INTERSECTION THEOREMS

Speaker: Mel Hochster
May 18, 2006

This is an edited version of the slides from a talk at the conference
entitled Homological Conjectures in honor of the sixtieth birthday
of Paul C. Roberts, held at Snowbird, Utah, May 18–19, 2006.

(PRINCIPAL IDEAL) THEOREM: Let R be a Noetherian ring and P a minimal prime of a principal ideal. Then the height of P is at most one.

(KRULL HEIGHT) THEOREM: Let R be a Noetherian ring and P a minimal prime of an ideal with n generators. Then the height of P is at most n.

Geometrically, every component of the intersection of a hypersurface X with Y irreducible has codimension at most one in Y.

Likewise, the intersection of two linear spaces X, Y of dimensions h and k in an n-dimensional vector space, if non-empty, is at least $h + k - n$.

Rephrased, the codimension of $X \cap Y$ in Y is at most the codimension of X in the ambient space.

This is another instance where intersecting does not decrease codimension.

Linearity is not the key point. In $\mathbb{C}^n$, if X and Y are varieties of dimensions h and k, every component of $X \cap Y$ has dimension at least $h + k - n$.

Proof: $X \cap Y \cong (X \times Y) \cap \Delta_n$, and Δ_n, the diagonal in $\mathbb{C}^n \times \mathbb{C}^n$, is defined by n equations. Apply the Krull height theorem.

Again, this can be rephrased to say that the codimension of $X \cap Y$ in Y is at most the codimension of X in the ambient space.

This is a more general instance where intersecting does not decrease codimension.

Define the superheight of the ideal I of the Noetherian ring R as $\sup \{\text{height}(IS) : IS \neq S\}$, as S runs through all Noetherian R-algebras with $IS \neq S$. One can take S to be a complete local domain here, and $\text{Rad}(IS) = m_S$.

(PRINCIPAL IDEAL) THEOREM: Superheight$(x\mathbb{Z}[x]) \leq 1$.

(KRULL HEIGHT) THEOREM : Superheight$((x_1, \ldots, x_n)\mathbb{Z}[x_1, \ldots, x_n]) \leq n$.

More generally (as we'll see):

(PESKINE-SZPIRO INTERSECTION) THEOREM: If R is local, M, $N \neq 0$ are finitely generated, and $\ell(M \otimes_R N) < \infty$, then $\dim N \leq \text{pd}_R M$.

This reduces at once to the case where $N = R/I$ is cyclic.

In an alternate form, it says that if $M \neq 0$ has finite projective dimension n with $I =\mathrm{Ann}_R M$, then superheight$(I) \leq n$.

That is, height$(IS) \leq \mathrm{pd}_R M$ when $IS \neq S$.

One may assume $R \to S$ complete local, IS primary to m_S, and $R \to S$ can be assumed onto after replacing R by a faithfully flat extension, so that $S = N = R/I$.

The principal ideal (resp., Krull height) theorem is the case $M = \mathbb{Z}[x]/(x)$ (resp., $\mathbb{Z}[\underline{x}]/(\underline{x})$ where $\underline{x} = x_1, \ldots, x_n$).

Peskine and Szpiro proved their intersection theorem in positive char. and in many cases in equal char. 0, by reduction to char. p. They also proved that it implies M. Auslander's zerodivisor conjecture and an affirmative answer to Bass's question. But all these remained open in mixed char. until the work of Paul Roberts.

Serre proved that in any regular ring, the superheight of a prime ideal P

is the same as the height of P. (Technically, what Serre did is the case where $R \to S$ is onto, but the general case follows.)

This can be viewed as an enormous generalization of the Krull height theorem.

The proof involves a notion of reduction to the diagonal after localization and completion that can be made to work even in mixed char. The most general case requires Serre's theory of intersection multiplicities and the help of some spectral sequences.

Note that codimension can increase. Let K be a field. In the ring $K[x, y, u, v]/(xv - yu)$, which has dimension 3, the ideal (x, y) has height one. When one kills (u, v), the height becomes 2. This is an isolated singularity.

Let X be a $2 \times n$ matrix of variables over K, and let $R = K[X]/I_2(X)$, where $K[X]$ is the polynomial ring in the entries of X and $I_2(X)$ is generated by 2×2 minors of X. $\dim R = n + 1$, and the ideal P generated by the entries of the 1st row is a height 1 prime. Kill the entries of the 2nd row: $\text{height}(P)$ increases to n !

Let $\underline{x} = x_1, \ldots, x_n$ and $\underline{y} = y_1, \ldots, y_n$ be indeterminates over $\mathbf{Z}$ and let $f_t = x_1^t x_2^t \cdots x_n^t - y_1 x_1^{t+1} - \cdots - y_n x_n^{t+1}$. Form the hypersurface $R = \mathbf{Z}[\underline{x}, \underline{y}]/(f_t)$. In this ring, the ideal $P = (\underline{x})R$ has height $n - 1$. The superheight is clearly at most n. Whether the superheight is $n - 1$ or n is a long standing open question!

The conjecture that the superheight is $n - 1$ is equivalent to the Direct Summand Conjecture, that regular Noetherian rings of dimension n are direct summands, over themselves, of their module-finite extension rings. This is known in equal characteristic, if $n \leq 3$, and in certain cases where $t \ll n$. We'll return to this problem later.

Independently, Peskine-Szpiro and Paul Roberts proved a strengthening of the intersection theorem (originally in char. p, but this implies the equal char. 0 case by very general methods) called the new intersection theorem. It readily implies the original Peskine-Szpiro intersection theorem.

(New Intersection)Theorem: Let R be a local ring and suppose that $0 \to G_n \to \cdots \to G_1 \to G_0 \to 0$ is a complex of finitely generated free R-modules all of whose homology modules have finite length, with $H_0(G_\bullet) \neq 0$. Then $\dim R \leq n$.

The intersection theorem follows:
let $F_\bullet$ be a minimal free resolution of M over R and apply the new intersection theorem over $\overline{R} = R/I$, where $I = \mathrm{Ann}_R N$, to $G_\bullet = \overline{R} \otimes_R F_\bullet$.

Paul Roberts also gave a proof of the new intersection theorem in equal characteristic 0 using the Grauert-Riemenschneider vanishing theorem. This is of great interest, since other proofs of similar results use reduction to char. $p > 0$.

Later, Paul Roberts settled the remaining case of the new intersection theorem, in mixed characteristic. This settled several other local homological conjectures. Our main focus for a while will be to give a partial description of this proof.

Ideas from intersection theory as developed in the paper of Baum, Fulton, and MacPherson and presented in Fulton's book *Intersection Theory* are needed. For an algebraic version see *Multiplicities and Chern Classes in local algebra* by Paul Roberts.

In particular, we need some knowledge of Chow groups and localized Chern characters. In the sequel, given local rings are assumed to be homomorphic images of a regular local ring.

The Chow groups $\mathcal{A}_i(R) = \mathcal{A}_i(X)$, where $X = \mathrm{Spec}(R)$, are obtained as follows.

Take the vector space $Z_i(R) = Z_i(X)$ over $\mathbb{Q}$ whose basis consists of closed subvarieties (these are reduced and irreducible) of dimension i in $\mathrm{Spec}(R)$ (corresponding to primes P such that the $\dim(R/P) = i$), and for each Q such that $\dim(R/Q) = i+1$, kill the span of the divisors of nonzero elements $f \in R/Q$.

Here, the divisor of $f \in D - \{0\}$ is a linear combination of basis elements corresponding to minimal primes P of f (so that D/P has dimension i), and the coefficient corresponding to P is the order of f in D_P, which is the length of the local ring D_P/fD_P.

If $\dim(R) = d$, we write $\mathcal{A}_*(X)$ for $\bigoplus_{i=0}^d \mathcal{A}_i(X)$.

We ambiguously write $[R/P]$ for the image of the class of $V(P)$ in any $\mathcal{A}_*(Z)$, where Z is closed and contains $V(P)$.

Let $F_\bullet$ be a bounded complex of finitely generated free R-modules. Let Z denote the *support* of $F_\bullet$, the reduced closed subscheme of X whose points correspond to all primes P such that the localization of $F_\bullet$ at P is not exact. That is, Z is the support of the total homology module $\bigoplus_i H_i(F_\bullet)$.

The *localized Chern character* $\mathrm{ch}(F_\bullet)$ of $F_\bullet$ is a sum of operators $\mathrm{ch}(F_\bullet) = \bigoplus_{i=0}^{d} \mathrm{ch}_i(F_\bullet)$, where $\mathrm{ch}_i(F_\bullet)$ gives a map of graded abelian groups of degree $-i$ from $\mathcal{A}_*(X) \to \mathcal{A}_*(Z)$. Thus, for all n, $\mathrm{ch}_i(F_\bullet) : \mathcal{A}_n(X) \to \mathcal{A}_{n-i}(Z)$.

If $Z \subseteq Z'$, a closed set, we have a map $\mathcal{A}_*(Z) \to \mathcal{A}_*(Z')$. Thus, we can have $\mathrm{ch}(F_\bullet)$ take values in $\mathcal{A}_*(Z')$. Moreover, if $F_\bullet$ is a complex of free modules on X, $\mathrm{ch}(F_\bullet)$ acts on $\mathcal{A}_*(Y)$ for every closed subscheme Y of X, for we may restrict the complex $F_\bullet$ to get a bounded free complex on Y.

The operators $\mathrm{ch}_i(F_\bullet)$ are extraordinarily well-behaved. They are compatible with proper push-forward and with intersection with Cartier divisors (e.g., with killing a nonzerodivisor in the ring).

Moreover, for bounded free complexes $F_\bullet, G_\bullet, H_\bullet$ with supports V, W, and Z, respectively:

(i) One has additivity: if $0 \to F_\bullet \to G_\bullet \to H_\bullet \to 0$ is exact then $\mathrm{ch}(G_\bullet) = \mathrm{ch}(F_\bullet) + \mathrm{ch}(H_\bullet)$ in $\mathcal{A}_*(V \cup Z)$.

(ii) One has a multiplicative property: $\mathrm{ch}(F_\bullet \otimes_A G_\bullet) = \mathrm{ch}(F_\bullet)\mathrm{ch}(G_\bullet)$ in $\mathcal{A}_*(V \cap W)$.

(iii) One has commutativity: for all i and j, $\mathrm{ch}_i(F_\bullet)\mathrm{ch}_j(G_\bullet) = \mathrm{ch}_j(G_\bullet)\mathrm{ch}_i(F_\bullet)$ in $\mathcal{A}_*(V \cap W)$.

(iv) There are Todd classes. A fin. gen. R-module M over has a Todd class $\tau(M)$ in $\mathcal{A}_*\big(\mathrm{Supp}(M)\big)$.

Let S be a regular local ring that maps onto R and let $F_\bullet$ be a finite free resolution of M over S. Then $\tau(M)$ is the result of applying $\mathrm{ch}(F_\bullet)$ to $[S]$ in $\mathcal{A}_*\big(\mathrm{Supp}(M)\big)$.

(v) One has a local Riemann-Roch formula. Let x be the closed point of X. Then $\mathcal{A}_*(x) = \mathcal{A}_0(x) = \mathbb{Q}x$, which we identify with $\mathbb{Q}$, mapping $x \mapsto 1$. If $F_\bullet$ has homology of finite length, then for every finitely generated R-module M, $\chi(F_\bullet \otimes_R M) = \mathrm{ch}(F_*)\big(\tau(M)\big)$.

The component $\tau_k(M)$ in degree $k = \dim(M)$ is a linear combination of classes $[R/P]$ where P is minimal in $\mathrm{Supp}(M)$ with $\dim(R/P) = k$. For such a P, the coefficient is the length of M_P over R_P.

In the case of (v) where $M = R$, we have $\chi(F_\bullet) = \mathrm{ch}(F_*)\big(\tau(M)\big) = \sum_i \mathrm{ch}_i(F_\bullet)\big(\tau_i(R)\big)$.

Let $d = \dim(R)$. It is crucial that the term $\mathrm{ch}_d(F_\bullet)\big(\tau_d(R)\big)$ exhibits behavior that is, in some ways, better than the behavior of the Euler characteristic.

It is straightforward to reduce to the case where R is a complete local domain of mixed char. $p > 0$ with perfect residue class field K.

Once we apply $\underline{\quad} \otimes_R /pR$, we are in char. p. Therefore, we may assume the length of the complex is exactly $d - 1$.

Let $\overline{G}_\bullet = G_\bullet \otimes_R (R/pR)$.

We have a commutative diagram:

$$
\begin{array}{ccc}
\mathcal{A}_d\big(\mathrm{Spec}\,(R)\big) & \xrightarrow{\ \mathrm{ch}_{d-1}(G_\bullet)\ } & \mathcal{A}_1(x) \\
\downarrow & & \downarrow \\
\mathcal{A}_{d-1}\big(\mathrm{Spec}\,(R/pR)\big) & \xrightarrow{\ \mathrm{ch}_{d-1}(\overline{G}_\bullet)\ } & \mathcal{A}_0(x)
\end{array}
$$

$\mathcal{A}_1(x) = 0$ implies that $\mathrm{ch}_{d-1}(\overline{G}_\bullet)([R/pR]) = 0$ in $\mathcal{A}_0(x) \cong \mathbb{Q}$.

If $G_\bullet$ is such a complex over a local ring A of prime char. $p > 0$, we let

$$
\chi_\infty(\overline{G}) = \lim_{e \to \infty} \frac{\chi\big(F^e(\overline{G}_\bullet)\big)}{p^{ne}}.
$$

(One can show this exists.) Here, F^e is the eth iteration of the Frobenius functor.

We can now complete the proof of the new intersection theorem by establishing two facts. First:

THEOREM: With A and $\overline{G}_\bullet$ as above, $\mathrm{ch}_n(\overline{G}_\bullet)([A]) = \chi_\infty(\overline{G}_\bullet)$.

This is proved with the help of the local Riemann-Roch theorem.

Second:

THEOREM: With A and $\overline{G}_\bullet$ as above, $\chi_\infty(\overline{G}_\bullet) > 0$.

Apply these results to the set-up we obtained earlier with $A = R/pR$. We have a contradiction, which proves the new intersection theorem!

The second theorem is purely a result in commutative algebra, but is still highly non-trivial.

Proof of the first theorem: The action of Frobenius in degree i is multiplication by p^{ei} — we assume this key fact here. Using this, one obtains: $\chi\big(F^e(\overline{G}_\bullet)\big) = \sum_{i=0}^{n} p^{ie}\mathrm{ch}_i(\overline{G}_\bullet)\big(\tau_i([A])\big)$. We divide both sides by p^{ne} and take the limit as $e \to \infty$.

All the terms in the sum corresponding to values of $i < n$ approach 0. This shows that $\chi_\infty(\overline{G}_\bullet)$ exists, and is equal to $\mathrm{ch}_h(\overline{G}_\bullet)\big(\tau_n([A])\big)$. Since $\tau_n([A]) = [A]$, the theorem is established.

The second theorem is proved by studying the spectral sequences of a certain double complex constructed from a dualizing complex. The spectral sequence argument enables one to reduce the argument to establishing the following:

Lemma: Let $(A,\, m,\, K)$ be a local ring of prime characteristic $p > 0$. Let $\overline{G}_\bullet$ be a finite free complex of finitely generated free modules whose homology has finite length and such that $H_0(\overline{G}_\bullet) \neq 0$.

(a) If M is a finitely generated R-module of dimension k, then there is a positive real constant C such that for all e,
$$\ell\Big(H_i\big(\mathrm{Hom}_A\big(F^e(\overline{G}_\bullet),\, M\big)\big)\Big) \leq Cp^{ke}.$$

(b) The quantity
$$\frac{\ell\Big(H_0\big(F^e(\overline{G}_\bullet)\big)\Big)}{p^{en}}$$
is bounded away from 0 as $e \to \infty$.

Part (b) is not difficult, but the part (a) is still a great deal of work. Similar results were proved by S. P. Dutta and G. Seibert.

Time limitations have forced us to omit a great deal, but we have given some indication of the main ingredients of the argument.

E. G. Evans and P. Griffith used a stronger form of the new intersection thm. to prove that a k th modules of syzygies of a module over a regular local ring, if not free, has rank $\geq k$. This is called the syzygy theorem.

In their "improved" new intersection thm., the hypothesis on $H_0(G_\bullet)$ is weakened: it need only have one minimal generator that is killed by a power of m.

Work of S. P. Dutta and myself implies, in fact, that the following three statements are equivalent:

(1) The direct summand conjecture.

(2) The canonical element conjecture (if (R, m, K) is local of dimension d, then the map $\mathrm{Ext}_R^d(K, \mathrm{syz}^d K) \to H_m^d(\mathrm{syz}^d K)$ is not 0.)

(3) The improved new intersection conjecture.

Thus, all three statements would follow from a "simple" result about codimension when intersecting with a fixed variety in a hypersurface! We also see that the syzygy theorem is an intersection theorem.

Briefly, what's new?

R. Heitmann has proved that the direct summand conjecture is true in dimension 3. For a domain R let R^+ denote the integral closure of R in an algebraic closure of its fraction field. He shows that if R is a three-dimensional complete local domain of mixed characteristic p, then $p^{1/N}$ kills $H_m^2(R^+)$ for arbitrarily large N.

Heitmann's result can be used to show the existence of big Cohen-Macaulay algebras in mixed char. in dimension 3.

Tight closure theory has given new insights and new theorems in this area. Let (R, m) be excellent local and equidimensional, for simplicity, and let $0 \to G_n \to \cdots \to G_0 \to 0$ be a complex which is a phantom resolution: the ranks behave properly, and the i th ideal of minors has height at least i for all i.

In characteristic p, this implies that the cycles are in the tight closure of the boundaries in all G_i for $i > 0$: this condition is called phantom acyclicity.

The following result is in *Phantom Homology* by Craig Huneke and myself.

(PHANTOM INTERSECTION) THEOREM Let $u \in H^0(G_\bullet) = M$ be killed by a power of m. Then either u is in the tight closure of 0 in M or $\dim(R) \le n$.

Since minimal generators are never in the tight closure of 0, this phantom intersection theorem strengthens the improved new intersection theorem.

The proof is easy if one has a big Cohen-Macaulay algebra B over R: just tensor with it. The complex becomes acyclic, and if it is too short, $B \otimes_R M$ cannot have depth 0, so that $1 \otimes u$ must be 0. In characteristic p, for example, R^+ is a big Cohen-Macaulay algebra.

The existence of big Cohen-Macaulay algebras in mixed characteristic remains unknown. Their existence is, literally, an intersection theorem: it is equivalent to bounding the superheight at $n-1$ of certain n generator ideals in specific finitely generated $\mathbb{Z}$-algebras.

One last open question. Let R be local and M, N fin. gen. modules such that $M \otimes_R N$ has finite length. If $\mathrm{pd}(M)$ is finite, is $\dim(M) + \dim(N) \leq \dim(R)$? This was asked by Peskine and Szpiro more than 35 years ago, and is still an open question even if R is a Cohen-Macaulay domain.