

TIGHT CLOSURE IN EQUAL CHARACTERISTIC ZERO

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There has been a change of numbering from versions of this manuscript dated prior to August 20, 1997. Section (1.5) has been added, and some of the material previously in Section (2.4) is now in Section (1.5). The previous (1.5) and (1.6) have become (1.6) and (1.7), respectively. Sections (3.6) and (3.7) of early versions have become Sections (4.1) and (4.2), respectively.

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PREFACE

The main objective of this monograph is to lay the foundations of tight closure theory for Noetherian rings containing a field of characteristic 0. However, we intend and hope that it will be useful in several other ways.

The first chapter contains, in effect, a survey of tight closure in characteristic p . Definitions and theorems are given in full, although for most proofs we simply give references. This chapter provides a quick introduction to characteristic p tight closure theory for newcomers, and may be a useful guide even for expert readers in finding material in the existing literature, which has become rather formidable. Moreover, it contains all of the equicharacteristic p material that is needed for the construction of the equicharacteristic 0 theory. For the most part, the equal characteristic zero theory only requires knowledge of the characteristic p theory for finitely generated algebras over a field and, in fact, the field can generally be taken to be finite. When no great cost is involved, we have frequently included results in greater generality, but we have not given the most general results known when too many technicalities would be involved. A few of the results in this section are new.

The equicharacteristic 0 theory rests heavily on the method of reduction to characteristic p . The results of the second chapter are aimed at supplying what is necessary in this direction: almost nothing here is new, but we had difficulty locating references for the facts that we needed in the right form and generality, and so we took this opportunity to give a self-contained treatment of what we required. It was our intention to make the method of reduction to characteristic p understandable and accessible to an audience with only a modest background in commutative algebra and algebraic geometry. While our

treatment was certainly influenced by the need to provide the tools that are required to develop tight closure theory, it was constantly in our minds that this chapter should enable many mathematicians to acquire the technique of reduction to characteristic p . Many of the results are concerned with comparing the behavior of the generic fiber of a map from an integral domain A with the behavior of “almost all” closed fibers, i.e., of comparing what happens when one tensors with the fraction field of A with what happens when one tensors with A/μ for μ a maximal ideal in a suitably small Zariski dense open subset of $\text{MaxSpec } A$. We have not made an effort to scour the literature to determine first sources for these results. Many can be extracted, in some form, from the massive Grothendieck-Diedonné treatise *Éléments de Géométrie Algébriques* — see for example [EGA1], [EGA2] in the Bibliography.

The third chapter develops the basic properties of tight closure in equal characteristic 0. We note here that there is a theory for Noetherian K -algebras for each fixed field K of characteristic 0. We do not know whether all these notions agree. The theory for $K = \mathbb{Q}$ gives the smallest tight closure, which we call *equational* tight closure. Another notion, *big equational* tight closure, is treated in the fourth chapter.

The fourth chapter contains a deeper exploration of properties of tight closure, including many of the most important and useful properties: that every ideal is tightly closed in a regular ring, that tight closure captures colons of parameter ideals (and so can be used to measure the failure of the Cohen-Macaulay property), that a ring in which every ideal (or even every parameter ideal) is tightly closed is Cohen-Macaulay (and normal), that direct summands of regular rings are Cohen-Macaulay (as a corollary of the first three characteristics of tight closure), that there is a tight closure version of the Briançon-Skoda theorem, and that there is a theory of “phantom homology” analogous to the characteristic p theory. The fourth chapter also contains a treatment of change of rings, and a brief treatment of rings with the property that every ideal is tightly closed (*weakly F -regular rings*) and those with the property that parameter ideals are tightly closed (*F -rational rings*) and their connection with rational singularities. There is also a likewise brief discussion of yet another notion of tight closure, *big equational* tight closure, as mentioned above. It gives

a larger tight closure than any of the other theories, but still has the crucial property that all ideals of regular rings are tightly closed. It was our original intention to give more detailed treatments of several of the topics in the fourth chapter, but we have decided that it is more important to make what is already written available at this time.

The Appendix contains a list of open questions. For the convenience of the reader we have provided a complete glossary of notation and terminology in (1.2.4). We have also included a very extensive bibliography in the hope that it will prove helpful to readers in dealing with the now voluminous literature on tight closure.

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CHAPTER 1.

PRELIMINARIES

In this chapter we first give an introduction that explains some of the motivations for studying tight closure, gives an overview of the entire monograph, and makes connections with the literature. The second section reviews many conventions concerning terminology and notation and also contains an alphabetical list showing where various terms and symbols are defined. The third section gives a summary of the main results of the entire monograph. The last three sections review the theory of tight closure and F -regularity in characteristic p .

(1.1) INTRODUCTION

In [HH4] (see also [HH1-3]) the authors introduced the notion of tight closure for Noetherian rings of characteristic p . The original motivation for studying tight closure in characteristic p is that it gives very simple proofs of a host of results that, before the development of this method, did not even seem related. These include:

- (1) A new proof that direct summands of regular rings are Cohen-Macaulay (hence, that rings of invariants of linearly reductive linear algebraic groups acting on regular rings are Cohen-Macaulay).
- (2) A new proof of the Briançon-Skoda theorem.
- (3) New proofs of various local homological conjectures.

Moreover, in every case, many new results are obtained, often very strong and quite unexpected generalizations of previous results. Already at the time that [HH4] was written the authors had worked out, in a preliminary form, a theory of tight closure in equal characteristic zero, provided that the base ring is a finitely generated algebra over a field. A definition for the tight closure of an ideal in this case is given in [HH4].

Our objective in this paper is to present, in a greatly improved form, the theory anticipated in [HH4]. One improvement over what is described in [HH4] is that this theory of tight closure is defined for *all* rings containing a field of characteristic zero. Moreover, it behaves sufficiently well to give counterparts for all of the results of types (1), (2), and (3) described above.

In the next section of this chapter we give some terminological and notational conventions, as well as an alphabetical index of terms and notations. In the following section we give a summary of some of the main results of the paper, and in the last three sections of this chapter we present a brief résumé of what is needed repeatedly throughout the paper from tight closure theory in characteristic p .

In Chapter 2 we develop a theory for affine algebras over a given field K of characteristic zero and then extend it in Chapter 3 to all Noetherian rings containing K . We indicate this kind of tight closure for a submodule $N \subseteq M$ by N^{*K} . If $K = \mathbb{Q}$, we also write $N^{*\text{eq}}$, and refer to this as the *equational* tight closure. Aspects of this theory are discussed further in Chapter 3 (where we shall also discuss the reason for the name: roughly speaking, when an element is in the equational tight closure it is there because a finite number of equations hold).

In general, $N^{*K} \subseteq N^{*L}$ whenever $K \subseteq L$, and so $N^{*\text{eq}} \subseteq N^{*K}$ for all fields K . We shall also introduce a variant notion of tight closure, $N^{*\text{EQ}}$, with essentially the same formal properties, such that $N^{*\text{eq}} \subseteq N^{*\text{EQ}}$ always, and whenever R happens to be a K -algebra, $N^{*\text{eq}} \subseteq N^{*K} \subseteq N^{*\text{EQ}}$. We call $^{*\text{EQ}}$ the *big* equational tight closure. We shall only discuss this notion briefly in this paper.

So far as we know, it is possible that $N^{*\text{eq}} = N^{*\text{EQ}}$ in all cases, which would simplify matters a good deal. For the moment, the philosophy is this:

(a) When we want to show that some sort of operation (whose definition does not depend on referring to tight closure) is contained in a tight closure (e.g., a colon ideal, a homology module, or an integral closure), then we want to show that it is contained in the $^{*\text{eq}}$ closure: since this is smallest, it gives the best result.

(b) When we prove an assertion such as, for example, that every submodule of ev-

ery finitely generated module over a certain ring is tightly closed, we want to prove this assertion, if we can, for ${}^{\ast}\text{EQ}$, since this is largest and gives the strongest result in this context.

(c) However, if we want to prove results of the sort that if a certain ideal or module is tightly closed then some ring or module has a certain kind of good behavior, then our preference is again to prove the result for ${}^{\ast}\text{eq}$, since this will give the strongest result.

(d) If the hypothesis and conclusion of a result both refer to tight closure, then the results for the different theories may well be incomparable: the ${}^{\ast}\text{EQ}$ theory version may have both a weaker hypothesis and a weaker conclusion than the ${}^{\ast}\text{eq}$ theory version (since $w \in W^{\ast}\text{EQ}$ is a potentially weaker statement than $w \in W^{\ast}\text{eq}$).

We hope that the situation will not prove unduly confusing. We feel that it is only a minor inconvenience to keep track of the parallel theories for ${}^{\ast}\text{eq}$, ${}^{\ast}K$, and ${}^{\ast}\text{EQ}$ until the situation is resolved. Moreover, we do get results like containments of colon ideals in tight closures and generalizations of the Briançon-Skoda theorem for ${}^{\ast}\text{eq}$, which is the most desirable situation according to (a). The ${}^{\ast}\text{eq}$ theory also suffices to prove that pure subrings of regular rings are Cohen-Macaulay in complete generality in equal characteristic: see Theorem (4.1.12).

It is worth noting that some of the results that can be obtained using tight closure theory can also be obtained using the weakly functorial existence of big Cohen-Macaulay algebras. The two subjects are intertwined in various ways. This is discussed in Chapter 8. We refer the reader to [HH7] (also see [HH5] and [Hu3]) and [HH11], as well as to [Ho8] and [Ho9] for more information.

For further information about tight closure theory in characteristic p , we refer the reader to [HH1-4, 6, 8-10], [Ho8, 9], [Hu1], [AHH], [FeW], [Ab1-4], [Gla1-3], [Sm1-9], [Hara1-5] [Si1-6], [Sw1, 2], [Vel], [W2], and [Wil].

(1.2) CONVENTIONS OF TERMINOLOGY AND NOTATION

In this section we first give several conventions about terminology and notation that are used throughout the paper. We also give an alphabetical list of notations and terms and

indicate where their definitions may be found.

(1.2.1) Basic terminology. We make the following conventions:

(a) Throughout this paper all rings are assumed to be commutative, associative, with identity, and modules are assumed to be unital. Ring homomorphisms are assumed to preserve the identity element. In a field or integral domain we assume $1 \neq 0$, and prime ideals are assumed to be proper, so that an ideal P of R is prime if and only if R/P is an integral domain. $\text{Spec } R$ is the topological space of prime ideals of R in the Zariski topology. $\text{Max Spec } R \subseteq \text{Spec } R$ is the subspace of all maximal ideals. Both $\text{Spec } R$ and $\text{Max Spec } R$ are empty precisely if $R = \{0\}$, and not otherwise. By a *local* ring (R, m, K) we mean always a Noetherian ring with a unique maximal ideal m and residue field $K = R/m$. A ring R with nilradical J is called *reduced* if $J = (0)$, and R_{red} denotes R/J .

The terms *finite type* and *finitely generated* for an R -algebra are synonymous. A localization of an R -algebra of finite type with respect to any multiplicative system is said to be *essentially of finite type over R* . An algebra that is finitely generated over a field K is referred to as an *affine algebra (over K)*.

(b) \mathbb{Q} denotes the field of rational numbers, $\mathbb{Z} \subseteq \mathbb{Q}$ the ring of integers, and $\mathbb{N} \subseteq \mathbb{Z}$ the set of nonnegative integers.

(c) If R is a ring we denote by R° the complement in R of the union of the minimal primes of R .

(d) If R is an integral domain we denote by R^+ the integral closure of R in an algebraic closure of its fraction field, which we refer to as an *absolute integral closure of R* . Cf. [Ar2], [HH7], [Sm1, 2], and [AH]. Note that R^+ is unique up to non-unique R -isomorphism.

(e) The *Krull dimension*, $\dim R$, of a ring R is the supremum of lengths n of chains $P_0 \subset \dots \subset P_n$ of distinct primes in R . The *height*, $\text{ht } P$, of a prime ideal P of R is $\dim R_P$, while the *height* of an ideal I is $+\infty$ if $I = R$ and $\inf \{\text{ht } P : P \text{ is prime and } P \supseteq I\}$ if I is proper.

Let R be Noetherian. We say that a proper ideal I is *unmixed* if all associated primes are minimal, i.e., if there are no embedded primes. We say that I has *pure height h* if all associated primes of I have height h . Let M be a finitely generated nonzero R -module.

Then $\dim M = \dim (R/\operatorname{Ann}_R M)$.

We say that a finitely generated R -module M has *pure dimension* d if, equivalently, for all $P \in \operatorname{Ass} M$ $\dim R/P = d$ or if every nonzero submodule of M has dimension d .

A *system of parameters* x_1, \dots, x_n for a local ring (R, m, K) is a sequence of $n = \dim R$ elements of m such that m is the radical of $(x_1, \dots, x_n)R$. (It is empty if $\dim R = 0$).

A local ring R is called *equidimensional* if for every minimal prime \mathfrak{p} of R , $\dim R/\mathfrak{p} = \dim R$. A Noetherian ring is called *locally equidimensional* if all of its local rings are equidimensional. (If R is catenary (see (1.2.1h), it suffices that this hold for local rings at maximal ideals.) A Noetherian ring R is called *biequidimensional* if for every maximal ideal m and minimal prime \mathfrak{p} contained in m , $\dim R_m/\mathfrak{p}R_m = \dim R$. If R is a finitely generated algebra over a field K , this holds if and only if for every minimal prime \mathfrak{p} of R , $\dim R/\mathfrak{p} = \dim R$.

(f) An A -algebra R is called *smooth* if it is finitely presented (which is equivalent to finitely generated if A is Noetherian) and formally smooth in the sense of, for example, [Iv] p. 33. (This is equivalent ([Iv], Proposition (3.3), p. 66) to the condition that R be A -flat and that the fibers be geometrically regular: cf. (2.3.1) and (2.3.3) for definitions.)

An A -algebra R is called *étale* if it is finitely presented and formally étale in the sense of, for example, [Iv] p. 33. See also (3.3.1).

(g) Let R be a ring and M an R -module. A *possibly improper regular sequence on M* (or *R -sequence on M* , or *M -sequence*) is a sequence of elements $x_1, \dots, x_n \in R$ such that x_{i+1} is not a zerodivisor on $M/(x_1, \dots, x_i)M$, $0 \leq i \leq n-1$. It is called a *regular sequence* (*R -sequence on M* , or *M -sequence*) if, moreover, $(x_1, \dots, x_n)M \neq M$. If $R \rightarrow S$ is a homomorphism of Noetherian rings, M is a finitely generated S -module, and I is an ideal of R , $\operatorname{depth}_I M$ the *depth* of M on I , is $+\infty$ if $IM = M$, and otherwise is the length of any maximal regular sequence on M in I (all of these will have the same length). We write $\operatorname{depth} I$ for $\operatorname{depth}_I R$.

A Noetherian ring R is *Cohen-Macaulay* if one (equivalently, all) of the following conditions hold:

- (1) For every maximal ideal m of R , $\operatorname{depth} mR_m = \dim R_m$.
- (2) For every prime ideal P of R , $\operatorname{depth} PR_P = \dim R_P$.

(3) For every ideal I of R , $\text{depth } I = \text{ht } I$.

(4) Every system of parameters in every local ring of R is a regular sequence.

The *type* of a Cohen-Macaulay local ring (R, m, K) is, equivalently, $\dim_K \text{Ann}_{R/I} m$, where I is the ideal generated by any system of parameters, or $\dim_K \text{Ext}_R^d(K, R)$, where $d = \dim R$.

A local ring R is called *Gorenstein* if, equivalently

(1) The injective dimension of R as an R -module is finite (in which case it is $\dim R$).

(2) R is Cohen-Macaulay of type 1.

A Noetherian ring R is called *Gorenstein* if all of its local rings (equivalently, all of its local rings at maximal ideals) are Gorenstein.

(h) A Noetherian ring R is called *catenary* if whenever $P \subseteq Q$ are prime ideals of the ring, all saturated chains of prime ideals joining P to Q have the same length. R is called *universally catenary* if every finitely generated R -algebra is catenary. A Noetherian ring R is called *excellent* if it is universally catenary, the set of regular primes $\{P \in \text{Spec } S : S_P \text{ is regular}\}$ is Zariski open in every finitely generated R -algebra S , and if for every local ring A of R the fibers of $A \rightarrow \hat{A}$ are geometrically regular (see (2.3.1) for the definition of geometrically regular, and (2.3.3) for the definition of fiber). Note that fields and, more generally, complete local rings are excellent, and that an algebra essentially of finite type over an excellent ring is excellent. The completion of a reduced (respectively, normal) excellent local ring is reduced (respectively, normal). Cf. [Mat], Chapter 13.

A Noetherian ring is called *locally excellent* if all of its local rings are excellent.

(1.2.2) Characteristic p conventions. (a) The letter “ p ” always denotes a positive prime integer (as usual, 1 is excluded). By a *ring of characteristic p* we always mean a ring of positive prime characteristic p .

In dealing with rings of positive prime characteristic we employ the following additional conventions:

(b) The letter “ e ” always denotes a nonnegative integer, and “ q ” always denotes p^e . Thus, “for some q ” means “for some $q = p^e$ with $e \in \mathbb{N}$ ” while “for all $q \gg 0$ ” means “for all q of the form p^e with e a sufficiently large element of \mathbb{N} .”

(c) If R is a ring of characteristic p , $F = F_R$ denotes the Frobenius endomorphism $F: R \rightarrow R$ defined by $F(r) = r^p$. $F^e = F_R^e$ denotes the e th iteration of F , so that $F(r) = r^q$. We usually omit the subscript R .

(d) $\mathbf{F} = \mathbf{F}_R$ denotes the *Frobenius* or *Peskine-Szpiro* functor from R -modules to R -modules. See (1.4.1) for more information. \mathbf{F}^e denotes its e th iteration. Also see (1.4.1) for a discussion of the notations $M^{[q]}$ and u^q when M is an R -module and u is an element of M .

(e) When R is reduced and of characteristic p , we write $R^{1/q}$ for the ring obtained by adjoining q th roots of all elements of R and R^∞ for $\cup_q R^{1/q}$. The maps of rings $R^q \subseteq R$, $R \subseteq R^{1/q}$ and $F^e: R \rightarrow R$ are isomorphic when R is reduced.

(1.2.3) Base change conventions. The following notational conventions will be very convenient:

When we have an A -algebra or A -module denoted either X or X_A and B is an A -algebra, we shall write X_B for $B \otimes_A X$ or $B \otimes_A X_A$. When we have a composite algebra homomorphism $A \rightarrow B \rightarrow C$ the isomorphism $C \otimes_B X_B \cong C \otimes_A X_A$ gives an essentially unambiguous meaning to X_C . If x or x_A is an element of X_A , we write x_B for $1 \otimes x$ or $1 \otimes x_A$ in X_B .

When $N \subseteq M$ or $N_A \subseteq M_A$ are modules we shall write $\langle N_B \rangle$ for the image of N_B in M_B . This notation is somewhat ambiguous, since it depends on the inclusion map $N \subseteq M$ or $N_A \subseteq M_A$ and not just on N_A . It will nonetheless be convenient, and will be clarified with appropriate comments as necessary. In fact, we occasionally write $\langle N \rangle$ for the image of the module N in some other module M when M and the map $N \rightarrow M$ are clear from the context.

(1.2.4) Index of definitions and notations. We include here an alphabetical list showing where the reader can find the definitions of various notations and terms. We have made a great effort to make this list complete. Locations are given by section number.

In the case of certain symbols the position in this list is somewhat arbitrary and there are some intentional redundancies: e.g., $+$ is treated as “plus” alphabetically while $*$ occurs twice, once as though it were “asterisk” and once as though it were “star” (with an

indication there to look for most $*$ notations as the “asterisk” spot).

- A . Usually, A denotes an integral domain with fraction field \mathfrak{F} .
- Absolute integral closure. (1.2.1d).
- Absolute domain. (2.3.1).
- Absolute prime. (2.3.1).
- Admissible (function of ideals). (4.5.1).
- Affine algebra (over a field). (1.2.1a).
- Affine progenitor. (3.1.1).
- Almost all (A -algebras B). (2.3.4).
- Almost all (fibers). (2.3.3c).
- Almost all (μ in $\text{Max Spec } A$). (2.2.2) (second paragraph).
- Approximately Gorenstein ring. (4.3.7).
- Approximation ring. (3.3.2).
- Artin approximation. (3.3.3).
- *. Characteristic p : (1.4.3).
- *eq. (3.4.3b).
- >*eq. (3.4.3b).
- *EQ.
- >*EQ.
- * K . For affine algebras: (2.2.3). General case: (3.4.3a).
- >* K . (3.2.1).
- */ A . (2.2.2).
- */ (A, Q) . (4.6.1).
- $-$ (as in I^- or \bar{I}). (1.6.3).
- Base change conventions (including M_B for $B \otimes_A M_A$ and $\langle N_B \rangle$). (1.2.3).
- Base ring (of an affine progenitor). (3.1.1).
- Biequidimensional. (1.2.1e).
- Big equational tight closure (*EQ). (4.6.1).

$[q]$. (1.4.1).

Briançon-Skoda theorem. (1.7.1), (4.1.5).

Canonical module or ideal. (4.3.10).

Catenary. (1.2.1h).

Closed fiber. (2.3.3a).

Cohen-Macaulay ring. (1.2.1g).

Commuting with tight closure. (4.2.9).

Complete local domain of a Noetherian ring. (2.3.10).

Completely stable (weak) test element. (1.4.5b,c).

$\langle \rangle$. (1.2.3).

$>^{eq}$. (3.4.3b). (N.B.: for other “ $>^{*?}$ ” notations see the “ $*?$ ” entries in the “a” portion of this list.)

Cyclically pure. (4.1.11).

Dense (weakly or strongly) F-regular type. (4.3.2).

Depth. (1.2.1g).

Descend(s). (2.1.10), second paragraph.

Descendably projective algebra. (4.2.12).

Descent; descent data. (2.1.1), (2.1.2).

$\dim R$. (1.2.1e).

Direct big equational tight closure ($>^{*EQ}$). (4.6.1).

Direct equational tight closure ($>^{eq}$). (3.4.3b).

Direct K -tight closure ($>^{*K}$). (3.2.1).

e . (1.2.2b).

Equational tight closure (eq). (3.4.3b).

Equidimensional (local ring). (1.2.1e).

Essentially of finite type (for an algebra). (1.2.1a).

Étale algebra. (1.2.1f) and (3.3.1).

Excellent ring. (1.2.1h).

\mathfrak{F} . Usually, \mathfrak{F} denotes the fraction field of the integral domain A .

F , F_R , F^e , F_R^e . (1.2.2c).

\mathbf{F} , \mathbf{F}_R , \mathbf{F}^e , \mathbf{F}_R^e . (1.2.2d), (1.4.1).

F . (1.5.6).

f^*K . (3.2.1).

F -finite. (1.5.6).

Fiber. (2.3.3a).

Fiberwise tightly closed. (4.3.9).

Filtered inductive limit. (4.2.1).

Filtered inductive limit of maps of algebras. (4.2.10).

Finite phantom projective dimension. (4.4.6).

Finite phantom resolution. (4.4.6).

Finite type (for an algebra). (1.2.1a).

For almost all. See “Almost all.”

Formal K -tight closure (f^*K). (3.2.1).

Formally very K -tightly closed. (4.2.11)

Formal minheight of an ideal. (4.4.2).

F -rational. (4.3.1).

F -rational type. (4.3.2).

F -regular. In characteristic p : (1.6.1). In characteristic 0: (4.3.1).

F -regular type. (4.3.2).

Frobenius closure (F). (1.5.6).

Frobenius or Peskine-Szpiro functors \mathbf{F} , \mathbf{F}^e . (1.4.1).

\mathbf{G}^e . Discussion preceding Question 3. in the Appendix.

General Néron desingularization. (4.2.1-3).

Generic fiber. (2.3.3a).

Generic freeness. (2.1.4).

Generically smooth. (1.4.8).

Geometrically connected. (2.3.1)

Geometrically normal. (2.3.1).

Geometrically reduced. (2.3.1).

Geometrically regular. (2.3.1), (4.2.1).

Gorenstein ring (1.2.1g).

Henselization. (3.3.1).

Hilbert ring. (4.3.14).

∞ . (1.2.2e).

Integral closure (of an ideal). (1.6.3).

Intersection-flat or \cap -flat. (4.2.16).

Jacobian ideal. (1.5.2).

Jacobson ring. (4.3.14).

$\kappa = \kappa(\mu)$. (2.2.2) (second paragraph).

$\kappa = \kappa(P)$. (2.3.3a).

Krull dimension. (1.2.1e).

K -tight closure ($*^K$). For affine algebras: (2.2.3). General case: (3.4.3a).

Lipman-Sathaye Theorem. (1.5.3).

Local ring (R, m, K) . (1.2.1a).

Locally stable (weak) test element. (1.4.5b,c).

Locally excellent ring. (1.2.1h).

Map of affine progenitors. (3.1.3a).

Minheight. (2.3.8).

Mnht. (2.3.8).

\ominus . (1.7.4), (4.1.7).

$\mathcal{M}(R')$. (3.1.3c).

M -sequence. (1.2.1g).

\mathcal{M}/\mathfrak{A} . (3.1.5f).

μ . (2.2.2) (second paragraph).

Nearly admissible (function of ideals). (4.5.1).

p . (1.2.2a)

$p = p(\mu)$ (2.2.2) (second paragraph).

Parameter ideal. (2.3.10).

Parameters. (2.3.10).

Persistence (of tight closure). Characteristic p : (1.4.13). Characteristic 0: (2.5.5k,l), (3.2.2k,l), (3.2.3k,l).

Peskine-Szpiro or Frobenius functors \mathbf{F} , \mathbf{F}^e . (1.4.1).

Phantom acyclicity. (4.4.1).

Phantom acyclicity criterion. (4.4.3).

Phantom homology. (4.4.1).

Phantom intersection theorem. (4.4.5).

Phantom projective dimension. (4.4.6).

Phantom resolution. (4.4.6).

$^+$. (1.2.1d).

Pointed étale extension. (3.3.1).

Possibly improper (regular sequence or M -sequence). (1.2.1g).

Pseudo-rational singularities. (4.3.16).

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\mathbb{Q} -Gorenstein. (4.3.10).

$q = p^e$. (1.2.2b).

$q = q(\mu)$. (2.2.2) (second paragraph).

q and q . (1.4.1).

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Rational singularities. (4.3.16).

$^{\text{reg}}$. (4.1.2)

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$R' \otimes \mathcal{M}$. (3.1.3b).

R° . (1.2.1c).

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. Characteristic p : (1.4.3). (N. B.: for other notations involving $$, see the “a” portion of the alphabetical list, i.e., treat “*” as “asterisk.”)

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(1.3) THE MAIN RESULTS

In this section we give a summary of some of the main results of the paper. In choosing results to be included here, we have been greatly influenced by the usefulness of the result. We have avoided results whose statements require technical definitions, and in many cases

we have not stated the strongest form, but have chosen to give instead a weaker result with a simpler statement. One of our goals has been to make a usable form of the results available to readers without their needing to read a great deal of the paper. The emphasis is very much on results that convey important properties of tight closure and on significant applications of tight closure rather than results that are used to build the theory. Results of the latter kind, such as Theorems (2.4.7), (2.4.9), (2.5.2), and (3.5.1) have been omitted here no matter how difficult they are and no matter how important they are in the development of the theory.

Thus, the title of this section is something of a misnomer, because, on the one hand, while we have included many of the main results, many have been omitted on the grounds of technicality, or because they are used more in building the theory than in the applications, while on the other hand we have included quite a few results, particularly in (1.3.1), which are rather straightforward, because of they are needed frequently when utilizing the theory.

In a number of instances we have repeated definitions in this section, particularly when they are short, that are given elsewhere. In most cases, we have given a footnote or other indication of where the reader can find a definition and relevant discussion of a notion being used.

The notion of K -tight closure discussed in the result below and referred to throughout this section is the same as the notion of *formal* K -tight closure of §(3.2).

The following result is essentially a restatement of Theorem (3.2.3).

(1.3.1) Theorem (existence and very basic properties of K -tight closure). *Let K be a field of characteristic zero. Then there is a closure operation, K -tight closure, defined on submodules N of finitely generated modules M over a Noetherian K -algebra S , with the properties listed below. The K -tight closure of N in M is denoted N_M^{*K} or simply N^{*K} .*

In what follows S is a Noetherian algebra over the characteristic zero field K , $N', N \subseteq M$ are finitely generated S -modules, I is an ideal of S , u is an element of M , and v denotes the image of u in M/N .

*Unless otherwise indicated, *K indicates K -tight closure in M .*

(a) N^{*K} is a submodule of M containing N .

(b) $u \in N^*_M$ if and only if $v \in 0^*_M/N$.

(c) The following three conditions are equivalent:

(1) $u \in N^{*K}$.

(2) For every complete local domain¹ B of R , we have that $u_B \in \langle N_B \rangle^{*K}_{M_B}$.

(3) For every complete local domain C to which R maps, we have that

$$u_C \in \langle N_C \rangle^{*K}_{M_C}.$$

(d) If $N \subseteq N' \subseteq M$ then $N^{*K}_M \subseteq N'^{*K}_M$ and $N^{*K}_{N'} \subseteq N^{*K}_M$.

(e) $(N^{*K})^{*K} = N^{*K}$.

(f) $(N \cap N')^{*K} \subseteq N^{*K} \cap N'^{*K}$.

(g) $(N + N')^{*K} = (N^{*K} + N'^{*K})^{*K}$.

(h) $(IN)^{*K}_M = ((I^{*K}_R)N^{*K}_M)^{*K}_M$.

(i) $(N :_M I)^{*K}_M \subseteq N^{*K} :_M I$ (respectively, $(N :_S N')^{*K} \subseteq N^{*K} :_S N'$). Hence, if $N = N^{*K}$ then $(N :_M I)^{*K} = N :_M I$ (respectively, $(N :_S N')^{*K} = N :_S N'$).

(j) If $N_i \subseteq M_i$ are finitely many finitely generated S -modules and we identify $N = \oplus_i N_i$ with its image in $M = \oplus_i M_i$ then the obvious injection $\oplus_i N_i^{*K}_{M_i} \hookrightarrow M$ maps $\oplus_i N_i^{*K}_{M_i}$ isomorphically onto N^{*K}_M .

(k) (Persistence of formal K -tight closure) Let L be a field containing K , let S' be a Noetherian L -algebra (hence, also, a K -algebra) and let $S \rightarrow S'$ be a K -algebra homomorphism. Let $u \in N^{*K}_M$. Then $1 \otimes u \in \langle S' \otimes_R N \rangle^{*K}_{S' \otimes_R M}$ over S' . In particular, this holds when $L = K$.

(l) (Persistence of formal K -tight closure: second version). Let L be a field containing K , let S' be a Noetherian L -algebra (hence, also, a K -algebra) and let $S \rightarrow S'$ be a K -algebra homomorphism. Let $N \subseteq M$ be finitely generated S -modules, and let $V \subseteq W$ be finitely generated S' -modules. Suppose that $u \in N^{*K}_M$. Suppose also that there is an R -homomorphism $\gamma : M \rightarrow W$ such that $\gamma(N) \subseteq V$. Then $\gamma(u) \in V^{*L}_W$.

(m) (Irrelevance of nilpotents) If J is the nilradical of S , then $J \subseteq (0)^{*K}$, and so $J \subseteq I^{*K}$ for all ideals I of S . Consequently, $JM \subseteq N^{*K}$. Moreover, if N^\sim denotes the image of

¹I.e., the quotient of the completion of a local ring of S by a minimal prime: see (2.3.10).

N in M/JM , then N^{*K} is the inverse image in M of the tight closure $(N^\sim)^{*K}_{M/JM}$, which may be computed either over S or over $S_{\text{red}} (= S/J)$.

- (n) Let $\mathfrak{p}^{(1)}, \dots, \mathfrak{p}^{(s)}$ be the minimal primes of S and let $S^{(i)} = R/\mathfrak{p}^{(i)}$. Let $M^{(i)} = S^{(i)} \otimes_S M$ and let $N^{(i)}$ be the image of $S^{(i)} \otimes_S N$ in $M^{(i)}$. Let $u^{(i)}$ be the image of u in $M^{(i)}$. Then $u \in N^{*K}$ if and only if $u^{(i)} \in (N^{(i)})^{*K}$ in $M^{(i)}$ over $S^{(i)}$, $1 \leq i \leq s$.
- (o) If $R = \prod_{i=1}^h R_i$ is a finite product and $M = \prod_i M_i$ and $N = \prod_i N_i$ are the corresponding product decompositions of M, N , respectively, then $u = (u_1, \dots, u_h) \in M$ is in N^{*K}_M over R if and only if for all i , $1 \leq i \leq h$, $u_i \in N_i^{*K}_{M_i}$.

Proof. See Theorem (3.2.2). \square

(1.3.2) Theorem. *If S is a locally excellent Noetherian K -algebra, where K is a field of characteristic 0, and $N \subseteq M$ are finitely generated S -modules such that $u \in N^{*K}_M$ for a certain element $u \in M$, then there exist a finitely generated K -algebra R , a K -homomorphism $R \rightarrow S$, a finitely generated R -module M_0 , a submodule N_0 of M_0 and an element $u_0 \in M_0$ such that u_0 is in the K -tight closure of N_0 in M_0 over R , such that $S \otimes_R M_0 \cong M$ and such that under that identification the image of $1 \otimes u_0$ is u and the image of $S \otimes_R N_0$ is N .*

Moreover, if S is a finitely generated K -algebra then the notion of K -tight closure agrees with that obtained by reduction to characteristic $p > 0$ discussed in (2.2.2).

Proof. See Theorem (3.4.1) and Corollary (3.4.2). \square

A Noetherian K -algebra is called *weakly F -regular* (respectively, *F -rational*) over the field K of characteristic zero if every ideal (respectively, every parameter ideal²) is K -tightly closed. A K -algebra is called *F -regular* over K if all of its localizations are weakly F -regular.

(1.3.3) Theorem (basic properties of weak F -regularity). *Let R be a Noetherian K -algebra over a field K of characteristic zero.*

- (a) *If R is regular then it is weakly F -regular, and, hence, F -regular.*

²See (2.3.10).

- (b) *If R is weakly F -regular over K then every submodule of every finitely generated R -module is tightly closed.*
- (c) *R is weakly F -regular over K if and only if its localization at every maximal ideal is tightly closed.*
- (d) *A local ring is weakly F -regular over K if and only if its completion is weakly F -regular.*
- (e) *A weakly F -regular ring over K and, more generally, an F -rational ring over K , is normal. A universally catenary F -rational ring over K and, hence, a universally catenary weakly F -regular ring over K , is Cohen-Macaulay.*

Proof. For part (a) see Theorem (4.3.6), for parts (b), (c), and (d) see Theorem (4.3.8), and for part (e) see Corollary (4.3.5). \square

The fact that F -rational and weakly F -regular rings are Cohen-Macaulay (under mild hypotheses) depends on some form of result concerning the fact that tight closure “captures” colon ideals involving parameters: the following is one result of this type.

(1.3.4) Theorem (tight closure captures colons). *Let K be a field of characteristic zero and let S be a Noetherian K -algebra. Let x_1, \dots, x_n be strong parameters³ in S . Then $(x_1, \dots, x_{n-1})^{*K} :_S x_n S = (x_1, \dots, x_{n-1})^{*K}$.*

Proof. See Theorem (4.1.7). \square

Although we do not restate here the rather lengthy descent result given in Theorem (3.5.1), we do want to emphasize that it plays a critical role in the proofs of the results on K -tight closure stated below.

We next note the following, which contains the result that a ring of invariants of a linearly reductive algebraic group acting on a regular ring is Cohen-Macaulay. Cf. [HR1], [Ke], and [Bou].

³See (2.3.10). This means that the x_i generate an ideal of height at least n modulo every minimal prime after localization and completion at any prime ideal of R . If R is universally catenary and locally equidimensional the condition simply means that the x_i are part of a system of parameters when one localizes at any prime containing them.

(1.3.5) Theorem. *If S is a regular Noetherian ring of equal characteristic zero and R is a subring of S that is a direct summand of S as an R -module (or, more generally, is pure⁴ in S) then R is Cohen-Macaulay (and normal — in fact the completion of every local ring of R is normal).*

Proof. See Theorem (4.1.12). \square

The main points leading up to the proof of Theorem (1.3.5) may be summarized as follows: because of the colon-capturing property of tight closure, under mild conditions one has that a ring in which every ideal is tightly closed is Cohen-Macaulay. But a regular ring has the property that every ideal is tightly closed (ultimately, because of the flatness of the Frobenius endomorphism once one passes to positive characteristic regular rings), and it is easy to show that the property that every ideal is tightly closed passes to direct summands (or pure subrings).

Recall that over a Noetherian ring S , if $N \subseteq M$ are finitely generated S -modules then an element $u \in M$ is said to be in the *regular closure* N^{reg}_M of N in M or, simply, in N^{reg} , if for every homomorphism of S to a regular Noetherian ring T , we have that the image of u in $T \otimes_S M$ is in the image of $T \otimes_S N$ in $T \otimes_S M$. See (4.1.2). Because the integral closure of an ideal I of S has a similar characterization in terms of homomorphisms to discrete valuation rings, we have at once that $I^{\text{reg}} \subseteq \bar{I}$.

(1.3.6) Theorem. *Let K be a field of characteristic zero and let S be a Noetherian K -algebra. Let $N \subseteq M$ be finitely generated S -modules. Then $N^{*K}_M \subseteq N^{\text{reg}}_M$.*

*Hence, if I is an ideal of S then $I^{*K} \subseteq I^{\text{reg}} \subseteq \bar{I}$.*

Proof. See (4.1.3) and (4.1.4). \square

(1.3.7) Theorem (generalized Briançon-Skoda theorem). *Let S be a Noetherian ring of equal characteristic zero and let I be an ideal of S generated by at most n elements. Then for every $k \in \mathbb{N}$, $(I^{n+k})^- \subseteq (I^{k+1})^{>*eq} \subseteq (I^{k+1})^{*eq}$.*

*Hence, if S is also a K -algebra for some field K then $(I^{n+k+1})^- \subseteq (I^{k+1})^{>*K} \subseteq (I^{k+1})^{*K}$.*

⁴See (4.1.11).

Proof. See Theorem (4.1.5). \square

Theorem (1.3.8) (phantom acyclicity criterion). *Let K be a field of characteristic 0 and let R be a Noetherian K -algebra. Let G_\bullet be a finite complex of finitely generated projective R -modules of constant rank.*

*Suppose that R is universally catenary and locally equidimensional and that $R_{\text{red}} \otimes_R G_\bullet$ satisfies the standard conditions⁵ on rank and height. Then G_\bullet is K -phantom acyclic, i.e., the cycles Z_i in G_i are in the K -tight closure $B_i^{*K}_{G_i}$ of the module of boundaries B_i in G_i .*

Proof. See Theorem (4.4.3), which gives a more general version that relaxes the conditions on the ring R . \square

The phantom acyclicity criterion has the following powerful corollary:

Theorem (1.3.9) (vanishing theorem for maps of Tor). *Let R be an equicharacteristic zero regular ring, let S be a module-finite extension of R that is torsion-free as an R -module (e.g., a domain), and let $S \rightarrow T$ be any homomorphism to a regular ring (or, if R is a K -algebra for some field K of characteristic 0, we may suppose instead that T is weakly F -regular over K). Then for every R -module M , the map $\text{Tor}_i^R(M, S) \rightarrow \text{Tor}_i^R(M, T)$ is 0 for all $i \geq 1$.*

Proof. See Theorem (4.4.4). \square

To see that this is a powerful theorem, note that it implies again that direct summands (and pure subrings) of regular rings are Cohen-Macaulay in equal characteristic zero. We may reduce to the case where S is complete local and a direct summand of a regular ring T . Then S is a module-finite extension of a regular ring R with system of parameters x_1, \dots, x_d . We may take $M = R/(x_1, \dots, x_d)R$ and conclude that the maps $\text{Tor}_i^R(M, S) \rightarrow \text{Tor}_i^R(M, T)$ are 0 for $i \geq 1$. But since S is a direct summand of T these maps are injective, and so this shows that $\text{Tor}_i^R(M, S) = 0$ for $i \geq 1$, which implies that x_1, \dots, x_d is a regular sequence in S and, hence, that S is Cohen-Macaulay.

⁵See (4.4.2).

In [HH11], §4, it is shown that if (1.3.9) were known in mixed characteristic it would imply the longstanding conjecture that regular rings are direct summands of their module-finite extensions (which is known in equal characteristic but not in mixed characteristic).

Theorem (1.3.10) (phantom intersection theorem). *Let K be a field of characteristic zero and let R be a Noetherian K -algebra that is locally equidimensional and universally catenary.⁶ Let G_\bullet be a complex of finitely generated projective R -modules of constant rank that satisfies the standard conditions⁷ on rank and height. Suppose that the complex G_\bullet is of length d . Let $z \in M = H_0(G_\bullet)$ be any element whose annihilator in R has height $> d$. Then $z \in 0_M^{*K}$. In consequence:*

- (1) *if (R, m, K) is local, z cannot be a minimal generator of M .*
- (2) *the image of z is 0 in $H_0(S \otimes_R G_\bullet)$ for any regular (or weakly F -regular) ring S to which R maps.*

Proof. See Theorem (4.4.5). \square

The following result greatly generalizes the colon-capturing property for tight closure given in (1.3.4). The notions of “admissible” and “nearly admissible” functions of ideals needed for the statement of the theorem are discussed in (4.5.1).

(1.3.11) Theorem. *Let K be a field of characteristic zero and let $A \rightarrow R$ be a homomorphism of Noetherian K -algebras such that A is regular. Suppose either that*

- (1) *A is the ring $\mathbb{Q}[x_1, \dots, x_n]$, \mathcal{I} is the set of all ideals of A generated by monomials in the variables x_1, \dots, x_n , and that for every integer h , $1 \leq h \leq n$, every h element subset of x_1, \dots, x_n consists of strong parameters⁸: this is equivalent to the hypothesis that every such subset generates an ideal of formal minheight⁹ at least h in R or*
- (2) *A is any regular ring, \mathcal{I} is the class of all ideals of A , and for every complete local domain S of R at a maximal ideal,¹⁰ if P is the contraction of the maximal ideal*

⁶These conditions are omitted in the more general version of this result given in Theorem (4.4.5), but then, wherever one has a hypothesis on the height of an ideal, one needs to make that hypothesis for the formal minheight of the ideal, which is discussed at the end of (4.4.2), instead.

⁷See (4.4.2).

⁸See (2.3.10).

⁹If R is universally catenary formal minheight coincides with minheight, and if R is locally equidimensional as well, formal minheight coincides with height, and this condition is much less technical.

¹⁰See (2.3.10).

of S to A then the height of PS is at least the height of P . ideal I of A the formal minheight of IR is at least the height of I .

Let \mathcal{F} be a nearly admissible function of k ideals. Then for any k ideals I_1, \dots, I_k in \mathcal{I} , $\mathcal{F}(I_1R, \dots, I_kR) \subseteq (\mathcal{F}(I_1, \dots, I_k)R)^{*K}$, and if \mathcal{F} is, moreover, admissible, then $\mathcal{F}(I_1R, \dots, I_kR)$ is trapped over $\mathcal{F}(I_1, \dots, I_k)$, i.e.,

$$\mathcal{F}(I_1, \dots, I_k)R \subseteq \mathcal{F}(I_1R, \dots, I_kR) \subseteq (\mathcal{F}(I_1, \dots, I_k)R)^{*K}.$$

Proof. See Theorem (4.5.3). \square

(1.3.12) Theorem (testing tight closure at maximal ideals. *Let R be a Noetherian ring containing a field K of characteristic 0, let $N \subseteq M$ be finitely generated R -modules, and let $u \in M$. Then $u \in N^{*K}_M$ if and only if for every complete local domain B of R at a maximal ideal, $u_B \in \langle N_B \rangle^{*K}_{M_B}$.*

Proof. See (4.2.6). \square

(1.3.13) Theorem (height-preserving extensions do not affect tight closure). *Let K be a field of characteristic 0 and let $R \rightarrow S$ be a homomorphism of Noetherian K -algebras such that*

(#) for every maximal ideal m of R and every minimal prime \mathfrak{p} of $(R_m)^\wedge$, there is a prime ideal Q of S lying over m and a prime ideal \mathfrak{q} of $(S_Q)^\wedge$ lying over \mathfrak{p} such that $\text{ht } P(S_Q)^\wedge/\mathfrak{q} \geq \dim(R_m)^\wedge/\mathfrak{p}$.

*Let $N \subseteq M$ be finitely generated R -modules and let $u \in M$. Then $u_S \in \langle N_S \rangle^{*K}_{M_S}$ if and only if $u \in N^{*K}_M$.*

*In fact, the conclusion that $u_S \in \langle N_S \rangle^{*K}_{M_S}$ if and only if $u \in N^{*K}_M$ is valid for a fixed pair of finitely generated modules $N \subseteq M$ if condition (#) holds for every maximal ideal m of R that is in the support of M/N .*

Proof. See Theorem (4.2.7). \square

The next three results are connected with the problem of when tight closure commutes with base change.

(1.3.14) Theorem (main theorem on geometrically regular base change). *Let K be a field of characteristic zero, and let R, S be Noetherian K -algebras such that S is locally excellent.*

- (a) *If $R \rightarrow S$ is a filtered inductive limit of K -algebra homomorphisms that commute with K -tight closure, then $R \rightarrow S$ commutes with K -tight closure.*
- (b) *If $R \rightarrow S$ is a homomorphism of finitely generated K -algebras that is smooth and such that S is descendably projective over R relative to K , then $R \rightarrow S$ commutes with K -tight closure.*
- (c) *If $R \rightarrow S$ is a filtered inductive limit of K -algebra homomorphisms satisfying the condition in (b) then $R \rightarrow S$ commutes with K -tight closure.*
- (d) *If S is a polynomial ring in finitely many variables over R , say $S = R[x_1, \dots, x_n]$, then $R \rightarrow S$ commutes with K -tight closure. Moreover, if I is any ideal of R that is K -tightly closed then the ideal of S generated by IS and any set W of monomials in the x 's is tightly closed in S .*
- (e) *If $N \subseteq M$ are finitely generated R -modules such that N is very K -tightly closed in M and S is geometrically regular over R , then N_S is very K -tightly closed in M_S .*
- (f) *If S is geometrically regular over R and if $N \subseteq M$ are finitely generated R -modules such that N is formally very K -tightly closed¹¹ in M , then N_S is K -tightly closed in M_S .*
- (g) *Let R be either a finitely generated K -algebra or a complete local ring,¹² and assume that R is reduced and equidimensional. Let I be an ideal of R that is generated by parameters.¹³ Let S be a geometrically regular R -algebra. Then $(IS)^{*K}$ (in S) is $I^{*K}S$, where I^{*K} is the K -tight closure of I in R . In particular, this holds when S is a localization of R .*

Proof. See Theorem (4.2.14). \square

¹¹See (4.2.10). A weaker condition suffices, namely it is sufficient that for each prime ideal P of R lying under a maximal ideal of S in the support of $(M/N)_S$, the image of N_P is very K -tightly closed in M_P with $B = (R_P)^\wedge$.

¹²As should be clear from the proof, the result holds somewhat more generally: what is needed is that R be a locally excellent Noetherian K -algebra that is a filtered inductive limit of reduced, equidimensional affine K -algebras, each of which contains a sequence of parameters that maps to the generators of the given ideal.

¹³See (2.3.10).

(1.3.15) Theorem. *Let K be a field of characteristic 0 and let $R \rightarrow S$ a K -algebra homomorphism. Suppose that S is locally excellent.*

- (a) *If R is a finitely generated K -algebra and $S = T \otimes_K R$, where T is regular, then $R \rightarrow S$ commutes with K -tight closure. In particular, this holds when T is any field extension of K .*
- (b) *If S is module-finite and smooth over R , then $R \rightarrow S$ commutes with K -tight closure. In particular this holds when L is any field contained in R , L' is a finite algebraic extension of L , and $S = L' \otimes_L R$.*
- (c) *If $S = L' \otimes_L R$ where L' is a possibly infinite algebraic extension of a field $L \subseteq R$, then $R \rightarrow S$ commutes with K -tight closure (but notice that we are assuming that $L' \otimes_L R$ is Noetherian and locally excellent: this is not automatic in this case).*

Proof. See (4.2.17). \square

(1.3.16) Theorem. *Let R be a Noetherian K -algebra, let m be a maximal ideal of R , let $L = R/m$ (thought of as an R -module), and let $N \subseteq M$ be finitely generated R -modules. let $R \rightarrow S$ be a homomorphism of Noetherian K -algebras such that $m' = mS$ is a maximal ideal of S and $R_m \rightarrow S_{m'}$ is faithfully flat. Suppose either that*

- (1) *M/N is an essential extension of L , or else that*
- (2) *M/N has a finite filtration by copies of L , and $L \rightarrow S/mS$ is an isomorphism.*

Then N is K -tightly closed in M over R if and only if N_S is K -tightly closed in M_S over S .

Proof. See (4.2.18). \square

The final three results listed discuss weak F -regularity, base change, and F -rationality. Weak F -regularity and F -rationality over K are defined in (4.3.1).

(1.3.17) Theorem (characterization of weak F -regularity). *Let K be a field of characteristic 0. Let R be a Noetherian K -algebra. Then the following conditions on R are equivalent:*

- (1) *R is weakly F -regular over K (i.e., every ideal of R is K -tightly closed).*
- (2) *For every maximal ideal of R , R_m is weakly F -regular over K .*

- (3) *For every maximal ideal of R , the completion of R_m is weakly F -regular over K .*
- (4) *R is normal (respectively, approximately Gorenstein), and for every maximal ideal m of R there is a sequence of m -primary irreducible ideals cofinal with the powers of m that are K -tightly closed.*
- (5) *For every pair of finitely generated R -modules $N \subseteq M$, N is tightly closed in M .*

Proof. See Theorem (4.3.8). \square

(1.3.18) Theorem (F -regularity and base change). *Let K be a field of characteristic 0 and let $R \rightarrow S$ be a flat homomorphism of Noetherian K -algebras.*

- (a) *If R is weakly F -regular over K , $R \rightarrow S$ is local, and the closed fiber is regular then S is weakly F -regular over K .*
- (b) *If R_P is weakly F -regular over K for every prime ideal P of R lying under a maximal ideal of S , and $R \rightarrow S$ is geometrically regular, then S is weakly F -regular over K .*
- (c) *If R is F -regular over K and $R \rightarrow S$ is geometrically regular then S is F -regular over K .*
- (d) *If R is a Hilbert ring¹⁴ (e.g., a finitely generated algebra over a field), $R \rightarrow S$ is smooth, and R is weakly F -regular over K , then S is weakly F -regular over K .*

Proof. See Theorem (4.3.14). \square

(1.3.19) Theorem (behavior of F -rational rings). *Let K be a field of characteristic 0 and let R be a locally excellent Noetherian K -algebra.*

- (a) *R is F -rational over K iff R_m is F -rational over K for every maximal ideal m of R .*
- (b) *If R is local, then R is F -rational over K iff \widehat{R} is F -rational over K .*
- (c) *(Localization and base change) If R is F -rational over K , then every localization of R is F -rational over K , and, more generally, if R is F -rational over K , $R \rightarrow S$ is geometrically regular, and S is locally excellent, then S is also F -rational over K .*
- (d) *If R is local and equidimensional, then R is F -rational over K if and only if the ideal generated by one system of parameters is K -tightly closed.*
- (e) *(F -rationality deforms) If R/fR is F -rational over K , where f is a nonzerodivisor in R , then R_P is F -rational over K for every prime ideal P containing f . In particular,*

¹⁴See (4.3.13).

if (R, m) is local and $f \in m$ is a nonzerodivisor such that R/fR is F -rational over K , then R is F -rational over K .

- (f) *If R is Gorenstein, then R is F -rational over K iff R is weakly F -regular over K iff R is F -regular over K .*

Proof. See Theorem (4.3.15). \square

(1.4) TIGHT CLOSURE THEORY AND TEST ELEMENTS IN POSITIVE CHARACTERISTIC

The reader may wish to review the characteristic p conventions discussed in (1.2.2).

(1.4.1) Frobenius (Peskin-Szpiro) functors. When $\theta: R \rightarrow S$ is a ring homomorphism the functor $S \otimes_R _$ is a covariant functor from R -modules to S -modules. It takes free modules to free modules of the same rank, projective modules to projective modules, and flat modules to flat modules. It preserves finite generation. If a module M has a finite presentation with matrix (r_{ij}) then $S \otimes_R M$ has such a presentation with matrix $(\theta(r_{ij}))$.

When $S = R$ and $\theta = F_R^e$, the e th iterate of the Frobenius endomorphism, we denote this functor \mathbf{F}_R^e . (See (1.2.1) and (1.2.2).) The subscript R is frequently omitted. All of the above remarks apply. Thus, if M has a finite presentation with matrix (r_{ij}) , $\mathbf{F}^e(M)$ has a finite presentation with matrix (r_{ij}^q) . Quite generally, there is a natural map $M \rightarrow S \otimes_R M$ sending $u \mapsto 1 \otimes u$, giving a natural transformation from the identity functor on R -modules to the composition of $S \otimes_R _$ with restriction of scalars from S to R . In the case where $\theta = F_R^e$, we denote the image of $u \in M$ under this map (in $\mathbf{F}_R^e(M)$) by u^q . With this notation we have that $(ru)^q = r^q(u^q)$ for $r \in R$ and $u \in M$.

When $N \subseteq M$ we denote by $N^{[q]}$ (or $N_M^{[q]}$ if greater precision is needed) the image of $\mathbf{F}^e(N)$ in $\mathbf{F}^e(M)$. This is the same as the R -span in $\mathbf{F}^e(M)$ of all the elements u^q for $u \in N$. It suffices to let u run through a set of generators for N over R . $N^{[q]}$ may also be viewed as the kernel of the map from $\mathbf{F}^e(M) \rightarrow \mathbf{F}^e(M/N)$, by the right exactness of tensor.

When M is free, $\mathbf{F}^e(M)$ may be identified (non-canonically) with M : if we choose a free basis $\{u_i\}$ for M we may establish the isomorphism by letting u_i correspond to $\{u_i^q\}$

in $\mathbf{F}^e(M)$. In coordinates, the map $u \mapsto u^q$ corresponds to the map sending the vector with coordinates r_i to the vector with coordinates r_i^q .

When $M = R$, $\mathbf{F}^e(M) = R$. If $I \subseteq R$ is an ideal, $I^{[q]}$ is the expansion of I under F^e , a standard notation, and is the ideal of R generated by all q th powers of elements of I . Note that for $u \in I$, the notation u^q just introduced coincides with the usual meaning of u^q . Of course, $\mathbf{F}^e(R/I) \cong R/I^{[q]}$.

The following observation is straightforward but often useful:

(1.4.2) Fact. *If $R \rightarrow S$ is a homomorphism of Noetherian rings of characteristic p then for all e there is an isomorphism of functors $S \otimes_R \mathbf{F}_R^e(_) \cong \mathbf{F}_S^e(S \otimes_R _)$ (both are covariant functors from R -modules to S -modules). \square*

(1.4.3) The definition of tight closure in characteristic p . Now suppose that R is a Noetherian ring of characteristic p , that $N \subseteq M$ are finitely generated R -modules, and that $u \in M$. We say that $u \in N^*$ (or N^*_M), the *tight closure* of N in M , if there exists an element $c \in R^\circ$ (see 1.2.1)) such that for all $q \geq 0$, $cu^q \in N^{[q]}_M$.

In the rest of this section we give some definitions and results that exhibit the properties of tight closure in characteristic p that will be needed throughout this manuscript.

(1.4.4) Theorem. *Let R be a Noetherian ring of characteristic p and let $N', N \subseteq M$ be R -modules. Let $I \subseteq R$ be an ideal. Let $c \in R$, let $u \in M$ and let v be the image of u in M/N . Unless otherwise specified, $*$ indicates tight closure in M .*

- (a) $cu^q \in N^{[q]}$ in $\mathbf{F}^e(M)$ if and only if $cv^q = 0$ in $\mathbf{F}^e(M/N)$.
- (b) $u \in N^*_M$ if and only if $v \in 0^*_{M/N}$.
- (c) N^* is a submodule of M and $(N^*)^* = N^*$.
- (d) If $N \subseteq N' \subseteq M$ then $N^* \subseteq N'^*$ and $N^*_{N'} \subseteq N^*_M$.
- (e) If J is the nilradical of R , then $JM \subseteq N^*$. Moreover, if N^\sim denotes the image $(N + JM)/JM$ of N in M/JM , then N^* is the inverse image in M of the tight closure $(N^\sim)^*_{M/JM}$, which may be computed either over R or over $R_{\text{red}} (= R/J)$.
- (f) If R is reduced or if $\text{Ann}_R(M/N)$ has positive height, then $u \in N^*$ if and only if there exists $c \in R^\circ$ such that $cu^q \in N^{[q]}$ for all $q = p^e$ (not just for $q \gg 0$).

- (g) Let $\mathfrak{p}_1, \dots, \mathfrak{p}_h$ be the minimal primes of R and let $R_i = R/\mathfrak{p}_i$. Let $M_i = R_i \otimes_R M$ and let N_i be the image of $R_i \otimes_R N$ in M_i . Let u_i be the image of u in M_i . Then $u \in N^*$ if and only if $u_i \in N_i^*$ in M_i over R_i , $1 \leq i \leq h$.
- (h) $(N \cap N')^* \subseteq N^* \cap N'^*$.
- (i) $(N + N')^* = (N^* + N'^*)^*$.
- (j) $(IN)^* = (I^* N^*)^*$.
- (k) $(N :_M I)^* \subseteq N^* :_M I$ and $(N :_R N')^* \subseteq N^* :_R N'$. Hence, if N is tightly closed in M , then so are $N :_M I$ (in R) and $N :_R N'$ (in M).
- (l) If $N_i \subseteq M_i$ are finitely many finitely generated R -modules and we identify $N = \bigoplus_i N_i$ with a submodule of $M = \bigoplus_i M_i$ in the obvious way, then $\bigoplus_i (N_i)^*_{M_i}$ is carried isomorphically onto N^*_M by the obvious map into M .
- (m) If $R = \prod_{i=1}^h R_i$ is a finite product and $M = \prod_i M_i$ and $N = \prod_i N_i$ are the corresponding product decompositions of M , N , respectively, then $u = (u_1, \dots, u_h) \in M$ is in N^*_M over R if and only if for all i , $1 \leq i \leq h$, $u_i \in N_i^*_{M_i}$.

Proof. (a) holds since $\mathbf{F}^e(M/N) \cong \mathbf{F}^e(M)/N^{[q]}$ and (b) follows from (a). For (c)-(f) cf. [HH4] Proposition (8.5) (a)-(c), (e), and (j). For (g) see [AHH] Lemma (2.10c). For (h), (i), (j), (k), (l) cf. [HH4] Proposition (8.5) (f), (g), (h), (k), (m). (Part (k) as stated here follows by applying part (k) of Proposition (8.5) of [HH4] with N replaced by N^* , for then we have $(N :_R I)^* \subseteq (N^* :_R I)^* = (N^* :_R I)$ and $(N :_M N')^* = N^* :_M N'$. Part (m) is immediate from the definitions, since $R^\circ = \prod_i R_i^\circ$ and $\mathbf{F}_R^e(M) \cong \prod_i \mathbf{F}_{R_i}^e(M_i)$, while $N^{[q]} = \prod_i N_i^{[q]}_{M_i}$ where $N_i^{[q]}_{M_i}$ is calculated over R_i .

(1.4.5) Definition. (a) Let R be a Noetherian ring of characteristic p and let $q' = p^{e'}$ for some integer $e' \in \mathbb{N}$. Then $c \in R^\circ$ is a q' -weak test element if for every finitely generated R -module M and every submodule $N \subseteq M$, an element $u \in M$ is in N^*_M if and only if $cu^q \in N^{[q]}$ for all $q \geq q'$.

(b) An element $c \in R^\circ$ is called a *locally* (respectively, *completely*) *stable q' -weak test element* if its image in (respectively, in the completion of) every local ring of R is a q' -weak test element.

(c) If c is a q' -weak test element for $q' = 1$ then c is called a *test element*. We make the

same convention for the locally stable and completely stable cases. If we do not want to specify the value of q' we may refer simply to a *weak test element*, or a *locally* or *completely stable weak test element*.

(d) We let $\tau(R) = \cap_M \text{Ann}_{R^0} R_M^*$. If R has a test element, then $\tau(R)$ is the ideal generated by the test elements and c is a test element if and only if $c \in \tau(R) \cap R^\circ$. See Definition (8.22) and Proposition (8.23) of [HH4]. Note that if $c \in \tau(R)$ and $u \in N^*_M$ then $cu^q \in N^{[q]}$ for all $q \geq 1$ whether $c \in R^\circ$ or not.

(e) We let $\widehat{\tau}(R)$ denote the ideal of all elements c of R such that for every ring B that is the completion of a local ring of R , the image of c in B is in $\tau(B)$. If R has a completely stable test element, then $\widehat{\tau}(R)$ is the ideal generated by the completely stable test elements for R . An element $c \in R$ is a completely stable test element for R if and only if $c \in \widehat{\tau}(R) \cap R^\circ$. (Cf. the discussion prior to Theorem (7.29) of [HH9].)

The following easy observation is used frequently:

(1.4.6) Fact. *If $R \rightarrow S$ is a flat homomorphism of arbitrary Noetherian rings, then R° maps into S° . \square (See (1.2.1c) for notation.)*

We record the following facts about test elements:

(1.4.7) Theorem. *Let R be a Noetherian ring of characteristic p .*

- (a) *c is a q' -weak test element for R if and only if its image in R_m is a q' -weak test element for R_m for every maximal ideal m of R .*
- (b) *A q' -weak locally stable test element is a q' -weak test element.*
- (c) *If $c \in R$ is a q' -weak test element (or a completely stable q' -weak test element) for a faithfully flat extension S of R , then it is a q' -weak test element (respectively, a completely stable q' -weak test element) for R .*
- (d) *In particular, if $c \in R$, R is local, and c is a q' -weak test element for \widehat{R} , then c is a q' -weak test element for R .*
- (e) *A completely stable q' -weak test element for R is a locally stable weak test element for R .*
- (f) *Let J be the nilradical of R and suppose that $J^{[q'']} = 0$, where q'' is a power of p . Let $c \in R$ and let c' be the image of c in $R_{\text{red}} = R/J$. If c is a q' -weak test element for*

R , then c' is a q' -weak test element for R_{red} . If c' is a q' -weak test element for R_{red} , then $c^{q''}$ is $q'q''$ -weak test element for R . The same statements are valid for the case of locally stable weak test elements, and for completely stable weak test elements if R has reduced formal fibers (e.g., if R is excellent).

(g) If R has a completely stable q' -weak test element c and $N \subseteq M$ are finitely generated R -modules with $u \in M$, then the following conditions are equivalent:

- (1) $u \in N^*_M$.
- (2) $u/1 \in (N_m)^*_{M_m}$ for every maximal ideal m of R .
- (3) For every B of the form $(R_m)^\wedge$, where m is a maximal ideal of R , $1 \otimes u \in \langle N_B \rangle^*_{M_B}$ (see (1.2.3) for notation).
- (4) For every ring C of the form $(R_m)^\wedge/\mathfrak{p}$, where m is a maximal ideal of R and \mathfrak{p} is a minimal prime of $(R_m)^\wedge$, $1 \otimes u \in \langle N_C \rangle^*_{M_C}$.

Proof. (a) is Proposition (8.13a) of [HH4]. (b) is immediate from (a). Part (c) follows from Lemma (6.14b,c) of [HH9], while (d) is immediate from (c) and (e) follows at once from (d). Part (f) follows from Proposition (8.13d) of [HH4] and Corollary (6.2c,d) of [HH4] as generalized to modules in the discussion following Proposition (8.13) of [HH4].

It remains to prove (g). Note first that if we have a map $R \rightarrow S$ such that R° maps into S° then it is trivial for finitely generated R -modules $N \subseteq M$ that N^* maps into $\langle N_S \rangle^*$ in M_S . Since $R^\circ \rightarrow S^\circ$ when $R \rightarrow S$ is flat (including localization and completion) and also when S is obtained by killing a minimal prime of R , it follows easily that (1) \Rightarrow (2) \Rightarrow (3) \Rightarrow (4). Since (3) \Leftrightarrow (4) by (1.4.4g), it suffices to show that (3) implies that $u \in N^*_M$. But if not, we can choose $q \geq q'$ such that $cu^q \notin N^{[q]}$, and this will be preserved when pass to R_m for a suitable maximal ideal m and then to $B = (R_m)^\wedge$, contradicting that c is a q' -weak test element in B . \square

(1.4.8) Discussion. We next record some results that we shall need concerning either the existence of test elements, or which we shall use later to prove the existence of test elements. Recall that $R^\infty = \cup_q R^{1/q}$ when R is reduced of characteristic p . (Cf. (1.2.2e).) If R is a finitely generated algebra over a reduced Noetherian domain A we say that R is *generically smooth* over A if there exists an element $a \in A^\circ$ such that R_a is smooth over

A_a . This is equivalent to the assertion that $(A^\circ)^{-1}R$ is smooth over $(A^\circ)^{-1}A$; the latter ring is the total quotient ring of A and is a product of fields, one for every minimal prime of A . If R is module-finite over A , this is equivalent to the condition that $(A^\circ)^{-1}R$ be étale over $(A^\circ)^{-1}A$. When $(A^\circ)^{-1}A$ is a field L , this simply says that $(A^\circ)^{-1}R$ is a finite product of finite separable field extensions of L .

The next result is Theorem (6.13) of [HH4] generalized to the module case as indicated in the discussion following Theorem (8.14) of [HH4]. For a closely related result in which the condition that c be in A° is relaxed, see Theorem (1.5.1) and the discussion that precedes it. Cf. also Exercise 2.5 of [Hu5].

(1.4.9) Theorem. *Let R be module-finite, torsion-free, and generically smooth over a regular domain A of characteristic p . Then every element $d \in A^\circ$ such that R_d is smooth over A_d has a power c that is a completely stable test element in R , and also in $B \otimes_R R$ for every A -flat regular domain $B \supseteq R$. A sufficient condition for $c \in A^\circ$ to have this property is that $cR^\infty \subseteq A^\infty[R]$. \square*

If A is regular but not necessarily a domain the situation does not change a great deal. It is then a finite product of regular domains, and R is a product in a corresponding way. By working in each coordinate separately we see at once:

(1.4.10) Corollary. *Let R be module-finite, torsion-free, and generically smooth¹⁵ over a regular ring A of characteristic p . Then every element $d \in A^\circ$ such that R_d is smooth over A_d has a power c that is a completely stable test element in R . A sufficient condition for $c \in A^\circ$ to have this property is that $cR^\infty \subseteq A^\infty[R]$. \square*

Corollary (1.4.10) has important uses in developing the theory of tight closure in equal characteristic zero: see, for example, Theorem (2.4.7), where it is applied.

The following result on the existence of test elements is very useful (see [HH9], Theorems (5.10) and (6.1)):

(1.4.11) Theorem. *Let R be a Noetherian ring of characteristic p and let $c \in R^\circ$ be such that $(R_{\text{red}})_c$ is regular. Suppose either that $F: R \rightarrow R$ is finite or that R is essentially of*

¹⁵See (1.4.8)

finite type over an excellent local ring. Then c has a power that is a completely stable weak test element. If R is reduced, then c has a power that is a completely stable test element.

In particular, any ring essentially of finite type over an excellent local ring has a completely stable weak test element, and such a ring has a completely stable test element if it is reduced. \square

(1.4.12) Remarks. Notice that the result applies to any algebra essentially of finite type over a field and to any excellent local ring (in particular, to any complete local ring). The result is also valid if one assumes only that $(R_{\text{red}})_c$ is weakly F -regular and Gorenstein, by [HH9], Theorem (7.32b).

We next note the following important result, which we refer to as the *persistence of tight closure*.

(1.4.13) Theorem (persistence of tight closure). *Let $R \rightarrow S$ be a homomorphism of Noetherian rings such that either R is essentially of finite type over an excellent local ring or such that S has a completely stable weak test element. Let $N \subseteq M$ be finitely generated R -modules and $u \in N^*_M$. Then $1 \otimes u \in \langle N_S \rangle^*_{M_S}$ over S . (See (1.2.3) for notation.) In particular, the conclusion is valid whenever either R or S is essentially of finite type over an excellent local ring. Note that when $M = R$ and $N = I$ is an ideal, this implies that $I^*S \subseteq (IS)^*$.*

Proof. If R is essentially of finite type over an excellent local ring the result follows from part (i) of Theorem (6.24) of [HH9] together with Proposition (6.23) of [HH9]. If S has a completely stable weak test element the result follows from part (ii) of Theorem (6.24) of [HH9] coupled with the last part of the remarks (6.26) of [HH9]. \square

(1.4.14) Theorem. *Let $R \rightarrow S$ be a homomorphism of Noetherian rings of characteristic p . Suppose that at least one of the following four conditions holds:*

- (1) R° maps into S° .
- (2) R° has a weak test element that maps into S° .
- (3) R is essentially of finite type over an excellent local ring.
- (4) S has a completely stable test element.

Suppose that M is a finitely generated R -module and W is a tightly closed submodule of $S \otimes_R M$. Let $N = \{u \in M : 1 \otimes u \in W\}$. Then N is tightly closed in M over R . In particular, if J is a tightly closed ideal of S , the contraction of J to R is tightly closed.

Proof. Suppose that $v \in N^*$. Each of the four conditions implies that when we pass to $S \otimes_R M$, the image u of v is in W^* (this is immediate from the definitions in the cases of conditions (1) and (2), and a consequence of (1.4.13) in cases (3) and (4)). Thus, $u \in W$ and so $v \in N$. \square

(1.4.15) Proposition. *Suppose that S is faithfully flat over R and that, moreover, the hypothesis of (1.4.14) holds. If c is an element of R such that $c \in \tau(S)$ (respectively, $\widehat{\tau}(S)$), then $c \in \tau(R)$ (respectively, $\widehat{\tau}(R)$). In particular, if $c \in S^\circ$ is a test element (respectively, a completely stable test element) for S , then it is a test element for R (respectively, a completely stable test element for R). See (1.4.5d,e) for notation.*

Proof. The flatness implies that $S^\circ \cap R \subseteq R^\circ$, and so the statement about test elements follows from the statements about the behavior of τ and $\widehat{\tau}$. First suppose that c is in $\tau(S)$. By persistence of tight closure, $u \in N_M^*$, where $N \subseteq M$ are finitely generated R -modules implies the same after one tensors with S , and it follows that cu is in the image of N_S (in M_S). Since S is faithfully flat over R , the result follows, since $N_S \cap M = N$.

Now suppose $c \in \widehat{\tau}(S)$, and let P be any prime of R . Then there is a prime Q of S lying over P . The map $R_P \rightarrow S_Q$ is faithfully flat, and so is the induced map of completions, from which the desired result follows. \square

The following result gives one case in which characteristic p tight closure always commutes with localization:

(1.4.16) Proposition. *Let R be a Noetherian ring of characteristic p and let $N \subseteq M$ be finitely generated R -modules.*

- (a) *Let W be a nonempty multiplicative system of R . Then $W^{-1}(N_M^*) \cong (W^{-1}N)_{W^{-1}M}^*$ (over $W^{-1}R$) provided that this holds for every multiplicative system of the form $R - P$, where P is a prime ideal of R disjoint from W . Hence, tight closure commutes with localization at an arbitrary multiplicative system for a given pair of finitely generated*

modules $N \subseteq M$ if and only if it commutes with localization at prime ideals.

- (b) Tight closure commutes with localization at W for the pair $N \subseteq M$ if and only if it commutes with localization at W for the pair $0 \subseteq M/N$.
- (c) If M/N has finite length, then tight closure commutes with localization in the sense of (a).

Proof. Part (a) is given by Lemma (3.5a) on p. 79 of [AHH], and part (b) is contained in (3.2) on p. 77 of [AHH]. For (c), note that by (b) we may assume that M has finite length and $N = 0$. by part (l) of (1.4.4) we may assume that M is killed by a power of a maximal ideal m (and $N = 0$). By part (a) we may assume that we are localizing at a prime ideal. If this prime ideal P is different from m , then both objects considered are 0, while the case where $P = m$ is Proposition (8.9) on p. 76 of [HH4]. \square

The discussion that follows fills a gap in the proof of Proposition (8.18b) on p. 81 of [HH4].

(1.4.17) Tight closures of submodules of projective modules over reduced rings:

a corrected proof. We begin re-examine the proof of (8.18b) in [HH4]. It contains an error that we correct here, although the statement is correct. The mistake is in the second line of the argument, where a reduction is made to the local case by choosing a maximal ideal m of R containing $N_G :_R x$ and passing to the localizations at m . This may not guarantee that the image of x remains outside $(N_m)^*_{G_m}$, however, since tight closure is not known to commute with localization at a maximal ideal except in special cases. If R has a locally stable q' -weak test element c , one can correct this line of argument as follows: since $x \notin N^*_G$, we can choose $q \geq q'$ such that $cx^q \notin N^{[q]}$, and we can preserve this condition after localizing at a maximal ideal m by choosing m to contain $N^{[q]} :_R x^q$. Since $c/1$ is also a q' -weak test element in R_m , we have that $x/1$ is not in $(N_m)^*_{G_m}$, as required.

However, we can give a different proof of (8.18b) of [HH4] without any additional hypothesis. First choose a finitely generated projective module G' such that $G \oplus_R G'$ is free. Then $N \subseteq G \oplus_R G' \subseteq F \oplus_R G'$ will also give a counterexample: since $N = N \oplus_R 0$ has a compatible direct sum decomposition, its tight closure in $G \oplus G'$ is $N^*_G \oplus_R 0$ while its tight closure in $F \oplus G'$ is $N^*_F \oplus_R 0$. Thus, we may assume without loss of generality that G is

free. The remainder of the argument given in the proof of (8.18b) in [HH4] is then valid without any changes in wording: the fact that R is local is not used. (Localization was only used to reach the case where G is free.) One replaces F by a quotient by a submodule maximal with respect to the property of not meeting G , so that $G \rightarrow F$ is essential. This implies that F is torsion-free and can be embedded in a free module having the same rank as G . Thus, we may assume that G, F are free of the same rank, say that both are R^h , and that the map between them is given by a size h square matrix α whose determinant D is not a zerodivisor in R . The rest of the calculation is word for word the same as what is given in the proof of (8.18b) in [HH4]. \square

(1.5) SOME NEW RESULTS ON TEST ELEMENTS IN POSITIVE CHARACTERISTIC

In this section we record some results on the existence of test elements in characteristic p that are not in the literature, although several of them can be proved by small modifications of existing arguments. Most of these results will be needed to develop the theory of test elements in the equal characteristic 0 case. Cf. (2.4).

The first result is a refinement of part of (1.4.10) stated earlier here, and also of part of Theorem (6.9) of [HH4]. The point is that in earlier versions of this result the element c is assumed to be in A or even A° , and this is not needed: the conclusion holds when c is simply an element of R . While in some sense this is a minor point, it was missed by the authors earlier, and it turns out to be very useful to have the stronger result available. Cf. Exercise 2.5 of [Hu5].

(1.5.1) Theorem. *Let A be regular Noetherian ring of characteristic p and let R be a module-finite extension of A that is torsion-free and generically smooth¹⁶ over A . Suppose that c is an element of R such that $cR^\infty \subseteq A^\infty[R]$. Then c is in the test ideal for R , and remains so after localization and completion. Thus, if $c \in R^\circ$ as well, then c is a completely stable test element for R .*

Proof. First note that A is a product of domains and there is a corresponding decomposition for R . We may therefore reduce to the case where A is a regular domain. To see that

¹⁶See (1.4.8)

c is in the test ideal for R we need to see that if $N \subseteq M$ are finitely generated modules and $u \in N^*$ then $cu \in N$ (cf. (1.4.5) (d)). The argument is exactly the same as in the proof that (d) \Rightarrow (e) in (6.9) or (8.14) of [HH4]. Since the details are written out explicitly only for (6.9) we refer to that proof. In the second line of that proof the subscript $_A$ in the expression “ $c \notin I^{[q]}:_A x^q$ ” should be changed to a subscript $_R$. Nothing else needs to be changed in that argument. Neither the fact that $c \in A$ nor the fact that $c \in A^\circ$ is used anywhere.

To see that if c is in R° then it is a completely stable test element let Q be a maximal ideal of R lying over the maximal ideal m of A , and let B be the completion of A_m . Our hypotheses are preserved when we replace A, R with $B, B \otimes_A R$. (Proceed in two steps: first localize at m and then complete. It is easy to verify that localization preserves the hypothesis. The least obvious point is that c still multiplies R^∞ into $A^\infty[R]$ after completion. To see this note that for each choice of q , since R is module-finite over A , $R^{1/q}$ is module-finite over $A^{1/q}$, and, hence, $R^{1/q}$ is module-finite over $A^{1/q}[R]$. For $q' \geq q$ sufficiently large, the product of c with each of the finitely many module generators of $R^{1/q}$ over $A^{1/q}[R]$ will lie in $A^{1/q'}[R]$, and the condition that c multiply $R^{1/q}$ into $A^{1/q'}[R]$ is preserved when we complete with respect to the maximal ideal of A (this gives the same topology on $A^{1/q}$ as the maximal ideal of $A^{1/q}$ does.) Because B is flat over A , the image of c in $S = B \otimes_A R$ is not in any minimal prime. The ring S is a finite product of local rings, one of which is the completion of R_Q , and the result follows. \square

We shall use this result to produce test elements in a number of ways: one is to combine it with a theorem of Lipman and Sathaye, (1.5.3) below.

(1.5.2) Discussion. In this discussion we do not make any assumption on the characteristic. Let $T \subseteq R$ be a module-finite extension, where T is a Noetherian domain, R is torsion-free as a T -module and the extension is generically smooth. Thus, if \mathcal{K} is the fraction field of T and $\mathcal{L} = \mathcal{K} \otimes_T R$ is the total quotient ring of R then $\mathcal{K} \rightarrow \mathcal{L}$ is a finite product of separable field extensions of \mathcal{K} . The *Jacobian ideal* $\mathcal{J}(R/T)$ is defined as the 0th Fitting ideal of the R -module of Kähler R -differentials $\Omega_{R/T}$, and may be calculated as follows: Write $R \cong T[X_1, \dots, X_n]/P$ and then $\mathcal{J}(R/T)$ is the ideal generated in R by the images of all the Jacobian determinants $\partial(g_1, \dots, g_n)/\partial(X_1, \dots, X_n)$ for n -tuples

g_1, \dots, g_n of elements of P . Moreover, to generate $\mathcal{J}(R/T)$ it suffices to take all the n -tuples of g_i from a fixed set of generators of P .

Now suppose in addition that T is regular. Let R' be the integral closure of R in \mathcal{L} , which is well known to be module-finite over T (the usual way to argue is that any discriminant multiplies it into a finitely generated free T -module: cf. (2.4.5), part (g)). Let $J = \mathcal{J}(R/T)$ and $J' = \mathcal{J}(R'/T)$. The result of Lipman and Sathaye ([LS], Theorem 2, p. 200) may be stated as follows:

(1.5.3) Theorem (Lipman-Sathaye). *With notation as above (in particular, there is no assumption about the characteristic, and T is regular), suppose also that R is an integral domain. If $u \in \mathcal{L}$ is such that $uJ' \subseteq R'$ then $uJR' \subseteq R$. In particular, we may take $u = 1$, and so $JR' \subseteq R$. \square*

This property of “capturing the normalization” will enable us to produce test elements here and universal test elements in (2.4).

(1.5.4) Corollary (existence of test elements via the Lipman-Sathaye theorem). *Let R be a domain module-finite and generically smooth over the regular domain A of characteristic p . Then every element c of $J = \mathcal{J}(R/A)$ is such that $cR^{1/q} \subseteq A^{1/q}[R]$ for all q , and, in particular, $cR^\infty \subseteq A^\infty[R]$. Thus, if $c \in J \cap R^\circ$, it is a completely stable test element.*

Proof. Since $A^{1/q}[R] \cong A^{1/q} \otimes_A R$, the image of c is in $\mathcal{J}(A^{1/q}[R]/A^{1/q})$, and so the Lipman-Sathaye theorem implies that c multiplies the normalization S of $A^{1/q}[R]$ into $A^{1/q}[R]$. Thus, it suffices to see that $R^{1/q}$ is contained in S . Since it is clearly integral over $A^{1/q}[R]$ (it is obviously integral over R), we need only see that the elements of $R^{1/q}$ are in the total quotient ring of $A^{1/q}[R]$, and for this purpose we may localize at A° . Thus, we may replace A by its fraction field and assume that A is a field, and then R is replaced by $(A^\circ)^{-1}R$, which is a separable field extensions. Thus, we come down to the fact that if $A \subseteq R$ is a finite separable field extension, then the injection $A^{1/q} \otimes_A R \rightarrow R^{1/q}$ (the map is an injection because separable and purely inseparable field extensions are linearly disjoint) is an isomorphism, which is immediate by a degree argument. \square

(1.5.5) Corollary (more test elements via the Lipman-Sathaye theorem). *Let K be a field of characteristic p and let R be a d -dimensional geometrically reduced¹⁷ domain over K that is finitely generated as a K -algebra. Let $R = K[x_1, \dots, x_n]/(g_1, \dots, g_r)$ be a presentation of R as a homomorphic image of a polynomial ring. Then the $(n-d) \times (n-d)$ minors of the Jacobian matrix $(\partial g_i / \partial x_j)$ are contained in the test ideal of R , and remain so after localization and completion. Thus, any element of the Jacobian ideal generated by all these minors that is in R° is a completely stable test element.*

Proof. We pass to $K(t) \otimes_K R$, if necessary, where $K(t)$ is a simple transcendental extension of K , to guarantee that the field is infinite. Our hypothesis remains the same, the Jacobian matrix does not change, and by (1.4.15), since $K(t) \otimes_K R$ is faithfully flat over R , it suffices to consider the latter ring. Thus, we may assume without loss of generality that K is infinite. The calculation of the Jacobian ideal is independent of the choice of indeterminates. We are therefore free to make a linear change of coordinates, which corresponds to choosing an element of $G = GL(n, K) \subseteq K^{n^2}$ to act on the one-forms of $K[x_1, \dots, x_n]$. For a dense Zariski open set U of $G \subseteq K^{n^2}$, if we make a change of coordinates corresponding to an element $\gamma \in U \subseteq G$ then, for every choice of d of the (new) indeterminates, if A denotes the K -subalgebra of R that these d new indeterminates generate, the two conditions listed below will hold:

- (1) R will be module-finite over A (and the d chosen indeterminates will then, per force, be algebraically independent) and
- (2) R will be generically smooth over A .

We may consider these two statements separately, for if each holds for a dense Zariski open subset of G we may intersect the two subsets. The first statement follows from the standard “linear change of variable” proofs of the Noether normalization theorem for affine K -algebras (these may be used whenever the ring contains an infinite field). For the second, we want each d element subset, say, after renumbering, x_1, \dots, x_d , of the variables to be a separating transcendence basis for the fraction field L of R over K . (The fact that R is geometrically reduced over K implies that L is separably generated over K .)

¹⁷See (2.3.1).

By, for example, either Theorem 5.10 (d) of [Ku3] or Proposition 5.4 of [Swan] a necessary and sufficient condition for x_1, \dots, x_d to be a separating transcendence basis is that the differentials of these elements dx_1, \dots, dx_d in $\Omega_{L/K} \cong L^d$ be a basis for $\Omega_{L/K}$ as an L -vector space. Since the differentials of the original variables span $\Omega_{L/K}$ over L , it is clear that the set of elements of G for which all d element subsets of the new variables have differentials that span $\Omega_{L/K}$ contains a Zariski dense open set.

Now suppose that a suitable change of coordinates has been made, and, as above, let A be the ring generated over K by some set of d of the elements x_i . Then the $n - d$ size minors of $(\partial g_i / \partial x_j)$ involving the $n - d$ columns of $(\partial g_i / \partial x_j)$ that correspond to variables not chosen as generators of A precisely generate $\mathcal{J}(R/A)$. R is module-finite over A by the general position argument, and since it is equidimensional and reduced, it is likewise torsion-free over A , which is a regular domain. It is generically smooth likewise, because of the general position of the variables. The result is now immediate from (1.5.4): as we vary the set of d variables, every $n - d$ size minor occurs as a generator of some $\mathcal{J}(R/A)$ \square

We record below some further facts about test elements in characteristic p .

(1.5.6) Discussion. We next prove a lemma in characteristic p that is a variation on Theorem (3.4) of [HH3] and Theorem (5.10) of [HH9] and that uses one of the ideas of [Ab2]. Recall that a Noetherian ring R of characteristic p is called *F-finite* if the Frobenius endomorphism $F: R \rightarrow R$ is such that R is module-finite over $F(R)$. Recall also that a reduced F -finite ring R is *strongly F-regular* if for every $d \in R^\circ$, there exists q such that (equivalently, for all sufficiently large q) the inclusion of the cyclic R -module $Rd^{1/q} \hookrightarrow R^{1/q}$ splits as a map of R -modules. We refer the reader to §3 of [HH3] and §5 of [HH9] for more detail. Strongly F -regular rings retain the property under localization, and strongly F -regular rings are F -regular. A Gorenstein F -finite ring is strongly F -regular if and only if it is weakly F -regular.

If $N \subseteq M$ the *Frobenius closure* N^F_M of N in M consists of all elements $u \in M$ such that $u^q \in N^{[q]}_M$ for some q . We define the *test ideal for Frobenius closure* to consist of all elements $d \in R$ such that d kills N^F_M/N for all pairs $N \subseteq M$ of finitely generated R -modules. Notice that if for every q there is an R -linear map $R^{1/q} \rightarrow R$ sending 1 to

d (or if there is an R -linear map $R^\infty \rightarrow R$ sending 1 to d) then d is in the test ideal for Frobenius closure.

(1.5.7) Theorem. *Let R be a reduced locally excellent Noetherian ring of characteristic p . Let \mathfrak{A} be an ideal of R such that for all $c \in \mathfrak{A}$ either (i) R_c is F -regular and Gorenstein or (ii) R is F -finite and R_c is strongly F -regular. Let \mathfrak{B} be an ideal of R contained in the test ideal for Frobenius closure. Then $\mathfrak{A}\mathfrak{B} \subseteq \hat{\tau}(R)$. (Cf. (1.4.5e) for the definition of $\hat{\tau}(R)$.) Hence, if \mathfrak{A} and \mathfrak{B} meet R° , then R has a completely stable weak test element.*

Proof. We must show that if $c \in \mathfrak{A}$ and $d \in \mathfrak{B}$ then $cd \in \hat{\tau}(R)$. All hypotheses are preserved by replacing R by its localization at a prime, and so we might as well assume that (R, m) is local. Since R is excellent local and reduced, \hat{R} is reduced, and for every q we have that $\hat{R} \otimes_R R^{1/q} \cong (\hat{R})^{1/q}$. (To see this, note that $\hat{R} \otimes_R R^{1/q} \cong (R^{1/q})^\wedge$; thus we have a map

$$(R^{1/q})^\wedge \cong \hat{R} \otimes_R R^{1/q} \rightarrow (\hat{R})^{1/q}.$$

The q th power of any element in $\hat{R} \otimes_R R^{1/q}$ is evidently in the image of \hat{R} . Since \hat{R} is reduced, so is $(R^{1/q})^\wedge$ (as a ring, it is isomorphic with \hat{R}), and we find that we have an injection $(R^{1/q})^\wedge \hookrightarrow (\hat{R})^{1/q}$. Both are module-finite over \hat{R} , and so $A = (R^{1/q})^\wedge \rightarrow (\hat{R})^{1/q} = B$ is module-finite. If x_1, \dots, x_h generate the maximal ideal of R then $x_1^{1/q}, \dots, x_h^{1/q}$ generate the maximal ideal in both A and B . The surjectivity now follows from the fact the induced map of residue fields is an isomorphism and Nakayama's lemma applied over A .) This yields that $\hat{R} \otimes_R R^\infty \cong \hat{R}^\infty$, and so applying $\hat{R} \otimes_R _$ to an R -linear map $R^\infty \rightarrow R$ whose value on 1 is d yields an \hat{R} -linear map $\hat{R}^\infty \rightarrow \hat{R}$ whose value on 1 is d . Moreover, \hat{R}_c is F -regular and Gorenstein if R_c is: the Gorenstein property follows because $R \rightarrow \hat{R}$ is flat with Gorenstein (in fact, regular) fibers, while the F -regularity follows from Theorem (7.25c) of [HH9]. Finally, $(\hat{R})_c$ is strongly F -regular if R_c is. (To see this, first pick $\gamma \in R^\circ$ so that R_γ is regular, and then replace γ by a power so that it is a completely stable test element for both R and \hat{R} (cf. Theorem (6.21) of [HH9]). Choose q such that $R\gamma^{1/q} \rightarrow R^{1/q}$ splits. Then $\hat{R}\gamma^{1/q} \rightarrow (\hat{R})^{1/q}$ splits (this is just the result of applying $\hat{R} \otimes_R _$). But then \hat{R} is strongly F -regular by Theorem (5.9a) of [HH9].

We have thus reduced to the case where R is complete local reduced and either (i) R_c

is Gorenstein and F -regular or (ii) R is F -finite and R_c is strongly F -regular, and d is in the test ideal for Frobenius closure. What we need to show is that $cd \in \tau(R)$.

First consider case (i). Fix a coefficient field of R , a p -base for the coefficient field, and consider the rings R^Γ as defined in the first paragraph of (6.11) of [HH9], where Γ is a cofinite subset of the p -base. By Lemma (6.13) of [HH9] for Γ sufficiently small, R^Γ is reduced. By Lemma (6.19) of [HH9], for Γ sufficiently small $(R^\Gamma)_c$ is F -regular and Gorenstein (the result is stated for $c \in R^\circ$, but the proof makes no use whatsoever of the condition that $c \in R^\circ$). But R^Γ is F -finite, purely inseparable over R , and faithfully flat over R by the second paragraph of (6.11) of [HH9]. Since $(R^\Gamma)_c$ is F -finite, Gorenstein and weakly F -regular, it follows that $(R^\Gamma)_c$ is strongly F -regular.

Thus, in both case (i) and case (ii) we may assume that R has a reduced local faithfully flat purely inseparable F -finite extension algebra R' (R' is R^Γ in case (i) and is R in case (ii)) such that R'_c is strongly F -regular.

It will suffice to show that if $M = R^t$ is a finitely generated free R -module, $N \subseteq M$ is a submodule and $u \in N^*$ then $cdu \in N$. Since $u \in N^*$ there is an element $f \in R^\circ$ such that $fu^q \in N^{[q]}$ for all q . Since R'_c is strongly F -regular the map $(R'_c)f^{1/q} \rightarrow (R'_c)^{1/q} \cong ((R')^{1/q})_c$ splits for some q . This yields an R'_c -linear mapping of $((R')^{1/q})_c \rightarrow R'_c$ sending $f^{1/q}$ to 1, and hence an R' -linear mapping ψ of $R'^{1/q} \rightarrow R'_c$ sending $f^{1/q}$ to 1. Since $R'^{1/q}$ is module-finite over R' we may multiply by a power of c to get an R' -linear map $R'^{1/q} \rightarrow R'$ sending $f^{1/q}$ to a power of c , say c^Q , and by increasing Q if necessary, we may assume that Q is a power of p . Now, $fu^{qQ} \in N^{[qQ]}$ and taking q th roots yields that $f^{1/q}u^Q \in N^{[Q]}R^{1/q}$ (i.e., the image of $R^{1/q} \otimes_R N^{[Q]} \rightarrow R^{1/q} \otimes_R \mathbf{F}^e(M)$). Since $R^{1/q} \subseteq R'^{1/q}$ we may now apply, componentwise, the R' -linear map $\psi : R'^{1/q} \rightarrow R'$ sending $f^{1/q}$ to c^Q to obtain that $c^Qu^Q \in N^{[Q]}R'$. Taking Q th roots again yields that $cu \in NR'^{1/Q} \subseteq NR^\infty$, and it follows that $cu \in NR^{1/q'}$ for some sufficiently large choice of q' , since only finitely many elements from R^∞ will be needed on the right. Then $(cu)^{q'} \in N^{[q']}$ and so $cu \in N^F$ and this yields that $cdu \in N$, as required. \square

If one knew that every weakly F -regular F -finite ring is strongly F -regular, then one could replace the conditions (i) and (ii) on c in the preceding theorem by the single weaker condition that R_c be weakly F -regular. This is an open question. Partial results, under

mild hypotheses, are obtained in [Wi1] (the case where the ring has dimension at most 3), [MacC] (the case where the ring has isolated non- \mathbb{Q} -Gorenstein points (where a Cohen-Macaulay ring is \mathbb{Q} -Gorenstein if the canonical module represents a torsion element of the divisor class group) and [LySm] (the case of finitely generated \mathbb{N} -graded algebras over a field).

(1.6) F -REGULARITY IN POSITIVE CHARACTERISTIC

(1.6.1) Definition. A Noetherian ring of characteristic p is called *weakly F -regular* if every ideal is tightly closed. It is called *F -regular* if its localization with respect to every multiplicative system is weakly F -regular.

(1.6.2) Theorem. *Let R be a Noetherian ring of characteristic p .*

- (a) *If R is regular, then R is F -regular.*
- (b) *If R is weakly F -regular then every submodule of every finitely generated module is tightly closed.*
- (c) *R is weakly F -regular if and only if its localization at every maximal ideal is weakly F -regular. R is F -regular if and only if its localization at every prime ideal is weakly F -regular.*
- (d) *If R is weakly F -regular then R is normal.*
- (e) *If R is either a homomorphic image of a Cohen-Macaulay ring or if R is locally excellent, and R is weakly F -regular, then R is Cohen-Macaulay.*
- (f) *If R is Gorenstein, then R is weakly F -regular if and only if R is F -regular.*
- (g) *A Gorenstein local ring is F -regular if and only if the ideal generated by one system of parameters is tightly closed.*

Proof. Parts (a), (b), (c) (first statement) and (d) are, respectively, Theorem (4.6), Proposition (8.7), Corollary (4.15), and Corollary (5.1), all from [HH4]. The second statement in (c) is immediate from the first. Part (e) follows from Theorem (3.4c) and Theorem (6.27b) of [HH9], while parts (f) and (g) follow from Theorem (4.2) of [HH9], parts (g), (f) and (d). \square

(1.6.3) Definition. If I is an ideal of a ring R , an element x of R is called *integral over* I if there exists a positive integer k and an equation

$$x^k + i_1 x^{k-1} + \cdots + i_{k-j} x^j + \cdots + i_{k-1} x + i_k = 0$$

where $i_j \in I^j$ for $1 \leq j \leq k$. The set of elements integral over I is an ideal, denoted I^- or \bar{I} , called the *integral closure of I* . I^- may also be characterized as follows: $(I^-)t$ is the degree one part of the integral closure of the *Rees ring* $R[It]$ in the polynomial ring in one variable, $R[t]$. If R is Noetherian, then $u \in I^-$ if and only if for every homomorphism $h: R \rightarrow V$, where V is a discrete valuation ring (equivalently, every such homomorphism whose kernel is a minimal prime of R), $h(x) \in IV$. Moreover, if S is an integral extension of the ring R and I is an ideal of R then $IS \cap R \subseteq I^-$. We refer the reader to [L] and [HH4], (5.1) for more background on integrally closed ideals.

The proof of (1.6.2d) depends on the following result, which is of considerable interest in its own right:

(1.6.4) Theorem. *Let R be a Noetherian ring of characteristic p .*

- (a) *If I is any ideal of R , then $I^* \subseteq I^-$. In particular, every integrally closed ideal and, hence, every radical ideal is tightly closed.*
- (b) *If I is a principal ideal then $I^- = I^*$.*

Proof. Part (a) is Theorem (5.2) of [HH4]. Part (b) in the case where the principal ideal is generated by an element of R° is Corollary (5.8) of [HH4]. The general case of (b) then follows from the fact that we may test modulo each minimal prime, and so we may assume that R is a domain. The result then follows from the case already discussed when $I \neq (0)$, while it is trivial if $I = (0)$. \square

(1.7) FURTHER TIGHT CLOSURE THEORY IN POSITIVE CHARACTERISTIC

We next observe:

(1.7.1) Theorem (generalized Briançon-Skoda theorem). *Let R be a Noetherian ring of characteristic p and I be an ideal of R generated by n elements. Then for every integer $k \in \mathbb{N}$,*

$$(I^{n+k})^- \subseteq (I^{k+1})^*.$$

Hence, if R is weakly F -regular (in particular, if R is regular), then

$$(I^{n+k})^- \subseteq I^{k+1}.$$

Proof. The second statement is obvious from the first. It suffices to prove the result modulo each minimal prime of R , so that we may assume that R is a domain. If $I = (0)$ the result is obvious, while otherwise we may apply Theorem (5.4) of [HH4]. \square

(1.7.2) Remarks. This is a somewhat improved version of Theorem (5.4) of [HH4]. Note that when $n = 1$ and $k = 0$ it may be used to prove Theorem (1.6.4b).

(1.7.3) Theorem. *Let R be a Noetherian ring of characteristic p and let S be an extension ring of R . Let N, M be finitely generated R -modules.*

- (a) *If S is module-finite over R then the inverse image of $\langle N_S \rangle_{M_S}^*$ (over S) in M is contained in N^*_M (over R).*
- (b) *If R has a completely stable weak test element and S is faithfully flat over R then the inverse image of $\langle N_S \rangle_{M_S}^*$ (over S) in M is contained in N^*_M (over R).*

Proof. Part (a) is Corollary (5.23) of [HH10]. To prove (b), suppose that $u \in M$ is such that its image in M_S is in $\langle N_S \rangle^*$ but that $u \notin N^*$. Since R has a completely stable weak test element we can choose a maximal ideal m of R such that the image of u in M_B is not in $\langle N_B \rangle^*$, where $B = (R_m)^\wedge$. We can choose a prime ideal Q of S lying over m , and it

follows that we have a counterexample with $R \rightarrow S$ replaced by $B \rightarrow (S_Q)^\wedge$. Thus, there is no loss of generality in assuming that $R \rightarrow S$ is a flat local homomorphism of complete local rings. Notice that if the original map $R \rightarrow S$ was étale, we are done by part (a), since the map of completions will be a module-finite extension.

In the general case we may use Theorem (5.9a) [Ho8], since tight closure and solid closure agree in this case by Theorem (8.6b) of [Ho8]. \square

(1.7.4) Theorem. *Let R be a universally catenary Noetherian ring of characteristic p and suppose that R has a completely stable weak test element. (Both conditions hold, for example, if R is essentially of finite type over an excellent local ring or if R is F -finite (module-finite over $F(R)$).)*

Let $n \geq 1$ be an integer and let x_1, \dots, x_n be elements of R such that for every minimal prime \mathfrak{p} of R and every prime ideal P containing $\mathfrak{p} + (x_1, \dots, x_n)R$, the images of the x 's form part of a system of parameters in $(R/\mathfrak{p})_P$. (If R is locally equidimensional it suffices, by (2.3.11d), that for every prime ideal P containing (x_1, \dots, x_n) , the images of the x 's in R_P form part of a system of parameters.

Then $(x_1, \dots, x_{n-1})^ :_S x_n = (x_1, \dots, x_{n-1})^*$, i.e., x_n is not a zerodivisor on the ideal $(x_1, \dots, x_{n-1})^*$. Hence, $(x_1, \dots, x_{n-1}) :_R x_n \subseteq (x_1, \dots, x_{n-1})^*$.*

Moreover, under the same hypotheses, if a_1, \dots, a_n and b_1, \dots, b_n are non-negative integers and $a \ominus b$ denotes $\max\{a - b, 0\}$, then

$$(x_1^{a_1}, \dots, x_n^{a_n})^* :_R x_1^{b_1} \cdots x_n^{b_n} = (x_1^{a_1 \ominus b_1}, \dots, x_n^{a_n \ominus b_n})^*.$$

Proof. Suppose that $ux_n \in (x_1, \dots, x_{n-1})^*$ but that $u \notin (x_1, \dots, x_{n-1})^*$. Let c be a completely stable q_0 -weak test element. Then there exists $q \geq q_0$ such that $cu^q \notin (x_1^q, \dots, x_{n-1}^q)$. This will be preserved after localization and completion at a suitable maximal ideal of R , and the parameter condition is also preserved (cf. (2.3.11b,d)). Since the image of c is still a q_0 -weak test element, it follows that we still have $u \notin (x_1, \dots, x_{n-1})^*$. Thus, there is no loss of generality in supposing that R is a complete local ring. Since tight closure may be tested modulo every minimal prime and since the hypothesis continues to hold after killing a minimal prime, there is no loss of generality in assuming that R is a

complete local domain. In this case R has a test element d . If $ux_n \in (x_1, \dots, x_{n-1})^*$ then $d(ux_n)^q \in (x_1^q, \dots, x_{n-1}^q)$ for all q , and then $du^q \in (x_1^q, \dots, x_{n-1}^q) :_R x_n^q$ for all q . By Theorem (7.15) of [HH4], this is contained in $(x_1^q, \dots, x_{n-1}^q)^*$, and since d is a test element we find that $d(du^q) = (d^2)u^q \in (x_1^q, \dots, x_{n-1}^q)$ for all q , which implies that $u \in (x_1, \dots, x_{n-1})^*$, as required.

The statement in the second paragraph is immediate from the statement in the first paragraph. It remains only to prove the final statement.

The left hand side is tightly closed by part (k) of Theorem (1.4.4), and the fact that it contains the right hand side is then immediate from the observation that $b_i + (a_i \ominus b_i) \geq a_i$ for all i . Thus, it suffices to show that if

$$x_1^{b_1} \cdots x_n^{b_n} u \in (x_1^{a_1}, \dots, x_n^{a_n})^*$$

then $u \in (x_1^{a_1 \ominus b_1}, \dots, x_n^{a_n \ominus b_n})^*$. Exactly as in the earlier part of the proof we may reduce to the case where R is a complete local domain, and so has a test element d . We then have that

$$x_1^{qb_1} \cdots x_n^{qb_n} u^q \in (x_1^{qa_1}, \dots, x_n^{qa_n})$$

for all q , from which we have that for all q

$$du^q \in (x_1^{qa_1}, \dots, x_n^{qa_n}) :_R x_1^{qb_1} \cdots x_n^{qb_n}$$

which is contained in $(x_1^{qa_1 \ominus qb_1}, \dots, x_n^{qa_n \ominus qb_n})^*$ by Theorem (7.15) of [HH4]. Note that $qa \ominus qb = q(a \ominus b)$ when $q \geq 0$. This yields

$$d^2 u^q \in (x_1^{a_1 \ominus b_1}, \dots, x_n^{a_n \ominus b_n})^{[q]}$$

for all q , and so $u \in (x_1^{a_1 \ominus b_1}, \dots, x_n^{a_n \ominus b_n})^*$, as required. \square

(1.7.5) Theorem. *Let $R \rightarrow S$ be a homomorphism of Noetherian rings of positive characteristic p and suppose that*

(#) for every maximal ideal m of R and minimal prime ideal \mathfrak{p} of $(R_m)^\wedge$ there exists a prime ideal Q of S lying over m and a prime ideal \mathfrak{q} of $(S_Q)^\wedge$ lying over \mathfrak{p} such that $\text{ht } m(S_Q)^\wedge / \mathfrak{q} \geq \dim(R_m)^\wedge / \mathfrak{p}$.

(Condition (#) holds, in particular, if S is module-finite over R or if S is faithfully flat over R .)

Suppose also that R has a completely stable weak test element.

Let $N \subseteq M$ be finitely generated R -modules and let $u \in M$. If $u_S \in \langle N_S \rangle_{M_S}^*$ then $u \in N_M^*$ over R . (Of course, the converse is also true under mild hypotheses, by the persistence of tight closure, Theorem (1.4.13).)

Proof. This is Corollary (8.8) of [Ho9], except for the parenthetical comment, which is discussed in the remarks below. \square

(1.7.6) Remarks. The condition (#) has a variant in which it is imposed for every prime ideal m of R , not just every maximal ideal. Both conditions make sense whenever $R \rightarrow S$ is a homomorphism of Noetherian rings, not just in the characteristic p case, and the remarks that follow apply to both conditions without any restrictions on the characteristic.

Note that whenever the specified inequality $\text{ht } m(S_Q)^\wedge/\mathfrak{q} \geq \dim(R_m)^\wedge/\mathfrak{p}$ holds, it is actually an equality, for if $\dim(R_m)^\wedge/\mathfrak{p} = n$, $m(R_m)^\wedge/\mathfrak{p}$ is the radical of an n generator ideal, and so the height of its expansion cannot exceed n .

To verify that the condition holds when $R \hookrightarrow S$ is module-finite, note that we may first replace $R \rightarrow S$ by $R_m \rightarrow S_m$ and so assume that (R, m) is local. Then, since the completion C of R is R -flat, $C \rightarrow S_C$ is a module-finite extension of C . S_C may be identified with the completion of S at mS : it is the product of the completions of S with respect to the finitely many maximal ideals lying over m . Given any minimal prime \mathfrak{p} of C , there is a prime ideal \mathfrak{q}_0 of S lying over it, and $C/\mathfrak{p} \hookrightarrow S_C/\mathfrak{q}_0$. Thus, S_C/\mathfrak{q}_0 is local, and must be a homomorphic image of one of the factors of S_C , i.e., it can be viewed as $(S_Q)^\wedge/\mathfrak{q}$ for a suitable maximal ideal Q of S lying over m .

If $R \rightarrow S$ is faithfully flat this is preserved when we localize R and S at m and then localize S at any prime Q lying over m , and it is preserved as well when we pass to the map of completions $(R_m)^\wedge \rightarrow (S_Q)^\wedge$ and then to $(R_m)^\wedge \rightarrow (S_Q)^\wedge$. We simplify notation: we assume that R, m is a complete local domain, that x_1, \dots, x_n is a system of parameters for R and that S is local and faithfully flat over R . We want to show that x_1, \dots, x_n is part of system of parameters for S , for then we can preserve this while killing a suitable

prime lying over (0) in R . But this is immediate from [Mat], Theorem 19 (3ii), p. 79, since a flat homomorphism satisfies going-down and a faithfully flat homomorphism induces a surjection $\operatorname{Spec} S \rightarrow \operatorname{Spec} R$.

The next two results record some of the facts about when base change commutes with tight closure in characteristic p . The first shows that for ideals that are, locally, generated by parameters, tight closure commutes with geometrically regular base change under mild conditions. We have not attempted to put this theorem in its most general form: the version stated will suffice for the applications to characteristic zero.

(1.7.7) Theorem. *Let R be a locally excellent Noetherian ring of characteristic p , and suppose that R is locally equidimensional, and possesses a weak test element. Suppose also that R is a homomorphic image of a Cohen-Macaulay ring, or that R is essentially of finite type over an excellent local ring, or that R is F -finite,¹⁸ and let $R \rightarrow S$ be a flat homomorphism with geometrically regular¹⁹ fibers. Let I be an ideal of R such that for every maximal ideal m of R , IR_m is generated by part of a system of parameters. Then $(IS)^* = (I^*)S$.*

Proof. Since R is locally equidimensional, the hypothesis on I implies that the minheight of I is equal to the number of generators of I . The result is now immediate from Corollary (8.5) of [AHH] (corresponding to part (b) of Theorem (8.3) of [AHH]). Note that the word “smooth” is used in [AHH] mean flat with geometrically regular fibers. \square

Second, for future reference we record the following result on geometrically regular base change from [HH9].

(1.7.8) Theorem. *Let $h : R \rightarrow S$ be a flat homomorphism of Noetherian rings of characteristic p with geometrically regular²⁰ fibers, and suppose that R is locally excellent (or that every local ring of R contains a test element for its completion). Let $N \subseteq M$ be finitely generated R -modules.*

- (a) *If N is tightly closed in M and remains so under localization, then $S \otimes_R N$ is tightly closed in $S \otimes_R M$ and remains so under localization.*

¹⁸See (1.5.6). Any of these three conditions implies that R is of *acceptable type* in the sense of [AHH]

¹⁹See (2.3.1) and (4.2.1).

²⁰See (2.3.1) and (4.2.1).

- (b) *If S is projective as an R -module and N is tightly closed in M then $S \otimes_R N$ is tightly closed in $S \otimes_R M$.*

Proof. This is Theorem (7.1) of [HH9]. \square

CHAPTER 2.

TIGHT CLOSURE IN AFFINE ALGEBRAS

The main objective of this chapter is the development of the definition (in the second section) and basic properties (in the fifth section) of tight closure in equal characteristic zero for an affine algebra R over a field K . This is achieved by the method of reduction to positive characteristic. The process involves studying, instead of the homomorphism $h: K \rightarrow R$, a homomorphism $h_A: A \rightarrow R_A$ instead, where A is a finitely generated \mathbb{Z} -subalgebra of K , R_A is a finitely generated A -subalgebra of R , the map $A \rightarrow R_A$ is induced by restricting $K \rightarrow R$, and $K \otimes_A h_A = h$. One then makes definitions in terms of the behavior of the fibers $R_\kappa (= \kappa \otimes_A R_A)$, where $\kappa = A/\mu$ for a maximal ideal μ varying in a Zariski dense open set of $\text{Max Spec } A$, i.e., the “general” closed fibers of $A \rightarrow R_A$.

The first section of this chapter contains a detailed study of the process of descent: there may be several K -algebras involved, modules over them, and maps between the various algebras and modules. As already mentioned, the definition of tight closure is given in the second section, but little is proved about it there. We first need to know a great deal about how the generic fibers (obtained by tensoring with the fraction field \mathfrak{F} of A) of various kinds of objects over A compare with the general closed fibers, and the results needed are established in the third section. The material in the first and third sections is not new, but it is difficult to find a convenient reference for it in the form that we need. The fourth section is devoted to a theory of “universal” test elements which enables us to show that apparently different notions of tight closure agree, and is useful for many technical reasons. As indicated above, in the fifth and final section we use the tools that have been developed to establish the fundamental properties of tight closure for affine K -algebras.

(2.1) DESCENT DATA AND DESCENT

(2.1.1) Discussion. Throughout this section R will denote a finitely generated algebra over a field K of characteristic 0, while $N \subseteq M$ will be finitely generated R -modules

and u an element of M . We want to discuss what it means to give *descent data* for the quintuple (K, R, M, N, u) . We also explore other aspects of descent: roughly speaking, what we mean by “descent” here is replacing K by a finitely generated \mathbb{Z} -subalgebra A , while replacing various finitely generated K -algebras and finitely generated modules over them by corresponding finitely generated algebras over A and modules over those algebras. One wants these “replacement” objects to be free as A -modules. Moreover, all this is to be done in such a way that the original objects are recovered when one applies $K \otimes_A -$. One may also wish to keep track of the behavior of various maps, and other information as well. The base change conventions discussed in (1.2.3) are used extensively in this section.

In the next section we shall explain when u is in the tight closure of N in M in terms of descent data. The reader interested in getting quickly to the definition of tight closure for affine algebras over fields of characteristic zero may read (2.1.2) and then (2.2.1-3), referring back to the further developments in (2.1) only as needed.

(2.1.2) Descent data. By *descent data* for a quintuple (K, R, M, N, u) as in (2.1.1) we mean a quintuple (A, R_A, M_A, N_A, u_A) satisfying the following conditions:

- (1) A is a finitely generated \mathbb{Z} -subalgebra of K . (Thus, R is an A -algebra.)
- (2) R_A is a finitely generated A -subalgebra of R such that the inclusion $R_A \subseteq R$ induces an isomorphism of R_K with R . Moreover, R_A is A -free.
- (3) M_A, N_A are finitely generated A -submodules of M, N respectively such that $N_A \subseteq M_A$, and all of the modules $M_A, N_A, M_A/N_A$ are A -free. Moreover the inclusion $M_A \subseteq M$ induces an isomorphism $M_K \cong M$ as R -modules (M_K becomes an R -module because of the identification of R_K with R).
- (4) The element u of M is in M_A and $u_A = u$.

The condition (3) is slightly redundant, in that the freeness of N_A and M_A/N_A forces the freeness of M_A (as an A -module).

The key point about descent data is that it always exists and is “stable” under enlargement.

(2.1.3) Discussion: the existence of descent data. Consider a quintuple

$$(K, R, M, N, u)$$

as above. We can write R as $K[x_1, \dots, x_n]/(G_1, \dots, G_s)$, where the x_i are indeterminates over K and G_1, \dots, G_s are finitely many polynomials. As a first approximation to giving descent data we can let A be the \mathbb{Z} -subalgebra of K generated by the coefficients of the G_i and we may let $R_A = A[x_1, \dots, x_n]/(G_1, \dots, G_s)$. Since we will have very frequent occasion to use it, we give here explicitly the lemma of generic freeness in the strong form developed in [HR1], Lemma (8.1), p. 146.

(2.1.4) Lemma (generic freeness). *Let A be a Noetherian domain, let R be a finitely generated A -algebra, let S be a finitely generated R -algebra, let W be a finitely generated S -module, let M be a finitely generated R -submodule of W and let N be a finitely generated A -submodule of W . Let $V = W/(M + N)$. Then there exists an element $a \in A - \{0\}$ such that V_a is free over A_a . \square*

This is quite important even in the special case where $R = S$ and M, N are both 0 (this case may be found in [Mat], §22.) In most cases where we make use of generic freeness, the flatness of the module or algebra under consideration would suffice. In fact, the most important use of (2.1.4) is the following:

(2.1.5) Observation. If $N_A \subseteq M_A$ and M_A/N_A is A -flat (no finiteness conditions are needed) then for every A -algebra B , we have an injection $N_B \rightarrow M_B$, and in this situation we identify N_B with its image in M_B . (The point is that $\text{Tor}_A^1(M_A/N_A, B)$ is zero.) \square

(2.1.3) continued. We may localize A at a single element to make R_A free over A . Then R_A injects into R_K and the latter is clearly isomorphic with R . Thus, we may take R_A to be a subring of R . We have then satisfied conditions (1) and (2) in the list we gave in the definition of descent data.

We next note that if B is any finitely generated \mathbb{Z} -subalgebra with $A \subseteq B \subseteq K$ then R_B satisfies all of these conditions. Moreover, we have:

$$(2.1.6) \quad R = R_K = \varinjlim_B R_B,$$

where B runs through the finitely generated \mathbb{Z} -subalgebras of K containing A and the direct limit is actually a directed union. Thus, for B with $A \subseteq B \subseteq K$ we have that $R_B \cong R_A[B] \subseteq R$ and we make this identification.

We have not yet addressed the construction of M_A, N_A , and u_A . We may describe M as the cokernel of a certain matrix (r_{ij}) whose entries may be represented by elements of $K[x]$: say r_{ij} is the image of f_{ij} . For A sufficiently large all of the f_{ij} will lie in $A[x]$. We may then define M_A as the cokernel of the matrix $(\overline{f_{ij}})$ which is the image of (f_{ij}) under the quotient map $A[x] \twoheadrightarrow R_A$. Again, we may localize at one element of $A - \{0\}$ such that M_A becomes A -free, by generic freeness. Then M_K is obviously isomorphic with M , since they have the same presentation, and $M_A \subseteq M_K$ since M_A is A -free. Thus, we may identify M_A with a submodule of M , and do so.

If we pass to R_B, M_B for B with $A \subseteq B \subseteq K$ then all this is preserved for choices of B finitely generated over \mathbb{Z} . Every M_B may be identified with $\text{Im}(M_B \rightarrow M)$, and once this is done we have:

$$(2.1.7) \quad M = M_K = \varinjlim_B M_B,$$

where B runs through the finitely generated \mathbb{Z} -subalgebras of K containing A and the direct limit is actually a directed union.

We have a presentation for both M_A and M as the cokernel of the matrix $(\overline{f_{ij}})$ (in the first case thought of as map of free R_A -modules, and in the second case thought of as a map of free R -modules). We can choose finitely many column vectors with entries in R (these are not necessarily columns of the matrix) whose images in M span N over K . After enlarging A further we can assume that there are polynomials g_{ij} in $A[x]$ such that their images in R_A give the entries of a matrix formed from these column vectors. Concatenating the matrices $(f_{ij}), (g_{ik})$ yields a matrix $(f_{ij}|g_{ik})$ with entries in $A[x]$ whose image in R_A gives a presentation of N . The image of this matrix over R_A gives a presentation of a module $(M/N)_A$ such that there is an obvious surjection $M_A \twoheadrightarrow (M/N)_A$. By localizing at one element of $A - \{0\}$, we may suppose that $(M/N)_A$ is A -free. We may let $N_A = \text{Ker}(M_A \twoheadrightarrow (M/N)_A)$: this is the same as the span in M_A of the images of the columns of the matrix $(\overline{g_{ik}})$. Localizing A once more we may assume that N_A is A -free. We have that $(M/N)_A \cong M_A/N_A$, and all three of M_A, N_A , and M_A/N_A are A -free. For every A -algebra B we have that $0 \rightarrow N_B \rightarrow M_B \rightarrow B \otimes_A (M_A/N_A) \rightarrow 0$ is exact. Applying this with $B = K$, we see that under the identification of M_K with M , N_K is carried to N ,

as we wanted. We also have:

$$(2.1.8) \quad N = N_K = \lim_{\longrightarrow B} N_B,$$

where B runs through the finitely generated \mathbb{Z} -subalgebras of K containing A and the direct limit is actually a directed union.

Evidently, from (2.1.7), if A is chosen sufficiently large the element u will be in M_A , and we may let $u_A = u$.

Clearly, we have constructed descent data: (1)–(4) are satisfied.

From this discussion we have:

(2.1.9) Proposition. *Given a quintuple (K, R, M, N, u) as in the first paragraph of this section there exist descent data (A, R_A, M_A, N_A, u_A) . Moreover, given such descent data the statements (2.1.6), (2.1.7), and (2.1.8) are valid, and the quintuple*

$$(B, R_B, M_B, N_B, u_B)$$

also gives descent data for every finitely generated \mathbb{Z} -subalgebra B of K containing A . \square

(2.1.10) Discussion: more elaborate descent. In many instances one has a field K , a finitely generated K -algebra R , and, instead of an inclusion of finitely generated R -modules $N \subseteq M$ and an element $u \in M$, a much more complicated set of information: a finite family of finitely generated R -modules, finitely many elements of those modules, finitely many maps among those modules, some of which are specified to take certain of the given elements of the modules to other given elements of the appropriate target modules, finitely many commutative diagrams involving those modules and maps, and finitely many exact sequences involving those modules and maps. One wants to “descend” all this. This situation is obviously much more general than the original one.

This means that we want to give a finitely generated \mathbb{Z} -subalgebra A of K , and R_A as before. We sometimes say that $A \rightarrow R_A$ *descends* $K \rightarrow R$ or that R_A *descends* R from K to A . For each module M in the finitely family we want to construct $M_A \subseteq M$, a finitely generated A -free R_A -submodule of M such that $M_K \rightarrow M$ is an isomorphism. Again, we say that M_A *descends* M when this is the case. We want the specified elements in a given

M to be elements of M_A . For each map $\phi: M \rightarrow M'$ in the family we want it to be the case that $\phi(M_A) \subseteq M'_A$, so that the restriction ϕ_A of ϕ to a map $M_A \rightarrow M'_A$ is defined, and such that ϕ_K is identified with ϕ (this will be automatic, since the two will agree on M_A , and M_A spans M over $R = R_K$). We shall also require that the kernel, image, and cokernel of every ϕ_A be A -free (but it is obvious that we may achieve this by localizing at one element). There will be diagrams over R_A corresponding to the specified diagrams over R : we shall want these to commute. Finally, we shall want that the sequences of modules over R_A corresponding to the exact sequences specified over R continue to be exact.

All of this can be done without difficulty. For each module M we can construct M_A from a presentation as in the proof of existence of descent data. Given a map $\phi: M \rightarrow M'$ it lifts to a map of finite presentations. Thus, we have two exact sequences $R^\nu \rightarrow R^\mu \rightarrow M \rightarrow 0$ and $R^{\nu'} \rightarrow R^{\mu'} \rightarrow M' \rightarrow 0$ where the left hand maps have matrices, say, α and α' respectively, and we have $\phi_0: R^\mu \rightarrow R^{\mu'}$ and $\phi_1: R^\nu \rightarrow R^{\nu'}$ (which we also think of as given by matrices) such that the diagram

$$\begin{array}{ccc}
 R^\nu & \xrightarrow{\phi_1} & R^{\nu'} \\
 \alpha \downarrow & & \alpha' \downarrow \\
 R^\mu & \xrightarrow{\phi_0} & R^{\mu'} \\
 \downarrow & & \downarrow \\
 M & \xrightarrow{\phi} & M'
 \end{array}$$

commutes. This says, in particular, that $\alpha'\phi_1 = \phi_0\alpha$ (as composition of maps or as products of matrices: our matrices act on the left). When A is large enough the entries of all these matrices are in R_A . M_A and M'_A are constructed as the cokernels of α, α' , respectively considered as maps of standard free modules over R_A , and the fact that $\alpha'\phi_1 = \phi_0\alpha$ (this obviously still holds) implies that when ϕ_0 is viewed as a map of free R_A -modules, it induces a well-defined map ϕ_A from $M_A \rightarrow M'_A$. By localizing A at one element of $A - \{0\}$ we may assume that both M_A, M'_A are A -free, and then M_A and M'_A inject into $M_K = M$, $M'_K = M'$, respectively. By construction, $K \otimes_A \phi_A$ is ϕ , and it follows that ϕ_A is the restriction of ϕ to M_A, M'_A .

If we enlarge A further, all this is preserved, and we are free to make any number of such enlargements, so long as we keep A a finitely generated \mathbb{Z} -algebra. In this way, we may enlarge A sufficiently that all the modules and maps under consideration descend.

In this situation, we observe the following: suppose that we have, originally, a sequence $M \xrightarrow{\phi} M' \xrightarrow{\psi} M''$ and also $M \xrightarrow{\theta} M''$ where $\theta = \psi\phi$. Suppose that we have arranged descent for the three modules and the three maps, so that we have $\phi_A: M_A \rightarrow M'_A$, $\psi_A: M'_A \rightarrow M''_A$ and $\theta_A: M_A \rightarrow M''_A$. Then $\theta_A = \psi_A\phi_A$ automatically, since each of ϕ_A, ψ_A, θ_A is the restriction of ϕ, ψ , or θ , respectively, and $\theta = \psi\phi$. It follows that the commutativity of the diagrams that held for the original maps is preserved automatically.

As in the discussion of descent data, we have for every one of the modules in the finite family under consideration that

$$(2.1.11) \quad M = M_K = \varinjlim_B M_B,$$

where B runs through the finitely generated \mathbb{Z} -subalgebras of K containing A and the direct limit is actually a directed union.

It is clear that we may choose A sufficiently large so that any given finite set of elements of M is in M_A . Since any descended map may be thought of as a restriction, the new maps will have the same values as before on the finite sets of specified elements.

It remains only to check that exactness can be preserved. Suppose that we have that $M' \xrightarrow{\phi} M \xrightarrow{\psi} M''$ is exact at M and that we have achieved descent. Let \mathfrak{F} be the fraction field of A . Then the sequence $M'_A \xrightarrow{\phi_A} M_A \xrightarrow{\psi_A} M''_A$ becomes exact at M_A when we apply $\mathfrak{F} \otimes_A _$, since the further base change to K is faithfully flat over \mathfrak{F} . (We already know that $\psi_A\phi_A = 0$ by our remarks on composition.) It follows that we can obtain exactness at M_A by localizing at *one* element of $A - \{0\}$. We can treat all of the (finitely many) questions of exactness with which we are concerned by the same method.

As always, all the conditions that we have specified to hold for the descent will continue to hold if we enlarge A further, replacing it by B finitely generated over \mathbb{Z} with $A \subseteq B \subseteq K$.

(2.1.12) Uniqueness. We continue discussion of the set-up of the preceding section. Our choices of A, R_A , and the various modules and maps M_A, ϕ_A , etc. are not unique. But

given two sets of choices, one indexed by $A \subseteq K$ and the other by $B \subseteq K$, they become the same for all sufficiently large finitely generated \mathbb{Z} -algebras C of K with $A \subseteq C$, $B \subseteq C$.

To see this, first suppose that we have $R_A \subseteq R$ and $R'_B \subseteq R$ (free over A , B respectively) such that $R_K \rightarrow R$ and $R'_K \rightarrow R$ are isomorphisms. Fix a finite set of generators θ_i for R_A over A and a finite set of generators θ'_j for R'_B over B . The freeness conditions imply that for any $C \supseteq (A \cup B)$ we have that $R_C = C[R_A] = C[\theta_i: i]$ and that $R'_C = C[R'_B] = C[\theta'_j: j]$. By (2.1.6) we know that for all sufficiently large C , the θ'_j are in R_C , so that $R_C = C[\theta_i, \theta'_j: i, j]$. But the same reasoning shows that $R'_C = C[\theta_i, \theta'_j: i, j]$.

Thus, given two choices of descent for K , R , and a family of modules, maps, etc., after suitably enlarging the rings we may assume that $A = B$ and that $R_A = R_B$. Now suppose that we have two choices of descent for a module M : these we may think of as finitely generated R_A -submodules $W, W' \subseteq M$, free over A , such that $K \otimes_A W \cong M$ and $K \otimes_A W' \cong M$. Choose finite sets of generators $\{w_i\}$ for W , and $\{w'_j\}$ for W' over R_A . For any finitely generated \mathbb{Z} -algebra B with $A \subseteq B \subseteq K$, $B \otimes_A W$ may be identified with the R_B -submodule of M generated by the w_i , and $B \otimes_A W'$ with the R_B -submodule of M generated by the w'_j . As in the earlier argument, for any sufficiently large B , the w'_j will be contained in $B \otimes_A W$ and the w_i in $B \otimes_A W'$, and for such a B we will have $B \otimes_A W = B \otimes_A W'$ (when they are identified with their images in M).

Evidently, the discussion of the above paragraph applies to all of the modules of the family. Thus, for all sufficiently large B the two choices of descent become the same for R_B and all of the modules. Since the maps of R_B -modules may then be viewed as restrictions, they too are the same.

(2.1.13) Preserving that a module is nonzero. We continue the discussion of the preceding paragraph. The point we want to make here is that if a certain module in our family is not zero, we preserve this when we descend: in fact, if $M \neq 0$, then M_A is a nonzero free A -module whose rank is the same as $\dim_K M$ (which is usually infinite). When $\phi: M \rightarrow M'$ is not onto or not injective, the same applies to ϕ_A : the kernel, image, and cokernel are all A -free after localization at one element of A° .

(2.1.14) Further refinements of descent. We list here several other observations about

what can be preserved by descent. A number of the details, which are straightforward, are left to the reader. In several cases we are simply elaborating the consequences of what has already been said above.

(a) In working with a chain of submodules we can preserve all the inclusions in the descent to a chain over R_A .

(b) If $N + N' = N''$ in M , we can arrange that $N_A + N'_A = N''_A$ in M_A . (Localize so that $N''_A/(N_A + N'_A)$ is A -free); it becomes zero when we apply $K \otimes _.$)

(c) If $N \cap N' = N''$ in M , we can arrange that $N_A \cap N'_A = N''_A$ in M_A .

(d) We can arrange that the homology of a specified finite complex of modules be preserved as we descend.

(e) We can specify the split exactness of a certain map as we descend (descend the auxiliary map in the opposite direction and preserve that the appropriate composition be the identity).

(f) We can preserve that a certain map be a specific finite R -linear combination of other maps. (Simply make sure that the coefficients from R needed are in R_A : this is then automatic from the fact that the maps descend via restriction.)

(g) The remarks in (a), (b), (c) apply to ideals of R . If $I \subseteq R$ is in the family then the A -algebra R_A/I_A may be viewed as solving the descent problem for the K -algebra R/I . Then, since $R_A/I_A \subseteq R/I$ it follows that if I is radical then I_A is radical, and if I is prime then I_A is prime. Also if I is primary to P in the family then I_A will be primary to P_A after we descend (since we shall have $I_A = I \cap R_A$, $P_A = P \cap R_A$).

Moreover, if N is any R -module in the family killed by I , so that it may be thought of as an (R/I) -module, then N_A is killed I_A (since $N_A \subseteq N$ and $I_A \subseteq I$) and so may thought of as an (R_A/I_A) -module. In fact, if $\text{Ann}_R N = I$ then $\text{Ann}_{R_A} N_A = I_A$ after localizing at one element of $A - \{0\}$.

(h) If x_1, \dots, x_h are elements of R that form a (possibly improper) regular sequence on M , then one can assume that these elements are in R_A and form a (possibly improper) regular sequence on M_A . By including all the modules $M/(x_1, \dots, x_t)M$ in the family one comes down to the case of preserving that a single element x not be a zerodivisor on M . This is clear since $M_A \subseteq M$. One can also preserve that the sequence is an R -sequence

(i.e., that $M(x_1, \dots, x_h)M \neq 0$) by (2.1.3). (One can also think in terms of preserving the homology of all of the Koszul complexes $K_\bullet(x_1, \dots, x_i; M)$.)

(2.1.15) Nilpotents. We continue the discussion of descent. Let J denote the ideal of nilpotents in R . Let $J_A = J \cap R_A$ be the ideal of nilpotents in R_A . If A is large enough J_A will contain generators of J . By localization at one element of $A - \{0\}$ we can arrange that all of $R_A, J_A, R_A/J_A$ are A -free. Thus, R_A/J_A will solve the descent problem for $R_{\text{red}} = R/J$, and J will be the expansion of J_A to R , which may be identified with J_K . This is preserved when we enlarge A .

(2.1.16) Minimal primes of R . We may assume that A has been chosen so large that the conclusions of (2.1.15) hold, with J, J_A defined as in (2.1.15). Let $\mathfrak{p}^{(1)}, \dots, \mathfrak{p}^{(h)}$ be the minimal primes of R .²¹ Then we can include them in the family and take A so large that $R_A/\mathfrak{p}_A^{(i)}$ solves the descent problem for $R/\mathfrak{p}^{(i)}$ for every i . One can preserve that $\cap_i \mathfrak{p}_A^{(i)} = J_A$. Since the $\mathfrak{p}^{(i)}$ are mutually incomparable, so are the $\mathfrak{p}_A^{(i)}$. It follows that the $\mathfrak{p}_A^{(i)}$ are the minimal primes of R_A , and that they are in bijective correspondence with the $\mathfrak{p}^{(i)}$ via expansion and contraction.

(2.1.17) Additional conditions that may be preserved while descending. We note the following additional conditions that may be imposed when we descend by localizing at one element of $A - \{0\}$.

(a) We may assume that A is regular, and, in fact, smooth over \mathbb{Z} . (Since A is an excellent domain we may localize at one element and so arrange that A be regular. Then $\mathbb{Q} \rightarrow \mathbb{Q} \otimes_{\mathbb{Z}} A$ is smooth, and so $\mathbb{Z}_s \rightarrow A_s$ is smooth for some $s \in \mathbb{Z} - \{0\}$, and then $\mathbb{Z} \rightarrow A_s$ is smooth.)

(b) We may assume that R_A/A is A -free. (This follows from the form of the generic freeness theorem given in Lemma (2.1.4).)

(2.1.18) Descent for several K -algebras. We now suppose that we are given finitely many K -algebras, and for each one finitely many modules, maps of modules, elements, etc. as before. In addition, we also consider finitely many K -algebra homomorphisms among

²¹These are not symbolic powers: the superscripts are simply indices. This kind of notation occurs frequently in the sequel, since we want to use the base ring (in the present instance, A) as a subscript.

them. Moreover, if R is one of the K -algebras, S is another, M is a finitely generated R -module, and W is a finitely generated S -module, we consider maps $M \rightarrow W$ that are R -linear (W becomes an R -module via restriction of scalars). Of course, we only consider finitely many such algebras. We discuss the new points that arise in this situation very briefly.

(a) We enlarge the family, if necessary, so that it contains the image of every K -algebra homomorphism, so that we may assume that every K -algebra map is a surjection followed by an injection.

(b) We may achieve descent for all the “data” over any one of these rings R_i using a suitably chosen \mathbb{Z} -subalgebra A_i of K . By passing to A containing all these A_i we may assume that we have achieved such descent using the same \mathbb{Z} -subalgebra A of K for all. Everything will be preserved as we enlarge A .

(c) We are, of course, assuming that if R is any of these K -algebras then R_A is A -free and we identify $R_K \cong R$ and think of R_A as a subring of R . Similarly, we think of $M_A \subseteq M$ when M is an R -module.

(d) If $\gamma: R \rightarrow S$ is a ring homomorphism then the restriction of γ to R_A will map it as an A -algebra to S_A for all sufficiently large A . To see this, choose a finite set of algebra generators $\{r_j\}$ for R_A over A . Each $\gamma(r_j)$ is expressible as a polynomial \mathcal{P}_j in elements of S_A with coefficients in K . For B containing A we still have that $R_B = B[r_j: j]$, and so if B contains the coefficients of all the \mathcal{P}_j then the image of R_B in $S_K = S$ will lie in S_B .

Thus, by choosing A sufficiently large, we may assume that all the algebra maps that we are concerned about descend to the A -algebras that we have chosen via restriction; moreover, we automatically recover the original algebra homomorphism when we apply $K \otimes_A _$. Compositions and values on specific elements are automatically preserved.

(e) By the lemma on generic freeness (2.1.4) we may localize at one element of A° and get $S_A/(\text{Im } R_A)$ to be A -free. Of course, we can also arrange that each $\text{Ker}(R_A \rightarrow S_A)$ be A -free. The injectivity or surjectivity of any of these maps is then unaffected by tensoring with the fraction field of A and, hence, by tensoring with K (over A).

(f) Once we have enlarged A so that we have an A -linear algebra homomorphism $R_A \rightarrow S_A$ we may then consider the problem of enlarging A further so that a given R -linear map

from a finitely generated R -module M to a finitely generated S -module W will induce a map $M_A \rightarrow W_A$ by restriction. The discussion is virtually identical to that already given for algebras: restriction will induce the desired map for all sufficiently large A . The lemma of generic freeness is still sufficient to guarantee that all kernels and cokernels of these maps are A -free, so that injectivity and surjectivity is unaffected by applying $K \otimes_-$. Compositions and values on specified elements are preserved automatically.

(g) The fact that $R \rightarrow S$ is a module-finite map is automatically preserved by descent for sufficiently large A , since it corresponds to the existence of an R -linear R -module surjection $R^t \twoheadrightarrow S$ for some t , and this map will descend to a surjection.

(h) By Noether normalization, a given K -algebra R can be written as a module-finite extension of a polynomial ring $K[x_1, \dots, x_n] = T$. Notice that we can use $A[x_1, \dots, x_n]$ as T_A , and this is preserved for any larger choice of A . By the remarks above, $T_A \rightarrow R_A$ will be module-finite and injective for all sufficiently large A .

(i) We continue the discussion in (h). The ring R is easily seen to have pure dimension n if and only if it has pure dimension n as a T -module. This means that it is torsion-free as a T -module and so is embeddable in the free T -module T^s for some positive integer s . When this is the case R_A will be embeddable in T_A^s for the same s , and after localizing at an element of A° we may assume that the cokernel is A -free. It then follows that for every A -algebra C , R_C is a module-finite extension of T_C and embeds in T_C^s . When C is a field, this implies that R_C has pure dimension n . Of particular importance later will be the case where $C = \kappa = A/\mu$ for some maximal ideal μ of A .

Other issues concerning descent will be addressed in the paper as they arise.

(2.2) TIGHT CLOSURE FOR AFFINE ALGEBRAS OVER FIELDS OF CHARACTERISTIC 0

(2.2.1) Discussion. Our approach to describing when an element $u \in M$ is in the tight closure of $N \subseteq M$, where $N \subseteq M$ are finitely generated modules over an algebra R finitely generated over a field K of characteristic zero is to first choose descent data (A, R_A, M_A, N_A, u_A) and then to make a definition in terms of this descent data. We therefore begin by studying the situation after descent.

(2.2.2) Tight closure over a finitely generated \mathbb{Z} -algebra relative to a subdo-

main. Throughout this section A is a domain finitely generated over \mathbb{Z} , and $A \subseteq R_A$ where R_A is finitely generated over A . We suppose that $N_A \subseteq M_A$ are finitely generated R_A -modules. We want to define $N_A^{*/A} M_A$, the *tight closure of N_A in M_A over R_A relative to A* . The reader should be warned that this is not a mixed characteristic notion. Whether an element of M_A is in this tight closure is unaffected by inverting any one element of A , for example. Rather, it is a notion that we shall use as a tool for defining tight closure over an affine K -algebra for some field $K \supseteq A$. (In fact, if K is the fraction field of A , we shall see later that $u_A \in N_A^{*/A} M_A$ if and only if $u_K \in N_K^{*K} M_K$, so that $N_A^{*/A} M_A$ is simply the contraction of $N_K^{*K} M_K$ to N_A . (Cf. Corollary (2.5.4).)

We shall adopt the following notational conventions. We shall write μ for a maximal ideal of A . Then $\kappa = \kappa(\mu)$ denotes A/μ . Note that κ is a finite field. We write $p = p(\mu)$ for the characteristic, and $q = q(\mu) = p(\mu)^e$ for some $e \in \mathbb{N}$. We almost always omit μ from the notation. The phrase “for almost all μ ” means for all μ in some Zariski dense open subset of $\text{Max Spec } A$.

Then we say that $u_A \in M_A$ is in $N_A^{*/A} M_A$ if for almost all $\mu \in \text{Max Spec } A$, $u_\kappa \in \langle N_\kappa \rangle^*_{M_\kappa}$.

Later, we shall see that $u_A \in M_A$ is in $N_A^{*/A} M_A$ if and only if there exists $c_A \in R_A^\circ$ such that for almost all $\mu \in \text{Max Spec } A$, $c_\kappa u_\kappa^q \in \langle N_\kappa \rangle^{[q]}$ (in $\mathbf{F}^e(M_\kappa)$). See Theorem (2.5.2). It is evident that $N_A^{*/A} M_A$ is a submodule of M_A containing N_A . Here, we are using, in a sense, a “uniform” multiplier (all the c_κ ’s are the images of a single c_A). In the definition given in the preceding paragraph, one is *a priori* permitted to use a different c_κ in every tight closure test as the maximal ideal μ varies. We note that tight closure for ideals was defined in [HH4] using the second version of the definition, but we have found that the form given here is usually more convenient.

Proving that the two notions agree will require a substantial effort, of which one ingredient is the theory of universal test elements developed in (2.4): one can use either discriminants or a theorem of Lipman and Sathaye [LS] to construct such elements.

(2.2.3) The definition of tight closure over an affine algebra over a field of characteristic zero. Let R be a finitely generated algebra over a field K of characteristic

zero. Let $N \subseteq M$ be finitely generated R -modules. We say that $u \in M$ is in the K -tight closure N_M^{*K} of N in M if there exist descent data (A, R_A, M_A, N_A, u_A) for (K, R, M, N, u) such that $u_A \in N_A^{*/A} M_A$ in M_A over R_A , as defined in (2.2.2). (As already noted in (2.2.2), this definition is different from the one given in [HH4], but agrees with it by Theorem (2.5.2).)

We want to establish certain basic facts about the tight closure *K : these will depend on the behavior of $^{*/A}$ and our ability to preserve certain facts about $R = R_K$ as we pass to the closed fibers R_κ of $A \rightarrow R_A$ for almost all $\mu \in \text{Max Spec } A$. We therefore postpone further discussion of the properties of $^{*/K}$ until we have established what we need about the behavior of these fibers in the next section, and also until we have proved certain facts about the existence of elements in R_A that serve as test elements in all fibers: this is done in (2.4). The reader may wish to skip one or both of the next two sections and refer back to them as necessary.

We conclude this section with a fairly detailed discussion of a non-trivial example of tight closure in equal characteristic zero.

(2.2.4) Example. Let K be any field of characteristic zero and let $R = K[X, Y, Z]/(f) = K[x, y, z]$, where $f = X^3 + Y^3 + Z^3$ and x, y, z denote the images of X, Y, Z , respectively, modulo (f) . Let $I = (x, y)R$. We want to see that $z^2 \in I^{*K}$ but that $z \notin I^{*K}$ here. Notice that R is a Cohen-Macaulay normal ring (there is an isolated singularity at the origin): in fact, it is a complete intersection, a surface in three-space, so that it is even Gorenstein as well. Thus, this example shows that ideals generated by parameters need not be tightly closed for such rings R . At the same time, the equation f shows that $z^3 \in I^3$ which implies that z is in the integral closure I^- of I (cf. (1.6.3)). Thus, in this example the tight closure is strictly smaller than the integral closure, which is quite the usual situation. (Simpler examples occur in regular rings, where every ideal is tightly closed: e.g., (x^2, y^2) is tightly closed in the polynomial ring $K[x, y]$, but its integral closure contains xy .)

To see why all these assertions hold, we choose descent data. We may take $A = \mathbb{Z}$ and $R_A = \mathbb{Z}[X, Y, Z]/(f)$. The closed fibers of $A \rightarrow R_A$ now correspond bijectively with the positive prime integers $p \in \mathbb{Z}$. We want to see that for almost all closed fibers (i.e., for all

but at most finitely many positive prime integers p), $z_\kappa^2 \in I_\kappa^*$. We also want to see that there are infinitely many choices of p such that $z_\kappa \notin I_\kappa^*$. Here, $\kappa = \kappa(p\mathbb{Z}) = \mathbb{Z}/p\mathbb{Z}$.

(This actually does show that $z \notin I^*$, and not just that $z \notin (IR_A)^{*/A}$, by virtue of Theorem (2.5.3), which asserts that the issue is independent of the choice of descent data.)

We exclude the fiber for $p = 3$. Fix some other prime p . Henceforth, we generally omit the subscript κ . It will suffice to see that there is a fixed positive integer a such that $x^a(z^{2q}) \in I^{[q]} = (x^q, y^q)$ for all q . We can write $2q = 3k + d$ where k is a nonnegative integer and $0 \leq d < 3$. Note that $1, z, z^2$ is a free basis for R_κ over $\kappa[x, y] = T$, so that every element has a unique representation as $\tau_0 + \tau_1 z + \tau_2 z^2$ with the $\tau_i \in T$. Then $x^a(z^2)^q$ becomes $x^a(z^3)^k z^d = x^a(x^3 + y^3)^k z^d$, and this will be in (x^q, y^q) provided that for all choices of nonnegative i, j with $i + j = k$, we have that at least one of the exponents in $x^{3i+a}y^{3j}$ is at least q . But if both exponents are only $q-1$, we have that $3i + a + 3j \leq 2q - 2$ or $3k + a \leq 2q - 2$. Since $3k$ is at least $2q - 2$, we see that even the choice $a = 1$ yields a contradiction.

By Theorem (1.4.11), x has a power that is a test element for R_κ . Thus, to see that $z \notin (IR_\kappa)^*$, it will suffice to show that for every fixed integer a we have $x^a z^q \notin (x^q, y^q)$ for $q \gg 0$. We leave this to the reader (the argument is given in (5.6) of [HH8]).

(2.3) COMPARISON OF FIBERS

We shall need the following:

(2.3.1) Definitions and discussion. Let K be a field, and \overline{K} an algebraic closure of K . A K -algebra R such that $\overline{K} \otimes_K R$ is a domain (respectively, reduced) is called an *absolute domain* (respectively, *geometrically reduced*) over K . It is equivalent to assume that $L \otimes_K R$ is a domain (respectively, reduced) for every extension field L of K . P is called an *absolute prime* ideal of R if R/P is an absolute domain. We shall refer to a finitely generated K -algebra R such that $\overline{K} \otimes_K R$ is regular (respectively, normal, respectively connected (i.e., has connected spectrum)) as *geometrically regular* (respectively, *geometrically normal*, respectively *geometrically connected*) over K . A finitely generated K -algebra is geometrically regular over K if and only if it is *smooth* over K : cf. (1.2.1f). Of course, if R is an absolute domain or geometrically reduced or geometrically normal

or geometrically regular then R is a domain (respectively, reduced, respectively normal, respectively regular). Moreover, an absolute prime ideal is prime.

We may define a not necessarily affine Noetherian K -algebra R to be *geometrically regular* over K if for every finite (equivalently, every finite purely inseparable) field extension K' of K , $K' \otimes_K R$ is regular. This is equivalent to the definition given in the preceding paragraph for the case where R is an affine K -algebra.

Notice that if K has characteristic zero or is perfect then R is geometrically regular (respectively, geometrically normal, respectively, geometrically reduced) if and only if it is regular (respectively, normal, respectively, reduced).²²

We also note the following:

(2.3.2) Fact. *Let K be a field. A finitely generated K -algebra R is geometrically reduced if and only if there is a nonzerodivisor f in R such that R_f is geometrically regular.*

(The condition is obviously sufficient since $S = \overline{K} \otimes_K R$ embeds in $\overline{K} \otimes_K R_f$, and the latter will be regular and, hence, reduced. Let T be the total quotient ring of R , a finite product of fields each finitely generated over K . If S is reduced then $\overline{K} \otimes_K T$, a localization of it, is reduced. Since this ring is integral over T , it is zero-dimensional, and, hence, regular. Thus, the localization of S at the set of nonzerodivisors W of R is regular, and it follows that W meets the defining ideal I of the non-regular locus of S . Choose f in $W \cap I$. \square)

(2.3.3) Definitions. (a) Let A be a ring. If P is any prime ideal of A , we write $\kappa = \kappa(P)$ for the field A_P/PA_P , which may be identified canonically with fraction field of A/P . When P is maximal, $\kappa(P)$ may be identified canonically with A/P . If $A \rightarrow R_A$ is any ring homomorphism and P is a prime ideal of A , we call the algebra $\kappa \rightarrow R_\kappa$ the *fiber* of $A \rightarrow R_A$ over P , where $\kappa = \kappa(P)$. If A is a domain with fraction field \mathfrak{F} we refer to $\mathfrak{F} \rightarrow R_{\mathfrak{F}}$

²²If we are studying whether $\overline{K} \otimes_K R$ is reduced then by a direct limit argument we may assume that R is finitely generated. Thus, we may assume that R is finitely generated in all cases. To see that $S = \overline{K} \otimes_K R$ is regular or reduced if R is it suffices to check the fibers, since $R \rightarrow S$ is flat. Thus, we need only consider the case where R is a field finitely generated over K . Then S is integral over R , and so 0-dimensional, and we need only check that it is reduced. But \overline{K} is a direct limit of finite separable extensions of K . For normality, note that if R is normal it will contain an ideal I of depth at least two generated by elements f such that R_f is regular. But then $I(\overline{K} \otimes_K R)$ also has depth at least two, and is generated by elements $f \in I$ such that $(\overline{K} \otimes_K R)_{1 \otimes f} \cong \overline{K} \otimes_K (R_f)$ is regular.

as the *generic fiber* of $A \rightarrow R_A$, while if $P = \mu$ is maximal we call the corresponding fiber a *closed fiber*.

(b) Given an R_A -module M_A , an A -algebra homomorphism $R_A \rightarrow S_A$, or such a homomorphism together with an R_A -linear map from an R_A -module M_A to an S_A -module W_A , we may also refer to their *fibers*, which are the R_κ -modules M_κ , the maps $R_\kappa \rightarrow S_\kappa$, or the R_κ -linear maps $M_\kappa \rightarrow S_\kappa$, as the case may be.

(c) Suppose that A is a domain. When a property holds for all fibers over primes not containing a certain element $a \in A^\circ$ (equivalently, for all fibers over primes in a Zariski dense open subset of $\text{Spec } A$), we shall say that the property holds for *almost all fibers*. This is equivalent to asserting that the property holds for all fibers after a base change from A to A_a (on all the algebras, modules, etc. being considered).

Of course, any property that holds for almost all fibers must hold for the generic fiber. We are interested in establishing the converse in a number of situations.

Notice that the notion of “almost all” here, when restricted to closed fibers, gives the same notion that we discussed earlier in the second paragraph of (2.2.2).

(2.3.4) Conventions and discussion. Throughout the rest of §(2.3), A will denote a Noetherian domain with fraction field \mathfrak{F} and K an extension field of \mathfrak{F} . R_A will denote a finitely generated A -algebra. Although in practice we shall be mainly interested in the case where A is a finitely generated \mathbb{Z} -algebra and the fibers are closed fibers, we shall not impose such restrictions for the moment. Our first basic result on passing from the generic fiber to almost all fibers is given just below: the conclusions hold not just for base changes to fields, but rather for base changes from A to an arbitrary Noetherian A -algebra B provided that they factor $A \rightarrow A_a \rightarrow B$ for a certain fixed element $a \in A^\circ$. (In fact, in many instances there is no need for B to be Noetherian, but we choose not to pursue this point.) We shall say that a result holds “for almost all B ” to mean “there exists $a \in A^\circ$ such that the result holds for all Noetherian A -algebras B such that a has invertible image in B .” More generally, given some specific condition \mathcal{C} (e.g., the condition that B be a domain), we shall say that the result holds “for almost all B satisfying \mathcal{C} ” to mean “there exists $a \in A^\circ$ such that the result holds for all Noetherian A -algebras B satisfying \mathcal{C} and

such that $a \in A^\circ$ has invertible image in B ."

(2.3.5) Theorem. *Let $A \rightarrow R_A$, and $\mathfrak{F} \subseteq K$ be as in (2.3.4) and let S_A be a finitely generated algebra over the Noetherian domain A . Let M_A be a finitely generated R_A -module and let W_A be a finitely generated S_A -module. We shall denote by B a varying Noetherian A -algebra. Note that the phrase "for almost all B " is defined in the preceding discussion.*

- (a) *Given an A -algebra homomorphism $R_A \rightarrow S_A$ and an R_A -linear map $M_A \rightarrow W_A$ with kernel N_A and cokernel C_A , then for almost all B , $M_B \rightarrow W_B$ has kernel N_B , and cokernel C_B . If $M_{\mathfrak{F}} \rightarrow W_{\mathfrak{F}}$ (or $M_K \rightarrow W_K$) is injective then for almost all B the map $M_B \rightarrow W_B$ is injective. In particular, for almost all fibers $M_\kappa \rightarrow W_\kappa$ is injective.*

If $M_{\mathfrak{F}} \rightarrow W_{\mathfrak{F}}$ (or $M_K \rightarrow W_K$) is surjective then for almost all B , $M_B \rightarrow N_B$ is surjective. In particular, for almost all fibers, $M_\kappa \rightarrow W_\kappa$ is surjective.

If $M_{\mathfrak{F}} \rightarrow W_{\mathfrak{F}}$ (or $M_K \rightarrow W_K$) is not surjective then for almost B , $M_B \rightarrow W_B$ is not surjective. In fact, for almost all B the cokernel is C_B , and is B -free of the same rank as the vector space dimension of $C_{\mathfrak{F}}$ over \mathfrak{F} (or C_K over K). In particular, these remarks apply to almost all fibers.

- (b) *If $R_{\mathfrak{F}} \rightarrow S_{\mathfrak{F}}$ (or $R_K \rightarrow S_K$) is injective then for almost all B , $R_B \rightarrow S_B$ is injective.*
- (c) *For almost all B , $\dim R_B = \dim B + \dim R_{\mathfrak{F}}$ ($= \dim B + \dim R_K$), where "dim" indicates Krull dimension here. In particular, for almost all fibers, $\dim R_\kappa = \dim R_{\mathfrak{F}}$ ($= \dim R_K$).*

After replacing A by A_a for some $a \in A^\circ$ we may assume that R_A is module-finite over a polynomial subring $T_A = A[x_1, \dots, x_n]$, and then R_B is module-finite over T_B for almost all B . Moreover, if $R_{\mathfrak{F}}$ or R_K is biequidimensional then for almost all B , R_B is embeddable in a finitely generated free T_B -module, where $n = \dim R_{\mathfrak{F}}$. In particular, for almost all fibers R_κ is biequidimensional.

- (d) *If $\mathcal{M}_{\bullet A}$ is a finite complex of finitely generated R_A -modules with homology $\mathcal{H}_{\bullet A}$ then for almost all B the homology of $\mathcal{M}_{\bullet B}$ is $\mathcal{H}_{\bullet B}$. Thus, if $\mathcal{M}_{\bullet \mathfrak{F}}$ (or $\mathcal{M}_{\bullet K}$) is exact (or acyclic), then so is $\mathcal{H}_{\bullet B}$ for almost all B . In particular, these remarks apply to almost all fibers.*

(e) For every given value of i , for almost all B ,

$$\mathrm{Tor}_i^{R_B}(M_B, W_B) \cong B \otimes_A \mathrm{Tor}_i^{R_A}(M_A, W_A)$$

and

$$\mathrm{Ext}_{R_B}^i(M_B, W_B) \cong B \otimes_A \mathrm{Ext}_{R_A}^i(M_A, W_A).$$

In particular, these results hold for almost all fibers.

(f) If $N_A, N'_A, N''_A \subseteq M_A$ are submodules and $N_K \cap N'_K = N''_K$ then $N_B \cap N'_B = N''_B$ for almost all B (K may be \mathfrak{F}). In particular, this result holds for almost all fibers.

(g) If $\mathrm{Ann}_{R_A} M_A = I_A$ then $\mathrm{Ann}_{R_B} M_B = I_B$ for almost all B . If $\mathrm{Ann}_{M_A} I_A = N_A$ then $\mathrm{Ann}_{M_B} I_B = N_B$ for almost all B . If $N_A :_{R_A} N'_A = I_A$ then $N_B :_{R_B} N'_B = I_B$ for almost all B . If $N_A :_{M_A} I_A = N'_A$ then $N_B :_{M_B} I_B = N'_B$ for almost all B . In particular, these results hold for almost all fibers.

(h) If a sequence of elements x_1, \dots, x_d of R_A forms a possibly improper regular sequence (respectively, a regular sequence) on $M_{\mathfrak{F}}$ (or M_K) then for almost all B its image in R_B forms a possibly improper regular sequence (respectively, a regular sequence) on M_B . In particular, this result holds for almost all fibers.

If I_A is an ideal of R_A then, for almost all B , $\mathrm{depth}_{I_B} M_B = \mathrm{depth}_{I_{\mathfrak{F}}} M_{\mathfrak{F}} = \mathrm{depth}_{I_K} M_K$. In particular, this holds for almost all fibers.

Proof. In every instance it is clear that a condition imposed for K implies the same condition for \mathfrak{F} (since the base changes from \mathfrak{F} to K are faithfully flat), and so we give proofs assuming the condition to hold for \mathfrak{F} .

All of the statements in (a) are immediate because we can localize A at a single element of A° so that all of the modules in the sequence $0 \rightarrow N_A \rightarrow M_A \rightarrow W_A \rightarrow C_A \rightarrow 0$ are A -free, by (2.1.4). Part (b) is a special case of (a).

To prove (c), we note that after localizing at one element of A° we have that R_A is module-finite over a polynomial subring $T_A = A[x_1, \dots, x_n]$. The injectivity of $T_A \rightarrow R_A$ will be preserved for almost all B by part (b). It follows that $\dim R_K = \dim R_{\mathfrak{F}} = n$, while $\dim R_B = \dim B[x_1, \dots, x_n]$.

Now suppose that R_k is biequidimensional. We may assume by (2.1.4) that R_A is A -free. An associated prime of (0) in R_A cannot, therefore, meet A , and so will yield an associated prime of 0 in $R_{\mathfrak{F}}$. Since $R_{\mathfrak{F}}$ is biequidimensional, it is torsion-free over $T_{\mathfrak{F}}$. It follows that R_A is torsion-free over T_A , and so can be embedded T_A -linearly in a finitely generated free T_A -module W_A . It follows that R_B embeds into W_B for almost all B , by part (a). When $B = \kappa$ is a field this implies that R_{κ} is a torsion-free module over T_{κ} and, hence, biequidimensional.

Part (d) follows because the complex and its homology can be fully described by finitely many short exact sequences, and we may localize at an element of A° so that all the modules occurring are A -free by (2.1.4).

To prove (e), choose a free resolution of M_A by finitely generated free modules and truncate it at the n th spot for some $n > i + 1$. Call this complex $G_{\bullet A}$. We may localize A so that all of the modules and their homology are A -free, as well as the modules of cycles and boundaries. It follows that for every B , $G_{\bullet B}$ gives a free resolution of M_B at least through degree $i + 1$, and so $\text{Tor}_i^{R_B}(M_B, W_B)$ (respectively, $\text{Ext}_{R_B}^i(M_B, W_B)$) can be computed as the (co)homology of the complex $G_{\bullet B} \otimes_{R_B} W_B$ (respectively, $\text{Hom}_{R_B}(G_{\bullet B}, W_B)$), which can be identified with $B \otimes_A (G_{\bullet A} \otimes_{R_A} W_A)$ (respectively, with $B \otimes_A \text{Hom}_{R_A}(G_{\bullet A}, W_A)$), and the result now follows from the first part applied to $G_{\bullet A} \otimes_{R_A} W_A$ (respectively, $\text{Hom}_{R_A}(G_{\bullet A}, W_A)$).

To prove (f) we may assume that $K = \mathfrak{F}$ as usual, and we may arrange that $N_A \cap N'_A = N''_A$ by localizing at one element of A° . This yields an exact sequence

$$0 \rightarrow N''_A \rightarrow M_A \xrightarrow{\alpha} M_A/N_A \oplus M_A/N'_A \rightarrow C_A \rightarrow 0$$

where α sends $u \mapsto (u + N_A, u + N'_A)$ and $C_A = \text{Coker } \alpha$. By the result of the first paragraph this remains exact for almost all B , which yields the result.

We obtain the first statement of (g) by observing that if $u_A^{(1)}, \dots, u_A^{(s)}$ are generators of M_A then there is an exact sequence

$$0 \rightarrow I_A \rightarrow R_A \xrightarrow{\beta} M_A^s \rightarrow D_A \rightarrow 0$$

where β sends r to $(ru_A^{(1)}, \dots, ru_A^{(s)})$ and D_A is the cokernel of β . It suffices to localize A so as to preserve the exactness of this sequence upon applying $B \otimes_A _$. The second

statement follows similarly by realizing $\text{Ann}_{M_A} I_A$ as the kernel of a map $M_A \xrightarrow{\gamma} M_A^s$, where γ is given by a matrix whose entries are generators for I_A . The statements about colons can then be deduced from these results and the observation that $N_A :_{M_A} I_A$ is the inverse image in M_A of $\text{Ann}_{M_A/N_A} I_A$ while $N_A :_{R_A} N'_A$ is $\text{Ann}_{R_A} (N_A + N'_A) / N_A$.

In the proof of the second part of (h) let x_1, \dots, x_d be generators of I_A . Then (h) follows by applying (d) to the Koszul complexes $K_\bullet(x_1, \dots, x_i; M_A)$ for $0 \leq i \leq d$. Note that $M_B / (x_1, \dots, x_d) M_B$ is zero or nonzero for almost all B , according as $M_{\mathfrak{F}} / (x_1, \dots, x_d) M_{\mathfrak{F}}$ is zero or nonzero, by part (a). For all Noetherian A -algebras B (which might be \mathfrak{F} , K , or κ) the depth δ is the smallest integer j such that

$$H_{n-j}(K_\bullet(x_1, \dots, x_d; M_B))$$

does not vanish. \square

We next observe:

(2.3.6) Theorem. *Let $A \rightarrow R_A$ and $\mathfrak{F} \subseteq K$ be as in (2.3.1).*

- (a) *If $\mathfrak{F} \rightarrow R_{\mathfrak{F}}$ (or $K \rightarrow R_K$) is geometrically regular then so is $\kappa \rightarrow R_\kappa$ for almost all fibers.*
- (b) *If $\mathfrak{F} \rightarrow R_{\mathfrak{F}}$ (or $K \rightarrow R_K$) is geometrically reduced then so is R_κ for almost all fibers.*
- (c) *If $\mathfrak{F} \rightarrow R_{\mathfrak{F}}$ (or $K \rightarrow R_K$) is an absolute domain then for almost all fibers, R_κ is an absolute domain.*

Proof. Part (a) is a consequence of the Jacobian criterion for smoothness: from the fact that $A \rightarrow R_A$ is smooth after localizing at A° , it follows that it becomes smooth after localizing at one element of A° . But then smoothness is preserved by an arbitrary base change, and the result follows.

We can now use this and (2.3.2) to prove (b). Since $R_{\mathfrak{F}}$ is geometrically reduced we can choose a nonzerodivisor f such that $(R_{\mathfrak{F}})_f$ is geometrically regular. By replacing A by a localization at an element of A° we may assume that $f = f_A$ is in R_A , and localizing again, if necessary, we may assume that $R_A / f R_A$ is A -free, so that f_κ will be a nonzerodivisor in every R_κ . Moreover, by localizing at one element of A° we may also assume that

$S_A = (R_A)_{f_A}$ is smooth over A . It then follows that for all fibers we have that f_κ is a nonzerodivisor in R_κ , so that $R_\kappa \subseteq (R_\kappa)_{f_\kappa}$ and $(R_\kappa)_{f_\kappa} \cong S_\kappa$ is smooth over κ .

It remains only to prove part (c). Let L be an algebraic closure of \mathfrak{F} and let $\lambda = \lambda(P)$ be an algebraic closure of $\kappa = \kappa(P)$. We are trying to show that if R_L is a domain then R_λ is a domain for all P in a dense open subset of $\text{Spec } R$.

We may assume that R_A is A -free and so $R_A \subseteq R_{\mathfrak{F}} \subseteq R_L$ is a domain. We are free to localize R_A at any one element $F_A \neq 0$, since by the theorem on generic freeness we can make $R_A/F_A R_A$ A -free, and then F_λ is a nonzerodivisor in every R_λ , so that $R_\lambda \subseteq (R_\lambda)_{F_\lambda}$, and R_λ is a domain if and only if $(R_\lambda)_{F_\lambda}$ is a domain. It follows that we may replace R_A by any birationally equivalent finitely generated A -algebra. Since R_L is a domain, for suitable $F = F_A$ the ring $(R_A)_F$ will contain a separating transcendence basis x_1, \dots, x_{n-1} for the fraction field of R_A over \mathfrak{F} . This fraction field is consequently the same as the fraction field of a \mathfrak{F} -algebra of the form $\mathfrak{F}[x_1, \dots, x_n]/(G)$, where G is a single polynomial, since the fraction field will be generated over $\mathfrak{F}(x_1, \dots, x_{n-1})$ by one element (since the extension will be separable). We can localize A at an element of A° so that it contains the coefficients of G .

We have therefore reduced to considering the case where $R = A[x_1, \dots, x_n]/(G)$, where G is a polynomial. What we must show is that if G is irreducible over $L[x]$, then its image G_λ is irreducible over $\lambda[x]$ for all P in a Zariski dense open set in $\text{Spec } A$, where $\lambda = \lambda(P)$ is an algebraic closure of $\kappa = \kappa(P)$.

Let $d = \deg G$. It will suffice to show that for each choice of positive integers $0 < a, b < d$ with $a + b = d$ that there is a Zariski dense set of P for which G_λ has no factorization as the product of a factor of degree a and a factor of degree b . The problem of finding a factorization for G over L can be attacked as follows: write down “general” polynomials $G_1(u, x)$, $G_2(v, x)$ of degrees a, b respectively in the x ’s with unknown coefficients u, v (where each of u, v denotes a string of indeterminates). Finding G_1, G_2 of the specified degrees with coefficients in a given ring B such that $G_1 G_2 = G$ translates into solving a finite system of polynomial equations

$$(\dagger) \quad \left\{ H_\nu(u, v) = c_\nu \right.$$

for u, v in B , where:

- (1) ν indexes the monomials of degree at most d in the x 's
- (2) c_ν is the coefficient of the monomial x^ν indexed by ν in G
- (3) $H_\nu(u, v)$ is the polynomial in the u 's and v 's with integer coefficients that is the coefficient of x^ν in $G_1(u, x)G_2(v, x)$.

Since the equations (\dagger) have no solution over L , by Hilbert's Nullstellensatz there are polynomials Q_j in u, v with coefficients in L such that $(\#) \sum_\nu Q_\nu(H_\nu - c_\nu) = 1$. The ring obtained by adjoining the coefficients of the Q_ν to A is contained in a module-finite extension A' of A_a for some $a \neq 0$ in A . We may replace A by A_a . Thus, we may assume that equations $(\#)$ hold with the Q_ν having coefficients in a module-finite extension A' of A . Now let P be any prime ideal of A . Then there is a prime ideal P' of A' lying over P , and we may view A/P as contained in A'/P' . The fraction field of A'/P' may be identified with a subfield of $\lambda = \lambda(P)$, the algebraic closure of $\kappa(P)$. The image of the equation $(\#)$ modulo P' shows the impossibility of factoring G_λ over λ into factors of the specified degrees a, b . Since we may further decrease the open set (but only finitely many times) to exclude every possible choice of a and b , the result is proved. \square

It will be convenient to characterize the height of an ideal in a Noetherian ring as follows:

(2.3.7) Facts. *Let R be a Noetherian ring and I an ideal of R .*

- (a) *If I is proper then I has height at least h if and only if there is a sequence of elements x_1, \dots, x_h in I such that for all i , $0 \leq i \leq h-1$, x_{i+1} is not in any minimal prime of $(x_1, \dots, x_i)R$.*
- (b) *I has height at most h if and only if there exists a proper ideal J containing I and an element y of R not a zerodivisor on J such that $yJ \subseteq \text{Rad } I$ and yJ is contained in the radical of an ideal generated by at most h elements.*

(In (a), the sequence is constructed by a trivial induction and standard prime avoidance. For (b) the sufficiency of the condition is clear, since IR_y and JR_y are the same up to radicals. For necessity, take J to be a minimal prime of I of height at most h : after localizing sufficiently at an element not in J , it will be the radical of an ideal generated by at most h elements, since the local ring of R at J will have a system of parameters. \square)

(2.3.8) Definition and discussion: minheight. If M is a module over a Noetherian ring R and I is an ideal of R we define the *minheight* of M on I , denoted $\text{mnht}_I M$, to be

$$\min\{\text{ht } I(R/P) : P \text{ is a minimal prime of } \text{Supp } M\}$$

so that $\text{mnht}_I R = \min\{\text{ht } I(R/P) : P \text{ is a minimal prime of } R\}$. By analogy with the conventions for “depth,” when I is an ideal of R by the *minheight* of I , $\text{mnht } I$, we mean $\text{mnht}_I R$. When R is locally equidimensional (cf. (1.2.1e)), the minheight of I and the height of I agree. In particular, minheight agrees with height in any biequidimensional finitely generated K -algebra, where K is a field.

We refer the reader to §2 of [HH8] for a more detailed treatment. Note that if S is faithfully flat over R and R is universally catenary then $\text{mnht}_{IS}(S \otimes_R M) = \text{mnht}_I M$; in particular, $\text{mnht } IS = \text{mnht } I$. This holds, for example, if S is the completion of the universally catenary local ring R . Cf. [HH8] Proposition (2.2f).

(2.3.9) Theorem. *Let $A \rightarrow R_A$ and $\mathfrak{F} \subseteq K$ be as in (2.3.4). Let I_A, P_A be ideals of R_A , and let M_A be a finitely generated R_A -module. Let B be a varying Noetherian A -algebra.²³*

- (a) *If x_A is an element of R_A such that x_K is not in any minimal prime of R_K (K may equal \mathfrak{F}), then for almost all B , x_B is not in any minimal prime of R_B . In particular, this holds for almost all fibers.*
- (b) *If x_1, \dots, x_h is a sequence of elements of R_A such that $(x_{i+1})_K$ is not in any minimal prime of $(x_1, \dots, x_i)R_K$ (K may equal \mathfrak{F}), $0 \leq i \leq h-1$, then for almost all B and all i , $0 \leq i \leq h-1$, $(x_{i+1})_B$ is not contained in any minimal prime of $(x_1, \dots, x_i)R_B$. In particular, this holds for almost all fibers.*
- (c) *For almost all B , the height of I_B is the same as the height of $I_{\mathfrak{F}}$ (or I_K), while $\dim(R_B/I_B) = \dim B + \dim(R_{\mathfrak{F}}/I_{\mathfrak{F}}) (= \dim B + \dim(R_K/I_K))$. In particular, for almost all fibers, $\text{ht } I_{\kappa} = \text{ht } I_{\mathfrak{F}}$ while $\dim(R_{\kappa}/I_{\kappa}) = \dim(R_{\mathfrak{F}}/I_{\mathfrak{F}})$.*
- (d) *If P_A is a prime ideal of R_A disjoint from A° such that height $P_{\mathfrak{F}} = h$ and $\dim R_{\mathfrak{F}}/P_{\mathfrak{F}} = d$, then, for almost all fibers, P_{κ} is an ideal of R_{κ} all of whose associated primes are minimal primes \mathfrak{q} of P_{κ} of height h , and for each of them $\dim R_{\kappa}/\mathfrak{q} = d$.*

²³Note that the phrase “for almost all B ” is defined in (2.3.4)

- (e) Let P_A, Q_A be incomparable primes of R_A disjoint from A° . Then for almost all fibers, every minimal prime of P_κ is incomparable with every minimal prime of Q_κ .
- (f) After localizing at one element of A° , the elements of $\text{Ass } M_A$ over R_A correspond bijectively with the elements of $\text{Ass } M_{\mathfrak{F}}$ over $R_{\mathfrak{F}}$ via contraction and expansion. Assume this localization has been done. Then for almost all B ,

$$\text{Ass } M_B = \bigcup_{P_A \in \text{Ass } M_A} \text{Ass}(R_B/P_B).$$

Notice that by part (d), for almost all fibers, the elements of $\text{Ass } R_\kappa/P_\kappa$ are the minimal primes \mathfrak{q} of P_κ , each of which has the same height as $P_{\mathfrak{F}}$ and has a quotient of the same dimension as $R_{\mathfrak{F}}/P_{\mathfrak{F}}$.

Moreover, for almost all fibers, the set of minimal primes of $\text{Ass } M_\kappa$ is

$$\bigcup_{P_A \text{ minimal in } \text{Ass } M_A} \text{Ass}(R_\kappa/P_\kappa).$$

If $M_{\mathfrak{F}}$ (or $R_{\mathfrak{F}}$) has no embedded primes, then the same is true for M_κ (or R_κ) for almost all fibers. If $M_{\mathfrak{F}}$ has pure dimension (i.e., for every associated prime $P_{\mathfrak{F}}$, $\dim R_{\mathfrak{F}}/P_{\mathfrak{F}} = \dim M_{\mathfrak{F}}$: cf. (1.2.1e) for a discussion of dimension), then so does M_κ for almost all fibers.

- (g) If I_K is unmixed (i.e., has no embedded primes: cf. (1.2.1e) again) or has pure height h (i.e., all primes in its primary decomposition have height h : cf. (1.2.1e) once more), then for almost all fibers I_κ is unmixed or has pure height h , as the case may be.
- (h) For almost all fibers, $\text{mnht}_{I_\kappa} M_\kappa = \text{mnht}_{I_{\mathfrak{F}}} M_{\mathfrak{F}} (= \text{mnht}_{I_K} M_K)$ and for almost all fibers the minheight of I_κ is the same as the minheight of I_K (or $I_{\mathfrak{F}}$).

Proof. To see (a), note that after we localize at one element of A° the minimal primes of R_A and $R_{\mathfrak{F}}$ will correspond bijectively via expansion and contraction, and that the ideal of all nilpotents J_A in R_A will expand to the ideal of all nilpotents in $R_{\mathfrak{F}}$. If x_A is not in any minimal prime of R_K , it will also not be in any minimal prime of $R_{\mathfrak{F}}$, since $R_{\mathfrak{F}} \rightarrow R_K$ is faithfully flat, and so x_A is a nonzerodivisor on R_A/J_A . By localizing at an element of A° such that $(R_A/J_A)/(x_A)$ is A -free, we can arrange that x_B will be a nonzerodivisor

in all the R_B/J_B . But all of the ideals J_B still consist of nilpotents, so that the minimal primes of R_B/J_B are simply the primes of the form \mathfrak{p}/J_B where \mathfrak{p} is a minimal prime of R_B . It follows that x_B is not in any minimal prime of R_B .

Part (b) is then immediate from the simultaneous application of (a) to all of the rings $R_A/(x_1, \dots, x_i)R_A$ and elements $(\bar{x}_{i+1})_A$, for $0 \leq i \leq h-1$, where the bar indicates images modulo $(x_1, \dots, x_i)R_A$.

The statement about dimensions in (c) is simply (2.3.5c) applied to R_A/I_A . Since $R_{\mathfrak{F}} \rightarrow R_K$ is faithfully flat, $\text{ht } I_K = \text{ht } I_{\mathfrak{F}}$, and we may work with $h = \text{ht } I_{\mathfrak{F}}$. The fact that $I_{\mathfrak{F}}$ is proper is preserved for almost all B . By (2.3.7a) we may construct a sequence of elements x_1, \dots, x_h in $I_{\mathfrak{F}}$ such that x_{i+1} is not in any minimal prime of $(x_1, \dots, x_i)R_{\mathfrak{F}}$ for all i , $0 \leq i \leq h-1$. After localizing A at one element of A° we may assume that these elements are in R_A . By part (b) this set-up is preserved when we pass to almost any R_B , and we see from a second application of (2.3.6a) that $\text{ht } I_B \geq h$ for almost all B . Similarly, by (2.3.7b) we may choose a proper ideal $J_{\mathfrak{F}}$ of $R_{\mathfrak{F}}$ containing $I_{\mathfrak{F}}$ such that $J_{\mathfrak{F}}$ is the radical of an ideal generated by at most h elements and such that there is an element $y_{\mathfrak{F}}$ of $R_{\mathfrak{F}}$ that is not a zerodivisor on $J_{\mathfrak{F}}$ and such that $y_{\mathfrak{F}}J_{\mathfrak{F}}$ is nilpotent modulo $I_{\mathfrak{F}}$. After replacing R_A by a suitable localization at one element of A° , we may assume that we have $J_A \subseteq R_A$, $y_A \in R_A$ with the same properties. We can then preserve this set-up while passing to almost all B , and it follows that $\text{ht } I_{\mathfrak{F}} \leq h$ for almost all fibers as well.

To prove (d), first note that for almost all κ , R_κ/P_κ is biequidimensional of dimension $\dim R_{\mathfrak{F}}/P_{\mathfrak{F}}$, by Theorem (2.3.5c). This shows that every associated prime of $P_{\mathfrak{F}}$ is a minimal prime \mathfrak{q} such that $\dim R_\kappa/\mathfrak{q} = \dim R_{\mathfrak{F}}/P_{\mathfrak{F}}$. Moreover, we know that P_κ itself has height h for almost all fibers, and so $\text{ht } \mathfrak{q} \geq h$ for any minimal prime \mathfrak{q} of P_κ . Notice that if R_A is a polynomial ring then the full statement about heights is obvious, since

$$\text{ht } \mathfrak{q} = \dim R_\kappa - \dim R_\kappa/\mathfrak{q} = \dim R_\kappa - \dim R_\kappa/P_\kappa = \dim R_{\mathfrak{F}} - \dim R_{\mathfrak{F}}/P_{\mathfrak{F}} = \text{ht } P_{\mathfrak{F}}$$

in that case.

Before completing the proof of (d), we prove (e). First note that we can map a polynomial ring onto R_A and replace P_A , Q_A by their inverse images in the polynomial ring without affecting any relevant issues. The fact that these are incomparable primes implies

that $\text{ht}(P_A + Q_A) > \max\{\text{ht } P_A, \text{ht } Q_A\}$. This will be preserved when we pass to almost any fiber. Moreover, if \mathcal{P} and \mathcal{Q} are minimal primes of P_κ, Q_κ respectively then for almost all fibers we have $\text{ht}(\mathcal{P} + \mathcal{Q}) \geq \text{ht}(P_\kappa + Q_\kappa) = \text{ht}(P_A + Q_A) > \max\{\text{ht } P_A, \text{ht } Q_A\} = \max\{\text{ht } P_\kappa, \text{ht } Q_\kappa\} = \max\{\text{ht } \mathcal{P}, \text{ht } \mathcal{Q}\}$, since we already know (d) in the polynomial ring case. This shows that \mathcal{P} and \mathcal{Q} are incomparable.

To see the other inequality on height in (d) in the general case, let $\mathfrak{p}_A^{(1)}, \dots, \mathfrak{p}_A^{(s)}$ denote the minimal primes of R_A . There is no loss of generality in assuming that R_A is reduced in studying this question, and we may assume that $\cap_j \mathfrak{p}_A^{(j)} = (0)$. We may consequently assume, using Theorem (2.3.5f), that $\cap_j \mathfrak{p}_\kappa^{(j)} = (0)$ for almost all κ . By renumbering, we may assume that $\mathfrak{p}_A^{(j)} \subseteq P_A$ for $j \leq t$ and that this fails for $j > t$. This too will be preserved for almost all fibers. Now, let \mathfrak{p} be a minimal prime of (0) in R_κ contained in \mathfrak{q} such that

$$\begin{aligned}
 (*) \quad \text{ht } \mathfrak{q} &= \text{ht } \mathfrak{q}/\mathfrak{p} \\
 &= \dim(R_\kappa/\mathfrak{p}) - \dim(R_\kappa/\mathfrak{q}) \\
 &= \dim(R_\kappa/\mathfrak{p}) - \dim(R_\kappa/P_\kappa)
 \end{aligned}$$

for almost all κ . Since $\cap_j \mathfrak{p}_\kappa^{(j)} = (0) \subseteq \mathfrak{p}$, we have that some $\mathfrak{p}_\kappa^{(j)} \subseteq \mathfrak{p}$ and, hence, by part (e), we may assume that $j \leq t$; if $j > t$ then \mathfrak{p} is a minimal prime of $\mathfrak{p}_\kappa^{(j)}$ comparable to \mathfrak{q} , a minimal prime of P_κ . Since P_A and $\mathfrak{p}_A^{(j)}$ are incomparable for $j > t$, (e) enables us to avoid this possibility on almost all fibers. But then $\dim R_\kappa/\mathfrak{p} = \dim R_\kappa/\mathfrak{p}_\kappa^{(j)} = \dim R_{\mathfrak{F}}/\mathfrak{p}_{\mathfrak{F}}^{(j)}$ on almost all fibers, with $j \leq t$. Then (*) above shows that $\text{ht } \mathfrak{q} = \dim R_\kappa/\mathfrak{p}_\kappa^{(j)} - \dim(R_\kappa/P_\kappa) = \dim R_{\mathfrak{F}}/\mathfrak{p}_{\mathfrak{F}}^{(j)} - \dim R_{\mathfrak{F}}/P_{\mathfrak{F}} = \text{ht } P_{\mathfrak{F}}/\mathfrak{p}_{\mathfrak{F}}^{(j)}$ (since $j \leq t$) $\leq \text{ht } P_{\mathfrak{F}}$, as required.

We next prove (f). It is obvious that we can localize A at one element of A° so that there is a bijection between the primes in $\text{Ass } M_A$ and those in $\text{Ass } M_k$. For each prime P_A in $\text{Ass } M_A$ fix an embedding of $R_A/P_A \rightarrow M_A$. Then this map remains injective for almost all fibers by (2.3.5a), and we have that R_B/P_B injects into M_B for almost all B , and this shows that $\text{Ass}(R_B/P_B) \subseteq \text{Ass } M_B$ for almost all B . On the other hand, there is a finite filtration of M_A such that every factor N_A is a torsion-free module over R_A/P_A for some choice of $P_A \in \text{Ass } M_A$, and we may also fix an embedding $N_A \subseteq G_A$ where G_A is free over R/P_A . Then for almost all fibers we have $\text{Ass } M_B \subseteq \cup_{N_A} \text{Ass } N_B$ as N_A runs through the various factors, and $\text{Ass } N_B \subseteq \text{Ass } G_B = \text{Ass}(R_B/P_B)$ for every N_A . This

shows that, for almost all B ,

$$\text{Ass } M_B = \bigcup_{P_A \in \text{Ass } M_A} \text{Ass } (R_B/P_B).$$

If $Q_A \in \text{Ass } M_A$, say $Q_A \supseteq P_A$ where $P_A \in \text{Ass } M_A$ and these survive in $\text{Ass } M_{\mathfrak{F}}$, then for almost all fibers, Q_{κ} contains P_{κ} and so every minimal prime \mathcal{Q} of Q_{κ} contains P_{κ} . The statements that follow are valid for almost all fibers. One has that Q_{κ} must contain some minimal prime of P_{κ} , and the set of minimal primes of P_{κ} is precisely $\text{Ass } (R_{\kappa}/P_{\kappa})$. Thus, every minimal element of $\text{Ass } M_{\kappa}$ is in $\text{Ass } (R_{\kappa}/P_{\kappa})$ where P_A is minimal in $\text{Ass } M_A$. It remains to see that if P_A is minimal in $\text{Ass } M_A$ then, for almost all fibers, all the elements of $\text{Ass } R_{\kappa}/P_{\kappa}$ are minimal in $\text{Ass } M_{\kappa}$. If $Q_A \in \text{Ass } M_A$ there are two cases. If $Q_{\mathfrak{F}} \supset P_{\mathfrak{F}}$ (strictly) then for almost all fibers the minimal primes of Q_{κ} all have too large a height to be contained in any minimal prime of P_{κ} , while if $Q_{\mathfrak{F}}$ and $P_{\mathfrak{F}}$ are incomparable then for almost all fibers every minimal prime of Q_{κ} is incomparable to every minimal prime of P_{κ} .

The statement about embedded primes is then obvious, and the statement about pure dimension is also obvious.

Part (g) is immediate: apply (f) to $M_A = R_A/I_A$.

It remains only to verify part (h). Since $\text{mnht}_{I_L} M_L$ is the same as the minheight of $I_L(R_L/J_L)$ in the ring (R_L/J_L) , where $J_L = \text{Ann}_{R_L} M_L$, it follows from (2.3.5g) that we need only prove the result for the case where $M_A = R_A$. Since $R_{\mathfrak{F}} \rightarrow R_K$ is faithfully flat and $R_{\mathfrak{F}}$ is universally catenary, we may assume that $K = \mathfrak{F}$ (cf. (2.3.8)). We may further assume that A has been localized so that the minimal primes $\mathfrak{p}_A^{(1)} \dots \mathfrak{p}_A^{(s)}$ of R_A are disjoint from A° and correspond bijectively with the minimal primes $\mathfrak{p}_{\mathfrak{F}}^{(1)} \dots \mathfrak{p}_{\mathfrak{F}}^{(s)}$ of $R_{\mathfrak{F}}$.

Now, for almost all fibers we have that

$$\begin{aligned} & \min\{\text{ht } I_{\kappa}(R_{\kappa}/\mathfrak{q}) : \mathfrak{q} \text{ is a minimal prime of } R_{\kappa}\} = \\ & \min\{\text{ht } I_{\kappa}((R_{\kappa}/\mathfrak{p}_{\kappa}^{(i)})/(\mathfrak{q}/\mathfrak{p}_{\kappa}^{(i)})) : 1 \leq i \leq s \text{ and } \mathfrak{q} \in \text{Ass } R_{\kappa}/\mathfrak{p}_{\kappa}^{(i)}\} = \\ & \min_i \{\min\{\text{ht } I_{\kappa}(R_{\kappa}/\mathfrak{q}) : \mathfrak{q} \text{ is a minimal prime of } \mathfrak{p}_{\kappa}^{(i)}\} = \\ & \min_i \{\text{mnht } I_{\kappa}(R_{\kappa}/\mathfrak{p}_{\kappa}^{(i)})\} = \\ & \min_i \{\text{ht } I_{\kappa}(R_{\kappa}/\mathfrak{p}_{\kappa}^{(i)})\} \text{ (since } R_{\kappa}/\mathfrak{p}_{\kappa}^{(i)} \text{ is locally equidimensional)} = \\ & \min_i \{\text{ht } I_{\mathfrak{F}}(R_{\mathfrak{F}}/\mathfrak{p}_{\mathfrak{F}}^{(i)})\} \text{ (by part (c))} = \end{aligned}$$

$\text{mnht } I_{\mathfrak{F}}$ (by definition). \square

(2.3.10) Definitions and discussion: parameters, strong parameters, and the complete local domains of a ring. (a) Let R be a Noetherian ring. We say that x_1, \dots, x_h are *parameters* (or is a *sequence of parameters*) if for every prime ideal P containing $I = (x_1, \dots, x_h)$, the images of the x 's form part of a system of parameters for the local ring R_P . An ideal generated by a sequence of parameters is called a *parameter ideal*.

We say that B is a *complete local domain of R (at the prime P)* if it is obtained from R by completing R_P and then killing a minimal prime. If P is a maximal ideal we refer to B as a *complete local domain of R at a maximal ideal*. We say that $x_1, \dots, x_h \in R$ are *strong parameters* (or is a *sequence of strong parameters*) if for every complete local domain B of R at a prime $P \supseteq (x_1, \dots, x_h)R = I$, the images of x_1, \dots, x_h in B are part of a system of parameters. Notice that if any element among the x_i is a unit, or if they generate the unit ideal, then the elements are both parameters and strong parameters: there are no primes containing $I = R$, and so the conditions hold vacuously.

(b) Also note that every complete local domain B of R arises as a complete local domain of R/\mathfrak{p} for some minimal prime \mathfrak{p} of R , and that, conversely, every complete local domain of R/\mathfrak{p} for \mathfrak{p} minimal in $\text{Spec } R$ is a complete local domain of R . Evidently, it suffices to see this when R is local. One key point is that any minimal prime \mathfrak{q} of \widehat{R} contracts to a minimal prime \mathfrak{p} of R , and then $\widehat{R}/\mathfrak{q} \cong (R/\mathfrak{p})^\wedge / (\mathfrak{q}/\mathfrak{p}\widehat{R})$, where $\mathfrak{q}/\mathfrak{p}\widehat{R}$ may be identified with a minimal prime of $(R/\mathfrak{p})^\wedge \cong \widehat{R}/\mathfrak{p}\widehat{R}$; on the other hand, any minimal prime of $(R/\mathfrak{p})^\wedge \cong \widehat{R}/\mathfrak{p}\widehat{R}$ for \mathfrak{p} minimal must correspond to a minimal prime \mathfrak{q} of \widehat{R} , for it will lie over \mathfrak{p} (elements of $R - \mathfrak{p}$ are nonzerodivisors in $(R/\mathfrak{p})^\wedge$) and $\mathfrak{q}\widehat{R}_{\mathfrak{q}}$ will be nilpotent on $\mathfrak{p}\widehat{R}_{\mathfrak{q}}$, which is nilpotent because $\mathfrak{p}R_{\mathfrak{p}}$ is nilpotent and expands to $\mathfrak{p}\widehat{R}_{\mathfrak{q}}$.

We note:

(2.3.11) Proposition. *Let $\underline{x} = x_1, \dots, x_h$ be elements of a Noetherian ring R and let $I = (x_1, \dots, x_h)R$.*

- (a) x_1, \dots, x_h are *parameters* if and only if $\text{ht } I \geq h$ (in which case $\text{ht } I$ is h or $+\infty$).
- (b) x_1, \dots, x_h are *strong parameters* for R if and only if for every prime P of R containing

$$I, \text{ mnht } I(R_P)^\wedge \geq h.$$

- (c) *If x_1, \dots, x_h are strong parameters in R , then they are strong parameters in every localization of R .*
- (d) *If R is universally catenary, then x_1, \dots, x_h are strong parameters iff $\text{mnht } I \geq h$ (in which case $\text{mnht } I$ is h or $+\infty$) iff x_1, \dots, x_h are parameters in R/\mathfrak{p} for every minimal prime \mathfrak{p} of R .*
- (e) *If x_1, \dots, x_h are strong parameters, then they are parameters in R and in R/\mathfrak{p} for every minimal prime \mathfrak{p} of R . Moreover, x_1, \dots, x_h are strong parameters in R if and only if they are strong parameters in R/\mathfrak{p} for every minimal prime \mathfrak{p} of R .*
- (f) *If x_1, \dots, x_h are part of a system of parameters for B whenever B is a complete local domain of R at a maximal ideal containing I , then x_1, \dots, x_h are strong parameters for R .*

Proof. We may assume in all parts that I is a proper ideal. For (a), if x_1, \dots, x_h are parameters then their images form a system of parameters after we localize at any minimal prime of I . This shows that every minimal prime of I has height h , and so $\text{ht } I = h$. On the other hand, suppose that $\text{ht } I \geq h$. Then this remains true when we localize at any prime P containing I . Thus, it will suffice to show that if R is local and $(x_1, \dots, x_h) \subseteq m$ is such that $\text{ht } I \geq h$, then \underline{x} is part of a system of parameters for R . Notice that it is clear that $\text{ht } I = h$ in this case. We use induction on $\dim R - h$. If m is a minimal prime of I the result is clear, for then $\dim R = \text{ht } m \leq h = \text{ht } I \leq \text{ht } m$. If not, choose $x_{h+1} \in m$ not in any minimal prime of (x_1, \dots, x_h) . Then it is clear that

$$h + 1 \geq \text{ht } (x_1, \dots, x_{h+1}) > \text{ht } I = h,$$

so that $\text{ht } (x_1, \dots, x_{h+1}) = h + 1$. By the induction hypothesis, x_1, \dots, x_{h+1} is part of a system of parameters for R .

Part (b) is immediate from the definition of strong parameters and the definition of minheight. Part (c) is clear since the complete local domains of a localization of R are a subset of the complete local domains of R .

The first equivalence in part (d) follows because when R_P is universally catenary, $\text{mnht } I(R_P)^\wedge = \text{mnht } IR_P$ and $\text{mnht } I = \inf \{ \text{mnht } IR_P : P \text{ is prime and } P \supseteq I \}$ by

[HH8], Proposition (2.2e,f). The second statement is a consequence of the fact that the completion B of a universally catenary equidimensional local ring is equidimensional (cf. [HIO], Theorem (18.17)).

The second statement in part (e) follows from the fact that the complete local domains of R are the same as the complete local domains of the rings R/\mathfrak{p} as \mathfrak{p} varies through the minimal primes of R , by (2.3.10b). For the first statement it evidently suffices to see that, in the local case, strong parameters are parameters, and since elements of a local ring R that are parameters in \widehat{R} are parameters in R , it suffices to see this when R is a complete local ring. But then it follows from the general fact that $\text{mnht } I \leq \text{ht } I$ (see [HH8] Proposition (2.2), part (g)).

To prove (f), let P be any prime ideal of R containing I and let m be a maximal ideal of R containing P . We must show that $\text{mnht } I(R_P)^\wedge \geq h$. Let $S = (R_m)^\wedge$. Since $R_m \rightarrow S$ is faithfully flat, we can choose a prime ideal Q of S lying over PR_m . Then $R_P \rightarrow S_Q$ is faithfully flat and so $(R_P)^\wedge \rightarrow (S_Q)^\wedge$ is faithfully flat. Since $(R_P)^\wedge$ is universally catenary and the map is faithfully flat, we have from [HH8] Proposition (2.2f) that $\text{mnht } I(R_P)^\wedge = \text{mnht } I(S_Q)^\wedge$, and so it will suffice to show that $\text{mnht } I(S_Q)^\wedge \geq h$. By the same result, $\text{mnht } I(S_Q)^\wedge = \text{mnht } IS_Q \geq \text{mnht } IS$ (by [HH8], Proposition (2.2d)) $\geq h$, by hypothesis. \square

(2.3.12) Corollary. *Let $A \rightarrow R_A$ and $\mathfrak{F} \subseteq K$ be as in (2.3.4). Suppose that $x_1, \dots, x_h \in R_A$ and their images in R_K (or $R_{\mathfrak{F}}$) are parameters (respectively, strong parameters). Then for almost all fibers, their images in R_κ are parameters (respectively, strong parameters).*

Proof. The problem is to preserve the height of (x_1, \dots, x_h) (or, for strong parameters, the minheight, since all the rings R_L , where L is a field, are universally catenary, and we may apply (2.3.11d)). The result is then immediate from (2.3.9c,h). \square

(2.3.13) Discussion. (a) When $(B, m_B, K_B) \rightarrow (C, m_C, K_C)$ is a flat local homomorphism of local rings and $\overline{C} = (C/m_B C, m_C/m_B C, K_C)$ is the closed fiber, we have that $\dim C = \dim B + \dim \overline{C}$ and $\text{depth } C = \text{depth } B + \text{depth } \overline{C}$. It follows that C is Cohen-Macaulay if and only if both B and \overline{C} are Cohen-Macaulay. Cf. [Mat] (20.A,B,C), pp. 152-4. In the same situation, if C (and, hence, B and \overline{C}) are Cohen-Macaulay, the type of

C (cf. (1.2.1g)) is the product of the types of B and \overline{C} (see [HeK], Satz 1.24, p. 6) so that C is Gorenstein if and only if B and \overline{C} are Gorenstein. It follows (globally) that if C is a Noetherian ring faithfully flat over a Noetherian ring B and C is Cohen-Macaulay (respectively, Gorenstein) then B is Cohen-Macaulay (respectively, Gorenstein).

(b) It is also worth noting that if R is a finitely generated algebra over a field K and L is an extension field of K then R is Cohen-Macaulay (respectively, Gorenstein) if and only if $L \otimes_K R$ is Cohen-Macaulay (respectively, Gorenstein). The “if” part follows because $R \rightarrow L \otimes_K R$ is faithfully flat; the “only if” part follows from the remarks above once we know that the fibers are Gorenstein, i.e., that if K' is a finitely generated field extension of K and L is any field extension of K , then $L \otimes_K K'$ is Gorenstein. But it is easy to see by induction on the number of field generators of K' over K that if L is a Gorenstein K -algebra and K' is a finitely generated field extension of K then $L \otimes_K K'$ is Gorenstein. (The problem reduces at once to the case of a single generator. If $K' = K(x)$ with x transcendental then $L \otimes_K K'$ is a localization of $L[x]$; if $K' = K[x]/(f)$ with f monic then $L \otimes_K K' = L[x]/(f)$ with f monic and, hence, not a zerodivisor in $L[x]$.) Cf. [WITO].

(c) We also note that the Cohen-Macaulay locus is Zariski open in $\text{Spec } R$ whenever R is a homomorphic image of a Cohen-Macaulay ring (cf. the discussion in [EGA1] Proposition (6.11.8) and Remarques (6.11.9)). Moreover, the Gorenstein locus in $\text{Spec } R$ is open whenever R is a homomorphic image of a Gorenstein ring: one can construct a global canonical module ω for R in that case, and if $\omega_P \cong R_P$ for some prime P (which characterizes when R_P is Gorenstein once it is known to be Cohen-Macaulay), then we can localize at one element $f \in R - P$ such that $\omega_f \cong R_f$.

The results of (2.3.14) just below are valid without the hypotheses that certain loci be open in $\text{Spec } R_A$: the more general result is given in (2.3.15).

(2.3.14) Lemma. *Let $A \rightarrow R_A$ and $\mathfrak{F} \subseteq K$ be as in (2.3.4).*

- (a) *Suppose that the Cohen-Macaulay locus $\{P \in \text{Spec } R_A : (R_A)_P \text{ is Cohen-Macaulay}\}$ is open. If R_K is Cohen-Macaulay, then so are almost all the fibers R_κ .*
- (b) *Suppose that the Gorenstein locus $\{P \in \text{Spec } R_A : (R_A)_P \text{ is Gorenstein}\}$ is open. If R_K is Gorenstein, then so are almost all the fibers R_κ .*

Proof. We give the proof of (a): the argument for (b) is identical.

Let $J \subseteq R_A$ define the non-Cohen-Macaulay locus. Since R_K is faithfully flat over R_k , if R_K is Cohen-Macaulay then so is $R_k = (A^\circ)^{-1}R_A$. It follows that A° meets J , and if we localize at $a \in A^\circ \cap J$, then R_B is Cohen-Macaulay, where $B = A_a$. Localizing further, if necessary, we may assume that R_B is free (hence, faithfully flat) over B . If P is a prime ideal of B , then every prime ideal \mathcal{Q} of R_κ , where $\kappa = \kappa(P)$, corresponds to a prime ideal Q of R_B lying over P , and $(R_\kappa)_\mathcal{Q}$ is isomorphic with the closed fiber of $B_P \rightarrow (R_B)_Q$, which will be Cohen-Macaulay since R_B is. \square

(2.3.15) Theorem. *Let $A \subseteq R_A$ and $\mathfrak{F} \subseteq K$ be as in (2.3.4). If R_K is Cohen-Macaulay (respectively, Gorenstein) then so is R_κ for almost all fibers.*

Proof. We are free to localize at one (or finitely many) elements of A . It is then clear that we can choose $B \subseteq A$ finitely generated over \mathbb{Z} and $R_B \subseteq R_A$ such that R_B is B -free and $R_A \cong A \otimes_B R_B$. Since R_B is a homomorphic image of a regular ring, the relevant loci are open. Since $K \otimes_B R_B \cong K \otimes_A R_A$, we see that $K \otimes_B R_B$ is Cohen-Macaulay (respectively, Gorenstein), and so we may apply (2.3.14) to conclude that after localizing B at one element of B° , we have that all fibers of $B \rightarrow R_B$ are Cohen-Macaulay (respectively, Gorenstein). We also localize A at this element so that we have homomorphisms $B \rightarrow A$ and $R_B \rightarrow R_A$ with $R_A \cong A \otimes_B R_B$. Let P be any prime of A and let Q be its contraction to B . Then the local homomorphism $B_Q \rightarrow A_P$ induces a homomorphism of residue fields, say $\lambda \rightarrow \kappa$. The fiber of R_A over P is $\kappa \otimes_A R_A \cong \kappa \otimes_A (A \otimes_B R_B) \cong \kappa \otimes_B R_B \cong \kappa \otimes_\lambda (\lambda \otimes_B R_B)$, and since $\lambda \otimes_B R_B$ is a Cohen-Macaulay (respectively, Gorenstein) ring finitely generated over λ , this remains true when we make a base change of the field, by (2.3.13b). \square

When the fraction field \mathfrak{F} of the domain A is of characteristic zero there are additional results on what properties of the generic fiber are preserved for almost all fibers. In the remainder of this section, we focus on these results. Formally, many of the proofs work also when k is a perfect field of characteristic p , but the fraction field of a Noetherian domain A of characteristic p cannot be perfect unless A is itself a field: consequently, this case is vacuous.

One can obtain interesting characteristic p versions of certain of the results below by

adding hypotheses in which certain rings are assumed to have the “geometric” version of one of the properties reduced, normal, etc. Since we do not need these technical results, we shall not pursue this point.

When the results obtained in the remainder of this section are applied in other parts of this paper, A will be a domain that is a finitely generated extension of \mathbb{Z} , so that it will in fact be the case that \mathfrak{F} has characteristic 0.

(2.3.16) Theorem. *Let $A \rightarrow R_A$ and $\mathfrak{F} \subseteq K$ be as in (2.3.4). Let I_A be an ideal of R_A . Assume also that \mathfrak{F} has characteristic zero. Let J_A be the ideal of all nilpotent elements of R_A and J the ideal of all nilpotent elements of R_K .*

- (a) *After localizing at one element of A , $J = J_K$. Moreover, for almost all fibers, J_κ is the ideal of all nilpotent elements in R_κ . If R_A is reduced, then R_κ is reduced for almost all fibers.*
- (b) *If Q_A is an ideal of R_A such that Q_K is the radical of I_K , then Q_κ is the radical of I_κ for almost all fibers.*
- (c) *If I_K is radical then I_κ is radical for almost all fibers. In particular, this holds when I_K is prime.*
- (d) *Let $\mathfrak{p}_A^{(1)}, \dots, \mathfrak{p}_A^{(s)}$ be the distinct minimal primes of A . Then for almost all fibers, $\mathfrak{p}_\kappa^{(1)}, \dots, \mathfrak{p}_\kappa^{(s)}$ are radical ideals of pure height 0 in R_κ , so that the minimal primes of R_κ are precisely the minimal primes of the $\mathfrak{p}_\kappa^{(i)}$, and each occurs as a minimal prime of precisely one of the $\mathfrak{p}_\kappa^{(i)}$.*

Proof. Let J' be the contraction of J to $R_{\mathfrak{F}}$, i.e., the ideal of all nilpotents in $R_{\mathfrak{F}}$. It is clear that after localizing at one element of A° we have that $J' = J_{\mathfrak{F}}$. It follows that the ring $R_{\mathfrak{F}}/J_{\mathfrak{F}}$ is reduced, and, hence, geometrically reduced. But then $K \otimes_{\mathfrak{F}} (R_{\mathfrak{F}}/J_{\mathfrak{F}})$ is reduced, whence it follows that $J = J_K$. This establishes the first statement in part (a). The second statement follows from the third statement applied to R_A/J_A (since it is clear that every J_κ consists of nilpotent elements of R_κ once we have localized enough so that R_A/J_A is free, so that it makes sense to view J_κ as an ideal of R_κ). Thus, it remains to show that if R_A is reduced, then so are almost all the fibers R_κ . But since \mathfrak{F} has characteristic zero, the fact that $R_{\mathfrak{F}} \subseteq R_K$ is reduced implies that it is geometrically reduced, and so we may

apply (2.3.6b).

Part (b) follows from (a) applied to R_A/I_A , and (c) is a special case. Part (d) is immediate from Theorem (2.3.9d,e,f) and the additional fact that $\mathfrak{p}_\kappa^{(i)}$ are radical ideals for almost all fibers. \square

We also have:

(2.3.17) Proposition. *Let $A \rightarrow R_A$ and $\mathfrak{F} \subseteq K$ be as in (2.3.4) and suppose that \mathfrak{F} has characteristic zero.*

If R_K is normal, then almost all the fibers R_κ are (geometrically) normal.

If R_K is reduced and S_K is its normalization then for almost all fibers, S_κ is the normalization of R_κ .

If $J_K \subseteq R_K$ is the integral closure of $I_K \subseteq R_K$ then J_κ is the integral closure of I_κ in R_κ for almost all fibers κ .

Proof. The condition for R_K to be normal is that the defining ideal of the smooth locus, which will be the same as the nonsingular locus since K has characteristic zero, has depth at least two. But this ideal may be obtained from the defining ideal \mathcal{J}_A of the smooth locus over A by taking the image of \mathcal{J}_K in R_K . (We may localize at one element of A° so that R_A/\mathcal{J}_A is A -free, so that for every field L to which A maps we have that \mathcal{J}_L injects into R_L and defines the smooth locus of R_L over L .) By Theorem (2.3.5h) we have that \mathcal{J}_κ has depth at least two as an ideal of R_κ for almost all fibers, and this is preserved when we pass to any extension field of κ . It follows that almost all the fibers are geometrically normal.

For the second part, since S_K is module-finite over R_K , we shall have that S_κ is module-finite over R_κ for almost all fibers, and by the first part it will be normal for almost all fibers. Moreover, for almost all fibers there will be a nonzerodivisor that multiplies it into R_κ .

For the third part it is routine to reduce to the case where R_K is reduced. Consider the Rees ring $R_K[I_K z] \subseteq R_K[z]$ and its normalization $T_K \subseteq S_K[z]$, whose degree 0 graded piece is S_K , and whose degree one graded piece has the form $J'_K z$, where J'_K is an ideal of S_K such that $J'_K \cap R_K = J_K$. By the lemma of generic freeness, almost all the fibers

of $R_A[I_A z]$ inject into $R_\kappa[z]$ and almost all the fibers of T_A inject into $S_\kappa[z]$. Almost all the T_κ are the normalizations of the corresponding $R_\kappa[I_\kappa z]$, with degree 0 graded piece S_κ , and it follows that for almost all fibers, $J'_\kappa \cap R_\kappa$ is the integral closure of I_κ . But since $R_K \rightarrow S_K/J'_K$ is injective, the lemma of generic freeness guarantees that $R_\kappa/J_\kappa \rightarrow S_\kappa/J'_\kappa$ will be injective for almost all fibers κ , which shows that $J'_\kappa \cap R_\kappa = J_\kappa$ for almost all κ . \square

(2.3.18) Discussion: decomposition into absolute primes. Throughout this discussion we assume that $A \rightarrow R_A$ and $\mathfrak{F} \subseteq K$ are as in (2.3.4) and that \mathfrak{F} has characteristic zero. Let P_A be a prime ideal of R_A . We know, for almost all fibers, that P_κ is a radical ideal by Theorem (2.3.16c), but we do not appear to have much control over its minimal primes: it is not even clear from what has been said so far that the number of them is bounded independent of κ .

On the other hand, if $P_{\mathfrak{F}}$ is an absolute prime then we know that P_κ is an absolute prime for almost all κ . The point we want to make here is that if, as well as localizing at one element of A° , one is willing to make a finite algebraic extension of \mathfrak{F} to, say, \mathfrak{F}' (and a corresponding module-finite extension A' of A) then one can decompose $P_{A'}$ as a finite intersection of primes of $R_{A'}$, say $P_{A'} = Q_{A'}^{(1)} \cap \cdots \cap Q_{A'}^{(t)}$, such that every $Q_{\mathfrak{F}}^{(i)}$, is an absolute prime of $R_{\mathfrak{F}'}$.

This is rather straightforward. Let L be an algebraic closure of \mathfrak{F} . Then P_L is radical, and has a primary decomposition $Q_L^{(1)} \cap \cdots \cap Q_L^{(t)}$ in R_L . By a direct limit argument there is a finite algebraic extension \mathfrak{F}' of \mathfrak{F} such that the contractions of these primes to $R_{\mathfrak{F}'}$, are distinct, and such that each of them is generated by its contraction to $R_{\mathfrak{F}'}$. Thus, for a sufficiently large but finite algebraic extension \mathfrak{F}' of \mathfrak{F} , we have that $P_{\mathfrak{F}'}$ has minimal primary decomposition $Q_{\mathfrak{F}'}^{(1)} \cap \cdots \cap Q_{\mathfrak{F}'}^{(t)}$, where every $Q_{\mathfrak{F}'}^{(i)}$ is absolutely prime over \mathfrak{F}' . Then we can find a ring A' finitely generated over A with $A \subseteq A' \subseteq \mathfrak{F}'$ such that all the $Q_{\mathfrak{F}'}^{(i)}$ arise by base change from their contractions $Q_{A'}^{(i)}$. Notice that, enlarging A' if necessary, we may think of it either as a module-finite extension domain of a localization of A at one element of A° or even as the localization at one element of A° of a module-finite extension domain of A (since every nonzero element of the extension domain has a nonzero multiple in A).

It follows that after localizing at one element of A° , we have that for all fibers of

$A' \rightarrow R_{A'}$, the minimal primary decomposition of $P'_{\kappa'}$ is given by $Q_{\kappa'}^{(1)} \cap \cdots \cap Q_{\kappa'}^{(t)}$.

Now suppose that we are concerned with the primary decomposition of the original P_A as we pass to fibers. Suppose that π is any prime ideal of A . Then there is a prime ideal π' of A' lying over π , and we get a finite algebraic extension of fields from $A_\pi/\pi A_\pi = \kappa$ to $\kappa' = A'_{\pi'}/\pi' A'_{\pi'}$. Then $P_{\kappa'}$ in $R_{\kappa'} \cong \kappa' \otimes_\kappa R_\kappa$ lies over P_κ in R_κ and so we have that $P_\kappa = (Q_{\kappa'}^{(1)} \cap R_\kappa) \cap \cdots \cap (Q_{\kappa'}^{(t)} \cap R_\kappa)$ is a primary decomposition of P_κ in R_κ . Note that since $R_{\kappa'}$ is faithfully flat over R_κ , every minimal prime of $P_{\kappa'} = P_\kappa R_{\kappa'}$, will contract to a minimal prime of P_κ in R_κ . Then the minimal primes of P_κ are simply the distinct primes of the form $Q_{\kappa'}^{(i)} \cap R_\kappa$ for $1 \leq i \leq t$, and so there are at most t of them. However, there may be duplications in that it is possible that $Q_{\kappa'}^{(i)} \cap R_\kappa = Q_{\kappa'}^{(j)} \cap R_\kappa$ even though $i \neq j$ and $Q_{\kappa'}^{(i)}$ and $Q_{\kappa'}^{(j)}$ are distinct. See the example below.

(2.3.19) Example. Suppose that $A = \mathbb{Z}$, $R_A = \mathbb{Z}[x, y]$ and that $P = (x^2 + y^2)$. This is prime but not an absolute prime. But if we let $A' = \mathbb{Z}[i]$ where i is a square root of -1 , we obtain the decomposition $P_{A'} = Q_{A'}^{(1)} \cap Q_{A'}^{(2)}$ where these two primes are generated by $x + yi$ and $x - yi$ respectively. In considering the closed fibers over \mathbb{Z} , which correspond to prime integers $p > 0$ in \mathbb{Z} , we get two distinct primes in the decomposition if p is odd and -1 is a square modulo p (i.e., $p \equiv 1$ modulo 4) and a single prime if p is odd and -1 is not a square modulo p (i.e., $p \equiv -1$ modulo 4). In the second case the two primes $Q_{\kappa'}^{(i)}$ have the same contraction to R_κ . If $p = 2$, P_κ is not radical: this is the fiber we exclude.

(2.4) UNIVERSAL TEST ELEMENTS

Before beginning the systematic study of the characteristic zero tight closure operation *K , we want to develop a method of constructing elements in a reduced finitely generated algebra R_A over a finitely generated \mathbb{Z} -algebra A that will turn out to be test elements for almost all the closed fibers R_κ . In fact, we shall see that even more is true.

(2.4.1) Conventions. Throughout §(2.4) let $A \supseteq \mathbb{Z}$ be a domain finitely generated over \mathbb{Z} with fraction field \mathfrak{F} , and let R_A be a finitely generated A -algebra. Throughout this section we shall frequently say “Let $A \rightarrow R_A$ be as in (2.4.1).”

(2.4.2) Definition: universal test elements. Let notation be as in (2.4.1). We shall

say that $c_A \in R_A$ is a *universal test element* for $A \rightarrow R_A$ if after A is replaced by a suitable localization at one element of A° the following conditions are satisfied:

- (1) $c_A \in R_A^\circ$. (This will hold after localizing at one element of A° if and only if $c_{\mathfrak{F}} \in R_{\mathfrak{F}}^\circ$.)
- (2) For every homomorphism $A \rightarrow \Lambda$, where Λ is a regular domain of positive characteristic, c_Λ is a completely stable test element for R_Λ .
- (2 $^\circ$) For every homomorphism $A \rightarrow \Lambda$, where Λ is a regular ring of positive characteristic, c_Λ is a completely stable test element for R_Λ .

Conditions (2) and (2 $^\circ$) above are equivalent: (2 $^\circ$) \Rightarrow (2) trivially. If (2) holds and Λ is any regular ring, Λ is a finite product of regular rings Λ_i , and R_Λ is the product of the rings R_{Λ_i} ; moreover, the i th component of c_Λ is c_{Λ_i} . It is then evident that c_Λ is a completely stable test element for R_Λ if and only if for all i , c_{Λ_i} is a completely stable test element for R_{Λ_i} .

Of course, since a field is a regular ring, when c_A is a universal test element it is a completely stable test element in R_κ for almost all closed fibers.

We note the following reformulation of (2.3.9a):

(2.4.3) Proposition. *Let $A \rightarrow R_A$ be as in (2.4.1). If $c_A \in R_A^\circ$ (respectively, is a nonzerodivisor), then after replacing A by its localization at one element of A° we have the following:*

For every homomorphism $A \rightarrow \Lambda$, where Λ is any Noetherian ring, $c_\Lambda \in R_\Lambda^\circ$ (respectively, is a nonzerodivisor). \square

Our next objective is to prove the existence of universal test elements when R_A is reduced. We want to reduce to the case where R_A is a domain, for which we need:

(2.4.4) Proposition. *Let $A \rightarrow R_A$ be as in (2.4.1). Suppose that $R_{\mathfrak{F}}$ is reduced and that $\mathfrak{p}_A^{(1)}, \dots, \mathfrak{p}_A^{(s)}$ are the minimal primes of R_A not meeting A . Suppose that $c_A^{(i)}$ is an element of R_A whose image in $R_A/\mathfrak{p}_A^{(i)}$ is a universal test element for $R_A/\mathfrak{p}_A^{(i)}$ for $1 \leq i \leq s$. Suppose also that for every i , $1 \leq i \leq s$, $b_A^{(i)}$ is chosen in $(\bigcup_{j \neq i} \mathfrak{p}_A^{(j)}) - \mathfrak{p}_A^{(i)}$, and also so that $b_A^{(i)} \mathfrak{p}_A^{(i)} = 0$ (since R_A is reduced and $\mathfrak{p}_A^{(i)}$ is a minimal prime, we can choose an element not in it that kills $\mathfrak{p}_A^{(i)}$, and we can multiply this for every $j \neq i$ by an element of $\mathfrak{p}_A^{(j)} - \mathfrak{p}_A^{(i)}$).*

Then $c_A = \sum_{i=1}^s b_A^{(i)} c_A^{(i)}$ is a universal test element for R_A .

Proof. We need to show that for almost every regular domain Λ of characteristic p to which A maps, c_Λ is a completely stable test element for R_Λ . We know for almost all²⁴ regular domains Λ of characteristic p to which A maps, for all i , $1 \leq i \leq s$, $c_\Lambda^{(i)}$ is a completely stable test element in $R_\Lambda/\mathfrak{p}_\Lambda^{(i)}$.

Now, it is clear that c_A is not in any of the $\mathfrak{p}_A^{(i)}$, since its image in $R_A/\mathfrak{p}_A^{(i)}$ is the same as the image of $b_A^{(i)} c_A^{(i)}$. It follows that c_Λ is in R_Λ° for almost all Λ . Let B be the completion of a local ring of R_Λ , and let $N \subseteq M$ be finitely generated B -modules. It will suffice to show that if $u \in N^*_M$ then $c_\Lambda u^q \in N^{[q]}$ in $\mathbf{F}^e(M)$ for all q . If we pass to $B/\mathfrak{p}_\Lambda^{(i)} B$ the persistence of tight closure (1.4.13) implies that the image of u is in the tight closure of $\langle N/\mathfrak{p}_\Lambda^{(i)} N \rangle$ in $M/\mathfrak{p}_\Lambda^{(i)} M$ over $B/\mathfrak{p}_\Lambda^{(i)} B$. Since the image of $c_\Lambda^{(i)}$ is a completely stable test element for $R_\Lambda/\mathfrak{p}_\Lambda^{(i)}$, its image in $B/\mathfrak{p}_\Lambda^{(i)} B$ is a test element, and it follows that for all q we have

$$(*_i) \quad c_\Lambda^{(i)} u^q \in N^{[q]} + \mathfrak{p}_\Lambda^{(i)} \mathbf{F}^e(M).$$

If we multiply the equation $(*_i)$ by $b_\Lambda^{(i)}$ then we see, since $b_\Lambda^{(i)} \mathfrak{p}_\Lambda^{(i)} = (0)$, that $b_\Lambda^{(i)} c_\Lambda^{(i)} u^q \in N^{[q]}$ for all i . Summing these equations over i yields the required result. \square

Our next main objective is to construct universal test elements in the reduced equidimensional case using discriminants. We first digress to give a discussion of discriminants.

(2.4.5) Discriminants: definition, discussion, and basic properties. (a) Let $T \subseteq R$ be a module-finite extension where T is a domain and let \mathcal{K} be the fraction field of T . Let $\theta = \theta_1, \dots, \theta_n$ be elements of R that form a basis for $\mathcal{L} = \mathcal{K} \otimes_T R$ over \mathcal{K} . We write $D = D(\theta) = D_{R/T}(\theta)$ for $\det(\text{tr } \theta_i \theta_j)$, where $\text{tr} = \text{tr}_{\mathcal{L}/\mathcal{K}}$ is the trace map from \mathcal{L} to \mathcal{K} . (Thus, the trace of u is the same as the trace of the \mathcal{K} -linear map $\mathcal{L} \rightarrow \mathcal{L}$ given by multiplication by u .) We refer to $D_{R/T}(\theta)$ as the *discriminant of $T \rightarrow R$ with respect to θ* . If we have a different basis θ' such that $\theta' = \alpha\theta$ (where θ, θ' are the two bases written as column vectors and α is a matrix of elements of \mathcal{K}), then the matrix of the bilinear form

²⁴See (2.3.4).

with respect to θ' is easily calculated to be $\alpha(\text{tr } \theta_i \theta_j) \alpha^{\text{tr}}$ (where $^{\text{tr}}$ indicates transpose), and so we have that $D(\theta') = (\det \alpha)^2 D(\theta)$.

(b) If T' is a domain with fraction field \mathcal{K}' containing T then the image of θ in $R' = T' \otimes_T R$ will be a basis for $\mathcal{K}' \otimes_{T'} R'$ over \mathcal{K}' , and it follows that $D_{R'/T'}(\text{Im } \theta)$ is the image of $D_{R/T}(\theta)$ in \mathcal{K}' .

(c) We next note that if T is a normal Noetherian domain, and u denotes any element of \mathcal{L} integral over T , then $\text{tr } u$ is an element of T . (If not, we can preserve the situation while replacing T by a discrete valuation ring V and killing the annihilator of V° in $V \otimes_T R$. Thus, we may assume that T is a DVR and that R is torsion-free over T . We may enlarge R to $R[u]$ and so assume that $u \in R$. Then R is torsion-free and hence free over T , and as our \mathcal{K} -basis for \mathcal{L} we may use a free basis for R over T . The matrix for multiplication by u with respect to this basis will have all entries in T , and so its trace is in T .)

(d) It follows that if T is a normal Noetherian domain then every entry of the matrix $(\text{tr } \theta_i \theta_j)$ is in T , and so $D_{R/T}(\theta)$ is always an element of T in this case.

(e) It is well known that a finite algebraic extension field \mathcal{L} of a field \mathcal{K} is separable if and only if for some (equivalently, every) basis $\theta = \theta_1, \dots, \theta_n$ for \mathcal{L} over \mathcal{K} , where $n = [\mathcal{L} : \mathcal{K}]$, $D = D_{\mathcal{L}/\mathcal{K}}(\theta)$ is not zero, i.e., if and only if $\tau: \mathcal{L} \otimes_{\mathcal{K}} \mathcal{L} \rightarrow \mathcal{K}$ via $\theta \otimes \theta' \mapsto \text{tr}_{\mathcal{L}/\mathcal{K}}(\theta \theta')$ is nondegenerate. (\mathcal{L}/\mathcal{K} is separable iff for some $\lambda \in \mathcal{L}$, $\text{tr}_{\mathcal{L}/\mathcal{K}} \lambda \neq 0$, and then for any θ , $\tau(\theta, \lambda/\theta) \neq 0$). It is easy to see, more generally, that a finite-dimensional (as a \mathcal{K} -vector space) \mathcal{K} -algebra \mathcal{L} is étale over \mathcal{K} if and only if the same condition holds. (Both conditions are invariant under making a base change to an algebraic closure of \mathcal{K} , so that we may assume that \mathcal{K} is algebraically closed. Then \mathcal{L} is étale if and only if $\mathcal{L} \cong \mathcal{K}^{\times n} = \mathcal{K} \times \dots \times \mathcal{K}$ (product ring) if and only if \mathcal{L} is reduced. It is trivial to verify that τ is nondegenerate if $\mathcal{L} = \mathcal{K}^{\times n}$ while if \mathcal{L} contains a nonzero nilpotent θ then $\text{tr}_{\mathcal{L}/\mathcal{K}} \theta \theta' = 0$ for all choices of θ' ($\theta \theta'$ is again nilpotent).)

(f) Thus, if $T \subseteq R$ as in (a), then R is generically smooth over T , i.e., \mathcal{L} is smooth (\equiv étale) over \mathcal{K} , if and only if $D_{R/T}(\theta) \neq 0$ for some (equivalently, every) choice of basis θ . Moreover, when T is normal, $D_{R/T}(\theta) \in T$.

(g) Again, let $T \subseteq R$ and $\mathcal{K} \subseteq \mathcal{L}$ be as in (a), with R torsion-free and generically smooth over T . Assume that T is normal, let θ be any basis for \mathcal{L} over \mathcal{K} in T , and let

$D = D_{R/T}(\theta)$, which we know to be a nonzero element of T . Let S denote the integral closure of R in \mathcal{L} , which may also be described as the integral closure of T in \mathcal{L} . Since \mathcal{L} is the total quotient ring of the reduced ring R , we may also think of S as the normalization of R in \mathcal{L} . Then $DS \subseteq R$. In fact, $DS \subseteq T\theta_1 + \cdots + T\theta_n$.²⁵

(h) Let $T \subseteq R$ be module-finite, torsion-free and generically smooth, where T is a Noetherian normal domain. If $D = D_{R/T}(\theta)$ is a unit of T then θ is a free basis for R over T , $T \rightarrow R$ is étale, and R is integrally closed in \mathcal{L} . Thus, $T_D \rightarrow R_D$ is étale. (Since D multiplies the integral closure of R in \mathcal{L} into $T\theta_1 + \cdots + T\theta_n \subseteq R$ it is clear that if D is a unit of T then θ is a free basis for R over T and R is integrally closed in \mathcal{L} . Since $T \rightarrow R$ is free it suffices to check that the fibers are étale. But after passing to any fiber the image θ_κ of θ is still a free basis for $\lambda = R_\kappa$ over κ , and the discriminant $D_{\lambda/\kappa}(\theta_\kappa)$ is the image of D in κ . Since D is a unit, the image is not zero.

(2.4.6) Discussion. Now suppose that $A \subseteq R_A$ and \mathfrak{F} are as in (2.4.1). Assume, moreover, that R_k is reduced and biequidimensional. After localizing at one element of A° , we may suppose that R_A is module-finite over a polynomial ring $T_A = A[x_1, \dots, x_d]$. Since $R_{\mathfrak{F}}$ is biequidimensional, after localizing A at one element of A° we may assume that R_A is torsion-free as a T_A -module. Since the fraction field $\mathcal{K} = \mathfrak{F}(x_1, \dots, x_d)$ of T_A has characteristic zero, we see that R_A is module-finite, torsion-free, and generically smooth over the regular domain T_A . We can fix a vector space basis θ_A for $\mathcal{L} \cong \mathcal{K} \otimes_{T_A} R_A$ over \mathcal{K} such that θ_A consists of elements $\theta_A^{(i)}$ in R_A . We shall write D_A for $D_{R_A/T_A}(\theta_A)$. Since $D_A \neq 0$ and is in T_A it is a nonzerodivisor on R_A .

Some of our main results on the existence of universal test elements are consequences of the following result, which carries through the plan indicated in the remark just prior to Lemma (6.5) of [HH4], p. 51:

²⁵This is well known: cf. the remark following the proof of Theorem 7 in Chapter V, §4 of [ZS], Vol. I, and, for that matter, Lemma (6.5) of [HH4], p. 51 (although the generality is somewhat greater here) but we sketch the proof. Let u be any element of S . We must show that $Du \in T\theta_1 + \cdots + T\theta_n$. We can write u uniquely as $u = \sum_{i=1}^n \lambda_i \theta_i$ with coefficients λ_i in \mathcal{K} . To complete the argument, it will suffice to prove that each of the elements $D\lambda_i$ is in T . But since $u = \sum_{i=1}^n \lambda_i \theta_i$, we have that for every j , $u\theta_j = \sum_{i=1}^n \lambda_i \theta_i \theta_j$, which yields the matrix equation $\beta\lambda = \gamma$, where $\beta = (\text{tr } \theta_i \theta_j)$, λ is the column with entries λ_i , and γ is the column with j th entry $\gamma_j = \text{tr}(u\theta_j) \in T$, by part (c). Since β also has entries in T , if we multiply both sides of this equation by the classical adjoint $\text{adj } \beta$ of β we obtain that $D\lambda = (\text{adj } \beta)\gamma$ has entries in T , and so the elements $D\lambda_i \in T$, as required.

(2.4.7) Theorem. *Let $A \subseteq R_A$ and \mathfrak{F} be as in (2.4.1) and suppose that $R_{\mathfrak{F}}$ is reduced and biequidimensional. Let θ and D_A be defined as in the preceding paragraph. Then D_A is a universal test element for $A \subseteq R_A$.*

Proof. Let $T_A = A[x_1, \dots, x_d]$ be as in the discussion preceding the statement of the theorem. Thus, we may assume that R_A is module-finite, torsion-free, and generically smooth over T_A and that we have fixed a T_A -linear embedding $\phi_A : R_A \hookrightarrow T_A^n$. Likewise, we have a T_A -linear embedding $\psi_A : T_A^n \hookrightarrow R_A$ that sends the standard free basis for T_A^n to the elements $\theta_A^{(1)}, \dots, \theta_A^{(n)}$ of the given basis θ_A (for \mathcal{L} over \mathcal{K}).

After localizing at one element of A° we may assume that the cokernels of ϕ_A, ψ_A and multiplication by D_A on R_A are A -free.

When we pass to (i.e., tensor over A with) almost any regular domain Λ of characteristic p such that A maps to Λ we have that ϕ_Λ and ψ_Λ are still injective. For almost every such Λ we have that R_Λ is module-finite and torsion-free over $T_\Lambda = \Lambda[x]$ and we also have that for almost all²⁶ Λ that $D_\Lambda \neq 0$. This shows also that θ_Λ is a basis over $\mathcal{K}(\Lambda)$, the fraction field of T_Λ , for $\mathcal{L}(\Lambda) = \mathcal{K}(\Lambda) \otimes_{T_\Lambda} R_\Lambda$. But then D_Λ is evidently $D_{R_\Lambda/T_\Lambda}(\theta_\Lambda)$, and it follows that R_Λ is generically smooth over T_Λ . Thus, for almost all Λ , R_Λ is a module-finite, torsion-free and generically smooth extension ring of the regular domain T_Λ , and D_Λ is a nonzero element of T_Λ . By (2.4.5h), localizing at D_Λ will make the extension $T_\Lambda \rightarrow R_\Lambda$ smooth.

Let p denote the characteristic of Λ and q denote p^e , as usual, with e varying in \mathbb{N} . By Theorem (1.4.9) it will suffice, to complete the proof, to show that for almost all Λ , D_Λ multiplies $(R_\Lambda)^\infty$ into $(T_\Lambda)^\infty[R_\Lambda] \subseteq (R_\Lambda)^\infty$, and for this it suffices to see that for all q , D_Λ multiplies $(R_\Lambda)^{1/q}$ into $(T_\Lambda)^{1/q}[R_\Lambda] \subseteq (R_\Lambda)^{1/q}$.

But we can argue almost exactly as in the proof of Corollary (1.5.4), using (2.4.5g) instead of the Lipman-Sathaye theorem. The only point that we need is that $(R_\Lambda)^{1/q}$ is in the normalization of $(T_\Lambda)^{1/q}[R_\Lambda]$ and we need to see that we have an inclusion after tensoring with the fraction field of T_Λ . The argument is the same as in the proof of (1.5.4), except that now when we localize R_Λ we get a finite product of separable field extensions instead of just one. \square

²⁶See (2.3.4).

We then have:

(2.4.8) Corollary. *Let $A \subseteq R_A$ and \mathfrak{F} be as in (2.4.1) and suppose that R_A is reduced. Then R_A has a universal test element.*

Proof. After localizing at one element of A° we have that the minimal primes of R_A correspond bijectively with those of $R_{\mathfrak{F}}$, and there is a universal test element modulo each minimal prime by (2.4.7). We may then apply (2.4.4). \square

It is also possible to obtain universal test elements in certain cases using the result of Lipman and Sathaye [LS], (1.5.3).

(2.4.9) Theorem (universal test elements via the Lipman-Sathaye theorem). *Let $A \subseteq R_A$ and \mathfrak{F} be as in (2.4.1). Suppose that R_A is module-finite over a regular ring T_A and that $R_{\mathfrak{F}}$ is an absolute domain. (Given R_A , after localizing A at one element of A° we may assume that A is regular and we may choose T_A , for example, so that it is a polynomial ring over A .) Let J_A denote the Jacobian ideal $\mathcal{J}(R_A/T_A)$. Then every nonzero element of J_A is a universal test element for $A \rightarrow R_A$.*

Proof. In this argument, Λ is a varying regular domain of characteristic p . Localize at one element of A° so that A is regular, $A \rightarrow T_A$ is smooth, and also so that R_A is A -free. For almost every²⁷field L to which A maps, R_L is a domain (cf. (2.3.6c)). If Λ is a domain with fraction field L , then since R_Λ is free (and, hence, torsion-free) over Λ it follows that for almost all Λ , R_Λ is a domain. For almost all Λ we have $T_\Lambda \subseteq R_\Lambda$ and the extension is module-finite. It follows that for almost all Λ , R_Λ is module-finite domain extension of T_Λ (hence, T_Λ is also a domain) and generically smooth over T_Λ . (Cf. (2.3.5), (2.3.6), and (2.3.9).) Moreover, $\mathcal{J}(R_\Lambda/T_\Lambda) = J_\Lambda \subseteq R_\Lambda$ for almost all Λ . The result is now immediate from Corollary (1.5.4). \square

(2.4.10) Corollary (more universal test elements via the Lipman-Sathaye theorem). *Let $A \subseteq R_A$ and \mathfrak{F} be as in (2.4.1). Suppose that $R_{\mathfrak{F}}$ is an absolute domain of dimension d . Let $R_A = A[x_1, \dots, x_n]/(f_1, \dots, f_r)$. Then every nonzero element of*

²⁷See (2.3.4).

the ideal generated by the $(n - d) \times (n - d)$ minors of the Jacobian matrix $(\partial f_i / \partial g_j)$ is a universal test element of R_A over A .

Proof. This follows from (2.4.9) in much the same way that (1.5.5) follows from (1.5.4). We replace the original x 's by sufficiently general linear combinations (we may think over \mathfrak{F} , which is infinite, but we actually need only invert one element of A°). We may then assume that R_A is module-finite over the subring generated over A by the images of any d element subset of the variables (again we may need to localize at one element of A°). We may then let T_A vary through the subrings of R_A generated over A by the various d element subsets of the variables, and now we may apply (2.4.9). \square

(2.4.11) Example. Let $A = \mathbb{Z}[1/3]$ and let $R_A = A[X, Y, Z]/(F) = A[x, y, z]$, where $F = X^3 + Y^3 + Z^3$. We may take $T_A = A[x, y]$, and we find that $(\partial F / \partial Z) = 3Z^2$, so that z^2 is a universal test element. By permuting the variables to take other choices of T_A , we see similarly that x^2, y^2 are universal test elements. Thus, x^2, y^2 , and z^2 are universal test elements. Note that the results of the discussion in [HH10] (8.22)–(8.24) in characteristic p can be sharpened similarly using (1.5.4).

If we take $A = \mathbb{Z}[1/d]$ where $d = d_1 \cdots d_n$, $n \geq 3$, and we let $R = A[X_1, \dots, X_n]/(F) = A[x_1, \dots, x_n]$, where $F = \sum_{i=1}^n X_i^{d_i}$, then the same method shows that for every i , $x_i^{d_i-1}$ is a universal test element. Cf. [HH10], (8.24), which likewise can be sharpened using (1.5.4).

(2.5) BASIC PROPERTIES OF TIGHT CLOSURE OVER AFFINE ALGEBRAS

Our first objective in this section is to show that the two conditions discussed in defining $N_M^{*/A}$ in (2.2.2) are equivalent. We then proceed to establish a number of basic results concerning the behavior of $*K$.

We shall make use of the results on universal test elements to show that the two conditions considered in (2.2.2) are equivalent in the reduced case. However, to handle the problems created by nilpotents we shall also need:

(2.5.1) Lemma. *Let A be a Noetherian domain whose fraction field \mathfrak{F} has characteristic*

zero and let $R \supseteq A$ be a finitely generated A -algebra, torsion-free over A , such that (0) has no embedded primes in R . Let $\mathfrak{p}_1, \dots, \mathfrak{p}_s$ be the minimal primes of R . Then there is a nonzerodivisor d in R and a module-finite extension S of R obtained by adjoining finitely many nilpotent elements to R such that:

- (1) d is not a zerodivisor in S
- (2) $dS \subseteq R$
- (3) The natural map $R_{\text{red}} \rightarrow S_{\text{red}}$ is an isomorphism.
- (4) The canonical surjection $S \twoheadrightarrow S_{\text{red}} \cong R_{\text{red}}$ splits as a map of A -algebras.

Proof. Let W be the multiplicative system $R - \cup_i \mathfrak{p}_i$, so that $W^{-1}R$ is a zero-dimensional ring, and, in fact, $W^{-1}R \cong \prod_i R_{\mathfrak{p}_i}$. Since every $R_{\mathfrak{p}_i}/\mathfrak{p}_i R_{\mathfrak{p}_i}$ contains a copy of \mathfrak{F} (\mathfrak{p}_i cannot meet A) and every $R_{\mathfrak{p}_i}$ is equal characteristic zero and 0-dimensional (hence, complete), it follows that every $R_{\mathfrak{p}_i}$ has a coefficient field $L_i \subseteq R_{\mathfrak{p}_i}$ such that $\mathfrak{F} \subseteq L_i$ (in equal characteristic zero, every subfield of a complete local ring can be enlarged to a coefficient field). Note that $L_i \rightarrow (R_{\mathfrak{p}_i})_{\text{red}}$ is an isomorphism, and so $\prod_i L_i \rightarrow (W^{-1}R)_{\text{red}} \cong W^{-1}(R_{\text{red}})$ is an isomorphism. We have a map $\eta : W^{-1}(R_{\text{red}}) \rightarrow W^{-1}R$ that splits the canonical surjection $\tau : W^{-1}R \rightarrow W^{-1}(R_{\text{red}})$. Because we chose every L_i to contain $\mathfrak{F} \supseteq A$, these maps are A -algebra homomorphisms. Let $\{\rho_j : j\}$ denote a finite set of generators for R_{red} over A . For every ρ_j , $\eta(\rho_j)$ is an element of $W^{-1}R$ that maps to ρ_j . We can choose $r_j \in R$ that maps to ρ_j , and then $\varepsilon_j = \eta(\rho_j) - r_j \in W^{-1}R$ is nilpotent in $W^{-1}R$. Let S be the subring of $W^{-1}R$ generated by all the ε_j . It is clear that S is module-finite over R , and that $S_{\text{red}} \rightarrow R_{\text{red}}$ is an isomorphism. Since $S \subseteq W^{-1}R$ but is module-finite over R , we can choose $d \in W$ such that $dS \subseteq R$. It is clear then that d is a nonzerodivisor in both R and S .

Finally, since S contains $R \supseteq A$ and also the images $r_j + \varepsilon_j = \eta(\rho_j)$ under η of the generators of R_{red} as an A -algebra, the restriction of η to R_{red} maps into S , and splits the map $S \rightarrow S_{\text{red}} \cong R_{\text{red}}$. \square

The following result reconciles the two possible definitions for $N^{*/A}$ discussed in (2.2.2).

(2.5.2) Theorem (a uniform multiplier can be used). *Let $A \supseteq \mathbb{Z}$ be a domain finitely generated over \mathbb{Z} , and let R_A be a finitely generated \mathbb{Z} -algebra.*

Then there is an element c_A of R_A° with the following property: if $N_A \subseteq M_A$ then $u_A \in N_A^{/A} M_A$ if and only if for almost all $\mu \in \text{Max Spec } A$ and $\kappa = \kappa(\mu)$, $c_\kappa u_\kappa^q \in N_\kappa^{[q]} M_\kappa$ for all $q \geq p$.*

Proof. First localize at one element of A° so that R_A is A -free and also so that there is a bijective correspondence between the associated primes of (0) in $R_{\mathfrak{F}}$ (\mathfrak{F} is the fraction field of A) and the associated primes of (0) in R_A , which may be assumed to be disjoint from A .

Note that it is obvious that if there exists $c_A \in R_A^\circ$ satisfying the condition given in the statement of the theorem then $u_\kappa \in N_\kappa^* M_\kappa$ for almost all fibers, since by (2.3.9a), c_κ will be in $(R_\kappa)^\circ$ for almost all fibers.

First case: R_A is reduced. If R_A is reduced choose c_A to be a universal test element for R_A . Then c_κ is a test element for almost all closed fibers. The result now follows from the fact that $u_\kappa \in (N_\kappa)^* M_\kappa$ if and only if $c_\kappa u_\kappa^q \in N_\kappa^{[q]} M_\kappa$ for all q (even $q = 1$ in this case).

Second case: the ideal (0) in R_A has no embedded primes. In this case we can apply the lemma above to choose $d_A \in R_A^\circ$ and $S_A \supseteq R_A$ satisfying the conditions (1)–(4) of Lemma (2.5.1). Let $\eta_A : (R_A)_{\text{red}} \rightarrow S_A$ be the splitting, and let $c_A^{(0)}$ be a universal test element for $(R_A)_{\text{red}}$. We shall prove that $c_A = d_A \eta_A(c_A^{(0)})$ has the required property. It suffices to consider the case where $N_A \subseteq M_A$, with $M_A = R_A^t$ is R_A -free. Let h be the degree of nilpotence of the nilradical of R_A . We may invert all positive prime integers p in A such that $p < h$. Let J_A denote the ideal of nilpotent elements of S_A and let \overline{R}_A denote the image $\eta_A((R_A)_{\text{red}})$ of $(R_A)_{\text{red}}$ in S_A . Thus, $S_A = \overline{R}_A + J_A$, and this is a direct sum of \overline{R}_A -modules. Here, $\eta_A : (R_A)_{\text{red}} \cong \overline{R}_A$. Suppose that $u_\kappa \in N_\kappa^*$ in M_κ for almost all closed fibers. The same holds when we pass to $(R_\kappa)_{\text{red}} \cong ((R_A)_{\text{red}})_\kappa \cong ((S_A)_{\text{red}})_\kappa \cong (S_\kappa)_{\text{red}}$ for almost all closed fibers by (1.4.4e), and we consider the corresponding problem over S_κ . The statements we make in the sequel are likewise valid for almost all closed fibers, by routine application of Theorems (2.3.5) and (2.3.9): we sometimes omit saying this.

In S_A we may write $u_A = v_A + \varepsilon_A$ where $v_A \in \overline{R}_A$ and $\varepsilon_A \in J_A$, and we may write the generators of N_A in the form $w_A^{(i)} + \zeta_A^{(i)}$ where the $w_A^{(i)} \in \overline{R}^t$ and the $\zeta_A^{(i)} \in J_A^t$ ($= J_A \oplus \dots \oplus J_A$). When we apply $\kappa \otimes_A _$ we still have that the h th power of J_κ is zero,

and so the q th power of any element of J_κ is zero for all $q \geq p$, since $p \geq h$. Killing the nilpotents in S_κ we have that v_κ is in the tight closure of the submodule N'_κ of \overline{R}_κ^t spanned by the $w_\kappa^{(i)}$ over \overline{R}_κ , and because $c_A^{(0)}$ is a universal test element for $(R_A)_{\text{red}}$, $d_A^{(0)} = \eta(c_A^{(0)})$ is a universal test element for \overline{R}_A . It follows that $d_\kappa^{(0)} v_\kappa^q \in N'_\kappa^{[q]}$ for all $q \geq 1$ over the reduced ring \overline{R}_κ . But then

$$(\#) \quad d_\kappa^{(0)} u_\kappa^q \in \langle S_\kappa \otimes_{R_\kappa} N_\kappa \rangle^{[q]} \quad \text{in} \quad \mathbf{F}^e(S_\kappa \otimes_{R_\kappa} M_\kappa) \cong S_\kappa^t \quad \text{for } q \geq p$$

because $q \geq p \geq h$ and raising to q th powers kills the nilpotent components.

Because $d_A S_A \subseteq R_A$ we have that $d_\kappa S_\kappa \subseteq R_\kappa$ for almost all fibers, while d_κ is also a nonzerodivisor in both R_κ and S_κ for almost all fibers. Multiplying by d_κ in $(\#)$ above yields that $(d_\kappa d_\kappa^{(0)}) u_\kappa^q \in N_\kappa^{[q]}$ over R_κ for all $q \geq p$, as required.

Third case: (0) has embedded primes in R_A . In this case let I_A denote the ideal of all elements that are killed by an element of $(R_A)^\circ$, which is contained in the nilradical of R_A . Choose $c_A^{(1)}$ in R_A such that its image in R_A/I_A (which satisfies the condition for the second case) solves the problem for R_A/I_A . One can also choose an element $c_A^{(2)}$ of $(R_A)^\circ$ that kills I_A : choose one such for each generator of I_A , and multiply them together. We claim that $c_A = c_A^{(1)} c_A^{(2)}$ solves the problem for R_A . To see this, suppose that u_A is such that $u_\kappa \in (N_\kappa)^*_{M_\kappa}$ for almost all fibers. By (1.4.4e) this condition holds after applying $(R_\kappa/I_\kappa) \otimes_{R_\kappa} _$, since I_κ may be identified with a subideal of the nilradical of R_κ for almost all fibers. By using the property of the image of $c_A^{(1)}$ in R_A/I_A we find that for almost all fibers we have that $c_\kappa^{(1)} u_\kappa^q \in N_\kappa^{[q]} + I_\kappa \mathbf{F}^e(M)$ for all $q \geq p = p(\kappa)$. Multiplying by $c_\kappa^{(2)}$ and using that $c_A^{(2)} I_A = (0)$ (which implies that $c_\kappa^{(2)} I_\kappa = (0)$), we obtain the required result. \square

We next want to see that in the definition of *K it does not matter how one chooses descent data.

(2.5.3) Theorem (independence of choice of descent). *Let K be a field, let R be a finitely generated K -algebra, let $N \subseteq M$ be finitely generated R -modules, and let $u \in M$.*

Let (A, R_A, M_A, N_A, u_A) and (B, R_B, M_B, N_B, u_B) be two possibly different choices of descent data for (K, R, M, N, u) .

Then $u_A \in N_A^{/A} M_A$ if and only if $u_B \in N_B^{*/B} M_B$ if and only if $u \in N^{*K} M$.*

Proof. First note that we are free to localize A at one element of A° without affecting any relevant issue, from the definition of $*/A$, and similarly for B . Thus, we may assume the usual consequences of such localization. Also, we have $u = u_A = u_B$.

If we know the first equivalence the second follows, since $u \in N^{*K} M$ means that $u_C \in N_C^{*/C} M_C$ for some choice of descent data (C, R_C, M_C, N_C, u_C) . Thus, it suffices to prove the first equivalence.

We know that for any sufficiently large finitely generated \mathbb{Z} -subalgebra C of K with $C \supseteq A, B$ that the identity maps on R and M respectively restrict to isomorphisms of $C \otimes_A R_A$ with $C \otimes_B R_B$ and of $C \otimes_A M_A$ with $C \otimes_B M_C$ over the “common” R_C , respectively, and that the latter isomorphism carries $C \otimes_A N$ to $C \otimes_B N_{B'}$ while $u = u_C = u_A = u_B$. Cf. (2.1.12). Thus, in proving the result it suffices to consider the case where $B = C \supseteq A$, since knowing this case permits us to compare each of $*/A$ and $*/B$ with $*/C$. Henceforth we may assume that $A \subseteq B$.

There is no loss of generality in localizing at one element of A so that B is free over A , and hence, faithfully flat over A . Since A is a Hilbert ring and B is a finitely generated A -algebra, every maximal ideal of B lies over a maximal ideal of A . Moreover, there is at least one maximal ideal of B lying over every maximal ideal of A .

The argument is now very simple. For any maximal ideal μ' of B , if $\mu = \mu' \cap A$ and $\kappa \hookrightarrow k'$ is the induced map of fields $A/\mu \rightarrow B/\mu'$ then the fiber of $B \rightarrow R_B$ over μ' is simply $\kappa \rightarrow R_{\kappa'}$ where $R_{\kappa'} = \kappa' \otimes_{\kappa} R_{\kappa}$. Since $\kappa \hookrightarrow \kappa'$ is a finite separable extension of fields (κ, κ' are finite fields and finite fields are perfect) we have that $u_{\kappa} \in N_{\kappa}^{*} M_{\kappa}$ if and only if $u_{\kappa'} \in N_{\kappa'}^{*} M_{\kappa'}$ (“only if” follows from the persistence of tight closure (1.4.13), while “if” follows from [HH9], Theorem (7.29a^o), part (ii)). Notice that every μ in $\text{Max Spec } A$ lies under at least one μ' in $\text{Max Spec } B$.

If $u_A \in N_A^{*/A}$ localize A, B at one element of A° so that $u_{\kappa} \in N_{\kappa}^{*}$ for all closed fibers, and then this will hold for all closed fibers of $B \rightarrow R_B$ by the discussion above. If $u_B \in N_B^{*/B}$ then first localize B at one element of B° so that $u_{\kappa'} \in N_{\kappa'}^{*}$ for all closed fibers of $B \rightarrow R_B$ and then localize A, B at one element of A° so that B becomes A -free again.

The fact that $u_{\kappa'} \in N_{\kappa'}^*$ for all closed fibers of $B \rightarrow R_B$ then implies that the same holds for all closed fibers of $A \rightarrow R_A$, by the discussion of the preceding paragraph. \square

(2.5.4) Corollary. *Let $A \rightarrow R_A$ and $\mathfrak{F} \subseteq K$ be as in (2.4.1), and let $N_A \subseteq M_A$ be finitely generated R_A -modules, and let $u_A \in M_A$. Then $u_A \in N_A^{*/A} M_A$ if and only if $u_K \in N_K^{*K} M_K$. In particular, this holds when $K = \mathfrak{F}$. \square*

The following result gives many of the basic properties of *K for affine algebras.

(2.5.5) Theorem. *Let K be a field of characteristic 0 and let R be a finitely generated K -algebra. Let $N', N \subseteq M$ be finitely generated R -modules. Let $u \in M$ and let v be the image of u in M/N . Let I be an ideal of R .*

*Unless otherwise indicated, *K without a following subscript stands for K -tight closure in M .*

- (a) $u \in N^{*K}_M$ if and only if $v \in 0^{*K}_{M/N}$.
- (b) N^{*K} is an R -submodule of M containing N .
- (c) Let $W = N^{*K}$ and fix descent data (A, R_A, M_A, N_A) for (K, R, M, N) as well as W_A for W . Then for almost all $\mu \in \text{Max Spec } A$, $W_\mu \subseteq N^*_{\mu}$ (in M_μ).
- (d) If $N \subseteq N' \subseteq M$ then $N^{*K} \subseteq N'^{*K}$ and $N^{*K}_{N'} \subseteq N^{*K}_M$.
- (e) $(N^{*K})^{*K} = N^{*K}$.
- (f) $(N \cap N')^{*K} \subseteq N^{*K} \cap N'^{*K}$.
- (g) $(N + N')^{*K} = (N^{*K} + N'^{*K})^{*K}$.
- (h) $(IN)^{*K}_M = ((I^{*K}_R)N^{*K}_M)^{*K}_M$.
- (i) $(N :_M I)^{*K}_M \subseteq N^{*K} :_M I$ (respectively, $(N :_R N')^{*K} \subseteq N^{*K} :_R N'$). Hence, if $N = N^{*K}$ then $(N :_M I)^{*K} = N :_M I$ (respectively, $(N :_R N')^{*K} = N :_R N'$).
- (j) If $N_i \subseteq M_i$ are finitely many finitely generated R -modules and we identify $N = \oplus_i N_i$ with its image in $M = \oplus_i M_i$ then the obvious injection of $\oplus_i N_i^{*K}_{M_i} \rightarrow M$ maps $\oplus_i N_i^{*K}_{M_i}$ isomorphically onto N^{*K}_M .
- (k) (Persistence of K -tight closure) Let L be a field containing K , let S be a finitely generated L -algebra (hence, also, a K -algebra) and let $R \rightarrow S$ be a K -algebra homomorphism. Let $u \in N^{*K}_M$. Then $1 \otimes u \in \langle S \otimes_R N \rangle^{*L}_{S \otimes_R M}$ over S . In particular, this holds when $L = K$.

- (l) (*Persistence of K -tight closure: second version.*) Let L be a field containing K , let S be a finitely generated L -algebra, let $h: R \rightarrow S$ be a K -homomorphism, let $N \subseteq M$ be finitely generated R -modules, and let $V \subseteq W$ be finitely generated S -modules. Suppose that $u \in N^{*K}_M$. Suppose also that there is an R -homomorphism $\gamma: M \rightarrow W$ such that $\gamma(N) \subseteq V$. Then $\gamma(u) \in V^{*L}_W$.
- (m) (*Irrelevance of nilpotents*) If J is the nilradical of R , then $J \subseteq (0)^{*K}$, and so $J \subseteq I^{*K}$ for all ideals I of R . Consequently, $JM \subseteq N^{*K}$. Moreover, if N^\sim denotes the image of N in M/JM , then N^{*K} is the inverse image in M of the tight closure $(N^\sim)^{*K}_{M/JM}$, which may be computed either over R or over $R_{\text{red}} (= R/J)$.
- (n) Let $\mathfrak{p}^{(1)}, \dots, \mathfrak{p}^{(s)}$ be the minimal primes of R and let $R^{(i)} = R/\mathfrak{p}^{(i)}$. Let $M^{(i)} = R^{(i)} \otimes_R M$ and let $N^{(i)}$ be the image of $R^{(i)} \otimes_R N$ in $M^{(i)}$. Let $u^{(i)}$ be the image of u in $M^{(i)}$. Then $u \in N^{*K}$ if and only if $u^{(i)} \in (N^{(i)})^{*K}$ in $M^{(i)}$ over $R^{(i)}$, $1 \leq i \leq s$.
- (o) If $R = \prod_{i=1}^h R_i$ is a finite product and $M = \prod_i M_i$ and $N = \prod_i N_i$ are the corresponding product decompositions of M , N , respectively, then $u = (u_1, \dots, u_h) \in M$ is in N^{*K}_M over R if and only if for all i , $1 \leq i \leq h$, $u_i \in N_i^{*K}_{M_i}$.

Proof. We can choose descent data not only for (K, R, M, N, u) but also for $I, N', N_i, M_i, v, W = N^{*K}$ and $M'' = M/N$, etc. as in (2.1.10). Henceforth we assume that such data (A, R_A, M_A, N_A, u_A) as well as N'_A, M''_A, v_a, W_A and so forth have been given.

(a) By (2.5.3), $u \in N^* \Leftrightarrow u_a \in N_A^{*/A} \Leftrightarrow u_\kappa \in N_\kappa^*$ for almost all fibers $\Leftrightarrow v_\kappa \in 0^*_{M''_\kappa}$ for almost all fibers (since for any given fiber the two conditions are equivalent by Theorem (1.4.4b)) $\Leftrightarrow v_A \in 0^{*/A}_{M''_A} \Leftrightarrow v \in 0^{*K}_{M''}$.

(b) If $u, w \in N^{*K}$ we can choose descent data with A sufficiently large that $u_A \in N_A^{*/A}$ and $w_A \in N_A^{*/A}$. But then for $r, s \in R$ (which descend to, say, $r_A, s_A \in R_A$) we have that for almost all fibers $r_\kappa u_\kappa + s_\kappa w_\kappa \in N_\kappa^*$ (since N_κ^* is a submodule of M_κ^*) and it follows that $r_A u_A + s_A w_A \in N_A^{*/A}$ and so $ru + sw \in N^{*K}$, as required. Thus, N^{*K} is a submodule of M , and it is clear that $N \subseteq N^{*K}$.

(c) Fix a finite set of generators $w_A^{(i)}$ for W_A . The condition clearly holds for each of these generators, for almost all closed fibers, and we may intersect the dense open subsets of $\text{Max Spec } A$ involved. Moreover, for almost all closed fibers the $w_\kappa^{(i)}$ generate W_κ .

(d) For almost all closed fibers we have that $N_\kappa \subseteq N'_\kappa \subseteq M_\kappa$, and the result is immediate

from the definition of *K and the corresponding facts in characteristic p (cf. (1.4.4d)).

(e) Since $N \subseteq N^{*K}$ by (b), (d) shows that $N^{*K} \subseteq (N^{*K})^{*K}$. Let $W = N^*$ and descend as in part (c). Then if $u \in (N^{*K})^{*K} = W^{*K}$ and $u = u_A \in M_A$ (this will be the case for sufficiently large A) then for almost all closed fibers $u_\kappa \in W_\kappa^* \subseteq (N_\kappa^*)^* = N_\kappa^*$ by the characteristic p result (cf. (1.4.4c), and so $u \in N_A^{*/A} M_A$. This shows that $u \in N^{*K}$.

(f) $(N \cap N')^{*K} \subseteq N^{*K}$ and $(N \cap N')^{*K} \subseteq N'^{*K}$ both follow from part (d).

(g) Since $N \subseteq N^{*K}$ and $N' \subseteq N'^{*K}$ we have that $N + N' \subseteq (N^{*K} + N'^{*K})$ and so $(N + N')^{K*} \subseteq (N^{K*} + N'^{K*})^{K*}$ by (d). But, also by (d), $N^{*K} \subseteq (N + N')^{*K}$ and $N'^{*K} \subseteq (N + N')^{*K}$ so that $N^{*K} + N'^{*K} \subseteq (N + N')^{*K}$ and another application of (d) yields that $(N^{*K} + N'^{*K})^{*K} \subseteq ((N + N')^{*K})^{*K} = (N + N')^{*K}$ by (e).

(h) Since $I \subseteq I^{*K}$ and $N \subseteq N^{*K}$ we have that $IN \subseteq I^{*K}N^{*K}$, and so $(IN)^{*K} \subseteq (I^{*K}N^{*K})^{*K}$ by (d). To prove the other inclusion let $L = I^{*K}$, let $W = N^{*K}$, let $u \in (I^{*K}N^{*K})^{*K}$ and choose descent data for K, R, M, N, W, I, L , and u , say $A, R_A, M_A, N_A, W_A, I_A, L_A$, and $u_A = u$. It is easy to see that after localizing at one element of A° , we may also assume that $L_A W_A \subseteq W_A$ may be used to descend LW and that $I_A N_A$ may be used to descend IN . Since $u \in (I^{*K}N^{*K})^{*K} = (LW)^{*K}$, it follows that for almost all closed fibers we have that $u_\kappa \in (L_A W_A)_\kappa^*$, and for almost all closed fibers this may be identified with $(L_\kappa W_\kappa)^*$. By two applications of part (c), for almost all closed fibers we have that $L_\kappa \subseteq I_\kappa^*$ and that $W_\kappa \subseteq N_\kappa^*$. This shows that for almost all closed fibers we have that $u_\kappa \in (I_\kappa^* N_\kappa^*)^*$, which, by Proposition (8.5h) of [HH4], is contained in $(I_\kappa N_\kappa)^*$, and for almost all fibers $I_\kappa N_\kappa = (I_A N_A)_\kappa$. It follows that for almost all closed fibers, $u_\kappa \in (I_A N_A)_\kappa^*$, as required, so that $u \in (IN)^{*K}$, as claimed.

(i) The second statement is immediate from the first statement, and the first statement follows formally from part (h). To see this, first note that what we must show is that $I(N :_M I)^{*K} \subseteq N^{*K}$ (respectively, $(N :_R N')^{*K} N' \subseteq N^*$) and with $W = N :_M I$ (respectively, $I' = N :_R N'$) we have that $IW^{*K} \subseteq I^{*K}W^{*K} \subseteq (IW)^{*K}$ (respectively, $I'^{*K}N' \subseteq I'^{*K}N'^{*K} \subseteq (I'N')^{*K}$) by (h), and, by definition $IW \subseteq N$ (respectively, $I'N' \subseteq N$).

(j) Take descent data $A, R_A, N_A^{(i)} \subseteq M_A^{(i)}$ for all i and then apply the definition of *K and part (m) of Proposition (8.5) of [HH4].

(k) Let $u \in N^{*K}$ and choose descent data (A, R_A, M_A, N_A, u_A) for (K, R, M, N, u) with $u = u_A$. Since $K \subseteq L$ we can choose descent data for (L, S) say (B, S_B) such that $A \subseteq B$. After localizing all objects at one element of A° we may assume that B is A -free, and so faithfully flat over A . Let W denote the image of $S \otimes_R M$. Let W_B be the image of $S_B \otimes_{R_A} N_A$ in $S_B \otimes_{R_A} M_A$. Let z_B denote the image of $1 \otimes u_A$ in $S_B \otimes_{R_A} M_A$. Then $(B, S_B, S_B \otimes_{R_A} M_A, W_B, z_B)$ is descent data over B for $(L, S, S \otimes_R M, W, 1 \otimes u)$. (Note here that whenever we localize B at one element of B° (so as, for example, to make some object B -free) we may also localize A at one element of A° so as to make the “new” B arising from this localization free over A again, and then every object that is B -free is also A -free.) The fact that these objects give descent data is easily verified using, primarily, the associativity of tensor:

$$\begin{aligned} L \otimes_B (S_B \otimes_{R_A} M_A) &\cong (L \otimes_B S_B) \otimes_{R_A} M_A \cong S \otimes_{R_A} M_A \cong (S \otimes_R R) \otimes_{R_A} M_A \cong \\ S \otimes_R (R \otimes_{R_A} M_A) &\cong S \otimes_R ((K \otimes_A R_A) \otimes_{R_A} M_A) \cong S \otimes_R (K \otimes_A M_A) \cong S \otimes_R M, \end{aligned}$$

while $L \otimes_B W_B$ is the same as the image of $L \otimes_B (S_B \otimes_{R_A} N_A)$ in $L \otimes_B (S_B \otimes_{R_A} M_A)$ (since L is B -flat) and by a similar sequence of identifications this is the image of $S \otimes_R M$.

We assume that A has been localized sufficiently that for all closed fibers, $u_\kappa \in N_\kappa^*$. Let μ' be any maximal ideal of B with residue field κ' , and suppose that μ' lies over μ in A with residue field κ . Since B is a finitely generated algebra over the Hilbert ring A , μ is maximal in $\text{Spec } A$. Here, $\kappa \subseteq \kappa'$ are both finite fields. Now $(S_B \otimes_{R_A} M_A)_{\kappa'} \cong \kappa' \otimes_B (S_B \otimes_{R_A} M_A) \cong S_{\kappa'} \otimes_{R_A} M_A \cong S_{\kappa'} \otimes_{R_\kappa} M_\kappa$, and the image of $W_{\kappa'} = \kappa' \otimes_B W_B$ will be the same for almost all μ' as the image of $\kappa' \otimes_B (S_B \otimes_{R_A} N_A)$ in $\kappa' \otimes_B (S_B \otimes_{R_A} M_A)$, which may be identified with the image of $S_{\kappa'} \otimes_{R_\kappa} N_\kappa \rightarrow S_{\kappa'} \otimes_{R_\kappa} M_\kappa$. What we need to show is that for almost all μ' , $z_{\kappa'}$, which may be identified with the image of $1 \otimes u_\kappa$ in $S_{\kappa'} \otimes_{\kappa'} M_\kappa$, is in $\langle S_{\kappa'} \otimes_{\kappa} N_\kappa \rangle^*$. But since $u_\kappa \in M_\kappa^*$, this is simply the persistence of tight closure in characteristic p under the homomorphism $R_\kappa \rightarrow S_{\kappa'}$, which is valid here because $S_{\kappa'}$ is finitely generated over the field κ' : see (1.4.13).

(l) By part (k) we have that $1 \otimes u$ is in $\langle S \otimes_R N \rangle^*$ in $S \otimes_R M$, and there is an S -linear map $S \otimes_R M \rightarrow W$ induced by γ such that $1 \otimes u$ maps to $\gamma(u)$ and $\langle S \otimes_R N \rangle$ maps into V . It follows that we may assume that $R = S$ and that $N \subseteq M$ are S -modules. Let N'

denote the inverse image of $V \subseteq W$ in M . Then $N' \supseteq N$, and so $u \in N'^{*L}_M$ by part (d). It follows that $\bar{x} \in 0^{*L}_{M/N'}$ (where the bar indicates reduction modulo N') by part (a). Since M/N' injects into W/V , it follows that the image of \bar{x} in W/V is in $0^{*L}_{W/V}$ by part (d), and so $\gamma(u) \in V^{*L}_W$ by another application of part (a).

(m) Choose descent data including an ideal $J_A \subseteq R_A$ descending the nilradical $J \subseteq R$. Since, for almost all closed fibers, J_κ is the nilradical of R_κ , it follows that $J_\kappa \subseteq (0)_\kappa^*$ for almost all closed fibers. Thus, $J \subseteq (0)^{*K} \subseteq I^{*K}$ for all ideals I , by (d). Moreover, $JM \subseteq (0)^{*K}M \subseteq ((0)M)^{*K}$ (by (h)) $= 0^{*K}_M \subseteq N^{*K}_M$ for all submodules N of M .

Now suppose that $u \in M$. We may assume that our descent data is such that $(R_A)_{\text{red}} = R_A/J_A$ descends R_{red} and that N^\sim_A , which we define as $(N_A + J_AM)/J_AM$, descends N^\sim (we may think of N^\sim_A as a module over either R_A or R_A/J_A). We also assume that $u = u_A \in M_A$. Then $u \in N^{*K} \Leftrightarrow u \in N^{*/A}_A \Leftrightarrow$ for almost all closed fibers, $u_\kappa \in N^*_\kappa$. Since for almost all closed fibers J_κ is the nilradical in R_κ , it follows from the irrelevance of nilpotents in characteristic p (cf. (1.4.4e)) that, for almost all closed fibers, $u_\kappa \in N^*_\kappa$ if and only if $u_\kappa \in (N^\sim_\kappa)^*$ calculated over R_κ , and also if and only if $u_\kappa \in (N^\sim_\kappa)^*$ calculated over $(R_\kappa)_{\text{red}}$. The first of these conditions is equivalent to the condition that u be in $(N^\sim)^{*K}$ calculated over R , and the second to the condition that it be in $(N^\sim)^{*K}$ calculated over R_{red} .

(n) The fact that if $u \in N^{*K}$ then $u^{(i)} \in N^{(i)*K}$ over $R^{(i)}$ is a simply a special case of (k) (with $L = K$ and $S = R^{(i)}$). Thus, it will suffice to show that if $u^{(i)} \in N^{(i)*K}$ for all i then $u \in N^{*K}$. Consider descent data (A, R_A, M_A, N_A, u_A) as usual with $u = u_A$ as well as minimal primes $\mathfrak{p}_A^{(i)}$ in R_A that give rise to the $\mathfrak{p}^{(i)}$: after localization at one element of A° , these may be assumed disjoint from A° . We know that for almost all closed fibers the minimal primes of R_κ are the same as the minimal primes of the various radical ideals $\mathfrak{p}_\kappa^{(i)}$, each occurring as a minimal prime of some $\mathfrak{p}_\kappa^{(i)}$ for a unique choice of i . By the persistence of tight closure in characteristic p , for almost all closed fibers we know that the image of $u_\kappa^{(i)}$ modulo every minimal prime $\mathfrak{q}^{(ij)}$ of $R_\kappa^{(i)}$ is in the tight closure of the image of $N_\kappa^{(i)}$ in $(R_\kappa^{(i)}/\mathfrak{q}^{(ij)}) \otimes_{R_\kappa^{(i)}} M_\kappa^{(i)}$, over $R_\kappa^{(i)}/\mathfrak{q}^{(ij)}$, which tells us that the image of u_κ is in the tight closure of the image of N_κ in $(R_\kappa/\mathfrak{q}) \otimes_{R_\kappa} M_\kappa$ for every minimal prime \mathfrak{q} of R_κ . This in turn yields that u_κ is in the tight closure of N_κ in M_κ by (1.4.4g).

(o) Choose descent data for each factor ring, and each pair consisting of a factor of M and the corresponding submodule of that factor. Taking products, we obtain descent data compatible with the product decompositions, and one has a corresponding decomposition for any closed fiber. The result is now immediate from Theorem (1.4.4m) (for the “if” part one needs that the intersection of a finite number of dense open subsets is a dense open subset). \square

(2.5.6) Definition. Suppose that R is a finitely generated algebra over a field K of characteristic zero and that $N \subseteq M$ are finitely generated R -modules. We refer to N^{*K} as the *K -tight closure of N in M* . If $N^{*K} = N$, we say that N is *K -tightly closed in M* .

(2.5.7) Remark. We continue the notation of (2.5.6). Suppose that N is K -tightly closed in M over R and we choose descent data A, R_A, M_A, N_A . It is natural to ask whether N_κ is tightly closed in M_κ over R_κ for almost all closed fibers. We do not know the answer. The problem is that, even for typical closed fibers, the elements in $N_\kappa^* - N_\kappa$ may be coming from different elements of M_A as the fiber varies. This leads to a variant notion, first discussed in [Kat], of when a submodule should be tightly closed over K : see (4.3.9).

(2.5.8) Remark. When $K \subseteq L$ are fields of characteristic zero, R is an affine L -algebra, and $N \subseteq M$ are finitely generated R -modules, we do not know in general whether $N^{*K}_M = N^{*L}_M$. Related issues are discussed in Chapter 4, Section 5.

We postpone further development of the properties of K -tight closure until the notion has been extended to all Noetherian K -algebras in Chapter 3. See §(4.1).

CHAPTER 3.

ARBITRARY NOETHERIAN ALGEBRAS OVER A FIELD

Our objective in this chapter is to extend the definition of K -tight closure, where K is a field of characteristic zero, to pairs of finitely generated modules $N \subseteq M$ over a Noetherian K -algebra R . There are no other finiteness conditions on R . We want the definition to agree with the notion defined for affine K -algebras R in Chapter 2. In the first section we give some preliminaries concerning descent of inclusions of modules to affine subalgebras. In the second section, we give definitions for two kinds of K -tight closure, *direct* and *formal*: eventually we shall drop the word “formal.” I.e., formal K -tight closure will be taken as the definition of K -tight closure, once we prove that it agrees with *K for affine algebras. (It is almost trivial that direct K -tight closure agrees with *K for affine K -algebras.) In the second section we also explore some of the most basic properties of the two notions.

In the third section we review a number of results concerning Artin approximation and the structure of formal power series rings, due to Artin and Rotthaus, that we shall use in critical ways in developing the theory of the new notion of K -tight closure. In the fourth section we prove that direct and formal K -tight closure agree for locally excellent K -algebras, and it is after we have proved this fact that we drop the word “formal” from the description. In the fifth section we develop a powerful result for descending information about a complete local ring S containing a field of characteristic zero K to an affine K -algebra R that maps to S : this result is a consequence of a theorem of Artin and Rotthaus [ArR] that is explained in the third section. The sixth section is devoted to further development of the properties of direct and formal K -tight closure, including parallels of a number of the fundamental results of the characteristic p theory. The last section deals with base change issues.

(3.1) MORE ABOUT DESCENT: AFFINE PROGENITORS

Throughout section (3.1) let K be a field, let S be a Noetherian K -algebra, let $N \subseteq M$ be finitely generated S -modules, and let \underline{u} be a finite sequence of elements of M .

(3.1.1) Definition: affine progenitors. By an *affine progenitor* for (S, M, N, \underline{u}) we shall mean a septuple $\mathcal{M} = (R, M_R, N_R, \underline{u}_R, h, \beta, \eta_R)$ where R is a finitely generated K -algebra, $h: R \rightarrow S$ is a K -homomorphism, M_R is a finitely generated S -module with an R -linear map $\beta: M_R \rightarrow M$ such that the induced map $\beta_*: S \otimes_R M_R \rightarrow M$ is an isomorphism (i.e., β induces $M_S \cong M$), \underline{u}_R is a finite sequence of elements of M_R such that β_* maps \underline{u}_R to \underline{u} , η_R is an R -linear map from N_R to M_R , and the induced map $N_S \rightarrow M_S \rightarrow M$, i.e., $\beta_* \eta_S$, maps N_S onto N . We do not require η_R to be injective, nor do we require that N_S be isomorphic to N . Also, we do not require that R be a subring of S . We refer to R as the *base ring* of the affine progenitor.

We shall usually drop h, β, η_R from the notation and refer to an affine progenitor $\mathcal{M} = (R, M_R, N_R, \underline{u}_R)$. The sequence \underline{u} often consists of a single element u , and in that case we write “ u ” and “ u_R ” instead of using a notation for a sequence with one element.

(3.1.2) Existence of affine progenitors. Affine progenitors always exist. One may see that by lifting the map $N \hookrightarrow M$ to a map of finite presentations (using standard free modules, so that all the maps among the free modules are given by matrices). Then choose R to be any affine K -subalgebra of S so large that it contains the entries of all the matrices, as well as the entries of vectors representing the elements in \underline{u} . We may then use the same matrices to give presentations of modules N_R, M_R and a map between them. It is obvious that if we apply $S \otimes_R _$ we get a map $S \otimes_R M_R \cong M$, and so we have a map $M_R \rightarrow S \otimes_R M_R \cong M$. In this choice of affine progenitor, R is contained in S as a K -subalgebra, and the map $N_S \rightarrow M_S$ carries N_S isomorphically onto N . We can replace $\eta_R: N_R \rightarrow M_R$ by $\eta_R(N_R) \subseteq M_R$ to obtain a new affine progenitor. The new η_R is an inclusion map and we still have surjections $N_S \twoheadrightarrow S \otimes_R \eta_R(N_R) \twoheadrightarrow N$ such that the composite map $N_S \rightarrow N$ is the same isomorphism as before. Thus, we can always choose an affine progenitor such that $R \subseteq S$, the map η_R is injective, and $N_S \cong N$.

(3.1.3) Maps of affine progenitors. (a) Suppose that we have affine progenitors $\mathcal{M} =$

$(R, M_R, N_R, \underline{u}_R, h, \beta, \eta_R)$ and $\mathcal{M}' = (R', M'_{R'}, N'_{R'}, \underline{u}'_{R'}, h', \beta', \eta'_{R'})$ for (S, M, N, \underline{u}) . By a map from \mathcal{M} to \mathcal{M}' we mean a triple (g, γ, δ) such that g is a K -algebra map $R \rightarrow R'$ with $h = h'g$, $\gamma: M_R \rightarrow M'_{R'}$ is R -linear and $\beta = \beta'\gamma$, γ maps \underline{u}_R to $\underline{u}'_{R'}$, and $\delta: N_R \rightarrow N'_{R'}$ R -linearly so that $\eta'_{R'}\delta = \gamma\eta_R$.

(b) Suppose that we are given an affine progenitor $\mathcal{M} = (R, M_R, N_R, \underline{u}_R, h, \beta, \eta_R)$ as above and also suppose that $h: R \rightarrow S$ factors $R \rightarrow R' \xrightarrow{h'} S$ as a map of K -algebras, where R' is an affine K -algebra. Then we may always form a new progenitor $R' \otimes_R \mathcal{M}$ to which \mathcal{M} maps: we have already specified R', h' , and we take $\eta_{R'}: N'_{R'} \rightarrow M'_{R'}$ to be $R' \otimes_R \eta_R$, $\underline{u}_{R'}$, to be the image of \underline{u}_R under $M_R \rightarrow M'_{R'}$ and β' to be the obvious map $R' \otimes_R M_R \rightarrow M$ sending $r' \otimes v \mapsto h'(r')\beta(v)$.

Thus, given an affine progenitor we can map it to one in which R is replaced by its image in S .

We can also replace $\eta_R: N_R \rightarrow M_R$ by $\eta_R(N_R) \hookrightarrow M_R$. Thus we can always map a given affine progenitor to one such that $R \subseteq S$ and η_R is an inclusion map. If $N_S \cong N$ this is preserved when we replace η_R by the inclusion $\eta_R(N_R) \subseteq M_R$.

Given an affine progenitor $(R, M_R, N_R, \underline{u}_R)$ we can always map it to an affine progenitor $(R', M'_{R'}, N'_{R'}, \underline{u}_{R'})$ such that $N'_S \cong N$. We first map it so that $R \subseteq S$. We may choose finitely many elements of N_S , say $\{w_i\}$, that span the kernel of $N_S \rightarrow N$. For R' sufficiently large, $N_{R'}$ will contain elements w'_i that map to the w_i . Let D denote the span over R' of the w'_i in $N_{R'}$. Then let $N'_{R'} = N_{R'}/D$. Combining this with the remarks above, we see that every affine progenitor maps to one such that $R \subseteq S$ as a K -subalgebra, $N_S \cong N$, and η_R is injective.

(c) If $\mathcal{M} = (R, M_R, N_R, \underline{u}_R)$ is an affine progenitor and we have a K -homomorphism $R \rightarrow R' \subseteq S$, we shall write $\mathcal{M}(R')$ for the affine progenitor $(R', M_{R'}, \langle N_{R'} \rangle, 1 \otimes \underline{u}_R)$.

(3.1.4) Mapping each of two progenitors to the same progenitor. We next observe that given two affine progenitors $\mathcal{M}, \mathcal{M}'$ for (S, M, N, \underline{u}) with notation as in (3.1.3a) there is an affine progenitor \mathcal{M}'' to which both map. Even if the two sequences are different, we can arrange for the larger progenitor to be such that \mathcal{M}'' contains a sequence of elements with subsequences to which each maps.

Choose $R'' \subseteq S$ containing the images of both R and R' . Then we can replace \mathcal{M} ,

\mathcal{M}' by $\mathcal{M}(R'')$, $\mathcal{M}'(R'')$. Thus, we can assume that \mathcal{M} , \mathcal{M}' have the same base ring R contained in S , and that $h = h'$. Since S is the directed union of its affine K -subalgebras and the maps $\beta : M_R \rightarrow M$, $\beta' : M'_R \rightarrow M$ induce isomorphisms $M_S \cong M$, $M'_S \cong M$, it is easy to see that for a suitable choice of affine K -algebra R_1 with $R \subseteq R_1 \subseteq S$ we have an isomorphism of $M_{R_1} \cong M'_{R_1}$ compatible with the isomorphisms $M_S \cong M \cong M'_S$. This implies that after replacing R by R_1 we may assume that $M_R = M'_R$ and that $\beta = \beta'$. Fix generators for the images of N_R and N'_R in M_R . As we enlarge R these elements continue to be generators. Each image of a generator of N_R in M_S is expressible as an S -linear combination of the images of the generators of N'_R , and conversely. A direct limit argument shows that this will be true when S is replaced by a certain enlargement R_2 of R . But then we may replace each of N_{R_2} and N'_{R_2} by its image in M_{R_2} , and the two images will be the same for R_2 sufficiently large. Finally, since \underline{u}_{R_2} and \underline{u}'_{R_2} have the same image in M_S , they will have the same image in M_{R_3} for R_3 sufficiently large.

The argument shows the following: suppose that we have two affine progenitors for $\mathcal{P} = (S, M, N, \underline{u})$, \mathcal{M} and \mathcal{M}' as above. Then for any sufficiently large K -subalgebra R'' of S containing the images in S of both R and R' , we have that $\mathcal{M}(R'') \cong \mathcal{M}'(R'')$.

Thus, every affine progenitor for \mathcal{P} maps to $\mathcal{M}(R'')$ for R'' sufficiently large.

(3.1.5) Further observations about affine progenitors. (a) If $(R, M_R, N_R, \underline{u}_R)$ is an affine progenitor of (S, M, N, \underline{u}) then for every Noetherian S -algebra S' it is also an affine progenitor of $S' \otimes_S (S, M, N, \underline{u})$, by which we mean $(S', S' \otimes_S M, \langle S' \otimes_S N \rangle, 1 \otimes \underline{u})$.

(b) Suppose that $(R, M_R, N_R, \underline{u}_R)$ is an affine progenitor for the quadruple (S, M, N, \underline{u}) . Then $(R, M_R/N_R, 0, \underline{v}_R)$ is an affine progenitor for $(S, M/N, 0, \underline{v})$ in an obvious way (where the replacement of “ u ” by “ v ” indicates that we are taking images modulo N_R or modulo N).

Let $W = M/N$. We note that every affine progenitor \mathcal{P} for $(S, W, 0, \underline{u})$ maps to one which has the form described in the preceding paragraph, since if we let \mathcal{M} denote $(R, M_R/N_R, 0, \underline{v}_R)$, \mathcal{P} will map to $\mathcal{M}(R'')$ for R'' sufficiently large.

(c) Suppose that M is free. For definiteness we fix a free basis for M and identify M with S^t . We can choose a finite set of generators for N over S , say $\{n_i\}$. We may let

R_0 be any affine K -subalgebra of S containing the entries of the vectors $\{n_i\}$ and the entries of the vectors in \underline{u} . Let $M_{R_0} = R_0^t$ and let N_{R_0} be the R_0 -span of the n_i . Let $\underline{u}_{R_0} = \underline{u}$. Then $\mathcal{M}_0 = (R_0, M_{R_0}, N_{R_0}, \underline{u}_{R_0})$ is an affine progenitor for (S, M, N, \underline{u}) . (This remains true if we increase N_{R_0} to $M_{R_0} \cap N$). For every larger R' , we obtain an affine progenitor $\mathcal{M}_0(R')$ to which this maps, namely $(R', R'^t, \text{Span}_{R'}\{n_i\}, \underline{u})$. (Alternatively, we could replace $\text{Span}_{R'}\{n_i\}$ by $R'^t \cap N_R$ here.)

Note that every affine progenitor for (S, M, N, \underline{u}) maps to one of the form $\mathcal{M}_0(R')$. (This is also true if we replace $\text{Span}_{R'}\{n_i\}$ by $R'^t \cap N_R$.)

(d) Suppose that we have $N \subseteq N' \subseteq M$ and a finite sequence of elements \underline{u} of M . We may have an affine progenitor for (S, M, N, \underline{u}) and an affine progenitor for $(S, M, N', \underline{u})$. We want to show that we can map both of these to a “common” affine progenitor in which we have that $N_R \rightarrow M_R$ factors $N_R \rightarrow N'_R \rightarrow M_R$. As usual we may assume that both rings R are contained in S and that after suitable enlargements the two rings are the same. Enlarging further we may assume that M_R and \underline{u}_R are the same for both affine progenitors. We still may not have that the given map $N_R \rightarrow M_R$ factors through the map $N'_R \rightarrow M_R$. But we can first enlarge R further and kill suitable submodules of N_R, N'_R if necessary so that $N_S \cong N, N'_S \cong N'$, and then the factorization will exist after a further enlargement of R . (Choose finitely many generators $\{v_j\}$ of N_R and consider their images in N' under the composite $N_R \rightarrow N_S = N \rightarrow N'$. After enlarging R there will be elements $\{w_j\}$ of N'_R that map to these images under $N'_R \rightarrow N'$. After enlarging R still further (to get the map to be well-defined) there will be a map of N_R to N'_R that takes the elements v_j to the corresponding w_j .)

(e) Now suppose that we are given finitely many modules $M^{(i)}$ over S and a submodule $N^{(i)}$ of each. Let M be the direct sum of the $M^{(i)}$ and let N be the direct sum of the $N^{(i)}$, which we identify with a submodule of N . Let \underline{u} be a sequence of elements in M and let $\underline{u}^{(i)}$ be the sequence of components in $M^{(i)}$. For every i , choose an affine progenitor $(R^{(i)}, M_{R^{(i)}}^{(i)}, N_{R^{(i)}}^{(i)}, \underline{u}_{R^{(i)}}^{(i)})$ for $(S, M^{(i)}, N^{(i)}, \underline{u}^{(i)})$ with $R^{(i)} \subseteq S$. Let R be an affine K -algebra containing all the $R^{(i)}$. We may map these progenitors so that the rings $R^{(i)}$ occurring are all equal to R . Thus, for every i we have an affine progenitor $(R, M_R^{(i)}, N_R^{(i)}, \underline{u}_R^{(i)})$

for $(S, M^{(i)}, N^{(i)}, \underline{u}^{(i)})$. Let $M_R = \oplus_i M_R^{(i)}$ and let $N_R = \oplus_i N_R^{(i)}$. Let \underline{u}_R be the sequence of elements of M_R whose component sequences are the $\underline{u}_R^{(i)}$. Then $\mathcal{M} = (R, M_R, N_R, \underline{u}_R)$ is an affine progenitor for (S, M, N, \underline{u}) that has a structure “compatible” with the direct sum decompositions of M, N . Moreover, by (3.1.4), every affine progenitor for (S, M, N, \underline{u}) maps to one of the form $\mathcal{M}(R')$, and each of these has the same kind of compatibility with the direct sum decompositions of M and N .

(f) Now suppose that $\mathfrak{B}_1, \dots, \mathfrak{B}_k$ are ideals of S . If $\mathcal{M} = (R, M_R, N_R, \underline{u}_R)$ is an affine progenitor for $\mathcal{Q} = (S, M, N, \underline{u})$ and \mathfrak{A}_i is the contraction of \mathfrak{B}_i to R , then $\mathcal{M}/\mathfrak{A}_i$, which we define to be $(R/\mathfrak{A}_i, M/\mathfrak{A}_i M, \langle N/\mathfrak{A}_i N \rangle, 1 \otimes \underline{u}_R)$, is an affine progenitor for $\mathcal{Q}_i = (S/\mathfrak{B}_i, M/\mathfrak{B}_i M, \langle N/\mathfrak{B}_i N \rangle, 1 \otimes \underline{u})$ for each i . Note that $R/\mathfrak{A}_i \hookrightarrow S/\mathfrak{B}_i$ whether or not $R \subseteq S$. Now suppose that we have an affine progenitor \mathcal{M}_0 for \mathcal{Q} and also an affine progenitor \mathcal{P}_i for each \mathcal{Q}_i . We want to show that \mathcal{M}_0 maps to an affine progenitor \mathcal{M} for \mathcal{Q} in such a way that for every i , \mathcal{P}_i maps to $\mathcal{M}/\mathfrak{A}_i$.

To see this, first note that we may assume that the base ring for every \mathcal{P}_i is a subring of S/\mathfrak{B}_i , and also that the base ring for \mathcal{M}_0 is contained in S . We can lift finitely many generators of each of these rings to S . We can then form an affine K -subalgebra R of S that contains all of these and the base ring for \mathcal{M}_0 . By mapping each progenitor further we may assume without loss of generality that the base ring for \mathcal{P}_i is the image in S/\mathfrak{B}_i of the base ring R_0 for \mathcal{M}_0 . Then $\mathcal{M}_0/\mathfrak{A}_i$ gives one progenitor for \mathcal{Q}_i and \mathcal{P}_i gives another. We now follow the procedure in (3.1.4) for mapping $\mathcal{M}_0/\mathfrak{A}_i$ and \mathcal{P}_i to a common target, except that we enlarge the base ring always by enlarging R and taking its image. We can do sufficient enlargement to handle all values of i simultaneously. Moreover, the enlargement in (3.1.4) has the form $\mathcal{M}_0(R)$ for $R \supseteq R_0$.

(g) We continue the notation of part (f). We simply want to note that if the \mathfrak{B}_i are the distinct minimal primes of S , then for all sufficiently large K -subalgebras R of S , the \mathfrak{A}_i are distinct and are the minimal primes of R . To guarantee this, note that since \mathfrak{B}_i is a minimal prime there exists an element s_i of $S - \mathfrak{B}_i$ and a positive integer $n(i)$ such that $s_i \mathfrak{B}_i^{n(i)} = 0$, since $\mathfrak{B}_i S_{\mathfrak{B}_i}$ is nilpotent. Simply take R large enough to contain all the s_i and at least one element of $\mathfrak{B}_i - \mathfrak{B}_j$ for all choices of i and j with $\mathfrak{B}_i \neq \mathfrak{B}_j$. Every \mathfrak{A}_i

is prime since it is the contraction of a prime. In R we still have that $s_i \mathfrak{A}_i^{n(i)} = 0$ with $s_i \notin \mathfrak{A}_i$, so that every \mathfrak{A}_i is a minimal prime of R . There are no containments $\mathfrak{A}_i \subseteq \mathfrak{A}_j$ if $\mathfrak{B}_i \neq \mathfrak{B}_j$. Finally, $\cap_i \mathfrak{A}_i \subseteq \cap_i \mathfrak{B}_i$ consists of nilpotents, which shows that the \mathfrak{A}_i must be all the minimal primes of R .

(3.2) DEFINITION AND BASIC PROPERTIES OF DIRECT AND FORMAL K -TIGHT CLOSURE

(3.2.1) Discussion and definition. Let S be a Noetherian K -algebra and $N \subseteq M$ be finitely generated S -modules. Let $u \in M$. We shall say that u is in the *direct K -tight closure* $N^{>*K}$ of N in M if there exists an affine progenitor (R, M_R, N_R, u_R) for (S, M, N, u) such that $u_R \in \langle N_R \rangle^{*K}_{M_R}$ in the sense of (2.2.3). Temporarily, we shall say that u is in the *formal K -tight closure* $N^{\text{f}*K}$ of N in M if for every complete local domain (cf. (2.3.10)) B of R , $1 \otimes u$ is in the direct K -tight closure of $\langle B \otimes_R N \rangle$ in $B \otimes_R M$.

Except for two basic results, Theorems (3.2.2) and (3.2.3) below, we postpone the exploration of these notions until we have digressed in the next section to discuss some needed results concerning approximation. In the following section we reconcile the new definitions with the definition for affine K -algebras given in (2.2.3): the three notions turn out to agree in that case. Once we have established that the formal K -tight closure agrees with the K -tight closure in the case of affine algebras we shall drop the word “formal”, i.e., we shall refer to the formal K -tight closure as the “ K -tight closure.” We shall also drop “f” from the notation at that point.

We shall, in fact, see in §(3.4) that the direct K -tight closure and the formal K -tight closure agree for any locally excellent Noetherian K -algebra when K is a field of characteristic zero.

The following two results are parallel to Theorem (2.5.5), although there isn’t a perfect correspondence of parts and corresponding parts may not have the same letters.

(3.2.2) Theorem (basic properties of direct K -tight closure). *Let S be a Noetherian algebra over a field K of characteristic zero. Let $N', N \subseteq M$ be finitely generated S -modules. Let $u \in M$ and let v be the image of u in M/N . Let I be an ideal of S .*

Unless otherwise indicated, $>^{*K}$ indicates direct K -tight closure in M .

- (a) $N^{>^{*K}}$ is a submodule of M containing N .
- (b) $u \in N^{>^{*K}}_M$ if and only if $v \in 0^{>^{*K}}_{M/N}$.
- (c) If $G = S^t$ is a finitely generated free module mapping onto M , H is the inverse image of N in G and $w \in H$ maps to u , then $u \in N^{>^{*K}}_M$ if and only if for some (equivalently, every sufficiently large) affine K -subalgebra R of S containing the entries of a set of generators $\{h_i\}$ for H and the entries of w (so that we may view w and the h_i as elements of $G_R = R^t \subseteq S^t = G$), we have that $w \in H_R^{*K}_{G_R}$ over R , where H_R is the R -span of the $\{h_i\}$. (The same result is valid if we take H_R to be $H \cap G_R$.)
- (d) If $N \subseteq N' \subseteq M$ then $N^{>^{*K}}_M \subseteq N'^{>^{*K}}_M$ and $N^{>^{*K}}_{N'} \subseteq N^{>^{*K}}_M$.
- (e) $(N^{>^{*K}})^{>^{*K}} = N^{>^{*K}}$.
- (f) $(N \cap N')^{>^{*K}} \subseteq N^{>^{*K}} \cap N'^{>^{*K}}$.
- (g) $(N + N')^{>^{*K}} = (N^{>^{*K}} + N'^{>^{*K}})^{>^{*K}}$.
- (h) $(IN)^{>^{*K}}_M = ((I^{>^{*K}}_R)N^{>^{*K}}_M)^{>^{*K}}_M$.
- (i) $(N :_M I)^{>^{*K}}_M \subseteq N^{>^{*K}} :_M I$ (respectively, $(N :_S N')^{>^{*K}} \subseteq N^{>^{*K}} :_S N'$). Hence, if $N = N^{>^{*K}}$ then $(N :_M I)^{>^{*K}} = N :_M I$ (respectively, $(N :_S N')^{>^{*K}} = N :_S N'$).
- (j) If $N_i \subseteq M_i$ are finitely many finitely generated S -modules and we identify $N = \oplus_i N_i$ with its image in $M = \oplus_i M_i$ then the obvious injection of $\oplus_i N_i^{>^{*K}}_{M_i} \rightarrow M$ maps $\oplus_i N_i^{>^{*K}}_{M_i}$ isomorphically onto $N^{>^{*K}}_M$.
- (k) (Persistence of direct K -tight closure) Let L be a field containing K , let S' be a Noetherian L -algebra (hence, also, a K -algebra) and let $S \rightarrow S'$ be a K -algebra homomorphism. Let $u \in N^{>^{*K}}_M$. Then $1 \otimes u \in \langle S' \otimes_R N \rangle^{>^{*L}}_{S' \otimes_R M}$ over S' . In particular, this holds when $L = K$.
- (l) (Persistence of direct K -tight closure: second version) Let L be a field containing K , let S' be a Noetherian L -algebra, let $h: S \rightarrow S'$ be a K -homomorphism, let $N \subseteq M$ be finitely generated S -modules, and let $V \subseteq W$ be finitely generated S' -modules. Suppose that $u \in N^{>^{*K}}_M$. Suppose also that there is an R -homomorphism $\gamma: M \rightarrow W$ such that $\gamma(N) \subseteq V$. Then $\gamma(u) \in V^{>^{*L}}_W$.
- (m) (Irrelevance of nilpotents) If J is the nilradical of S , then $J \subseteq (0)^{>^{*K}}$, and so $J \subseteq I^{>^{*K}}$ for all ideals I of S . Consequently, $JM \subseteq N^{>^{*K}}$. Moreover, if N^\sim denotes

the image of N in M/JM , then $N^{>*K}$ is the inverse image in M of the tight closure $(N^\sim)^{>*K}_{M/JM}$, which may be computed either over S or over $S_{\text{red}} (= S/J)$.

- (n) Let $\mathfrak{p}^{(1)}, \dots, \mathfrak{p}^{(s)}$ be the minimal primes of S and let $S^{(i)} = S/\mathfrak{p}^{(i)}$. Let $M^{(i)} = S^{(i)} \otimes_R M$ and let $N^{(i)}$ be the image of $S^{(i)} \otimes_S N$ in $M^{(i)}$. Let $u^{(i)}$ be the image of u in $M^{(i)}$. Then $u \in N^{>*K}$ if and only if $u^{(i)} \in (N^{(i)})^{>*K}$ in $M^{(i)}$ over $S^{(i)}$, $1 \leq i \leq s$.
- (o) If $R = \prod_{i=1}^h R_i$ is a finite product and $M = \prod_i M_i$ and $N = \prod_i N_i$ are the corresponding product decompositions of M , N , respectively, then $u = (u_1, \dots, u_h) \in M$ is in $N^{>*K}_M$ over R if and only if for all i , $1 \leq i \leq h$, $u_i \in N_i^{>*K}_{M_i}$.

Proof. (a) Given $u, u' \in N^{>*K}$ we can choose affine progenitors $\mathcal{M}, \mathcal{M}'$ such that $u_R \in N_R^{*K}_{M_R}$ in the first and $u' \in N_{R'}^{*K}_{M_{R'}}$ in the second. The key point is that by (3.1.4) these map to a progenitor $(R'', M''_R, N''_{R''}, \underline{u}_{R''})$ (where $\underline{u} = u, u'$), and by the persistence of $*K$ we have that both $u_{R''}$ and $u'_{R''}$ are in $\langle N''_{R''} \rangle^{*K}_{M''_{R''}}$. It then follows that

$$r_1 u_{R''} + r_2 u'_{R''} \in \langle N''_{R''} \rangle^{*K}_{M''_{R''}},$$

since this is a submodule of $M''_{R''}$. Thus, $N^{>*K}$ is a submodule of M , and it is obvious that it contains N .

(b) If $u \in N^{>*K}_M$ then we have that $u_R \in \langle N_R \rangle^{*K}_{M_R}$ for some affine progenitor (R, M_R, N_R, u_R) of (S, M, N, u) . It is then immediate that $v_R \in 0^{*K}_{M_R/N_R}$ (with notation as in (3.1.5b)). Now suppose that $W = M/N$ and that we have chosen an affine progenitor for $(S, W, 0, \underline{v})$, say $(R, W_R, Y_R, \underline{v}_R)$, such that $\underline{v}_R \in \langle \langle Y_R \rangle \rangle^{*K}_{W_R}$. We can map this progenitor to one of the form $(R, M_R/N_R, 0, V_R)$ as in (3.1.5b). Then v_R is in $0^{*K}_{M_R/N_R}$ by the persistence of tight closure, and then $u_R \in N_R^{*K}_{M_R}$.

(c) By part (b), $u \in N^{>*K}_M$ iff $v \in 0^{>*K}_{M/N}$ iff $\bar{w} \in 0^{>*K}_{G/H}$ (where \bar{w} is the image of w in G/H) iff $w \in H^{>*K}_G$. For sufficiently large $R \subseteq S$, (R, G_R, H_R, w_R) as described with $w_R = w$ is an affine progenitor of (S, G, H, w) . (It does not matter which version of H_R we use: $H \cap G_R$ may be larger, but the additional elements are R' -linear combinations of the generators of H_R when R is enlarged suitably to R' .) It is clear that the condition given is sufficient for w to be in $H^{>*K}_G$ (and once this holds over R , it will hold over every larger R' by the persistence of K -tight closure in the affine case). The condition is necessary

because if $w \in H^{>*K}_G$ then $w \in \langle H_R \rangle^{*K}_{G_R}$ for some affine progenitor (R, G_R, H_R, w_R) of (S, G, H, w) , and this affine progenitor maps to one of the form specified here by (3.1.5c).

(d) This is immediate from (3.1.5d), the definitions, and the persistence of tight closure.

(e) It is clear that $N^{>*K} \subseteq (N^{>*K})^{>*K}$. We must prove the other inclusion. Let $W = N^{>*K}$. Of course, $N \subseteq W \subseteq M$. Let $v \in W^{>*K}$. We want to show that $v \in W$. Choose a sequence of elements \underline{u} in W that generate W over S , and let \underline{v} denote \underline{u} together with v . We can choose affine progenitors for (S, M, N, \underline{v}) and (S, M, W, \underline{v}) so that in the first all the elements representing the elements of \underline{u} are in the K -tight closure of the image of N_R in M_R , and in the second the element representing v is in the K -tight closure of the image of W_R . Moreover, we can assume that the elements in $\underline{u}_{R'}$ generate $W_{R'}$. By (3.1.5d) we can map both progenitors to a “common” affine progenitor such that we have $N_R \rightarrow W_R \rightarrow M_R$ and the other conditions that we have discussed continue to hold. It follows that the image of W_R in M_R is in the K -tight closure of the image of N_R , and that the image of v_R is in the K -tight closure of the image of W_R in M_R . This implies that the image of v_R is in the K -tight closure of the image of N_R in M_R , and, hence, that the image of v_R is in the K -tight closure of the image of N_R in M_R . But this shows that $v \in N^{>*K} = W$, as required.

(f), (g) These parts follow formally from what has already been shown precisely as in the proofs of parts (f) and (g) of Theorem (2.5.5).

(h) Since $I \subseteq I^{>*K}$ and $N \subseteq N^{>*K}$ we have that $IN \subseteq (I^{>*K})N^{>*K}$ and so $(IN)^{>*K} \subseteq ((I^{>*K})N^{>*K})^{>*K}$. It remains to prove the other inclusion. Let i be an element of $I^{>*K}$, and let $u \in N^{>*K}$. It will suffice to show that $iu \in (IN)^{>*K}$, for this yields $(I^{>*K})N^{>*K} \subseteq (IN)^{>*K}$ and hence $((I^{>*K})N^{>*K})^{>*K} \subseteq ((IN)^{>*K})^{>*K} = (IN)^{>*K}$ by part (e). There is an affine progenitor (R_0, R_0, I_0, i) for (S, S, I, i) such that $R_0 \subseteq S$, $I_0 \subseteq I$, $I_0 S = I$, and i is in I_0^{*K} over R_0 , by part (c). There is also an affine progenitor (R, M_R, N_R, u_R) for (S, M, N, u) such that u_R is in $\langle N_R \rangle^{*K}_{M_R}$. By mapping further we may assume that $R \subseteq S$, that $R \supseteq R_0$, and that $N_R \subseteq M_R$. Then $(R, M_R, (I_0 R)N_R, iu_R)$ is an affine progenitor of (S, M, IN, iu) . Since $i \in (I_0 R)^{*K}_R$ and $u_R \in N_R^{*K}_{M_R}$ over R , we have that $iu_R \in ((I_0 R)N_R)^{*K}_{M_R}$ over K by (2.5.5h), and so $iu \in (IN)^{>*K}_M$ as required.

(i) This follows formally from part (h) and the preceding parts precisely as in the proof of Theorem (2.5.5i).

(j) This is immediate from the definition of $^{>*K}$, the discussion in (3.1.5e) and Theorem (2.5.5j).

(k) This is immediate from the definition of direct K -tight closure, (3.1.5a), and the persistence of *K .

(l) This follows from (k) and the earlier parts precisely as in the proof of part (l) of Theorem (2.5.5).

(m) Given any element j of J , it will be nilpotent in any affine K -subalgebra R of S that contains j , and then $(R, R, (0), j)$ is an affine progenitor of $(S, S, (0), j)$. That $j \in (0)^{*K}_R$ follows from (2.5.5m), and so $j \in (0)^{>*K}_S$. That $JM \subseteq N^{>*K}$ now follows by precisely the same argument as in the proof of (2.5.5m).

Now suppose that $u \in N^{>*K}_M$. It is clear that $1 \otimes u \in (N^\sim)^{>*K}_{M/JM}$ over S since

$$(M/JM)/N^\sim \cong M/(N + JM)$$

and $u \in (N + JM)^{>*K}_M$ over S . The fact that $1 \otimes u \in (N^\sim)^{>*K}_{M/JM}$ over S/J is a special case of (k). On the other hand, suppose that $1 \otimes u \in (N^\sim)^{>*K}_{M/JM}$ either over S or over S/J . We must show that $u \in N^{>*K}_M$. But if this holds over S then it will hold over S/J as well, applying (k) again. Thus, we might as well assume that $1 \otimes u \in (N^\sim)^{>*K}_{M/JM}$ over S/J . Choose an affine progenitor of $(S/J, M/JM, \langle N/JN \rangle, 1 \otimes u)$ that exhibits the tight closure relation. (Note that $\langle N/JN \rangle = N^\sim$.) By the result of (3.1.5f), we may map this progenitor to one of the form $\mathcal{M}/(J \cap R)$, where \mathcal{M} is an affine progenitor of for (S, M, N, u) . But then (2.5.5m) shows that $u \in N^{>*K}_M$, since $J \cap R$ is nilpotent.

(n) We know from (k) that $u^{(i)}$ is in $N^{(i)>*K}_{M^{(i)}}$ over $S^{(i)}$ for all i . Now suppose that this condition holds for all i and we want to prove that $u \in N^{(i)>*K}$. By the result of (3.1.5f) we can choose an affine progenitor $\mathcal{M} = (R, M_R, N_R, u_R)$ of (S, M, N, u) such that for all i , $1 \otimes u_R$ is in $\langle N_R/\mathfrak{A}^{(i)}N \rangle^{*K}_{M_R/\mathfrak{A}^{(i)}M_R}$ over $R/\mathfrak{A}^{(i)}$. (For every i we can arrange a progenitor of $(S/\mathfrak{p}^{(i)}, M/\mathfrak{p}^{(i)}M, \langle N/\mathfrak{p}^{(i)} \rangle, 1 \otimes u)$ that satisfies the tight closure condition, and we can then map all of these separate progenitors simultaneously to ones arising from a single progenitor of (S, M, N, u) by the method of (3.1.5f).) By (3.1.5g) for R large

enough the \mathfrak{A}_i are simply the minimal prime ideals of R , and the result now follows from (2.5.5n).

(o) Choose an affine progenitor \mathcal{M}_i for every (R_i, M_i, N_i, u_i) and form a progenitor \mathcal{P} for (R, M, N, u) by taking products. If every $u_i \in M_i^{>*K}_{N_i}$ choose the \mathcal{M}_i so that this holds, in the sense of affine K -tight closure, in every progenitor. By (2.5.5o) this gives a progenitor for (R, M, N, u) which displays the fact that $u \in N^{>*K}_M$. On the other hand, if $u \in N^{>*K}_M$ form \mathcal{P} as above, but also choose another affine progenitor in which the K -tight closure condition holds (in the affine sense). These both map to an affine progenitor of the form $\mathcal{P}(R')$, which will be a product of progenitors for the factors, and in which the K -tight closure condition will hold. Now apply (2.5.5o). \square

We now give a parallel result for formal K -tight closure.

(3.2.3) Theorem (basic properties of formal K -tight closure). *Let S be a Noetherian algebra over a field K of characteristic zero. Let $N', N \subseteq M$ be finitely generated S -modules. Let $u \in M$ and let v be the image of u in M/N . Let I be an ideal of S .*

*Unless otherwise indicated, ${}^{\text{f}*K}$ indicates formal K -tight closure in M .*

- (a) $N^{\text{f}*K}$ is a submodule of M containing N , and, in fact, containing $N^{>*K}_M$. If S is a complete local domain then $N^{\text{f}*K} = N^{>*K}_M$.
- (b) $u \in N^{\text{f}*K}_M$ if and only if $v \in 0^{\text{f}*K}_{M/N}$.
- (c) The following three conditions are equivalent:
 - (1) $u \in N^{\text{f}*K}$.
 - (2) For every complete local domain B of R , we have that $u_B \in \langle N_B \rangle^{\text{f}*K}_{M_B}$.
 - (3) For every complete local domain C to which R maps, we have that

$$u_C \in \langle N_C \rangle^{\text{f}*K}_{M_C}.$$

- (d) If $N \subseteq N' \subseteq M$ then $N^{\text{f}*K}_M \subseteq N'^{\text{f}*K}_M$ and $N^{\text{f}*K}_{N'} \subseteq N^{\text{f}*K}_M$.
- (e) $(N^{\text{f}*K})^{\text{f}*K} = N^{\text{f}*K}$.
- (f) $(N \cap N')^{\text{f}*K} \subseteq N^{\text{f}*K} \cap N'^{\text{f}*K}$.
- (g) $(N + N')^{\text{f}*K} = (N^{\text{f}*K} + N'^{\text{f}*K})^{\text{f}*K}$.
- (h) $(IN)^{\text{f}*K}_M = ((I^{\text{f}*K}_R)N^{\text{f}*K}_M)^{\text{f}*K}_M$.

- (i) $(N :_M I)^{f*K}_M \subseteq N^{f*K} :_M I$ (respectively, $(N :_S N')^{f*K} \subseteq N^{f*K} :_S N'$). Hence, if $N = N^{f*K}$ then $(N :_M I)^{f*K} = N :_M I$ (respectively, $(N :_S N')^{f*K} = N :_S N'$).
- (j) If $N_i \subseteq M_i$ are finitely many finitely generated S -modules and we identify $N = \oplus_i N_i$ with its image in $M = \oplus_i M_i$ then the obvious injection $\oplus_i N_i^{f*K}_{M_i} \hookrightarrow M$ maps $\oplus_i N_i^{f*K}_{M_i}$ isomorphically onto N^{f*K}_M .
- (k) (Persistence of formal K -tight closure) Let L be a field containing K , let S' be a Noetherian L -algebra (hence, also, a K -algebra) and let $S \rightarrow S'$ be a K -algebra homomorphism. Let $u \in N^{f*K}_M$. Then $1 \otimes u \in \langle S' \otimes_R N \rangle^{>*K}_{S' \otimes_R M}$ over S' . In particular, this holds when $L = K$.
- (l) (Persistence of formal K -tight closure: second version). Let L be a field containing K , let S' be a Noetherian L -algebra (hence, also, a K -algebra) and let $S \rightarrow S'$ be a K -algebra homomorphism. Let $N \subseteq M$ be finitely generated S -modules, and let $V \subseteq W$ be finitely generated S' -modules. Suppose that $u \in N^{f*K}_M$. Suppose also that there is an R -homomorphism $\gamma : M \rightarrow W$ such that $\gamma(N) \subseteq V$. Then $\gamma(u) \in V^{f*L}_W$.
- (m) (Irrelevance of nilpotents) If J is the nilradical of S , then $J \subseteq (0)^{f*K}$, and so $J \subseteq I^{f*K}$ for all ideals I of S . Consequently, $JM \subseteq N^{f*K}$. Moreover, if N^\sim denotes the image of N in M/JM , then N^{f*K} is the inverse image in M of the tight closure $(N^\sim)^{f*K}_{M/JM}$, which may be computed either over S or over $S_{\text{red}} (= S/J)$.
- (n) Let $\mathfrak{p}^{(1)}, \dots, \mathfrak{p}^{(s)}$ be the minimal primes of S and let $S^{(i)} = R/\mathfrak{p}^{(i)}$. Let $M^{(i)} = S^{(i)} \otimes_S M$ and let $N^{(i)}$ be the image of $S^{(i)} \otimes_S N$ in $M^{(i)}$. Let $u^{(i)}$ be the image of u in $M^{(i)}$. Then $u \in N^{f*K}$ if and only if $u^{(i)} \in (N^{(i)})^{f*K}$ in $M^{(i)}$ over $S^{(i)}$, $1 \leq i \leq s$.
- (o) If $R = \prod_{i=1}^h R_i$ is a finite product and $M = \prod_i M_i$ and $N = \prod_i N_i$ are the corresponding product decompositions of M , N , respectively, then $u = (u_1, \dots, u_h) \in M$ is in N^{f*K}_M over R if and only if for all i , $1 \leq i \leq h$, $u_i \in N_i^{f*K}_{M_i}$.

Proof. (a) If $u, v \in N^{f*K}$ and $s_1, s_2 \in S$, then the image of $s_1 u + s_2 v$ in M_B for any complete local domain B of R is in $\langle N_B \rangle^{>*K}_{M_B}$ since $\langle N_B \rangle^{>*K}_{M_B}$ is a submodule of M_B . This shows that N^{f*K} is a submodule of M . The fact that $N^{>*K} \subseteq N^{f*K}$ is immediate from the persistence of $^{>*K}$ applied to the maps from S to its various complete local domains.

The fact stated in the final sentence follows from the general inclusion $N^{>*K} \subseteq N^{f*K}$

and the fact that when S is a complete local domain it itself is one of the complete local domains of S , which forces $N^{\text{f}*K} \subseteq N^{>*K}$ in this case.

(b) We have that $u \in N^{\text{f}*K}_M$ if and only if $u_B \in \langle N_B \rangle^{>*K}_{M_B}$ for all complete local domains B of S if and only if $v_B \in 0^{>*K}_{M_B/\langle N_B \rangle}$ for all complete local domains B of S , where v_B is the image of u_B modulo $\langle N_B \rangle$. Since $M_B/\langle N_B \rangle \cong (M/N)_B$, this is equivalent to the condition that $v_B \in 0^{>*K}_{(M/N)_B}$ for all complete local domains B of R , which in turn is equivalent to the condition that $v \in 0^{\text{f}*K}_{M/N}$.

(c) Suppose that $u \in N^{\text{f}*K}$ and let C be a complete local domain to which R maps. To show that $u_C \in \langle N_C \rangle^{\text{f}*K}_{M_C}$ we must show that for every complete local domain D of C , $u_D \in \langle N_D \rangle^{>*K}_{M_D}$. Let Q be the contraction of the maximal ideal of D to S . Then $(S_Q)^\wedge$ maps to D , and the kernel must contain some minimal prime of $(S_Q)^\wedge$. It follows that some complete local domain B of S maps to D . We know that $u_B \in \langle N_B \rangle^{>*K}_{M_B}$ and the result now follows from the persistence of direct K -tight closure applied to the homomorphism $B \rightarrow D$. Thus, (1) \Rightarrow (3), while (3) \Rightarrow (2) is obvious, and (2) \Rightarrow (1) is immediate from the definition and the final statement in part (a).

(d) If $u \in N^{\text{f}*K}_M$ then for all complete local domains B of R we have that $u_B \in \langle N_B \rangle^{>*K}_{M_B} \subseteq \langle N'_B \rangle^{>*K}_M$, and so $u \in N'^{\text{f}*K}_M$. If $u \in N^{\text{f}*K}_{N'}$ then for all complete local domains B of R we have that $u_B \in \langle N_B \rangle^{>*K}_{N'_B}$ (where $u_B \in N'_B$, and $\langle N_B \rangle$ is the image of N_B in N'_B). Let $\langle\langle N_B \rangle\rangle$ denote the image of N_B in M_B . Since the map $N'_B \rightarrow M_B$ carries the pair $(\langle N_B \rangle, N'_B)$ into $(\langle\langle N_B \rangle\rangle, M_B)$ it follows from Theorem (3.2.2) that the image of u_B in M_B is in $\langle\langle N_B \rangle\rangle^{>*K}$.

(e) Let $W = N^{\text{f}*K}$. It will suffice to show that if $u \in W^{\text{f}*K}$ then $u \in N^{\text{f}*K}$. Let $\{w_i\}$ be a finite set of generators of W . For every complete local domain B of R we have that $u \in \langle W_B \rangle^{>*K}_{M_B}$ and also that the image of every w_i is in $\langle N_B \rangle^{>*K}_{M_B}$. But the images of the w_i generate $\langle W_B \rangle$, and so $\langle W_B \rangle \subseteq \langle N_B \rangle^{>*K}_{M_B}$. But then $u_B \in (\langle N_B \rangle^{>*K}_{M_B})^{>*K} = \langle N_B \rangle^{>*K}_{M_B}$ by Theorem (3.2.2e), and since this holds for all B we have that $u \in N^{\text{f}*K}_M$, as required.

(f), (g) These parts now follow formally from the earlier parts just as in the proofs of parts (f), (g) of Theorem (2.5.5).

(h) As in the proof of (2.5.5h) it suffices to show that $I^{\text{f}*K}N^{\text{f}*K} \subseteq (IN)^{\text{f}*K}$. Let

$i \in I^{f*K}$ and $u \in N^{f*K}$. Then for all complete local domains B of S we have that $i_B \in (IB)^{>*K}_B$ and that $u_B \in N_B^{>*K}_{M_B}$, and it follows from Theorem (3.2.2h) that $(iu)_B = i_B u_B \in \langle (IN)_B \rangle^{>*K}$, since $\langle (IN)_B \rangle = I_B N_B$.

(i) This follows formally from (h) and the earlier parts as in the proof of Theorem (2.5.5i).

(j) This is immediate from the definition of $f*K$ and Theorem (3.2.2j).

(k) This is immediate from (c), since S maps to every complete local domain of S' .

(l) This follows from (k) and the earlier parts precisely as in the proof of part (l) of Theorem (2.5.5).

(m) This is obvious, since the complete local domains of S_{red} and S are the same.

(n) This is clear from (2.3.10b).

(o) The assertion follows because every complete local domain of R is a complete local domain of one of the factor rings, and every complete local domain of one of the factor rings is a complete local domain of R . \square

We note the following immediate corollary of part (l) of Theorems (3.2.2) and (3.2.3).

(3.2.4) Corollary. *Let L be a field of characteristic zero and let S be a Noetherian L -algebra. Let $K \rightarrow L$ be a field homomorphism. Let $N \subseteq M$ be finitely generated S -modules. Then $N_M^{>*K} \subseteq N_M^{>*L}$ and $N_M^{f*K} \subseteq N_M^{f*L}$. \square*

(3.2.5) Remark. As already noted in (2.5.8), we do not know whether these inclusions can be strict, even in the affine case.

We defer further investigation of the properties of these tight closure operations until we have shown that they agree in the case of a locally excellent K -algebra (§(3.4)) and proved a critical result on descent from complete local rings to affine K -algebras (§(3.5)). Before proceeding we need to discuss some results concerning Artin approximation and its generalizations.

(3.3) ARTIN APPROXIMATION AND THE STRUCTURE OF POWER SERIES RINGS

This section contains an exposition of some material related to Artin approximation and its generalizations that is needed in the two following sections. We begin with a brief discussion of Henselization.

(3.3.1) Remarks on Henselization and étale extensions. By an *étale algebra* we mean a formally étale algebra that is finitely presented. Throughout the rest of this paragraph assume that (R, m, K) is local, i.e., Noetherian with a unique maximal ideal. By a *pointed étale extension* of R we mean a local homomorphism $(R, m, K) \rightarrow (S, n, L)$ such that S is a localization of an étale algebra over R at a prime ideal lying over m and such that the induced map of residue fields $K \rightarrow L$ is an isomorphism. The *Henselization* R^h of R is a direct limit of pointed étale extensions: moreover, every pointed étale extension of R has a unique local R -algebra homomorphism to R^h . Note that R^h is faithfully flat over R , Noetherian, with the same residue field, and has maximal ideal mR^h . The map $R \rightarrow R^h$ induces an isomorphism of the completions, so that R^h may always be thought of as a subring of \widehat{R} , and this subring gives a canonical choice of Henselization. R^h is regular if and only if R is regular.

Every pointed étale extension of R has the form $(R[x]/(F))_Q$ where:

- (1) $F = F(x)$ is a monic polynomial in x .
- (2) Q is a maximal ideal of $R[x]$ generated by m and a single linear polynomial of the form $x - r$, where r is an element of R .
- (3) If λ is the residue of r in K , then $\overline{F}'(\lambda) \neq 0$, where F' is the derivative of F with respect to x , and the bar indicates that the coefficients are to be reduced modulo m (this is equivalent to the requirement that the image of F' in S be invertible).

We refer the reader to [Ray] for a detailed treatment of these ideas. Note that in this situation S may also be viewed as the direct limit of the rings $(R[x]/(F))_G$ where G is an

element of $R[x] - Q$. If G is taken sufficiently “big” (i.e., having sufficiently many factors) then the image of F' will be invertible in $(R[x]/(F))_G$, which is then a *standard étale extension* of R . Moreover, if G is taken sufficiently big then $(R[x]/(F))_G$ will inject into R^h . It follows that R^h is a directed union of subrings of the form $(R[x]/(F))_G$ where F is monic, and the image of F' is invertible in $(R[x]/(F))_G$.

(3.3.2) Approximation rings. A local ring (R, m, K) is called an *approximation ring* if every system of polynomial equations in finitely many variables with coefficients in R that has a solution in \widehat{R} has a solution in R . A stronger property is immediate: given a solution over \widehat{R} then for every positive integer N one can find a solution in R congruent to the given solution modulo $m^N \widehat{R}$ (one may keep track of the congruence condition using auxiliary variables and equations).

Note that R is an approximation ring if and only if every finitely generated R -algebra S that admits an R -algebra homomorphism to \widehat{R} admits an R -algebra homomorphism to R . (To see this, think of S as $R[X_1, \dots, X_n]/(F_1, \dots, F_m)$, consider the system of polynomial equations

$$\left\{ F_j(X) = 0, \ 1 \leq j \leq m, \right.$$

and use the fact that the R -algebra homomorphisms to an R -algebra C correspond bijectively to the solutions of the system of equations above for the X 's in C , whether $C = R$ or $C = \widehat{R}$.) Likewise, if R is a local ring such that R^h (its Henselization) is an approximation ring, then every finitely generated R -algebra S that admits an R -homomorphism to \widehat{R} admits an R -homomorphism to R^h .

By the *Artin approximation* theorem we mean the following deep result of M. Artin [Ar1]:

(3.3.3) Theorem (Artin approximation). *The Henselization of a local ring essentially of finite type over V , where V is either a field or an excellent discrete valuation ring, is an approximation ring. \square*

We note that by [EGA2] (18.7.2)-(18.7.5), the Henselization of a universally catenary local ring is excellent if and only if the local ring is excellent; in particular, the Henselization of an excellent local ring is excellent.

We shall also need the following equal characteristic zero generalization of (3.3.3), due to C. Rotthaus [Rot].

(3.3.4) Theorem (Rotthaus). *Every excellent Henselian local ring of equal characteristic zero is an approximation ring.*

Hence, if a finitely generated algebra S over an equicharacteristic 0 excellent local ring R admits an R -homomorphism into \widehat{R} , it admits an R -homomorphism into R^h . \square

The hypothesis that the ring be of equal characteristic 0 is not needed here: one may show that every excellent Henselian local ring is an approximation ring using general Néron desingularization: see (4.2). However, we shall only need the equal characteristic 0 case here.

Finally, we shall also make substantial use of the following result proved in [ArR].

(3.3.5) Theorem (Artin-Rotthaus). *Let K denote either a field or an excellent discrete valuation ring. Let $T = K[[x_1, \dots, x_n]]$ be the formal power series ring in n variables over K . Then every K -algebra homomorphism of a finitely generated K -algebra R to T factors $R \rightarrow S \rightarrow T$ where the maps are K -algebra homomorphisms and S has the form $(K[x_1, \dots, x_n, y_1, \dots, y_t]_m)^h$, where the x_i are as above, the x_i and y_j are algebraically independent elements, over K , of the maximal ideal of T , m is the ideal of the polynomial ring $K[x, y]$ generated by (x, y) and, if K is a DVR, by the generator of the maximal ideal of K , and h denotes Henselization. \square*

This is a very powerful theorem that easily implies (3.3.3). Note that t will vary depending on the subalgebra: there is no bound, since $K[[x_1, \dots, x_n]]$ has infinite transcendence degree over K unless $n = 0$.

Note that it suffices, in proving such a result, to consider the case where R is a subring of T , since in the general case we may replace R by its image in T . Notice, however, that even when R is a subring of T this result does not assert that S can be taken to be a subring of T .

Also note that (3.3.5) is a particular case of general Néron desingularization (applied to the geometrically regular map $A \rightarrow \widehat{A}$, where A is the localization of $K[x]$ at the maximal ideal generated by the x 's and, if K is a DVR, the generator of its maximal ideal), which

is discussed in detail in (4.2.1–3). However, we have made an effort to avoid using the full strength of general Néron desingularization where possible, and, since the proof of (3.3.5) is given in a short and self-contained paper, whenever it suffices we refer to (3.3.5) instead.

(3.3.6) Remark. By the remarks in (3.3.1), we may use, in (3.3.5), instead of the Henselization of $K[x, y]_m$, an étale extension of $K[x, y]$ contained in the Henselization. This will be an affine K -algebra, a regular domain, and the x 's will form a regular sequence.

(3.4) THE LOCALLY EXCELLENT CASE

The main result of this section is that formal K -tight closure and direct K -tight closure agree for locally excellent Noetherian algebras over a field K of characteristic 0. It is then an easy corollary that both these notions agree with our original notion *K for finitely generated K -algebras.

Here is the precise statement.

(3.4.1) Theorem. *Let S be a locally excellent Noetherian algebra over a field K of characteristic 0. Let $N \subseteq M$ be finitely generated S -modules. Then the following three conditions on an element $u \in M$ are equivalent:*

- (1) $u \in N^{>{}^{*K}}_M$.
- (2) *For every maximal ideal m of S , if $C = (S_m)^\wedge$ then $u_C \in \langle N_C \rangle^{>{}^{*K}}_{M_C}$.*
- (3) $u \in N^{\text{f}{}^{*K}}_M$.

Hence, $N^{>{}^{*K}}_M = N^{\text{f}{}^{*K}}_M$.

Proof. The final statement is the same as the equivalence of (1) and (3).

By Theorem (3.2.3a), (1) \Rightarrow (3). (3) clearly implies (2), since it implies that the same condition holds when C is replaced by any of its quotients by a minimal prime, and we may apply Theorem (3.2.2n). Thus, it will suffice to show that (2) \Rightarrow (1). By Theorems (3.2.2) and (3.2.3) we may assume without loss of generality that $M = S^t$ is free. First choose an affine progenitor $\mathcal{M} = (R, M_R, N_R, u_R)$ of $\mathcal{P} = (S, M, N, u)$ as in (3.1.5c), so that $M_R = R^t$, $N_R \subseteq N$ is spanned over R by finitely many generators of N and $u_R = u$. We shall repeatedly enlarge R in the sequel: when we replace R by an affine K -algebra

R' with $R \subseteq R' \subseteq S$ we replace \mathcal{M} by $\mathcal{M}(R')$ (see (3.1.3c) and (3.1.4)), which is an affine progenitor for \mathcal{P} satisfying the same conditions.

For every maximal ideal m of S we can choose a finitely generated R -subalgebra of $C = (S_m)^\wedge$, call it ${}_m\mathcal{R}$, such that $u_{{}_m\mathcal{R}}$ is in $\langle N_{{}_m\mathcal{R}} \rangle^{*K}$ in $M_{{}_m\mathcal{R}}$, since u_C is in $\langle N_C \rangle^{>*K}_{M_C}$. The affine K -algebra ${}_m\mathcal{R}$ then admits an R -algebra homomorphism into $(S_m)^\mathbf{h}$, the Henselization, by Theorem (3.3.4). It follows that ${}_m\mathcal{R}$ admits an R -algebra homomorphism into a standard étale extension ${}_mS = (S_{mf}[X]/{}_mG)_{mH}$ where ${}_mf \in S - m$, ${}_mG$ is monic in X with coefficients in S_{mf} , ${}_mH \in S_{mf}[X]$ has degree strictly smaller than that of ${}_mG$ (we can subtract off a multiple of ${}_mG$ to make this true) and has at least one coefficient not in ${}_mS_{mf}$, and the image of the derivative ${}_mG'$ is a unit of ${}_mS$. We can localize S further at an element not in m (i.e., we can replace ${}_mf$ by a multiple not in m) so that we may assume as well that at least one coefficient of ${}_mH$ is a unit of S_{mf} .

These conditions guarantee that ${}_mS$ is faithfully flat and étale over S_{mf} .

The set of elements $\{{}_mf : m \in \text{Max Spec } S\}$ must generate the unit ideal of S , since for every maximal ideal of S at least one of them is not in that maximal ideal. Choose finitely many maximal ideals $\{m_i\}$ such that the ${}_mif$ generate the unit ideal in S . We shall use the subscript i to index objects that we were formerly indexing by the finitely many maximal ideals m_i . Thus, ${}_if = {}_{m_i}f$, ${}_iS = {}_{m_i}S$, ${}_i\mathcal{R} = {}_{m_i}\mathcal{R}$, etc. We can enlarge R (and all the ${}_i\mathcal{R}$ along with it) so that all of the following conditions are satisfied:

- (i) The ${}_if$ are in R , and all the coefficients of the ${}_iG$, ${}_iH$ are in R_{if} .
- (ii) The ${}_if$ generate the unit ideal in R .
- (iii) Every ${}_iH$ has a coefficient that is a unit of R_{if} .
- (iv) Every derivative ${}_iG'$ is invertible in $(R_{if}[X]/({}_iG))_{iH}$.

We let ${}_iR = (R_{if}[X]/({}_iG))_{iH}$. Then ${}_iR$ is étale and faithfully flat over R_{if} and ${}_iS \cong S \otimes_R {}_iR$.

Since every ${}_i\mathcal{R}$ is finitely generated over K we can enlarge R further (and all the ${}_iR$ along with it) so that:

- (v) For every i , the R -algebra homomorphism ${}_i\mathcal{R} \rightarrow {}_iS$ factors ${}_i\mathcal{R} \rightarrow {}_iR \rightarrow {}_iS$.

Then $u_{{}_iR}$ is in the (affine) K -tight closure of $\langle N_{{}_iR} \rangle$ in $M_{{}_iR}$ for all i , by the persistence

of *K . Let $R' = \Pi_i({}_iR)$. (Then $R \rightarrow R'$ is étale and faithfully flat, since the ${}_if$ generate the unit ideal of R .) To complete the proof, it will suffice to show u_R is in the (affine) K -tight closure of $\langle N_R \rangle$ in M_R .

We proceed by choosing a finitely generated \mathbb{Z} -subalgebra A of K and descent data for R , M_R , $\langle N_R \rangle \subseteq M_R$, all the algebras ${}_iR$, and the maps $R \rightarrow {}_iR$. One can choose A so large that the ${}_if$ are in R_A (so that ${}_if_A = {}_if$), and the ${}_iG$ and ${}_iH$ have all their coefficients in $(R_A)_{{}_if}$, which we may think of as $(R_{{}_if})_A$. We take ${}_iG_A = {}_iG$, ${}_iH_A = {}_iH$, and we take ${}_iR_A = ((R_A)_{{}_if}[X]/({}_iG))_{{}_iH}$. For A large enough we will have that the conditions (i)–(iv) above hold with every element and ring subscripted by A in the obvious way. We then let $R'_A = \Pi_i({}_iR_A)$.

By construction, $R_A \rightarrow R'_A$ is an étale map of algebras that is faithfully flat after we apply $K \otimes_A _$. It follows that it is also faithfully flat after we apply $\mathfrak{F} \otimes_A _$ (\mathfrak{F} is the fraction field of A). We want to see that it will be faithfully flat after we localize at one element of A° . Since the map is étale the image is open in $\text{Spec } R_A$: let \mathfrak{A} be the defining ideal of the complement of the image. Each minimal prime of \mathfrak{A} must contain an element of A° , since none of them survives when we localize at A° . It follows that there is an element of A° in $\text{Rad } I$ and, hence, in I . After we localize at this element the map $\text{Spec } R'_A \rightarrow \text{Spec } R_A$ is onto, and so $R_A \rightarrow R'_A$ is faithfully flat. Thus, for almost all closed fibers, we have that $R_\kappa \rightarrow R'_\kappa$ is faithfully flat and étale, and $R'_\kappa = \Pi_i({}_iR_\kappa)$. Moreover, for almost all closed fibers we know that $u_{{}_iR_\kappa} \in \langle N_{{}_iR_\kappa} \rangle^*_{M_{{}_iR_\kappa}}$ for all i , which implies by Theorem (1.4.4m) that for almost all closed fibers we have that $u_{R'_\kappa} \in \langle N_{R'_\kappa} \rangle^*_{M_{R'_\kappa}}$. Since for almost all closed fibers we have that R'_κ is faithfully flat over R_κ , by Theorem (1.7.3b) we have that for almost all closed fibers $u_{R_\kappa} \in \langle N_{R_\kappa} \rangle^*_{M_{R_\kappa}}$. It follows that $u_R \in N_R^{*K}_{M_R}$ and so $u \in N^{>*K}_M$, as required. \square

(3.4.2) Corollary. *Let R be a finitely generated algebra over a field K of characteristic zero. Let $N \subseteq M$ be finitely generated R -modules.*

$$\text{Then } N^{*K}_M = N^{>*K}_M = N^{\text{f}*K}_M.$$

Proof. The fact that $N^{*K} \subseteq N^{>*K}$ is obvious, since for any $u \in M$ we can take the quadruple (R, M, N, u) as an affine progenitor of itself. The other inclusion is obvious

from the persistence of *K . The second equality is the result of (3.4.1). \square

(3.4.3) Definition. Let R be a Noetherian K -algebra, where K is a field of characteristic zero, and let $N \subseteq M$ be finitely generated R -modules.

(a) We define the K -tight closure N^{*K}_M of N in M to be the formal K -tight closure of N in M . By Corollary (3.4.2), this agrees with our definition for affine algebras.

(b) Every Noetherian ring R of equal characteristic zero is (uniquely) a \mathbb{Q} -algebra. When $K = \mathbb{Q}$ we shall refer to the direct \mathbb{Q} -tight closure of N in M as the *direct equational tight closure of N in M* , and we denote it $N^{>*eq}_M$. We refer to the \mathbb{Q} -tight closure of N in M as the *equational tight closure of N in M* and denote it N^{*eq}_M .

(3.4.4) Remarks. The equational tight closure gives us a very well-behaved notion defined for all Noetherian rings of equal characteristic zero. We shall see in the next chapter that there is a competing notion, the big equational tight closure. We do not know whether they are really different.

If R is locally excellent, the equational tight closure is the same as the direct equational tight closure. All instances where an element is in a direct equational tight closure are then the result of mapping from an instance of \mathbb{Q} -tight closure over an affine \mathbb{Q} -algebra R . Since R can be written as $\mathbb{Q}[X_1, \dots, X_n]/(F_1, \dots, F_m)$, where the F_i are finitely many polynomials with rational coefficients, all these instances of equational tight closure are somehow forced by the equations F_j , and this is the reason for the name.

(3.5) HEIGHT-PRESERVING DESCENT FROM COMPLETE LOCAL RINGS OVER K TO AFFINE K -ALGEBRAS

Our main objective in this section is to prove the following result on descent from complete local rings containing a field K of characteristic zero to affine K -algebras.

(3.5.1) Theorem. *Let K be a field of characteristic zero and let (S, m, L) be a complete local ring that is a K -algebra. Assume that S is equidimensional and unmixed.*

Suppose that R_0 is a subring of S that is finitely generated as a K -algebra. We also assume given finitely many sequences of elements $\{z_t^{(i)}\}_t$ in R_0 , each of which is part of a system of parameters for S .

Then there is a finitely generated K -algebra R such that the homomorphism $R_0 \hookrightarrow S$ factors $R_0 \hookrightarrow R \rightarrow S$ and such that the following conditions are satisfied:

- (1) R is biequidimensional.
- (2) The image of each sequence $\{z_t^{(i)}\}_t$ in R is a sequence of strong parameters.
- (3) If \mathfrak{m} is the contraction of \mathfrak{m} to R , then $\dim R_{\mathfrak{m}} - \text{depth } R_{\mathfrak{m}} = \dim S - \text{depth } S$. In particular, $R_{\mathfrak{m}}$ is Cohen-Macaulay iff S is Cohen-Macaulay.
- (4) If S is a reduced (respectively, a domain) then so is R .

(N.B. In general, $\dim R_{\mathfrak{m}}$ is substantially bigger than $\dim S$.)

Proof. Extend K to a coefficient field L for S . Fix a system of parameters x_1, \dots, x_n for S and view S as module-finite over $T = L[[x_1, \dots, x_n]]$. The method of proof that we shall use is to transfer the descent problem to a problem over T and then use Theorem (3.3.5) to solve the problem over T . We first solve the problem assuming that $K = L$ and then descend from an L -algebra solution to a K -algebra solution.

Note that making R_0 larger only makes the problem harder. We first extend each of the sequences $\{z_t^{(i)}\}_t$ to a full system of parameters for S . We can include x_1, \dots, x_n among them. Enlarge R_0 to contain all of these elements. Second, each element of each of these sequences has a power that is an S -linear combination of the x_j . Moreover, for each sequence $\{z_t^{(i)}\}_t$ each x_j has a power that is an S -linear combination of the $z_t^{(i)}$. We may enlarge R_0 to contain all the coefficients in these linear combinations. Thus, in R_0 , the ideals (x_1, \dots, x_n) and $(z_1^{(i)}, \dots, z_n^{(i)})$ may be assumed to have the same radical, and this is preserved when the ideals are expanded to any ring to which R_0 maps. Thus, it suffices to satisfy condition (2) for the image of the sequence x_1, \dots, x_n , and we no longer need concern ourselves with the behavior of the $z_t^{(i)}$.

Let θ denote a set of module generators $\theta_1 = 1, \dots, \theta_b$ for S over T . We may assume these are minimal generators, and that each of $\theta_2, \dots, \theta_b$ is in the maximal ideal of S and therefore has a power in the ideal generated by the x 's. Because S is equidimensional and unmixed it is torsion-free as a T -module, and so we can choose an embedding of S in a free T -module T^η . Choose a minimal free resolution for S over T , beginning with a free module whose free basis is mapped to $\theta_1, \dots, \theta_b$. Thus, we have a finite free acyclic

complex \mathcal{G}_\bullet over T :

$$0 \rightarrow T^{b(\rho+1)} \xrightarrow{\alpha_{\rho+1}} \dots \xrightarrow{\alpha_3} T^{b(2)} \xrightarrow{\alpha_2} T^b \xrightarrow{\alpha_1} T^\eta$$

where α_1 factors $T^b \twoheadrightarrow S \hookrightarrow T^\eta$. Here, the matrices α_i have entries in the maximal ideal \mathfrak{M} of T for $i \geq 2$, and $\rho = \text{pd}_T S$ since the resolution of S is minimal. We may write $b(1) = b$ and $b(0) = \eta$.

By (3.3.5) (and the remark (3.3.6)), every finitely generated L -algebra that maps to T maps to such an L -algebra that is a regular domain in which x_1, \dots, x_n is a permutable regular sequence. We shall call such an L -algebra together with an L -algebra map of it to T *permissible*. Given any finite set of elements of T we may choose D so that these elements have liftings to D . Given any finite set of polynomial equations over L holding on finitely many elements of T we may map D further to obtain a permissible choice of D in which the specified relations hold on liftings of the specified elements. In the sequel we shall often refer to “enlarging” D (or D'), by which we shall mean mapping D (or D') to an affine L -algebra D'' such that $D \rightarrow T$ (or $D' \rightarrow T$) factors $D \rightarrow D'' \rightarrow T$ (or $D' \rightarrow D'' \rightarrow T$) and such that $D'' \rightarrow T$ is permissible.

Each entry of each of the matrices α_i for $i \geq 2$ is a linear combination in T of the x 's. Thus, we may choose a sufficiently large permissible L -algebra $D \rightarrow T$ so that:

(a) All the entries of the α_i have liftings to D for $i \geq 1$ (we also use α_i to denote the lifting of the matrix α_i), and the (lifted) entries of α_i for $i \geq 2$ are in $(x_1, \dots, x_n)D$.

By the acyclicity criterion of [BE] we know that, if $r_i = \sum_{j \geq i} (-1)^{j-i} b(j)$ for $1 \leq i \leq \rho+1$, then the largest nonvanishing ideal of minors of α_i is the ideal of r_i size minors, and that this ideal has depth at least i in T , where $1 \leq i \leq \rho+1$. Thus, for every such i we can choose i T -linear combinations of the minors of size r_i in α_i such that they form part of a full system of parameters $\zeta^{(i)}$ for T . Each of these will be a linear combination of the x_i , and each x_i will have a power that is in the ideal generated by the elements $\zeta^{(i)}$ in T . We may assume that all of the elements of T that we have mentioned have liftings to D , so that:

(b) For all i , $1 \leq i \leq \rho+1$ there are i elements of D that are linear combinations in D

of the size r_i minors of α_i and such that these i elements can be extended to a sequence $\zeta^{(i)}$ of length n in D that has the same radical, in D , as $(x_1, \dots, x_n)D$.

We shall denote by $\mathfrak{G}_\bullet(D)$ the finite free complex:

$$0 \rightarrow D^{b(\rho+1)} \xrightarrow{\alpha_{\rho+1}} \dots \xrightarrow{\alpha_3} D^{b(2)} \xrightarrow{\alpha_2} D^b \xrightarrow{\alpha_1} D^\eta.$$

Once D is so large that (a) and (b) hold and since $(x_1, \dots, x_n)D$ has depth n , it follows that $\mathfrak{G}_\bullet(D)$ is acyclic, by the acyclicity criterion of [BE]. If \mathcal{D} is any D -algebra we write $\mathfrak{G}_\bullet(\mathcal{D})$ for $\mathcal{D} \otimes_D \mathfrak{G}_\bullet(D)$. Thus, $\mathcal{G}_\bullet = \mathfrak{G}_\bullet(T)$. We shall write $\alpha_i(\mathcal{D})$ for α_i viewed as map from $\mathcal{D}^{b(i)} \rightarrow \mathcal{D}^{b(i-1)}$.

For any permissible D' with $D \rightarrow D' \rightarrow T$ we may let $R_{D'} = \text{Coker } \alpha_2(D') \cong \text{Im } \alpha_1(D') \subseteq (D')^\eta$. *A priori*, $R_{D'}$ is simply a finitely generated D' -module. If we localize at $\mathfrak{m} = \mathfrak{M} \cap D'$ we see that $\mathfrak{G}_\bullet(D')_{\mathfrak{m}}$, with the 0th term dropped, yields a minimal finite free resolution of $(R_{D'})_{\mathfrak{m}}$. Hence,

$$\begin{aligned} (\#) \quad \dim (R_{D'})_{\mathfrak{m}} - \text{depth } (R_{D'})_{\mathfrak{m}} &= \text{pd}_{R'_{\mathfrak{m}}} (R_{D'})_{\mathfrak{m}} = \text{pd}_T S = \dim T - \text{depth } S \\ &= \dim S - \text{depth } S. \end{aligned}$$

Moreover, for all permissible $D' \rightarrow T$ we have a commutative diagram (but we shall need to give an argument to show that we can fill in the middle vertical arrow):

$$(3.5.2) \quad \begin{array}{ccccc} T^b & \twoheadrightarrow & S & \hookrightarrow & T^\eta \\ \uparrow & & \uparrow & & \uparrow \\ D'^b & \twoheadrightarrow & R_{D'} & \hookrightarrow & D'^\eta \end{array}$$

Evidently, the image of D'^b in D'^η maps into the image S of T^b in T^η , so that for all permissible D' with $D \rightarrow D' \rightarrow T$ we have that $R_{D'}$ maps into S . (Note also that if $D' \rightarrow T$ is injective, then both of the vertical arrows on the ends are injective, and it then follows that the vertical arrow in the middle is injective as well, so that in this case R'_D injects into S . We shall not use this in the proof, but see Remark (3.5.3) following the argument.)

Note that for all large permissible D' we have that $R_{D'}$ is simply $\sum_{j=1}^b D'\theta_j$, where we are writing θ_j for the image in $R_{D'}$ of the j^{th} generator of D'^b : these θ_j map to the original θ_j spanning S over T .

For all j, k we have that $\theta_j \theta_k = \sum_{\nu=1}^b \tau_{\nu jk} \theta_\nu$ for elements $\tau_{\nu jk} \in T$. Now choose a permissible D' so large that:

(c) All of the elements $\tau_{\nu jk}$ have liftings, which we denote by the same letters, to D' , and there is a commutative ring structure on $R_{D'}$ such that for all j, k we have $\theta_j \theta_k = \sum_{\nu=1}^b \tau_{\nu jk} \theta_\nu$ in $R_{D'}$.

The last part of (c) requires some explanation. We first need to know that there is a well-defined D' -bilinear map $R_{D'} \times R_{D'} \rightarrow R_{D'}$ such that (θ_j, θ_k) maps to $\sum_{\nu=1}^b \tau_{\nu jk} \theta_\nu$ for all j, k . There is obviously such a map of $D'^b \times D'^b \rightarrow R'_D$. To get a well-defined map when D'^b is replaced by $R_{D'}$ we need to know that each generator of $\text{Ker}(D'^b \twoheadrightarrow R'_D)$ when paired on either side with a generator of D'^b , is killed. This is true when we pass from D' to T , and so it becomes true when D' is large enough so that it contains liftings of a certain finite set of elements of T and a certain finite set of relations on these hold.

This gives a “multiplication” on $R_{D'}$ for large D' which may fail to be commutative or associative. However, because any two generators commute when we pass to T , this also holds when D' is suitably large, and because associativity holds for any three generators when we pass to T it also holds for suitably large D' . Thus, for all sufficiently large permissible D' with $D \rightarrow D' \rightarrow T$ we have that $R_{D'}$ is a commutative, associative ring with identity, module-finite over the domain D' , that $S \cong T \otimes_{D'} R_{D'}$, and that, in fact, a finite free resolution of S over T may be obtained from a finite free resolution of $R_{D'}$ over D' by tensoring over D' with T .

For any finite set of elements of S and finite set of polynomial relations over L holding among them, for D' sufficiently large these elements and their relations will lift to $R_{D'}$: one can write each element as a T -linear combination of the θ_j , and choose D' to contain liftings of the coefficients from T that are needed. The equations over S then translate into equations over T . In consequence, for all sufficiently large permissible D' we have:

(d) Each of $\theta_2, \dots, \theta_b$ has a power in the ideal generated by the x 's in $R_{D'}$.

(e) $R_0 \rightarrow S$ factors $R_0 \rightarrow R_{D'} \rightarrow S$.

It follows that, with $\mathfrak{m} = \mathfrak{M} \cap D'$, we have that $\text{Rad } \mathfrak{m}R_{D'}$ contains all the θ 's except θ_1 , and it follows that $R_{D'}$ has a unique maximal ideal lying over \mathfrak{m} , which must be

$m \cap R_{D'} = \mathfrak{m}$. Thus, $(R_{D'})_{\mathfrak{m}} = (R_{D'})_{\mathfrak{m}}$ and it follows that we may take $R = R_{D'}$ and all of the conditions (1) – (3) of the theorem will hold. Note that R is embedded in D'^{η} and so has pure dimension as a D -module, which shows that it is biequidimensional. The fact that the x 's form a regular sequence on D then implies that they form strong parameters in R , while (3) follows from (#) above. (Note that, by the parenthetical comment at the end of the paragraph following the commutative diagram (3.5.2), when $D' \subseteq T$, we have that R'_D may be identified with a subring of S .)

It remains to explain why, when S is reduced (respectively, a domain), we can guarantee that $R_{D'}$ has the same property. First suppose that S is reduced. Choose a subset of the θ_j that form a free basis for S over T after we tensor with the fraction field \mathcal{F} of T . By renumbering, we may assume that these are $\theta_1, \dots, \theta_h$. Thus, the quotient of S by the free T -submodule spanned by these elements is a T -torsion module, so that each of the θ_j for $j > h$ has a nonzero T -multiple that is a T -linear combination of $\theta_1, \dots, \theta_h$. For D' large enough this will continue to hold in $R_{D'}$. Moreover, we can choose a nonzero element t of T such that entries of the matrix

$$t(\mathrm{tr}_{\mathcal{F} \otimes_T S / \mathcal{F}}(\theta_j \theta_k)),$$

where $1 \leq j, k \leq h$, are in T , and the fact that S is reduced implies that the determinant of this matrix is a nonzero element of T . For sufficiently large D' we shall have that $t \in D'$ and the calculation of the discriminant for $R_{D'}$ over D' will be the same, which implies that D' is reduced.

Now suppose, moreover, that S is a domain. The argument above shows that $R_{D'}$ is reduced for D' sufficiently large. With \mathcal{F} the fraction field of T as before, we may choose a primitive element λ for $\mathcal{F} \otimes S$, the fraction field of the domain S , over \mathcal{F} . There is no loss of generality in assuming that λ is in S . Consider the minimal monic polynomial (of degree h) satisfied by λ : we may clear denominators and so obtain a polynomial $t_0 z^h + \dots$ satisfied by λ with t_0 a nonzero element of T . We may replace λ by $t_0 \lambda$: its minimal monic polynomial $G(z)$ then has coefficients in T . There is no loss of generality in including the elements λ^i , $0 \leq i \leq h-1$, among the θ_i , so that the corresponding powers of a lifting of λ are among the generators for $R_{D'}$, and we may even assume for large D' that

$R_{D'}/D'[\lambda]$ is a D' -torsion module. Since $R_{D'}$ may be assumed torsion-free over D' , it will be a domain provided that $D'[\lambda]$ is a domain. We may assume that D' contains liftings of the coefficients of the minimal monic polynomial $G(z)$, and, enlarging D' further if necessary we may assume that $G(\lambda) = 0$ in $R_{D'}$, where we are writing G for the lifting of the original G to $R_{D'}$. To complete the argument, it suffices to show that G is irreducible over the fraction field of D' . But, since D' is regular it is normal, and this implies that if G is reducible over the fraction field of D' then it is the product of two monic polynomials of lower degree over D' itself. But then we get a corresponding factorization over T by applying the map $D' \rightarrow T$ to the coefficients, and this is a contradiction.

This completes the argument when $L = K$. Now suppose that we have constructed an affine L -algebra R with $R_0 \hookrightarrow R \rightarrow S$ satisfying (1) – (3). Fix finitely many generators of R_0 over K . By the results of §(2.1) we can find descent data (A, D_A, R_A) for (L, D, R) such that A is a finitely generated \mathbb{Z} -subalgebra of L and the finitely many generators of R_0 with which we are concerned are in R_A . Here, as usual, we have that $D_A \subseteq D, R_A \subseteq R$ and that R_A is A -free. We can arrange that A be regular, that $A \rightarrow D_A$ be smooth, that the θ_i generate R_A as a D_A -module and that $R_A \subseteq D_A^\eta$ with an A -free cokernel. Let B denote the K -subalgebra of L generated by \mathbb{Z} -generators for A . Thus, $B = K[A]$ is an affine K -subalgebra of L . By localizing at one more element of B we may also assume that B is regular. Then $R_B \subseteq R_L = R$ is an affine K -subalgebra of R that will have all of the required properties. Since it is a subalgebra, it will also be reduced (respectively, a domain) if R is. \square

(3.5.3) Remark. At this time we do not whether Theorem (3.3.5) can be strengthened to assert that the maps from the Henselized rings S to the complete ring T can be taken to be injective (our notation is that of (3.3.5)). If this is true, then the rings discussed in (3.3.6) could also be taken to be subrings of T , and this would mean that in the proof of (3.5.1) just above we would only need to consider injective maps $D \rightarrow T$ and $D' \rightarrow T$. We mention this because in this case the proof becomes simpler in several ways. By the parenthetical remark in the paragraph following the commutative diagram (3.5.2), we may assume that $R_{D'}$ embeds in S , and so we may think of the rings $R_{D'}$ as subrings of S . This would make the arguments given to establish (4) unnecessary: evidently, in this case,

if S is reduced or a domain then so is $R_{D'}$.

CHAPTER 4.

FURTHER PROPERTIES OF TIGHT CLOSURE

In the first section of this chapter we consider a number of very important theorems that can be proved using tight closure techniques. A few results of lesser significance are also included because they indicate the success of the current theory in obtaining parallel results to those of the positive characteristic theory.

The second section contains results on the extent to which tight closure is preserved by change of rings in two cases: that is when one has a ring extension that preserves height in a certain sense, and the second is when the ring extension is geometrically regular.

The third section deals with phantom homology, the fourth with ring-theoretic properties defined by the requirement that some family of ideals of the ring be tightly closed, and the fifth with another notion of tight closure for the equal characteristic case.

(4.1) SOME MAJOR APPLICATIONS

We are now ready to establish a number of important properties of *K that parallel the properties of tight closure in characteristic p .

(4.1.1) Theorem. *Let K be a field of characteristic zero, let S be a regular Noetherian K -algebra and let $N \subseteq M$ be finitely generated S -modules. Then $N^{*K}_M = N$.*

Hence, for every regular Noetherian ring S of equal characteristic zero, if $N \subseteq M$ are finitely generated S -modules then $N^{\text{eq}}_M = N$.*

Proof. The second statement is just the case where $K = \mathbb{Q}$. For the first statement, suppose that we have a counterexample, so that $u \in M - N$ while $u \in N^{*K}$. The first condition can be preserved while passing from S to a complete local domain of S , and the second condition is automatically preserved by the persistence of K -tight closure. Thus, we may assume without loss of generality that (S, m, L) is a complete regular local domain. Second, we may replace N by $N + m^t M$ for $t \gg 0$ while preserving that $u \in M - N$ if we

take t sufficiently large. In fact, taking N maximal with respect to not containing u , we may assume that M/N has finite length with the image of u generating the socle. We may replace M, N, u by $M/N, 0, u + N$. Thus, we can assume that $N = 0$, and that M is an essential extension of Ku of finite length. Then the injective hull of M is the same as the injective hull of the residue field, and so if x_1, \dots, x_n is a regular system of parameters for S , we see that we may assume that M embeds in $S/(x_1^t, \dots, x_n^t)S$ for t sufficiently large and that we may take u to be the image of $(x_1 \cdots x_n)^{t-1}$, since this element generates the socle. Thus, it will suffice to show that $u = (x_1 \cdots x_n)^{t-1}$ is not in the K -tight closure of $(x_1^t, \dots, x_n^t)S$.

Choose L to be a coefficient field of S containing K . Then it will suffice to show that u is not in the L -tight closure of $(x_1^t, \dots, x_n^t)S$, and since S is complete this is the same as the direct L -tight closure. Now, by (3.3.5) and (3.3.6), if there is an affine progenitor that forces u into the direct L -tight closure, there is one whose base ring R is étale over $L[x, y]$, and hence, a regular ring in which the x 's form a regular sequence. Thus, in R we still have $u \notin (x_1^t, \dots, x_n^t)R$, and it will suffice to see that $u \notin (x_1^t, \dots, x_n^t)R^{*L}$ (in the affine sense). But when we choose descent data, since R is smooth over L , almost all closed fibers are regular, and so it follows that $u_\kappa \notin (x_1^t, \dots, x_n^t)R_\kappa$ for almost κ . \square

(4.1.2) Definition. Let $N \subseteq M$ be finitely generated modules over a Noetherian ring S . We shall say that $u \in M$ is in the *regular closure* N^{reg}_M of N in M if for every regular ring T to which S maps, $u_T \in \langle N_T \rangle$ (in M_T). This is slightly different from the notion considered in [HH4] and [HH8], where it was required that S° map into T° . This regular closure is *a priori* smaller than the one considered in [HH4] and [HH8] (although we do not know an example where it is actually strictly smaller). This makes the following Corollary slightly stronger than if it were stated for the notion of [HH4] and [HH8].

(4.1.3) Corollary. *Let K be a field of characteristic zero and let S be a Noetherian K -algebra. Let $N \subseteq M$ be finitely generated S -modules. Then $N^{*K}_M \subseteq N^{\text{reg}}_M$.*

Proof. Let $u \in N^{*K}_M$ and suppose that S maps to a regular Noetherian ring T . By the persistence of K -tight closure, $u_T \in \langle N_T \rangle^*_{M_T} = \langle N_T \rangle$ (since T is regular). Thus, $u \in N^{\text{reg}}_M$. \square

(4.1.4) Corollary. *Let K be a field of characteristic zero and let S be a Noetherian K -algebra. Let I be any ideal of S . Then $I^* \subseteq I^-$, the integral closure of I . Hence, all radical ideals of S and, in particular, all prime ideals of S are K -tightly closed.*

Proof. An element is in I^- if and only if it is in IV for all maps of R to discrete valuation rings V , which shows that $I^{\text{reg}} \subseteq I^-$, and we may apply (4.1.3) \square

(4.1.5) Theorem (generalized Briançon-Skoda theorem). *Let S be a Noetherian ring of equal characteristic zero and let I be an ideal of S generated by at most n elements. Then for every $k \in \mathbb{N}$, $(I^{n+k})^- \subseteq (I^{k+1})^{>* \text{eq}} (\subseteq (I^{k+1})^{* \text{eq}})$.*

*Hence, if S is also a K -algebra for some field K then $(I^{n+k+1})^- \subseteq (I^{k+1})^{>*K} (\subseteq (I^{k+1})^{*K})$.*

Proof. Fix generators of I , say $I = (u_1, \dots, u_n)$. It is clear that if an element z is in $((u_1, \dots, u_n)^{n+k})^-$ then this remains true when S is replaced by a suitable affine \mathbb{Q} -subalgebra containing z and u_1, \dots, u_n . We therefore reduce at once to the case where S is an affine \mathbb{Q} -algebra.

For sufficiently large descent data we shall continue to have the equation of integral dependence, and we can take the image of this equation in every closed fiber. The result is now immediate from the definition of K -tight closure in the affine case and the fact that the generalized Briançon-Skoda theorem holds for all the closed fibers.

The final statement is then obvious. \square

(4.1.6) Corollary. *Let K be a field of characteristic zero, let S be a Noetherian K algebra and let I be a principal ideal of S . Then $I^{*K} = I^-$. In particular, $I^{* \text{eq}} = I^-$.*

Proof. $I^{*K} \subseteq I^-$ by Corollary (4.1.4) and the other inclusion follows from the generalized Briançon-Skoda theorem (4.1.5) in the case where $n = 1$ and $k = 0$. \square

The following very important result gives a taste of the subject matter of Section (4.5): Theorem (4.5.3) is a substantial generalization.

(4.1.7) Theorem (tight closure captures colons). *Let K be a field of characteristic zero and let S be a Noetherian K -algebra. Let x_1, \dots, x_n be strong parameters in S . Then $(x_1, \dots, x_{n-1})^{*K} :_S x_n S = (x_1, \dots, x_{n-1})^{*K}$.*

Hence, $(x_1, \dots, x_{n-1}) :_S x_n S \subseteq (x_1, \dots, x_{n-1})^{*eq}$.

Under the same hypotheses, if a_1, \dots, a_n and b_1, \dots, b_n are non-negative integers and $a \ominus b$ denotes $\max\{a - b, 0\}$, then

$$(x_1^{a_1}, \dots, x_n^{a_n})^{*K} :_S x_1^{b_1} \cdots x_n^{b_n} = (x_1^{a_1 \ominus b_1}, \dots, x_n^{a_n \ominus b_n})^{*K}.$$

Proof. Suppose that $x_n u \in (x_1, \dots, x_{n-1})^{*K}$. We must show that $u \in (x_1, \dots, x_{n-1})^{*K}$. It suffices to show that after passing to a complete local domain B of S , we have that $u_B \in ((x_1, \dots, x_{n-1})B)^{>*K}$. Since the strong parameter hypothesis is preserved when we pass to B , and since $x_n u$ is still in the K -tight closure after we pass to B , we may assume that S is a complete local domain. If any of x_1, \dots, x_{n-1} is a unit, or if x_n is a unit, the result is obvious. Thus, we may assume that x_1, \dots, x_n is part of a system of parameters for the complete local domain S .

Since $x_n u \in (x_1, \dots, x_{n-1})^{>*K}$ we know that there is an affine K -subalgebra R of S containing x_1, \dots, x_{n-1}, x_n , and u such that $x_n u \in ((x_1, \dots, x_{n-1})R)^{*K}$ (in the affine sense). By Theorem (3.5.1), we can give a K -algebra factorization $R \rightarrow R_1 \rightarrow S$ of $R \rightarrow S$ such that R_1 is a domain finitely generated over K and such that the images of x_1, \dots, x_n are a sequence of parameters in R_1 . We change notation and write R for R_1 (we no longer know that $R_1 \rightarrow S$ is injective, but we shall not need this). When we take descent data and pass to closed fibers (indexed by κ) we have for almost all closed fibers that the images of x_1, \dots, x_n are a sequence of parameters such that, if we use a bar to indicate images in R_κ , then $\bar{x}_n \bar{u} \in ((\bar{x}_1, \dots, \bar{x}_{n-1})R_\kappa)^*$. It follows from the characteristic p version of this result, (1.7.4), that $\bar{u} \in ((\bar{x}_1, \dots, \bar{x}_{n-1})R_\kappa)^*$ for almost all closed fibers.

The statement in the second paragraph is immediate from the result of the first paragraph. We now consider the statement of the final paragraph. It is easy to see that the left hand side contains the right hand side. (The left hand side is K -tightly closed by Theorem (3.2.3), part (i), and so it suffices to show that it contains $(x_1^{a_1 \ominus b_1}, \dots, x_n^{a_n \ominus b_n})$. Thus, we want to show that

$$(x_1^{a_1}, \dots, x_n^{a_n}) :_S x_1^{b_1} \cdots x_n^{b_n} \supseteq (x_1^{a_1 \ominus b_1}, \dots, x_n^{a_n \ominus b_n}),$$

and so it is enough to show that

$$x_1^{b_1} \cdots x_n^{b_n} x_i^{a_i \ominus b_i} \in (x_1^{a_1}, \dots, x_n^{a_n})$$

for every i , which follows from the observation that $b_i + (a_i \ominus b_i) \geq a_i$ for every i .)

Now suppose that $x_1^{b_1} \cdots x_n^{b_n} u \in (x_1^{a_1}, \dots, x_n^{a_n})^{*K}$. We must show that u is in the ideal $(x_1^{a_1 \ominus b_1}, \dots, x_n^{a_n \ominus b_n})^{*K}$. It suffices to show that after passing to a complete local domain B of S , we have that

$$u_B \in ((x_1^{a_1 \ominus b_1}, \dots, x_n^{a_n \ominus b_n})B)^{*K}.$$

Since the strong parameter hypothesis is preserved when we pass to B , and since the element $x_1^{b_1} \cdots x_n^{b_n} u$ is still in the K -tight closure after we pass to B , we may assume that S is a complete local domain. If any of the a_i is 0, or if any of the x_i is a unit, then both ideals are the unit ideal and the result is obvious. Thus, we may assume that x_1, \dots, x_n is part of a system of parameters for the complete local domain S .

The rest of the argument is very similar to the one given for the proof of the statement in the first paragraph of the theorem. Since $x_1^{b_1} \cdots x_n^{b_n} u \in (x_1^{a_1}, \dots, x_n^{a_n})^{*K}$ we know that there is an affine K -subalgebra R of S containing x_1, \dots, x_n , and u such that $x_1^{b_1} \cdots x_n^{b_n} u \in ((x_1^{a_1}, \dots, x_n^{a_n})R)^{*K}$ (in the affine sense). By Theorem (3.5.1), we can give a K -algebra factorization $R \rightarrow R_1 \rightarrow S$ of $R \rightarrow S$ such that R_1 is a domain finitely generated over K and such that the images of x_1, \dots, x_n are a sequence of parameters in R_1 . We change notation and write R for R_1 . When we take descent data and pass to closed fibers (indexed by κ) we have for almost all closed fibers that the images of x_1, \dots, x_n are a sequence of parameters such that, if we use a bar to indicate images in R_κ , then

$$\bar{x}_1^{b_1} \cdots \bar{x}_n^{b_n} \bar{u} \in ((\bar{x}_1^{a_1}, \dots, \bar{x}_n^{a_n})R_\kappa)^*.$$

It follows from the characteristic p version of this result, (1.7.4), that

$$\bar{u} \in ((\bar{x}_1^{a_1 \ominus b_1}, \dots, \bar{x}_n^{a_n \ominus b_n})R_\kappa)^*$$

for almost all fibers. \square

(4.1.8) Corollary. *If S is a Noetherian ring of equal characteristic zero in which every parameter ideal is tightly closed in the sense of *eq , then S is normal. Moreover, if S is universally catenary, then S is Cohen-Macaulay.*

Proof. We first note that if $S \cong S_1 \times S_2$ is a product, then the property cited is inherited by each factor, for if x_1, \dots, x_n are parameters in, say, S_1 , then the elements

$(x_1, 1), \dots, (x_n, 1)$ are parameters in S . If $I = (x_1, \dots, x_n)S_1$, it follows from (3.2.3o) that I is tightly closed in S_1 , since the tight closure of $I \times S_2 = ((x_1, 1), \dots, (x_n, 1))S$ in S is $I^{\text{eq}}_{S_1} \times S_2$. Thus, we may assume without loss of generality that S is not a product.

We first establish normality. Since (0) is tightly closed, S is reduced. If S is zero-dimensional it must be a field. Otherwise, we have that every principal ideal of height one is integrally closed, since such an ideal is generated by a parameter, and integral closure coincides with equational tight closure for principal ideals, by (4.1.6). But Lemma (5.9) of [HH4] implies that if $\text{Spec } S$ is connected of positive dimension and every principal ideal of height one is integrally closed, then S is normal.

Finally, we prove that if S is universally catenary, then S is Cohen-Macaulay. Let m be any maximal ideal of S (we now may assume that $\text{Spec } S$ is connected and that S is normal, so that S is a domain), and suppose that m has height n . Then by standard prime avoidance we may construct a sequence of elements x_1, \dots, x_n in m such that for all i , $1 \leq i \leq n$, x_1, \dots, x_i generates an ideal of height i . By (2.3.11d), x_1, \dots, x_i consists of strong parameters, $1 \leq i \leq n$, and it follows from (4.1.7) above that x_1, \dots, x_n is a regular sequence. Thus, R_m is Cohen-Macaulay for every maximal ideal m . \square

(4.1.9) Proposition. *Let H be an additive subsemigroup of \mathbb{Z}^s containing 0, and let $R = \bigoplus_{h \in H} R_h$ be a Noetherian K -algebra of equal characteristic 0 graded by H , where K maps into R_0 . Let $N \subseteq M$ be finitely generated H -graded modules. Then N^{*K}_M is an H -graded submodule of M .*

Proof. We may assume $H = \mathbb{Z}^s$, thinking of the additional graded components as zero. Let U denote the multiplicative group of units of R_0 . Then U^s acts on R by ring automorphisms as follows: given $v = (u_1, \dots, u_s) \in U^s$ and $r \in R_h$, u sends r to $u^h r$, where if $h = (h_1, \dots, h_s)$ then $u^h = u_1^{h_1} \cdots u_s^{h_s}$. There are corresponding actions of U^s on M and N by automorphisms (the action on N is induced by the action on M), and $u(rm) = (u(r))(u(m))$. All of these actions are evidently K -linear.

Suppose we denote the action of u by θ_u . Quite generally, if $\theta: R \rightarrow R'$ is a K -isomorphism, $\theta: M \rightarrow M'$ is an isomorphism such that for all $r \in R$ and $m \in M$, $\theta(rm) = \theta(r)\theta(m)$, and $N \subseteq M$, then θ obviously induces an isomorphism of N^{*K}_M

calculated over R with $\theta(N)^{*K}_{M'}$ calculated over R' . We may apply this with $\theta = \theta_u$, $R' = R$, and $M' = M$ to conclude that N^{*K}_M is stable under the action of U^h . Since U contains $\mathbb{Q} - \{0\}$, a standard argument using the invertibility of Van der Monde matrices shows that N^{*K}_M is H -graded: cf. (7.30) in [HH9] and (4.1) in [HH10]. \square

(4.1.10) Corollary. *Let K be a field of characteristic zero and let S be a universally catenary Noetherian K -algebra.*

Suppose that S is local and x_1, \dots, x_n are in the maximal ideal or that S is \mathbb{N} -graded and that x_1, \dots, x_n are forms of positive degree. Also suppose that x_1, \dots, x_n are parameters modulo every minimal prime ideal of R .

*If $(x_1, \dots, x_n)^{*K} = (x_1, \dots, x_n)$ then $(x_1, \dots, x_i)^{*K} = (x_1, \dots, x_i)$, $0 \leq i \leq n$, and x_1, \dots, x_n is a regular sequence.*

*If $(x_1, \dots, x_{n-1})^{*K} = (x_1, \dots, x_{n-1})$ then x_1, \dots, x_n is a regular sequence.*

Proof. First note that x_1, \dots, x_{n-1} are still parameters modulo every minimal prime \mathfrak{p} : if not, in $D = R/\mathfrak{p}$ we can choose a (homogeneous) minimal prime P of height at most $n-2$ containing the images of these elements, and then there is a (homogeneous) prime Q minimal over $P + x_n D$. Then $P \subseteq Q$ are consecutive, and since D is a catenary domain, $\text{ht } Q = \text{ht } P + 1 \leq n-1$, contradicting the assumption that x_1, \dots, x_n are parameters in D . It follows by reverse induction that x_1, \dots, x_t are parameters modulo every minimal prime for $0 \leq t \leq n$.

For $0 \leq t \leq n$ let $I_t = (x_1, \dots, x_t)$. We next show that if $I_{t+1}^{*K} = I_{t+1}$ for some value of t , $0 \leq t \leq n-1$, then $I_t^{*K} = I_t$. To see this, note that $I_t^{*K} \subseteq I_{t+1}^{*K} = I_{t+1} = I_t + x_{t+1}R$, and the element $r \in R$ needed to represent an element of I_t^{*K} in the form $i + x_{t+1}r$ with $i \in I_t$ must be in $I_t^{*K} :_R x_{t+1} = I_t^{*K}$ by (4.1.7), so that $I_t^{*K} = I_t + x_{t+1}I_t^{*K}$. But then $I_t^{*K} = I_t$ by Nakayama's lemma (from (4.1.9) we know that I_t^{*K} is graded if I_t is).

Thus, from either hypothesis, we know by reverse induction that $I_t^{*K} = I_t$ for $0 \leq t \leq n-1$, and by (4.1.7), this implies that x_{t+1} is not a zerodivisor on I_t for $0 \leq t \leq n-1$. \square

Note that some hypothesis such as “local” or “graded” is needed here, or the argument in the first paragraph will fail: in $K[x, y, z]$, where every ideal is tightly closed, the elements $xz, yz, 1-y$ are strong parameters (and form a regular sequence in a different order,

namely, $1 - y, xz, yz$), but they do not form a regular sequence (and the first two are not parameters) in the original order.

(4.1.11) Definition and discussion: purity. We say that a map of R -modules $N \rightarrow M$ is *pure* if $W \otimes_R N \rightarrow W \otimes_R M$ is injective for every R -module W . Since W may be equal to R , this implies that $N \rightarrow M$ is injective. If M/N is finitely presented, then $N \hookrightarrow M$ is pure if and only if it splits. It follows that if R is Noetherian, $N \hookrightarrow M$ is pure if and only if $N \rightarrow M'$ splits for all $M' \subseteq M$ containing the image of N such that M'/N is finitely generated. Cf. [HR1], §6, [HR2] §5(a) and [HH11], Lemma (2.1) for further discussion. We shall very often be interested in the condition that a ring homomorphism $R \rightarrow S$ be pure (over R), in which case we shall say that R is a *pure subring* of S . We are particularly interested in this condition when R is a Noetherian ring. The condition that a ring homomorphism $R \rightarrow S$ be pure implies that every ideal of R is contracted from S .

We shall say that $N \rightarrow M$ is *cyclically pure* if $N/IN \rightarrow M/IM$ is injective for every ideal I of R . Again, this implies that the original map is injective, since I may be (0) . This is the same as the condition for purity in the preceding paragraph with W restricted to be a cyclic R -module.

Note that both purity and cyclic purity are preserved by localization at an arbitrary multiplicative system W of R (note that every ideal J of $W^{-1}R$ is of the form $W^{-1}I$, where I is an ideal of R that is contracted with respect to the multiplicative system W).

On the face of it, the condition that a ring extension $R \hookrightarrow S$ be cyclically pure is weaker than condition that it be pure, but they are often equivalent: cf. [Ho4] and the discussion in (8.6) of [HH4], where these conditions are shown to be equivalent for a Noetherian ring R if R is normal or if R is excellent and reduced.

We now prove a considerable strengthening of the main result of [HR1] on the Cohen-Macaulay property for rings of invariants of linearly reductive groups G acting on regular rings: the key point is that in the situations described in [HR1], S is regular and the fixed ring S^G is a pure subring of S . The situation just below is therefore much more general.

(4.1.12) Theorem. *Every pure subring of an equicharacteristic regular ring is a Cohen-Macaulay ring (and normal: in fact, the completion of each of its local rings is normal).*

Proof. The characteristic p case is proved in §7 of [HR1], and by a tight closure argument similar to the one given below in [HH4], Theorem (4.10). We therefore assume equal characteristic 0.

If R is universally catenary we may argue as follows: every ideal is tightly closed, because if I is an ideal of R and $r \in I^{*\text{eq}}_R$ then $r \in (IS)^{*\text{eq}}_S = IS$, since S is regular, and $IS \cap R = I$. Notice that we have only used cyclic purity. We may now apply (4.1.8).

In the general case, we make the same reduction as in the beginning of [HR1], §7. We first note that the issue is local on R , and we may assume that (R, m) is local. We then replace R, S by their m -adic and (mS) -adic completions, respectively. S remains regular, and purity is preserved by Corollary (6.13) of [HR1]. Thus, the completion of every local ring of R is Cohen-Macaulay and normal, by the argument in the first paragraph, and it follows that R is Cohen-Macaulay and normal. \square

There is no real gain in generality in assuming that R is cyclically pure in S instead of pure: when R is normal, the two conditions are equivalent by the main results of [Ho4] and the discussion in (8.6) of [HH4].

The next result is aimed at showing that the tight closure of a submodule of a projective module over a normal ring is independent of how it is embedded in a projective module. It is parallel to Theorem (8.18)²⁸ of [HH4] (the characteristic p case) and to Proposition (5.11) of [Ho8] (the case of solid closure).

(4.1.13) Theorem. *Let R be a reduced Noetherian K -algebra, where K is a field of characteristic 0, and let M, N, F, G be finitely generated R -modules.*

- (a) *If M/N is torsion-free, then N is tightly closed in M . More generally, N^{*K}_M may be identified with a submodule of $N' = \text{Ker}(M \rightarrow (R^\circ)^{-1}(M/N))$. If N is torsion-free, then $N' \subseteq (R^\circ)^{-1}N$.*
- (b) *If $N \subseteq G \subseteq F$, where G is projective and F is any module, then $N^{*K}_F \cap G = N^{*K}_G$. Hence, if G is tightly closed in F , then $N^{*K}_F = N^{*K}_G$.*
- (c) *If R is normal, and $G \subseteq F$ with G projective and F torsion-free, then G is K -tightly closed in F . If an arbitrary torsion-free module N has embeddings in two*

²⁸The proof of (8.18b) of [HH4] has a gap that is corrected in (1.4.17) of this manuscript.

*possibly distinct finitely generated projective modules F and G , then $N^{*K}_F \cong N^{*K}_G$ canonically.*

Proof. (a) The proof is verbatim the same as the proof of Proposition (8.18a) in [HH4].

(b) We may use exactly the same argument given in the corrected proof of (8.18b) of [HH4] in (1.4.17) of this manuscript to reduce to the case where $G \hookrightarrow F$ is the map $R^h \hookrightarrow R^h$ given by a matrix α whose determinant D is not a zerodivisor. Suppose that $u \in (G \cap N^{*K}_F) - N^{*K}_G$. Since u is not in N^{*K}_G there is at least one complete local domain B of R such that $u_B \notin \langle N_B \rangle^{>*K}_{G_B}$ over B . Suppose that $B = (R_P)^\wedge / \mathfrak{p}$, where P is a prime of R and \mathfrak{p} is a minimal prime of $(R_P)^\wedge$. Since $(R_P)^\wedge$ is R -flat, the image of D is not a zerodivisor in this ring, and so is not in \mathfrak{p} . Thus, D has nonzero image in B . We now obtain a new counterexample, for direct K -tight closure, by replacing R , $N \hookrightarrow G \hookrightarrow F$, and $u \in G$ by B , $\langle N_B \rangle \hookrightarrow G_B \hookrightarrow F_B$, and $u_B \in G_B$. (The map $G_B \rightarrow F_B$ is still injective because the image of D in B is not zero.) We change notation and henceforth assume that R is a complete local domain and that we are working with direct K -tight closure, while $G \hookrightarrow F$ is the map $R^h \hookrightarrow R^h$ with matrix α . Let v be the image of $u \in G$ in F . Since $v \in N^{>*K}_F$ we can choose an affine progenitor (R_0, F_0, N_0, v_0) for (R, F, N, v) such that $R_0 \subseteq R$ is an affine K -algebra that is a domain, $N_0 \subseteq F_0 = R_0^h \subseteq R^h = F$, $v = v_0$, and such that $v_0 \in N_0^{*K}_{F_0}$. By enlarging R_0 by adjoining finitely many more elements of R , if necessary, we may assume that R_0 contains all the entries of α , and that v is in the column space of α (since this is true over R). Thus, we may factor $N_0 \hookrightarrow F_0 = R_0^h$ through $G_0 = R_0^h$, where the map $G_0 \hookrightarrow F_0$ has matrix α , in such a way that $v = v_0$ is the image of an element u_0 , in G_0 , and we may assume that $u_0 = u$.

We must still have that $u_0 \notin N_0^{*K}_{G_0}$, for if it were in this tight closure it would follow that $u \in N^{*K}_G$ over R , a contradiction. We now change notation by dropping the subscript “0”: we have obtained a counterexample over an affine K -algebra R that is a domain. As in previous counterexamples, we have that $G \hookrightarrow F$ is given by a matrix α whose determinant is nonzero, i.e., a parameter. We now take descent data, replacing K by a finitely generated \mathbb{Z} -subalgebra A , R by R_A , and so forth. For almost all closed fibers, R_κ is reduced, D_κ is still a parameter and, hence, a nonzerodivisor, and $u_\kappa \in N_\kappa^{*K}_{F_\kappa}$. It then follows from the characteristic p version of the result that for almost all closed fibers,

$u_\kappa \in N_\kappa^* G_\kappa$, and so $u \in N^{*K}_G$, a contradiction. This completes the proof of (4.1.13b).

(c) The proof is the same as that of Proposition (8.18c) in [HH4]. \square

(4.2) CHANGE OF RINGS

Let $N \subseteq M$ be finitely generated modules over a Noetherian K -algebra R and let S be a Noetherian R -algebra.

In this section we prove results of two kinds: one kind asserts that if the map $R \rightarrow S$ preserves heights “sufficiently well” (we shall give a precise condition in (4.2.7) below), then an element $u \in M$ is in N^{*K}_M over R iff $u_S \in \langle N_S \rangle^{*K}_{M_S}$. Of course, “only if” is automatic: this is the persistence of K -tight closure. The interesting part of the implication, the part that needs some condition on preserving heights, is the “if” part.

The second kind asserts that if $R \rightarrow S$ is smooth (or *geometrically regular* — see (4.2.1)) then the K -tight closure of the image of N_S in M_S is the image of $S \otimes_R N^{*K}_M$. However, while this may be true in very great generality, at the moment we are limited to making the assertion only under quite a few restrictions:²⁹ see Theorems (4.2.14) and (4.2.15), and Proposition (4.2.17) and (4.2.18). One of the main difficulties is that we do not know whether tight closure commutes with localization, even for affine algebras over an algebraically closed field of characteristic p .

A number of the results of this section depend on general Néron desingularization in equal characteristic zero. This result is first stated and then discussed below, in (4.2.2) and (4.2.3).

(4.2.1) Discussion and definition. Flat homomorphisms of Noetherian rings with geometrically regular fibers are most often referred to as “regular” in the literature, but in [HH9] are called “smooth.” In this paper the term “smooth” is reserved for finitely presented algebras. To minimize any possible ambiguity, we shall refer to a flat homomorphism with geometrically regular fibers as *geometrically regular*. Thus, while we shall

²⁹In an earlier version of this manuscript, the authors thought that they could also obtain the result for flat homomorphisms of complete local rings with a regular closed fiber, using the beautiful results on the structure of flat and smooth affine algebras obtained in [RG], and a result of Bass (Corollary (4.5), p. 31 of [Bass]) which asserts that infinitely generated projective modules over a Noetherian ring with connected spectrum are free. However, we have not as yet been able to carry through the details of this program.

speak of “smooth” homomorphisms and “geometrically regular” homomorphisms, we shall avoid the term “regular homomorphism.” Note that $R \rightarrow S$ is smooth if and only if it is geometrically regular and S is finitely presented over R (of course, if R is Noetherian, S is finitely presented if and only if it is finitely generated).

If S is an R -algebra and \mathcal{P} is a property of R -algebras we shall say that S is a *filtered inductive limit* of R -algebras with property \mathcal{P} if for every homomorphism of a finitely generated R -algebra S_0 to S there exists an R -algebra S_1 such that S_1 has property \mathcal{P} and the R -algebra homomorphism $S_0 \rightarrow S$ factors $S_0 \rightarrow S_1 \rightarrow S$, where the maps are R -algebra homomorphisms. This property is unaffected if we only require the conclusion to hold when $S_0 \subseteq S$, since in the more general case we may apply it to the image of S_0 in S instead. Note however, that we do *not* require that $S_1 \rightarrow S$ be injective. We shall be applying this terminology primarily in the case where the property is smoothness.

Note that a filtered inductive limit of smooth algebras is a direct limit of smooth algebras. To see this, let Λ_0 be the set of all finitely generated R -subalgebras of S and for every $\lambda \in \Lambda_0$ choose a smooth R -algebra R_λ together with a factorization $\lambda \rightarrow R_\lambda \rightarrow S$. Now suppose that $\Lambda_0, \dots, \Lambda_i$ have been constructed and that for every j with $0 \leq j \leq i$ and every $\lambda \in \Lambda_j$ we have a smooth R -algebra R_λ together with a map $R_\lambda \rightarrow S$. Also suppose that if $0 \leq j < i$ then we have a relation $<$ on $\Lambda_j \times \Lambda_{j+1}$ and for every pair (λ, μ) with $\lambda < \mu$ we have an R -algebra homomorphism $R_\lambda \rightarrow R_\mu$ compatible with the maps to S which kills the kernel of the map $R_\lambda \rightarrow S$ and such that for all $\lambda, \lambda' \in \Lambda_j$ with $j < i$ there exists μ in Λ_{j+1} such that $\lambda < \mu$ and $\lambda' < \mu$, and if $u \in R_\lambda$ and $v \in R_{\lambda'}$ have the same image in S then they have the same image in R_μ . Then one can construct Λ_{i+1} as follows: let Λ_{i+1} be the set of all subsets of Λ_i containing either 2 elements or 1 element, let $<$ on $\Lambda_i \times \Lambda_{i+1}$ be such that $\lambda < \mu$ precisely if $\lambda \in \mu$, and for all μ in Λ_{i+1} define $R_\mu \rightarrow S$ taking R' to be the subring of S generated by the images of the (one or two) rings R_λ for $\lambda \in \mu$ and choosing $R_\mu \rightarrow S$ with R_μ smooth over R such that $R' \hookrightarrow S$ factors $R' \rightarrow R_\mu \rightarrow S$.

Thus, recursively, one obtains a countably infinite sequence Λ_i with the properties specified above. The disjoint union of the Λ_i , with the partial order generated by the relations on the various $\Lambda_j \times \Lambda_{j+1}$, becomes a directed set Λ , and in this way one obtains

a directed family of smooth R -algebras R_λ , each with an R -algebra map to S . It is easy to verify that the direct limit of the R_λ is S . (Call the direct limit S' . Clearly, we have a map $S' \rightarrow S$. Since every element of S is contained in some λ in Λ_0 , it is clear that $S' \rightarrow S$ is onto. Suppose that $u \in R_\lambda$ and $v \in R_{\lambda'}$ have the same image in S , where $\lambda \in \Lambda_j$ and $\lambda' \in \Lambda_i$. We must show that they have the same image in S' . We may suppose without loss of generality that $j \leq i$. Then λ is less than or equal to some element in Λ_i , so that we may assume that $j = i$. But then when we may take μ to be the element in Λ_{i+1} that dominates both λ and λ' , and we have that u and v have the same image in R_μ , which shows that they are equal in S' . Thus, $S' \rightarrow S$ is injective.)

(4.2.2) Theorem (general Néron desingularization). *Let $R \rightarrow S$ be a geometrically regular homomorphism of Noetherian rings. Let C be a finitely generated R -algebra with an R -algebra homomorphism to S . Then there exist a (finitely presented) smooth R -algebra D and R -algebra homomorphisms $C \rightarrow D \rightarrow S$ whose composition is the given R -algebra homomorphism $C \rightarrow S$. In other words, S is a filtered inductive limit of smooth R -algebras.*

(4.2.3) Discussion: general Néron desingularization. It is also true, conversely, that a filtered inductive limit of smooth R -algebras, if it is Noetherian, is geometrically regular: this is much easier.

A version of (4.2.2) is given in [Po1], [Po2], but some experts had difficulty following the arguments, and doubt was expressed. Remedies are offered in [Og] and [And], while a proof along different lines is given in [Sp]. All questions have been resolved by the complete treatment given in the expository paper [Swan].

We are now ready to begin our discussion of ring extensions that “preserve” height: we begin with module-finite extensions.

(4.2.4) Lemma. *Let K be a field of characteristic 0 and let $R \rightarrow S$ be a module-finite extension of Noetherian K -algebras. Let $N \subseteq M$ be finitely generated R -modules and let $u \in M$. Then $u \in N^{*K}_M$ over R iff $u_S \in \langle N_S \rangle^{*K}_{M_S}$.*

Proof. “Only if” follows from the persistence of K -tight closure, so we need only prove that if $u \notin N^{*K}_M$ then $u_S \notin \langle N_S \rangle^{*K}_{M_S}$. We suppose otherwise and obtain a contradiction. Since $u \notin N^{*K}_M$ this must be preserved for some complete local domain C/\mathfrak{p} of R , where

C is the completion of a local ring of R and \mathfrak{p} is a minimal prime of C . We replace R, S, M, N and u by C, S_C, M_C, N_C and u_C respectively: C is R -flat, and so $N_C \subseteq M_C$ and S_C is a module-finite extension of C . Choose a minimal prime ideal \mathfrak{q} of S_C lying over \mathfrak{p} . We may then replace C by C/\mathfrak{p} and S_C by S_C/\mathfrak{q} . We may thus assume without loss of generality that both R and S are complete local domains, and we change back to our original notation. Note that in making the replacement of S we are using the persistence of K -tight closure.

Then u is in the direct K -tight of N in M over S , and we can choose an affine progenitor for $(S, M_S, \langle N_S \rangle, u)$ that demonstrates this. The base ring S_0 of this affine progenitor may be chosen to be an affine K -subalgebra of S , and we may choose this progenitor such that the M_S, N_S are the cokernels of matrices with entries in $R \cap S_0$, such that $N_S \rightarrow M_S$ is induced by a map given by a matrix with entries in $R \cap S_0$, and such that the element mapping to u is the image of a vector with entries in $R \cap S_0$. Each of the generators of S_0 as a K -algebra will satisfy an equation of integral dependence on R , and we may choose an affine K -subalgebra of R_0 of R so large that it contains all the coefficients of these equations, as well as coefficients needed to give presentations of M, N , the map $N \rightarrow M$ and the element that maps to u .

We drop the 0 subscripts: what we have shown is that there is a counterexample for K -tight closure in the affine sense where R, S are domains that are affine K -algebras and S is a module-finite extension of R . We now perform descent as described in Chapter 1, replacing K, R, S, M, N, u by $A, R_A, S_A, M_A, N_A, u_A$, where A is a finitely generated \mathbb{Z} -subalgebra of K . After localizing A suitably at one element of A° we have that for all closed fibers κ , $R_\kappa \in S_\kappa$ is module-finite and $N_\kappa \subseteq M_\kappa$, while $\kappa \otimes_A \langle S_A \otimes_{R_A} N_A \rangle$ may be identified with $\langle S_\kappa \otimes_{R_\kappa} N_\kappa \rangle$. Then $1 \otimes u_\kappa \in \langle S_\kappa \otimes_{R_\kappa} N_\kappa \rangle^*_{S_\kappa \otimes_{R_\kappa} M_\kappa}$ over S_κ for almost all closed fibers, and it follows from the characteristic p version of this theorem, which is a special case of (1.7.5), that $u_\kappa \in N_\kappa^*_{M_\kappa}$ for almost all fibers, so that $u \in N^{*K}_M$ after all. \square

Note that this gives another proof, in the case where a is a nonzerodivisor, that the integral closure of aR is contained in the K -tight closure: if b is integral over the ideal aR an equation of integral dependence for b on aR shows that b/a is integral over R within

the total quotient ring, and adjoining this element therefore does not affect whether b is in the K -tight closure of aR . But $b \in aR[b/a]$.

The next result, Lemma (4.2.5), needs a special case of general Néron desingularization, namely that if $L \subseteq L'$ are fields of characteristic 0 and $C = L[[x_1, \dots, x_n]]$, $D = L'[[x_1, \dots, x_n]]$ are formal power series rings with $C \subseteq D$, then D is a filtered inductive limit of smooth C -algebras.

Lemma (4.2.5) is nearly parallel to a result proved for tight closure in characteristic p , namely Corollary (8.8) of [Ho8], which is reproduced in §1 as Theorem (1.7.5), except that in Corollary (8.8) of [Ho8] one only needs the hypothesis (#) for *maximal* ideals P , not for all prime ideals P . The full parallel is obtained later, in Theorem (4.2.7). Note that there is some discussion of the condition (#) in (1.7.6).

(4.2.5) Lemma. *Let K be a field of characteristic 0 and let $R \rightarrow S$ be a homomorphism of Noetherian K -algebras such that*

(#) for every prime ideal P of R and every minimal prime \mathfrak{p} of $(R_P)^\wedge$, there is a prime ideal Q of S lying over P and a prime ideal \mathfrak{q} of $(S_Q)^\wedge$ lying over \mathfrak{p} such that

$$\text{ht } P(S_Q)^\wedge/\mathfrak{q} \geq \dim(R_P)^\wedge/\mathfrak{p}.$$

*Let $N \subseteq M$ be finitely generated R -modules and let $u \in M$. Then $u_S \in \langle N_S \rangle^{*K}_{M_S}$ if and only if $u \in N^{*K}_M$.*

Proof. As usual, it suffices to prove that if $u \notin N^{*K}_M$ then this is preserved when we pass to S , and we assume the contrary. Then the image of u will fail to be in the direct K -tight closure after passing to $B = (R_P)^\wedge/\mathfrak{p}$ for a suitable prime ideal P of R and minimal prime \mathfrak{p} of $(R_P)^\wedge$. Choose Q and \mathfrak{q} as in the hypothesis. Then $B \rightarrow C = (S_Q)^\wedge$ is an injection of complete local domains. Then we may pass to a new counterexample, replacing R , M , N , u , and S by B , M_B , $\langle N_B \rangle$, u_B , and C . Changing back to our original notation, we may assume that $(R, P) \hookrightarrow (S, Q)$ is a local homomorphism of complete local domains such that $\text{ht } PS \geq \dim R$. (Note that we are using the persistence of K -tight closure when we pass from the original S to C .)

Suppose that $\dim R = N$, and let x_1, \dots, x_n be a system of parameters for R . Then $P = \text{Rad}(x_1, \dots, x_n)R$ in R , and so $\text{ht } PS = \text{ht}(x_1, \dots, x_n)S$. This was assumed to be at least

n , but it now follows that it is exactly n . Then x_1, \dots, x_n is part of a system of parameters for S , and may be extended to a full system of parameters, say $x_1, \dots, x_n, y_1, \dots, y_r$. A minimal prime \mathcal{Q} of $(y_1, \dots, y_r)S$ will have height r , and so $T = S/\mathcal{Q}$ will be a complete local domain of dimension n in which the images of the elements x_1, \dots, x_n are a system of parameters. This implies that the map $R \rightarrow T$ is injective: if there is a kernel, then since R is a domain and its image is a domain, the dimension of the image will be smaller than n , and so the maximal ideal of the image will be the radical of an ideal generated by $n - 1$ or fewer elements. But then the images of the x 's are all in the radical of an $n - 1$ generator ideal, and this implies that the same holds for the maximal ideal of T , a contradiction, since $\dim T = n$.

We therefore may replace S by T and still have a counterexample. Thus, we may assume that $R \rightarrow S$ is a local injection of complete local domains of the same dimension and that x_1, \dots, x_n is a system of parameters for both R and S .

We next replace S by its normalization (the persistence of tight closure allows us to do this), and we may then replace R by its normalization, which may be identified with a subring of S . By Lemma (4.2.4) this does not affect the issues. Thus, we may assume that $R \hookrightarrow S$ is a local injection of complete normal local domains and that x_1, \dots, x_n is a system of parameters for both rings.

Let L be a coefficient field for R containing the image of K and let L' be a coefficient field for S containing the image of L . Without loss of generality we may assume that $K \subseteq R \subseteq S$ and $K \subseteq L \subseteq L'$. Then R is module-finite over its regular subring $D = L[[x_1, \dots, x_n]]$, while S is module-finite over its regular subring $D' = L'[[x_1, \dots, x_n]]$. Let $W = D' \otimes_D R$. Since D' and R are both D -subalgebras of S , we can factor $R \subseteq S$ as $R \rightarrow D' \rightarrow S$.

Since L has characteristic 0, $D \rightarrow D'$ is geometrically regular (see, for example, Theorem (7.38c) of [HH9]), and so general Néron desingularization in characteristic 0 implies that D' is a filtered inductive limit of smooth D -algebras. Applying $-\otimes_R$, we see that W is a filtered inductive limit of smooth R -algebras. Since R is normal, W is normal. Since R is module-finite over D , W is module-finite over the complete regular local ring D' . Since $W/(x_1, \dots, x_n)W$ is isomorphic with

$$(D'/(x_1, \dots, x_n)D') \otimes_{D/(x_1, \dots, x_n)D} (R/(x_1, \dots, x_n)R) \cong L' \otimes_L (R/(x_1, \dots, x_n)R)$$

it follows easily that W is local, and, since W is normal, it is a domain in which x_1, \dots, x_n is a system of parameters. It follows that $D' \rightarrow S$ is injective.

Since S is module-finite over D' , it is certainly module-finite over W . By Lemma (4.2.4), we may replace S by D' . We change notation again and assume that $S = D'$. Thus, we have reduced to the case where S is a filtered inductive limit of smooth extensions of R . Both rings are still local. We are about to make a reduction of a different sort in which we will lose the property of being local, but we shall still keep track of the maximal ideals of R and S in a limited way.

The fact that $u \in \langle N_S \rangle^{*K}_{M_S}$ implies that it is in the direct K -tight closure, and so there is a finitely generated K -subalgebra S_0 of S which is the base ring for an affine progenitor $(S_0, M_{S_0}, N_{S_0}, u_{S_0})$ for $(S, M_S, \langle N_S \rangle, u_S)$ such that $u_{S_0} \in M_{S_0}^{*K}_{N_{S_0}}$. Moreover, by enlarging S_0 if necessary we may assume that it contains an affine K -subalgebra R_0 with $R_0 \subseteq R$ such that we have presentations of N_{S_0} , M_{S_0} , and $N_{S_0} \rightarrow M_{S_0}$ as well as a vector representing u_{S_0} , all with entries in R_0 . The ring $R[S_0]$ is a finitely generated R -subalgebra of S , and so we may choose a smooth R -algebra T and a factorization $R[S_0] \rightarrow T \rightarrow S$, which yields $S_0 \rightarrow T \rightarrow S$. Fix a finite algebra presentation for T over R . We may enlarge R_0 to an affine K -subalgebra R_1 of R that contains all the coefficients used in this presentation, and we may let $T(R_1)$ denote the R_1 -algebra with the same presentation, so that $T \cong R \otimes_{R_1} T(R_1)$. If R_2 is any affine K -subalgebra of R containing R_1 then we may define $T(R_2)$ as $R_2 \otimes_{R_1} T(R_1)$, and T is the direct limit of the algebras $T(R_2)$. The algebra $T(R_2)$ is smooth over R_2 for R_2 sufficiently large (REF needed) and for R_2 sufficiently large we shall also have that $S_0 \rightarrow T$ factors $S_0 \rightarrow T(R_2) \rightarrow T$. We write S_2 for $T(R_2)$.

Let m_2 denote the contraction of the maximal ideal m of R to $R_2 \subseteq R$. Then $m_2 S_2$ does not contain 1, since its image in S under $S_2 \rightarrow T \rightarrow S$ is contained in mS and $R \rightarrow S$ is a local homomorphism. Thus, m_2 is in the image of $\text{Spec } S_2 \rightarrow \text{Spec } R_2$, and since $R_2 \rightarrow S_2$ is smooth, that image is Zariski open. It follows that we may choose $f \in R_2 - m_2$ such that $\text{Spec } (S_2)_f \rightarrow \text{Spec } (R_2)_f$ is surjective, and, of course, since (R, m) is local, $R_3 = (R_2)_f$ is an affine K -algebra of R . Let $S_3 = (S_2)_f$. Then $R_3 \rightarrow S_3$ is a smooth and faithfully flat map of affine K -algebras. Moreover, there are modules $N_3 \subseteq M_3$ over R_3 and an element $u_3 \in M_3$ such that $u \notin N_3^{*K}_{M_3}$ (over R_3 , and in the sense of affine K -tight closure), while

$u_{S_3} \in \langle (N_3)_{S_3} \rangle^{*K}_{(M_3)_{S_3}}$ (over S_3 , in the sense of affine K -tight closure).

We shall now drop the subscript “3”: we shall not need to refer to the original local case again. We reach a contradiction quickly by descent of the type described in Chapter 1: we replace K by a sufficiently large finitely generated \mathbb{Z} -subalgebra A , and we choose descent data $R_A \rightarrow S_A$ (a smooth homomorphism), $N_A \subseteq M_A$, $u_A \in M_A$, etc. The complement of the image of $\text{Spec } S_A$ in $\text{Spec } R_A$ is closed: suppose that it is defined by I . Since the map becomes faithfully flat when we tensor with K over A , it also becomes faithfully flat when we tensor with the fraction field \mathfrak{F} of A , and it follows that I expands to the unit ideal when we localize at A° , i.e., that I meets A° . Thus, we may replace A by its localization at one element of A° and so assume that $R_A \rightarrow S_A$ is both smooth and faithfully flat. It follows that $R_\kappa \rightarrow S_\kappa$ is faithfully flat for all closed fibers, and for almost all closed fibers we may identify $\langle N_\kappa \rangle$ with $\langle N_A \rangle_\kappa$. We can now conclude from Theorem (1.7.5) (in the faithfully flat case) that $u_\kappa \in N_\kappa^{*K}_{M_\kappa}$ for almost all closed fibers, since this becomes true after the faithfully flat base change $R_\kappa \rightarrow S_\kappa$. \square

(4.2.6) Corollary. *Let R be a Noetherian ring containing a field K of characteristic 0, let $N \subseteq M$ be finitely generated R -modules, and let $u \in M$. Then $u \in N^{*K}_M$ if and only if for every complete local domain B of R at a maximal ideal, $u_B \in \langle N_B \rangle^{*K}_{M_B}$.*

Proof. Consider any prime ideal P of R and choose a maximal ideal m of R containing P . Since $R_m \rightarrow (R_m)^\wedge$ is faithfully flat, there is a prime ideal Q of $(R_m)^\wedge$ lying over P , and then $C = (R_P)^\wedge \rightarrow (((R_m)^\wedge)_Q)^\wedge = D$ is faithfully flat. It suffices to show that $u_C \in \langle N_C \rangle^{*K}_{M_C}$, and by (4.2.5) it suffices to prove this when C is replaced by D . But R_m^\wedge maps to D , so that this follows from the persistence of tight closure. \square

We immediately obtain an improved form of (4.2.5).

(4.2.7) Theorem. *Let K be a field of characteristic 0 and let $R \rightarrow S$ be a homomorphism of Noetherian K -algebras such that*

(#) for every maximal ideal m of R and every minimal prime \mathfrak{p} of $(R_m)^\wedge$, there is a prime ideal Q of S lying over m and a prime ideal \mathfrak{q} of $(S_Q)^\wedge$ lying over \mathfrak{p} such that
 $\text{ht } P(S_Q)^\wedge/\mathfrak{q} \geq \dim (R_m)^\wedge/\mathfrak{p}.$

Let $N \subseteq M$ be finitely generated R -modules and let $u \in M$. Then $u_S \in \langle N_S \rangle^{*K}_{M_S}$ if and only if $u \in N^{*K}_M$.

In fact, the conclusion that $u_S \in \langle N_S \rangle^{*K}_{M_S}$ if and only if $u \in N^{*K}_M$ is valid for a fixed pair of finitely generated modules $N \subseteq M$ if condition $(\#)$ holds for every maximal ideal m of R that is in the support of M/N .

Proof. By (4.2.6), in the second sentence of the proof of (4.2.5) we may assume that $P = m$ is maximal, and then, since $(\#)$ holds for this m , the rest of the argument is exactly the same as the proof of (4.2.5). For the final statement note that not only can we choose $P = m$ to be maximal, but it obviously must be in the support of M/N or else $N_m = M_m$, which will imply that $\langle N \rangle_B = M_B$ and so we cannot preserve that the image of u is not in the tight closure of $\langle N \rangle_B$ in M_B over B in this case. \square

(4.2.8) Discussion. Throughout the rest of Section (4.2) we shall let K be a field of characteristic 0, and we shall focus on the problem of whether K -tight closure commutes with geometrically regular base change. We believe that this should be true without further hypothesis for locally excellent K -algebras, but we have not been able to prove it in complete generality, because we do not know that tight closure commutes with localization even in characteristic p . The results that follow do establish that K -tight closure will commute with geometrically regular base change in characteristic 0 in many important cases, and also establish that if tight closure can be shown to commute with localization in characteristic p (even for affine domains over a finite field), then K -tight closure commutes with geometrically regular base change for locally excellent rings of characteristic 0: see Theorem (4.2.14e), from which this follows at once.

For simplicity we shall freely assume in the sequel that given rings are locally excellent Noetherian K -algebras. We first need to give several relevant definitions.

(4.2.9) Definition: commuting with tight closure. We shall say that a flat K -algebra homomorphism $R \rightarrow S$ commutes with K -tight closure if for every inclusion of finitely generated R -modules $N \subseteq M$ we have that $(N_S)^{*K}_{M_S} = (N^{*K}_M)_S$, where, as usual, the subscript S indicates the tensor product with S over R . We always have a natural inclusion $(N^{*K}_M)_S \subseteq (N_S)^{*K}_{M_S}$. Note that the condition for the inclusion $N \subseteq M$ is equivalent to the

condition for the inclusion $0 \subseteq M/N$. The condition obviously implies that if N is K -tightly closed in M , then N_S is K -tightly closed in M_S . Conversely, if whenever N is K -tightly closed in M then N_S is K -tightly closed in M_S then $R \rightarrow S$ commutes with K -tight closure, since N_S and $(N_M^{*K})_S$ will have the same K -tight closure in M_S .

(4.2.10) Definition: Filtered inductive limits of maps of algebras. We shall say that a K -algebra homomorphism $R \rightarrow S$ is a *filtered inductive limit* of the K -algebra homomorphisms in the family $\{R_\lambda \rightarrow S_\lambda\}_{\lambda \in \Lambda}$ (or having some property \mathcal{P} , in which case we may think of the family as consisting of the maps of K -algebras with property \mathcal{P}) if every commutative diagram

$$\begin{array}{ccc} R & \longrightarrow & S \\ \uparrow & & \uparrow \\ R_0 & \longrightarrow & S_0 \end{array}$$

such that R_0, S_0 are finitely generated K -algebras can be enlarged (factored) to a commutative diagram

$$\begin{array}{ccc} R & \longrightarrow & S \\ \uparrow & & \uparrow \\ R_\lambda & \longrightarrow & S_\lambda \\ \uparrow & & \uparrow \\ R_0 & \longrightarrow & S_0 \end{array}$$

where $R_\lambda \rightarrow S_\lambda$ is in the specified family.

Note that by general Néron desingularization, every geometrically regular map of K -algebras is a filtered inductive limit of smooth maps of finitely generated K -algebras. The point is that S is a filtered inductive limit of smooth R -algebras, and any smooth R -algebra has the form $R \otimes_{R_0} S_0$ where R_0 is a finitely generated subalgebra of R and S_0 is smooth over R_0 .

(4.2.11) Definition: very tightly closed ideals and submodules. Let $N \subseteq M$ be finitely generated modules over a Noetherian ring R of characteristic p . We shall say that

N is *very tightly closed* in M if, for every prime ideal P of R , N_P is tightly closed in N_P over R_P . In particular, this applies to the case where $M = R$ and $N = I$ is a ideal. Note that once this holds for primes, it holds for localization at every multiplicative system, and that whether the condition holds really only depends on M/N : see (1.4.16). Note also that if M/N has finite length and N is tightly closed in M , then it is very tightly closed, by (14.16c).

Now suppose that R is a finitely generated K -algebra, where K is a field of characteristic 0, and that $N \subseteq M$ are finitely generated modules over R . We say that N is *very K -tightly closed* in M if there exists descent data (A, R_A, M_A, N_A) where A is a finitely generated \mathbb{Z} -subalgebra of K such that for almost all maximal ideals μ of A , if $\kappa = A/\mu$ then N_κ is very tightly closed in M_κ . By localizing A at one nonzero element we may actually assume that this condition holds for all maximal ideals of A , and, as usual, we can guarantee that R_A and S_A are A -flat (even A -free).

We claim that, once the above conditions are satisfied, for every finitely generated A -algebra B (and, hence, for any sufficiently large choice of A), the condition will continue to hold. (If μ_1 is a maximal ideal of B and μ its contraction to A , then since both $\kappa = A/\mu$ and $\kappa_1 = B/\mu_1$ are finite, $\kappa \rightarrow \kappa_1$ is a finite separable extension. Then $(R_B)_{\kappa_1} \cong \kappa_1 \otimes_\kappa R_\kappa$, and the result follows from (1.7.8a).)

More generally, we shall say the N is *very K -tightly closed* in M over a locally excellent ring R if every map $R_0 \rightarrow R$, where R_0 is finitely generated over K , factors $R_0 \rightarrow R_1 \rightarrow R$, where R_1 is a finitely generated K -algebra and is the base ring of an affine progenitor (R_1, M_1, N_1) for (R, M, N) such that N_1 is very K -tightly closed in M_1 . Finally, we shall say that N is *formally very K -tightly closed* in M if for every complete local ring B arising as the completion of R_P for some prime ideal P of R , the image of N_B in M_B is very K -tightly closed in M_B .

(4.2.12) Definition: descendably projective algebras. If S is a finitely generated R -algebra and R is finitely generated over the field K , we say the S is descendably projective over R relative to K if there exists descent data $(A, R_A, S_A, R_A \rightarrow S_A)$ such that S_A is projective over R_A . Then, for every A -algebra B , S_B is projective over R_B . In particular, S must be projective over R if it is descendably projective. It then follows that S_κ is

projective over R_κ for all fibers, including, of course, all closed fibers. (We do not know whether, if S is simply assumed to be projective over R , it is necessarily descendably projective.) Note that once we have a choice of descent data such that S_A is projective over R_A , S_B is projective over R_B for any larger choice. Thus, the condition might be stated in terms of a requirement for all sufficiently large choices of A , instead of the weaker requirement that at least one choice of descent data exist.

We shall need the following:

(4.2.13) Lemma.. *Let K be a field of characteristic zero, let R, S be Noetherian K -algebras with S locally excellent, and suppose that $R \rightarrow S$ is flat and is a filtered inductive limit of flat maps $R_\lambda \rightarrow S_\lambda$. Let $R_0 \rightarrow S_0$ be a K -algebra map such that for all λ the diagram*

$$\begin{array}{ccc} R & \longrightarrow & S \\ \uparrow & & \uparrow \\ R_\lambda & \longrightarrow & S_\lambda \\ \uparrow & & \uparrow \\ R_0 & \longrightarrow & S_0 \end{array}$$

commutes, and let $N_0 \subseteq M_0$ be finitely generated R_0 -modules. For each λ let $N_\lambda = R_\lambda \otimes_{R_0} N_0$ and $M_\lambda = R_\lambda \otimes_{R_0} M_0$. Let $N = R \otimes_{R_0} N_0$ and $M = R \otimes_{R_0} M_0$. If $(N_\lambda)_{S_\lambda}$ is K -tightly closed in $(M_\lambda)_{S_\lambda}$ over S_λ , then N_S is K -tightly closed in M_S over S .

*More generally, if for every λ , the K -tight closure over S_λ of the image of $(N_\lambda)_{S_\lambda}$ in $(M_\lambda)_{S_\lambda}$ is $(N_\lambda^{*K}_{M_\lambda})_{S_\lambda}$, then the K -tight closure of the image of N_S in M_S over S is $(N^{*K}_M)_S$.*

Proof. The final statement implies the earlier one. Let $u \in M_S$ be in the tight closure of N_S . We must show that u is in $(N^{*K}_M)_S$. Since S is locally excellent, there exists an affine progenitor in which the element w mapping to u is in the tight closure. By mapping the affine progenitor further we may assume that there is a map of affine K -algebras $R' \rightarrow S'$ such that the diagram

$$\begin{array}{ccc}
R & \longrightarrow & S \\
\uparrow & & \uparrow \\
R_\lambda & \longrightarrow & S_\lambda \\
\uparrow & & \uparrow \\
R' & \longrightarrow & S'
\end{array}$$

commutes, and the map $N \subseteq M$ is defined over R' , i.e., arises as the tensor product of S over R' with a map of finitely generated R' -modules $N' \subseteq M'$. We factor the above diagram so that it has an extra row $R_\lambda \rightarrow S_\lambda$ as in the definition (4.2.10). But then the image of w in $S_\lambda \otimes_{R'} M'$ is in the K -tight closure of $(N_\lambda)_{S_\lambda}$, and so in $(N_\lambda^{*K})_{S_\lambda}$, and it follows that u is in $(N^{*K})_S$. \square

(4.2.14) Theorem (main theorem on geometrically regular base change). *Let K be a field of characteristic zero, and let R, S be Noetherian K -algebras such that S is locally excellent.*

- (a) *If $R \rightarrow S$ is a filtered inductive limit of K -algebra homomorphisms that commute with K -tight closure, then $R \rightarrow S$ commutes with K -tight closure.*
- (b) *If $R \rightarrow S$ is a homomorphism of finitely generated K -algebras that is smooth and such that S is descendably projective over R relative to K , then $R \rightarrow S$ commutes with K -tight closure.*
- (c) *If $R \rightarrow S$ is a filtered inductive limit of K -algebra homomorphisms satisfying the condition in (b) then $R \rightarrow S$ commutes with K -tight closure.*
- (d) *If S is a polynomial ring in finitely many variables over R , say $S = R[x_1, \dots, x_n]$, then $R \rightarrow S$ commutes with K -tight closure. Moreover, if I is any ideal of R that is K -tightly closed then the ideal of S generated by IS and any set W of monomials in the x 's is tightly closed in S .*
- (e) *If $N \subseteq M$ are finitely generated R -modules such that N is very K -tightly closed in M and S is geometrically regular over R , then N_S is very K -tightly closed in M_S .*
- (f) *If S is geometrically regular over R and if $N \subseteq M$ are finitely generated R -modules such that N is formally very K -tightly closed³⁰ in M , then N_S is K -tightly closed in M_S .*

³⁰See (4.2.10). A weaker condition suffices, namely it is sufficient that for each prime ideal P of R

M_S .

- (g) *Let R be either a finitely generated K -algebra or a complete local ring,³¹ and assume that R is reduced and equidimensional. Let I be an ideal of R that is generated by parameters.³² Let S be a geometrically regular R -algebra. Then $(IS)^{*K}$ (in S) is $I^{*K}S$, where I^{*K} is the K -tight closure of I in R . In particular, this holds when S is a localization of R .*

Proof. (a) follows immediately from (4.2.13). (b) follows at once from the fact that we can use descent data such that S_A is projective over R_A , so that projectivity is preserved when we pass to fibers, for then tight closure commutes with base change in characteristic p for every fiber by (1.7.8b). Then (c) is obvious from (a) and (b).

The fact that tight closure commutes with base change for finite polynomial extensions is immediate from (c). For the second statement we note that by (4.1.9) the tight closure is monomially graded and so we may assume, if the result is false, that there is an element of the K -tight closure of $IS + (W)S$ of the form rv , where v is a monomial in the x 's not divisible by any element of W and $r \in R$ is not in I . Since S is locally excellent this holds for a finitely generated K -subalgebra of S containing generators of I , and after enlargement we may assume that it is of the form $R_0[x_1, \dots, x_n]$ where R_0 is a finitely generated K -subalgebra of R . (We replace R, S, I by $R_0, S_0 = R_0[x_1, \dots, x_n], I \cap R_0$ while retaining the same W . Thus, we may assume that R is affine over K . We set up descent data, and the result is then immediate from the corresponding result in characteristic p . (We may pass to the case where R is reduced, and choose a test element $c \in R^\circ$ for both R and S . Then $c(r^q v^q) \in I^{[q]} + (W)^{[q]}$ together with $v \notin W$ implies that $cr^q \in I^{[q]}$ for all q , contradicting $r \notin I = I^*$.)

We next prove (e). Since S is a filtered inductive limit of smooth R -algebras, it suffices to prove the result when S is smooth over R . It therefore suffices to show in the affine case

lying under a maximal ideal of S in the support of $(M/N)_S$, the image of N_B is very K -tightly closed in M_B with $B = (R_P)^\wedge$.

³¹As should be clear from the proof, the result holds somewhat more generally: what is needed is that R be a locally excellent Noetherian K -algebra that is a filtered inductive limit of reduced, equidimensional affine K -algebras, each of which contains a sequence of parameters that maps to the generators of the given ideal.

³²See (2.3.10).

that if N is very K -tightly closed in M over an affine K -algebra R , and S is smooth over R , then the same is true for N_S in M_S . Choose descent data over R such that N_κ is very tightly closed in M_κ for all fibers, and, after enlarging A if necessary, compatible descent data for S . The result is then immediate from (1.7.8a).

(f) We prove the weaker version described in the footnote. Suppose that there is an element of the K -tight closure of $\langle N \rangle_S$ in M_S not in N_S . We can choose a maximal ideal \mathcal{M} of S such that this remains true when we pass to the completion of the local ring at that maximal ideal, and it will necessarily be in the support of $(M/N)_S$. Suppose that this maximal ideal lies over a prime P in R . Then we get a contradiction by replacing R, S, N, M by $B = (R_P)^\wedge, (S_\mathcal{M})^\wedge, N_B, M_B$, respectively, and applying part (e).

(g) Since S is a filtered inductive limit of smooth R -algebras we may reduce to the case where S is smooth over R by (4.2.13). The case of affine algebras follows from the corresponding fact (1.7.7) in characteristic p , since we have that tight closure commutes with smooth base change in this situation in every fiber. Finally, in the case where R is complete, the result now follows from (4.2.13) together with the fact that R is a filtered inductive limit of reduced equidimensional K -algebras R_λ each of which contains elements that map to the generating parameters and which are themselves parameters in R_λ , by (3.5.). \square

(4.2.15) Theorem. *Let K be a field of characteristic 0 and let R be a Noetherian K -algebra. Let $N \subseteq M$ be finitely generated modules such that the support of M/N consists of a finite set of maximal ideals $\{m_1, \dots, m_r\}$ of R , and suppose that for each of these maximal ideals the field R/m_i is algebraic over K (this is automatic if R is finitely generated over an algebraic extension of K). Then if N is tightly closed in M , it is very K -tightly closed in M .*

*Hence, in this situation, the K -tight closure of N_S in M_S over S is $(N_M^{*K})_S$.*

Proof. First consider the case where R is affine. The fact that the support of M/N is a finite set of maximal ideals (in this case, the hypothesis amounts to the assumption that M/N has finite length) can be preserved for almost all fibers (and, hence, after localizing A at one element, for all fibers) for suitable descent data. Thus, we only need that N is

very tightly closed in M when M/N has finite length in the characteristic p case, which is (1.4.16c).

In the general case the key point is that one can choose “arbitrarily large” affine progenitors for (R, M, N) such the support of M_0/N_0 is contained only in maximal ideals, because the contraction of a maximal ideal m of R such that R/m is algebraic over K to any affine K -subalgebra of R is still maximal. After first choosing any affine progenitor one can map to one where the base ring $R_0 \subseteq R$, and one can further map so that the annihilator I_0 of M_0/N_0 is precisely the contraction of the annihilator I of M/N to R_0 while keeping R_0 as the base ring: one simply adds on relations that make generators of I_0 kill each element of M_0 modulo N_0 . Now suppose that I the primary decomposition of I is $Q_1 \cap \cdots \cap Q_r$ where every Q_i is primary to a maximal ideal m_i of R such that R/m_i is algebraic over K . Then $I_0 = \bigcap_i (Q_i \cap R_0)$, and each $Q_i \cap R_0$ is primary to $m_i \cap R_0$. Since $K \subseteq R_0/(m_i \cap R_0) \subseteq R/m_i$, where the last is an algebraic field extension of K , it follows that R_0/m_i is also an algebraic field extension of K , so that $m_i \cap R_0$ is a maximal ideal of R_0 . Thus, the support of M_0/N_0 is a finite set of maximal ideals. \square

(4.2.16) Discussion. Let R be a finitely generated K -algebra, where K is a field of characteristic 0. Let $N \subseteq M$ be finitely generated K -modules. We do not know whether N_M^{*K} is the intersection of all the K -tightly closed submodules N' of M such that $N \subseteq N'$ and M/N' has finite length. If this were true, it would follow that K -tight closure commutes with base changes $R \rightarrow S$ whenever S is geometrically regular over R and *intersection-flat* over R or *\cap -flat* over R) (This means that $R \rightarrow S$ is flat, and for every family of submodules $\{M_\lambda\}_{\lambda \in \Lambda}$ of every finitely generated R -module M , we have $S \otimes_R (\bigcap_\lambda M_\lambda) \cong \bigcap_\lambda (S \otimes_R M_\lambda)$ by the obvious map.)

We next observe:

(4.2.17) Proposition. *Let K be a field of characteristic 0 and let $R \rightarrow S$ a K -algebra homomorphism. Suppose that S is locally excellent.*

- (a) *If R is a finitely generated K -algebra and $S = T \otimes_K R$, where T is regular, then $R \rightarrow S$ commutes with K -tight closure. In particular, this holds when T is any field extension of K .*

- (b) *If S is module-finite and smooth over R , then $R \rightarrow S$ commutes with K -tight closure. In particular this holds when L is any field contained in R , L' is a finite algebraic extension of L , and $S = L' \otimes_L R$.*
- (c) *If $S = L' \otimes_L R$ where L' is a possibly infinite algebraic extension of a field $L \subseteq R$, then $R \rightarrow S$ commutes with K -tight closure (but notice that we are assuming that $L' \otimes_L R$ is Noetherian and locally excellent: this is not automatic in this case).*

Proof. To prove part (a) first note that since T is regular and K has characteristic 0, $K \rightarrow R$ is geometrically regular, and so R is a filtered inductive limit of K -smooth algebras. Hence, by (4.2.14a), we may assume that T is smooth over K . Now choose descent data for R, T , say R_A, T_A , where $A \subseteq K$ is a finitely generated \mathbb{Z} -subalgebra of K and T_A is smooth over A . By generic freeness, we may localize at one element of A° and so guarantee that T_A is A -free. Now, we may use $S_A = T_A \otimes_A R_A$ to descend S , and it follows that S is descendably projective over R relative to K , so that (4.2.14b) applies.

For (b), we may write $R \rightarrow S$ as a direct limit of maps $R_\lambda \rightarrow S_\lambda$ where every R_λ is finitely generated over K and contained in R , and every S_λ is module-finite and smooth over R_λ . Thus, we may assume, by (4.2.14a), that R is affine. Choose a map of R^s onto S for some s , and a splitting of the surjection $R^s \rightarrow S$. In choosing descent data A, R_A, S_A we may also descend the map to obtain $R_A^s \rightarrow S_A$, which will be surjective after localizing at one element of A° , and a map $S_A \rightarrow R_A^s$, which will be a splitting of $R_A^s \rightarrow S_A$ after localizing at one more element of A° . Thus, S is descendably projective over R relative to K , and the result follows from (4.2.14b).

Part (c) is immediate from part (b) by a direct limit argument (we apply (4.2.14a) again). \square

Finally, we want to record the following fact, which in some ways is more elementary than our other results on base change. However, it is quite important in the the proofs of the results concerning the effect of geometrically regular base change on various kinds of F -regularity in Section (4.3).

(4.2.18) Proposition. *Let R be a Noetherian K -algebra, let m be a maximal ideal of R , let $L = R/m$ (thought of as an R -module), and let $N \subseteq M$ be finitely generated R -modules.*

let $R \rightarrow S$ be a homomorphism of Noetherian K -algebras such that $m' = mS$ is a maximal ideal of S and $R_m \rightarrow S_{m'}$ is faithfully flat. Suppose either that

- (1) M/N is an essential extension of L , or else that
- (2) M/N has a finite filtration by copies of L , and $L \rightarrow S/mS$ is an isomorphism.

Then N is K -tightly closed in M over R if and only if N_S is K -tightly closed in M_S over S .

Proof. Let $L' = S/mS$. We replace $N \subseteq M$ by $0 \subseteq M/N$ and so that $M = M/N$ is an essential extension of L , and the only prime in its support is m . Thus M is killed by a power of m , and M may be thought of as a module over R_m , while M_S may be thought of as a module over $S_{m'}$. Since $R_m \rightarrow S_{m'}$ is flat, the annihilator of m in M_S is spanned by the annihilator of m in M , and so if u is an element of M that generates the socle in M , then the image v of u in M_S spans the socle (since $mS = m'$). Since u is in every submodule of M that is strictly larger than 0, we have that u is in the K -tight closure of 0 if and only if 0 is not K -tightly closed in M . Similarly, v is in the K -tight closure of 0 in M_S iff 0 is not K -tightly closed in M_S . The result now follows at once from Lemma (4.2.7), since the fact that $S_{m'}$ is faithfully flat over R_m implies that condition (#) is satisfied for every prime in the support of $M/N = M$.

We now consider part (2), which is quite similar. As in the proof of part (1), the socle in M_S is the expansion of the socle of M , but now, since we are assuming that $L \cong S/mS$, we see that the socle in M is isomorphic with the socle in M_S . (In fact, our hypothesis implies that $R/m^t \cong S/m^tS$ for all t , and so $M \cong M_S$.) Hence, 0 is tightly closed in M over R if and only if every element $u \neq 0$ of the socle is not in the tight closure of 0 in M over R , and 0 is tightly closed in M_S if and only if for every $u \neq 0$ in the socle of M over R , the image v of u in M_S is not in the tight closure of 0 in M_S . The result now follows again from Lemma (4.2.7). \square

(4.2.19) Remark. Note that either hypothesis in (4.2.18) enables us to test tight closure after tensoring with S by checking elements that come from the module M under the map $M \rightarrow M_S$. When the socle in M/N is two-dimensional (or higher) and $R/m \subseteq S/mS$ is a proper field extension, this technique is not available, and we do not know how to

overcome this difficulty.

(4.3) *F*-REGULARITY AND *F*-RATIONALITY

In this section we fix a field K of characteristic 0 and then consider several important classes of rings defined in terms of K -tight closure: the definitions are given in (4.3.1) and (4.3.2) just below.

(4.3.1) Definition. Let R be a Noetherian K -algebra. R is called *weakly F -regular* if every ideal is K -tightly closed. R is called *F -regular* if every localization of R is weakly F -regular. R is called *F -rational* if every parameter ideal³³ is K -tightly closed. If the choice of field is not clear we add “over K ”, e.g., we may speak of a ring that is “weakly F -regular over K .”

(4.3.2) Definition. Let R be a finitely generated K -algebra. Then R is said to be of *weakly F -regular type* (respectively, of *F -regular type*, of *strongly F -regular type*, or of *F -rational type*) if for some choice of descent data (A, R_A) , where A is a finitely generated \mathbb{Z} -subalgebra of K , almost all closed fibers R_κ of $A \rightarrow R_A$ are weakly F -regular (respectively, F -regular, strongly F -regular, or F -rational) in the characteristic p sense. We add the words “over K ” if the choice of field is not clear from context. If one of these conditions holds for a Zariski dense set of closed fibers instead of for almost all closed fibers we modify terminology by inserting the word “dense,” so that we may speak of dense F -regular type, dense F -rational type, and so forth.

(4.3.3) Proposition. Let $K \subseteq L$ be fields of characteristic 0 and let R be a Noetherian L -algebra. If R is weakly F -regular (respectively, F -regular over L , or F -rational over L) then it also has this property over K .

Proof. The result is immediate from the fact that the K -tight closure of an ideal is contained in the L -tight closure: cf. (3.2.4) (recall that formal K -tight closure is K -tight closure). \square

³³See (2.3.10).

(4.3.4) Proposition. *Let R be a Noetherian K -algebra where K is a field of characteristic 0.*

(a) *$(R \text{ is } F\text{-regular over } K) \Rightarrow (R \text{ is weakly } F\text{-regular over } K) \Rightarrow (R \text{ is } F\text{-rational over } K).$*

(b) *Suppose that R is finitely generated over K . Then:*

$(R \text{ is of strongly } F\text{-regular type over } K) \Rightarrow (R \text{ is of } F\text{-regular type over } K) \Rightarrow (R \text{ is of weakly } F\text{-regular type over } K) \Rightarrow R \text{ is of } F\text{-rational type over } K.$

(c) *Suppose again that R is finitely generated over K . Then:*

$(R \text{ is of weakly } F\text{-regular type over } K) \Rightarrow (R \text{ is } F\text{-regular over } K) \text{ /smallskip and } (R \text{ is of } F\text{-rational type over } K) \Rightarrow (R \text{ is } F\text{-rational over } K).$

Proof. Part (a) is immediate from the definitions, and part (b) from the definitions coupled with the fact that the corresponding string of implications holds in characteristic p . To prove part (c), we first consider the case of weakly F -regular type. Note that one can show that an element not in a given ideal is not in the tight closure by choosing descent data and observing that the element is still not in the ideal in almost all fibers. But since almost all fibers are weakly F -regular, the element cannot be in the tight closure. The argument for the case of F -rational type is similar, except that we restrict attention to parameter ideals, and need to observe that, after descent, a sequence of parameters remains a sequence of parameters in almost all fibers, which is true by Corollary (2.3.12). \square

(4.3.5) Corollary. *Let R be a Noetherian K -algebra satisfying any of the conditions described in (4.3.4) above, i.e., any form of R -regularity or F -rationality. Then R is F -rational over K and, hence, F -rational over \mathbb{Q} . It follows that R is normal and, if R is universally catenary (which is always the case for locally excellent rings, and, in particular, for finitely generated K -algebras), then R is Cohen-Macaulay.*

Proof. That R is F -rational over K follows from (4.3.4), and that R is F -rational over \mathbb{Q} follows from (4.3.3), while the last sentence is essentially the content of (4.1.8). \square

Moreover, since regularity localizes, we may restate Theorem (4.1.1) as follows:

(4.3.6) Theorem. *Every regular Noetherian K -algebra is F -regular over K . \square*

(4.3.7) Discussion and definition: approximately Gorenstein rings. A local Noetherian ring (R, m) is called *approximately Gorenstein* if there are irreducible m -primary ideals contained in every power of m . It follows that there is a sequence of m -primary irreducible ideals cofinal with the powers of m . A Noetherian ring is called *approximately Gorenstein* if all of its local rings at maximal ideals are approximately Gorenstein. The notion is developed in [Ho4], and all of the results cited in this discussion are from that paper. If the Krull dimension is 0, approximately Gorenstein rings are precisely the same as Gorenstein rings. If the Krull dimension is one or more, the condition for a ring to be approximately Gorenstein is mild. A local ring R is easily seen to be approximately Gorenstein if and only if its completion is approximately Gorenstein. A complete local ring R of dimension at least one is approximately Gorenstein if and only if it has depth at least one, and for each prime $P \in \text{Ass } R$ such that $\dim R/P = 1$, the module $R/P \oplus R/P$ is not embeddable in R . This is one of the main results of [Ho4]. It easily follows that if R is a Noetherian ring that is normal, or excellent and reduced, then R is approximately Gorenstein.

(4.3.8) Theorem (characterization of weak F -regularity). *Let K be a field of characteristic 0. Let R be a Noetherian K -algebra. Then the following conditions on R are equivalent:*

- (1) *R is weakly F -regular over K (i.e., every ideal of R is K -tightly closed).*
- (2) *For every maximal ideal of R , R_m is weakly F -regular over K .*
- (3) *For every maximal ideal of R , the completion of R_m is weakly F -regular over K .*
- (4) *R is normal (respectively, approximately Gorenstein), and for every maximal ideal m of R there is a sequence of m -primary irreducible ideals cofinal with the powers of m that are K -tightly closed.*
- (5) *For every pair of finitely generated R -modules $N \subseteq M$, N is tightly closed in M .*

Proof. (1) \Rightarrow (4) since by (4.3.5) R is normal, and by the discussion in (4.3.7), normal Noetherian rings are approximately Gorenstein. Since (5) \Rightarrow (1) is obvious, if we can show that (4) \Rightarrow (5) then the equivalence of (1), (4), and (5) will have been established. Assume (4). Let $N \subseteq M$ be finitely generated modules, and suppose that x as an element of M not

in the K -tight closure of N . Then we can replace N by a submodule of M maximal with respect to the property of not containing x , and we can replace $N \subseteq M$ by $0 \subseteq M/N$ and x by its image in M/N . The hypothesis that every nonzero submodule of M contains x then implies that the annihilator of x is a maximal ideal m of R and that M is an essential extension of $Rx \cong R/m = K$, say, and it follows some power of m kills x . By (4) we can choose an irreducible m -primary ideal I such that I is K -tightly closed and $Ix = 0$. Then M may be viewed as an essential extension of K that happens to be an R/I -module, and since R/I is a zero-dimensional Gorenstein ring, M embeds in R/I . But since I is K -tightly closed in R , it follows that 0 is K -tightly closed in R/I , and, hence, in M , a contradiction.

The equivalence of (2) and (3) with the other conditions now follows from the observation that for an irreducible m -primary ideal I such that x generates the socle in R/I , where m is a maximal ideal in R , the ideal I is K -tightly closed in R (i.e., x is not in the K -tight closure) iff the image of x is not in the K -tight closure of IB , where B may denote either R_m or its completion, by (4.2.18). Moreover, the m -primary ideals of R correspond via expansion and contraction bijectively with those of B (whether B is R_m or its completion). \square

(4.3.9) Remark and definition: fiberwise tight closure. Let K be a field of characteristic 0. Over an affine K -algebra R one may define a variant notion of when a submodule is tight closed. The notion is first described in [Kat1]. Let $N \subseteq M$ be finitely generated R -modules and define N to be *fiberwise tightly closed* in M (in the sense of K -tight closure) if for some (equivalently) every choice of descent data (the proof of the equivalence is easy) one has that N_κ is tightly closed in M_κ for almost all closed fibers. This implies very easily that N is tightly closed in M . We do not know whether, over an affine K -algebra R , the condition that $N \subseteq M$ is K -tightly closed implies that N is fiberwise tightly closed in M .

We next prove a result about the behavior of algebras satisfying one of the variations of F -regular or F -rational type. Many such results are immediate from corresponding results in characteristic p , and we have not tried to give an exhaustive account here. Proposition (4.3.11) gives a sampling. We first need a definition.

(4.3.10) Definitions: canonical modules and the \mathbb{Q} -Gorenstein property. A Cohen-Macaulay local ring R is said to have a *canonical module* ω if ω is a finitely generated R -module whose depth is equal to the dimension of R (i.e., a maximal Cohen-Macaulay module), and such that if x_1, \dots, x_d is a system of parameters of R , $\omega/(x_1, \dots, x_d)\omega$ has a socle that is a one-dimensional vector space over the residue field. An equivalent condition on the finitely generated module ω is that its Matlis dual is the highest local cohomology module of the ring. For basic facts about canonical modules we refer the reader to [GrHa] and [HeK]. In this local case, ω is unique up to isomorphism. Such a canonical module ω exists if and only if R is a homomorphic image of a Gorenstein local ring S , in which case $\omega \cong \text{Ext}_S^h(R, S)$, where h may be characterized either as $\dim S - \dim R$ or as the height of the ideal I such that $R \cong S/I$. The least number of generators of ω is the type of R .

More generally, if ω is a finitely generated R -module over a Cohen-Macaulay ring R such that for every maximal (equivalently, prime) ideal P of R , ω_P is a canonical module for R_P , then ω is called a *canonical module* for R . If $R = S/I$ where S is Gorenstein and all minimal primes of I have height h , then $\text{Ext}_S^h(R, S)$ is again a canonical module for R . A canonical module for R , if it exists, is unique up to tensoring with a module that is locally free of rank one. If R is a domain, there is always an embedding $\omega \subseteq R$, so that ω is isomorphic as a module with some ideal J of R . J is called a *canonical ideal* for R . It automatically has pure height one. Thus, if R is normal, we may think of the class of ω or J in the divisor class group of R . The positive multiples of this class correspond to the symbolic powers $J^{(r)}$ of J , where $J^{(r)}$ is the contraction of $W^{-1}J^r \subseteq W^{-1}R$ to R : here, W is the complement of the union of the minimal primes of J (equivalently, W is the set of elements of R that are not zerodivisors on J).

We shall say that a Cohen-Macaulay normal ring R is \mathbb{Q} -Gorenstein if it has a canonical module $\omega_R = \omega$, and there exists a positive multiple of the element represented by ω in the divisor class group of R that is locally principal. Note that a Gorenstein normal ring is always \mathbb{Q} -Gorenstein. Note also that the property of being \mathbb{Q} -Gorenstein is independent of the choice of ω .

(4.3.11) Proposition. *Let K be a field of characteristic 0 and let R be a finitely generated K -algebra.*

- (a) *If R is regular, then R is of strongly F -regular type over K .*
- (b) *If R is of F -rational type over K and Gorenstein, then R is of strongly F -regular type over K .*
- (c) *If R is of weakly F -regular type over K and either*
 - (1) *R is \mathbb{Q} -Gorenstein except possibly at isolated closed points, or*
 - (2) *R is \mathbb{N} -graded,**then R is of strongly F -regular type over K .*
- (d) *If R is of weakly F -regular type over K and $g \in R$, then R_g is of weakly F -regular type over K .*
- (e) *If R is of F -rational type over K and $g \in R$, then R_g is of F -rational type over K .*

Proof. In all parts we may assume that we have chosen descent data, so that we have R_A where A is a domain that is a finitely generated \mathbb{Z} -algebra. As usual, we denote by \mathfrak{F} the field of fractions of A . It follows that all closed fibers R_κ are finitely generated algebras over a finite field κ , and, hence F -finite. Each part follows from the fact that one can preserve the hypothesis while passing to descent data and then to almost all fibers, and then apply a corresponding fact in characteristic p .

For part (a), since R is regular over a field of characteristic 0, we know that R_A can be made smooth over A by localizing at one element of A° . The fibers are then F -finite and regular, and the result follows from Theorem (3.1)(c) of [HH3].

For part (b), note that almost all fibers are F -rational (by the definition of F -rational type), F -finite by our initial remark, and Gorenstein by (2.3.15).

For part (c) we need to show that the respective hypotheses (1) and (2) are preserved for almost all closed fibers R_κ , for then we may apply the main result of [MacC] in case (1) or the main result of [LySm] in case (2) to obtain the desired conclusion. In case (2) it is obvious that the \mathbb{N} -grading is preserved when we pass to fibers. We focus on case (1). The ring is normal and therefore a finite product of domains. We can therefore assume that we are in the domain case, so that all maximal ideals have height equal to the dimension of the ring. Since R is Cohen-Macaulay, R_A will be Cohen-Macaulay after localizing at one element of A° , and we can write $R_A = A[X_1, \dots, X_n]/\mathcal{P}_A$ where \mathcal{P}_A is a prime ideal of $S_A = R[X_1, \dots, X_n]$, a polynomial ring mapping onto R_A . Let h be the height of $c\mathcal{P}_A$.

By (2.1.15) we can localize at one element of A° so that all the closed fibers will be Cohen-Macaulay, and of the same dimension as R , by (2.3.9). Let $\omega_A = \text{Ext}_{S_A}^h(R_A, S_A)$. Since we can localize at an element of A° so that A becomes Gorenstein or even regular, this is a canonical module for R_A . More important is that ω_A is a relative canonical module in the sense that, after localizing further, at one element of A° , we may assume that for any map $A \rightarrow L$, where L is a field, $\omega_L = L \otimes_A \omega_A$ is a canonical module for R_L , since by (2.3.5) part (e), after one such localization we may identify $L \otimes_A \text{Ext}_{S_A}^h(R_A, S_A)$ with $\text{Ext}_{S_L}^h(R_L, S_L)$ for any field L to which A maps, and R_L will be S_L/\mathcal{P}_L where cP_L has pure height h in the polynomial ring S_L . We may choose an embedding of $\omega_{\mathfrak{F}}$, which is a canonical module for $R_{\mathfrak{F}}$, as an ideal $J_{\mathfrak{F}}$ of $R_{\mathfrak{F}}$. After localizing at one element of A° we may assume that this embedding arises from an inclusion $J_A \subseteq R_A$ by localization, where there is an isomorphism $\omega_A \cong J_A$ that gives rise to the isomorphism $\omega_{\mathfrak{F}} \cong J_{\mathfrak{F}}$ by localization. Then J_K is a canonical ideal for R . The locus

$$\{P \in \text{Spec } R : (J_K^{(h!)})_P \text{ is not principal}\}$$

(not principal is equivalent to not free here) is closed and non-increasing with h , hence is stable for all $h \gg 0$. Since the ring is \mathbb{Q} -Gorenstein except at finitely many closed points, it follows that we can choose $r = h!$ such that $J_K^{(r)}$ is locally principal except at finitely many closed points, say those corresponding to the maximal ideals $\{m_1, \dots, m_s\}$ of R . Because the extension of fields $\mathfrak{F} \subseteq K$ may be transcendental, these maximal ideals need not lie over maximal ideals of $R_{\mathfrak{F}}$. However, by enlarging A , if necessary, we can get them to lie over maximal ideals of $R_{\mathfrak{F}}$. By localizing further at one nonzero element of the enlargement of A , if necessary, we can assume that our previous suppositions continue to hold.

It now follows that $J_{\mathfrak{F}}^{(r)}$ is locally principal except at finitely many closed points of $R_{\mathfrak{F}}$, those lying under the m_i : if we localize at any element g not in the union of these maximal ideals, $(J_{\mathfrak{F}}^{(r)})_g$ becomes locally free of rank one once we tensor with K over \mathfrak{F} , and so it must have been locally free of rank one originally.

We can localize at one element of A° to guarantee that $\mathfrak{F} \otimes_A J_A^{(r)} \cong J_{\mathfrak{F}}^{(r)}$, and also that R_A/J_A and $R_A/J_A^{(r)}$ are A -flat. Let $I_{\mathfrak{F}}$ denote the ideal of $R_{\mathfrak{F}}$ generated by the elements

$g \in R_{\mathfrak{F}}$ such that $(J_{\mathfrak{F}}^{(r)})_g$ is principal. If we localize at a prime ideal Q containing I , $(J_{\mathfrak{F}}^{(r)})_Q$ is not principal (or we could find another such element). It follows that $V(I_{\mathfrak{F}})$ is a finite set of maximal ideals of $R_{\mathfrak{F}}$. Choose generators g_1, \dots, g_t of $I_{\mathfrak{F}}$ such that every $(J_{\mathfrak{F}}^{(r)})_{g_i}$ is principal. It follows that there is a large integer N and for every i an element u_i of $J_{\mathfrak{F}}^{(r)}$ such that $g_i^N J_{\mathfrak{F}}^{(r)} \subseteq u_i R_{\mathfrak{F}}$. By localizing at one element of A we can assume that all the $g_i \in R_A$, that all the $u_i \in J_A^{(r)}$, and that all the closed fibers of $A \rightarrow R_A/I_A$ are zero-dimensional, where $I_A = (g_1, \dots, g_s)R_A$ (since this is true after we pass to \mathfrak{F}). Finally, we can assume that the calculation of symbolic powers commutes with passing to closed fibers as follows: Choose one element $f \in R_{\mathfrak{F}}$, not a zerodivisor on $J_{\mathfrak{F}}$, such that $fJ_{\mathfrak{F}}^{(r)} \subseteq J_{\mathfrak{F}}^r$, and note that $R_{\mathfrak{F}}/J_{\mathfrak{F}}^{(r)}$ has no embedded primes. After localizing at one element of A° we may assume that $f \in R_A$ is not a zerodivisor on R_A/J_A and that $fJ_A^{(r)} \subseteq J_A^r$, while $R_A/J_A^{(r)}$ has no embedded primes. After localizing at one element of A° we have that all this is preserved when we pass to closed fibers, and it follows that for each closed fiber $J_\kappa^{(r)}$ is simply $\kappa \otimes_A J_A^{(r)}$.

We have now shown that for suitably large descent data, the condition that the ring be \mathbb{Q} -Gorenstein except at finitely many closed points is preserved when we pass to almost all closed fibers, which is what we needed to complete the proof of part (1) of (c).

Parts (d) and (e) follow from the corresponding facts (REF) for affine algebras of characteristic p . \square

We now return to the study of various kinds of F -regularity and F -rationality. We first note:

(4.3.12) Theorem. *Let K be a field of characteristic 0 and let $R \rightarrow S$ be a homomorphism of Noetherian K -algebras. If R is cyclically pure³⁴ in S , and S is weakly F -regular (respectively, F -regular) then so is R .*

Proof. First suppose that S is weakly F -regular over K . If u is in the K -tight closure of $I \subseteq R$ in R then the image of u in S is in the K -tight closure of IS in S , by persistence

³⁴This means that every ideal of R is contracted from S , and is true when S is faithfully flat over R , when R is a direct summand of S as an R -module, or when R is pure in S . (Each of the three conditions listed is weaker than its predecessors.) See (4.1.11).

of tight closure, and so the image of u is in the contraction of IS to R , which is I by hypothesis. Thus, every ideal of R is K -tightly closed.

The case where S is F -regular over K now follows from the preceding case and the fact that our hypothesis is preserved when we localize at any multiplicative system of R (cf. (4.1.11)). \square

(4.3.13) Discussion and definition: Hilbert rings. We shall refer to a commutative ring R as a *Hilbert ring* (the term *Jacobson ring* is also used) if every prime ideal is an intersection of maximal ideals. The property passes from R to homomorphic images of R and to finitely generated R -algebras, and so a finitely generated algebra over a field is a Hilbert ring. If R is a Hilbert ring and S is a finitely generated R -algebra, then every maximal ideal of S lies over a maximal ideal of R . See Section 1–3 of [Kap] (our definition differs from the one given there, but see Exercise 9.(c) on p. 20).

(4.3.14) Theorem (F -regularity and base change). *Let K be a field of characteristic 0 and let $R \rightarrow S$ be a flat homomorphism of Noetherian K -algebras.*

- (a) *If R is weakly F -regular over K , $R \rightarrow S$ is local, and the closed fiber is regular then S is weakly F -regular over K .*
- (b) *If R_P is weakly F -regular over K for every prime ideal P of R lying under a maximal ideal of S , and $R \rightarrow S$ is geometrically regular, then S is weakly F -regular over K .*
- (c) *If R is F -regular over K and $R \rightarrow S$ is geometrically regular then S is F -regular over K .*
- (d) *If R is a Hilbert ring³⁵ (e.g., a finitely generated algebra over a field), $R \rightarrow S$ is smooth, and R is weakly F -regular over K , then S is weakly F -regular over K .*

Proof. For part (a) we note that we may complete both R and S and all our hypotheses are preserved. By part (3) of (4.3.8) our hypotheses are preserved, and to show that a local ring is weakly F -regular over K , it suffices to show this for the completion. Thus, there is no loss of generality in assuming that $R \rightarrow S$ is a flat local homomorphism of complete local rings with a regular closed fiber. Choose a coefficient field K' of R containing K and extend its image in S to a coefficient field L for S . We may represent R as a module-finite

³⁵See (4.3.13).

image of $A = K'[[x_1, \dots, x_n]]$, a formal power series ring, and let y_1, \dots, y_d be elements of S whose images in the closed fiber $S/M_R S$ form a regular system of parameters. Let $B = L[[x_1, \dots, x_n, y_1, \dots, y_d]]$ is a formal power series ring that is contained in S such that S is module-finite over the regular ring B , and if we let $T = B \otimes_A R$, there is an obvious map $T \rightarrow S$. This map is an isomorphism: the argument is given parenthetically beginning on the last line of p. 36 of [HH9].

We can make the ring extension from R to S in two steps: in the first we tensor with $B_0 = L[[x_1, \dots, x_n]]$ (so that $d = 0$). Then, in the second step, we simply adjoin the formal power series indeterminates y_1, \dots, y_d . This reduces the problem to considering the two steps separately.

First consider the case where $d = 0$, so that we are simply extending the field. Choose a sequence of irreducible m_R -primary ideals of R cofinal with the powers of m_R . These are K -tightly closed, and when expanded to S they have the same properties: they are irreducible and cofinal with the powers of S . By (4.3.8) it suffices to show that these expanded ideals remain K -tightly closed, which follows from (4.2.18), with hypothesis as in part (1).

We now consider the case where we are adjoining formal power series indeterminates (but we may assume the residue field does not change). In $S = R[[y_1, \dots, y_d]]$ the ideals of the form $IS + (y_1^t, \dots, y_d^t)S$, where I is an irreducible M_R -primary ideal, are irreducible and cofinal with the powers of M_S . It suffices to show that these ideals are K -tightly closed. Let $T = R[y_1, \dots, y_d]$. The ideals $IT + (y_1^t, \dots, y_d^t)T$ are K -tightly closed by Theorem (4.2.14), part (d). By (4.2.18), the fact that $J = IT + (y_1^t, \dots, y_d^t)T$ is K -tightly closed is preserved when we pass to the completion of T at $m_R + (y_1, \dots, y_d)T$, since J is primary to this maximal ideal of T . (Since J is irreducible while there is no field extension, the hypotheses of both (1) and (2) of (4.2.18) hold here — either suffices.) \square

The following theorem contains a number of important results on the behavior of F -rationality. The theory is better behaved in several ways than the theory of weak F -regularity.

(4.3.15) Theorem (behavior of F -rational rings). *Let K be a field of characteristic*

0 and let R be a locally excellent Noetherian K -algebra.

- (a) R is F -rational over K iff R_m is F -rational over K for every maximal ideal m of R .
- (b) If R is local, then R is F -rational over K iff \widehat{R} is F -rational over K .
- (c) (Localization and base change) If R is F -rational over K , then every localization of R is F -rational over K , and, more generally, if R is F -rational over K , $R \rightarrow S$ is geometrically regular, and S is locally excellent, then S is also F -rational over K .
- (d) If R is local and equidimensional, then R is F -rational over K if and only if the ideal generated by one system of parameters is K -tightly closed.
- (e) (F -rationality deforms) If R/fR is F -rational over K , where f is a nonzerodivisor in R , then R_P is F -rational over K for every prime ideal P containing f . In particular, if (R, m) is local and $f \in m$ is a nonzerodivisor such that R/fR is F -rational over K , then R is F -rational over K .
- (f) If R is Gorenstein, then R is F -rational over K iff R is weakly F -regular over K iff R is F -regular over K .

Proof. We begin by proving that the condition given in part (d) is sufficient for the ring to be F -rational. Since the ring is equidimensional, the given system of parameters, say x_1, \dots, x_d , is a system of parameters modulo every minimal prime. It follows from Corollary (4.1.10) that the ring is Cohen-Macaulay, and reduced, since (0) is tightly closed. We next observe that the ideal (x_1^t, \dots, x_d^t) is tightly closed for every positive integer t . To see this, note that if not, there would be an element of $(x_1^t, \dots, x_d^t) :_R m$ and not in (x_1^t, \dots, x_d^t) in the tight closure, corresponding to a nonzero socle element in $R/(x_1^t, \dots, x_d^t)$. But then this element has the form $(x_1 \cdots x_d)^{t-1}u$, where the element $u \in (x_1, \dots, x_d) :_R m$ but $u \notin (x_1, \dots, x_d)$. Since

$$x_1^{t-1} \cdots x_d^{t-1}u \in (x_1^t, \dots, x_d^t)^{*K}$$

we have that

$$u \in (x_1^t, \dots, x_d^t)^{*K} :_R x_1^{t-1} \cdots x_d^{t-1}$$

and so, by the final statement in Theorem (4.1.7), we have that $u \in (x_1, \dots, x_d)^{*K} = (x_1, \dots, x_d)$ by hypothesis, and so $(x_1 \cdots x_d)^{t-1}u \in (x_1^t, \dots, x_d^t)$ after all.

If y_1, \dots, y_d is any other system of parameters, then from the fact that the highest local cohomology module can be viewed as either the direct limit of the modules $R/(x_1^t, \dots, x_d^t)$ or the direct limit of the modules $R/(y_1^s, \dots, y_d^s)$, with the maps in the direct limit system being injective in either case, it follows that $R/(vctyd)$ embeds in $R/(x_1^t, \dots, x_d^t)$ for t sufficiently large. Since 0 is K -tightly closed in the latter, it is K -tightly closed in the former, which shows that (y_1, \dots, y_d) K -is tightly closed in R for any system of parameters y_1, \dots, y_d .

Finally, suppose that x_1, \dots, x_k is part of a system of parameters x_1, \dots, x_d . The fact that (x_1, \dots, x_k) is K -tightly closed then follows from (4.1.10), or one may deduce it instead from the fact that

$$(x_1, \dots, x_k) = \bigcap_t (x_1, \dots, x_k, x_{k+1}^t, \dots, x_d^t).$$

This completes the proof of part (d).

Part (b) now follows because an ideal of R generated by a system of parameters is K -tightly closed in R if and only if its expansion to \widehat{R} is K -tightly closed in \widehat{R} , by (4.2.18): the second part of the hypothesis holds.

To prove (a) first suppose that every R_m is F -rational, and that x_1, \dots, x_k are parameters in R generating an ideal I such that $u \in I^{*K}$ while $u \notin I$. Choose a maximal ideal m containing $I :_R u$. Then $I \subseteq m$, and I is a parameter ideal of R_m , while $u \notin IR_m$. But $u \in (IR_m)^{*K}$ by the persistence of K -tight closure, a contradiction.

To prove the other half of (a) suppose that R is F -rational and that m is a given maximal ideal of R . Choose a maximal parameter ideal I of R generated by x_1, \dots, x_d inside m , and let J be the contraction of IR_m to R . Then $J = I :_R f$ for a suitable element $f \in R - m$. Then I is K -tightly closed by hypothesis, and so J is K -tightly closed by Theorem (3.2.3), part (i), and since J is m -primary, $JR_m = (x_1, \dots, x_d)R_m$ is K -tightly closed in R_m by (4.2.18) (hypothesis (2) holds), and so R_m is F -rational over K by part (d), proved above. This completes the proof of (a).

We next prove (c). We first prove that localizations $W^{-1}R$ of F -rational rings R over K are F -rational over K . Since each local ring of $W^{-1}R$ is a local ring of R , it suffices to show that R_P is F -rational for every prime P . We may first replace R by R_m for a

maximal ideal $m \supseteq P$, by part (a), and so we may assume that (R, m) is local. Choose a prime ideal Q of the completion \widehat{R} that is a minimal prime of $P\widehat{R}$ and so lies over P , so that $R_P \rightarrow \widehat{R}_Q$ is a faithfully flat map of local rings such that the maximal ideal PR_P of R_P expands to an ideal primary to the maximal ideal of \widehat{R}_Q . Then a system of parameters for R_P is also a system of parameters for \widehat{R}_Q . It follows that if \widehat{R}_Q is F -rational, then so is R_P , since the contraction of a K -tightly closed ideal is K -tightly closed. We know that \widehat{R} is F -rational by part (b). Thus, we may assume without loss of generality that R is complete (and F -rational), and we need to show that the localization at any prime ideal Q remains F -rational. We may choose parameters x_1, \dots, x_k in Q such that their images in R_Q form a system of parameters. Then (x_1, \dots, x_k) is K -tightly closed by hypothesis, and the expansion of this ideal to R_Q remains tightly closed by Theorem (4.2.14), part (g), since R is a complete domain. Thus, R_Q is F -rational by part (d), which was established in the first part of this proof.

Now suppose that $R \rightarrow S$ is geometrically regular and that R is F -rational over K . It suffices to show that the completion of every local ring of S is F -rational, and so we may replace S by such a completed localization and assume that S is a complete local ring. We may replace R by its localization at the contraction of the maximal ideal of S , by what has already been proved, and we may then complete R . The result is a flat local homomorphism of complete local rings $(R, m) \rightarrow (S, n)$ such that R is F -rational. Moreover, the closed fiber S/mS is regular, i.e., a formal power series ring. Then, exactly as in the proof of (4.3.14) part (a), we may consider two cases, one in which we are enlarging the residue field and the other in which we are adjoining formal power series indeterminates. In the first case a system of parameters for R is also a system of parameters for S , and remains tightly closed by Theorem (4.2.14), part (g). In the second case we may assume that S is $R[[y_1, \dots, y_k]]$. Let x_1, \dots, x_d be a system of parameters for R , so that $\text{vect} x_d, y_1, \dots, y_k$ is a system of parameters for S . Then the ideal generated by $x_1, \dots, x_d, y_1, \dots, y_k$ in the polynomial ring $R[[y_1, \dots, y_d]]$ is tightly closed by Theorem (4.2.14), part (d), and remains tightly closed when we pass to S by Proposition (4.2.18) (the hypothesis (2) holds). This completes the proof of (c).

To prove (e), first note that because we can localize the F -rational ring R/fR and

preserve F -rationality, we may assume that R is local. Extend f to a system of parameters for R , say $f = x_1, \dots, x_d$. Then the ideal generated by the images of x_2, \dots, x_d in R/fR is K -tightly closed in R/fR , and so its contraction to R , which is $(x_1, \dots, x_d)R$ is K -tightly closed in R , and we may apply part (d).

Finally, suppose that R is Gorenstein and F -rational. Since we know that these properties are preserved by localization, it will suffice to prove that R is weakly F -regular. For this purpose we may replace R by its localization at a maximal ideal, and so we may assume that (R, m) is local. Let x_1, \dots, x_d be a system of parameters. Then each of the ideals $x_1^t, \dots, x_d^t)R$ is generated by parameters and so is K -tightly closed in R . These are irreducible m -primary ideals cofinal with the powers of m , and so it follows that R is weakly F -regular by condition (4) of Theorem (4.3.8). \square

(4.3.16) Discussion and definitions: rational and pseudo-rational singularities.

A ring R essentially of finite type over a field of equal characteristic zero is said to have *rational singularities* if R is normal and for some (equivalently, every) desingularization $Y \rightarrow X$ of $X = \operatorname{Spec} R$ (desingularizations exist by virtue of the main result of [Hir]), $H^i(Y, \mathcal{O}_Y) = 0$ for all $i \geq 1$. This condition is local on X and so extends to schemes. It holds when R is regular, and roughly speaking means that in some cohomological sense R is very much like a regular ring. The condition implies that R is Cohen-Macaulay as well as normal, but is strictly stronger.

There are many other characterizations of rational singularities in equal characteristic zero. For example, if R is a local ring essentially of finite type over a field K of characteristic zero, then R has rational singularities iff R is normal, Cohen-Macaulay, and the direct image of the canonical sheaf of a desingularization Y (which should be taken to be $\Omega_{Y/K}^d$ with $d = \dim Y$, the sheaf of highest order differentials), is ω_X , the canonical sheaf on $X = \operatorname{Spec} R$. Cf. [KKMS], pp. 50–51. The proof of this fact uses the Grauert-Riemenschneider vanishing theorem [GR].

Lipman and Teissier define a local ring (R, m) of dimension d to be *pseudo-rational* if it is normal, Cohen-Macaulay, analytically unramified (i.e., the completion is reduced — this is automatic for excellent normal local rings), and if for any proper birational map

$\pi: Y \rightarrow X = \operatorname{Spec} R$ such that Y is normal, if $E = \pi^{-1}(\{m\})$, the closed fiber, then the canonical map of local cohomology³⁶ $H_m^d(\pi_* \mathcal{O}_Y) = H_m^d(R) \xrightarrow{\delta_\pi^d} H_E^d(\mathcal{O}_Y)$ is injective.

It is not difficult to show, by considering a dual notion, that having pseudo-rational singularities is equivalent to having rational singularities for algebras essentially of finite type over a field of characteristic 0. We now record the following results of K. E. Smith:

(4.3.17) Theorem (K. E. Smith). *Let R be an excellent local ring of positive prime characteristic p . If R is F -rational, then R has pseudo-rational singularities.*

We refer the reader to [Sm1] or [Sm3] for the proof.

From this it is not difficult to show:

(4.3.18) Theorem (K. E. Smith). *Let R be an algebra of finite type over a field K of characteristic 0. If R has dense F -rational type³⁷ over K then R has rational singularities.*

Again, we refer the reader to [Sm1] and [Sm3] for details. The converse result was proved by N. Hara in [Hara5] using the results of [DI]:

(4.3.19) Theorem (N. Hara). *Let R be an algebra essentially of finite type over a field of characteristic 0. If R has rational singularities, then R has F -rational type over K .*

We refer to the reader to [Hara5] for the proof. Combining these two results we have:

(4.3.20) Theorem (Hara-Smith). *An algebra of finite type over a field K has rational singularities if and only if it has dense F -rational type over K if and only if it has F -rational type over K . \square*

Hence:

(4.3.21) Theorem. *If R is a Gorenstein \mathbb{N} -graded finitely generated K -algebra, then R is of strongly F -regular type over K if and only if the ideal generated by one homogeneous system of parameters is tightly closed.*

³⁶Here $H_Z^i(_)$ is the i th right derived functor of $H_Z^0(_)$, where $H_Z^0(\mathcal{G})$ consists of the global sections of \mathcal{G} whose support is contained in Z ; however, we write m instead of Z for local cohomology with supports in the closed set defined by m .

³⁷See (4.3.2).

Proof. Fix a homogenous system of parameters f_1, \dots, f_d and a homogeneous element u representing the socle generator modulo these parameters. Choose descent data such that the f_i and u are in R_A . After localizing at one element of A° we may assume that the images of the f 's are a homogeneous system of parameters in each closed fiber, and that the image of u is a socle generator modulo the f 's in each closed fiber. We may likewise assume that there is no dense open set such that u is in the tight closure of the parameter ideal, and it follows that the closed fibers where the parameter ideal is tightly closed are dense. But since this is a graded Gorenstein ring, for each such fiber the ring is strongly F -regular, and, in particular, F -rational. Thus, the ring has rational singularities and now it follows that almost all fibers are F -rational and so (since they are Gorenstein and F -finite) strongly F -regular. \square

(4.3.22) Discusssion. It seems likely that an \mathbb{N} -graded Cohen-Macaulay ring that if a finitely generated K -algebra is of F -rational type if and only if the ideal generated by one homogeneous system of parameters is tightly closed. Theorem (4.2.1) is the case where the ring has type 1. But there appear to be serious difficulties when the socle has dimension 2 or more. The fact that the parameter ideal is tightly closed implies that, after descent, any given socle element can be excluded from the tight closure of the image of the ideal for almost all fibers, but this does not show, *a priori* that the image of the ideal is tightly closed for almost all fibers. As the closed fiber varies an element of the tight closure of the ideal may be “hopping around” in the socle in an uncontrolled way.

We conclude with some important examples of rings of (strongly) F -regular type.

(4.3.23) Theorem. *Let K be a field of characteristic 0 and let $X = (x_{ij})$ be an $r \times s$ matrix of indeterminates over K with say $r \leq s$. Let $K[X]$ denote the polynomial ring over K in the indeterminates x_{ij} .*

- (a) *Let I denote the ideal of $K[X]$ generated by the size t minors of X for some t , $1 \leq t \leq r$. Then $R = K[X]/I$ is of strongly F -regular type over K .*
- (b) *Let S denote the subring of $K[X]$ generated by the size r minors of X , which is a homogeneous coordinate ring of a Grassmann variety. Then S is of strongly F -regular type over K .*

Proof. In both cases, after descent and passage to fibers one gets a ring of the same kind over a finite field κ . Thus, it suffices to see that in the case of a finite field these rings are strongly F -regular. (Since κ is finite these rings are F -finite.) The F -regularity of these rings is proved in Theorem (7.14) of [HH10]. The rings in part (b) are Gorenstein and so strong F -regularity follows. For part (a) one may use the same idea as in the proof of Theorem (7.14) of [HH10]. One may think of X as the first r rows of a possibly larger square matrix of indeterminates Y , and then $\kappa[X]/I$ is an algebra retract of $T = \kappa[Y]/J$, where J is the ideal generated by the size t minors of Y . Since Y is square, T is Gorenstein and, hence, strongly F -regular. But $\kappa[X]/I$ is an algebra retract of $\kappa[Y]/J$: one obtains the retraction by killing the indeterminates in the bottom $s - r$ rows of Y , and so $\kappa[X]/I$ is strongly F -regular as well. \square

(4.4) PHANTOM HOMOLOGY AND HOMOLOGICAL THEOREMS

In this section we prove several results that greatly generalize several of what used to be known as the “local homological conjectures” — many of these are now theorems: cf. [PS1,2], [Ho1–3,5–7], [Ro1–4], [Du], and [EvG1–3]. This extends to equal characteristic zero a program that has already been carried out in characteristic p .

Throughout this section let K be a field of characteristic 0 and let R be a Noetherian K -algebra.

(4.4.1) Definition: phantom homology. A complex of finitely generated R -modules

$$\cdots \rightarrow M_{i+1} \rightarrow M_i \rightarrow M_{i-1} \cdots$$

is said to have K -phantom homology (or simply *phantom homology*) at the i th spot if the kernel Z_i of $M_i \rightarrow M_{i-1}$ is in the K -tight closure of the image B_i of $M_{i+1} \rightarrow M_i$ in M_i . If K is understood from context we may simply refer to *phantom homology*. *Phantom cohomology* is defined analogously. A left complex

$$\cdots \rightarrow M_i \rightarrow \cdots \rightarrow M_1 \rightarrow M_0$$

is called K -phantom acyclic (or simply *phantom acyclic*, if K is understood), if its homology is K -phantom for all $i > 0$.

(4.4.2) Discussion: terminology and notation for complexes. Let G_\bullet denote a finite complex over R consisting of projective modules that are locally free of constant rank: say that G_i has rank b_i . Let α_i denote the i th map, $1 \leq i \leq d$. In the local case, or when we otherwise know that every G_i is free, we shall assume that every G_i has been identified with R^{b_i} , and in this case we shall also let α_i denote a matrix of the i th map, $1 \leq i \leq d$. In the free case we shall let r_i be the determinantal rank of α_i , and let I_i be the ideal generated by the size r_i minors of α_i ($\alpha_{d+1} = 0$ and $r_{d+1} = 0$). Let \mathcal{H} denote a function on the ideals of R such as height or depth or minheight (see (2.3.8)) that takes on non-negative integer values on proper ideals and the value $+\infty$ on the unit ideal. Recall that G_\bullet satisfies *the standard conditions on rank and \mathcal{H}* if for $1 \leq i \leq d$, $b_i = r_{i+1} + r_i$ and $\text{ht } I_i \geq i$. Thus, we may speak about “the standard conditions on rank and height” or “the standard conditions on rank and depth.”

We can extend these notions to complexes of projective modules of constant rank once we have defined the analogues of the ideals of minors $I_r(\alpha)$. We make this extension as follows. Identify $R \cong \text{Hom}_R(R, R)$ in the obvious way. Given a map $\alpha : G \rightarrow G'$ of projective modules we define $I_1(\alpha)$ to be the ideal of R generated by all $a \in R$ corresponding to elements of $\text{Hom}_R(R, R)$ obtained as composite maps of the form $\gamma \circ \alpha \circ \beta$ where $\beta \in \text{Hom}_R(R, G)$ and $\gamma \in \text{Hom}_R(G', R)$. It suffices to use the elements obtained as β runs through a set of generators for $\text{Hom}_R(R, G)$ and γ runs through a set of generators for $\text{Hom}_R(G', R)$. It is easy to see that when G, G' are free, this ideal is the same as the ideal generated by the entries of a matrix for the map $G \rightarrow G'$. Moreover, the construct commutes with base change and, in particular, with localization. Now we may define $I_r(\alpha)$ for maps α of projective modules for all $r \geq 1$ as $I_1(\wedge^r \alpha)$.

We define the *formal minheight* of an ideal I to be the smallest of the minheights of the ideals IB , where B is the completion of the local ring R_P of R at a prime ideal P containing I , or $+\infty$ if $I = R$.

We can now prove several analogues of results from characteristic p tight closure theory. The first is:

Theorem (4.4.3) (phantom acyclicity criterion). *Let K be a field of characteristic*

0 and let R be a Noetherian K -algebra. Let G_\bullet be a finite complex of finitely generated projective R -modules of constant rank.

Suppose that R is universally catenary and locally equidimensional and that $R_{\text{red}} \otimes_R G_\bullet$ satisfies the standard conditions on rank and height. Then G_\bullet is K -phantom acyclic.

More generally, if R is universally catenary and $R_{\text{red}} \otimes_R G_\bullet$ satisfies the standard conditions on rank and minheight³⁸ then G_\bullet is K -phantom acyclic.

Still more generally, for any Noetherian K -algebra R , if $R_{\text{red}} \otimes_R G_\bullet$ satisfies the standard conditions on rank and formal minheight then G_\bullet is phantom acyclic.

Proof. Consider a cycle in degree $i \geq 1$: we must show that it is in the tight closure of the boundaries. By (4.2.6) it suffices to verify this after localizing at a prime ideal of R , completing, and killing a minimal prime. In the locally equidimensional case height and minheight agree, while in the universally catenary case, completion does not affect the minheight of an ideal of a local ring. Thus, in the first case height and minheight agree with formal minheight and in the second, minheight agrees with formal minheight.

Thus, it suffices to prove the assertion in the third case. Localizing, completing, and killing a minimal prime will not decrease the formal minheight of the expansion of $I_{r_j}(\alpha_j)$ to the ring obtained, $j \geq 1$. (Nilpotents are automatically killed in the process.) In particular, $I_{r_j}(\alpha_j)$ does not become 0, $j \geq 1$, and it follows that the ranks of the maps do not change.

We have consequently reduced to the case where R is a complete local domain and G_\bullet itself satisfies the standard conditions on rank and height. Let z denote the cycle in G_i , $i \geq 1$, corresponding to the cycle we started with.

We may assume that every $G_j \cong R^{b_j}$ for a suitable choice of b_j , $j \geq 0$, and that the maps in the complex are given by matrices α_j , $j \geq 1$.

By Theorem (3.5.1) we can find a finitely generated K -algebra R' mapping to R such that (1) R' is a domain and (2) R' contains elements that map to the entries of the matrices α_j . Note that since the condition is equational we can also choose the lifting matrices α'_j so that (3) the product $\alpha'_j \alpha'_{j+1}$ is 0 for every $j \geq 1$.

³⁸See (2.3.8).

This means that the lifting G'_\bullet of G_\bullet that we get is a complex. Since it is again an equational condition we can also choose the lifting such that (4) all $r_j + 1$ size minors of the matrix α'_j vanish for every $j \geq 1$.

Moreover, Theorem (3.5.1) guarantees that in addition to the conditions listed above we may also choose R' and the lifting such that (5) the ideal generated by the r_j size minors of α'_j has the same height, which is at least j , as the ideal of r_j size minors of α_j , $j \geq 1$.

Finally, we may also assume that (6) there is a cycle z' in G'_i that maps to z when we apply $R \otimes_{R'} _$ to G'_\bullet .

Of course, applying $R \otimes_{R'} _$ to G'_\bullet gives our original complex back. Thus, it will suffice to show that z' is in the K -tight closure of the boundaries in G'_i in the affine K -algebra sense over R' .

Now one descends to an affine A -subalgebra R'_A of R' , where A is a finitely generated of K , as in Chapter 2. One chooses A large enough so that the entries of the matrices in the complex G'_\bullet and the coordinates of z' are in R'_A . By Theorem (2.3.9c) the heights of the ideals of rank size minors will be the same for almost all closed fibers as they were for R' : moreover, for almost all closed fibers the ring will be reduced and equidimensional (cf. (2.3.9), 92.3.16)). By the characteristic p version of the phantom acyclicity criterion, e.g., Theorem (9.8) of [HH4], the image of z' is in the tight closure of the image of the boundaries for almost all closed fibers, and the result follows. \square

This yields at once a very powerful result:

Theorem (4.4.4) (vanishing theorem for maps of Tor). *Let R be an equicharacteristic zero regular ring, let S be a module-finite extension of R that is torsion-free as an R -module (e.g., a domain), and let $S \rightarrow T$ be any homomorphism to a regular ring (or, if R is a K -algebra for some field K of characteristic 0, we may suppose instead that T is weakly F -regular over K). Then for every R -module M , the map $\mathrm{Tor}_i^R(M, S) \rightarrow \mathrm{Tor}_i^R(M, T)$ is 0 for all $i \geq 1$.*

Proof. Since Tor commutes with direct limits we may reduce to the case where the module M is finitely generated. If the image of the map is nonzero, then we may localize at a maximal ideal of T in the support. Note that T remains regular (or weakly F -regular over

K in the second case, by Theorem (4.3.8)). We may then replace T by its completion (in the second case, T remains weakly F -regular over K by Theorem (4.3.8) again). We may then replace R by the completion B of its localization at the contraction of the maximal ideal of T and S by $B \otimes_R S$. Thus, we may assume that R is a complete regular local ring. Note that if T is regular we may still view this as the second case by taking $K = \mathbb{Q}$, and we shall do this in the remainder of the proof.

A minimal free resolution G_\bullet of M over R satisfies the standard conditions on rank and depth by the acyclicity criterion of [BE], and so it satisfies the standard conditions on rank and height when we pass to $S \otimes_R G_\bullet$, which, by (4.4.3), is then K -phantom acyclic: any given cycle in degree $i > 0$ (representing a typical element of $\mathrm{Tor}_i^R(M, S)$) is in the tight closure of the boundaries. This is preserved when we tensor further and pass to $T \otimes_R G_\bullet$. Since T is weakly F -regular over K the result follows. \square

The mixed characteristic version (with T assumed regular) of (4.4.4) is an important open question. If it is true, the consequences are dramatic. It implies that regular rings are direct summands of their module-finite extensions and that pure subrings of regular rings are Cohen-Macaulay, both of which are open questions in mixed characteristic. These issues are explored in [HH11], §4, where (4.4.4) is proved by in equal characteristic by a different method. See also the discussion following (1.3.9).

Finally, we mention the following analogue of a characteristic p result from [HH8] (see Theorems (6.5) and (6.6)).

Theorem (4.4.5) (phantom intersection theorem). *Let K be a field of characteristic zero and let R be a Noetherian K -algebra. Let G_\bullet be a complex of finitely generated projective R -modules of constant rank that satisfies the standard conditions on rank and formal minheight. (If R is locally equidimensional and universally catenary it is equivalent to assume that R satisfies the standard conditions on rank and height.) Suppose that the complex G_\bullet is of length d . Let $z \in M = H_0(G_\bullet)$ be any element whose annihilator in R has formal minheight $> d$. (Again, if R is locally equidimensional and universally catenary we may assume, equivalently, that the height of the annihilator is greater than d .) Then $z \in 0_M^{*K}$. In consequence:*

- (1) if (R, m, K) is local, z cannot be a minimal generator of M .
- (2) the image of z is 0 in $H_0(S \otimes_R G_\bullet)$ for any regular (or weakly F -regular) ring S to which R maps.

Proof. To prove that $z \in 0_M^*$ it suffices to establish this after passing to the quotient of a completed local ring of R by a minimal prime. The standard conditions, which may now be thought of as standard conditions on rank and height, will continue to hold. The height of the annihilator will still be greater than d .

Thus, we have reduced to the case where R is a complete local domain. We now descend the complex G_\bullet to a complex of the same length G'_\bullet over an affine K -algebra R' , which may be taken to be a domain, mapping to R , exactly as in the proof of (4.4.3). We can do this so that there is an element z' of $M' = H_0(G'_\bullet)$ that maps to z when we apply $R \otimes_{R'} -$ (which gives G_\bullet back). Moreover, we can keep track of generators for the annihilator of z in R , and guarantee, by Theorem (3.5.1), that this ideal has at least the same height that it did over R . Finally, we descend further, replacing R' by a finitely generated subalgebra R'_A over a finitely generated \mathbb{Z} -algebra $A \subseteq K$ in such a way that all relevant entries of matrices and coordinates are in R'_A . For almost all closed fibers the image of the complex will satisfy the standard conditions on rank and height, the ring will be reduced and equidimensional, and the height of the annihilator of the image of z' will be greater than d . It follows from the characteristic p version of the phantom intersection theorem, Theorem (6.5) of [HH8].

Part (1) then follows because mM is tightly closed in M (M/mM is a direct sum of copies of R/m , and m is tightly closed in R), while part (2) is obvious. \square

Theorem (4.4.5) is a strengthening of the “improved” new intersection theorem (discussed, for example, in [Ho7]).

(4.4.6) Phantom resolutions and finite phantom projective dimension. Let R be a Noetherian ring of equal characteristic 0. We assume for simplicity that $\text{Spec } R$ is connected so that finitely generated projective modules are automatically locally free of constant rank. (If not, one may study each component of $\text{Spec } R$ separately.) Let G_\bullet be a finite free complex of finitely generated projective modules satisfying the standard conditions on rank and formal minheight (if R is universally catenary this is the same as

minheight, and if R is equidimensional as well it is the same as height). We call G_\bullet a *finite phantom resolution* of $M = H_0(G_\bullet)$ and say that M has *finite phantom projective dimension*. There is a theory in equal characteristic zero containing analogues of many of the results of [AB1] and [AHH], but we shall not do a detailed study here.

(4.5) ITERATED OPERATIONS AND CONSTRAINTS ON PARAMETERS

In this section we discuss results of the following type: Suppose that one has ideals that are either generated by monomials in parameters or else are expanded from a regular ring R to a ring S , where $R \rightarrow S$ either is module-finite or preserves heights sufficiently well. Suppose further that one performs a sequence of operations on these ideals, which might include taking colon ideals and intersections. Under suitable hypotheses, it turns out that the result of performing the operations is in the tight closure of an ideal that can be constructed in a simple way: in the monomial case, in the tight closure of the monomial ideal one would get if the parameters were actually a permutable R -sequence, while in the case of ideals expanded from a regular ring, in the tight closure of the expansion of the ideal obtained by performing the same operations in the regular ring. This theory is parallel to the one developed in §7 of [HH4].

Throughout the rest of this section we fix a field K of characteristic 0, and we assume that given rings are K -algebras. Moreover, tight closure will be assumed to be K -tight closure.

(4.5.1) Discussion and definition: admissible and nearly admissible functions of ideals. Let U_1, U_2, U_3, \dots denote variable ideals. We define recursively a class of ideal-valued functions of several ideals, which we refer to as the *admissible functions*, as follows. The projections sending (U_1, \dots, U_k) to U_i , $1 \leq i \leq k$, are admissible, and if \mathcal{F}, \mathcal{G} are admissible then so are the sum $\mathcal{F} + \mathcal{G}$, the product, $\mathcal{F}\mathcal{G}$, the intersection, $\mathcal{F} \cap \mathcal{G}$, as well as the function \mathcal{F}^{*K} (this assigns to (U_1, \dots, U_k) the K -tight closure of $\mathcal{F}(U_1, \dots, U_k)$), and the function $\mathcal{F} : U_j$.

Thus, the functions defined by $\mathcal{F}(U_1, U_2, U_3) = (U_1 + U_2)^{*K} : U_3$ and

$$\mathcal{G}(U_1, U_2, U_3, U_4, U_5, U_6) = (U_1 : U_2) + (U_3 : U_4)^{*K}(U_5 : U_6)$$

are admissible. This definition differs from the one given in §7 of [HH4] in allowing K -tight closures. Note that $U_1 : (U_2 + U_3)$ is not *a priori* admissible, but can be rewritten as $(U_1 : U_2) \cap (U_1 : U_3)$, which is. Similarly, $U_1 : (U_2 U_3)$ is not *a priori* admissible, but can be rewritten as $(U_1 : U_2) : U_3$, which is. However, $U_1 : (U_2 \cap U_3)$ is not admissible.

We define *nearly admissible* ideal-valued functions of ideals recursively as follows. All admissible functions are nearly admissible, and if \mathcal{F} and \mathcal{G} are nearly admissible then so are $\mathcal{F} + \mathcal{G}$, $\mathcal{F}\mathcal{G}$, $\mathcal{F} \cap \mathcal{G}$, $(\mathcal{F})^{*K}$, and, *provided as well* that \mathcal{G} is admissible, $\mathcal{F} : \mathcal{G}$ is nearly admissible.

Notice that in the definition of “admissible” only the variables themselves are allowed in the denominator of the colon operator. In the definition of “nearly admissible” all admissible functions are allowed in the denominator of the colon operator, but not functions that are only nearly admissible. The notion defined here again differs from that defined in §7 of [HH4] in allowing K -tight closures. Thus,

$$U_1 : (U_2 \cap U_3) + U_4 : (U_5 \cap U_6)$$

is nearly admissible, but

$$U_7 : (U_1 : (U_2 \cap U_3) + U_4 : (U_5 \cap U_6))$$

is not, because in the latter the denominator, while nearly admissible, is not admissible.

(4.5.2) Definition: trapped ideals. Let $A \rightarrow R$ be a map of K -algebras. We shall say that an ideal J of R is *trapped* over an ideal I of A if $IR \subseteq J \subseteq (IR)^{*K}$.

The main result of this section is the following theorem, which greatly generalizes Theorem (4.1.7).

(4.5.3) Theorem. *Let K be a field of characteristic zero and let $A \rightarrow R$ be a homomorphism of Noetherian K -algebras such that A is regular. Suppose either that*

- (1) A is the ring $\mathbb{Q}[x_1, \dots, x_n]$, \mathcal{I} is the set of all ideals of A generated by monomials in the variables x_1, \dots, x_n , and that for every integer h , $1 \leq h \leq n$, every h element subset of x_1, \dots, x_n consists of strong parameters³⁹: this is equivalent to the hypothesis that every such subset generates an ideal of formal minheight⁴⁰ at least h in R or
- (2) A is any regular ring, \mathcal{I} is the class of all ideals of A , and for every complete local domain S of R at a maximal ideal,⁴¹ if P is the contraction of the maximal ideal of S to A then the height of PS is at least the height of P . ideal I of A the formal minheight of IR is at least the height of I .

Let \mathcal{F} be a nearly admissible function of k ideals. Then for any k ideals I_1, \dots, I_k in \mathcal{I} , $\mathcal{F}(I_1R, \dots, I_kR) \subseteq (\mathcal{F}(I_1, \dots, I_k)R)^{*K}$, and if \mathcal{F} is, moreover, admissible, then $\mathcal{F}(I_1R, \dots, I_kR)$ is trapped over $\mathcal{F}(I_1, \dots, I_k)$, i.e.,

$$\mathcal{F}(I_1, \dots, I_k)R \subseteq \mathcal{F}(I_1R, \dots, I_kR) \subseteq (\mathcal{F}(I_1, \dots, I_k)R)^{*K}.$$

Proof. We first consider the result for admissible functions. It is immediate from the recursive nature of the definition that the theorem follows by induction (on the number of steps needed to construct the function in question recursively) provided that we can show the that if $J \subseteq R$ is trapped over I and $J' \subseteq R$ is trapped over I' , where I, I' are in \mathcal{I} , then

- (a) $J + J'$ is trapped over $I + I'$,
- (b) JJ' is trapped over II' ,
- (c) $J \cap J'$ is trapped over $I \cap I'$,
- (d) J^{*K} is trapped over I , and
- (e) $J' :_R IR$ is trapped over $I' :_A I$.

Parts (a), (b) and (d) are immediate from parts (f), (g) and (e), respectively, of Theorem (3.2.3). It remains to prove parts (c) and (e). The inclusions $(I \cap I')R \subseteq J \cap J'$ and

³⁹See (2.3.10).

⁴⁰If R is universally catenary formal minheight coincides with minheight, and if R is locally equidimensional as well, formal minheight coincides with height, and this condition is much less technical.

⁴¹See (2.3.10).

$(I' :_A I)R \subseteq J' :_R IR$ are obvious. To complete the proof, it will suffice to show that if I and I' are in \mathcal{I} then

- (f) $(IR)^{*K} \cap (I'R)^{*K} \subseteq ((I \cap I')R)^{*K}$ and
- (g) $(I'R)^* :_R IR \subseteq ((I' :_A I)R)^{*K}$.

Let u be an element of the ideal on the left hand side. We must show that u is in the K -tight closure of $(I \cap I')R$ (respectively, $(I' :_A I)R$). It suffices to prove this after we replace R by a complete local domain of R at a maximal ideal. The hypothesis (1) or (2) is preserved. Thus, we may assume that R is a complete local domain.

We first consider the situation under hypothesis (2). We replace A first by the completion of its localization at the contraction of the maximal ideal of R . We extend a coefficient field $L \subseteq K$ for A to a coefficient field L' for R . We extend a regular system of parameters x_1, \dots, x_d for A to a system of parameters x_1, \dots, x_n for R . We may then replace $A \cong L[[x_1, \dots, x_d]]$ by its flat extension $A' = L'[[x_1, \dots, x_n]]$, and the ideals I, I' by their extensions to A' . We may thus assume that $A \rightarrow R$ is a module-finite extension of a complete regular local ring, and that R is a domain. We then can view $A \rightarrow R$ as a direct limit of maps $A_0 \rightarrow R_0$ where A_0 is a smooth affine K -algebra and R_0 is a module-finite domain extension of A_0 . The result now follows, after descent, from the characteristic p case.

The proof under the hypothesis (1) is similar. Some of the elements x_i may become invertible when we pass to the complete local domain of R at a maximal ideal. We can then study the problem after “removing” any power of such a variable from any monomial in which it occurs. The x_i that do not become invertible will be part of a system of parameters for the complete local domain. This may be extended to a full system of parameters, and we may reduce to the case of a module-finite domain extension of a regular ring with the x_i as a regular system of parameters, which follows from (2).

Now consider the case of nearly admissible functions. The same argument proves the result. Note that in the case of a colon, one needs that if J is trapped over I and $J' \subseteq (I'R)^{*K}$ then $J' :_R J \subseteq ((I' :_A I)R)^{*K}$. This is true because, since $J \supseteq IR$, $J' :_R J \subseteq J' :_I R$. \square

(4.6) BIG EQUATIONAL TIGHT CLOSURE

In this section we discuss briefly one more notion of tight closure for Noetherian rings containing the rationals. It is *a priori* larger than any of the K -tight closures discussed previously (which are defined when the ring is a K -algebra). We do not know whether this notion is truly distinct from equational tight closure. If not, then proving that all ideals are tightly closed in a ring in the sense of big equational tight closure gives additional information.

(4.6.1) Discussion, notation, and the definition of big equational tight closure

Let A denote a finitely generated \mathbb{Z} -algebra and Q a prime disjoint from \mathbb{Z}° . Let $N_A \subseteq M_A$ be finitely generated A -modules. We shall, momentarily, define a closure operation, denoted $*/(A, Q)$, on $N_A \subseteq M_A$. It will turn out to be unaffected by localization at nonzero elements of $\mathbb{Z} \subseteq A$, and also unaffected by localization at elements of $A - Q$, so that one can really think of it as a closure operation on submodules of a finitely generated module over A_Q , which is a typical local ring of a finitely generated \mathbb{Q} -algebra.

To define $*/(A, Q)$, first localize at an element of \mathbb{Z}° so that A/Q is \mathbb{Z} -smooth (but we do not change notation), and let W_p , where p is a positive prime integer, denote the multiplicative system of all nonzerodivisors on $Q(A/pA)$ in A/pA . Let $B(p) = W_p^{-1}(A/pA)$, and let $v \in (N_A^{*/(A, Q)})_{M_A}$ if for almost all p $v_{B(p)} \in \langle N_{AB(p)} \rangle_{M_{AB(p)}}^*$.

Now let R be a Noetherian \mathbb{Q} -algebra, let $N \subseteq M$ be finitely generated R -modules and let $u \in M$. We define u to be in $N^{>*EQ}$ (the *direct* big equational tight closure of N in M) if for every local ring S of R (so that S is the localization of R at some prime ideal), there is a map $A_Q \rightarrow S$ for some finitely generated \mathbb{Z} -algebra A and prime Q and there exist A -modules $N_A \subseteq M_A$ and an element $u_A \in M_A$ satisfying

- (1) the inclusion $S \otimes_A N \subseteq S \otimes_A M_A$ is isomorphic with the inclusion $S \otimes_R N \subseteq S \otimes_R M$ in such a way that $1 \otimes u_A$ corresponds to $1 \otimes u$ and
- (2) $u_A \in (N_A^{*/(A, Q)})_{M_A}$.

Thus, roughly speaking, any instance where $u \in N_M^{>*EQ}$ arises because, locally, it is the result of a base change from an instance of an element being in a $*/(A, Q)$ closure.

Finally, if R is any Noetherian ring, $N \subseteq M$ are finitely generated R -modules, and $u \in M$, we define $u \in N_M^*$ if for every complete local domain B of R , the image of $1 \otimes u$ in $B \otimes_R M$ is in the direct biq equational tight closure of the image of $B \otimes_R N$ in $B \otimes_R M$ over B .

Virtually all of the theory developed for $*K$ -tight closure has parallel results for big equational tight closure. However, we shall not give a detailed development here.

However, we do note the following analogue of Theorem (4.1.1) which shows that the big equational tight closure is still rather small.

(4.6.2) Theorem. *Let R any regular Noetherian ring and $N \subseteq M$ finitely generated modules. Then $N_M^{*EQ} = N$.*

Proof (sketch). First, if u were in $N_M^{*EQ} - N$ we could preserve this while localizing and completing. Thus, we need only consider the case where R is a complete regular local ring, say $R = K[[x_1, \dots, x_n]]$. Exactly as in the proof of Theorem (4.1.1) we may reduce to studying the case where $N = (x_1^t, \dots, x_n^t)R$ for some positive integer t and $M = R$, and we need only be concerned about whether $(x_1 \cdots x_n)^{t-1} \in (x_1^t, \dots, x_n^t)^{*EQ}$.

Suppose this happens for a certain $A \rightarrow R$ and prime Q of A . Use the Artin-Rotthaus theorem to factor $A_Q \rightarrow R$ through an affine ring $B_{Q'} \rightarrow R$ such that B is and elements y_i lifting the x_i are part of a regular system of parameters for B_Q . By the persistence of tight closure in characteristic p for algebras essentially of finite type over a field, one will have that for almost all p there will be a regular ring of characteristic p containing the images of the y_i as part of a system of parameters such that $(y_1 \cdots y_n)^{t-1} \in (y_1^t, \dots, y_n^t)^* = (y_1^t, \dots, y_n^t)$, which gives the desired contradiction. \square

We already know that $N^{*eq} \subseteq N^{*K}$ when the latter makes sense.

(4.6.3) Theorem. *Let K be a field of characteristic zero and let R be a Noetherian K -algebra. Let $N \subseteq M$ be finitely generated R -modules. Then $N^{*K} \subseteq N^{*EQ}$.*

Sketch of the proof. Let $u \in N^{*K}$. We must show that $u \in N^{*EQ}$. By virtue of the definition this immediately reduces to the case where R has been replaced by one of the

complete local domains of R . By choosing an affine progenitor mapping to R we may reduce to the case where R is a domain finitely generated over K , $N \subseteq M$ are finitely generated R -modules, and $u \in M$ is such that $u \in N_M^*$ in the affine K -algebra sense. Then we can find an affine \mathbb{Z} -subalgebra A of R and descent data (A, R_A, M_A, N_A, u_A) as in Chapter 2.

We have $N_A \subseteq M_A$, $u_A \in M_A$, and we know that $u_\kappa \in N_\kappa^*$ for almost all closed fibers κ . Localize A so much that $u_\kappa \in N_\kappa^*$ for all closed fibers. Pick c_A to be a universal test element for $A \rightarrow R_A$. Localize A so much that if we map A to any regular ring T of characteristic p , c_T is a test element in R_T . Also localize \mathbb{Z} so much (call the localization $\tilde{\mathbb{Z}}$) that $\tilde{\mathbb{Z}} \rightarrow A$ is faithfully flat.

We want to show that $u = u_K$ is in $N_K^{*\text{EQ}}$ over $R = R_K$, and it suffices to show that if Q is a prime ideal of R_A disjoint from A° then $u_A \in R_A^{*/(R_A, Q)}$, i.e., for almost all $p \in \mathbb{Z}$, we have that $u_{B(p)} \in N_{B(p)}^*$. Since the rings $B(p)$ are regular, if the condition we want fails then for some p we can choose q such that $c_{B(p)}u^q$ is not in $N_{B(p)}^{[q]}$ in some fiber over \mathbb{Z} . This implies that $c_{A/(p)}u^q$ is not in $N_{A/(p)}^{[q]}$ and also is not a W_p -torsion element in $\mathbf{F}^e(M_{A/(p)})/N_{A/(p)}^{[q]}$. This implies that we can localize at one element of $(A/(p))^\circ$ and make all the modules in the sequence

$$0 \rightarrow c_{A/(p)}u^q \rightarrow \mathbf{F}^e(M_{A/(p)}/N_{A/(p)}) \rightarrow D \rightarrow 0$$

free over $A/(p)$, with the first one nonzero. But then this is preserved when we kill a maximal ideal of $A/(p)$, yielding $c_\kappa u_\kappa^q \notin \mathbf{F}^e(M_\kappa/N_\kappa)$, a contradiction. \square

We note that in [Ho8], Theorem (11.4) it is shown that the solid closure contains the big equational tight closure for Noetherian rings R of equal characteristic zero.

APPENDIX

QUESTIONS

We give here a list of open questions connected with tight closure theory that we consider intriguing.

Throughout these questions, unless otherwise specified, R is a locally excellent Noetherian ring and $N \subseteq M$ are finitely generated R -modules. K, L always denote fields and $K \subseteq L$. If R is local then it has maximal ideal m and residue field K .

1. Does tight closure commute with localization in characteristic p ? In characteristic zero? (Cf. [AHH] for a thorough discussion of the localization problem.)
2. Is it true that weakly F -regular rings are F -regular? This is an open question in characteristic p and in characteristic zero. It is known in the Gorenstein case (and in characteristic p in dimension at most three, using the results of [Wil]: cf. 4. below), but is open in characteristic p even if R is an affine algebra over an algebraically closed field.

Over a Noetherian ring R of characteristic p , if M is a finitely generated R -module we shall denote by $\mathbf{G}^e(M)$ the module $\mathbf{F}^e(M)/0^*$, where 0^* is taken in $\mathbf{F}^e(M)$.

3. If R is complete local, is there a positive constant integer b such that for all $e \in \mathbb{N}$, m^{bp^e} kills $H_m^0(\mathbf{F}^e(M))$? Is this true even when R is a complete local weakly F -regular ring and $M = R/I$, where I is primary to a prime P such that $\dim R/P = 1$? An affirmative answer to the second question would yield that weakly F -regular implies F -regular for locally excellent rings R . (Cf. [AHH].) One may ask the same question for $H_m^0(\mathbf{G}^e(M))$, and an affirmative answer has the same consequence. Arguments like this are given in §7 of [AHH].
4. In characteristic p , is every F -finite weakly F -regular ring strongly F -regular? (Then weakly F -regular would imply F -regular even without the hypothesis F -finite.) This is known:

- (1) in the Gorenstein case and
- (2) if $\dim R \leq 3$ (cf. [Wil])
- (3) if R has a canonical module that represents a torsion element of the divisor class group except at isolated points (cf. [MacC]).

5. Let R be a complete local Cohen-Macaulay domain and let J be an ideal of R that is isomorphic as a module with a canonical module for R . Fix a system of parameters x_1, \dots, x_n for R . For every $q = p^e$ and every positive integer t let

$$\theta_{e,t}: J^{[q]}/(x_1^q, \dots, x_n^q)J^{[q]} \rightarrow J^{[q]}/(x_1^{qt}, \dots, x_n^{qt})J^{[q]}$$

be the map induced by multiplication by $(x_1 \cdots x_n)^{qt-q}$ on $J^{[q]}$. Can one always choose a system of parameters x_1, \dots, x_n and a positive integer t_0 such that $\text{Ker } \theta_{e,t}$ is the same for all $t \geq t_0$? For fixed e , these kernels increase as t increases. Note that t_0 is to be independent of e . This and related problems are studied in [Wil], [Kat], and [MacC]. An affirmative answer in the special case where R is F -finite and weakly F -regular would suffice to show that weakly F -regular is equivalent to strongly F -regular for F -finite rings, and that weakly F -regular is equivalent to F -regular for locally excellent rings of characteristic p .

6. Is the weakly F -regular locus open? (This is an open question for affine rings over algebraically closed fields both in characteristic p and characteristic 0.) This would follow for algebras essentially of finite type over an excellent local ring of characteristic p if weakly F -regular F -finite rings are strongly F -regular, because the strongly F -regular locus is known to be open.
7. Suppose that R has characteristic p . Let $T = \bigcup_e \text{Ass}(\mathbf{G}^e(M)/0^*)$ (see the discussion before Question 3.). Is T finite? Does it have only finitely many maximal elements? An affirmative answer would reduce the question of whether tight closure commutes with localization to the case where R is complete local and one is localizing at a prime P with $\dim R/P = 1$. Cf. [Kat], [AHH]. In [Kat] it is shown that $\bigcup_e \text{Ass}(\mathbf{F}^e(M)/0^*)$ need not have only finitely many maximal elements.
8. Is characteristic p tight closure for an ideal I in a locally excellent domain R the same as $IR^+ \cap I$ (called the *plus closure* of I)? This is known for parameter ideals [Sm1, 2] (and

the result of [Sm2] coupled with a result of [Ab3] shows that the answer is affirmative for $N \subseteq M$ whenever M/N has a finite phantom projective resolution). An affirmative answer implies that tight closure commutes with localization. Whether tight closure is the same as plus closure is an open question even in dimension two, in fact, even if the ring is a cubical cone $K[X, Y, Z]/(X^3 + Y^3 + Z^3)$. Cf. [McD]. Also see Question **22**.

A closely related question is this: in characteristic p , in a ring with a weak completely stable test element, is every instance of tight closure the result of an instance in a algebra of finite type over $\mathbb{Z}/p\mathbb{Z}$ and the persistence of tight closure? (I.e., if $u \in N^*_M$ over S is there a characteristic p affine progenitor (R, M_0, N_0, u_0) such that R is a finitely generated $(\mathbb{Z}/p\mathbb{Z})$ -algebra and $u_0 \in N_0^*_{M_0}$.) This reduces to the case of a complete local ring.

9. When does weak F -regularity deform, i.e., if R is a local domain, $x \neq 0$ and R/xR is weakly F -regular, under what conditions must R be weakly F -regular? Some conditions are needed: this was shown in [Si7]. (Deformation does hold when R is Gorenstein.)
10. For affine algebras in characteristic 0, does weakly F -regular imply weakly F -regular type? Does F -rational imply F -rational type?
11. Let K have characteristic 0, let A be a finitely generated \mathbb{Z} -subalgebra of K , and let (A, R_A, M_A, N_A) be descent data for (K, R, M, N) . Suppose that N is K -tightly closed in M . Is it true that for almost all closed fibers, N_κ is tightly closed in M_κ ?
12. Is tight closure over a local ring, in all characteristics, simply the contracted expansion from a higher level balanced big Cohen-Macaulay algebra? (See also question 8. above: note that in the case of an excellent local ring R of characteristic p , R^+ is a balanced higher level big Cohen-Macaulay algebra for R .) Cf. [Ho8], [Ho9], and [HH11].
13. If R is an affine L -algebra, does $N^{*K}_M = N^{*L}_M$? More generally, if R is an arbitrary Noetherian ring containing \mathbb{Q} does $N^{*\text{eq}}_M = N^{*\text{EQ}}_M$? (It would suffice to know this when R is a local ring of a finitely generated \mathbb{Q} -algebra.)
14. If a flat homomorphism of rings $R \rightarrow S$ has an F -regular base R and geometrically F -regular fibers, is S F -regular? This is known in good cases in characteristic p if the fibers are geometrically regular. (Cf. [HH9].)

15. If R is excellent, reduced, characteristic p , and of finite Krull dimension, does R have a test element? (Cf. [Ab2].)
16. Under mild conditions on a characteristic p ring R (e.g., if R is reduced and finitely generated as an algebra over an excellent local ring) does formation of the ideal of test elements commute with localization? With completion? With geometrically regular base change (where *geometrically regular* means flat with geometrically regular fibers)?
17. Let R be a characteristic p local ring such that one system of parameters generates a tightly closed ideal. Is R F -rational? This is known ([HH9] §4) if R is equidimensional. Some positive results are given in [Sgh].
18. The following are open questions both in characteristic p and in characteristic zero. If a module has finite phantom projective dimension locally does it have finite phantom projective dimension globally? Does a direct summand of a module of finite phantom projective dimension have finite phantom projective dimension? Is it possible to characterize having finite phantom projective dimension without referring to a specific phantom resolution? Over an affine algebra of characteristic zero, if a module has finite phantom projective dimension over almost all closed fibers after choosing descent data, does it have a finite phantom resolution in equal characteristic zero? Cf. [Ab1], [AHH].
19. Let R be an F -finite Noetherian domain of characteristic p . When is it true that for some q , there is an R -linear map $\theta: R^{1/pq} \rightarrow R^{1/q}$ sending 1 to 1? When is it true that for every element $d \in R^\circ$ (or at least for one test element d) there is some q and an R -linear map $\phi: R^{1/q} \rightarrow R^{1/pq}$ such that $\phi(d^{1/q}) = d^{1/pq}$.

The existence of q and θ implies that $u \in M$ is in the Frobenius closure of N in M iff $u^q \in N^{[q]}$ (in $\mathbf{F}^e(M)$). The existence of q and ϕ for a test element d implies, for $u \in M$, that $u \in N^*_M$ iff $du^q \in \mathbf{F}^e(N)$. Thus, the usual infinite family of tests one needs to check whether an element is in a tight closure is replaced by a single test.

If ϕ exists for some test element d and some q one can easily show that tight closure commutes with localization over R .

These ideas are explored in [McD].

20. Under mild hypotheses on a Noetherian ring R containing \mathbb{Q} , is it true that if R_c is

regular than c has a fixed power that kills N_M^*/N for every pair of finitely generated modules M, N over R with $N \subseteq M$? There are versions of this question for each form of tight closure in equal characteristic zero. One case of great interest is that where R is essentially of finite type over an excellent local ring.

21. Let R_A be a finitely generated algebra over a Noetherian domain A of positive characteristic p , and suppose that the generic fiber $L \rightarrow R_L$ is separable, where L is the fraction field of A . If R_L is weakly F -regular (respectively, F -regular) does the same hold for almost every fiber? One may raise a corresponding question for tight closure of ideals: if I_A is an ideal of R_A such that I_L is tightly closed in R_L , does the same hold for almost all fibers? This is proved for certain ideals in [McC], and it is also shown that if R_{L^∞} is strongly F -regular, then for almost all fibers $\kappa \rightarrow R_\kappa$, R_{κ^∞} is strongly F -regular.
22. Let R be a locally excellent K -algebra, where K has characteristic 0. Is every K -tightly closed ideal an intersection of K -tightly closed ideals that are primary to maximal ideals? Likewise, if N is K -tightly closed in a finitely generated R -module M , is N the intersection of K -tightly closed submodules N' of M such that M/N' has finite length?
23. Let $R = K[X, Y, Z]/(X^3 + Y^3 + Z^3) = K[x, y, z]$. If K has characteristic congruent to 2 modulo 3, is tight closure in this ring the same as Frobenius closure?
24. Let R be a Noetherian ring of positive characteristic p . Under mild conditions on R , and, in particular, if R is essentially of finite type over an excellent local ring, is it true that every test element is completely stable?
25. Let A be a finitely generated algebra over the integers and let R_A be a reduced, finitely generated A -algebra. Let I_A denote the ideal of R generated by the universal test elements for R (cf. (2.4.2)). Is it true that for almost all closed fibers $\kappa \rightarrow R_\kappa$ of $A \rightarrow R_A$, the image I_κ of I_A in R_κ is the test ideal for R_κ ?

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