Topics in Tight Closure Theory

by

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Dedicated to my advisor, Mel Hochster, for making this possible

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CHAPTER I

Introduction

Among recent developments in commutative algebra, tight closure theory, introduced in 1988 by Hochster and Huneke, stands apart because of the multitude and variety of its applications.

The first "tight closure" proofs were discovered before the notion was formally defined and studied, but in retrospective the presence of tight closure ideas can be clearly seen. Among the results using tight closure techniques we mention a few: the existence of big Cohen-Macaulay modules; rings of invariants of linearly reductive groups acting on regular rings are Cohen-Macaulay; the integral closure of the nth power of an n generated ideal of a regular ring is contained in the ideal (the Briançon-Skoda theorem); the monomial conjecture and the syzygy theorem (HH1).

Tight closure is a priori defined in positive characteristic p using the action of the Frobenius endomorphism, which acts by raising each element of the ring to the pth power. A notion for finitely generated algebras over a field of characteristic zero is obtained by reduction to characteristic p; this allows us to prove theorems in equal characteristic zero using characteristic p methods. Various notions have been proposed in mixed characteristic, but they do not share all the good properties that allow us to exploit tight closure in the equal characteristic case.

Some of the important features of tight closure include the fact that every ideal in a regular ring is tightly closed, and the tight closure of an ideal is contained in the integral closure. If R is a local ring and $x_1, \ldots, x_n$ are part of a system of parameters, then, under mild hypotheses, $(x_1, \ldots, x_{n-1}) : x_n$ is contained in the tight closure of the ideal $(x_1, \ldots, x_{n-1})$.

One of the most important theoretical breakthroughs in tight closure theory is the existence of test elements proved in [HH3]. Test elements can be thought of as uniform annihilators for tight closure, and their existence makes tight closure computations possible in some situations.

Although the applications of tight closure are numerous and well understood, very little is known about tight closure itself. It is difficult to find the tight closure of an arbitrary ideal even in very simple rings, such as the cubical cone ring

$$R = \frac{k[[X, Y, Z]]}{(X^3 + Y^3 + Z^3)},$$

and there is no general algorithm available. The reason for this difficulty is the fact that the definition of tight closure requires infinitely many tests in order to decide whether a given element is in the tight closure of an ideal. Empirical evidence suggests that in "simple" rings such as the one above there might be a bound for the number of tests one needs to perform. The existence of such a bound would considerably improve our understanding of tight closure, making calculations possible, as well as answering the long standing open question of whether tight closure commutes with localization.

An Outline of the Thesis

Chapter 2 gives a brief review of the definitions and notations used in this work, as well as of the basic results in tight closure theory.

In Chapter 3 we study the problem of finding a constant b such that m^{bq} annihilates the 0th local cohomology of $R/I^{[q]}$, where (R,m) is a characteristic p ring, and I is an ideal with $\dim(R/I) = 1$. We prove that such a bound exists when R is a graded ring, and I is a homogeneous ideal. The relevance of this result to tight closure theory lies in the connections with the localization problem. For example, one consequence of this result is the fact that for any homogeneous ideal of height $d-1$ in a graded ring of dimension d, tight closure commutes with localization at any multiplicative system.

In Chapter 4 we restrict attention to the case of the cubical cone ring

$$R = \frac{k[[x,y,z]]}{(x^3+y^3+z^3)},$$

where k is a field of characteristic $p \equiv 2 \pmod 3$, or sometimes to the larger class of Gorenstein local rings (R,m) with test ideal m, and in which the tight closure $\underline{\boldsymbol{a}}^*$ equals the Frobenius closure $\underline{\boldsymbol{a}}^F$ for every ideal $\underline{\boldsymbol{a}}$ generated by a system of parameters.

It is conjectured that in any ring satisfying these assumptions, and in particular in the cubical cone rings described above, we have $I^* = I^F$ for every ideal I. The principal focus of Chapter 4 is studying this problem for the particular case of m–primary ideals. The main cases where the result is known are the case of irreducible ideals contained in some parameter ideal, and the case of ideals of finite projective dimension. The question is still not settled, but we do give several criteria for when $I^* = I^F$, some of which involve rather subtle conditions on lengths (see Thm. 4.5.7, Prop. 4.5.8, Prop. 4.5.9).

Since the class of weakly F–regular rings (those rings for which $I^* = I$ for every ideal I) and the class of F–pure rings (for which $I^F = I$ for every ideal I) play an important role in the classification of singularities, it would be of considerable interest to study the class of rings for which $I^* = I^F$ for every ideal I.

We introduce the notion of corner powers $I^{<q>}$ of an m-primary ideal I, which can be thought of as the duals of the Frobenius powers $I^{[q]}$. These corner power ideals provide examples of ideals for which the Frobenius closure equals the tight closure, and equals the whole socle of the ideal. One of our main results (Thm. 4.5.7) states that the equality of tight closure and Frobenius closure implies, and in the case of the cubical cone ring of characteristic 2 is equivalent to the statement that for any m-primary ideal I and for all $q \gg 0$, we have $(I^{<q>})^* = (I^{<q>})^F = I^{<q>} : m$.

We also study some properties of the ring R^∞ obtained from R by adjoining all p^eth roots of elements of R. In general, this is not a Noetherian ring. It is noticed that whenever R^∞ is coherent, the equality of tight closure and Frobenius closure holds, but we also prove that for the ring

$$R = \frac{k[[x, y, z]]}{(x^3 + y^3 + z^3)},$$

with $\mathrm{char}(k) = 2$, R^∞ is not coherent.

There are various more subtle conditions that are weaker than the coherence of R^∞, but that still imply that $I^* = I^F$ for every ideal I (see Prop. 4.5.9, Cor. 5.2.11). The corner powers, and also the notion of special tight closure, introduced in Chapter 5, play an important role in formulating these conditions. We do not know whether these conditions hold in general in the cubical cone rings.

In Chapter 5 we introduce two new notions: $*$–independence, and special tight closure. The importance of $*$–independence lies in the fact that any ideal is contained in the tight closure of a $*$–independent ideal, therefore the tight closure of an arbitrary ideal can be identified with the tight closure of a $*$–independent ideal. We give a criterion for $*$–independence, and study various properties of $*$–independent ideals.

The special tight closure of an ideal is a distinguished subideal of its tight closure. Our main result in this section says that in the case of a normal Cohen–Macaulay graded domain over a perfect field, the tight closure of any $*$–independent ideal I admits a decomposition mod mI as the direct sum of the ideal itself and its special tight closure. Here, m is the homogeneous maximal ideal of R. This is striking because if one thinks of I^*/I as a kind of cohomology, the result gives canonical representatives for cohomology classes. It would be of great interest to know whether this phenomenon also exists in equal characteristic 0.

Among the consequences of the results in Chapter 5, we mention some equivalent formulations of the equality of tight closure and Frobenius closure in the class of rings considered in Chapter 4 (see Prop. 5.1.4, Cor. 5.2.7, Cor. 5.2.10). Although the theory of $*$–independence and special tight closure is valid in a more general context than that of the rings considered in Chapter 4, the study of tight closure in these rings, and in particular the problem of showing that it is equal to the Frobenius closure, was the initial motivation for introducing these new notions.

Let R be an excellent local domain, and R^+ be the integral closure in an algebraic closure of its fraction field. It is an important open question whether $I^+ := IR^+ \cap R$ is equal to I^*. In general, $I^+ \subset I^*$, and equality is known for parameter ideals (see [S1]). This question is also open in the cubical cone rings in characteristic $p \equiv 2 \pmod 3$; the similar question with Frobenius closure is stronger.

We hope that the notion of special tight closure might be helpful in answering questions about tight closure. Many open questions in tight closure theory involve a closure operation which is a priori smaller than the tight closure, but might in fact be equal to the tight closure. The case of the Frobenius closure studied in Chapter 4 is an example, in special classes of rings, but in an arbitrary ring, there is hope that

$I^* = I^+$ for every ideal I. This characterization of tight closure would be of great theoretical importance. We hope that using the special tight closure I^{*sp} instead of the whole tight closure will be helpful in studying whether I^* is contained in I^+ or I^F, since one will only need to prove the containment for the smaller ideal I^{*sp}.

Another topic of interest that we discuss in Chapter 5 is the existence of strong test ideals; we show that in any reduced complete local ring, the test ideal is a strong test ideal. Strong test ideals have been introduced by Huneke in 1997, and their existence was shown to have a number of interesting applications, but it was not known until now that strong test ideals exist in this generality.

CHAPTER II

Tight Closure

In this chapter we provide a brief review of the theory of tight closure. We shall concentrate mainly on the tight closure of ideals in a commutative Noetherian ring of characteristic $p > 0$.

Tight closure can also be defined for submodules of a module over a Noetherian ring, and for characteristic zero rings.

Throughout this chapter, R will denote a commutative Noetherian ring of characteristic p, and q will denote a power of the characteristic, $q = p^e$.

2.1 Notations

Definition: The *Frobenius endomorphism* $F : R \to R$ is defined by $F(r) = r^p$ for all $r \in R$; the eth iteration of the Frobenius endomorphism is denoted F^e: $F^e(r) = r^{p^e}$.

Define ${}^eR$ to be the (R, R)-bimodule equal to R as a group, with the usual action given by multiplication in R on the right, and the action induced by restriction of scalars via F^e on the left:

If $r \in R$, $s \in {}^eR$ we have $r * s = r^{p^e}s$ and $s * r = rs$.

The *Frobenius functor* is the functor on the category of R-modules defined by base change via F:

$F(M) = M \otimes_R {}^1R$ for every R-module M.

We can also define $F^e(M) = M \otimes_R {}^eR$.

Definition: If R is a reduced ring of characteristic p we can define $R^{1/q}$ to be the ring obtained from R by adjoining qth roots of all elements of R, where $q = p^e$.

The inclusion $R \hookrightarrow R^{1/q}$ is isomorphic with $F^e : R \to R$, and ${}^eR$ may be identified with $R^{1/q}$. We write R^∞ for the union $\cup_q R^{1/q}$. This is almost never a Noetherian ring, but from the perspective of tight closure the study of its properties is relevant even if one is mainly interested in Noetherian rings.

Definition: If I is an ideal of R generated by elements $f_1 \ldots, f_n$, then we define $I^{[q]}$ to be the ideal generated by $f_1^q, \ldots, f_n^q$.

Examples: If $M = R/I$ where I is an ideal, then $F^e(M) = R/I^{[q]}$.

If M has a presentation $R^m \xrightarrow{A} R^n \longrightarrow M$, then $F^e(M)$ is the cokernel of the map $R^m \xrightarrow{F^e(A)} R^n$, where the matrix of $F^e(A)$ is obtained from the matrix of A by raising each entry to the power $q = p^e$.

2.2 Tight closure

Definition: If R is a ring, we define $R^0 = \{x \in R | x \text{ is not in any minimal prime of R}\}$.

Definitions: Let I be an ideal of R, and $x \in R$. We say that x is in the *tight closure* of I and we write $x \in I^*$ if there exists an element $c \in R^0$ such that $cx^q \in I^{[q]}$ for all $q \gg 0$.

I is called *tightly closed* if $I = I^*$.

The ring R is called *weakly F–regular* if every ideal is tightly closed.

R is called *F–regular* if every localization $W^{-1}R$ of R is weakly F–regular.

It is conjectured that the two notions of F–regularity coincide. The equivalence of these two notions is known for Gorenstein rings. Moreover, there is a third notion, F–rationality, that is also equivalent with F–regularity in the Gorenstein case.

Definition: R is called *F–rational* if all parameter ideals are tightly closed (an ideal generated by n elements is a parameter ideal if it has height at least n).

We summarize a few facts about F–regular and F–rational rings:

Proposition 2.2.1. *a). A regular ring is F–regular.*

b). R is weakly F–regular if and only if R_m is weakly F–regular for all maximal ideals m.

c). If R_m is F–rational for all maximal ideals m, then R is F–rational.

d). An excellent local ring (R, m) is F–rational if and only if some ideal generated by a system of parameters is tightly closed.

e). If R is locally excellent and F–rational, then so is $W^{-1}R$, where W is any multiplicative system.

f). A Gorenstein ring is F–regular if and only if it is F–rational.

Proof: a). is Thm 4.4. [HH1], b). Cor 4.15 [HH1], c)., d)., e). Prop 1.4.3 (ii), (v), (vii) of [S1], f). is Thm 4.2 (g) of [HH5]. $\square$

Now we outline some of the basic properties of tight closure

Proposition 2.2.2. *Let R be a Noetherian ring of characteristic $p > 0$, and let I and J be ideals in R.*

a). I^* is an ideal containing I.

b). $(I^*)^* = I^*$.

c). If $I \subset J$, then $I^* \subset J^*$.

d). If the height of I is positive, or if R is reduced, then $x \in I^* \iff \exists\, c \in R^0$ such that $cx^q \in I^{[q]}$ for all $q = p^e$.

e). For any ideal I, I^* is the inverse image in R of $(IR_{\mathrm{red}})^*$

f). If I is tightly closed, then $I : J$ is tightly closed for all J.

g).$I^* \subset \overline{I}$, where $\overline{I}$ denotes the integral closure of I .

h). $x \in I^* \Leftrightarrow$ the image of x in R/P is in the tight closure of $(I+P)/P$ for every minimal prime P of R.

Proof: See [HH1]; a).-f). are in prop. 4.1; g), is theorem 5.2; h). is prop. 6.25 (a). $\square$

It follows from h). that we can always reduce tight closure computations to the domain case.

2.3 Test elements

Definition: $c \in R^0$ is a *test element* for R if for all ideals $I \subset R$ we have $u \in I^* \Leftrightarrow cu \in I$. $c \in R^0$ is a *completely stable test element* if its image in every local ring of R is a test element, and its image in the completion of every local ring of R is a test element.

Theorem 2.3.1. *Assume that R is a reduced excellent local ring. If $c \in R^0$ is such that R_c is regular, then c has a power which is a completely stable test element.*

This theorem is proved in [HH3].

In particular, completely stable test elements always exist in excellent reduced local rings.

When R has a test element, we define $\tau(R) = \tau$ to be the ideal generated by the test elements; it is called the *test ideal.*

One consequence of the previous theorem is that excellent local rings (R, m) that are isolated singularities have m-primary test ideals. Knowing that the test ideal is large is often helpful in computations.

CHAPTER III

The Localization Problem

In this chapter we present the localization problem, one of the most important open questions in tight closure theory. We concentrate on attempts to find a bound for the powers of the maximal ideal which annihilate the local cohomology of Frobenius powers.

Proving that such a bound exists in general would imply an affirmative answer to a weaker form of the localization problem, namely the equivalence of F–regularity and weak F–regularity. The main result of this chapter is the existence of a bound for local cohomology in the graded case.

3.1 The localization problem

Conjecture 3.1.1. *Let R be a Noetherian ring of characteristic p, W a multiplicative system, and I an ideal. Then*

$$I^*(W^{-1}R) = (IW^{-1}R)^*.$$

Note that the inclusion $I^*(W^{-1}R) \subset (IW^{-1}R)^*$ is obvious; indeed, if $cx^q \in I^{[q]}$ for all $q \gg 0$, with $c \in R^0$, then $c/1 \in (W^{-1}R)^0$, and $c/1(x/1)^q \in I^{[q]}(W^{-1}R) = (IW^{-1}R)^{[q]}$.

If this conjecture were true, then in particular the notions of F-regularity and weak F-regularity would coincide. Therefore, the following is a weaker conjecture:

Conjecture 3.1.2. *If R is a weakly F-regular ring and W is a multiplicative system, then $W^{-1}R$ is weakly F-regular.*

In the rest of this section, we shall review the present state of knowledge related to these two conjectures.

Proposition 3.1.3. *If m is a maximal ideal and I is an m-primary, then $(IR_m)^* = I^*R_m$.*

Proof: This is Prop. 4.14 of [HH1].

Proposition 3.1.4. *If R is locally excellent and locally equidimensional, then for every ideal I generated by parameters and for every multiplicative system W, we have $I^*(W^{-1}R) = (IW^{-1}R)^*$*

More generally, tight closure commutes with localization for $N \subset M$, where N and M are modules such that M/N has finite phantom projective dimension (see [AHH]).

Conjecture 3.1.2 has a positive answer for the case when R is a positively graded algebra over a field ([LS]).

3.2 Local cohomology

Definition: Let M be a module over R, and m a maximal ideal. Then

$$H^0_m(M) = \{x \in M | m^N x = 0 \text{ for some positive integer N}\}.$$

If M is finitely generated, then there is a positive integer N (depending on M and m) such that $m^N H^0_m(M) = 0$. The higher local cohomology modules $H^i_m(M)$ can be defined as the right derived functors of H^0_m, but we shall not study them here.

Conjecture 3.2.1. *Let (R, m) be a local Noetherian ring of characteristic p, and let I be an ideal primary to a prime P with $\dim(R/P) = 1$.*

Then there is a constant b depending only on I such that for all $q = p^e$ we have

$$m^{bq} H^0_m \left(\frac{R}{I^{[q]}} \right) = 0.$$

This question has been asked by Hochster and Huneke in [HH1]; their motivation came from the observation that Conjecture 3.2.1 implies Conjecture 3.1.2 (see the discussion after cor. 4.16 in [HH1]).

Moreover, if Conjecture 3.2.1 holds for a certain ideal I as above, then we can show that tight closure commutes with localization for I.

Proposition 3.2.2. *Let (R, m) be a local Noetherian ring of characteristic p, I an ideal primary to a prime P with $\dim(R/P) = 1$, and assume that Conjecture 3.2.1 holds for I. Then for every multiplicative system W, we have $I^*(W^{-1}R) = (IW^{-1}R)^*$.*

Proof: It is sufficient to check for $W = R \backslash P$. (See [AHH, Lemma 3.5 a).].)

Assume that $u/1 \in (IR_P)^*$ with $u \in R$; then there is a $c \in R^0$ such that $cu^q \in I^{[q]}R_P$. Hence for every $q = p^e$, there is an $f_q \in R \backslash P$ such that $cf_q u^q \in I^{[q]}$.

Since $I^{[q]} + (f_q)$ is an m-primary ideal, it follows that for all q, the class of cu^q in $R/I^{[q]}$ is in $H^0_m(R/I^{[q]})$, and our assumption implies that $m^{bq}cu^q \subset I^{[q]}$, where b is constant. Fix an element $f \in m \backslash P$; then we have

$$cf^{bq}u^q \in I^{[q]} \Rightarrow f^b u \in I^* \Rightarrow u \in I^* R_P. \qquad \square$$

In the rest of this chapter, we will focus in proving Conjecture 3.2.1 in certain special cases.

We will frequently use the following:

Observation : Given (R, m) local ring of Krull dimension $n+1$, M a finitely generated module with $\dim(M) = 1$, we can choose a system of parameters $x_1, x_2, \ldots, x_n, z$ for R, with $x_1, \ldots, x_n \in \mathrm{Ann(M)}$. There is an h such that $m^h \subset (x_1, \ldots, x_n, z)$, and therefore $m^{(n+1)hq} \subset (x_1^q, \ldots, x_n^q, z^q)$. Since $F^e(M)$ is annihilated by $x_1^q, \ldots, x_n^q$, we have:

$$H_m^0(F^e(M)) = \bigcup_N \mathrm{Ann}_{F^e(M)}(z^N),$$

and if c is such that for every $q = p^e$,

$$\bigcup_N \mathrm{Ann}_{F^e(M)}(z^N) = \mathrm{Ann}_{F^e(M)}(z^{cq}),$$

then the conclusion of the conjecture is satisfied, with $b = ch(n+1)$.

The following notation will be useful:

Notation: If I and J are ideals of a Noetherian ring R, let $I : J^\infty := \bigcup_N (I : J^N)$. With this notation,

$$H_m^0\left(\frac{R}{I^{[q]}}\right) = \frac{I^{[q]} : m^\infty}{I^{[q]}},$$

and the previous observation shows that if $\mathrm{ht}(I) = \dim(R) - 1$ and z is not in any minimal prime of I, then

$$H_m^0\left(\frac{R}{I^{[q]}}\right) = \frac{I^{[q]} : z^\infty}{I^{[q]}}.$$

The conjecture can be settled when R is a 2-dimensional normal ring:

Proposition 3.2.3. *If R is normal Noetherian ring of dimension 2 and characteristic $p > 0$, I is an ideal, and m a maximal ideal containing I, then there is a constant b such that $\forall q = p^e$, $H_m^0(R/I^{[q]})$ is annihilated by m^{bq}.*

Proof: We may replace R by R_m, I by IR_m, and therefore assume that (R, m) is local. If $I = 0$ or I is m-primary, the statement is clearly true. Assume that I has height 1, and let $p_1, \ldots, p_n$ be the minimal primes of I. For each i, R_{p_i} is a

DVR, hence IR_{p_i} is a principal ideal; choose $u_i \in I$ such that $IR_{p_i} = (u_i)R_{p_i}$. Then $I^{[q]}R_{p_i} = (u_i^q)R_{p_i}$. For each i, choose z_i in all $q \in \mathrm{Min(R/u_i)}$ except p_i. Since R is Noetherian, for each i we can find c_i such that $(u_i) : z_i^\infty = (u_i) : z_i^{c_i}$; this implies $(u_i^q) : z_i^\infty = (u_i^q) : z_i^{qc_i}$ for all q. This follows because $R/(u^q)$ has a filtration with factors $(u^n)/(u^{n+1}), n = 0, 1, \ldots, q-1$, and $(u^n)/(u^{n+1}) \cong R/(u)$.

We have

$$I^{[q]} : m^\infty = \{f \in R | \exists\ z \text{ not in any minimal prime of } I \text{ such that } fz \in I^{[q]}\}$$

$$= I^{[q]}R_{p_1} \cap \ldots \cap I^{[q]}R_{p_n} \cap R.$$

But $I^{[q]}R_{p_i} = (u_i^q)R_{p_i} \cap R = (u_i^q) : z_i^\infty$ since $p_i + (z_i)$ is m-primary. It follows that for all $f \in I^{[q]} : m^\infty$ and for all i, we have $f \in (u_i^q) : z_i^{qc_i}$, hence $f \in (u_1^q, \ldots, u_n^q) : (z_1^{qc_1}, \ldots, z_n^{qc_n}) \subset I^{[q]} : (I + (z_1^{c_1}, \ldots z_n^{c_n}))^{[q]} \subset I^{[q]} : m^{cq}$ for some c, since $I + (z_1^{c_1}, \ldots z_n^{c_n})$ is m-primary. $\square$

3.3 Main result

In this section we will concentrate on proving Conjecture 3.2.1 for the particular case where R is a graded affine algebra, and I is a homogeneous ideal. More precisely, we will prove the following:

Theorem 3.3.1. *Let k be a field of characteristic $p > 0$, and let q denote a power of p. Let R be a finitely generated, positively graded algebra over k, of Krull dimension $d = n + 1$, and I a homogeneous ideal of height n. Then there is a constant b such that for all $q = p^e$, $H_m^0(R/I^{[q]})$ is annihilated by m^{bq}, where m is the homogeneous maximal ideal.*

More generally, if M is a finitely generated graded R-module with $\dim(M) = 1$, *then there is a constant b such that for all $q = p^e$, $H_m^0(F^e(M))$ is annihilated by m^{bq}.*

This result (in the ideal case) has also been recently obtained by C. Huneke in [Hu3], using very different methods.

We need the following preliminary results:

Lemma 3.3.2. *Let $\mathcal{F}, \mathcal{G}$ be finitely generated free $\mathbb{Z}$- graded modules over $k[Z]$, where the grading on $k[Z]$ is obtained by assigning some positive degree to the variable Z, and $A : \mathcal{F} \to \mathcal{G}$ be a linear map. Let $\{f_1, f_2, \dots, f_n\}$ and $\{g_1, g_2, \dots, g_m\}$ be homogeneous bases for $\mathcal{F}$, respectively $\mathcal{G}$.*

Assume that $A(f_i)$ is homogeneous for all i.

Then there is a choice of bases for $\mathcal{F}$ and $\mathcal{G}$ such that A is given by a diagonal matrix, and the diagonal entries are homogeneous of degree at most $N := \max(\deg(Af_i)) - \min(\deg(g_j))$. (Therefore they are monomials in which the exponent on Z is bounded by $N/\deg Z$.)

Proof: By defining a new grading on $\mathcal{F}$, we can assume without loss of generality that A preserves degrees of homogeneous elements, therefore $N = \max(\deg(f_i)) - \min(\deg(g_j))$. For each i, we can write

$$A(f_i) = \sum_j \alpha_{ij} Z^{b_{ij}} g_j,$$

where $\alpha_{ij} \in k$, and the b_{ij}'s are nonnegative integers. After renumbering the f's and g's, we may assume that $\alpha_{11} \neq 0$, and b_{11} is minimal among all b_{ij} with $\alpha_{ij} \neq 0$. Say that

$$A(f_1) = (\alpha_{11} Z^{b_{11}}) g_1 + \cdots$$

with $\alpha_{11} \in k \backslash \{0\}$, and for every $i > 0$

$$A(f_i) = (\alpha_{i1} Z^{bi1}) g_1 + \cdots .$$

After the change of basis

$$f_i' = f_i - Z^{b_{i1}-b_{11}} \frac{\alpha_{i1}}{\alpha_{11}} f_1,$$

$A(f_i')$ will not contain g_1, and f_i' is also homogeneous of the same degree as f_i. Indeed, we have

$$\begin{aligned} \deg f_i &= \deg(A(f_i)) = (\deg Z) b_{i1} + \deg g_1, \\ \deg f_1 &= \deg(A(f_1)) = (\deg Z) b_{11} + \deg g_1, \quad \Rightarrow \\ \deg f_i &= \deg f_1 + (\deg Z)(b_{i1} - b_{11}). \end{aligned}$$

Also define

$$g_1' = g_1 + \frac{A(f_1) - \alpha_{11} Z^{b_{11}} g_1}{\alpha_{11} Z^{b_{11}}}.$$

g_1' is equal to g_1 plus a linear combination of $g_2, \ldots, g_m$, therefore $\{g_1', g_2, \ldots, g_m\}$ is again a basis for $\mathcal{G}$, and g_1' is homogeneous of the same degree as g_1. Indeed, we get that $A(f_1) - \alpha_{11} Z^{b_{11}} g_1$ is homogeneous of degree $(\deg Z) b_{11} + \deg g_1$, and so $(A(f_1) - \alpha_{11} Z^{b_{11}} g_1)/\alpha_{11} Z^{b_{11}}$ is homogeneous of degree equal to the degree of g_1.

Now we have $A(f_1) = \alpha_{11} Z^{b_{11}} g_1'$. In the new bases $\{f_1', \ldots, f_n'\}$ and $\{g_1', g_2, \ldots, g_m\}$, the matrix of A has only zeroes on the first row and column, except on the position $(1,1)$, and the entry on this position is $\alpha_{11} Z^{b_{11}}$, which has degree $b_{11} \deg Z = \deg(f_1) - \deg(g_1) \leq N$. Now look at the remaining part of the matrix. Since the degrees are preserved for the new bases, the result follows by induction on n and m.

Lemma 3.3.3. *Let R be finitely generated algebra over a field k of positive characteristic p, with dim(R) $= n+1$. Let M be a finitely generated R-module of dimension 1. Assume that $x_1, \ldots, x_n, z \in R$ are such that R is module-finite over the subalgebra $k[x_1, \ldots, x_n, z]$, that $x_1, \ldots, x_n, z$ are algebraically independent over k, and that $x_1, \ldots, x_n \in \mathrm{Ann(M)}$. If R is graded, take $x_1, \ldots, x_n, z$ to be a homogeneous system of parameters. Then there exist constants C, a_1, a_2 such that $\forall e > 0, F^e(M)$ has a* *presentation as a $k[Z]$-module given by a matrix of size $a_1q^n \times a_2q^n$, in which the degree in Z of each entry is at most Cq, where $q = p^e$.*

If moreover R and M are graded and $x_1, \ldots, x_n, z$ are homogeneous, then the presentation of $F^e(M)$ satisfies the conditions in Lemma 1.

Proof: R is module-finite over the polynomial ring

$$R_0 = k[X_1, \ldots, X_n, Z] \cong k[x_1, \ldots, x_n, z];$$

write

$$R = \frac{R_0e_1 \oplus \cdots \oplus R_0e_s}{< r_1, \ldots, r_t >},$$

where $r_i = \sum_j r_{ij}e_j$, with $r_{ij} \in R_0$.

If M has a presentation over R given by a $N \times k$ matrix A, then $F^e(M)$ has a presentation given by the matrix $F^e(A)$ in which every entry of A is raised to the qth power. If $f_1, \ldots, f_N$ are the generators of M, then the generators of $F^e(M)$ are $1 \otimes f_1, \ldots, 1 \otimes f_N$ (recall that $F^e(M) = S \otimes M$, where S is the R-algebra obtained by the action of the Frobenius morphism on R); if $f_1, \ldots, f_N$ are homogeneous of degrees $d_1, \ldots, d_N$, then $1 \otimes f_1, \ldots, 1 \otimes f_N$ are homogeneous of degrees $qd_1, \ldots, qd_N$. To simplify notation, we shall write f'_j instead of $1 \otimes f_j$.

We can write

$$F^e(M) = \frac{Rf'_1 \oplus \cdots \oplus Rf'_N}{< a_1^q, \ldots, a_k^q >},$$

where $a_i^q = \sum_{j=1}^N a_{ij}^q f'_j$, with $a_{ij} \in R$. In the graded case, all generators and relations are homogeneous.

Since $X_1^q, \ldots, X_n^q$ kill $F^e(M)$, we can also write

$$F^e(M) = \frac{Rf'_1 \oplus \cdots \oplus Rf'_N}{< a_1^q, \ldots, a_k^q, X_1^q f'_1, \ldots, X_n^q f'_1, \ldots, X_1^q f'_N, \ldots, X_n^q f'_N >} = \frac{\mathcal{F}}{\mathcal{G}}.$$

In order to get a presentation of $F^e(M)$ as an R_0-module, note that

$$\mathcal{F} = Rf_1' \oplus \cdots \oplus Rf_N' = \frac{\bigoplus R_0 e_i f_j'}{< r_1 f_1', \ldots, r_n f_1', \ldots, r_1 f_N', \ldots, r_n f_N' >},$$

and as an R_0-submodule of $\mathcal{F}, \mathcal{G}$ is generated by

$$a_1^q e_1, \ldots, a_k^q e_1, \ldots, a_1^q e_s, \ldots, a_k^q e_s,$$

and the $X_\alpha^q f_\beta' e_\gamma$'s, $\forall \alpha, \beta, \gamma$.

In order for this to give a presentation over R_0, we need to express each $a_i^q e_\tau = \sum a_{ij}^q f_j' e_\tau$ as an R_0-linear combination of the $f_j' e_i$'s; this can be done by writing each a_{ij} as an R_0-linear combination of e_α's, and each $e_\alpha e_\beta = \sum_\gamma s_{\alpha\beta\gamma} e_\gamma$, with $s_{\alpha\beta\gamma} \in R_0$.

Since $F^e(M)$ is annihilated by $X_1^q, \ldots, X_n^q$, we can give a presentation of it as an R_1-module, where

$$R_1 = \frac{k[X_1, \ldots, X_n, Z]}{(X_1^q, \ldots, X_n^q)},$$

by reducing all coefficients in R_0 modulo $(X_1^q, \ldots, X_n^q)$:

$$F^e(M) = \frac{\mathcal{F}'}{\mathcal{G}'},$$

where $\mathcal{F}' = \bigoplus R_1 e_i f_j'$, and $\mathcal{G}'$ is the submodule generated by

$$\overline{r_1} f_1', \ldots, \overline{r_n} f_1', \ldots, \overline{r_1} f_N', \ldots, \overline{r_n} f_N', \overline{a_1^q e_1}, \ldots, \overline{a_k^q e_1}, \ldots, \overline{a_1^q e_s}, \ldots, \overline{a_k^q e_s},$$

where $\overline{a_i^q e_j}$ means that we first express $a_i^q e_j$ as an R_0-linear combination of the $f_i' e_j$'s, and then reduce each coefficient modulo $(X_1^q, \ldots, X_n^q)$.

R_1 is a finitely generated free $k[Z]$-module with basis

$$\{X_1^{i_1} \ldots X_n^{i_n} | 0 \le i_1, \ldots, i_n < q\}.$$

To get a presentation for $F^e(M)$ as a $k[Z]$-module, we need to express $\mathcal{F}', \mathcal{G}'$ as $k[Z]$-modules: $\mathcal{F}'$ is the free $k[Z]$-module with basis

$$\{X_1^{i_1} \ldots X_n^{i_n} e_i f_j' | 0 \le i_1, \ldots, i_n < q; 1 \le i \le s; 1 \le j \le N\},$$

and $\mathcal{G}'$ is the submodule of $\mathcal{F}'$ generated as a $k[Z]$-module by all $\overline{r_i f_j' X_1^{i_1} \dots X_n^{i_n}}$, and $\overline{a_i^q e_j X^{i_1} \dots X_n^{i_n}}$. We can write

$$F^e(M) = \mathrm{Coker}(A : G_{q1}^{\oplus t} \oplus G_{q2}^{\oplus k} \longrightarrow F_q),$$

where $F_q = \mathcal{F}'$,

$$G_{q1} = \bigoplus_{0 \le i_1, \dots, i_n < q; 1 \le j \le N} k[Z] X_1^{i_1} \dots X_n^{i_n} f_j',$$
$$G_{q2} = \bigoplus_{0 \le i_1, \dots, i_n < q; 1 \le j \le s} k[Z] X_1^{i_1} \dots X_n^{i_n} e_j,$$

and if we denote the generators in the i^{th} copy of G_{q1}, respectively G_{q2} by $(X_1^{i_1} \dots X_n^{i_n} f_j')_i$, respectively $(X_1^{i_1} \dots X_n^{i_n} e_j)_i$, we have

$$A((X_1^{i_1} \dots X_n^{i_n} f_j')_i) = \overline{r_i X_1^{i_1} \dots X_n^{i_n}} f_j',$$

$$A((X_1^{i_1} \dots X_n^{i_n} e_j)_i) = \overline{a_i^q e_j X_1^{i_1} \dots X_n^{i_n}},$$

where the overline means that each product is represented as R_0-linear combination of $e_i f_j'$'s, and the coefficients are reduced modulo $(X_1^q, \dots, X_n^q)$, such that the result is in F_q.

Note that F_q has rank sNq^n, and $G_{q1}^{\oplus t} \oplus G_{q2}^{\oplus k}$ has rank $(tN + ks)q^n$, and t, N, k, s, n are independent of q.

In the graded case, we have

$$\deg(A(X_1^{i_1} \dots X_n^{i_n} f_j')_i) = \deg(r_i) + \deg((X_1^{i_1} \dots X_n^{i_n} f_j')_i), \text{ or}$$

$$A((X_1^{i_1} \dots X_n^{i_n} f_j')_i) = 0,$$

and

$$\deg(A(X_1^{i_1} \dots X_n^{i_n} e_j)_i) = \deg(a_i^q) + \deg((X_1^{i_1} \dots X_n^{i_n} e_j)_i), \text{or}$$

$$A((X_1^{i_1} \dots X_n^{i_n} e_j)_i) = 0.$$

In the general case, we need to keep track of the exponent on Z. The exponent on Z in $\overline{r_i X^{i_1} \dots X_n^{i_n} f'_j}$ is bounded by the highest exponent appearing in the r_i's; call it c_1.

In order to evaluate the exponent on Z in $\overline{a_i^q e_j X_1^{i_1} \dots X_n^{i_n}}$, we need to look at $a_i^q e_j$, expressed as an R_0-linear combination of $e_\tau f_\nu$'s.

Recall that $a_i^q = \sum a_{ij}^q f'_j$, with $a_{ij} \in R$; $a_{ij} = \sum a_{ij\tau} e_\tau$ with $a_{ij\tau} \in R_0$; then $a_{ij}^q = \sum a_{ij\tau}^q e_\tau^q$, and therefore $a_i^q e_j = \sum a_{i\eta\tau}^q e_\tau^q e_j f_\eta$.

For all τ, let $e_\tau^p = \sum v_{\tau\alpha} e_\alpha$, with $v_{\tau\alpha} \in R_0$. Then

$$e_\tau^{p^2} = \sum v_{\tau\alpha}^p e_\alpha^p = \sum_{\alpha,\beta} v_{\tau\alpha}^p v_{\alpha\beta} e_\beta.$$

If c_2 is the highest exponent on Z appearing in the $v_{\alpha\beta}$'s, we can show by induction on e that the exponents in the coefficients of $e_\tau^{p^e}$ are bounded by $c_2(p^{e-1}+p^{e-2}+\dots+1) \leq c_3 p^e = c_3 q$, where $c_3 = c_2/(p-1)$.

Let b_1 be the highest exponent in the $a_{ij\tau}$'s, and b_2 be the highest exponent in the $s_{ij\tau}$'s, where $e_i e_j = \sum s_{ij\tau} e_\tau$, $s_{ij\tau} \in R_0$. Then the exponent on Z in $a_i^q e_j$ is bounded by $qb_1 + c_3 q + b_2 \leq Cq$, where $C = b_1 + b_2 + c_3$.

Proof of Theorem: The first statement follows from the second.

Let $x_1, \dots, x_n, z$ be a homogeneous system of parameters, with $x_1, \dots, x_n \in$ Ann(M).

From the observation, it suffices to find $c > 0$ such that for every $q = p^e, y \in F^e(M)$, $z^N y = 0 \Rightarrow z^{cq} y = 0$.

By Noether normalization, R is module-finite over the subalgebra generated by $x_1, \dots, x_n, z$, which is isomorphic to the polynomial ring $k[X_1, \dots, X_n, Z]$. Write:

$$R = \frac{k[X_1, \dots, X_n, Z]e_1 \oplus \dots \oplus k[X_1, \dots, X_n, Z]e_s}{(r_1, \dots, r_t)}$$

with $e_1, \dots, e_s$ homogeneous, and $r_1, \dots, r_t$ homogeneous relations in $e_1, \dots, e_s$. Let α_0 be the degree of z, $\alpha_1, \alpha_2, \dots, \alpha_n$ be the degrees of $x_1, \dots, x_n$. Let $f_1, \dots, f_N$ be homogeneous generators for M. Then $f'_1, \dots, f'_N$ (as in the proof of Lemma 3.3.3, $f'_j = 1 \otimes f_j$) are homogeneous generators for $F^e(M)$, of degrees equal to

$$q\text{deg}(f_1), \dots, q\text{deg}(f_N).$$

Let $a_1^q, \dots, a_k^q$ be homogeneous relations among these generators (as in the proof of Lemma 3.3.3). Let $\alpha = \max(\alpha_i)$, $r = \max(\deg(r_i))$, $e = \max(\deg(e_i)), a = \max(\deg(a_i)), f = \max(\deg(f_i))$. Consider $k[Z]$ with the grading obtained by $\text{deg}Z = \alpha_0$.

Let F_q be the free graded $k[Z]$- module on the generators $\{X_1^{i_1} \dots X_n^{i_n} e_i f'_j\}$, with $0 \leq i_1, \dots, i_n < q$, $1 \leq i \leq s$, $1 \leq j \leq N$. The degrees of these generators are

$$0 \leq \alpha_1 i_1 + \ldots + \alpha_n i_n + \deg(e_i) + q\text{deg}(f_j) \leq \alpha n q + e + qf$$

By Lemma 2, we know that $F^e(M)$ has a presentation as a $k[Z]$-module given by a matrix

$$A : G_{q1}^{\oplus t} \oplus G_{q2}^{\oplus k} \longrightarrow F_q,$$

where G_{q1}, G_{q2} are as in the proof of Lemma 2.

Apply Lemma 1 with $\mathcal{F}$=$G_{q1}^{\oplus t} \oplus G_{q2}^{\oplus k}$, $\mathcal{G}$=F_q. The proof of Lemma 2 also shows that the N_i's we use in Lemma 1 are in the set

$$\{\deg(r_1), \dots, \deg(r_t), q\text{deg}(a_1), \dots, q\text{deg}(a_k)\}.$$

Therefore $N \leq r + aq$, where r and a are constants, depending only on R and M.

Lemma 1 shows that the degrees of the entries in the diagonal form of the matrix are bounded by

$$N + n\alpha q + e + qf \leq (n\alpha + a + f)q + r + e,$$

and therefore the exponents on Z are bounded by $((n\alpha + a + f)q + r + e)/\alpha_0$, which is a linear function of q.

But if these degrees are $b_1, \ldots, b_l$, we have

$$F^e(M) = (\text{free } k[Z] - \text{module}) \oplus \frac{k[Z]}{(Z^{b_1})} \oplus \cdots \oplus \frac{k[Z]}{(Z^{b_l})},$$

and hence this bound also bounds the power of Z that is needed to kill the Z-torsion in $F^e(M)$.

The following observation is the best result we know so far in the non-graded case, and was first proved by P. Roberts (unpublished) using very different methods. The proof we present here is more elementary.

Proposition 3.3.4. *Let R be a finitely generated algebra over a field k of characteristic $p > 0$, M a finitely generated 1-dimensional R-module. Then for every maximal ideal $m \in \mathrm{Supp(M)}$, if $d = \dim(R_m)$, there is a constant B such that for all $q = p^e, m^{Bq^d}$ kills $H^0_m(F^e(M_m))$.*

Proof: Since $\dim(R_m) = d$, we may choose $f \in R\backslash m$ such that $\dim(R_f) = d$. Indeed, if $\dim(R) = d$, we are done; if $\dim(R) > d$, then for any $P \in \mathrm{Min(R)}$ with $\dim(R/P) > d$, we have $P \not\subset m$. Choose f_1 in all such P but not in m; then $\dim(R_{f_1}) < \dim(R)$; replace R by R_{f_1}, and continue until the condition is satisfied.

Let $I = \mathrm{Ann}(M)$. By Noether normalization, we can pick $x_1, \ldots, x_{d-1} \in I$, $z \in m$, such that R is module-finite over $k[x_1, \ldots, x_{d-1}, z]$.

By Lemma 2, we have a presentation of $F^e(M)$ as a $k[z]$-module, given by a matrix A in which the degree of each entry is bounded by Cq, where C is a constant (independent of q.) We also know that the size of this matrix is $a_1q^{d-1} \times a_2q^{d-1}$, where a_1, a_2 are constants.

The matrix A is row and column equivalent to a diagonal matrix, in which the nonzero diagonal entry on the i^{th} row and column is

$$\frac{\text{gcd(all minors of size } i)}{\text{gcd(all minors of size } (i-1))},$$

and obviously has degree at most Ciq.

Let $a = \max(a_1, a_2)$. Then the degrees of the diagonal entries are bounded by Caq^d, let $B' = Ca$.

This shows that the $z^{B'q^d}$ kills the z-torsion of $F^e(M)$, hence it kills the z-torsion of $F^e(M)_m = F^e(M_m)$, which is exactly $H^0_m(F^e(M_m))$. Therefore there is a constant B such that m^{Bq^d} kills $H^0_m(F^e(M_m))$.

CHAPTER IV

Tight Closure in Cubical Cone Rings

4.1 Motivation

Due to the fact that deciding whether an element is in the tight closure of an ideal involves infinitely many computations, tight closure is in general very difficult to compute.

The only class of rings of dimension at least 2 for which we can explicitly describe the tight closure of an arbitrary ideal is the class of weakly F-regular rings, in which every ideal is equal to its tight closure.

Attempting a description of tight closure even for very simple rings, such as the cubical cone ring

$$R = \frac{k[[X, Y, Z]]}{(X^3 + Y^3 + Z^3)},$$

gives rise to many interesting questions.

In this chapter we will concentrate on finding various characterizations of tight closure in this ring, where k is a field of positive characteristic p. Some of the results are in fact valid for more general rings, as we will specify later.

The main question studied in this chapter is the relationship between tight closure and Frobenius closure for ideals in the cubical cone ring

$$R = \frac{k[[X, Y, Z]]}{(X^3 + Y^3 + Z^3)},$$

where k is a field of characteristic $p \equiv 2 \pmod 3$. It is conjectured that the two closure operations coincide in this case. This question has also been studied by M. McDermott in her thesis [Mc]. We find some classes of ideals for which equality holds (see Thm. 4.3.13, Prop. 4.4.2, and Cor. 4.4.6), but the general question is still unsettled.

For the case of m-primary ideals, the problem admits equivalent formulations in terms of Hilbert-Kunz functions. When R is the cubical cone ring in characteristic 2, the Hilbert-Kunz function of an m-primary ideal I can be explicitly described using the length of the Tor with $R^{1/2}$. This observation allows us to obtain several equivalent formulations of this problem (Thm. 4.5.7, Prop. 4.5.8, Prop. 4.5.9), which involve the notion of corner powers of an ideal.

If the characteristic of k is $p \equiv 1 \pmod 3$, then it is known that tight closure and Frobenius closure do not coincide, since in this case it is known from [HR] that R is F-pure, i.e. every ideal is equal to its Frobenius closure, while for example $z^2 \in (x,y)^*$, and therefore we have $z^2 \in (x,y)^* \backslash (x,y)^F$.

We are also studying the same question for a more general class of rings, namely all local Gorenstein rings (R,m) of positive characteristic in which the test ideal is equal to the maximal ideal m and the equality of tight closure and Frobenius closure is known to hold for every ideal generated by a system of parameters.

Definition If (R,m) is a local ring of Krull dimension d, an ideal $\underline{\boldsymbol{a}}$ is called a *parameter ideal* if it is m-primary and generated by d elements (forming a system of parameters).

In Gorenstein rings it is reasonable to expect that a property related to tight closure that holds for ideals generated by parameters will hold for every ideal; for example this is the case for the property of the ideal being equal to its tight closure.

The hypothesis on the test ideal is helpful in the sense that it restricts the set of elements which could be in the tight closure of a given ideal; under this assumption, we have

$$I^* = \bigcap_q \{x \in R | x^q \in I^{[q]} : m\}.$$

Definition: The ideal $I : m$ is called the socle of the ideal I.

The above characterization of tight closure says that, assuming that the test ideal is m, $x \in I^*$ if and only if x^q is in the socle of $I^{[q]}$ for all $q = p^e$.

Another interesting question, for which there is some experimental evidence, at least in the cubical cone ring in characteristic 2, is whether there is a $q_0 = p^{e_0}$ independent of I such that $x \in I^*$ if and only if $x^{q_0} \in I^{[q_0]} : m$. The experimental evidence we have so far suggests that for the cubical cone ring in characteristic 2, $q_0 = 4$ would work.

A positive answer to the above question would also solve the localization problem.

Our main focus in this chapter is the study of tight closure of m-primary ideals. The existence of a bound as above would allow us to generalize the results we know for m-primary ideals to results about arbitrary ideals.

4.2 Preliminary results

In this section we will prove that the ring

$$R = \frac{k[[X, Y, Z]]}{(X^3 + Y^3 + Z^3)}$$

where k is a field of characteristic $p \equiv 2 \pmod 3$ has the three properties claimed in the previous section. The lower case letters will denote the images of the upper case letters in R. Let $m = (x, y, z)$.

Most of the results in this section were previously known, but we include the proofs for completeness.

We make the following convention:

Convention: In the rest of this chapter, $\underline{\boldsymbol{a}}$, $\underline{\boldsymbol{b}}$, $\underline{\boldsymbol{c}}$ will denote parameter ideals.

R is clearly a Gorenstein ring, since it is a hypersurface. It is also known that in this ring the test ideal is equal to m; see [Mc]. It remains to check that the tight closure of parameter ideals is equal to the Frobenius closure.

Lemma 4.2.1. *Let* $R = k[[X, Y, Z]]/(X^3 + Y^3 + Z^3)$.

If $p = 3k+2$, *then* $z^{2p} \in (x^p, y^p)$, *and therefore we have* $(x, y)^* = (x, y)^F = (x, y) : m$.

Proof: $z^{2p} = -(x^3 + y^3)^{2k+1}z$, and therefore it suffices to see that each term in the expansion of $(x^3 + y^3)^{2k+1}$ is divisible by one of x^p OR y^p.

Look at $x^{3i}y^{3(2k+1-i)}$. If this term is not divisible by x^p, then $3i < p = 3k + 2 \Rightarrow i \leq k \Rightarrow 3(2k + 1 - i) \geq 3k + 3 = p + 1$, and therefore y^p divides it. $\square$

The following lemma allows us to reduce questions about ideals generated by a more complicated system of parameters to questions about ideals generated by a simple system of parameters. We will state it in greater generality that needed here for future reference.

Lemma 4.2.2. *If R is any Cohen-Macaulay ring of characteristic p, $x_1, \dots, x_d$ a system of parameters, then for each $t > 0$ we have an injective map*

$$\frac{R}{(x_1, \dots, x_d)} \xrightarrow{(x_1 \dots x_d)^{t-1}} \frac{R}{(x_1^t, \dots, x_d^t)}$$

If $I \supset (x_1, \dots, x_d)$ and $u \in R$, then we have

$$u \in I^* \Leftrightarrow (x_1 \dots x_d)^{t-1}u \in \left((x_1^t, \dots, x_d^t) + (x_1 \dots x_d)^{t-1}I\right)^*.$$

The similar statement for Frobenius closure is also true.

Proof: ($\Rightarrow$) is immediate.

($\Leftarrow$) If the statement on the right hand side holds, then there exists $c \in R^0$ such that $\forall q = p^e$, $\exists i_q \in I^{[q]}$, such that

$$c(x_1 \dots x_d)^{q(t-1)}u^q - (x_1 \dots x_d)^{q(t-1)} i_q \in (x_1^{tq}, \dots, x_d^{tq})$$

The latter holds

$$\Leftrightarrow cu^q - i_q \in (x_1^{tq}, \dots, x_d^{tq}) : (x_1 \dots x_d)^{q(t-1)} = (x_1^q, \dots, x_d^q),$$

i.e. $\Leftrightarrow cu^q \in I^{[q]}$. since we have this for all q, $u \in I^*$. The proof for Frobenius closure is similar. $\square$

The following is now immediate from Lemma 4.2.1 and Lemma 4.2.2.

Corollary 4.2.3. *Let R be the cubical cone ring in characteristic $p = 3k+2$; then for every $t > 0$, we have $(x^{t-1}y^{t-1}z^2)^p \in (x^{tp}, y^{tp})$, and therefore*

$$(x^t, y^t)^* = (x^t, y^t)^F = (x^t, y^t) : m.$$

The next step is to prove this property for an ideal generated by an arbitrary system of parameters; in fact we can do even better: we can prove it for any m-primary irreducible ideal that is contained in some parameter ideal. Before we state the next result, we will review some facts about irreducible ideals, which will be used throughout this chapter.

Definition Let R be a Noetherian ring. An ideal I of R is called *irreducible* if whenever $I = I_1 \cap I_2$ with I_1, I_2 ideals, we must have $I_1 = I$ or $I_2 = I$.

Facts: 1. If (R, m) is a local ring, and I is an m-primary ideal, then I is an irreducible ideal if and only if $l((I : m)/I) = 1$.

2. If (R, m) is a Gorenstein local ring, and I is an ideal, then I is irreducible if and only if R/I is Gorenstein. Any parameter ideal in a Gorenstein ring is irreducible.

3. If (R, m) is a local ring with residue field k, $E_R(k)$ denotes the injective hull of k over R. If R is a zero-dimensional Gorenstein ring, then $E_R(k) = R$. If $J \subset R$ is an ideal, then $E_{R/J}(k) = \mathrm{Ann}_{E_R(k)} J$.

Observation 4.2.4. *If (R, m) is a local ring of characteristic p with test ideal equal to m, and I is an irreducible ideal, then the only possibilities for the tight closure of I are $I^* = I$ or $I^* = I : m$*

Proof: The assumption that m is the test ideal implies that $I \subset I^* \subset I : m$, and since I is irreducible, one of the inclusions must be equality. □

Proposition 4.2.5. *Let (R, m) be a Gorenstein local ring with residue class field $k = R/m$.*

a). Let I and J be irreducible m-primary ideals with $J \subset I$. then we have $I = J : f$ for some $f \in R$.

b). If I is an irreducible m-primary ideal, and $J \supset I$ is an arbitrary ideal, then we have $I : (I : J) = J$.

Proof: I, J are m-primary irreducible $\Rightarrow R/I, R/J$ are zero-dimensional Gorenstein rings, and R/J maps onto R/I. Therefore

$$R/I \cong E_{R/I}(k) = \mathrm{Ann}_{E_{R/J}(k)} I = \mathrm{Ann}_{R/J} I = \frac{J : I}{J}$$

$\Rightarrow (J : I)/J$ is a cyclic R-module, hence $\exists\ f \in R$ such that $J : I = J + fR$.

For any local ring A with residue field k, we know that there is a duality between ideals of R and submodules of $E_A(k)$, given by

$$\underline{\boldsymbol{a}} \to \mathrm{Ann}_{E_A(k)} \underline{\boldsymbol{a}},\ \ E \subset E_A(k) \to \mathrm{Ann}_A E \text{ and } \underline{\boldsymbol{a}} = \mathrm{Ann}_A(\mathrm{Ann}_{E_a(k)} \underline{\boldsymbol{a}}).$$

When $A = R/J$, $\underline{\boldsymbol{a}} = I/J$, we get that

$$I/J = \mathrm{Ann}_{R/J}(\mathrm{Ann}_{R/J} I/J) = \mathrm{Ann}_{R/J} \frac{J : I}{J} = \mathrm{Ann}_{R/J} \frac{J + fR}{J} = \frac{J : f}{J},$$

and therefore $I = J : f$.

Part b). is just a restatement of the duality mentioned above. $\square$

Corollary 4.2.6. *If (R, m) is a Gorenstein local ring in which the test ideal is equal to m and we have $\underline{\boldsymbol{a}}^* = \underline{\boldsymbol{a}}^F$ for one parameter ideal $\underline{\boldsymbol{a}}$, the same holds for any parameter ideal $\underline{\boldsymbol{b}}$.*

In particular, if R is the cubical cone ring in characteristic $p \equiv 2 \pmod 3$, and (a, b) is any parameter ideal, then $(a, b)^* = (a, b)^F$.

Proof: Assume that R is not F-rational; otherwise, the matter is trivial. Therefore, for any parameter ideal $\underline{\boldsymbol{b}}$, we have $\underline{\boldsymbol{b}}^* = \underline{\boldsymbol{b}} : m$.

Let $\underline{\boldsymbol{b}} = (a_1, \ldots, a_d)$, where $d = \dim(R)$. Pick $q \gg 0$ such that $\underline{\boldsymbol{b}}^{[q]} \subset \underline{\boldsymbol{a}}$. Notice that $\underline{\boldsymbol{b}}^{[q]}$ is still a parameter ideal. Let u be a generator for the socle of $\underline{\boldsymbol{b}}$; then $a_1^{q-1} \ldots a_d^{q-1} u$ generates the socle of $\underline{\boldsymbol{b}}^{[q]}$, and we have seen in Lemma 4.2.2 that $u \in \underline{\boldsymbol{b}}^F \Leftrightarrow a_1^{q-1} \ldots a_d^{q-1} u \in {\underline{\boldsymbol{b}}^{[q]}}^F$.

We can write $\underline{\boldsymbol{a}} = \underline{\boldsymbol{b}}^{[q]} : f$ for some $f \in R$, and we have the following commuting diagram:

$$\begin{array}{ccc} \frac{R}{\underline{\boldsymbol{a}}} & \overset{f}{\hookrightarrow} & \frac{R}{\underline{\boldsymbol{b}}^{[q]}} \\ \downarrow & & \downarrow \\ \frac{R}{\underline{\boldsymbol{a}}^{[p]}} & \xrightarrow{f^p} & \frac{R}{\underline{\boldsymbol{b}}^{[pq]}} \end{array}$$

where the vertical maps are the maps that raise each element to the p^{th} power. $a_1^{q-1} \ldots a_d^{q-1} u$ is the image in $R/\underline{\boldsymbol{b}}^{[q]}$ of some socle generator v for $\underline{\boldsymbol{a}}$, therefore

$$a_1^{q-1} \ldots a_d^{q-1} u = fv \Rightarrow (a_1^{q-1} \ldots a_d^{q-1} u)^p = f^p v^p \in f^p \underline{\boldsymbol{a}}^{[pq]} \subset \underline{\boldsymbol{b}}^{[qp]}. \quad \square$$

Corollary 4.2.7. *Let (R,m) is a local Gorenstein ring in which the test ideal is m and $\underline{\boldsymbol{a}}^* = \underline{\boldsymbol{a}}^F$ for one parameter ideal $\underline{\boldsymbol{a}}$. Then the same holds for any m-primary irreducible ideal which is contained in some parameter ideal $\underline{\boldsymbol{b}}$.*

In particular, this applies if R is the cubical cone ring in characteristic $p \equiv 2 \bmod 3$.

Proof: The previous corollary tells us that $\underline{\boldsymbol{b}}^* = \underline{\boldsymbol{b}}^F$. The same proof as for the previous corollary works, using $\underline{\boldsymbol{b}}$ instead of $\underline{\boldsymbol{a}}$. $\square$

Corollary 4.2.8. *Let (R,m) be Gorenstein local ring with test ideal m. Assume that R is not F-rational. Then for every m-primary ideal I in R, $I : m$ is tightly closed, and $mI = mI^*$.*

Proof: Since the test ideal is m and the ring is not F-rational, it follows that for every parameter ideal $\underline{\boldsymbol{a}}$, we have $\underline{\boldsymbol{a}}^* = \underline{\boldsymbol{a}} : m$.

We can write $I = \underline{\boldsymbol{a}} : (f_1, \dots, f_n)$, with $\underline{\boldsymbol{a}} \subset I$ parameter ideal. Then

$$I : m = \underline{\boldsymbol{a}} : m(f_1, \dots, f_n) = (\underline{\boldsymbol{a}} : mf_1) \cap \dots \cap (\underline{\boldsymbol{a}} : mf_n) = (\underline{\boldsymbol{a}}^* : f_1) \cap \dots \cap (\underline{\boldsymbol{a}}^* : f_n)$$

is an intersection of tightly closed ideals, and therefore it is tightly closed.

The inclusion $mI \subset mI^*$ is clear, and we know that $mI : m$ is a tightly closed ideal containing I, so it must contain I^*. Therefore $mI^* \subset mI$. $\square$

4.3 Corner powers

In this section, we will define the corner powers of an ideal I, denoted $I^{<q>}$, where q is a power of the characteristic p. These can be thought of as being dual in some sense to the Frobenius powers.

We will show that these operation produces a large class of ideals for which tight closure and Frobenius closure coincide, and they also coincide with the socle of the ideal. The socle dimension of these ideals can be arbitrarily large.

For the cubical cone in characteristic 2, we will show that the problem of the equality of Frobenius closure and tight closure can be translated into an equivalent problem which involves corner powers instead of Frobenius powers. In other characteristics, this second property might be in general weaker.

In order to be able to state the definition, we need some preliminary results. The following fact is well known, but we will include a proof for completeness.

Proposition 4.3.1. *Let (R, m) be d dimensional local Cohen-Macaulay. Let $a_1, \dots, a_d$ and $b_1, \dots, b_d$ be two systems of parameters. Let $\underline{\boldsymbol{a}} = (a_1, \dots, a_d)$, and $\underline{\boldsymbol{b}} = (b_1, \dots, b_d)$.*

Assume that $\underline{\boldsymbol{a}} \subset \underline{\boldsymbol{b}}$, and write $(a_1, \dots, a_d) = (b_1, \dots, b_d)A$, where A is a $d \times d$ matrix with entries in R.

Let $\delta = \det(A)$. Then $\underline{\boldsymbol{b}} = \underline{\boldsymbol{a}} : \delta$.

Proof: Let $\boldsymbol{K}(\underline{\boldsymbol{a}}; R)$, $\boldsymbol{K}(\underline{\boldsymbol{b}}; R)$ be the Koszul complex resolutions of $R/\underline{\boldsymbol{a}}$ and $R/\underline{\boldsymbol{b}}$. We can define a map of complexes $\boldsymbol{K}(\underline{\boldsymbol{a}}; R) \longrightarrow \boldsymbol{K}(\underline{\boldsymbol{b}}; R)$ which extends the natural projection $R/\underline{\boldsymbol{a}} \longrightarrow R/\underline{\boldsymbol{b}}$, by letting $\Lambda^i(A) : \Lambda^i(R) \longrightarrow \Lambda^i(R)$. Here, Λ^i denotes the ith exterior power.

The last component of this map of complexes is $\varphi : R \longrightarrow R$, where φ denotes multiplication by δ. φ induces a map $\overline{\varphi} : \mathrm{Ext}^d_R(R/\underline{\boldsymbol{b}}, R) \longrightarrow \mathrm{Ext}^d_R(R/\underline{\boldsymbol{a}}, R)$.

This map can be canonically identified with the map $R/\underline{\boldsymbol{b}} \longrightarrow R/\underline{\boldsymbol{a}}$ defined by multiplication by δ. We need to prove that this map is injective.

Choose n large enough so that $\underline{\boldsymbol{c}} := (b_1^n, \dots, b_d^n) \subset \underline{\boldsymbol{a}}$; then the map

$$\Phi : \mathrm{Ext}^d_R(R/\underline{\boldsymbol{c}}, R) \longrightarrow \mathrm{Ext}^d_R(R/\underline{\boldsymbol{b}}, R)$$

factors through $\overline{\varphi}$; hence it is enough to show that Φ is injective.

But Φ can be canonically identified with the map $R/\underline{\boldsymbol{c}} \longrightarrow R/\underline{\boldsymbol{b}}$ given by multiplication by $(b_1 \ldots b_d)^{n-1}$, and it is clear that this is injective. $\square$

Proposition 4.3.2. ***(Definition of corner powers)*** *Let (R, m) be a d dimensional local Gorenstein ring of characteristic p. For every $q = p^e$, the following operation is well-defined for m-primary ideals:*

If J is an m-primary ideal, let $\underline{\boldsymbol{a}} \subset J$ be a parameter ideal, and let $I = \underline{\boldsymbol{a}} : J$. Then we can write $J = \underline{\boldsymbol{a}} : I$. Define $J^{<q>} := \underline{\boldsymbol{a}}^{[q]} : I^{[q]}$.

If $\underline{\boldsymbol{a'}} \subset J$ is another parameter ideal, and $J = \underline{\boldsymbol{a'}} : I'$ for some ideal I', we have

$$J^{<q>} = \underline{\boldsymbol{a}}^{[q]} : I^{[q]} = \underline{\boldsymbol{a'}}^{[q]} : I'^{[q]}.$$

Proof: If $\underline{\boldsymbol{a}} \subset J$ is fixed, and we can write

$$J = \underline{\boldsymbol{a}} : I = \underline{\boldsymbol{a}} : I' \Rightarrow I + \underline{\boldsymbol{a}} = I' + \underline{\boldsymbol{a}} \ (= \underline{\boldsymbol{a}} : J)$$

$$\Rightarrow I^{[q]} + \underline{\boldsymbol{a}}^{[q]} = I'^{[q]} + \underline{\boldsymbol{a}}^{[q]} \Rightarrow \underline{\boldsymbol{a}}^{[q]} : I^{[q]} = \underline{\boldsymbol{a}}^{[q]} : I'^{[q]}.$$

If $\underline{\boldsymbol{a}}$, $\underline{\boldsymbol{a'}} \subset J$ are two parameter ideals, we can find a parameter ideal $\underline{\boldsymbol{a_0}} \subset \underline{\boldsymbol{a}} \cap \underline{\boldsymbol{a'}}$.

Let $I = \underline{\boldsymbol{a}} : J,\ I_0 = \underline{\boldsymbol{a_0}} : J$. It suffices to show that

$$\underline{\boldsymbol{a}}^{[q]} : I^{[q]} = \underline{\boldsymbol{a_0}}^{[q]} : I_0^{[q]}.$$

Since $\underline{\boldsymbol{a_0}} \subset \underline{\boldsymbol{a}}$, we can write $(\underline{\boldsymbol{a_0}})^T = A(\underline{\boldsymbol{a_0}})^T$, where $\underline{\boldsymbol{a}}^T$ denotes the column vector formed by the generators of $\underline{\boldsymbol{a}}$, and A is a $d \times d$ matrix.

Proposition 4.3.1 shows that $\underline{\boldsymbol{a}} = \underline{\boldsymbol{a_0}} : \delta$, where $\delta = \det(\mathrm{A})$.

We also have $(\underline{\boldsymbol{a}}^q)^T = A^q(\underline{\boldsymbol{a_0}}^q)^T$, and $\det(A^q) = \delta^q$. Here, $(\underline{\boldsymbol{a}}^q)^T$ is the column vector formed by the qth powers of the generators of $\underline{\boldsymbol{a}}$, and A^q is the matrix obtained by raising each entry of A to the qth power.

Therefore we have $\underline{\boldsymbol{a}}^{[q]} = \underline{\boldsymbol{a_0}}^{[q]} : \delta^q$.

It follows that we have

$$J = \underline{\boldsymbol{a}} : I = \underline{\boldsymbol{a_0}} : I_0 \Rightarrow \underline{\boldsymbol{a_0}} : \delta I = \underline{\boldsymbol{a_0}} : I_0 \Rightarrow$$

$$\delta I + \underline{\boldsymbol{a_0}} = I_0 + \underline{\boldsymbol{a_0}} \Rightarrow \delta^q I^{[q]} + \underline{\boldsymbol{a_0}}^{[q]} = I_0^{[q]} + \underline{\boldsymbol{a_0}}^{[q]}$$

$$\Rightarrow \underline{\boldsymbol{a_0}}^{[q]} : \delta I^{[q]} = \underline{\boldsymbol{a_0}}^{[q]} : I_0^{[q]} = \underline{\boldsymbol{a}}^{[q]} : I^{[q]}. \quad \square$$

Convention: In the following, unless otherwise stated, we will always assume that J is an m-primary ideal, $\underline{\boldsymbol{a}} \subset J$ a parameter ideal, and $I = \underline{\boldsymbol{a}} : J$. Then by definition, we have $J^{<q>} = \underline{\boldsymbol{a}}^{[q]} : I^{[q]}$.

Observation 4.3.3. *Let R be a Gorenstein ring, I an ideal, and $\underline{\boldsymbol{a}} \subset I$ a parameter ideal. With notations as in the convention, we also have $I^{[q]} = \underline{\boldsymbol{a}}^{[q]} : J^{<q>}$.*

Observation 4.3.4. *Let R be a Gorenstein local ring. If J_1, J_2 are m-primary ideals, we have*

a). $(J_1 \cap J_2)^{<q>} = J_1^{<q>} \cap J_2^{<q>}$.

b). $J_1 \subset J_2 \Rightarrow J_1^{<q>} \subset J_2^{<q>}$.

c). $(J_1 + J_2)^{<q>} \supset J_1^{<q>} + J_2^{<q>}$. This inclusion is usually strict.

Proof:a). Let $\underline{\boldsymbol{a}} \subset J_1 \cap J_2$ parameter ideal, $I_1 = \underline{\boldsymbol{a}} : J_1,\ I_2 = \underline{\boldsymbol{a}} : J_2$. Then

$$J_1 \cap J_2 = \underline{\boldsymbol{a}} : (I_1 + I_2) \Rightarrow (J_1 \cap J_2)^{<q>} = \underline{\boldsymbol{a}}^{[q]} : (I_1 + I_2)^{[q]} =$$

$$(\underline{\boldsymbol{a}}^{[q]} : I_1^{[q]}) \cap (\underline{\boldsymbol{a}}^{[q]} : I_2^{[q]}) = J_1^{<q>} \cap J_2^{<q>}.$$

b). and c). are similar.

Observation 4.3.5. *Let (R, m) be a Gorenstein local ring. If $\underline{\boldsymbol{a}}$ is an ideal generated by a system of parameters, then $\underline{\boldsymbol{a}}^{<q>} = \underline{\boldsymbol{a}}^{[q]}$.*

Proposition 4.3.6. *Let (R, m) be a Gorenstein local ring. If J is an m-primary ideal, then $l(R/J^{<q>})$ behaves like a Hilbert-Kunz function, that is, there is a real constant $c_J > 0$ such that*

$$l\left(\frac{R}{J^{<q>}}\right) = c_J q^d + \mathcal{O}(q^{d-1}),$$

where $d = \dim\, R$.

More precisely, we have

$$l(R/J^{<q>}) = l(R/\underline{\boldsymbol{a}}^{[q]}) - l(R/I^{[q]}),$$

where $\underline{\boldsymbol{a}} \subset J$ is a parameter ideal, and $I = \underline{\boldsymbol{a}} : J$.

Proof: We have $J^{<q>} = \underline{\boldsymbol{a}}^{[q]} : I^{[q]}$. Note that $R/\underline{\boldsymbol{a}}^{[q]}$ maps onto $R/I^{[q]}$.

Since R is Gorenstein and $\underline{\boldsymbol{a}}$ is a parameter ideal, $R/\underline{\boldsymbol{a}}^{[q]}$ is a Gorenstein ring; hence we have

$$\frac{R}{\underline{\boldsymbol{a}}^{[q]}} = E_{R/\underline{\boldsymbol{a}}^{[q]}}(k) \;\Rightarrow\; \frac{J^{<q>}}{\underline{\boldsymbol{a}}^{[q]}} = \mathrm{Ann}_{R/\underline{\boldsymbol{a}}^{[q]}}\left(\frac{I^{[q]}}{\underline{\boldsymbol{a}}^{[q]}}\right) = E_{R/I^{[q]}}(k),$$

and therefore

$$l\left(\frac{J^{<q>}}{\underline{\boldsymbol{a}}^{[q]}}\right) = l\left(\frac{R}{I^{[q]}}\right).$$

This can be rewritten as

$$l\left(\frac{R}{\underline{\boldsymbol{a}}^{[q]}}\right) - l\left(\frac{R}{J^{<q>}}\right) = l\left(\frac{R}{I^{[q]}}\right),$$

and therefore $l(R/J^{<q>})$ can be written as a difference of Hilbert-Kunz functions.□

Proposition 4.3.7. *Let (R, m) be a Gorenstein local ring with test ideal equal to m. Assume that R is excellent and analytically irreducible.*

a). If $J = m$, then $J^{<q>} = R$ or m. More precisely, if for every parameter ideal $\underline{\boldsymbol{a}}$ we have $\underline{\boldsymbol{a}}^* = \underline{\boldsymbol{a}}^F$, then for $q \gg 0$, $m^{<q>} = R$; otherwise, $m^{<q>} = m$ for all q.

b). If J is an ideal strictly contained in m, then there is a q_0 such that $J^{<q>} \subset m^{[q/q_0]}$ for all $q \geq q_0$.

Proof: a). Let $\underline{\boldsymbol{a}} \subset m$ be a parameter ideal; since R is not F-rational, $\underline{\boldsymbol{a}} : m = \underline{\boldsymbol{a}}^*$. If $\underline{\boldsymbol{a}}^* = \underline{\boldsymbol{a}}^F$, let q_0 be such that $(\underline{\boldsymbol{a}} : m)^{[q_0]} = \underline{\boldsymbol{a}}^{[q_0]}$. For $q \geq q_0$ we have $m^{<q>} = \underline{\boldsymbol{a}}^{[q]} : (\underline{\boldsymbol{a}} : m)^{[q]} = R$.

If $\underline{\boldsymbol{a}}^* \neq \underline{\boldsymbol{a}}^F$, then for all q we have $m^{<q>} = \underline{\boldsymbol{a}}^{[q]} : (\underline{\boldsymbol{a}} : m)^{[q]} = m$.

b). Let $\underline{\boldsymbol{a}} \subset J$ be a parameter ideal, and let $I = \underline{\boldsymbol{a}} : J$. The assumptions imply that $\underline{\boldsymbol{a}} : m = \underline{\boldsymbol{a}}^*$ is strictly contained in I. Let $f \in I \backslash \underline{\boldsymbol{a}}^*$. Then the Proposition 5.1.1 shows that there is a q_0 such that

$$J^{<q>} = \underline{\boldsymbol{a}}^{[q]} : I^{[q]} \subset \underline{\boldsymbol{a}}^{[q]} : f^q \subset m^{[q/q_0]}$$

for every $q \geq q_0$. $\square$

Observation 4.3.8. *The previous proposition shows that when J is strictly contained in m, the corner powers $J^{<q>}$ are contained in higher and higher powers of the maximal ideal as q increases; however, it is not true in general that $J^{<q>} \subset J$.*

Example: Let

$$R = \frac{k[[X, Y, Z]]}{(X^3 + Y^3 + Z^3)},$$

and assume $\mathrm{char}(k) = 2$. Let $J = (x^2, y^2, z^2)$ and $\underline{\boldsymbol{a}} = (x^2, y^2)$. Then

$$I = (x^2, y^2) : J = (x^2, y^2) : z^2 = (x^2, y^2, z).$$

Then $J^{<2>} = (x^4, y^4) : z^2 = (x^4, y^4, xyz)$, and we have $xyz \in J^{<2>} \backslash J$. $\square$

Observation 4.3.9. *Let (R, m) be a Gorenstein local ring. For every m-primary ideal J, we have $J^{<p>} \supset J^{[p]}$.*

Proof: Let $J = \underline{\boldsymbol{a}} : I$; then for every $f \in J$, we have $fI \subset \underline{\boldsymbol{a}} \Rightarrow f^p I^{[p]} \subset \underline{\boldsymbol{a}}^{[p]} \Rightarrow f^p \in J^{<p>}$. $\square$

In the rest of this section, we will assume in addition to (R, m) Gorenstein local, that the test ideal of R is m, and that for every parameter ideal $\underline{\boldsymbol{a}}$ we have $\underline{\boldsymbol{a}}^* = \underline{\boldsymbol{a}}^F = \underline{\boldsymbol{a}} : m$.

This implies that for each $\underline{\boldsymbol{a}}$ there is a $q_0 = p^{e_0}$ such that $\underline{\boldsymbol{a}}^{[q_0]} = (\underline{\boldsymbol{a}} : m)^{[q_0]}$. It can be shown that q_0 does not depend on $\underline{\boldsymbol{a}}$, and moreover the same property holds if $\underline{\boldsymbol{a}}$ is an m-primary irreducible ideal contained in some parameter ideal; see the proof of Cor. 4.2.6 and Cor. 4.2.7.

Proposition 4.3.10. *Let (R, m) be a Gorenstein local ring with test ideal equal to m, and assume that q_0 is such that $\underline{\boldsymbol{a}}^{[q_0]} = (\underline{\boldsymbol{a}} : m)^{[q_0]}$ for every parameter ideal $\underline{\boldsymbol{a}}$. Let I be an m-primary ideal; then $(I : m)^{[q_0]} \subset I^{<q_0>}$. Moreover, for $f \in R$ we have $f^{q_0} \in I^{<q_0>} \Leftrightarrow f \in I : m \Leftrightarrow f^q \in I^{<q>} : m$ for some value of q.*

Proof: Write $I = \underline{\boldsymbol{a}} : J$, where $\underline{\boldsymbol{a}}$ is a parameter ideal. Then $I^{<q_0>} = \underline{\boldsymbol{a}}^{[q_0]} : J^{[q_0]}$.

$$f^{q_0} \in I^{<q_0>} \Leftrightarrow (fJ)^{[q_0]} \subset \underline{\boldsymbol{a}}^{[q_0]} \Leftrightarrow fJ \subset \underline{\boldsymbol{a}}^F = \underline{\boldsymbol{a}} : m \Leftrightarrow f \in (\underline{\boldsymbol{a}} : m) : J = I : m.$$

To prove the second equivalence, note that $f^q \in I^{<q>} : m \Leftrightarrow mf^q \in \underline{\boldsymbol{a}}^{[q]} : J^{[q]} \Leftrightarrow (fJ)^{[q]} \subset \underline{\boldsymbol{a}}^{[q]} : m = (\underline{\boldsymbol{a}}^{[q]})^* \Leftrightarrow fJ \subset \underline{\boldsymbol{a}}^* = \underline{\boldsymbol{a}} : m \Leftrightarrow f \in I : m$. $\square$

For the purpose of studying tight closure of an ideal, it is convenient to introduce the following notation:

Notation: $I^{1/q} = \{u \in R \mid u^q \in I\}$.

Corollary 4.3.11. *Let (R, m) be a Gorenstein local ring with test ideal equal to m, and assume that q_0 is such that $\underline{\boldsymbol{a}}^{[q_0]} = (\underline{\boldsymbol{a}} : m)^{[q_0]}$ for every parameter ideal $\underline{\boldsymbol{a}}$.*

Write $q_0 = p^{n_0}$, and fix an $r \in \{0, 1, \ldots, n_0 - 1\}$. For every m-primary I, we have

$$I^* = \bigcap_{q = p^{in_0+r}} (I^{[q]} : m)^{1/q} = \bigcap_{q = p^{in_0+r}} (mI^{[q]} : m)^{1/q},$$

and both intersections are decreasing.

Proof: $u \in I^* \Leftrightarrow mu^q \in I^{[q]}$ for all $q = p^{in_0+r}$, hence $u \in I^* \Leftrightarrow u \in (I^{[q]} : m)^{1/q}$ for all q of this form; we also know from Cor. 4.2.8 that $u \in I^* \Rightarrow u^q \in (I^{[q]})^* \Rightarrow mu^q \in mI^{[q]}$, since $mJ^* = mJ$ for all m-primary J, so $u \in \bigcap_q (mI^{[q]} : m)^{1/q}$.

To prove that both intersections are decreasing, it suffices to show that

$$u^{q_0 q} \in I^{[q_0 q]} : m \Rightarrow u^q \in mI^{[q]} : m.$$

Write $I = \underline{\boldsymbol{a}} : J$, with $\underline{\boldsymbol{a}} \subset mI$; using Obs. 4.3.3 and Prop. 4.3.10, we get

$$u^{q_0 q} \in I^{[q_0 q]} : m = \underline{\boldsymbol{a}}^{[q_0 q]} : mJ^{<q_0 q>} \Rightarrow u^{q_0 q}(J^{<q>} : m)^{[q_0]} \subset u^{q_0 q} J^{<q_0 q>} \subset \underline{\boldsymbol{a}}^{[q_0 q]} : m = \underline{\boldsymbol{a}}^{[q_0 q]^*}.$$

This implies that

$$u^q (J^{<q>} : m) \subset \underline{\boldsymbol{a}}^{[q]^*} = \underline{\boldsymbol{a}}^{[q]} : m, \text{ hence } \Rightarrow u^q \in \underline{\boldsymbol{a}}^{[q]} : m(J^{<q>} : m).$$

Equivalently,

$$mu^q \in \underline{\boldsymbol{a}}^{[q]} : (J^{<q>} : m) = mI^{[q]}.$$

The last equality follows from Prop. 4.2.5 b)., because $\underline{\boldsymbol{a}}^{[q]} \subset mI^{[q]}$. $\square$

Comment: The previous corollary shows that $I^* = \bigcap (I^{[q]} : m)^{1/q}$ is a decreasing intersection of ideals containing the m-primary ideal I^*, and therefore it is stationary. This means that for all I there is a q_I such that $I^* = (I^{[q_I]} : m)^{1/q_I}$; in other words, it suffices to perform tight closure tests up to the q_I^{th} power in order to decide whether a certain element is in the tight closure of I.

If we could prove that whenever $(I^{[q]})^{1/q} = (I^{[qq_0]})^{1/qq_0} \Rightarrow (I^{[q]})^{1/q} = (I^{[Q]})^{1/Q}$ for every $Q = qq_0^n$, we would have an algorithm for computing tight closure: just compute $(I^{[q]} : m)^{1/q}$ with $q = p^{r+kn_0}$ until we get two such consecutive ideals that are equal; they would have to equal the tight closure of I.

The following proposition shows that the property that $J_1 \subset J_2 \subset J_1^*$, where J_1 and J_2 are m-primary ideals, is unchanged by linkage with respect to a suitable parameter ideal.

In particular, in order to show that tight closure equals Frobenius closure for every m-primary ideal, it is sufficient to show that this holds for one representative in each linkage class.

Proposition 4.3.12. *Let (R, m) be a Gorenstein local ring with test ideal equal to m. Let J_1 be an m-primary ideal containing a parameter ideal $\underline{\boldsymbol{a}}$, and let $J_2 = J_1 + (f)$, with $f \in J_1^*$.*

Let $I_1 = \underline{\boldsymbol{a}} : J_1$, $I_2 = \underline{\boldsymbol{a}} : J_2$. Note that $I_2 \subset I_1$. If $\underline{\boldsymbol{b}}$ is a parameter ideal contained in I_2 and we let $J_1' = \underline{\boldsymbol{b}} : I_1$, $J_2' = \underline{\boldsymbol{b}} : I_2$, then we have $J_2' = J_1' + (f')$ for some $f' \in (J_1')^*$.

Moreover, $f \in J_1^F \Leftrightarrow f' \in (J_1')^F$.

Proof: Applying the second part of Obs. 4.3.6 gives us

$$l\left(\frac{J_2^{[q]}}{J_1^{[q]}}\right) = l\left(\frac{I_1^{<q>}}{I_2^{<q>}}\right) = l\left(\frac{{J_2'}^{[q]}}{{J_1'}^{[q]}}\right).$$

Since the test ideal is m, $f \in J_1^* \Leftrightarrow \forall q,\ l(J_2^{[q]}/J_1^{[q]}) \leq 1$. This implies that $l({J_2'}^{[q]}/{J_1'}^{[q]}) \leq 1$ for all q, and hence $J_2' = J_1' + (f')$, for some f', and $f' \in {J_1'}^*$. Similarly, $f \in J_1^F \Leftrightarrow\ l(J_2^{[q]}/J_1^{[q]}) = 0$ for $q \gg 0\ \Leftrightarrow f' \in {J_1'}^F$. $\square$

The following result shows that the high corner powers of m-primary ideals have the property that their socle is contained in the Frobenius closure; in particular, tight closure equals Frobenius closure for any ideal of this form.

Recall that q_0 is such that $\underline{\boldsymbol{a}}^{[q_0]} = (\underline{\boldsymbol{a}} : m)^{[q_0]}$ for every ideal $\underline{\boldsymbol{a}}$ generated by a system of parameters. It follows that the same statement is true if I is an irreducible m-primary ideal which is contained in some parameter ideal.

Theorem 4.3.13. *Let (R, m) be a Gorenstein local ring, and assume that there is a q_0 such that $(\underline{\boldsymbol{a}} : m)^{[q_0]} = \underline{\boldsymbol{a}}^{[q_0]}$ for every parameter ideal $\underline{\boldsymbol{a}}$. Let $J = \underline{\boldsymbol{a}} + (f_1, \dots, f_n)$ be an m-primary ideal, with $\underline{\boldsymbol{a}}$=parameter ideal, and let $I = \underline{\boldsymbol{a}} : J$. Assume that for all $i = 1, \dots, n$ we have*

$$f_i \notin \Big(\underline{\boldsymbol{a}} + (f_1, \dots, \hat{f_i}, \dots, f_n)\Big)^*.$$

Then for all $q \gg 0$, we have $(I^{<q>} : m)^{[q_0]} = (I^{<q>})^{[q_0]}$.

The same conclusion holds under the more general hypothesis that

$$f_i \in \Big(\underline{\boldsymbol{a}} + (f_1, \dots, \hat{f_i}, \dots, f_n)\Big)^* \Rightarrow f_i \in \Big(\underline{\boldsymbol{a}} + (f_1, \dots, \hat{f_i}, \dots, f_n)\Big)^F.$$

In particular, if we know that tight closure equals Frobenius closure for every ideal, then we can say that for every m-primary I and for all $q \gg 0$, we have

$$(I^{<q>} : m)^{[q_0]} = (I^{<q>})^{[q_0]}.$$

Proof: It is enough to prove the second statement. Assume that q_1 is such that whenever $f_i \in (\underline{\boldsymbol{a}} + (f_1, \dots, \hat{f_i}, \dots, f_n))^* \Rightarrow f_i^{q_1} \in (\underline{\boldsymbol{a}} + (f_1, \dots, \hat{f_i}, \dots, f_n))^{[q_1]}$, and therefore we can exclude those $f_i^{q_1}$ from the set of generators of $J^{[q_1]}$.

After replacing J by $J^{[q]}$, I by $I^{<q>}$ and $\underline{\boldsymbol{a}}$ by $\underline{\boldsymbol{a}}^{[q]}$ with $q \gg 0$

Pick $q \gg 0$ such that $q \geq q_1$ and

$$f_i \notin \Big(\underline{\boldsymbol{a}} + (f_1, \dots, \hat{f_i}, \dots, f_n)\Big)^* \Rightarrow f_i^q \notin \Big(\underline{\boldsymbol{a}}^{[q]} + (f_1^q, \dots, \hat{f_i}^q, \dots, f_n^q)\Big) : m.$$

After replacing J by $J^{[q]}$, I by $I^{<q>}$ and $\underline{\boldsymbol{a}}$ by $\underline{\boldsymbol{a}}^{[q]}$, we can assume that $J = \underline{\boldsymbol{a}} + (f_1, \dots, f_n)$ is such that for all $i = 1, \dots, n$ we have

$$f_i \notin \Big(\underline{\boldsymbol{a}} + (f_1, \dots, \hat{f}_i, \dots f_n)\Big) : m.$$

Write $I = I_1 \cap \dots \cap I_n$, where each $I_i := \underline{\boldsymbol{a}} : f_i$ is irreducible. Our set up implies that for all i, $I_1 \cap \dots \cap \hat{I}_i \cap \dots \cap I_n \not\subset I_i : m$.

The socle dimension of I is at most n, since I is an intersection of n irreducible ideals. For each $i = 1, \dots, n$ we can pick $g_i \in I_1 \cap \dots \cap \hat{I}_i \cap \dots \cap I_n \setminus I_i : m$.

Then $I : g_i = I_i : g_i$ is irreducible, and strictly contained in m. Note that we have $R/(I : g_i) \overset{g_i}{\hookrightarrow} R/I$, and therefore the image of the socle element in $R/(I : g_i)$ is a socle element of R/I; call this element u_i. Note that $u_i \in I_1 \cap \dots \cap \hat{I}_i \cap \dots \cap I_n \backslash I_i$,and therefore $\forall i, u_i \notin I + (u_1, \cdots, \hat{u}_i, \cdots u_n)$; hence $u_1, \cdots, u_n$ form a basis for the socle in R/I.

If $I : g_i$ is not contained in a parameter ideal, replace I by $I^{<q>}$ for $q \gg 0$, and g_i by g_i^q. All the above assumptions continue to hold: we still have
$g_i^q \notin I_i^{<q>} : m$ since otherwise Obs. 4.3.10 shows that $g_i \in I_i : m$, contradiction. In addition, the second part of Prop. 4.3.7 shows that $I^{<q>} : g_i^q = (I : g_i)^{<q>}$ will be contained in some parameter ideal, and therefore the ${q_0}^{\text{th}}$ power of the socle generator is in $(I^{<q>} : g_i^q)^{[q_0]}$ by Corollary 4.2.7.

If u_i denotes the socle generator of $R/I^{<q>}$ corresponding to g_i^q as described above, it follows that that $u_i^{q_0} \in I^{<q>[q_0]}$. Since $u_1, \dots, u_n$ generate the socle of $I^{<q>}$, it follows that $(I^{<q>} : m)^{[q_0]} = (I^{<q>})^{[q_0]}$. □

Comment: We will show later (see 4.5.7) that in the cubical cone in characteristic 2, the converse of this theorem is also true:

If for all m-primary ideal I and all $q \gg 0$ we have $(I^{<q>} : m)^{[p]} = (I^{<q>})^{[p]}$, then for every m-primary ideal J we have $J^* = J^F$.

4.4 Coherence

In this section we will concentrate on the study of R^∞, which is defined as the direct limit of all the $R^{1/q}$, where $q = p^e$. R will be assumed to be a reduced Noetherian ring of characteristic p.

It is easy to see that R^∞ is not in general a Noetherian ring, but whenever it is coherent (this property is weaker than Noetherian), we must have $I^* = I^F$ for every ideal $I \subset R$.

However, we will prove that if R is the cubical cone ring in characteristic $p = 2$, then R^∞ is not a coherent ring. The failure of coherence in this ring comes from the fact that there are ideals J for which

$$\mathrm{Tor}_1^R(R/J^{[q]}, R^{1/2}) \neq 0$$

for every value of $q = 2^e$.

The study of this Tor module as a function of q, for a fixed ideal J will constitute the subject of the next section, and in characteristic 2 it will be seen to be related to the study of the Hilbert-Kunz function of J.

Definition: A ring S is *coherent* if every finitely-generated ideal I is finitely presented, i.e., the module of relations on the generators of I is finitely generated.

Note that an ideal I of R^∞ is finitely generated if and only if it can be written as $I_0 R^\infty$, where I_0 is an ideal in R^{1/q_0} for some q_0. Since $R \cong R^{1/q_0}$, it will be sufficient to consider the case $I = I_0 R^\infty$, where I_0 is an ideal in R.

Proposition 4.4.1. *Let R be any Noetherian reduced ring of characteristic p. If I is an ideal in R, IR^∞ is finitely presented as an R^∞ module $\Leftrightarrow$ there exists a $q_0 = p^{e_0}$*

such that for all $q \geq q_0$ we have

$$\mathrm{Tor}_1^R\left(\frac{R}{I^{[q]}}, R^{1/p}\right) = 0.$$

Proof: Let $I = (f_1, \ldots, f_n)$. Look at a resolution of $R/I^{[q]}$ over R and tensor it with $R^{1/p}$. The effect of this operation is that the entries of the matrices giving the maps in the resolution are raised to the pth power. Hence we can write

$$\mathrm{Tor}_1^R\left(\frac{R}{I^{[q]}}, R^{1/p}\right) = \frac{\text{relations over } R \text{ in } f_1^{pq}, \ldots, f_n^{pq}}{\text{submodule generated by the } p^{th} \text{ powers of relations in } f_1^q, \ldots, f_n^q}$$

$$= \frac{\text{relations over } R^{1/pq} \text{ in } f_1, \ldots, f_n}{\text{submodule of relations coming from } R^{1/q}}.$$

Let N be the module of relations in $f_1, \ldots, f_n$ over R^∞; this is equal to the direct union of all the modules of relations over $R^{1/q}$, as q varies through all powers of p. Therefore, N is finitely generated if and only if it is generated by relations coming from R^{1/q_0} for some q_0.

This is equivalent to saying that for all $q \geq q_0$, all the relations in $f_1, \ldots, f_n$ over $R^{1/pq}$ belong to the submodule of relations coming from $R^{1/q}$, and therefore it is equivalent to $\mathrm{Tor}_1^R\left(\frac{R}{I^{[q]}}, R^{1/p}\right) = 0$ for all $q \geq q_0$. $\square$

Proposition 4.4.2. *Let (R, m) be a reduced local Noetherian ring of characteristic p. Let I be an ideal for which there is a q_0 such that $\mathrm{Tor}_1(R/I^{[q]}, R^{1/p}) = 0$ for all $q \geq q_0$.*

Then if $I_0 \subset I$ is such that $I \subset I_0^*$, we must have $I^{[q_0]} \subset I_0^{[q_0]}$, hence $I \subset I_0^F$.

Proof: By enlarging I_0, we can assume that $I = I_0 + (f)$, and $f \in I_0^*$; we also have $f^{q_0} \in (I_0^{[q_0]})^*$. Consider the short exact sequence

$$0 \longrightarrow \frac{R}{I_0^{[q_0]} : f^{q_0}} \xrightarrow{f^{q_0}} \frac{R}{I_0^{[q_0]}} \longrightarrow \frac{R}{I^{[q_0]}} \longrightarrow 0.$$

By tensoring this repeatedly with $R^{1/p}$ and using the hypothesis that $\mathrm{Tor}_1^R\left(\frac{R}{I^{[q]}}, R^{1/p}\right) = 0$ for all $q \geq q_0$, we get

$$0 \longrightarrow \frac{R}{(I_0^{[q_0]} : f^{[q_0]})^{[q]}} \xrightarrow{f^{qq_0}} \frac{R}{I_0^{[qq_0]}} \longrightarrow \frac{R}{I^{[qq_0]}} \longrightarrow 0$$

for all q, and therefore $I_0^{[qq_0]} : f^{qq_0} = (I_0^{[q_0]} : f)^{[q]}$. Assume that $f^{q_0} \notin I_0^{[q_0]}$, so $I_0^{[q_0]} : f^{q_0} \subset m$; it follows that $\bigcap_q (I_0^{[qq_0]} : f^{qq_0}) = 0$, contradiction with $f^{q_0} \in I_0^{[q_0]^*}$. $\square$

Comment: If R^∞ is coherent, then for every ideal I in R there is a q_0 such that $\mathrm{Tor}_1^R\left(\frac{R}{I^{[q]}}, R^{1/p}\right) = 0$ for all $q \geq q_0$. It follows that tight closure and Frobenius closure must be the same for every ideal.

Indeed, assume that there exists I such that $J = I^* \neq I^F$. By the previous observation, there is a q_0 such that $J^{[q_0]} = I^{[q_0]}$, contradiction.

Comment: If R is a hypersurface and I is an m-primary ideal, then $\mathrm{Tor}_1(R/I^{[q]}, R^{1/p}) = 0$ for all $q \geq q_0$ is equivalent to $\mathrm{pd}_R(R/I^{[q_0]}) < \infty$. This follows from Herzog's result in [He] , since if $\mathrm{Tor}_1(R/I^{[q_0]}, R^{1/p}) = 0$, then for $q = pq_0$, $\mathrm{Tor}_1(R/I^{[q]}, R^{1/p}) = 0 \Leftrightarrow \mathrm{Tor}_1(R/I^{[q_0]}, R^{1/p^2}) = 0$, and so on. Also note that in the case of a hypersurface ring, when I is an m-primary ideal, $\mathrm{Tor}_1(R/I^{[q]}, R^{1/p}) = 0$ implies $\mathrm{Tor}_i(R/I^{[q]}, R^{1/p}) = 0$ for all $i > 0$ (see 4.4.3).

From now on, assume that R is a hypersurface; more precisely, R can be written as $R = T/fT$, where (T, m) is a regular local ring, and $f \in T$. We will frequently use the theory of free resolutions over such rings (see [Ei]), in particular the fact that the resolution of any Cohen-Macaulay module M of depth equal to the dimension of R is periodic with period 2. We will apply this result for $M = R^{1/q}$.

Another fact that will be frequently used is that any finite acyclic complex of free modules over a characteristic p local ring R stays acyclic when tensored with $R^{1/q}$

over R (see [PS]), for every $q = p^e$. In particular, if M is a module of finite projective dimension, $F^e(M)$ also has finite projective dimension for all e, and a minimal free resolution of $F^e(M)$ is obtained by tensoring the minimal resolution of M with $R^{1/q}$. This also implies that $\mathrm{Tor}_i(M, R^{1/q}) = 0$, for all $i > 0$ and for every finitely generated module$M$ of finite projective dimension.

Note that any hypersurface is a Gorenstein ring, hence the theory of corner powers developed in the previous section applies.

Observation 4.4.3. *Let $R = T/fT$ be a hypersurface. If the minimal free resolution $\ldots \longrightarrow R^n \xrightarrow{A} R^n \xrightarrow{B} R^n \longrightarrow M$ of M is periodic of period 2, then for every finite length module N we have*

$$l\left(\mathrm{Tor}_1(M,N)\right) = l\left(\mathrm{Tor}_2(M,N)\right).$$

Proof: Let $\overline{A}$, $\overline{B}$ be the results of tensoring the maps A, B with N. Then we have

$$\mathrm{Tor}_1(M,N) = \frac{\ker(\overline{B})}{\mathrm{im}(\overline{A})},$$

and

$$\mathrm{Tor}_2(M,N) = \frac{\ker(\overline{A})}{\mathrm{im}(\overline{B})},$$

and the equality follows because

$$l(\ker(\overline{A})) + l(\mathrm{im}(\overline{A})) = l(\ker(\overline{B})) + l(\mathrm{im}(\overline{B})) = l(N). \qquad \square$$

Proposition 4.4.4. *Let $R = T/fT$ be a hypersurface of positive characteristic p, and let I be an m-primary ideal. Let q be a power of p. Then*

$$l\left(\frac{I^{<q>}}{I^{[q]}}\right) = l\left(\mathrm{Tor}_1^R\left(\frac{R}{I}, R^{1/q}\right)\right).$$

Proof: Write $I = \underline{\boldsymbol{a}} : (f_1, \ldots, f_n)$, with $\underline{\boldsymbol{a}}$ is an ideal generated by a system of parameters. We have the following short exact sequence:

$$0 \longrightarrow \frac{R}{I} \overset{(f_1,\ldots,f_n)}{\longrightarrow} \bigoplus_n \left(\frac{R}{\underline{\boldsymbol{a}}}\right) \longrightarrow N \longrightarrow 0,$$

where N is a finite length module.

Tensor with $R^{1/q}$ and look at the resulting long exact sequence. The periodicity of projective resolutions over hypersurfaces and the long exact sequence imply that

$$l\left(\mathrm{Tor}_1^R(N, R^{1/q})\right) = l\left(\mathrm{Tor}_2^R(R/I, R^{1/q})\right) = l\left(\mathrm{Tor}_1^R(R/I, R^{1/q})\right).$$

Here we are using the fact that ideals generated by parameters have finite projective dimension, and therefore have vanishing Tor with $R^{1/q}$.

We also have the exact sequence

$$0 \longrightarrow \mathrm{Tor}_1^R(N, R^{1/q}) \longrightarrow \frac{R}{I^{[q]}} \overset{(f_1^q,\ldots,f_n^q)}{\longrightarrow} \bigoplus_n \left(\frac{R}{\underline{\boldsymbol{a}}^{[q]}}\right) \longrightarrow N \otimes R^{1/q} \longrightarrow 0$$

Since the kernel of the map $R/I^{[q]} \overset{(f_1^q,\cdots,f_n^q)}{\longrightarrow} \bigoplus_n (R/\underline{\boldsymbol{a}}^{[q]})$ is $(\underline{\boldsymbol{a}}^{[q]} : (f_1^q, \ldots, f_n^q))/I^{[q]} = I^{<q>}/I^{[q]}$, it follows that $l(I^{<q>}/I^{[q]}) = l(\mathrm{Tor}_1^R(N, R^{1/q})) = l(\mathrm{Tor}_1^R(R/I, R^{1/q}))$. $\square$

Corollary 4.4.5. *Let (R, m) be a characteristic p hypersurface. Let I be an m-primary ideal, and let $\underline{\boldsymbol{a}} \subset I$ be a parameter ideal. If $J = \underline{\boldsymbol{a}} : I$, and q is a power of p, then we have*

$$l\left(\mathrm{Tor}_1^R\left(\frac{R}{I}, R^{1/q}\right)\right) = l\left(\mathrm{Tor}_1^R\left(\frac{R}{J}, R^{1/q}\right)\right).$$

Therefore, the length of the tor with $R^{1/p}$ is an invariant of the linkage class of an m-primary ideal.

We will see later that if R is the cubical cone ring in characteristic 2, there are infinitely many values for this length, when I varies through all m-primary ideals, and therefore there are infinitely many linkage classes of m-primary ideals in this ring.

Proof: We have $I^{<q>} = \underline{\boldsymbol{a}}^{[q]} : J^{[q]}$ and $I^{[q]} = \underline{\boldsymbol{a}}^{[q]} : J^{<q>}$; this implies that

$$l\left(\frac{I^{<q>}}{I^{[q]}}\right) = l\left(\frac{J^{<q>}}{J^{[q]}}\right),$$

and the claim follows immediately from Prop. 4.4.4. □

The next result shows that whenever R is a characteristic p hypersurface, in which we know that the test ideal is m and that tight closure equals Frobenius closure for parameter ideals, then the same must be true for any m-primary ideal having finite projective dimension.

Corollary 4.4.6. *Let (R, m) be a hypersurface of characteristic p. Assume that the test ideal is m and that there is a q_0 such that for every parameter ideal $\underline{\boldsymbol{a}}$ we have $(\underline{\boldsymbol{a}} : m)^{[q_0]} = \underline{\boldsymbol{a}}^{[q_0]}$. If I is m-primary and R/I has finite projective dimension, then $(I : m)^{[q_0]} = I^{[q_0]}$. Hence the socle of I is in the Frobenius closure, and therefore tight closure and Frobenius closure are equal.*

Proof: Since I has finite projective dimension, we know that $\mathrm{Tor}_1^R(R/I, R^{1/q_0}) = 0$, and according to Prop. 4.4.4, this implies $I^{<q_0>} = I^{[q_0]}$. By Prop. 4.3.10, we also know that $(I : m)^{[q_0]} \subset I^{<q_0>} = I^{[q_0]}$. □

The rest of this section will be dedicated to showing that if

$$R = \frac{k[[X, Y, Z]]}{(X^3 + Y^3 + Z^3)},$$

where k is a field of characteristic $p = 2$, then R^∞ is not a coherent ring.

In order to do this, it suffices to find an ideal I such that for all $q = 2^e$ we have

$$\mathrm{Tor}_1(R/I^{[q]}, R^{1/2}) \neq 0.$$

We start with a some preliminary observations:

Lemma 4.4.7. *Let*

$$R = \frac{k[[X,Y,Z]]}{(X^3+Y^3+Z^3)},$$

where k is a field of characteristic $p=2$. Let $m=(x,y,z)$. For $q \geq 4$, $R/m^{[q]}$ has finite projective dimension, and therefore

$$\mathrm{Tor}_i^R(R/m^{[q]}, R^{1/q}) = 0,$$

for all $i \geq 1,\ q = p^e$.

Proof: Note that $(x^4,y^4) : z^4 = (xy, z^2)$, and therefore we have the short exact sequence

$$0 \longrightarrow \frac{R}{(xy,z^2)} \xrightarrow{z^4} \frac{R}{(x^4,y^4)} \longrightarrow \frac{R}{m^{[4]}} \longrightarrow 0$$

where the first map is the multiplication by z^4. Since $R/(x^4,y^4)$ and $R/(xy,z^2)$ have finite projective dimension, the claim follows. $\square$

Notation: To simplify the notation, for every ideal J we write

$$\mathrm{tor}\left(\frac{R}{J}\right) := l\left(\mathrm{Tor}\left(\frac{R}{J}, R^{1/2}\right)\right).$$

Proposition 4.4.8. *Let $R = k[[X,Y,Z]]/(X^3+Y^3+Z^3)$, with characteristic of $k=2$, and $I=(y^2, x+y, z)$. Let $q_n = 4^n$; then for all n we have*

$$l\left(\mathrm{Tor}\left(\frac{R}{I^{[q_n]}}, R^{1/2}\right)\right) = 8.$$

Proof: Let $N_0 = 2,\ N_t = 4N_{t-1} - 4$, and $I_t = (y^{N_t}, x+y, z)$. Note that $I_0 = I$. Then we claim that

$$\mathrm{tor}\left(\frac{R}{I^{[q_n]}}\right) = \mathrm{tor}\left(\frac{R}{I_n}\right). \tag{4.1}$$

This is trivial for $n=0$; in order to prove the claim for $n \geq 1$, it suffices to show that for all $t = 0, \dots, n$ and for all q we have

$$\mathrm{tor}\left(\frac{R}{I_{t+1}^{[q]}}\right) = \mathrm{tor}\left(\frac{R}{I_t^{[4q]}}\right). \tag{4.2}$$

If this holds, then

$$\operatorname{tor}\left(\frac{R}{I^{[q_n]}}\right) = \operatorname{tor}\left(\frac{R}{I_1^{[q_{n-1}]}}\right) = \operatorname{tor}\left(\frac{R}{I_2^{[q_{n-2}]}}\right) = \cdots = \operatorname{tor}\left(\frac{R}{I_n}\right).$$

For all t, let $J_t = (y^{N_t}, x^2+y^2, (x+y)z)$. It is easy to check that $I_t^{[4]} : y^4 = J_{t+1}$: for any positive integer N, if $y^4 f \in (y^{4N}, x^4+y^4, z^4)$, then $y^4(f - y^{4N-4}g) \in (x^4+y^4, z^4)$ for some g, hence $f - y^{4N-4}g \in (x^4+y^4, z^4) : y^4 = (x^2+y^2, (x+y)z)$ (this has been obtained using MACAULAY), and therefore $f \in (y^{4N-4}, (x^2+y^2, (x+y)z)$.

Since $I_t^{[4]} + (y^4) = m^{[4]}$, which has finite projective dimension, we have the short exact sequence

$$0 \longrightarrow \frac{R}{J_{t+1}} \longrightarrow \frac{R}{I_t^{[4]}} \longrightarrow \frac{R}{m^{[4]}} \longrightarrow 0,$$

which moreover stays exact upon tensoring with $R^{1/q}$ for all q. The long exact sequence for tors shows that

$$\operatorname{tor}\left(\frac{R}{I_t^{[4q]}}\right) = \operatorname{tor}\left(\frac{R}{J_{t+1}^{[q]}}\right). \tag{4.3}$$

On the other hand, note that

$$(y^{N_{t+1}}, x^2+y^2) : (x+y)z = (y^{N_{t+1}}, x+y) : z,$$

and since corner powers are well-defined, this implies that

$$(y^{N_{t+1}}, x^2+y^2)^{[q]} : (x+y)^q z^q = (y^{N_{t+1}}, x+y)^{[q]} : z^q \text{ for all } q.$$

Therefore we have the following short exact sequences:

$$0 \longrightarrow \frac{R}{(y^{N_{t+1}}, x+y)^{[q]} : z^q} \xrightarrow{(x^q+y^q)z^q} \frac{R}{(y^{N_{t+1}}, x^2+y^2)^{[q]}} \longrightarrow \frac{R}{J_{t+1}^{[q]}} \longrightarrow 0$$

$$0 \longrightarrow \frac{R}{(y^{N_{t+1}}, x+y)^{[q]} : z^q} \xrightarrow{z^q} \frac{R}{(y^{N_{t+1}}, x+y)^{[q]}} \longrightarrow \frac{R}{I_{t+1}^{[q]}} \longrightarrow 0$$

Since $(y^{N_{t+1}}, x^2+y^2)$ and $(y^{N_{t+1}}, x+y)$ are parameter ideals, and therefore have finite projective dimension, it follows that

$$\operatorname{tor}\left(\frac{R}{I_{t+1}^{[q]}}\right) = \operatorname{tor}\left(\frac{R}{(y^{N_{t+1}}, x+y)^{[q]} : z^q}\right) = \operatorname{tor}\left(\frac{R}{J_{t+1}^{[q]}}\right) = \operatorname{tor}\left(\frac{R}{I_t^{[4q]}}\right).$$

The last equality follows from equation 4.3. Therefore equation 4.2 is proved, which in turn proves 4.1. Now it is easy to compute

$$\operatorname{tor}\left(\frac{R}{I_n}\right) = l\left(\frac{I_n^{<2>}}{I_n^{[2]}}\right).$$

Note that $I_n = (y^{N_n}, x+y) : z^2$, and the above expression is equal to

$$= l\left(\frac{(y^{2N_n}, x^2+y^2) : z^4}{(y^{2N_n}, x^2+y^2, z^2)}\right) = l\left(\frac{(y^{2N_n-2}, z^2, x+y)}{(y^{2N_n}, x^2+y^2, z^2)}\right)$$

$$= l\left(\frac{(y^{2N_n}, x^2+y^2, z^2) + (x+y)}{(y^{2N_n}, x^2+y^2, z^2)}\right) + l\left(\frac{(y^{2N_n}, x+y, z^2) + (y^{2N_n-2})}{(y^{2N_n}, x+y, z^2)}\right)$$

$$= l\left(\frac{R}{(y^{2N_n}, x^2+y^2, z^2) : (x+y)}\right) + l\left(\frac{R}{(y^{2N_n}, x+y, z^2) : y^{2N_n-2}}\right) = 8$$

since $(y^{2N_t}, x^2+y^2, z^2) : (x+y) = (y^{2N_t}, x+y, z^2) : y^{2N_t-2} = (x^2, z^2, x+y)$ (it is easy to check that $(x^2, z^2, x+y)$ is contained in both colon ideals, and anything not in $(x^2, z^2, x+y)$ can be written modulo $(x^2, z^2, x+y)$ as a linear combination with coefficients in k of x, y, z, and xz. It can be checked that noting of this form is in the colon ideals on the left-hand side). $\square$

4.5 Hilbert-Kunz functions and characteristic 2 results

Due to the particular form of the minimal free resolution of $R^{1/2}$ as an R-module, we will be able to obtain some results that are specific to characteristic 2 hypersurfaces.

In this section $R = T/fT$ will be a hypersurface, where $T = k[[X_1, \ldots, X_n]]$ is a formal power series ring.

We start by noting the following:

Proposition 4.5.1. *Let $R = T/fT$ be a hypersurface of characteristic p, where $T = k[[X_1, \dots, X_n]]$, and let q be a power of p. Then the minimal free resolution of $R^{1/q}$ over R has the following form:*

$$\cdots \xrightarrow{A} R^N \xrightarrow{B} R^N \longrightarrow R^{1/q} \longrightarrow 0$$

where A is the map corresponding to multiplying the generators of $R^{1/q}$ (as an R-module) by $f^{1/q}$, and B corresponds to multiplication by $f^{q-1/q}$.

In particular, if $p = 2$ and we consider the minimal free resolution of $R^{1/2}$ over R, we get $A = B$, therefore this resolution is periodic of period one.

Proof: since $f^{1/q} f^{q-1/q} = f$, it suffices to check that B gives a presentation of $R^{1/q}$ over R. Let $x_1, \dots, x_n$ denote the images of $X_1, \dots X_n$ in R. Then the generators of $R^{1/q}$ over R are $(x_1^{i_1} \dots x_n^{i_n})^{1/q}$, where $0 \le i_1, \dots, i_n < q$.

If we have a relation $\sum a_{i_1 \dots i_n} (x_1^{i_1} \dots x_n^{i_n})^{1/q} = 0$, with $a_{i_1 \dots i_n} \in R$, it follows that $\sum a^q_{i_1 \dots i_n} (X_1^{i_1} \dots X_n^{i_n})$ is a multiple of f in T, hence $\sum a_{i_1 \dots i_n} (x_1^{i_1} \dots x_n^{i_n})^{1/q}$ is a multiple of $f^{1/q}$. $\square$

The main application of this observation is the connection between Hilbert-Kunz function of an m-primary ideal I and the values of $l(\mathrm{Tor}_1(R/I^{[q]}, R^{1/2}))$ as a function of q.

We review some facts about Hilbert-Kunz functions. These can be thought of as the analogue of Hilbert functions, where we replace ordinary powers by Frobenius powers. Their behavior is much more subtle; they were studied extensively by P. Monsky ([Mo1], [Mo2], [HM]).

Definition: Let (R, m) be a Noetherian local ring of characteristic p, and let I be an m-primary ideal. *The Hilbert-Kunz function* of I is defined as $l_I(q) = l(R/I^{[q]})$, where q varies through all powers of p.

We summarize a few of the main results concerning Hilbert-Kunz functions.

Theorem 4.5.2. *Let (R, m) be a Noetherian local ring of characteristic p and let I be an m-primary ideal.*

1). If $\dim(R) = d$, then there exists a positive real number e_I such that

$$l_I(q) = e_I q^d + \mathcal{O}(q^{d-1}).$$

e_I is called the Hilbert-Kunz multiplicity of the ideal I.

2). If R is a complete intersection and $\mathrm{pd}_R(R/I) < \infty$, then $l_I(q) = q^d l(R/I)$. In general, $l_I(q) \le p^d l_I(q/p)$.

3). If R is a two-dimensional normal domain, then there exists a real number λ such that $l_I(q) = e_I q^2 + \lambda q + \mathcal{O}(1)$.

If R is the homogeneous coordinate ring of a smooth projective curve over a field, then λ=0.

1). is in [Mo1], 2). is Thm. 1.9 [Du] , and 3). is in [Mo2].

It is conjectured that e_I is always rational number; this is known in only a few cases. Most of the results in this direction are for the maximal ideal of certain classes of rings.

It is also conjectured that if R is as in Thm 4.5.2 3). then the bounded term $\mathcal{O}(1)$ is a periodic function.

The connection between tight closure and Hilbert-Kunz functions is given by the following result:

Theorem 4.5.3. *Let (R, m) be an excellent reduced equidimensional Noetherian local ring, and let $I_1 \subset I_2$ be m-primary ideals. Then $I_2 \subset I_1^*$ if and only if $e_{I_1} = e_{I_2}$.*

Proof: See Thm. 8.4 in [HH1]. $\square$

If R is a characteristic 2 hypersurface, we note the following formula:

Proposition 4.5.4. *Let (R, m) be a characteristic 2 hypersurface, and let I be an m-primary ideal. Then we have*

$$l\left(\frac{R}{I^{[2]}}\right) = \frac{N}{2} l\left(\frac{R}{I}\right) + \frac{1}{2} l\left(\mathrm{Tor}_1\left(\frac{R}{I}, R^{1/2}\right)\right).$$

where N is the minimal number of generators of $R^{1/2}$ as an R-module.

Note in particular that $l\left(\mathrm{Tor}_1(R/I, R^{1/2})\right)$ must always be an even number.

In the particular case when R is the cubical cone ring in characteristic 2, this formula becomes

$$l\left(\frac{R}{I^{[2]}}\right) = 4l\left(\frac{R}{I}\right) + \frac{1}{2} l\left(\mathrm{Tor}_1\left(\frac{R}{I}, R^{1/2}\right)\right).$$

Proof: Consider the minimal free resolution of $R^{1/2}$ over R as described in Prop. 4.5.1; let $\overline{A}$ be the result of tensoring A with R/I. When the resolution is tensored over R with R/I, we get the following complex:

$$\cdots \xrightarrow{\overline{A}} \left(\frac{R}{I}\right)^N \xrightarrow{\overline{A}} \left(\frac{R}{I}\right)^N \longrightarrow \frac{R}{I^{[2]}} \longrightarrow 0,$$

The length of the homology of this complex (except at the 0th spot, where it equals $l(R/I^{[2]})$) is equal to l$(\mathrm{Tor}_1(R/I, R^{1/2}))$; this follows from Obs. 4.4.3.

Therefore we have

$$l\left(\frac{R}{I^{[2]}}\right) = l\left(\mathrm{coker}(\overline{\mathrm{A}})\right) = l\left(\ker(\overline{\mathrm{A}})\right) = l\left(\mathrm{Tor}_1\left(\frac{R}{I}, R^{1/2}\right)\right) + l\left(\mathrm{im}(\overline{\mathrm{A}})\right)$$
$$= l\left(\mathrm{Tor}_1(\frac{R}{I}, R^{1/2})\right) + Nl\left(\frac{R}{I}\right) - l\left(\mathrm{coker}(\overline{\mathrm{A}})\right),$$

and the equality follows. □

Definition: Let I be an m-primary ideal, and let $l_I(q)$ be the Hilbert-Kunz function of I. Then the function $t_I(q) := l_I(pq) - p^2 l_I(q)$ will be called *the tail of the Hilbert-Kunz function of I.*

If R is the homogeneous coordinate ring of a smooth projective curve, then we have $l_I(q) = e_I q^2 + b_I(q)$, where $b_I(q)$ is bounded, hence $t_I(q) = b_I(pq) - p^2 b_I(q)$ is also bounded, and moreover $b_I(q)$ is periodic if and only if $t_I(q)$ is periodic.

Comment: If d is the dimension of R, and $T = k[[X_1, \ldots, X_{d+1}]]$, then we have $N = 2^{d+1}$, since a minimal set of generators for $R^{1/2}$ over R is given by all the $(x_{i_1} \ldots x_{i_k})^{1/2}$, where $\{i_1, \ldots, i_k\} \subset \{1, \ldots, d+1\}$; therefore the previous corollary can be interpreted as saying that $\frac{1}{2} l(\mathrm{Tor}_1(R/I^{[q]}, R^{1/2})$ gives the tail of the Hilbert-Kunz function of I.

For the case when $d = 2$, whenever we know that the length of the above tor modules is constant as a function of q (or more generally periodic, and we know the set of values it takes), we can explicitly compute the Hilbert-Kunz function; in this situation, the Hilbert-Kunz multiplicity can be seen to be a rational number.

For example, this is the case if R is the cubical cone ring in characteristic 2, and $I = (x^2, x+y, z)$.

We now restrict our attention to the case R is the cubical cone ring in characteristic $p = 2$.

Proposition 4.5.5. *Let (R, m) be the cubical cone ring in characteristic 2, and let I be an m-primary ideal. Then*

$$l\left(\mathrm{Tor}_1^R\left(\frac{R}{I^{[q]}}, R^{1/2}\right)\right)$$

is bounded as a function of q.

Moreover, if we let $l_I(q) = e_I q^2 + b_I(q)$, where $b_I(q)$ is a bounded function, we have

$$t_I(q) = b_I(2q) - 4b_I(q) = \frac{1}{2} l\left(\mathrm{Tor}_1\left(\frac{R}{I^{[q]}}, R^{1/2}\right)\right).$$

Proof: The first part was proved by Monsky (and is true in any 2-dimensional normal domain).

The second part follows from the formula given in Thm. 4.5.4. $\square$

Comment: In view of this result, the conjecture that $b_I(q)$ is eventually periodic as a function of q is equivalent (in this particular ring) to the conjecture that $l\left(\mathrm{Tor}_1(R/I^{[q]}, R^{1/2})\right)$ is periodic as a function of q.

However, using MACAULAY, we have found examples of ideals in this ring for which the first 9 values of the above function do not contain any repetition. For example, for $I = m^9$, the first 9 values are: 90, 96, 84, 108, 78, 120, 102, 72, 132.

Another interesting observation is that for an m-primary ideal I in the cubical cone ring in characteristic 2, there is a connection between the length of $\mathrm{Tor}_1(R/I, R^{1/2})$ and the third (stable) Betti number of R/I.

This shows that the study of Hilbert-Kunz functions in this ring is equivalent to the study of the minimal free resolution of $R/I^{[q]}$, as q varies.

Observation 4.5.6. *Let R be the cubical cone ring in characteristic 2, and let I be an m-primary ideal. Then we have*

$$l\left(\mathrm{Tor}_1(\frac{R}{I}, R^{1/2})\right) = 2b_3\left(\frac{R}{I}\right).$$

Proof: This can be seen by comparing the minimal free resolutions of R/m and $R^{1/2}$. The minimal resolution of R/m has the form

$$\cdots \xrightarrow{B} R^4 \xrightarrow{C} R^4 \xrightarrow{B} R^4 \longrightarrow R^3 \longrightarrow R \longrightarrow R/m \longrightarrow 0$$

with

$$B = \begin{bmatrix} x & 0 & z^2 & y^2 \\ z & y^2 & x^2 & 0 \\ y & z^2 & 0 & x^2 \\ 0 & x & y & z \end{bmatrix},$$

$$C = \begin{bmatrix} z^2 & y^2 & x^2 & 0 \\ y & z & 0 & x^2 \\ x & 0 & z & y^2 \\ 0 & x & y & z^2 \end{bmatrix}$$

(the resolution of R/m has been obtained using MACAULAY, but it can also be checked directly). We know from Prop. 4.5.1 that the minimal resolution of $R^{1/2}$ is of the form

$$\cdots \xrightarrow{A} R^8 \xrightarrow{A} R^8 \xrightarrow{A} R^{1/2} \longrightarrow 0,$$

where A is the matrix obtained by multiplying the generators

$$1, x^{1/2}, y^{1/2}, z^{1/2}, (xy)^{1/2}, (xz)^{1/2}, (yz)^{1/2}, (xyz)^{1/2}$$

of $R^{1/2}$ as an R-module by $x^{3/2} + y^{3/2} + z^{3/2}$; more precisely,

$$A = \begin{bmatrix} 0 & x^2 & y^2 & z^2 & 0 & 0 & 0 & 0 \\ x & 0 & 0 & 0 & y^2 & z^2 & 0 & 0 \\ y & 0 & 0 & 0 & x^2 & 0 & z^2 & 0 \\ z & 0 & 0 & 0 & 0 & x^2 & y^2 & 0 \\ 0 & y & x & 0 & 0 & 0 & 0 & z^2 \\ 0 & z & 0 & x & 0 & 0 & 0 & y^2 \\ 0 & 0 & z & y & 0 & 0 & 0 & x^2 \\ 0 & 0 & 0 & 0 & z & y & x & 0 \end{bmatrix}$$

and after rearranging the rows and columns, it can be seen that $A = B \oplus C$.

Let $\overline{B}$, $\overline{C}$ denote the result of tensoring B, respectively C with M.

Therefore we have

$$l\left(\mathrm{Tor}_1(M, R^{1/2})\right) = l\left(\frac{\ker(\overline{\mathrm{B}} \oplus \overline{\mathrm{C}})}{\mathrm{im}(\overline{\mathrm{B}} \oplus \overline{\mathrm{C}})}\right) = l\left(\frac{\ker(\overline{B})}{\mathrm{im}(\overline{C})}\right) + l\left(\frac{\ker(\overline{C})}{\mathrm{im}(\overline{B})}\right) =$$

$$b_3(M) + b_4(M) = 2b_3(M)$$

Recall that for any module M over a 2-dimensional hypersurface, we have $b_i(M) = b_3(M)$ for all $i \geq 3$, because of the periodicity of the resolution, hence the last equality follows. $\square$

In the rest of this section, we will present several equivalent formulations of the fact that tight closure equals Frobenius closure for every m-primary ideal in the cubical cone ring in characteristic 2.

First, we prove that the converse of Thm. 4.3.13 holds in this ring

Theorem 4.5.7. *Let R be the cubical cone ring in characteristic 2, and assume that for every m-primary ideal I, and for all $q \gg 0$, we have $(I^{<q>} : m)^{[2]} = (I^{<q>})^{[2]}$. Then for every m-primary ideal J, we have $J^* = J^F$.*

The same statement is true under the more general assumption that (R, m) is a hypersurface in characteristic 2, in which the test ideal is m, and Frobenius closure equals tight closure for parameter ideals.

Proof: First note that, given the assumption that the test ideal is m, the problem of proving tight closure equals Frobenius closure for m-primary ideals admits a simple reformulation in terms of Hilbert-Kunz functions:

If J is m-primary and $f \in J^* \backslash J^F$, let $J' = J + (f)$. Since we know that $f^q \in J^{[q]} : m$ for all q, it follows that $l_J(q) - l_{J'}(q) = 1$ for all q. We must show that this is not possible. In particular, it is sufficient to show that the above difference of Hilbert-Kunz functions evaluated at q is an even number for all $q \gg 0$.

Let $\underline{\boldsymbol{a}} \subset J$ be a parameter ideal, and let $I = \underline{\boldsymbol{a}} : J,\ I' = \underline{\boldsymbol{a}} : J'$. Then Prop. 4.3.6 shows that for all q we have

$$l\left(\frac{I^{<q>}}{I'^{<q>}}\right) = l\left(\frac{J'^{[q]}}{J^{[q]}}\right).$$

We know that $mJ^{[q]} = mJ'^{[q]}$, and therefore we have $I^{<q>} : m = I'^{<q>} : m$. By assumption, it follows that if $q \gg 0$, $(I^{<q>})^{[2]} = (I'^{<q>})^{[2]}$, and hence

$$l\left(\frac{I^{<2q>}}{I'^{<2q>}}\right) = l\left(\frac{I^{<2q>}}{I^{<q>[2]}}\right) - l\left(\frac{I'^{<2q>}}{I'^{<q>[2]}}\right) = l\left(\mathrm{Tor}_1^R(R/I^{<q>}, R^{1/2})\right) - l\left(\mathrm{Tor}_1^R(R/I'^{<q>}, R^{1/2})\right)$$

and therefore is an even number. □

Proposition 4.5.8. *Let R be the cubical cone ring in characteristic 2. Let $J = (f_1, \dots, f_n)$ be an m-primary ideal such that for all i, $f_i \notin (f_1, \dots, \hat{f_i}, \dots, f_n)^*$. Assume that $f \in J^*$, and let $J' = J + (f)$. If there exists a parameter ideal $(a, b) \subset J$ such that if we let $I' = (a, b) : J'$, we have*

$$l\left(\frac{(I'^{<q>} : m)^{[2]}}{(I'^{<q>})^{[2]}}\right) \neq 7$$

for at least one large enough value of q, then $f \in J^F$.

Proof: Let $I = (a, b) : J$. The assumption on the generators of J guarantees that Thm 4.3.13 applies: we know that for $q \gg 0$, $(I^{<q>} : m)^{[2]} = (I^{<q>})^{[2]}$.

On the other hand, if we assume by contradiction that $f \notin J^F$, then $l(J'^{[q]}/J^{[q]}) = 1$ for all q, hence we have $l(I^{<q>}/I'^{<q>}) = 1$ for all q.

Using the formula in 4.5.4, and the fact that $I^{<q>} : m = I'^{<q>} : m$, we get

$$l\left(\frac{(I'^{<q>} : m)^{[2]}}{(I'^{<q>})^{[2]}}\right) = l\left(\frac{(I'^{<q>} : m)^{[2]}}{(I'^{<q>})^{[2]}}\right) + l\left(\frac{(I^{<q>})^{[2]}}{(I'^{<q>})^{[2]}}\right) =$$

$$= 4\left(\frac{I'^{<q>}}{I^{<q>}}\right) + \frac{1}{2}\left(l\left(\mathrm{Tor}_1(R/I'^{<q>}, R^{1/2})\right) - l\left(\mathrm{Tor}_1(R/I^{<q>}, R^{1/2})\right)\right)$$

$$= 4 + \frac{1}{2}\left(l\left(\mathrm{Tor}_1(R/J'^{[q]}, R^{1/2})\right) - l\left(\mathrm{Tor}_1(R/J^{[q]}, R^{1/2})\right)\right). \tag{4.4}$$

The last equality follows from the Cor. 4.4.5, which shows that

$$l(\mathrm{Tor}_1^R(R/I^{<q>}, R^{1/2})) = l(\mathrm{Tor}_1^R(R/J^{[q]}, R^{1/2})).$$

Our assumptions imply that

$$l\left(\mathrm{Tor}_1(R/J'^{[q]}, R^{1/2})\right) - l\left(\mathrm{Tor}_1(R/J^{[q]}, R^{1/2})\right) = 6, \text{ since}$$

$$1 = l\left(\frac{J'^{[2q]}}{J^{[2q]}}\right) = 4l\left(\frac{J'^{[q]}}{J^{[q]})}\right) - \frac{1}{2}\left(l\left(\mathrm{Tor}_1(R/J'^{[q]}, R^{1/2})\right) - l\left(\mathrm{Tor}_1(R/J^{[q]}, R^{1/2})\right)\right)$$
$$= 4 - \frac{1}{2}\left(l\left(\mathrm{Tor}_1(R/J'^{[q]}, R^{1/2})\right) - l\left(\mathrm{Tor}_1(R/J^{[q]}, R^{1/2})\right)\right).$$

Hence, equation 4.4 shows that

$$l\left(\frac{(I'^{<q>} : m)^{[2]}}{(I'^{<q>})^{[2]}}\right) = 7 \ \forall q \gg 0, \text{ contradiction.} \quad \square$$

Proposition 4.5.9. *Let R be the cubical cone ring in characteristic 2, and let $J = (f_1, \dots, f_n)$ be an m-primary ideal, and assume that for all i, $f_i \notin (f_1, \dots, \hat{f}_i, \dots, f_n)^*$. Let $J' = J + (f)$, with $f \in J^*$. If there exists a parameter ideal $(a, b) \subset J$ such that for $I' = (a, b) : J'$ there is a large enough value of q such that*

$$(I'^{<q>} : m)^{[2]} \subset (I'^{<q>})^{[2]} : m,$$

then $f \in J^F$.

Note that this condition is weaker than the condition in Thm. 4.5.7, which was shown to be equivalent to the equality of tight closure and Frobenius closure for m-primary ideals.

Proof: Assume by contradiction that $f \notin J^F$. Let $I = (a, b) : J$. We know that $l(I^{<q>}/I'^{<q>}) = 1$, hence we can write $I^{<q>} = I'^{<q>} + (u)$, where $u \in I'^{<q>} : m$.

Also, we know that $l\left((I^{<q>})^{[2]}/(I'^{<q>})^{[2]}\right) = 7$, and therefore

$$l(R/((I'^{<q>})^{[2]} : u^2)) = 7.$$

This implies that

$$(I'^{<q>})^{[2]} : u^2 = (x^2, y^2, z^2, xyz),$$

contradiction with the assumption that $u^2 \in (I'^{<q>})^{[2]} : m$. $\square$

CHAPTER V

$*$–Independence and Special Tight Closure

In this chapter we introduce and study two new notions: $*$–independence and special tight closure.

The tight closure of any ideal is equal to the tight closure of a *-independent ideal, so the study of $*$–independent ideals can reduce questions about the tight closure of an arbitrary ideal to the same questions for simpler ideals.

The special tight closure of an ideal can be thought of as the complement of the ideal itself inside its tight closure, at least in a certain class of rings, in which our main result is shown to hold. This class includes the cubical cone rings in any characteristic.

Special tight closure allows us to formulate an (apparently) weaker conjecture, which is shown to be equivalent to the equality of tight closure and Frobenius closure for m-primary ideals (see Cor. 5.2.7, Cor. 5.2.10).

5.1 $*$–Independence

Definition: Let R be a characteristic p ring. We say that $f_1, \dots, f_n \in R$ are $*$*–independent* if for any $i = 1, \dots, n$ we have $f_i \notin (f_1, \dots, \hat{f_i}, \dots, f_n)^*$.

The reason for introducing this notion is the observation that every ideal can be thought of as being in the tight closure of a $*$–independent ideal. Indeed, let $I = (f_1, \dots, f_n)$. Assume that $f_1, \dots, f_n$ are not $*$–independent; by renumbering, we can assume that $f_n \in (f_1, \dots, f_{n-1})^*$, and the assertion follows by induction on n.

The following result will be used throughout this chapter; the result is known to specialists, but we include a proof for lack of a convenient reference.

Theorem 5.1.1. *Let (R, m) be a excellent, analytically irreducible local ring of characteristic p, let I be an ideal, and let $f \in R$.*

Assume that $f \notin I^*$; then there exists $q_0 = p^{e_0}$ such that for all $q \geq q_0$ we have $I^{[q]} : f^q \subset m^{[q/q_0]}$.

We need the following lemma:

Lemma 5.1.2. *Let $R \subset R'$ be an extension of local characteristic p domains such that R' is either module-finite or purely inseparable over R (i.e., there is a Q such that $R' \subset R^{1/Q}$). Let m_R, $m_{R'}$ be the maximal ideals of R, respectively R'. Assume that R has a test element.*

Assume that I is an ideal of R, and $f \in R$.

a). $f \notin I^* \Rightarrow f \notin (IR')^*$.

b). If there is a q_1 such that $I^{[q]}R' :_{R'} f^q \subset m_{R'}^{[q/q_1]}$ for all $q > q_1$, then there is a q_0 such that $I^{[q]} :_R f^q \subset m_R^{[q/q_0]}$ for all $q > q_0$.

Proof of Lemma: a). For the module-finite case, we know that $(IR')^* \cap R \subset I^*$ (see [HH1]).

Assume that R' is purely inseparable over R, and $f \in (IR')^*$. Let $c \in R$ be such that $cf^q \in I^{[q]}R'$ for all $q \Rightarrow (cf^q)^Q \subset I^{[qQ]}$, hence $cf^q \in I^{[q]^*}$ for all q, and this implies that $f \in I^*$.

b). Note that the usual powers are cofinal with the Frobenius powers, that is there is a constant r such that for all q we have $m^{rq} \subset m^{[q]} \subset m^q$, hence $I^{[q]} : f^q \subset m_R^{[q/q_0]}$ for all $q > q_0$ is equivalent to $I^{[q]} : f^q \subset m_R^{q/q_0'}$ for all $q \geq q_0'$, where q_0' is a (different) constant, and similarly $I^{[q]}R' :_{R'} f^q \subset m_{R'}^{[q/q_1]} \Leftrightarrow I^{[q]}R' :_{R'} f^q \subset m_{R'}^{q/q_1'}$.

If $R \subset R'$ is module-finite, then there is a q_1'' such that $m_{R'}^{q_1''} \subset m_R R'$; it follows that

$$I^{[q]} :_R f^q \subset (I^{[q]}R' :_{R'} f^q) \cap R \subset m_{R'}^{q/q_1'} \cap R \subset m_R^{q/q_1'q_1''} R' \cap R.$$

The Artin-Rees Lemma implies that there is a q_0' such that $m_R^{q/q_1'q_1''} R' \cap R \subset m_R^{q/q_0'}$ for all $q \geq q_0'$, hence the desired conclusion holds.

If R' is a purely inseparable extension of R, then $m_{R'}^{[Q]} \subset m_R$; it follows that $m_{R'}^{[q/q_1]} \subset m_R^{[q/q_1 Q]}$, and again the conclusion follows. $\square$

Proof of Theorem: Replace R by $\hat{R}$ and I by $I\hat{R}$; it is still true that $f \notin I\hat{R}^*$, since the hypothesis implies that R has a completely stable test element; moreover,

$$I^{[q]} : f^q = (I^{[q]}\hat{R} : f^q) \cap R,$$

hence it will be sufficient to prove that the conclusion holds for $I\hat{R}$.

Therefore, we can write R as a module-finite, domain extension of a regular local ring A. Replace R by $R[A^{1/q}]$ for $q \gg 0$, such that $R[A^{1/q}]$ is generically smooth over $A^{1/q}$, and replace A by $A^{1/q}$, which is still a regular local ring.

By Lemma 5.1.2, it is still true that $f \notin (IR[A^{1/q_0}])^*$, and moreover it is sufficient to prove that the conclusion holds for $IR[A^{1/q_0}]$. We can moreover replace this new R by a module-finite extension which is Galois over A.

Now we can use Thm. 6.9 of [HH1] for the new A and R: there is a constant $r > 0$ such that for all $\epsilon \in R^\infty$ with $\epsilon f \in IR^\infty$ we have ord $(N(\epsilon)) > r$, where $N : R \to A$ denotes the norm of the module-finite extension $A \subset R$, and for every

$a \in A$, $\mathrm{ord}_A(a) = n$ if $a \in m_A^n \backslash m_A^{n+1}$.

Therefore

$$u \in I^{[q]} : f^q \Rightarrow u^{1/q} \in (IR^\infty : f) \Rightarrow \mathrm{ord}_A(N(u)) = q\mathrm{ord}_A(N(u^{1/q})) > rq.$$

Let $u = u_1, \dots, u_d$ be all the conjugates of u in R, so that $N(u) = u_1 \dots u_d$. Let $\mathrm{ord}_R(u) = n$ if $u \in m^n \backslash m^{n+1}$. Since there are automorphisms of R which interchange the u_i', given by the action of the Galois group, we have $\mathrm{ord}_R u_1 = \dots = \mathrm{ord}_R u_d = \mathrm{ord}_R(u)$.

We will use a theorem of Izumi (see [Iz], [R]), which states that if R is analytically irreducible, there are constants C', C'' such that for every $x, y \in R$ we have $\mathrm{ord}_R(xy) \leq C'\mathrm{ord}_R(x) + \mathrm{ord}_R(y) + C'$. Induction on d shows that there is a constant C (depending on the ring R only) such that $\mathrm{ord}_R(N(u)) = \mathrm{ord}_R(u_1 \dots u_d) \leq C\mathrm{ord}_R(u)$.

The Artin-Rees lemma and the fact that $m_A R$ is an m-primary ideal imply that there is a constant c such that $m^n \cap R \subset m_A^{n/c}$ for all $n \geq c$, and therefore $\mathrm{ord}_R(N(u)) \geq \mathrm{ord}_A(N(u))/c$.

It follows that $\mathrm{ord}_R(u) > 1/cC\mathrm{ord}(N(u)) > (r/cC)q$.

If q_1 is a power of p such that $r/cC \geq 1/q_1$ this implies that $u \in m^{q/q_1}$.

Note that when (R, m) is the localization at the irrelevant ideal of a graded domain, or the completion of a ring of this form, ord_R defines a valuation on R, and we can apply Thm. 3.1 of [HH3] for a simpler proof.

Proposition 5.1.3. *Let (R, m) be an analytically irreducible excellent local ring of characteristic pLet $f_1, \dots, f_n$ and $g_1, \dots, g_n$ be two minimal sets of generators for an ideal J. Then $f_1, \dots, f_n$ are $*$–independent if and only if $g_1, \dots, g_n$ are $*$–independent.*

Proof: Assume that $f_1, \dots, f_n$ are $*$–independent. Let A be an invertible $n \times n$

matrix with entries in R such that $\underline{f}A = \underline{g}$, where $\underline{f}$ is the vector $(f_1, \ldots, f_n)$.

Since A is invertible and R is local, A is row and column equivalent to a diagonal matrix (at least one entry of A must be a unit; we can perform row and column operations to make the rest of the elements on that row and column zero; the remaining matrix must still contain a unit, so the process can be continued). This means that we can obtain $\underline{g}$ from $\underline{f}$ by a finite sequence of transformations of the form $f_1 \to f_1, \ldots, f_{i-1} \to f_{i-1}, f_i \to f_i + df_{i+1}, f_{i+1} \to f_{i+1}, \ldots, f_n \to f_n$, plus permutations of the f_i's.

Therefore it suffices to consider the case $g_1 = f_1 + df_2$, $g_2 = f_2, \ldots, g_n = f_n$ for some $d \in R$. We want to prove that $g_1, \ldots, g_n$ are $*$–independent.

For $i \geq 3$, the fact that $g_i \notin (g_1, \ldots, \hat{g}_i, \ldots, g_n)^*$ is clearly equivalent with the corresponding assumption for f_i.

Assume that $g_1 \in (g_2, \ldots, g_n)^*$, hence there is a $c \in R^0$ such that for all $q = p^e$, we have

$$c(f_1 + df_2)^q \in (f_2^q, \ldots, f_n^q).$$

This implies that $\forall\ q,\ cf_1^q \in (f_2^q, f_3^q, \ldots, f_n^q)$, which contradicts $f_1 \notin (f_2, \ldots f_n)^*$.

Assume that $g_2 \in (g_1, g_3, \ldots g_n)^*$, hence $\exists\ c \in R$ such that for all q,

$$cf_2^q \in ((f_1 + df_2)^q, f_3^q, \ldots, f_n^q) \Rightarrow\ \exists\ a_q \in R \text{ such that}$$

$$a_q f_1^q + (a_q d^q - c) f_2^q \in (f_3^q, \ldots, f_n^q).$$

There are two cases:

1. There is a positive integer N such that for all q, $a_q \notin m^N$; from Prop. 5.1.1, it follows that $f_1 \in (f_2, \ldots, f_n)^*$, contradiction.

2. For any positive integers N, there is a q_N such that $a_{q_N} \in m^N$. Fix an integer N_0 such that $c \notin m^{N_0}$; then it follows that for all for all $N > N_0$, we have

$a_{q_N} d^{q_N} - c \notin m^{N_0}$, and therefore from Prop. 5.1.1 it follows that $f_2 \in (f_1, f_3, \dots, f_n)^*$, contradiction. $\square$

This result shows that in a local ring, given an ideal I, the property that a minimal system of generators for I is $*$–independent does not depend on the choice of the system of generators. Therefore, in this situation it makes sense to say that the ideal I is $*$–independent if it admits a $*$–independent system of generators.

Definition: Let R be a local ring of positive characteristic. An ideal I is called *$*$–independent* if it has a system of generators consisting of $*$–independent elements (equivalently, if every minimal system of generators consists of $*$–independent elements).

Proposition 5.1.4. *Let (R, m) be an analytically irreducible excellent local ring of characteristic pLet $I = (f_1, \dots, f_n)$ be an ideal of R, and let $\mathcal{I}$ be the family of ideals generated by all subsets $\{i, i_1, \dots, i_k\}$ of $\{1, \dots, n\}$. Then every ideal in $\mathcal{I}$ has a $*$–independent Frobenius power if and only if the following condition holds:*

$(**)$ For every subset $\{i, i_1, \dots, i_k\}$ of $\{1, \dots, n\}$, we have either $f \notin (f_{i_1}, \dots, f_{i_n})^*$, or $f \in (f_{i_1}, \dots, f_{i_n})^F$.

In particular, the assertion that tight closure equals Frobenius closure for every ideal of R is equivalent to the assertion that for every ideal I there is a Frobenius power $I^{[q_0]}$ which is $*$–independent.

Proof: Assume that condition$(**)$ holds, but I is not $*$–independent. Without loss of generality, we may assume $f_n \in (f_1, \dots, f_{n-1})^*$, hence by assumption there is a q_0 such that $f_n^{q_0} \in (f_1^{q_0}, \dots, f_{n-1}^{q_0})$. We can write $I^{[q_0]} = (f_1^{q_0}, \dots, f_{n-1}^{q_0})$. Since $f_1^{q_0}, \dots, f_{n-1}^{q_0}$ continue to satisfy the hypothesis, the claim follows by induction on n.

Conversely, assume that every ideal in $\mathcal{I}$ has a $*$–independent Frobenius power. We need to show that condition$(**)$ holds. Assume for example that $f_1 \in (f_2, \dots, f_n)^*$.

Let $J = (f_2, \dots, f_n)$. We need to show that $f_1 \in J^F$.

By assumption, there is a q_0 such that $I^{[q_0]}$ is $*$–independent, therefore any minimal system of generators consists of $*$–independent elements. Since $f_1^{q_0} \in (f_2^{q_0}, \dots, f_n^{q_0})^*$, it follows that $f_1^{q_0}, \dots, f_n^{q_0}$ cannot be a minimal system of generators of $I^{[q_0]}$.

Case 1: There is a minimal system of generators of $I^{[q_0]}$ consisting of a subset of $\{f_2^{q_0}, \dots f_n^{q_0}\}$. This implies that $f_1^{q_0} \in J^{[q_0]}$, hence $f \in J^F$.

Case 2: There is a $*$–independent system of generators for $I^{[q_0]}$ consisting of $f_1^{q_0}$ and a subset of $\{f_2^{q_0}, \dots, f_n^{q_0}\}$. By renumbering, we may assume that $f_1^{q_0}, f_2^{q_0}, \dots, f_k^{q_0}$ is a minimal system of generators for $I^{[q_0]}$, where $k < n$.

For every $i = 2, \dots, n-k$, we can write $f_{k+i}^{q_0} \equiv \alpha_i f_1^{q_0} \bmod (f_2^{q_0}, \dots, f_k^{q_0})$ for some $\alpha_i \in R$.

If one of the α_i's is a unit, it follows that $f_1^{q_0} \in J^{[q_0]}$, hence $f_1 \in J^F$.

Assume all $\alpha_i \in m$. Since $f_1 \in J^*$, we can fix an element $c \in R^0$ such that $cf_1^{qq_0} \in J^{[qq_0]}$ for every $q = p^e$, and therefore $cf_1^{qq_0} \in (f_2^{qq_0}, \dots f_k^{qq_0}, \alpha_1^q f_1^{qq_0}, \dots \alpha_{n-k}^q f_1^{qq_0})$.

Let b be such that $c \notin m^b$; for all $q \gg 0$, it follows that there is an element not in m^b which multiplies $f_1^{qq_0}$ in the ideal $(f_2^{qq_0}, \dots, f_k^{qq_0})$; from Prop. 5.1.1, it follows that $f_1 \in (f_2, \dots, f_k)^*$. Since $k < n$, this is a contradiction. $\square$

The following result allows us to characterize $*$–independent ideals, and study some of their properties.

Proposition 5.1.5. *Let (R, m) be an analytically irreducible excellent local ring of characteristic pLet $J = (f_1, \dots, f_n)$ be a minimal system of generators for J.*

a). If $f_1, \dots, f_n$ are $*$–independent, and $\mathfrak{b}$ is any m-primary ideal, then there is *a q_0 depending on $\mathfrak{b}$ such that for all $q \geq q_0$ we have*

$$l\left(\frac{J^{[q]}}{\mathfrak{b}J^{[q]}}\right) = nl\left(\frac{R}{\mathfrak{b}}\right), \text{ and} \tag{5.1}$$

$$l\left(\frac{J^{[q]}}{\mathfrak{b}^{[q/q_0]}J^{[q]}}\right) = nl\left(\frac{R}{\mathfrak{b}^{[q/q_0]}}\right). \tag{5.2}$$

b). Assume, in addition, that R has an m-primary test ideal τ, and $\mathfrak{b}$ is an m-primary ideal such that $\mathfrak{b} \subset \tau$, $\mathfrak{b} \neq \tau$. If, moreover, there is a q_0 such that

$$l\left(\frac{J^{[q_0]}}{\mathfrak{b}J^{[q_0]}}\right) = nl\left(\frac{R}{\mathfrak{b}}\right),$$

then $f_1, \dots, f_n$ are $*$–independent.

Note that in this result it suffices to check that the length condition is satisfied for one choice of the m-primary ideal $\mathfrak{b}$ and for one value of q_0, and it follows that the condition is satisfies for every m-primary ideal, and for all $q \gg 0$ (where the meaning of $q \gg 0$ depends on the choice of $\mathfrak{b}$).

Proof: Note that in general if $J = (f_1, \dots, f_n)$ and $\mathfrak{b}$ is an m-primary ideal, we have

$$l\left(\frac{J}{\mathfrak{b}J}\right) = \sum_{i=1}^{n} l\left(\frac{R}{(\mathfrak{b}f_1, \dots, \mathfrak{b}f_i, f_{i+1}, \dots, f_n) : f_i}\right),$$

and each term in the sum is at most equal to $l(R/\mathfrak{b})$, since

$$(\mathfrak{b}f_1, \dots, \mathfrak{b}f_i, f_{i+1}, \dots, f_n) : f_i \supset \mathfrak{b}.$$

Hence, equality holds if and only if for all $i = 1, \dots, n$ we have

$$(\mathfrak{b}f_1, \dots, \mathfrak{b}f_i, f_{i+1}, \dots, f_n) : f_i = \mathfrak{b}.$$

a). Assume that $f_1, \dots, f_n$ are $*$–independent, and let $\mathfrak{b}$ be a fixed m-primary ideal.

Let q_1 be such that $m^{[q_1]} \subset \mathfrak{b}$, and let q_2 be such that for all $i = 1, \dots n$, and all $q \geq q_2$, we have

$$(f_1^q, \dots, \hat{f_i^q}, \dots, f_n^q) : f_i^q \subset m^{[q/q_2]}$$

(the hypothesis implies that such a q_2 exists). Let $q_0 = q_1 q_2$. Then for all $q \geq q_0$, we have

$$(f_1^q, \dots, \hat{f_i^q}, \dots, f_n^q) : f_i^q \subset \mathfrak{b}^{[q/q_0]}$$

This implies that for each i and all $q \geq q_0$, we have

$$(\mathfrak{b}^{[q/q_0]} f_1^q, \dots, \mathfrak{b}^{[q/q_0]} f_i^q, f_{i+1}^q, \dots, f_n^q) : f_i = \mathfrak{b}^{[q/q_0]}, \text{ and}$$

$$(\mathfrak{b} f_1^q, \dots, \mathfrak{b} f_i^q, f_{i+1}^q, \dots, f_n^q) : f_i^q = \mathfrak{b}.$$

Indeed, if

$$u \in (\mathfrak{b}^{[q/q_0]} f_1^q, \dots, \mathfrak{b}^{[q/q_0]} f_i^q, f_{i+1}^q, \dots, f_n^q) : f_i^q$$

$$(\text{respectively } u \in (\mathfrak{b} f_1^q, \dots, \mathfrak{b} f_i^q, f_{i+1}^q, \dots, f_n^q) : f_i^q)$$

then there is a $b \in \mathfrak{b}^{[q/q_0]}$ (respectively, $b \in \mathfrak{b}$), such that

$$u - b \in (f_1^q, \dots, \hat{f_i^q}, \dots, f_n^q) : f_i^q \subset \mathfrak{b}^{[q/q_0]} \Rightarrow u \in \mathfrak{b}^{[q/q_0]}$$

(respectively, $u \in \mathfrak{b}$).

b). Assume that $f_1, \dots, f_n$ are not $*$–independent; after renumbering, we may assume that $f_1 \in (f_2, \dots, f_n)^*$. Then for all q we have

$$\tau \subset (f_2^q, \dots, f_n^q) : f_1^q \subset (\mathfrak{b} f_1^q, f_2^q, \dots, f_n^q) : f_1^q.$$

If we choose $\mathfrak{b}$ such that $l(R/\mathfrak{b}) > l(R/\tau)$, we see that for all q, $l\left(J^{[q]}/\mathfrak{b}J^{[q]}\right) < nl(R/\mathfrak{b})$, contradiction. $\square$

Proposition 5.1.6. *Let (R, m) be an analytically irreducible characteristic p local ring. Let I be an arbitrary ideal; then there exist $*$–independent ideals $I_0 \subset I$ such that $I \subset I_0^*$.*

If I_0, I_1 are two such ideals, and we assume in addition that I/I_0 and I/I_1 have finite length (this is always true if the test elements generate an m-primary ideal, or if I is m-primary), then I_0 and I_1 have the same minimal number of generators.

Proof: The first statement is clear.

Let n_0, n_1 be the number of generators for I_0, respectively I_1. Let d be the Krull dimension of R.

Let $\mathfrak{a}$ be any m-primary ideal. Thm. 5.1.5 shows that there exists a q_0 such that for all $q \geq q_0$ we have

$$l\left(\frac{I_0^{[q]}}{\mathfrak{a}^{[q/q_0]}I_0^{[q]}}\right) = n_0 l\left(\frac{R}{\mathfrak{a}^{[q/q_0]}}\right), \quad l\left(\frac{I_1^{[q]}}{\mathfrak{a}^{[q/q_0]}I_1^{[q]}}\right) = n_1 l\left(\frac{R}{\mathfrak{a}^{[q/q_0]}}\right). \tag{5.3}$$

Our assumptions imply that there is a constant C such that for all q,

$$l\left(\frac{I^{[q]}}{I_0^{[q]}}\right) \leq Cq^{d-1}, \quad l\left(\frac{I^{[q]}}{I_1^{[q]}}\right) \leq Cq^{d-1}. \tag{5.4}$$

We also know that $\mathfrak{a}I^{[q_0]} \subset (\mathfrak{a}I_0^{[q_0]})^*$, and $\mathfrak{a}I^{[q_0]} \subset (\mathfrak{a}I_1^{[q_0]})^*$. Since $\mathfrak{a}I^{[q_0]}/\mathfrak{a}I_0^{[q_0]}$, $\mathfrak{a}I^{[q_0]}/\mathfrak{a}I_1^{[q_0]}$ also have finite length, we may assume that the choice of C is such that for all q,

$$l\left(\frac{\mathfrak{a}^{[q/q_0]}I^{[q]}}{\mathfrak{a}^{[q/q_0]}I_0^{[q]}}\right) \leq Cq^{d-1}, \qquad l\left(\frac{\mathfrak{a}^{[q/q_0]}I^{[q]}}{\mathfrak{a}^{[q/q_0]}I_1^{[q]}}\right) \leq Cq^{d-1}. \tag{5.5}$$

Here we are using Thm. 8.17 a). in [HH1].

From the in(equalities) in (3), (4), and (5), we get

$$n_0 l\left(\frac{R}{\mathfrak{a}^{[q/q_0]}}\right) - Cq^{d-1} \leq l\left(\frac{I^{[q]}}{\mathfrak{a}^{[q/q_0]}I^{[q]}}\right) = l\left(\frac{I^{[q]}}{I_0^{[q]}}\right) + l\left(\frac{I_0^{[q]}}{\mathfrak{a}^{[q/q_0]}I_0^{[q]}}\right) - l\left(\frac{\mathfrak{a}^{[q/q_0]}I^{[q]}}{\mathfrak{a}^{[q/q_0]}I_0^{[q]}}\right)$$

$$\leq n_0 l\left(\frac{R}{\mathfrak{a}^{[q/q_0]}}\right) + Cq^{d-1}, n_1 l\left(\frac{R}{\mathfrak{a}^{[q/q_0]}}\right) - Cq^{d-1} \leq l\left(\frac{I^{[q]}}{\mathfrak{a}^{[q/q_0]}I^{[q]}}\right) =$$
$$l\left(\frac{I^{[q]}}{I_1^{[q]}}\right) + l\left(\frac{I_1^{[q]}}{\mathfrak{a}^{[q/q_0]}I_1^{[q]}}\right) - l\left(\frac{\mathfrak{a}^{[q/q_0]}I^{[q]}}{\mathfrak{a}^{[q/q_0]}I_1^{[q]}}\right) \leq n_1 l\left(\frac{R}{\mathfrak{a}^{[q/q_0]}}\right) + Cq^{d-1},$$

and therefore for all q we have

$$n_0 l\left(\frac{R}{\mathfrak{a}^{[q/q_0]}}\right) \leq n_1 l\left(\frac{R}{\mathfrak{a}^{[q/q_0]}}\right) + 2Cq^{d-1}. \tag{5.6}$$

For sufficiently large values of q, we have $l(R/\mathfrak{a}^{[q/q_0]}) > 2Cq^{d-1}$, and then (6) implies that $n_0 \leq n_1$. By symmetry, it follows that $n_0 = n_1$. $\square$

Next, we obtain some equivalent characterizations of $*$–independence, and we discuss the connection between this notion and the problem of showing that tight closure equals Frobenius closure in the cubical cone rings.

Corollary 5.1.7. *Let R be an excellent local 2-dimensional Cohen-Macaulay analytically irreducible ring of characteristic p, and I an m-primary ideal generated by n $*$–independent elements. If $\underline{\boldsymbol{a}}$ is a fixed parameter ideal, then for all $q \gg 0$ we have*

$$l\left(\frac{I^{[q]}:\underline{\boldsymbol{a}}}{I^{[q]}}\right) = (n-1)l\left(\frac{R}{\underline{\boldsymbol{a}}}\right).$$

Proof: Since $R/\underline{\boldsymbol{a}}$ has projective dimension 2, we have

$$l\left(\frac{R}{I^{[q]}} \otimes_R \frac{R}{\underline{\boldsymbol{a}}}\right) - l\left(\mathrm{Tor}_1^R\left(\frac{R}{I^{[q]}}, \frac{R}{\underline{\boldsymbol{a}}}\right)\right) + l\left((\mathrm{Tor}_2^R\left(\frac{R}{I^{[q]}}, \frac{R}{\underline{\boldsymbol{a}}}\right)\right) = 0.$$

This follows by tensoring the minimal free resolution of $R/\underline{\boldsymbol{a}}$ with $R/I^{[q]}$, and breaking the resulting complex into short exact sequences. Each of these short exact sequences consists of modules of finite length, and the length is additive.

For $q \gg 0$, $R/I^{[q]} \otimes R/\underline{\boldsymbol{a}} = R/\underline{\boldsymbol{a}}$, and equation 5.1 now implies that

$$l\left(\mathrm{Tor}_2^R\left(\frac{R}{I^{[q]}}, \frac{R}{\underline{\boldsymbol{a}}}\right)\right) = nl\left(\frac{R}{\underline{\boldsymbol{a}}}\right) - l\left(\frac{R}{\underline{\boldsymbol{a}}}\right).$$

On the other hand, we can compute $\mathrm{Tor}_2^R(R/I^{[q]}, R/\underline{\boldsymbol{a}})$ by using the Koszul resolution of $R/\underline{\boldsymbol{a}}$, and see that it is equal to $(I^{[q]}:\underline{\boldsymbol{a}})/I^{[q]}$. $\square$

Corollary 5.1.8. *Let R be an excellent local 2-dimensional Cohen-Macaulay analytically irreducible ring, let $\underline{\boldsymbol{a}}$ be a parameter ideal, and I a $*$–independent ideal. Consider the beginning of the minimal free resolution of $R/I^{[q]}$:*

$$\cdots \longrightarrow R^b \xrightarrow{B} R^m \xrightarrow{A} R^n \longrightarrow R \longrightarrow 0. \tag{5.7}$$

Assume that $R/I^{[q]}$ does not have finite projective dimension. Then for $q \gg 0$, the ideal generated by the entries of the matrix B is not contained in $\underline{\boldsymbol{a}}$ (while the ideal generated by the entries of A is contained in higher and higher powers of the maximal ideal, as q increases).

Proof: Tensor the resolution 5.7 with $R/\underline{\boldsymbol{a}}$; let $\overline{A}$, $\overline{B}$ denote the maps obtained form A, B after tensoring. For $q \gg 0$, Prop. 5.1.1 and the assumption that I is $*$–independent imply that the entries of A are contained in $\underline{\boldsymbol{a}}$, hence $\overline{A} = 0$. Assume by contradiction that all the entries of B are also contained in $\underline{\boldsymbol{a}}$. Then Cor. 5.1.7 implies that

$$(n-1)l\left(\frac{R}{\underline{\boldsymbol{a}}}\right) = l\left(\mathrm{Tor}_2^R\left(\frac{R}{I^{[q]}}, \frac{R}{\underline{\boldsymbol{a}}}\right)\right) = l\left(\ker(\overline{B})\right) = ml\left(\frac{R}{\underline{\boldsymbol{a}}}\right),$$

and therefore $m = n-1$. But this contradicts the assumption that $R/I^{[q]}$ does not have finite projective dimension. $\square$

Observation 5.1.9. *Let (R, m) be an analytically irreducible excellent local ring of characteristic pLet J be an ideal generated by $*$–independent elements $f_1, \ldots, f_n$. Let $\mathfrak{b}$ and $\mathfrak{a}$ be fixed ideals, where $\mathfrak{a}$ is assumed to be m-primary.*

Then there exists a q_0 such that for all $q \gg 0$ we have $(\mathfrak{a}J^{[q]} : \mathfrak{b}) \cap J^{[q]} = (\mathfrak{a} : \mathfrak{b})J^{[q]}$ and $(\mathfrak{a}^{[q/q_0]}J^{[q]} : \mathfrak{b}) \cap J^{[q]} = (\mathfrak{a}^{[q/q_0]} : \mathfrak{b})J^{[q]}$.

Proof: By the Prop. 5.1.1, there is a q_1 such that for all $q \geq q_1$ and for all $i = 1, \ldots, n$ we have $(f_1^q, \ldots, \hat{f_i^q}, \ldots, f_n^q) : f_i^q \subset m^{[q/q_1]}$. Let $q_0 \geq q_1$ such that $m^{[q_0]} \subset \mathfrak{a}^{[q_1]}$, and therefore $m^{[q/q_1]} \subset \mathfrak{a}^{[q/q_0]}$ for all $q \gg 0$.

Assume that $u = \alpha_1 f_1^q + \ldots + \alpha_n f_n^q \in (\mathfrak{a}^{[q/q_0]} J^{[q]} : \mathfrak{b}) \cap J^{[q]}$ (respectively, $u \in \mathfrak{a} J^{[q]} : \mathfrak{b}$). Then for every $b \in \mathfrak{b}$ we can write $b\alpha_1 f_1^q + \ldots + b\alpha_n f_n^q = a_1 f_1^q + \ldots + a_n f_n^q$, with $a_1, \ldots, a_n \in \mathfrak{a}^{[q/q_0]}$ (respectively, $a_1, \ldots, a_n \in \mathfrak{a}$).

Since $(b\alpha_1 - a_1) f_1^q + \ldots + (b\alpha_n - a_n) f_n^q = 0$, we have

$$b\alpha_i - a_i \in (f_1^q, \ldots, \hat{f_i^q}, \ldots f_n^q) : f_i^q \subset \mathfrak{a}^{[q/q_0]}$$

for all $i = 1, \ldots, n$.

Since this holds for every $b \in \mathfrak{b}$, it follows that $\alpha_i \in \mathfrak{a}^{[q/q_0]} : \mathfrak{b}$ (respectively, $\alpha_i \in \mathfrak{a} : \mathfrak{b}$). $\square$

Proposition 5.1.10. *Let $R = k[[x, y, z]]/(F)$, where $F = z^N + f(x, y)$ for some positive integer N. Let $m = (x, y, z)$. Assume that R is a domain. Let I be an m-primary $*$–independent ideal with the minimal number of generators equal to n. For every $q = p^e$ let $s_q := l((I^{[q]} : m)/I^{[q]})$ be the socle dimension of $R/I^{[q]}$. Then for $q \gg 0$, we have $n - 1 \leq s_q \leq N(n - 1)$.*

Assume in addition that $\underline{\boldsymbol{a}}^{[p]} = (\underline{\boldsymbol{a}} : m)^{[p]}$ for every parameter ideal $\underline{\boldsymbol{a}}$. Then, if $q \gg 0$ and $s_q = n - 1$, we have $(I^{[q]} : m)^{[p]} = I^{[qp]}$.

Proof: Let $\underline{\boldsymbol{a}} = (x, y)$; note that $l(R/\underline{\boldsymbol{a}}) = N$. Apply Cor, 5.1.7: for $q \gg 0$ we have $l((I^{[q]} : \underline{\boldsymbol{a}})/I^{[q]}) = N(n - 1)$.

But $(I^{[q]} : \underline{\boldsymbol{a}})/I^{[q]}$ is a module over $R/\underline{\boldsymbol{a}} = k[[z]]/(z^N)$, so it has a decomposition into cyclic modules of the form $R/(x, y, z)$, $R/(x, y, z^2)$, ..., $R/(x, y, z^N) = R/(x, y)$.

Let k_i be the number of terms of the form $R/(x, y, z^i)$, where $i = 1, \ldots, N$. Then we have

$$s_q = k_1 + k_2 + \ldots + k_N, \text{ and } N(n - 1) = k_1 + 2k_2 + \ldots + Nk_N,$$

and the first claim follows.

Note that $s_q = n-1$ implies that $k_1 = \ldots = k_{N-1} = 0$, and therefore $(I^{[q]} : \underline{\boldsymbol{a}})/I^{[q]}$ is a direct sum of cyclic modules isomorphic to $R/\underline{\boldsymbol{a}}$. It follows that

$$\frac{I^{[q]} : m}{I^{[q]}} \cong \bigoplus \frac{\underline{\boldsymbol{a}} : m}{\underline{\boldsymbol{a}}}$$

and the last claim follows from the fact that $(\underline{\boldsymbol{a}} : m)^{[p]} = \underline{\boldsymbol{a}}^{[p]}$. $\square$

Proposition 5.1.11. *Let (R, m) be an analytically irreducible excellent local ring of characteristic pLet I be an ideal of finite projective dimension.*

a). I is $*$–independent.

b). If we assume in addition that R is 2-dimensional Cohen-Macaulay, and $\underline{\boldsymbol{a}}$ is a fixed parameter ideal, then for all $q \gg 0$, $\underline{\boldsymbol{a}}I^{[q]}$ and $I^{[q]} : \underline{\boldsymbol{a}}$ have finite projective dimension.

c). If $R = k[[x, y, z]]/(x^3 + y^3 + z^3)$, where k is a field of positive characteristic, then for all $q \gg 0$ we have

$$\dim \left(\frac{I^{[q]} : m}{I^{[q]}} \right) = n - 1,$$

where n is the minimal number of generators of I.

Proof: a). Let $f_1, \ldots, f_n$ be a minimal system of generators for I. Then for any $i = 1, \ldots, n$ the short exact sequence

$$0 \longrightarrow \frac{R}{(f_1, \ldots, \hat{f_i}, \ldots, f_n) : f_i} \longrightarrow \frac{R}{(f_1, \ldots, \hat{f_i}, \ldots, f_n)} \longrightarrow \frac{R}{I} \longrightarrow 0$$

stays exact when tensored with $R^{1/q}$ for all $q = p^e$. This shows that

$$(f_1^q, \ldots, \hat{f_i^q}, \ldots, f_n^q) : f_i^q = ((f_1, \ldots, \hat{f_i}, \ldots, f_n) : f_i)^{[q]},$$

hence $f_i \notin (f_1, \ldots, \hat{f_i}, \ldots, f_n)^*$.

b) Note that Prop. 5.1.5 implies that for all $q \gg 0$ we have

$$\frac{I^{[q]}}{\underline{\boldsymbol{a}}I^{[q]}} = \bigoplus_n \frac{R}{\underline{\boldsymbol{a}}},$$

where n is the minimal number of generators of I.

The short exact sequence

$$0 \longrightarrow \bigoplus_n \frac{R}{\underline{\boldsymbol{a}}} \longrightarrow \frac{R}{I^{[q]}} \longrightarrow \frac{R}{\underline{\boldsymbol{a}} I^{[q]}} \longrightarrow 0$$

implies that $I^{[q]}/\underline{\boldsymbol{a}}I^{[q]}$ has finite projective dimension, and the short exact sequence

$$0 \longrightarrow \frac{R}{I^{[q]} : \underline{\boldsymbol{a}}} \longrightarrow \frac{R}{I^{[q]}} \oplus \frac{R}{I^{[q]}} \longrightarrow \frac{\underline{\boldsymbol{a}}}{\underline{\boldsymbol{a}} I^{[q]}}$$

implies that $I^{[q]} : \underline{\boldsymbol{a}}/I^{[q]}$ has finite projective dimension.

c). Let $\underline{\boldsymbol{a}} = (x, y)$. Write $(I^{[q]} : \underline{\boldsymbol{a}})/I^{[q]}$ as a direct sum of cyclic modules of the form $R/(x, y)$, $R/(x, y, z^2)$, and $R/(x, y, z)$ as in the proof of Prop. 5.1.10. Since $(I^{[q]} : \underline{\boldsymbol{a}})/I^{[q]}$ has finite projective dimension, this implies that $(I^{[q]} : \underline{\boldsymbol{a}})/I^{[q]}$ is a direct sum of copies of $R/\underline{\boldsymbol{a}}$, and $\dim(I^{[q]} : m/I^{[q]})$ equals the number of copies in the direct sum.

Cor. 5.1.7 shows that there are $n-1$ copies, hence the desired equality. □

Lemma 5.1.12. *Let (R, m) be an analytically irreducible excellent local ring of characteristic pLet $\mathfrak{a}$ and I be $*$–independent ideals, with $\mathfrak{a}$ m-primary. Then for $q \gg 0$, $\mathfrak{a}I^{[q]}$ is also $*$–independent.*

Proof: Let $I = (f_1, \dots, f_n)$, $\mathfrak{a} = (a_1, \dots, a_m)$. We need to show that for $q \gg 0$, $a_1 f_1^q, \dots, a_m f_1^q, \dots, a_1 f_n^q, \dots, a_m f_n^q$ are $*$–independent. Fix q_0 such that $(f_2^Q, \dots, f_n^Q) : f_1^Q \subset \mathfrak{a}^{[Q/q_0]}$ for all $Q \geq q_0$. This is possible because I is $*$–independent and $\mathfrak{a}$ is m-primary. Let $q > q_0$.

Assume by contradiction that

$$a_1 f_1^q \in (a_2 f_1^q, \dots, a_m f_1^q, f_2^q, \dots, f_n^q)^*.$$

Fix a nonzero element c such that

$$c a_1^Q f_1^{qQ} \in (a_2^Q f_1^{qQ}, \dots, a_m^Q f_1^{qQ}, f_2^{qQ}, \dots, f_n^{qQ}) \text{ for all } Q.$$

Then for all Q there exist $d_2, \ldots, d_m \in R$ such that

$$ca_1^Q + d_2a_2^Q + \ldots + d_m a_m^Q \in (f_2^{qQ}, \ldots, f_n^{qQ}) : f_1^{qQ} \subset \mathfrak{a}^{[qQ/q_0]}.$$

It follows that for all Q there exists a $d \in R$ such that

$$(c - da_1^{qQ/q_0 - Q}) \in (a_2^{qQ/q_0}, \ldots, a_m^{qQ/q_0}) : a_1^Q \subset (a_2^Q, \ldots, a_m^Q) : a_1^Q.$$

Since $\mathfrak{a}$ is $*$–independent, this colon ideal is contained in higher and higher powers of the maximal ideal as Q increases, contradiction with the assumption that c is a fixed nonzero element. $\square$

Corollary 5.1.13. *Let $R = k[[x, y, z]]/(F)$ be a two-dimensional hypersurface, with $F = z^N + f(x, y)$, and let I be a $*$–independent ideal. Assume that R is a domain. Let $m = (x, y, z)$. Then there is a q_0 such that for all $q \gg 0$ we have $m^{[q/q_0]}I^{[q]} : m \not\subset I^{[q]}$.*

Proof: Let $I = (f_1, \ldots f_n)$. By assumption, we can choose q_1 such that for all $i = 1, \ldots n$ and for all $q \geq q_1$, we have $(f_1^q, \ldots, \hat{f_i}^q, \ldots f_n^q) : f_i^q \subset m^{[q/q_1]}$. Let $\underline{\boldsymbol{a}} = (a, b)$ be a fixed parameter ideal, and let q_2 be such that $m^{[q_2]} \subset \underline{\boldsymbol{a}}^{[q_1]}$, and therefore $m^{[q/q_1]} \subset \underline{\boldsymbol{a}}^{[q/q_2]}$ for all $q \gg 0$.

Apply Corollary 5.1.9 with $\mathfrak{b} = m$; there is a q_3 such that $(\underline{\boldsymbol{a}}^{[q/q_3]}I^{[q]} : m) \cap I^{[q]} = (\underline{\boldsymbol{a}}^{[q/q_3]} : m)I^{[q]}$. Let $q_0 > \max(q_2, q_3)$. Note that if q_3 works in Cor. 5.1.9, then any $q_0 \geq q_3$ will also work.

Assume by contradiction that $\underline{\boldsymbol{a}}^{[q/q_0]}I^{[q]} : m \subset I^{[q]}$; then it follows that

$$\underline{\boldsymbol{a}}^{[q/q_0]}I^{[q]} : m = (\underline{\boldsymbol{a}}^{[q/q_0]} : m)I^{[q]},$$

and therefore we have

$$l\left(\frac{\underline{\boldsymbol{a}}^{[q/q_0]}I^{[q]} : m}{\underline{\boldsymbol{a}}^{[q/q_0]}I^{[q]}}\right) = l\left(\frac{(\underline{\boldsymbol{a}}^{[q/q_0]} : m)I^{[q]}}{\underline{\boldsymbol{a}}^{[q/q_0]}I^{[q]}}\right) = nl\left(\frac{\underline{\boldsymbol{a}}^{[q/q_0]} : m}{\underline{\boldsymbol{a}}^{[q/q_0]}}\right) = n.$$

On the other hand, the ideal $\underline{\boldsymbol{a}}I^{[q_0]}$ has a minimal system of generators consisting of $2n$ elements: $af_1^{q_0}, \dots, af_n^{q_0}, bf_1^{q_0}, \dots, bf_n^{q_0}$.

From Lemma 5.1.12, it follows that $af_1^{q_0}, \dots, af_n^{q_0}, bf_1^{q_0}, \dots, bf_n^{q_0}$ are $*$–independent. By the Prop. 5.1.10, it follows that the socle dimension of $\underline{\boldsymbol{a}}^{[q/q_0]}I^{[q]}$ is at least $2n-1$ for all $q \gg 0$, contradiction. $\square$

Other properties of $*$–independent ideals considered in this section are related to the study of certain properties that are preserved when $I^{[q]}$ is replaced by $\underline{\boldsymbol{a}}I^{[q]}$ for every parameter ideal $\underline{\boldsymbol{a}}$.

In the next section, we will see that in a certain class of rings, which includes the cubical cone, it suffices to consider tight closure of ideals of the form $\underline{\boldsymbol{a}}I^{[q]}$ in order to obtain information about the tight closure of an arbitrary m-primary ideal I.

Proposition 5.1.14. *Let (R, m) be a d-dimensional local Cohen-Macaulay ring. Assume that R is excellent and analytically irreducible. Let I be a $*$–independent ideal, and let $\underline{\boldsymbol{a}}$ be a parameter ideal.*

Then for all $q \gg 0$ we have :

a). $l\left(\mathrm{Tor}_i\left(R^{1/p}, R/I^{[q]}\right)\right) = l\left(\mathrm{Tor}_i\left(R^{1/p}, R/\underline{\boldsymbol{a}}I^{[q]}\right)\right)$ for all $i > 0$.

b). $b_i(I^{[q]}) = b_i(\underline{\boldsymbol{a}}I^{[q]})$ for all $i > d$, where b_i denotes the ith Betti number.

Proof: From Proposition 5.1.5, we see that for all $q \gg 0$ we have

$$\frac{I^{[q]}}{\underline{\boldsymbol{a}}I^{[q]}} = \bigoplus_{i=1}^{n} \frac{R}{\underline{\boldsymbol{a}}},$$

where n is the minimal number of generators of I.

Both claims follow by looking at the long exact sequences obtained by tensoring

$$0 \longrightarrow \frac{I^{[q]}}{\underline{\boldsymbol{a}}I^{[q]}} \longrightarrow \frac{R}{\underline{\boldsymbol{a}}I^{[q]}} \longrightarrow \frac{R}{I^{[q]}} \longrightarrow 0$$

with $R^{1/p}$, and respectively R/m. $\square$

Proposition 5.1.15. *If R is a characteristic 2 excellent analytically irreducible hypersurface, I is a $*$-independent m-primary ideal, and $\mathfrak{a}$ is m-primary, then there is a $q_0 = 2^{e_0}$ such that for $q \geq q_0$ we have*

$$l\left(\operatorname{Tor}_1^R\left(\frac{R}{\mathfrak{a}I^{[q]}}, R^{1/2}\right)\right) = l\left(\operatorname{Tor}_1^R\left(\frac{R}{I^{[q]}}, R^{1/2}\right)\right) + nl\left(\operatorname{Tor}_1^R\left(\frac{R}{\mathfrak{a}}, R^{1/2}\right)\right), \text{ and} \tag{5.8}$$

$$l\left(\operatorname{Tor}_1^R\left(\frac{R}{\mathfrak{a}^{[q/q_0]}I^{[q]}}, R^{1/2}\right)\right) = l\left(\operatorname{Tor}_1^R\left(\frac{R}{I^{[q]}}, R^{1/2}\right)\right) + nl\left(\operatorname{Tor}_1^R\left(\frac{R}{\mathfrak{a}^{[q/q_0]}}, R^{1/2}\right)\right), \tag{5.9}$$

where n is the minimal number of generators of I.

Proof: Let N be the minimal number of generators of $R^{1/2}$ as an R-module. Recall that for every m-primary ideal J we have

$$l\left(\frac{R}{J^{[2]}}\right) = \frac{N}{2}l\left(\frac{R}{J}\right) + \frac{1}{2}l\left(\operatorname{Tor}_1^R(R/J, R^{1/2})\right)$$

(see Thm. 4.5.4). Apply this formula for $J = I^{[q]}$ and $J = \mathfrak{a}I^{[q]}$ (respectively, $J = \mathfrak{a}^{[q/q_0]}$), where $q \gg 0$ is such that equation 5.1 (respectively, equation 5.2) holds for $\mathfrak{b} = \mathfrak{a}^{[2]}$. we get

$$l\left(\frac{R}{I^{[2q]}}\right) = \frac{N}{2}l\left(\frac{R}{I^{[q]}}\right) + \frac{1}{2}l\left(\operatorname{Tor}_1^R(R/I^{[q]}, R^{1/2})\right), \tag{5.10}$$

$$l\left(\frac{R}{\mathfrak{a}^{[2]}I^{[2q]}}\right) = \frac{N}{2}l\left(\frac{R}{\mathfrak{a}I^{[q]}}\right) + \frac{1}{2}l\left(\operatorname{Tor}_1^R(R/\mathfrak{a}I^{[q]}, R^{1/2})\right), \tag{5.11}$$

$$\left(\text{respectively, } l\left(\frac{R}{\mathfrak{a}^{[2q/q_0]}I^{[2q]}}\right) = \frac{N}{2}l\left(\frac{R}{\mathfrak{a}^{[q/q_0]}I^{[q]}}\right) + \frac{1}{2}l\left(\operatorname{Tor}_1^R(R/\mathfrak{a}^{[q/q_0]}I^{[q]}, R^{1/2})\right)\right). \tag{5.12}$$

From 5.10 and 5.11 (respectively, 5.12) it follows that

$$l\left(\frac{I^{[2q]}}{\mathfrak{a}^{[2]}I^{[2q]}}\right) = \frac{N}{2}l\left(\frac{I^{[q]}}{\mathfrak{a}I^{[q]}}\right) + \frac{1}{2}\left(l(\operatorname{Tor}_1^R(R/\mathfrak{a}I^{[q]}, R^{1/2})) - l(\operatorname{Tor}_1^R(R/I^{[q]}, R^{1/2}))\right). \tag{5.13}$$

$$\left(l\left(\frac{I^{[2q]}}{\mathfrak{a}^{[2q/q_0]}I^{[2q]}}\right)=\frac{N}{2}l\left(\frac{I^{[q]}}{\mathfrak{a}^{[q/q_0]}I^{[q]}}\right)+\frac{1}{2}\left(l(\mathrm{Tor}_1^R(R/\mathfrak{a}^{[q/q_0]}I^{[q]},R^{1/2}))-l(\mathrm{Tor}_1^R(R/I^{[q]},R^{1/2}))\right)\right). \tag{5.14}$$

Equation 5.13 (respectively, equation 5.14) and equation 5.1 imply that

$$nl\left(\frac{R}{\mathfrak{a}^{[2]}}\right)=\frac{N}{2}nl\left(\frac{R}{\mathfrak{a}}\right)+\frac{1}{2}\left(l(\mathrm{Tor}_1^R(R/\mathfrak{a}I^{[q]},R^{1/2}))-l(\mathrm{Tor}_1^R(R/I^{[q]},R^{1/2}))\right),$$

$$nl\left(\frac{R}{\mathfrak{a}^{[2q/q_0]}}\right)=\frac{N}{2}nl\left(\frac{R}{\mathfrak{a}^{[q/q_0]}}\right)+\frac{1}{2}\left(l(\mathrm{Tor}_1^R(R/\mathfrak{a}^{[q/q_0]}I^{[q]},R^{1/2}))-l(\mathrm{Tor}_1^R(R/I^{[q]},R^{1/2}))\right).$$

But

$$l\left(\frac{R}{\mathfrak{a}^{[2]}}\right)=\frac{N}{2}l\left(\frac{R}{\mathfrak{a}}\right)+\frac{1}{2}l\left(\mathrm{Tor}_1^R(R/\mathfrak{a},R^{1/2})\right),$$

$$l\left(\frac{R}{\mathfrak{a}^{[2q/q_0]}}\right)=\frac{N}{2}l\left(\frac{R}{\mathfrak{a}^{[q/q_0]}}\right)+\frac{1}{2}l\left(\mathrm{Tor}_1^R(R/\mathfrak{a}^{[q/q_0]},R^{1/2})\right),$$

hence the claim follows.

The proof for the second equality is similar. □

Corollary 5.1.16. *Let R be the cubical cone ring in characteristic 2. Then there are infinitely many values possible for $l(\mathrm{Tor}_1^R(R/I,R^{1/2}))$ when I varies through all m-primary ideals of R. In particular, there are infinitely many linkage classes of m-primary ideals in this ring.*

Proof: let $\mathfrak{a}=(x^2,x+y,z)$; we know that $l(\mathrm{Tor}_1^R(R/\mathfrak{a}^{[q]},R^{1/2}))=8$ for all $q=4^n$. Let I be any $*$–independent m-primary ideal, and let n be the minimal number of generators of I. Then there is a q_0 such that for $q\geq q_0$, we have

$$l\left(\mathrm{Tor}_1^R(R/\mathfrak{a}^{[q/q_0]}I^{[q]},R^{1/2})\right)\geq 8n,$$

and by Lemma 5.1.12 we know that $\mathfrak{a}^{[q/q_0]}I^{[q]}$ is a $*$–independent ideal with $3n$ generators. By replacing I by $\mathfrak{a}^{[q/q_0]}I^{[q]}$ and continuing as above, we get a sequence of $*$–independent m-primary ideals I_n for which that values of $l(\mathrm{Tor}_1^R(R/I_n,R^{1/2}))$ are unbounded. □

5.2 Special tight closure

In this section, we define the special tight closure of an ideal, and we prove that in a certain class of rings, which includes the cubical cone rings, the special tight closure of an m-primary ideal I can be thought of as the complement of the ideal itself inside its tight closure.

As a consequence of this result, many questions about tight closure may be reduced to questions about special tight closure, which may simplify certain problems.

Definition: Let R be a characteristic p ring, with test ideal τ, and let J be any ideal. We say that u is in the special tight closure of an ideal I with respect to J ($u \in I^{*J}$) if there is a q_0 such that the following equivalent conditions hold:

1. There is a $c \in R^0$ such that for all $q \geq q_0$, $cu^q \in J^{[q/q_0]}I^{[q]}$
2. $u^{q_0} \in (JI^{[q_0]})^*$.
3. For all $q \geq q_0$, $\tau u^q \in J^{[q/q_0]}I^{[q]}$.

If (R, m) is local, let $I^{*sp} = I^{*m}$.

Clearly, $mI \subset I^{*sp} \subset I^*$. We will investigate a class of rings in which $I^* = I^{*sp} + I$, and therefore for the purpose of studying I^*/I, it suffices to look at elements of I^{*sp}. To illustrate the importance of this property, we prove the following:

Proposition 5.2.1. *Let (R, m) be an analytically irreducible excellent local ring of characteristic p, or graded domain with irrelevant ideal m. If $I^* = I + I^{*sp}$, and I is $*$–independent, we have the following direct sum decomposition:*

$$\frac{I^*}{mI} = \frac{I}{mI} \oplus \frac{I^{*sp}}{mI}.$$

Proof: Let $f_1, \ldots, f_n$ be minimal generators for I. It suffices to prove that $I \cap I^{*sp} = mI$. Let $f = a_1 f_1 + \ldots + \ldots + a_n f_n \in I \cap I^{*sp}$, and assume for example that a_1 is a unit. Since $f \in I^{*sp}$, $\exists c \in R^0$ such that for all $q \geq q_0$ we have

$$c(a_1^q f_1^q + \ldots + a_n^q f_n^q) \in m^{[q/q_0]} I^{[q]} \Rightarrow (ca_1^q + d) f_1^q \in (f_2^q, \ldots, f_n^q),$$

where $d \in m^{[q/q_0]}$. Assume $c \in m^b \backslash m^{b+1}$. Since $ca_1^q + d \in m^b \backslash m^{b+1}$ for all $q \gg 0$, this implies that $f_1 \in (f_2, \ldots f_n)^*$, contradiction. $\square$

Lemma 5.2.2. *Let (R, m) be an analytically irreducible excellent local ring of characteristic pIf $f \in (f_1, \ldots, f_n)^*$ and $f_i \notin (f_1, \ldots, \hat{f}_i, \ldots, f_n, f)^*$ for all $i = 1, \ldots, n$, then $f \in (f_1, \ldots, \hat{f}_i, \ldots, f_n)^{*sp}$.*

Proof: Let ν be the number of generators of m. Let $c \in R^0$ be such that for all q, $cf^q \in (f_1^q, \ldots, f_n^q)^*$. Then we can write

$$cf^q = a_{1q} f_1^q + \ldots + a_{nq} f_n^q.$$

For all i, we must have

$$a_{iq} \in (f_1^q, \ldots, \hat{f}_i^q, \ldots, f_n^q, f^q) : f_i^q,$$

and since $f_i \notin (f', f_1, \ldots, \hat{f}_i, \ldots, f_n)^*$, there is a q_0 such that $a_{iq} \in m^{[q/q_0]}$. which implies $cf^q \in m^{[q/q_0]}(f_1^q, \ldots, f_n^q)$, hence $f \in (f_1, \ldots, f_n)^{*sp}$. $\square$

We will use the following strategy to find classes of rings in which for every m-primary ideal I, we have $I^* \subset I^{*sp} + I$: given an element $f \in I^*$, and generators $f_1, \ldots, f_n$ for I, we will find a linear combination $f' = \alpha_1 f_1 + \ldots + \alpha_n f_n \in I$, such that $f + f', f_1, \ldots, f_n$ satisfy the hypothesis of Lemma 5.2.2.

We will prove the following:

Theorem 5.2.3. *Let R be a normal Cohen–Macaulay $\mathbb{N}$-graded domain; write $R = R_0 \oplus R_1 \oplus R_2 \oplus \cdots$, where $R_0 = K$ is a perfect field of characteristic p. Let $m = R_1 \oplus R_2 \oplus \ldots$.*

Then for every ideal I, we have $I^* = I + I^{*sp}$, and hence if I is $*$–independent, we have $I^*/mI = I/mI \oplus I^{*sp}/mI$.

We will use the following result, due to K. E. Smith:

Theorem 5.2.4. *Let R be a normal $\mathbb{N}$-graded ring over a perfect field of prime characteristic p, and let I_1, I_2 be ideals generated by homogeneous elements of degrees strictly less that δ and greater or equal to δ respectively. Let $z \in R$ be homogeneous of degree δ. Then $z \in (I_1 + I_2)^* \Rightarrow z \in I_1^* + I_2$.*

Proof: This is Thm. C of [S4]. $\square$

Before staring the proof of Theorem 5.2.3, we need some preliminary results:

Lemma 5.2.5. *Let R be a normal $\mathbb{N}$-graded domain; write $R = R_0 \oplus R_1 \oplus R_2 \oplus \cdots$, where $R_0 = K$ is a perfect field of characteristic p. Let $m = R_1 \oplus R_2 \oplus \ldots$. Let $\mathfrak{a}$ be a homogeneous m-primary ideal, and let $h,\ l \in R$. Assume that the following conditions hold:*

(1) $h \in (\mathfrak{a}, l)^*$, and

(2) $l \in (\mathfrak{a}, h + \alpha l)^*$ for all $\alpha \in K$.

Then $h, l \in \mathfrak{a}^*$.

Proof: First note that conditions (1) and (2) imply $h \in \mathfrak{a}^* \Leftrightarrow l \in \mathfrak{a}^*$.

Consider $\mathfrak{a}$ fixed, and assume by contradiction that there is a $l \notin \mathfrak{a}^*$ such that (1) and (2) hold for some choice of h. Let $(\mathfrak{a}, l)$ be a minimal ideal of this type containing $\mathfrak{a}$, for which there exists an h as above, but $l \notin \mathfrak{a}^*$.

For every $c \in m$, conditions (1) and (2) continue to hold for cl, ch. By the minimality assumption, it follows that $cl,\ ch \in \mathfrak{a}^*$. Since this must be true for every $c \in m$, it follows that $h,\ l \in \mathfrak{a}^* : m \backslash \mathfrak{a}^*$.

Let

$$\mathfrak{a}^* : m = \mathfrak{a}^* + (e_1, \dots, e_n),$$

where $e_1, \dots, e_n$ are minimal generators of $\mathfrak{a}^* : m/\mathfrak{a}^*$. Note that we can choose $e_1, \dots, e_n$ to be homogeneous, and arrange them in increasing order of their degrees:

$$\deg(e_1) \leq \ldots \leq \deg(e_n).$$

Write

$$h \equiv \alpha_1 e_1 + \ldots + \alpha_n e_n \mod(\mathfrak{a}^*),$$

$$l \equiv \beta_1 e_1 + \ldots + \beta_n e_n \mod(\mathfrak{a}^*),$$

with $\alpha_i,\ \beta_i \in K$; we may replace h, l by $\alpha_1 e_1 + \ldots + \alpha_n e_n$, respectively $\beta_1 e_1 + \ldots + \beta_n e_n$, in the sense that $h \in (\mathfrak{a}, l)^* \Rightarrow \alpha_1 e_1 + \ldots + \alpha_n e_n \in (\mathfrak{a}^*, \beta_1 e_1 + \ldots + \beta_n e_n)^*$.

We will prove by induction on i that conditions (1) and (2) imply that $\alpha_i = 0$ for all i, hence $h \in \mathfrak{a}^*$, which is a contradiction.

Note that for all i we have $e_i \notin (\mathfrak{a}^*, e_{i+1}, \dots, e_n)^*$, since otherwise Thm. 5.2.4 would imply that $e_i \in \mathfrak{a}^* + (e_{i+1}, \dots, e_n)$, which is false.

For $i = 1$, assume that $\alpha_1 \neq 0$. If $\beta_1 = 0$, condition (1) implies that $e_1 \in (\mathfrak{a}^*, e_2, \dots, e_n)^*$, contradiction.

If $\beta_1 \neq 0$, let $\alpha = -\alpha_1/\beta_1$; condition (2) implies $e_1 \in (\mathfrak{a}^*, e_2, \dots, e_n)^*$, contradiction.

Assume that $\alpha_1 = \ldots = \alpha_{i-1} = 0$; condition (2) with $\alpha = 0$ implies that $\beta_1 = \ldots = \beta_{i-1} = 0$. Assume that $\alpha_i \neq 0$. If $\beta_i = 0$, condition (1) implies that $e_i \in (\mathfrak{a}^*, e_{i+1}, \dots, e_n)^*$, contradiction.

If $\beta_i \neq 0$, let $\alpha = -\alpha_i/\beta_i$; condition (2) implies that $e_i \in (\mathfrak{a}^*, e_{i+1}, \dots, e_n)^*$, contradiction.

By induction, the claim is proved for all i, which contradicts the assumption that $h \notin \mathfrak{a}^*$. $\square$

Lemma 5.2.6. *Let $R = R_0 \oplus R_1 \oplus \cdots$ be a Noetherian graded ring, and let $I = (f_1, \dots, f_n)$ be an ideal, and $f \in R$ an element such that $f \notin I$.*

If d is the highest degree of any homogeneous component appearing in $f, f_1, \dots, f_n$, then $f \notin I + R_{\geq 2d+1}$.

Proof: Assume that $f \in I + R_{\geq 2d+1}$. Let $a_1, \dots, a_n$ be such that

$$f - a_1 f_1 - \ldots - a_n f_n \in R_{\geq 2d+1}.$$

Write f, the f_i's and the a_i's as sums of homogeneous components:

$$f = f^0 + \ldots + f^d,$$

$$f_i = f_i^0 + \ldots + f_i^d,$$

$$a_i = a_i^0 + \ldots + a_i^N,$$

where the upper index denotes the degree of the respective homogeneous component. Identifying homogeneous components yields:

$$f^j = \sum_{k=0}^{j} a_1^k f_1^{j-k} + \ldots + \sum_{k=0}^{j} a_n^k f_n^{j-k}$$

for all $j \leq d$, and the homogeneous components of degrees $d+1, \dots, 2d$ appearing in $a_1 f_1 + \ldots + a_n f_n$ must be zero.

If we let $a_i' = a_i^0 + \ldots + a_i^d$ for all $i = 1, \dots, n$, this shows that

$$f = a_1' f_1 + \ldots + a_n' f_n,$$

hence $f \in I$. $\square$

Proof of Thm. 5.2.3: Let $I = (f_1, \dots, \hat{f_i}, \dots, f_n)$, and let $f \in I^*$. Replace $\{f_1, \dots, f_n\}$ by a minimal subset $\{f_{i_1}, \dots, f_{i_s}\}$ such that $f \in (f_{i_1}, \dots, f_{i_s})^*$.

Therefore we can assume that $\forall\ i \in \{1, \dots, n\}$,

$$f \notin (f_1, \dots, \hat{f_i}, \dots, f_n)^*.$$

We want to show that for each $i = 1, \dots, d$, we can find $\alpha_i \in K$ such that $f_i \notin (f', f_1, \dots, \hat{f_i}, \dots, f_n)^*$, where $f' = f + \sum_i \alpha_i f_i$.

Assume that we can find $\alpha_1 \in K$ such that $f_1 \notin (f_2, \dots, f_n, f + \alpha_1 f_1)^*$; replace f by $f + \alpha_1 f_1$. Next, find $\alpha_2 \in K$ such that $f_2 \notin (f_1, f_3, \dots f_n, f + \alpha_2 f_2)^*$. Note that replacing f by $f + \alpha_2 f_2$ preserves the fact that $f_1 \notin (f_2, \dots, f_n, f)^*$.

This argument shows that it is enough to find an $\alpha_i \in K$ for each $i = 1, \dots, n$ such that $f_i \notin (f_1, \dots, f_{i-1}, f + \alpha_i f_i, f_{i+1}, \dots, f_n)^*$.

Assume by contradiction that, for example, for $i = 1$, we have

$$f_1 \in (f_2, \dots, f_n, f + \alpha f_1)^* \ \forall\ \alpha \in K.$$

Let d be a test element for R. Since $f \notin (f_2, \dots, f_n)^*$, we can pick q such that

$$df^q \notin (f_2, \dots, f_n)^{[q]}.$$

The proof of Thm. 4.6 in [HH4] shows that for every N there exist homogeneous m-primary irreducible ideals contained in m^N (see also [Ho]). Lemma 5.2.6 now shows that we can find a homogeneous m-primary irreducible ideal $\mathfrak{a}$ such that $df^q \notin (f_2, \dots, f_n)^{[q]} + \mathfrak{a}$.

This implies that

$$\mathfrak{a} : (f_2^q, \dots, f_n^q) \not\subset \mathfrak{a} : df^q.$$

Therefore we can pick an element $c \in \mathfrak{a} : (f_2^q, \dots, f_n^q) \backslash \mathfrak{a} : df^q$.

Since $f \in (f_1, \ldots, f_n)^*$ and $cf_2^q, \ldots, cf_n^q \in \mathfrak{a}$, it follows that we have

$$cf^q \in (cf_1^q, \ldots, cf_n^q)^* \subset (\mathfrak{a}, cf_1^q)^*, \tag{5.15}$$

and similarly, for all $\alpha \in K$, the assumption that $f_1 \in (f + \alpha f_1, f_2, \ldots, f_n)^*$ implies that

$$cf_1^q \in (\mathfrak{a}, c(f^q + \alpha^q f_1^q))^*.$$

Since the field K is assumed to be perfect, every $\beta \in K$ can be written as α^q for some α, and therefore we can also say that for all $\beta \in K$, we have

$$cf_1^q \in (\mathfrak{a}, c(f^q + \beta f_1^q))^*. \tag{5.16}$$

Lemma 5.2.5 shows that equations 5.15 and 5.16 imply that $cf^q \in \mathfrak{a}^*$, but this is a contradiction, since c was chosen in such a way that $d(cf^q) \notin \mathfrak{a}$, and d is a test element. $\square$

We note some immediate corollaries of this theorem:

Corollary 5.2.7. *Let (R, m) be graded normal Cohen-Macaulay domain with $R_0 = K$ a perfect filed, and with irrelevant ideal m. Let I be an ideal such that there exists a large enough value of q for which $m(I^{[q]})^* \subset (I^{[q]})^F$. Then $I^* \subset I^F$.*

In particular, in order to prove that tight closure equals Frobenius closure for every ideal, it suffices to show that we have $(mI)^* \subset I^F$ for every ideal I.

Proof: We know that $I^* = I + I^{*sp}$; it suffices to show that $I^{*sp} \subset I^F$. But for $q \gg 0$, $(I^{*sp})^{[q]} \subset (mI^{[q]})^* \subset (I^{[q]})^F \Rightarrow I^{*sp} \subset I^F$. $\square$

Corollary 5.2.8. *Let (R, m) be graded normal Cohen-Macaulay domain with $R_0 = K$ a perfect filed, and with irrelevant ideal m. Assume that there is an m-primary test ideal τ. Then for every m-primary ideal $\mathfrak{a} \subset \tau$, and for every ideal I, there is a q_0 such that $\mathfrak{a}(I^*)^{[q_0]} = \mathfrak{a}I^{[q_0]}$.*

Proof Let $f \in I^*$; write $f = g + h$ where $g \in I$ and $h \in I^{*sp}$. Then for $q \gg 0$, $\tau h^q \subset \mathfrak{a}I^{[q]}$. $\square$

Corollary 5.2.9. *Let R be a normal Cohen-Macaulay $\mathbb{N}$-graded domain; write $R = R_0 \oplus R_1 \oplus R_2 \oplus \cdots$, where $R_0 = K$ is a perfect field of characteristic p. Let $m = R_1 \oplus R_2 \oplus \ldots$.*

If the test ideal τ is m-primary, and $d = \dim(R)$, then for every ideal I we have $f \in I^* \Rightarrow f^d \in I^F$.

Proof: Let $\underline{\boldsymbol{a}} \subset \tau$ be a parameter ideal, and let q_0 be such that $\underline{\boldsymbol{a}}(I^*)^{[q_0]} = \underline{\boldsymbol{a}}I^{[q_0]}$. Such a q_0 exists from the previous corollary. Then the usual determinant trick shows that $f^{q_0 d} \in I^{[q_0]}$.

Indeed, let $\underline{\boldsymbol{a}} = (a_1, \ldots, a_d)$, and let $f \in I^*$. Then there exists a $d \times d$ matrix A with entries in $I^{[q_0]}$ such that $f^{q_0}(a_1, \ldots, a_d)^T = A(a_1, \ldots, a_d)^T$, and therefore $\det(A - f^{q_0} I_d) = 0$. This shows that f^{q_0} satisfies an equation of integral dependence over $I^{[q_0]}$, of degree d, and in particular $f^{dq_0} \in I^{[q_0]}$. $\square$

As an application of special tight closure, we discuss some conditions that would imply that tight closure equals Frobenius closure in the cubical cone rings studied in Chapter 4. We do not know whether these conditions hold in general in these rings.

Corollary 5.2.10. *Let $R = k[[X, Y, Z]]/(X^3 + Y^3 + Z^3)$ be the cubical cone ring in characteristic $p = 2$, with k a perfect field. Let I be an m-primary ideal, and let $I' = I + (f)$, with $f \in I^*$.*

If there exists a parameter ideal $\underline{\boldsymbol{a}}$ such that $b_3(\underline{\boldsymbol{a}}I'^{[q]}) \geq b_3(I'^{[q]})$ for all $q \gg 0$, then we must have $f \in I^F$.

Proof: We can assume without loss of generality that I is $*$–independent, and therefore for $q \gg 0$ we have $b_3(\underline{\boldsymbol{a}}I^{[q]}) = b_3(I^{[q]})$, by part b). of Prop. 5.1.14.

Assume by contradiction that $f \in I^* \backslash I^F$. Then by Obs. 4.5.6 it follows that for all $q = 2^e$ we have

$$l\left(\mathrm{Tor}_1\left(R/I'^{[q]}, R^{1/2}\right)\right) = l\left(\mathrm{Tor}_1\left(R/I^{[q]}, R^{1/2}\right)\right) + 6,$$

and by Cor. 4.5.4 it follows that $b_3(I'^{[q]}) = b_3(I^{[q]}) + 3 = b_3(\underline{\boldsymbol{a}}I^{[q]}) + 3 = b_3(\underline{\boldsymbol{a}}I'^{[q]})$, contradiction. $\square$

Corollary 5.2.11. *Let (R, m) be a Gorenstein local ring of characteristic p in which $\underline{\boldsymbol{a}}^* = \underline{\boldsymbol{a}}^F$ for every parameter ideal $\underline{\boldsymbol{a}}$, and $I^* = I + I^{*sp}$ for every ideal I.*

Assume that I is such that for any $q_0 = p^{e_0}$ there exist $q \gg 0$ such that

$$(m^{[q/q_0]} I^{[q]})^{<p>} \subset I^{[pq]}. \tag{5.17}$$

Then $I^* = I^F$.

Proof: It is enough to show that $I^{*sp} \subset I^F$. Let $u \in I^{*sp}$. Then there exists q_0 such that $u^q \in (m^{[q/q_0]} I^{[q]}) : \tau$ for all $q \gg 0$, where τ is the test ideal.

This implies that $u^{pq} \in (m^{[q/q_0]} I^{[q]})^{<p>} \subset I^{[pq]}$, hence $u \in I^F$. $\square$

Obs: Note that the condition that for every I, q_0 there exists q such that equation 5.17 is satisfied is weaker than the condition that R^∞ is coherent, since the latter condition implies that for all $q \gg 0$,$m^{[q/q_0]} I^{[q]}$ has finite projective dimension, hence $(m^{[q/q_0]} I^{[q]})^{<p>} = (m^{[pq/q_0]} I^{[pq]} \subset I^{[pq]}$.

In the rest of this chapter, we will discuss some results related to the existence of strong test ideals and some applications. Strong test ideals were first introduced in [Hu2], but their existence was not known in the generality discussed here.

Definition: Let R be a characteristic p ring. An ideal T is called a strong test ideal if for every ideal I we have $TI = TI^*$.

Proposition 5.2.12. *Let (R, m) be a local reduced ring of characteristic p in which the test ideal commutes with completion, i.e. $\tau_R \hat{R} = \tau_{\hat{R}}$. Then for every ideal I, we have $\tau I^* = \tau I$, i.e τ is a strong test ideal.*

Proof: It is enough to show that the equality holds after completing, hence we may assume that R is complete.

It suffices to show that for every ideal I, $I : \tau$ is tightly closed. Indeed, applying this to τI we get that $I \subset \tau I : \tau \Rightarrow I^* \subset \tau I : \tau$, hence $\tau I^* = \tau I$.

Let $k = R/m$, and let E be the injective hull of k over R. Then Prop. 8.23 of [HH1] shows that

$$\tau = \mathrm{Ann}_R 0_E^{*fg},$$

where 0_E^{*fg} denotes the finitistic tight closure of 0. This implies that $0_E^{*fg} = \mathrm{Ann}_E \tau$.

Let I be an m-primary irreducible ideal of R. Then $R/I \hookrightarrow E$. We have $\mathrm{Ann}_{R/I}\tau = R/I \cap \mathrm{Ann}_E \tau = R/I \cap 0_E^{*fg}$, hence it is tightly closed in R/I. This implies that $I : \tau$ is a tightly closed ideal, whenever I is irreducible m-primary. Since any m-primary ideal can be written as intersection of m-primary irreducible ideals, and every ideal can be written as intersection of m-primary ideals, it follows that $I : \tau$ is tightly closed for any ideal I. $\square$

Corollary 5.2.13. *Let R be a reduced locally excellent ring, and let*

$$k = \sup\{\nu(\tau_{\hat{R}_m} | m = \text{maximal ideal}\}.$$

Then for every ideal I and every $u \in I^*$, we have

$$u^k \in Iu^{k-1} + i^2 u^{k-2} + \cdots + I^k.$$

Proof: It suficess to check that the conclusion holds after we replace R by $\hat{R}_m$ for every maximal ideal m. Since our assumptions imply persistence of tight closure, we still have $u \in (I\hat{R}_m)^*$ for every m.

Hence, we may assume that R is complete local, with test ideal τ and the number of generators of τ is at most k. Let $\tau = (c_1, \dots, c_k)$ By Prop. 5.2.12, we have $u\tau \subset \tau I$, hence for every $i = 1, \dots k$ there exist $a_{ij} \in I$ such that

$$xc_i = \sum_{j=1}^{k} a_i j c_j.$$

Let $A = (a_{ij}) = k \times k$ matrix with entries in I; then $\det(A - xI_k) = 0$ provides an equation of integral dependence of x over I of degree k. $\square$

In a similar vein, we note the following result, which provides a large class of tightly closed ideals, under additional assumptions on the ring:

Proposition 5.2.14. *Let R be a normal $\mathbb{N}$-graded Gorenstein domain; write $R = R_0 \oplus R_1 \oplus R_2 \oplus \cdots$, where $R_0 = K$ is a perfect field of characteristic p.*

Let $m = R_1 \oplus R_2 \oplus \cdots$ and assume that $R = K[R_1]$. Assume that the test ideal τ has a minimal system of generators consisting of homogeneous elements of the same degree δ.

Let $\underline{\boldsymbol{a}}$ be an ideal generated by a homogeneous system of parameters. Then for any $k \geq 0$, every ideal $\underline{\boldsymbol{a}} : \tau m^k \subset I \subset \underline{\boldsymbol{a}} : \tau m^{k+1}$ is tightly closed.

Proof: First, we claim that $\underline{\boldsymbol{a}} : \tau m^{k+1} = \underline{\boldsymbol{a}} : \tau m^k + (e_{k1}, \dots, e_{kN_k})$, with $e_{k1}, \dots, e_{kN_k}$ homogeneous of the same degree d_k.

We prove this statement by induction on k. For $k = 0$, note that $\underline{\boldsymbol{a}} : m = \underline{\boldsymbol{a}} + (e_0)$, with e_0 homogeneous, and if we write $\underline{\boldsymbol{a}} : \tau m = \underline{\boldsymbol{a}} : \tau + (e_{01}, \dots, e_{0N_0})$ with $e_{01}, \dots, e_{0N_0}$ homogeneous not in $\underline{\boldsymbol{a}} : \tau$, it follows that for all $i = 0, \dots, N_0$ there exists an element $c \in \tau$ of degree δ such that ce_{0i} has the same degree as e_0. Since this must hold for every i, it follows that $e_{01}, \dots, e_{0N_0}$ have the same degree

Assume that the statement holds for k, and write

$$\underline{\boldsymbol{a}} : \tau m^{k+1} = \underline{\boldsymbol{a}} : \tau m^k + (e_{k1}, \dots, e_{kN_k}),$$

with $e_{k1}, \ldots, e_{kN_k} \in \underline{\boldsymbol{a}} : \tau m^{k+1} \backslash \underline{\boldsymbol{a}} : \tau m^k$, and homogeneous.

Since m is generated by forms of degree 1, for every $i = 1, \ldots, N_k$ we can find a $c_i \in R_1$ such that $c_i e_{ki} \in \underline{\boldsymbol{a}} : \tau m^k \backslash \underline{\boldsymbol{a}} : m^{k-1}$, hence modulo $\underline{\boldsymbol{a}} : \tau m^k$, $c_i e_{ki}$ is equal to a linear combination of $e_{k-11}, \ldots, e_{k-1N_{k-1}}$ with coefficients in K. It follows that $c_i e_{ki}$ is homogeneous of degree d_{k-1}, and therefore e_{ki} is homogeneous of degree $d_{k-1} - 1$ for all i, hence the claim follows by induction.

Any ideal $\underline{\boldsymbol{a}} : \tau m^k \subset I \subset \underline{\boldsymbol{a}} : \tau m^{k+1}$ is of the form

$$I = \underline{\boldsymbol{a}} : \tau m^k + (g_1, \ldots, g_n)$$

where $g_1, \ldots, g_n$ are linear combinations of $e_{k1}, \ldots e_{kN_k}$, with coefficients in K. Therefore, $g_1, \ldots, g_n$ are homogeneous of the same degree d_k.

Since $\underline{\boldsymbol{a}} : \tau m^{k+1}$ is tightly closed, $I^* \subset \underline{\boldsymbol{a}} : \tau m^{k+1}$; if I is not tightly closed, there is an $f =$ linear combination of $e_{k1}, \ldots, e_{kN_k}$ with coefficients in K such that $f \in I^*$.

Note that $\deg(f) = \deg(g_1) = \ldots = \deg(g_n)$. Thm. 5.2.4 (see also Thm. 2.7 of [S3]) implies $f \in (g_1, \ldots, g_n) + (\underline{\boldsymbol{a}} : \tau m^k)^* = I$. $\square$

As an immediate corollary of this result, we note the following:

Corollary 5.2.15. *Let R be a normal $\mathbb{N}$-graded Gorenstein domain; write $R = R_0 \oplus R_1 \oplus R_2 \oplus \cdots$, where $R_0 = K$ is a perfect field of characteristic p.*

Let $m = R_1 \oplus R_2 \oplus \cdots$ and assume that $R = K[R_1]$. Assume that the test ideal τ has a minimal system of generators consisting of homogeneous elements of the same degree δ.

If $\mathfrak{b}$ is any m-primary irreducible ideal, for all $k \geq 1$, every ideal $\mathfrak{b} : \tau m^k \subset I \subset \mathfrak{b} : \tau m^{k+1}$ is tightly closed.

Proof: Let $\mathfrak{b}$ be an m-primary irreducible ideal, and let $\underline{\boldsymbol{a}} \subset \mathfrak{b}$ be a homogeneous parameter ideal; then there is an $f \in R$ such that $\mathfrak{b} = \underline{\boldsymbol{a}} : f$.

For any $h \in R$, $h \in I^* \Rightarrow fh \in (\underline{\boldsymbol{a}} : \tau m^k + fI)^*$. Since $\underline{\boldsymbol{a}} : \tau m^k \subset \underline{\boldsymbol{a}} : \tau m^k + fI \subset \underline{\boldsymbol{a}} : \tau m^{k+1}$, Lemma 5.2.14 implies that $fh \in \underline{\boldsymbol{a}} : \tau m^k + fI$, hence $f(h-i) \in \underline{\boldsymbol{a}} : \tau m^k$ for some $i \in I$, and therefore $h - i \in \mathfrak{b} : \tau m^k \subset I$, so $h \in I$. $\square$

Corollary 5.2.16. *Let R be a normal $\mathbb{N}$-graded Gorenstein domain; write $R = R_0 \oplus R_1 \oplus R_2 \oplus \cdots$, where $R_0 = K$ is a perfect field of characteristic p.*

Let $m = R_1 \oplus R_2 \oplus \cdots$ and assume that $R = K[R_1]$. Assume that the test ideal τ has a minimal system of generators consisting of homogeneous elements of the same degree δ.

Then for any $k \geq 1$, any ideal $\tau m^{k+1} \subset \mathfrak{b} \subset \tau m^k$ is a strong test ideal, i.e. we have $\mathfrak{b} I^* = \mathfrak{b} I$ for every ideal I.

Another application of the existence of strong test ideals is the following:

Proposition 5.2.17. *Assume that R is a characteristic p ring which has a strong test ideal $T = (c_1, \ldots, c_k)$. Then any ideal I such that*

$$I \supset (c_1, \ldots, c_{k-1})I : c_k,$$

is tightly closed. In particular, if (R, m) is local, any ideal I containing $m(c_1, \ldots, c_{k-1}) : c_k$ is tightly closed.

Proof: Let $u \in I^*$. Then we have $uT \subset IT$, hence there exist $a_i \in I$ such that

$$c_k u = c_1 a_1 + \ldots + c_k a_k,$$

hence

$$u - a_k \in (c_1, \ldots, c_{k-1})I : c_k \subset I \Rightarrow u \in I. \quad \square$$

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ABSTRACT

Tight Closure and Local Cohomology

by

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This thesis deals with two issues in tight closure theory: one is connected with the problem of whether tight closure commutes with localization, and the second with understanding and computing tight closure, especially in cubical cones and related rings.

It is shown that if R is a graded affine algebra over a characteristic p ring, and M a finitely generated graded module over R of Krull dimension one, then one can bound the exponent on the maximal ideal m needed to annihilate the local cohomology of Frobenius images of M by a fixed positive constant times q. One consequence of this result is the fact that if R is as above and I is a homogeneous ideal with $\dim(R/I) = 1$, then tight closure commutes with localization for I, at any multiplicative system W.

We study the problem of computing tight closure for an arbitrary m-primary ideal, if R is the completed cubical cone ring over a field of characteristic $p \equiv 2$

(mod 3). We find various conditions which imply that in rings of this form, the tight closure of any ideal is equal to the Frobenius closure. We also consider a more general class of rings in which most of the methods developed here apply: Gorenstein local rings with test ideal equal to the maximal ideal m, and in which tight closure equal Frobenis closure for parameter ideals.

We construct an example to show that coherence of R^∞ does not hold in these rings.

We introduce two new notions: a notion of independence related to tight closure, and special tight closure, and show that the study of these notions contributes to a better understanding of tight closure.