

[Preliminary Version]

TEST EXPONENTS FOR MODULES WITH FINITE PHANTOM PROJECTIVE DIMENSION

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ABSTRACT. Let (R, \mathfrak{m}) be an equidimensional excellent local ring of prime characteristic p . We give an alternate proof of the existence of a test exponent for any given $c \in R^\circ$ and all ideals generated by (full or partial) systems of parameters. This follows from a more general result about the existence of a test exponent for any given Artinian R -module. If we further assume R is Cohen-Macaulay, then we show that there exists a test exponent for any given $c \in R^\circ$ and all finite length modules with finite projective dimension. We then apply the results to F -rational rings and relate the results to a *new* kind of F -rational signature of R .

0. INTRODUCTION

Throughout this paper R is a Noetherian ring of prime characteristic p . By (R, \mathfrak{m}, k) , we indicate that R is a local ring with maximal ideal \mathfrak{m} and residue field $R/\mathfrak{m} = k$.

Also, we always use $q = p^e, Q = p^E, q_0 = p^{e_0}, q' = p^{e'}, q'' = p^{e''}$, etcetera, to denote varying powers of p with $e, E, e_0, e', e'' \in \mathbb{N}$.

Let M be an R -module. Then for any $e \geq 0$, we can derive a left R -module structure on the set M by $r \cdot m := r^{p^e} m$ for any $r \in R$ and $m \in M$. For technical reasons, we keep the original right R -module structure on M by default. We denote the derived R - R -bimodule by ${}^e M$. Thus, in ${}^e M$, we have $r \cdot m = m \cdot r^{p^e}$, which is equal to $r^q m$ in the original M . If R is reduced, then ${}^e R$, as a left R -module, is isomorphic to $R^{1/q} := \{r^{1/p} \mid r \in R\}$. We use $\lambda^l(-), \lambda^r(-)$ to denote the left and right lengths of a bimodule. It is easy to see that $\lambda^l({}^e M) = q^{\alpha(R)} \lambda^r({}^e M) = q^{\alpha(R)} \lambda(M)$ for any finite length module M over (R, \mathfrak{m}, k) , in which $\alpha(R) = \log_p[k : k^p]$.

We say that R is F -finite if ${}^1 R$ (or, equivalently, ${}^e R$ for all e) is finitely generated as an left R -module.

For any R -module M and e , we can always form a new R -module $F^e(M)$ by scalar extension via $F^e : R \rightarrow R$ by $r \mapsto r^q$. In other words, $F^e(M)$ has the R -module structure that is determined by the right R -module structure of $M \otimes_R {}^e R$; and it is this R -module structure of $F^e(M)$ that we mean unless otherwise specified. If $h \in \text{Hom}_R(M, N)$, then we correspondingly have $F^e(h) : \text{Hom}_R(F^e(M), F^e(N))$. Sometimes, especially when both M and N are free, we may write $F^e(h)$ as $h^{[q]}$.

A very important concept in studying rings of characteristic p is tight closure. Tight closure was first studied and developed by Hochster and Huneke in the 1980's.

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Definition 0.1 (Hochster-Huneke, [HH1]). Let R be a Noetherian ring of prime characteristic p and $N \subseteq M$ be R -modules. The tight closure of N in M , denoted by N_M^* , is defined as follows: An element $x \in M$ is said to be in N_M^* if there exists an element $c \in R^\circ$ such that $x \otimes c \in N_M^{[q]} \subseteq M \otimes_R {}^eR$ for all $e \gg 0$, where R° is the complement of the union of all minimal primes of the ring R and $N_M^{[q]}$ denotes the (right) R -submodule of $F_R^e(M) = M \otimes_R {}^eR$ generated by $\{x \otimes 1 \in M \otimes_R {}^eR \mid x \in N\}$. The element $x \otimes 1 \in M \otimes_R {}^eR$ is denoted by $x_M^{p^e} = x_M^q$. (By our convention on $F_R^e(M)$, we have $cx_M^q = x \otimes c \in N_M^{[q]}$.)

Definition 0.2 ([HH2]). Let R be a Noetherian ring of prime characteristic p , $q_0 = p^{e_0}$ and let $N \subseteq M$ be R -modules. We say $c \in R^\circ$ is a q_0 -weak test element for $N \subseteq M$ if $c(N_M^*)^{[q]} \subseteq N_M^{[q]}$ for all $q \geq q_0$. In case $N = 0$, we may simply call it a test element for M . By a q_0 -weak test element, we simply mean a q_0 -weak test element for all R -modules. If a q_0 -weak test element c remains a q_0 -weak test element under every localization, then we call c a locally stable q_0 -weak test. Finally, in case $q_0 = 1$, we simply call c a test element or locally stable test element.

Definition 0.3 ([HH3]). Let R be a Noetherian ring of prime characteristic p , $c \in R$, and $N \subseteq M$ (finitely generated) R -modules. We say that $Q = p^E$ is a test exponent for c and $N \subseteq M$ (over R) if, for any $x \in M$, the occurrence of $cx^q \in N_M^{[q]}$ for one single $q \geq Q$ implies $x \in N_M^*$. In case $N = 0$, we may simply call it a test exponent for c and M .

Remark 0.4. (1) It is easy to check the following statements: To say $c \in R^\circ$ is a test element for $N \subseteq M$ is the same as to say c is a test element for $(0 \subseteq) M/N$. Similarly, to say $Q = p^E$ is a test exponent for c and $N \subseteq M$ is the same as to say Q is a test exponent for c and $(0 \subseteq) M/N$.

(2) However, by ‘a (q_0 -weak) test element for an ideal I ’, we usually mean ‘a (q_0 -weak) test element for $I \subseteq R$ ’ rather than ‘a (q_0 -weak) test element for $0 \subseteq I$ ’. Similarly, when we say ‘a test exponent for c and an ideal I ’, we usually mean ‘a test exponent for c and $I \subseteq R$ ’ rather than ‘a test exponent for c and $0 \subseteq I$ ’.

Under mild conditions, test elements exist.

Theorem 0.5. *Let R be F -finite or essentially of finite type over an excellent local ring (A, \mathfrak{n}) of characteristic p . Say $\sqrt{0}^{[q_0]} = 0$, where $\sqrt{0}$ is the nilradical of R .*

- (1) *There exists a completely stable q_0 -weak test element for all finitely generated R -modules. (See [HH2].)*
- (2) *In fact, there exists a completely stable q_0 -weak test element for all (not necessarily finitely generated) R -modules. (It suffices to prove the case where R (and hence A) is reduced. Under the assumption that R is F -finite, this was proved in the thesis of Haggai Elitzur, [E]. From this we can see the remaining case via a faithfully flat extension $R \rightarrow R \otimes_A \widehat{A} \rightarrow R \otimes_A \widehat{A}^\Gamma$, where \widehat{A} is the \mathfrak{n} -adic completion of A and \widehat{A}^Γ is a suitable (F -finite) Γ -extension of \widehat{A} . See [HH2] for details about Γ -extensions.)*

If there exists a test exponent for a locally stable test element $c \in R^\circ$ and (finitely generated) R -modules $N \subseteq M$, then the tight closure of N in M commutes with localization. This result is implicit in [McD] and is explicitly stated in [HH3, Proposition 2.3]. Moreover, Hochster and Huneke showed in [HH3] that the converse is true as below.

Theorem 0.6 ([HH3]). *Let R be a Noetherian ring of prime characteristic p with a given locally stable test element c , and $N \subseteq M$ finitely generated R -modules. Assume that the tight closure of N in M commutes with localization. Then there exists a test exponent for c and $N \subseteq M$.*

In [HH3], Hochster and Huneke asked, among other questions, whether there exists a uniform test exponent for a given test element and all ideals generated by systems of parameters. This question has been recently answered positively by R. Y. Sharp.

Theorem 0.7 (Sharp, [Sh, Theorem 3.2]). *Let (R, \mathfrak{m}) be an equidimensional excellent local ring of prime characteristic p and $c \in R^\circ$. Then there exists a test exponent for c and all ideals generated by (partial or full) systems of parameters of R .*

In Theorem 2.4, we use the Artinian property of $H_{\mathfrak{m}}^{\dim(R)}(R)$ and colon-capturing to give an alternate proof of the above Theorem 0.7.

Inspired by Sharp's result, we then naturally ask whether there is a uniform test exponent for a given $c \in R^\circ$ and all finitely generated R -modules with (finite length and) finite phantom projective dimension. While this question remains unsettled, we can give an affirmative answer in case R is Cohen-Macaulay or in case $\dim(R) \leq 2$. Throughout this paper, we use $\lambda(M)$ and $\text{ppd}(M)$ to denote the length and phantom projective dimension of an R -module M respectively.

Theorem (Corollary 3.3, Corollary 3.4). *Let (R, \mathfrak{m}) be an equidimensional Noetherian excellent local ring of prime characteristic p . Assume either that R is Cohen-Macaulay or $\dim(R) \leq 2$. Then, for any $c \in R^\circ$, there is a test exponent for c and all R -modules M with $\lambda(M) < \infty$, $\text{ppd}(M) < \infty$.*

For any finitely generated R -modules L, M such that $\lambda(L) < \infty$, we denote

$$e_{HK}(L, M) = \lim_{q \rightarrow \infty} \lambda(F^e(L) \otimes_R M) / q^{\dim(R)} = \lim_{q \rightarrow \infty} \lambda^r(L \otimes_R {}^eM) / q^{\dim(R)},$$

whose existence is ensured by a result of G. Seibert in [Se]. In case $L = R/I$, we often denote $e_{HK}(R/I, M)$ by $e_{HK}(I, M)$ in spite of the remote possibility of confusion. It is easy to see that $e_{HK}(I, M) = \lim_{q \rightarrow \infty} \lambda(M/I^{[q]}M) / q^{\dim(R)}$ and the existence of $e_{HK}(I, M)$ was first proved by P. Monsky in [Mo].

Next we connect the above results about uniform test exponents for all modules of finite length and of finite phantom projective dimension to results about *the F -rational signature* of R . Here we mention that $r_R(M)$ has been defined and studied in [HY]. Throughout this paper, s.o.p. is short for "system of parameters."

Definition 0.8. Let (R, \mathfrak{m}) be a local ring of prime characteristic p and M a finitely generated R -module.

(1) Define (see [HY])

$$r_R(M) = \inf\{e_{HK}((\underline{x}), M) - e_{HK}(J, M) \mid \underline{x} \text{ is a s.o.p. for } R \text{ and } (\underline{x}) \subsetneq J\}.$$

In particular, $r_R(R)$ is called the F -rational signature of R .

(2) In case (R, \mathfrak{m}) is such that $\text{ppd}(R/(\underline{x})) < \infty$ for every system of parameters \underline{x} of R (e.g., R is an equidimensional homomorphic image of a Cohen-Macaulay ring or R is an equidimensional excellent ring), we define

$$r'_R(M) = \inf\{e_{HK}(L, M) - e_{HK}(L/K, M) \mid \text{ppd}(L) < \infty, \lambda(L) < \infty, 0 \neq K \subseteq L\}.$$

Otherwise, we define $r'_R(M) = 0$. We call $r'_R(R)$ the phantom F -rational signature of R .

Theorem (Theorem 4.2). *Let (R, \mathfrak{m}) be a Noetherian local ring of prime characteristic p . Assume there exists a common weak parameter test element for R and \widehat{R} (e.g., R is excellent). Then R is F -rational $\iff \widehat{R}$ is F -rational $\iff r'(R) > 0 \iff r'(M) > 0$ for every (or, equivalently, some) finitely generated R -module with $\dim(M) = \dim(R)$.*

1. SOME PRELIMINARY RESULTS ABOUT TEST EXPONENTS

We first observe the following easy lemma about test exponents, although it is not directly used in the sequel.

Lemma 1.1. *Let R be a Noetherian ring of characteristic p . For any $b, c \in R^\circ$ and R -modules $N \subseteq M$, the following are true.*

- (1) *If Q is a test exponent for bc and $N \subseteq M$, then Q is a test exponent for c and $N \subseteq M$.*
- (2) *If, for some $q_0 = p^{e_0}$, Q is a test exponent for c^{q_0} and $N_M^{[q_0]} \subseteq F_R^{e_0}(M)$, then Q is a test exponent for c and $N \subseteq M$.*

Proof. (1). If $cx^q \in N_M^{[q]} \subseteq F_R^e(M)$ for some $x \in M$ and $p^e = q \geq Q$, then $bcx^q \in N_M^{[q]} \subseteq F_R^e(M)$ and hence $x \in N_M^*$.

(2). Suppose $cx^q \in N_M^{[q]} \subseteq F_R^e(M)$ for some $x \in M$ and $p^e = q \geq Q$. Then $c^{q_0}x^{q_0q} \in N_M^{[q_0q]} \subseteq F_R^{e_0+e}(M)$, or, in other words, $c^{q_0}(x^{q_0})^q \in (N_M^{[q_0]})_{F_R^{e_0}(M)}^{[q]} \subseteq F_R^e(F_R^{e_0}(M))$. This implies $x^{q_0} \in (N_M^{[q_0]})_{F_R^{e_0}(M)}^*$, which forces $x \in N_M^*$. \square

For simplicity, we state the next two results (i.e., Lemma 1.2 and Lemma 1.3) in terms of test exponent for c and $(0 \subseteq)M$ only. It is an easy task to give the corresponding statements in terms of test exponents for c and $N \subseteq M$.

Lemma 1.2. *Let R be a Noetherian ring of characteristic p with the set of minimal primes $\min(R) = \{P_1, P_2, \dots, P_r\}$ so that $\sqrt{0} = \bigcap_{i=1}^r P_i$. For any $c \in R^\circ$ (or simply $c \in R$) and any (finitely generated) R -module M , the following statements are true.*

- (1) *If Q is a test exponent for $c + P_i$ and M/P_iM over R/P_i for all $i = 1, 2, \dots, r$, then Q is a test exponent for c and M .*
- (2) *If Q is a test exponent for $c + \sqrt{0}$ and $M/\sqrt{0}M$ over $R/\sqrt{0}$, then Q is a test exponent for c and M .*

Proof. (1). Suppose $cx^q = 0 \in F_R^e(M)$ for some $x \in M$ and $p^e = q \geq Q$. Then, $(c + P_i)(x + P_iM)_{M/P_iM}^q = 0 \in F_{R/P_i}^e(M/P_iM)$, which implies $x + P_iM \in 0_{M/P_iM}^*$ for every $i = 1, 2, \dots, r$. This forces $x \in 0_M^*$ (see [HH1]).

(2). This follows similarly. \square

The next lemma deals with module-finite and pure ring extensions. In particular, the lemma applies to any reduced Nagata (e.g., excellent) ring and its integral closure in its total quotient ring.

Lemma 1.3. *Let $R \subseteq S$ be an extension of Noetherian rings of characteristic p , $c \in R$, and let M be a (finitely generated) R -module. Assume either (1) $R \subseteq S$ is module-finite, or (2) $R \subseteq S$ is a pure extension with a common weak test element in R . If Q is a test exponent for c and $0 \subseteq M \otimes_R S$ over S , then Q is a test exponent for c and $0 \subseteq M$.*

Proof. Suppose $cx^q = 0 \in F_R^e(M)$ for some $x \in M$ and $p^e = q \geq Q$. Then $c(x \otimes 1)^q = 0 \in F_S^e(M \otimes_R S)$ and hence $x \otimes 1 \in 0_{M \otimes_R S}^*$, which implies $x \in 0_M^*$. \square

The next lemma relies on the ‘colon-capturing’ property of tight closure, which is systematically studied in [HH1, Section 7].

Lemma 1.4. *Let (R, \mathfrak{m}) be a Noetherian local ring of characteristic p , $\dim(R) = d$, and $\underline{x} = x_1, x_2, \dots, x_d$ and $\underline{y} = y_1, y_2, \dots, y_d$ be two systems of parameters such that $(\underline{y}) \subseteq (\underline{x})$. For each $j = 1, 2, \dots, d$, say $y_j = \sum_{i=1}^d x_i a_{ij}$ with $a_{ij} \in R$. Denote the resulting $d \times d$ matrix $(a_{ij})_{d \times d}$ by A . Then*

$$(1) (\underline{y})^* :_R (\underline{x}) \supseteq (\det(A))^* \text{ and } (\underline{y})^* :_R \det(A) \supseteq (\underline{x})^*.$$

Further assume that (R, \mathfrak{m}) is equidimensional and, moreover, that R is either excellent or a homomorphic image of a Cohen-Macaulay ring. Then

$$(2) (\underline{y})^* :_R (\underline{x}) = ((\underline{y}) + (\det(A)))^* \text{ and } (\underline{y})^* :_R \det(A) = (\underline{x})^*.$$

(3) *For any $c \in R$, if Q is a test exponent for c and $(\underline{y}) \subseteq R$, then Q is a test exponent for c and $(\underline{x}) \subseteq R$.*

Proof. (1). This is straightforward (cf. [HH1, Proposition 4.1(b)(k)]).

To prove (2) and (3), we may assume (R, \mathfrak{m}) is an equidimensional homomorphic image of a Cohen-Macaulay ring without loss of generality. (Indeed, in case R is equidimensional and excellent, it suffices to prove (2) and (3) for \widehat{R} .)

(2). This can be proved similarly as [HH1, Theorem 7.9]. In case there is no explicit proof available in the literature, we mention that this can be proved, as usual, by lifting everything up to the Cohen-Macaulay ring of which R is a homomorphic image.

(3). Suppose $cx^q \in (\underline{x})^{[q]}$ for some $x \in R$ and $q \geq Q$. Then $c(\det(A)x)^q = \det(A)^q cx^q \in (\underline{y})^{[q]}$ and hence $\det(A)x \in (\underline{y})^*$, which implies $x \in (\underline{y})^* :_R \det(A) = (\underline{x})^*$ by part (2) above. \square

AN ALTERNATE PROOF OF SHARP'S THEOREM

We first show result about the existence of a test exponent for Artinian modules. Although the argument can be traced back to [HH3] (for modules of finite length), we include a proof here for the sake of convenience and completeness.

Proposition 2.1 (Compare with [HH3, Proposition 2.6]). *Let R be a Noetherian ring of prime characteristic p and $N \subseteq M$ be R -modules such that M/N is Artinian. Assume there exists $d \in R^\circ$ that is a q_0 -weak test element for $N_M^{[q]} \subseteq F_R^e(M)$ for all $q \gg 0$. Then, for any $c \in R^\circ$, there exists a test exponent for c and $N \subseteq M$.*

Proof. For every e , let $N_e = \{u \in M \mid cu^q \in (N_M^{[q]})_{F^e(M)}^F\}$. Then, as shown in the proof of [HH3, Proposition 2.6], $N_1 \supseteq N_2 \supseteq \cdots \supseteq N_e \supseteq N_{e+1} \supseteq \cdots \supseteq N$ and hence there exists $Q = p^E$ such that $N_e = N_E$ for all $e \geq E$.

Suppose $cx^{q'} \in N_M^{[q']}$ for some $x \in M$ and $q' \geq Q$. Then $x \in N_{e'}$ and thus $x \in N_e$ for all $e \geq E$. This means $cx^q \in (N_M^{[q]})_{F^e(M)}^F \subseteq (N_M^{[q]})_{F^e(M)}^*$ for all $q \geq Q$. Consequently, $dc^{q_0}x^{qq_0} = d(cx^q)^{q_0} \in (N_M^{[q]})_{F^e(M)}^{[q_0]} = N_M^{[qq_0]}$ for all $q \gg Q$, which implies $x \in N_M^*$. \square

In light of Theorem 0.5, we get the following consequence of Proposition 2.1.

Theorem 2.2. *Let R be an algebra essentially of finite type over an excellent local ring of characteristic p , $c \in R^\circ$, and M an Artinian R -module. Then there exists a test exponent for c and M .*

Proof. This follows immediately from Theorem 0.5(2) and Proposition 2.1. \square

We may refine Proposition 2.1 as follows when the Artinian R -module is the highest local cohomology.

Proposition 2.3. *Let (R, \mathfrak{m}) be a Noetherian local ring of prime characteristic p and $c \in R^\circ$. Assume (R, \mathfrak{m}) has the colon-capturing property and there exists a q_0 -weak test element $b \in R^\circ$ for all parameter ideals of R . Then there exists a test exponent for c and $0 \subset H_{\mathfrak{m}}^{\dim(R)}(R)$.*

Proof. Say $\dim(R) = d$. Then $H_{\mathfrak{m}}^d(R) = \varinjlim_{\underline{x}: \text{s.o.p.}} \frac{R}{(\underline{x})R}$. For any $u \in R$ and any system of parameters $\underline{x} = x_1, \dots, x_d$ of R , denote the image of $\frac{u}{(x_1, \dots, x_d)}$ in $H_{\mathfrak{m}}^d(R)$ by $[\frac{u}{(x_1, \dots, x_d)}]$. Recall that, for any $e \in \mathbb{N}$, there is a canonical isomorphism $F_R^e(H_{\mathfrak{m}}^d(R)) \cong H_{\mathfrak{m}}^d(R)$, under which we may simply write $[\frac{u}{(x_1, \dots, x_d)}]_{H_{\mathfrak{m}}^d(R)}^q = [\frac{u^q}{(x_1^q, \dots, x_d^q)}]$. By colon-capturing, we see that $[\frac{u}{(x_1, \dots, x_d)}] \in 0_{H_{\mathfrak{m}}^d(R)}^*$ if and only if $u \in (x_1, \dots, x_d)_R^*$ (cf. [Sm, Proposition 2.5]). This implies that b is a weak test element for $0 \subset H_{\mathfrak{m}}^d(R)$. (Indeed, for any $[\frac{u}{(x_1, \dots, x_d)}] \in 0_{H_{\mathfrak{m}}^d(R)}^*$, we have $u \in (x_1, \dots, x_d)_R^*$. Then $bu^q \in (x_1, \dots, x_d)^{[q]}$ for all $q \geq q_0$, which implies $b[\frac{u}{(x_1, \dots, x_d)}]_{H_{\mathfrak{m}}^d(R)}^q = [\frac{bu^q}{(x_1^q, \dots, x_d^q)}] = 0 \in F^e(H_{\mathfrak{m}}^d(R))$ for all $q \geq q_0$.) Consequently, b is a weak test element for $0 \subset F^e(H_{\mathfrak{m}}^d(R))$ for all $e \in \mathbb{N}$. Thus, by Proposition 2.1, there exists a test exponent, say $Q = p^E$, for c and $H_{\mathfrak{m}}^d(R)$. \square

Now we are ready to give a new proof of Sharp's result about a uniform test exponent for $c \in R^\circ$ and all ideals generated by systems of parameters.

Theorem 2.4 (R. Y. Sharp, [Sh, Theorem 3.2]). *Let (R, \mathfrak{m}) be an equidimensional excellent local ring of prime characteristic p and $c \in R^\circ$. Then there exists a test exponent for c and all ideals generated by (partial or full) systems of parameters of R .*

Proof. Say $\dim(R) = d$. By Proposition 2.3, there is a test exponent Q for c and $H_{\mathfrak{m}}^d(R)$. Here we keep the same usage of $[\frac{u}{(x_1, \dots, x_d)}]$ as in the above proof of Proposition 2.3.

Now, it suffices to show that Q is a test exponent for c and $(x_1, \dots, x_i) \subseteq R$ for any (partial or full) system of parameters $\underline{x} = x_1, \dots, x_i$ of R . But, then, it suffices to verify the case where $\underline{x} = x_1, \dots, x_d$ is any full system of parameters of R via a standard technique (see the last paragraph of the proof of [Sh, Theorem 3.2]).

Finally, for any $u \in R$ and $q \geq Q$, suppose $cu^q \in (\underline{x})^{[q]} = (x_1^q, \dots, x_d^q)$. This implies $c[\frac{u}{(x_1, \dots, x_d)}]_{H_{\mathfrak{m}}^d(R)}^q = 0 \in F_R^e(H_{\mathfrak{m}}^d(R))$. Thus, by the choice of Q , $[\frac{u}{(x_1, \dots, x_d)}] \in 0_{H_{\mathfrak{m}}^d(R)}^*$, which forces $u \in (x_1, \dots, x_d)_R^*$ by colon-capturing as in Proposition 2.3 (cf. [Sm, Proposition 2.5]). \square

Next, we state a corollary of the theorem above.

Corollary 2.5. *Let (R, \mathfrak{m}) be an equidimensional excellent local ring of prime characteristic p and $c \in R^\circ$. Then there exists a test exponent for $c/1$ and all ideals generated by (partial or full) systems of parameters of R_P (over R_P) for all $P \in \text{Spec}(R)$.*

Proof. By Theorem 2.4, there exists a test exponent, $Q = p^E$, for c and all ideals generated by (partial or full) systems of parameters of R . Fix an arbitrary $P \in \text{Spec}(R)$. It suffices to show that Q is a test exponent for $c/1$ and all ideals generated by (partial or full) systems of parameters of R_P (over R_P). Then, again, it suffices show that Q is a test exponent for c and all ideals generated by (full) systems of parameters of R_P (over R_P).

Say $\dim(R_P) = h$. Then by prime avoidance, there exists $\underline{x} = x_1, \dots, x_h \in P$ such that \underline{x} is a (partial) system of parameters of R . Then, for any $0 < n \in \mathbb{N}$, $\underline{x}^n := x_1^n, \dots, x_h^n$ is also a (partial) system of parameters of R and, moreover, $x_1^n/1, \dots, x_h^n/1$ is a (full) system of parameters of R_P .

Let $\underline{y} = y_1, \dots, y_h$ be any full system of parameters of R_P . We need to prove that Q is a text exponent for $c/1$ and $(\underline{y}) \subseteq R_P$ in order to finish the proof. As there exists a positive integer $n \in \mathbb{N}$ such that $(x_1^n, \dots, x_h^n)R_P \subseteq (\underline{y})$, it suffices to prove that Q is a text exponent for $c/1$ and $(x_1^n, \dots, x_h^n)R_P \subseteq R_P$ by Lemma 1.4(3).

Now suppose $(c/1)v^q \in (x_1^n, \dots, x_h^n)^{[q]}R_P$ for some $v \in R_P$ and $q \geq Q$. Without loss of generality, we may assume $v = u/1$ with $u \in R$. That is, there exists $s \in R \setminus P$ such that $scu^q \in (x_1^n, \dots, x_h^n)^{[q]}R$. Hence $c(su)^q \in (x_1^n, \dots, x_h^n)^{[q]}R$, which implies $su \in (x_1^n, \dots, x_h^n)_R^*$. Therefore, $v = u/1 \in (x_1^n, \dots, x_h^n)_R^*R_P \subseteq ((x_1^n, \dots, x_h^n)R_P)_{R_P}^*$. \square

3. MODULES WITH FINITE (PHANTOM) PROJECTIVE DIMENSION

Question 3.1. Assume (R, \mathfrak{m}) is an equidimensional local ring of prime characteristic p that is either excellent or a homomorphic image of a Cohen-Macaulay ring. For

a given $c \in R^\circ$, does there exist a test exponent for c and all finitely generated R -modules of finite phantom projective dimension?

If R is Cohen-Macaulay, then it is known that phantom projective dimension is the same as projective dimension. For this reason, the following theorem may be viewed as a partial answer to the above question.

Theorem 3.2. *Let (R, \mathfrak{m}) be a Cohen-Macaulay Noetherian local ring of prime characteristic p with $\dim(R) = d$. Fix any $c \in R$, if $Q = p^E$ is a test exponent for c and all ideals generated by (full) systems of parameters of R , then Q is a test exponent for c and all R -modules of finite length and of finite projective dimension.*

Proof. Let $M \neq 0$ be a typical R -module such that $\lambda(M) < \infty$ and $\text{pd}(M) < \infty$. Suppose $cu^{q'} = 0 \in F^{e'}(M)$ for some $u \in M, q' \geq Q$. We need to show $u \in 0_M^*$.

Fix a minimal projective resolution G_\bullet of M as follows

$$G_\bullet : \quad 0 \longrightarrow G_d \xrightarrow{\phi_d} G_{d-1} \xrightarrow{\phi_{d-1}} \cdots \xrightarrow{\phi_2} G_1 \xrightarrow{\phi_1} G_0 \longrightarrow 0.$$

Then choose a system of parameters \underline{x} of R such that $(\underline{x}) \subseteq \text{Ann}_R(u)$ and construct the Koszul complex $K_\bullet(\underline{x}, R)$ as follows

$$K_\bullet(\underline{x}, R) : \quad 0 \longrightarrow K_d \xrightarrow{\psi_d} K_{d-1} \xrightarrow{\psi_{d-1}} \cdots \xrightarrow{\psi_2} K_1 \xrightarrow{\psi_1} K_0 \longrightarrow 0,$$

where $K_i = R^{\binom{d}{i}}$. In particular, ψ_d is represented by matrix (x_1, x_2, \dots, x_d) and the 0-th homology of $K_\bullet(\underline{x}, R)$ is $R/(\underline{x})$. Thus the R -linear map $h : R/(\underline{x}) \rightarrow M = H_0(G_\bullet)$ sending the class of 1 to u can be lifted to a chain map $g_\bullet : K_\bullet(\underline{x}, R) \rightarrow G_\bullet$. Denote $g_0(1) = y$. Then $cy^{q'} \in (\text{Image}(\phi_1))_{G_0}^{[q']}$ and we now only need to show $y \in (\text{Image}(\phi_1))_{G_0}^*$.

For every q , there is an induced R -linear chain map $g_\bullet^{[q]} : F^e(K_\bullet(\underline{x}, R)) \rightarrow F^e(G_\bullet)$. Now the fact that $cy^{q'} \in (\text{Image}(\phi_1))_{G_0}^{[q']}$ (i.e., $cu^{q'} = 0$) implies that the chain map $cg_\bullet^{[q]}$ is homotopic to the zero chain map. In particular, there exists $\delta_{d-1} \in \text{Hom}_R(F^{e'}(K_{d-1}), F^{e'}(G_d))$ such that $cg_d^{[q]} = \delta_{d-1} \circ \psi_d^{[q]}$. Applying $\text{Hom}_R(-, R)$, we get

$$\begin{aligned} c(\text{Image}(\text{Hom}(g_d, R)))_{K_d}^{[q]} &= \text{Image}(\text{Hom}(cg_d^{[q]}, R)) \\ &\subseteq \text{Image}(\text{Hom}(\psi_d^{[q]}, R)) = (\underline{x})^{[q]}, \end{aligned}$$

which implies $\text{Image}(\text{Hom}(g_d, R)) \subseteq (\underline{x})_R^*$ since $q' \geq Q$. That is to say that there exists $b \in R^\circ$ such that

$$\begin{aligned} \text{Image}(\text{Hom}(bg_d^{[q]}, R)) &= b \text{Image}(\text{Hom}(g_d^{[q]}, R)) \\ &= b(\text{Image}(\text{Hom}(g_d, R)))_R^{[q]} \subseteq (\underline{x})^{[q]} = \text{Image}(\text{Hom}(\psi_d^{[q]}, R)) \end{aligned}$$

for all $q \gg 0$. Therefore, the chain maps

$$\text{Hom}(bg_\bullet^{[q]}, R) : \text{Hom}(F^e(G_\bullet), R) \rightarrow \text{Hom}(F^e(K_\bullet(\underline{x}, R)), R)$$

are homotopic to 0 for all $q \gg 0$. Hence, there exist $\epsilon_1^{[q]} \in \text{Hom}_R(F^e G_1, F^e(K_0))$ such that $\text{Hom}(bg_0^{[q]}, R) = \epsilon_1^{[q]} \circ \text{Hom}(\phi_1^{[q]}, R)$ for all $q \gg 0$. This, after going through

$\text{Hom}(-, R)$, would in turn imply

$$by^q \in b(\text{Image}(g_0))_{G_0}^{[q]} = \text{Image}(bg_0^{[q]}) \subseteq \text{Image}(\phi_1^{[q]}) = (\text{Image}(\phi_1))_{G_0}^{[q]},$$

for all $q \gg 0$. We now conclude that $y \in (\text{Image}(\phi_1))_{G_0}^*$ and the proof is complete. We also remark that the above argument of using homotopy to determine membership in the tight closure has appeared in [Ab]. \square

Corollary 3.3. *Let (R, \mathfrak{m}) be a Cohen-Macaulay Noetherian excellent local ring of prime characteristic p . Then, for any $c \in R^\circ$, there is a test exponent for c and all R -modules of finite length and of finite (phantom) projective dimension.*

Proof. This follows from Theorem 0.7 and Theorem 3.2. \square

We also notice that Question 3.1 reduces to the Cohen-Macaulay case if $\dim(R) \leq 2$.

Corollary 3.4. *Let (R, \mathfrak{m}) be an equidimensional excellent Noetherian local ring of prime characteristic p with $\dim(R) \leq 2$. Then, for any given $c \in R^\circ$, there exists a test exponent for c and all R -modules of finite length and of finite phantom projective dimension.*

Proof. By [HH1, Definition 9.1], we observe that any R -module of finite length and of finite phantom projective dimension over R remains so after we extend the scalar to the integral closure of R/P in its fraction field for every $P \in \min(R)$. Therefore, by Lemma 1.2 and Lemma 1.3, we may assume R is normal without loss of generality. (We may assume R is complete as well.) But now R is excellent Cohen-Macaulay and the claim follows from Corollary 3.3. \square

4. A CONNECTION WITH F -RATIONAL SIGNATURE

The F -rational signature, $r_R(M)$, has been studied in [HY]. In this section, we investigate the behavior of $r'_R(M)$ (cf. Definition 0.8(2)). We start with a remark.

Remark 4.1. First, from definition, it is immediate to see that $r(M) \geq r'(M)$ for any finitely generated R -module M .

Second, suppose that $\text{ppd}(R/\langle \underline{x} \rangle) < \infty$ for every system of parameters \underline{x} of R . Then we observe that $r'_{\widehat{R}}(M \otimes_R \widehat{R}) \leq r'_R(M)$ for any finitely generated R -module M . The reason is that any R -module of finite length and of finite phantom projective dimension remain so considered as an \widehat{R} -module.

It turns out that $r(M)$ and $r'(M)$ behave quite similarly. For example, assuming R is excellent, the positiveness of $r'(R)$ characterizes F -rationality.

Theorem 4.2. *Let (R, \mathfrak{m}) be a Noetherian local ring of prime characteristic p . And we use M to denote a finitely generated R -module. Consider*

- (1) $r(R) > 0$; (1') $r'(R) > 0$;
- (2) $r(M) > 0$ for every M with $\dim(M) = \dim(R)$;
- (2') $r'(M) > 0$ for every M with $\dim(M) = \dim(R)$;
- (3) $r(M) > 0$ for some M ; (3') $r'(M) > 0$ for some M ;
- (4) \widehat{R} is F -rational;

(5) R is F -rational.

Then $(1) \Leftrightarrow (1') \Leftrightarrow (2) \Leftrightarrow (2') \Leftrightarrow (3) \Leftrightarrow (3') \Leftrightarrow (4) \Rightarrow (5)$. If, moreover, R is excellent or there exists a common weak parameter test element for R and \widehat{R} , then $(1) \Leftrightarrow (1') \Leftrightarrow (2) \Leftrightarrow (2') \Leftrightarrow (3) \Leftrightarrow (3') \Leftrightarrow (4) \Leftrightarrow (5)$.

Proof. The implication $(4) \Rightarrow (5)$ is always true. And $(4) \Leftrightarrow (5)$ holds if there exists a common parameter test element for R and \widehat{R} , which is the case when R is excellent. It remains to show $(1) \Leftrightarrow (1') \Leftrightarrow (2) \Leftrightarrow (2') \Leftrightarrow (3) \Leftrightarrow (3') \Leftrightarrow (4)$.

If \widehat{R} is not F -rational, then $r_R(M) = 0$, which forces $r'_R(M) = 0$, for every finitely generated R -module M (see [HY, Theorem 4.1]). Thus any single one of (1), (1'), (2), (2'), (3), or (3') will imply (4).

It is straightforward to see that (3') implies R is an F -rational domain, which in turn implies statements (1), (1'), (2), (2'), and (3).

Now it remains to prove $(4) \Rightarrow (3')$ in order to complete the proof. So we assume \widehat{R} is F -rational. Thus R is Cohen-Macaulay and, by Remark 4.1, $r'_{\widehat{R}}(M \otimes_R \widehat{R}) \leq r'_R(M)$ for any finitely generated R -module M . As a result, we may assume R is a complete F -rational domain (hence Cohen-Macaulay). But now it is enough to show $r'(R) > 0$ as $r'(M) = \text{rank}_R(M)r'(R)$ where $\text{rank}_R(M)$ denotes the torsion-free rank of M over R .

One of the ingredients in this proof is the following Fact, which is explicitly stated in [HY, Theorem 3.3].

Fact. *Let (R, \mathfrak{m}) be a complete local domain of prime characteristic p . There exist $c \in R^\circ, c' \in R^\circ$ and $q'' = p^{e''}$ such that, for any R -modules L with $\lambda(L) < \infty$, if q''' is a test exponent for cc' and $0 \subseteq L$, then $e_{HK}(L, R) - e_{HK}(L/K, R) \geq 1/(q''q''')^{\dim(R)}$ for every K such that $0_L^* \not\subseteq K \subseteq L$.*

Now we come back to the proof of Theorem 4.2. Remember that R is a complete F -rational local ring (hence a Cohen-Macaulay domain) now. Choose $c, c' \in R^\circ$ as in the above Fact. In [Ab], Aberbach has shown that $0_L^* = 0$ for every R -module L such that $\text{ppd}_R(L) < \infty$. Also, from Corollary 3.3, there is a test exponent, say q'' , for cc' and all R -modules L such that $\text{ppd}_R(L) < \infty$ and $\lambda(L) < \infty$, which forces $e_{HK}(L, R) - e_{HK}(L/K, R) \geq 1/(q''q''')^{\dim(R)}$ for every $0 \neq K \subseteq L$ by the above Fact. Now it follows from Definition 0.8 that $r'(R) \geq 1/(q''q''')^{\dim(R)} > 0$. \square

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