Lower Bounds for Betti Numbers of Special Extensions

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This paper is written, on the occasion of the sixty-fifth birthday of Wolmer Vasconcelos, in celebration of his enormous contributions to the field of commutative algebra.

1 Introduction

Let (R, m, k) be a regular local ring of Krull dimension n and let M be a finite length R-module. We will write a minimal free resolution of M as

$$0 \to R^{\alpha_n^M} \to \cdots \to R^{\alpha_1^M} \to R^{\alpha_0^M} \to M \to 0,$$

where $\{\alpha_0^M, \ldots, \alpha_n^M\}$ are the Betti numbers of M. Giving lower bounds for these Betti numbers has been a long standing problem in commutative algebra. In fact, in 1977, Buchsbaum and Eisenbud made the following conjecture [BE]:

Conjecture 1. Let R be a regular local ring of Krull dimension n and M be a finite length R-module. Then $\alpha_t^M \geq \binom{n}{t}$ for $t = 0, \ldots, n$.

Shortly after Buchsbaum and Eisenbud's paper appeared, this problem was submitted to Hartshorne's problem list by M. Horrocks [Ha]. For this reason, Conjecture 1 is often referred to as Horrocks' Problem. In this paper we will refer to it as the BEH Conjecture.

Buchsbaum and Eisenbud originally showed [BE] that if a resolution of M has an associative multiplicative structure (such as the Koszul complex has, for instance), then the t^{th} Betti number of M is larger than the t^{th} binomial coefficient for all $t = 0, \ldots, n$. They also pointed out, however, that in an appendix due to V. Khimich in a paper by Avramov [Av] it is shown that there exists a finite length module for which no associative multiplicative resolution can exist.

Further progress was made on BEH in [EG] by Evans and Griffith. They showed that if M is a direct sum of quotients of monomial ideals, then the Betti numbers of M satisfied the conjecture. In 1987, Huneke and Ulrich [HU] showed that the Betti numbers of M must satisfy the bound if M is cyclic and $\mathrm{Ann}(M)$ is in the linkage class of a complete intersection. Charalambous [Ch] and Santoni [S] did independent work which proved the multigraded case in general. Charalambous, in fact, showed that in the monomial case, the bound is never sharp except when M is a complete intersection.

In 1997, Chang [C] proved the following theorem:

Theorem 1.1 (Chang). *Let (R, m, k) be a regular local ring of Krull dimension n and let M be a finite length R-module such that $m^2M = 0$. Then the Betti numbers α_t^M of M satisfy $\alpha_t^M \geq \binom{n}{t}$, $1 \leq t \leq n$.*

This case of the BEH conjecture is the starting point for our investigation of the substantially more general situation handled in this paper. To understand the connection note that m^2 kills M if and only if there are non-negative integers a, b such that the sequence

$$0 \to (R/m)^a \to M \to (R/m)^b \to 0$$

is exact. Now it is clear that our result, stated immediately following, is an extension of Chang's work:

Theorem 1.2 (Main Theorem). *Let (R, m, k) be a regular local ring of Krull dimension n and let M be an R-module of finite length. Suppose that there is a short exact sequence*

$$0 \to \bigoplus_{\nu=1}^{a} (R/I_\nu) \to M \to \bigoplus_{\eta=1}^{b} (R/J_\eta) \to 0,$$

where $I_1, \ldots, I_a$, and $J_1, \ldots, J_b$ are ideals generated by R-sequences. If $J_\eta \subseteq I_\nu$ for all $\nu = 1, \ldots, a$ and $\eta = 1, \ldots, b$, or if $I_\nu \subseteq J_\eta$ for all ν, η, then the Betti numbers α_t^M of M satisfy $\alpha_t^M \geq \binom{n}{t}$, $1 \leq t \leq n$.

Thus, the BEH conjecture holds for the class of modules described in the statement of the theorem. The proof of this result, is given at the end of §3.

We are able to remove the inclusion requirements on the J_η and I_ν in the case that one of the positive integers a, b, is very much larger[1] than the other. This elementary combinatorial argument is given in §5: see Theorem 5.1.

The idea of the proof of 1.2 is much more difficult. It depends on developing a method for passing from M to a new extension,

$$0 \to \bigoplus_{\nu=1}^{a}(R/m) \to M' \to \bigoplus_{\eta=1}^{b}(R/m) \to 0,$$

in such a way that M' is guaranteed to have Betti numbers that are at least as small as the Betti numbers of M. We may then apply Chang's theorem to complete the argument.

We produce M' by making certain generic choices. The key effort is showing that the connecting homomorphisms in the long exact sequence for $\mathrm{Ext}(\_, k)$ for the extension defining M are given by a matrix of polynomial functions in a certain large number of coefficients needed to make explicit all the information in the hypothesis of the theorem. For the detailed statement see Theorem 3.2. We lay the groundwork for this argument in §2. We complete this program in Section 3 utilizing the relationship between the Yoneda pairing and connecting homomorphisms for Ext. Generic choices of the coefficients mentioned determine an extension

$$0 \to \bigoplus_{\nu=1}^{a}(R/m) \to M' \to \bigoplus_{\eta=1}^{b}(R/m) \to 0$$

in such a way that the connecting homomorphisms in the long exact sequence for $\mathrm{Ext}_R(\_, k)$ have maximal rank. We use the fact that the Betti numbers can be calculated as the k-vector space dimension of the modules $\mathrm{Ext}^t_R(\_, k)$ to show that we can guarantee that $\alpha_t^M \geq \alpha_t^{M'}$ for all $t = 0, \ldots, n$.

Our hope is that the methods used here may be applied much more generally to reduce many cases of the theorem to other, more tractable ones. Note, for example, that in our passage from the M to M' the length of the module is typically reduced dramatically, and the structure of the module M' is substantially simpler than that of M. In §4 we use related ideas to single out a class of modules of particular interest for the BEH conjecture.

[1] It suffices if $a \geq \frac{n-2}{3}b + 1$ or $b \geq \frac{n-2}{3}a + 1$.

2 Preliminaries

Let (R, m, k) be a regular local ring and M, A, and B be R-modules. Note that we can calculate the Betti numbers of each of these modules using $\mathrm{Ext}_R(\_, k)$, that is, $\alpha_t^M = \dim_k \mathrm{Ext}_R^t(M, k)$, $\alpha_t^A = \dim_k Ext_R^t(A, k)$, and $\alpha_t^B = \dim_k \mathrm{Ext}_R^t(B, k)$ for all t. If we have a short exact sequence,

$$0 \to B \to M \to A \to 0, \tag{1}$$

then we form the long exact sequence in $\mathrm{Ext}_R(\_, k)$, which is

$$\cdots \to k^{\alpha_{t-1}^B} \xrightarrow{\delta_{t-1}^M} k^{\alpha_t^A} \to k^{\alpha_t^M} \to k^{\alpha_t^B} \xrightarrow{\delta_t^M} k^{\alpha_{t+1}^A} \to \cdots .$$

Note that we reference the connecting homomorphism, δ_t^M, with the middle module in the short exact sequence (1), although a more precise indicator would be the element of $\mathrm{Ext}_R^1(B, A)$ that the short exact sequence represents.

This yields the equality

$$\alpha_t^M = \alpha_t^A + \alpha_t^B - \mathrm{rank}(\delta_{t-1}^M) - \mathrm{rank}(\delta_t^M). \tag{2}$$

To find a module which has Betti numbers at most equal to those of M, it is enough to find an extension for which the flanking modules are known to have Betti numbers at most equal to those of A and B respectively, while the new connecting homomorphisms have rank the same as or larger than the original ones. The pursuit of this idea will eventually permit us to reduce the proof of Theorem 1.2 to Chang's theorem.

Remark 2.1. We want to apply this discussion to the situation in the hypothesis of Theorem 1.2. We require the inclusions $J_\eta \subseteq I_\nu$ for all $\eta = 0, \ldots, b$ and $\nu = 0, \ldots, a$ because when they hold the calculation of the relevant Ext module is greatly simplified, giving a very useful structure that enables us to calculate as well the connecting homomorphisms of all extensions of this sort simultaneously in terms of certain parameters. The remarks that follow begin the elaboration if this idea.

Remark 2.2. Let R be a ring (it need not be Noetherian) and let θ be an R-linear map from a free R-module on a finite set of generators Σ_1 and a free R-module on another finite set of generators Σ_2, where the two sets of generators have not necessarily been given linear orderings. In this case, by the *matrix* of θ we mean the function from $\Sigma_1 \times \Sigma_2$ to R whose value on (σ_1, σ_2) is the coefficient of σ_2 in R when $\theta(\sigma_1)$ is written as an R-linear combination of the elements of Σ_2.

Remark 2.3. Now let $f_1, \ldots, f_n$ be a regular sequence in R, and denote by $J = (f_1, \ldots, f_n)R$ the ideal they generate. Suppose that $J \subseteq I$. Then $\mathrm{Ext}^j_R(R/J, R/I) \cong (R/I)^{\binom{n}{j}}$. *Moreover, once the generators* $f_1, \ldots, f_n$ *for* J *are fixed, this Ext module has a "canonical"* R/I*-free basis indexed by the* j *element subsets of* $\{1, \ldots, n\}$. We refer to this basis as the *standard* basis.

To see this, we may use the Koszul complex $K_\bullet(f_1, \ldots, f_n; R)$ as a free resolution of R/J: if $U_1, \ldots, U_n$ is a basis for the free module in degree one such that U_j maps to f_j, then the elements $U_{i_1} \wedge \cdots \wedge U_{i_j}$ such that $i_1 < \cdots < i_j$ are a free basis for the free R-module $K_j = K_j(f_1, \ldots, f_n; R)$. Since I contains J, when we apply $\mathrm{Hom}_R(\_, R/I)$ the maps in the complex all become 0, and the j th module may be identified with $\mathrm{Hom}_R(K_j, R/I) \cong \mathrm{Hom}_R(K_j, R/J) \otimes R/I$. We may take as an (R/I)-free basis the image of the "dual" basis for $\mathrm{Hom}_R(K_j, R/I)$. In particular, $\mathrm{Ext}^1_R(R/J, R/I) \cong (R/I)^n$, and we have a canonical or standard free basis *given* that we have a specific regular sequence $f_1, \ldots, f_n$ generating J. (In fact, we can replace R/I by any R-module N that is killed by J: in that case the same argument shows that $\mathrm{Ext}^j_r(R/J, N) \cong N^{\oplus \binom{n}{j}}$, which we may think of $\mathrm{Ext}^j_r(R/J, R/J) \otimes_R N$.)

In the next section we show how to use this basis information to calculate the connecting homomorphism in the cases with which we are concerned.

3 Extensions

The Yoneda pairing for Ext is treated in great detail in MacLane's book on Homology [Mac], especially in Chapter III, §6 and §9 (Theorem 9.1 gives the relationship with connecting homomorphisms). The following description (cf. [Mac], Ch., III, §6, Exercise 2.) will be useful for us here: given R modules A, B, C and integers j, $h \geq 0$ we want to give a map

$$\mathrm{Ext}^h_R(A, B) \otimes_R \mathrm{Ext}^j_R(B, C) \to \mathrm{Ext}^{j+h}_R(A, C).$$

The Yoneda pairing may be constructed by specifying for each element $\zeta \in \mathrm{Ext}^h_R(A, B)$ a map $\mathrm{Ext}^j_R(B, C) \to \mathrm{Ext}^{j+h}_R(A, C)$. Let $P_\bullet$ and $Q_\bullet$ be projective resolutions of A and B respectively over R. Then ζ is represented by $\eta \in \mathrm{Hom}_R(P_h, B)$ mapping to 0 in $\mathrm{Hom}_R(P_{h+1}, B)$. The map η lifts to a map $P_h \to Q_0$, and this extends to a map of complexes:

$$\begin{array}{ccccccccccc} \cdots & \longrightarrow & P_{h+j} & \longrightarrow & \cdots & \longrightarrow & P_{h+1} & \longrightarrow & P_h & \longrightarrow & 0 \\ & & \downarrow & & & & \downarrow & & \downarrow & & \\ \cdots & \longrightarrow & Q_j & \longrightarrow & \cdots & \longrightarrow & Q_1 & \longrightarrow & Q_0 & \longrightarrow & 0 \end{array}$$

Thus, we have a map $P_{h+j} \to Q_j$ for all $j \geq 0$. When we apply $\mathrm{Hom}_R(\_, C)$ and take cohomology, we get induced maps

$$\lambda_\zeta : \mathrm{Ext}^j_R(B, C) \to \mathrm{Ext}^{h+j}_R(A, C).$$

The Yoneda pairing sends $\zeta \otimes \beta$ to $\lambda_\zeta(\beta)$. This map is independent of the choices that we made, and is R-bilinear.

Remark 3.1. We need to comment on what happens in case one of the three modules is a finite direct sum. If A (respectively, C) is a finite direct sum, then since Ext commutes with finite direct sum in either variable module, both sides in the Yoneda pairing split into corresponding terms, and the pairing is simply the direct sum of the pairings coming from the individual terms. However, if B is a finite direct sum of, say, s terms B_i, the right hand side splits into the sum of s^2 terms while the left hand side is unaffected. In this case the pairing is 0 on summands of the form $\mathrm{Ext}^h_R(A, B_i) \otimes_R \mathrm{Ext}^j_R(B_{i'}, C)$ for $i \neq i'$ and is the Yoneda pairing for A, B_i, C for summands of the form $\mathrm{Ext}^h_R(A, B_i) \otimes_R \mathrm{Ext}^j_R(B_i, C)$. This is easily deduced from the construction described above and the corresponding facts for Hom.

The following theorem shows that there is a sort of "universal" matrix for certain Yoneda pairings when the modules have a certain form, and constitutes one of the key ingredients for proving the main result. It will allow us, likewise, to calculate connecting homomorphisms in a general setting.

Before stating the theorem we need to fix a considerable amount of notation. Let t, n be fixed integers, $0 \leq t \leq n$, where $n \geq 1$. Let u_{ij}, v_{ij} for $1 \leq i, j \leq n$ be $2n^2$ indeterminates over $\mathbb{Z}$. Suppose that R is any ring (it need not be Noetherian), that $x_1, \ldots, x_n$, $f_1, \ldots, f_n$, and $g_1, \ldots, g_n$ are regular sequences in R, that f is the $n \times 1$ column matrix (f_i), g is the $n \times 1$ column matrix (g_i), and x is the $n \times 1$ column matrix (x_i). Let (r_{ij}), $(s_{i,j})$

be two $n \times n$ matrices over R such that $f = (r_{i,j})g$ and $g = (s_{i,j})x$. Let I, J, and $\mathcal{M}$ be the ideals $(g_1, \ldots, g_n)R$, $(f_1, \ldots, f_n)R$, and $(x_1, \ldots, x_n)R$, respectively. Note that $\mathrm{Ext}^1_R(R/J, R/I) \otimes_R \mathrm{Ext}^r_R(R/I, R/\mathcal{M})$ is free over $R/\mathcal{M}$, and has a free basis corresponding to the Cartesian product of the free bases for the two factor modules provided by Remark 2.3. Likewise, $\mathrm{Ext}^1_R(R/I, R/\mathcal{M})$ is free over $R/\mathcal{M}$, and has a free basis provided by Remark 2.3.

Theorem 3.1 (universal polynomials for Yoneda pairings). *Let t, n, u_{ij}, and v_{ij} be as in the preceding paragraph. Then there is an $\binom{n}{t+1} \times n\binom{n}{t}$ matrix Γ^n_t with entries $P_{ij} \in \mathbb{Z}[u_{ij}, v_{ij}]$ with the following property. If R is a ring as in the preceding paragraph, then for all choices of $f_1, \ldots, f_n$, $g_1, \ldots, g_n$, $x_1, \ldots, x_n$, (r_{ij}), and (s_{ij}) as above, and with all notations as in the preceding paragraph, the matrix of the Yoneda pairing*

$$\mathrm{Ext}^1_R(R/J, R/I) \otimes \mathrm{Ext}^t_R(R/I, R/\mathcal{M}) \to \mathrm{Ext}^{t+1}_R(R/J, R/\mathcal{M}),$$

with respect to the bases indicated in the preceding paragraph, is $\Gamma^n_t(r, s)$, the result of substituting $u_{ij} = r_{ij}$ and $v_{ij} = s_{ij}$ in the various polynomial entries of the matrix Γ^n_t.

Proof. Let $X_1, \ldots, X_n$ be additional indeterminates over $\mathbb{Z}$ and let $T = \mathbb{Z}[u_{ij}, v_{ij}, X_i]$. Let X be the $n \times 1$ column vector X_i, let G be the $n \times 1$ column vector $(v_{ij})X$, with entries $G_1, \ldots, G_n$, and let F be the $n \times 1$ column vector $(u_{ij})G$, with entries $F_1, \ldots, F_n$. By a result, for example, of [Ho], $F_1, \ldots, F_n$ and $G_1, \ldots, G_n$ are regular sequences in T. Consider the ring homomorphism $\phi : T \to R$ that sends X_i to x_i, u_{ij} to r_{ij} and v_{ij} to s_{ij} for all i, j.

We have the Yoneda pairing

$$\mathrm{Ext}^1_T\big(T/(G), T/(F)\big) \otimes_T \mathrm{Ext}^j_T\big(T/(F), T/(X)\big) \to \mathrm{Ext}^{j+1}_T\big(T/(G), T/(X)\big)$$

over T. Since both sides are free over $T/(X) \cong \mathbb{Z}[u, v]$, the map will be given by a matrix Γ of polynomials in u, v. We need to check that when when we specialize the u_{ij} and v_{ij} under the map $T \to R$, that we get the Yoneda pairing

$$\mathrm{Ext}^1_R(R/J, R/I) \otimes \mathrm{Ext}^t_R(R/I, R/m) \to \mathrm{Ext}^{t+1}_R(R/J, R/m)$$

over R. For this purpose, we think of the pairing as constructed in Remark 2.3 over T, using Koszul complexes to resolve $T/(F)$ and $T/(G)$ respectively. A key point is that when we apply $R \otimes_T \_$, these complexes become free resolutions of R/J and R/I respectively over R. It suffices to see that the Yoneda pairing "specializes" correctly for each fixed choice of free generator of $\mathrm{Ext}^1_T\big(T/(F),\, T/(G)\big)$, using the standard generators of Remark 2.3. The i^{th} free generator w_i corresponds to the map sending the U_i to 1 in $T/(G)$ and the other U_j, $j \neq i$, to 0. These "specialize" to the correspondingly numbered standard generators of $\mathrm{Ext}^1_R(R/J,\, R/I)$: for each of the $w_i : K_1(F_1, \ldots, F_n; T) \to T/(G)$, when one applies $R \otimes_T \_$ one gets the corresponding map of $K_1(f_1, \ldots, f_n; R)$ to R/I, sending U_i to $1 \in R/I$ and the other U_j, $j \neq i$, to 0. Now, once one has constructed a map of complexes over T that gives the Yoneda pairing, if one applies $R \otimes_T \_$, one gets a map of complexes over R that gives the corresponding Yoneda pairing over R, and the result follows. $\square$

Remark 3.2. It is important to note that M *does not appear* anywhere in the above statement. As indicated earlier, Theorem 3.1 provides a calculation of the Yoneda pairing in great generality.

It is now quite straightforward, although notationally cumbersome, to generalize Theorem 3.1 to the case where each complete intersection is replaced by a direct sum of complete intersections. As in the case of Theorem 3.1 we give a preliminary discussion of the multiple items needed to state the result.

Let t, n be fixed integers, $0 \leq t \leq n$, where $n \geq 1$. Let a and b be positive integers. Let $u_{ij}^{(\nu,\eta)}$, $v_{ij}^{(\nu)}$ for $1 \leq i,\, j \leq n$, $1 \leq \nu \leq a$, $1 \leq \eta \leq b$ be $abn^2 + an^2$ indeterminates over $\mathbb{Z}$. Suppose that R is any ring (it need not be Noetherian), that for all η, ν (varying as above) $x_1, \ldots, x_n$, $f_1^{(\eta)}, \ldots, f_n^{(\eta)}$ and $g_1^{(\nu)}, \ldots, g_n^{(\nu)}$ are regular sequences in R, that $f^{(\eta)}$ is the $n \times 1$ column matrix $(f_i^{(\eta)})$, that $g^{(\nu)}$ is the $n \times 1$ column matrix $(g_i^{(\nu)})$, and that x is the $n \times 1$ column matrix (x_i). Suppose also that for all η, ν that $(r_{i,j}^{(\nu,\eta)})$ and $(s_{i,j}^{(\nu)})$ are $n \times n$ matrices over R such that $f^{(\eta)} = (r_{i,j}^{(\nu,\eta)})g^{(\nu)}$ and $g^{(\nu)} = (s_{i,j}^{(\nu)})x$. Let I_ν, J_η, and $\mathcal{M}$ be the ideals $(g_1^{(\nu)}, \ldots, g_n^{(\nu)})R$, $(f_1^{(\eta)}, \ldots, f_n^{(\eta)})$, and $(x_1, \ldots, x_n)R$, respectively. Note that every

$$\mathrm{Ext}^1_R(R/J_\eta,\, R/I_\nu) \otimes_R \mathrm{Ext}^t_R(R/I_{\nu'},\, R/\mathcal{M})$$

is free over $R/\mathcal{M}$, and has a free basis corresponding to the Cartesian product of the free bases for the two factor modules provided by Remark 2.3. Likewise, $\mathrm{Ext}^1_R(R/I_\nu,\, R/\mathcal{M})$ is free over $R/\mathcal{M}$, and has a free basis provided by Remark 2.3. Since Ext commutes with finite direct sums in either variable module, we get a free basis over $R/\mathcal{M}$ for

$$\mathrm{Ext}^1_R(\bigoplus_{\eta=1}^{b} R/J_\eta,\ \bigoplus_{\nu=1}^{a} R/I_\nu) \otimes_R \mathrm{Ext}^t_R(\bigoplus_{\nu=1}^{a} R/I_\nu,\ R/\mathcal{M})$$

and for $\mathrm{Ext}^{t+1}_R(\bigoplus_{\eta=1}^{b} R/J_\eta,\ R/\mathcal{M})$.

Theorem 3.2 (Universal Yoneda polynomials for sums). *Let t, n, a, b, $u_{ij}^{(\nu,\eta)}$, and $v_{ij}^{(\nu)}$ be as in the preceding paragraph. Then there is an $b\binom{n}{t+1} \times a^2bn\binom{n}{t}$ matrix $\Gamma_t^{(n,a,b)}$ with entries $Q_{ij} \in \mathbb{Z}[u_{ij}^{(\nu,\eta)}, v_{ij}^{(\nu,\eta)}]$ with the following property. If R is a ring as in the preceding paragraph, then for all choices of $f_1^{(\eta)}, \ldots, f_n^{(\eta)}, g_1^{(\nu)}, \ldots, g_n^{(\nu)}, x_1, \ldots, x_n, (r_{ij}^{(\nu,\eta)})$, and $(s_{ij}^{(\nu)})$ as above, and with all notations as in the preceding paragraph, the matrix of the Yoneda pairing*

$$\mathrm{Ext}^1_R\Big(\bigoplus_{\eta=1}^{b} R/J_\eta,\ \bigoplus_{\nu=1}^{a} R/I_\nu\Big) \otimes_R \mathrm{Ext}^t_R\Big(\bigoplus_{\nu=1}^{a} R/I_\nu,\ R/\mathcal{M}\Big)$$

$$\to \mathrm{Ext}^{t+1}_R\Big(\bigoplus_{\eta=1}^{b} R/J_\eta,\ R/\mathcal{M}\Big),$$

with respect to the bases indicated in the preceding paragraph, is $\Gamma_t^{(n,a,b)}(r,\ s)$, the result of substituting $u_{ij}^{(\nu,\eta)} = r_{ij}^{(\nu,\eta)}$ and $v_{ij}^{(\nu)} = s_{ij}^{(\nu)}$ in the various polynomial entries of the matrix $\Gamma_t^{(n,a,b)}$.

Proof. Because Ext distributes over finite direct sums in either variable and the Yoneda pairing is bilinear, by Remark 3.1 this reduces at once to Theorem 3.1: the difference is that now we need to keep track of a matrix for every choice of η and ν that gives the expressions for the generators of J_η in terms of those of I_ν, and a matrix for every ν that gives expressions for the generators of I_ν in terms of the generators of $\mathcal{M}$. □

We are now ready to prove our main result.

Proof. (Of the Main Theorem 1.2.) By taking Matlis duals (equivalently, applying $\mathrm{Ext}^n_R(\_,\ R)$) we may interchange the roles of the two direct sums of ideals in the extension. Thus, we may assume without loss of generality that every J_η is contained in every I_ν rather than the other way round.

Since we may replace R by $R(z)$, the localization of $R[z]$ at $mR[z]$, we may assume without loss of generality that K is infinite.

We shall show that for every module M arising as an extension of the form under consideration, there is an extension producing a module M' that has Betti numbers at least as large as those of M, and such that M' satisfies the hypotheses of Chang's theorem. Evidently, this implies our theorem.

We adopt the notations of Theorem 3.2 to describe the extension

$$0 \to \bigoplus_{\nu=1}^{a}(R/I_\nu) \to M \to \bigoplus_{\eta=1}^{b}(R/J_\eta) \to 0,$$

with $(R,\, m,\, K)$ the regular local ring and with $\mathcal{M} = m = (x_1,\, \ldots,\, x_n)R$, where the x_i are a minimal set of generators of m. Choose a regular sequence $g_1^{(\nu)},\, \ldots,\, g_n^{(\nu)}$ generating I_ν for every ν, and a regular sequence $f_1^{(\eta)},\, \ldots,\, f_n^{(\eta)}$ generating J_η for every η. Because of the containments $J_\eta \subseteq I_\nu$, for every (ν, η) we can choose a matrix $(r_{ij}^{(\nu,\eta)})$ over R such that $f^{(\eta)} = (r_{ij}^{(\nu,\eta)})g^{(\nu)}$, in the notations of Theorem 3.2. Likewise, with the same notations, since every $I_\nu \subseteq m$, for every ν we can choose a matrix $(s_{ij}^{(\nu)})$ over R such that $g^{(\nu)} = (s_{ij}^{(\nu)})x$. We have a standard set of generators for

$$\mathrm{Ext}^1_R\big(\bigoplus_{\eta=1}^{b}(R/J_\eta),\, \bigoplus_{\nu=1}^{a}(R/I_\nu)\big)$$

by Remark 2.3, and the identification of the displayed Ext with

$$\bigoplus_{\eta,\nu}\mathrm{Ext}^1_R(R/J_\eta,\, R/I_\nu).$$

Each of the ab terms in this latter direct sum has a standard free basis, with n elements, over R/I_ν. The element ϵ of this Ext corresponding to the extension with which we are working can therefore be represented by giving abn coefficients $c_{\nu,\eta,i}$ from R needed for these standard generators (they are not unique, of course).

By Theorem 9.1 of [Mac], the connecting homomorphism δ_t^M in the long exact sequence for $\mathrm{Ext}^\bullet_R(\_,\, K)$ obtained from the short exact sequence

$$0 \to \bigoplus_{\nu=1}^{a}(R/I_\nu) \to M \to \bigoplus_{\eta=1}^{b}(R/J_\eta) \to 0$$

is given by taking the Yoneda pairing

$$\operatorname{Ext}^1_R\Big(\bigoplus_{\eta=1}^{b} R/J_\eta,\, \bigoplus_{\nu=1}^{a} R/I_\nu\Big) \otimes_R \operatorname{Ext}^t_R\Big(\bigoplus_{\nu=1}^{a} R/I_\nu,\, R/m\Big)$$

$$\to \operatorname{Ext}^{t+1}_R\Big(\bigoplus_{\eta=1}^{b} R/J_\eta,\, R/m\Big),$$

and restricting it to the element $\epsilon \in \operatorname{Ext}^1_R\big(\bigoplus_{\eta=1}^{b} R/J_\eta,\, \bigoplus_{\nu=1}^{a} R/I_\nu\big)$ corresponding to M. That is $\delta_t^M(\gamma)$ is the image of $\epsilon\otimes\gamma$ under the Yoneda pairing. We introduce abn new indeterminates $w_{\nu,\eta,i}$ corresponding to the $c_{\nu,\eta,i}$. It follows from Theorem 3.2 and this discussion that there is a $bn\binom{n}{t} \times a^2bn\binom{n}{t+1}$ matrix $\Delta_t^{(n,a,b)}$ with entries in $\mathbb{Z}[u_{ij}^{(\nu,\eta)},\, v_{ij}^{(\nu)},\, w_{\nu,\eta,i}]$ such that the matrix of δ_t^M is obtained from $\Delta_t^{(n,a,b)}$ by specializing the $u_{ij}^{(\nu,\eta)}$, $v_{ij}^{(\nu)}$ and $w_{\nu,\eta,i}$ to $r_{ij}^{(\nu,\eta)}$, $s_{ij}^{(\nu)}$, and $c_{\nu,\eta,i}$, respectively, and then applying $K \otimes_R \_$. In particular, the ranks of the δ_t^M only depend on the images of the the elements $r_{ij}^{(\nu,\eta)}$, $s_{ij}^{(\nu)}$, and $c_{\nu,\eta,i}$ modulo m.

We now want to think in terms of constructing new extensions (and consequently new modules M' to replace M) by allowing the elements r, s, and c to vary (we are omitting the various subscripts and superscripts). We think of the residues of these elements modulo m as varying in $K^{abn^2+an^2+abn}$, an affine algebraic variety over K. We may assume that the elements r, s, c have been chosen so that their residues are in "general position," by which we simply mean that they have been chosen to avoid finitely many closed subsets of the affine space $K^{abn^2+an^2+abn}$. Specifically, we want all of the matrices $(r_{ij}^{(\nu,\eta)})$ and $(s_{ij}^{(\nu)})$ to be invertible modulo m, and we want the ranks of all of the finitely many connecting homomorphisms (each coming from a matrix of polynomials $\Delta_t^{(n,a,b)}(u,\, v,\, w)$ considered mod m) to be as large as possible. When we do this we get new ideals I_ν with new sets of n generators: but all of these new ideals are equal to m, because of the invertibility conditions that we have placed on the matrices. Likewise, the new ideals J_η that we get are also all equal to m.

All these are still generated by regular sequences, and so the ranks of the connecting homomorphisms corresponding to the extension of direct sums that comes from this new "general position" data are determined by the $\Delta_t^{(n,a,b)}(u,\, v,\, w)$. Therefore we have constructed an extension

$$0 \to K^a \to M' \to K^b \to 0$$

such that when one forms the long exact sequence for $\mathrm{Ext}_R^\bullet(\_,\, K)$ coming from it, the ranks of the connecting homomorphisms are all at least as large as they were for the original short exact sequence. As already noted in §2, formula (2), for a short exact sequence

$$0 \to B \to M \to A \to 0$$

one has

$$\alpha_t^M = \alpha_t^A + \alpha_t^B - \mathrm{rank}(\delta_{t-1}^M) - \mathrm{rank}(\delta_t^M).$$

In the transition from M to M', α_t^A and α_t^B do not change, while the ranks of the connecting homomorphisms can only increase. Thus, $\alpha_t^M \geq \alpha_t^{M'}$ for all t. The result now follows from Chang's Theorem 1.1 applied to M'. □

4 Ext-General Modules

It will be convenient in this section to assume that the residue class field of the local ring $(R,\, m,\, K)$ is infinite. There is no loss of generality in making this assumption in studying the BEH conjecture.

We shall say that a finite length module M over a regular local ring $(R,\, m,\, K)$ is *Ext-general* if for every submodule B of M, with $A = M/B$, the connecting homomorphisms in the long exact sequence for $\mathrm{Ext}_R^\bullet(\_,\, K)$ coming from the short exact sequence $0 \to B \to M \to A \to 0$ all have maximum rank. Let ϵ be the element of $\mathrm{Ext}_R^1(A,\, B)$ corresponding to the short exact sequence. As noted earlier, by Theorem 9.1 of [Mac] the connecting homomorphisms are induced by the Yoneda pairings

$$\mathrm{Ext}_R^1(A,\, B) \otimes_R \mathrm{Ext}_R^t(B,\, K) \to \mathrm{Ext}_R^{t+1}(A,\, K)$$

by restricting the first element in the pairing to be ϵ. Clearly, the ranks only depend on the image of ϵ in $K \otimes_R \mathrm{Ext}_R^1(A,\, B)$, and maximum rank for *all* connecting homomorphisms will evidently be achieved on a Zariski dense open subset of the affine variety corresponding to the vector space $K \otimes_R \mathrm{Ext}_R^1(A,\, B)$.

Given any R-module M that is not Ext-general, we can construct an R-module of the same length that has Betti numbers no larger than those of M, and with at least one of them strictly smaller. For we can choose

$$0 \to B \to M \to A \to 0$$

such that at least one of the associated connecting homomorphisms does not have maximum possible rank. With ϵ as above we simply replace this extension by another

$$0 \to B \to M' \to A \to 0$$

in which the element ϵ has been chosen so that its image has general position in $K \otimes_R \mathrm{Ext}^1_R(A,\, B)$. This cannot decrease the ranks of *any* of the connecting homomorphisms, and will increase at least the one that was not already maximum. This will produce an extension such that the module M' will have Betti numbers less than or equal to those of M, by formula (2) of §2, with at least one of them strictly smaller. We shall say that M' is an *immediate generalization* of M. A finite sequence of immediate generalizations will produce a module of the same length as M that we call a *generalization* of M. Given any finite length module M, it has a generalization, necessarily of the same length, which is Ext-general, for a sequence of successive generalizations is bounded in length by the sum of the Betti numbers of the original module: the sum decreases by at least one at each stage.

In particular, if there is a counterexample to the BEH conjecture, there is a counterexample in which the module is Ext-general. A study of Ext-general modules may be helpful in attacking the conjecture. Of course, these ideas are closely related to the proof of the main results here.

5 A Combinatorial Case

We desire greatly, of course, to remove the condition from Theorem 1.2 that the $J_\eta \subseteq I_\nu$. Currently we can do this only by adding conditions on a and b. We record the following elementary result.

Theorem 5.1. *Suppose that $I_1, \ldots, I_a$, and $J_1, \ldots, J_b \subseteq R$ are ideals generated by R-sequences and that M occurs in a short exact sequence*

$$0 \to \bigoplus_{\nu=1}^{a} (R/I_\nu) \to M \to \bigoplus_{\eta=1}^{b} (R/J_\eta) \to 0.$$

Then $\alpha_i^M \geq \binom{n}{i}$ provided that either $a \geq b(\frac{n-i}{i+1}) + 1$ or $b \geq a(\frac{n-i}{i+1}) + 1$. In particular, if $a \geq b(\frac{n-2}{3}) + 1$ or $b \geq a(\frac{n-2}{3}) + 1$ then $\alpha_i^M \geq \binom{n}{i}$ for all $i = 0, 1, \ldots, n$.

Proof. By taking Matlis duals (equivalently, applying $\mathrm{Ext}^n_R(\_, R)$) we may interchange the roles of a and b. Therefore, we may assume without loss of generality that $b \geq a$. Since the BEH conjecture is known if $i = 0, 1, n-1$, or n, we assume throughout the rest of the argument that $2 \leq i \leq n-2$.

Since the Betti numbers of the first and third modules in the short exact sequence of the theorem are $a\binom{n}{i}$ and $b\binom{n}{i}$, respectively, we have from (2) at the beginning of §2 that

$$\alpha_i^M = a\binom{n}{i} + b\binom{n}{i} - \mathrm{rank}(\delta_{i-1}^M) - \mathrm{rank}(\delta_i^M).$$

We use the naive estimates that $\mathrm{rank}(\delta_i^M) \leq a\binom{n}{i+1}$ and $\mathrm{rank}(\delta_{i-1}^M) \leq a\binom{n}{i}$ for $2 \leq i \leq n-2$ to obtain that $\alpha_i^M \geq a\binom{n}{i}+b\binom{n}{i}-a\binom{n}{i+1}-a\binom{n}{i} = b\binom{n}{i}-a\binom{n}{i+1} = \binom{n}{i}\big(b - a(\frac{n-i}{i+1})\big)$, which is $\geq \binom{n}{i}$ provided that $b \geq a(\frac{n-i}{i+1}) + 1$, as claimed. The final statement now follows because $\frac{n-i}{i+1}$ decreases as i increases, and we need only consider the range $2 \leq i \leq n-2$. □

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