positive, $r_\nu = 0$. A special power series may alternatively be described as one which is a finite sum of elements from the power series rings over R in the proper subsets of the variables $y_1, \ldots, y_n$.

Thus, a special power series is simply one that does not "honestly involve" all the variables in any one of its nonzero monomial terms. The special power series are the obvious R-module complement in $R[[\mathbf{y}]]$ for the principal ideal $(y_1 \cdots y_n)$.

(9.2) Lemma. *Let notation be as in (9.1). Let $r_1, \ldots, r_n$, $r \in R$, and let*

$$f = r - \sum_{i+1}^{n} r_i X_i.$$

Let $z = y_1 \cdots y_n$ and let $z_i = \prod_{j \neq i} y_j$, so that $z = y_i z_i$ for every i. Then an R-module homomorphism $\phi\colon R[\mathbf{X}] \to R$ kills the ideal $fR[\mathbf{X}]$ (and so corresponds to an R-module *homomorphism of $S = R[\mathbf{X}]/fR[\mathbf{X}]$ to R) if and only if the corresponding power series g has the property that $(rz - \sum_{i+1}^{n} r_i z_i)g$ is special in the sense of Definition (9.1).*

Proof. Let $e_1, \dots, e_n$ be the standard free basis for $\mathbb{Z}^n$. The ideal $(r - \sum_{i+1}^{n} r_i X_i)R[\mathbf{X}]$ is spanned as an R-module by the elements

$$X^\nu(r - \sum_{i+1}^{n} r_i X_i) = rX^\nu - \sum_{i+1}^{n} r_i X^{\nu+e_i}$$

as ν runs through $\mathbb{N}^n$. Thus, the power series $g = \sum_\nu a_\nu y^\nu$ represents a map that factors through S if and only if

$$ra_\nu = \sum_i r_i a_{\nu+e_i}$$

for all $\nu \in \mathbb{N}^n$. This condition can be described in terms of the power series g as follows: $rg \equiv \sum r_i y_i^{-1} g$ modulo terms on the right with a negative exponent. Note that the terms with a negative exponent will actually involve only one y_i to a negative power, and that negative power will be -1. Let T_i denote the ring of formal power series over R in the variables y_j for $j \neq i$. Then g satisfies the condition cited above if and only if we have

$$rg = \sum_i r_i y_i^{-1} g + \sum_i y_i^{-1} h_i$$

with $h_i \in T_i$. Multiplying through by $z = y_1 \cdots y_n$ yields $(rz - \sum_i r_i z_i)g \in \sum_i z_i T_i$. Thus, if g corresponds to a map $S \to R$ then $(rz - \sum_i r_i z_i)g$ is special. On the other hand, if $(rz - \sum_i r_i z_i)g$ is special, so that it is in $\sum_i T_i$, it is clear that the value must actually be in $\sum_i z_i T_i$: since z and the z_i's are all in the ideal generated by the z_i's, every monomial term in the expansion of $(rz - \sum_i r_i z_i)g$ must be divisible by at least one of the z_i (i.e., $(z_1, \dots, z_n)R[[y]] \cap \sum_i T_i = \sum_i z_i T_i$). $\square$

(9.3) Theorem. *Let (R, m, K) be a complete local domain, let $I = (r_1, \dots, r_n)R$ be an ideal and let $r \in R$. Let $\mathbf{y} = y_1, \dots, y_n$ be analytic indeterminates over R. Let $z = y_1 \cdots y_n$ and for $1 \le i \le n$ let $z_i = \prod_{j \neq i} y_j$. If $r \neq 0$ or if $I \neq (0)$ then $r \in I^\bigstar$ if and only if $rz - \sum_i r_i z_i$ has a nonzero multiple in $R[[\mathbf{y}]]$ that is special in the sense of (9.1).*

Proof. Let $S = R[X_1, \dots, X_n]/(r - \sum_{i=1}^{n} r_i X_i)$, which is a generic forcing algebra for (R, I, r). The result follows at once from Lemma (9.2), since S is solid (i.e., has a nonzero R-module homomorphism to R) if and only if the power series g corresponding to the induced map $R[X] \twoheadrightarrow S \to R$ is nonzero and $(rz - \sum_i r_i z_i)g$ is special. Since $R[[\mathbf{y}]]$ is a domain and the elements r, r_i are not all zero, g is nonzero if and only if $(rz - \sum_i r_i z_i)g$ is nonzero. $\square$

(9.4) *Remarks.* (a) Evidently, one can use the criterion of Theorem 9.3 to define solid closure for ideals in complete local domains. This definition has the advantage of being very elementary, in a certain sense: one does not even have to know what an R-module homomorphism is to understand it. However, virtually every other property of solid closure becomes more obscure from this point of view. It is not even immediately clear that $I \subseteq I^\bigstar$. (One can see this directly as follows: if, in fact, $r = \sum_i a_i r_i$ then

$$rz - \sum_i r_i z_i = \sum_i a_i r_i z - \sum_i r_i z_i = \sum_i (1 - a_i y_i) r_i z_i,$$

and multiplication by $\prod_i (1 - a_i y_i)^{-1}$ produces

$$\sum_i (\prod_{j \neq i} (1 - a_j y_j)^{-1}) r_i z_i.$$

The i^{th} term in the sum does not involve y_i, and so we have produced the required special multiple.)

(b) Let notation be as in the proof of Theorem (9.3), so that S is a generic forcing algebra for (R, I, r). The proof of (9.2) shows that $\mathrm{Hom}_R(S, R)$ is simply the R-module of power series $g \in R[[y]]$ such that $(rz - \sum_i r_i z_i)g$ is special.

When $\dim R = d$ and (R, m, K) is Gorenstein (as well as being a complete local domain), and, in particular, when R is regular, $\mathrm{Hom}_R(S, R)$ is the same as the dual, into an injective hull of K over R, of $H^d_m(S) \cong S \otimes_R H^d_m(R)$ (the injective hull of K over R may be identified with $H^d_m(R)$). Thus, in this complete Gorenstein local domain case, we have identified the dual of $H^d_m(S)$ with the R-module of power series g such that $(rz - \sum_i r_i z_i)g$ is special.

(c) We may also characterize tight closure for modules in similar terms. Let $A = (r_{ij})$ (an $h \times n$ matrix over R) and ρ (an $h \times 1$ column vector over R) be data for a generic forcing algebra for a triple (M, N, u). The question of when u is in the solid closure of N in M is equivalent to the question of when the column vector ρ is in the solid closure of the column space of A in R^h. Let r_i be the i^{th} entry of ρ. The generic forcing algebra S may be written as $R[X_1, \dots, X_n]/(f_1, \dots, f_h)$ where f_i is the polynomial $\sum_j r_{ij} X_j - r_i$. The R-module maps from S to R may be viewed as the maps from $R[\mathbf{X}]$ to R that kill the ideal of $R[\mathbf{X}]$ generated by the f_i. Since this ideal is the sum (as an abelian group) of the principal ideals generated by the f_i, it is necessary and sufficient that the map $R[\mathbf{X}] \to R$ kill all of the ideals $f_i R[\mathbf{X}]$. As before, the map $R[\mathbf{X}] \to R$ is represented by a formal power series $g \in R[[\mathbf{y}]]$, and the condition on g is that for all i, $1 \le i \le k$, $(r_i z - \sum_j r_{ij} z_j)g$ be special. Thus, u is in the solid closure of N in M if and only if there is a nonzero power series g such that all of the products $(r_i z - \sum_j r_{ij} z_j)g$ are special.

10. Shadow homology

We introduce the notion of shadow homology, which is parallel to the notion of phantom homology in tight closure theory (cf. [HH4], §9 and [HH8]). We show that certain of the results of [HH4] and [HH8] concerning phantom homology have analogues for shadow homology. For example, the conditions developed for the phantom acyclicity of a finite free complex in Theorem (9.8) of [HH4] and Theorem 3.22 of [HH8] are essentially valid for shadow acyclicity if the ring contains a field. However, the proof of this fact in the equal characteristic zero case uses the existence of big Cohen-Macaulay algebras!

We begin with a definition.

(10.1) Definition. *Let $M_\bullet$ denote a complex of modules*

$$\cdots \to M_{i+1} \xrightarrow{d} M_i \xrightarrow{e} M_{i-1} \to \cdots$$

over a Noetherian ring R. We shall say that that an element $\overline{u} \in H_i(M_\bullet)$ is a shadow element of the homology group if for some (equivalently, every) element of $\mathrm{Ker}\, e$ that represents it is in the solid closure of $\mathrm{Im}\, d$ in M_i. We shall say that $M_\bullet$ has shadow homology at the i^{th} spot if every element of $H_i(M_\bullet)$ is a shadow element. We shall say that a left complex (so that M_i vanishes for $i < 0$) is shadow acyclic if $M_\bullet$ has shadow homology at the i^{th} spot for all $i \geq 1$.

The following result is trivial but of great usefulness:

(10.2) Proposition. *Let $R \to S$ be a homomorphism of Noetherian rings and let $M_\bullet$ be a complex of finitely generated R-modules. Then the induced map $H_i(M_\bullet) \to H_i(S \otimes_R M_\bullet)$ sends shadow elements to shadow elements (now working over S).*

In particular, if $M_\bullet$ has shadow homology at the i^{th} spot and S is weakly S-regular then the map $H_i(M_\bullet) \to H_i(S \otimes_R M_\bullet)$ is zero.

Proof. Suppose that $u \in Z = \mathrm{Ker}\,(M_i \to M_{i-1})$ represents a shadow element in the homology, so that it is in the solid closure of $B = \mathrm{Im}\,(M_{i+1} \to M_i)$. Then by Theorem (5.6) on the persistence of solid closure, $1 \otimes u$ will be in the solid closure of the image of $S \otimes_R B$ in $S \otimes_R M_i$, and this is the same as the image of $S \otimes_R M_{i+1}$ in $S \otimes_R M_i$.

The second statement is then immediate. $\square$

To pursue the theory further we shall make use of the the existence of big Cohen-Macaulay algebras in the equal characteristic case.

(10.3) Definition. *If (R, m, K) is a local ring we shall say that an R-module M is a big Cohen-Macaulay module if there is some system of parameters $x_1, \ldots, x_d$ for R that is a regular sequence on M, by which we mean not only that x_i is a nonzerodivisor on $M/(x_1, \ldots, x_{i-1})M$ for $1 \leq i \leq d-1$, but also that $(x_1, \ldots, x_d)M \neq M$. If every system of parameters is a regular sequence on M, then R is called a balanced big Cohen-Macaulay module (cf.* [Sh]*).*

An R-algebra S is called a (balanced) big Cohen-Macaulay algebra if it is a (balanced) big Cohen-Macaulay R-module.

The existence of big Cohen-Macaulay modules for equicharacteristic local rings was established in [Ho2]. Quite recently, the existence of big Cohen-Macaulay algebras in the equicharacteristic case was proved as well, in [HH7] (see also [HH5], [Hu2] and [HH12]):

(10.4) Theorem. *If (R, m, K) is an equicharacteristic local ring, then there is a ring homomorphism $R \to S$ such that $mS \neq S$ and such that every system of parameters for R is a regular sequence on S. Thus, S is a balanced big Cohen-Macaulay algebra for R. Moreover, S may be chosen to be quasilocal (and $R \to S$ local).*

If R is a complete (or excellent) local domain of positive prime characteristic p, one may choose S to be the integral closure R^+ of R in an algebraic closure of its fraction field. (Note that $R \to R^+$ is local if R is complete). $\square$

The connection with solid closure comes from the following observation:

(10.5) Proposition. *Let (R, m, K) be a local ring of dimension d. Let M be a big Cohen-Macaulay module (or algebra) for R.*

(a) *$\widehat{R} \otimes_R M$ is a big Cohen-Macaulay module (or algebra) for $\widehat{R}$.*

(b) *$(\widehat{R}/\mathfrak{p}) \otimes_R M$ is solid over $\widehat{R}/\mathfrak{p}$ for at least one minimal prime $\mathfrak{p}$ of $\widehat{R}$ such that $\dim \widehat{R}/\mathfrak{p} = \dim \widehat{R}$ $(= \dim R)$.*

(c) *If $\widehat{R}$ has only one minimal prime, then M is formally solid.*

Proof. (a) Since $\widehat{R}$ is a faithfully flat extension of R and a system of parameters for R is a system of parameters for $\widehat{R}$, $\widehat{R} \otimes_R M$ is a big Cohen-Macaulay module for $\widehat{R}$.

(b) We may replace R, M by $\widehat{R}$, $\widehat{R} \otimes_R M$. Thus, we may assume that R is complete. Let $\mathbf{x} = x_1, \dots, x_d$ be a system of parameters for R that is a regular sequence on M. Then $H^d_m(M) \cong H^d_{(\mathbf{x})}(M)$ is nonzero, since the maps

$$M/(x_1^t, \dots, x_d^t)M \to M/(x_1^{t+1}, \dots, x_d^{t+1})M$$

in the direct limit system which may be used to compute $H^d_{(\mathbf{x})}(M)$ (these maps are induced by multiplication by $x_1 \cdots x_d$ on the numerators) are all injective. Then R has a finite filtration

$$R = J_0 \supseteq J_1 \supseteq \cdots \supseteq J_h = (0)$$

in which each factor J_i/J_{i+1} occurring is of the form R/P_i for some prime ideal P_i of R, and there is a corresponding filtration for M, namely

$$M = J_0M \supseteq J_1M \supseteq \cdots \supseteq J_hM = (0).$$

It follows that $H^d_m(J_iM/J_{i+1}M) \neq 0$ for at least one choice of i. Since the surjection $J_i \otimes_R M \twoheadrightarrow J_iM$ sends the image of $J_{i+1} \otimes_R M$ to $J_{i+1}M$, it follows that there is a surjection of $(J_i/J_{i+1}) \otimes_R M \cong (R/P_i) \otimes_R M$ onto $J_{i+1}M/J_iM$. Thus, there is a prime ideal P_i of R such that $H^d_m\big((R/P_i) \otimes_R M\big) \neq 0$. But since $(R/P_i) \otimes_R M$ may be viewed as a module over R/P_i for the purpose of computing local cohomology (replacing m by $m(R/P_i)$), this cannot happen unless $\dim R/P = d$, i.e., unless $P_i = \mathfrak{p}$ is a minimal prime of R such that $\dim R/\mathfrak{p} = d$.

(c) This is immediate from (b). $\square$

Thus:

(10.6) Corollary. *Let R be a complete local domain. Then an R-algebra that has an R-algebra homomorphism to a big Cohen-Macaulay algebra for R is solid.* $\square$

It is reasonable to ask, whether, conversely, when R is a complete local domain, every solid R-algebra can be mapped to a big Cohen-Macaulay algebra for R. This is true if $\dim R \leq 2$: see §12. But if $\dim R \geq 3$ it is false, in general, when R contains a field of characteristic 0, as the examples just below show. We do not know what the situation is when R has characteristic p or is of mixed characteristic.

(10.7) *Examples.* (a) Let $G = SL(n, K)$, where K is a field of characteristic zero, and let $X = (x_{ij})$ denote an $n \times (n+1)$ matrix of indeterminates over K. Let $S = K[x_{ij}]_{ij}$ be the polynomial ring in the entries of X, and let G act K-linearly on R by sending the entries of X to the entries of αX for all $\alpha \in G$. Then the ring of invariants $R = S^G$ is well known (cf. [We]) to be generated by the $n+1$ size n minors $\Delta_1, \dots, \Delta_{n+1}$ of X, and these are algebraically independent. To be precise, we shall use Δ_j to denote the product of $(-1)^j$ with the determinant of the $n \times n$ matrix obtained by deleting the j^{th} column of X. Every row of X gives a relation on the Δ_j's, namely

$$(\#) \qquad \sum_{j=1}^{n+1} x_{ij}\Delta_j = 0.$$

The Reynolds operator is a K-linear retraction ρ that sends the h^{th} graded piece $[S]_h$ of S to the h^{th} graded piece $[R]_h$ of R. (It kills the sum of irreducible G-submodules of $[S]_h$ on which G acts non-trivially.) Moreover, ρ yields a degree preserving R-module retraction $S \to R$. We refer to [We] and [Mum] for details. Let $\widehat{R}$, $\widehat{S}$ denote the completions of R and S with respect to the ideals $(\Delta_j)_j R$ and $(x_{ij})_{ij} S$, respectively. Evidently, ρ also induces a retraction $\widehat{S} \to \widehat{R}$ as $\widehat{R}$-modules, so that $\widehat{S}$ is a solid $\widehat{R}$-algebra.

We want to observe that S cannot, however, be embedded in an R-algebra such that $\Delta_1, \dots, \Delta_{n+1}$ is a regular sequence if $n \geq 2$. If T were such an algebra then we would have from the relations (#) given above that each x_{ij} is in the ideal of T generated by the Δ_ν for $\nu \neq j$ (since x_{ij} multiplies Δ_j into that ideal). But then $(x_{ij})_{ij}T \subseteq (\Delta_j)_j T$. On the other hand, each Δ_j is in the n^{th} power of the ideal $(x_{ij})_{ij}R$ even over R, so that $(\Delta_j)_j T \subseteq \big((x_{ij})_{ij}R\big)^n$. But this shows that $(\Delta_j)_j T \subseteq \big((\Delta_j)_j T\big)^n$, which is impossible if the Δ_j form a regular sequence on T.

(b) In the example considered in Remark (8.11), S cannot be mapped to a big Cohen-Macaulay R-algebra T, since $x_1^2x_2^2x_3^2$ will still be in the ideal (x_1^3, x_2^3, x_3^3) in T, and this cannot happen if x_1, x_2, x_3 is a regular sequence on T.

(10.8) *Finite projective complexes and formal minheight.* If $\alpha\colon G \to G'$ is any map of finitely generated projective modules let $I_1(\alpha)$ denote the ideal that is the image of the map $G \otimes_R \operatorname{Hom}_R(G', R) \to R$ sending $g \otimes f$ to $f(\alpha(g))$, and let $I_t(\alpha) = I_1(\wedge^t \alpha)$ for $t \geq 0$ with the convention $I_0(\alpha) = R$. Let rank α denote the largest integer t such that $I_t(\alpha) \neq 0$ (this is determinantal rank: when G, G' are free, $I_t(\alpha)$ is the ideal generated by the size t minors of a matrix for α).

Let $G_\bullet$ denote a finite complex of finitely generated projective modules over the nonzero Noetherian ring R, say

$$0 \to G_d \to \cdots \to G_i \xrightarrow{\alpha_i} G_{i-1} \to \cdots \to G_0 \to 0,$$

and assume for simplicity that each G_i is locally free of constant rank b_i (the ranks are automatically constant if $\operatorname{Spec} R$ is connected). Let $r_i = \sum_{j=i}^{d}(-1)^{j-i}b_i$. Let δ be a function from the ideals of R to $\mathbb{N} \cup \{\infty\}$, such as depth, height, or minheight. We say that the complex $G_\bullet$ *satisfies the standard conditions on rank and* δ if for all i, $1 \leq i \leq d$, rank $\alpha_i = r_i$ and $\delta\big(I_{r_i}(\alpha_i)\big) \geq i$. Thus, a complex satisfies the standard conditions on rank and depth if and only if it is acyclic ([BE]). Moreover, by the results of [HH4] §9, [HH8]

§3, and, especially, Proposition (5.4) of [AHH], if R is a homomorphic image of a Cohen-Macaulay ring and of characteristic p, then $G_\bullet$ and all its images under the iterates of the Frobenius functor have phantom homology in positive degree (i.e., $G_\bullet$ is *stably phantom acyclic*) if and only if $R_{red} \otimes_R G_\bullet$ satisfies the standard conditions on rank and minheight. (The *minheight* $\operatorname{mnht} I$ of an ideal I is defined as

$$\min \{\operatorname{ht} I(R/\mathfrak{p}) : \mathfrak{p} \text{ is a minimal prime of } R\}.$$

See [HH8], §2.) Thus, one would hope to have an analogous result for solid closure, and this is the case in equal characteristic. We do not know whether the analogous result in mixed characteristic holds. The equal characteristic result is remarkably simple to prove, given the existence of big Cohen-Macaulay algebras. To avoid issues arising from pathology of the ring, we shall work with formal minheight of the ideals: we define the *formal minheight* of I as

$$\inf\{\operatorname{mnht} I(R_m)\widehat{\ }: m \text{ is a maximal ideal of } R\}.$$

Evidently, it suffices to consider $m \supseteq I$. If R is universally catenary then minheight does not change upon completion, and the formal minheight of I is the same as the minheight of I. We refer the reader to §2 of [HH8].

(10.9) Theorem. *Let R be a Noetherian ring of equal characteristic and let $G_\bullet$ be a finite complex finitely generated projective modules of constant rank. Suppose that $R_{red} \otimes_R G_\bullet$ satisfies the standard conditions on rank and formal minheight. Then G has shadow homology in positive degree.*

Proof. If we tensor with a complete local domain of R the standard conditions on rank and formal minheight continue to hold (note that the nilpotents are automatically killed; the ranks cannot go up, and do not decrease because all the ideals of minors have formal minheight at least one). Hence, it suffices to consider the case where R is a complete local domain, so that minheight agrees with height. Let S denote a balanced big Cohen-Macaulay algebra for R. By Appendix B of [Nor] (cf. also Theorem (1.2.3) of [Ab1]), $S \otimes_R G_\bullet$ is acyclic. It follows that, for $i \geq 1$, every cycle in G_i is in the solid closure of the boundaries, since it is in the expansion of the boundaries to $S \otimes_R G_i$, and S is a solid R-algebra by (10.5). □

(10.10) *Remark.* The conclusion of (10.9) is valid in mixed characteristic if $\dim R \leq 2$, or, more generally, whenever every complete local domain of R has a big Cohen-Macaulay algebra (in dimension two, one may use the normalization).

Similarly, we have:

(10.11) Theorem. *Let R be a Noetherian ring of equal characteristic and let $x_1, \ldots, x_n$ be elements of R such that the formal minheight of $I = (x_1, \ldots, x_n)R$ is at least n. Let $J = (x_1, \ldots, x_{n-1})R$. Then $J :_R x_n R \subseteq J^\bigstar$.*

Proof. It suffices to see this once we expand I, J to a complete local domain of R. The result is clear if any element of J becomes a unit or if x_n becomes a unit. Thus, we may assume that R is a complete local domain and $x_1, \ldots, x_n$ are in the maximal ideal. The desired conclusion now follows by expanding to a big Cohen-Macaulay algebra S for R, as in (10.9). □

(10.12) *Remark.* Suppose that a finite complex $G_\bullet$ of finitely generated projective R-modules is such that $R_{red} \otimes_R G_\bullet$ satisfies the standard conditions on rank and formal minheight. (We may work with minheight if the ring is universally catenary.) Then this is preserved when one tensors with an R-algebra S provided that the map $R \to S$ preserves formal minheight, i.e., provided that for every ideal I of R, the formal minheight of IS is at least as big as the formal minheight of I. For example, if R is a complete local domain and the standard conditions on rank and height hold over R, then they hold after applying $S \otimes_R \_$ for any complete local domain S module-finite over R.

(10.13) *Remark.* One would really like to have much stronger results than (10.11) comparable to the results on iterated operations for tight closure obtained in §7 of [HH4]. For example, suppose for simplicity that $x_1, \ldots, x_n$ is a system of parameters in a complete local domain R. One would like to have that for integers $t \geq 1$,

$$(\dagger) \qquad (x_1, \ldots, x_{n-1}, x_n^t)^\bigstar :_R x_n \subseteq (x_1, \ldots, x_{n-1}, x_n^{t-1})^\bigstar$$

While this can be proved in characteristic p using tight closure results, it is false in equal characteristic zero, and we do not know whether it holds in mixed characteristic, even if the ring is regular. In fact, if the ring is regular, the iterated use of instances of (†) can be used to show that

$$(x_1^t, \ldots, x_n^t)^\bigstar :_R (x_1 \cdots x_n)^{t-1} \subseteq (x_1, \ldots, x_n)^\bigstar \neq R,$$

for all $t \geq 1$, and then it follows that that $(x_1^t, \ldots, x_n^t)$ is solidly closed for all t. This in turn implies that regular rings are S-regular. Because of the example of Roberts [Ro6], we know that regular rings of dimension three or more are not S-regular in equal characteristic zero, and this shows that (†) fails in general in dimension three in equal characteristic zero. We do not know what happens in complete local domains of mixed characteristic.

11. Big Cohen-Macaulay algebras and tight closure in equal characteristic zero

Let $N \subseteq M$ be finitely generated modules over a locally excellent domain R of characteristic p. It is an open question whether

(‡) an element $u \in M$ is in $N^*{}_M$ if and only if it is in the expansion of N to $R^+ \otimes_R M$, where R^+ is the integral closure of R in an algebraic closure of its fraction field.

Some evidence for this can be found in [Sm1, Sm2], where it is shown to be true for ideals of R generated by parameters. (More generally, it is established in [Ab3] that, given the result of [Sm2], then (‡) holds whenever M/N has finite phantom projective dimension.)

It should be noted that the question can be reduced to studying complete local domains. For a complete local domain R of characteristic p, R^+ is a balanced big Cohen-Macaulay module. The following result can therefore be viewed as a weakened version of (‡). However, the result is of considerable interest in its own right: for example, it gives a new proof that balanced big Cohen-Macaulay algebras exist in characteristic p. However, this proof does not yield the existence of big Cohen-Macaulay algebras in the weakly functorial sense discussed in detail in [HH12], which is very important for certain applications.

(11.1) Theorem. *Let R be a complete local domain (or analytically irreducible excellent local domain) of characteristic p, and let $N \subseteq M$ be finitely generated R-modules. Let $u \in M$. Then $u \in N^{\bigstar}{}_M$ $(= N^*{}_M)$ if and only if there exists a balanced big Cohen-Macaulay algebra S over R such that $1 \otimes u \in \langle N_S \rangle$.*

Proof. The "if" direction is clear, since a big Cohen-Macaulay algebra over an analytically irreducible domain is formally solid: see (10.5). To prove "only if" we begin with the generic forcing algebra T over R for the triple (M, N, u) and perform successive algebra modifications with respect to relations on various systems of parameters for R (the precise meaning is given below). If no finite sequence of such modifications

$$T = T_0 \to T_1 \to \cdots \to T_r$$

is such that

$$(\#) \qquad 1 = \sum_{j=1}^{n} y_j \theta_j$$

with the θ_j in T_r and the y's a set of generators for the maximal ideal of R, then a certain direct limit of such modifications will serve as a balanced big Cohen-Macaulay module for R. The details of this kind of argument are discussed at length in [HH12] (see §§(3.1)–(3.7) of [HH12], especially (3.6) and (3.7)), and here we shall only give the proof of the key point, that a finite sequence of modifications cannot lead to the relation (#). Here, when we say that T_{i+1} is an *algebra modification* of T_i with respect to a relation on parameters for R we mean that for that value of i there exists part of a system of parameters

$$x_1^{(i)}, \dots, x_{k_i}^{(i)}, x_{k_i+1}^{(i)}$$

for R and a relation

$$x_{k_i+1}^{(i)} s^{(i)} = \sum_{t=1}^{k_i} x_t^{(i)} s_t^{(i)}$$

with $s^{(i)}, s_t^{(i)} \in T_i$ such that

$$T_{i+1} = T_i[z_1^{(i)}, \dots, z_{k_i}^{(i)}]/(s^{(i)} - \sum_{t=1}^{k_i} x_t^{(i)} z_t^{(i)})$$

with the $z_t^{(i)}$ indeterminates over T_i.

We can assume without loss of generality that M is free here, and that N is the column space of a matrix $\alpha = (a_{ij})$ which is, say, ν by μ. Then we may take T to be the algebra $R[z_1^{(0)}, \dots, z_\mu^{(0)}]/\mathfrak{A}$ where $\mathfrak{A}$ is the ideal generated by the entries of the column matrix $\alpha Z^{(0)} - u$: here, $Z^{(0)}$ is the $\mu \times 1$ column whose entries are the $z_j^{(0)}$ and $u \in M$ has been written as a $\nu \times 1$ column.

For $0 \le i \le r$ we define a certain finite subset $\Sigma_i \subseteq T_i$ and an integer $b(j) \in \mathbb{N}$ by reverse induction on i in the following manner:

(1) Σ_r is the set whose elements are the θ's occurring in the relation $(\#)$ $1 = \sum_{j=1}^n y_j\theta_j$ in T_r.

(2) If $0 < j < r$ and $\Sigma_i \subseteq T_i$ has been defined for $i \ge j$ while $b(i)$ has been defined for $r \ge i > j$ define Σ_{j-1} and $b(j)$ as follows. Express each element of Σ_j as a polynomial over T_{j-1} in the standard generators (the images of the $z_t^{(j)}$) for T_j over T_{j-1}. Let $b(j)$ be the greatest degree of any of these polynomials. Let Σ_{j-1} consist of the coefficients of these polynomials and the elements $s^{(j-1)}$, $s_t^{(j-1)}$ occurring in the relation used in the construction of T_j from T_{j-1}.

(3) Express the elements of Σ_0 as polynomials in the standard generators (the images of the $z_t^{(0)}$) of T_0 over R, and let $b(0)$ be the greatest degree of any of these polynomials.

Thus, (1) gets the Σ's started. Repeated application of (2) enables one to construct $b(r)$ and Σ_{r-1}, $b(r-1)$ and Σ_{r-2}, $\dots$, $b(1)$ and Σ_0. Finally, (3) specifies $b(0)$.

We next define two "intertwined" sequences of integers

$$\beta(0), \dots, \beta(r) \in \mathbb{N} \text{ and } B(0), \dots, B(r) \in \mathbb{N}$$

from the $b(i)$ by the following recursive rules:

(i) $\beta(0) = 1$ and $B(0) = b(0)$.

(ii) For $0 < i \le r$, $\beta(i+1) = B(i) + 1$ and $B(i+1) = b(i+1)\beta(i+1) + B(i)$.

We let $B = B(r)$.

Next, choose a test element c for tight closure in R: this is possible by Theorem (6.1a) of [HH9]. Then for any sequence of elements $x_1, \dots, x_{k+1}$ of R that are part of a system of parameters, if $w \in (x_1, \dots, x_k) :_R x_{k+1}$ then $cw^q \in (x_1^q, \dots, x_k^q)R$ for all $q = p^e$, since tight closure "captures" colon ideals for parameters: see, for example, Theorem (4.7) of [HH4] or the much more extended discussion in §7 of [HH4]. Of course, since $u \in N^*{}_M$ we also have that for all q, $cu^q \in N^{[q]}$. Taking q^{th} roots we can assert instead that the following two conditions hold:

(i) For all q, $c^{1/q}u \in NR^{1/q} \subseteq R^{1/q} \otimes_R M$ and

(ii) For all q, if $x_1, \dots, x_{k+1}$ is part of a system of parameters for R then

$$w \in (x_1, \dots, x_k)R :_R x_{k+1} \text{ implies that } c^{1/q}w \in (x_1, \dots, x_k)R^{1/q}.$$

Now let $R^\infty = \bigcup_q R^{1/q}$. We note the following fact:

(ii°) For all q, if $x_1, \dots, x_{k+1}$ is part of a system of parameters for R then

$$w \in (x_1, \dots, x_k)R^\infty :_{R^\infty} x_{k+1} \text{ implies that } c^{1/q}w \in (x_1, \dots, x_k)R^\infty.$$

To see why, suppose that $wx_{k+1} = \sum_{i=1}^k x_iw_i$ with $w, w_1, \dots, w_k \in R^\infty$. The point is that we may choose a power of p, say Q, so large that $Q \ge q$ and such that all the elements w^Q, w_i^Q are in R. Taking Q^{th} powers we see that $w^Q \in (x_1^Q, \dots, x_k^Q)R :_R x_{k+1}^Q$.

It follows that $cw^Q \in (x_1^Q, \dots, x_k^Q)R$ from (ii). Since $Q/q \geq 1$ we have as well that $c^{Q/q}w^Q \in (x_1^Q, \dots, x_k^Q)R$. Taking Q^{th} roots yields that $c^{1/q}w \in (x_1, \dots, x_k)R^\infty$.

The next point is this: we shall show that for every integer $q = p^e$ there are R-algebra homomorphisms $\psi_i: T_i \to R_c^\infty$, $0 \leq i \leq r$, such that the images of the generators of T_i over T_{i-1} (or over R if $i = 0$) are contained in the cyclic R^∞-submodule of R_c^∞ generated by $c^{-\beta(i)/q}$, while the image of $\Sigma_i \subseteq T_i$ is contained in the cyclic R^∞-submodule of R_c^∞ spanned by $c^{-B(i)/q}$. Moreover, each ψ_{i+1} extends ψ_i, $0 \leq i < r$.

Notice that we defined the integers $\beta(i)$, $B(i)$, and, in particular, the integer $B = B(r)$ before introducing q, so that, evidently, these integers are independent of q.

To define $\psi_0: T \to R_c^\infty$ we note that the fact that $c^{1/q}u \in NR^{1/q}$ yields a solution of the matrix equation $\alpha Z_0 = u$ for the z_t^0 in $c^{-1/q}R^{1/q}$. We let ψ_0 be the unique R-algebra map sending z_t^0 to the element of $c^{-1/q}R^{1/q}$ given by this solution. Since the elements of Σ_0 are polynomials of degree $b(0)$ in the z_t^0, they map into $c^{-b(0)/q}R^{1/q}$.

Now suppose that we have constructed $\psi_i: T_i \to R_c^\infty$ such that the generators map into $c^{-\beta(i)/q}R^\infty$ and Σ_i maps into $c^{-B(i)/q}R^\infty$, where $0 \leq i < r$. We want to extend ψ_i to

$$(\#\#) \qquad T_{i+1} = T_i[z_1^{(i)}, \dots, z_{k_i}^{(i)}]/(s^{(i)} - \sum_{t=1}^{k_i} x(i)_t z_t^{(i)}),$$

where there is a relation

$$x_{k_i+1}^{(i)} s^{(i)} = \sum_{t=1}^{k_i} x_t^{(i)} s_t^{(i)}.$$

In the sequel we sometimes omit the superscripts $^{(i)}$ and write $k = k_i$. By the construction of the Σ's, the elements $s^{(i)}$, $s_t^{(i)}$ are in Σ_i. It follows that we obtain a relation

$$x_{k+1}\psi_i(s) = \sum_{t=1}^{k} x_t \psi_i(s_t),$$

where the elements $\psi_i(s)$, $\psi_i(s_t)$ are all in $c^{-B(i)/q}R^\infty$. Let $\psi_i(s) = \sigma C^{-B(i)/q}$ and $\psi_i(s_t) = \sigma_t c^{-B(i)/q}$, with σ, σ_t in R^∞. Then

$$x_{k+1}\sigma = \sum_{t=1}^{k} x_t \sigma_t$$

in R^∞, and so by condition (ii°) above we have that

$$c^{1/q}\sigma = \sum_{t=1}^{k} x_t \tau_t,$$

where the elements $\tau_t \in R^\infty$. This yields an equation

$$\sigma = \sum_{t=1}^{k} x_t (\tau_t c^{-1/q})$$

and hence

$$\begin{aligned}\psi_i(s) &= \sigma c^{-B(i)/q}\\ &= \sum_{t=1}^{k} x_t(\tau_t c^{-B(i)/q})(c^{-1/q})\\ &= \sum_{t=1}^{k} x_t(\tau_t c^{-\beta(i+1)/q}).\end{aligned}$$

By the formula (##) defining T_{i+1} we can extend ψ_i from T_i to T_{i+1} by sending each $z_t^{(i)}$ to $\tau_t c^{-\beta(i+1)/q}$. The generators are mapping into $c^{-\beta(i+1)/q}R^\infty$, as required. Now each element of Σ_{i+1} can be written as a polynomial of degree at most $b(i+1)$ in these generators with coefficients in Σ_i, and the value of such a polynomial will lie in

$$(c^{-\beta(i+1)/q})^{b(i+1)}c^{-B(i)/q}R^\infty = c^{-B(i+1)/q}R^\infty,$$

as required.

We are now ready for the dénouement. We apply ψ_r to the equation (#) $1 = \sum_{j=1}^n y_j\theta_j$ to obtain the equation

$$1 = \sum_{j=1}^{n} y_j\psi_r(\theta_j)$$

holding in R_c^∞. The value of $\psi_r(\theta_j)$ lies in $c^{-B/q}R^\infty$, where $B = B(r)$. Multiplying by $c^{B/q}$ we obtain that $c^{B/q} \in mR^\infty$ for every $q = p^e$. Taking q^{th} powers, we have that $c^B \in m^{[q]}R^\infty \cap R$ for every $q = p^e$. Since $IR^\infty \cap R \subseteq I^*$ for any ideal I ($u \in IR^\infty \cap R$ implies that $1 \cdot u^{qq'} \in I^{[qq']}$ for all q'), we find that $c^B \in (m^{[q]})^*$ for all q. We emphasize again that $B = B(r)$ is independent of q. Since c is a test element, we have that

$$c \cdot c^B = c^{B+1} \in \bigcap_q m^{[q]} \subseteq \bigcap_q m^q = (0),$$

a contradiction. □

(11.2) *Remark.* We may apply (11.1) in instances where we know that the element u is in N^* or even N. For example, we can take $M = R$, $N = R$, and $u = 0$. The result of (11.1) then shows that R has a balanced big Cohen-Macaulay algebra. This is a new proof of the existence of big Cohen-Macaulay algebras in characteristic p, independent of the results of [HH7].

(11.3) *Comparison of solid closure with tight closure in equal characteristic zero.* We next want to show that that if $N \subseteq M$ are finitely generated modules over a Noetherian ring R containing $\mathbb{Q}$ then $N^*{}_M \subseteq N^\bigstar{}_M$ insofar as * is defined. To this end we want to use the "biggest" version of $N^*{}_M$ considered in [HH11]. The appropriate notation is N^{*EQ}: we refer the reader to [HH11] for details, but we give a brief description below.

There is no loss of generality in assuming that M is free and that N is the span of the columns of a $\nu\times\mu$ matrix $\alpha=(a_{ij})$ over R. The following then gives the "biggest" version of tight closure: $u\in {N^{*EQ}}_M$ if for every map of R to a complete local domain S there exists a finitely generated $\mathbb{Z}$-algebra B, a prime ideal Q of B, and a map of B_Q to S such that:

(1) B has a $\nu\times\mu$ matrix α_B that maps to the image of α over S, so that if we define $M_B=B^\nu$ and N_B as the column space of α_B in M_B, then $S\otimes_B(M_B/N_B)\cong S\otimes_R(M/N)$. Moreover, M_B has an element u_B that maps to $1\otimes u$ in $S\otimes_R M$.

(2) B/Q is smooth over $\mathbb{Z}$: this may be achieved by localizing B at one element not in Q and $\mathbb{Z}$ at one nonzero integer h. (It then follows that for every prime p not dividing h, $B/(Q+pB)$ is regular of characteristic p.) For p not dividing h we denote by W_p the multiplicative system in B consisting of elements not in any minimal prime of the radical ideal $Q+pB$, and we let $B\langle p\rangle$ denote $W_p^{-1}B/pB$. We write $M_{B\langle p\rangle}$ for $B\langle p\rangle\otimes_B M_B$ and $N_{B\langle p\rangle}$ for the image of $B\langle p\rangle\otimes_B N_B$ in $M_{B\langle p\rangle}$.

(3) For all but finitely many primes p of $\mathbb{Z}$ with p not dividing h, we have that $1\otimes u\in {N_{B\langle p\rangle}^{\;*}}_{M_{B\langle p\rangle}}$ over $B\langle p\rangle$.

(11.4) Theorem. *Let R be any Noetherian ring containing $\mathbb{Q}$. Let $N\subseteq M$ be finitely generated R-modules. Then ${N^{*EQ}}_M\subseteq {N^{\bigstar}}_M$. Moreover, if R is a complete local domain and $u\in {N^{*EQ}}_M$, then there is a balanced big Cohen-Macaulay R-algebra $\mathcal{T}$ such that $1\otimes u$ is in the expansion of N to $\mathcal{T}\otimes_R M$.*

Proof. Let $u\in {N^{*EQ}}_M$ and let S be a complete local domain of R. It suffices to show the result after replacing R, M, N, u by S, $S\otimes_R M$, $\langle N_S\rangle$, $1\otimes u$. Thus, we may assume that R is a complete local domain. We have $B\to B_Q\to R$ as in the definition of ^{*EQ}. The contraction of the maximal ideal of R to B is a prime Q_0 contained in Q. It is easy to see that we can preserve (1), (2), (3) working with B_{Q_0} instead of B_Q. Thus, we may assume that $B_Q\to R$ is a local homomorphism. We may replace B_Q by its image in R without affecting any relevant issues. We may therefore also assume that the map $B_Q\to R$ is injective. We shall complete the argument by proving the final assertion: since a big Cohen-Macaulay algebra over a complete local domain is solid, this will suffice.

We note that if we factor $B_Q\to B'_{Q'}\to R$ (where the maps are local and B' is a finitely generated $\mathbb{Z}$-algebra) then the image $u_{B'}$ of u_B continues to be in ${N_{B'}^{\;*EQ}}_{M'_B}$, where $M_{B'}=B'\otimes_B M$ and $N_{B'}$ is $\mathrm{Im}\,(B'\otimes_B N_B\to M_{B'})$.

Our objective is to show that u is forced into the expansion of N to a big Cohen-Macaulay algebra. As in the proof of (11.1), if not, a generic forcing algebra $T=T^{(0)}$ for (M,N,u) will have a sequence of algebra modifications with respect to relations on segments of various systems of parameters,

$$T^{(0)}\to\cdots\to T^{(r)},$$

such that $1=\sum_{j=1}^n y_i\theta_i$, where the y_i generate the maximal ideal of R and the $\theta_i\in T^{(r)}$.

Because R is the direct limit of local rings $B'_{Q'}$ where B' is a finitely generated $\mathbb{Z}$-subalgebra of R and Q' is the contraction to B' of the maximal ideal of R, we have that the map $B_Q \to R$ factors $B_Q \to B'_{Q'} \to R$ in such a way that all the systems of parameters involved, the y's, all coefficients from R needed in the construction of the sequence $T^{(i)}$, all coefficients from R needed to describe the θ's in terms of the standard generators of the T's, etc. are actually in B'. Moreover, it is shown in [HH11], Theorem (3.5.1), using the main result of [ArR], that $B'_{Q'} \to R$ factors $B'_{Q'} \to B''_{Q''} \to R$ where B'' is a domain of the same sort in which a specified set of sequences of parameters (i.e., each sequence is part of a system of parameters) retain the property of being sequences of parameters. (Here, the dimension of B'' may be much larger than that of R.) We do not assume that the map $B'' \to R$ is necessarily injective. We change notation and write B_Q for $B''_{Q''}$. Note that by localizing B at one element not in Q we may further assume that any given finite set of elements of B_Q is in B. Thus, we may assume that we have a sequence of algebras

$$T_B = T_B^{(0)} \to T_B^{(1)} \to \cdots \to T_B^{(r)}$$

over B such that

(1) T_B is a generic forcing algebra for (M_B, N_B, u_B).
(2) For $0 \le i < r$, $T_B^{(i+1)}$ is an algebra modification of $T_B^{(i)}$ with respect to a relation on *parameters* for Q (by which we mean that the "parameters" can be extended to a sequence of ht Q elements that generate an ideal whose radical is Q).
(3) $T^{(i)} \cong R \otimes_B T_B^{(i)}$.
(4) $1 = \sum_{i=1}^{n} y_i\theta_i$ in $T_B^{(r)}$.

Notice that the map $B \to R$ will factor $B' \to R''$ where B' is the normalization of B and R'' is a suitable module-finite extension of R. Thus, there is no loss of generality in assuming that B is normal: these replacements do not affect any relevant issues. It follows that for all but finitely many primes p of $\mathbb{Z}$, B/pB is normal (although it may not be a domain): see, for example, [HH11], Proposition (2.3.17).

We now get a contradiction by passing to characteristic p for suitable p: we only require that

(1) B/pB be normal,
(2) the image of each sequence of parameters used in defining the sequence of algebra modifications $T^{(i)}$ remain a sequence of parameters, and
(3) $u_{B\langle p\rangle}$ be in the tight closure of $N_{B\langle p\rangle}$ in $M_{B\langle p\rangle}$.

We consider B/pB localized at a minimal prime P of $Q(B/pB)$: call this ring C. Notice that C is a localization of $B\langle p\rangle$, so that u_C is in the tight closure of N_C in M_C where $M_C = C \otimes_{B\langle p\rangle} M_{B\langle p\rangle}$, and u_C, N_C indicate the images of $u_{B\langle p\rangle}$, $C \otimes_{B\langle p\rangle} N_{B\langle p\rangle}$ respectively in M_C. Then $T_C = C \otimes_B T_B$ is a generic forcing algebra for (M_C, N_C, u_C) and the algebras $T_C^{(i)} = C \otimes_B T_B^{(i)}$ form a sequence of modifications of exactly the sort considered in (11.1). The image of the relation $1 = \sum_{i=1}^{n} y_i\theta_i$ then contradicts (11.1), since C is normal and, hence, analytically irreducible. $\square$

(11.5) Corollary (Briançon-Skoda theorem for solid closure). *Let I be an ideal of an equicharacteristic ring R generated by at most d elements and let $k \in \mathbb{N}$. Then* $\overline{I^{d+k}} \subseteq (I^{k+1})^\star$.

Proof. The issues are unaffected by passing to a complete local domain. In characteristic p the result follows Theorem (5.4) of [HH4]. In equal characteristic zero the result follows from the fact that $(I^{k+1})^{*EQ} \subseteq (I^{k+1})^{\bigstar}$ just established and the results of [HH11]. □

(11.6) Remark. Likewise, whenever a result for tight closure in equal characteristic zero shows that it is "big enough" to contain a certain ideal or module, the same will hold for solid closure. On the other hand, we generally do not understand the situation in mixed characteristic.

12. The case of dimension two

In this section we make a detailed exploration of solid algebras and solid closure in dimension two. The main result here is that an algebra over a two-dimensional complete local domain is solid if and only if it cam be mapped to a big Cohen-Macaulay algebra: see Theorem (12.5). We first need:

(12.1) Definition. *Let R be a ring, let $x, y \in R$ and let M be an R-module. Let $I = (x, y)R$. We denote by $\Theta = \Theta(x, y; M)$ the submodule*

$$\{u \in M_{xy} : \textit{for some } h \in \mathbb{N}, I^h u \in \mathit{Im}\, M\} \subseteq M_{xy}.$$

Thus, $\Theta/(\mathit{Im}\, M) \subseteq M_{xy}/(\mathit{Im}\, M)$ is $H^0_I\big(M_{xy}/(\mathit{Im}\, M)\big)$.

Note that given an arbitrary R-linear map $M \to N$ there is an induced map $M_{xy} \to N_{xy}$, and that this map carries $\Theta(x, y; M)$ to $\Theta(x, y; N)$. Thus, $\Theta(x, y; \_)$ is a covariant functor from R-modules to R-modules. Notice also that if we have a homomorphism $R \to R'$ carrying x, y to x', y' and M is an R'-module, then

$$\Theta(x', y'; {}_{R'}M) \cong \Theta(x, y; {}_{R}M),$$

where the subscript on the left of M indicates over which ring we are considering it as a module.

If $M = R$ is an integral domain with fraction field L and $\mathfrak{A}$ is an ideal of R, then Nagata defines the $\mathfrak{A}$-transform $S(\mathfrak{A}; R)$ as

$$\{f \in L : \text{for some } h \in \mathbb{N}, \mathfrak{A}^h f \subseteq R\}.$$

In case $\mathfrak{A} = I = (x, y)R$ this agrees with $\Theta(x, y; R)$. Nagata makes a deep study of the $\mathfrak{A}$-transform in Chapter V of [N1], pp. 41-60, including examples where it is not finitely generated as an R-algebra, even though R is an affine domain over a field.

(12.2) *Discussion.* We want to make some observations about regular sequences x, y of length two on an R-module M. Let $I = (x, y)$.

(a) If $xu = yv$ then $u \in yM$ and $v \in xM$. If $x^s u = y^t v$, where s, t are positive integers, then $u \in y^t M$ and $v \in x^s M$. (That $v \in xM$ is part of the definition of regular sequence. But if $v = xv'$ then $x(u - yv') = 0$ and since x is not a zerodivisor on M, $u = yv'$. The second statement follows because x^s, y^t is also a regular sequence on M.)

(b) If x, y is a regular sequence on M then y, x is a regular sequence on M if and only if y is not a zerodivisor on M. (The other condition that is needed is proved in (a)).

(c) If x, y is a regular sequence on M, then x, y and y, x are both regular sequences on the I-adic completion $\widehat{M}$ $(= \varprojlim_t M/I^tM)$ of M. (Let $\{m_n\}_n$ be a Cauchy sequence in M representing an element μ of the completion. If $y\mu = 0$ then ym_n is eventually in $(x^t, y^{t+1})M$ for any given t, say $ym_n = x^tu_n + y^{t+1}v_n$. Then $y(m_n - y^tv_n) = x^tu_n$ shows that $m_n - y^tv_n$ is in x^tM and so, for all t, m_n is eventually in $(x^t, y^t)M$. The argument for x is the same. Now suppose that $\{m_n\}_n$ represents $\mu \in \widehat{M}$ and $\{p_n\}_n$ represents $\pi \in \widehat{M}$. Suppose that $x\mu = y\pi$. For all t, $xm_n - yp_n$ is in $(x^{t+1}, y^{t+1})M$ for all sufficiently large n, so that we may choose $n(t)$ arbitrarily large such that

$$xm_{n(t)} - yp_{n(t)} = x^{t+1}u_{n(t)} + y^{t+1}v_{n(t)},$$

and then

$$x(m_{n(t)} - x^tu_{n(t)}) = y(p_{n(t)} + y^tv_{n(t)}).$$

Thus we may choose an increasing sequence $n(t)$ such that the preceding equation holds, and $\{m_{n(t)} - x^tu_{n(t)}\}_t$, $\{p_{n(t)} - y^tv_{n(t)}\}_t$ are Cauchy sequences indexed by t with limits μ, π as before. Using these to replace the original sequences, we see that there is no loss of generality in assuming that $xm_n = yp_n$ for all n. It follows that $p_n = xz_n$ for all n (notice that z_n is uniquely determined). For all t, $xz_n - xz_{n+1} \in (x^{t+1}, y^t)M$ for large n, say

$$x(z_n - z_{n+1}) = x^{t+1}g_n + y^th_n,$$

so that $x(z_n - z_{n+1} - x^tg_n) = y^th_n$, and then $z_n - z_{n+1} - x^tg_n \in y^tM$, which shows that for all t and for all sufficiently large n, $z_n - z_{n+1} \in (x^t, y^t)M$. Thus, $\{z_n\}_n$ is a Cauchy sequence, and it follows that $x\mu = y\pi$ implies that $\pi \in x\widehat{M}$. Since we have already seen that x, y are both nonzerodivisors on $\widehat{M}$, x, y is a permutable regular sequence on $\widehat{M}$).

(d) There is an obvious map $M \to \widehat{M}$ whose kernel is $\bigcap_t I^tM$.

We also note:

(12.3) Corollary. *Let (R, m) be a local ring of dimension two, let x, y be a system of parameters, and let M be an R-module such that x, y is a regular sequence on M. Then every system of parameters for R is a regular sequence on M.*

Proof. Given systems of parameters x, y and u, v, we can choose z so that x, z and u, z are both systems of parameters. The chain of systems of parameters

$$x,y;\ x,z;\ z,x;\ z,u;\ u,z;\ u,v$$

shows that it suffices to consider the case where we permute the two parameters, handled in (12.2c) (note that every pair of elements determines the same topology on M), and the case where the two systems of parameters have their first elements in common, say x, y and x, z. But this case is clear, since, modulo xR, y has a power that is divisible by z. □

With these preliminaries out of the way, we next observe:

(12.4) Lemma. *Let R be a ring, let $x, y \in R$, and let M be an R-module. Let $\Theta = \Theta(x, y; M)$. Let $I = (x, y)R$. Then:*

(a) *x, y are nonzerodivisors on Θ and x, y is a possibly improper regular sequence on Θ.*

(b) *If M is an R-algebra, then Θ is an R-subalgebra of M_{xy}.*

(c) *If N is any R-module such that x, y are nonzerodivisors on N and x, y is a possibly improper regular sequence on N, then every R-module map $\phi: M \to N$ factors uniquely $\phi: M \to \Theta \to N$. Moreover:*

  (1) *If, in addition, N is I-adically separated and there exists a nonzero R-module map $\phi: M \to N$, then x, y is a regular sequence on Θ. (It suffices that $Im\,\phi$ not be contained in $\bigcap_t I^t N$.)*

  (2) *If, instead, we assume in addition that M, N are R-algebras and the map $M \to N$ is an R-algebra homomorphism, then $\Theta \to N$ is also an R-algebra homomorphism.*

(d) *The following conditions are equivalent:*

  (i) *M can be mapped R-homomorphically to an R-module N on which x, y form a regular sequence in such a way that the image of M is not contained in $\bigcap_t I^t N$.*

  (ii) *There is a nonzero R-linear map from M to an I-adically separated R-module N on which x, y form a regular sequence.*

  (iii) *There is a nonzero R-linear map from M to an I-adically complete and separated R-module N on which x, y form a regular sequence.*

  (iv) *$Im(M \to \Theta)$ is not contained in $\bigcap_t I^t \Theta$.*

(e) *If M is an R-algebra, then M has an R-algebra homomorphism to an R-algebra S such that x, y is a regular sequence on S if and only if $(x, y)\Theta \neq \Theta$, i.e., if and only if $1 \notin (x, y)\Theta$. In this case, Θ itself is an R-algebra on which x, y is a regular sequence.*

Proof. The map ϕ kills the kernel of the map $M \to M_{xy}$. There is no loss of generality in replacing M by its quotient by this kernel, and we henceforth assume that x, y are nonzerodivisors on M, so that $M \subseteq M_{xy}$. This does not change Θ.

(a) It is clear that x, y are nonzerodivisors on $\Theta \subseteq M_{xy}$. If $x\theta = y\theta'$ with $\theta, \theta' \in \Theta$ then $\tau = \theta/y = \theta'/x$ is multiplied into M by a power of I, since both $x\tau$ and $y\tau$ are multiplied into M by I^s for $s \gg 0$, and then $I^{s+1}\tau \subseteq M$. Thus, $\tau \in \Theta$, which shows that $\theta' \in x\Theta$.

(b) If a power of I multiplies each of θ, θ' into the image of M, then the sum of the two exponents will yield a power of I that multiplies $\theta + \theta'$ (respectively, $\theta\theta'$) into the image of M.

(c) The map $\phi: M \to N$ induces a map $\psi : M_{xy} \to N_{xy}$. The key point is that the image of Θ is contained in N. For suppose $\theta \in \Theta$. Then for large t, $x^t\theta = u$ and $y^t\theta = v$ are in M. Then

$$x^t\psi(\theta) = \psi(x^t\theta) = \psi(u) = \phi(u) = u' \in N$$

and, similarly, $y^t\psi(\theta) = v' \in N$. But then $y^t u' = x^t v'$ shows that u' is a multiple of x^t in N, say $u' = x^t u''$, and then $x^t\psi(\theta) = x^t u''$ shows that $\psi(\theta) = u'' \in N$. The uniqueness is obvious, since xy is not a zerodivisor on N.

To prove (1), note that if $I\Theta = \Theta$ then $I^t\Theta = \Theta$ for all $t \in N$, and then the same will be true for the image of Θ in N. Since N is I-adically separated, this implies that the image of Θ is 0, a contradiction, since the image of M is not zero. (The parenthetical comment follows from the fact that we may replace N by $\widehat{N}$, and the induced map $\Theta \to \widehat{N}$ will be nonzero provided that $\operatorname{Im}\Theta$ is not contained in $\bigcap_t I^t N$.)

(2) is a consequence of the fact that the map is constructed as the restriction of the induced may $\psi: M_{xy} \to N_{xy}$, and ψ is evidently an algebra homomorphism.

(d) (iii) $\Rightarrow$ (ii) $\Rightarrow$ (i) is obvious. Given (i), we replace N by $\widehat{N}$: since $\operatorname{Ker}(N \to \widehat{N}) = \bigcap_t I^t N$, we obtain a complete module as required. Thus, (i), (ii), and (iii) are equivalent. Part (c) (1) shows that (ii) $\Rightarrow$ (iv). To complete the proof, it will suffice to show that (iv) $\Rightarrow$ (iii). But given (iv), we have that the induced map $M \to \widehat{\Theta}$ is not zero, and so $\widehat{\Theta} \neq 0$. It is then automatic that $\widehat{\Theta} \neq I\widehat{\Theta}$, and x, y is a regular sequence on $\widehat{\Theta}$ by part (a) and (12.2c).

(e) If $1 \notin (x,y)\Theta$ then it is clear from part (a) that x, y is a regular sequence on Θ, since $(x,y)\Theta \neq \Theta$. On the other hand, from part (c) (2) the R-algebra map $M \to S$ factors $M \to \Theta \to S$, where these are maps of R-algebras. If $(x,y)S \neq S$ then $1 \notin (x,y)S$, from which it is clear that $1 \notin (x,y)\Theta$. □

(12.5) Theorem. *Let (R,m,K) be a complete local domain of dimension at most two with system of parameters x, y. Then an R-algebra S is solid if and only if it has an R-algebra homomorphism $S \to T$ such that x, y are nonzerodivisors on T and x, y is a regular sequence on T, so that T is a big Cohen-Macaulay algebra for R. We may choose $T = \Theta(x,y;R' \otimes_R S)$ where R' may be chosen to be either the normalization of R or the S_2-ification of R (the latter may be identified with $\Theta(x,y;R)$ and is a subring of the normalization of R). We may also choose $T = \Theta(x,y;S)$.*

Proof. We have that $R' \otimes_R S$ is solid over R' and admits an R'-linear map ϕ to R' such that the image of 1 is not zero. It then follows from Lemma (12.4) that x, y are nonzerodivisors on $T = \Theta(x,y;R' \otimes_R S)$ and that x, y is a regular sequence on T, since x, y is a regular sequence on R', since R' is $(x,y)R'$-adically separated, and since the map ϕ is not zero.

Since $R \to S \to \Theta(x,y;R' \otimes_R S) = T$ factors $R \to S \to T' \to T$ with $T' = \Theta(x,y;S)$, we must also have that $1 \notin (x,y)T'$, and so T' is a big Cohen-Macaulay algebra over R to which S maps as well. □

(12.6) Corollary. *Let (R,m,K) be a complete local domain of dimension two with system of parameters $x, y \in m$, let $N \subseteq M$ be finitely generated R-modules, let $u \in M$, and let S be a generic forcing algebra for the triple (M,N,u). Then u is in the solid closure of N if and only if with $\Theta = \Theta(x,y;S)$ (defined in (12.1)) we have that $(x,y)\Theta \neq \Theta$, i.e., $1 \notin (x,y)\Theta$.* □

We use this criterion in an example:

(12.7) *Example.* Let $R = K[[X,Y,Z]]/(g) = K[[x,y,z]]$, where K is a field of characteristic zero and $g = X^3 + Y^3 + Z^3$, and let $I = (x,y)R$. Then $I^{*EQ} = I^{\bigstar}$. First note that $z^2 \in I^{*eq} \subseteq I^{*EQ}$: see [HH11], Example (2.2.4). Since we know that $I^{*EQ} \subseteq I^{\bigstar}$, if we show that $z \notin I^{\bigstar}$ then it will follow that $I^{*eq} = I^{\bigstar} = (x,y,z^2)$ (and also that $I^{*EQ} = (x,y,z^2)$), for any ideal strictly larger than (x,y,z^2) must contain z. But $z \in I^{\bigstar}$ if and only if R can be mapped to an algebra S on which x, y is a regular sequence and $z \in IS$.

Thus, it will suffice to show that if $x^3+y^3+z^3=0$ in S, with 3 invertible in S (actually, we know that $S \supseteq \mathbb{Q}$ here) and $z = ux+vy$, then x, y cannot be a regular sequence in S (we have changed notation, using the same letters x, y, z for the images of the original x, y, z in S). But then $x^3+y^3+(ux+vy)^3 = 0$ which can be rewritten as:

$$(u^3+1)x^3 + (3u^2v)(x^2y) + (3uv^2)(xy^2) + (v^3+1)y^3 = 0.$$

If x, y is a regular sequence on S then the relations on the elements x^3, x^2y, xy^2, y^3 are spanned by the quadruples

$$(y,-x,0,0),\ (0,y,-x,0) \quad \text{and} \quad (0,0,y,-x).$$

This implies that

$$(u^3+1, 3u^2v, 3uv^2, v^3+1) = a(y,-x,0,0) + b(0,y,-x,0) + c(0,0,y,-x)$$

with $a, b, c \in S$, yielding

$$\begin{array}{lrcl}
(1) & u^3+1 &=& ay \qquad \text{or} \qquad (1^\circ) \quad u^3 = -1+ay \\
(2) & 3u^2v &=& -ax+by \\
(3) & 3uv^2 &=& -bx+cy \\
(4) & v^3+1 &=& -cx \qquad \text{or} \qquad (4^\circ) \quad v^3 = -1-cx
\end{array}$$

Multiplying (1°) and (4°) together, we obtain that $u^3v^3 \in 1+(x,y)S$ while equation (2) (or (3)) shows that $3u^3v^3 \in (x,y)S$, and, since 3 is invertible, $u^3v^3 \in (x,y)S$. Thus, $1 \in (x,y)S$, a contradiction. □

(12.8) *Remark.* We do not know, in general, whether, over a complete local domain of dimension two and equal characteristic zero, the big equational tight closure, ^{*EQ}, agrees with the solid closure, ${}^{\bigstar}$.

13. Regular rings revisited

We have already noted (cf. (7.24)) that, as a consequence of a local cohomology calculation of P. Roberts, in the regular rings $K[x,y,z]$ and $K[[x,y,z]]$, where K is a field of characteristic zero, we have that $x^2y^2z^2 \in (x^3,y^3,z^3)^{\bigstar}$. Nonetheless, there are several positive results on when ideals in regular rings are solidly closed that yield useful information.

(13.1) Theorem. *Let R be a regular Noetherian ring. Let $I \subseteq R$ be an ideal and let $N \subseteq M$ be finitely generated R-modules.*

(a) *Suppose that p is a prime integer and that R/pR is regular. Then every ideal of R containing p is solidly closed in R. More generally, if M/N is killed by p, then N is solidly closed in M.*

(b) *Suppose that R is a finitely generated $\mathbb{Z}$-algebra of characteristic zero. Then every ideal I of R such that R/I is torsion-free over $\mathbb{Z}$ is solidly closed. More generally, if M/N is torsion-free over $\mathbb{Z}$, then N is solidly closed in M.*

Proof. (a) It suffices to prove the final statement. We may assume without loss of generality that $N = 0$. Suppose that $u \in M$ is in the solid closure of 0 in M. Then this remains true when we take images in $S \otimes_R M$, where $S = R/pR$. Since S is regular of characteristic p and $S \otimes_R M = M$, we have that u is in $0^{\bigstar}{}_M$ over S, which is 0, and so $u = 0$ as required.

(b) By part (a), it will suffice to prove that $N = \bigcap_p (N + pM)$ as p runs through the positive prime integers such that R/pR is regular, since for every such p, $N + pM$ is solidly closed in M. Since R is regular, $\mathbb{Q} \otimes_{\mathbb{Z}} R$ is smooth over $\mathbb{Q}$, and so for some $b \in \mathbb{Z} - \{0\}$, R_b is smooth over $\mathbb{Z}_b$. It follows that for all positive prime integers p not dividing b that R/pR is smooth over $\mathbb{Z}/p\mathbb{Z}$, and, hence, regular. Thus, the set of p such that R/pR is regular is infinite. The result now follows from (13.2), given immediately below, applied to M/N. $\square$

(13.2) Theorem. *Let M be a finitely generated module over R, where R is a finitely generated $\mathbb{Z}$-algebra, and suppose that M is torsion-free as a $\mathbb{Z}$-module. Then $\bigcap_{p \in P} pM = 0$ for any infinite set P of prime integers.*

Proof. By the lemma of generic freeness (see [HR1], Lemma (8.1), p. 146), we may choose a nonzero integer $a \in \mathbb{Z}$ such that M_a is free over $\mathbb{Z}_a$. Since M is torsion-free over $\mathbb{Z}$, $M \subseteq M_a$. Let Q be the set of primes in P not dividing a: it is still infinite. We have that $\bigcap_{p \in P} pM \subseteq \bigcap_{p \in Q} pM_a$, which is zero simply because M_a is Z_a-free and $\bigcap_{p \in Q} pZ_a = 0$. $\square$

(13.3) Corollary. *Let R be a regular domain finitely generated over $\mathbb{Z}$, with $\mathbb{Z} \subseteq R$, and suppose that S is a solid R-algebra. Then $\mathbb{Q} \otimes_{\mathbb{Z}} R \to \mathbb{Q} \otimes_{\mathbb{Z}} S$ is pure.*

Proof. Suppose that $T \subseteq U$ where T is a normal Noetherian domain and U is an extension ring. We claim that $T \hookrightarrow U$ is pure if and only if every ideal of T is contracted from U (of course, "only if" is trivial). To see this, note that the issue is local on the maximal ideals m of T, and so it suffices that, for every such m, T_m be pure in U_m. Since T_m is normal, it is approximately Gorenstein by the results of [Ho4], and it follows from the results of that paper that T_m is pure in U_m if every ideal $\mathfrak{B}$ of T_m primary to mT_m is contracted from U_m, and $\mathfrak{B}$ will be the expansion of its contraction $\mathfrak{A}$ to T, which is primary to m. But if $t/f \in \mathfrak{B}U_m = \mathfrak{A}U_m$ with $t \in T - \mathfrak{A}$ and $f \in R - m$, then gt is in the contraction of $\mathfrak{A}U$ to T for some $g \in T - m$, and so if $\mathfrak{A}$ is contracted from U it follows that $gt \in \mathfrak{A}$. Since $\mathfrak{A}$ is m-primary, it follows that $t \in \mathfrak{A}$, a contradiction. This establishes the claim.

We now apply this with $T = \mathbb{Q} \otimes_{\mathbb{Z}} R$ and $U = \mathbb{Q} \otimes_{\mathbb{Z}} S$.

Thus, it suffices to show that every ideal of $\mathbb{Q} \otimes_{\mathbb{Z}} R$ is contracted from $\mathbb{Q} \otimes_{\mathbb{Z}} S$. Such an ideal J will have the form $I(\mathbb{Q} \otimes_{\mathbb{Z}} R)$, where I is its contraction to R. Since R/I embeds into $\mathbb{Q} \otimes_{\mathbb{Z}} (R/I)$, it follows that R/I is torsion-free over $\mathbb{Z}$. Thus, by (13.1b), I is solidly closed. Since $R \to S$ is solid, it is injective; moreover, since I is solidly closed, I is contracted from S. But if $u \in J(\mathbb{Q} \otimes_{\mathbb{Z}} S) \cap (\mathbb{Q} \otimes_{\mathbb{Z}} R)$ then a multiple of u by some nonzero integer h will be an element of R, and, multiplying further by a nonzero integer k, we have that $khu \in IS \cap R = I$. Since kh is a unit of $\mathbb{Q} \otimes_{\mathbb{Z}} R$, we have that $u \in J$. $\square$

(13.4) *Remark.* In the case where S is module-finite over R this follows rather trivially from a trace argument, but it does not appear that a trace argument can be used to recover the general case of (13.3).

(13.5) Proposition. *Let R be any Noetherian ring that does not contain $\mathbb{Q}$, and let $X_1, \ldots, X_n$ be formal indeterminates over R. Let $\mathcal{M}_1, \ldots, \mathcal{M}_h$, and $\mathcal{M}$ be monomials in* *the X_i in $S = R[[X_1, \dots, X_n]]$, and let $I = (\mathcal{M}_1, \dots, \mathcal{M}_h)S$. Then $\mathcal{M} \in I^\bigstar$ in S if and only if $\mathcal{M} \in I$.*

Proof. Since $\mathbb{Q}$ is not contained in R, for some positive prime integer p, p has no inverse in R. Then pR is contained in a maximal ideal m of R, where $K = R/m$ has characteristic p. Given a counterexample, it remains a counterexample when we pass to $K[[X_1, \dots, X_n]]$, a contradiction, since this is a regular ring of characteristic p. □

(13.6) *Example.* In $\mathbb{Z}[X_1, X_2, X_3]$ we have that (X_1^3, X_2^3, X_3^3) is solidly closed, by (13.1b). If V is a discrete valuation ring of mixed characteristic p, then in $V[[X_1, X_2, X_3]$ we do not know whether (X_1^3, X_2^3, X_3^3) is solidly closed, but we know at least that $X_1^2X_2^2X_3^2$ is not in the solid closure, by (13.5). Both these examples show that solid closure does not commute with localization, since $X_1^2X_2^2X_3^2$ is in the solid closure after we localize at $\mathbb{Z} - \{0\}$ (respectively, $V - \{0\}$). Cf. (7.24).

(13.7) *Remarks.* Let (V, pV) be a discrete valuation ring of mixed characteristic p such that p generates the maximal ideal and let $R = V[[X_1, \dots, X_n]]$. Note that an ideal generated by monomials in p, $X_1, \dots, X_n$ is solidly closed provided that p is in the ideal (by (13.1a)), and that we know something about the behavior of the solid closure of an ideal generated by monomials in the X's only (i.e., with no positive powers of p occurring) from (13.5). However, we do not know whether, when $n = 3$, there is a positive integer h such that $p^hX_1^2X_2^2X_3^2$ is in $(X_1^3, X_2^3, X_3^3)^\bigstar$.

Some weak results about the solid closures of monomial ideals can be obtained by mapping to rings of lower dimension. To illustrate the technique, we prove:

(13.8) Theorem. *Let x, y, z be elements generating an ideal of height three in a Noetherian ring R. Assume either*

(i) *that $2 \cdot 1_R$ is not in any height three minimal prime of $(x, y, z)R$ (this is automatic if $2 \cdot 1_R$ is invertible in R) or*
(ii) *that R_{red} has characteristic 2.*

Then $xyz \notin (x^2, y^2, z^2)^\bigstar$.

Proof. If we have a counterexample we can localize at a suitable minimal prime of (x, y, z) (not containing $2 \cdot 1_R$ in case (i)) complete, and kill a minimal prime. By the persistence of solid closure, we still have a counterexample. Thus, there is no loss of generality in assuming that x, y, z is part of a system of parameters in a complete local domain R, and that either (i) $2 = 2 \cdot 1_R$ is a unit or (ii) R has characteristic 2. In the second case the result is immediate from Theorem (7.15a) of [HH4] and the fact that solid closure agrees with tight closure in this case, and we henceforth assume that we are in case (i), where 2 is a unit of R. Now map R to

$$S = R[[U, V]]/\big(x - (U^2 + V^2), y - (U^2 - V^2), z - UV\big).$$

Since R has dimension 3, $R[[U, V]]$ has dimension 5, and it is clear that U, V, x, y, z is a system of parameters, whence

$$U, V, x - (U^2 + V^2), y - (U^2 - V^2), z - UV$$

is a system of parameters for $R[[U, V]]$. It follows that the images u, v of U, V in S are a system of parameters for the two-dimensional complete local ring S. In S, x, y, z map to $u^2 + v^2$, $u^2 - v^2$, and uv. Thus, it will suffice to show that if S is a complete local ring of dimension two in which 2 is a unit and u, v is a system of parameters, then with $x = u^2 + v^2$, $y = u^2 - v^2$, $z = uv$ we have that $xyz \notin (x^2, y^2, z^2)^\star$ in S. As before, we may kill a suitable minimal prime. Thus, we may assume that S is a two-dimensional complete local domain. But then if $xyz \in (x^2, y^2, z^2)^\star$ we can find a big Cohen-Macaulay algebra T over S such that u, v is a regular sequence on T and $xyz \in (x^2, y^2, z^2)T$, by (12.5). This says that $(u^4 - v^4)uv \in (u^4 + v^4, u^2v^2)T$. Moreover, we may complete T with respect to the $(u, v)T$-adic topology (equivalently, with respect to the mT-adic topology, where m is the maximal ideal of S, since each of $(u, v)S$ and m has a power contained in the other). It follows (cf. Corollary (12.3)) that every system of parameters for S is a regular sequence on T. Now, since uv, $u^4 + v^4$ is also a system of parameters for S it is a regular sequence on T, and so

$$u^4 - v^4 \in \big(u^4 + v^4, (uv)^2\big) :_T uv = (u^4 + v^4, uv)$$

which yields

$$u^4 - v^4 = a(u^4 + v^4) + buv$$

or

$$(a - 1)u^4 + buv + (a + 1)v^4 = 0.$$

It follows that

$$(a - 1, b, a + 1) = c(v, -u^3, 0) + d(0, -v^3, u),$$

i.e., $a = 1 + cv$, $b = -cu^3 - dv^3$, $a = -1 + du$. Subtracting the third equation from the first shows that $2 \in (u, v)T$ is not a unit of T, and, hence, not a unit of R, a contradiction. $\square$

(13.9) *Remark.* It is hopeless to try to show, *by mapping to lower dimensional regular rings*, that if f, g, h are a system of parameters in a local ring of mixed characteristic and of dimension 3 then $f^2g^2h^2 \notin (f^3, g^3, h^3)^\star$. The key point is that, by the Briançon-Skoda theorem, one has that $f^2g^2h^2 \in (f^3, g^3, h^3)$ for any three elements f, g, h of a regular ring of dimension at most two. A detailed explanation is given in (6.8) of [Ho8].

14. Questions

In this concluding section we discuss a number of questions about the behavior of solid closure which, so far as the author knows, are open.

(14.1) Question. *Is every ideal of a regular local ring of mixed characteristic solidly closed?*

Roberts' calculation, discussed in (7.23) and (7.24), is discouraging, but the question seems to remain open in mixed characteristic. An affirmative answer would show that a regular local ring of mixed characteristic is pure in any solid algebra over it, and, in particular, is a direct summand of every module-finite extension algebra.

It would be interesting to know, given an ideal I generated by monomials in a regular system of parameters for a regular local ring R, precisely when another such monomial is in the solid closure of I. We understand the situation in characteristic p, where every ideal is solidly closed in a regular ring, and Roberts' calculation shows that the answer may be complicated in the equal characteristic case. Mixed characteristic remains mysterious. But we may ask:

(14.2) Question. *If it is not true that every ideal of a mixed characteristic regular local ring is solidly closed, can one find a variant closure operation for which this is true and that is sufficiently well-behaved to yield a proof of the direct summand conjecture?*

Suppose we denote this enigmatic closure for an ideal $I \subseteq R$ by $I^{\diamondsuit}$. If we know that when S is a module-finite extension domain of a complete local domain R then $IS \cap R \subseteq I^{\diamondsuit}$ and that $I^{\diamondsuit} = I$ if the ring R is regular, then the operation is "sufficiently well-behaved."

(14.3) Question. *If a complete local domain (R, m) has residual characteristic p, is it true that an R-algebra S is solid if and only if S can be mapped to a balanced big Cohen-Macaulay algebra T for R?*

This is true in dimension two without restriction on the characteristic, by Theorem (12.5), and false in equal characteristic zero in dimension greater than or equal to three, by Examples (10.7a,b). The author feels that it is much more likely to be true in characteristic p than in mixed characteristic.

The following is a closely related question:

(14.4) Question. *Let (R, m) be a complete local domain of residual characteristic p and let S be a solid R-algebra. Is every algebra modification of S with respect to a system of parameters for R again solid over R?*

See the first paragraph of the proof of Theorem (11.1) for a discussion of algebra modifications. Briefly, if $\mathbf{x} = x_1, \dots, x_n$ is a system of parameters for a complete local domain R, and S is an R-algebra, by an *algebra modification* (*of type* k, where $0 \le k < n$) with respect to $\mathbf{x}$ of S over R we mean an S-algebra T of the form

$$R[z_1, \dots, z_k]/(s_{k+1} - \sum_{j=1}^{k} x_j z_j),$$

where $s_{k+1} \in S$ is such that there exist $s_1, \dots, s_k \in S$ satisfying $x_{k+1}s_{k+1} = \sum_{j=1}^{k} x_j s_j$ in S.

In fact, (14.3) and (14.4) can easily be shown to be equivalent. Assume (14.3). Then S can be mapped to a balanced big Cohen-Macaulay algebra T for R, and it is easy to see that any algebra modification S' of S can also be mapped to T. Since T is solid, it follows that S' is solid. On the other hand, assume that (14.4) holds. Then every finite sequence of algebra modifications of S (the systems of parameters and types may vary) yields a solid R-algebra T, and for such an algebra we have that m is contracted from T, i.e., $1 \notin mT$. But then, as in §3 of [HH12], we may construct a balanced big Cohen-Macaulay algebra for R to which S maps as a direct limit of algebras obtained by finite sequences of algebra modifications of S.

We note that (14.3) reduces to the case where S is finitely generated over R. (It suffices to check the "only if" part. But if S cannot be mapped to a balanced big Cohen-Macaulay algebra for R then it has a finite sequence of algebra modifications, $S = T_0 \to \cdots \to T_h$ such that $1 \in mT_h$. It is easy to see that some finitely generated subalgebra S_0 of S will also have such a sequence of modifications, and S_0 is also solid over R if S is.)

Any finitely generated solid algebra over R maps onto a minimal solid algebra (cf. (6.1)), also finitely generated over R. If we view R as a finite module over a complete regular local ring A, the minimal solid algebras over R are also minimal solid algebras over A (cf. (6.2e)). We are therefore led to the following question:

(14.5) Question. *What are the minimal solid algebras over a complete regular local domain, especially in characteristic p and in mixed characteristic?*

Of course, the question is answered in dimension one by Theorem (6.4).

Finally, we ask:

(14.6) Question. *Is the Briançon-Skoda theorem for "solid closure" valid in arbitrary Noetherian rings? That is, given an ideal I generated by n elements in an arbitrary Noetherian ring of mixed characteristic p, is $\overline{I^n} \subseteq I^\bigstar$? (One may also ask whether various versions of the Briançon-Skoda theorem hold, e.g., is $\overline{I^{n+k}} \subseteq (I^{k+1})^\bigstar$ for every nonnegative integer k.)*

To prove this for a given ring R, it suffices to prove it for the complete local domains of R. The result is known in equal characteristic, by Corollary (11.5). Thus, one may reduce to the case of a complete local domain of mixed characteristic.

The above questions give just a small taste of what remains to be done in the study of solid algebras and solid closure.

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