Classical Algebraic Geometry: a modern view.I

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Preface

The main purpose of the present treatise is to give an account of some of the topics in algebraic geometry, which, while having occupied the minds of many mathematicians in previous generations, have fallen out of fashion in modern times. Often in the history of mathematics, new ideas and techniques make the work of previous generations of researchers obsolete. This mainly refers to the foundations of the subject and the fundamental general theoretical facts used heavily in research. Even the greatest achievements of the past generations, which can be found, for example, in the work of F. Severi on algebraic cycles or in the work of O. Zariski in the theory of algebraic surfaces, have been greatly generalized and clarified so that they now remain only of historical interest. In contrast, the fact that a nonsingular cubic surface has 27 lines or a plane quartic has 28 bitangents cannot be improved and continues to fascinate modern geometers. One of the goals of this present work is then to save from oblivion the work of many mathematicians who discovered these classic tenets and many other beautiful results.

In writing this book the greatest challenge the author has faced was distilling the material down to what should be covered. The number of concrete facts, examples of special varieties, and beautiful geometric constructions that have accumulated during the classical period of the development of algebraic geometry is enormous, and what the reader is going to find in the book is only the tip of the iceberg; a work that is like a taste sampler of classical algebraic geometry. It avoids most of the material found in other modern books on subject, such as, for example, [10], where one can find many classical results on algebraic curves. Instead, it tries to assemble or, in other words, to create a compendium of material that either cannot be found, is too dispersed to be found easily, or is not treated adequately by contemporary research papers. On the other hand, while most of the material treated in the book exists in classical treatises in algebraic geometry, their somewhat archaic terminology, and what is by now completely forgotten background knowledge makes these books useful to but a handful of experts in classical literature. Lastly, one must admit that the author's personal taste also has much sway in the choice of material.

The reader should be warned that the book is by no means an introduction to algebraic geometry. Although some of the exposition can be followed with only a minimum background in algebraic geometry, for example, based on Shafarevich's book [708] it often relies on current cohomological techniques, such as those found in Hartshorne's book [379]. The idea was to reconstruct a result by using modern techniques but not necessarily its original proof. For one, the ingenious geometric constructions in those proofs were often beyond the author's abilities to follow them completely. Understandably, the price of this was often to replace a beautiful geometric argument with a dull cohomological one. For those looking for a less demanding sample of some of the topics covered in the book, the recent beautiful book [54] may be of great use.

No attempt has been made to give a complete bibliography. To give an idea of such an enormous task, one could mention that the report on the status of topics in algebraic geometry submitted to the National Research Council in Washington in 1928 [715] contains more than 500 items of bibliography by 130 different authors only on the subject of planar Cremona transformations (covered in one of the chapters of the present book.) Another example is the bibliography on cubic surfaces compiled by J. E. Hill [394] in 1896 which alone contains 205 titles. Meyer's article [515] cites around 130 papers published between 1896 and 1928. The title search in MathSciNet reveals more than 200 papers refereed since 1940, many of them published only in the past 20 years. How sad it is when one considers the impossibility of saving from oblivion so many names of researchers of the past who have contributed so much to our subject.

A word about exercises: some of them are easy and follow from the definitions, and some are hard; they are included to provide additional facts not covered in the main text. In this case, we sometimes indicate the sources for the statements and solutions.

I am very grateful to many people for their comments and corrections to many previous versions of the manuscript. I am especially thankful to Sergey Tikhomirov, whose help in the mathematical editing of the book was essential for getting rid of many mistakes in the previous versions. The author bears sole responsibility for all the errors still found in the book.

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We will take $\mathbb{C}$ as the ground field, although many constructions in this book works over an arbitrary algebraically closed field.

We will usually denote by E a vector space of dimension $n+1$ and denote by $|E|$ the projective space of lines in E. Its dual vector space will be denoted by $E^\vee$. In Grothendieck's notation, the projective space $\mathbb{P}(E)$ is equal to $|E^\vee|$.

A basis $(\xi_0, \dots, \xi_n)$ in E defines an isomorphism $E \cong \mathbb{C}^{n+1}$ and identifies $|E|$ with the projective space $\mathbb{P}^n := |\mathbb{C}^{n+1}|$. For any nonzero vector $v \in E$, we denote by $[v]$ the corresponding point in $|E|$. If $E = \mathbb{C}^{n+1}$ and $v = (a_0, \dots, a_n) \in \mathbb{C}^{n+1}$ we set $[v] = [a_0, \dots, a_n]$. We call $[a_0, \dots, a_n]$ the *projective coordinates* of a point $[a] \in \mathbb{P}^n$. Another common notation for the projective coordinates of $[a]$ is $(a_0 : a_1 : \dots : a_n)$, or simply $(a_0, \dots, a_n)$, if no confusion arises.

We denote by $\mathbf{S}(E)$ the *symmetric algebra* of E, the quotient of the tensor algebra $T(E) = \oplus_{d \geq 0} E^{\otimes d}$ by the two-sided ideal generated by tensors of the form $v \otimes w - w \otimes v$. The symmetric algebra is a graded commutative algebra

$$\mathbf{S}(E) = \bigoplus_{d=0}^{\infty} S^d(E),$$

where $S^d(E)$ is the image of $E^{\otimes d}$ in the quotient algebra. The vector space $S^d(E)$ is called the *dth symmetric power* of E. Its dimension is equal to $\binom{d+n}{d}$. The image of a tensor $v_1 \otimes \cdots \otimes v_d$ in $S^d(E)$ is denoted by $v_1 \cdots v_d$.

The projective space $|E|$ comes with the tautological invertible sheaf $\mathcal{O}_{|E|}(1)$ whose space of global sections is identified with $E^\vee$. Its d-th tensor power is denoted by $\mathcal{O}_{|E|}(d)$. Its space of global sections is identified with $S^d(E^\vee)$.

For any $f \in S^d(E^\vee)$, $d > 0$, we denote by $V(f)$ the corresponding effective divisor from $|\mathcal{O}_{|E|}(d)|$, considered as a closed subscheme of $|E|$, not necessarily reduced. We call $V(f)$ a *hypersurface* of degree d in $|E|$ defined by equation $f = 0$* A hypersurface of degree 1 is a *hyperplane*. By definition, $V(0) = |E|$ and $V(1) = \emptyset$. The projective space $|S^d(E^\vee)|$ can be viewed as the projective space of hypersurfaces in $|E|$. It is equal to the complete linear system $|\mathcal{O}_{|E|}(d)|$. Using isomorphism (e2 1.2), we may identify the projective space $|S^d(E)|$ of hypersurfaces of degree d in $|E^\vee|$ with the dual of the projective space $|S^d E^\vee|$. A hypersurface of degree d in $|E^\vee|$ is classically known as an *envelope* of *class* d.

The symmetric group $\mathfrak{S}_d$ has a natural linear representation in $E^{\otimes d}$ via permuting the factors. The symmetrization operator $\oplus_{\sigma\in\mathfrak{S}_d}\sigma$ is the projector operator onto the subspace of symmetric tensors

$$S_d(E) := (E^{\otimes d})^{\mathfrak{S}_d}$$

multiplied by $d!$. It factors through $S^d(E)$ and defines a natural isomorphism

$$S^d(E) \to S_d(E).$$

Replacing E by its dual space $E^\vee$, we obtain a natural isomorphism

$$\mathfrak{p}_d : S^d(E^\vee) \to S_d(E^\vee). \tag{1.1}$$ e1

Under the identification of $(E^\vee)^{\otimes d}$ with the space $(E^{\otimes d})^\vee$, we will be able to identify $S_d(E^\vee)$ with the space $\mathrm{Hom}(E^d, \mathbb{C})^{\mathfrak{S}_d}$ of symmetric d-multi-linear functions $E^d \to \mathbb{C}$. The isomorphism $\mathfrak{p}_d$ is classically known as the *total polarization map*.

Next, we use that the quotient map $E^{\otimes d} \to S^d(E)$ is the universal symmetric d-multi-linear map; that is, any symmetric linear map $E^{\otimes d} \to F$ with values in some vector space F factors through a linear map $S^d(E) \to F$. If $F = \mathbb{C}$, this gives a natural isomorphism

$$(E^{\otimes d})^\vee = S_d(E^\vee) \to S^d(E)^\vee.$$

Composing it with $\mathfrak{p}_d$, we get a natural isomorphism

$$S^d(E^\vee) \to S^d(E)^\vee. \tag{1.2}$$ e2

It can be viewed as a perfect bilinear pairing, the *polarity pairing*

$$\langle,\rangle : S^d(E^\vee) \otimes S^d(E) \to \mathbb{C}. \tag{1.3}$$ eeq2

*This notation should not be confused with the notation of the closed subset in Zariski topology defined by the ideal (f). It is equal to $V(f)_{\mathrm{red}}$.

This pairing extends the natural pairing between E and $E^\vee$ to the symmetric powers. Explicitly,

$$\langle l_1 \cdots l_d, w_1 \cdots w_d \rangle = \sum_{\sigma \in \mathfrak{S}_d} l_{\sigma^{-1}(1)}(w_1) \cdots l_{\sigma^{-1}(d)}(w_d).$$

One can extend the total polarization isomorphism to a *partial polarization map*

$$\langle , \rangle : S^d(E^\vee) \otimes S^k(E) \to S^{d-k}(E^\vee), \quad k \le d, \tag{1.4}$$

eq2

$$\langle l_1 \cdots l_d, w_1 \cdots w_k \rangle = \sum_{1 \le i_1 \le \ldots \le i_k \le n} \langle l_{i_1} \cdots l_{i_k}, w_1 \cdots w_k \rangle \prod_{j \ne i_1, \ldots, i_k} l_j.$$

In coordinates, if we choose a basis $(\xi_0, \ldots, \xi_n)$ in E and its dual basis $t_0, \ldots, t_n$ in $E^\vee$, then we can identify $\mathbf{S}(E^\vee)$ with the polynomial algebra $\mathbb{C}[t_0, \ldots, t_n]$ and $S^d(E^\vee)$ with the space $\mathbb{C}[t_0, \ldots, t_n]_d$ of homogeneous polynomials of degree d. Similarly, we identify $S^d(E)$ with $\mathbb{C}[\xi_0, \ldots, \xi_n]_d$. The polarization isomorphism extends by linearity of the pairing on monomials

$$\langle t_0^{i_0} \cdots t_n^{i_n}, \xi_0^{j_0} \cdots \xi_n^{j_n} \rangle = \begin{cases} i_0! \cdots i_n! & \text{if } (i_0, \ldots, i_n) = (j_0, \ldots, j_n), \\ 0 & \text{otherwise.} \end{cases}$$

One can give an explicit formula for pairing (1.4) in terms of differential operators. Since $\langle t_i, \xi_j \rangle = \delta_{ij}$, it is convenient to view a basis vector ξ_j as the partial derivation operator $\partial_j = \frac{\partial}{\partial t_j}$.The pairing (1.4) becomes

$$\langle \psi(\xi_0, \ldots, \xi_n), f(t_0, \ldots, t_n) \rangle = D_\psi(f),$$

where

$$D_\psi = \psi(\partial_0, \ldots, \partial_n).$$

For any monomial $\boldsymbol{\partial}^{\mathbf{i}} = \partial_0^{i_0} \cdots \partial_n^{i_n}$ and any monomial $\mathbf{t}^{\mathbf{j}} = t_0^{j_0} \cdots t_n^{j_n}$, we have

$$\boldsymbol{\partial}^{\mathbf{i}}(\mathbf{t}^{\mathbf{j}}) = \begin{cases} \frac{\mathbf{j}!}{(\mathbf{j}-\mathbf{i})!} \mathbf{t}^{\mathbf{j}-\mathbf{i}} & \text{if } \mathbf{j} - \mathbf{i} \ge 0, \\ 0 & \text{otherwise.} \end{cases} Then, \tag{1.5}$$

eq4

Here and later, we use the vector notation:

$$\mathbf{i}! = i_0! \cdots i_n!, \quad \binom{k}{\mathbf{i}} = \frac{k!}{\mathbf{i}!}, \quad |\mathbf{i}| = i_0 + \cdots + i_n.$$

The total polarization $\tilde{f}$ of a polynomial f is given explicitly by the following formula:

$$\tilde{f}(v_1, \ldots, v_d) = D_{v_1 \cdots v_d}(f) = (D_{v_1} \circ \ldots \circ D_{v_d})(f).$$

Taking $v_1 = \ldots = v_d = v$, we get

$$\tilde{f}(v, \ldots, v) = d!f(v) = D_{v^d}(f) = \sum_{|\mathbf{i}|=d} \binom{d}{\mathbf{i}} \mathbf{a}^{\mathbf{i}} \boldsymbol{\partial}^{\mathbf{i}} f. \tag{1.6}$$ exx4

symbolic1 *Remark* 1.1.1. The polarization isomorphism was known in the classical literature as the *symbolic method*. Suppose $f = l^d$ is a d-th power of a linear form. Then, $D_v(f) = dl(v)l^{d-1}$ and

$$D_{v_1} \circ \ldots \circ D_{v_k}(f) = d(d-1)\cdots(d-k+1)l(v_1)\cdots l(v_k)l^{d-k}.$$

In classical notation, a linear form $\sum a_i x_i$ on $\mathbb{C}^{n+1}$ is denoted by a_x and the dot-product of two vectors a, b is denoted by (ab). Symbolically, one denotes any homogeneous form by a_x^d and the right-hand side of the previous formula reads as $d(d-1)\cdots(d-k+1)(ab)^k a_x^{d-k}$.

Let us take $E = S^m(U^\vee)$ for some vector space U and consider the linear space $S^d(S^m(U^\vee)^\vee)$. Using the polarization isomorphism, we can identify $S^m(U^\vee)^\vee$ with $S^m(U)$. Let $(\xi_0, \ldots, \xi_r)$ be a basis in U and $(t_0, \ldots, t_{r+1})$ be the dual basis in $U^\vee$. Then, we can take for a basis of $S^m(U)$ the monomials $\boldsymbol{\xi}^{\mathbf{j}}$. The dual basis in $S^m(U^\vee)$ is formed by the monomials $\frac{1}{\mathbf{i}!}\mathbf{x}^{\mathbf{i}}$. Thus, for any $f \in S^m(U^\vee)$, we can write

$$m!f = \sum_{|\mathbf{i}|=m} \binom{m}{\mathbf{i}} a_{\mathbf{i}} \mathbf{x}^{\mathbf{i}}. \tag{1.7}$$ symbol2

In symbolic form, $m!f = (a_x)^m$. Consider the matrix

$$\Xi = \begin{pmatrix} \xi_0^{(1)} & \ldots & \xi_0^{(d)} \\ \vdots & \vdots & \vdots \\ \xi_r^{(1)} & \ldots & \xi_r^{(d)} \end{pmatrix},$$

where $(\xi_0^{(k)}, \ldots, \xi_r^{(k)})$ is a copy of a basis in U. Then, the space $S^d(S^m(U))$ is equal to the subspace of the polynomial algebra $\mathbb{C}[(\xi_j^{(i)})]$ in $d(r+1)$ variables $\xi_j^{(i)}$ of polynomials which are homogeneous of degree m in each column of the matrix and symmetric with respect to permutations of the columns. Let $J \subset \{1, \ldots, d\}$ with $\#J = r+1$ and (J) be the corresponding maximal minor of the matrix Ξ. Assume $r+1$ divides dm. Consider a product of $k = \frac{dm}{r+1}$ such minors in which each column participates exactly m times. Then, a sum of such products which is invariant with respect to permutations of columns represents an element from $S^d(S^m(U))$ which has an additional property that it is invariant with respect to the group $\mathrm{SL}(U) \cong \mathrm{SL}(r+1, \mathbb{C})$ We can interpret elements of $S^d(S^m(U^\vee)^\vee)$ as polynomials in coefficients of $a_{\mathbf{i}}$ of a homogeneous form of degree d in $r+1$ variables written in the form (1.7). We write symbolically an invariant in the form $(J_1)\cdots(J_k)$ meaning that it is obtained as the sum of such products with some coefficients. If the number d is small, we can use letters, say $a, b, c, \ldots$, instead of numbers $1, \ldots, d$. For example, $(12)^2(13)^2(23)^2 = (ab)^2(bc)^2(ac)^2$ represents an element in $S^3(S^4(\mathbb{C}^2))$.

In a similar way, one considers the matrix

$$\begin{pmatrix} \xi_0^{(1)} & \ldots & \xi_0^{(d)} & t_0^{(1)} & \ldots & t_0^{(s)} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \xi_r^{(1)} & \ldots & \xi_r^{(d)} & t_r^{(1)} & \ldots & t_r^{(s)} \end{pmatrix}.$$

The product of k maximal minors such that each of the first d columns occurs exactly k times and each of the last s columns occurs exactly p times represents a *covariant* of degree p and order k. For example, $(ab)^2 a_x b_x$ represents the *Hessian determinant*

$$\mathrm{He}(f) = \det\begin{pmatrix} \frac{\partial^2 f}{\partial x_1^2} & \frac{\partial^2 f}{\partial x_1 \partial x_2} \\ \frac{\partial^2 f}{\partial x_2 \partial x_1} & \frac{\partial^2 f}{\partial x_2^2} \end{pmatrix}$$

of a ternary cubic form f.

The natural isomorphism

$$(E^\vee)^{\otimes d} \cong H^0(|E|^d, \mathcal{O}_{|E|}(1)^{\boxtimes d}),\ S_d(E^\vee) \cong H^0(|E|^d, \mathcal{O}_{|E|}(1)^{\boxtimes d})^{\mathfrak{S}_d}$$

allows one to give the following geometric interpretation of the polarization isomorphism. Consider the diagonal embedding $\delta_d : |E| \hookrightarrow |E|^d$. Then, the total polarization map is the inverse of the isomorphism

$$\delta_d^* : H^0(|E|^d, \mathcal{O}_{|E|}(1)^{\boxtimes d})^{\mathfrak{S}_d} \to H^0(|E|, \mathcal{O}_{|E|}(d)).$$

We view $a_0\partial_0 + \cdots + a_n\partial_n \neq 0$ as a point $a \in |E|$ with projective coordinates $[a_0, \ldots, a_n]$.

Definition 1.1.2. *Let $X = V(f)$ be a hypersurface of degree d in $|E|$ and $x = [v]$ be a point in $|E|$. The hypersurface*

$$P_{a^k}(X) := V(D_{v^k}(f))$$

of degree $d-k$ is called the k-th polar hypersurface *of the point a with respect to the hypersurface $V(f)$ (or of the hypersurface with respect to the point).*

exaa *Example* 1.1.3. Let $d = 2$, i.e.,

$$f = \sum_{i=0}^{n} \alpha_{ii} t_i^2 + 2 \sum_{0 \le i < j \le n} \alpha_{ij} t_i t_j$$

is a quadratic form on $\mathbb{C}^{n+1}$. For any $x = [a_0, \dots, a_n] \in \mathbb{P}^n$, $P_x(V(f)) = V(g)$, where

$$g = \sum_{i=0}^{n} a_i \frac{\partial f}{\partial t_i} = 2 \sum_{0 \le i \le j \le n} a_i \alpha_{ij} t_j, \quad \alpha_{ji} = \alpha_{ij}.$$

The linear map $v \mapsto D_v(f)$ is a map from $\mathbb{C}^{n+1}$ to $(\mathbb{C}^{n+1})^\vee$ which can be identified with the *polar bilinear form* associated to f with matrix $2(\alpha_{ij})$.

Let us give another definition of the polar hypersurfaces $P_{x^k}(X)$. Choose two different points $a = [a_0, \dots, a_n]$ and $b = [b_0, \dots, b_n]$ in $\mathbb{P}^n$ and consider the line $\ell = \overline{ab}$ spanned by the two points as the image of the map

$$\varphi : \mathbb{P}^1 \to \mathbb{P}^n, \quad [u_0, u_1] \mapsto u_0 a + u_1 b := [a_0 u_0 + b_0 u_1, \dots, a_n u_0 + b_n u_1]$$

(a parametric equation of ℓ). The intersection $\ell \cap X$ is isomorphic to the positive divisor on $\mathbb{P}^1$ defined by the degree d homogeneous form

$$\varphi^*(f) = f(u_0 a + u_1 b) = f(a_0 u_0 + b_0 u_1, \dots, a_n u_0 + b_n u_1).$$

Using the Taylor formula at $(0, 0)$, we can write

$$\varphi^*(f) = \sum_{k+m=d} \frac{1}{k!m!} u_0^k u_1^m A_{km}(a, b), \tag{1.8}$$ eq1.6

where

$$A_{km}(a, b) = \frac{\partial^d \varphi^*(f)}{\partial u_0^k \partial u_1^m}(0, 0).$$

Using the Chain Rule, we get

$$A_{km}(a, b) = \sum_{|\mathbf{i}|=k, |\mathbf{j}|=m} \binom{k}{\mathbf{i}} \binom{m}{\mathbf{j}} \mathbf{a}^{\mathbf{i}} \mathbf{b}^{\mathbf{j}} \boldsymbol{\partial}^{\mathbf{i}+\mathbf{j}} f = D_{a^k b^m}(f). \tag{1.9}$$ 1.7

Observe the symmetry

$$A_{km}(a, b) = A_{mk}(b, a). \tag{1.10}$$ sym

Fixing a and letting b vary in $\mathbb{P}^n$, we obtain a hypersurface $V(A_{km}(a, x))$ of degree $d - k$ which is the k-th polar hypersurface of $X = V(f)$ with respect to the point a. When we fix b and vary a in $\mathbb{P}^n$, we obtain the m-th polar hypersurface $V(A_{km}(x, b))$ of X with respect to the point b.

Note that

$$D_{a^k b^m}(f) = D_{a^k}(D_{b^m}(f)) = D_{b^m}(a) = D_{b^m}(D_{a^k}(f)) = D_{a^k}(f)(b). \tag{1.11}$$ sympol

This gives the symmetry property of polars:

$$b \in P_{a^k}(X) \Leftrightarrow a \in P_{b^{d-k}}(X). \tag{1.12}$$ symm

Since we are in characteristic 0, if $m \le d$, $D_{a^m}(f)$ cannot be zero for all a. To see this, we use the *Euler formula*:

$$df = \sum_{i=0}^{n} t_i \frac{\partial f}{\partial t_i}.$$

Applying this formula to the partial derivation, we obtain

$$d(d-1)\cdots(d-k+1)f = \sum_{|\mathbf{i}|=k} \binom{k}{\mathbf{i}} \mathbf{t}^{\mathbf{i}} \boldsymbol{\partial}^{\mathbf{i}} f \tag{1.13}$$ euler

(also called the Euler formula). It follows from this formula that, for all $k \le d$,

$$a \in P_{a^k}(X) \Leftrightarrow a \in X. \tag{1.14}$$ 1.11

ex:1.1.4 *Example* 1.1.4. Let M_d be the vector space of complex square matrices of size d with coordinates t_{ij}. We view the determinant function $\det : M_d \to \mathbb{C}$ as an element of $S^d(M_d^\vee)$, i.e.,a polynomial of degree d in the variables t_{ij}. Let $C_{ij} = \frac{\partial \det}{\partial t_{ij}}$. We have $C_{ij}(A) = \frac{\partial \det}{\partial t_{ij}}(A)$. For any point $A = (a_{ij})$ in M_d the value of C_{ij} at A is equal to the ij-th cofactor of A. Applying (1.6) [exx4], for any $B = (b_{ij}) \in M_d$, we obtain

$$\begin{aligned} D_{A^{d-1}B}(\det) &= D_A^{d-1}(D_B(\det)) \\ &= D_A^{d-1}(\sum b_{ij}C_{ij}) = (d-1)! \sum b_{ij}C_{ij}(A). \end{aligned} \tag{1.15}$$

Thus, $D_A^{d-1}(\det)$ is a linear function $\sum t_{ij}C_{ij}(A)$ on M_d. The linear map

$$S^{d-1}(M_d) \to M_d^\vee, \quad A \mapsto \frac{1}{(d-1)!} D_A^{d-1}(\det),$$

can be identified with the function $A \mapsto \mathrm{adj}(A)$, where $\mathrm{adj}(A)$ is the cofactor matrix (classically called the *adjugate matrix* of A, but not the adjoint matrix, as it is often called in modern textbooks).

1.1.2 First polars

SS:1.1.2

Let us consider some special cases. Let $X = V(f)$ be a hypersurface of degree d. Obviously, any 0-th polar of X is equal to X and, by (1.12) [symm], the d-th polar $P_{a^d}(X)$ is empty if $a \notin X$, and equals $\mathbb{P}^n$ if $a \in X$. Now, take $k = 1, d-1$. Using (1.6) [exx4], we obtain

$$D_a(f) = \sum_{i=0}^{n} a_i \frac{\partial f}{\partial t_i},$$

$$\frac{1}{(d-1)!} D_{a^{d-1}}(f) = \sum_{i=0}^{n} \frac{\partial f}{\partial t_i}(a) t_i.$$

Together with (1.12), this implies the following:

Theorem 1.1.5. *For any smooth point $x \in X$, we have*

$$P_{x^{d-1}}(X) = \mathbb{T}_x(X).$$

If x is a singular point of X, $P_{x^{d-1}}(X) = \mathbb{P}^n$. Moreover, for any $a \in \mathbb{P}^n$,

$$X \cap P_a(X) = \{x \in X : a \in \mathbb{T}_x(X)\}.$$

Here and later on, we denote by $\mathbb{T}_x(X)$ the *embedded tangent space* of a projective subvariety $X \subset \mathbb{P}^n$ at its point x. It is a linear subspace of $\mathbb{P}^n$ equal to the projective closure of the affine Zariski tangent space $T_x(X)$ of X at x (see [375], p. 181).

In classical terminology, the intersection $X \cap P_a(X)$ is called the *apparent boundary* of X from the point a. If one projects X to $\mathbb{P}^{n-1}$ from the point a, Then, the apparent boundary is the ramification divisor of the projection map.

The following picture illustrates the polar line of a conic.

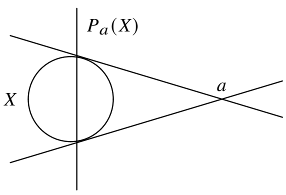


Figure 1.1 Polar line of a conic

The set of first polars $P_a(X)$ defines a linear system contained in the complete linear system $\left|\mathcal{O}_{\mathbb{P}^n}(d-1)\right|$. The dimension of this linear system $\le n$.

Proposition 1.1.6. *The dimension of the linear system of first polars $\le r$ if and only if, after a linear change of variables, the polynomial f becomes a polynomial in $r+1$ variables.*

Proof Let $X = V(f)$. It is obvious that the dimension of the linear system of first polars $\le r$ if and only if the linear map $E \to S^{d-1}(E^\vee), v \mapsto D_v(f)$ has kernel of dimension $\ge n-r$. Choosing an appropriate basis, we may assume that the kernel is generated by vectors $(1,0,\ldots,0)$, *etc*. Now, it is obvious that f does not depend on the variables $t_0,\ldots,t_{n-r-1}$. □

It follows from Theorem 1.1.5 that the first polar $P_a(X)$ of a point a with respect to a hypersurface X passes through all singular points of X. One can say more.

P1.1.3 **Proposition 1.1.7.** *Let a be a singular point of X of multiplicity m. For each $r \le \deg X - m$, $P_{a^r}(X)$ has a singular point at a of multiplicity m and the tangent cone of $P_{a^r}(X)$ at a coincides with the tangent cone $\mathrm{TC}_a(X)$ of X at a. For any point $b \neq a$, the r-th polar $P_{b^r}(X)$ has multiplicity $\ge m - r$ at a and its tangent cone at a is equal to the r-th polar of $\mathrm{TC}_a(X)$ with respect to b.*

Proof Let us prove the first assertion. Without loss of generality, we may assume that $a = [1, 0, \dots, 0]$. Then, $X = V(f)$, where

$$f = t_0^{d-m} f_m(t_1, \dots, t_n) + t_0^{d-m-1} f_{m+1}(t_1, \dots, t_n) + \cdots + f_d(t_1, \dots, t_n). \quad (1.16)$$ newref

The equation $f_m(t_1, \dots, t_n) = 0$ defines the *tangent cone* of X at b. The equation of $P_{a^r}(X)$ is

$$\frac{\partial^r f}{\partial t_0^r} = r! \sum_{i=0}^{d-m-r} \binom{d-m-i}{r} t_0^{d-m-r-i} f_{m+i}(t_1, \dots, t_n) = 0.$$

It is clear that $[1, 0, \dots, 0]$ is a singular point of $P_{a^r}(X)$ of multiplicity m with the tangent cone $V(f_m(t_1, \dots, t_n))$.

Now, we prove the second assertion. Without loss of generality, we may assume that $a = [1, 0, \dots, 0]$ and $b = [0, 1, 0, \dots, 0]$. Then, the equation of $P_{b^r}(X)$ is

$$\frac{\partial^r f}{\partial t_1^r} = t_0^{d-m} \frac{\partial^r f_m}{\partial t_1^r} + \cdots + \frac{\partial^r f_d}{\partial t_1^r} = 0.$$

The point a is a singular point of multiplicity $\ge m - r$. The tangent cone of $P_{b^r}(X)$ at the point a is equal to $V(\frac{\partial^r f_m}{\partial t_1^r})$ and this coincides with the r-th polar of $\mathrm{TC}_a(X) = V(f_m)$ with respect to b. □

We leave it to the reader to see what happens if $r > d - m$.

Keeping the notation from the previous proposition, consider a line ℓ through the point a such that it intersects X at some point $x \neq a$ with multiplicity greater than one. The closure $\mathrm{EC}_a(X)$ of the union of such lines is called the *enveloping cone* of X at the point a. If X is not a cone with vertex at a, the branch divisor of the projection $p : X \setminus \{a\} \to \mathbb{P}^{n-1}$ from a is equal to the projection of the enveloping cone. Let us find the equation of the enveloping cone.

As above, we assume that $a = [1, 0, \dots, 0]$. Let H be the hyperplane $t_0 = 0$. Write ℓ in a parametric form $ua + vx$ for some $x \in H$. Plugging in the equation newref (1.16), we get

$$P(t) = t^{d-m} f_m(x_1, \dots, x_n) + t^{d-m-1} f_{m+1}(x_1, \dots, x_m) + \cdots + f_d(x_1, \dots, x_n) = 0,$$

where $t = u/v$.

We assume that $X \neq \mathrm{TC}_a(X)$, i.e., X is not a cone with vertex at a (otherwise, by definition, $\mathrm{EC}_a(X) = \mathrm{TC}_a(X)$). The image of the tangent cone under the projection $p : X \setminus \{a\} \to H \cong \mathbb{P}^{n-1}$ is a proper closed subset of H. If $f_m(x_1, \ldots, x_n) \neq 0$, then a multiple root of $P(t)$ defines a line in the enveloping cone. Let $\mathsf{D}_k(A_0, \ldots, A_k)$ be the discriminant of a general polynomial $P = A_0T^k + \cdots + A_k$ of degree k. Recall that

$$A_0\mathsf{D}_k(A_0, \ldots, A_k) = (-1)^{k(k-1)/2}\mathrm{Res}(P, P')(A_0, \ldots, A_k),$$

where $\mathrm{Res}(P, P')$ is the resultant of P and its derivative P'. It follows from the known determinant expression of the resultant that

$$\mathsf{D}_k(0, A_1, \ldots, A_k) = (-1)^{\frac{k^2-k+2}{2}} A_0^2 \mathsf{D}_{k-1}(A_1, \ldots, A_k).$$

The equation $P(t) = 0$ has a multiple zero with $t \neq 0$ if and only if

$$\mathsf{D}_{d-m}(f_m(x), \ldots, f_d(x)) = 0.$$

So, we see that

$$\mathrm{EC}_a(X) \subset V(\mathsf{D}_{d-m}(f_m(x), \ldots, f_d(x))), \qquad (1.17)$$
$$\mathrm{EC}_a(X) \cap \mathrm{TC}_a(X) \subset V(\mathsf{D}_{d-m-1}(f_{m+1}(x), \ldots, f_d(x))).$$

It follows from the computation of $\frac{\partial^r f}{\partial t_0^r}$ in the proof of the previous proposition that the hypersurface $V(\mathsf{D}_{d-m}(f_m(x), \ldots, f_d(x)))$ is equal to the projection of $P_a(X) \cap X$ to H.

Suppose $V(\mathsf{D}_{d-m-1}(f_{m+1}(x), \ldots, f_d(x)))$ and $\mathrm{TC}_a(X)$ do not share an irreducible component. Then

$$V(\mathsf{D}_{d-m}(f_m(x), \ldots, f_d(x))) \setminus \mathrm{TC}_a(X) \cap V(\mathsf{D}_{d-m}(f_m(x), \ldots, f_d(x)))$$

$$= V(\mathsf{D}_{d-m}(f_m(x), \ldots, f_d(x))) \setminus V(\mathsf{D}_{d-m-1}(f_{m+1}(x), \ldots, f_d(x))) \subset \mathrm{EC}_a(X),$$

gives the opposite inclusion of (1.17), and we get

$$\mathrm{EC}_a(X) = V(\mathsf{D}_{d-m}(f_m(x), \ldots, f_d(x))). \qquad (1.18)$$ eqec

Note that the discriminant $\mathsf{D}_{d-m}(A_0, \ldots, A_k)$ is an invariant of the group $\mathrm{SL}(2)$ in its natural representation on degree k binary forms. Taking the diagonal subtorus, we immediately infer that any monomial $A_0^{i_0} \cdots A_k^{i_k}$ entering in the discriminant polynomial satisfies

$$k\sum_{s=0}^{k} i_s = 2\sum_{s=0}^{k} s i_s.$$

It is also known that the discriminant is a homogeneous polynomial of degree $2k-2$. Thus, we get

$$k(k-1) = \sum_{s=0}^{k} s i_s.$$

In our case, $k = d - m$ and we obtain that

$$\deg V(\mathsf{D}_{d-m}(f_m(x), \ldots, f_d(x))) = \sum_{s=0}^{d-m} (m+s) i_s$$

$$= m(2d-2m-2) + (d-m)(d-m-1) = (d+m)(d-m-1).$$

This is the expected degree of the enveloping cone.

Example 1.1.8. Assume $m = d - 2$, then

$$\mathsf{D}_2(f_{d-2}(x), f_{d-1}(x), f_d(x)) = f_{d-1}(x)^2 - 4 f_{d-2}(x) f_d(x),$$
$$\mathsf{D}_2(0, f_{d-1}(x), f_d(x)) = f_{d-2}(x) = 0.$$

Suppose $f_{d-2}(x)$ and f_{d-1} are coprime. Then, our assumption is satisfied, and we obtain

$$\mathrm{EC}_a(X) = V(f_{d-1}(x)^2 - 4 f_{d-2}(x) f_d(x)).$$

Observe that the hypersurfaces $V(f_{d-2}(x))$ and $V(f_d(x))$ are everywhere tangent to the enveloping cone. In particular, the quadric tangent cone $\mathrm{TC}_a(X)$ is everywhere tangent to the enveloping cone along the intersection of $V(f_{d-2}(x))$ with $V(f_{d-1}(x))$.

For any nonsingular quadric Q, the map $x \mapsto P_x(Q)$ defines a projective isomorphism from the projective space to the dual projective space. This is a special case of a correlation.

According to the classical terminology, a projective automorphism of $\mathbb{P}^n$ is called a *collineation*. An isomorphism from $|E|$ to its dual space $\mathbb{P}(E)$ is called a *correlation*. A correlation $\mathfrak{c} : |E| \to \mathbb{P}(E)$ is given by an invertible linear map $\phi : E \to E^\vee$ defined uniquely up to proportionality. A correlation transforms points in $|E|$ to hyperplanes in $|E|$. A point $x \in |E|$ is called *conjugate* to a point $y \in |E|$ with respect to the correlation $\mathfrak{c}$ if $y \in \mathfrak{c}(x)$. The transpose of the inverse map ${}^t\phi^{-1} : E^\vee \to E$ transforms hyperplanes in $|E|$ to points in $|E|$. It can be considered as a correlation between the dual spaces $\mathbb{P}(E)$ and $|E|$. It is denoted by $\mathfrak{c}^\vee$ and is called the *dual correlation*. It is clear that $(\mathfrak{c}^\vee)^\vee = \mathfrak{c}$. If H is a hyperplane in $|E|$ and x is a point in H, then point $y \in |E|$ conjugate to x under $\mathfrak{c}$ belongs to any hyperplane H' in $|E|$ conjugate to H under $\mathfrak{c}^\vee$.

A correlation can be considered as a line in $(E \otimes E)^\vee$ spanned by a nondegenerate bilinear form, or, in other words, as a nonsingular correspondence of type $(1,1)$ in $|E| \times |E|$. The dual correlation is the image of the divisor under the switch of the factors. A pair $(x, y) \in |E| \times |E|$ of conjugate points is just a point on this divisor.

We can define the *composition of correlations* $\mathfrak{c}' \circ \mathfrak{c}^\vee$. Collineations and correlations form a group $\Sigma\mathrm{PGL}(E)$ isomorphic to the group of outer automorphisms of $\mathrm{PGL}(E)$. The subgroup of collineations is of index 2.

A correlation $\mathfrak{c}$ of order 2 in the group $\Sigma\mathrm{PGL}(E)$ is called a *polarity*. Its linear representative ϕ, satisfies ${}^t\phi = \lambda\phi$ for some nonzero scalar λ. After transposing, we obtain $\lambda = \pm 1$. The case $\lambda = 1$ corresponds to the (quadric) polarity with respect to a nonsingular quadric in $|E|$ which we discussed in this section. The case $\lambda = -1$ corresponds to a *null-system* (or *null polarity*) which we will discuss in Sections 2.1 and 10.2. In terms of bilinear forms, a correlation is a quadric polarity (resp. null polarity) if it can be represented by a symmetric (skew-symmetric) bilinear form.

T1.1.4 **Theorem 1.1.9.** *Any projective automorphism is equal to the product of two quadric polarities.*

Proof Choose a basis in E to represent the automorphism by a Jordan matrix J. Let $J_k(\lambda)$ be its block of size k with λ at the diagonal. Let

$$B_k = \begin{pmatrix} 0 & 0 & \dots & 0 & 1 \\ 0 & 0 & \dots & 1 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 1 & \dots & 0 & 0 \\ 1 & 0 & \dots & 0 & 0 \end{pmatrix}.$$

Then,

$$C_k(\lambda) = B_k J_k(\lambda) = \begin{pmatrix} 0 & 0 & \dots & 0 & \lambda \\ 0 & 0 & \dots & \lambda & 1 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \lambda & \dots & 0 & 0 \\ \lambda & 1 & \dots & 0 & 0 \end{pmatrix}.$$

Observe that the matrices B_k^{-1} and $C_k(\lambda)$ are symmetric. Thus, each Jordan block of J can be written as the product of symmetric matrices. Hence, J is the product of two symmetric matrices. It follows from the definition of composition in the group $\Sigma\mathrm{PGL}(E)$ that the product of the matrices representing the bilinear forms associated to correlations coincides with the matrix representing the projective transformation equal to the composition of the correlations. □

1.1.3 Polar quadrics

SS:1.1.3

A $(d-2)$-polar of $X = V(f)$ is a quadric, called the *polar quadric* of X with respect to $a = [a_0, \ldots, a_n]$. It is defined by the quadratic form

$$q = D_{a^{d-2}}(f) = \sum_{|\mathbf{i}|=d-2} \binom{d-2}{\mathbf{i}} \mathbf{a}^{\mathbf{i}} \boldsymbol{\partial}^{\mathbf{i}} f.$$

Using (1.9), we obtain

$$q = \sum_{|\mathbf{i}|=2} \binom{2}{\mathbf{i}} \mathbf{t}^{\mathbf{i}} \boldsymbol{\partial}^{\mathbf{i}} f(a).$$

By (1.14), each $a \in X$ belongs to the polar quadric $P_{a^{d-2}}(X)$. Also, by Theorem 1.1.5,

$$\mathbb{T}_a(P_{a^{d-2}}(X)) = P_a(P_{a^{d-2}}(X)) = P_{a^{d-1}}(X) = \mathbb{T}_a(X). \tag{1.19}$$ tang2

This shows that the polar quadric is tangent to the hypersurface at the point a.

Consider the line $\ell = \overline{ab}$ through two points a, b. Let $\varphi : \mathbb{P}^1 \to \mathbb{P}^n$ be its parametric equation, i.e., a closed embedding with the image equal to ℓ. It follows from (1.8) and (1.9) that

$$i(X, \overline{ab})_a \geq s+1 \Longleftrightarrow b \in P_{a^{d-k}}(X), \quad k \leq s. \tag{1.20}$$ multt

For $s = 0$, the condition means that $a \in X$. For $s = 1$, by Theorem 1.1.5, this condition implies that b, and hence ℓ, belongs to the tangent plane $\mathbb{T}_a(X)$. For $s = 2$, this condition implies that $b \in P_{a^{d-2}}(X)$. Since ℓ is tangent to X at a, and $P_{a^{d-2}}(X)$ is tangent to X at a, this is equivalent to that ℓ belongs to $P_{a^{d-2}}(X)$.

It follows from (1.20) that a is a singular point of X of multiplicity $\geq s+1$ if and only if $P_{a^{d-k}}(X) = \mathbb{P}^n$ for $k \leq s$. In particular, the quadric polar $P_{a^{d-2}}(X) = \mathbb{P}^n$ if and only if a is a singular point of X of multiplicity ≥ 3.

Definition 1.1.10. *A line is called an* inflection tangent *to X at a point a if*

$$i(X, \ell)_a > 2.$$

flextangent **Proposition 1.1.11.** *Let ℓ be a line through a point a. Then, ℓ is an inflection tangent to X at a if and only if it is contained in the intersection of $\mathbb{T}_a(X)$ with the polar quadric $P_{a^{d-2}}(X)$.*

Note that the intersection of an irreducible quadric hypersurface $Q = V(q)$ with its tangent hyperplane H at a point $a \in Q$ is a cone in H over the quadric $\bar{Q}$ in the image $\bar{H}$ of H in $|E/[a]|$.

flexcone **Corollary 1.1.12.** *Assume $n \geq 3$. For any $a \in X$, there exists an inflection tangent line. The union of the inflection tangents containing the point a is the cone $\mathbb{T}_a(X) \cap P_{a^{d-2}}(X)$ in $\mathbb{T}_a(X)$.*

Example 1.1.13. Assume a is a singular point of X. By Theorem 1.1.5, this is equivalent to that $P_{a^{d-1}}(X) = \mathbb{P}^n$. By (1.19), the polar quadric Q is also singular at a and therefore it must be a cone over its image under the projection from a. The union of inflection tangents is equal to Q.

exa4 *Example* 1.1.14. Assume a is a nonsingular point of an irreducible surface X in $\mathbb{P}^3$. A tangent hyperplane $\mathbb{T}_a(X)$ cuts out in X a curve C with a singular point a. If a is an ordinary double point of C, there are two inflection tangents corresponding to the two branches of C at a. The polar quadric Q is nonsingular at a. The tangent cone of C at the point a is a cone over a quadric $\bar{Q}$ in $\mathbb{P}^1$. If $\bar{Q}$ consists of two points, there are two inflection tangents corresponding to the two branches of C at a. If $\bar{Q}$ consists of one point (corresponding to non-reduced hypersurface in $\mathbb{P}^1$), then we have one branch. The latter case happens only if Q is singular at some point $b \neq a$.

SS:1.1.4

1.1.4 The Hessian hypersurface

Let $Q(a)$ be the polar quadric of $X = V(f)$ with respect to some point $a \in \mathbb{P}^n$. The symmetric matrix defining the corresponding quadratic form is equal to the *Hessian matrix* of second partial derivatives of f

$$\mathrm{He}(f) = \Big(\frac{\partial^2 f}{\partial t_i \partial t_j}\Big)_{i,j=0,\ldots,n},$$

evaluated at the point a.The quadric $Q(a)$ is singular if and only if the determinant of the matrix is equal to zero (the locus of singular points is equal to the projectivization of the null-space of the matrix). The hypersurface

$$\mathrm{He}(X) = V(\det \mathrm{He}(f))$$

describes the set of points $a \in \mathbb{P}^n$ such that the polar quadric $P_{a^{d-2}}(X)$ is singular. It is called the *Hessian hypersurface* of X. Its degree is equal to $(d-2)(n+1)$ unless it coincides with $\mathbb{P}^n$.

mistake **Proposition 1.1.15.** *The following is equivalent:*

(i) $\mathrm{He}(X) = \mathbb{P}^n$;
(ii) there exists a nonzero polynomial $g(z_0, \ldots, z_n)$ such that

$$g(\partial_0 f, \ldots, \partial_n f) \equiv 0.$$

Proof This is a special case of a more general result about the *Jacobian determinant* (also known as the *functional determinant*) of $n+1$ polynomial functions $f_0, \ldots, f_n$ defined by

$$J(f_0, \ldots, f_n) = \det\big(\big(\frac{\partial f_i}{\partial t_j}\big)\big).$$

Suppose $J(f_0, \ldots, f_n) \equiv 0$. Then, the map $f : \mathbb{C}^{n+1} \to \mathbb{C}^{n+1}$ defined by the functions $f_0, \ldots, f_n$ is degenerate at each point (i.e., df_x is of rank $< n+1$ at each point x). Thus, the closure of the image is a proper closed subset of $\mathbb{C}^{n+1}$. Hence, there is an irreducible polynomial that vanishes identically on the image.

Conversely, assume that $g(f_0, \ldots, f_n) \equiv 0$ for some polynomial g which we may assume to be irreducible. Then,

$$\frac{\partial g}{\partial t_i} = \sum_{j=0}^{n} \frac{\partial g}{\partial z_j}(f_0, \ldots, f_n)\frac{\partial f_j}{\partial t_i} = 0, \; i = 0, \ldots, n.$$

Since g is irreducible, its set of zeros is nonsingular on a Zariski open set U. Thus, the vector

$$\big(\frac{\partial g}{\partial z_0}(f_0(x), \ldots, f_n(x)), \ldots, \frac{\partial g}{\partial z_n}(f_0(x), \ldots, f_n(x)\big)$$

is a nontrivial solution of the system of linear equations with matrix $(\frac{\partial f_i}{\partial t_j}(x))$, where $x \in U$. Therefore, the determinant of this matrix must be equal to zero. This implies that $J(f_0, \ldots, f_n) = 0$ on U, hence it is identically zero. □

noether *Remark* 1.1.16. It was claimed by O. Hesse that the vanishing of the Hessian implies that the partial derivatives are linearly dependent. Unfortunately, his attempted proof was wrong. The first counterexample was given by P. Gordan and M. Noether in Gordan [346]. Consider the polynomial

$$f = t_2t_0^2 + t_3t_1^2 + t_4t_0t_1.$$

Note that the partial derivatives

$$\frac{\partial f}{\partial t_2} = t_0^2, \quad \frac{\partial f}{\partial t_3} = t_1^2, \quad \frac{\partial f}{\partial t_4} = t_0t_1$$

are algebraically dependent. This implies that the Hessian is identically equal to zero. We have

$$\frac{\partial f}{\partial t_0} = 2t_0t_2 + t_4t_1, \quad \frac{\partial f}{\partial t_1} = 2t_1t_3 + t_4t_0.$$

Suppose that a linear combination of the partials is equal to zero. Then,

$$c_0t_0^2 + c_1t_1^2 + c_2t_0t_1 + c_3(2t_0t_2 + t_4t_1) + c_4(2t_1t_3 + t_4t_0) = 0.$$

Collecting the terms in which t_2, t_3, t_4 enter, we get

$$2c_3t_0 = 0, \quad 2c_4t_1 = 0, \quad c_3t_1 + c_4t_0 = 0.$$

This gives $c_3 = c_4 = 0$. Since the polynomials t_0^2, t_1^2, t_0t_1 are linearly independent, we also get $c_0 = c_1 = c_2 = 0$.

The known cases when the assertion of Hesse is true are $d = 2$ (any n) and $n \leq 3$ (any d) (see [346], [497], [138]).

Recall that the set of singular quadrics in $\mathbb{P}^n$ is the *discriminant hypersurface* $\mathsf{D}_2(n)$ in $\mathbb{P}^{n(n+3)/2}$ defined by the equation

$$\det\begin{pmatrix} t_{00} & t_{01} & \dots & t_{0n} \\ t_{01} & t_{11} & \dots & t_{1n} \\ \vdots & \vdots & \vdots & \vdots \\ t_{0n} & t_{1n} & \dots & t_{nn} \end{pmatrix} = 0.$$

By differentiating, we easily find that its singular points are defined by the determinants of $n \times n$ minors of the matrix. This shows that the singular locus of $\mathsf{D}_2(n)$ parameterizes quadrics defined by quadratic forms of rank $\leq n-1$ (or corank ≥ 2). We say that a quadric Q is of rank k if the corresponding quadratic form is of this rank. Note that

$$\dim \mathrm{Sing}(Q) = \mathrm{corank}\, Q - 1.$$

Assume that $\mathrm{He}(f) \neq 0$. Consider the rational map $p : |E| \to |S^2(E^\vee)|$ defined by $a \mapsto P_{a^{d-2}}(X)$. Note that $P_{a^{d-2}}(f) = 0$ implies $P_{a^{d-1}}(f) = 0$ and hence $\sum_{i=0}^n b_i \partial_i f(a) = 0$ for all b. This shows that a is a singular point of X. Thus, p is defined everywhere except maybe at singular points of X. So the map p is regular if X is nonsingular, and the pre-image of the discriminant hypersurface is equal to the Hessian of X. The pre-image of the singular locus $\mathrm{Sing}(\mathsf{D}_2(n))$ is the subset of points $a \in \mathrm{He}(f)$ such that $\mathrm{Sing}(P_{a^{d-2}}(X))$ is of positive dimension.

Here, is another description of the Hessian hypersurface.

steinerian **Proposition 1.1.17.** *The Hessian hypersurface* $\mathrm{He}(X)$ *is the locus of singular points of the first polars of* X.

Proof Let $a \in \mathrm{He}(X)$ and let $b \in \mathrm{Sing}(P_{a^{d-2}}(X))$. Then,

$$D_b(D_{a^{d-2}}(f)) = D_{a^{d-2}}(D_b(f)) = 0.$$

Since $D_b(f)$ is of degree $d-1$, this means that $\mathbb{T}_a(P_b(X)) = \mathbb{P}^n$, i.e., a is a singular point of $P_b(X)$.

Conversely, if $a \in \mathrm{Sing}(P_b(X))$, then $D_{a^{d-2}}(D_b(f)) = D_b(D_{a^{d-2}}(f)) = 0$. This means that b is a singular point of the polar quadric with respect to a. Hence, $a \in \mathrm{He}(X)$. □

Let us find the affine equation of the Hessian hypersurface. Applying the Euler formula (1.13), we can write

$$t_0 f_{0i} = (d-1)\partial_i f - t_1 f_{1i} - \cdots - t_n f_{ni},$$

$$t_0 \partial_0 f = df - t_1 \partial_1 f - \cdots - t_n \partial_n f,$$

where f_{ij} denote the second partial derivative. Multiplying the first row of the Hessian determinant by t_0 and adding to it the linear combination of the remaining rows taken with the coefficients t_i, we get the following equality:

$$\det(\mathrm{He}(f)) = \frac{d-1}{t_0} \det \begin{pmatrix} \partial_0 f & \partial_1 f & \dots & \partial_n f \\ f_{10} & f_{11} & \dots & f_{1n} \\ \vdots & \vdots & \vdots & \vdots \\ f_{n0} & f_{n1} & \dots & f_{nn} \end{pmatrix}.$$

Repeating the same procedure,but, this time, with the columns, we finally get

$$\det(\mathrm{He}(f)) = \frac{(d-1)^2}{t_0^2} \det \begin{pmatrix} \frac{d}{d-1} f & \partial_1 f & \dots & \partial_n f \\ \partial_1 f & f_{11} & \dots & f_{1n} \\ \vdots & \vdots & \vdots & \vdots \\ \partial_n f & f_{n1} & \dots & f_{nn} \end{pmatrix}. \tag{1.21}$$

affine

Let $\phi(z_1, \dots, z_n)$ be the dehomogenization of f with respect to t_0, i.e.,

$$f(t_0, \dots, t_d) = t_0^d \phi(\frac{t_1}{t_0}, \dots, \frac{t_n}{t_0}).$$

We have

$$\frac{\partial f}{\partial t_i} = t_0^{d-1} \phi_i(z_1, \dots, z_n), \ \frac{\partial^2 f}{\partial t_i \partial t_j} = t_0^{d-2} \phi_{ij}(z_1, \dots, z_n), \quad i, j = 1, \dots, n,$$

where

$$\phi_i = \frac{\partial \phi}{\partial z_i}, \quad \phi_{ij} = \frac{\partial^2 \phi}{\partial z_i \partial z_j}.$$

Plugging these expressions in (1.21), we obtain that, up to a nonzero constant factor,

$$t_0^{-(n+1)(d-2)}\det(\mathrm{He}(\phi)) = \det\begin{pmatrix} \frac{d}{d-1}\phi(z) & \phi_1(z) & \dots & \phi_n(z) \\ \phi_1(z) & \phi_{11}(z) & \dots & \phi_{1n}(z) \\ \vdots & \vdots & \vdots & \vdots \\ \phi_n(z) & \phi_{n1}(z) & \dots & \phi_{nn}(z) \end{pmatrix}, \qquad (1.22)$$ affine2

where $z = (z_1, \dots, z_n), z_i = t_i/t_0,\ i = 1, \dots, n$.

Remark 1.1.18. If $f(x, y)$ is a real polynomial in three variables, the value of (1.22) at a point $v \in \mathbb{R}^n$ with $[v] \in V(f)$ multiplied by $\frac{-1}{f_1(a)^2+f_2(a)^2+f_3(a)^2}$ is equal to the *Gauss curvature* of $X(\mathbb{R})$ at the point a (see [304]).

SS:1.1.5

1.1.5 Parabolic points

Let us see where $\mathrm{He}(X)$ intersects X. We assume that $\mathrm{He}(X)$ is a hypersurface of degree $(n+1)(d-2) > 0$. A glance at the expression (1.22) reveals the following fact.

obv **Proposition 1.1.19.** *Each singular point of X belongs to* $\mathrm{He}(X)$.

Let us see now when a nonsingular point $a \in X$ lies in its Hessian hypersurface $\mathrm{He}(X)$.

By Corollary 1.1.12, the inflection tangents in $\mathbb{T}_a(X)$ sweep the intersection of $\mathbb{T}_a(X)$ with the polar quadric $P_{a^{d-2}}(X)$. If $a \in \mathrm{He}(X)$, then the polar quadric is singular at some point b.

If $n = 2$, a singular quadric is the union of two lines, so this means that one of the lines is an inflection tangent. A point a of a plane curve X such that there exists an inflection tangent at a is called an *inflection point* of X.

If $n > 2$, the inflection tangent lines at a point $a \in X \cap \mathrm{He}(X)$ sweep a cone over a singular quadric in $\mathbb{P}^{n-2}$ (or the whole $\mathbb{P}^{n-2}$ if the point is singular). Such a point is called a *parabolic point* of X. The closure of the set of parabolic points is the *parabolic hypersurface* in X (it could be the whole X).

flex **Theorem 1.1.20.** *Let X be a hypersurface of degree $d > 2$ in $\mathbb{P}^n$. If $n = 2$, then* $\mathrm{He}(X) \cap X$ *consists of inflection points of X. In particular, each nonsingular curve of degree ≥ 3 has an inflection point, and the number of inflection points is either infinite or less than or equal to $3d(d-2)$. If $n > 2$, then the set* $X \cap \mathrm{He}(X)$ *consists of parabolic points. The parabolic hypersurface in X is either the whole X or a subvariety of degree $(n+1)d(d-2)$ in $\mathbb{P}^n$.*

Example 1.1.21. Let X be a surface of degree d in $\mathbb{P}^3$. If a is a parabolic point of X, then $\mathbb{T}_a(X) \cap X$ is a singular curve whose singularity at a is of multiplicity higher than 3 or it has only one branch. In fact, otherwise X has at least two distinct inflection tangent lines which cannot sweep a cone over a singular quadric in $\mathbb{P}^1$. The converse is also true. For example, a nonsingular quadric has no parabolic points, and all nonsingular points of a singular quadric are parabolic.

A generalization of a quadratic cone is a *developable surface*. It is a special kind of a *ruled surface* which is characterized by the condition that the tangent plane does not change along a ruling. We will discuss these surfaces later in Chapter 10. The Hessian surface of a developable surface contains this surface. The residual surface of degree $2d-8$ is called the *pro-Hessian surface*. An example of a developable surface is the quartic surface

$$(t_0t_3 - t_1t_2)^2 - 4(t_1^2 - t_0t_2)(t_2^2 - t_1t_3) = -6t_0t_1t_2t_3 + 4t_1^3t_3 + 4t_0t_2^3 + t_0^2t_3^2 - 3t_1^2t_2^2 = 0.$$

It is the surface swept out by the tangent lines of a rational normal curve of degree 3. It is also the *discriminant surface* of a binary cubic, i.e., the surface parameterizing binary cubics $a_0u^3 + 3a_1u^2v + 3a_2uv^2 + a_3v^3$ with a multiple root. The pro-Hessian of any quartic developable surface is the surface itself Cayley3 [114].

Assume now that X is a curve. Let us see when it has infinitely many inflection points. Certainly, this happens when X contains a line component; each of its points is an inflection point. It must be also an irreducible component of $\mathrm{He}(X)$. The set of inflection points is a closed subset of X. So, if X has infinitely many inflection points, it must have an irreducible component consisting of inflection points. Each such component is contained in $\mathrm{He}(X)$. Conversely, each common irreducible component of X and $\mathrm{He}(X)$ consists of inflection points.

We will prove the converse in a little more general form taking care of not necessarily reduced curves.

common **Proposition 1.1.22.** *A polynomial $f(t_0,t_1,t_2)$ divides its Hessian polynomial* $\mathrm{He}(f)$ *if and only if each of its multiple factors is a linear polynomial.*

Proof Since each point on a non-reduced component of $X_{\mathrm{red}} \subset V(f)$ is a singular point (i.e.,all the first partials vanish), and each point on a line component is an inflection point, we see that the condition is sufficient for $X \subset \mathrm{He}(f)$. Suppose this happens and let R be a reduced irreducible component of the curve X which is contained in the Hessian. Take a nonsingular point of R and consider an affine equation of R with coordinates (x,y). We may assume that $O_{R,x}$ is included in $\hat{O}_{R,x} \cong \mathbb{C}[[t]]$ such that $x=t$, $y=t^r\epsilon$, where $\epsilon(0)=1$. Thus, the equation of R looks like

$$f(x,y) = y - x^r + g(x,y), \qquad (1.23)$$ affeq

where $g(x,y)$ does not contain terms $cy, c \in \mathbb{C}$. It is easy to see that $(0,0)$ is an inflection point if and only if $r > 2$ with the inflection tangent $y = 0$.

We use the affine equation of the Hessian (1.22) and obtain that the image of

$$h(x,y) = \det\begin{pmatrix} \frac{d}{d-1}f & f_1 & f_2 \\ f_1 & f_{11} & f_{12} \\ f_2 & f_{21} & f_{22} \end{pmatrix}$$

in $\mathbb{C}[[t]]$ is equal to

$$\det\begin{pmatrix} 0 & -rt^{r-1} + g_1 & 1 + g_2 \\ -rt^{r-1} + g_1 & -r(r-1)t^{r-2} + g_{11} & g_{12} \\ 1 + g_2 & g_{12} & g_{22} \end{pmatrix}.$$

Since every monomial entering in g is divisible by y^2, xy or $x^i, i > r$, we see that $\frac{\partial g}{\partial y}$ is divisible by t and $\frac{\partial g}{\partial x}$ is divisible by t^{r-1}. Also, g_{11} is divisible by t^{r-1}. This shows that

$$h(x,y) = \det\begin{pmatrix} 0 & at^{r-1} + \cdots & 1 + \cdots \\ at^{r-1} + \cdots & -r(r-1)t^{r-2} + \cdots & g_{12} \\ 1 + \cdots & g_{12} & g_{22} \end{pmatrix},$$

where $\cdots$ denotes terms of higher degree in t. We compute the determinant and see that it is equal to $r(r-1)t^{r-2} + \cdots$. This means that its image in $\mathbb{C}[[t]]$ is not equal to zero unless the equation of the curve is equal to $y = 0$, i.e., the curve is a line. □

In fact, we have proved more. We say that a nonsingular point of X is an inflection point of *order* $r-2$ and denote the order by $\mathrm{ordfl}_x X$ if one can choose an equation of the curve as in (1.23) with $r \geq 3$. It follows from the previous proof that $r-2$ is equal to the multiplicity $i(X,\mathrm{He})_x$ of the intersection of the curve and its Hessian at the point x. It is clear that $\mathrm{ordfl}_x X = i(\ell, X)_x - 2$, where ℓ is the inflection tangent line of X at x. If X is nonsingular, we have

$$\sum_{x \in X} i(X,\mathrm{He})_x = \sum_{x \in X} \mathrm{ordfl}_x X = 3d(d-2). \tag{1.24}$$

flexorder

SS:1.1.6

1.1.6 The Steinerian hypersurface

Recall that the Hessian hypersurface of a hypersurface $X = V(f)$ is the locus of points a such that the polar quadric $P_{a^{d-2}}(X)$ is singular. The *Steinerian hypersurface* $\mathrm{St}(X)$ of X is the locus of singular points of the polar quadrics. Thus,

$$\mathrm{St}(X) = \bigcup_{a \in \mathrm{He}(X)} \mathrm{Sing}(P_{a^{d-2}}(X)). \tag{1.25}$$

steinerian2

The proof of Proposition 1.1.17 shows that it can be equivalently defined as

$$\mathrm{St}(X) = \{a \in \mathbb{P}^n : P_a(X) \text{ is singular}\}. \tag{1.26}$$

We also have

$$\mathrm{He}(X) = \bigcup_{a \in \mathrm{St}(X)} \mathrm{Sing}(P_a(X)). \tag{1.27}$$

A point $b = [b_0, \ldots, b_n] \in \mathrm{St}(X)$ satisfies the equation

$$\mathrm{He}(f)(a) \cdot \begin{pmatrix} b_0 \\ \vdots \\ b_n \end{pmatrix} = 0, \tag{1.28}$$

where $a \in \mathrm{He}(X)$. This equation defines a subvariety

$$\mathrm{HS}(X) \subset \mathbb{P}^n \times \mathbb{P}^n \tag{1.29}$$

given by $n+1$ equations of bidegree $(d-2, 1)$.When the Steinerian map (see below) is defined, it is just its graph. The projection to the second factor is a closed subscheme of $\mathbb{P}^n$ with support at $\mathrm{St}(X)$. This gives a scheme-theoretical definition of the Steinerian hypersurface which we will accept from now on. It also makes clear why $\mathrm{St}(X)$ is a hypersurface, not obvious from the definition. The expected dimension of the image of the second projection is $n-1$.

The following argument confirms our expectation. It is known (see, for example, [325]) that the locus of singular hypersurfaces of degree d in $|E|$ is a hypersurface

$$\mathsf{D}_d(n) \subset |S^d(E^\vee)|$$

of degree $(n+1)(d-1)^n$ defined by the *discriminant* of a general degree d homogeneous polynomial in $n+1$ variables (the *discriminant hypersurface*). Let L be the projective subspace of $|S^{d-1}(E^\vee)|$ that consists of first polars of X. Assume that no polar $P_a(X)$ is equal to $\mathbb{P}^n$. Then,

$$\mathrm{St}(X) \cong L \cap \mathsf{D}_n(d-1).$$

So, unless L is contained in $\mathsf{D}_n(d-1)$, we get a hypersurface. Moreover, we obtain

$$\deg(\mathrm{St}(X)) = (n+1)(d-2)^n. \tag{1.30}$$

Assume that the quadric $P_{a^{d-2}}(X)$ is of corank 1. Then, it has a unique singular point b with the coordinates $[b_0, \ldots, b_n]$ proportional to any column or a row of the adjugate matrix $\mathrm{adj}(\mathrm{He}(f))$ evaluated at the point a. Thus, St(X) coincides with the image of the Hessian hypersurface under the rational map

$$\mathrm{st} : \mathrm{He}(X) \dashrightarrow \mathrm{St}(X), \quad a \mapsto \mathrm{Sing}(P_{a^{d-2}}(X)),$$

given by polynomials of degree $n(d-2)$. We call it the *Steinerian map*. Of course, it is not defined when all polar quadrics are of corank > 1. Also, if the first polar hypersurface $P_a(X)$ has an isolated singular point for a general point a, we get a rational map

$$\mathrm{st}^{-1} : \mathrm{St}(X) \dashrightarrow \mathrm{He}(X), \quad a \mapsto \mathrm{Sing}(P_a(X)).$$

These maps are obviously inverse to each other. It is a difficult question to determine the sets of indeterminacy points for both maps.

degst **Proposition 1.1.23.** *Let X be a reduced hypersurface. The Steinerian hypersurface of X coincides with $\mathbb{P}^n$ if X has a singular point of multiplicity ≥ 3. The converse is true if we additionally assume that X has only isolated singular points.*

Proof Assume that X has a point of multiplicity ≥ 3. We may harmlessly assume that the point is $p = [1, 0, \dots, 0]$. Write the equation of X in the form

$$f = t_0^k g_{d-k}(t_1, \dots, t_n) + t_0^{k-1} g_{d-k+1}(t_1, \dots, t_n) + \cdots + g_d(t_1, \dots, t_n) = 0, \tag{1.31}$$ po

where the subscript indicates the degree of the polynomial. Since the multiplicity of p is greater than or equal to 3, we must have $d - k \geq 3$. Then, a first polar $P_a(X)$ has the equation

$$a_0 \sum_{i=0}^{k} (k-i) t_0^{k-1-i} g_{d-k+i} + \sum_{s=1}^{n} a_s \sum_{i=0}^{k} t_0^{k-i} \frac{\partial g_{d-k+i}}{\partial t_s} = 0. \tag{1.32}$$ po2

It is clear that the point p is a singular point of $P_a(X)$ of multiplicity $\geq d - k - 1 \geq 2$.

Conversely, assume that all polars are singular. By Bertini's Theorem (see [375], Theorem 17.16), the singular locus of a general polar is contained in the base locus of the linear system of polars. The latter is equal to the singular locus of X. By assumption, it consists of isolated points, hence we can find a singular point of X at which a general polar has a singular point. We may assume that the singular point is $p = [1, 0, \dots, 0]$ and (1.31) is the equation of X. Then, the first polar $P_a(X)$ is given by (1.32). The largest power of t_0 in this expression is at most k. The degree of the equation is $d - 1$. Thus, the point p is a singular point of $P_a(X)$ if and only if $k \leq d - 3$, or, equivalently, if p is at least triple point of X. □

Example 1.1.24. The assumption on the singular locus is essential. First, it is easy to check that $X = V(f^2)$, where $V(f)$ is a nonsingular hypersurface has no points of multiplicity ≥ 3 and its Steinerian coincides with $\mathbb{P}^n$. An example of a reduced hypersurface X with the same property is a surface of degree 6 in $\mathbb{P}^3$ given by the equation

$$(\sum_{i=0}^{3} t_i^3)^2 + (\sum_{i=0}^{3} t_i^2)^3 = 0.$$

Its singular locus is the curve $V(\sum_{i=0}^3 t_i^3) \cap V(\sum_{i=0}^3 t_i^2)$. Each of its points is a double point on X. Easy calculation shows that

$$P_a(X) = V\big((\sum_{i=0}^{3} t_i^3)\sum_{i=0}^{3} a_i t_i^2 + (\sum_{i=0}^{3} t_i^2)^2 \sum_{i=0}^{3} a_i t_i\big).$$

and

$$V(\sum_{i=0}^{3} t_i^3) \cap V(\sum_{i=0}^{3} t_i^2) \cap V(\sum_{i=0}^{3} a_i t_i^2) \subset \mathrm{Sing}(P_a(X)).$$

By Proposition 1.1.7, $\mathrm{Sing}(X)$ is contained in $\mathrm{St}(X)$. Since the same is true for $\mathrm{He}(X)$, we obtain the following.

Proposition 1.1.25. *The intersection* $\mathrm{He}(X) \cap \mathrm{St}(X)$ *contains the singular locus of* X.

One can assign one more variety to a hypersurface $X = V(f)$. This is the *Cayleyan variety*. It is defined as the image $\mathrm{Cay}(X)$ of the rational map

$$\mathrm{HS}(X) \dashrightarrow G_1(\mathbb{P}^n), \quad (a,b) \mapsto \overline{ab},$$

where $G_r(\mathbb{P}^n)$ denotes the Grassmannian of r-dimensional subspaces in $\mathbb{P}^n$.

Note that in the case $n = 2$, the Cayleyan variety is a plane curve in the dual plane, the *Cayleyan curve* of X.

caydeg **Proposition 1.1.26.** *Let X be a general hypersurface of degree $d \geq 3$. Then,*

$$\deg \mathrm{Cay}(X) = \begin{cases} \sum_{i=1}^n (d-2)^i \binom{n+1}{i}\binom{n-1}{i-1} & \text{if } d > 3, \\ \frac{1}{2}\sum_{i=1}^n \binom{n+1}{i}\binom{n-1}{i-1} & \text{if } d = 3, \end{cases}$$

where the degree is considered with respect to the Plücker embedding of the Grassmannian $G_1(\mathbb{P}^n)$.

Proof Since $\mathrm{St}(X) \neq \mathbb{P}^n$, the correspondence $\mathrm{HS}(X)$ is a complete intersection of $n+1$ hypersurfaces in $\mathbb{P}^n \times \mathbb{P}^n$ of bidegree $(d-2, 1)$. Since

$a \in \mathrm{Sing}(P_a(X))$ implies that $a \in \mathrm{Sing}(X)$, the intersection of $\mathrm{HS}(X)$ with the diagonal is empty. Consider the regular map

$$r : \mathrm{HS}(X) \to G_1(\mathbb{P}^n), \quad (a, b) \mapsto \overline{ab}. \tag{1.33}$$ re

It is given by the linear system of divisors of type $(1, 1)$ on $\mathbb{P}^n \times \mathbb{P}^n$ restricted to $\mathrm{HS}(X)$. The genericity assumption implies that this map is of degree one onto the image if $d > 3$ and of degree two if $d = 3$ (in this case the map factors through the involution of $\mathbb{P}^n \times \mathbb{P}^n$ that switches the factors).

It is known that the set of lines intersecting a codimension 2 linear subspace Λ is a hyperplane section of the Grassmannian $G_1(\mathbb{P}^n)$ in its Plücker embedding. Write $\mathbb{P}^n = |E|$ and $\Lambda = |L|$. Let $\omega = v_1 \wedge \ldots \wedge v_{n-1}$ for some basis $(v_1, \ldots, v_{n-1})$ of L. The locus of pairs of points $(a, b) = ([w_1], [w_2])$ in $\mathbb{P}^n \times \mathbb{P}^n$ such that the line $\overline{ab}$ intersects Λ is given by the equation $w_1 \wedge w_2 \wedge \omega = 0$. This is a hypersurface of bidegree $(1, 1)$ in $\mathbb{P}^n \times \mathbb{P}^n$. This shows that the map (1.33) is given by a linear system of divisors of type $(1, 1)$. Its degree (or twice of the degree) is equal to the intersection $((d-2)h_1 + h_2)^{n+1}(h_1 + h_2)^{n-1}$, where h_1, h_2 are the natural generators of $H^2(\mathbb{P}^n \times \mathbb{P}^n, \mathbb{Z})$. We have

$$((d-2)h_1 + h_2)^{n+1}(h_1 + h_2)^{n-1} =$$

$$\Big(\sum_{i=0}^{n+1} \tbinom{n+1}{i}(d-2)^i h_1^i h_2^{n+1-i}\Big)\Big(\sum_{j=0}^{n-1} \tbinom{n-1}{j} h_1^{n-1-j} h_2^j\Big)$$

$$= \sum_{i=1}^{n} (d-2)^i \tbinom{n+1}{i}\tbinom{n-1}{i-1}.$$

□

For example, if $n = 2, d > 3$, we obtain a classical result

$$\deg \mathrm{Cay}(X) = 3(d-2) + 3(d-2)^2 = 3(d-2)(d-1),$$

and $\deg \mathrm{Cay}(X) = 3$ if $d = 3$.

Remark 1.1.27. The homogeneous forms defining the Hessian and Steinerian hypersurfaces of $V(f)$ are examples of *covariants* of f. We already discussed them in the case $n = 1$. The form defining the Cayleyan of a plane curve is an example of a *contravariant* of f.

SS:1.1.7

1.1.7 The Jacobian hypersurface

In the previous sections we discussed some natural varieties attached to the linear system of first polars of a hypersurface. We can extend these constructions to arbitrary n-dimensional linear systems of hypersurfaces in $\mathbb{P}^n = |E|$.

We assume that the linear system has no fixed components, i.e., its general member is an irreducible hypersurface of some degree d. Let $L \subset S^d(E^\vee)$ be a linear subspace of dimension $n+1$ and $|L|$ be the corresponding linear system of hypersurfaces of degree d. Note that, in the case of linear system of polars of a hypersurface X of degree $d+1$, the linear subspace L can be canonically identified with E and the inclusion $|E| \subset |S^d(E^\vee)|$ corresponds to the polarization map $a \mapsto P_a(X)$.

Let $\mathsf{D}_d(n) \subset |S^d(E^\vee)|$ be the discriminant hypersurface.The intersection

$$\mathsf{D}(|L|) = |L| \cap \mathsf{D}_d(n)$$

is called the *discriminant hypersurface* of $|L|$. We assume that it is not equal to $\mathbb{P}^n$, i.e., not all members of $|L|$ are singular. Let

$$\widetilde{\mathsf{D}}(|L|) = \{(x, D) \in \mathbb{P}^n \times |L| : x \in \mathrm{Sing}(D)\}$$

with two projections $p : \widetilde{\mathsf{D}} \to \mathsf{D}(|L|)$ and $q : \widetilde{\mathsf{D}} \to |L|$. We define the *Jacobian hypersurface* of $|L|$ as

$$\mathrm{Jac}(|L|) = q(\widetilde{\mathsf{D}}(|L|)).$$

It parameterizes singular points of singular members of $|L|$. Again, it may coincide with the whole $\mathbb{P}^n$. In the case of polar linear systems, the discriminant hypersurface is equal to the Steinerian hypersurface, and the Jacobian hypersurface is equal to the Hessian hypersurface.

The *Steinerian hypersurface* $\mathrm{St}(|L|)$ is defined as the locus of points $x \in \mathbb{P}^n$ such that there exists $a \in \mathbb{P}^n$ such that $x \in \cap_{D \in |L|} P_{a^{n-1}}(D)$. Since $\dim L = n+1$, the intersection is empty, unless there exists D such that $P_{a^{n-1}}(D) = 0$. Thus, $P_{a^n}(D) = 0$ and $a \in \mathrm{Sing}(D)$, hence $a \in \mathrm{Jac}(|L|)$ and $D \in \mathsf{D}(|L|)$. Conversely, if $a \in \mathrm{Jac}(|L|)$, then $\cap_{D \in |L|} P_{a^{n-1}}(D) \neq \emptyset$ and it is contained in $\mathrm{St}(|L|)$. By duality (1.12),

$$x \in \bigcap_{D \in |L|} P_{a^{n-1}}(D) \Leftrightarrow a \in \bigcap_{D \in |L|} P_x(D).$$

Thus, the Jacobian hypersurface is equal to the locus of points which belong to the intersection of the first polars of divisors in $|L|$ with respect to some point $x \in \mathrm{St}(X)$. Let

$$\mathrm{HS}(|L|) = \{(a, b) \in \mathrm{He}(|L|) \times \mathrm{St}(|L|) : a \in \bigcap_{D \in |L|} P_b(D)\}$$

$$= \{(a, b) \in \mathrm{He}(|L|) \times \mathrm{St}(|L|) : b \in \bigcap_{D \in |L|} P_{a^{d-1}}(D)\}.$$

It is clear that $\mathrm{HS}(|L|) \subset \mathbb{P}^n \times \mathbb{P}^n$ is a complete intersection of $n+1$ divisors of type $(d-1, 1)$. In particular,

$$\omega_{\mathrm{HS}(|L|)} \cong \mathrm{pr}_1^*(\mathcal{O}_{\mathbb{P}^n}((d-2)(n+1))). \tag{1.34}$$

One expects that, for a general point $x \in \mathrm{St}(|L|)$, there exists a unique $a \in \mathrm{Jac}(|L|)$ and a unique $D \in \mathsf{D}(|L|)$ as above. In this case, the correspondence $\mathrm{HS}(|L|)$ defines a birational isomorphism between the Jacobian and Steinerian hypersurface. Also, it is clear that $\mathrm{He}(|L|) = \mathrm{St}(|L|)$ if $d = 2$.

Assume that $|L|$ has no base points. Then, $\mathrm{HS}(|L|)$ does not intersect the diagonal of $\mathbb{P}^n \times \mathbb{P}^n$. This defines a map

$$\mathrm{HS}(|L|) \to G_1(\mathbb{P}^n), \quad (a, b) \mapsto \overline{ab}.$$

Its image $\mathrm{Cay}(|L|)$ is called the *Cayleyan variety* of $|L|$.

A line $\ell \in \mathrm{Cay}(|L|)$ is called a *Reye line* of $|L|$. It follows from the definitions that a Reye line is characterized by the property that it contains a point such that there is a hyperplane in $|L|$ of hypersurfaces tangent to ℓ at this point. For example, if $d = 2$ this is equivalent to the property that ℓ is contained is a linear subsystem of $|L|$ of codimension 2 (instead of expected codimension 3).

The proof of Proposition caydeg 1.1.26 applies to our more general situation to give the degree of $\mathrm{Cay}(|L|)$ for a general n-dimensional linear system $|L|$ of hypersurfaces of degree d.

$$\deg \mathrm{Cay}(|L|) = \begin{cases} \sum_{i=1}^n (d-1)^i \binom{n+1}{i}\binom{n-1}{i-1} & \text{if } d > 2, \\ \frac{1}{2}\sum_{i=1}^n \binom{n+1}{i}\binom{n-1}{i-1} & \text{if } d = 2. \end{cases} \tag{1.35}$$ caydeg2

Let $\underline{f} = (f_0, \ldots, f_n)$ be a basis of L. Choose coordinates in $\mathbb{P}^n$ to identify $S^d(E^\vee)$ with the polynomial ring $\mathbb{C}[t_0, \ldots, t_n]$. A well-known fact from the complex analysis asserts that $\mathrm{Jac}(|L|)$ is given by the determinant of the Jacobian matrix

$$J(\underline{f}) = \begin{pmatrix} \partial_0 f_0 & \partial_1 f_0 & \ldots & \partial_n f_0 \\ \partial_0 f_1 & \partial_1 f_1 & \ldots & \partial_n f_1 \\ \vdots & \vdots & \vdots & \vdots \\ \partial_0 f_n & \partial_1 f_n & \ldots & \partial_n f_n \end{pmatrix}.$$

In particular, we expect that

$$\deg \mathrm{Jac}(|L|) = (n+1)(d-1).$$

If $a \in \mathrm{Jac}(|L|)$, then a nonzero vector in the null-space of $J(\underline{f})$ defines a point x such that $P_x(f_0)(a) = \ldots = P_x(f_n)(a) = 0$. Equivalently,

$$P_{a^{n-1}}(f_0)(x) = \ldots = P_{a^{n-1}}(f_n)(x) = 0.$$

This shows that $\mathrm{St}(|L|)$ is equal to the projectivization of the union of the null-spaces $\mathrm{Null}(\mathrm{Jac}(\underline{f}(a))), a \in \mathbb{C}^{n+1}$. Also, a nonzero vector in the null space of the transpose matrix ${}^tJ(\underline{f})$ defines a hypersurface in $\mathsf{D}(|L|)$ with singularity at the point a.

Let $\mathrm{Jac}(|L|)^0$ be the open subset of points where the corank of the jacobian matrix is equal to 1. We assume that it is a dense subset of $\mathrm{Jac}(|L|)$. Then, taking the right and the left kernels of the Jacobian matrix, defines two maps

$$\mathrm{Jac}(|L|)^0 \to \mathsf{D}(|L|), \quad \mathrm{Jac}(|L|)^0 \to \mathrm{St}(|L|).$$

Explicitly, the maps are defined by the nonzero rows (resp. columns) of the adjugate matrix $\mathrm{adj}(\mathrm{He}(\underline{f}))$.

Let $\phi_{|L|} : \mathbb{P}^n \dashrightarrow |L^\vee|$ be the rational map defined by the linear system $|L|$. Under some assumptions of generality, which we do not want to spell out, one can identify $\mathrm{Jac}(|L|)$ with the ramification divisor of the map and $\mathsf{D}(|L|)$ with the dual hypersurface of the branch divisor.

Let us now define a new variety attached to a n-dimensional linear system in $\mathbb{P}^n$. Consider the inclusion map $L \hookrightarrow S^d(E^\vee)$ and let

$$L \hookrightarrow S^d(E)^\vee, \quad f \mapsto \tilde{f},$$

be the restriction of the total polarization map (1.2) to L. Now, we can consider $|L|$ as a n-dimensional linear system $\widetilde{|L|}$ on $|E|^d$ of divisors of type $(1,\dots,1)$. Let

$$\mathrm{PB}(|L|) = \bigcap_{D \in \widetilde{|L|}} D \subset |E|^d$$

be the base scheme of $\widetilde{|L|}$. We call it the *polar base locus* of $|L|$. It is equal to the complete intersection of $n+1$ effective divisors of type $(1,\dots,1)$. By the adjunction formula,

$$\omega_{\mathrm{PB}(|L|)} \cong \mathcal{O}_{\mathrm{PB}(|L|)}.$$

If smooth, $\mathrm{PB}(|L|)$ is a *Calabi-Yau variety* of dimension $(d-1)n-1$.

For any $f \in L$, let $N(f)$ be the set of points $x = ([v^{(1)}],\dots,[v^{(d)}]) \in |E|^d$ such that

$$\tilde{f}(v^{(1)},\dots,v^{(j-1)},v,v^{(j+1)},\dots,v^{(d)}) = 0$$

for every $j = 1,\dots,d$ and $v \in E$. Since

$$\tilde{f}(v^{(1)},\dots,v^{(j-1)},v,v^{(j+1)},\dots,v^{(d)}) = D_{v^{(1)}\dots v^{(j-1)}v^{(j+1)}\dots v^{(d)}}(D_v(f)),$$

This can be also expressed in the form

$$\widetilde{\partial_j f}(v^{(1)},\dots,v^{(j-1)},v^{(j+1)},\dots,v^{(d)}) = 0, \ j = 0,\dots,n. \tag{1.36}$$ add

Choose coordinates $u_0, \dots, u_n$ in L and coordinates $t_0, \dots, t_n$ in E. Let $\underline{\tilde{f}}$ be the image of a basis $\underline{f}$ of L in $(E^\vee)^d$. Then, $\mathrm{PB}(|L|)$ is a subvariety of $(\mathbb{P}^n)^d$ given by a system of d multi-linear equations

$$\tilde{f}_0(t^{(1)}, \dots, t^{(d)}) = \dots = \tilde{f}_n(t^{(1)}, \dots, t^{(d)}) = 0,$$

where $t^{(j)} = (t_0^{(j)}, \dots, t_n^{(j)}), j = 1, \dots, d$. For any $\lambda = (\lambda_0, \dots, \lambda_n)$, set $\tilde{f}_\lambda = \sum_{i=0}^n \lambda_i \tilde{f}_i$.

gkz **Proposition 1.1.28.** *The following is equivalent:*

(i) $x \in \mathrm{PB}(|L|)$ *is a singular point,*
(ii) $x \in N(\tilde{f}_\lambda)$ *for some* $\lambda \neq 0$.

Proof The variety $\mathrm{PB}(|L|)$ is smooth at a point x if and only if the rank of the $d(n+1) \times (n+1)$-size matrix

$$(a_{ij}^k) = \Big(\frac{\partial \tilde{f}_k}{\partial t_i^{(j)}}(x)\Big)_{i,k=0,\dots,n,j=1,\dots,d}$$

is equal to $n+1$. Let $\tilde{f}_u = u_0\tilde{f}_0 + \dots + u_n\tilde{f}_n$, where $u_0, \dots, u_n$ are unknowns. Then, the nullspace of the matrix is equal to the space of solutions $u = (\lambda_0, \dots, \lambda_n)$ of the system of linear equations

$$\frac{\partial \tilde{f}_u}{\partial u_0}(x) = \dots = \frac{\partial \tilde{f}_u}{\partial u_n}(x) = \frac{\partial \tilde{f}_u}{\partial t_i^{(j)}}(x) = 0. \tag{1.37}$$ sys1

For a fixed λ, in terminology of [325] (GKZ), p. 445, the system has a solution x in $|E|^d$ if $\tilde{f}_\lambda = \sum \lambda_i \tilde{f}_i$ is a *degenerate multilinear form*. By Proposition 1.1 from Chapter 14 of loc.cit., $\tilde{f}_\lambda$ is degenerate if and only if $N(\tilde{f}_\lambda)$ is non-empty. This proves the assertion. □

For any non-empty subset I of $\{1, \dots, d\}$, let Δ_I be the subset of points $x \in |E|^d$ with equal projections to i-th factors with $i \in I$. Let Δ_k be the union of Δ_I with $\#I = k$. The set Δ_d is denoted by Δ (the small diagonal).

Observe that $\mathrm{PB}(|L|) = \mathrm{HS}(|L|)$ if $d = 2$ and $\mathrm{PB}(|L|) \cap \Delta_{d-1}$ consists of d copies isomorphic to $\mathrm{HS}(|L|)$ if $d > 2$.

Definition 1.1.29. *A n-dimensional linear system* $|L| \subset |S^d(E^\vee)|$ *is called* regular *if* $\mathrm{PB}(|L|)$ *is smooth at each point of* Δ_{d-1}.

Proposition 1.1.30. *Assume* $|L|$ *is regular. Then*

(i) $|L|$ *has no base points,*
(ii) $\widetilde{\mathsf{D}}(|L|)$ *is smooth.*

Proof (i) Assume that $x = ([v_0], \dots, [v_0]) \in \mathrm{PB}(|L|) \cap \Delta$. Consider the linear map $L \to E$ defined by evaluating $\tilde{f}$ at a point $(v_0, \dots, v_0, v, v_0, \dots, v_0)$, where $v \in E$. This map factors through a linear map $L \to E/[v_0]$, and hence has a nonzero f in its kernel. This implies that $x \in N(f)$, and hence x is a singular point of $\mathrm{PB}(|L|)$.

(ii) In coordinates, the variety $\widetilde{\mathsf{D}}(|L|)$ is a subvariety of type $(1, d-1)$ of $\mathbb{P}^n \times \mathbb{P}^n$ given by the equations

$$\sum_{k=0}^{n} u_k \frac{\partial f_k}{\partial t_0} = \dots = \sum_{k=0}^{n} u_k \frac{\partial f_k}{\partial t_n} = 0.$$

The tangent space at a point $([\lambda], [a])$ is given by the system of $n+1$ linear equations in $2n+2$ variables $(X_0, \dots, X_n, Y_0, \dots, Y_n)$

$$\sum_{k=0}^{n} \frac{\partial f_k}{\partial t_i}(a) X_k + \sum_{j=0}^{n} \frac{\partial^2 f_\lambda}{\partial t_i \partial t_j}(a) Y_j = 0, \quad i = 0, \dots, n, \tag{1.38}$$ sys3

where $f_\lambda = \sum_{k=0}^{n} \lambda_k f_k$. Suppose $([\lambda], [a])$ is a singular point. Then, the equations are linearly dependent. Thus, there exists a nonzero vector $v = (\alpha_0, \dots, \alpha_n)$ such that

$$\sum_{i=0}^{n} \alpha_i \frac{\partial f_k}{\partial t_i}(a) = D_v(f_k)(a) = \tilde{f}_k(a, \dots, a, v) = 0, \ k = 0, \dots, n$$

and

$$\sum_{i}^{n} \alpha_i \frac{\partial^2 f_\lambda}{\partial t_i \partial t_j}(a) = D_v\Big(\frac{\partial f_\lambda}{\partial t_j}\Big)(a) = D_{a^{d-2}v}\Big(\frac{\partial f_\lambda}{\partial t_j}\Big) = 0, \ j = 0, \dots, n,$$

where $f_\lambda = \sum \lambda_k f_k$. The first equation implies that $x = ([a], \dots, [a], [v])$ belongs to $\mathrm{PB}(|L|)$. Since $a \in \mathrm{Sing}(f_\lambda)$, we have $D_{a^{d-1}}(\frac{\partial f_\lambda}{\partial t_j}) = 0, \ j = 0, \dots, n$. By (add 1.36), this and the second equation now imply that $x \in N(f_\lambda)$. By Proposition gkz 1.1.28, $\mathrm{PB}(|L|)$ is singular at x, contradicting the assumption.

□

Corollary 1.1.31. *Suppose $|L|$ is regular. Then, the projection*

$$q : \widetilde{\mathsf{D}}(|L|) \to \mathsf{D}(|L|)$$

is a resolution of singularities.

Consider the projection $p : \widetilde{\mathsf{D}}(|L|) \to \mathrm{Jac}(|L|), (D, x) \mapsto x$. Its fibres are linear spaces of divisors in $|L|$ singular at the point $[a]$. Conversely, suppose $\mathsf{D}(|L|)$ contains a linear subspace, in particular, a line. Then, by Bertini's Theorem all singular divisors parameterized by the line have a common singular point. This implies that the morphism p has positive dimensional fibres. This simple observation gives the following.

Proposition 1.1.32. *Suppose* $\mathsf{D}(|L|)$ *does not contain lines. Then,* $\widetilde{\mathsf{D}}(|L|)$ *is smooth if and only if* $\mathrm{Jac}(|L|)$ *is smooth. Moreover,* $\mathrm{HS}(|L|) \cong \mathrm{St}(|L|) \cong \mathrm{Jac}(|L|)$.

appl *Remark* 1.1.33. We will prove later in Example ex:discrim 1.2.3 that the tangent space of the discriminant hypersurface $\mathsf{D}_d(n)$ at a point corresponding to a hypersurface $X = V(f)$ with only one ordinary double point x is naturally isomorphic to the linear space of homogeneous forms of degree d vanishing at the point x modulo $\mathbb{C}f$. This implies that $\mathsf{D}(|L|)$ is nonsingular at a point corresponding to a hypersurface with one ordinary double point unless this double point is a base point of $|L|$. If $|L|$ has no base points, the singular points of $\mathsf{D}(|L|)$ are of two sorts: either they correspond to divisors with worse singularities than one ordinary double point, or the linear space $|L|$ is tangent to $\mathsf{D}_d(n)$ at its nonsingular point.

Consider the natural action of the symmetric group $\mathfrak{S}_d$ on $(\mathbb{P}^n)^d$. It leaves $\mathrm{PB}(|L|)$ invarian. The quotient variety

$$\mathrm{Rey}(|L|) = \mathrm{PB}(|L|)/\mathfrak{S}_d$$

is called the *Reye variety* of $|L|$. If $d > 2$ and $n > 1$, the Reye variety is singular.

Example 1.1.34. Assume $d = 2$. Then, $\mathrm{PB}(|L|) = \mathrm{HS}(|L|)$ and $\mathrm{Jac}(|L|) = \mathrm{St}(|L|)$. Moreover, $\mathrm{Rey}(|L|) \cong \mathrm{Cay}(|L|)$. We have

$$\deg \mathrm{Jac}(|L|) = \deg \mathsf{D}(|L|) = n+1, \quad \deg \mathrm{Cay}(|L|) = \sum_{i=1}^{n} \binom{n+1}{i}\binom{n-1}{i-1}.$$

The linear system is regular if and only if $\mathrm{PB}(|L|)$ is smooth. This coincides with the notion of regularity of a web of quadrics in $\mathbb{P}^3$ discussed in CossecReye [172].

A Reye line ℓ is contained in a codimension 2 subspace $\Lambda(\ell)$ of $|L|$, and is characterized by this condition. The linear subsystem $\Lambda(\ell)$ of dimension $n-2$ contains ℓ in its base locus. The residual component is a curve of degree $2^{n-1}-1$ which intersects ℓ at two points. The points are the two ramification points of the pencil $Q \cap \ell, Q \in |L|$. The two singular points of the base locus of $\Lambda(\ell)$ define two singular points of the intersection $\Lambda(\ell) \cap \mathsf{D}(|L|)$. Thus, $\Lambda(\ell)$ is a codimension 2 subspace of $|L|$ which is tangent to the determinantal hypersurface at two points.

If $|L|$ is regular and $n = 3$, $\mathrm{PB}(|L|)$ is a K3 surface, and its quotient $\mathrm{Rey}(|L|)$ is an Enriques surface. The Cayley variety is a congruence (i.e., a surface) of lines in $G_1(\mathbb{P}^3)$ of order 7 and class 3 (this means that there are 7 Reye lines through a general point in $\mathbb{P}^3$ and there 3 Reye lines in a general plane). The Reye lines are bitangents of the quartic surface $\mathsf{D}(|L|)$. The quartic surface has 10 nodes and is called *Cayley quartic symmetroid*. We refer for the details to [172], [259, Chapter 9]. The *Reye congruence* of lines is also discussed in [360].

1.2 The Dual Hypersurface

S:1.2

1.2.1 The polar map

SS:1.2.1

Let $X = V(f)$ for some $f \in S^d(E^\vee)$. We assume that it is not a cone. The polarization map

$$E \to S^{d-1}(E^\vee), \quad v \mapsto D_v(f),$$

allows us to identify $|E|$ with an n-dimensional linear system of hypersurfaces of degree $d-1$. This linear system defines a rational map

$$\mathsf{p}_X : |E| \dashrightarrow \mathbb{P}(E).$$

It follows from (1.12) that the map is given by assigning to a point a the linear polar $P_{a^{d-1}}(X)$. We call the map p the *polar map* defined by the hypersurface X. In coordinates, the polar map is given by

$$[x_0, \ldots, x_n] \mapsto \Big[\frac{\partial f}{\partial t_0}, \ldots, \frac{\partial f}{\partial t_n}\Big].$$

Recall that a hyperplane $H_a = V(\sum a_i \xi_i)$ in the dual projective space $(\mathbb{P}^n)^\vee$ is the point $a = [a_0, \ldots, a_n] \in \mathbb{P}^n$. The pre-image of the hyperplane H_a under p_X is the polar $P_a(X) = V(\sum a_i \frac{\partial f}{\partial t_i})$.

If X is nonsingular, the polar map is a regular map given by polynomials of degree $d-1$. Since it is a composition of the Veronese map and a projection, it is a finite map of degree $(d-1)^n$.

Proposition 1.2.1. *Assume X is nonsingular. The ramification divisor of the polar map is equal to* $\mathrm{He}(X)$.

Proof Note that, for any finite map $\phi : X \to Y$ of nonsingular varieties, the ramification divisor $\mathrm{Ram}(\phi)$ is defined locally by the determinant of the linear map of locally free sheaves $\phi^*(\Omega^1_Y) \to \Omega^1_X$. The image of $\mathrm{Ram}(\phi)$ in Y is called the *branch divisor*. Both of the divisors may be nonreduced. We have the *Hurwitz formula*

$$K_X = \phi^*(K_Y) + \mathrm{Ram}(\phi), \qquad (1.39)$$ hurwitz2

The map ϕ is étale outside $\mathrm{Ram}(\phi)$, i.e., for any point $x \in X$ the homomorphism of local ring $\mathcal{O}_{Y,\phi(x)} \to \mathcal{O}_{X,x}$ defines an isomorphism of their formal completions. In particular, the pre-image $\phi^{-1}(Z)$ of a nonsingular subvariety $Z \subset Y$ is nonsingular outside the support of $\mathrm{Ram}(\phi)$. Applying this to the polar map we see that the singular points of $P_a(X) = p_X^{-1}(H_a)$ are contained in the ramification locus $\mathrm{Ram}(p_X)$ of the polar map. On the other hand, we know that the set of singular points of first polars is the Hessian $\mathrm{He}(X)$. This shows that $\mathrm{He}(X) \subset \mathrm{Ram}(\mathsf{p}_X)$. Applying the Hurwitz formula for the canonical sheaf

$$K_{\mathbb{P}^n} = p_X^*(K_{(\mathbb{P}^n)^\vee}) + \mathrm{Ram}(\mathsf{p}_X).$$

we obtain that $\deg(\mathrm{Ram}(\mathsf{p}_X)) = (n+1)(d-2) = \deg(\mathrm{He}(X))$. This shows that $\mathrm{He}(X) = \mathrm{Ram}(\mathsf{p}_X)$. □

Let us describe the branch divisor. One can show that the pre-image of a hyperplane H_a in $\mathbb{P}(E)$ corresponding to a point $a \in |E|$ is singular if and only if its intersection with the branch divisor is not transversal. This means that the dual hypersurface of the branch divisor is the Steinerian hypersurface. Equivalently, the branch divisor is the dual of the Steinerian hypersurface.

SS:1.2.2

1.2.2 Dual varieties

Recall that the *dual variety* $X^\vee$ of a subvariety X in $\mathbb{P}^n = |E|$ is the closure in the dual projective space $(\mathbb{P}^n)^\vee = |E^\vee|$ of the locus of hyperplanes in $\mathbb{P}^n$ which are tangent to X at some nonsingular point of X.

The dual variety of a hypersurface $X = V(f)$ is the image of X under the rational map given by the first polars. In fact, the point $[\partial_0 f(x), \ldots, \partial_n f(x)]$ in $(\mathbb{P}^n)^\vee$ is the hyperplane $V(\sum_{i=0}^n \partial_i f(x) t_i)$ in $\mathbb{P}^n$ which is tangent to X at the point x.

The following result is referred to as the *Reflexivity theorem*. Its proof can found in many modern text-books (e.g. [325], [375], [747], [814]).

reflexivitythm **Theorem 1.2.2** (Reflexivity Theorem)**.**

$$(X^\vee)^\vee = X.$$

It follows from any proof in loc. cit. that, for any nonsingular point $y \in X^\vee$ and any nonsingular point $x \in X$,

$$\mathbb{T}_x(X) \subset H_y \Leftrightarrow \mathbb{T}_y(X^\vee) \subset H_x.$$

Here, we continue to identify a point a in $|E|$ with a hyperplane H_a in $\mathbb{P}(E)$. The set of all hyperplanes in $(\mathbb{P}^n)^\vee$ containing the linear subspace $\mathbb{T}_y(X^\vee)$ is the dual linear space of $\mathbb{T}_y(X^\vee)$ in $\mathbb{P}^n$. Thus, the fiber of the *duality map* (or *Gauss map*)

$$\gamma : X^{\mathrm{ns}} \to X^\vee, \quad x \mapsto \mathbb{T}_x(X), \tag{1.40}$$

over a nonsingular point $y \in X^\vee$ is an open subset of the projective subspace in $\mathbb{P}^n$ equal to the dual of the tangent space $\mathbb{T}_y(X^\vee)$. Here and later on, X^{ns} denotes the set of nonsingular points of a variety X. In particular, if $X^\vee$ is a hypersurface, the dual space of $\mathbb{T}_y(X^\vee)$ must be a point, and hence, the map γ is birational.

Let us apply this to the case where X is a nonsingular hypersurface. The polar map is a finite map, hence, the dual of a nonsingular hypersurface is a hypersurface. The duality map is a birational morphism

$$\mathsf{p}_X|_X : X \to X^\vee.$$

The degree of the dual hypersurface $X^\vee$ (if it is a hypersurface) is called the *class* of X. For example, the class of any plane curve of degree > 1 is well-defined.

For example, if X is a nonsingular quadric, the dual hypersurface is also a nonsingular quadric given by the adjugate matrix $\mathrm{adj}(A)$, where A is the symmetric matrix (a_{ij}) defined by its equation from Example exaa 1.1.3.

ex:discrim *Example* 1.2.3. Let $\mathsf{D}_d(n)$ be the discriminant hypersurface in $|S^d(E^\vee)|$. We would like to describe explicitly the tangent hyperplane of $\mathsf{D}_d(n)$ at its nonsingular point. Let

$$\tilde{\mathcal{D}}_d(n) = \{(X, x) \in |\mathcal{O}_{\mathbb{P}^n}(d)| \times \mathbb{P}^n : x \in \mathrm{Sing}(X)\}.$$

Let us see that $\tilde{\mathcal{D}}_d(n)$ is nonsingular and the projection to the first factor

$$\pi : \tilde{\mathcal{D}}_d(n) \to \mathsf{D}_d(n) \tag{1.41}$$ projj

is a resolution of singularities. In particular, π is an isomorphism over the open set $\mathsf{D}_d(n)^{\mathrm{ns}}$ of nonsingular points of $\mathsf{D}_d(n)$.

The fact that $\tilde{\mathcal{D}}_d(n)$ is nonsingular follows easily from considering the projection to $\mathbb{P}^n$. For any point $x \in \mathbb{P}^n$ the fiber of the projection is the projective space of hypersurfaces which have a singular point at x (this amounts to $n+1$ linear conditions on the coefficients). Thus, $\tilde{\mathcal{D}}_d(n)$ is a projective bundle over $\mathbb{P}^n$, and hence, is nonsingular.

Let us see where π is an isomorphism. Let $A_{\mathbf{i}}, |\mathbf{i}| = d$, be the projective coordinates in $\left|\mathcal{O}_{\mathbb{P}^n}(d)\right| = |S^d(E^\vee)|$ corresponding to the coefficients of a hypersurface of degree d and let $t_0, \ldots, t_n$ be projective coordinates in $\mathbb{P}^n$.

Then, $\tilde{\mathcal{D}}_d(n)$ is given by $n+1$ bihomogeneous equations of bidegree $(1, d-1)$:

$$\sum_{|\mathbf{i}|=d} i_s A_{\mathbf{i}} \mathbf{t}^{\mathbf{i}-e_s} = 0, \quad s = 0, \ldots, n. \tag{1.42}$$

tangent

Here, e_s is the s-th unit vector in $\mathbb{Z}^{n+1}$.

A point $(X, [v_0]) = (V(f), [v_0]) \in |\mathcal{O}_{\mathbb{P}^n}(d)| \times \mathbb{P}^n$ belongs to $\tilde{\mathcal{D}}_d(n)$ if and only if, replacing $A_{\mathbf{i}}$ with the coefficient of f at $\mathbf{t}^i$ and t_i with the i-th coefficient of v_0, we get the identities.

We identify the tangent space of $|S^d(E^\vee)| \times |E|$ at a point $(X, [v_0])$ with the space $S^d(E^\vee)/\mathbb{C}f \oplus E/\mathbb{C}v_0$. In coordinates, a vector in the tangent space is a pair $(g, [v])$, where $g = \sum_{|\mathbf{i}|=d} a_{\mathbf{i}} \mathbf{t}^{\mathbf{i}}$, $v = (x_0, \ldots, x_n)$ are considered modulo pairs $(\lambda f, \mu v_0)$. Differentiating equations (1.42), we see that the tangent space is defined by the $(n+1) \times \binom{n+d}{d}$-matrix

$$\begin{pmatrix} \ldots & i_0 x^{\mathbf{i}-e_0} & \ldots & \sum_{|\mathbf{i}|=d} i_0 i_0 A_{\mathbf{i}} x^{\mathbf{i}-e_0-e_0} & \ldots & \sum_{|\mathbf{i}|=d} i_0 i_n A_{\mathbf{i}} x^{\mathbf{i}-e_0-e_n} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \ldots & i_n x^{\mathbf{i}-e_n} & \ldots & \sum_{|\mathbf{i}|=d} i_n i_0 A_{\mathbf{i}} x^{\mathbf{i}-e_n-e_0} & \ldots & \sum_{|\mathbf{i}|=d} i_n i_n A_{\mathbf{i}} x^{\mathbf{i}-e_n-e_n} \end{pmatrix}$$

where $x^{\mathbf{i}-e_s} = 0$ if $\mathbf{i} - e_s$ is not a non-negative vector. It is easy to interpret solutions of these equations as pairs (g, v) from above such that

$$\nabla(g)(v_0) + \mathrm{He}(f)(v_0) \cdot v = 0. \tag{1.43}$$

eeeq

Since $[v_0]$ is a singular point of $V(f)$, $\nabla(f)([v_0]) = 0$. Also $\mathrm{He}(f)(v_0) \cdot v_0 = 0$, as follows from Theorem 1.1.20. This confirms that pairs $(\lambda f, \mu v_0)$ are always solutions. The tangent map $d\pi$ at the point $(V(f), [v_0])$ is given by the projection $(g, v) \mapsto g$, where (g, v) is a solution of (1.43). Its kernel consists of the pairs $(\lambda f, v)$ modulo pairs $(\lambda f, \mu v_0)$. For such pairs the equations (1.43) give

$$\mathrm{He}(f)(v_0) \cdot v = 0. \tag{1.44}$$

zzzz

We may assume that $v_0 = (1, 0, \ldots, 0)$. Since $[v_0]$ is a singular point of $V(f)$, we can write $f = t_0^{d-2} f_2(t_1, \ldots, t_n) + \ldots$. Computing the Hessian matrix at the point v_0, we obtain that it is equal to

$$\begin{pmatrix} 0 & \ldots & \ldots & 0 \\ 0 & a_{11} & \ldots & a_{1n} \\ \vdots & \vdots & \vdots & \vdots \\ 0 & a_{n1} & \ldots & a_{nn} \end{pmatrix}, \tag{1.45}$$

m1

where $f_2(t_1, \ldots, t_n) = \sum_{0 \le i,j \le n} a_{ij} t_i t_j$. Thus, a solution of (1.44), not proportional to v_0 exists if and only if $\det \mathrm{He}(f_2) = 0$. By definition, this means that the singular point of X at x is not an ordinary double point. Thus, we obtain that the projection map (1.41) is an isomorphism over the open subset of $\mathsf{D}_d(n)$ representing hypersurfaces with an isolated ordinary singularity.

We can also find the description of the tangent space of $\mathsf{D}_d(n)$ at its point $X = V(f)$ representing a hypersurface with a unique ordinary singular point x. It follows from calculation of the Hessian matrix in (1.45), that its corank at the ordinary singular point is equal to 1. Since the matrix is symmetric, a vector in its nullspace is orthogonal to the column of the matrix. We know that $\mathrm{He}(f)(v_0) \cdot v_0 = 0$. Thus, the dot-product $\nabla(g)(v_0) \cdot v_0$ is equal to zero. By Euler's formula, we obtain $g(v_0) = 0$. The converse is also true. This proves that

$$T(\mathsf{D}_d(n))_X = \{g \in S^d(E^\vee)/\mathbb{C}f : g(x) = 0\}. \tag{1.46}$$ tandiscr

Now, we are ready to compute the dual variety of $\mathsf{D}_d(n)$. The condition $g(b) = 0$, where $\mathrm{Sing}(X) = \{b\}$ is equivalent to $D_{b^d}(f) = 0$. Thus, the tangent hyperplane, considered as a point in the dual space $|S^d(E)| = |S^d(E^\vee)^\vee|$ corresponds to the envelope $b^d = (\sum_{s=0}^n b_s \partial_i)^d$. The set of such envelopes is the *Veronese variety* V_d^n, the image of $|E|$ under the *Veronese map* $\mathsf{v}_d : |E| \to |S^d(E)|$. Thus,

$$\mathsf{D}_d(n)^\vee \cong \mathsf{v}_d(\mathbb{P}^n), \tag{1.47}$$ dualdiscr

Of course, it is predictable. Recall that the Veronese variety is embedded naturally in $|\mathcal{O}_{\mathbb{P}^n}(d)|^\vee$. Its hyperplane section can be identified with a hypersurface of degree d in $\mathbb{P}^n$. A tangent hyperplane is a hypersurface with a singular point, i.e., a point in $\mathsf{D}_d(n)$. Thus, the dual of V_d^n is isomorphic to $\mathsf{D}_d(n)$, and hence, by duality, the dual of $\mathsf{D}_d(n)$ is isomorphic to V_d^n.

ex:1.2.4 *Example* 1.2.4. Let $Q = V(q)$ be a nonsingular quadric in $\mathbb{P}^n$. Let $A = (a_{ij})$ be a symmetric matrix defining q. The tangent hyperplane of Q at a point $[x] \in \mathbb{P}^n$ is the hyperplane

$$t_0 \sum_{j=0}^n a_{0j} x_j + \cdots + t_n \sum_{j=0}^n a_{nj} x_j = 0.$$

Thus, the vector of coordinates $\mathbf{y} = (y_0, \ldots, y_n)$ of the tangent hyperplane is equal to the vector $A \cdot x$. Since A is invertible, we can write $x = A^{-1} \cdot y$. We have

$$0 = x \cdot A \cdot x = (y \cdot A^{-1}) \cdot A \cdot (A^{-1} \cdot y) = y \cdot A^{-1} \cdot y = 0.$$

Here, we treat x or y as a row-matrix or as a column-matrix in order the matrix multiplication makes sense. Since $A^{-1} = \det(A)^{-1}\mathrm{adj}(A)$, we obtain that the dual variety of Q is also a quadric given by the adjugate matrix $\mathrm{adj}(A)$.

The description of the tangent space of the discriminant hypersurface from Example 1.2.3 has the following nice application (see also Remark 1.1.33).

tangste

Proposition 1.2.5. *Let X be a hypersurface of degree d in $\mathbb{P}^n$. Suppose a is a nonsingular point of the Steinerian hypersurface* $\mathrm{St}(X)$. *Then,* $\mathrm{Sing}(P_a(X))$ *consists of an ordinary singular point b and*

$$\mathbb{T}_a(\mathrm{St}(X)) = P_{b^{d-1}}(X).$$

SS:1.2.3

1.2.3 Plücker formulas

Let $X = V(f)$ be a nonsingular irreducible hypersurface that is not a cone. Fix $n-1$ general points $a_1, \ldots, a_{n-1}$ in $\mathbb{P}^n$. Consider the intersection

$$X \cap P_{a_1}(X) \cap \ldots \cap P_{a_{n-1}}(X) = \{b \in \mathbb{P}^n : a_1, \ldots, a_{n-1} \in \mathbb{T}_b(X)\}.$$

The set of hyperplanes through a general set of $n-1$ points is a line in the dual space. This shows that

$$\deg X^\vee = \#X \cap P_{a_1}(X) \cap \ldots \cap P_{a_{n-1}}(X) = d(d-1)^{n-1}. \qquad (1.48)$$ degreedual

The computation does not apply to singular X since all polars $P_a(X)$ pass through singular points of X. In the case when X has only isolated singularities, the intersection of $n-1$ polars with X contains singular points which correspond to hyperplanes that we excluded from the definition of the dual hypersurface. So, we get the following formula:

$$\deg(X^\vee) = d(d-1)^{n-1} - \sum_{x \in \mathrm{Sing}(X)} i(X, P_{a_1}(X), \ldots, P_{a_{n-1}}(X))_x. \qquad (1.49)$$ genplucker

To state an explicit formula, we need some definition. Let $\underline{\phi} = (\phi_1, \ldots, \phi_k)$ be a set of polynomials in $\mathbb{C}[z_1, \ldots, z_n]$. We assume that the holomorphic map $\mathbb{C}^n \to \mathbb{C}^k$ defined by these polynomials has an isolated critical point at the origin. Let $J(\underline{\phi})$ be the jacobian matrix. The ideal $\mathcal{J}(\underline{\phi})$ in the ring of formal power series $\mathbb{C}[[z_1, \ldots, z_n]]$ generated by the maximal minors of the Jacobian matrix is called the *Jacobian ideal* of $\underline{\phi}$. The number

$$\mu(\underline{\phi}) = \dim \mathbb{C}[[z_1, \ldots, z_n]]/\mathcal{J}(\underline{\phi})$$

is called the *Milnor number* of $\underline{\phi}$. Passing to affine coordinates, this definition easily extends to the definition of the Milnor number $\mu(X,x)$ of an isolated singularity of a complete intersection subvariety X in $\mathbb{P}^n$.

We will need the following result of Lê Dũng Tráng Le [484], Theorem 3.7.1.

Le **Lemma 1.2.6.** *Let Z be a complete intersection in $\mathbb{C}^n$ defined by polynomials $\phi_1,\ldots,\phi_k$ with isolated singularity at the origin. Let $Z_1 = V(\phi_1,\ldots,\phi_{k-1})$. Then*

$$\mu(\phi_1,\ldots,\phi_{k-1}) + \mu(\phi_1,\ldots,\phi_{k-1},\phi_k)$$

$$= \dim \mathbb{C}[[z_1,\ldots,z_n]]/(\phi_1,\ldots,\phi_{k-1},\mathcal{J}(\phi_1,\ldots,\phi_k)).$$

Now, we can state and prove the *Plücker–Teissier formula* for a hypersurface with isolated singularities:

teissier **Theorem 1.2.7.** *Let X be a hypersurface in $\mathbb{P}^n$ of degree d. Suppose X has only isolated singularities. For any point $x \in \mathrm{Sing}(X)$, let*

$$e(X,x) = \mu(X,x) + \mu(H \cap X, x),$$

where H is a general hyperplane section of X containing x.Then,

$$\deg X^\vee = d(d-1)^{n-1} - \sum_{x \in \mathrm{Sing}(X)} e(X,x).$$

.

Proof We have to show that $e(X,x) = i(X, P_{a_1}(X),\ldots,P_{a_{n-1}}(X))_x$. We may assume that $x = [1,0,\ldots,0]$ and choose affine coordinates with $z_i = t_i/t_0$. Let $f(t_0,\ldots,t_n) = t_0^d g(z_1,\ldots,z_n)$. Easy calculations employing the Chain Rule, give the formula for the dehomogenized partial derivatives

$$x_0^{-d}\frac{\partial f}{\partial t_0} = dg + \sum \frac{\partial g}{\partial z_i} z_i,$$
$$x_0^{-d}\frac{\partial f}{\partial t_i} = \frac{\partial g}{\partial z_i},\ i = 1,\ldots,n.$$

Let $H = V(h)$ be a general hyperplane spanned by $n-1$ general points $a_1,\ldots,a_{n-1}$, and $h : \mathbb{C}^n \to \mathbb{C}$ be the projection defined by the linear function $h = \sum \alpha_i z_i$. Let

$$F : \mathbb{C}^n \to \mathbb{C}^2, \quad z = (z_1,\ldots,z_n) \mapsto (g(z), h(z)).$$

Consider the Jacobian determinant of the two functions (g,h)

$$J(g,h) = \begin{pmatrix} \frac{\partial g}{\partial z_1} & \ldots & \frac{\partial g}{\partial z_n} \\ \alpha_1 & \ldots & \alpha_n \end{pmatrix}.$$

The ideal $(g, J(g, h))$ defines the set of critical points of the restriction of the map F to $X \setminus V(t_0)$. We have

$$(g, J(g, h)) = (g, \alpha_i \frac{\partial g}{\partial z_j} - \alpha_j \frac{\partial g}{\partial z_i})_{1 \le i < j \le n},$$

The points $(0, \ldots, 0, \alpha_j, 0, \ldots, 0, -\alpha_i, 0, \ldots, 0)$ span the hyperplane H. We may assume that these points are our points $a_1, \ldots, a_{n-1}$. So, we see that $(g, J(g, h))$ coincides with the ideal in the completion of local ring $\mathcal{O}_{\mathbb{P}^n, x}$ generated by f and the polars $P_{a_i}(f)$. By definition of the index of intersection, we have

$$i(X, P_{a_1}(X), \ldots, P_{a_{n-1}}(X))_x = \mu(g, h).$$

It remains to apply Lemma 1.2.6, where $Z = V(g)$ and $Z_1 = V(g) \cap V(h)$. □

ex:1.2.8 *Example* 1.2.8. An isolated singular point x of a hypersurface X in $\mathbb{P}^n$ is called an A_k*-singularity* (or a singular point of type A_k) if the formal completion of $\mathcal{O}_{X,x}$ is isomorphic to $\mathbb{C}[[z_1, \ldots, z_n]]/(z_1^{k+1} + z_2^2 + \cdots + z_n^2)$. If $k = 1$, it is an ordinary quadratic singularity (or a *node*), if $k = 2$, it is an *ordinary cusp*. We get

$$\mu(X, x) = k, \ \mu(X \cap H, x) = 1.$$

This gives the Plücker formula for hypersurfaces with s singularities of type $A_{k_1}, \ldots, A_{k_s}$

$$\deg X^\vee = d(d-1)^{n-1} - (k_1 + 1) - \cdots - (k_s + 1). \tag{1.50}$$ plucker1

In particular, when X is a plane curve C with δ nodes and κ ordinary cusps, we get a familiar *Plücker formula*

$$\deg C^\vee = d(d-1) - 2\delta - 3\kappa. \tag{1.51}$$ plucker

Note that, in case of plane curves, $\mu(H \cap X, x)$ is always equal to $\mathrm{mult}_x X - 1$, where $\mathrm{mult}_x X$ is the multiplicity of X at x.

$$\deg C^\vee = d(d-1) - \sum_{x \in \mathrm{Sing}(X)} (\mu(X, x) + \mathrm{mult}_x X - 1). \tag{1.52}$$ pluckercurves

Note that the dual curve $C^\vee$ of a nonsingular curve C of degree $d > 2$ is always singular. This follows from the formula for the genus of a nonsingular plane curve a nd the fact that C and $C^\vee$ are birationally isomorphic. The polar map $C \to C^\vee$ is equal to the normalization map. A singular point of $C^\vee$ corresponds to a line which is either tangent to C at several points, or is an inflection tangent. We skip a local computation which shows that a line which is an inflection tangent at one point with ordfl = 1 (an *honest inflection tangent*) gives an ordinary cusp of $C^\vee$ and a line which is tangent at two points which are not inflection points (*honest bitangent*) gives a node. Thus, we obtain that the number $\check{\delta}$ of nodes of $C^\vee$ is equal to the number of honest bitangents of C and the number $\check{\kappa}$ of ordinary cusps of $C^\vee$ is equal to the number of honest inflection tangents to $C^\vee$.

Assume that C is nonsingular and $C^\vee$ has no other singular points except ordinary nodes and cusps. We know that the number of inflection points is equal to $3d(d-2)$. Applying Plücker formula (1.51) to $C^\vee$, we get that

$$\check{\delta} = \frac{1}{2}\big(d(d-1)(d(d-1)-1) - d - 9d(d-2)\big) = \frac{1}{2}d(d-2)(d^2-9). \quad (1.53)$$

This is the (expected) number of bitangents of a nonsingular plane curve. For example, we expect that a nonsingular plane quartic has 28 bitangents.

We refer for discussions of Plücker formulas to many modern text-books (e.g. [303], [315], [360], [325]). A proof of Plücker–Teissiere formula can be found in [743]. A generalization of the Plücker–Teissier formula to complete intersections in projective space was given by S. Kleiman [451]

1.3 Polar s-Hedra

1.3.1 Apolar schemes

We continue to use E to denote a complex vector space of dimension $n+1$. Consider the polarization pairing (1.2)

$$S^d(E^\vee) \times S^k(E) \to S^{d-k}(E^\vee), \quad (f, \psi) \mapsto D_\psi(f).$$

Definition 1.3.1. *$\psi \in S^k(E)$ is called* apolar *to $f \in S^d(E^\vee)$ if $D_\psi(f) = 0$. We extend this definition to hypersurfaces in the obvious way.*

Lemma 1.3.2. *For any $\psi \in S^k(E), \psi' \in S^m(E)$ and $f \in S^d(E^\vee)$,*

$$D_{\psi'}(D_\psi(f)) = D_{\psi\psi'}(f).$$

Proof By linearity and induction on the degree, it suffices to verify the assertion in the case when $\psi = \partial_i$ and $\psi' = \partial_j$. In this case, it is obvious. □

Corollary 1.3.3. *Let $f \in S^d(E^\vee)$. Let $\mathrm{AP}_k(f)$ be the subspace of $S^k(E)$ spanned by forms of degree k apolar to f. Then,*

$$\mathrm{AP}(f) = \bigoplus_{k=0}^{\infty} AP_k(f)$$

is a homogeneous ideal in the symmetric algebra $\mathbf{S}(E)$.

Definition 1.3.4. *The quotient ring*

$$A_f = \mathbf{S}(E)/\mathrm{AP}(f)$$

is called the apolar ring *of* f.

The ring A_f inherits the grading of $\mathbf{S}(E)$. Since any polynomial $\psi \in S^r(E)$ with $r > d$ is apolar to f, we see that A_f is annihilated by the ideal $\mathfrak{m}_+^{d+1} = (\partial_0, \dots, \partial_n)^{d+1}$. Thus, A_f is an Artinian graded local algebra over $\mathbb{C}$. Since the pairing between $S^d(E)$ and $S^d(E^\vee)$ has values in $S^0(E^\vee) = \mathbb{C}$, we see that $\mathrm{AP}_d(f)$ is of codimension 1 in $S^d(E)$. Thus, $(A_f)_d$ is a vector space of dimension 1 over $\mathbb{C}$ and coincides with the *socle* of A_f, i.e., the ideal of elements of A_f annihilated by its maximal ideal.

Note that the latter property characterizes Gorenstein graded local Artinian rings, see [281], [422].

Proposition 1.3.5 (F. S. Macaulay)**.** *The correspondence* $f \mapsto A_f$ *is a bijection between* $|S^d(E^\vee)|$ *and graded Artinian quotient algebras* $\mathbf{S}(E)/J$ *with 1-dimensional socle.*

Proof Let us show how to reconstruct $\mathbb{C}f$ from $\mathbf{S}(E)/J$. The multiplication of d vectors in E composed with the projection to $S^d(E)/J_d$ defines a linear map $S^d(E) \to S^d(E)/J_d \cong \mathbb{C}$. Choosing a basis $(\mathbf{S}(E)/J)_d$, we obtain a linear function f on $S^d(E)$. It corresponds to an element of $S^d(E^\vee)$. □

Recall that any closed non-empty subscheme $Z \subset \mathbb{P}^n$ is defined by a unique saturated homogeneous ideal I_Z in $\mathbb{C}[t_0, \dots, t_n]$. Its locus of zeros in the affine space $\mathbb{A}^{n+1}$ is the affine cone C_Z of Z isomorphic to $\mathrm{Spec}(\mathbb{C}[t_0, \dots, t_n]/I_Z)$.

Definition 1.3.6. *Let* $f \in S^d(E^\vee)$. *A subscheme* $Z \subset |E^\vee| = \mathbb{P}(E)$ *is called* apolar to f if its homogeneous ideal I_Z is contained in $\mathrm{AP}(f)$, or, equivalently, $\mathrm{Spec}(A_f)$ is a closed subscheme of the affine cone C_Z of Z.

This definition agrees with the definition of an apolar homogeneous form ψ. A homogeneous form $\psi \in S^k(E)$ is apolar to f if and only if the hypersurface $V(\psi)$ is apolar to $V(f)$.

Consider the natural pairing

$$(A_f)_k \times (A_f)_{d-k} \to (A_f)_d \cong \mathbb{C} \tag{1.54}$$

defined by multiplication of polynomials. It is well defined because of Lemma 1.3.2. The left kernel of this pairing consists of $\psi \in S^k(E) \mod \mathrm{AP}(f) \cap S^k(E)$ such that $D_{\psi\psi'}(f) = 0$ for all $\psi' \in S^{d-k}(E)$. By Lemma 1.3.2,

$D_{\psi\psi'}(f) = D_{\psi'}(D_\psi(f)) = 0$ for all $\psi' \in S^{d-k}(E)$. This implies $D_\psi(f) = 0$. Thus, $\psi \in \mathrm{AP}(f)$, and hence, is zero in A_f. This shows that the pairing (pairing3 1.54) is a perfect pairing. This is one of the nice features of a Gorenstein Artinian algebra (see Eisenbud [281], 21.2).

It follows that the Hilbert polynomial

$$H_{A_f}(t) = \sum_{i=0}^{d} \dim(A_f)_i t^i = a_d t^d + \cdots + a_0$$

is a reciprocal monic polynomial, i.e., $a_i = a_{d-i}, a_d = 1$. It is an important invariant of a homogeneous form f.

E1.3.1 *Example* 1.3.7. Let $f = l^d$ be the d-th power of a linear form $l \in E^\vee$. For any $\psi \in S^k(E) = (S^k(E)^\vee)^\vee$ we have

$$D_\psi(l^d) = d(d-1)\cdots(d-k+1)l^{d-k}\psi(l) = d!l^{[d-k]}\psi(l),$$

where we set

$$l^{[i]} = \begin{cases} \frac{1}{i!}l^i & \text{if } k \le d, \\ 0 & \text{otherwise.} \end{cases}$$

Here, we view $\psi \in S^d(E)$ as a homogeneous form on $E^\vee$. In coordinates, $l = \sum_{i=0}^n a_i t_i, \psi = \psi(\partial_0, \ldots, \partial_n)$ and $\psi(l) = d!\psi(a_0, \ldots, a_n)$. Thus, we see that $AP_k(f), k \le d$, consists of polynomials of degree k vanishing at l. Assume, for simplicity, that $l = t_0$. The ideal $AP(t_0^d)$ is generated by $\partial_1, \ldots, \partial_n, \partial_0^{d+1}$. The Hilbert polynomial is equal to $1 + t + \cdots + t^d$.

1.3.2

1.3.2 Sums of powers

For any point $a \in |E^\vee|$, we continue to denote by H_a the corresponding hyperplane in $|E|$.

Suppose $f \in S^d(E^\vee)$ is equal to a sum of powers of nonzero linear forms

$$f = l_1^d + \cdots + l_s^d. \tag{1.55}$$ sum2

This implies that, for any $\psi \in S^k(E)$,

$$D_\psi(f) = D_\psi(\sum_{i=1}^s l_i^d) = \sum_{i=1}^s \psi(l_i) l_i^{[d-k]}. \tag{1.56}$$ apol

In particular, taking $d = k$, we obtain that

$$\langle l_1^d, \ldots, l_s^d \rangle_{S^d(E)}^\perp = \{\psi \in S^d(E) : \psi(l_i) = 0, i = 1, \ldots, s\} = (I_Z)_d,$$

where Z is the closed reduced subscheme of points $\{[l_1], \ldots, [l_s]\} \subset |E^\vee|$ corresponding to the linear forms l_i, and I_Z denotes its homogeneous ideal.

This implies that the codimension of the linear span $\langle l_1^d, \dots, l_s^d \rangle$ in $S^d(E^\vee)$ is equal to the dimension of $(I_Z)_d$, hence the forms $l_1^d, \dots, l_s^d$ are linearly independent if and only if the points $[l_1], \dots, [l_s]$ impose independent conditions on hypersurfaces of degree d in $\mathbb{P}(E) = |E^\vee|$.

Suppose $f \in \langle l_1^d, \dots, l_s^d \rangle$, then $(I_Z)_d \subset \mathrm{AP}_d(f)$. Conversely, if this is true, we have

$$f \in \mathrm{AP}_d(f)^\perp \subset (I_Z)_d^\perp = \langle l_1^d, \dots, l_s^d \rangle.$$

If we additionally assume that $(I_{Z'})_d \not\subset \mathrm{AP}_d(f)$ for any proper subset Z' of Z, we obtain, after replacing the forms $l_i's$ by proportional ones, that

$$f = l_1^d + \cdots + l_s^d.$$

Definition 1.3.8. *A* polar s-hedron *of f is a set of hyperplanes $H_i = V(l_i), i = 1, \dots, s$, in $|E|$ such that*

$$f = l_1^d + \cdots + l_s^d,$$

and, considered as points $[l_i]$ in $\mathbb{P}(E)$, the hyperplanes H_i impose independent conditions in the linear system $|\mathcal{O}_{\mathbb{P}(E)}(d)|$. A polar s-hedron is called nondegenerate *if the hyperplanes $V(l_i)$ are in general linear position (i.e., no $n+1$ hyperplanes intersect).*

Note that this definition does not depend on the choice of linear forms defining the hyperplanes. Also, it does not depend on the choice of the equation defining the hypersurface $V(f)$. We can also view a polar s-hedron as an unordered set of points in the dual space. In the case $n = 2$, it is often called a *polar s-gon*, although this terminology is somewhat confusing since a polygon comes with an order of its set of vertices.. Also, in dimension two, we can employ the terminology of *s-laterals*.

The following propositions follow from the discussion above.

apollemma **Proposition 1.3.9.** *Let $f \in S^d(E^\vee)$. Then, $Z = \{[l_1], \dots, [l_s]\}$ is a polar s-hedron of f if and only if the following properties are satisfied*

(i) $I_Z(d) \subset \mathrm{AP}_d(f)$*;*

(ii) $I_{Z'}(d) \not\subset \mathrm{AP}_d(f)$ *for any proper subset Z' of Z.*

prop1 **Proposition 1.3.10.** *A set $Z = \{[l_1], \dots, [l_s]\}$ is a polar s-hedron of $f \in S^d(E^\vee)$ if and only if Z, considered as a closed subscheme of $|E^\vee|$, is apolar to f but no proper subscheme of Z is apolar to f.*

SS:1.3.3

1.3.3 Generalized polar s-hedra

Proposition 2.45 allows one to generalize the definition of a polar s-hedron. A polar s-hedron can be viewed as a reduced closed subscheme Z of $\mathbb{P}(E) = |E|^\vee$ consisting of s points. Obviously,

$$h^0(\mathcal{O}_Z) = \dim H^0(\mathbb{P}(E), \mathcal{O}_Z) = s.$$

More generally, we may consider non-reduced closed subschemes Z of $\mathbb{P}(E)$ of dimension 0 satisfying $h^0(\mathcal{O}_Z) = s$. The set of such subschemes is parameterized by a projective algebraic variety $\mathrm{Hilb}^s(\mathbb{P}(E))$ called the *punctual Hilbert scheme* of $\mathbb{P}(E)$ of 0-cycles of length s.

Any $Z \in \mathrm{Hilb}^s(\mathbb{P}(E))$ defines the subspace

$$I_Z(d) = \mathbb{P}(H^0(\mathbb{P}(E), \mathcal{I}_Z(d))) \subset H^0(\mathbb{P}(E), \mathcal{O}_{\mathbb{P}(E)}(d)) = S^d(E).$$

The exact sequence

$$0 \to H^0(\mathbb{P}(E), \mathcal{I}_Z(d)) \to H^0(\mathbb{P}(E), \mathcal{O}_{\mathbb{P}(E)}(d)) \to H^0(\mathbb{P}(E), \mathcal{O}_Z) \quad (1.57)$$
$$\to H^1(\mathbb{P}(E), \mathcal{I}_Z(d)) \to 0$$

exseq

shows that the dimension of the subspace

$$\langle Z \rangle_d = \mathbb{P}(H^0(\mathbb{P}(E), \mathcal{I}_Z(d))^\perp) \subset |S^d(E^\vee)| \quad (1.58)$$

spanz

is equal to $h^0(\mathcal{O}_Z) - h^1(\mathcal{I}_Z(d)) - 1 = s - 1 - h^1(\mathcal{I}_Z(d))$. If Z is reduced and consists of points $p_1, \ldots, p_s$, then $\langle Z \rangle_d = \langle v_d(p_1), \ldots, v_d(p_s) \rangle$, where $v_d : \mathbb{P}(E) \to \mathbb{P}(S^d(E))$ is the Veronese map. Hence, $\dim\langle Z \rangle_d = s - 1$ if the points $v_d(p_1), \ldots, v_d(p_s)$ are linearly independent. We say that Z is *linearly d-independent* if $\dim\langle Z \rangle_d = s - 1$.

Definition 1.3.11. *A* generalized s-hedron *of $f \in S^d(E^\vee)$ is a linearly d-independent subscheme $Z \in \mathrm{Hilb}^s(\mathbb{P}(E))$, which is apolar to f.*

Recall that Z is apolar to f if, for each $k \geq 0$,

$$I_Z(k) = H^0(\mathbb{P}(E), \mathcal{I}_Z(k)) \subset \mathrm{AP}_k(f). \quad (1.59)$$

apolarz

According to this definition, a polar s-hedron is a reduced generalized s-hedron. The following is a generalization of Proposition 1.3.9.

mainlemma

Proposition 1.3.12. *A linearly d-independent subscheme $Z \in \mathrm{Hilb}^s(\mathbb{P}(E))$ is a generalized polar s-hedron of $f \in S^d(E^\vee)$ if and only if*

$$I_Z(d) \subset \mathrm{AP}_d(f).$$

Proof We have to show that the inclusion in the assertion implies $I_Z(d) \subset \mathrm{AP}_k(f)$ for any $k \le d$. For any $\psi' \in S^{d-k}(E)$ and any $\psi \in I_Z(k)$, the product $\psi\psi'$ belongs to $I_Z(k)$. Thus, $D_{\psi\psi'}(f) = 0$. By the duality, $D_\psi(f) = 0$, i.e., $\psi \in \mathrm{AP}_k(f)$. □

kanev *Example* 1.3.13. Let $Z \in \mathrm{Hilb}^s(\mathbb{P}(E))$ be the union of k *fat points* p_k, i.e., at each $p_i \in Z$ the ideal $\mathcal{I}_{Z,p_i}$ is equal to the m_i-th power of the maximal ideal. Obviously,

$$s = \sum_{i=1}^{k} \binom{n+m_i-1}{m_i-1}.$$

Then, the linear system $|\mathcal{I}_Z(d)|$ consists of hypersurfaces of degree d with points p_i of multiplicity $\ge m_i$. One can show (see Iarrobino [422], Theorem 5.3) that Z is apolar to f if and only if

$$f = l_1^{d-m_1+1} g_1 + \cdots + l_k^{d-m_k+1} g_k,$$

where $p_i = V(l_i)$ and g_i is a homogeneous polynomial of degree $m_i - 1$ or the zero polynomial.

Remark 1.3.14. It is not known whether the set of generalized s-hedra of f is a closed subset of $\mathrm{Hilb}^s(\mathbb{P}(E))$. It is known to be true for $s \le d+1$ since in this case $\dim I_Z(d) = t := \dim S^d(E) - s$ for all $Z \in \mathrm{Hilb}^s(\mathbb{P}(E))$ (see Iarrobino [422], p.48). This defines a regular map of $\mathrm{Hilb}^s(\mathbb{P}(E))$ to the Grassmannian $G_{t-1}(|S^d(E)|)$ and the set of generalized s-hedra equal to the pre-image of a closed subset consisting of subspaces contained in $\mathrm{AP}_d(f)$. Also, we see that $h^1(\mathcal{I}_Z(d)) = 0$, hence, Z is always linearly d-independent.

SS:1.3.4 1.3.4 Secant varieties and sums of powers

Consider the Veronese map of degree d

$$\mathsf{v}_d : |E| \to |S^d(E)| = |S^d(E^\vee)^\vee|, \quad [v] \mapsto [v^d],$$

defined by the complete linear system $|S^d(E^\vee)|$ of hypersurfaces of degree d in $|E|$. The image of this map is the *Veronese variety* V_d^n of dimension n and degree d^n. It is isomorphic to $\mathbb{P}^n$. By choosing a monomial basis $\mathbf{t}^{\mathbf{i}}$ in the linear space of homogeneous polynomials of degree d we obtain that the Veronese variety is isomorphic to the subvariety of $\mathbb{P}^{\binom{n+d}{d}-1}$ given by equations

$$A_{\mathbf{i}} \cdot A_{\mathbf{j}} - A_{\mathbf{k}} A_{\mathbf{m}} = 0, \quad \mathbf{i}+\mathbf{j} = \mathbf{k}+\mathbf{m},$$

where $A_{\mathbf{i}}$ are dual coordinates in the space of polynomials of degree d. The image of $\mathbb{P}^n$ under the map defined by a choice of a basis of the complete linear system of hypersurfaces of degree d is called a n-dimensional Veronese variety of degree d^n.

normcurve *Example* 1.3.15. Let us specialize and discuss rational normal curves in $\mathbb{P}^n$, classically known as *norm-curves*. We will frequently use them. Let U be a two-dimensional linear space with a basis (e_0, e_1) and the dual basis (u_0, u_1), the coordinates in U. The space $S^d(U)$ (resp. $S^d(U^\vee)$) has a natural monomial basis $(e_0^d, e_0^{d-1}e_1, \ldots, e_1^d)$ (resp. $(u_0^d, u_0^{d-1}u_1, \ldots, u_1^d)$). The polarization isomorphism

$$S^d(U)^\vee \to S^d(U)^\vee$$

assigns to $u_0^{d-i}u_1^i$ the linear function on $S^d(U)$ that takes the value $\frac{1}{d!}(d-i)!i!$ on $e_0^{d-i}e_1^i$ and zero on all other monomials. This shows that the basis $(\binom{d}{i}u^{d-i}v^i)_{i=0,\ldots,d}$ is the dual basis of $(e_0^d, e_0^{d-1}e_1, \ldots, e_1^d)$. It is also a basis in $S^d(U^\vee)$. Thus, any *binary form* $b_d \in S^d(U^\vee)$ can be written as

$$b_d = \sum_{i=0}^{d} \binom{d}{i} a_i u_0^{d-i}u_1^i, \tag{1.60}$$ binary

so that $(a_0, \ldots, a_d)$ are the natural coordinates in the space $S^d(\mathbb{U}^\vee)$. Symbolically, we can write

$$b_d = (a_u)^d.$$

Let us clarify the coordinate-free definition of the Veronese map

$$\mathsf{v}_d : |U| \to |S^d(U)|.$$

It is defined by assigning to $\alpha e_o + \beta e_1 \in U$ the linear function $f \mapsto f(\alpha, \beta)$ on $S^d(U^\vee) = S^d(U)^\vee$. It follows that

$$\mathsf{v}_d([\alpha e_0 + \beta e_1]) = [\sum_{i=0}^{d} \binom{d}{i}\alpha^{d-i}\beta^i e_0^{d-i}e_1^i] = [(\alpha e_0 + \beta e_1)^d].$$

In coordinates, this is the map

$$(u_0, u_1) \mapsto (u_0^d, u_0^{d-1}u_1, \ldots, u_0 u_1^{d-1}, u_1^d). \tag{1.61}$$ par1

Passing to the projective space, we see that the Veronese map v_d is given by the complete linear system $|S^d(U)^\vee| = |\mathcal{O}_{|U|}(d)|$. The image $R_d \subset |S^d(U)|$ of this map is the *Veronese curve* of degree d, or the *rational normal curve* of degree d. Its image under any isomorphism $|S^d(U) \to \mathbb{P}^d$ is a Veronese curve or a rational normal curve. If we re-denote the coordinates $u_0^{d-i}u_1^i$ by $(x_0, \ldots, x_d)$, a hyperplane $V(\sum_{i=0}^d a_i x_i)$ intersects R_d along the closed subscheme isomorphic, under the Veronese map v_d, to the closed subscheme $V(\sum_{i=0}^d a_i u_0^{d-i}u_1^i)$ of $|U|$.

Dually, we have the Veronese map

$$\mathsf{v}_d^\vee : |U^\vee| \to |S^d(U^\vee), \quad [au_0 + bu_1] \to [(au_0 + bu_1)^d].$$

One can use the isomorphism $|U| \to |U^\vee|$ defined by the pairing $U \times U \to \bigwedge^2 U$ and a choice of an isomorphism $\bigwedge^2 U \to \mathbb{C}$. We make this choice by requiring that it sends $e_0 \wedge e_1$ to 1. In this case $(e_0, e_1) \mapsto (-u_1, u_0)$. Composing $^\vee_{\mathsf{d}}$ with the isomorphism $|U| \to |U^\vee|$, we obtain the *dual Veronese map*

$$\mathsf{v}_d : |U| \to |S^d(U^\vee)|. \tag{1.62}$$

dualveronese

The image of this map is the *dual Veronese curve*.

The dual Veronese map is given by

$$[\alpha, \beta] \mapsto [-\beta u_1 + \alpha u_0)^d].$$

Since the basis of $S^d(U^\vee)$ is formed by $\binom{d}{k} u_0^k u_1^{d-k}$, the map is given by

$$[\alpha, \beta] \mapsto [(-1)^d \beta^d, (-1)^{d-1} \beta^{d-1}\alpha, \ldots, \alpha^d].$$

The hyperplane in $|S^d(U)|$ corresponding to a point on the dual Veronese curve R_d^* cuts out R_d at one point $\mathsf{v}_d([\alpha e_0 + \beta e_1])$ with multiplicity d. We say that the dual Veronese curve is the locus of *d-osculating hyperplanes* of R_d (see the general definition of osculating hyperplanes in Section 10.4.

One can combine the Veronese map and the Segre map to define a *Segre-Veronese variety* $\mathsf{V}_{n_1,\ldots,n_k}(d_1, \ldots, d_k)$. It is equal to the image of the map $\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k}$ defined by the complete linear system $|\mathcal{O}_{\mathbb{P}^{n_1}}(d_1) \boxtimes \cdots \boxtimes \mathcal{O}_{\mathbb{P}^{n_k}}(d_k)|$.

The notion of a polar s-hedron acquires a simple geometric interpretation in terms of the secant varieties of the Veronese variety V_d^n. If a set of points $[l_1], \ldots, [l_s]$ in $|E|$ is a polar s-hedron of f, then $[f] \in \langle [l_1^d], \ldots, [l_s^d] \rangle$, and hence, $[f]$ belongs to the $(s-1)$-secant subspace of V_d^n. Conversely, a general point in this subspace admits a polar s-hedron. Recall that for any irreducible nondegenerate projective variety $X \subset \mathbb{P}^N$ of dimension r its *t-secant variety* $\mathrm{Sec}_t(X)$ is defined to be the Zariski closure of the set of points in $\mathbb{P}^N$ which lie in the linear span of dimension t of some set of $t + 1$ linearly independent points in X.

Counting constants, easily gives

$$\dim \mathrm{Sec}_t(X) \le \min(r(t+1) + t, N).$$

The subvariety $X \subset \mathbb{P}^N$ is called *t-defective* if the inequality is strict. An example of a 1-defective variety is a Veronese surface in $\mathbb{P}^5$.

A fundamental result about secant varieties is the following Lemma whose modern proof can be found in [814], Chapter II, and in [197]

Lemma 1.3.16 (A. Terracini)**.** *Let $p_1, \ldots, p_{t+1}$ be general $t+1$ points in X and p be a general point in their span. Then,*

$$\mathbb{T}_p(\mathrm{Sec}_t(X)) = \overline{\mathbb{T}_{p_1}(X), \ldots, \mathbb{T}_{p_{t+1}}(X)}.$$

The inclusion part

$$\overline{\mathbb{T}_{p_1}(X), \ldots, \mathbb{T}_{p_{t+1}}(X)} \subset \mathbb{T}_p(\mathrm{Sec}_t(X))$$

is easy to prove. We assume for simplicity that $t = 1$. Then, $\mathrm{Sec}_1(X)$ contains the cone $C(p_1, X)$ which is swept out by the lines $\overline{p_1 q}, q \in X$. Therefore, $\mathbb{T}_p(C(p_1, X)) \subset \mathbb{T}_p(\mathrm{Sec}_1(X))$. However, it is easy to see that $\mathbb{T}_p(C(p_1, X))$ contains $\mathbb{T}_{p_1}(X)$.

Corollary 1.3.17. $\mathrm{Sec}_t(X) \neq \mathbb{P}^N$ *if and only if, for any $t+1$ general points of X, there exists a hyperplane section of X singular at these points. In particular, if $N \leq r(t+1)+t$, the variety X is t-defective if and only if, for any $t+1$ general points of X, there exists a hyperplane section of X singular at these points.*

ex1.1.3 *Example* 1.3.18. Let $X = \mathsf{V}_d^n \subset \mathbb{P}^{\binom{d+n}{n}-1}$ be a Veronese variety. Assume $n(t+1)+t > \binom{d+n}{n}-1$. A hyperplane section of X is isomorphic to a hypersurface of degree d in $\mathbb{P}^n$. Thus, $\mathrm{Sec}_t(\mathsf{V}_d^n) \neq |S^d(E^\vee)|$ if and only if, for any $t+1$ general points in $\mathbb{P}^n$, there exists a hypersurface of degree d singular at these points.

Consider a Veronese curve $\mathsf{V}_d^1 \subset \mathbb{P}^d$. Assume $2t+1 \geq d$. Since $d < 2t+2$, there are no homogeneous forms of degree d which have $t+1$ multiple roots. Thus, the Veronese curve $R_d = \nu_d(\mathbb{P}^1) \subset \mathbb{P}^d$ is not t-degenerate for $t \geq (d-1)/2$.

Take $n = 2$ and $d = 2$. For any two points in $\mathbb{P}^2$ there exists a conic singular at these points, namely the double line through the points. This explains why a Veronese surface V_2^2 is 1-defective.

Another example is $\mathsf{V}_4^2 \subset \mathbb{P}^{14}$ and $t = 4$. The expected dimension of $\mathrm{Sec}_4(X)$ is equal to 14. For any five points in $\mathbb{P}^2$, there exists a conic passing through these points. Taking it with multiplicity 2, we obtain a quartic which is singular at these points. This shows that V_4^2 is 4-defective.

The following Corollary of Terracini's Lemma is called the *First Main Theorem on apolarity* in [277] (EhrenborgRota). The authors gave an algebraic proof of this Theorem without using Terracini's Lemma.

Corollary 1.3.19. *A general homogeneous form in $S^d(E^\vee)$ admits a polar s-hedron if and only if there exist linear forms $l_1, \ldots, l_s \in E^\vee$ such that, for any nonzero $\psi \in S^d(E)$, the ideal $AP(\psi) \subset \mathbf{S}(E^\vee)$ does not contain $\{l_1^{d-1}, \ldots, l_s^{d-1}\}$.*

Proof A general form in $S^d(E^\vee)$ admits a polar s-hedron if and only if the secant variety $\mathrm{Sec}_{s-1}(\mathsf{V}_d^n)$ is equal to the whole space. This means that the span of the tangent spaces at some points $q_i = V(l_i^d), i = 1, \ldots, s$, is equal to the whole space. By Terracini's Lemma, this is equivalent to that the tangent spaces of the Veronese variety at the points q_i are not contained in a hyperplane defined by some $\psi \in S^d(E) = S^d(E^\vee)^\vee$. It remains to use that the tangent space of the Veronese variety at q_i is equal to the projective space of all homogeneous forms $l_i^{d-1}l, l \in E^\vee \setminus \{0\}$ (see Exercise 1.18). Thus, for any nonzero $\psi \in S^d(E)$, it is impossible that $P_{l_i^{d-1}l}(\psi) = 0$ for all l and for all i. But $P_{l_i^{d-1}l}(\psi) = 0$ for all l if and only if $P_{l_i^{d-1}}(\psi) = 0$. This proves the assertion. □

The following fundamental result is due to J. Alexander and A. Hirschowitz [7]. A simplified proof can be found in [71] or [124].

Theorem 1.3.20. *If $d > 2$, the Veronese variety V_d^n is t-defective if and only if*

$$(n, d, t) = (2, 4, 4), (3, 4, 8), (4, 3, 6), (4, 4, 13).$$

In all these cases the secant variety $\mathrm{Sec}_t(\mathsf{V}_d^n)$ is a hypersurface. The Veronese variety V_2^n is t-defective only if $1 \le t \le n$. Its t-secant variety is of dimension $n(t+1) - \frac{1}{2}(t-2)(t+1) - 1$.

For the sufficiency of the condition, only the case $(4, 3, 6)$ is not trivial. It asserts that for 7 general points in $\mathbb{P}^3$ there exists a cubic hypersurface which is singular at these points. To see this, we use a well-known fact that any $n + 3$ general points in $\mathbb{P}^n$ lie on a Veronese curve of degree n (see, for example, [375], Theorem 1.18). So, we find such a curve R through 7 general points in $\mathbb{P}^4$ and consider the 1-secant variety $\mathrm{Sec}_1(R)$. It is a cubic hypersurface given by the catalecticant invariant of a binary quartic form. It contains the curve R as it singular locus.

Other cases are easy. We have seen already the first two cases. The third case follows from the existence of a quadric through nine general points in $\mathbb{P}^3$. The square of its equation defines a quartic with 9 points. The last case is similar. For any 14 general points, there exists a quadric in $\mathbb{P}^4$ containing these points. In the case of quadrics, we use that the variety of quadrics of corank r is of codimension $r(r + 1)/2$ in the variety of all quadrics.

Obviously, if $\dim \mathrm{Sec}_{s-1}(\mathsf{V}_d^n) < \dim |S^d E^\vee| = \binom{n+d}{n} - 1$, a general form $f \in S^d(E^\vee)$ cannot be written as a sum of s powers of linear forms. Since $\dim \mathrm{Sec}_{s-1}(\mathsf{V}_d^n) \le \min\{(n+1)s - 1, \binom{n+d}{n} - 1\}$, the minimal number $s(n, d)$ of powers needed to write f as a sum of powers of linear forms satisfies

$$s(n, d) \ge \left\lceil \frac{1}{n+1}\binom{n+d}{n} \right\rceil. \tag{1.63}$$

If V_d^n is not $(s-1)$-defective, then the equality holds. Applying Theorem 1.3.20, we obtain the following.

C1.3.11 **Corollary 1.3.21.**

$$s(n,d) = \left\lceil \frac{1}{n+1}\binom{n+d}{n} \right\rceil$$

unless $(n,d) = (n,2),(2,4),(3,4),(4,3),(4,4)$. *In these exceptional cases* $s(n,d) = n+1, 6, 10, 8, 15$ *instead of expected* $\lceil \frac{n-1}{2} \rceil, 5, 9, 8, 14$.

Remark 1.3.22. If $d > 2$, in all the exceptional cases listed in the previous corollary, $s(n,d)$ is larger by one than the expected number. The variety of forms of degree d that can be written as the sum of the expected number of powers of linear forms is a hypersurface in $|\mathcal{O}_{\mathbb{P}^n}(d)|$. In the case $(n,d,t) = (2,4,5)$, the hypersurface is of degree 6, and it is given by the catalecticant matrix which we will discuss later in this chapter. The curves parameterized by this hypersurface are Clebsch quartics which we will discuss in Chapter 6. The case $(n,d) = (4,3)$ was studied only recently in [564]. The hypersurface is of degree 15. In the other two cases, the equation expresses that the second partials of the quartic are linearly dependent (see [326], pp. 58-59.)

One can also consider the problem of a representation of several forms $f_1,\ldots,f_k \in S^d(E^\vee)$ as a sum of powers of the same set (up to proportionality) of linear forms $l_1,\ldots,l_s$. This means that the forms share a common polar s-hedron. For example, a well-known result from linear algebra states that two general quadratic forms q_1,q_2 in k variables can be simultaneously diagonalized. In our terminology, this means that they have a common polar k-hedron. More precisely, this is possible if the $\det(q_1+\lambda q_2)$ has $n+1$ distinct roots (we will discuss this later in Chapter 8 while studying del Pezzo surfaces of degree 4).

Suppose

$$f_j = \sum_{i=1}^{s} a_i^{(j)} l_i^d, \quad j = 1,\ldots,k. \tag{1.64}$$ toeplitz2

We view this as an element $\phi \in U^\vee \otimes S^d(E^\vee)$, where $U = \mathbb{C}^k$. The map ϕ is the sum of s linear maps ϕ of rank 1 with the images spanned by l_i^d. So, we can view each ϕ as a vector in $U^\vee \otimes S^d(E^\vee)$ equal to the image of a vector in $U^\vee \otimes E^\vee$ embedded in $U^\vee \otimes E^\vee$ by $u \otimes l \mapsto u \otimes l^d$. Now, everything becomes clear. We consider the Segre-Veronese embedding

$$|U^\vee| \times |E^\vee| \hookrightarrow |U^\vee| \times |S^d(E^\vee)| \hookrightarrow |U^\vee \otimes S^d(E^\vee)|$$

defined by the linear system of divisors of type $(1,d)$ and view $[\phi]$ as a point in the projective space $|U^\vee \otimes S^d(E^\vee)|$ which lies on the $(s-1)$-secant variety of $\mathsf{V}_{k-1,n}(1,d)$.

For any linear map $\phi \in \mathrm{Hom}(U, S^d(E^\vee))$, consider the linear map

$$\mathfrak{T}_\phi : \mathrm{Hom}(U,E) \to \mathrm{Hom}(\bigwedge^2 U, S^{d-1}(E^\vee)),$$

defined by

$$\mathfrak{T}_\phi(\alpha) : u \wedge v \mapsto D_{\alpha(u)}(\phi(v)) - D_{\alpha(v)}(\phi(u)).$$

We call this map the *Toeplitz map*. Suppose that ϕ is of rank 1 with the image spanned by l^d, then $\mathfrak{T}_\phi$ is of rank equal to $\dim \bigwedge^2 U - 1 = (k-2)(k+1)/2$. If we choose a basis $u_1, \ldots, u_k$ in U and coordinates $t_0, \ldots, t_n$ in E, then the image is spanned by $l^{d-1}(a_i u_i - a_j u_j)$, where $l = \sum a_i t_i$. This shows that, if ϕ belongs to $\mathrm{Sec}_{s-1}(|U^\vee| \times |E^\vee|)$,

$$\mathrm{rank}\, \mathfrak{T}_\phi \le s(k-2)(k+1)/2. \tag{1.65}$$ toeplitz1

The expected dimension of $\mathrm{Sec}_{s-1}(|U^\vee| \times |E^\vee|)$ is equal to $s(k+n)-1$. Thus, we expect that $\mathrm{Sec}_{s-1}(|U^\vee| \times |E^\vee|)$ coincides with $|U^\vee \otimes S^d(E^\vee)|$ when

$$s \ge \left\lceil \frac{k}{k+n}\binom{n+d}{n} \right\rceil. \tag{1.66}$$ toeplitz3

If this happens, we obtain that a general set of k forms admits a common polar s-hedron. Of course, as in the case $k=1$, there could be exceptions if the secant variety is $(s-1)$-defective.

E1.3.4 *Example* 1.3.23. Assume $d=2$ and $k=3$. In this case, the matrix of $\mathfrak{T}_\phi$ is a square matrix of size $3(n+1)$. Let us identify the spaces $U^\vee$ and $\bigwedge^2 U$ by means of the volume form $u_1 \wedge u_2 \wedge u_3 \in \bigwedge^3 U \cong \mathbb{C}$. Also identify $\phi(u_i) \in S^2(E^\vee)$ with a square symmetric matrix A_i of size $n+1$. Then, an easy computation shows that one can represent the linear map $\mathfrak{T}_\phi$ by the skew-symmetric matrix

$$\begin{pmatrix} 0 & A_1 & A_2 \\ -A_1 & 0 & A_3 \\ -A_2 & -A_3 & 0 \end{pmatrix}. \tag{1.67}$$ toeplitz

Now, the condition (1.65) for

$$s = \left\lceil \frac{k\binom{n+d}{n}}{k+n} \right\rceil = \left\lceil \frac{3(n+2)(n+1)}{2(n+3)} \right\rceil = \begin{cases} \frac{1}{2}(3n+2) & \text{if } n \text{ is even,} \\ \frac{1}{2}(3n+1) & \text{if } n \text{ is odd} \ge 3, \\ 3 & \text{if } n = 1 \end{cases}$$

becomes equivalent to the condition

$$\Lambda = \mathrm{Pf}\begin{pmatrix} 0 & A_1 & A_2 \\ -A_1 & 0 & A_3 \\ -A_2 & -A_3 & 0 \end{pmatrix} = 0. \tag{1.68}$$

It is known that the secant variety $\mathrm{Sec}_{s-1}(|U| \times |E|)$ of the Segre-Veronese variety is a hypersurface if $n \geq 3$ is odd and the whole space if n is even (see [729], Lemma 4.4). It implies that, in the odd case, the hypersurface is equal to $V(\Lambda)$. Its degree is equal to $3(n+1)/2$. Of course, in the even case, the pfaffian vanishes identically.

In the case $n = 3$, the pfaffian Λ was introduced by E. Toeplitz [749]. It is an invariant of the net† of quadrics in $\mathbb{P}^3$ that vanishes on the nets with common polar pentahedron. Following [334], we call Λ the *Toeplitz invariant*. Let us write its generators f_1, f_2, f_3 in the form (1.64) with $n = 3$ and $s = \frac{1}{2}(3n+1) = 5$. Since the four linear forms l_i are linearly dependent, we can normalize them by assuming that $l_1 + \cdots + l_5 = 0$ and assume that $l_1, \ldots, l_5$ span a 4-dimensional subspace. Consider a cubic form

$$F = \frac{1}{3}\sum_{i=1}^{5} l_i^3,$$

and find three vectors v_i in $\mathbb{C}^4$ such that

$$(l_1(v_j), \ldots, l_5(v_j)) = (a_1^{(j)}, \ldots, a_5^{(j)}), \quad j = 1, 2, 3.$$

Now, we check that $f_j = D_{v_j}(F)$ for $j = 1, 2, 3$. This shows that the net spanned by f_1, f_2, f_3 is a net of polar quadrics of the cubic F. Conversely, we will see later that any general cubic form in 4 variables admits a polar pentahedron. Thus, any net of polars of a general cubic surface admits a common polar pentahedron. So, the Toeplitz invariant vanishes on a general net of quadrics in $\mathbb{P}^3$ if and only if the net is realized as a net of polar quadrics of a cubic.

Remark 1.3.24. Let (n, d, k, s) denote the numbers for which we have strict inequality in (1.66). We call such 4-tuples exceptional. Examples of exceptional 4-tuples are $(n, 2, 3, \frac{1}{2}(3n+1))$ with odd $n \geq 2$. The secant hypersurfaces in these cases are given by the Toeplitz invariant Λ. The case $(3, 2, 3, 5)$ was first discovered by G. Darboux [200].‡ It has been rediscovered and extended to any odd n by G. Ottaviani [563]. There are other two known examples. The case $(2, 3, 2, 5)$ was discovered by F. London [492]. The secant variety

†We employ classical terminology calling a 1-dimensional (resp. 2-dimensional, resp. 3-dimensional) linear system a *pencil* (resp. a *net*, resp. a *web*).

‡Darboux claimed wrongly that the case $(3, 2, 4, 6)$ is exceptional, the mistake was pointed out by Terracini [744] without proof, a proof is in [90].

is a hypersurface given by the determinant of order 6 of the linear map $\mathfrak{T}_\phi$ (see Exercise 1.30). The examples $(3,2,5,6)$ and $(5,2,3,8)$ were discovered recently by E. Carlini and J. Chipalkatti [90]. The secant hypersurface in the second case is a hypersurface of degree 18 given by the determinant of $\mathfrak{T}_\phi$. There are no exceptional 4-tuples $(n,2,2,s)$ [90] and no exceptional 4-tuples (n,d,k,s) for large n (with some explicit bound)[1]. We refer to [131], where the varieties of common polar s-hedra are studied.

Remark 1.3.25. Assume that one of the matrices A_1,A_2,A_3 in (1.67) is invertible, say let it be A_2. Then,

$$\begin{pmatrix} I & 0 & 0\\ 0 & I & -A_1A_2^{-1}\\ 0 & 0 & I\end{pmatrix}\cdot\begin{pmatrix} 0 & A_1 & A_2\\ -A_1 & 0 & A_3\\ -A_2 & -A_3 & 0\end{pmatrix}\cdot\begin{pmatrix} I & 0 & 0\\ 0 & I & 0\\ 0 & -A_2^{-1}A_1 & I\end{pmatrix} = \begin{pmatrix} 0 & 0 & A_2\\ 0 & B & A_3\\ -A_2 & -A_3 & 0\end{pmatrix},$$

where

$$B = A_1A_2^{-1}A_3 - A_3A_2^{-1}A_1.$$

This shows that

$$\text{rank}\begin{pmatrix} 0 & A_1 & A_2\\ -A_1 & 0 & A_3\\ -A_2 & -A_3 & 0\end{pmatrix} = \text{rank}\, B + 2n + 2.$$

The condition $\text{rank}B \le 2$ is known in the theory of vector bundles over the projective plane as *Barth's condition* on the net of quadrics in $\mathbb{P}^n$. It does not depend on the choice of a basis of the net of quadrics spanned by the quadrics with matrices A_1,A_2,A_3. Under Barth's condition, the discriminant curve $\det(z_0A_1+z_1A_2+2A_3)=0$ of singular quadrics in the net is a *Darboux curve* of degree $n+1$ (see [32]).

1.3.5 The Waring problem

The well-known Waring problem in number theory asks about the smallest number $s(d)$ such that each natural number can be written as a sum of $s(d)$ d-th powers of natural numbers. It also asks in how many ways it can be done. Its polynomial analog asks about the smallest number $s(n,d)$ such that a general homogeneous polynomial of degree d in $n+1$ variables can be written as a sum of s d-th powers of linear forms. Corollary (1.3.21) solves this problem.

Other versions of the Waring problem ask the following questions:

- (W1) Given a homogeneous forms $f \in S^d(E^\vee)$, study the *variety of sums of powers* $\mathrm{VSP}(f,s)^o$, i.e., the subvariety of $\mathbb{P}(E)^{(s)}$ that consists of polar

s-hedra of f or, more general, the subvariety $\mathrm{VSP}(f,s)$ of $\mathrm{Hilb}^s(\mathbb{P}(E))$ parameterizing generalized polar s-hedra of f.

- (W2) Given s, find the equations of the closure $\mathrm{PS}(s,d;n)$ in $S^d(E^\vee)$ of the locus of homogeneous forms of degree d which can be written as a sum of s powers of linear forms.

We can also ask similar questions for several forms in $S^d(E^\vee)$.

Note that $\mathrm{PS}(s,d;n)$ is the affine cone over the secant variety $\mathrm{Sec}_{s-1}(\mathsf{V}_d^n)$. In the language of secant varieties, the variety $\mathrm{VSP}(f,s)^o$ is the set of linearly independent sets of s points $p_1,\ldots,p_s$ in V_d^n such that $[f]\in\langle p_1,\ldots,p_s\rangle$ and does not belong to the span of the proper subset of the set of these points. The variety $\mathrm{VSP}(f,s)$ is the set of linearly independent $Z\in\mathrm{Hilb}^s(\mathbb{P}(E))$ such that $[f]\in\langle Z\rangle$. Note that there is a natural map

$$\mathrm{VSP}(f,s)\to G(s,S^d(E)),\quad Z\mapsto\langle Z\rangle_d,$$

where $G(s,S^d(E))=G_{s-1}(|S^d(E)|)$ is the Grassmannian of s-dimensional subspaces of $S^d(E)$. This map is not injective in general.

Also, note that for a general form f, the variety $\mathrm{VSP}(f,s)$ is equal to the closure of $\mathrm{VSP}(f,s)^o$ in the Hilbert scheme $\mathrm{Hilb}^s(\mathbb{P}(E))$ (see [422], 7.2). It is not true for an arbitrary form f. One can also embed $\mathrm{VSP}(f;s)^o$ in $\mathbb{P}(S^d(E))$ by assigning to $\{l_1,\ldots,l_s\}$ the product $l_1\cdots l_s$. Thus, we can compactify $\mathrm{VSP}(f,s)^o$ by taking its closure in $\mathbb{P}(S^d(E))$. In general, this closure is not isomorphic to $\mathrm{VSP}(f,s)$.

Remark 1.3.26. If (d,n) is not one of the exceptional cases from Corollary 1.3.21 and $\binom{n+d}{d}=(n+1)s$ for some integer s, then a general form of degree d admits only finitely many polar s-hedra. How many? The known cases are given in the following table.

d	n	s	#	reference
2s-1	1	s	1	J. Sylvester [738]
5	2	7	1	D. Hilbert [393], H. Richmond [622] F. Palatini [569]
7	2	12	5	A. Dixon and T. Stuart [227], K. Ranestad and F.-O. Schreyer [607]
8	2	15	16	K. Ranestad and F.-O. Schreyer [607]
3	3	5	1	J. Sylvester [738]

The list contains three cases when there is a unique polar s-hedron. It is conjectured that there are no more of such cases. The evidence for this conjecture can be found in papers by M. Mella [510], [511], where it is proven that there are no new cases when $n = 2, d \geq 5$, and $n \geq 3$ and n divides $\binom{n+d}{n-1}$.

An explicit description of positive-dimensional varieties of sums of powers $\mathrm{VSP}(f, s)$ is known only in a few cases (d, n, s). We will discuss the cases $(d, n, s) = (2s-1, 1, s), (3, 3, 5)$ later. For other cases, see papers [424] $((d, n, s) = (3, 5, 10))$, [535], $((d, n, s) = (6, 2, 10))$, [235] $((d, n, s) = (3, 2, 4))$ and [607] $((d, n, s) = (3, 4, 8), (2, 3, 4), (6, 2, 10))$,

1.4 Dual Homogeneous Forms

1.4.1 Catalecticant matrices

Let $f \in S^d(E^\vee)$. Consider the linear map (the *apolarity map*)

$$\mathrm{ap}_f^k : S^k E \to S^{d-k}(E^\vee), \quad \psi \mapsto D_\psi(f). \tag{1.69}$$

Its kernel is the space $\mathrm{AP}_k(f)$ of forms of degree k which are apolar to f.

Assume that $f = \sum_{i=1}^s l_i^d$ for some $l_i \in E^\vee$. It follows from (1.56) that

$$\mathrm{ap}_f^k(S^k(E)) \subset \langle l_1^{d-k}, \ldots, l_s^{d-k} \rangle,$$

and hence,

$$\mathrm{rank}(\mathrm{ap}_f^k) \leq s. \tag{1.70}$$

Since any f can be written as a sum of s powers of linear forms if $2s > d$, we assume that

$$s \leq k \leq d - k.$$

If we choose a basis in E and a basis in $E^\vee$, then ap_f^k is given by a matrix of size $\binom{k+n}{k} \times \binom{n+d-k}{d-k}$ whose entries are linear forms in coefficients of f.

Choose a basis $\xi_0, \ldots, \xi_n$ in E and the dual basis $t_0, \ldots, t_n$ in $E^\vee$. Consider a monomial lexicographically ordered basis in $S^k(E)$ (resp. in $S^{d-k}(E^\vee)$). The matrix of ap_f^k with respect to these bases is called the k-th *catalecticant matrix* of f and is denoted by $\mathrm{Cat}_k(f)$. Its entries $c_{\mathbf{uv}}$ are parameterized by pairs $(\mathbf{u}, \mathbf{v}) \in \mathbb{N}^{n+1} \times \mathbb{N}^{n+1}$ with $|\mathbf{u}| = d - k$ and $|\mathbf{v}| = k$. If we write

$$f = \sum_{|\mathbf{i}|=d} \binom{d}{\mathbf{i}} a_{\mathbf{i}} \mathbf{t}^{\mathbf{i}},$$

then

$$c_{\mathbf{uv}} = a_{\mathbf{u}+\mathbf{v}}.$$

This follows easily from formula (1.5).

Considering $a_{\mathbf{i}}$ as independent variables $t_{\mathbf{i}}$, we obtain the definition of a general catalecticant matrix $\mathrm{Cat}_k(d, n)$.

ex:1.4.1 *Example* 1.4.1. Let $n = 1$. Write $f = \sum_{i=0}^{d} \binom{d}{i} a_i t_0^{d-i} t_1^i$. Then

$$\mathrm{Cat}_k(f) = \begin{pmatrix} a_0 & a_1 & \dots & a_k \\ a_1 & a_2 & \dots & a_{k+1} \\ \vdots & \vdots & \vdots & \vdots \\ a_{d-k} & a_{d-k+1} & \dots & a_d \end{pmatrix}.$$

A square matrix of this type is called a *circulant matrix*, or a *Hankel matrix*. It follows from (1.70) that $f \in \mathrm{PS}(s, d; 1)$ implies that all $(s+1)\times(s+1)$ minors of $\mathrm{Cat}_k(f)$ are equal to zero. Thus, we obtain that $\mathrm{Sec}_{s-1}(\mathsf{V}_d^1)$ is contained in the subvariety of $\mathbb{P}^d$ defined by $(s+1)\times(s+1)$-minors of the matrices

$$\mathrm{Cat}_k(d, 1) = \begin{pmatrix} t_0 & t_1 & \dots & t_k \\ t_1 & t_2 & \dots & t_{k+1} \\ \vdots & \vdots & \vdots & \vdots \\ t_{d-k} & t_{d-k+1} & \dots & t_d \end{pmatrix}.$$

For example, if we take $s = 1$, we obtain that the Veronese curve $\mathsf{V}_d^1 \subset \mathbb{P}^d$ satisfies the equations $t_a t_b - t_c t_d = 0$, where $a + b = c + d$. If we take, $k = 1$, we get the well-known equations of a rational normal curve

$$\mathrm{rank}(\begin{pmatrix} t_0 & t_1 & \cdots & a_{d-1} \\ t_1 & t_2 & \cdots & a_d \end{pmatrix} = 1, \qquad (1.71)$$ eqnratnormal

(see, for example, [375]). As is easy to see, if we take we take $k > 1$, we get the same equations.

Assume $d = 2k$. Then, the Hankel matrix is a square matrix of size $k + 1$. Its determinant vanishes if and only if f admits a nonzero apolar form of degree k. The set of such f's is a hypersurface in the space of binary forms of degree $2k$. It contains the Zariski open subset of forms which can be written as a sum of k powers of linear forms (see section 1.60).

For example, take $k = 2$. Then, the equation

$$\det \begin{pmatrix} a_0 & a_1 & a_2 \\ a_1 & a_2 & a_3 \\ a_2 & a_3 & a_4 \end{pmatrix} = 0 \qquad (1.72)$$ hankelq

describes binary quartics

$$f = a_0 t_0^4 + 4a_1 t_0^3 t_1 + 6a_2 t_0^2 t_1^2 + 4a_3 t_0 t_1^3 + a_4 t_1^4$$

which lie in the Zariski closure of the locus of quartics represented in the form $(\alpha_0 t_0 + \beta_0 t_1)^4 + (\alpha_1 t_0 + \beta_1 t_1)^4$. Note that a quartic of this form has simple roots unless it has a root of multiplicity 4. Thus, any binary quartic with simple roots satisfying equation (1.72) can be represented as a sum of two powers of linear forms.

The determinant (1.72) is an invariant of a binary quartic. The cubic hypersurface in $\mathbb{P}^4$ defined by equation (1.72) is equal to the 1-secant variety of a rational normal curve R_4 in $\mathbb{P}^4$.

Note that

$$\dim \mathrm{AP}_i(f) = \dim \mathrm{Ker}(\mathrm{ap}^i_f) = \binom{n+i}{i} - \mathrm{rank}\ \mathrm{Cat}_i(f).$$

Therefore,

$$\dim(A_f)_i = \mathrm{rank}\ \mathrm{Cat}_i(f),$$

and

$$H_{A_f}(t) = \sum_{i=0}^{d} \mathrm{rank}\ \mathrm{Cat}_i(f) t^i. \tag{1.73}$$

Since the ranks of ap^i_f and its transpose are the same, we obtain

$$\mathrm{rank}\ \mathrm{Cat}_i(f) = \mathrm{rank}\ \mathrm{Cat}_{d-i}(f)$$

confirming that $H_{A_f}(t)$ is a reciprocal monic polynomial.

Suppose $d = 2k$ is even. Then, the coefficient at t^k in $H_{A_f}(t)$ is equal to the rank of $\mathrm{Cat}_k(f)$. The matrix $\mathrm{Cat}_k(f)$ is a square matrix of size $\binom{n+k}{k}$. One can show that for a general f, this matrix is invertible. A polynomial f is called *degenerate* if $\det(\mathrm{Cat}_k(f)) = 0$. It is called *nondegenerate* otherwise. Thus, the set of degenerate polynomials is a hypersurface (*catalecticant hypersurface*) given by the equation

$$\det(\mathrm{Cat}_k(2k, n)) = 0. \tag{1.74}$$

The polynomial $\det(\mathrm{Cat}_k(2k, n))$ in variables $t_{\mathbf{i}}$, $|\mathbf{i}| = d$, is called the *catalecticant determinant*.

Example 1.4.2. Let $d = 2$. It is easy to see that the catalecticant polynomial is the discriminant polynomial. Thus, a quadratic form is degenerate if and only if it is degenerate in the usual sense. The Hilbert polynomial of a quadratic form f is

$$H_{A_f}(t) = 1 + rt + t^2,$$

where r is the rank of the quadratic form.

Example 1.4.3. Suppose $f = t_0^d + \cdots + t_s^d, s \le n$. Then, $t_0^i, \ldots, t_s^i$ are linearly independent for any i, and hence, rank $\mathrm{Cat}_i(f) = s$ for $0 < i < d$. This shows that

$$H_{A_f}(t) = 1 + s(t + \cdots + t^{d-1}) + t^d.$$

Let $\mathcal{P}$ be the set of reciprocal monic polynomials of degree d. One can stratify the space $S^d(E^\vee)$ by setting, for any $p \in \mathcal{P}$,

$$S^d(E^\vee)_p = \{f \in S^d(E^\vee) : H_{A_f} = p\}.$$

If $f \in \mathrm{PS}(s, d; n)$ we know that

$$\text{rank } \mathrm{Cat}_k(f) \le h(s, d, n)_k = \min(s, \tbinom{n+k}{n}, \tbinom{n+d-k}{n}).$$

One can show that, for a general enough f, the equality holds (see Iarrobino [422], p.13). Thus, there is a Zariski open subset of $\mathrm{PS}(s, d; n)$ which is contained in the strata $S^d(E^\vee)_p$, where $p = \sum_{i=0}^d h(s, d, n)_i t^i$.

1.4.2 Dual homogeneous forms

SS:1.4.2

In Subsection SS:1.2.2 1.2.2, we introduced the introduced the notion of the dual variety of a hypersurface, and, in particular, the dual quadric. Using the notion of the catalecticant matrix, for any homogeneous form of even degree $f \in S^{2k}(E^\vee)$, in a similar fashion one can define the dual homogeneous form $f^\vee \in S^{2k}(E)$. If the degree is greater than two, the hypersurface $V(f)$ differs from the dual hypersurface $V(f)^\vee$.

Consider the pairing

$$\Omega_f : S^k(E) \times S^k(E) \to \mathbb{C}, \tag{1.75}$$ conjj

defined by

$$\Omega_f(\psi_1, \psi_2) = \mathrm{ap}_f^k(\psi_1)(\psi_2) = D_{\psi_2}(\mathrm{ap}_f^k(\psi_1)) = D_{\psi_1\psi_2}(f),$$

where we identify the spaces $S^k(E^\vee)$ and $S^k(E)^\vee$. The pairing can be considered as a symmetric bilinear form on $S^k(E)$. Its matrix with respect to a monomial basis in $S^k(E)$ and its dual monomial basis in $S^k(E^\vee)$ is the catalecticant matrix $\mathrm{Cat}_k(f)$.

Let us identify Ω_f with the associated quadratic form on $S^k(E)$ (the restriction of Ω_f to the diagonal). This defines a linear map

$$\Omega : S^{2k}(E^\vee) \to S^2(S^k(E)^\vee), \quad f \mapsto \Omega_f.$$

There is also the natural left inverse map of Ω

$$P : S^2(S^k(E)^\vee) \to S^{2k}(E^\vee)$$

defined by multiplication $S^k(E^\vee) \times S^k(E^\vee) \to S^{2k}(E^\vee)$. All these maps are $\mathrm{GL}(E)$-equivariant and realize the linear representation $S^{2k}(E^\vee)$ as a direct summand in the representation $S^2(S^k(E^\vee))$.

Definition 1.4.4. *Assume that $f \in S^{2k}(E^\vee)$ is nondegenerate. The dual quadratic form $\Omega_f^\vee$ of Ω_f is called the* dual homogeneous form *of f. We will identify it with the polar bilinear form on S^kV.*

Remark 1.4.5. Note that, contrary to the assertion of Theorem 2.3 in [243], $\Omega_f^\vee$ is not equal, in general, to $\Omega_{f^\vee}$ for some $f^\vee \in S^{2k}(V)$. We thank Bart van den Dries for pointing out that the adjugate matrix of the catelecticant matrix is not, in general, a catalecticant matrix as was wrongly asserted in the proof.

Recall that the locus of zeros of a quadratic from $q \in S^2(E^\vee)$ consists of vectors $v \in E$ such that the value of the polarized bilinear form $b_q : E \to E^\vee$ at v vanishes at v. Dually, the set of zeros of $q^\vee \in S^2(E)$ consists of linear functions $l \in E^\vee$ such that the value of $b_{q^\vee} : E^\vee \to E$ at l is equal to zero. The same is true for the dual form $\Omega_f^\vee$. Its locus of zeros consists of linear forms l such that $\Omega_f^{-1}(l^k) \in S^k(E)$ vanishes on l. The degree k homogeneous form $\Omega_f^{-1}(l^k)$ is classically known as the *anti-polar* of l (with respect to f).

Definition 1.4.6. *Two linear forms $l, m \in E^\vee$ are called* conjugate *with respect to a nondegenerate form $f \in S^{2k}(E^\vee)$ if*

$$\Omega_f^\vee(l^k, m^k) = 0.$$

Proposition 1.4.7. *Suppose f is given by (1.55), where the powers l_i^k are linearly independent in $S^k(E^\vee)$. Then, each pair l_i, l_j is conjugate with respect to f.*

Proof Since the powers l_i^k are linearly independent, we may include them in a basis of $S^k(E^\vee)$. Choose the dual basis in $S^k(E)$. Then, the catalecticant matrix of f has the upper corner matrix of size s equal to the diagonal matrix. Its adjugate matrix has the same property. This implies that $l_i^k, l_i^j, i \neq j$, are conjugate with respect to $\Omega_f^\vee$. □

1.4.3 The Waring rank of a homogeneous form

Since any quadratic form q can be reduced to a sum of squares, one can define its rank as the smallest number r such that

$$q = l_1^2 + \cdots + l_r^2$$

for some $l_1, \ldots, l_r \in E^\vee$.

Definition 1.4.8. *Let $f \in S^d E^\vee$. Its* Waring rank $\mathrm{wrk}(f)$ *is the smallest number r such that*

$$f = l_1^d + \cdots + l_r^d \tag{1.76}$$

sum

for some linear forms $l_1, \ldots, l_r \in E^\vee$.

The next result follows immediately from the proof of Proposition conj 1.4.7.

prop3 **Proposition 1.4.9.** *Let Ω_f be the quadratic form on $S^k(E)$ associated to $f \in S^{2k}(E^\vee)$. Then, the Waring rank of f is greater than or equal to the rank of Ω_f.*

Let f be a nondegenerate form of even degree $2k$. By Corollary C1.3.11 1.3.21,

$$\mathrm{wrk}(f) = s(2k, n) \geq \left\lceil \frac{1}{n+1}\binom{n+d}{d} \right\rceil,$$

with strict inequality only in the following cases:

- $d = 2, \mathrm{wrk}(f) = \mathrm{rank}\ \Omega_f = n + 1$;
- $n = 2, d = 4, \mathrm{wrk}(f) = \mathrm{rank}\ \Omega_f = 6$;
- $n = 3, d = 4, \mathrm{wrk}(f) = \mathrm{rank}\ \Omega_f = 10$;
- $n = 4, d = 4, \mathrm{wrk}(f) = \mathrm{rank}\ \Omega_f = 15$.

In all non-exceptional cases,

$$\mathrm{wrk}(f) \geq \frac{1}{n+1}\binom{n+2k}{n} = \binom{n+k}{n}\frac{(n+2k)\cdots(n+k)}{2k\cdots(k+1)(n+1)} \geq \mathrm{rank}\ \Omega_f.$$

In most cases, we have strict inequality.

1.4.4 Mukai's skew-symmetric bilinear form

SS:1.4.4

Let $\omega \in \bigwedge^2 E$ be a skew-symmetric bilinear form on $E^\vee$. It admits a unique extension to a Poisson bracket $\{,\}_\omega$ on $S(E^\vee)$ which restricts to a skew-symmetric bilinear form

$$\{,\}_\omega : S^{k+1}(E^\vee) \times S^{k+1}(E^\vee) \to S^{2k}(E^\vee). \tag{1.77}$$

poisson

Recall that a *Poisson bracket* on a commutative algebra A is a skew-symmetric bilinear map $A \times A \to A, (a, b) \mapsto \{a, b\}$ such that its left and right partial maps $A \to A$ are derivations.

Let $f \in S^{2k}(E^\vee)$ be a nondegenerate form and $\Omega_f^\vee \in S^2(S^k(E))$ be its dual form. For each ω as above, define $\sigma_{\omega,f} \in \bigwedge^2 S^{k+1}(E)$ by

$$\sigma_{\omega,f}(g, h) = \Omega_f^\vee(\{g, h\}_\omega).$$

mukai **Theorem 1.4.10.** *Let f be a nondegenerate form in $S^{2k}(E^\vee)$ of Waring rank $N = \operatorname{rank}\Omega_f = \binom{n+k}{n}$. For any $Z = \{[\ell_1], \dots, [\ell_N]\} \in \mathrm{VSP}(f, N)^o$, let $\langle Z\rangle_{k+1}$ be the linear span of the powers l_i^{k+1} in $S^{k+1}(E^\vee)$. Then,*

(i) $\langle Z\rangle_{k+1}$ *is isotropic with respect to each form* $\sigma_{\omega,f}$*;*
(ii) $\mathrm{ap}_f^{k-1}(S^{k-1}E) \subset \langle Z\rangle_{k+1}$*;*
(iii) $\mathrm{ap}_f^{k-1}(S^{k-1}E)$ *is contained in the radical of each* $\sigma_{\omega,f}$*.*

Proof To prove the first assertion it is enough to check that, for all i, j, one has $\sigma_{\omega,f}(l_i^{k+1}, l_j^{k+1}) = 0$. We have

$$\sigma_{\omega,f}(l_i^{k+1}, l_j^{k+1}) = \Omega_f^\vee(\{l_i^{k+1}, l_j^{k+1}\}_\omega) = \Omega_f^\vee(l_i^k, l_j^k)\omega(l_i, l_j).$$

Since ℓ_i^k are linearly independent, by Proposition 1.4.7, $\Omega_f^\vee(l_i^k, l_j^k) = 0$. This checks the first assertion.

For any $\psi \in S^{k-1}(E)$,

$$D_\psi(f) = D_\psi\Big(\sum_{i=1}^N l_i^{2k}\Big) = \sum_{i=1}^N D_\psi(l_i^{2k}) = \tfrac{(2k)!}{(k+1)!}\sum_{i=1}^N D_\psi(l_i^{k-1})l_i^{k+1}.$$

This shows that $\mathrm{ap}_f^{k-1}(S^{k-1}(E))$ is contained in $\langle Z\rangle_{k+1}$. It remains for to check that $\sigma_{\omega,f}(D_\psi(f), g) = 0$ for any $\psi \in S^{k-1}(E), g \in S^{k+1}(E^\vee), \omega \in \bigwedge^2 E$. Choose coordinates $t_0, \dots, t_n$ in $E^\vee$ and the dual coordinates $\xi_0, \dots, \xi_n$ in E. The space $\bigwedge^2 E$ is spanned by the forms $\omega_{ij} = \xi_i \wedge \xi_j$. We have

$$\{D_\psi(f), g\}_{\omega_{ij}} = D_{\xi_i}(D_\psi(f))D_{\xi_j}(g) - D_{\xi_j}(D_\psi(f))D_{\xi_i}(g)$$

$$= D_{\xi_i\psi}(f)D_{\xi_j}(g) - D_{\xi_j\psi}(f)D_{\xi_i}(g) = D_{\psi\xi_i}(f)D_{\xi_j}(g) - D_{\psi\xi_j}(f)D_{\xi_i}(g).$$

For any $g, h \in S^k(E^\vee)$,

$$\Omega_f^\vee(g, h) = \langle\Omega_f^{-1}(g), h\rangle.$$

Thus,

$$\sigma_{\omega_{ij},f}(D_\psi(f), g) = \Omega_f^\vee(D_{\psi\xi_i}(f), D_{\xi_j}(g)) - \Omega_f^\vee(D_{\psi\xi_j}(f), D_{\xi_i}(g))$$

$$= \langle\psi\xi_i, D_{\xi_j}(g)\rangle - \langle\psi\xi_j, D_{\xi_i}(g)\rangle = D_\psi(D_{\xi_i\xi_j}(g) - D_{\xi_j\xi_i}(g)) = D_\psi(0) = 0.$$

□

Since $\mathrm{ap}_f^{k-1}(E)$ is contained in the radical of $\sigma_{\omega,f}$, we have the induced skew-symmetric form on $S^{k+1}(E^\vee)/\mathrm{ap}_f^{k-1}(E)$. By Lemma 1.3.2,

$$S^{k+1}(E^\vee)/\mathrm{ap}_f^{k-1}(E) = \mathrm{AP}_{k+1}(f)^\vee.$$

If no confusion arises, we denote the induced form by $\sigma_{\omega,f}$ and call it the *Mukai's skew-form*.

One can also consider the collection of the Mukai skew-forms $\sigma_{\omega,f}$ as a linear map

$$\sigma_f : \bigwedge^2 E \to \bigwedge^2 \mathrm{AP}_{k+1}(f), \quad \omega \mapsto \sigma_{\omega,f},$$

or, its transpose

$$^t\sigma_f : \bigwedge^2 \mathrm{AP}_{k+1}(f)^\vee \to \bigwedge^2 E^\vee. \tag{1.78}$$ mu2

Let $Z = \{[l_1], \ldots, [l_s]\} \in \mathrm{VSP}(f, s)^o$ be a polar s-hedron of a nondegenerate form $f \in S^{2k}(E^\vee)$ and, as before, let $\langle Z\rangle_{k+1}$ be the linear span of $(k+1)$-th powers of the linear forms l_i. Let

$$L(Z) = \langle Z\rangle_{k+1}/\mathrm{ap}_f^{k-1}(S^{k-1}(E)). \tag{1.79}$$ mukai2

It is a subspace of $S^{k+1}(E^\vee)/\mathrm{ap}_f^{k-1}(S^{k-1}(E))$ which we identify with the dual space $\mathrm{AP}_{k+1}(f)^\vee$ of $\mathrm{AP}_{k+1}(f)$.

Now, observe that $\langle Z\rangle_{k+1}^\perp$ is equal to $I_Z(k+1)$, where we identify Z with the reduced closed subscheme of the dual projective space $\mathbb{P}(E)$. This allows one to extend the definition of $L(Z)$ to any generalized polar s-hedron $Z \in \mathrm{VSP}(f; s)$:

$$L(Z) = I_Z(k+1)^\perp/\mathrm{ap}_f^{k-1}(S^{k-1}(E)) \subset S^{k+1}(E^\vee)/\mathrm{ap}_f^{k-1}(S^{k-1}(E)).$$

inj **Proposition 1.4.11.** *Let f be a nondegenerate homogeneous form of degree $2k$ of Waring rank equal to $N_k = \binom{n+k}{k}$. Let $Z, Z' \in \mathrm{VSP}(f, N_k)$. Then,*

$$L(Z) = L(Z') \Longleftrightarrow Z = Z'.$$

Proof It is enough to show that

$$I_Z(k+1) = I_{Z'}(k+1) \Longrightarrow Z = Z'.$$

Suppose $Z \neq Z'$. Choose a subscheme Z_0 of Z of length $N_k - 1$ that is not a subscheme of Z'. Since $\dim I_{Z_0}(k) \geq \dim S^k(E^\vee) - h^0(\mathcal{O}_Z) = \binom{n+k}{k} - N_k + 1 = 1$, we can find a nonzero $\psi \in I_{Z_0}(k)$. The sheaf $\mathcal{I}_Z/\mathcal{I}_{Z_0}$ is supported at one point x and is annihilated by the maximal ideal $\mathfrak{m}_x$. Thus, $\mathfrak{m}_x\mathcal{I}_{Z_0} \subset \mathcal{I}_Z$. Let $\xi \in E$ be a linear form on $E^\vee$ vanishing at x but not vanishing at any proper closed subscheme of Z'. This implies that $\xi\psi \in I_Z(k+1) = I_{Z'}(k+1)$ and hence, $\psi \in I_{Z'}(k) \subset \mathrm{AP}_k(f)$ contradicting the nondegeneracy of f. □

Lemma 1.4.12. *Let $f \in S^{2k}(E^\vee)$ be a nondegenerate form of Waring rank $N_k = \binom{k+n}{n}$. For any $Z \in \mathrm{VSP}(f, N_k)^o$,*

$$\dim L(Z) = \binom{n+k-1}{n-1}.$$

Proof Counting constants, we see that

$$\dim\langle Z\rangle_{k+1} \geq \dim S^{k+1}(E) - N_k,$$

and hence,

$$\dim L(Z) = \dim\langle Z\rangle_{k+1}^{\perp} - \dim \mathrm{ap}_f^{k-1}(S^{k-1}(E)) \leq N_k - \binom{n+k-1}{n} = \binom{n+k-1}{n-1}.$$

We have to consider the exceptional cases where $\mathrm{wrk}(f) = \mathrm{rank}\,\Omega_f$. The assertion is obvious in the case $k = 1$. The space $L(Z)$ is of expected dimension unless $l_1^2, \ldots, l_{n+1}^2$ are linearly dependent. This implies that f is a quadratic form of rank less than $n+1$, contradicting the assumption.

Assume $n = 2, k = 2$ and $\dim L(Z) > 3$, or, equivalently, $\dim\langle Z\rangle_3 > 4$. Since $\mathrm{AP}_2(f) = \{0\}$, there are no conics passing through Z. In particular, no four points are collinear. Let C be a conic through the points $[l_1], \ldots, [l_5]$ and let x_1, x_2 be two additional points on C such that each irreducible component of C contains at least four points. Since $\dim\langle Z\rangle_3 > 4$, we can find a two-dimensional linear system of cubics through $[l_1], \ldots, [l_5], x_1, x_2$. By Bezout's Theorem, C belongs to the fixed part of the linear system. The residual part is a 2-dimensional linear system of lines through $[l_6]$, an obvious contradiction.

Similar arguments check the assertion in the cases $n = 2, k = 3, 4$. In the remaining case $n = 3, k = 2$, we argue as follows. We have $N_2 = 10$. Assume $L(Z) < 6$, or, equivalently, $\dim\langle Z\rangle_3 > 10$. Since $\mathrm{AP}_2(f) = \{0\}$, no 4 lines are collinear (otherwise, a quadric through 3 points on the line, and the remaining 6 points, will contain all ten points). Choose three non-collinear points p_1, p_2, p_3 among the ten points and two general points on each line $\overline{p_i p_j}$ and one general point in the plane containing the three points. Then, we can find a 3-dimensional linear system of cubics in $|\langle Z\rangle_3|$ passing through the additional 7 points. It contains the plane through p_1, p_2, p_3. The residual linear system consists of quadrics through the remaining 7 points in Z. Since no four lines are collinear, it is easy to see that the dimension of the linear system of quadrics through 7 points is of dimension 2. This contradiction proves the assertion. □

C1.4.9 **Corollary 1.4.13.** *Let $f \in S^{2k}(E^\vee)$ be a nondegenerate form of Waring rank $N_k = \binom{n+k}{n}$. Let $\mathrm{VSP}(f, N_k)^o$ be the variety of polar polyhedra of f. Then, the map $Z \mapsto L(Z)$ is an injective map*

$$\mathrm{VSP}(f, N_k)^o \to G(\tbinom{n+k-1}{n-1}, \mathrm{AP}_{k+1}(f)^\vee).$$

Its image is contained in the subvariety of subspaces isotropic with respect to all Mukai's skew forms $\sigma_{\omega,f}$ on $\mathrm{AP}_{k+1}(f)^\vee$.

Example 1.4.14. Assume $n = 2$. Then, $\mathrm{wrk}(f) = \mathrm{rank}\,\Omega_f = \binom{k+2}{2}$ if and only if $k = 1, 2, 3, 4$. In these cases the Corollary applies. We will consider the cases $k = 1$ and $k = 2$ later. If $k = 3$, we obtain that $\mathrm{VSP}(f, 10)^o$ embeds in $G(4, 9)$. Its closure is a K3 surface [535], [607]. If $k = 4$, $\mathrm{VSP}(f, 15)^o$ embeds in $G(5, 15)$. It consists of 16 points [607].

1.4.5 Harmonic polynomials

SS:1.4.5

Let $q \in S^2(E^\vee)$ be a nondegenerate quadratic form on E. For convenience of notation, we identify q with the apolarity map $\mathrm{ap}_q^1 : E \to E^\vee$. By the universal property of the symmetric power, the isomorphism $q : E \to E^\vee$ extends to a linear isomorphism $S^k(q) : S^k(E) \to S^k(E^\vee)$ which defines a symmetric nondegenerate pairing

$$(\, , \,)_k : S^k(E) \times S^k(E) \to \mathbb{C}. \tag{1.80}$$

newpair

It is easy to check that, for any $\xi \in S^k(E)$ and $v \in E$,

$$(\xi, v^k) = k!\xi(l_v^k),$$

where $l_v \in E^\vee$ is the linear function $\mathrm{ap}_q^1(v)$.

Let us compare the pairing 1.80 with the pairing Ω_{q^k} from (1.75). Choose a basis $(\eta_0, \dots, \eta_n)$ in E and the dual basis $(t_0, \dots, t_n)$ in $E^\vee$ such that $q = \frac{1}{2}(\sum t_i^2)$, so that $q(\eta_i) = t_i$. Then,

$$S^k(q)(\boldsymbol{\eta}^{\boldsymbol{i}}) = \boldsymbol{t}^{\boldsymbol{i}}.$$

However,

$$\mathrm{ap}_{q^k}^k(\boldsymbol{\eta}^{\boldsymbol{i}}) = k!\boldsymbol{t}^{\boldsymbol{i}} + qg,$$

for some $g \in S^{k-2}(E^\vee)$. Thus,

$$(S^k(q) - \frac{1}{k!}\mathrm{ap}_{q^k}^k)(S^k(E)) \subset qS^{k-2}(E^\vee).$$

Let

$$\mathcal{H}_q^k(E) = (qS^{k-2}(E^\vee))^\perp \subset S^k(E)$$

be the subspace of *q-harmonic symmetric tensors*. In more convenient language, exchanging the roles of E and $E^\vee$, and replacing q with the dual form $q^\vee \in S^2(E)$, we have

$$\mathcal{H}_q^k(E^\vee) = \mathrm{Ker}(D_{q^\vee} : S^k(E^\vee) \to S^{k-2}(E^\vee)).$$

In the previous choice of coordinates, the operator $D_{q^\vee}$ is the *Laplace operator*

$\frac{1}{2}\sum\frac{\partial^2}{\partial t_i^2}$. Restricting $\mathrm{ap}^k_{q^k}$ to the subspace $\mathcal{H}^k_q(E)$, we obtain a nondegenerate symmetric pairing

$$\mathcal{H}^k_q(E)\times\mathcal{H}^k_q(E)\to\mathbb{C}$$

which coincides with the restriction of $\frac{1}{k!}\Omega_{q^k}$ to the same subspace. Changing E to $E^\vee$, we also obtain a symmetric nondegenerate pairing

$$\mathcal{H}^k_q(E^\vee)\times\mathcal{H}^k_q(E^\vee)\to\mathbb{C}$$

which can be defined either by the restriction of the pairing (1.80) or by the quadratic form $\frac{1}{k!}\Omega_{(q^\vee)^k}$. Note that all these pairings are equivariant with respect to the orthogonal group $\mathrm{O}(E,q)$, i.e.,can be considered as pairings of the linear representations of $\mathrm{O}(E,q)$. We have the direct sum decomposition of linear representations

$$S^k(E)=\mathcal{H}^k_q(E)\oplus q^\vee S^{k-2}(E). \quad (1.81)$$ harmpair

The summand $q^\vee S^{k-2}(E)$ coincides with $\mathrm{ap}^{k-2}_{q^\vee}(S^{k-2}(E^\vee))$. The linear representation $\mathcal{H}^k_q(E)$ is an irreducible representation of $\mathrm{O}(E,q)$ (see [345]).

Next, let us see that, in the case when f is a power of a nondegenerate quadratic polynomial, the Mukai form coincides, up to a scalar multiple, with the skew form on the space of harmonic polynomials studied by N. Hitchin in [401] and [402].

The Lie algebra $\mathfrak{o}(E,q)$ of the orthogonal group $\mathrm{O}(E,q)$ is equal to the Lie subalgebra of the Lie algebra $\mathfrak{gl}(E)$ of endomorphisms of E that consists of operators $A:E\to E$ such that the composition $A\circ q^{-1}:E^\vee\to E\to E$ is equal to the negative of its transpose. This defines a linear isomorphism of vector spaces

$$\bigwedge^2 E^\vee\to\mathfrak{o}(E,q),\quad \omega\mapsto\tilde{\omega}=q^{-1}\circ\omega:E\to E^\vee\to E.$$

Now, taking $\omega\in\bigwedge^2 E^\vee$, and identifying $S^{k+1}(E^\vee)/\mathrm{ap}^{k-1}_{q^k}(S^{k-1}(E))$ with $\mathcal{H}^{k+1}_q(E^\vee)$, we obtain the Mukai pairing

$$\sigma_{\omega,q^k}:\mathcal{H}^{k+1}_q(E^\vee)\times\mathcal{H}^{k+1}_q(E^\vee)\to\mathbb{C}$$

on the space of harmonic $k+1$-forms on E.

P1.4.8 **Proposition 1.4.15.** *For any $g,h\in\mathcal{H}^{k+1}_q(E^\vee)$ and any $\omega\in\bigwedge^2 E^\vee$,*

$$\sigma_{\omega,q^k}(g,h)=\frac{(k+1)^2}{k!}(\tilde{\omega}\cdot g,h)_{k+1},$$

where $(,)_{k+1}:S^{k+1}(E^\vee)\times S^{k+1}(E^\vee)\to\mathbb{C}$ is the symmetric pairing defined by $S^{k+1}(q^{-1})$.

Proof It is known that the space $\mathcal{H}_q^{k+1}(E^\vee)$ is spanned by the forms $q(v)^{k+1}$, where v is an isotropic vector for q, i.e.,$[v] \in V(q)$ (see Goodman [345], Proposition 5.2.6). So, it is suffices to check the assertion when $g = q(v)^{k+1}$ and $h = q(w)^{k+1}$ for some isotropic vectors $v, w \in E$. Choose a basis $(\xi_0, \ldots, \xi_n)$ in E and the dual basis $t_0, \ldots, t_n$ in $E^\vee$ as in the beginning of this subsection. An element $u \in \mathfrak{o}(E, q)$ can be written in the form $\sum a_{ij} t_i \frac{\partial}{\partial t_j}$ for some skew-symmetric matrix (a_{ij}). We identify (a_{ij}) with the skew 2-form $\omega \in \bigwedge^2 E$. We can also write $g = (\alpha \cdot t)^{k+1}$ and $h = (\beta \cdot t)^{k+1}$, where we use the dot-product notation for the sums $\sum \alpha_i t_i$. We have

$$(u \cdot g, h)_{k+1} = (k+1)!\big(\sum a_{ij} t_i \frac{\partial}{\partial t_j} (\alpha \cdot t)^{k+1}\big)(\beta) = (k+1)!(k+1)(\alpha \cdot \beta)^k \omega(\alpha \cdot t, \beta \cdot t).$$

The computations from the proof of Theorem mukai 1.4.10, show that

$$\sigma_{\omega, q^k}(g, h) = \Omega^\vee_{q^k}((\alpha \cdot t)^k, (\beta \cdot t)^k) \omega(\alpha \cdot t, \beta \cdot t).$$

It is easy to see that $\Omega^\vee_{q^k}$ coincides with $\Omega_{(q^\vee)^k}$ on the subspace of harmonic polynomials. We have

$$\Omega_{(q^\vee)^k}\big((\alpha \cdot t)^k, (\beta \cdot t)^k\big) = D_{(\alpha \cdot t)^k}\big(\tfrac{1}{2} \sum \xi_i^2\big)^k((\beta \cdot t)^k)$$

$$= k! D_{(\alpha \cdot \xi)^k}((\beta \cdot t)^k) = (k!)^2 (\alpha \cdot \beta)^k.$$

This checks the assertion. □

Computing the catalecticant matrix of q^k we find that q^k is a nondegenerate form of degree $2k$. Applying Corollary C1.4.9 1.4.13, we obtain that in the cases listed in Corollary C1.3.11 1.3.21, there is an injective map

$$\mathrm{VSP}(q^k, \tbinom{n+k}{n}) \to G(\tbinom{n+k-1}{n-1}, \mathcal{H}_q^{k+1}(E^\vee)). \tag{1.82}$$ inj2

Its image is contained in the subvariety of subspaces isotropic with respect to the skew-symmetric forms $(g, h) \mapsto (u \cdot g, k)_{k+1}, u \in \mathfrak{o}(E, q)$.

The following Proposition gives a basis in the space of harmonic polynomials (see Miles [518]). We assume that $(E, q) = (\mathbb{C}^{n+1}, \frac{1}{2} \sum t_i^2)$.

Proposition 1.4.16. *For any set of non-negative integers $(b_0, \ldots, b_n)$ such that $b_i \le 1$ and $\sum b_i = k$, let*

$$H^k_{b_0, \ldots, b_n} = \sum (-1)^{[a_0/2]} \frac{k![a_0/2]!}{\prod_{i=0}^n a_i! \prod_{i=1}^n (\frac{b_i - a_i}{2})!} \prod_{i=0}^n t_i^{a_i},$$

where the summation is taken over the set of all sequences of non-negative integers $(a_0, \ldots, a_n)$ such that

- $a_i \equiv b_i \mod 2, i = 0, \ldots, n,$

- $\sum_{i=0}^{n} a_i = k$,
- $a_i \le b_i, i = 1, \ldots, n$.

Then, the polynomials $H^k_{b_0,\ldots,b_n}$ *form a basis of the space* $\mathcal{H}^k_q(\mathbb{C}^{n+1})$.

For any polynomial $f \in \mathbb{C}[t_0, \ldots, t_n]$ one can find the projection Hf to the subspace of harmonic polynomials. The following formula is taken from [790].

$$Hf = f - \sum_{s=1}^{[k/2]} (-1)^{s+1} \frac{q^s \Delta^s f}{2^s s!(n-3+2k)(n-5+2k)\cdots(n-2s-1+2k)}, \tag{1.83}$$

where $\Delta = \sum \frac{\partial^2}{\partial t_i^2}$ is the Laplace operator.

Example 1.4.17. Let $n = 2$ so that $\dim E = 3$. The space of harmonic polynomials $\mathcal{H}^k_q(E^\vee)$ is of dimension $\binom{k+2}{2} - \binom{k}{2} = 2k+1$. Since the dimension is odd, the skew form σ_{ω,q^k} is degenerate. It follows from Proposition 1.4.15 that its radical is equal to the subspace of harmonic polynomials g such that $\tilde{\omega} \cdot g = 0$ (recall that $\tilde{\omega}$ denotes the element of $\mathfrak{o}(E, q)$ corresponding to $\omega \in \bigwedge^2 E$). In coordinates, a vector $u = (u_0, u_1, u_2) \in \mathbb{C}^3$ corresponds to the skew-symmetric matrix

$$\begin{pmatrix} 0 & u_0 & u_1 \\ -u_0 & 0 & u_2 \\ -u_1 & -u_2 & 0 \end{pmatrix}$$

representing an endomorphism of E, or an element of $\bigwedge^2 E$. The Lie bracket is the cross-product of vectors. The action of a vector u on $f \in \mathbb{C}[t_0, t_1, t_2]$ is given by

$$u \cdot f = \sum_{i,j,k=0}^{2} \epsilon_{ijk} t_i u_j \frac{\partial f}{\partial t_k},$$

where $\epsilon_{i,j,k} = 0$ is totally skew-symmetic with values equal to $0, 1, -1$.

For any $v \in E$, let us consider the linear form $l_v = q(v) \in E^\vee$. We know that $q(v)^k \in \mathcal{H}^k_q(E^\vee)$ if $[v] \in V(q)$. If $[v] \notin V(q)$, then we can consider the projection f_v of $(l_v)^k$ to $\mathcal{H}^k_q(E^\vee)$. By (1.83), we get

$$f_v = l_v^k + \sum_{s=1}^{[k/2]} (-1)^s \frac{k(k-1)\cdots(k-2s+1)}{2^s s!(2k-1)\cdots(2k-2s+1)} q(v)^s q^s l_v^{k-2s}. \tag{1.84}$$

We have

$$u \cdot l_v = l_{u \times v}.$$

Since $f \mapsto u \cdot f$ is a derivation of $\mathrm{Sym}(E^\vee)$ and $u \cdot q = 0$, we obtain

$$u \cdot f_v = l_{u\times v}\big(kl_v^{k-1} + \sum_{s=1}^{[k/2]}(-1)^s \frac{k(k-1)\cdots(k-2s+1)(k-2s)q(v)^s l_v^{k-2s-2}}{2^s s!(2k-1)\cdots(2k-2s+1)}\big). \tag{1.85}$$

This implies that the harmonic polynomial f_u satisfies $u \cdot f_u = 0$ and hence, belongs to the radical of the skew form σ_{u,q^k}. The Lie algebra $\mathfrak{so}(3)$ is isomorphic to the Lie algebra $\mathfrak{sl}(2)$ and its irreducible representation on the space of degree k harmonic polynomials is isomorphic to the representation of $\mathfrak{sl}(2)$ on the space of binary forms of degree $2k$. It is easy to see that the space of binary forms invariant under a nonzero element of $\mathfrak{sl}(2)$ is one-dimensional. This implies that the harmonic polynomial f_u spans the radical of σ_{u,q^k} on $\mathcal{H}_q^k(E^\vee)$.

Let $f \in H^k(E^\vee)$ be a nonzero harmonic polynomial of degree k. The orthogonal complement $f^\perp$ of f with respect to $(\ ,\ )_k : \mathcal{H}_q^k(E^\vee)\times\mathcal{H}_q^k(E^\vee) \to \mathbb{C}$ is of dimension $2k$. The restriction of the skew-symmetric form σ_{u,q^k} to $f^\perp$ is degenerate if and only if $f_u \in f^\perp$, i.e., $(f_u, f)_k = (l_u^k, f) = f(u) = 0$. Here, we used that the decomposition (1.81) is an orthogonal decomposition with respect to $(\ ,\ )_k$. Let Pf be the pfaffian of the skew form σ_{u,q^k} on $f^\perp$. It is equal to zero if and only if the form is degenerate. By above, it occurs if and only if $f(u) = 0$. Comparing the degrees, this gives

$$V(f) = V(\mathrm{Pf}).$$

So, every harmonic polynomial can be expressed in a canonical way as a pfaffian of a skew-symmetric matrix with entries linear forms, a result due to N. Hitchin [403].

1.5 First Examples

1.5.1 Binary forms

Let U be a 2-dimensional linear space and $f \in S^d(U^\vee)\setminus\{0\}$. The hypersurface $X = V(f)$ can be identified with a positive divisor $\mathrm{div}(f) = \sum m_i x_i$ of degree d on $|U| \cong \mathbb{P}^1$. Since $\bigwedge^2 U \cong \mathbb{C}$, we have a natural isomorphism $U \to U^\vee$ of linear representations of $\mathrm{SL}(U)$. It defines a natural isomorphism between the projective line $|U|$ and its dual projective line $\mathbb{P}(U)$. In coordinates, a point $a = [a_0, a_1]$ is mapped to the hyperplane $V(a_1t_0 - a_0t_1)$ whose zero set is equal to the point a. If X is reduced (i.e., f has no multiple roots), then, under the identification of $|U|$ and $\mathbb{P}(U)$, X coincides with its dual $X^\vee$. In general, $X^\vee$ consists of simple roots of f. Note that this is consistent with the Plücker–Teissier formula. The degrees of the Hessian and the Steinerian coincide, although they are different if $d > 3$. Assume that X is reduced. The partial derivatives of f define the polar map $g : |U| \to |U|$ of degree $d-1$. The ramification divisor $\mathrm{He}(X)$ consists of $2d-4$ points and it is mapped bijectively onto the branch divisor $\mathrm{St}(X)$.

Example 1.5.1. We leave the case $d = 2$ to the reader. Consider the case $d = 3$. In coordinates,

$$f = a_0t_0^3 + 3a_1t_0^2t_1 + 3a_2t_0t_1^2 + a_3t_1^3.$$

All invariants are powers of the *discriminant invariant*

$$\Delta = a_0^2a_3^2 + 4a_0a_2^3 + 4a_1^3a_3 - 6a_0a_1a_2a_3 - 3a_1^2a_2^2. \tag{1.86}$$ cubinv

whose symbolic expression is $(12)^2(13)(24)(34)^2$ (see Turnbull [768], p. 244). The Hessian covariant

$$H = (a_0a_2 - a_1^2)t_0^2 + (a_0a_3 - a_1a_2)t_0t_1 + (a_1a_3 - a_2^2)t_1^2.$$

Its symbolic expression is $(ab)a_xb_y$. There is also a cubic covariant

$$J = J(f,H) = \det\begin{pmatrix} t_0^3 & 3t_0^2t_1 & 3t_0t_1^2 & t_1^3 \\ a_2 & -2a_1 & a_0 & 0 \\ a_3 & -a_2 & -a_1 & a_0 \\ 0 & -a_3 & -2a_2 & a_1 \end{pmatrix}$$

with symbolic expression $(ab)^2(ac)^2b_xc_x^2$. The covariants f, H and J form a complete system of covariants, i.e.,generate the module of covariants over the algebra of invariants.

E1.5.2 *Example* 1.5.2. Consider the case $d = 4$. In coordinates,

$$f = a_0t_0^4 + 4a_1t_0^3t_1 + 6a_2t_0^2t_1^2 + 4a_3t_0t_1^3 + a_4t_1^4.$$

There are two basic invariants S and T on the space of quartic binary forms. Their symbolic expression are $S = (12)^4$ and $T = (12)^2(13)^2(23)^2$. Explicitly,

$$\begin{aligned} S &= a_0a_4 - 4a_1a_3 + 3a_2^2, \\ T &= a_0a_2a_4 + 2a_1a_2a_3 - a_0a_3^2 - a_1^2a_4 - a_2^3. \end{aligned} \tag{1.87}$$

Note that T coincides with the determinant of the catalecticant matrix of f. Each invariant is a polynomial in S and T. For example, the discriminant invariant is equal to

$$\Delta = S^3 - 27T^2.$$

The Hessian $\mathrm{He}(X) = V(H)$ and the Steinerian $S(X) = V(K)$ are both of degree 4. We have

$$H = (a_0a_2 - a_1^2)t_0^4 + 2(a_0a_3 - a_1a_2)t_0^3t_1 + (a_0a_4 + 2a_1a_3 - 3a_2^2)t_0^2t_1^2$$
$$+2(a_1a_4 - a_2a_3)t_0t_1^3 + (a_2a_4 - a_3^2)t_1^4.$$

and

$$K = \Delta((a_0t_0 + a_1t_1)x^3 + 3(a_1t_0 + a_2t_1)x^2y + 3(a_2t_0 + a_3t_1)xy^2 + (a_3t_0 + a_4t_1)y^3).$$

Observe that the coefficients of H (resp. K) are of degree 2 (resp. 4) in coefficients of f. There is also a covariant $J = J(f, H)$ of degree 6 and the module of covariants is generated by f, H, J over $\mathbb{C}[S, T]$. In particular, $K = \alpha Tf + \beta SH$, for some constants α and β. By taking f in the form

$$f = t_0^4 + 6mt_0^2t_1^2 + t_1^4, \tag{1.88}$$

and comparing the coefficients, we find

$$2K = -3Tf + 2SH. \tag{1.89}$$

Under identification $|U| = \mathbb{P}(U)$, a generalized k-hedron Z of $f \in S^d(U^\vee)$ is the zero divisor of a form $g \in S^k(U)$ which is apolar to f. Since

$$H^1(|E|, \mathcal{I}_Z(d)) \cong H^1(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(d-k)) = 0, \quad k \geq d+1,$$

any Z is automatically linearly independent. Identifying a point $[g] \in |S^k(U)|$ with the zero divisor $\mathrm{div}(g)$, we obtain

Theorem 1.5.3. *Assume $n = 1$. Then,*

$$\mathrm{VSP}(f; k) = |\mathrm{AP}_k(f)|.$$

Note that the kernel of the map

$$S^k(U) \to S^{d-k}(U^\vee), \quad \psi \mapsto D_\psi(f)$$

is of dimension $\geq \dim S^k(U) - \dim S^{d-k}(U^\vee) = k + 1 - (d - k + 1) = 2k - d$. Thus, $D_\psi(f) = 0$ for some nonzero $\psi \in S^k(U)$, whenever $2k > d$. This shows that f has always generalized polar k-hedron for $k > d/2$. If d is even, a binary form has an apolar $d/2$-form if and only if $\det \mathrm{Cat}_{d/2}(f) = 0$. This is a divisor in the space of all binary d-forms.

Example 1.5.4. Take $d = 3$. Assume that f admits a polar 2-hedron. Then,

$$f = (a_1t_0 + b_1t_1)^3 + (a_2t_0 + b_2t_1)^3.$$

It is clear that f has 3 distinct roots. Thus, if $f = (a_1t_0 + b_1t_1)^2(a_2t_0 + b_2t_1)$

has a double root, it does not admit a polar 2-hedron. However, it admits a generalized 2-hedron defined by the divisor $2p$, where $p = (b_1, -a_1)$. In the secant variety interpretation, we know that any point in $|S^3(E^\vee)|$ either lies on a unique secant or on a unique tangent line of the rational cubic curve. The space $\mathrm{AP}_2(f)$ is always one-dimensional. It is generated either by a binary quadric $(-b_1\xi_0 + a_1\xi_1)(-b_2\xi_0 + a_2\xi_1)$, or by $(-b_1\xi_0 + a_1\xi_1)^2$.

Therefore, $\mathrm{VSP}(f,2)^o$ consists of one point or empty but $\mathrm{VSP}(f,2)$ always consists of one point. This example shows that $\mathrm{VSP}(f,2) \neq \overline{\mathrm{VSP}(f,2)^o}$ in general.

SS:1.5.2

1.5.2 Quadrics

It follows from Example ex1.1.3 1.3.18 that $\mathrm{Sec}_t(\mathsf{V}_2^n) \neq |S^2(E^\vee)|$ if and only if there exists a quadric with $t+1$ singular points in general position. Since the singular locus of a quadric $V(q)$ is a linear subspace of dimension equal to $\mathrm{corank}(q)-1$, we obtain that $\mathrm{Sec}_n(\mathsf{V}_2^n) = |S^2(E^\vee)|$. Therefore, any general quadratic form can be written as a sum of $n+1$ squares of linear forms $l_0, \ldots, l_n$. Of course, linear algebra gives more. Any quadratic form of rank $n+1$ can be reduced to sum of squares of the coordinate functions. Assume that $q = t_0^2 + \cdots + t_n^2$. Suppose we also have $q = l_0^2 + \cdots + l_n^2$. Then, the linear transformation $t_i \mapsto l_i$ preserves q, and hence, is an orthogonal transformation. Since polar polyhedra of q and λq are the same, we see that the projective orthogonal group $\mathrm{PO}(n+1)$ acts transitively on the set $\mathrm{VSP}(f, n+1)^o$ of polar $(n+1)$-hedra of q. The stabilizer group G of the coordinate polar polyhedron is generated by permutations of coordinates and diagonal orthogonal matrices. It is isomorphic to the semi-direct product $2^n \rtimes \mathfrak{S}_{n+1}$ (the Weyl group of root systems of types B_n, D_n), where we use the notation 2^n for the 2-elementary abelian group $(\mathbb{Z}/2\mathbb{Z})^n$. Thus, we obtain

T1.5.3 **Theorem 1.5.5.** *Let q be a quadratic form in $n+1$ variables of rank $n+1$. Then,*

$$\mathrm{VSP}(q, n+1)^o \cong \mathrm{PO}(n+1)/2^n \rtimes \mathfrak{S}_{n+1}.$$

The dimension of $\mathrm{VSP}(q, n+1)^o$ *is equal to* $\frac{1}{2}n(n+1)$.

Example 1.5.6. Take $n = 1$. Using the Veronese map $\mathsf{v}_2 : \mathbb{P}^1 \to \mathbb{P}^2$, we consider a nonsingular quadric $Q = V(q)$ as a point p in $\mathbb{P}^2$ not lying on the conic $C = V(t_0t_2 - t_1^2)$. A polar 2-gon of q is a pair of distinct points p_1, p_2 on C such that $p \in \langle p_1, p_2\rangle$. The set of polar 2-gons can be identified with the pencil of lines through p with the two tangent lines to C deleted. Thus, $W(q,2)^o = \mathbb{P}^1 \setminus \{0, \infty\} = \mathbb{C}^*$. There are two generalized 2-gons $2p_0$ and $2p_\infty$

defined by the tangent lines. Each of them gives the representation of q as $l_1 l_2$, where $V(l_i)$ are the tangents. We have $\mathrm{VSP}(f,2) = \overline{\mathrm{VSP}(f,2)^o} \cong \mathbb{P}^1$.

Let $q \in S^2(E^\vee)$ be a nondegenerate quadratic form. We have an injective map (1.82)

$$\mathrm{VSP}(q, n+1)^o \to G(n, \mathcal{H}^2_q(E)) \cong G(n, \tbinom{n+2}{2} - 1). \qquad (1.90)$$ mapq

Its image is contained in the subvariety $G(n, \mathcal{H}^2_q(E))_\sigma$ of subspaces isotropic with respect to the Mukai skew forms.

Recall that the Grassmann variety $G(m, W)$ of linear m-dimensional subspaces of a linear space W of dimension N carries the natural rank n vector bundle $\mathcal{S}$, the *universal subbundle*. Its fiber over a point $L \in G(m, W)$ is equal to L. It is a subbundle of the trivial bundle $W_{G(m,W)}$ associated to the vector space W. We have a natural exact sequence

$$0 \to \mathcal{S} \to W^\vee_{G(m,W)} \to \mathcal{Q} \to 0,$$

where $\mathcal{Q}$ is the *universal quotient sheaf*, such that the fiber of $\mathcal{Q}^\vee \subset W_{G(m,W)}$ over the point $L \in G(m.W)$ is equal to $L \subset W$ (we will discuss the Grassmannians later with great details in Section 2.4 and in Chapters 10 and 11.

By restriction, we can view the Mukai form $\sigma_q : \bigwedge^2 E \to \bigwedge^2 \mathcal{H}^2_q(E^\vee)$ as a section of the vector bundle $\bigwedge^2 \mathcal{S}^\vee \otimes \bigwedge^2 E^\vee$. The image of $\mathrm{VSP}(q, n+1)$ is contained in the zero locus of a section of this bundle defined by σ_q. Since the rank of the vector bundle is equal to $\binom{n}{2}\binom{n+1}{2}$, we expect that the dimension of its zero locus is equal to

$$\dim G(n, \tbinom{n+2}{2} - 1) - \binom{n}{2}\binom{n+1}{2} = n(\tbinom{n+2}{2} - 1 - n) - \binom{n}{2}\binom{n+1}{2}.$$

Unfortunately, this number is ≤ 0 for $n > 2$, so the expected dimension is wrong. However, when $n = 2$, we obtain that the expected dimension is equal to $3 = \dim \mathrm{VSP}(q, 3)$. We can view $\sigma_{\omega,q}$ as a hyperplane in the Plücker embedding of $G(2, \mathcal{H}^2_q(E)) \cong G(2, 5)$. So, $\mathrm{VSP}(q, 3)$ embeds into the intersection of 3 hyperplane sections of $G(2, 5)$.

chap **Theorem 1.5.7.** *Let q be a nondegenerate quadratic form on a 3-dimensional vector space E. Then, the image of* $\mathrm{VSP}(q, 3)$ *in $G(2, \mathcal{H}^2_q(E))$, embedded in the Plücker space, is a smooth irreducible 3-fold equal to the intersection of $G(2, \mathcal{H}^2_q(E))$ with a linear space of codimension 3.*

Proof We have $\dim \mathcal{H}^2_q(E) = 5$, so $G(2, \mathcal{H}^2_q(E)) \cong G(2, 5)$ is of dimension 6. Hyperplanes in the Plücker space are elements of the space $|\bigwedge^2 \mathcal{H}^2_q(E)^\vee|$. Note that the functions $s_{q,\omega}$ are linearly independent. In fact, a basis ξ_0, ξ_1, ξ_2

in E gives a basis $\omega_{01} = \xi_0 \wedge \xi_1, \omega_{02} = \xi_0 \wedge \xi_2, \omega_{12} = \xi_1 \wedge \xi_2$ in $\bigwedge^2 E$. Thus, the space of sections $s_{q,\omega}$ is spanned by 3 sections s_{01}, s_{02}, s_{12} corresponding to the forms ω_{ij}. Without loss of generality, we may assume that $q = t_0^2 + t_1^2 + t_2^2$. If we take $a = t_0t_1 + t_2^2, b = -t_0^2 + t_1^2 + t_2^2$, we see that $s_{01}(a,b) \neq 0, s_{12}(a,b) = 0, s_{02}(a,b) = 0$. Thus, a linear dependence between the functions s_{ij} implies the linear dependence between two functions. It is easy to see that no two functions are proportional. So our 3 functions $s_{ij}, 0 \le i < j \le 2$ span a 3-dimensional subspace of $\bigwedge^2 \mathcal{H}_q^2(E^\vee)$, and hence, define a codimension 3 projective subspace L in the Plücker space $|\bigwedge^2 \mathcal{H}_q^2(E)|$. The image of $\mathrm{VSP}(q,3)$ under the map (1.90) is contained in the intersection $G(2,E) \cap L$. This is a 3-dimensional subvariety of $G(2, \mathcal{H}_q^2(E))$, and hence, contains $\mu(\mathrm{VSP}(q,3))$ as an irreducible component. We skip an argument, based on counting constants, which proves that the subspace L belongs to an open Zariski subset of codimension 3 subspaces of $\bigwedge^2 \mathcal{H}_q^2(E)$ for which the intersection $L \cap G(2, \mathcal{H}_q^2(E))$ is smooth and irreducible (see [243]). □

It follows from the adjunction formula and the known degree of $G(2,5)$ that the closure of $\mathrm{VSP}(q,3)^o$ in $G(2, \mathcal{H}_q^2(E))$ is a smooth Fano variety of degree 5. We will discuss it again in the next chapter.

Remark 1.5.8. One can also consider the varieties $\mathrm{VSP}(q,s)$ for $s > n+1$. For example, we have

$$t_0^2 - t_2^2 = \tfrac{1}{2}(t_0+t_1)^2 + \tfrac{1}{2}(t_0-t_1)^2 - \tfrac{1}{2}(t_1+t_2)^2 - \tfrac{1}{2}(t_1^2 - t_2)^2,$$
$$t_0^2 + t_1^2 + t_2^2 = (t_0+t_2)^2 + (t_0+t_1)^2 + (t_1+t_2)^2 - (t_0+t_1+t_2)^2.$$

This shows that $\mathrm{VSP}(q, n+2), \mathrm{VSP}(q, n+3)$ are not empty for any nondegenerate quadric Q in $\mathbb{P}^n, n \ge 2$.

Exercises

1.1 Suppose X is a plane curve and $x \in X$ is its ordinary double point. Show that the pair consisting of the tangent line of the first polar $P_a(X)$ at x and the line $\overline{ax}$ is harmonically conjugate (see section 2.1.2) to the pair of tangents to the branches of X at x in the pencil of lines through x. If x is an ordinary cusp, then show that the polar line of $P_a(X)$ at x is equal to the cuspidal tangent of X at x.

1.2 Show that a line contained in a hypersurface X belongs to all polars of X with respect to any point on this line.

1.3 Find the multiplicity of the intersection of a plane curve C with its Hessian at an ordinary double point and at an ordinary cusp of C. Show that the Hessian has a triple point at the cusp.

1.4 Suppose a hypersurface X in $\mathbb{P}^n$ has a singular point x of multiplicity $m > 1$. Prove that $\mathrm{He}(X)$ has this point as a point of multiplicity $\ge (n+1)m - 2n$.

1.5 Suppose a hyperplane is tangent to a hypersurface X along a closed subvariety Y
ex:1.5 of codimension 1. Show that Y is contained in $\mathrm{He}(X)$.

1.6 Suppose f is the product of d distinct linear forms $l_i(t_0, \ldots, t_n)$. Let A be the matrix of size $(n+1) \times d$ whose i-th column is formed by the coefficients of l_i (defined, of course up to proportionality). Let Δ_I be the maximal minor of A corresponding to a subset I of $[1, \ldots, d]$ and f_I be the product of linear forms $l_i, i \notin I$. Show that

$$\mathrm{He}(f) = (-1)^n (d-1) f^{n-1} \sum_I \Delta_I^2 f_I^2$$

Muir
ex:1.6 ([531], p. 660).

1.7 Find an example of a reduced hypersurface whose Hessian surface is nowhere
ex:1.7 reduced.

1.8 Show that the locus of points on the plane where the first polars of a plane curve X are tangent to each other is the Hessian of X and the set of common tangents
ex:1.8 is the Cayleyan curve.

1.9 Show that each inflection tangent of a plane curve X, considered as a point in the
e:1.9 dual plane, lies on the Cayleyan of X.

1.10 Show that the class of the Steinerian $\mathrm{St}(X)$ of a plane curve X of degree d is
ex:1.10 equal to $3(d-1)(d-2)$ but its dual is not equal to $\mathrm{Cay}(X)$.

1.11 Let $\mathsf{D}_{m,n} \subset \mathbb{P}^{mn-1}$ be the image in the projective space of the variety of $m \times n$ matrices of rank $\leq \min\{m, n\} - 1$.

$$\tilde{\mathcal{D}}_{m,n} = \{(A, x) \in \mathbb{P}^{mn-1} \times \mathbb{P}^n : A \cdot x = 0\}$$

ex:1.11 is a resolution of singularities of $\mathsf{D}_{m,n}$. Find the dual variety of $\mathsf{D}_{m,n}$.

ex:1.12 1.12 Find the dual variety of the Segre variety $s(\mathbb{P}^n \times \mathbb{P}^n) \subset \mathbb{P}^{n^2+2n}$.

1.13 Let X be the union of k nonsingular conics in general position. Show that $X^\vee$ is
ex:1.13 also the union of k nonsingular conics in general position.

1.14 Let X has only δ ordinary nodes and κ ordinary cusps as singularities. Assume that the dual curve $X^\vee$ has also only $\check{\delta}$ ordinary nodes and $\check{\kappa}$ ordinary cusps as
ex:1.14 singularities. Find $\check{\delta}$ and $\check{\kappa}$ in terms of d, δ, κ.

ex:1.15 1.15 Give an example of a self-dual (i.e.,$X^\vee \cong X$) plane curve of degree > 2.

1.16 Show that the Jacobian of a net of plane curves has a double point at each simple base point unless the net contains a curve with a triple point at the base point
Enriques-Chisini
ex:1.16 [292].

1.17 Let $|L|$ be a general n-dimensional linear system of quadrics in $\mathbb{P}^n$ and $|L|^\perp$ be the $(\binom{n+2}{2} - n - 2)$-dimensional subspace of apolar quadrics in the dual space. Show that the variety of reducible quadrics in $|L|^\perp$ is isomorphic to the Reye
1.17 variety of $|L|$ and has the same degree.

1.18 Show that the embedded tangent space of the Veronese variety V_d^n at a point represented by the form l^d is equal to the projectivization of the linear space of
ex:1.18 homogeneous polynomials of degree d of the form $l^{d-1}m$.

1.19 Using the following steps, show that V_3^4 is 6-defective by proving that for any seven general points p_i in $\mathbb{P}^4$, there is a cubic hypersurface with singular points at the p_i's.

(i) Show that there exists a Veronese curve R_4 of degree four through the seven points.

(ii) Show that the secant variety of R_4 is a cubic hypersurface which is singular along R_4.

1.20 Let q be a nondegenerate quadratic form in $n+1$ variables. Show that $\mathrm{VSP}(q, n+1)^o$ embedded in $G(n, E)$ is contained in the linear subspace of codimension n.

1.21 Compute the catalecticant matrix $\mathrm{Cat}_2(f)$, where f is a homogeneous form of degree 4 in 3 variables.

1.22 Let $f \in S^{2k}(E^\vee)$ and Ω_f be the corresponding quadratic form on $S^k(E)$. Show that the quadric $V(\Omega_f)$ in $|S^k(E)|$ is characterized by the following two properties:

- Its pre-image under the Veronese map $\mathsf{v}_k : |E| \to |S^k(E)|$ is equal to $V(f)$;
- Ω_f is apolar to any quadric in $|S^k(E^\vee)|$ which contains the image of the Veronese map $|E^\vee| = \mathbb{P}(E) \to |S^k(E^\vee)| = \mathbb{P}(S^k(E))$.

1.23 Let C_k be the locus in $|S^{2k}(E^\vee)|$ of hypersurfaces $V(f)$ such that $\det \mathrm{Cat}_k(f) = 0$. Show that C_k is a rational variety. [Hint: Consider the rational map $C_k \dashrightarrow |E|$) which assigns to $V(f)$ the point defined by the subspace $\mathrm{AP}_k(f)$ and study its fibres].

1.24 Give an example of a polar 4-gon of the cubic $t_0t_1t_2 = 0$.

1.25 Find all binary forms of degree d for which $\mathrm{VSP}(f, 2)^o = \emptyset$.

1.26 Let f be a form of degree d in $n+1$ variables. Show that $\mathrm{VSP}(f, \binom{n+d}{d})^o$ is an irreducible variety of dimension $n\binom{n+d}{d}$.

1.27 Describe the variety $\mathrm{VSP}(f, 4)$, where f is a nondegenerate quadratic form in 3 variables.

1.28 Show that a smooth point y of a hypersurface X belongs to the intersection of the polar hypersurfaces $P_x(X)$ and $P_{x^2}(X)$ if and only if the line connecting x and y intersects X at the point y with multiplicity ≥ 3.

1.29 Show that the vertices of two polar tetrahedra of a nonsingular quadric in $\mathbb{P}^3$ are base points of a net of quadrics. Conversely, the set of 8 base points of a general net of quadrics can be divided in any way into two sets, each of two sets is the set of vertices of a polar tetrahedron of the same quadric [717].

1.30 Suppose two cubic plane curves $V(f)$ and $V(g)$ admit a common polar pentagon. Show that the determinant of the 6×6-matrix $[\mathrm{Cat}_1(f)\ \mathrm{Cat}_1(g)]$ vanishes [307].

Historical Notes

Although some aspects of the theory of polarity for conics were known to mathematicians of Ancient Greece, the theory originates in projective geometry, in the works of G. Desargues, G. Monge and J. Poncelet. For Desargues the polar of a conic was a generalization of the diameter of a circle (when the pole is taken at infinity). He referred to a polar line as a "transversale de l'ordonnance". According to the historical accounts found in [292], vol. II, and [174], p. 60, the name "polaire" was introduced by J. Gergonne. Apparently, the polars of curves of higher degree appear first in the work of E. Bobillier [62] and then, with the introduction of projective coordinates, in the works of J. Plücker [596]. These geometers were the first to realize the duality property of polars: if a point x

belongs to the s-th polar of a point y with respect to a curve of degree d, then y belongs to the $(d-s)$-th polar of x with respect to the same curve. Many properties of polar curves were stated in a purely geometric way by J. Steiner [723]. As customary for him, with no proofs. Good historical accounts can be found in [57] and [577], p.279.

The Hessian and the Steinerian curves with their relations to the theory of polars were first studied by J. Steiner [723] who called them *conjugate Kerncurven*. The current name for the Hessian curve was coined by J. Sylvester [739] in honor of O. Hesse who was the first to study the Hessian of a ternary cubic [387] under the name *der Determinante* of the form. The current name of the Steinerian curve goes back to G. Salmon [652] and L. Cremona [182]. The Cayleyan curve was introduced by A. Cayley in [102] who called it the *pippiana*. The current name was proposed by L. Cremona. Most of the popular classical text-books in analytic geometry contain an exposition of the polarity theory (e.g. [150], [292], [652]).

The theory of dual varieties and the generalization of Plücker formulae to arbitrary dimension is still a popular subject of modern algebraic geometry. It is well-documented in modern literature, so this topic is barely touched here (see, for example, [449]).

The theory of apolarity was a trendy topic of classical algebraic geometry. It originates from the works of Rosanes [635] who called apolar forms of the same degree *conjugate forms* and Reye [615]. who introduced the term "apolar". The condition of polarity $D_\psi(f) = 0$ was viewed as vanishing of the simultaneous bilinear invariant of a form f of degree d and a form ψ of class d. It was called the *harmonizant*. We refer to a survey of classical results to [577] and to a modern exposition of some of these results to [243], which we followed here.

The Waring problem for homogeneous forms originates from a more general problem of finding a canonical form for a homogeneous form. Sylvester's result about reducing a cubic form in four variables to the sum of 5 powers of linear forms is one of the earliest examples of solution of the Waring problem. We will discuss this later in the book. F. Palatini was the first who recognized the problem as a problem about the secant variety of the Veronese variety [567], [568] and as a problem of the existence of envelopes with a given number of singular points (in less general form, the relationship was found earlier by J. E. Campbell [82]). T he Alexander-Hirschowitz theorem was claimed by J. Bronowski [75] in 1933. However, citing C. Ciliberto [136], he had only a plausibility argument. The case $n = 2$ was first established by F. Palatini [568], and the case $n = 3$ was solved by A. Terracini [745]. Terracini was the first to recognize the exceptional case of cubic hypersurfaces in $\mathbb{P}^4$ [744]. The original proof of Terracini's Lemma can be found in [746]. We also refer to [326] for a modern survey of the problem. An excellent historical account and in-depth theory of the Waring problems and the varieties associated with it can be found in the book of A. Iarrobino and V. Kanev [422].

The fact that a general plane quintic admits a unique polar 7-gon was first mentioned by D. Hilbert in his letter to C. Hermite [393]. The proofs were given later by Palatini [570] and H. Richmond [622], [624]

In earlier editions of his book [653] G. Salmon mistakenly applied counting constants to assert that three general quadrics in $\mathbb{P}^3$ admit a common polar pentahedron. G. Darboux [200] was the fist to show that the counting of constants is wrong. W. Frahm [307] proved that the net of quadrics generated by three quadrics with a common polar pentahedron must be a net of polars of a cubic surface and also has the property that its discriminant curve is a Lüroth quartic, a plane quartic which admits an inscribed pentagon. In [749] E.Toeplitz (the father of Otto Toeplitz) introduced the invariant Λ of three quadric surfaces whose vanishing is necessary and sufficient for the existence of a common polar pentahedron. The fact that two general plane cubics do not admit a common polar pentagon was first discovered by F. London [492]. The Waring Problem continues to attract the attention of contemporary mathematicians. Some references to modern literature can found in this chapter.

2
Conics and Quadric Surfaces

Ch2

2.1 Self-polar Triangles

S:2.1

2.1.1 Veronese quartic surfaces

SS:2.1.1

Let $\mathbb{P}^2 = |E|$ and $|S^2(E^\vee)| \cong \mathbb{P}^5$ be the space of conics in $\mathbb{P}^2$. Recall, for this special case, the geometry of the *Veronese quartic surface* V_2^2, the image of the Veronese map

$$\mathsf{v}_2 : |E^\vee| \to |S^2(E^\vee)|, \quad [l] \mapsto [l^2].$$

If we view $S^2(E^\vee)$ as the dual space of $S^2(E)$, then the Veronese surface parameterizes hyperplanes H_l in $S^2(E)$ of conics passing through the point $[l]$ in the dual plane $|E^\vee|$. The Veronese map v_2 is given by the complete linear system $|\mathcal{O}_{|E^\vee|}(2)| = |S^2(E)|$. Thus, the pre-image of a hyperplane in $|S^2(E^\vee)|$ is a conic in the plane $|E^\vee|$. The conic is singular if and only if the hyperplane is tangent to the Veronese surface. There are two possibilities, either the singular conic C is the union of two distinct lines (a line-pair), or it is equal to a double line. In the first case, the hyperplane is tangent to the surface at a single point. The point is the image of the singular point $[l]$ of the conic. In the second case, the hyperplane is tangent to the Veronese surface along a curve R equal to the image of the line C_{red} under the restriction of the Veronese map. It follows that the curve R is a conic cut out on the Veronese surface by a plane. In this way, we see that the *dual variety of the Veronese surface* is isomorphic to the discriminant cubic hypersurface $\mathsf{D}_2(2)$ parameterizing singular conics. The tangent plane to the Veronese surface at a point $[l]^2$ is the intersection of hyperplanes which cut out a conic in $|E^\vee|$ with singular point $[l]$. The plane of conics in $|E|$ apolar to such conics is the plane of reducible conics with one component equal to the line $V(l)$.

The cubic hypersurface $D_2(2)$ has two 2-dimensional families of planes. A plane from the first family is a net of conics with a base line $V(l)$. It contains one point $[l^2]$ in V_2^2, and it is tangent to the surface at this point. A plane from the second family is a net of conics with one base point of multiplicity three. Its members are the line-pairs with a fixed singular point. The intersection of this plane with the Veronese surface is the set of double lines passing through the base point. It is a conic in the plane.

Two planes from the first family intersect at one point not lying on V_2^2. The point is the union of their base lines. Two families from the second family intersect at one points lying on V_2^2. It is the double line joining their base points. The two families of planes in $\mathbb{P}^5$ are examples of the largest irreducible families of mutually intersecting planes in $\mathbb{P}^5$ classified by U. Morin Morin1 [525], Morin2 [526].

Choosing a basis in E, we can identify the space $S^2(E^\vee)$ with the space of symmetric 3×3-matrices. The Veronese surface V_2^2 in $|S^2(E^\vee)|$ is identified with matrices of rank 1. Its equations are given by 2×2-minors. The variety of matrices of rank ≤ 2 is the cubic hypersurface whose equation is given by the determinant.

Replacing E with $E^\vee$, we obtain the definition of the *dual Veronese surface*, the image of $|E|$ in $|S^2(E)|$ of the map given by the complete linear system of conics in $|E|$. Its points are hyperplanes of conics apolar to a conic of rank one. Choosing a basis $(x^2, xy, xz, y^2, yz, z^2)$ of $S^2(E^\vee)$ as projective coordinates $(t_0, t_1, t_2, t_3, t_4, t_5)$ in $|S^2(E)|$. In these coordinates, the equation of the dual Veronese surface is given by

$$\operatorname{rank}\begin{pmatrix} t_0 & t_1 & t_2 \\ t_1 & t_2 & t_3 \\ t_2 & t_4 & t_5 \end{pmatrix} = 1. \tag{2.1}$$

eqveronese

The cubic hypersurface $\mathsf{D}_2(2)$ is given by the determinant of the matrix.

The dual basis $(\frac{1}{2}\xi_0^2, \xi_0\xi_1, \xi_0\xi_2, \frac{1}{2}\xi_1^2, \xi_1\xi_2, \frac{1}{2}\xi_2^2)$ (with respect to polarity pairing (eeq2 1.3)) gives the equation of the Veronese surface V_2^2 in $|S^2(E^\vee)$ in coordinates $(u_0, \ldots, u_5)$:

$$\operatorname{rank}\begin{pmatrix} 2u_0 & u_1 & u_2 \\ u_1 & 2u_2 & u_3 \\ u_2 & u_4 & 2u_5 \end{pmatrix} = 1. \tag{2.2}$$

eqveronese

Since any quadratic form of rank 2 in E can be written as a sum of quadratic forms of rank 1, the secant variety $\mathrm{Sec}_1(\mathsf{V}_2^2)$ coincides with $\mathsf{D}_2(2)$. Also, it coincides with the *tangential variety* $\mathrm{Tan}(\mathsf{V}_2^2)$, the union of tangent planes $\mathbb{T}_x(\mathsf{V}_2^2), \mathsf{x} \in \mathsf{V}_2^2$. It is singular along the Veronese surface.

Let us look at a possible projection of V_2^2 to $\mathbb{P}^4$. It is given by a linear subsystem $|V|$ of $|S^2(E)|$. Let K be the apolar conic to all conics from $|V|$. It is a point $\mathfrak{o}$ in the dual space $|S^2(E^\vee)|$ equal to the center of the projection. The conic K could be nonsingular, a line-pair, or a double line. In the first two cases $\mathfrak{o} \notin \mathsf{V}_2^2$. The image of the projection is a quartic surface in $\mathbb{P}^4$, called a *projected Veronese surface*. If K is nonsingular, $\mathfrak{o}$ does not lie on $\mathrm{Sec}_1(\mathsf{V}_2^2)$, hence the projected Veronese surface is a nonsingular quartic surface in $\mathbb{P}^4 = \mathbb{P}(V)$. If K is a line-pair, then $\mathfrak{o}$ lies on a tangent plane of V_2^2 at some point $[l^2]$. Hence, it lies on the plane spanning a conic contained in V_2^2. The restriction of the projection map to this conic is of degree 2, and its image is a double line on the projected Veronese surface. Two ramification points are mapped to two *pinch points* of the surface. Finally, $\mathfrak{o}$ could be on V_2^2. The image of the projection is a cubic surface S in $\mathbb{P}^4$. All conics on V_2^2 containing $\mathfrak{o}$ are projected to lines on S. So, S is a nonsingular cubic scroll in $\mathbb{P}^4$ isomorphic to the blow-up of V_2^2, hence of $\mathbb{P}^2$, at one point. In our notation for rational normal scrolls from Subsection CAG-2:SS:8.1.1 8.1.1 it is the scroll $S_{1,4}$.

Let us now project V_2^2 further to $\mathbb{P}^3$. This time, the linear system $|V|$ defining the projection is of dimension 3. Its apolar linear system is a pencil, a line ℓ in $|S^2(E^\vee)|$. Suppose the apolar pencil does not intersect V_2^2. In this case the pencil of conics does not contain a double line, hence contains exactly three line-pairs. The three line-pairs correspond to the intersection of ℓ with the cubic hypersurface $\mathrm{Sec}_1(\mathsf{V}_2^2)$. As we saw in above, this implies that the image S of the projection is a quartic surface with three double lines. These lines are concurrent. In fact, a pencil of plane sections of S containing one of the lines has residual conics singular at the points of intersection with the other two lines. Since the surface is irreducible, this implies that the other two lines intersect the first one. Changing the order of the lines, we obtain that each pair of lines intersect. This is possible only if they are concurrent (otherwise they are coplanar, and plane containing the lines intersect the quartic surface along a cubic taken with multiplicity 2).

The projection of a Veronese surface from a line not intersecting V_2^2 is called a *Steiner quartic*. Choose coordinates t_0, t_1, t_2, t_3 such that the equations of the singular lines are $t_1 = t_2 = 0, t_1 = t_3 = 0$ and $t_2 = t_3 = 0$. Then, the equation of a Steiner surface can be reduced to the form $t_0t_1t_2t_3 + g_4 = 0$. By taking the partial derivatives at the point $[1,0,0,0]$ and general points of the singular lines, we find that g_4 is a linear combination of the monomial $t_1^2t_2^2, t_1^2t_3^2, t_2^2t_3^2$. Finally, by scaling the coordinates, we reduce the equation to the form

$$t_0t_1t_2t_3 + t_1^2t_2^2 + t_1^2t_3^2 + t_2^2t_3^2 = 0. \quad (2.3)$$

steinerequation

An explicit birational map from $\mathbb{P}^2$ onto the surface is given by

$$[y_0, y_1, y_2] \mapsto [(-y_0+y_1+y_2)^2, (y_0-y_1+y_2)^2, (y_0+y_1-y_2)^2, (y_0+y_1+y_2)^2]. \quad (2.4)$$

steinerparametrization

Next, we assume that the center of the projection is line ℓ intersecting V_2^2. In this case, the image of the projection is a cubic scroll, the projection of the rational normal scroll $S_{1,4}$ to $\mathbb{P}^3$. There are two possibilities, the pencil of conics defined by ℓ has two singular members, or one singular member, a double line. This gives two possible cubic scrolls. We will give their equations in the next Chapter.

Replacing E with $|E^\vee|$, we can define the Veronese surface in $|S^2(E)|$, the image of the plane $|E|$ under the map given by the complete linear system of conics. We leave it to the reader to "dualize" the statements from above.

2.1.2 Polar lines

Let $C = V(q)$ be a nonsingular conic. For any point $a \in \mathbb{P}^2$, the first polar $P_a(C)$ is a line, the *polar line* of a. For any line ℓ there exists a unique point a such that $P_a(C) = l$. The point a is called the *pole* of ℓ. The point a considered as a line in the dual plane is the polar line of the point ℓ with respect to the dual conic $\check{C}$.

One can also define the polar line with pole $a = [v]$ as the set of points *conjugate* to a with respect to C; that is, points $b = [w]$ such that $b_q(v, w) = 0$, where b_q is the associated symmetric bilinear form of q. Dually, one defines *conjugate lines* with respect to C.

Borrowing terminology from the Euclidean geometry, we call three non-collinear lines in $\mathbb{P}^2$ a *triangle*. The lines themselves will be called the *sides* of the triangle. The three intersection points of pairs of sides are called the *vertices* of the triangle.

A set of three non-collinear lines ℓ_1, ℓ_2, ℓ_3 is called a *self-polar triangle* with respect to C if each ℓ_i is the polar line of C with respect to the opposite vertex. It is easy to see that it suffices that only two sides are polar to the opposite vertices.

Proposition 2.1.1. *Three lines $\ell_i = V(l_i)$ form a self-polar triangle for a conic $C = V(q)$ if and only if they form a polar triangle of C.*

Proof Let $\ell_i \cap \ell_j = [v_{ij}]$. If $q = l_1^2 + l_2^2 + l_3^2$, then $D_{v_{ij}}(q) = 2l_k$, where $k \neq i, j$. Thus, a polar triangle of C is a self-conjugate triangle. Conversely, if $V(D_{v_{ij}}(q)) = \ell_k$, then $D_{v_{ik}v_{ij}}(q) = D_{v_{jk}v_{ij}}(q) = 0$. This shows that the conic C is apolar to the linear system of conics spanned by the reducible conics $\ell_i + \ell_j$. It coincides with the linear system of conics through the three points ℓ_1, ℓ_2, ℓ_3 in the dual plane. Applying Proposition 2.45, we obtain that the self-conjugate triangle is a polar triangle.

Of course, we can prove the converse by computation. Let

$$2q = a_{00}t_0^2 + a_{11}t_1^2 + a_{22}t_2^2 + 2a_{01}t_0t_1 + 2a_{02}t_0t_2 + 2a_{12}t_1t_2 = 0.$$

Choose projective coordinates in $\mathbb{P}^2$ such that $\ell_i = V(t_i)$. Then,

$$P_{[1,0,0]}(X) = \ell_1 = V(\frac{\partial q}{\partial t_0}) = V(a_{00}t_0 + a_{01}t_1 + a_{02}t_2), \tag{2.5}$$

$$P_{[0,1,0]}(X) = \ell_2 = V(\frac{\partial q}{\partial t_1}) = V(a_{11}t_1 + a_{01}t_0 + a_{12}t_2),$$

$$P_{[0,0,1]}(X) = \ell_2 = V(\frac{\partial q}{\partial t_2}) = V(a_{22}t_2 + a_{02}t_0 + a_{12}t_1)$$

implies that $q = \frac{1}{2}(t_0^2 + t_1^2 + t_2^2)$. □

Remark 2.1.2. Similarly, one can define a self-polar $(n+1)$-hedron of a quadric in $\mathbb{P}^n$ and about the reduction of a quadratic form to principal axes in linear algebra.

Let $Q = V(q)$ and $Q' = V(q')$ be two quadrics in $\mathbb{P}^1$. We say that Q and Q' are *harmonically conjugate* if the dual quadric of Q is apolar to Q'. In other words, if $D_{q^\vee}(q') = 0$. In coordinates, if

$$q = \alpha t_0^2 + 2\beta t_0t_1 + \gamma t_1^2, \quad q' = \alpha' t_0^2 + 2\beta' t_0t_1 + \gamma' t_1^2.$$

then $q^\vee = \gamma\eta_0^2 - 2\beta\eta_0\eta_1 + \alpha\eta_1^2$, and the condition becomes

$$-2\beta\beta' + \alpha\gamma' + \alpha'\gamma = 0. \tag{2.6}$$ harmcong

It shows that the relation is symmetric (one can extend it to quadrics in higher-dimensional spaces but it will not be symmetric).

Of course, a quadric in $\mathbb{P}^1$ can be identified with a set of two points in $\mathbb{P}^1$, or one point with multiplicity 2. This leads to the classical definition of *harmonically conjugate* $\{a, b\}$ and $\{c, d\}$ in $\mathbb{P}^1$. We will see later many other equivalent definitions of this relation.

Let $\mathbb{P}^1 = |U|$, where $\dim U = 2$. Since $\dim \bigwedge^2 U = 1$, we can identify $|E|$ with $\mathbb{P}(E)$. Explicitly, a point with coordinates $[a, b]$ is identified with a point $[-b, a]$ in the dual coordinates. Under this identification, the dual quadric $q^\vee$ vanishes at the zeros of q. Thus, (2.6) is equivalent to the polarity condition

$$D_{cd}(q) = D_{ab}(q') = 0, \tag{2.7}$$ polcond

where $V(q) = \{a, b\}, V(q') = \{c, d\}$.

wref **Proposition 2.1.3.** *Let ℓ_1, ℓ_2, ℓ_3 be a triangle with vertices $a = \ell_1 \cap \ell_2, b = \ell_1 \cap \ell_3$ and $c = \ell_2 \cap \ell_3$. Then, the triangle is a self-polar triangle of a conic C*

if and only if $a \in P_b(C) \cap P_c(C)$ and the pairs of points $C \cap \ell_3$ and (b, c) are harmonically conjugate.

Proof Consider the pair $C \cap \ell_3$ as a quadric q in ℓ_3. We have $c \in P_b(C)$, thus $D_{bc}(q) = 0$. Restricting to ℓ_3 and by using (2.7), we see that the pairs b, c and $C \cap \ell_3$ are harmonically conjugate. Conversely, if $D_{bc}(q) = 0$, the polar line $P_b(C)$ contains a and intersects ℓ_3 at c, hence coincides with $\overline{ac}$. Similarly, $P_c(C) = \overline{ab}$. □

Any triangle in $\mathbb{P}^2$ defines the dual triangle in the dual plane $(\mathbb{P}^2)^\vee$. Its sides are the pencils of lines with the base point of one of the vertices.

Corollary 2.1.4. *The dual of a self-polar triangle of a conic C is a self-polar triangle of the dual conic $\check{C}$.*

2.1.3 The Variety of self-polar triangles

Here, we will discuss a compactification of the variety $\mathrm{VSP}(q, 3)$ of polar triangles of a nondegenerate quadratic form in three variables.

Let C be a nonsingular conic. The group of projective transformations of $\mathbb{P}^2$ leaving C invariant is isomorphic to the projective complex orthogonal group

$$\mathrm{PO}(3) = \mathrm{O}(3)/(\pm I_3) \cong \mathrm{SO}(3).$$

It is also isomorphic to the group $\mathrm{PSL}(2)$ via the Veronese map

$$\mathsf{v}_2 : \mathbb{P}^1 \to \mathbb{P}^2, \quad [t_0, t_1] \mapsto [t_0^2, t_0 t_1, t_1^2].$$

Obviously, PO_3 acts transitively on the set of self-polar triangles of C. We may assume that $C = V(\sum t_i^2)$. The stabilizer subgroup of the self-polar triangle defined by the coordinate lines is equal to the subgroup generated by permutation matrices and orthogonal diagonal matrices. It is easy to see that it is isomorphic to the semi-direct product $(\mathbb{Z}/2\mathbb{Z})^2 \rtimes \mathfrak{S}_3 \cong \mathfrak{S}_4$. Thus, we obtain the following:

Theorem 2.1.5. *The set of self-polar triangles of a nonsingular conic has a structure of a homogeneous space SO_3/Γ, where Γ is a finite subgroup isomorphic to $\mathfrak{S}_4$.*

A natural compactification of the variety of self-conjugate triangles of a nondegenerate conic q is the variety $\mathrm{VSP}(q, 3)$ which we discussed in the previous chapter. In Theorem 1.5.7, we have shown that it is isomorphic to the intersection of the Grassmannian $G(3, 5)$ with a linear subspace of codimension 3. Let us see this construction in another way, independent of the theory developed in the previous chapter.

Let $V = \mathsf{V}_2^2$ be a Veronese surface in $\mathbb{P}^5$. We view $\mathbb{P}^5$ as the projective space of conics in $\mathbb{P}^2$ and V_2^2 as its subvariety of double lines. A trisecant plane of V is spanned by three linearly independent double lines. A conic $C \in \mathbb{P}^5$ belongs to this plane if and only if the corresponding three lines form a self-polar triangle of C. Thus, the set of self-polar triangles of C can be identified with the set of trisecant planes of the Veronese surface which contain C. The latter will also include *degenerate self-polar triangles* corresponding to the case where the trisecant plane is tangent to the Veronese surface at some point. Projecting from C to $\mathbb{P}^4$, we will identify the set of self-polar triangles (maybe degenerate) with the set of trisecant lines of the projected Veronese surface V_4. This is a closed subvariety of the Grassmann variety $G_1(\mathbb{P}^4)$ of lines in $\mathbb{P}^4$.

Let E be a linear space of odd dimension $2k+1$ and let $G(2,E) := G_1(|E|)$ be the Grassmannian of lines in $|E|$. Consider the Plücker embedding $\bigwedge^2$: $G(2,E) \hookrightarrow G_1(\bigwedge^2 E) = |\bigwedge^2 E|$. Any nonzero $\omega \in (\bigwedge^2 E)^\vee = \bigwedge^2 E^\vee$ defines a hyperplane H_ω in $|\bigwedge^2 E|$. Consider ω as a linear map $\alpha_\omega : E \to E^\vee$ defined by $\alpha_\omega(v)(w) = \omega(v,w)$. The map α_ω is skew-symmetric in the sense that its transpose map coincides with $-\alpha_\omega$. Thus, its determinant is equal to zero, and $\mathrm{Ker}(\alpha_\omega) \neq \{0\}$. Let v_0 be a nonzero element of the kernel. Then, for any $v \in E$, we have $\omega(v_0, v) = \alpha_\omega(v)(v_0) = 0$. This shows that ω vanishes on all decomposable 2-vectors $v_0 \wedge v$. This implies that the intersection of the hyperplane H_ω with $G(2,E)$ contains all lines which intersect the linear subspace $C_\omega = |\mathrm{Ker}(\alpha_\omega)| \subset |E|$, which we call the *pole* of the hyperplane H_ω.

Now, recall the following result from linear algebra (see Exercise 2.1). Let A be a skew-symmetric matrix of odd size $2k+1$. Its principal submatrices A_i of size $2k$ (obtained by deleting the i-th row and the i-th column) are skew-symmetric matrices of even size. Let Pf_i be the pfaffians of A_i (i.e. $\det(A_i) = \mathrm{Pf}_i^2$). Assume that $\mathrm{rank}(A) = 2k$, or, equivalently, not all Pf_i vanish. Then, the system of linear equations $A \cdot x = 0$ has one-dimensional null-space generated by the vector $(a_1, \ldots, a_{2k+1})$, where $a_i = (-1)^{i+1}\mathrm{Pf}_i$.

Let us go back to Grassmannians. Suppose we have an $s+1$-dimensional subspace W in $\bigwedge^2 E^\vee$ spanned by $\omega_0, \ldots, \omega_s$. Suppose that, for any $\omega \in W$, rank $\alpha_\omega = 2k$, or, equivalently, the pole C_ω of H_ω is a point. It follows from the theory of determinant varieties that the subvariety

$$\{\mathbb{C}\omega \in |\bigwedge^2 E^\vee| : \mathrm{corank}\ \alpha_\omega \geq i\}$$

is of codimension $\binom{i}{2}$ in $|\bigwedge^2 E^\vee|$ (see [376], [458]). Thus, if $s < 4$, a general W satisfies the assumption. Consider a regular map $\Phi : |W| \to |E|$ defined by $\omega \mapsto C_\omega$. If we take $\omega = t_0\omega_0 + \cdots + t_s\omega_s$ so that $t = (t_0, \ldots, t_s)$ are projective coordinate functions in $|W|$, we obtain that Φ is given by $2k+1$ principal pfaffians of the matrix A_t defining ω.

We shall apply the preceding to the case where $\dim E = 5$. Take a general 3-dimensional subspace W of $\bigwedge^2 E^\vee$. The map $\Phi : |W| \to |E| \cong \mathbb{P}^4$ is defined by homogeneous polynomials of degree 2. Its image is a projected Veronese surface S. Any trisecant line of S passes through three points on S, which are the poles of elements w_1, w_2, w_3 from W. These elements are linearly independent, otherwise their poles lie on the conic image of a line under Φ. But no trisecant line can be contained in a conic plane section of S. We consider $\omega \in W$ as a hyperplane in the Plücker space $|\bigwedge^2 E|$. Thus, any trisecant line is contained in all hyperplanes defined by W. Now, we are ready to prove the following.

Theorem 2.1.6. *Let $\bar{X}$ be the closure in $G_1(\mathbb{P}^4)$ of the locus of trisecant lines of a projected Veronese surface. Then, $\bar{X}$ is equal to the intersection of $G_1(\mathbb{P}^4)$ with three linearly independent hyperplanes. In particular, $\bar{X}$ is a Fano 3-fold of degree 5 with canonical sheaf $\omega_{\bar{X}} \cong \mathcal{O}_{\bar{X}}(-2)$.*

Proof As we observed above, the locus of poles of a general 3-dimensional linear space W of hyperplanes in the Plücker space is a projected Veronese surface V and its trisecant variety is contained in $Y = \cap_{w\in W} H_w \cap G_1(\mathbb{P}^4)$. So, its closure $\bar{X}$ is also contained in Y. On the other hand, we know that $\bar{X}$ is irreducible and 3-dimensional (it contains an open subset isomorphic to the homogeneous space $X = \mathrm{SO}(3)/\mathfrak{S}_4$). By Bertini's Theorem, the intersection of $G_1(\mathbb{P}^4)$ with a general linear space of codimension 3 is an irreducible 3-dimensional variety. This proves that $Y = \bar{X}$. By another Bertini's Theorem, Y is smooth. The rest is the standard computation of the canonical class of the Grassmann variety and the adjunction formula. It is known that the canonical class of the Grassmannian $G = G_m(\mathbb{P}^n)$ of m-dimensional subspaces of $\mathbb{P}^n$ is equal to

$$K_G = \mathcal{O}_G(-n-1). \tag{2.8}$$ cangras

By the adjunction formula, the canonical class of $\bar{X} = G_1(\mathbb{P}^4) \cap H_1 \cap H_2 \cap H_3$ is equal to $\mathcal{O}_{\bar{X}}(-2)$. □

Corollary 2.1.7. *The homogeneous space $X = \mathrm{SO}(3)/\mathfrak{S}_4$ admits a smooth compactification $\bar{X}$ isomorphic to the intersection of $G_1(\mathbb{P}^4)$, embedded via Plücker in $\mathbb{P}^9$, with a linear subspace of codimension 3. The boundary $\bar{X} \setminus X$ is an anti-canonical divisor cut out by a hypersurface of degree 2.*

Proof The only unproven assertion is one about the boundary. To check this, we use that the 3-dimensional group $G = \mathrm{SL}(2)$ acts transitively on a three-dimensional variety X minus the boundary. For any point $x \in X$, consider the map $\mu_x : G \to X, g \mapsto g \cdot x$. Its fiber over the point x is the isotropy subgroup G_x of x. The differential of this map defines a linear map $\mathfrak{g} = T_e(G) \to T_x(X)$. Letting x vary in X, we obtain a map of vector bundles

$$\phi : \mathfrak{g}_X = \mathfrak{g} \times X \to T(X).$$

Now, take the determinant of this map

$$\bigwedge^3 \phi = \bigwedge^3 \mathfrak{g} \times X \to \bigwedge^3 T(X) = K_X^\vee,$$

where K_X is the canonical line bundle of X. The left-hand side is the trivial line bundle over X. The map $\bigwedge^3 \phi$ defines a section of the anti-canonical line bundle. The zeros of this section are the points where the differential of the map μ_x is not injective, i.e., where $\dim G_x > 0$. But this is exactly the boundary of X. In fact, the boundary consists of orbits of dimension smaller than 3, hence the isotropy of each such orbit is of positive dimension. This shows that the boundary is contained in our anti-canonical divisor. Obviously, the latter is contained in the boundary. Thus, we see that the boundary is equal to the intersection of $G_1(\mathbb{P}^4)$ with a quadric hypersurface.

□

rmk:2.1.8 *Remark* 2.1.8. There is another construction of the variety $\mathrm{VSP}(q, 3)$ due to S. Mukai and H. Umemura [532]. Let V_6 be the space of homogeneous binary forms $f(t_0, t_1)$ of degree 6. The group $\mathrm{SL}(2)$ has a natural linear representation in V_6 via linear change of variables. Let $f = t_0 t_1 (t_0^4 - t_1^4)$. The zeros of this polynomial are the vertices of a regular octahedron inscribed in $S^2 = \mathbb{P}^1(\mathbb{C})$. The stabilizer subgroup of f in $\mathrm{SL}(2)$ is isomorphic to the binary octahedron group $\Gamma \cong \mathfrak{S}_4$. Consider the projective linear representation of $\mathrm{SL}(2)$ in $|V_6| \cong \mathbb{P}^5$. In the loc. cit. it is proven that the closure $\bar{X}$ of this orbit in $|V_6|$ is smooth and $B = \bar{X} \setminus X$ is the union of the orbits of $[t_0^5 t_1]$ and $[t_0^6]$. The dimension of the first orbit is equal to 2. Its isotropy subgroup is isomorphic to the multiplicative group $\mathbb{C}^*$. The second orbit is one-dimensional, and it is contained in the closure of the first one. The isotropy subgroup is isomorphic to the subgroup of upper triangular matrices. One can also show that B is equal to the image of $\mathbb{P}^1 \times \mathbb{P}^1$ under a $\mathrm{SL}(2)$-equivariant map given by a linear system of curves of bidegree $(5, 1)$. Thus, B is of degree 10, hence is cut out by a quadric. The image of the second orbit is a smooth rational curve in B and is equal to the singular locus of B. The fact that the two varieties are isomorphic follows from the theory of Fano 3-folds. It can be shown that there is a unique Fano threefold V with $\mathrm{Pic}(V) = \mathbb{Z}\frac{1}{2}K_V$ and $K_V^3 = 40$ (see [425, Theorem 3.3.1]).

SS:2.1.4

2.1.4 Conjugate triangles

Let $C = V(f)$ be a nonsingular conic. Given a triangle with sides ℓ_1, ℓ_2, ℓ_3, the poles of the sides are the vertices of a triangle, called the *conjugate triangle*. Its sides are the polar lines of the vertices of the original triangle. It is clear that this defines a duality in the set of triangles. Clearly, a triangle is *self-conjugate* if and only if it is a self-polar triangle.

The following is an example of conjugate triangles. Let ℓ_1, ℓ_2, ℓ_3 be three tangents to C at the points p_1, p_2, p_3, respectively. They form a triangle which can be viewed as a *circumscribed triangle*. It follows from Theorem T11 1.1.5 that the conjugate triangle has vertices p_1, p_2, p_3. It can be viewed as an *inscribed triangle*. The lines $\ell_1' = \overline{p_2p_3}, \ell_2' = \overline{p_1p_3}, \ell_3' = \overline{p_1p_2}$ are polar lines with respect to the vertices q_1, q_2, q_3 of the circumscribed triangle (see the picture).

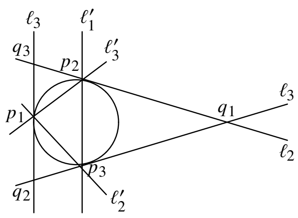


Figure 2.1 Special conjugate triangles Tpqr

In general, let a side ℓ_i of a triangle Δ intersect the conic C at p_i and p_i'. Then, the vertices of the conjugate triangle are the intersection points of the tangents of C at the points p_i, p_i'.

Two lines in $\mathbb{P}^2$ are called *conjugate* with respect to C if the pole of one of the lines belongs to the other line. It is a reflexive relation on the set of lines. Obviously, two triangles are conjugate if and only if each of the sides of the first triangle is conjugate to a side of the second triangle.

Recall the basic notion of projective geometry, the *perspectivity*. Two triangles are called *perspective* from a line (resp. from a point) if there exists a bijection between their sets of sides (resp. vertices) such that the intersection points of the corresponding sides (resp. the lines joining the corresponding points) lie on the same line (resp. intersect at one point). The line is called the *line of perspectivity* or *perspectrix*, and the point is called the *center of perspectivity* or *perspector*. *Desargues' Theorem* asserts that the properties of being perspective from a line or from a point are equivalent.

T2.7.1 **Theorem 2.1.9** (M. Chasles)**.** *Two conjugate triangles with no common vertex are perspective.*

Proof Choose coordinates such that the sides ℓ_1, ℓ_2, ℓ_3 of the first triangle are $t_0 = 0, t_1 = 0, t_2 = 0$, respectively. Then, the vertices of the first triangle $\ell_2 \cap \ell_3 = p_1 = [1,0,0], \ell_1 \cap \ell_3 = p_2 = [0,1,0]$ and $\ell_1 \cap \ell_3 = p_3 = [0,0,1]$. Let

$$A = \begin{pmatrix} a & b & c \\ b & d & e \\ c & e & f \end{pmatrix} \tag{2.9}$$ m11

be the symmetric matrix defining the conic. Then, the polar lines ℓ_i' of the point p_i is given by the equation $\alpha t_0 + \beta t_1 + \gamma t_2 = 0$, where (α, β, γ) is the i-th column of A. The vertices of the conjugate triangle are $\ell_1 \cap \ell_1' = (0, c, -b), \ell_2 \cap \ell_2' = (e, 0, -b)$ and $\ell_3 \cap \ell_3' = (e, -c, -0)$. The condition that the points are collinear is the vanishing of the determinant

$$\det \begin{pmatrix} 0 & c & -b \\ e & 0 & -b \\ e & -c & 0 \end{pmatrix}.$$

Computing the determinant, we verify that it indeed vanishes. □

Now, let us consider the following problem. Given two triangles $\{\ell_1, \ell_2, \ell_3\}$ and $\{\ell_1', \ell_2', \ell_3'\}$ without common sides, find a conic C such that the triangles are conjugate to each other with respect to C.

Since $\dim \bigwedge^3 E = 1$, we can define a natural isomorphism $|\bigwedge^2 E^\vee| \to |E|$. Explicitly, it sends the line $[l \wedge l']$ to the intersection point $[l] \cap [l']$. Suppose the two triangles are conjugate with respect to a conic C. Let $|E| \to |E^\vee|$ be the isomorphism defined by the conic. The composition $|\bigwedge^2 E^\vee| \to |E| \to |E^\vee|$ must send $\ell_i \wedge \ell_j$ to ℓ_k'. Let $\ell_i = [l_i], \ell_i' = [l_i']$. Choose coordinates t_0, t_1, t_2 in E, and let X, Y be the 3×3-matrices with the jth row equal to coordinates of ℓ_i and ℓ_i', respectively. Of course, these matrices are defined by the triangles only up to scaling the columns. It is clear that the k-column of the inverse matrix X^{-1} can be taken for the coordinates of the point $\ell_i \cap \ell_j$ (here $i \neq j \neq k$). Now, we are looking for a symmetric matrix A such that $AX^{-1} = {}^tY$. The converse is also true. If we find such a matrix, the rows of X and Y would represent two conjugate triangles with respect to the conic defined by the matrix A. Fix some coordinates of the sides of the two triangles. This will fix the matrices X, Y. Now, we are looking for a diagonal invertible matrix D such that

$$QA = {}^tYDX \quad \text{is a symmetric matrix.} \tag{2.10}$$ tepl1

There are three linear conditions $a_{ij} = a_{ji}$ for a matrix $A = (a_{ij})$ to be symmetric. So, we have three equations with three unknowns, the entries of the matrix D. The condition for the existence of a solution must be given in terms of a determinant whose entries depend on the coordinates of the sides of the triangles. We identify l_i and l_i' with vectors in $\mathbb{C}^3$ and use the dot-product in $\mathbb{C}^3$ to get the following three equations with unknowns $\lambda_1, \lambda_2, \lambda_3$:

$$\lambda_1 l_1 \cdot l_2' - \lambda_2 l_2 \cdot l_1' = 0$$
$$\lambda_1 l_1 \cdot l_3' - \lambda_3 l_3 \cdot l_1' = 0$$
$$\lambda_2 l_2 \cdot l_3' - \lambda_3 l_3 \cdot l_2' = 0.$$

The matrix of the coefficients of the system of linear equations is equal to

$$M = \begin{pmatrix} l_1 \cdot l_2' & -l_2 \cdot l_1' & 0 \\ l_1 \cdot l_3' & 0 & -l_3 \cdot l_1' \\ 0 & l_2 \cdot l_3' & -l_3 \cdot l_2' \end{pmatrix}.$$

The necessary condition is that

$$\det M = (l_3 \cdot l_1')(l_1 \cdot l_2')(l_2 \cdot l_3') - (l_2 \cdot l_1')(l_1 \cdot l_3')(l_3 \cdot l_2') = 0. \qquad (2.11)$$ tepl2

We also need a solution with nonzero coordinates. It is easy to check (for example, by taking coordinates in which X or Y is the identity matrix), that the existence of a solution with a zero coordinate implies that the triangles have a common vertex. This contradicts our assumption.

Note that condition (2.10) is invariant with respect the action of $\mathrm{GL}(E)$ since any $G \in \mathrm{GL}(E)$ transforms X, Y to GX, GY, and hence, transforms A to ${}^tGAG$ which is still symmetric. Taking $l_1 = t_0, l_2 = t_1, l_3 = t_2$, we easily check that condition (2.11) is equivalent to the condition that the two triangles with sides defined by l_1, l_2, l_3 and l_1', l_2', l_3' are perspective from a line. Thus, we obtain the following:

Corollary 2.1.10. *Two triangles with no common side are conjugate triangles with respect to some conic if and only if they are perspective triangles.*

Taking the inverse of the matrix A from (2.10), we obtain that $X^{-1}D^{-1t}B^{-1}$ is symmetric. It is easy to see that the j-th column of X^{-1} can be taken for the coordinates of the side of the triangle opposite the vertex defined by the j-th column of X. This shows that the dual triangles are conjugate with respect to the dual quadric defined by the matrix A^{-1}. This proves *Desargues' Theorem*, we used before.

Theorem 2.1.11. *Two triangles are perspective from a point if and only if they are perspective from a line.*

Let C be a nonsingular conic and $\mathfrak{o}$ be a point in the plane but not in C. The projection from $\mathfrak{o}$ defines an involution $\tau_{\mathfrak{o}}$ on C with two fixed points equal to the set $P_{\mathfrak{o}}(C) \cap C$. This involution can be extended to the whole plane such that $\mathfrak{o}$ and the polar line $P_{\mathfrak{o}}$ is its set of fixed points. To show this, we may assume C is the conic $V(t_0t_2 - t_1^2)$, image of the Veronese map $\mathsf{v}_2 : \mathbb{P}^1 \to C$, $[u_0, u_1] \mapsto [u_0^2, u_0u_1, u_1^2]$. We identify a point $x = [x_0, x_1, x_2]$ in the plane with a symmetric matrix

$$X = \begin{pmatrix} x_0 & x_1 \\ x_1 & x_2 \end{pmatrix},$$

so that the conic is given by the equation $\det X = 0$. Consider the action of $G \in SL(2)$ on $\mathbb{P}^2$ which sends X to ${}^tGXG$. This defines an isomorphism from PSL(2) to the subgroup of PGL(3) leaving the conic C invariant. In this way, any automorphism of C extends to a projective transformation of the plane leaving C invariant. Any nontrivial element of finite order in PGL(3) is represented by a diagonalizable matrix, and hence, its set of fixed points consists of either a line plus a point, or three isolated points. The first case occurs when there are two equal eigenvalues, and the second one occurs when all eigenvalues are distinct. In particular, an involution belongs to the first case. It follows from the definition of the involution τ that the two intersection points of $P_{\mathfrak{o}}(C)$ with C are fixed under the extended involution $\tilde{\tau}$. So, the point $\mathfrak{o}$, being the intersection of the tangents to C at these points, is fixed. Thus, the set of fixed points of the extended involution $\tilde{\tau}$ is equal to the union of the line $P_{\mathfrak{o}}(C)$ and the point $\mathfrak{o}$.

As an application, we get a proof of the following *Pascal's Theorem* from projective geometry.

Theorem 2.1.12. *Let $p_1, \ldots, p_6$ be the set of vertices of a hexagon inscribed in a nonsingular conic C. Then, the intersection points of the opposite sides $\overline{p_ip_{i+1}} \cap \overline{p_{i+3}p_{i+4}}$, where i is taken modulo 3, are collinear.*

Proof A projective transformation of $\mathbb{P}^1$ is uniquely determined by the images of three distinct points. Consider the transformation of the conic C (identified with $\mathbb{P}^1$ by a Veronese map) which transforms p_i to $p_{i+3}, i = 1, 2, 3$. This transformation extends to a projective transformation τ of the whole plane leaving C invariant. Under this transformation, the pairs of the opposite sides $\overline{p_ip_{i+3}}$ are left invariant. Thus, their intersection point is fixed. A projective transformation with three fixed points on a line fixes the line pointwise. So, all three intersection points lie on a line. □

The line joining the intersection points of opposite sides of a hexagon is called the *Pascal line*. Changing the order of the points, we get 60 Pascal lines associated with six points on a conic.

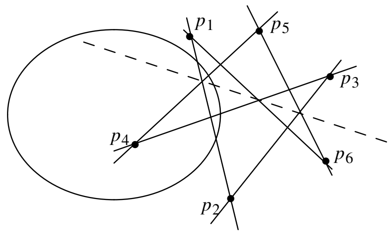


Figure 2.2 Pascal's Theorem

pqr

One can see that the triangle Δ_1 with sides $\overline{p_1p_2}, \overline{p_1p_6}, \overline{p_2p_3}$ and the triangle Δ_2 with sides $\overline{p_4p_5}, \overline{p_3p_4}, \overline{p_5p_6}$ are in perspective from the Pascal line. Hence, they are perspective from the pole of the Pascal line with respect to the conic. Note that not all vertices of the triangles are on the conic.

Via duality, we obtain *Brianchon's Theorem.*

Theorem 2.1.13. *Let $p_1, \ldots, p_6$ be the set of vertices of a hexagon whose sides touch a nonsingular conic C. Then, the diagonals $\overline{p_ip_{i+3}}, i = 1, 2, 3$ intersect at one point.*

We leave it to the reader to find two perspective triangles in this situation.

We view a triangle as a point in $(\mathbb{P}^2)^3$. Thus, the set of ordered pairs of conjugate triangles is an open subset of the hypersurface in $(\mathbb{P}^2)^3 \times (\mathbb{P}^2)^3 = (\mathbb{P}^2)^6$ defined by Equation (2.11). The equation is multilinear and is invariant with respect to the projective group $\mathrm{PGL}(3)$ acting diagonally, with respect to the cyclic group of order 3 acting diagonally on the product $(\mathbb{P}^2)^3 \times (\mathbb{P}^2)^3$, and with respect to the switch of the factors in the product $(\mathbb{P}^2)^3 \times (\mathbb{P}^2)^3$. It is known from the invariant theory that the determinant of the matrix M, considered as a section of the sheaf $H^0((\mathbb{P}^2)^6, \mathcal{O}_{\mathbb{P}^2}^{\boxtimes 6})$ must be a linear combination of the products of the maximal minors (ijk) of the matrix whose columns are the six vectors $l_1, l_1', l_2, l_2', l_3, l_3'$ such that each columns occurs in the product once. We use that $\det M = 0$ expresses the condition that the intersection points $\ell_i \cap \ell_i'$ are collinear.

Fix a basis in $\Lambda^3(E)$ to define a natural isomorphism

$$\bigwedge^2 (\bigwedge^2 E) \to E, \ (v_1 \wedge v_2, w_1 \wedge w_2) \mapsto (v_1 \wedge v_2 \wedge w_1)w_2 - (v_1 \wedge v_2 \wedge w_2)w_1.$$

This corresponds to the familiar identity for the vector product of 3-vectors

$$(v_1 \times v_2) \times (w_1 \wedge w_2) = (v_1 \times v_2 \times w_1)w_2 - (v_1 \times v_2 \times w_2)w_1.$$

If we apply this formula to $E^\vee$ instead of E, we obtain that the line spanned by the points $\ell_1 \cap \ell_1'$ and $\ell_2 \cap \ell_2'$ has equation $\det(l_1, l_1', l_2)l_2' - \det(l_1, l_1', l_2')l_2 = 0$. The condition that this line also passes through the intersection point $\ell_3 \cap \ell_3'$ is

$$\det(l_3, l_3', \det(l_1, l_1', l_2)l_2' - \det(l_1, l_1', l_2')l_2)$$

$$= \det(l_1, l_1', l_2)\det(l_3, l_3', l_2') - \det(l_1, l_1', l_2')\det(l_3, l_3', l_2) = 0.$$

This shows that the determinant in (2.11) can be written in symbolic form as

$$(12, 34, 56) := (123)(456) - (124)(356). \tag{2.12}$$

Remark 2.1.14. Let X be be the symmetric product $(\mathbb{P}^2)^{(3)} = (\mathbb{P}^2)^3/\mathfrak{S}_3$ of three copies of $\mathbb{P}^2$. Let $\mathcal{T} \subset X$ be the open subset of X that consists of triangles; that is, non-collinear unordered triples of points. Its complement is a hypersurface in X. Fix a smooth conic C and consider the open subset $\mathcal{T}'$ of $\mathcal{T}$ that consists of triangles with no vertex on C and no side is tangent to C. Its complement is the union of two irreducible hypersurfaces in T. Intersecting the sides of a triangle $T \in \mathcal{T}'$ with C, we obtain three pairs of points on C. Conversely, a triple of pairs of points on C defines a triangle from $\mathcal{T}'$. The group $\mathrm{Aut}(\mathbb{P}^2, C)$ acts on $\mathcal{T}'$ with finite stabilizers isomorphic to $\mathfrak{S}_4$. Let Y be the orbits pace. It is a 3-dimensional variety. A triple of pairs of points on C defines a hyperelliptic curve, the double cover of C branched at the six points. Each pair of points defines a 2-torsion divisor class in $\mathrm{Jac}(C)$, and the three pairs define an isotropic plane in the subgroup $\mathrm{Jac}(C)[2] \cong \mathbb{F}_2^4$ of 2-torsion points equipped with the structure of a symplectic linear space over $\mathbb{F}_2$ (see Subsection 5.2.2). Let $\mathcal{M}_2^{\mathrm{iso}}$ be the moduli space of hyperelliptic curves with a choice of an isotropic plane in $\mathrm{Jac}(C)[2]$. This is an étale cover of $\mathcal{M}_2$ of degree 15 (the number of isotropic planes). It follows from above that the variety $Y = \mathcal{T}'/\mathrm{PO}(3)$ is isomorphic to $\mathcal{M}_2^{\mathrm{iso}}$. Now, we use that, taking the conjugate triangle with respect to T defines a biregular involution $\mathfrak{R}$ of $\mathcal{T}'$. It is known classically as the *Richelot involution*. It is a generalization of Gauss' arithmetic-geometric mean for elliptic curvets. We refer to this and related constructions to [253, Example 2].

Note that the locus of fixed points of the Richelot involution $\mathfrak{R}$ consists of one point represented by the orbit of a self-conjugate triangle. If we choose the conic C to be $V(x^2 - y^2 + z^2)$ and take $V(xyz)$ as a self-conjugate triangle, we obtain the hyperelliptic curve

$$w^2 - (u^4 - v^4)uv = 0$$

together with the isotropic plane defined by the pairs of the Weierstrass points $\{(1, 0, 0), (0, 1, 0)\}, \{(1, \pm 1, 0)\}, \{(1, \pm i, 0\}$. The curve is the unique, up to isomorphism, hyperelliptic curve with the group of automorphisms isomorphic to the dihedral group D_8 of order 8 (Type (VI) in Bolza's classification of the automorphism groups of hyperelliptic curves of genus 2 [64]).

2.2 Poncelet relation

2.2.1 Darboux's Theorem

Let C be a conic, and let $T = \{\ell_1, \ell_2, \ell_3\}$ be a circumscribed triangle. A conic C' which has T as an inscribed triangle is called the *Poncelet related conic*. Since passing through a point impose one condition, we have ∞^2 Poncelet related conics corresponding to a fixed triangle T. Varying T, we expect to get ∞^5 conics, so that any conic is Poncelet related to C with respect to some triangle. But surprisingly, this is wrong! Darboux's Theorem asserts that there is a pencil of divisors $p_1 + p_2 + p_3$ such that the triangles T with sides tangent to C at the points p_1, p_2, p_3 define the same Poncelet related conic.

We shall prove this here. In fact, we shall prove a more general result, in which triangles are replaced with n-polygons. An *n-polygon* P in $\mathbb{P}^2$ is an ordered set of $n \geq 3$ points $(p_1, \ldots, p_n)$ in $\mathbb{P}^2$ such that no three points p_i, p_{i+1}, p_{i+2} are collinear. The points p_i are the *vertices* of P, the lines $\overline{p_i, p_{i+1}}$ are called the *sides* of P (here $p_{n+1} = p_1$). The group $\mathfrak{S}_n$ acts transitively on the set of n-polygons with the stabilizer subgroup generated by the permutation $(n, n-1, \ldots, 1)$ and the cyclic subgroup of order n. Therefore, the number of n-gons with the same set of vertices is equal to $n!/2n = (n-1)!/2$.

We say that P circumscribes a nonsingular conic C if each side is tangent to C. Given any ordered set $(q_1, \ldots, q_n)$ of n points on C, let ℓ_i be the tangent lines to C at the points q_i. Then, they are the sides of the n-gon P with vertices $p_i = \ell_i \cap \ell_{i+1}, i = 1, \ldots, n$ $(\ell_{n+1} = \ell_1)$. The n-gon P circumscribes C. This gives a one-to-one correspondence between n-gons circumscribing C and ordered sets of n points on C.

Let $P = (p_1, \ldots, p_n)$ be a n-gon that circumscribes a nonsingular conic C. A conic S is called *Poncelet n-related* to C with respect to P if all points p_i lie on C.

Let us start with any two conics C and S. We choose a point p_1 on S and a tangent ℓ_1 to C passing through p_1. It intersects S at another point p_2. We repeat this construction. If the process stops after n steps (i.e. we are not getting new points p_i), we get an inscribed n-gon in S which circumscribes C. In this case S is Poncelet related to C. The *Darboux Theorem* which we will prove later says that, if the process stops, we can construct infinitely many n-gons with this property starting from an arbitrary point on S.

Consider the following correspondence on $C \times S$:

$$R = \{(x, y) \in C \times S : \overline{xy} \text{ is tangent to } C \text{ at } x\}.$$

Since, for any $x \in C$ the tangent to C at x intersects S at two points, and, for any $y \in S$ there are two tangents to C passing through y, so we get that R is of bidegree $(2, 2)$. This means if we identify C, S with $\mathbb{P}^1$, then R is a curve of bidegree $(2, 2)$. As is well-known, R is a curve of arithmetic genus one.

nonsingular **Lemma 2.2.1.** *The curve R is nonsingular if and only if the conics C and S intersect at four distinct points. In this case, R is isomorphic to the double cover of C (or S) ramified over the four intersection points.*

Proof Consider the projection map $\pi_S : R \to S$. This is a map of degree 2. A branch point $y \in S$ is a point such that there is only one tangent to C passing through y. Obviously, this is possible only if $y \in C$. It is easy to see that R is nonsingular if and only if the double cover $\pi_S : R \to S \cong \mathbb{P}^1$ has four branch points. This proves the assertion. □

Note that, if R is nonsingular, the second projection map $\pi_C : R \to C$ must also have 4 branch points. A point $x \in C$ is a branch point if and only if the tangent of C at x is tangent to S. So we obtain that two conics intersect transversally if and only if there are four different common tangents.

Take a point $(x[0], y[0]) \in R$, and let $(x[1], y[1]) \in R$ be defined as follows: $y[1]$ is the second point on S on the tangent to $x[0]$, $x[1]$ is the point on C different from $x[0]$ at which a line through $y[1]$ is tangent to C. This defines a map $\tau_{C,S} : R \to R$. This map has no fixed points on R, and hence, if we fix a group law on R, is a translation map t_a with respect to a point a. Obviously, we get an n-gon if and only if t_a is of order n, i.e. the order of a in the group law is n. As soon as this happens, we can use the automorphism for constructing n-gons starting from an arbitrary point $(x[0], y[0])$. This is the Darboux Theorem which we have mentioned above.

gdarboux **Theorem 2.2.2** (G. Darboux)**.** *Let C and S be two nondegenerate conics intersecting transversally. Then, C and S are Poncelet n-related if and only if the automorphism $\tau_{C,S}$ of the associated elliptic curve R is of order n. If C and S are Poncelet n related, then starting from any point $x \in C$ and any point $y \in S$ there exists an n-gon with a vertex at y and one side tangent to C at y which circumscribes C and inscribed in S.*

To check explicitly whether two conics are Poncelet related, one needs to recognize when the automorphism $\tau_{C,S}$ is of finite order. Let us choose projective coordinates such that C is the Veronese conic $t_0t_2 - t_1^2 = 0$, the image of $\mathbb{P}^1$ under the Veronese map $[t_0, t_1] \mapsto [t_0^2, t_0t_1, t_2^2]$. By using a projective transformation leaving C invariant we may assume that the four intersection points p_1, p_2, p_3, p_4 of C and S are the images of the points $0, 1, \infty, a$. Then, R is isomorphic to the elliptic curve given by the affine equation

$$y^2 = x(x-1)(x-a).$$

The conic S belongs to the pencil of conics with base points $p_1, \ldots, p_4$:

$$(t_0t_2 - t_1^2) + \lambda t_1(at_0 - (1+a)t_1 + t_2) = 0.$$

We choose the zero point in the group law on R to be the point $(x[0], y[0]) = (p_4, p_4) \in C \times S$. Then, the automorphism $\tau_{C,S}$ sends this point to $(x[1], y[1])$, where

$$y[1] = (\lambda a, \lambda(1+a)+1, 0), \quad x[1] = ((a+1)^2\lambda^2, 2a(1+a)\lambda, 4a^2).$$

Thus, $x[1]$ is the image of the point $(1, \frac{2a}{(a+1)\lambda}) \in \mathbb{P}^1$ under the Veronese map. The point $y[1]$ corresponds to one of the two roots of the equation

$$y^2 = \frac{2a}{(a+1)\lambda}\Big(\frac{2a}{(a+1)\lambda} - 1\Big)\Big(\frac{2a}{(a+1)\lambda} - a\Big).$$

We need a criterion characterizing points $(x, \pm\sqrt{x(x-1)(x-a)})$ of finite order. Note that different choice of the sign corresponds to the involution $x \mapsto -x$ of the elliptic curve. So, the order of the points corresponding to two different choices of the sign are the same. We have the following result of A. Cayley.

Theorem 2.2.3 (A. Cayley)**.** *Let R be an elliptic curve with affine equation*

$$y^2 = g(x),$$

where $g(x)$ is a cubic polynomial with three distinct nonzero roots. Let $y = \sum_{i=0}^{\infty} c_i x^i$ be the formal power Taylor expansion of y in terms of the local parameter x at the point $p = (0, \sqrt{g(0)})$. Then, p is of order $n \geq 3$ if and only if

$$\begin{vmatrix} c_2 & c_3 & \ldots & c_{k+1} \\ c_3 & c_4 & \ldots & c_{k+2} \\ \vdots & \vdots & \vdots & \vdots \\ c_{k+1} & c_{k+2} & \ldots & c_{2k} \end{vmatrix} = 0, \quad n = 2k+1,$$

$$\begin{vmatrix} c_3 & c_4 & \ldots & c_{k+1} \\ c_4 & c_5 & \ldots & c_{k+2} \\ \vdots & \vdots & \vdots & \vdots \\ c_{k+1} & c_{k+2} & \ldots & c_{2k-1} \end{vmatrix} = 0, \quad n = 2k.$$

Proof Let ∞ be the point at infinity of the affine curve $y^2 - g(x) = 0$. The rational function x (resp. y) has pole of order 2 (resp. 3) at ∞. If $n = 2k+1$, the rational functions $1, x, \ldots, x^k, y, xy, \ldots, x^{k-1}y$ form a basis of the linear space $H^0(C, \mathcal{O}_C(n\infty))$. If $n = 2k$, the same is true for the functions $1, x, \ldots, x^k, y, xy, \ldots, x^{k-2}y$. A point $p = (0, c_0)$ is a n-torsion point if and only if there is a linear combination of these functions which vanishes at this point with order n. Since x is a local parameter at the point p, we can expand y in a formal power series $y = \sum_{k=0}^{\infty} c_k x^k$. Let us assume $n = 2k+1$, the other case is treated similarly. We need to find some numbers $(a_0, \ldots, a_{2k})$ such that, after plugging in the formal power series,

$$a_0 + a_1 x + \ldots + a_k x^k + a_{k+1} y + \ldots + a_{2k} x^{k-1} y$$

is divisible by x^{2k+1}. This gives a system of n linear equations

$$a_i + a_{k+1} c_i + \cdots + a_{k+1+i} c_0 = 0, \ i = 0, \ldots, k,$$
$$a_{2k} c_{2+i} + a_{2k-1} c_{3+i} + \cdots + a_{k+1} c_{k+1+i} = 0, \ i = 0, \ldots, k-1.$$

The first $k+1$ equations allow us to eliminate $a_0, \ldots, a_k$. The last k equations have a solution for $(a_{k+1}, \ldots, a_{2k})$ if and only if the first determinant in the assertion of the Theorem vanishes.

□

To apply the Proposition we have to take

$$\alpha = \frac{2a}{(a+1)\lambda}, \quad \beta = 1 + \frac{2a}{(a+1)\lambda}, \quad \gamma = a + \frac{2a}{(a+1)\lambda}.$$

Let us consider the variety $\mathcal{P}_n$ of pairs of conics (C, S) such that S is Poncelet n-related to C. We assume that C and S intersect transversally. We already know that $\mathcal{P}_n$ is a hypersurface in $\mathbb{P}^5 \times \mathbb{P}^5$. Obviously, $\mathcal{P}_n$ is invariant with respect to the diagonal action of the group SL(3) (acting on the space of conics). Thus, the equation of $\mathcal{P}_n$ is an invariant of a pair of conics. This invariant was computed by F. Gerbardi [329]. It is of bidegree $(\frac{1}{4}T(n), \frac{1}{2}T(n))$, where $T(n)$ is equal to the number of elements of order n in the abelian group $(\mathbb{Z}/n\mathbb{Z})^2$.

Let us look at the quotient of $\mathcal{P}_n$ by PSL(3). Consider the rational map $\beta : \mathbb{P}^5 \times \mathbb{P}^5 \to (\mathbb{P}^2)^{(4)}$ which assigns to (C, S) the point set $C \cap S$. The fiber of β over a subset B of four points in general linear position is isomorphic to an open subset of $\mathbb{P}^1 \times \mathbb{P}^1$, where $\mathbb{P}^1$ is the pencil of conics with base point B. Since we can always transform such B to the set of points $\{[1,0,0], [0,1,0], [0,0,1], [1,1,1]\}$, the group PSL(3) acts transitively on the open subset of such 4-point sets. Its stabilizer is isomorphic to the permutation group $\mathfrak{S}_4$ generated by the following matrices:

$$\begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 & -1 \\ 0 & -1 & -1 \\ 0 & 0 & -1 \end{pmatrix}.$$

The orbit space $\mathcal{P}_n/\mathrm{PSL}(3)$ is isomorphic to a curve in an open subset of $\mathbb{P}^1 \times \mathbb{P}^1/\mathfrak{S}_4$, where $\mathfrak{S}_4$ acts diagonally. By considering one of the projection maps, we obtain that $\mathcal{P}_n/\mathrm{PSL}(3)$ is an open subset of a cover of $\mathbb{P}^1$ of degree N equal to the number of Poncelet n-related conics in a given pencil of conics with 4 distinct base points with respect to a fixed conic from the pencil. This number was computed by F. Gerbardi Gerbardi [329] and is equal to $\frac{1}{2}T(n)$. A modern account of Gerbardi's result is given in Barth1 [34]. A smooth compactification of $\mathcal{P}_n/\mathrm{PSL}(3)$ is the modular curve $X^0(n)$ that parameterizes the isomorphism classes of the pairs (R, e), where R is an elliptic curve and e is a point of order n in R.

gonethree2 **Proposition 2.2.4.** *Let C and S be two nonsingular conics. Consider each n-gon inscribed in C as a subset of its vertices, and also as a positive divisor of degree n on C. The closure of the set of n-gons inscribed in C and circumscribing S is either empty, or a g_n^1, i.e. a linear pencil of divisors of degree n.*

Proof First, observe that two polygons inscribed in C and circumscribing S which share a common vertex must coincide. In fact, the two sides passing through the vertex in each polygon must be the two tangents of S passing through the vertex. They intersect C at another two common vertices. Continuing in this way, we see that the two polygons have the same set of vertices. Now, consider the Veronese embedding v_n of $C \cong \mathbb{P}^1$ in $\mathbb{P}^n$. An effective divisor of degree n is a plane section of the Veronese curve $\mathsf{V}_n^1 = \mathsf{v}_n(\mathbb{P}^1)$. Thus, the set of effective divisors of degree n on C can be identified with the dual projective space $(\mathbb{P}^n)^\vee$. A hyperplane in $(\mathbb{P}^n)^\vee$ is the set of hyperplanes in $\mathbb{P}^n$ which pass through a fixed point in $\mathbb{P}^n$. The degree of an irreducible curve $X \subset (\mathbb{P}^n)^\vee$ of divisors is equal to the cardinality of the set of divisors containing a fixed general point of V_n^1. In our case, it is equal to one. □

SS:2.2.2 2.2.2 Poncelet Curves and Vector Bundles

Let C and S be two Poncelet n-related conics in the plane $\mathbb{P}^2 = |E|$. Recall that this means that there exist n points $p_1, \ldots, p_n$ on C such that the tangent lines $\ell_i = \mathbb{T}_{p_i}(C)$ meet on S. One can drop the condition that S is a conic. We say that a plane curve S of degree $n-1$ is *Poncelet-related* to the conic C if there exist n points, as above, such that the tangents to C at these points meet on S.

We shall prove an analog of the Darboux Theorem for Poncelet-related curves of degree larger than 2. First, we have to remind some constructions in the theory of vector bundles over the projective plane.

Let $\mathbb{P}^1 = |U|$, where U is a two-dimensional vector space and let $\mathbb{P}^2 = |V|$, where V is a three-dimensional vector space. A closed embedding $\mathsf{v} : \mathbb{P}^1 \hookrightarrow \mathbb{P}^2$ has the image isomorphic to a nonsingular conic, a Veronese curve. This defines an isomorphism

$$E^\vee = H^0(|E|, \mathcal{O}_{|E|}(1)) \cong H^0(|U|, \mathcal{O}_{|U|}(2)) = S^2(U^\vee).$$

Its transpose defines an isomorphism $E \cong S^2(U)$. This gives a bijective correspondence between nonsingular conics and linear isomorphisms $E \to S^2(U)$. Also, since $\dim \bigwedge^2 U = 1$, a choice of a basis in $\bigwedge^2 U$ defines a linear isomorphism $U \cong U^\vee$. This gives an isomorphism of projective spaces $|U| \cong |U|^\vee$ that does not depend on a choice of a basis in $\bigwedge^2 U$. Thus, a choice of a nonsingular conic in $|E|$ also defines an isomorphism $|E^\vee| \to |S^2(U)|$ which must be given by a nonsingular conic in $|E^\vee|$. This is, of course, the dual conic.

Fix an isomorphism $\mathbb{P}^2 \cong |S^2(U)|$ defined by a choice of a conic C in $\mathbb{P}^2$. Consider the multiplication map $S^2(U) \otimes S^{n-2}(U) \to S^n(U)$. It defines a rank 2 vector bundle $\mathcal{S}_{n,C}$ on $\mathbb{P}^2$ whose fiber at the point $x = [q] \in |S^2(U)|$ is equal to the quotient space $S^n(U)/qS^{n-2}(U)$. One easily sees that it admits a resolution of the form

$$0 \to S^{n-2}(U)(-1) \to S^n(U) \to \mathcal{S}_{n,C} \to 0, \qquad (2.13)$$ schbundle

where we identify a vector space V with the vector bundle π^*V, where π is the structure map to the point. The vector bundle $\mathcal{S}_{n,C}$ is called the *Schwarzenberger vector bundle* associated to the conic C. Its dual bundle has the fiber over a point $x = [q]$ equal to the linear space

$$(S^n(U)/qS^{n-2}(U))^\vee = \{f \in S^n(U^\vee) : D_q(f) = 0\}. \qquad (2.14)$$ newap

Embedding $|U^\vee|$ in $|S^n(U^\vee)|$ by means of the Veronese map, we will identify the divisor of zeros of q with a divisor $V(q)$ of degree 2 on the Veronese curve $R_n \subset |S^n(U^\vee)|$, or, equivalently, with a 1-secant of R_n. A hyperplane containing this divisor is equal to $V(qg)$ for some $g \in S^{n-2}(U)$. Thus, the linear space (2.14) can be identified with the projective span of $V(q)$. In other words, the fibres of the dual projective bundle $\mathcal{S}_{n,C}^\vee$ are equal to the secants of the Veronese curve R_n.

It follows from (2.13) that the vector bundle $\mathcal{S}_{n,C}$ has the first Chern class of degree $n-1$ and the second Chern class is equal to $n(n-1)/2$. Thus, we expect that a general section of $\mathcal{S}_{n,C}$ has $n(n-1)/2$ zeros. We identify the space of sections of $\mathcal{S}_{n,C}$ with the vector space $S^n(U)$. A point $[s] \in |S^n(U)|$

can be viewed as a hyperplane H_s in $|S^n(U^\vee)|$. Its zeros are the secants of R_n contained in H_s. Since H_s intersects R_n at n points $p_1, \ldots, p_n$, any 1-secant $\overline{p_i p_j}$ is a 1-secant contained in H_s. The number of such 1-secants is equal to $n(n-1)/2$.

Recall that we can identify the conic with $|U|$ by means of the Veronese map $\mathsf{v}_2 : |U| \to |S^2(U)|$. Similarly, the dual conic $C^\vee$ is identified with $|U^\vee|$. By using the Veronese map $\mathsf{v}_n : |U^\vee| \to |S^n(U^\vee)|$, we can identify $C^\vee$ with R_n. Now, a point on R_n is a tangent line on the original conic C, hence n points $p_1, \ldots, p_n$ from above are the sides ℓ_i of an n-gon circumscribing C. A secant $\overline{p_i p_j}$ from above is a point in $\mathbb{P}^2$ equal to the intersection point $q_{ij} = \ell_i \cap \ell_j$. And the $n(n-1)/2$ points q_{ij} represent the zeros of a section s of the Schwarzenberger bundle $\mathcal{S}_{n,C}$.

For any two linearly independent sections s_1, s_2, their determinant $s_1 \wedge s_2$ is a section of $\bigwedge^2 \mathcal{S}_{n,C}$, and hence, its divisor of zeros belongs to the linear system $|\mathcal{O}_{\mathbb{P}^2}(n-1)|$. When we consider the pencil $\langle s_1, s_2 \rangle$ spanned by the two sections, the determinant of each member $s = \lambda s_1 + \mu s_2$ has the zeros on the same curve $V(s_1 \wedge s_2)$ of degree $m-1$.

Let us summarize this discussion by stating and proving the following generalization of the Darboux Theorem.

Theorem 2.2.5. *Let C be a nonsingular conic in $\mathbb{P}^2$ and let $\mathcal{S}_{n,C}$ be the associated Scwarzenberger rank 2 vector bundle over $\mathbb{P}^2$. Then, n-gons circumscribing C are parameterized by $|H^0(\mathcal{S}_{n,C})|$. The vertices of the polygon Π_s defined by a section s correspond to the subscheme $Z(s)$ of zeros of the section s. A curve of degree $n-1$ passing through the vertices corresponds to a pencil of a sections of $\mathcal{S}_{n,C}$ containing s and is equal to the determinant of a basis of the pencil.*

Proof A section s with the subscheme of zeros $Z(s)$ with ideal sheaf $\mathcal{I}_{Z(s)}$ defines the exact sequence

$$0 \to \mathcal{O}_{\mathbb{P}^2} \xrightarrow{s} \mathcal{S}_{n,C} \to \mathcal{I}_Z(n-1) \to 0.$$

A section of $\mathcal{I}_Z(n-1)$ is a plane curve of degree $n-1$ passing through $Z(s)$. The image of a section t of $\mathcal{S}_{n,C}$ in $H^0(\mathcal{I}_Z(n-1))$ is the discriminant curve $s \wedge t$. Any curve defined by an element from $H^0(\mathcal{I}_Z(n-1))$ passes through the vertices of the n-gon Π_s and is uniquely determined by a pencil of sections containing s. □

One can explicitly write the equation of a Poncelet curve as follows. First we choose a basis ξ_0, ξ_1 of the space U and the basis $(\xi_0^d, \xi_0^{d-1}\xi_1, \ldots, \xi_1^d)$ of the space $S^d(U)$. The dual basis in $S^n(U^\vee)$ is $(\binom{d}{i} t_0^{d-i} t_1^i)_{0 \le i \le d}$. Now, the coordinates in the plane $|S^2(U)|$ are $t_0^2, 2t_0t_1, t_2^2$, so a point in the plane is a binary conic $Q = a\xi_0^2 + 2b\xi_0\xi_1 + c\xi_1^2$. For a fixed $x = [Q] \in |S^2(U)|$, the matrix of the multiplication map $S^{n-2}(U) \to S^n(U), G \mapsto QG$ is

$$K(x) = \begin{pmatrix} a & & & & \\ 2b & a & & & \\ c & 2b & \ddots & & \\ & c & \ddots & \ddots & \\ & & \ddots & \ddots & a \\ & & & \ddots & 2b \\ & & & & c \end{pmatrix}.$$

A section of $\mathcal{S}_{n,C}$ is given by $f = \sum_{i=0}^n c_i \xi_0^{n-i}\xi_1^i \in S^n(U)$. Its set of consists of points x such that the vector $\mathbf{c}$ of the coefficients belongs to the column subspace of the matrix $K(x)$. Now, we vary f in a pencil of binary forms whose coefficient vector $\mathbf{c}$ belongs to the nullspace of some matrix A of size $(n-1) \times (n+1)$ and rank $n-1$. The determinant of this pencil of sections is the curve in the plane defined by the degree $n-1$ polynomial equation in $x = [a, b, c]$

$$\det\bigl(K(x) \cdot A\bigr) = 0.$$

Note that the conic C in our choice of coordinates is $V(t_1^2 - t_0t_2)$.

Remark 2.2.6. Recall that a section of $\mathcal{S}_{n,C}$ defines a n-gon in the plane $|S^2(U)|$ corresponding to the hyperplane section $H_s \cap R_n$. Its vertices is the scheme of zeros $Z(s)$ of the section s. Let $\pi : X(s) \to \mathbb{P}^2$ be the blow-up of $Z(s)$. For a general s, the linear system of Poncelet curves through $Z(s)$ embeds the surface $X(s)$ in $|S^n(U^\vee)|$ with the image equal to $H_s \cap \mathrm{Sec}_1(R_n)$. The exceptional curves of the blow-up are mapped onto the secants of R_n which are contained in H_s. These are the secants $\overline{p_ip_j}$, where $H_s \cap R_n = \{p_1, \ldots, p_n\}$. The linear system defining the embedding is the proper transform of the linear system of curves of degree $n-1$ passing through $\frac{1}{2}n(n-1)$ points of $Z(s)$. This implies that the embedded surface $X(s)$ has the degree equal to $(n-1)^2 - \frac{1}{2}n(n-1) = \frac{1}{2}(n-1)(n-2)$. This is also the degree of the secant variety $\mathrm{Sec}_1(R_n)$. For example, take $n = 4$ to get that the secant variety of R_4 is a cubic hypersurface in $\mathbb{P}^4$ whose hyperplane sections are cubic surfaces isomorphic to the blow-up of the six vertices of a complete quadrilateral.

2.2.3 Complex Circles

SS:2.2.3

Fix two points in the plane and consider the linear system of conics passing through the two points. It maps the plane to $\mathbb{P}^3$ with the image equal to a nonsingular quadric $Q = V(q)$. Thus, we may identify each conic from the linear system with a hyperplane in $\mathbb{P}^3$, or using the polarity defined by Q, with a point. When the two points are the points $[0, 1, \pm i]$ in the real projective plane with the line at infinity $t_0 = 0$, a real conic becomes a circle, and we obtain that the geometry of circles can be translated into the orthogonal geometry of real 3-dimensional projective space. In coordinates, the rational map $\mathbb{P}^2 \dashrightarrow \mathbb{P}^3$ is given by

$$[t_0, t_1, t_2] \mapsto [x_0, x_1, x_2, x_3] = [t_1^2 + t_2^2, t_0t_1, t_0t_2, t_0^2].$$

Its image is the quadric

$$Q = V(x_0x_3 - x_1^2 - x_2^2).$$

Explicitly, a point $[v] = [\alpha_0, \alpha_1, \alpha_2, \alpha_3] \in \mathbb{P}^3$ defines the *complex circle*

$$S(v) : \alpha_0(t_1^2 + t_2^2) - 2t_0(\alpha_1 t_1 + \alpha_2 t_2) + \alpha_3 t_0^2 = 0. \qquad (2.15)$$ sphere

By definition, its center is the point $c = [\alpha_0, \alpha_1, \alpha_2]$, its radius square r^2 is defined by the formula

$$\alpha_0^2 r^2 = \alpha_1^2 + \alpha_2^2 - \alpha_0\alpha_3 = q(\alpha). \qquad (2.16)$$ radius1

Let us express the property that two circles are tangent to each other. It applies to complex circles as well.

P:2.2.7 **Proposition 2.2.7.** *Let $[v], [w]$ be two points in $\mathbb{P}^3$, and let $S(v), S(w)$ be two complex circles corresponding to planes in $\mathbb{P}^3$ which are polar to the points with respect to the quadric $Q = V(q)$. Then, the two circles touch each other if and only if*

$$(v, v)(w, w) - (v, w)^2 = 0, \qquad (2.17)$$ circle

where (v, w) denotes the bilinear form associated to the quadratic form q.

Proof Let $\ell = V(\lambda v + \mu w)$ be the line spanned by the points $[v]$ and $[w]$. Via polarity, it corresponds to a pencil of planes in $\mathbb{P}^3$. The pre-images of two planes are tangent if and only if the pencil contains a plane tangent to the quadric Q. Dually this means that the line ℓ is tangent to Q. This is equivalent to the binary form

$$q(\lambda v + \mu w) = \lambda^2(v, v) + 2(v, w)\lambda\mu + \mu^2(w, w)$$

having has a double root. Of course, this happens if and only if (2.17) holds.

$\square$

Note that relation (2.17) is of degree 2 in v and w. If we identify the space of circles with $\mathbb{P}^3$, this implies that the pairs of touching complex circles is a hypersurface in $\mathbb{P}^3 \times \mathbb{P}^3$ of bidegree $(2,2)$. It is easy to see that the diagonal of $\mathbb{P}^3 \times \mathbb{P}^3$ is the double locus of the hypersurface.

Fix two complex irreducible circles $S_1 = S(v)$ and $S_2 = S(w)$ and consider the variety F of complex circles $S(x)$ touching S_1 and S_2. It is equal to the quartic curve E, the intersection of two quadrics Q_{S_1} and Q_{S_2} of conics touching S_1 and S_2'. It is given by equations:

$$(v,v)(x,x) - (v,x)^2 = (w,w)(x,x) - (w,x)^2 = 0.$$

The quadrics S_1 and S_2 are cones with vertices $p_1 = [v]$, $p_2 = [w]$, respectively. Eliminating (x,x) from the equations, we obtain that R is the union of two planes

$$\Pi^{\pm}_{S_1,S_2} = V(\sqrt{(v,v)}(w,x) \pm \sqrt{(w,w)}(v,x)).$$

The planes intersect along the line $(v,x) = (w,x) = 0$. It intersects S_1 and S_2 at two points lying on the null-quadric Q. Each point corresponds to the *null-circle*, i.e. $\alpha_0^2 r^2 = 0$ in (2.16). It is the union of two lines, each through the ideal points on $V(t_0)$ intersecting at one of the intersection points of the circles S_1 and S_2.

In the case where S and S' touch each other, the pencil spanned by S_1 and S_2 is a line component of the quartic curve F. It contains the vertices of the cones, and hence, enters with multiplicity 2. The residual component is an irreducible conic.

Theorem 2.2.8 (J. Steiner)**.** *Suppose S_1 and S_2 are in general position. Let $C_1, \ldots, C_n$ be a sequence of complex circles tangent to S_1 and S_2 such that C_{i+1} is tangent to C_i for all $i = 1, \ldots, n$ (where $C_{n+1} = C_1$). Suppose, after m steps, S_m is equal to S_1. Then, there is a such sequence starting from any circel C_1' tangent to S_1 and S_2.*

Proof Recall that we denoted by F the quartic curve of conics tangent to S_1 and S_2. Let F_0 be one of the irreducible components of F. It is a conic isomorphic to $\mathbb{P}^1$.

$$X = \{(C, C') \in F_0 \times F_0 : C \text{ touches } C'\}.$$

The pre-image of $C \in F_0$ under the projection to the first factor is equal to the intersection of F_0 with the plane tangent to the cone Q_C and another plane. It consists of two equal points and another couple of points a, b. The double point obviously corresponds to the diagonal in $F_0 \times F_0$. So, if we delete the diagonal Δ from X, we obtain a curve $X^0 \subset F_0 \times F_0$ of bidegree $(2, 2)$.

Let $\alpha : X_0 \to F_0$ be the first projection. It is a degree 2 map. The only case where the fiber $\alpha^{-1}(C_1)$ consists of one point is when C_1 is one of the two null-lines touching S_1 and S_2. Thus, we see that X_0 has only two branch points for each of the two projections $X \to R$. Since its arithmetic genus is equal to one, it must consist of two irreducible curves V_1 and V_2 of bidegree $(1, 1)$ intersecting at two points. Since X_0 is obviously invariant with respect to the switch s of the factors, $s(V_1) = V_2$. Let

$$\tau : V_1 \setminus \Delta \to V_1 \setminus \Delta$$

be the map that sends $(C_1, C_2) \in V_1$ to $(C_2, C_3) \in V_1$. Since $V_1 \cong \mathbb{P}^1 \setminus \{a, b\} \cong \mathbb{C}^*$, the automorphism τ of V_1 must be the multiplication by a constant λ. The property that the sequence $C_1, C_2, \ldots, C_n$ obtained by iteration of τ terminates is equivalent to that $\lambda^n = 1$. Since this property is independent of a choice of C_1, the assertion follows.

□

Remark 2.2.9. We followed the proof from [34]. When S and S' are concentric real circles, the assertion is evident. The general case of real conics can be reduced to this case (see [305], [661]). Poncelet's and Steiner's Theorems are examples of a *porism*, which can be loosely stated as follows. If one can find one object satisfying a certain special property then there are infinitely many such objects. There are some other poristic statements for complex circles: Emch' Theorem and the *zig-zag theorem* discussed in [34].

S:2.3

2.3 Quadric Surfaces

SS:2.3.1

2.3.1 Polar Properties of Quadrics

Many of the polar properties of conics admit extension to nonsingular quadrics in higher-dimensional $\mathbb{P}^n$. For example, a *self-polar* $(n+1)$*-hedron* is defined as a collection of $n+1$ ordered hyperplanes $V(l_i)$ in general linear position such that the pole of each plane $V(l_i)$ is equal to the intersection point of the remaining hyperplanes. Similarly to the case of conics, one proves that a self-polar $(n+1)$-hedron is the same as a polar $(n+1)$-hedron of the quadric.

The definition of the conjugate $(n+1)$-hedra is a straightforward extension of the definition of conjugate triangles. We say that two simplexes Σ and Σ' are *mutually polar* with respect to a quadric Q if the poles of the facets of T' are vertices of T. This implies that the images of k-dimensional faces of T under the polarity defined by Q are the opposite $(n-k)$-dimensional facets of Σ'. The condition (2.10) extends to any dimension. However, it does not translate to a single equation on the coefficients of the linear forms defining the polyhedra. This time we have a system of $n(n+1)/2$ linear equations with $n+1$ unknowns and the condition becomes the rank condition.

We adopt the terminology of convex geometry to call the set of $n+1$ linearly independent hyperplanes a *simplex*. The intersection of a subset of k hyperplanes will be called an $(n-k)$-dimensional *face*. If $k = n$, this is a *vertex* , if $k = n-1$, this is an *edge*, if $n = 0$ this is a *facet* .

The notion of perspectivity of triangles extends to quadrics of any dimension. We say that two simplexes are *perspective* from a point $\mathfrak{o}$ if there is a bijection between the sets of vertices such that the lines joining the corresponding vertices pass through the point $\mathfrak{o}$. We say that the two simplexes are perspective from a hyperplane if this hyperplane contains the intersections of corresponding facets. We have also an extension of Desargues' Theorem.

Theorem 2.3.1 (G. Desargues)**.** *Two simplexes are perspective from a point if and only if they are perspective from a hyperplane.*

Proof Without loss of generality, we may assume that the first simplex Σ is the coordinate simplex with vertices $p_i = [e_i]$ and it is perspective from the point $\mathfrak{o} = [e] = [1, \ldots, 1]$. Let $q_i = [v_i]$ be the vertices of the second simplex Σ_2. Then, we have $v_i = e + \lambda_i e_i$ for some scalars λ_i. After subtracting, we obtain $v_i - v_j = \lambda_i e_i - \lambda_j e_j$. Thus, any two edges $\overline{p_i p_j}$ and $\overline{q_i q_j}$ meet at a point r_{ij} which lies in the hyperplane $H = V(\sum_{i=0}^n \frac{1}{\lambda_i} t_i)$. Since the intersection of the facet of Σ_1 opposite the point p_k with the facet of Σ_2 opposite the point q_k contains all points r_{ij} with $i, j \neq k$, and they span the intersection, we get that the two simplexes are perspective from H. The converse assertion follows by duality. □

Remark 2.3.2. As remarked [702], p.252, the previous assertion is a true space generalization of the classical Desargues's Theorem. Other generalization applies to two space triangles and asserts that the perspectivity from a point implies that the intersection points of the corresponding sides (which automatically intersect) are collinear.

Let $b_q : E \to E^\vee$ be an isomorphism defined by a nonsingular quadric $Q = V(q)$. For any linear subspace L of E, the subspace $b_q(L)^\perp$ of E is called the *polar* of L with respect to Q. It is clear that the dimensions of a subspace and its polar subspace add up to the dimension of $|E|$. Two subspaces Λ and Λ' of the same dimension are called *conjugate* if the polar subspace of Λ intersects Λ'.

These classical definitions can be rephrased in terms of standard definitions of multilinear algebra. Let Λ (resp. Λ') be spanned by $[v_1], \ldots, [v_k]$ (resp. $[w_1], \ldots, [w_k]$). For any two vectors $v, w \in E$, let $(v, w)_q$ denote the value of the polar bilinear form b_q of q on (v, w).

L2.2.14 **Lemma 2.3.3.** *Λ and Λ' are conjugate with respect to Q if and only if*

$$\det \begin{pmatrix} (v_1, w_1)_q & (v_2, w_1)_q & \ldots & (v_k, w_1)_q \\ (v_1, w_2)_q & (v_2, w_2)_q & \ldots & (v_k, w_2)_q \\ \vdots & \vdots & \vdots & \vdots \\ (v_1, w_k)_q & (v_2, w_k)_q & \ldots & (v_k, w_k)_q \end{pmatrix} = 0.$$

Proof Let $b_q : E \to E^\vee$ be the linear isomorphism defined by b_q. The linear funtions $b_q(v_1), \ldots, b_q(v_k)$ form a basis of a k-dimensional subspace L of $E^\vee$ whose dual $L^\perp$ is an $(n-k)$-dimensional subspace of E. It is easy to see that the spans of $v_1, \ldots, v_k$ and $w_1, \ldots, w_k$ have a common nonzero vector if and only if $L^\perp$ intersects non-trivially the latter span. The condition for this is that, under the natural identification $\bigwedge^k(E^\vee)$ and $\bigwedge^k(E)^\vee$, we have

$$b_q(v_1) \wedge \ldots \wedge b_q(v_k)(w_1 \wedge \ldots \wedge w_k) = \det((v_i, w_j)_q) = 0.$$

□

It follows from the lemma that the relation to be conjugate is symmetric.

From now on, until the end of this section, we assume that $n = 3$.

A tetrahedron in $\mathbb{P}^3$ with conjugate opposite edges is called *self-conjugate*. It is clear that a polar tetrahedron of Q is self-conjugate, but the converse is not true.

Let T be a tetrahedron with vertices $p_1 = [v_1], \ldots, p_4 = [v_4]$. Suppose that two pairs of opposite edges are conjugate with respect to some quadric Q. Then, T is self-conjugate (see [736, Bd. III, p. 135], or [717, 7.381]). The proof is immediate. Suppose the two conjugate pairs of edges are $(\overline{p_1p_2}, \overline{p_3p_4})$ and $(\overline{p_1p_3}, \overline{p_2p_4})$. For brevity, let us denote $(v_i, v_j)_q$ by (ij). Then, $(13)(24) - (14)(23) = 0$, and $(12)(34) - (14)(23) = 0$ imply, after subtraction, $(13)(24) - (12)(34) = 0$. This means that the remaining pair $(\overline{p_1p_3}, \overline{p_2p_3})$ is conjugate.

We know from Theorem 2.1.9 that two conjugate triangles are perspective. In the case of quadrics we have a weaker property expressed in the following Chasles' Theorem.

Chasles **Theorem 2.3.4.** *[M. Chasles] Let T and T' be two mutually polar tetrahedra with respect to a quadric Q. Suppose no two opposite edges of T are conjugate.*

Then, the lines joining the corresponding vertices belong to the same ruling of lines of some nonsingular quadric Q'.

Proof Let p_1, p_2, p_3, p_4 be the vertices of T and let q_1, q_2, q_3, q_4 be the vertices of T'. In the following, $\{i, j, k, l\} = \{1, 2, 3, 4\}$. By definition, q_l is a pole of the plane spanned by p_i, p_j, p_k and the matching between the vertices is $p_i \mapsto q_i$. Suppose the edge $\overline{p_i p_j}$ is not conjugate to the opposite edge $\overline{p_k p_l}$. This means that it does not intersect the edge $\overline{q_i q_j}$. This implies that the lines $\overline{p_i q_i}$ and $\overline{p_j q_j}$ do not intersect. By symmetry of the conjugacy relation, we also obtain that the lines $\overline{p_k q_k}$ and $\overline{p_l q_l}$ do not intersect. Together this implies that we may assume that the first three lines $\ell_i = \overline{p_i q_i}$ are not coplanar.

Without loss of generality, we may assume that the first tetrahedron T is the coordinate tetrahedron. Let $A = (a_{ij})$ be a symmetric matrix defining the quadric Q and let $C = \mathrm{adj}(A) = (c_{ij})$ be the adjugate matrix defining the dual quadric. The coordinates of facets of T are columns of $A = (a_{ij})$. The coordinates of the intersection point of three facets defined by three columns A_i, A_j, A_k of A are equal to the column C_m of C, where $m \neq i, j, k$. Thus, a general point on the line generated by the point $[1, 0, 0, 0]$ has coordinates $[\lambda, \mu c_{12}, c_{13}, c_{14}]$, and similar for other three lines. Recall that by Steiner's construction (see [360, p. 528], one can generate a nonsingular quadric by two projectively equivalent pencils of planes through two skew lines. The quadric is the union of the intersection of the corresponding planes. Apply this construction to the pencil of planes through the first two lines. They projectively matched by the condition that the corresponding planes in the pencils contain the same point $[c_{31}, c_{32}, \lambda, c_{41}]$ on the third line. The two planes from each pencil are defined by the equations

$$\det\begin{pmatrix} t_0 & t_1 & t_2 & t_3 \\ 1 & 0 & 0 & 0 \\ \lambda c_{11} & c_{12} & c_{13} & c_{14} \\ c_{31} & c_{32} & \lambda & c_{34} \end{pmatrix}$$

$$= t_1 c_{13} c_{34} + t_2(c_{14} c_{32} - c_{12} c_{34}) - t_3 c_{13} c_{32} + \lambda(t_3 c_{12} - t_1 c_{14}) = 0,$$

$$\det\begin{pmatrix} t_0 & t_1 & t_2 & t_3 \\ 0 & 1 & 0 & 0 \\ c_{21} & c_{22} & c_{23} & c_{24} \\ c_{31} & c_{32} & \lambda & c_{34} \end{pmatrix}$$

$$= t_0 c_{23} c_{34} + t_2(c_{24} c_{31} - c_{21} c_{34}) - t_3 c_{23} c_{31} + \lambda(t_3 c_{21} - t_1 c_{24}) = 0,$$

Eliminating λ, we find the equation of the quadric

$$(c_{12}c_{34} - c_{24}c_{13})(c_{23}t_0t_3 + c_{14}t_1t_2) + (c_{13}c_{24} - c_{14}c_{23})(c_{12}t_2t_3 + c_{34}t_0t_1)$$

$$+(c_{14}c_{23} - c_{12}c_{34})(c_{13}t_1t_3 + c_{24}t_0t_2) = 0.$$

By definition, the quadric contains the first three lines. It is immediately checked that a general point $[c_{41}, c_{42}, c_{43}, \lambda]$ on the fourth line lies on the quadric. □

The following result follows from the beginning of the proof.

P2.2.17 **Proposition 2.3.5.** *Let T and T' be two mutually polar tetrahedra. Assume that T, and hence T', is self-conjugate. Then, T and T' are in perspective from the intersection points of the lines joining the corresponding vertices and perspective from the polar plane of this point.*

One can think that the covariant quadric Q' constructed in the proof of Chasles' Theorem 2.3.4 degenerates to a quadratic cone. Counting parameters, it is easy to see that the pairs of perspective tetrahedra depend on the same number 19 of parameters as pairs of tetrahedra mutually polar with respect to some quadric. It is claimed in [29, Volume 3, p. 45], that any two perspective tetrahedra are, in fact, mutually polar with respect to some quadric. Note that the polarity condition imposes three conditions, and the self-conjugacy condition imposes two additional conditions. This agrees with counting constants ($5 = 24 - 19$).

One can apply the previous construction to the problem of writing a quadratic form q as a sum of five squares of linear forms. Suppose we have two self-conjugate tetrahedra T and T' with respect to a quadric Q that are also mutually polar with respect to Q. By Proposition 2.3.5, they are perspective. Choose coordinates such that T is the coordinate tetrahedron and let $A = (a_{ij})_{0\le i,j\le 3}$ be a symmetric matrix defining Q. We know that the equations of facets H_i of T' are $V(\sum_{j=0}^{3} a_{ij}t_j)$. Since T is self-conjugate, the intersection lines $H_0 \cap H_1$ meet the coordinate lines $t_0 = t_1 = 0$. This means that the equations $a_{20}t_2 + a_{30}t_3 = 0$ and $a_{21}t_2 + a_{31}t_3 = 0$ have a nonzero solution, i.e. $a_{20}a_{31} = a_{21}a_{30}$. Similarly, we get that $a_{10}a_{32} = a_{30}a_{12}$ and $a_{01}a_{32} = a_{02}a_{31}$. Using the symmetry of the matrix, this implies that the six products are equal. Hence, $a_{03}a_{13}/a_{12} = a_{23}a_{03}/a_{02} = a_{03}a_{13}/a_{01}$ are all equal to some number α. Then, the equation of the quadrics can be written as a sum of five squares

$$\sum_{i=0}^{3} a_{ii}t_i^2 + 2\sum_{0\le i<j\le 3} a_{ij}t_it_j$$

$$= \sum_{i=0}^{2}(a_{ii} - \alpha a_{i3})t_i^2 + (a_{33} - \alpha)t_3^2 + \alpha^{-1}(\sum_{i=0}^{2} a_{i3}t_i + \alpha t_3)^2 = 0.$$

Here, we assume that A is general enough. The center of the perspectivity of the two tetrahedra is the pole of the plane $V(a_{03}t_0 + a_{13}t_1 + a_{23}t_2 + \alpha t_3)$.

The pentad of points consisting of the vertices of a self-conjugate tetrahedron with regard to a quadric Q and the center of the perspectivity $\mathfrak{o}$ of the tetrahedron and its polar tetrahedron form a *self-conjugate pentad* (and pentahedron in the dual space). This means that the pole of each plane spanned by three vertices lies on the opposite edge. As follows from above, the pentad of points defined by a self-conjugate tetrahedron defines a polar polyhedron of Q consisting of the polar planes of the pentad.

Proposition 2.3.6. *Let $H_i = V(l_i), i = 1, \ldots, 5$, form a nondegenerate polar pentahedron of a quadric $Q = V(q)$. Let $p_1, \ldots, p_5$ be the poles of the planes $V(l_i)$ with respect to Q. Then, the pentad $p_1, \ldots, p_5$ is self-conjugate and is a polar polyhedron of the dual quadric.*

Proof Let x_i be the pole of H_i with respect to Q. Then, the pole of the plane spanned by x_i, x_j, x_k is the point $x_{ijk} = H_i \cap H_j \cap H_k$. We may assume that $q = \sum_{i=0}^{4} l_i^2$. Then, $P_{x_{ijk}}(Q)$ belongs to the pencil $\mathcal{P}$ generated by the remaining two planes H_r, H_s. When we vary a point along the edge $\overline{x_r x_s}$, the polar plane of the point belongs to the pencil $\mathcal{P}$. For one of the points, the polar plane will be equal to the plane $P_{x_{ijk}}(Q)$, hence this points coincide with x_{ijk}. By definition, the pentad is self-conjugate.

The second assertion can be checked by straightforward computation. Since the polar pentahedron is nondegenerate, we can choose coordinates such that the polar pentahedron of Q is to equal to the union of the coordinate tetrahedron and the plane $V(\sum t_i)$. We can write

$$2q = \sum_{i=0}^{3} \lambda_i t_i^2 + (\sum_{i=0}^{3} t_i)^2$$

for some nonzero scalars λ_i. For any $v = (a_0, a_1, a_2, a_3) \in \mathbb{C}^4$, we have

$$D_v(q) = \sum_{i=0}^{3}(a + \lambda_i a_i)t_i$$

where $a = \sum_{i=0}^{3} a_i$. Let $\xi_i = a + \lambda_i a_i$ be considered as coordinates in the dual space. We can express a_i in terms of ξ_i by solving a system of linear equations with the coefficient matrix

$$\begin{pmatrix} \lambda_0 & 1 & 1 & 1 \\ 1 & \lambda_1 & 1 & 1 \\ 1 & 1 & \lambda_2 & 1 \\ 1 & 1 & 1 & \lambda_3 \end{pmatrix}.$$

Write $a_j = L_j(\xi_0, \dots, \xi_3) = \sum_{j=0}^3 c_{ij}\xi_j$, where (c_{ij}) is the inverse matrix. Let $v_j^* = (c_{0j}, c_{1j}, c_{2j}, c_{3j})$. The dual quadric consists of points $(\xi_0, \xi_1, \xi_2, \xi_3)$ such that $q(a_0, a_1, a_2, a_3) = 0$. This gives the equation of the dual quadric

$$Q^\vee = V\Big(\sum_{i=0}^3 \lambda_i L_i(\xi_0, \xi_1, \xi_2, \xi_3)^2 + \big(\sum_{i=0}^3 L_i(\xi_0, \xi_1, \xi_2, \xi_3)\big)^2\Big).$$

So, we see that the dual quadric has the polar polyhedron defined by the planes $V(L_i), V(\sum L_i)$. We have

$$D_{v_j^*}(q) = \sum_{i=0}^3 (\lambda_i a_i + a) c_{ij} t_i = t_j, \ j = 0, 1, 2, 3,$$

hence $D_{\sum v_j^*}(q) = \sum t_j$. This checks that the points of the pentad are poles of the planes of the polar pentahedron of Q. □

Remark 2.3.7. Let $\Pi_1, \dots, \Pi_N$ be sets of m-hedra in $\mathbb{P}^n, n > 1$, with no common elements. Suppose that these polyhedra, considered as hypersurfaces in $\mathbb{P}^n$ of degree m (the unions of their hyperplanes), belong to the same pencil. It is easy to see that this is equivalent to that the first two m-hedra Π_1, Π_2 are perspective from each hyperplane of $\Pi_3, \dots, \Pi_k$. The open problem is as follows.

What is the maximal possible number $N(n, m)$ of such polyhedra?

By taking a general hyperplane, we get $N(n, m) \leq N(2, m)$. It is known that $N(2, m) \geq 3$ and $N(2, 2), N(2, 3) = 4$. It was proven by J. Stipins [728] (see also [813]) that $N(2, m) \leq 4$ for all m and it is conjectured that $N(2, m) = 3$ for $m \neq 3$.

In the next chapter, we will consider the case $n = 2, m = 3, N = 4$. In the case $n = 3, m = 4, N = 3$, the three tetrahedra are called *desmic desmic* (the name is due to C. Stephanos [727]). The configuration of the 12 planes forming three desmic tetrahedra has a beautiful geometry (see, for example, [507], [508]). A general member of the pencil generated by three desmic tetrahedra is a *desmic quartic surface*. It has 12 singular points and represents a special embedding of a Kummer surface of the product of two isomorphic elliptic curves. We refer to [416] for some modern treatment of desmic quartic surfaces. We will discuss this later in Example 12.3.11.

SS:2.3.2

2.3.2 Invariants of a Pair of Quadrics

Let $Q_1 = V(f)$ and $Q_2 = V(g)$ be two quadrics in $\mathbb{P}^n$ (not necessarily nonsingular). Consider the pencil $V(t_0 f + t_1 g)$ of quadrics spanned by C and S. The zeros of the discriminant equation $D = \mathrm{discr}(t_0 f + t_1 g) = 0$ correspond to singular quadrics in the pencil. In coordinates, if f, g are defined by symmetric matrices $A = (a_{ij}), B = (b_{ij})$, respectively, then $D = \det(t_0 A + t_1 B)$ is a homogeneous polynomial of degree $\leq n + 1$. Choosing different system of coordinates replaces A, B by $Q^T AQ, Q^T BQ$, where Q is an invertible matrix. This replaces D with $\det(Q)^2 D$. Thus, the coefficients of D are invariants on the space of pairs of quadratic forms on $\mathbb{C}^{n+1}$ with respect to the action of the group $\mathrm{SL}(n+1)$. To compute D explicitly, we use the following formula for the determinant of the sum of two $m \times m$ matrices $X + Y$:

$$\det(X + Y) = \sum_{1\leq i_1<\ldots<i_k\leq n} \Delta_{i_1,\ldots,i_k}, \qquad (2.18)$$ formdet

where $\Delta_{i_1,\ldots,i_k}$ is the determinant of the matrix obtained from X by replacing the columns $X_{i_1}, \ldots, X_{i_k}$ with the columns $Y_{i_1}, \ldots, Y_{i_k}$. Applying this formula to our case, we get

$$D = \Theta_0 t_0^{n+1} + \sum_{i=1}^{n} \Theta_i t_0^{n+1-i} t^i + \Theta_{n+1} t_n^{n+1} \qquad (2.19)$$ salmon

where $\Theta_0 = \det A, \Theta_{n+1} = \det B$, and

$$\Theta_k = \sum_{1\leq i_1<\ldots<i_k\leq n+1} \det(A_1 \ldots B_{i_1} \ldots B_{i_k} \ldots A_{n+1}),$$

where $A = [A_1 \ldots A_{n+1}], B = [B_1 \ldots B_{n+1}]$. We immediately recognize the geometric meanings of vanishing of the first and the last coefficients of D. The coefficient Θ_0 (resp. Θ_{n+1}) vanishes if and only if Q_1 (resp. Q_2) is a singular conic.

propx **Proposition 2.3.8.** *Let Q_1 and Q_2 be two general quadrics. The following conditions are equivalent.*

(i) $\Theta_1 = 0$;
(ii) *Q_2 is apolar to the dual quadric $Q_1^\vee$;*
(iii) *Q_1 admits a polar simplex with vertices on Q_2.*

Proof First note that

$$\Theta_1 = \mathrm{Tr}(B\mathrm{adj}(A)). \qquad (2.20)$$

Now, $\mathrm{adj}(A)$ is the matrix defining $Q_1^\vee$ and the equivalence of (i) and (ii) becomes clear.

Since Θ_i are invariants of (Q_1, Q_2), we may assume that $Q_1 = V(\sum_{i=0}^n t_i^2)$. Suppose (iii) holds. Since the orthogonal group of C acts transitively on the set of polar simplexes of Q_1, we may assume that the coordinate simplex is inscribed in Q_2. Then, the points $[1, 0, \ldots, 0], \ldots, [0, \ldots, 0, 1]$, must be on Q_2. Hence,

$$Q_2 = V(\sum_{0 \le i < j \le n} a_{ij} t_i t_j),$$

and the condition (i) is verified.

Now, suppose (i) holds. Choose coordinates such that $Q_1 = V(\sum \alpha_i t_i^2)$. Start from any point on Q_2 but not on Q_1, and choose a projective transformation that leaves Q_1 invariant and sends the point to the point $p_1 = [1, 0, \ldots, 0]$. The quadric Q_2 transforms to a quadric with an equation in which the coefficient at x_0^2 is equal to 0. The polar line of p_1 with respect to Q_1 is $V(\sum_{i=1}^n \alpha_i t_i)$. It intersects Q_2 along a quadric of dimension $n-2$ in the hyperplane $t_0 = 0$. Using a transformation leaving $V(t_0)$ and Q_1 invariant, we transform Q_2' to another quadric such that the point $p_2 = [0, 1, 0, \ldots, 0]$ belongs to $V(t_0) \cap Q_2'$. This implies that the coefficients of the equation of Q_2' at t_0^2 and t_1^2 are equal to zero. Continuing in this way, we may assume that the equation of Q_2 is of the form $a_{nn} t_n^2 + \sum_{0 \le i < j \le n}^n a_{ij} t_i t_j = 0$. The trace condition is $a_{nn} \alpha_n^{-1} = 0$. It implies that $a_{nn} = 0$, and hence the point $p_{n+1} = [0, \ldots, 0, 1]$ is on Q_2. The triangle with vertices $[1, 0, \ldots, 0], \ldots, [0, \ldots, 0, 1]$ is a polar simplex of Q_1 which is inscribed in Q_2. □

Observe that, if $Q_1 = V(\sum t_i^2)$, the trace condition means that the conic Q_2 is defined by a harmonic polynomial with respect to the Laplace operator.

Definition 2.3.9. *A quadric Q_1 is called* apolar *to a quadric Q_2 if one of the equivalent conditions in Proposition* propx 2.3.8 *holds. If Q_1 is apolar to Q_2 and vice versa, the quadrics are called* mutually apolar.

The geometric interpretation of other invariants Θ_i is less clear. First note that a quadratic form q on a vector space E defines a quadratic form $\Lambda^k q$ on the space $\bigwedge^k E$. Its polar bilinear form is the map $\bigwedge^k b_q : \bigwedge^k E \to \bigwedge^k E^\vee = (\bigwedge^k E)^\vee$, where $b_q : E \to E^\vee$ is the polar bilinear form of q. Explicitly, the polar bilinear form $\bigwedge^k b_q$ is defined by the formula

$$(v_1 \wedge \ldots \wedge v_k, w_1 \wedge \ldots \wedge w_k) = \det(b_q(v_i, w_j))$$

which we already used in Lemma L2.2.14 2.3.3.

If A is the symmetric matrix defining q, then the matrix defining $\bigwedge^k q$ is denoted by $A^{(k)}$ and is called the *k-th compound matrix* of A. If we index the rows and the columns of $A^{(k)}$ by an increasing sequence $J = (j_1, \ldots, j_k) \subset$

$\{1,\ldots,n+1\}$, then the entry $A^{(k)}_{J,J'}$ of $A^{(k)}$ is equal to the (J,J')-minor $A_{J,J'}$ of A. Replacing each $A^{(k)}_{J,J'}$ with the minor $A_{\overline{J'},\overline{J}}$ taken with the sign $(-1)^{\epsilon(J,J')}$, we obtain the definition of the *adjugate k-th compound matrix* $\mathrm{adj}^{(k)}(A)$ (not to be confused with $\mathrm{adj}(A^{(k)})$). The Laplace formula for the determinant gives

$$A^{(k)}\mathrm{adj}^{(k)}(A) = \det(A)I.$$

If A is invertible, then $A^{(k)}$ is invertible and $(A^{(k)})^{-1} = \frac{1}{\det A}\mathrm{adj}(A^{(k)})$.

We leave it to the reader to check the following fact.

Proposition 2.3.10. *Let $Q_1 = V(q), Q_2 = V(q')$ be defined by symmetric matrices A, B and let $A^{(k)}$ and $B^{(k)}$ be their k-th compound matrices. Then,*

$$\Theta_k(A,B) = \mathrm{Tr}(A^{(n+1-k)}\mathrm{adj}(B^{(k)})).$$

exa:2.3.11 *Example* 2.3.11. Let $n = 3$. Then, there is only one new invariant to interpret. This is $\Theta_2 = \mathrm{Tr}(A^{(2)}\mathrm{adj}(B^{(2)}))$. The compound matrices $A^{(2)}$ and $B^{(2)}$ are 6×6 symmetric matrices whose entries are 2×2-minors of A and B taken with an appropriate sign. Let $A = (a_{ij})$. The equation of the quadric defined by $A^{(2)}$ is given by the *bordered determinant*

$$\det\begin{pmatrix} a_{00} & a_{01} & a_{02} & a_{03} & \xi_0 & \eta_0 \\ a_{10} & a_{11} & a_{12} & a_{13} & \xi_1 & \eta_1 \\ a_{20} & a_{21} & a_{22} & a_{23} & \xi_2 & \eta_2 \\ a_{30} & a_{31} & a_{32} & a_{33} & \xi_3 & \eta_3 \\ \xi_0 & \xi_1 & \xi_2 & \xi_3 & 0 & 0 \\ \eta_0 & \eta_1 & \eta_2 & \eta_3 & 0 & 0 \end{pmatrix} = 0. \tag{2.21}$$ bd1

The equation is called the *line-equation* or *complex equation* of the quadric Q defined by the matrix A. If we take the minors $\xi_i\eta_j - \xi_j\eta_i$ as Plücker coordinates in $|\bigwedge^2 \mathbb{C}^4|$, the line-equation parameterizes lines in $\mathbb{P}^3$ which are tangent to the quadric Q. This can be immediately checked by considering a parametric equation of a line $\lambda(\xi_0,\xi_1,\xi_2,\xi_3) + \mu(\eta_0,\eta_1,\eta_2,\eta_3)$, inserting it in the equation of the quadric and finding the condition when the corresponding quadratic form in λ,μ has a double root. In matrix notation, the condition is $(\xi A\xi)(\eta A\eta) - (\xi A\eta)^2 = 0$, which can be easily seen rewritten in the form of the vanishing of the bordered determinant. The intersection of the quadric defined by the matrix $A^{(2)}$ with the Klein quadric defining the Grassmannian of lines in $\mathbb{P}^3$ is an example of a *quadratic line complex* associated to a quadric. We will discuss this and other quadratic line complexes in Chapter 10.

Take $Q = V(\sum t_i^2)$. Then, the bordered determinant becomes equal to

$$(\sum_{i=0}^{3} \xi_i^2)(\sum_{i=0}^{3} \eta_i) - (\sum_{i=0}^{3} \xi_i\eta_i)^2 = \sum_{0\le i<j\le 3} (\xi_i\eta_j - \xi_j\eta_i)^2 = \sum_{0\le i<j\le 3} p_{ij}^2,$$

where p_{ij} are the Plücker coordinates. We have

$$\Theta_2(A,B) = \mathrm{Tr}(B_2) = \sum_{0\le i<j\le 3} (b_{ij}b_{ji} - b_{ii}b_{jj}).$$

The coordinate line $t_i = t_j = 0$ touches the quadric Q_2 when $b_{ij}b_{ji} - b_{ii}b_{jj} = 0$. Thus, Θ_2 vanishes when a polar tetrahedron of Q_1 has its edges touching Q_2.

It is clear that the invariants Θ_k are bihomogeneous of degree $(k, n+1-i)$ in coefficients of A and B. We can consider them as invariants of the group $\mathrm{SL}(n+1)$ acting on the product of two copies of the space of square symmetric matrices of size $n+1$. One can prove that the $n+1$ invariants Θ_i form a complete system of polynomial invariants of two symmetric matrices. This means that the polynomials Θ_i generate the algebra of invariant polynomials (see [768, p. 304]).

One can use the invariants Θ_i to express different mutual geometric properties of two quadrics. We refer to [717] for many examples. We give only one example.

Theorem 2.3.12. *Two quadrics touch each other if and only if*

$$J = \mathsf{D}(\Theta_0, \ldots, \Theta_{n+1}) = 0,$$

where D *is the discriminant of a binary form of degree* $n+1$.

Proof This follows from the description of the tangent space of the discriminant hypersurface of quadratic forms. The line defining the pencil of quadrics generated by the two quadrics does not intersect the discriminant hypersurface transversally if and only if one of quadrics in the pencil is of corank ≥ 2, or one of the quadrics has a singular point at the base locus of the pencil (see (1.46)). In the case of pencils the first condition implies the second one. Thus, the condition for tangency is that one of the roots of the equation $\det(t_0A + t_1B) = 0$ is a multiple root. □

The invariant J is called the *tact-invariant* of two quadrics.* Note that two quadrics touch each other if and only if their intersection has a singular point.

Corollary 2.3.13. *The degree of the hypersurface of quadrics in* $\mathbb{P}^n$ *touching a given nonsingular quadric is equal to* $n(n+1)$.

*The terminology is due to A. Cayley, taction = tangency.

Proof This follows from the known property of the discriminant of a binary form $\sum_{i=0}^{d} a_i t_0^{d-i} t_1^i$. If we assign the degree $(d-i, i)$ to each coefficient a_i, then the total degree of the discriminant is equal to $d(d-1)$. This can be checked, for example, by computing the discriminant of the form $a_0 t_0^d + a_d t_1^d$, which is equal to $d^d a_0^{d-1} a_d^{d-1}$ (see [325, p. 406]). In our case, each Θ_k has bidegree $(n+1-k, k)$, and we get that the total bidegree is equal to $(n(n+1), n(n+1))$. Fixing one of the quadrics, we obtain the asserted degree of the hypersurface. □

2.3.3 Invariants of a Pair of Conics

In this case, we have four invariants $\Theta_0, \Theta_1, \Theta_2, \Theta_3$, which are traditionally denoted by $\Delta, \Theta, \Theta', \Delta'$, respectively.

The polynomials

$$(R_0, R_1, R_2, R_3) = (\Theta\Theta', \Delta\Delta', \Theta'^3\Delta, \Theta^3\Delta')$$

are bihomogeneous of degrees $(3,3), (3,3), (6,6), (6,6)$. They define a rational map $\mathbb{P}^5 \times \mathbb{P}^5 \dashrightarrow \mathbb{P}(1,1,2,2)$. We have the obvious relation $R_0^3 R_1 - R_2 R_3 = 0$. After dehomogenization, we obtain rational functions

$$X = R_1/R_0^2,\ Y = R_2/R_0,\ Z = R_3/R_0^2$$

such that $X = YZ$. The rational functions

$$Y = \Theta'\Delta/\Theta^2, \quad Z = \Theta\Delta'/\Theta'^2$$

generate the field of rational invariants of pairs of conics (see [716, p. 280]). The polynomials R_0, R_1, R_2, R_3 generate the algebra of bi-homogeneous invariants on $\mathbb{P}^5 \times \mathbb{P}^5$ with respect to the diagonal action of $\mathrm{SL}(4)$ and the GIT-quotient is isomorphic to the rational surface $V(t_0^3 t_1 - t_2 t_3)$ in the weighted projective space $\mathbb{P}(1,1,2,2)$. The surface is a normal surface with one rational double point $[0,1,0,0]$ of type A_2. The singular point corresponds to a unique orbit of a pair of nonsingular conics (C, S) such that $C^\vee$ is apolar to S and $S^\vee$ is apolar to C. It is represented by the pair

$$t_0^2 + t_1^2 + t_2^2 = 0, \quad t_0^2 + \epsilon t_1^2 + \epsilon^2 t_2^2 = 0,$$

where $\epsilon = e^{2\pi i/3}$. The stabilizer subgroup of this orbit is a cyclic group of order 3 generated by a cyclic permutation of the coordinates.

Recall that the GIT-quotient parameterizes minimal orbits of semi-stable points. In our case, all semi-stable points are stable, and unstable points correspond to a pairs of conics, one of which has a singular point on the other conic.

Using the invariants $\Delta, \Theta, \Theta', \Delta'$, one can express the condition that the two conics are Poncelet related.

cayley **Theorem 2.3.14.** *Let C and S be two nonsingular conics. A triangle inscribed in C and circumscribing S exists if and only if*

$$\Theta'^2 - 4\Theta\Delta' = 0.$$

Proof Suppose there is a triangle inscribed in C and circumscribing S. Applying a linear transformation, we may assume that the vertices of the triangle are the points $[1,0,0]$, $[0,1,0]$ and $[0,0,1]$ and $C = V(t_0t_1 + t_0t_2 + t_1t_2)$. Let $S = V(g)$, where

$$g = at_0^2 + bt_1^2 + ct_2^2 + 2dt_0t_1 + 2et_0t_2 + 2ft_1t_2. \tag{2.22}$$ gggg

The triangle circumscribes S when the points $[1,0,0], [0,1,0], [0,0,1]$ lie on the dual conic $\check{S}$. This implies that the diagonal entries $bc - f^2, ac - e^2, ab - d^2$ of the matrix $\mathrm{adj}(B)$ are equal to zero. Therefore, we may assume that

$$g = \alpha^2t_0^2 + \beta^2t_1^2 + \gamma^2t_2^2 - 2\alpha\beta t_0t_1 - 2\alpha\gamma t_0t_2 - 2\beta\gamma t_1t_2. \tag{2.23}$$ g

We get

$$\Theta' = \mathrm{Tr}\left(\begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix} \cdot \begin{pmatrix} 0 & 2\alpha\beta\gamma^2 & 2\alpha\gamma\beta^2 \\ 2\alpha\beta\gamma^2 & 0 & 2\beta\gamma\alpha^2 \\ 2\alpha\gamma\beta^2 & 2\beta\gamma\alpha^2 & 0 \end{pmatrix}\right) = 4\alpha\beta\gamma(\alpha+\beta+\gamma),$$

$$\Theta = \mathrm{Tr}\left(\begin{pmatrix} \alpha^2 & -\alpha\beta & -\alpha\gamma \\ -\alpha\beta & \beta^2 & -\beta\gamma \\ -\alpha\gamma & -\beta\gamma & \gamma^2 \end{pmatrix}\begin{pmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{pmatrix}\right) = -(\alpha+\beta+\gamma)^2,$$

$$\Delta' = -4(\alpha\beta\gamma)^2.$$

This checks that $\Theta'^2 - 4\Theta\Delta' = 0$.

Let us prove the sufficiency of the condition. Take a tangent line ℓ_1 to S intersecting C at two points x, y and consider the tangent lines ℓ_2, ℓ_3 to S passing through x and y, respectively. The triangle with sides ℓ_1, ℓ_2, ℓ_3 circumscribes S and has two vertices on C. Choose the coordinates such that this triangle is the coordinate triangle. Then, we may assume that $C = V(at_0^2 + 2t_0t_1 + 2t_1t_2 + 2t_0t_2)$ and $S = V(g)$, where g is as in (2.23). Computing $\Theta'^2 - 4\Theta\Delta'$, we find that it is equal to zero if and only if $a = 0$. Thus, the coordinate triangle is inscribed in C. □

Darboux's Theorem is another example of a poristic statement, with respect to the property of the existence of a polygon inscribed in one conic and circumscribing the other conic. Another example of a poristic statement is one of the equivalent properties of a pair of conics from Proposition 2.3.8: Given two nonsingular conics C and S, there exists a polar triangle of C inscribed in S, or, in other words, C is apolar to S.

Recall from Theorem 1.1.9 that any projective automorphism of $\mathbb{P}^n = |E|$ is a composition of two polarities $\phi, \psi : |E| \to |E^\vee|$.

Proposition 2.3.15. *Let C and S be two different nonsingular conics and $g \in \mathrm{Aut}(\mathbb{P}^2)$ be the composition of the two polarities defined by the conics. Then, g is of order 3 if and only if C and S are mutually apolar.*

Proof Let A, B be symmetric 3×3 matrices corresponding to C and S. The conics C and S are mutually apolar if and only if $\mathrm{Tr}(AB^{-1}) = \mathrm{Tr}(BA^{-1}) = 0$. The projective transformation g is given by the matrix $X = AB^{-1}$. This transformation is of order 3 if and only if the characteristic polynomial $|X - \lambda I_3|$ of the matrix X has zero coefficients at λ, λ^2. Since $\mathrm{Tr}(X) = 0$, the coefficient at λ^2 is equal to zero. The coefficient at λ is equal to zero if and only if $\mathrm{Tr}(X^{-1}) = \mathrm{Tr}(BA^{-1}) = 0$. Thus, g is of order 3 if and only if $\mathrm{Tr}(AB^{-1}) = \mathrm{Tr}(BA^{-1}) = 0$. □

Remark 2.3.16. It is immediate that any set of mutually apolar conics is linearly independent. Thus, the largest number of mutually apolar conics is equal to six. The first example of a set of six mutually apolar conics was given by F. Gerbardi [327]. The following is a set of mutually apolar conics given by P. Gordan [349]:

$$
\begin{aligned}
&t_0^2 + \epsilon t_1^2 + \epsilon^2 t_2^2 = 0,\\
&t_0^2 + \epsilon^2 t_1^2 + \epsilon t_2^2 = 0,\\
&r^2(t_0^2 + t_1^2 + t_2^2) + r\sqrt{3}(t_0t_1 + t_0t_2 + t_1t_2) = 0,\\
&r^2(t_0^2 + t_1^2 + t_2^2) + r\sqrt{3}(-t_0t_1 - t_0t_2 + t_1t_2) = 0,\\
&r^2(t_0^2 + t_1^2 + t_2^2) + r\sqrt{3}(-t_0t_1 + t_0t_2 - t_1t_2) = 0,\\
&r^2(t_0^2 + t_1^2 + t_2^2) + r\sqrt{3}(t_0t_1 - t_0t_2 - t_1t_2) = 0,
\end{aligned}
$$

where $\eta = e^{2\pi i/3}, r = \frac{-\sqrt{3}+\sqrt{-5}}{4}$. These six quadrics play an important role in the theory of invariants of the *Valentiner group* G, the subgroup of $\mathrm{PGL}(3)$ isomorphic to the alternating group $\mathfrak{A}_6$. in $\mathbb{C}^3$ with the algebra of invariants generated by three polynomials of degrees 6, 12 and 30. The invariant of degree 6 is the sum of cubes of the six mutually apolar quadratic forms. The invariant of degree 12 is their product. The invariant of degree 30 is also expressed in terms of the six quadratic forms but in a more complicated way (see [328], [349]). We refer to [335] for further discussion of mutually apolar conics.

Consider the set of polar triangles of C inscribed in S. We know that this set is either empty or of dimension ≥ 1. We consider each triangle as a set of its three vertices, i.e. as an effective divisor of degree 3 on S.

Proposition 2.3.17. *The closure X of the set of self-polar triangles with respect to C which are inscribed in S, if not empty, is a g_3^1, i.e. a linear pencil of divisors of degree 3.*

Proof First, we use that two self-polar triangles with respect to C and inscribed in S which share a common vertex must coincide. In fact, the polar line of the vertex must intersect S at the vertices of the triangle. Then, the assertion is proved using the argument from the proof of Proposition 2.2.4. □

Note that a general g_3^1 contains four singular divisors corresponding to ramification points of the corresponding map $\mathbb{P}^1 \to \mathbb{P}^1$. In our case these divisors correspond to four intersection points of C and S.

Another example of a poristic statement is the following.

Theorem 2.3.18. *Let T and T' be two different triangles. The following assertions are equivalent:*

(i) there exists a conic S containing the vertices of the two triangles;
(ii) there exists a conic Σ touching the sides of the two triangles;
(iii) there exists a conic C with polar triangles T and T'.

Moreover, when one of the conditions is satisfied, there is an infinite number of triangles inscribed in S, circumscribed around Σ, and all of these triangles are polar triangles of C.

Proof (iii)⇔ (ii) According to Proposition 1.3.9, a conic C admits T as a polar triangle if the conics in the dual plane containing the sides of the triangle are all apolar to C. If T and T' are polar triangles of C, then the two nets of conics passing through the sides of the first and the second triangle intersect in the 4-dimensional space of apolar conics. The common conic is the conic Σ from (ii). Conversely, if Σ exists, the two nets contain a common conic, and hence, are contained in a four-dimensional space of conics in the dual plane. The apolar conic is the conic C from (iii).

(iii)⇔ (i) This follows from the previous argument applying Corollary 2.1.4.

Let us prove the last assertion. Suppose one of the conditions of the Theorem is satisfied. Then, we have the conics C, S, Σ with the asserted properties with respect to the two triangles T, T'. By Proposition 2.3.17, the set of self-polar triangles with respect to C inscribed in S is a g^1_3. By Proposition 2.2.4, the set of triangles inscribed in S and circumscribing Σ is also a g^1_3. Two g^1_3's with 2 common divisors coincide. □

Recall from Theorem 2.3.12 that the condition that two conics C and S touch each other is

$$27\Delta^2\Delta'^2 - 18\Theta\Theta'\Delta\Delta' + 4\Delta\Theta'^3 + 4\Delta'\Theta^3 - \Theta'^2\Theta^2 = 0. \tag{2.24}$$

The variety of pairs of touching conics is a hypersurface of bidegree $(6,6)$ in $\mathbb{P}^5 \times \mathbb{P}^5$. In particular, conics touching a given conic is a hypersurface of degree 6 in the space of conics. This fact is used for the solution of the famous *Apollonius problem* in enumerative geometry: find the number of nonsingular conics touching five fixed general conics (see [315], Example 9.1.9).

Remark 2.3.19. Choose a coordinate system such that $C = V(t_0^2+t_1^2+t_2^2)$. Then, the condition that S is Poncelet-related to C with respect to triangles is easily seen to be equal to

$$c_2^2 - c_1c_3 = 0,$$

where

$$\det(A - tI_3) = (-t)^3 + c_1(-t)^2 + c_2(-t) + c_3$$

is the characteristic polynomial of a symmetric matrix A defining S. This is a quartic hypersurface in the space of conics. The polynomials c_1, c_2, c_3 generate the algebra of invariants of the group $\mathrm{SO}(3)$ acting on the space $V = S^2((\mathbb{C}^3)^\vee)$. If we use the decomposition $V = \mathcal{H}_q \oplus \mathbb{C}q$, where $q = t_0^2 + t_1^2 + t_2^2$ and $\mathcal{H}_q$ is the space of harmonic quadratic polynomials with respect to q, then the first invariant corresponds to the projection $\mathcal{H}_q \oplus \mathbb{C}q \to \mathbb{C}q$. Let $\mathsf{v}_2 : \mathbb{P}^1 \to \mathbb{P}^2$ be the Veronese map with the image equal to C. Then, the pull-back map

$$\nu^* : V = H^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(2)) \to H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(4))$$

defines an isomorphism of the representation $\mathcal{H}_q$ of $\mathrm{SO}(3)$ with the representation $S^4((\mathbb{C}^2)^\vee)$ of $\mathrm{SL}(2)$. Under this isomorphism, the invariants c_2 and c_3 correspond to the invariants S and T on the space of binary quartics from Example 1.5.2. In particular, the fact that a harmonic conic is Poncelet-related to C is equivalent to the corresponding binary quartic admiting an apolar binary quadric. Also, the discriminant invariant of degree 6 of binary quartics corresponds to the condition that a harmonic conic touches C.

2.3.4 The Salmon conic

One can also look for *covariants* or *contravariants* of a pair of conics, that is, rational maps $|\mathcal{O}_{\mathbb{P}^2}(2)| \times |\mathcal{O}_{\mathbb{P}^2}(2)| \dashrightarrow |\mathcal{O}_{\mathbb{P}^2}(d)|$ or $|\mathcal{O}_{\mathbb{P}^2}(2)| \times |\mathcal{O}_{\mathbb{P}^2}(2)| \dashrightarrow |\mathcal{O}_{\mathbb{P}^2}(d)|^\vee$ which are defined geometrically, i.e. not depending on a choice of projective coordinates.

Recall the definition of the *cross ratio* of four distinct ordered points $p_i = [a_i, b_i]$ on $\mathbb{P}^1$

$$R(p_1p_2;p_3p_4) = \frac{(p_1-p_2)(p_3-p_4)}{(p_1-p_3)(p_2-p_4)}, \tag{2.25}$$ crossratio

where

$$p_i - p_j = \det\begin{pmatrix} a_i & b_i \\ a_j & b_j \end{pmatrix} = a_ib_j - a_jb_i.$$

It is immediately checked that the cross ratio does not take the values $0, 1, \infty$. It does not depend on the choice of projective coordinates. It is also invariant under a permutation of the four points equal to the product of two commuting transpositions. The permutation (12) changes R to $-R/(1-R)$ and the permutation (23) changes R to $1/R$. Thus, there are at most six possible cross ratios for an ordered set of four points

$$R,\ \frac{1}{R},\ 1-R,\ \frac{1}{1-R},\ \frac{R}{R-1},\ \frac{R-1}{R}.$$

The number of distinct cross ratios may be reduced to three or two. The first case happens if and only if one of them is equal to -1 (the other ones will be 2 and $1/2$). The unordered set of four points in this case is called a *harmonic quadruple*. The second case happens when R satisfies $R^2+R+1 = 0$, i.e. R is one of two cubic roots of 1 not equal to 1. In this case we have an *equianharmonic quadruple*.

If we identify the projective space of binary forms of degree 2 with the projective plane, the relation (harmcong 2.6) can be viewed as a symmetric hypersurface H of bidegree $(1,1)$ in $\mathbb{P}^2 \times \mathbb{P}^2$. In particular, it makes sense to speak about harmonically conjugate pairs of maybe coinciding points. We immediately check that a double point is harmonically conjugate to a pair of points if and only if it coincides with one of the roots of this form.

We can extend the definition of the cross ratio to any set of points no three of which coincide by considering the cross ratios as the point

$$\mathbf{R} = [(p_1-p_2)(p_3-p_4), (p_1-p_3)(p_2-p_4)] \in \mathbb{P}^1. \tag{2.26}$$

It is easy to see that two points coincide if and only if $\mathbf{R} = [0,1], [1,1], [1,0]$. This corresponds to $R = 0, 1, \infty$.

Two pairs of points $\{p_1, p_2\}$ and $\{q_1, q_2\}$ are harmonically conjugate in the sense of definition (2.6) if and only if $R(p_1q_1; q_2p_2) = -1$. To check this, we may assume that p_1, p_2 are roots of $f = \alpha t_0^2 + 2\beta t_0 t_1 + \gamma t_1^2$ and q_1, q_2 are roots of $g = \alpha' t_0^2 + 2\beta' t_0 t_1 + \gamma' t_1^2$, where, for simplicity, we may assume that $\alpha, \alpha' \neq 0$ so that, in affine coordinates, the roots x, y of the first equations satisfy $x + y = -2\beta/\alpha, xy = \gamma/\alpha$ and similarly the roots of the second equation x', y' satisfy $x' + y' = -2\beta'/\alpha', x'y' = \gamma'/\alpha'$. Then,

$$R(xx'; y'y) = \frac{(x - x')(y' - y)}{(x - y')(x' - y)} = -1$$

if and only if

$$(x - x')(y' - y) + (x - y')(x' - y) = (x + y)(x' + y') - 2xy - 2x'y$$

$$= \frac{4\beta\beta'}{\alpha\alpha'} - \frac{2\gamma}{\alpha} - \frac{2\gamma'}{\alpha'} = -2\frac{\alpha\gamma' + \alpha'\gamma - 2\beta\beta'}{\alpha\alpha'} = 0.$$

So, we see that the two pairs of roots form a harmonic quadruple if and only if (2.6) holds.

The expression $\alpha\gamma' + \alpha'\gamma - 2\beta\beta''$ is an invariant of a pair (f, g) of binary quadratic forms. It is equal to the coefficient at t for the discriminant of $f + tg$. It is analogous to the invariants Θ and Θ' for a pair of conics.

The *Salmon conic* associated to a pair of conics C and C' is the locus $\mathsf{S}(C, C')$ of points x in $\mathbb{P}^2$ such that the pairs of the tangents through x to C and to C' are harmonically conjugate. Note that it makes sense even when x lies on one of the conics. In this case, one considers the corresponding tangent as the double tangent.

Let A be a square symmetric 3×3-matrix. The entries of the adjugate matrix $\mathrm{adj}(A)$ are quadratic forms in the entries of A. By polarization, we obtain

$$\mathrm{adj}(\lambda_0 A + \lambda_1 B) = \lambda_0^2 \mathrm{adj}(A) + \lambda_0\lambda_1 \mathrm{adj}(A, B) + \lambda_1^2 \mathrm{adj}(B),$$

where $(A, B) \to \mathrm{adj}(A, B)$ is a bilinear function of A and B.

Theorem 2.3.20. *Let $C = V(q), C' = V(q')$, where q and q' are quadratic forms defined by symmetric matrices $A = (a_{ij})$ and $B = (b_{ij})$. Then, the Salmon conic* $\mathsf{S}(C, C')$ *is defined by the matrix* $\mathrm{adj}(\mathrm{adj}(A), \mathrm{adj}(B))$.

Proof By duality, the pencil of lines through a point $x = [x_0, x_1, x_2]$ corresponds to the line $\ell_x = V(x_0u_0 + x_1u_1 + x_2u_2)$ in the dual plane with dual coordinates u_0, u_1, u_2. Without loss of generality, we may assume that $x_2 = -1$. Let $C^\vee, C'^\vee$ be the dual conics defined by the matrices $\mathrm{adj}(A) = (A_{ij}), \mathrm{adj}(B) = (B_{ij})$. The intersection of the line ℓ_x with $C^\vee$ is equal to two points $[u_0, u_1, x_0u_0 + x_1u_1]$ such that

$$(A_{00} + A_{02}x_0 + A_{22}x_0^2)u_0^2 + (A_{11} + A_{12}x_1 + A_{22}x_1^2)u_1^2$$

$$+2(A_{22}x_0t_1 + A_{02}x_1 + A_{12}x_0 + A_{01})u_0u_1 = 0.$$

Replacing A with B, we get the similar formula for the intersection of ℓ with $C'^{\vee}$. The intersection points $[u_0, u_1, x_0u_0 + x_1u_1]$ correspond to the tangent lines to C and C' passing through the point x. By (2.6) harmconj, they are harmonically conjugate if and only if

$$(A_{00} + A_{02}x_0 + A_{22}x_0^2)(B_{11} + B_{12}x_1 + B_{22}x_1^2)$$

$$+(B_{00} + B_{02}x_0 + B_{22}x_0^2)(A_{11} + A_{12}x_1 + A_{22}x_1^2)$$

$$-2(A_{22}x_0t_1 + A_{02}x_1 + A_{12}x_0 + A_{01})(B_{22}x_0x_1 + B_{02}x_1 + B_{12}x_0 + B_{01}) = 0.$$

This gives the equation of the Salmon conic $S(C, C')$:

$$(A_{22}B_{11} + A_{11}B_{22} - 2A_{12}B_{12})x_0^2 + (A_{00}B_{22} + A_{22}B_{00} - 2A_{02}B_{02})x_1^2$$

$$+(A_{00}B_{11} + A_{11}B_{00} - 2A_{01}B_{01})x^2 + 2(A_{02}B_{12} + A_{12}B_{02} - A_{22}B_{02} - A_{02}B_{22})x_0x_1$$

$$+2(A_{02}B_{11} + A_{11}B_{02} - A_{12}B_{01} - A_{01}B_{12})x_0x_2$$

$$+2(A_{00}B_{12} + A_{12}B_{00} - A_{02}B_{01} - A_{01}B_{02})x_1x_2 = 0.$$

It is easy to see that the symmetric matrix defining this quadratic form is equal to $\mathrm{adj}(\mathrm{adj}(A), \mathrm{adj}(B))$. □

Let $\mathsf{S}(C, C') = V(s)$. Consider the pencil generated by $C^{\vee}$ and $C'^{\vee}$. In matrix notation, it is equal to the pencil of matrices $\mathrm{adj}(A) + t\mathrm{adj}(B)$. The dual conics of this pencil form a quadratic family of conics defined by the matrices $\mathrm{adj}(\mathrm{adj}(A) + t\mathrm{adj}(B)) = |A|A + tS + t^2|B|B$, where S is the matrix defining the Salmon conic. Its members are tangent to the quartic curve $V(s^2 - 4|A||B|qq')$. Since the members of the linear pencil pass through the four points $C^{\vee} \cap C'^{\vee}$, all members of the quadratic family are tangent to the four common tangents of C and C'. Thus,

$$V(s^2 - 4|A||B|qq') = V(l_1l_2l_3l_4), \tag{2.27}$$ salrel

where $V(l_i)$ are the common tangents. This implies the following remarkable property of the Salmon conic.

coolidge **Theorem 2.3.21.** *Let C and C' be two conics such that the dual conics intersect at four distinct points representing the four common tangents of C and S. Then, the eight tangency points lie on the Salmon conic associated with C and C'.*

Here, is another proof of the theorem that does not use (2.27). Let x be a point where the Salmon conic meets C. Then, the tangent line ℓ through x to C represents a double line in the harmonic pencil formed by the four tangents through x to C and S. As we remarked before, the conjugate pair of lines must contain ℓ. Thus, ℓ is a common tangent to C and S, and hence, x is one of the eight tangency points. Conversely, the argument is reversible and shows that every tangency point lies on the Salmon conic.

The Salmon conic represents a covariant of pairs of conics. A similar construction gives a contravariant conic in the dual plane, called the *Salmon envelope conic* $\mathsf{S}'(C, C')$. It parameterizes lines which intersect the dual conics C and C' at two pairs of harmonically conjugate points. We leave it to the reader to show that its equation is equal to

$$(a_{22}b_{11} + a_{11}b_{22} - 2a_{12}b_{12})u_0^2 + (a_{00}b_{22} + a_{22}b_{00} - 2a_{02}b_{02})u_1^2$$

$$+(a_{00}b_{11} + a_{11}b_{00} - 2a_{01}b_{01})u_2^2 + 2(a_{02}b_{12} + a_{12}b_{02} - a_{22}b_{02} - a_{02}b_{22})u_0u_1$$

$$+2(a_{02}b_{11} + a_{11}b_{02} - a_{12}b_{01} - a_{01}b_{12})u_0u_2$$

$$+2(a_{00}b_{12} + a_{12}b_{00} - a_{02}b_{01} - a_{01}b_{02})u_1u_2 = 0.$$

If we write $\mathsf{S}'(C, C') = V(s')$, we find, as above, that $V(s'^2 - q^\vee q'^\vee)$ is equal to the union of four lines corresponding to intersection points of $C \cap C'$. This implies that the Salmon envelope conic passes through the eight points corresponding to the eight tangents of C and C' at the intersection points.

The equation of the Salmon conic is greatly simplified if we simultaneously diagonalize the quadrics q and q' defining C and C'. Assume $q = t_0^2+t_1^2+t_2^2$, $q' = at_0^2 + bt_1^2 + ct_2^2$. Then, the equation of $\mathsf{S}(C, C')$ becomes

$$a(b + c)t_0^2 + b(c + a)t_1^2 + c(a + b)t_2^2 = 0,$$

and the equation of $\mathsf{S}'(C, C')$ becomes

$$(b + c)u_0^2 + (c + a)u_1^2 + (a + b)u_2^2 = 0.$$

By passing to the dual conic, we see that the dual conic $\mathsf{S}'(C, C')^\vee$ is different from $\mathsf{S}(C, C')$. Its equation is

$$(a + c)(a + b)t_0^2 + (a + b)(b + c)t_1^2 + (a + b)(b + c)t_2^2 = 0.$$

It can be expressed as a linear combination of the equations of C, C' and $\mathsf{S}(C, C')$

$$(a+c)(a+b)t_0^2+(a+b)(b+c)t_1^2+(a+b)(b+c)t_2^2 = (ab+bc+ac)(t_0^2+t_1^2+t_2^2)$$

$$+(a + b + c)(at_0^2 + bt_1^2 + ct_2^2) - (a(b + c)t_0^2 + b(c + a)t_1^2 + c(a + b)t_2^2).$$

Remark 2.3.22. The full system of covariants, and contravariants of a pair of conics is known (see [351], p. 286.) The curves $C, C', \mathsf{S}'(C, C')$, the Jacobian of C, C', and $\mathsf{S}(C, C')$ generate the algebra of covariants over the ring of invariants. The envelopes $C^\vee, C'^\vee, \mathsf{S}'(C, C')$, the Jacobian $C^\vee, C'^\vee$, and $\mathsf{S}'(C, C')$ generate the algebra of contravariants.

S:2.4

2.4 Enumerative Theory of Quadrics

In this section, we discuss a natural compactification of the projective space of smooth quadrics such that many enumerative problems on quadrics follow from the intersection theory of this space.

SS:2.4.1

2.4.1 Projective Bundles, Grassmannians, and Tangent Cones

Here we introduce some general background, on which we rely in this section and also later in Chapter 7. We refer to [315, Appendix B.6] for the details.

First, we remind the reader of the notions of a *vector bundle* and a *projective bundle*, which will be used often in the book (see [379]).

According to modern terminology, a *vector bundle* over a base scheme S is a locally free sheaf $\mathcal{E}$ of $\mathcal{O}_S$-modules. If S is connected, the rank of $\mathcal{E}$ is defined. A vector bundle of rank one is called an invertible sheaf, or a *line bundle*. A geometric version of this is the notion of a *geometric vector bundle* defined as the S-scheme

$$\mathbb{V}(\mathcal{E}) := \mathrm{Spec}(\mathbf{S}(\mathcal{E})),$$

where $\mathbf{S}(\mathcal{E})$ is the symmetric algebra of $\mathcal{E}$. This is a scheme over S, so it comes with a structure morphism $p : \mathbb{V}(\mathcal{E}) \to S$ such that $p_*(\mathcal{O}_{\mathbb{V}(\mathcal{E})}) = \mathcal{E}$.

By definition of the affine spectrum, a homomorphism of invertible sheaves $u : \mathcal{F} \to \mathcal{E}$ defines a morphism $f_u : \mathbb{V}(\mathcal{E}) \to \mathbb{V}(\mathcal{F})$. It is called a morphism of vector bundles. It corresponds to a homomorphism $u : \mathcal{F} \to \mathcal{E} = p_*\mathcal{O}_{\mathbb{V}(\mathcal{E})}$ that defines $f_u^*(\mathcal{F}) \to \mathcal{O}_{\mathbb{V}(\mathcal{E})}$.

A *subbundle* of a vector bundle $\mathcal{E}$ is a locally free subsheaf $\mathcal{F}$ of $\mathcal{E}^\vee$ which is locally split; in other words, the quotient $\mathcal{E}^\vee/\mathcal{F}$ is locally free. Passing to the duals, we get a surjective homomorphism $\mathcal{E} \to \mathcal{F}^\vee$ that defines a closed embedding $\mathbb{V}(\mathcal{F}^\vee) \hookrightarrow \mathbb{V}(\mathcal{E})$.

By definition, a local section $s : U \to \mathbb{V}(\mathcal{E})$ is defined by a section of the dual sheaf $\mathcal{E}^\vee = \mathrm{Hom}(\mathcal{E}, \mathcal{O}_U)$ over U. For any S-scheme X, a morphism $X \to \mathbb{V}(\mathcal{E})$ is defined by a homomorphism of $\mathcal{O}_X$-modules $f^*\mathcal{E} \to \mathcal{O}_X$. In particular, taking $X \to S$ to be the inclusion $i : s = \mathrm{Spec}(\kappa(x)) \hookrightarrow S$ of a closed point s with the residue field $\kappa(s)$, we obtain a homomorphism of vector spaces $\kappa(s)^{n+1} \cong i^*\mathcal{E} \to \kappa(s)$ that defines a point of the closed fiber $\mathbb{V}(\mathcal{E})_s = \mathrm{Spec}(i^*(\mathcal{E})) = \mathrm{Spec}(\mathbf{S}(\kappa(s)^{n+1}) \cong \mathbb{A}^{n+1}_{\kappa(s)}$. Thus, we can informally say that a geometric vector bundle over S is an algebraic family of vector spaces parameterized by a scheme S.

A *projective bundle* associated with $\mathcal{E}$ is defined by

$$\mathbb{P}(\mathcal{E}) = \mathrm{Proj}(\mathbf{S}(\mathcal{E})).$$

It is a projective scheme $p : \mathbb{P}(\mathcal{E}) \to S$ over S. For any open affine set $U = \mathrm{Spec}(A)$ of S, $p^{-1}(U) = \mathrm{Proj}(\mathbf{S}(\mathcal{E}_U))$. If $\mathcal{E}$ is trivialized over U, $p^{-1}(U) \cong \mathrm{Proj}(\mathbf{S}(A^{n+1}) = \mathrm{Proj}(A[t_0, \ldots, t_n]) := \mathbb{P}^n_A$. Here, $t_0, \ldots, t_n$ is a basis of the free A-module $\mathcal{E}_U$, or the dual basis (coordinates) of the dual A-module $\mathcal{E}_U^\vee$.

By definition of the projective spectrum of a graded algebra, $\mathbb{P}(\mathcal{E})$ comes with an invertible sheaf $\mathcal{O}_{\mathbb{P}(\mathcal{E})}(1)$ (sometimes denoted just by $\mathcal{O}(1)$)) satisfying

$$p_*(\mathcal{O}_{\mathbb{P}(\mathcal{E})}(m)) \cong S^m(\mathcal{E}), \quad m \geq 0,$$

where $\mathcal{O}_{\mathbb{P}(\mathcal{E})}(m) := \mathcal{O}_{\mathbb{P}(\mathcal{E})}(1)^{\otimes m}$. As is customary, for any coherent sheaf $\mathcal{F}$ on $\mathbb{P}(\mathcal{E})$, we set

$$\mathcal{F}(k) := \mathcal{F} \otimes \mathcal{O}(k).$$

Note that, for any invertible sheaf $\mathcal{L}$ on S, $\mathbb{P}(\mathcal{E}) \cong \mathbb{P}(\mathcal{E} \otimes \mathcal{L})$, however the sheaves $\mathcal{O}(1)$ are different, namely, they differ by $\otimes p^*(\mathcal{L})$.

For any scheme $\pi : X \to S$ over S, a morphism of S-schemes $f : X \to \mathbb{P}(\mathcal{E})$ is defined by an invertible sheaf $\mathcal{L}$ over X and a surjection $\phi : f^*\mathcal{E} \to \mathcal{L}$. When we trivialize $\mathbb{P}(\mathcal{E})$ over $U = \mathrm{Spec}(A)$, the surjection ϕ defines $r+1$ sections of $\mathcal{L}|p^{-1}(U)$. This gives a local map $x \mapsto [s_0(x), \ldots, s_r(x)]$ from $f^{-1}(U)$ to $p^{-1}(U) = \mathbb{P}^n_A$. These maps are glued together to define a global map. We have $\mathcal{L} = f^*\mathcal{O}(1)$.

The identity morphism $\mathrm{id}_{\mathbb{P}(\mathcal{E})}$ corresponds to a canonical surjection

$$p^*\mathcal{E} \to \mathcal{O}(1). \tag{2.28}$$

globalgeneration

In particular, the closed embedding of a closed point $i_s : \mathrm{Spec}(\kappa(s)) \hookrightarrow S$ defines a surjection $\mathcal{E} \to i_s^*\mathcal{E} = \mathcal{E}(s)$ and the closed embedding $\mathbb{P}(\mathcal{E}(s)) \hookrightarrow \mathbb{P}(\mathcal{E})$ of the fiber $p^{-1}(s)$ of $p : \mathbb{P}(\mathcal{E}) \to S$. So, the projective bundle can be viewed as an algebraic family of n-dimensional projective spaces parameterized by S.

veroneserel *Example* 2.4.1. The surjection $p^*\mathcal{E} \to \mathcal{O}(1)$ defines a surjection $p^*S^d(\mathcal{E}) \to \mathcal{O}(d)$. The corresponding morphism $\mathsf{v}_d : \mathbb{P}(\mathcal{E}) \to \mathbb{P}(S^d(\mathcal{E}))$ is called the dth *Veronese map*. By definition, $\mathsf{v}_d^*\mathcal{O}_{\mathbb{P}(S^d(\mathcal{E}))}(1) = \mathcal{O}_{\mathbb{P}(\mathcal{E})}(d)$. The map of local sections is defined by the canonical map $\mathcal{E}^\vee \to S^2(\mathcal{E}^\vee)$, $s \mapsto s^2$.

Another special case is when we have a surjective homomorphism $u : \mathcal{E} \to \mathcal{E}'$ of locally free sheaves. It defines. a surjection $\mathbf{S}(\mathcal{E}) \to \mathbf{S}(\mathcal{E}')$, and hence, a closed embedding $j_u : \mathbb{P}(\mathcal{E}') \hookrightarrow \mathbb{P}(\mathcal{E})$. We have $j_u^*(\mathcal{O}_{\mathbb{P}(\mathcal{E}')}(1)) = \mathcal{O}_{\mathbb{P}(\mathcal{E})}(1)$ and a surjection $j_u^*(\mathcal{E}) \to \mathcal{O}_{\mathbb{P}(\mathcal{E}')}(1)$ is equal to $j_u^*\mathcal{E} \to \mathcal{E}' \to \mathcal{O}_{\mathbb{P}(\mathcal{E}')}(1)$.

Taking $\mathcal{E}'$ to be an invertible sheaf $\mathcal{L}$, we obtain a bijection between sections of $\mathbb{P}(\mathcal{E})$ and surjections $\mathcal{E} \to \mathcal{L}$.

The closed embedding $j_u : \mathbb{P}(\mathcal{F}) \hookrightarrow \mathbb{P}(\mathcal{E})$ is a regular embedding. It is defined by its sheaf ideal $\mathcal{I}$ such that the sheaf $\mathcal{I}/\mathcal{I}^2$ is locally free of rank equal to the difference between the ranks of $\mathcal{E}$ and $\mathcal{F}$. For any closed subscheme Z of a scheme X, the sheaf $\mathcal{I}_Z/\mathcal{I}_Z^2$ is called the *conormal sheaf* of X. Its dual sheaf $\mathcal{N}_{Z/X} = \mathcal{H}om(\mathcal{I}_Z/\mathcal{I}_Z^2, \mathcal{O}_Z)$ is called the *normal sheaf* of Z. .

The following proposition gives a geometric interpretation of the sheaf $\mathrm{Ker}(u)$ Sernesi [704, Proposition 4.6.2].

sernesi **Proposition 2.4.2.** *Let $u : \mathcal{E} \to \mathcal{F}$ be a surjection of locally free sheaves of $\mathcal{O}_S$-modules and $j_u : Y = \mathbb{P}(\mathcal{F}) \hookrightarrow X = \mathbb{P}(\mathcal{E})$ be the corresponding regular closed embedding of S-schemes. Then*

$$\mathcal{N}_{Y/X} \cong j_u^*(p^*\mathrm{Ker}(u)^\vee)(1).$$

Let $\Omega^i_{X/Y}$ denote the sheaf or relative Kahler differentials for a morphism of schemes $X \to Y$. We have

$$\Omega^i_{\mathbb{V}(\mathcal{E})/S} = p^*(\bigwedge^i \mathcal{E}) \tag{2.29}$$ canclassproj

and, for any $i \geq 1$, an exact sequence

$$0 \to \Omega^i_{\mathbb{P}(\mathcal{E})/S} \to p^*(\bigwedge^i \mathcal{E})(-i) \to \Omega^{i-1}_{\mathbb{P}(\mathcal{E})/S} \to 0. \tag{2.30}$$

(see DeligneSGA [211]). Taking $i = 1$, we obtain an exact sequence

$$0 \to \Omega^1_{\mathbb{P}(\mathcal{E})/S} \to p^*(\mathcal{E})(-1) \to \mathcal{O}_{\mathbb{P}(\mathcal{E})} \to 0. \tag{2.31}$$ eulerseq0

passing to the duals sheaves, we get the *Euler exact sequence*

$$0 \to \mathcal{O}_{\mathbb{P}(\mathcal{E})} \to p^*(\mathcal{E}^\vee)(1) \to \Theta_{\mathbb{P}(\mathcal{E})/S} \to 0, \tag{2.32}$$ eulerseq1

where $\Theta_{\mathbb{P}(\mathcal{E})/S}$ is the relative tangent sheaf.

We also need to recall the definition of the *Grassmannian bundles* which generalize projective bundles.

Let $\mathcal{E}$ be a vector bundle over a scheme S as before. The Grassmannian bundle $G(k, \mathcal{E})$ represents the functor on the category of schemes over S that assigns to a scheme $f : T \to S$ the set of equivalence classes of locally free quotients of rank k of $f^*\mathcal{E}^\vee$ on T. Two such quotients are equivalent if the kernels of the surjections onto the quotients coincide. The identity morphism $\mathrm{id}_{G(k,\mathcal{E})}$ defines a projection $p : G(k,\mathcal{E}) \to S$ and a surjection $p^*\mathcal{E} \to \mathcal{R}$, where $\mathcal{R}$ is the universal locally free quotient of rank k, called the *universal quotient bundle*. Taking it kernel, we get a canonical exact sequence of locally free sheaves on $G(k,\mathcal{E})$:

$$0 \to \mathcal{K} \to p^*\mathcal{E}^\vee \to \mathcal{R} \to 0. \tag{2.33}$$ universal1

The kernel $\mathcal{K}$ is a locally free sheaf of rank $n+1-k$, called the universal subbundle. †

The surjection $p^*\mathcal{E}^\vee \to \mathcal{R}$ defines a closed embedding

$$\mathbb{P}(\mathcal{R}) \hookrightarrow \mathbb{P}(p^*\mathcal{E}^\vee) := |\mathcal{E}|_{G(k,\mathcal{E})}.$$

Passing to the closed fibers, we get that each closed point $x \in G(k,\mathcal{E})$ defines a closed embedding of the fibers $\mathbb{P}(\mathcal{R}))_x = \mathbb{P}(\mathcal{R}(x)) \hookrightarrow |\mathcal{E}|_x = |\mathcal{E}(x)|$. Thus, a closed point of $G(k,\mathcal{E})$ can be identified with a $k-1$-dimensional projective subspace of $|\mathcal{E}_x|$, or, via projective duality, a codimension $k-1$ subspace of $\mathbb{P}(\mathcal{E})_x$. In a linear algebra interpretation, a closed point of $G(k,\mathcal{E})$ is a k-dimensional linear subspace of $\mathcal{E}(x)$, or, equivalently, a quotient of $\mathcal{E}(x)^\vee$ of dimension k. For this reason, one uses $G_{k-1}(\mathbb{P}(\mathcal{E}))$ as another notation for $G(k,\mathcal{E})$. Also, this explains the reason of saying that $\mathbb{P}(\mathcal{R})$ is the *tautological subbundle* over $G(k,\mathcal{E})$.

This can be expressed by drawing a diagram

$$\begin{array}{ccc} & \mathbb{P}(\mathcal{R}) & \\ {}^{p}\swarrow & & \searrow^{q} \\ \mathbb{P}(\mathcal{E}^\vee) & & G(k,\mathcal{E}), \end{array}$$

where the projection p is defined by the projection $\mathbb{P}(\mathcal{E})_{G(k,\mathcal{E}} = \mathbb{P}(\mathcal{E}) \times_S G(k,\mathcal{E}) \to \mathbb{P}(\mathcal{E})$.

It is clear that

$$G(1,\mathcal{E}) = \mathbb{P}(\mathcal{E}), \quad \mathcal{R} = \mathcal{O}(1),$$
$$G(n,\mathcal{E}) = G(1,\mathcal{E}^\vee) = \mathbb{P}(\mathcal{E}^\vee) := |\mathcal{E}|, \quad \mathcal{K}^\vee = \mathcal{O}_{|E|}(1).$$

More generally, passing to the duals in exact sequence (2.33), we get the duality isomorphism

$$G(k,\mathcal{E}) \to G(n+1-k,\mathcal{E}^\vee). \tag{2.34}$$

†The standard notations are Q for the universal quotient bundle, and $\mathcal{S}$ for the universal subbundle, we changed them to avoid some confusion with the notation used in the sequel.

On fibers, it is the canonical duality map that assigns to a linear subspace L of a vector space E its dual subspace $E^\perp$ of linear functions vanishing on L.

The surjection $p^*\mathcal{E} \to \mathcal{R}$ defines a. surjection

$$p^* \bigwedge^k \mathcal{E} \to \bigwedge^k \mathcal{R},$$

where $\bigwedge^k \mathcal{R}$ is an invertible sheaf on $G(k, \mathcal{E})$. It defines a closed embedding

$$j : G(k, \mathcal{E}) \hookrightarrow \mathbb{P}(p^* \bigwedge^k \mathcal{E}). \tag{2.35}$$

This canonical embedding is called the *Plücker embedding*, and the sheaf $\det(\mathcal{R}) = \bigwedge^k \mathcal{E}$ is denoted by $\mathcal{O}_{G(k,\mathcal{E})}(1)$.

We leave it to the reader to specialize everything in the case where $S = \mathrm{Spec}(\Bbbk)$ is a point and $\mathcal{E}$ is a vector space over $\Bbbk$.

flag *Example* 2.4.3. For any $m \le k$, the Grassmannian bundle $F(m,k) = G(m, \mathcal{R}_k)$, where $\mathcal{R}_k$ is the universal quotient bundle over $G(k, \mathcal{E})$, is called the *flag variety*. Its closed points over $s \in S$ are flags if subspaces $L' \subset L$ of dimensions m and k in the projective space $\mathbb{P}(\mathcal{E})_s$. It comes with the structural projections $p_1 : F(m,k,\mathcal{E}) \to G(k,\mathcal{E})$ of a scheme over $G(k,\mathcal{E})$. There is also the canonical universal exact sequence

$$0 \to \mathcal{K}_{m,k} \to p_1^*\mathcal{R}_k \to \mathcal{R}_{m,k} \to 0,$$

where $\mathcal{R}_{m,k}$ is a locally free sheaf of rank k. The surjection $p_1^*\mathcal{E}^\vee \to p_1^*\mathcal{R}_k \to \mathcal{R}_{m,k}$ defines the second projection $p_2 : F(m,k,\mathcal{E}) \to G(k,\mathcal{E})$, such that $\mathcal{R}_{m,k} = p_2^*(\mathcal{R}_m)$.

Note the special cases: $F(1,k,\mathcal{E}) = \mathbb{P}(\mathcal{R}_k)$, $F(k-1,k) = \mathbb{P}(\mathcal{R}_k^\vee)$.

Recall that, in Subsection SS:1.1.2 1.1.2, we introduced the tangent cone $\mathrm{TC}_x(X)$ of a hypersurface $X = V(f) \subset \mathbb{P}^n$ at its closed point. More generally, let Z be a closed subscheme of a scheme Y with the ideal sheaf $\mathcal{I}_Z$. We define the *normal cone*

$$\mathrm{C}_Z Y := \mathrm{Spec}\,(\bigoplus_{k=0}^\infty \mathcal{I}_Z^k/\mathcal{I}_Z^{k+1})$$

and *projective normal cone*

$$\mathbb{P}\mathrm{C}_Z Y := \mathrm{Proj}(\bigoplus_{k=0}^\infty \mathcal{I}_Z^k/\mathcal{I}_Z^{k+1})$$

(see Fulton [315, Appendix B.6]). The natural surjection of algebras

$$\bigoplus_{k=0}^\infty \mathcal{I}_Z^k/\mathcal{I}_Z^{k+1} \to \mathcal{O}_Y/\mathcal{I}_Z = \mathcal{O}_Z$$

defines the closed embedding $Z \hookrightarrow \mathrm{C}_Z Y$. The image is called the *vertex* of $\mathrm{C}_Z Y$.

In particular, if $Z = \{y\}$ is a closed point of Y, the normal cone is called the *tangent cone* (resp. *the projective tangent cone*). It is denoted by $\mathrm{TC}_y(Y)$ (resp. $\mathbb{T}\mathrm{C}_y(Y)$). It is immediate to check that, in the case where Y is a hypersurface and x its closed point, the new definition agrees with our earlier definition of the tangent cone. The vertex of the tangent cone is the point x.

Recall that the sheaf $\mathcal{I}_Z/\mathcal{I}_Z^2$ is called the *conormal sheaf* of Z in Y, and its dual $\mathcal{N}_{Z/Y} := (\mathcal{I}_Z/\mathcal{I}_Z^2)^\vee$ is called the *normal sheaf* of Z in Y. If Z and Y are smooth, we have an exact sequence

$$0 \to \mathcal{I}_Z/\mathcal{I}_Z^2 \to \Omega_Y^1 \otimes \mathcal{O}_Z \to \Omega_Z^1 \to 0,$$

and, passing to the duals, we have an exact sequence

$$0 \to \Theta_Z \to \Theta_Y \otimes \mathcal{O}_Z \to \mathcal{N}_{Z/Y} \to 0, \tag{2.36}$$

definitionnormal

which justifies the name for $\mathcal{N}_{Z/Y}$. If $i : Z = \{y\} \hookrightarrow Y$ is a point of Y, (not necessarily nonsingular), $\Theta_Y = \{0\}_x$, and $\mathcal{N}_{y/Y} = i_*((\mathfrak{m}_{y,Y}/\mathfrak{m}_{y,Y}^2)^\vee)$, where $T_y(Y) = (\mathfrak{m}_{y,Y}/\mathfrak{m}_{y,Y}^2)^\vee$ is the Zariski tangent space of Y at y (see [379, p. 37]). The surjection $\mathbf{S}(\mathfrak{m}_{y,Y}/\mathfrak{m}_{y,Y}^2) \to \oplus_{k=0}^\infty \mathfrak{m}_{y,Y}^k/\mathfrak{m}_{y,Y}^{k+1}$ gives the closed embedding of $\mathrm{TC}_y(Y)$ in the affine scheme $\mathrm{Spec}(\mathbf{S}(\mathfrak{m}_{y,Y}/\mathfrak{m}_{y,Y}^2))$ that can be identified with the Zariski tangent space.

Recall that the *blow-up* scheme of Y along Z, or with center Z, is defined by

$$\mathsf{Bl}_Z(Y) := \mathrm{Spec}(\bigoplus_{k=0}^\infty \mathcal{I}_Z^k).$$

It comes with a birational morphism $\sigma : \mathsf{Bl}_Z(Y) \to Y$ which is called the *blowing up morphism*. Its *exceptional divisor* is the scheme-theoretical pre-image of Y which is equal to

$$\mathsf{E} = \mathrm{Proj}(\bigoplus_{k=0}^\infty \mathcal{I}_Z^k/\mathcal{I}_Z^{k+1}).$$

If $Z \hookrightarrow Y$ is a regular embedding, e.g. both Z and Y are regular, the sheaf algebra $\bigoplus_{k=0}^\infty \mathcal{I}_Z^k/\mathcal{I}_Z^{k+1}$ coincides with the symmetric algebra $\mathbf{S}(\mathcal{I}_Z/\mathcal{I}_Z^2)$ and

$$\mathsf{E} = \mathrm{Proj}(\mathbf{S}(\mathcal{I}_Z/\mathcal{I}_Z^2)) = \mathbb{P}(\mathcal{N}_{Z/Y}^\vee).$$

As we see, E coincides with the projective normal cone. Let

$$\begin{array}{ccc} \mathsf{E} & \xrightarrow{j} & X \\ \downarrow{\scriptstyle \sigma_{\mathsf{E}}} & & \downarrow{\scriptstyle \sigma} \\ Z & \xrightarrow{i} & Y \end{array} \tag{2.37}$$ blowup

be the commutative diagram, where E is the exceptional divisor of σ.

The tautological invertible sheaf $\mathcal{O}_{\mathsf{E}}(1)$ coincides with the conormal sheaf $\mathcal{I}_{\mathsf{E}}/\mathcal{I}_{\mathsf{E}}^2$ of the Cartier divisor E in $\mathsf{Bl}_Z(Y)$. By definition of $\mathcal{O}_{\mathsf{E}}(1)$, we have

$$(g_{\mathsf{E}})_*(\mathcal{O}_{\mathsf{E}}(1)) = \mathcal{I}_Z/\mathcal{I}_Z^2.$$

Since the ideal sheaf $\sigma(\mathcal{I}_Z)$ generates the ideal sheaf $\mathcal{I}_{\mathsf{E}}$, there is a surjection

$$g^*(\mathbf{S}(f^*\mathcal{I}_Z/\mathcal{I}_Z^2)) \to \bigoplus_{k=0}^{\infty} \mathcal{I}_{\mathsf{E}}^k/\mathcal{I}_Z^{k+1}.$$

After applying $(g_E)_*$ and taking the affine spectrum, we obtain a closed embedding

$$C_Z Y \hookrightarrow \mathbb{V}(\mathcal{N}_{Z/Y}^\vee) := \mathrm{Spec}(\mathbf{S}(\mathcal{I}_Z/\mathcal{I}_Z^2)).$$

It follows from the definitions that $\mathbb{P}\mathrm{C}_Z Y$ coincides with the exceptional divisor of $\mathsf{Bl}_Z(\mathrm{C}_Z Y) \to \mathrm{C}_Z Y$, where we identify Y with the vertex of the normal cone $\mathrm{C}_Z Y$. Of course, this is a special case of a more general fact: if $X = \mathrm{Proj}(A)$, where $A = \oplus_{i=0}^{\infty} A_i$ is a graded algebra over an algebra K with $A_0 = K$ and generated by A_1, then $C = \mathrm{Spec}\, A$ is called the affine cover over X, $C_0 = \mathrm{Spec}\, A_0$ is its vertex and the exceptional divisor of the blow-up $\mathsf{Bl}_{C_0}(C)$ is isomorphic to X.

Let $f : Y' \to Y$ be a morphism, $g : Z' = Z \times_Y Y' = f^{-1}(Z) \to Z$. Then,

$$\mathrm{C}_{Z'} Y' = g^* \mathrm{C}_Z Y = \mathrm{C}_Z Y \times_Z Z'. \tag{2.38}$$ conechange

In particular, taking f to be a closed embedding $f : Y' \hookrightarrow Y$, we obtain that

$$\mathsf{Bl}_{Z \cap Y'}(Y') \cong \mathsf{Bl}_Z(Y) \times_Y Y'. \tag{2.39}$$ propertransform1

If $Y' \subset Z$, then $\mathsf{Bl}_{Z\cap Y'}(Y') = \sigma^{-1}(Y')$. Otherwise, it is equal to the proper transform $\bar{Y}'$ of Y' in $\mathsf{Bl}_Z(Y)$. The exceptional divisor E' of $\sigma' : \mathsf{Bl}_{Z\cap Y'}(Y') \to Y'$ is equal $\mathsf{E} \times_Z Y'$.

Recall that, if $f : X \to Y$ is any birational morphism, and $Y' \subset Y$ such that $f^{-1}(Y') \to Y'$ is an isomorphism over an open dense subset U of Y', the *proper transform* of Y' under f is defined to be the closure of $f^{-1}(U)$ in X.

Keep the notation from the previous paragraph, assume that $Z \subset Y \subset X$ are regular embeddings, and let $\mathcal{N}_{\bar{Y}/\mathsf{Bl}_Z(X)}$ be the normal bundle of the proper transform $\bar{Y}$ of Y in $\mathsf{Bl}_Z(X)$. Then the restrictionof $\sigma : \mathsf{Bl}_Z(X) \to X$ to $\bar{Y}$ is isomorphic to the blow-up $\nu : \mathsf{Bl}_Z(Y) \to Y$. Moreover,

$$\mathcal{N}_{\bar{Y}/\mathsf{Bl}_Z(X)} \cong \nu^*(\mathcal{N}_{Y/X}) \otimes \mathcal{O}_{\mathsf{Bl}_Z(Y)}(-\mathsf{F}), \tag{2.40}$$ exseqnormal1

where $F \subset E$ is the exceptional divisor of ν.

Another useful fact is the existence of the exact sequence of locally free sheaves on Z:

$$i^*\mathcal{N}^\vee_{Y'/Y} \to \mathcal{N}^\vee_{Z/Y} \to \mathcal{N}^\vee_{Z/Y'} \to 0, \tag{2.41}$$

where $i : Z \hookrightarrow Y'$ and $Y' \hookrightarrow Y$ are closed regular embeddings. If both closed embeddings are regular, the first arrow is injective, and passing to the duals, we get an exact sequence

$$0 \to \mathcal{N}_{Z/Y'} \to \mathcal{N}^\vee_{Z/Y} \to i^*\mathcal{N}^\vee_{Y'/Y} \to 0. \tag{2.42}$$ exseqnormal2

The surjection $\mathcal{N}^\vee_{Z/Y} \to \mathcal{N}^\vee_{Z/Y'}$ defines a closed embedding of the exceptional divisor of $\mathsf{Bl}_Z(Y')$ in the exceptional divisor of $\mathsf{Bl}_Z(Y)$.

For later reference, let us add the following well-known assertion about the behavior of the canonical class under the blow-up (see [379, Chapter II, Exercise 8.5]).

canblowup **Proposition 2.4.4.** *Consider the diagram (2.50), where Y and Z are smooth varieties over a field. Then*

$$K_Y = \sigma^*(K_X) + (d-1)E,$$

where d is the codimension of Z in Y.

2.4.2 Intersection theory and Segre classes

SS:2.4.2

Let us recall the definition of the Chow group $A(X)$ of algebraic cycles modulo rational equivalence.

Let X be an algebraic scheme over a field $\Bbbk$. We say that X is an algebraic variety if it is irreducible and reduced An algebraic *k-cycle* is an element of the free abelian group $Z_k(X)$ generated by the set of points $x \in X$ of dimension k (i.e., the residue field $\kappa(x)$ of x is of algebraic dimension k over $\Bbbk$). We identify a point x with its closure $\overline{\{x\}}$ in X. It is a closed subscheme of X, which is an algebraic variety of dimension k (a subvariety). We write $[V]$ for V considered as an element of $Z_k(X)$. Two k-cycles Z and Z' are called *rationally equivalent* if the difference is equal to the projections of a cycle $\mathcal{Z}(0) - \mathcal{Z}(\infty)$ on $X \times \mathbb{P}^1$ for some cycle $\mathcal{Z}$ on the product.

One can give an equivalent definition as follows. A prime divisor $\mathfrak{p}$ of height one in an integral domain A defines a function $\mathrm{ord}_{\mathfrak{p}} : A \setminus \{0\} \to \mathbb{Z}$ by setting $\mathrm{ord}_{\mathfrak{p}}(a) = \mathrm{length}(A_{\mathfrak{p}}/(a_{\mathfrak{p}}))$, where $a_{\mathfrak{p}}$ is the image of a in the local (one-dimensional) ring $A_{\mathfrak{p}}$. This function is extended to a unique homomorphism $Q(A)^* \to \mathbb{Z}$, where $Q(A)$ is the field of fractions of A. By globalizing, we obtain a function $\mathrm{ord}_x : R(X) \to \mathbb{Z}$, where x is a point on X of codimension 1 and $R(X)$ is the field of rational functions on a variety X‡. Now, we define the subgroup $B_k(X)$ of rationally equivalent k-cycles as the group generated by cycles of the form $\sum_x \mathrm{ord}_x \phi x$, where ϕ is a rational function on a subvariety $Y \subset X$ of dimension $k+1$. The quotient group $A_k(X) = Z_k(X)/B_k(X)$ is called the *Chow group* of k-cycles on X. We set

$$A_*(X) = \oplus_k A_k(X).$$

For any $\alpha = \sum n_x x \in Z_0(X)$, we define

$$\int_X \alpha = \sum_x n_x [\kappa(x) : \mathbb{k}].$$

When X is proper, this extends to $A_0(X)$, and, to the whole $A_*(X)$, where, by definition, $\int_X \alpha = 0$ if $\alpha \in A_k(X), k > 0$.

For any scheme X, one defines its *fundamental class* by

$$[X] = \sum n_V [V],$$

where V is an irreducible component of X and n_V is its multiplicity, the length of $\mathcal{O}_{X,\eta}$, where η is a generic point of V.

For any proper morphism $f : X \to Y$ of schemes, one defines the push-forward homomorphism

$$f_* : A_*(X) \to A_*(Y)$$

by setting, for any subvariety V,

$$\sigma_*[V] = \deg(V/\sigma(V))[\sigma(V)]$$

and extending the definition by linearity. Note that $\deg(V/\sigma(V)) = 0$ if the map $V \to \sigma(V)$ is not of finite degree. One checks that rationally equivalent to zero cycles go to zero, so the definition is legal. The homomorphism σ_* preserves the grading of $A_*(X)$.

The pull-back $\sigma^* : A_*(Y) \to A_*(X)$ is defined only for flat morphisms, regular closed embeddings, and their compositions. For a flat morphism σ, one sets, for any subvariety V, $\sigma^*[V] = [\sigma^{-1}(V)]$. It shifts the degree by increasing it by the relative dimension of σ.

Recall that a *Weil divisor* on a normal variety X of dimension n is an element

‡From now, on a variety over a field $\mathbb{k}$ means an integral algebraic scheme over $\mathbb{k}$

of $Z_{n-1}(X)$. It defines a reflexive sheaf $\mathcal{O}_X(D)$ of rank one (see Subsection SS:4.1.2 4.1.2). A *Cartier divisor* on X is a section of the sheaf $\mathcal{R}_X/\mathcal{O}_X^*$, where $\mathcal{R}_X$ is the constant sheaf of total rings of fractions. The function $\mathrm{ord}_x : \mathcal{R}(X) \to \mathbb{Z}$ factors through $\mathcal{O}_X^*$ and defines a homomorphism

$$\mathrm{CDiv}(X) \to \mathrm{WDiv}(X), \quad D \mapsto [D] = \sum_x \mathrm{ord}_x(D)x,$$

where $\mathrm{CDiv}(X)$ (resp. $\mathrm{WDiv}(X)$) is the group of Cartier (resp. Weil) divisors on X. A Weil divisor D is a Cartier divisor if and only if $\mathcal{O}_X(D)$ is locally free.

Let D be a Cartier divisior on X. One can restrict it to any subvariety V of X, and set

$$D \cdot [V] = [j^*(D)],$$

where $j : V \hookrightarrow X$ is the inclusion morphism of V to X. It is considered as a cycle on V and also as a cycle on X by means of $j_* : A_*(V) \to A_*(X)$. This extends by linearity to the intersection $D \cdot \alpha$ of D with any cycle class $\alpha \in A_*(X)$, so we can consider any divisor D as an endomorphism $\alpha \mapsto D \cdot \alpha$ of $A_*(X)$. It depends only on the linear equivalence class of D. By iterating the endomorphism, we can define, for any Cartier divisors $D_1, \ldots, D_k$ and $\alpha \in A_m(X)$, the intersection

$$D_1 \cdot \cdots \cdot D_k \cdot \alpha \in A_{m-k}(X).$$

For any closed subvariety Y containing $\mathrm{Supp}(D) \cap V$, we can identify $D \cdot [V]$ with an element of $A_*(Y)$. In particular, for any Y as above, any Cartier divisor D on X can be considered as a homomorphism

$$A_k(Y) \to A_{k-1}(\mathrm{Supp}(D) \cap Y), \quad \alpha \mapsto D \cdot \alpha.$$

By definition,

$$D_1 \cdots D_k = D_1 \cdot \cdots \cdot D_k \cdot [X]. \qquad (2.43)$$ defD

If X is of pure dimension n, this is an element of $A_{n-k}(X)$. We abbreviate $D^k = D \cdots D$ (k times). We also identify $A_0(X)$ with $\mathbb{Z}$ if X is a smooth irreducible algebraic variety. In particular, $D^n \in \mathbb{Z}$, where $n = \dim X$.

The intersection product of Cartier divisors is commutative and associative; the projection formula holds and depends only on the linear equivalence classes.

The following property allows one to compute the intersection of divisors on its resolution of singularities.

pullbackintersection **Proposition 2.4.5.** *Let f"$X' \to X$ be a proper morphism of complete algebraic varieties, $D_1, \ldots, D_k$ be Cartier divisors on X and $\alpha \in A_k(X')$.. Then,*

$$f^*(D_1) \cdot \ldots \cdot f^*(D_k) \cdot \alpha = D_1 \cdot \ldots \cdot D_k \cdot f_*\alpha.$$

In particular, if $\alpha = [X']$,

$$f^*(D_1) \cdot \ldots \cdot f^*(D_n) \cdot [X'] = r(D_1 \cdot \ldots \cdot D_n \cdot [X]),$$

where r is the degree of f (zero if $\dim X' > \dim X$).

In the case X is smooth, one can extend the intersection theory of divisors to the intersection theory of all rational equivalence classes of algebraic cycles. There is a unique product structure on $A(X)$ satisfying

1. For any two subvarieties V and W of X intersecting transversally,
$$[V] \cdot [W] = [V \cap W].$$
2. If $A(X)$ is graded by the codimension of cycles, then the product preserves the grading, i.e.,
$$A^k \times A^m \to A^{k+m}, \quad (\alpha, \beta) \mapsto \alpha \cdot \beta$$
that defines a pairing
$$A^c(X) \times A_k(X) \to A_{k-c}(X), \quad (\alpha, \beta) \mapsto \alpha \cap \beta.$$
3. For any morphism $f : Y \to X$ of smooth varieties, there is a unique homomorphism $f^* : A(X) \to A(Y)$ of graded rings such that
$$f^*([V]) = [f^{-1}(V)],$$
where V is a subvariety of X of codimension c such that $f^{-1}(V)$ is reduced and of codimension c in Y.
4. There is the *projection formula*:
$$f_*(f^*\alpha \cdot \beta) = \alpha \cdot f_*(\beta).$$

We refer to Fulton [315] for the intersection theory on singular varieties.

The las property gives an important corollary. Let $f : X' \to X$ be a birational morphism, for example a resolution of singularities of X. The $f^*(D)$ is a Cartier divisor on X' if D is. We have

$$f_*(f^*([D_1]) \cdot \cdots \cdot f^*([D_k]) \cdot [X']) = D_1 \cdot \cdots \cdot [D_k] \cdot [X].$$

This shows that we can compute the intersection of divisors on X' instead of X.

For any graded sheaf of algebras $\mathcal{A} = \oplus_{i \geq 0} \mathcal{A}_i$ over a scheme Y one defines the cone $C = \mathrm{Spec}(\mathcal{A})$, the projective cone $\mathbb{P}\mathrm{C} = \mathrm{Proj}(\mathcal{A})$, and its projective completion $\widehat{C} = \mathrm{Proj}(\mathcal{A}[z])$. The latter comes with a projection $p : \widehat{C} \to Y$ and an invertible sheaf $\mathcal{O}(1)$. The closed subscheme $V(z)$ of $\widehat{C}$ is isomorphic to

$\mathbb{P}$C, and there is an open embedding $j : C \hookrightarrow \widehat{C}$ with complement isomorphic to $V(z)$.

The Segre class $s(C)$ is defined by

$$s(C) = p_*(\sum_{i\geq 0} c_1(O(1))^i \cdot [\widehat{C}]) \in A_*(Y). \tag{2.44}$$

segrecone

We will be interested in the special case, where $\mathcal{A} = \oplus \mathcal{I}_Z^i/\mathcal{I}_Z^{i+1}$ and $C = C_Z(X)$ is the normal cone of a closed subscheme Z in X defined by the Ideal $\mathcal{I}_Z$. By definition,

$$s(Z, X) := \sum_{i\geq 0} s_i(Z, X) := s(C_Z(X)) = \sum_i s_i(C \in A_*(Z).$$

The following are the fundamental properties of Segre classes.

- If $Z \hookrightarrow X$ is a regular embedding (equivalently, the conormal bundle $\mathcal{N}_{Y/X}^\vee = \mathcal{I}_Y/\mathcal{I}_Y^2$ is locally free), then

$$s(Y, X) = c(\mathcal{N}_{Y/X})^{-1}, \tag{2.45}$$

prop1

where $c(\mathcal{F})$ denotes the total Chern class of aa coherent sheaf $\mathcal{F}$.

- If $f : X' \to X$ is a proper morphism of irreducible schemes and $g : Z' = f^{-1}(Z) \to Z$ is the restriction of f to Z', then

$$g_*(s(Z', X')) = \deg(X'/X)s(Z, X). \tag{2.46}$$

prop2

- If $f : X' \to X$ is flat, and $Z' = f^{-1}(Z)$, then

$$g^*(s(Z, X)) = s(Z', X'). \tag{2.47}$$

prop3

- If $\sigma : \tilde{X} = \mathsf{Bl}_Z(X) \to X$ is the blow-up of a proper closed subscheme Z in X, and E is the exceptional divisor with the projection $\sigma_{\mathsf{E}} : E \to Z$, then

$$s(Z, X) = \sum_{i\geq 1}(-1)^{i-1}(\sigma_{\mathsf{E}})_*([E]^i) = \sum_{i\geq 0}(\sigma_{\mathsf{E}})_*(\eta^i), \tag{2.48}$$

prop4

In the last formula, $\eta = c_1(O_E(1)) \in A^1(E)$, and we identify η^i with $\eta^i \cap [E]$. Moreover, we consider E as a Cartier divisor on X and take $[E]^i$ in the sense of the definition (defD 2.43). We have $j_*(\eta^i) = (-1)^i[E]^{i+1}$ by Theorem cohblowup 2.4.6, where we consider E as a divisor of X and take the intersection product $E^{i+1} \in A^{i+1}(X)$. Then

$$\sigma_*(E^i) = i_*(\sigma_{\mathsf{E}})_*([E]^i) = (-1)^{i-1}i_*((\sigma_{\mathsf{E}})_*\eta^{i-1}). \tag{2.49}$$

Even in the case where X is a smooth algebraic variety, the Segre classes of arbitrarily closed subscheme Z of X are difficult to compute. However, if

$Z \hookrightarrow X$ is a regular embedding, the property $s(Z, X) = c(\mathcal{N}_{Z/X})^{-1}$ makes the computation much easier.

We will use the diagram (2.50) Let

$$\begin{array}{ccc} \mathsf{E} & \xrightarrow{j} & X \\ \big\downarrow{\scriptstyle \sigma_{\mathsf{E}}} & & \big\downarrow{\scriptstyle \sigma} \\ Z & \xrightarrow{i} & Y \end{array} \tag{2.50}$$

blowup

for the standard notations for a blow-up and use Theorem 2.4.6 for the intersection products on the blow-up.

We denote by $s_i(Z, X)$ the part of $s(Z, X)$ from $A_i(Z)$. It follows that

$$s_{n-i}(Z, X) = (-1)^{i-1}(\sigma_{\mathsf{E}})_*([E]^i) = (\sigma_{\mathsf{E}})_*(\eta^{i-1}), \quad n = \dim X. \tag{2.51}$$

In particular, $(\sigma_{\mathsf{E}})_*([E]^i) = 0, i < d := \mathrm{codim}(Z, X)$. If $i = d$, we get

$$s_{n-d}(Z, X) = (-1)^{d-1}(\sigma_{\mathsf{E}})_*([E]^d) = (\sigma_{\mathsf{E}})_*(\eta^{d-1}) = e_Z(X)[Z],$$

where $e_Z(X)$ is the multiplicity of Z in X (see [315, 4.3]). In particular, if $Z \hookrightarrow X$ is a regular embedding $s_{n-d}(Z, X) = c_0(\mathcal{N}_{Z/X}) = [Z]$, so $e_Z(X) = 1$.

Since $s_{n-d-1}(Z, X) = -c_1(\mathcal{N}_{Z/X}) \in A^1(Z) = A_{n-d-1}(Z)$ if $Z \hookrightarrow X$ is a regular embedding, we get

$$c_1(\mathcal{N}_{Z/X}) = (-1)^{d-1}(\sigma_{\mathsf{E}})_*([E]^{d+1}) = (\sigma_{\mathsf{E}})_*(\eta^d).$$

A final note, before we go to examples is that, assuming that i is a regular embedding, the homomorphism

$$(\sigma_{\mathsf{E}})_* : A_0(E) \to A_0(Z)$$

is bijective. This follows from [315, Proposition 6.7 (d)]. So, we can identify $(-E)^n$ with η^{n-1} and $s_0(Z, X)$.

cohblowup **Theorem 2.4.6.** *Assume $i : Z \hookrightarrow Y$ is a regular embedding. Then, $A_*(X)$ is generated by $\sigma^*(A_*(Y))$ and $j_*(A_*(\mathsf{E}))$. The multiplication rules, defined by the cup-product, are the following:*

$$\sigma^*(\alpha) \cdot \sigma^*(\beta) = \sigma^*(\alpha \cdot \beta), \tag{2.52}$$

$$\sigma^*(\alpha) \cdot j_*(\gamma) = j_*(\gamma \cdot \sigma^*(i^*\alpha)), \tag{2.53}$$

$$j_*(\gamma) \cdot j_*(\delta) = -j_*(\gamma \cdot \delta \cdot \eta), \tag{2.54}$$

where $\alpha, \beta \in A_*(Y), \gamma, \delta \in A_*(\mathsf{E})$ and $\eta = c_1(\mathcal{O}_{\mathsf{E}}(1))$. Moreover,

$$A_*(\mathsf{E}) = \sigma^*(A_*(Z))[\eta],$$

where η satisfies a single relation

$$\sum_{i=0}^{d}(-\eta)^i \sigma_{\mathsf{E}}^*(c_{d-i}(\mathcal{N}_{Z/Y})) = 0. \tag{2.55}$$

defchern

Moreover, there is an exact sequence that describes the relations between elements in $\sigma_{\mathsf{E}}^*(A_*(Z)), \sigma^*(A_*(Y))$ and $j_*(A_*(E))$:

$$0 \to A_*(Z) \xrightarrow{(i_*,g)} A_*(Y) \oplus A_*(E) \xrightarrow{(\sigma^*,j_*)} A_*(X) \to 0, \tag{2.56}$$

relationblowup

where $g : A_*(Z) \to A_*(E)$ is defined by the formula

$$g(\alpha) = -\sigma_{\mathsf{E}}^*(\alpha)\cdot((-\eta)^{d-1} + \sigma_{\mathsf{E}}^*(c_1(\mathcal{N}_{Z/X}))\cdot(-\eta)^{d-2} + \cdots + \sigma_{\mathsf{E}}^*(c_{d-1}(\mathcal{N}_{Z/X})).$$

Note that the formula (2.55) is a special case of the formula that defines the Chern classes of any locally free sheaf $\mathcal{E}$ of rank r

$$\sum_{i=0}^{r}(-\eta)^i p^*(c_{d-i}(\mathcal{E})) = 0. \tag{2.57}$$

cohprojbundle

where $\eta = c_1(\mathcal{O}_{\mathbb{P}(\mathcal{E})}(1))$.

Let e be the algebraic class of the exceptional divisor E. It is equal to $j_*([\mathsf{E}])$, where $[\mathsf{E}]$ is the fundamental class of E. It also follows from the third relation that

$$e^{i+1} = (-1)^i j_*(\eta^i).$$

iskmanin *Example* 2.4.7. Let X be a smooth projective variety of dimension n and Z be a closed point in X. Then, $\mathsf{E} \cong \mathbb{P}^{n-1}$ and η is the class of a hyperplane in E. We have $\mathsf{E}^k = (-1)^{k-1} j_*(\eta^{k-1})$. In particular, $\mathsf{E}^n = (-1)^{n-1}$, where we, as always, identify $A_0(X)$ with $\mathbb{Z}$.

Suppose now that Z is a smooth subvariety of Y of codimension $d = 2$. Then $c_1(\mathcal{N}_{Z/Y}) = c_1(Y)\cdot Z - c_1(Z)\eta$. Applying (2.56), we get

$$\begin{aligned} \mathsf{E}^3 &= -c_1(\mathcal{N}_{Z/Y}),\\ \mathsf{E}^2 &= -j_*(\eta) = -\sigma^*(Z) + j_*(\sigma_{\mathsf{E}}^*(\mathcal{N}_{Z/Y})). \end{aligned} \tag{2.58}$$

iskformula

For example, if $\dim Y = 3$, and Z is a smooth curve of genus g, we get $\mathsf{E}^2 = -\sigma^*(Z) + (2g - 2 + c_1(X)\cdot Z)j_*(\mathfrak{f})$, where $\mathfrak{f}$ is the class of a fiber of σ_{E} in $A_1(\mathsf{E})$.

Finally, formula (2.58) gives in this case

$$\mathsf{E}\cdot j_*(\mathfrak{f}) = -1. \tag{2.59}$$

The *Segre classes* $s_i(Z,Y) \in A_i(Z)$ are defined by the equality

$$s_i(Z,Y) := (\sigma_{\mathsf{E}})_*(\eta^{i+d-1}) = (-1)^{i+d-1}\sigma_*(j_*(e^{i+d})). \tag{2.60}$$

defsegreclass

We set

$$s(Z,Y) = \sum_{i\ge 0} s_i(Z,Y) \in A_*(Z).$$

In the special case when $Z \hookrightarrow Y$ is a regular embedding of smooth varieties

$$s(Z,Y) = 1 + s_1(Z,Y) + \cdots + s_{\dim Z}(Z,Y) \in A_*(Z),$$

such that

$$s(Z,Y)\cdot c(\mathcal{N}_{Z/Y}) = [Z],$$

where $c(\mathcal{N}_{Z/Y}) = \sum_{i=0}^{\dim Z} c_i(\mathcal{N}_{Z/Y}) \in A_*(Z)$ is the Chern class of the normal bundle $\mathcal{N}_{Z/Y}$.

ex:blowuppoint *Example* 2.4.8. Let $\sigma : Y = \mathsf{Bl}_x(X) \to X$ be the blow-up of a closed point on a smooth algebraic variety X of codimension n. Then $\mathcal{N}_{x/X} \cong (i_x)_*\kappa(x)^n$ is the sky-scrapper sheaf supported at x. We have $c_0(\mathcal{N}_{x/X}) = [x], c_i(\mathcal{N}_{x/X}) = 0, i \ne 0$, and hence, $s(x,X) = [x]$. This gives

$$[E]^n = (-1)^{n-d}. \tag{2.61}$$ excurve

This is a familiar formula for the blow-up of a point a smooth algebraic surface X. The exceptional curve E was classically called an *exceptional curve of the first kind* and, nowadays, we just say that E is a (-1)-curve.

Of course, the formula (2.61) can be proved by an elementary argument. The formula is local, so we may assume that X is a projective subvariety of $\mathbb{P}^N$. Take a hyperplane section H_0 that contains x. We embed X, then $\sigma^*([H_0]) = [\bar{H}_0] + [E]$, where $\bar{H}$ is the proper transform of H. Replacing H_0 by a hyperplane section not containing x, we get

$$0 = \sigma^*([H])\cdot[E] = \sigma^*[H_0]\cdot[E] = ([\bar{H}_0]+[E])\cdot[E] = j_*\eta + [E]^2.$$

Thus, $[E]^2 = -j_*(\eta)$ (in agreement with Theorem 2.4.6). Next, we get $[\bar{H}_0]\cdot[E]^2 = -[\bar{H}_0]\cdot j_*\eta = -j_*(\eta\cdot i^*([H_0]+[E]) = -j_*(\eta\cdot i^*([E]) = j_*(\eta^2)$. Continuing in this way, we get

$$E^{k+1} = (-1)^k j_*(\eta^k).$$

ex:blowupsubspace *Example* 2.4.9. Let P_k be a codimension k subspace of $\mathbb{P}^n$. Using the exact sequence (2.42) for normal sheaves, we get, by induction on k,

$$\mathcal{N}_{P_k/\mathbb{P}^n} \cong \mathcal{O}_P(1)^{\oplus k}. \tag{2.62}$$

It follows that

$$s(P_k,\mathbb{P}^n) = \frac{1}{(1+h)^k} = (1-h+h^2+\cdots+(-1)^i h^i+\cdots)^k = \sum_{i=0}^{n-k}\binom{n-i-1}{k-1}(-h)^{i-k+1},$$

where $h = c_1(\mathcal{O}_P(1)) \in A_*(P)$. This gives

$$s_i(P_k, \mathbb{P}^n) = (-1)^{n-k-i}\binom{n-i-1}{k-1}h^{n-k-i}.$$

Since $\dim P_k = n-k$, $h^{m-k+1} \in A_{n-m-1}(P_k)$. Let $E = \mathbb{P}(\mathcal{N}^\vee_{P_k/\mathbb{P}^n}) = \mathbb{P}(\mathcal{O}_P(-1)^{\oplus k})$ be the exceptional divisor of the blow up $Y = \mathsf{Bl}_{P_k}(\mathbb{P}^n)$. Then,

$$(\sigma_{\mathsf{E}})_*([E]^i) = 0, \quad i < k-1,$$

and

$$(\sigma_{\mathsf{E}})_*([E]^i) = (-1)^{i-1}s(Z,X)_{n-i} = (-1)^{k-1}\binom{i-1}{k-1}h^{i-k+1}$$

for $k-1 \le i \le n-1$.

In particular,

$$[E]^n = (-1)^{k-1}\binom{n-1}{k-1}. \tag{2.63}$$

The simple relation between the Segre class $s(Z,Y)$ and $c(\mathcal{N}_{Z/Y}$ fails if $Z \hookrightarrow Y$ is not a regular embedding. For example, if $Z = \{y\}$ is a closed point of Y of multiplicity $m > 1$, then $s(Z,Y)_0 = m[Z]$ and $c(\mathcal{N}_{y,Y}) = [Z]$.

A resolution of indeterminacy of a rational map are often obtained by a sequence of blow-ups of smooth subvarieties and their proper transforms. The following proposition show how the the Segre class changes under the proper transform.

proptransformcoh **Proposition 2.4.10.** *Let $i : Y' \hookrightarrow Y$ is a closed embedding of smooth varieties and $\bar{Y}'$ be the proper transform of Y' in $X = \mathsf{Bl}_Z(Y)$. Then*

$$[\bar{Y}'] = \sigma^*([Y']) - j_*([c(\mathcal{E}) \cdot \sigma_{\mathsf{E}}^*(s(Y' \cap Z, Y')]_k),$$

where $\mathcal{E} = \sigma_{\mathsf{E}}^*(\mathcal{N}_{Z/Y})/\mathcal{O}_{\mathsf{E}}(-1)$ and $k = \dim Y'$.

In the special case where $Z \subset Y'$, we get

$$c(\mathcal{E}) = c(\sigma_{\mathsf{E}}^*(\mathcal{N}_{Z/Y})/(1-\eta),$$

and, applying exact sequence (2.42) exseqnormal2, we obtain

$$\sigma_{\mathsf{E}}^*(s(Y' \cap Z, Y')) = \sigma_{\mathsf{E}}^*(s(Z, Y')) = s(Z,Y)/i^*(s(Y',Y)).$$

This gives

$$[\bar{Y}'] = \sigma^*([Y']) - j_*(c(\sigma_{\mathsf{E}}^*(\mathcal{N}_{Z/Y})/(1-\eta) \cdot \sigma_{\mathsf{E}}^*(s(Z,Y))i^*(s(Y',Y))$$

$$= \sigma^*([Y']) - j_*([\sigma_{\mathsf{E}}^*(i^*(c(\mathcal{N}_{Y'/Y}))(1+\eta+\eta^2+\cdots)]_{k-1}).$$

In particular, if $k = 1$, we get $[\tilde{Y}'] = \sigma^*([Y']) - [E]$, as expected.

Another special case is when $\dim Y' \cap Z \le k - d$, for example, Y' intersects transversally Z. In this case, we obtain

$$[\bar{Y}'] = \sigma^*([Y']). \tag{2.64}$$ fulton6.7.2

Fulton
[315, Corollary 6.7.2].

Example 2.4.11. Let ℓ be a line in $\mathbb{P}^n$ and $Z = \{x_1, \dots, x_m\} \in \ell$ be the closed reduced subset consisting of m distinct points in ℓ. Let $\sigma_1 : X_1 = \mathsf{Bl}_Z(\mathbb{P}^n) \to \mathbb{P}^n$ and $\sigma_2 : X_2 = \mathsf{Bl}_{\bar{\ell}}(X_1) \to X_1$. The proper transform $\bar{\ell}$ is isomorphic to $\mathbb{P}^1$, let h be the divisor class of a point on $\bar{\ell}$. The exceptional divisor E of σ_1 intersects $\bar{\ell}$ at m points. This gives

$$\mathcal{N}_{\bar{\ell}/X_1} = \sigma_1^*(\mathcal{O}_\ell(1)^{\oplus n-1}) \otimes \mathcal{O}_{\bar{\ell}}(-m).$$

Hence,

$$s(\bar{\ell}, X_1) = c(\mathcal{O}_{\bar{\ell}}(1-m)^{\oplus n-1})^{-1} = \frac{1}{(1+(1-m)h)^{n-1}},$$

We have

$$s_0(\bar{\ell}, X_1) = (n-1)(m-1)h, \quad s_1(\bar{\ell}, X_1) = [\bar{\ell}]. \tag{2.65}$$

Let E_2 be the exceptional divisor of σ_2 and $\bar{E}_1$ be the proper transform of the exceptional divisor E_1 of σ_1. We obtain

$$[\bar{E}_1]^n = \sigma_2^*[E_1] = (-1)^{n-1}m, \quad [E_2]^n = (-1)^n(n-1)(m-1).$$

2.4.3 The Discriminant Variety of Quadrics

SS:2.4.3

We fix a base scheme T over an algebraically closed field $\Bbbk$ of characteristic $p \neq 2$ and a locally free sheaf $\mathcal{E}$ of rank $n+1$ over T. For most applications, T is equal to $\mathrm{Spec}(\Bbbk)$ and $\mathcal{E}$ is a vector space E of dimension $n+1$, however we will need this generality for the proofs.

Definition 2.4.12. *A* quadratic form *on $\mathcal{E}$ is defined in one of the following equivalent ways:*

(i) a section q of $S^2(\mathcal{E}^\vee)$;

(ii) a homomorphism $q : \mathcal{E} \to \mathcal{E}^\vee$ such that its transpose $\mathcal{E} = (\mathcal{E}^\vee)^\vee \to \mathcal{E}^\vee$ coincides with q.

(iii) a homomorphism $q : S^2(\mathcal{E}) \to \mathcal{O}_T$.

The set $S^2(\mathcal{E}^\vee)$ of quadratic forms on $\mathcal{E}$ is a free $\mathcal{O}_T$-module of rank $\frac{1}{2}(n+1)(n+2)$. Let $\mathbb{P} = \mathbb{P}(\mathcal{E}^\vee)$, a quadratic form $q \in S^2(\mathcal{E}^\vee)$ defines a section of $\mathcal{O}_{\mathbb{P}}(2)$. An element of the linear system $|\mathcal{O}_{\mathbb{P}}(2)| = \mathbb{P}(H^0(\mathbb{P}, \mathcal{O}_{\mathbb{P}}(2))^\vee)$ is called a *quadric* in $\mathbb{P}$ and denoted by $V(q)$.

The projective space $\mathbb{P}(S^2(\mathcal{E}^\vee)) = |\mathcal{O}_{\mathbb{P}}(2)|$ of quadrics on $\mathbb{P}$ will be denoted by $\mathsf{Q}_{\mathcal{E}}$. We have

$$S^2(\mathcal{E}^\vee) = H^0(\mathsf{Q}_{\mathcal{E}}, \mathcal{O}_{\mathsf{Q}_{\mathcal{E}}}(1)).$$

For any $q \in S^2(\mathcal{E}^\vee)$, let $q_t : \mathcal{E}(t) \to \mathcal{E}(t)^\vee$ be the corresponding quadratic form on the fibers. It is a linear map of vector spaces of dimension $n+1$. Let

$$D_c(q) := \{t \in T : \operatorname{corank}(q_t) \geq c\}$$

After we trivialize $\mathcal{E}$ on some open affine set $U = \operatorname{Spec}(A)$ of T, and choose a basis $(e_1, \ldots, e_{n+1})$ of $\mathcal{E}$ and the dual basis $(e_1^*, \ldots, e_{n+1}^*)$ of $\mathcal{E}^\vee$, we obtain that q is given by a symmetric matrix of size $n+1$ with coefficients in A. It follows that $D_c(q)$ is a closed subscheme of T defined by minors of size $n+2-c$.

Considered q as a linear map $q : \mathcal{E} \to \mathcal{E}^\vee$, we can define its exterior power

$$\wedge^k(q) : \bigwedge^k \mathcal{E} \to \bigwedge^k \mathcal{E})^\vee.$$

This is a quadratic form on $\bigwedge^k \mathcal{E}$. Then, $D_c(q)$ is the scheme of zeros of the section $\wedge^{n+2-c}(q)$.

For any scheme $f : T' \to T$ and a sheaf $\mathcal{F}$ of $\mathcal{O}_T$-modules on T, we denote by $\mathcal{F}_{T'}$ the pull-back $f^*\mathcal{F}$ of $\mathcal{F}$ on T'. Similarly, for any scheme X over T, we denote by $X_{T'}$ the base change $X \times_T T'$ considered as a scheme over T'.

Let

$$\mathsf{u}_{\mathcal{E}} : \mathcal{E}_{\mathsf{Q}_{\mathcal{E}}} \to \mathcal{E}^\vee_{\mathsf{Q}_{\mathcal{E}}} \otimes \mathcal{O}_{\mathsf{Q}_{\mathcal{E}}}(1) \qquad (2.66)$$ universalquadric

be the homomorphism of locally free sheaves on $\mathsf{Q}_{\mathcal{E}}$ such that, after tensoring with $\mathcal{O}_{\mathsf{Q}_{\mathcal{E}}}(-1)$, its direct image under the structural projection $\mathsf{Q}_{\mathcal{E}} \to T$ is equal to the tautological homomorphism $\mathcal{E} \otimes S^2(\mathcal{E}^\vee) \to \mathcal{E}^\vee$. We call $\mathsf{u}_{\mathcal{E}}$ the *universal quadratic form*.

The property of universality is explained as follows. First, we slightly generalize the notion of a quadratic form by introducing a *twisted quadratic form* over a base T to be a section of $S^2(\mathcal{E}^\vee) \otimes \mathcal{L}$, or a symmetric map $\mathcal{E} \to \mathcal{E}^\vee \otimes \mathcal{L}$, where $\mathcal{L}$ is an invertible sheaf on T. Then, for any T-scheme $f : X \to T$ and a nonzero quadratic form on $\mathcal{E}_X \to \mathcal{E}_X^\vee \otimes \mathcal{L}$ with values in some invertible sheaf $\mathcal{L}$, there exists a unique morphism of T-schemes $\phi : X \to \mathsf{Q}_{\mathcal{E}}$ such that $q = f^*(\mathsf{u}_{\mathcal{E}})$ (up to multiplication by a scalar) and $\mathcal{L} = f^*\mathcal{O}_{\mathsf{Q}_{\mathcal{E}}}(1)$.

Every coherent sheaf on a scheme S defines a unique stratification $\{Z(c)\}_{c \geq 0}$ with the following properties:

1. The restriction of $\mathcal{F}$ to $Z(c)$ is a locally free sheaf of rank c.
2. The closure $\bar{Z}(c)$ is contained in $\cup_{c' \geq c} Z(c')$.
3. Any morphism $f : S' \to S$ such that $f^*\mathcal{F}$ is locally free factors through $\coprod_c Z(c) \subset S$.
4. The stratification commutes with the base change.

(see [704, Theorem 4.2.7]).

We apply this to the case where $\mathcal{F} = \mathrm{Coker}(\mathsf{u}_{\mathcal{E}})$. Then,

$$Z(c) = \{V(q) \in \mathsf{Q}_{\mathcal{E}} : \mathrm{Coker}(\mathsf{u}_{\mathcal{E}}) \text{ is locally free of rank } c\}.$$

For each $t \in T$, the fiber of $Z(c)_t$ consists of quadrics $V(q_t)$ such that $q_t : \mathcal{E}(t) \to \mathcal{E}^\vee(t)$ is of corank c. We set

$$\mathsf{Q}_{\mathcal{E}}(c)^\circ := Z(c),$$

and

$$\mathsf{Q}_{\mathcal{E}}(c) := \bar{Z}(c).$$

By trivializing, $\mathcal{E}$ and taking fibers over T, we find that $\mathsf{Q}_{\mathcal{E}}(c)$ is defined by the zero scheme of the section $\wedge^{n+2-c}(\mathsf{u}_{\mathcal{E}})$ of $S^2(\wedge^{n+2-c}(\mathcal{E}^\vee))$.

There is a stratification by closed subschemes:

$$\mathsf{Q}_{\mathcal{E}} = \mathsf{Q}_{\mathcal{E}}(0) \supset \mathsf{Q}_{\mathcal{E}}(1) \supset \cdots \supset \mathsf{Q}_{\mathcal{E}}(n).$$

The last open strata $\mathsf{Q}_{\mathcal{E}}(n)^\circ$ is closed. It is equal to the image of the Veronese map

$$v_2 : \mathbb{P}(\mathcal{E}^\vee) = |\mathcal{E}| \to \mathbb{P}(S^2(\mathcal{E}^\vee)) = |S^2\mathcal{E}|$$

defined by the linear system $|\mathcal{O}_{|\mathcal{E}|}(2)|$.

Let $Q = V(q)$ be a quadric of corank c in a projective space $|E|$ and

$$L = \mathrm{Ker}(q) = \mathrm{Im}(q)^\perp = \mathrm{Coker}(q)^\vee$$

be the radical of q, so that $|L| = \mathrm{Sing}(Q)$. The quadric Q is equal to the cone over a quadric $\bar{Q}$ in $|E/L|$ with the vertex $|L|$. The singular locus of the cone Q' over any quadric $\bar{Q}'$ in $|E/L|$ is a subspace $|L'|$ containing $|L|$. The variety of all quadrics Q' of corank $c' \geq c$ with $|L| \subset \mathrm{Sing}(Q')$ can be identified with $\mathsf{Q}_{E/L}(c'-c)| = \mathsf{Q}_{(L^\perp)^\vee}(c'-c)$.

Varying L in the Grassmannian $G(c, E)$, we obtain $(E/L)^\vee = (\mathcal{K}_c^\vee)(L)$ that defines a surjection $S^2(\mathcal{E}_{G(c,\mathcal{E})}) \to S^2(\mathcal{K}_c^\vee)$ and the closed embedding $\mathsf{Q}_{\mathcal{K}_c^\vee} = \mathbb{P}(S^2(\mathcal{K}_c^\vee)) \hookrightarrow \mathbb{P}(S^2(\mathcal{E}_{G(c,E)})) = \mathsf{Q}_{\mathcal{E}} \times G(c, E)$. After composing it with the projection $\mathcal{E}_{G(c,E)} \to \mathsf{Q}_{\mathcal{E}}$, we get a morphism $p_2 : \mathsf{Q}_{\mathcal{K}_c^\vee} \to \mathsf{Q}_{\mathcal{E}}$ and a commutative diagram:

$$\begin{array}{ccccccc}
 & & \mathsf{Q}_{\mathcal{K}^\vee_{c'}} = \mathsf{Q}_{\mathcal{K}^\vee_c}(c'-c)^\circ & \hookrightarrow & \mathsf{Q}_{\mathcal{K}^\vee_c} & & \\
 & \swarrow^{p_1^{c'}} & \downarrow{\scriptstyle p_2^{c'}} & & \downarrow{\scriptstyle p_2} & \searrow^{p_1} & \\
G(c',E) & \xleftarrow{\ s_{c'}\ } & \mathsf{Q}_E(c')^\circ & \hookrightarrow & \mathsf{Q}_E(c) & \overset{\mathfrak{s}_c}{\dashrightarrow} & G(c,E)
\end{array}$$

where p_1 is the structural projection and

$$\mathfrak{s}_c : \mathsf{Q}_{\mathcal{E}}(c)^\circ \to \mathbb{G}, \quad Q \mapsto \mathrm{Sing}(Q),$$

considered as a rational map from $\mathsf{Q}_E(c)$ to $G(c,E)$).

Let $F(c,c',E) \cong G(c,\mathcal{R}_{c'})$ be the flag variety from Example flag 2.4.3. The pair of morphisms $p_1 : \mathsf{Q}_{\mathcal{K}_c}(c') \to G(c,E)$, $p_1^{c'} : \mathsf{Q}_{\mathcal{K}_c}(c'-c) \to G(c',E)$ define a map $\alpha : \mathsf{Q}_{\mathcal{K}_c}(c'-c) \to F(c,c',E)$ such that

$$\mathsf{Q}_{\mathcal{K}_c}(c'-c) \cong \mathbb{P}(S^2(\mathcal{K}^\vee_{c,c'})) = \mathsf{Q}_{\mathcal{K}^\vee_{c,c'}},$$

as schemes over $F(c,c',E)$. We have the cartesian diagram:

$$\begin{array}{ccccc}
\mathsf{Q}_E(c')^\circ & \xleftarrow{\ p_2\ } & \mathsf{Q}_{\mathcal{K}^\vee_c}(c'-c)^\circ & = & \mathsf{Q}_{\mathcal{K}^\vee_{c,c'}} \\
\downarrow{\scriptstyle \mathfrak{s}_{c'}} & & \downarrow{\scriptstyle \alpha} & & \\
G(c',E) & \longleftarrow & F(c,c',E) & &
\end{array}$$

We leave it to the reader to extend the previous discussion to the relative case where E is replaced by a locally free sheaf $\mathcal{E}$ over some base T that leads to the following:

vainsencher4 **Proposition 2.4.13.** *Let $\mathcal{K}_c$ be the universal subsheaf over $G(c,\mathcal{E})$ and $p_1 : \mathsf{Q}_{\mathcal{K}_c^\vee} \to G(c,\mathcal{E})$ be the projective bundle of quadrics in $\mathbb{P}(\mathcal{K}_c)$. Let*

$$p_2 : \mathsf{Q}_{\mathcal{K}^\vee_c}(c'-c)^\circ \hookrightarrow \mathbb{P}(S^2(\mathcal{K}_c^\vee)) \hookrightarrow \mathbb{P}(S^2(\mathcal{E})_{G(c,\mathcal{E})}) = \mathsf{Q}_{\mathcal{E}} \times G(c,\mathcal{E}) \to \mathsf{Q}_{\mathcal{E}}$$

be the composition of the closed embeddings and the projection morphism. For any $c' \geq c$, there is an isomorphism of $\mathsf{Q}_{\mathcal{E}}(c')^\circ$-schemes

$$\mathsf{Q}'_{\mathcal{K}^\vee_{c'}} \cong p_2^{-1}(\mathsf{Q}_{\mathcal{E}}(c')^\circ) \cong \mathsf{Q}_{\mathcal{E}}(c')^\circ \times_{G(c',\mathcal{E})} G(c,\mathcal{R}_{c'}),$$

where $\mathsf{Q}_{\mathcal{E}}(c')^\circ \to G(c',\mathcal{E})$ is equal to the morphism $s_{c'}$.

cor:two **Corollary 2.4.14.** *The projection*

$$p_2 : \mathsf{Q}_{\mathcal{K}^\vee_c} \to \mathsf{Q}_{\mathcal{E}}(c)$$

is isomorphic to the blow-up of the closed subscheme $\mathsf{Q}_{\mathcal{E}}(c+1)$ of $\mathsf{Q}_{\mathcal{E}}(c)$.

It is also isomorphic to the proper transform of $\mathsf{Q}_{\mathcal{E}}(c)$ *under the blow-up* $\mathsf{Bl}_{\mathsf{Q}_{\mathcal{E}}(c+1)}(\mathsf{Q}_{\mathcal{E}})$.

Proof For simplicity, we assume that $\mathcal{E}$ is a vector space over $T = \mathrm{Spec}(\Bbbk)$. We leave the general case to the reader. The subscheme $\mathsf{Q}_{\mathcal{K}_c^\vee}(c+1) \subset \mathsf{Q}_{\mathcal{K}_c}$ is the relative discriminant hypersurface. In particular, the pre-image of the closed subscheme $\mathsf{Q}_E(c+1)$ of $\mathsf{Q}_E(c)$ in $\mathsf{Q}_{\mathcal{K}_c}(c+1)$ is a divisor. By the universal property of the blow-up [379, Chapter II, Proposition 7.14], there is a birational morphism $\phi : \mathsf{Q}_{\mathcal{K}_c^\vee} \to \mathsf{Bl}_{\mathsf{Q}_E(r-1)}(\mathsf{Q}_E)$. The rational map $\mathfrak{s}_c : \mathsf{Q}_E(c) \dashrightarrow G(c,E) \subset |\bigwedge^{n+2-c} E|$ is given by a linear system of hypersurfaces in Q_E generated by the $(n+2-c)$-minors of the symmetric matrix (A_{ij}) of projective coordinates on Q_E. Its base locus is equal to the subscheme $\mathsf{Q}_E(c+1)$. It is known that the blow-up $\mathsf{Bl}_{\mathsf{Q}_E(c+1)}(\mathsf{Q}_E)$ resolves the indeterminacy of the rational map [379, Chapter II, Example 7.17.4]. Since the projections $\mathsf{Q}_{\mathcal{K}_c^\vee} \to \mathsf{Q}_E(c)$ and $\mathsf{Q}_{\mathcal{K}_c^\vee} \to G(c,E)$ also reserve the indeterminacy of this map, the schemes $\mathsf{Q}_{\mathcal{K}_c^\vee}$ and $\mathsf{Bl}_{\mathsf{Q}_E(r-1)}(\mathsf{Q}_E)$ are isomorphic over the open subset $\mathsf{Q}_E(c+1)^\circ$. It follows that ϕ is a small contraction. However, $\mathrm{Pic}(G(c,E)) \cong \mathbb{Z}$, hence $\mathrm{Pic}(\mathsf{Q}_{\mathcal{K}_c^\vee}) \cong \mathbb{Z}^2$. The projections to $G(c,E)$ and $\mathsf{Q}_E(c+1)$ are two extremal divisorial contractions of $\mathsf{Q}_{\mathcal{K}_c^\vee}$, there is no small contractions.

The second assertion follows from (2.39). □

cor:two **Corollary 2.4.15.** *Assume* $c' = c+1$. *There is an isomorphism*

$$\mathsf{Q}_{\mathcal{K}_c^\vee}(1)^\circ = p_2^{-1}(\mathsf{Q}_{\mathcal{E}}(c')^\circ) \cong \mathsf{Q}_{\mathcal{E}}(c')^\circ \times_{G(c',\mathcal{E})} \mathbb{P}(\mathcal{R}_{c'}).$$

The image of the closed embedding $\mathsf{Q}_{\mathcal{K}_c^\vee}(1)^\circ \hookrightarrow \mathsf{Q}_{\mathcal{K}_c^\vee}$ *is the relative discriminant hypersurface in* $\mathsf{Q}_{\mathcal{K}_c^\vee}$. *In particular, if* $c = n-1$, $\mathsf{Q}_{\mathcal{K}_c^\vee}(1)^\circ \cong \mathbb{P}(\mathcal{K}_{n-1}^\vee)$ *and the embedding coincides with the Veronese map.*

The following proposition is proven in [778, Proposition 4.4]. We will give another proof.

normalbundle1 **Proposition 2.4.16.** *The normal bundle* $\mathcal{N}_{\mathsf{Q}_{\mathcal{E}}(c)^\circ/\mathsf{Q}_{\mathcal{E}}}$ *is isomorphic to* $\mathfrak{s}_c^* S^2(\mathcal{R}_c)(1)$.

Proof For simplicity of the notation, let us put $G = G(c,\mathcal{E}), P = \mathsf{Q}_{\mathcal{K}_c^\vee}, \mathsf{Q} = \mathsf{Q}_{\mathcal{E}}$ We identify P with the closure of the graph $\Gamma_{\mathfrak{s}_c} \subset \mathsf{Q}(c)^\circ \to G$. By definition, it is the graph of the rational map $\mathfrak{s}_c : \mathsf{Q}(c) \dashrightarrow G$.

Let $j : \mathsf{Q}(c) \times G \hookrightarrow \mathsf{Q} \times G$ be the closed embedding. We use exact sequence (2.42) to obtain an exact sequence

$$0 \to \mathcal{N}_{P/\mathsf{Q}(c)\times G} \to \mathcal{N}_{P/\mathsf{Q}\times G} \to j^*(\mathcal{N}_{\mathsf{Q}(c)\times G/\mathsf{Q}\times G}) \qquad (2.67)$$ firstexseq

where the last homomorphism is surjective after the restriction to $\mathsf{Q}(c)^\circ \times G$.

Let $P' = p_2^{-1}(\mathsf{Q}(c)^\circ) \cong \mathsf{Q}(c)^\circ$. It is the graph of $\mathfrak{s}_2$. Since the graph is the pre-image of the diagonal under the map $\mathfrak{s}_c \times \mathrm{id}_{\mathbb{G}} : \Gamma \to \mathbb{G} \times \mathbb{G}$, and the conormal sheaf of the diagonal is isomorphic to Ω_G^1 under any of the two projections $G \times G \to G$, we see that

$$\mathcal{N}_{P'/\mathsf{Q}(c)^\circ \times G} \cong \mathfrak{s}_c^* \Theta_G. \qquad (2.68)$$ secondiso

By Proposition sernesi 2.4.2,

$$\mathcal{N}_{\Gamma/\mathsf{Q}\times G} \cong \mathfrak{s}_c^* \mathrm{Ker}(S^2(E)_G \to S^2(\mathcal{K}_c^\vee))^\vee \otimes \mathcal{O}(1).$$

The invertible sheaf $\mathcal{O}(1)$ is equal to $p_2^* \mathcal{O}_{\mathsf{Q}}(1)$. We use the exact sequence

$$0 \to S^2 \mathcal{R}_c^\vee \to \mathrm{Ker}(S^2(E)_G \to S^2(\mathcal{K}_c^\vee)) \to \mathcal{R}_c^\vee \otimes \mathcal{K}_c^\vee \to 0.$$

Passing to the duals and twisting by $\mathcal{O}(1)$, we get an exact sequence

$$0 \to \mathcal{K}_c \otimes \mathcal{R}_c(1) \to (\mathrm{Ker}(S^2(E)_G \to S^2(\mathcal{K}_c^\vee))^\vee(1) \to S^2(\mathcal{R}_c)(1) \to 0.$$

The universal quadratic form $\mathsf{u}_E : E_{\mathsf{Q}} \to E_{\mathsf{Q}}^\vee(1)$ defines an isomorphism $\mathcal{K}_c^\vee = \mathcal{E}/\mathcal{R}_c^\vee \to \mathcal{K}_c(1) \subset \mathcal{E}(1)$ over $\mathsf{Q}(c)^\circ$. This gives an exact sequence

$$0 \to \mathfrak{s}_c^*(\mathcal{K}_c^\vee \otimes \mathcal{R}_c) \to \mathfrak{s}_c^*(\mathrm{Ker}(S^2(E)_G \to S^2(\mathcal{K}_c^\vee))^\vee \otimes \mathcal{O}_{\mathsf{Q}}(1)) \to \mathfrak{s}_c^* S^2(\mathcal{R}_c)(1) \to 0.$$

Now, we use that $\Theta_G \cong \mathcal{K}_c^\vee \otimes \mathcal{R}_c$ (see Lemma CAG-2:tang 10.1.1 in Chapter 10), and applying (secondiso 2.68), we obtain an exact sequence

$$0 \to \mathcal{N}_{\Gamma/\mathsf{Q}(c)^\circ \times G} \to \mathcal{N}_{\Gamma/\mathsf{Q}\times G} \to \mathfrak{s}_c^* S^2(\mathcal{R}_c)(1) \to 0.$$

Comparing it with exact sequence (firstexseq 2.67), we find that

$$\mathfrak{s}_c^* S^2(\mathcal{R}_c)(1) \cong j^*(\mathcal{N}_{\mathsf{Q}(c)^\circ \times G/\mathsf{Q}\times G}) \cong \mathcal{N}_{\mathsf{Q}(c)^\circ/\mathsf{Q}}.$$

□

cor:one **Corollary 2.4.17.** *(i)* $\mathrm{codim}(\mathsf{Q}_{\mathcal{E}}(c), \mathsf{Q}_{\mathcal{E}}) = \frac{1}{2}c(c+1)$.

(ii) $\mathrm{codim}(\mathsf{Q}_{\mathcal{E}}(c+1), \mathsf{Q}_{\mathcal{E}}(c)) = c+1$.

(iii) $\mathsf{Q}_{\mathcal{E}}(c)^\circ$ *is the largest open subscheme of* $\mathsf{Q}_{\mathcal{E}}(c)$ *which is smooth over* T.

(iv) The relative multiplicity $\mathrm{mult}_{\mathsf{Q}_{\mathcal{E}}(c+1)^\circ} \mathsf{Q}_{\mathcal{E}}(c)$ *of* T*-subscheme* $\mathsf{Q}_{\mathcal{E}}(c+1)^\circ$ *of* $\mathsf{Q}_{\mathcal{E}}(c)$ *is equal to* 2^c.

Proof We know that

$$\begin{aligned}
\mathrm{codim}(\mathsf{Q}_E(c), \mathsf{Q}_E) &= \dim_T \mathsf{Q}_{\mathcal{E}} - \dim_T \mathsf{Q}_{\mathcal{E}}(c) \\
&= (\tfrac{1}{2}(n+2)(n+1) - 1) - (\dim_T \mathsf{Q}_{\mathcal{K}_c} + \dim_T G(c, \mathcal{E})) \\
&= \tfrac{1}{2}((n+2)(n+1) - r(r+1) - 2r(n+1-r)) \\
&= \tfrac{1}{2}(n+1-r)(n+2-r) = \tfrac{1}{2}c(c+1).
\end{aligned} \qquad (2.69)$$

This proves (i) and also (ii). Since $\mathbb{P}(S^2(\mathcal{K}_c^\vee))$ is smooth over T and p_2 is an isomorphism over $\mathsf{Q}_{\mathcal{E}}(c)^\circ$, we obtain that $\mathsf{Q}_{\mathcal{E}}(c)^\circ$ is smooth over T.

Passing to fibers over T, we may assume that $\mathcal{E} = E$ is a vector space. By Proposition 2.4.16. the exceptional divisor of $\mathsf{Bl}_{\mathsf{Q}_E(c+1)^\circ}(\mathsf{Q}_E)$ is equal to $\mathbb{P}(\mathfrak{s}_{c+1}^* S^2(\mathcal{R}_{c+1}))$. We know from Corollary 2.4.15 that the proper transform of $\mathsf{Q}_E(c)^\circ$ in $\mathsf{Bl}_{\mathsf{Q}_E(c+1)}(\mathsf{Q}_E)$ is the projective bundle $\mathbb{P}(\mathfrak{s}_{c+1}^* R_{c+1}^\vee))$. It embeds i via the Veronese map $v_2 : \mathbb{P}(\mathfrak{s}_c^* \mathcal{R}_{c+1}) \to \mathbb{P}(\mathfrak{s}_c^* S^2(\mathcal{R}_c))$. Thus the multiplicity of $\mathsf{Q}_E(c+1)^\circ$ in $\mathsf{Q}_E(c)$ is equal to the degree of the Veronese variety in each fiber of the projective bundles over $\mathsf{Q}_E(c+1)^\circ$. It is equal to 2^c. This proves (iii) and (iv).

□

Applying [315, Theorem 14.4], we get the following additional property of the subschemes $\mathsf{Q}_{\mathcal{E}}(c)$:

Corollary 2.4.18. *Suppose that the base T of $\mathcal{E}$ is a Cohen-Macaulay scheme. Then the subschemes $\mathsf{Q}_{\mathcal{E}}(c)$ of $\mathsf{Q}_{\mathcal{E}}$ are Cohen-Macaulay.*

Note that, the proof uses the fact due to Eagon and Hochster that, over a field, the schemes $\mathsf{Q}_E(c)$ are Cohen-Macaulay.

jz *Remark* 2.4.19. For any Cohen-Macaulay base scheme T, the subscheme $D_c(\mathcal{E})$ is of expected codimension $\frac{1}{2}c(c+1)$. The same cited Theorem of Fulton implies that $D_k(\mathcal{E})$ is Cohen-Macaulay if the equality holds. If T is regular, one can also compute the class $[D_k(\mathcal{E})]$ in $A(T)$:

$$[D_\sigma(\mathcal{E})] = 2^c \det \begin{pmatrix} c_k & c_{k+1} & c_{k+2} & \cdots & c_{2k-1} \\ c_{k-2} & c_{k-1} & c_k & \cdots & c_{2k-3} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ & & & & c_1 \end{pmatrix}, \tag{2.70}$$

where c_i are the Chern classes $c_i(\mathcal{E})$ of $\mathcal{E}$ (see [376], [433]).

For example

$$[D_2(\mathcal{E})] = 2^2 \det \begin{pmatrix} c_2 & c_3 \\ c_0 & c_1 \end{pmatrix}, \quad [D_3(\mathcal{E})] = 2^3 \det \begin{pmatrix} c_3 & c_4 & c_5 \\ c_1 & c_2 & c_3 \\ 0 & c_0 & c_1 \end{pmatrix}.$$

Using this formula, one can compute the degrees of the varieties $\mathsf{Q}_E(c)$ in the case $\mathcal{E} = E$ is a vector space. We get

$$\deg(\mathsf{Q}_E(c)) = \prod_{0 \le i \le c-1} \frac{\binom{n+1+i}{c-i}}{\binom{2i+1}{i}}. \tag{2.71}$$

For example, if $n = 3$,

$$\deg(\mathsf{Q}_E(1)) = 4, \quad \deg(\mathsf{Q}_E(2)) = 10, \quad \deg(\mathsf{Q}_E(3)) = 8.,$$

and, if $n = 4$,

$$\deg(\mathsf{Q}_E(1)) = 5,\ \deg(\mathsf{Q}_E(2)) = 20,\ \deg(\mathsf{Q}_E(3)) = 35,\ \deg(\mathsf{Q}_E(4)) = 16.$$

Recall that, in Subsection SS:1.1.2 1.1.2, we defined the embedded tangent space $\mathbb{T}_x(X)$ of a variety at its closed point x. They are fibers of the *embedded tangent bundle* $\mathbb{PT}(X) = \mathbb{P}(\mathcal{P}_X)$, where $\mathcal{P}$ is defined by the exact sequence

$$0 \to \Omega^1_X \to \mathcal{P} \to \mathcal{O}_X \to 0.$$

Here $\mathcal{P} = \mathcal{P}_X(\mathcal{O}_X)$ is the locally free sheaf of principal 1-parts of $\mathcal{O}_X$ (see Subsection CAG-2:SS:10.4.3 10.4.3).

cor:2.4.14 **Corollary 2.4.20.** *There is an isomorphism* $\mathbb{PT}(\mathsf{Q}_E(c)^\circ) \cong \mathbb{P}(\mathcal{E}^\vee)$*, where*

$$\mathcal{E} = \mathfrak{s}_c^*\mathrm{Ker}(S^2(E^\vee)_{\mathsf{Q}} \to S^2(\mathcal{R}_c)(1).$$

In particular, there is a canonical isomorphism between $\mathbb{T}(\mathsf{Q}_E(c)^\circ)_Q$ *and the linear system* $|\mathcal{O}_{|E|}(2) - \mathrm{Sing}(Q)|$ *of quadrics vanishing on* $\mathrm{Sing}(Q)$.

Proof The exact sequence

$$0 \to \mathcal{N}^\vee_{\mathsf{Q}_E(c)^\circ/\mathsf{Q}_E} \to \Omega^1_{\mathsf{Q}_E} \otimes \mathcal{O}_{\mathsf{Q}(c)^\circ} \to \Omega^1_{\mathsf{Q}_E(c)^\circ} \to 0$$

extends to an exact sequence

$$0 \to \mathcal{N}^\vee_{\mathsf{Q}_E(c)^\circ/\mathsf{Q}_E} \to \mathcal{P}_{\mathsf{Q}_E} \otimes \mathcal{O}_{\mathsf{Q}(c)^\circ} \to \mathcal{P}_{\mathsf{Q}_E(c)^\circ} \to 0.$$

We have $\mathcal{P}_{\mathsf{Q}_E} \otimes \mathcal{O}_{\mathsf{Q}(c)^\circ} \cong S^2(E)_{\mathsf{Q}_E}(-1) \otimes \mathcal{O}_{\mathsf{Q}(c)^\circ}$. This gives

$$\mathcal{P}_{\mathsf{Q}_E(c)^\circ} \cong \mathfrak{s}_c^*\mathrm{Coker}(S^2(\mathcal{R}^\vee)(-1) \to S^2(E)_{\mathsf{Q}_E}(-1)) \cong \mathfrak{s}_c^*\mathrm{Ker}(S^2(E^\vee_{\mathsf{Q}}) \to S^2(\mathcal{R}_c))(1).$$

□

In the case $c = 1$, the assertion about the embedded tangent spaces was proved in Example ex:discrim]1.2.3.

Let $\tilde{\mathsf{Q}}(c)$ be the proper transform of $\mathsf{Q}(c)$ in $\mathsf{Bl}_{\mathsf{Q}(c+1)^\circ}(\mathsf{Q})$. Applying (propertransform1 2.39) to the inclusions $\mathsf{Q}(c+1) \subset \mathsf{Q}(c) \subset \mathsf{Q}$, we find a

$$\tilde{\mathsf{Q}}(c) = \mathsf{Bl}_{\mathsf{Q}(c+1)^\circ}(\mathsf{Q}(c)) \subset \mathsf{Bl}_{\mathsf{Q}(c+1)^\circ}(\mathsf{Q}).$$

Let E' be the exceptional divisor of $\mathsf{Bl}_{\mathsf{Q}(c+1)^\circ}(\mathsf{Q}(c))$ and E be the exceptional divisor of $\mathsf{Bl}_{\mathsf{Q}(c+1)}(\mathsf{Q})$. The fiber of E' over $Q \in \mathsf{Q}(c+1)$ is the dual space $\mathrm{Sing}(Q)^\circ$ and the fiber of Q in E is the space of quadrics in $\mathrm{Sing}(Q)$. On each fiber, the embedding $\mathrm{Sing}(Q)^\vee$ in the space of quadrics on $\mathrm{Sing}(Q)$ is the Veronese embedding.

Proposition 2.4.21. *Let $Q_0 \in \mathsf{Q}_E(c')^\circ$ and $\Pi = \mathrm{Sing}(Q_0)$ and $\mathbb{P}\mathrm{C}_{Q_0}(\mathsf{Q}_E(c))$ be the projective tangent cone of Q_0 in $\mathsf{Q}_E(c), c < c'$, that is, the fiber of the projective tangent cone $\mathbb{P}\mathrm{C}_{\mathsf{Q}_E(c+1)}\mathsf{Q}_E(c)$ at Q_0. Then,*

$$\mathbb{P}\mathrm{C}_{Q_0}(\mathsf{Q}_E(c)) = \{Q \in \mathsf{Q}_E(c) : \mathrm{corank}(Q \cap \mathrm{Sing}(Q_0)) \geq c\}.$$

The vertex of the normal cone $C_{Q_0}(\mathsf{Q}_E(c))$ is equal to $\mathbb{T}_{Q_0}(\mathsf{Q}_E)$.

Proof By Proposition 2.4.13 and Corollary 2.4.15, $\mathbb{P}\mathrm{C}_{Q_0}(\mathsf{Q}_E(c))$ is equal to the fiber of the projection $p_2 : \mathsf{Q}_{\mathcal{K}_c(c')} \to \mathsf{Q}_E(c)$ over Q_0. It consists of quadrics $Q \in \mathsf{Q}_E(c)$ with $\mathrm{Sing}(Q) \subset \mathrm{Sing}(Q_0)$. The subspace $\mathrm{Sing}(Q_0)$ is tangent to Q along a quadric in $\mathrm{Sing}(Q_0)$ with singular locus of dimension $\geq c$. □

Consider the restriction of the universal quadratic form (2.66) to $\mathsf{Q}_{\mathcal{E}}(c)^\circ$ m

$$\mathsf{u}_{\mathcal{E}}(c) : \mathcal{E}_{\mathsf{Q}_{\mathcal{E}}(c)^\circ} \to \mathcal{E}^\vee_{\mathsf{Q}_{\mathcal{E}}(c)^\circ}(1)$$

and let

$$\mathcal{V}_c := \mathrm{Coker}(\mathsf{u}_{\mathcal{E}}(c)).$$

Its fiber over $Q = V(q) \in \mathsf{Q}_{\mathcal{E}}(c)^\circ$ is equal to the dual space of the radical of the quadratic form q. It follows that

$$\mathcal{V}_c = \mathfrak{s}_c^*(\mathcal{R}_c)(1),$$

where $\mathcal{R}_c$ is the universal quotient sheaf over $G(c, \mathcal{E})$. Applying Proposition 2.4.13, we get

$$\mathcal{N}_{\mathsf{Q}(c)^\circ/\mathsf{Q}} \cong S^2(\mathcal{V}_c)(1). \tag{2.72}$$

This important fact was first proven by A. Tjurin [754, Theorem 1] (see also [778, Proposition 4.4]).

2.4.4 The Space of Complete Quadrics

We keep the notation from the previous subsections. Let $\det(\mathcal{E}) := \bigwedge^s \mathcal{E}$ be the determinant invertible sheaf of $\mathcal{E}$. Using the non-degenerate pairing

$$\bigwedge^{s} \mathcal{E} \otimes \bigwedge^{n+1-s} \mathcal{E} \to \det(\mathcal{E}),$$

we can identify $\bigwedge^n \mathcal{E}$ with $\mathcal{E}^\vee \otimes \det(\mathcal{E})$ and $\bigwedge^n \mathcal{E}^\vee$ with $\mathcal{E} \otimes \det(\mathcal{E})^{\otimes -1}$. Then, $\wedge^n(q)$ defines a quadratic form

$$\check{q} : (\mathcal{E} \otimes \det(\mathcal{E}))^\vee \to \mathcal{E} \otimes \det(\mathcal{E}).$$

We call it the dual quadratic form. Trivializing $\mathcal{E}$ and $\det(\mathcal{E})$ over some open affine set $U = \mathrm{Spec}(A)$ of T, we obtain that $\check{q}$ is given by the adjugate matrix $\mathrm{adj}(M) = (c_{ij})$ of cofactors of a symmetric matrix $M = (a_{ij})$ with entries in A defining q over U.

In particular, we see that $\wedge^n(b_q) = 0$ if the corank of b_q is greater than one and, if it is equal to one, the corank of $\check{q}$ is equal to n.

Let

$$\check{\mathcal{E}} := (\mathcal{E} \otimes \det(\mathcal{E}))^\vee = \mathcal{E}^\vee \otimes \det(\mathcal{E})^{\otimes -1},$$

and

$$\mathrm{Adj}_{\mathcal{E}} : \mathsf{Q}_{\mathcal{E}} \dashrightarrow \mathsf{Q}_{\check{\mathcal{E}}} \qquad (2.73)$$ adjmap

be the rational map defined by $V(q) \mapsto V(\check{q})$. Of course, if $\mathcal{E} = \Bbbk^{n+1}$, the map coincides with the map on the projective space of symmetric matrices that takes a metric M to its inverse M^{-1}.

It is immediate to check that

$$\check{\check{\mathcal{E}}} = \mathcal{E}$$

and

$$\mathrm{Adj}_{\check{\mathcal{E}}} \circ \mathrm{Adj}_{\mathcal{E}} = \mathrm{id}_{\mathsf{Q}_{\mathcal{E}}}, \quad \mathrm{Adj}_{\mathcal{E}} \circ \mathrm{Adj}_{\check{\mathcal{E}}} = \mathrm{id}_{\mathsf{Q}_{\check{\mathcal{E}}}} \qquad (2.74)$$

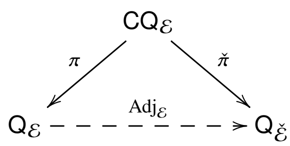
In particular, when $\mathcal{E} = \mathcal{O}_T^{n+1}$ is a free $\mathcal{O}_T$-module, $\mathrm{Adj}_{\mathcal{E}}$ is a birational involution.

The next theorem which we leave without proof describes a smooth resolution of indeterminacy of the rational map $\mathrm{Adj}_{\mathcal{E}}$ (see Vainsencher [778]).

vainsencher1 **Theorem 2.4.22.** *Let*

$$\pi := \pi_{n-1} \circ \cdots \circ \pi_1 : \mathsf{CQ}_{\mathcal{E}} = \mathsf{Q}^n_{\mathcal{E}} \to \mathsf{Q}^{n-1}_{\mathcal{E}} \to \cdots \to \mathsf{Q}^1_{\mathcal{E}} \to \mathsf{Q}_{\mathcal{E}} = \mathsf{Q}^0_{\mathcal{E}}$$

be the composition of the blow-up $\pi_1 : \mathsf{Q}^1_{\mathcal{E}} \to \mathsf{Q}_{\mathcal{E}}$ *of* $\mathsf{Q}_{\mathcal{E}}(n)$*, the blow-up* $\pi_2 : \mathsf{Q}^2_{\mathcal{E}} \to \mathsf{Q}^1_{\mathcal{E}}$ *of the proper transform of* $\overline{\mathsf{Q}}_{\mathcal{E}}(n-1)$*, and so on. Then,* $\mathsf{CQ}_{\mathcal{E}}$ *is a smooth resolution of the birational map* $\mathrm{Adj}_{\mathcal{E}}$*.*



Definition 2.4.23. *A closed point in* CQ_E *is called a* complete quadric *in* $|\mathcal{E}|$ *of dimension* $n-1$*. The variety* CQ_E *is called the variety of complete quadrics in* $|\mathcal{E}|$ *of dimension* $n-1$.

There is another description of the variety $\mathsf{CQ}_{\mathcal{E}}$, which is in the case $T = \mathrm{Spec}(\Bbbk)$ due to J. Semple. Let

$$Q^{\wedge k} = V(\wedge^k(q)) \subset \mathbb{P}(\bigwedge^k E))$$

In the case where $\mathcal{E}$ is a vector space, we know from Example exa:2.3.11 2.3.11 that the quadric $Q^{\wedge k}$ intersects the Grassmannian variety $G(k, E) \subset |\bigwedge^k(E)|$ along the subvariety of linear k-dimensional subspaces tangent to $Q = V(q)$. We refer to Chapter 10, where we will discuss the special case where $k = 2$, known as the tangential quadratic complex of lines.

laksov **Theorem 2.4.24.** *Let $\mathsf{Q}'_{\mathcal{E}}$ be the closure of the image of the map*

$$\mathsf{Q}_{\mathcal{E}}(0)^\circ \to \prod_{i=1}^{n} \mathsf{Q}_{\bigwedge^i \mathcal{E}}, \quad Q \to (Q, Q^{\wedge 2}, \ldots, Q^{\wedge n}).$$

Then, $\mathsf{Q}_{\mathcal{E}}(0)^\circ$ is isomorphic to $\mathsf{CQ}_{\mathcal{E}}$. Let

$$\rho_k : \mathsf{CQ}_{\mathcal{E}} \to \mathbb{P}(S^2(\bigwedge^k \mathcal{E}))$$

be the composition of this isomorphism and the projection map

$$\prod_{i=1}^{n} \mathbb{P}(S^2(\bigwedge^i(E)) \to \mathbb{P}(S^2(\bigwedge^k(E)).$$

Then, $\rho_1 = \pi, \rho_n = \check{\pi}$, and the commutative diagram

$$\begin{array}{ccc} & \mathsf{CQ}_{\mathcal{E}} & \\ {\scriptstyle \rho_1} \swarrow & & \searrow {\scriptstyle \rho_k} \\ \mathsf{Q}_{\mathcal{E}} & \dashrightarrow & \mathsf{Q}_{\bigwedge^k \mathcal{E}} \end{array}$$

resolves the rational map $Q \mapsto Q^{\wedge^k}$.

Let $E_i, i = 1, \ldots, n-1$, be the exceptional divisor of π_i, E_i be the proper transforms of E_i in $\mathsf{CQ}_{\mathcal{E}}$, and E_n be the proper transform of the discriminant hypersurface $\mathsf{Q}_{\mathcal{E}}(1)$ in $\mathsf{CQ}_{\mathcal{E}}$. We call them the *boundary divisors* in $\mathsf{CQ}_{\mathcal{E}}$.

Since $\mathrm{Adj}_{\mathcal{E}} \circ \mathrm{Adj}_{\check{\mathcal{E}}} = \mathrm{id}_{\mathsf{Q}_{\mathcal{E}}}$, the resolution $(\pi, \check{\pi})$ satisfies the following property:

Proposition 2.4.25. *Let E_i be the exceptional divisor of $\pi_i : \mathsf{CQ}^i_{\mathcal{E}} \to \mathsf{CQ}^{i-1}_{\mathcal{E}}$ and $\check{E}_i$ be the exceptional divisor of $\check{\pi}_i : \mathsf{CQ}^i_{\check{\mathcal{E}}} \to \mathsf{CQ}^{i-1}_{\check{\mathcal{E}}}$. Then*

$$\check{\pi}(E_i) = \check{E}_{n+1-i}, \quad \pi(\check{E}_i) = E_{n+1-i}.$$

We denote by $Z_r, 2 \le r \le n$ the proper transform of $\mathsf{Q}_{\mathcal{E}}(n+1-r)$ under π_r, and set $Z_1 = E_1$.

We see that

$$\mathsf{CQ}_{\mathcal{E}} \setminus \pi^{-1}(\mathsf{Q}_{\mathcal{E}}^{\circ}) \cup \mathsf{E}_1 \cup \cdots \mathsf{E}_n \cap \mathsf{E}_{n+1},$$

and there are 2^n different types of complete quadrics in $|\mathcal{E}|$. They correspond to different intersections of $0 \le k \le n$ of the boundary divisors. It is proven in [778] that the boundary divisors intersect transversally, so that

$$\mathsf{E}_{r_1,\ldots,r_k} = \mathsf{E}_{r_1} \cap \cdots \cap \mathsf{E}_{r_k}, \quad 1 \le r_1 < \cdots < r_k \le n$$

is a smooth codimension k closed subvariety of $\mathsf{CQ}_{\mathcal{E}}$.

For the future use, let us introduce the following notation:

$$\mathsf{E}^{\circ}_{r_1,\ldots,r_k} := \mathsf{E}_{r_1,\ldots,r_k} \setminus \bigcup_{r \notin \{r_1,\ldots,r_k\}} \mathsf{E}_{r_1,\ldots,r_k,r}. \tag{2.75}$$

Let us give a geometric meaning of a complete quadric in $\mathsf{E}_{r_1,\ldots,r_k}$. For simplicity, we assume that $\mathcal{E}$ is a linear spaces E of dimension $n+1$ and leave the general case to the reader.

First of all, we identify a complete quadric $\tilde{Q}$ with its projection Q_E if the projection is an isomorphism at $\tilde{Q}$. Thus, a complete quadric of rank $n+1$ is a nonsingular quadric, and a complete quadric $Q \in \mathsf{E}_n \setminus (\mathsf{E}_n \cap \mathsf{E})$ is an irreducible quadric cone.

Let $\tilde{Q} \in \mathsf{E}_{r_1,\ldots,r_k}$. Since $\tilde{Q} \in \mathsf{E}_{r_1}$, the image of $\tilde{Q}$ under $\pi_n \circ \cdots \pi_{r_1} : \mathsf{CQ}_E \to \mathsf{CQ}_E^{r_1}$ belongs to Z_{r_1}. Its projection Q under $\pi_{r_1-1} \circ \cdots \circ \pi_1 : \mathsf{CQ}_E^{r_1} \to \mathsf{Q}_{\mathcal{E}}$ belongs to $\mathsf{Q}_{\mathcal{E}}(r_1)$. Let $F_{r_1} = \mathrm{Sing}(Q) \cong \mathbb{P}^{r_1-1}$. Since $\tilde{Q} \in \mathsf{E}_{r_2}$, we, similarly, find that the image of $\tilde{Q}$ under $\pi_n \circ \cdots \pi_{r_2} : \mathsf{CQ}_E \to \mathsf{CQ}_E^{r_2-1}$ belongs to Z_{r_2}, the proper transform of $\mathsf{Q}_{\mathcal{E}}(r_2)$ in $\mathsf{CQ}_E^{r_2-1}$. By Proposition 2.4.13, it defines a quadric of corank $r_2 - r_1$ in F_{r_1}. Its singular locus is a subspace of F_{r_1} of dimension $r_2 - r_1 - 1$. Continuing in this way, we find a flag

$$F_{r_k} \subsetneq \ldots \subsetneq F_{r_1}$$

of linear subspaces in $|E|$ of relative dimension $n - r_k$. A complete quadric $\tilde{Q} \in \mathsf{E}_{r_1,\ldots,r_k}$ can be viewed as a collection of quadrics $Q_{r_i} \subset F_{r_i}$ with $\mathrm{Sing}(Q_{r_i}) = F_{r_i+1}$. We set

$$\mathrm{rank}(\tilde{Q}) := (r_1+1,\ldots,r_k+1) \text{ or} (r_k+1,\ldots,r_1+1)$$

and $\mathrm{rank}(\tilde{Q}) = n+1$, if $\pi(\tilde{Q})$ is a smooth quadric.

As we explained earlier, each Q_{r_i} can be considered as a smooth quadric in $\mathcal{F}^{\perp}_{r_i+1} \subset \mathbb{P}(E)$. So, we may consider $\tilde{Q}$ as a collection of a flag of linear subspaces $F^{\perp}_{r_1} \subset \cdots \subset F^{\perp}_{r_k} \subset \mathbb{P}(E)$ and a smooth quadrics in each $F^{\perp}_{r_i}$.

Also, note that a complete quadric can be defined inductively as a nonsingular quadric in $F_1^\perp$ and a complete quadric in F_1 of rank $(r_2+1,\ldots,r_k+1)$.

Replacing π with $\check{\pi}$, we may consider $\tilde{Q}$ as a complete quadric $\tilde{Q}^\vee$ in $\mathbb{P}(E)$ of rank $(n+1-r_k,\ldots,n+1-r_1)$. It is defined by a flag $F_{r_1}^\perp \subset \cdots \subset F_{r_k}^\perp \subset \mathbb{P}(E)$ and a smooth quadric in each F_{r_i}.

Let us now consider examples in small dimensions.

completeconics *Example* 2.4.26. Assume $n=2$ and $\mathcal{E}=E$ is a three-dimensional linear space. The space CQ_E of complete conics is just the blow-up of the Veronese quartic surface $\mathsf{Q}_2(1)$ of double lines in $\mathbb{P}^5=\mathsf{Q}_E$. The boundary in CQ_E consists of one divisor $\mathsf{E}_1\cup\mathsf{E}_2$.

So, there are four different types of complete conics

1. Q is a smooth conic and $\tilde{Q}$ can be identified with Q. The dual complete quadric is the dual conic $Q^\vee$. The flag is just $|E|$ and the rank is (3).
2. $\tilde{Q}\in\mathsf{E}_2^\circ$ and Q is a line-pair with $\mathrm{Sing}(Q)=F_2\cong\mathbb{P}^0$. We can identify $\tilde{Q}$ with Q. The dual complete conic $\tilde{Q}^\vee$ is defined by the flag $F_2^\perp\subset\mathbb{P}(E)$ and a pair of distinct points on the line $F_2^\perp$.
3. $\tilde{Q}\in\mathsf{E}_1^\circ$ and Q is a double line. The flag is $F_1=\mathrm{Sing}(Q)\subset|E|$ and $\tilde{Q}$ is defined by a pair of distinct points on the line F_1. The rank of $\tilde{Q}$ is (1). The dual complete quadric $\tilde{Q}^\vee$ is defined by the flag $F_1^\perp\subset\mathbb{P}(E)$. It is the point $F_1^\perp$ which we agreed to identify with a line-pair.
4. $\tilde{Q}\in\mathsf{E}_1\cap\mathsf{E}_2$ and Q is a double line. The flag is $F_2\subset F_1\subset|E|$ and $\tilde{Q}$ is a quadric in $F_1\cong\mathbb{P}^1$ singular at the point F_2, i.e., a line and a double point on it. The rank of $\tilde{Q}$ is equal to $(1,2)=(2,1)$. The dual complete quadric $\tilde{Q}^\vee$ is defined by the flag $F_1^\perp\subset F_2^\perp\subset\mathbb{P}(E)$ and a quadric in $F_2^\perp$ with the singular point $F_1^\perp$, i.e., a double point on a line.

   We see that types 1 and 4 are self-dual, and types 2 and 3 are dual to each other.

Example 2.4.27. Assume $n=3$ and $\mathcal{E}=E$ is a linear space of dimension 4. The space CQ_E of complete quadrics in $\mathbb{P}^3$ is the composition of two blowing ups of $\mathsf{Q}_E\to\mathsf{Q}_E^1\to\mathsf{Q}_E\cong\mathbb{P}^9$. The exceptional divisor E_1 of the first blow-up $\pi_1:\mathsf{CQ}_E^1\to\mathsf{Q}_E$ is a projective $\mathbb{P}^5$-bundle over $\mathsf{Q}_E(3)\cong\mathsf{v}_2(\mathbb{P}^3)$. It is isomorphic to $\mathbb{P}(S^2(\mathcal{R}_3))$, where $\mathcal{R}_3\cong\Theta_{\mathbb{P}(E)}(-1)$ is the universal rank 3 quotient bundle over $G(3,E)\cong\mathbb{P}(E)$.

The projection $\check{\pi}:\mathsf{E}_1\to\mathsf{Q}_{E^\vee}(1)$ is a resolution of singularities of the discriminate hypersurface in the dual space. It blows down $\mathsf{E}_1\cap\mathsf{E}_2$ to $\mathsf{Q}_{\mathcal{E}^\vee}(2)$ and $\mathsf{E}_1\cap\mathsf{E}_2\cap\mathsf{E}_3$ to $\mathsf{Q}_{\mathcal{E}^\vee}(1)$. We see that its fibers are three-dimensional that shows that the resolution is different from the resolution of singularities of the discriminant hypersurface of quadrics from Example ex:discrfm 1.2.3.

There are 8 types of complete quadrics in $\mathbb{P}^3$.

1. $\tilde{Q} = Q$ is a smooth quadric and $\tilde{Q}^\vee = Q^\perp$.
2. $\tilde{Q} \in \mathsf{E}_3^\circ$ identified with the irreducible quadric cone Q. Its rank is (3). The dual quadric is defined by the flag $\mathrm{Sing}(Q)^\perp \subset \mathbb{P}(E)$ and a smooth conic in the plane $\mathrm{Sing}(Q)^\perp$.
3. $\tilde{Q} \in \mathsf{E}_2^\circ$ and Q is a plane-pair. The flag is $F_2 = \mathrm{Sing}(Q) \subset |E|$ and $\tilde{Q}$ defines a smooth quadric in $F_2 \cong \mathbb{P}^1$. The rank of $\tilde{Q}$ is equal to (2). We can also view it as the dual complete quadric defined by the flag $F_2^\perp \subset \mathbb{P}(E)$ and a smooth quadric in $F_2^\perp \cong \mathbb{P}^1$. This can be considered as the dual complete quadric.
4. $\tilde{Q} \in \mathsf{E}_1^\circ$ and Q is a double plane. The flag is $F_1 = \mathrm{Sing}(Q) \subset |E|$ and $\tilde{Q}$ defines a smooth conic in $F_1 \cong \mathbb{P}^2$. We agreed to identify it with an irreducible cone in $\mathbb{P}(E)$. The rank of $\tilde{Q}$ is equal to (1). We can also view it as the dual complete quadric defined by the flag $F_1^\perp \subset \mathbb{P}(E)$ and a smooth quadric in $F_1^\perp \cong \mathbb{P}^0$.
5. $\tilde{Q} \in (\mathsf{E}_3 \cap \mathsf{E}_2)^\circ$ and Q is a plane-pair. The flag is $F_2 \subset F_3 \subset |E|$ and $\tilde{Q}$ defines a quadric in $F_3 \cong \mathbb{P}^1$ with a singular point at F_2. Dually, we can view $\tilde{Q}$ as a smooth conic in $F_2^\perp \cong \mathbb{P}^2$ and a smooth quadric in $F_1^\perp \cong \mathbb{P}^1$. The rank of $\tilde{Q}$ is equal to $(2,3)$.
6. $\tilde{Q} \in (\mathsf{E}_3 \cap \mathsf{E}_1)^\circ$ and Q is a double plane. The flag is $F_3 \subset F_1 \subset |E|$ and $\tilde{Q}$ defines a conic in $F_1 \cong \mathbb{P}^2$ which is singular at F_3. The rank is $(1,3)$. The dual complete quadric is defined by the flag $F_1^\perp \subset F_3^\perp$ and defines a conic in $F_3^\perp \cong \mathbb{P}^2$ singular at the point $F_1^\perp$.
7. $\tilde{Q} \in (\mathsf{E}_1 \cap \mathsf{E}_2)^\circ$ and Q is a double plane. The flag is $F_2 \subset F_1 \subset |E|$ and $\tilde{Q}$ defines a conic in $F_1 \cong \mathbb{P}^2$ with double line $F_2 \cong \mathbb{P}^1$. The rank is $(1,2)$. Dually, we can view $\tilde{Q}$ as a quadric in $F_2^\perp \cong \mathbb{P}^1$ singular at the point $F_1^\perp$.
8. $\tilde{Q} \in \mathsf{E}_1 \cap \mathsf{E}_2 \cap \mathsf{E}_3$ and Q is a double plane. The flag is $F_3 \subset F_2 \subset F_1 \subset |E|$ and $\tilde{Q}$ defines a conic in $F_1 \cong \mathbb{P}^2$ singular along the line F_2 and a quadric in F_2 singular at the point F_3. Its rank is $(1,2,3)$. The dual quadric is defined by the flag $F_1^\perp \subset F_2^\perp \subset F_3^\perp \subset \mathbb{P}(E)$ and a quadric in $F_3^\perp \cong \mathbb{P}^2$ singular along the line $F_2^\perp$ and a point on $F_2^\perp$ taken with multiplicity two.

We see that the types 1, 3, 6, and 8 are self-dual. The pairs $(2,4)$, $(5,7)$ are dual to each other.

SS:2.4.7

2.4.5 The Intersection Theory on the Space of Complete Quadrics

Suppose we are given $N = \dim \mathsf{Q}_E = \frac{1}{2}(n^2 + 3n)$ geometric conditions on quadrics in $\mathbb{P}^n$ such that the quadrics satisfying one of the conditions form an irreducible divisor G_i in Q_E. A solution of the enumerative problem gives the number of quadrics satisfying all conditions. Since we expect that N divisors in Q_E intersect at finitely many points, we say that the problem is well-defined when the intersection of the divisors is transversal and their intersection number could be taken as the solution of the problem.

Unfortunately, this rarely happens. For example, consider the geometric conditions such that each $G_1, \dots, G_k$ consists of quadrics passing through a point $p_1, \dots, p_k$ and the remaining divisors $G'_1, \dots, G'_{N-k}$ are the divisors tangent to hyperplanes $H_1, \dots, H_{N-k}$. Even, if we additionally assume that the points and hyperplanes in a general mutual incidence position, when $N - k \geq 3$, our problem is not well-defined. In fact, the linear system of quadrics passing through $p_1, \dots, p_k$ is of dimension $N - k \geq 3$, hence its intersection with the codimension 3 divisor $\mathsf{Q}_E(n)$ is not empty. A quadric Q from the intersection contains a line in its singular locus. Hence, Q is tangent to all hyperplanes $H_1, \dots, H_{N-k}$. So, the correct solution must consist of the number of residual intersection points. To find it, we replace Q_E with the space CQ_E, and consider the proper transforms $\bar{G}_i$ of the divisors G_i in CQ_E. If the intersection $\bar{G}_1 \cap \cdots \bar{G}_N$ is transversal, the number of such quadrics is the solution to our problem.

To find $\#\bar{G}_1 \cap \cdots \cap \bar{G}_N$, we use the intersection theory in CQ_E.

Let $\mathsf{E}_1, \dots, \mathsf{E}_{n-1}$ be the proper transforms to CQ_E of the exceptional divisors E_r of each blow-up $\mathsf{CQ}^r \to \mathsf{CQ}^{r-1}$ from Theorem vainsencher1 2.4.22. We apply Theorem cohblowup 2.4.6 to each blow-up, and obtain that $A(\mathsf{CQ})$ is freely generated by the divisor classes $\mu_1, e_1, \dots, e_{n-1}$ of the divisors $\mathsf{E}_1, \dots, \mathsf{E}_{n-1}$ and the class $\mu_1 = \pi^*(h)$, where $h = c_1(\mathsf{Q}_E(1))$ is the divisor class of a hyperplane in Q_E. Moreover, $(\mu_1, e_1, \dots, e_{n-1})$ is a free basis of $A^1(\mathsf{CQ}) \cong \mathrm{Pic}(\mathsf{CQ})$

We can use another basis formed by $\mu_1, \dots, \mu_n$, where $\mu_k = \rho_i^*(c_1(O_{\mathsf{Q}_{\bigwedge^k E}}))$, where p_i is the projection of CQ_E to $\mathsf{Q}_{\bigwedge^k E}$ from Theorem laksov 2.4.24.

prop:twobases **Proposition 2.4.28.** *Let $A_*(\mathsf{CQ}_E)$ be the Chow ring of CQ_E. Then $\mathrm{Pic}(\mathsf{CQ}_E) = A_1(\mathsf{CQ}_E) = H^2(\mathsf{CQ}_E, \mathbb{Z}) \cong \mathbb{Z}^n$. The the divisor classes of the boundary divisors $(e_1, \dots, e_n)$ form a basis in $\mathrm{Pic}(\mathsf{CQ}_E)$. Another basis $(\mu_1, \dots, \mu_n)$ is formed by the pre-images μ_i of the divisor classes of a hyperplane in $\mathbb{P}(S^2(\bigwedge^i E)$ under the projection $\pi_i : \mathsf{CQ}_E \to \mathbb{P}(S^2(\bigwedge^i E)$. The two bases are related as follows:*

$$\mu_k = k\mu_1 - (k-1)e_1 - \cdots - e_{k-1}, \quad k = 1, \dots, n, \tag{2.76}$$

$$e_k = 2\mu_k - \mu_{k-1} - \mu_{k+1}, \quad k = 1, \dots, n, \tag{2.77}$$

where $\mu_0 = \mu_{n+1} = 0$

Proof Consider the rational map

$$\Phi_k = \pi_k \circ \pi^{-1} : \mathsf{Q}_E \dashrightarrow \mathbb{P}(S^2(\bigwedge^k E))$$

which sends a quadric $Q = V(q)$ to the quadric $Q^{\wedge^k} := V(\wedge^k q)$. If A is a symmetric matrix associated to b_q, then $V(\wedge^k q)$ corresponds to the symmetric matrix $A_{(k)}$ as defined in Subsection 2.3.2. It follows from formula (2.21) and its immediate extension to arbitrary k, that the map Φ_k is given by $k \times k$-minors of A. They are of multiplicity $k-i$ on the locus of zeros of $(k-i)\times(k-i)$-minors [281]. This shows that the map Φ_k is given by the linear system $|\pi^*(\mathcal{O}_{\mathsf{Q}_E}(k)) - \mathsf{E}_{k-1} - 2\mathsf{E}_{k-2} - \cdots - k\mathsf{E}_1|$, hence

$$\mu_k = k\mu_1 - (k-1)e_1 - \cdots - e_{k-1}, \quad k = 1, \ldots, n.$$

Inverting the matrix

$$M = \begin{pmatrix} 1 & 0 & \ldots & \ldots & 0 & 0 & 0 \\ 2 & -1 & \ldots & \ldots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ k & -k+1 & \ldots & -2 & -1 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ n & -n+1 & \ldots & \ldots & \ldots & -2 & -1 \end{pmatrix},$$

we obtain the second set of relations for $k = 1, \ldots, n-1$. The last relation comes from the expression $e_n = (n+1)\mu_1 - ne_1 - \cdots - e_{n-1}$. □

Remark 2.4.29. An attentive reader will notice that the matrix expressing e_i in terms of $\mu_1, \ldots, \mu_n$ coincides with the Cartan matrix of a simple root system of type A_n. This is not a coincidence. The space of complete quadrics is a special case of a *wonderful compactification* of homogeneous spaces of semi-simple algebraic groups of the form G/H, where H is the set of fixed points of an involution in G. In our case, the homogeneous space is $\mathrm{SL}(n+1)/\mathrm{O}(n+1)$ and the involution is the transpose involution. The cohomology ring of the wonderful compactification has two bases correspond to a set of simple roots and a set of weights of the Lie algebra of $\mathrm{SL}(n+1)$. We refer to [206] for the exposition of this theory.

Now, any geometric condition G_i, considered as a divisor on CQ_E can be expressed as a linear combination $\sum_{j=1}^n \alpha_{ij}\mu_j$. The coefficients α_{ij} is classically known as *characteristic numbers* of the geometric condition G_i. The solution of an enumerative problem consists of the evaluation of the number

$$\prod_{i=1}^{N}\Big(\sum_{j=1}^{n}\alpha_{ij}\mu_j\Big) = \sum_{i_1+\cdots+i_n=N}^{n} a_{i_1,\ldots,i_n}\mu_1^{i_1}\cdots\mu_n^{i_n}.$$

Of course, it is a formidable task, and the explicit computations can be made only for small n.

To compute all the intersection numbers one needs to know the Segre classes $s(Z_{r+1}, \mathsf{CQ}^r)$. The rather complicated commutation of the normal bundle $\mathcal{N}_{Z_{r+1}/\mathsf{CQ}^r}$ to Vainsencher [778]. For our modest need, that is, consider the case $n = 2$ and $n = 3$, we need only the following two lemmas which we will prove.

segreclass1

Lemma 2.4.30. *Identifying $\mathsf{Q}_{\mathcal{E}}(n)$ with $\mathbb{P}(\mathcal{E})$ via the Veronese map v_2, we have the following exact sequence on $\mathbb{P}(\mathcal{E})$:*

$$0 \to \mathcal{E}_{\mathbb{P}(\mathcal{E}^\vee)}(1) \to \mathsf{v}_2^*(S^2(\mathcal{E}^\vee)_{\mathbb{P}(\mathcal{E})})(2) \to \mathcal{N}_{\mathsf{Q}_{\mathcal{E}}(n)/\mathsf{Q}_{\mathcal{E}}} \to 0.$$

In particular, if $\mathcal{E} = E$ is a vector space over $\Bbbk$, we get

$$s(\mathsf{Q}_E(n), \mathsf{Q}_E) = \frac{(1+H)^{n+1}}{(1+2H)^{\frac{1}{2}(n+2)(n+1)}},$$

where $H = c_1(\mathcal{O}_{\mathbb{P}(\mathcal{E})}(1))$.

Proof The variety $\mathsf{Q}_E(n)$ is the Veronese variety in $\mathbb{P}(S^2(\mathcal{E}))$. Let us denote it by V and $\mathbb{P}(S^2(\mathcal{E}))$ by $\mathbb{P}$. Let $h = c_1(\mathcal{O}_{\mathbb{P}}(1)$ and $H = c_1(\mathcal{O}_{|\mathcal{E}|}(1)$. Under the Veronese map $\mathsf{v}_2 : |\mathcal{E}| \to \mathbb{P}$, $\mathsf{v}_2^*(h) = 2H$.

We use exact sequence definitionnormal (2.36)

$$0 \to \Theta_{V/T} \to \Theta_{\mathbb{P}/T} \otimes \mathcal{O}_V \to \mathcal{N}_{V,\mathbb{P}} \to 0$$

and the Euler exact sequences

$$0 \to \mathcal{O}_{\mathbb{P}} \to S^2\mathcal{E}_{\mathbb{P}}^\vee(1) \to \Theta_{\mathbb{P}/T} \to 0,$$

and

$$0 \to \mathcal{O}_{\mathbb{P}(\mathcal{E})} \to \mathcal{E}_{\mathbb{P}(\mathcal{E})}^\vee(1) \to \Theta_{\mathbb{P}(\mathcal{E})/T} \to 0.$$

Applying v_2^* to the second exact sequence, we get a commutative diagram:

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & \Theta_{|\mathcal{E}|/T} & \longrightarrow & \mathsf{v}_2^*(\Theta_{\mathbb{P}/T}) & \longrightarrow & \mathsf{v}_2^*(\mathcal{N}_{V/\mathbb{P}}) & \longrightarrow & 0 \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & \mathcal{E}^\vee(1)_{\mathbb{P}(\mathcal{E})} & \longrightarrow & \mathsf{v}_2^*(S^2(\mathcal{E}^\vee)_{\mathbb{P}(\mathcal{E})})(2) & \longrightarrow & \mathcal{N}_{V/\mathbb{P}} & \longrightarrow & 0 \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & \mathcal{O}_{\mathbb{P}(\mathcal{E})} & \longrightarrow & \mathcal{O}_{\mathbb{P}(\mathcal{E})} & \longrightarrow & 0 & \longrightarrow & 0 \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

The assertion about the normal bundle is given by the middle horizontal exact sequence. The assertion about the Segre class follows from the standard properties of Chern classes. □

Remark 2.4.31. The same proof computes the normal sheaf of the Veronese variety $\mathsf{V}_d^n = \mathsf{v}_d(\mathbb{P}(\mathcal{E})) \subset \mathbb{P}(S^d(\mathcal{E}))$ for any $d \geq 2$. We have an exact sequence

$$0 \to \mathcal{E}_{\mathbb{P}(\mathcal{E}^\vee)}(1) \to \mathsf{v}_d^*(S^2(\mathcal{E}^\vee)_{\mathbb{P}(\mathcal{E})})(d) \to \mathcal{N}_{\mathsf{V}_d^n/\mathbb{P}(S^d(\mathcal{E}))} \to 0.$$

normalone **Lemma 2.4.32.**

$$\mathcal{N}_{Z_2/\mathsf{CQ}^1_{\mathcal{E}}} \cong p_1^* S^2(\mathcal{R}_2) \otimes p_2^* \mathcal{O}_{\mathsf{Q}_E(2)}(1) \otimes \mathcal{O}_{Z_2}(-E_1),$$

where $(p_2, p_1) : Z_2 \to \mathsf{Q}_E \times G(2, \mathcal{E})$ *resolves the indeterminacy of* $\mathfrak{s}_2 : \mathsf{Q}_E(2) \dashrightarrow G(2, E)$.

Proof We denote $G(2, \mathcal{E})$ by $\mathbb{G}$ and $\mathsf{CQ}^1_{\mathcal{E}}$ by CQ^1. It follows from the proof of Proposition 2.4.16 that both sides coincide over $\mathsf{Q}_{\mathcal{E}}(2)^\circ$. Also, there is an exact sequence

$$0 \to \mathcal{K}_2 \otimes \mathcal{R}_2 \to \mathcal{N}_{Z_2/\mathsf{Q}_{\mathcal{E}} \times \mathbb{G}} \to p_1^* S^2(\mathcal{R}_2)(1) \to 0.$$

Let $\nu = (\pi_1, \mathrm{id}_{\mathbb{G}}) : \mathsf{CQ}^1 \times \mathbb{G} \to \mathsf{Q}_{\mathcal{E}} \times \mathbb{G}$. Applying (2.40), we obtain

$$\mathcal{N}_{Z_2, \mathsf{CQ}^1} \cong \nu^*(\mathcal{N}_{Z_2/\mathsf{Q}_{\mathcal{E}} \times G})(-E_1).$$

Since $Z_2 \cap E_1$ is a divisor in Z_2, the pre-image of Z_2 in $\mathsf{CQ}^1 \times \mathbb{G}$ is isomorphic to $Z_2 \times \mathbb{G}$. The surjection $\mathcal{N}_{Z_2/\mathsf{Q}_{\mathcal{E}} \times \mathbb{G}} \to S^2(\mathcal{R}_2)(1)$ defines a surjection

$$\mathcal{N}_{Z_2, \mathsf{CQ}^1 \times \mathbb{G}} \to f^* S^2(\mathcal{R}_2)(-E_1),$$

where $f : \mathsf{CQ}^1 \times \mathbb{G} \to \mathsf{CQ}^1$ is the first projection. The pull-back of both sides of the assertion under f^* is a surjection of locally free sheaves of the same rank whose restriction to $\mathsf{CQ}^1 \setminus E_1$ are isomorphic. Hence they are isomorphic, and the assertion follows.

□

SS:2.4.6

2.4.6 Enumerative geometry of conics

We start with the case $n = 2$. We denote CQ_E by CQ_2.

Applying Proposition prop:twobases 2.4.28, we obtain that $\mathrm{Pic}(\mathsf{Q}_2)$ has two bases (μ_1, μ_2) and (μ_1, e_1). They are related by $\mu_2 = 2\mu_1 - e_1$. We also know that $e_2 = 3\mu_1 - 2e_1 = 2\mu_2 - \mu_1$.

Let $p \in |E|$ be a point and $\mathsf{Q}_{2,p}$ be the linear system of conics with base point p. Its proper transform to CQ_2 is a hypersurface with divisor class μ_1.

Every geometric divisorial condition is a divisor D in in some linear system $|a\mu_1 + b\mu_2|$, where (a, b) are the characteristic numbers. A solution of an enumerative problem consists of computation of the intersection index of five divisors $D_1, \ldots, D_5$. To compute the intersection number, we need to know the numbers $\mu_1^i \mu_2^{5-i}$. Since $\mu_2 = 2\mu_1 - e_1$, it is enough to compute the intersection numbers $\mu_1^i \cdot e_1^{5-i}$. The following lemma makes it easy.

Lemma 2.4.33.

$$\mu_1^i e_1^{5-i} = 0, \quad i \geq 3.$$

Proof By the projection formula

$$\mu_1^i e_1^{5-i} = \pi^*(h_1^i) \cdot e_1^{5-i} = h_1^i \pi_*(e_1^{5-i}).$$

Since $\mathrm{codim}(\mathsf{Q}_2(1), \mathsf{Q}_2) = 3$, we get $\pi_*(e_1^{5-i}) \in A_{2-i}(\mathsf{Q}_2(1))$, hence $\pi_*(e_1^{5-i}) = 0, i = 3, 4, 5$. □

Obviously, $\mu_1^5 = \mu_2^5 = h_1^5$. Now, we have

$$\mu_1^4 \mu_1 = \mu_1^4 \cdot (2\mu_1 - e_1) = 2\mu_1^5 = 2, \quad \mu_1^3 \cdot \mu_2^2 = \mu_1^3 \cdot (2\mu_1 - e_1)^2 = 4.$$

The remaining numbers $\mu_1^i \mu_1^{5-i}$ are computed via the duality. We have

$$\mu_1^2 \cdot \mu_2^3 = \mu_1^3 \cdot \mu_2^2 = 2, \quad \mu_1 \cdot \mu_2^4 = \mu_1^4 \cdot \mu_2 = 2.$$

The number $\mu_1^k \mu_2^{5-k}$ gives the solution to the enumerative problem of counting the number of complete conics that pass through k general points and tangent to $5 - k$ general lines, provided we proof that the corresponding divisors intersect transversally. This follows from Kleiman's transversality theorem KleimanTr [448]. To apply Kleiman' theorem we use that the group $\mathrm{Aut}(\mathbb{P}^2)$ acts transitively on the hypersurfaces representing the classes μ_1 and μ_2. One can also check that the transversal intersection of five enumerative divisors consists of smooth conics (see [87]).

Let us add one more condition: a conic is tangent to a fixed smooth conic C_1. Choosing a Veronese map $\mathsf{v}_2 : \mathbb{P}^1 \to C_1$, the pre-image of any conic C different from C_1 is a divisor of degree 4 in $\mathbb{P}^1$. The conic C is tangent to C_1 if this divisor is not reduced. The discriminant variety of binary forms of degree 4 is hypersurface $H(C_1)$ in $|\mathcal{O}_{\mathbb{P}^1}(4)| \cong \mathbb{P}^4$ of degree 6 (see Subsection 1.5.1). It is a cone of degree 6 in $\mathsf{Q}_2 \cong \mathbb{P}^5$ with the vertex $\{C_1\}$. We leave it to the reader to check that $\mathsf{Q}_2(1)$ is the double locus of $H(C_1)$. This implies that the $\pi^*(H(C_1)$ represents the divisor class $6\mu_1 - 2e_1$

Using our computations from the previous example, we get

$$(2\mu_1 + 2\mu_2)^5 = 2^5(\mu^5 + 5\mu^4\nu + 10\mu^3\nu^2 + 10\mu^2\nu^3 + 5\mu\nu^4 + +\nu^5)$$

$$= 2^5(1 + 5 \cdot 2 + 10 \cdot 4 + 10 \cdot 4 + 10 \cdot 4 + 5 \cdot 2 + 1) = 3264.$$

This gives the solution of the classical enumerative problem of finding the number of conics tangent to a set of five conics. We refer for the history of this problem to [450].

Similar computations give the answer to the enumerative problem: find the number of conics passing through k points, tangent to l lines and $5 - k - l$ conics (see [40]).

$\frac{k}{l}$	0	1	2	3	4	5
0	3264	816	184	36	6	1
1	816	224	56	12	2	
2	184	56	16	4		
3	36	12	4			
4	6	2				
5	1					

Table 2.1 *Number of conics passing through k points and tangent to l lines and* $5 - k - l$ *conics*

Example 2.4.34. Fix three conics in a general position that pass through two fixed points in the planes. By definition, the conics are complex circles (see subsection 2.2.3). It follows from Proposition 2.2.7 that the number of circles touching the three circles is equal to the intersection of 3 quadrics in the space of quadrics. It is expected to be equal to 8. The fact that it is equal to 8 for a general choice of three conics can be proven by a picture (see the cover of the book [283]. Our enumerative problem gives that the number must be equal $\mu_1^2(2\mu_1 + 2\mu_2)^3 = 184$. The discrepancy is explained by the fact that three conics are special: all of them pass through two fixed points. So, from the point of view of enumerative geometry of conics, the three fixed conics are not in a general position. Another special position is when the three fixed real circles are pairwise tangent. Then the number of real circles that touch the three fixed circles C_1, C_2, C_3) drops to 2. If k_1, k_2, k_3 denote the curvatures of C_1, C_2, C_3 (the inverses of their radii), the curvatures of two new circles satisfy the *Descartes' equation*

$$t^2 - 2t(k_1 + k_2 + k_3) - 2(k_1k_2 + k_1k_3 + k_2k_3) = 0.$$

There is a choice of a solution determined by the order of the set of curvatures. For example, choosing C_4 with the largest curvature, we obtain a new set of three overlapping circles (C_2, C_3, C_4). Continuing in this way, we get a *circle packing* of the plane, called the *Apollonian circle packing*. similar constructions lead to *Apollonian sphere packings*. Because of the great importance of Apollonian sphere packings in hyperbolic geometry, dynamics and number theory, there is enormous literature about them. We refer to [254], where one can get some references to the literature and some applications to algebraic geometry.

Remark 2.4.35. We have to warn the reader that not any enumerative problem of conics can be reduced to the intersection theory on the variety of complete conics. The proper transforms of the hypersurfaces defining some geometrical conditions in CQ_2 may still have a common subvariety of complete cognacs $(C, C^\vee)$, where C is a line with a marked point (see [703, B3] and [450]). To solve such a enumerative problem one has to use the intersection theory on some blow-up of CQ_2.

vainsencher2 *Remark* 2.4.36. One can also develop the enumerative theory of conics in a projective space $\mathbb{P}^n$ with $n > 2$ [778]. Since any conic is contained in a plane, one replaces the 3-dimensional vector space E by the tautological vector bundle $\mathcal{E}$ over the Grassmannian $G_2(\mathbb{P}^n)$ of planes in $\mathbb{P}^n$. So, a conic becomes a point in the projective bundle $\mathbb{P}(\mathcal{E}^\vee)$ over $G_2(\mathbb{P}^n)$. The birational map Adj_2 becomes a birational map over $G_2(\mathbb{P}^n)$ and one can smoothly resolve indeterminacy of the rational map Adj_2 to obtain the variety of complex conics $\mathsf{CQ}_{2,\mathbb{P}^n}$ over $G_2(\mathbb{P}^n)$. It comes with two projections

$$\begin{array}{ccc} & \mathsf{CQ}_2(\mathbb{P}^n) & \\ \swarrow & & \searrow \\ \mathbb{P}(S^2(\mathcal{E}^\vee)) & \overset{\mathrm{Adj}_{\mathcal{E}}}{\dashrightarrow} & \mathbb{P}(S^2(\mathcal{E})) \end{array}$$

Its fiber over a point $\Pi \subset \mathbb{P}^n$ is isomorphic to CQ_2. The cohomology ring $H^*(\mathbb{P}(S^2(\mathcal{E}^\vee)), \mathbb{Z})$ can also be computed. For example, if $n = 3$, $G_2(\mathbb{P}^3) = (\mathbb{P}^3)^\vee$, and $\mathbb{P}(\mathcal{E}^\vee)$ is a projective 5-bundle over $\mathbb{P}^3$. Conics of rank one form a subvariety of $\mathbb{P}(S^2(\mathcal{E}^\vee)$ and the variety of complete conics is isomorphic to its blow-up. The second cohomology group is now generated by three cohomology classes: the class e of the exceptional divisor, the class ρ of the pre-image of a hyperplane in the base of the bundle, and $\mu_1 = c_1(\mathcal{O}_{\mathbb{P}(\mathcal{E}^\vee)}(1))$. We have $2\mu_1 - e = \nu$, where $\mu_2 = (\pi^*)^*(c_1(\mathcal{O}_{\mathbb{P}(S^2(\mathcal{E}))}(1)))$.

Remark 2.4.37. The enumerative problems for complex conics extends to conics over any algebraically closed field of characteristic $\neq 2$. However, in characteristic 2, the duality fails since the dual of a smooth conic is a double line. For example, the number of conics tangent to 5 general conics becomes 51 instead of 3264 [777].

Remark 2.4.38. One can also consider higher codimension (multiple) geometric conditions on conics. For example, the codimension two condition that conics touch a fixed line at a fixed point. Since the fixed point lies on the line, it cannot be considered as the product $\mu_1 \cdot \mu_2$ of two codimension-one conditions. In fact, the group $A^2(\mathsf{CQ})$ of codimension two algebraic cycles is generated by $\mu_1^2, \mu_1 \cdot \mu_2, \mu_2$ only after tensoring with $\mathbb{Q}$. The double condition we considered is represented by the class s such that $2s = \mu_1 \cdot \mu_2$. For example, the number of conics tangent to a fixed conic and tangent to two lines at fixed points is equal to $\frac{1}{4}\mu_1^2\mu_2^2(2\mu_1 + 2\mu_2) = \frac{1}{2}(\mu_1^2\mu_2^3 + \mu_1^3\mu_2^2) = 4$.

2.4.7 Enumerative geometry of quadrics in $\mathbb{P}^3$

We continue to use the notations $\mathsf{Q} = \mathsf{Q}_E, \mathsf{CQ}_E^i = \mathsf{CQ}^i$. The Picard group $\mathrm{Pic}(\mathsf{CQ})$ has a basis (μ_1, μ_2, ρ) or (μ_1, e_1, e_2), related by

$$\mu_2 = 2\mu_1 - e_1, \quad \mu_3 = 3\mu_1 - 2\epsilon_1 - e_2. \tag{2.78}$$

For convenience in consulting the classical literature, let us switch to the classical notations:

$$(\mu, \nu, \rho) := (\mu_1, \mu_2, \mu_3).$$

As we explained in the previous subsection, any enumerative problem for quadrics in $\mathbb{P}^3$ requires the computation of the numbers $\mu^a \cdot \nu^b \cdot \rho^c$, $a+b+c = 9$. We denote these numbers by $[a, b, c]$.

First, note that the cohomology class e_1 of the proper transform of the exceptional divisor of $\pi_1 : \mathsf{CQ}^1 \to \mathsf{Q}$ in CQ is equal to $\pi_2^*([E_1])$. This follows from [315, Corollary 6.7.2] because $5 = \dim E_1 \cap Z_2 \leq \dim E_1 -$

$\mathrm{codim}(Z_2, \mathsf{CQ}^1) = 5$. This implies

$$\begin{aligned}\mu^a \cdot e_1^b \cdot e_2^c &= \pi_2^*(\pi_1^*(h_1^a) \cdot [E_1]^b) \cdot e_2^c \\ &= \pi_1^*(h_1^a) \cdot [E_1]^b \cdot (\pi_2)_*(e_2^c) = (-1)^{c-1} \pi_1^*(h_1^a) \cdot [E_1]^b s_{c-3}(Z_2, \mathsf{CQ}^1).\end{aligned} \tag{2.79}$$

obsevation1

From this we infer

$$\begin{aligned}\mu^a \nu^b \rho^c &= \mu^a \cdot (2\mu - e_1)^b \cdot (3\mu - 2e_1 - e_2)^c \\ &= \mu^a \cdot \nu^b \cdot (3\mu - 2e_1)^c = \mu^a \cdot \nu^b \cdot (2\nu - \mu)^c,\end{aligned} \tag{2.80}$$

obsevation2

if $c = 1, 2$. We get

$$\begin{aligned}\mu^a \nu^b &= \mu^a (2\mu - e_1)^b = \sum_{i=0}^{b} (-1)^i 2^i \mu^{a+i} e_1^{b-i}, \\ \mu^a \nu^b \rho &= \mu^a \cdot \nu^b \cdot (2\nu - \mu) = 2\mu^a \nu^{b+1} - \mu^{a+1} \nu^b, \\ \mu^a \nu^b \rho^2 &= \mu^a \cdot \nu^b \cdot (2\nu - \mu)^2 = \mu^a \nu^{b+2} - 4\mu^{a+1} \nu^{b+1} + \mu^{a+2} \nu^b.\end{aligned} \tag{2.81}$$

observation3

Next, we use the duality

$$\mu^a \nu^b \rho^c = \mu^c \nu^b \rho^a$$

that shows that one needs only to compute

$$\begin{aligned}&\mu^a \nu^{9-a}, \quad \mu^a \nu^{8-a} \rho, \quad \mu^a \nu^{7-a} \rho^2, \\ &\mu^6 \rho^3, \quad \mu^5 \nu \rho^3, \quad \mu^4 \nu^2 \rho^3, \quad \mu^3 \nu^3 \rho^3, \quad \mu^5 \rho^4, \quad \mu^4 \nu \rho^4.\end{aligned} \tag{2.82}$$

rest

To compute the numbers $\mu^a \nu^b$, it suffices to compute $\mu^a e_1^{9-a}$. We use that

$$\mu^a e_1^{9-a} = \pi_1^*(h_1^a) \cdot [E_1]) = h_1^a \cdot (\pi_1)_*([E_1]^{9-a}) = (-1)^{a-1} h_1^a s_{3-a}(\mathsf{Q}(3), \mathsf{Q}).$$

This immediately gives

$$\mu^m \cdot \nu^n = 2^b h_1^{m+n} - \sum_{k=0}^{n-6} \binom{n}{k} 2^k h_1^{m+k} \cdot s_{n-6-k}(\mathsf{Q}(3), \mathsf{Q}). \tag{2.83}$$

observation4

By Proposition segreclass1 2.4.30,

$$s(\mathsf{Q}(3), \mathsf{Q}) = \frac{(1+H)^4}{(1+2H)^{10}} = 1 - 16H + 146H^2 - 996H^3. \tag{2.84}$$

Here, $H = c_1(\mathcal{O}_{\mathsf{Q}(3)}(1)$, where $\mathsf{Q}(3)$ is identified with $\mathbb{P}(E)$, Using the formulas from above, we get

$$\begin{aligned}
\nu^9 &= 2^9 - 8 \cdot 84 \cdot 8 + 4 \cdot 36 \cdot 16 \cdot 4 - 2 \cdot 146 \cdot 2 \cdot 2 + 998 = 92,\\
\mu^1\nu^8 &= 2^8 - 4 \cdot 28\mu^3 \cdot [\mathsf{Q}(1)] + 16 \cdot 16\mu^2 \cdot H + 146\mu \cdot H^2 = 92,\\
\mu^2\nu^7 &= 2^7 - 14\mu^3 \cdot [\mathsf{Q}(1)] - 16\mu^2 \cdot H = 128 - 112 + 64 = 80,\\
\mu^3\nu^6 &= 2^6 - \mu^3 \cdot [\mathsf{Q}(1)] = 64 - 8 = 56,\\
\mu^9 &= h_1^9 = 1.
\end{aligned} \tag{2.85}$$

Of course, the last equality is obvious. Similarly, we compute the remaining numbers

$$\mu^4\nu^5 = 32, \quad \mu^5\nu^4 = 16, \quad \mu^6\nu^3 = 8, \quad \mu^7\nu^2 = 4, \quad \mu^8\nu^1 = 2. \tag{2.86}$$

Using these numbers, and applying (2.81), we obtain

$$\begin{aligned}
\mu^3\nu^5\rho &= \mu^3\nu^5\mu_3 = 2\mu^3\nu^6 - \mu^4\nu^5 = 112 - 32 = 80,\\
\mu^3\nu^4\rho^2 &= \mu^3\nu^4\rho^2 = 4\mu^3\nu^6 - 4\mu^4\nu^5 + \mu^5\nu^4 = 112.
\end{aligned} \tag{2.87}$$

Similarly, we compute

$$\begin{aligned}
&\mu\nu^7\rho = 104, \quad \mu^2\nu^6\rho = 104, \quad \mu^4\nu^4\rho = 48,\\
&\mu^5\nu^3\rho = 24, \quad \mu^5\nu^2\rho = 18, \quad \mu^7\nu\rho = 6, \quad \mu^8\rho = 3,\\
&\mu^7\rho^2 = 9, \quad \mu^6\nu\rho^2 = 12, \quad \mu^5\nu^2\rho^2 = 36,\\
&\mu^4\nu^3\rho^2 = 72, \quad \mu^3\nu^4\rho^2 = 112, \quad \mu^2\nu^5\rho^2 = 128.
\end{aligned} \tag{2.88}$$

It remains to compute the remaining numbers $\mu^a\nu^b\rho^c$ with $c = 3, 4$ from (2.82). Let us assume that $c = 3$. Then

$$\mu^6\rho^3 = \mu^6\rho^3 = \mu^6(3\mu - 2e_1 - e_2)^3 = \mu^6(2\nu - \mu - e_2)^3$$

$$\mu^6(-\mu^3 + 6\mu^2\nu - 12\mu\nu^2 + 8\nu^3) - \mu^6(-3\mu^2 + 12\mu\nu - 12\nu)e_2 + (-3\mu + 6\nu)e_2^2 - e_2^3.$$

We know from above how to compute the sum in the first bracket. We have

$$\mu^6 e_2^3 = \pi_1^*(h_1^6)(\pi_2)_*(e_2^3) = \pi_1^*(h_1^6)s(Z_2, \mathsf{CQ}^1)_0 = \pi_1^*(h_1)[Z_2] = h_1^6[\mathsf{Q}(2)) = \deg(\mathsf{Q}(2)) = 10.$$

Similarly, we get $\mu^7 e_2^2 = \pi_1^*(h_1^6)(\pi_2)_*(e_2^2) = 0$ and

$$\mu^a\nu^b e_2 = \mu^a(2\mu - e_1)^b e_2 = \pi_1^*(h_1)(2\tau^*(h_1) - [E_1])^b(\pi_2)_*(e_2).$$

So, we obtain

$$\mu^6\rho^3 = \mu^6(-\mu^3 + 6\mu^2\nu - 12\mu\nu^2 + 8\nu^3) - \pi_1^*(h_1) \cdot e_2 = 27 - \deg(\mathsf{Q}(2)) = 17.$$

Similarly, we get

$$\mu^5\nu\rho^3 = 34, \quad \mu^4\nu^2\rho^3 = 68, \quad \mu^3\nu^3\rho^3 = 104. \tag{2.89}$$

It remains to compute $\mu^5\rho^4$ and $\mu^4\nu\rho^4$. For this, we will need to compute the Segre class $s_1(Z_2, \mathsf{CQ}^1)$. This yields

$$s_1(Z_2, \mathsf{CQ}^1) = -c_1(p_1^* S^2\mathcal{R}_2) \otimes p_2^*(\mathcal{O}_{\mathsf{Q}(2)}(1) \otimes \mathcal{O}_{Z_2}(-E_1)).$$

Let $p_1 : Z_2 = \mathbb{P}(S^2\mathcal{K}_2^\vee) \to G(2, E)$ be the structural projection of the projective bundle and $p_2 : Z_2 \to \mathsf{Q}(2)$ be the projection of the blow-up $Z_2 = \mathsf{Bl}_{\mathsf{Q}(3)}(\mathsf{Q}(2))$. The closed embedding inclusion $i : \mathbb{P}(S^2\mathcal{K}_2^\vee) \hookrightarrow \mathbb{P}(S^2(E))_{G(2,E)}$ shows that

$$\eta := \mathcal{O}_{Z_2}(1) = i^*\mathcal{O}_{\mathbb{P}(S^2(E)_{G(2,E)})}(1) = p_2^*(h_1).$$

The Chern classes of the sheaf $\mathcal{R}_2$ are known. We will show in Subsection CAG-2:SS:10.1.2 10.1.2 that

$$c_1(\mathcal{R}_2) = \sigma_1 := c_1(\mathcal{O}_{\mathbb{G}}(1)).$$

Applying Fulton [315, Examples 3.2.2 and 14.5.2], we find that $c_1(S^2\mathcal{R}_2) = 1 + 3\sigma_1$. Hence,

$$s_1(Z_2, \mathsf{CQ}^1) = -3(p_1^*(\sigma_1) + \pi_1^*(h_1) - [E_1 \cap Z_2].$$

We have

$$\begin{aligned} &\mu^5\rho^4 = \mu^5\rho^4 = \mu^5(2\nu - \mu - e_2)^4 \\ &\quad = \mu^5(\mu^4 - 8\mu^3\nu + 24\mu^2\nu^2 - 32\mu\nu^3 + 16\nu^4) - 4\mu^5(2\nu - \mu)e_2^3 + \mu^5 e_2^4. \end{aligned} \tag{2.90}$$

Here, we used that, by above, the terms containing e_2 in power one or two are equal to zero.

We have

$$\mu^5(\mu^4 - 8\mu^3\nu + 24\mu^2\nu^2 - 32\mu\nu^3 + 16\nu^4) = 81,$$

$$4\mu^5(2\nu - \mu)e_2^3 = 4\mu^5(3\mu - 2e_1)e_2^3 = 12\deg(\mathsf{Q}(2)) - 8\mu^4 e_1 \cdot [Z_2])$$

$$= 120 - 8h_1^4 \cdot (\pi_1)_*([E_1 \cap Z_2] = 120,$$

Here, we used that $Z_2 \cap E_1 \to \mathsf{Q}(3)$ is a of positive relative dimension, so $(\pi_1)_*([E_1 \cap Z_2] = 0$. It remains to compute the last summand. We have

$$\mu^5 e_2^4 = -\pi_1^*(h_1)s_1(Z_2, \mathsf{CQ}^1) = 3(\pi_1^*(h_1^5)(p_1^*(\sigma_1) + \pi_1^*(h_1) - [E_1 \cap Z_2])$$

$$= 3(\pi_1^*(h_1^6) \cdot [Z_2] + \pi_1^*(h_1^5) \cdot p_1^*(\sigma_1)) = 30 + \eta^5 \cdot p_1^*(\sigma_1)$$

To compute $\eta^5 \cdot p_1^*(\sigma_1)$, we use that $p^*(\sigma_1) = p_1^*(c_1(\mathcal{O}_{G(2,E)}(1)) = \pi_2^*(\nu) \cdot [Z_2]$, hence

$$\pi_1^*(h_1^5)\cdot p_1^*(\sigma_1) = \pi_1^*(h_1^5)\cdot\nu\cdot[Z_2] = \pi_1^*(h_1^5)\cdot\pi_1^*(3h_1 - 2[E_1]) - (\pi_2)_*(e_2))\cdot[Z_2]$$

$$= 3\pi_1^*(h_1)\cdot[Z_2] = 30.$$

Adding up the summands, we get

$$\mu^5\rho^4 = 81 - 120 + 60 = 21.$$

We leave it to the reader to do a similar computation to obtain that

$$\mu^4\nu\rho^4 = 42.$$

Knowing the numbers $[a, b, c]$, we can solve other enumerative problems.

Example 2.4.39. In this example we compute the number N_3 of quadrics tangent to nine quadrics in a general position. The cohomology class of hypersurface of complete quadrics tangent to the fixed quadrics is equal to $2(\mu+\nu+\rho)$ [207]. Thus, we need to compute

$$2^9(\mu + \nu + \rho)^9 = 2^9(\sum_{a+b+c=9} \frac{9!}{a!b!c!}[a, b, c]$$

After a lengthy computation, we find that

$$N_3 = 666,841,088.$$

Note that the loc.cit. paper contains the numbers N_4 (resp. N_5) of quadrics in $\mathbb{P}^4$ (resp. $\mathbb{P}^5$) that are tangent to 14 (resp. 20) quadrics in general position. The numbers are

$$N_4 = 48,942,189,946,470,400,$$

$$N_5 = 641,211,464,734,373,953,791,690,014,720.$$

The Table 2.2 summarizes our compyations. One can also find it Schubert's book [667, p. 105] or Semple-Roth's book [701, Chapter XI, §5]

$\mu^9 = 1$	$\mu^8\nu = 2$	$\mu^7\nu^2 = 4$	$\mu^6\nu^3 = 8$	$\mu^5\nu^4 = 16$
$\mu^4\nu^5 = 32$	$\mu^3\nu^6 = 56$	$\mu^2\nu^7 = 80$	$\mu\nu^8 = 92$	$\nu^9 = 92$
$\mu^8\rho = 3$	$\mu^7\nu\rho = 6$	$\mu^6\nu^2\rho = 12$	$\mu^5\nu^3\rho = 24$	$\mu^4\nu^4\rho = 48$
$\mu^3\nu^5\rho = 80$	$\mu^2\nu^6\rho = 104$	$\mu\nu^7\rho = 104$	$\mu^7\rho^2 = 9$	$\mu^6\nu\rho^2 = 18$
$\mu^5\nu^2\rho^2 = 36$	$\mu^4\nu^3\rho^2 = 72$	$\mu^3\nu^4\rho^2 = 112$	$\mu^2\nu^5\rho^2 = 128$	$\mu^6\rho^3 = 17$
$\mu^5\nu\rho^3 = 34$	$\mu^4\nu^2\rho^3 = 68$	$\mu^3\nu^3\rho^3 = 104$	$\mu^5\rho^4 = 81$	$\mu^4\nu\rho^4 = 42$

schubertquadrics

Table 2.2 *Characteristic numbers for quadrics in* $\mathbb{P}^3$

In Chapter 11, we will also need to know the characteristic numbers for quadric cones in $\mathbb{P}^3$. A natural guess is that the numbers must be equal to $e_3\mu^a\nu^b\rho^c, a+b+c=8$. However, the hypersurface representing the divisor class ρ consists of singular quadrics touching a fixed plane Π. It is equal to the proper transform in CQ_3 of the intersection of the discriminant quartic hypersurface $\Delta = \mathsf{Q}_3(3)$ with its first polar with respect to the point dual to Π. It intersects Δ along a codimension one subvariety of Δ of degree 6 taken with multiplicity two. Thus, the correct numbers are $2^{-c}e_3\mu^a\nu^b\rho^c$.

Applying Proposition prop:twobases 2.4.28, we get $e_3 = 2\rho - \nu = 4\rho - \nu$, hence

$$2^{-c}e_3\mu^a\nu^b\mu^c = 2^{-c}(2\rho-\nu)(\mu^a\nu^b\rho^{c+1} - \mu^a\nu^{b+1}\rho^c).$$

Using the previous table, we can compute the characteristic numbers for cones. For brevity, we set $2^{-c}e_3\mu^a\nu^b\rho^c := \mu^a\nu^b\rho^c, a+b+c=8$.. In Table schubertcones 2.3 below, we use that the characteristic numbers $\mu^a\nu^b\rho^c = 0$ if $c > 3$. This is because the vertex of the cone cannot lie in more than 3 different planes.

$\mu^8 = 4$	$\mu^7\nu = 8$	$\mu^6\nu^2 = 16$	$\mu^5\nu^3 = 32$	$\mu^4\nu^4 = 64$
$\mu^3\nu^5 = 104$	$\mu^2\nu^6 = 128$	$\mu\nu^7 = 116$	$\nu^8 = 92$	
$\mu^7\rho = 6$	$\mu^6\nu\rho = 12$	$\mu^5\nu^2\rho = 24$	$\mu^4\nu^3\rho = 48$	$\mu^3\nu^4\rho = 72$
$\mu^2\nu^5\rho = 76$	$\mu\nu^6\rho = 52$	$\nu^7\rho = 34$	$\mu^6\rho^2 = 4$	$\mu^5\nu\rho^2 = 8$
$\mu^4\nu^2\rho^2 = 16$	$\mu^3\nu^3\rho^2 = 24$	$\mu^2\nu^4\rho^2 = 24$	$\mu\nu^5\rho^2 = 14$	$\mu^5\rho^3 = 16$
$\mu^4\nu\rho^3 = 2$	$\mu^3\nu^2\rho^3 = 4$	$\mu^2\nu^3\rho^3 = 4$		

schubertcones Table 2.3 *Characteristic numbers for cones in* $\mathbb{P}^3$

Also, note that, via duality, the same table gives the characteristic numbers for space conics. The condition $\mu^a\nu^b\rho^c, a+b+c=8$, gives the number of space conics $C \subset \Pi$ with the spanning plane Π passing through c general points, intersecting b general lines, and tangent to a general planes. Table refschubertcones can be found in SchubertBook [667, p. 95] and in SR [701, Chapter XI, §5].

A general one-dimensional family of quadrics in $\mathbb{P}^3$ (and in any $\mathbb{P}^n$) contains only quadrics of corank ≤ 1. On the other hand, a general one-dimensional family of quadric cones in $\mathbb{P}^3$ lifted to CQ_3 is contained in E_3 and intersects $\mathsf{E}_2\circ$ (resp. E_1°) at finitely many points δ (resp. η) corresponding to reducible cones (resp. point-pairs).

prop:sturm1 **Proposition 2.4.40.** *Let $\mathcal{K}$ be an irreducible algebraic family of quadric cones whose proper transform in CQ_3 has the cohomologically class $\mu^a\nu^b\rho^c, a+b=c=8$. Let $\alpha = \mu^{a+1}\nu^b\rho^c, \beta = \mu^a\nu^{b+1}\rho^c, \gamma = \mu^a\nu^b\rho^{c+1}$ be the number of quadric cones in the family passing through a general point, tangent to a* *general line, and tangent to a general plane, respectively. Then*

$$2\beta = \gamma + 2\alpha + \eta, \quad 2\rho = \beta + \delta.$$

Proof We have

$$\delta = e_2\mu^a\nu^b\rho^c, \quad \eta = e_3\mu^a\nu^b\rho^c.$$

prop:twobases

Applying Proposition 2.4.28, we obtain

$$\begin{aligned} \delta &= (2\nu - \mu - \rho)\mu^a\nu^b\rho^c = 2\mu^a\nu^{b+1}\rho^{c+1} - \mu^{a+1}\nu^b\nu^c - \mu^a\nu^b\rho^{c+1}, \\ \eta &= (2\rho - \nu)\mu^a\nu^b\rho^c = 2\mu^a\nu^b\rho^{c+1} - \mu^a\nu^{b+1}\rho^c. \end{aligned} \tag{2.91}$$

This proves the proposition. □

E:2

Exercises

2.1 Let E be a vector space of even dimension $n = 2k$ over a field $\Bbbk$ of characteristic 0 and $(e_1, \ldots, e_n)$ be a basis in E. Let $\omega = \sum_{i<j} a_{ij} e_i \wedge e_j \in \bigwedge^2 E^\vee$ and $A = (a_{ij})_{1\le i\le j\le n}$ be the skew-symmetric matrix defined by the coefficients a_{ij}. Let $\wedge^k\omega = \omega \wedge \cdots \wedge \omega = ak!e_1 \wedge \cdots \wedge e_n$ for some $a \in \Bbbk$. The element a is called the *pfaffian* of A and is denoted by $\mathrm{Pf}(A)$.

(i) Show that

$$\mathrm{Pf}(A) = \sum_{S\in\mathcal{S}} \epsilon(S) \prod_{(i,j)\in S} a_{ij},$$

where S is a set of pairs $(i_1, j_1), \ldots, (i_k, j_k)$ such that $1 \le i_s < j_s \le 2k$, $s = 1, \ldots, k$, $\{i_1, \ldots, i_k, j_1, \ldots, j_k\} = \{1, \ldots, n\}$, $\mathcal{S}$ is the set of such sets S, $\epsilon(S) = 1$ if the permutation $(i_1, j_1, \ldots, i_k, j_k)$ is even and -1 otherwise.

(ii) Compute $\mathrm{Pf}(A)$ when $n = 2, 4, 6$.

(iii) Show that, for any invertible matrix C,

$$\mathrm{Pf}({}^tC \cdot A \cdot C) = \det(C)\mathrm{Pf}(A).$$

(iv) Using (iii) prove that

$$\det(A) = \mathrm{Pf}(A)^2.$$

(iv) Show that

$$\mathrm{Pf}(A) = \sum_{i=1}^{n} (-1)^{i+j-1}\mathrm{Pf}(A_{ij})a_{ij},$$

where A_{ij} is the matrix of order $n-2$ obtained by deleting the i-th and j-th rows and columns of A.

(v) Let B be a skew-symmetric matrix of odd order $2k-1$ and B_i be the matrix of order $2k-2$ obtained from B by deleting the i-th row and i-th column. Show that the vector $(\mathrm{Pf}(B_1), \ldots, (-1)^{i+1}\mathrm{Pf}(B_i), \ldots, \mathrm{Pf}(B_{2k-1}))$ is a solution of the equation $B \cdot x = 0$.

(vi) Show that the rank of a skew-symmetric matrix A of any order n is equal to the largest m such that there exist $i_1 < \ldots < i_m$ such that the matrix $A_{i_1\ldots i_m}$ obtained from A by deleting i_j-th rows and columns, $j = 1, \ldots, m$, has nonzero pfaffian .

ex:2.1

2.2 Let $\mathsf{V} = \mathsf{v}_2(\mathbb{P}^2)$ be a Veronese surface in $\mathbb{P}^5$, where $\mathbb{P}^5$ is considered as the space of conics in $\mathbb{P}^2$.

(i) Let Λ be a plane in $\mathbb{P}^5$ and $\mathcal{N}_\Lambda$ be the net of conics in $\mathbb{P}^2$ cut out by hyperplanes containing Λ. Show that Λ is a trisecant plane if and only if the set of base points of $\mathcal{N}_\Lambda$ consists of ≥ 3 points (counting with multiplicities). Conversely, a net of conics through three points defines a unique trisecant plane.

(ii) Show that the nets of conics with two base points, one of them is infinitely near, forms an irreducible divisor in the variety of trisecant planes.

(iii) Using (ii), show that the anti-canonical divisor of degenerate triangles is irreducible.

(iv) Show that the trisecant planes intersecting the Veronese surface at one point (corresponding to net of conics with one base point of multiplicity 3) define a smooth rational curve in the boundary of the variety of self-polar triangles. Show that this curve is equal to the set of singular points of the boundary.

ex:2.2

2.3 Let $U \subset (\mathbb{P}^2)^{(3)}$ be the subset of the symmetric product of $\mathbb{P}^2$ parameterizing the sets of three distinct points. For each set $Z \in U$ let L_Z be the linear system of conics containing Z. Consider the map $f : U \to G_2(\mathbb{P}^5), Z \mapsto L_Z \subset |\mathcal{O}_{\mathbb{P}^2}(2)|$.

(i) Consider the divisor D in U parameterizing sets of 3 distinct collinear points. Show that $f(D)$ is a closed subvariety of $G_2(\mathbb{P}^5)$ isomorphic to $\mathbb{P}^2$.

(ii) Show that the map f extends to the Hilbert scheme $(\mathbb{P}^2)^{[3]}$ of 0-cycles Z with $h^0(\mathcal{O}_Z) = 3$.

(iii) Show that the closure $\bar{D}$ of $\pi^{-1}(D)$ in the Hilbert scheme is isomorphic to a $\mathbb{P}^3$-bundle over $\mathbb{P}^2$ and the restriction of f to $\bar{D}$ is the projection map to its base.

(iv) Define the map $\tilde{f} : \mathcal{P} \to |\mathcal{O}_{\mathbb{P}^2}(2)|$, which assigns to a point in the fiber $p^{-1}(Z)$ the corresponding conic in the net of conics though Z. Show that the fiber of $\tilde{f}$ over a nonsingular conic C is isomorphic to the Fano variety of self-polar triangles of the dual conic $C^\vee$.

(v) Let $\mathcal{P}^s = \tilde{f}^{-1}(\mathsf{D}_2(2))$ be the pre-image of the hypersurface of singular conics. Describe the fibres of the projections $p : \mathcal{P}^s \to (\mathbb{P}^2)^{[3]}$ and $\tilde{f} : \mathcal{P}^s \to \mathsf{D}_2(2)$.

2.3

2.4 Identify $\mathbb{P}^1$ with its image under the Veronese map $\mathsf{v}_2 : \mathbb{P}^1 \to \mathbb{P}^2$.

(i) Show that any involution of $\mathbb{P}^1$ (i.e. an automorphism of order 2) coincides with the involution of the Veronese conic obtained by projection from a point not lying on the conic (called the center of the involution).

(ii) Show that two involutions of $\mathbb{P}^1$ without common fixed points commute if and only if the two pairs of fixed points are harmonically conjugate.

(iii) Show that the product of three involutions is an involution if their centers are collinear (J. Valles). The converse is known for any odd number of involutions.

ex:2.4

2.5 Prove that two unordered pairs $\{a, b\}, \{c, d\}$ of points in $\mathbb{P}^1$ are harmonically conjugate if and only if there is an involution of $\mathbb{P}^1$ with fixed points a, b that switches c and d. ex:2.5

2.6 Prove the following *Hesse's Theorem*. If two pairs of opposite vertices of a quadrilateral are each conjugate for a conic, then the third pair is also conjugate. Such a quadrilateral is called a *Hesse quadrilateral*. Show that four lines form a polar quadrilateral for a conic if and only if it is a Hesse quadrilateral. ex:2.6

2.7 A tetrad of points p_1, p_2, p_3, p_4 in the plane is called *self-conjugate* with respect to a nonsingular conic if no three points are collinear and the pole of each side $\overline{p_i p_j}$ lies on the opposite side $\overline{p_k p_l}$.

(i) Given two conjugate triangles, show that the vertices of one of the triangles together with the center of perspectivity form a self-conjugate tetrad.

(ii) Show that the four lines with poles equal to p_1, p_2, p_3, p_4 form a polar quadrilateral of the conic and any nondegenerate polar quadrilateral is obtained in this way from a self-conjugate tetrad.

(iii) Show that any polar triangle of a conic can be extended to a polar quadrilateral. ex:2.7

2.8 Extend Darboux's Theorem to the case of two tangent conics. ex:2.8

2.9 Show that the secant lines of a Veronese curve R_m in $\mathbb{P}^m$ are parameterized by the surface in the Grassmannian $G_1(\mathbb{P}^m)$ isomorphic to $\mathbb{P}^2$. Show that the embedding of $\mathbb{P}^2$ into the Grassmannian is given by the Schwarzenberger bundle. ex:2.9

2.10 Let U be a 2-dimensional vector space. Use the construction of curves of degree $n-1$ Poncelet-related to a conic to exhibit an isomorphism of linear representations $\bigwedge^2(S^n U)$ and $S^{n-1}(S^2 U)$ of $\mathrm{SL}(U)$. ex:2.10

2.11 Assume that the pencil of sections of the Schwarzenberger bundle $\mathcal{S}_{n,C}$ has no base points. Show that the Poncelet curve associated to the pencil is nonsingular at a point x defined by a section s from the pencil if and only if the scheme of zeros $Z(s)$ is reduced. ex:2.11

2.12 Find a geometric interpretation of vanishing of the invariants Θ, Θ' from (2.19) salmon in the case where C or S is a singular conic. ex:2.12

2.13 Let p_1, p_2, p_3, p_4 be four distinct points on a nonsingular conic C. Show that the triangle with the vertices $A = \overline{p_1 p_3} \cap \overline{p_2 p_4}$, $B = \overline{p_1 p_2} \cap \overline{p_3 p_4}$ and $C = \overline{p_1 p_4} \cap \overline{p_2 p_3}$ is a self-conjugate triangle with respect to C. ex:2.13

2.14 Show that two pairs $\{a, b\}, \{c, d\}$ of points in $\mathbb{P}^1$ with a common point are never harmonically conjugate. ex:2.14

2.15 Let (a, b, c, d) be a quadrangle in $\mathbb{P}^2$, and p, q be the intersection points of two pairs of opposite sides $\overline{ab}, \overline{cd}$ and $\overline{bc}, \overline{ad}$. Let p', q' be the intersection points of the line $\overline{pq}$ with the diagonals $\overline{ac}$ and $\overline{bd}$. Show that the pairs (p, q) and (p', q') are harmonically conjugate. ex:2.15

2.16 Show that the pair of points on a diagonal of a complete quadrilateral defined by its sides is harmonically conjugate to the pair of points defined by intersection with other two diagonals. ex:2.16

2.17 Show that a general net of conics admits a common polar quadrangle. ex:2.17

2.18 Show that four general conics admit a unique common polar quadrangle. ex:2.18

2.19 Find the condition on a pair of conics expressing that the associate Salmon conic is degenerate. ex:2.19

2.20 Show that the triangle formed by any three tangents to two general conics is in perspective with any three of common points. ex:2.20

2.21 Show that the set of $2n+2$ vertices of two self-polar $(n+1)$-hedra of a quadric in

$\mathbb{P}^n$ impose one less condition on quadrics. In particular, two self-polar triangles lie on a conic, two self-polar tetrahedra are the base points of a net of quadrics. ex:2.21

2.22 A hexad of points in $\mathbb{P}^3$ is called self-conjugate with respect to a nonsingular quadric if no four are on the plane and the pole of each plane spanned by three points lies on the plane spanned by the remaining three points. Show that the quadric admits a nondegenerate polar hexahedron whose planes are polar planes of points in the hexad. Conversely, any nondegenerate polar hexahedron of the quadric is obtained in this way from a self-conjugate tetrad.

2.23 Show that the variety of sums of five powers of a nonsingular quadric surface is isomorphic to the variety of self-conjugate pentads of points in $\mathbb{P}^3$. ex:2.23

2.24 Consider 60 Pascal lines associated with a hexad of points on a conic. Prove the following properties of the lines.

(i) There are 20 points at which three of Pascal lines intersect, called the *Steiner points*.

(ii) The 20 Steiner points lie on 15 lines, each containing four of the points (the *Plücker lines*).

(iii) There are 60 points each contained in three Plücker lines (the *Kirkman points*). ex:2.24

2.25 Prove the following generalization of Pascal's Theorem. Consider the 12 intersection points of a nonsingular quadric surface Q with six edges of a tetrahedron T with vertices p_1, p_2, p_3, p_4. For each vertex p_i choose one of the 12 points on each edge $\overline{p_i p_j}$ and consider the plane Λ_i spanned by these three points. Show that the four lines in which each of these four planes meats the opposite face of the tetrahedron are rulings of a quadric. This gives 32 quadrics associated to the pair (T, Q) ChaslesAperçu [129], p. 400, BakerBook [29], v. 3, Ex. 15, Salmon-Fiedler [654], p. 362. ex:2.25

2.26 Let $\Theta_0, \ldots, \Theta_4$ be the invariants of a pair of quadric surfaces.

(i) Show that the five products $\Theta_2, \Theta_0\Theta_4, \Theta_1\Theta_3, \Theta_1^2\Theta_4, \Theta_3^2\Theta_0$ generate the algebra of invariants of bidegrees (m, n) with $m = n$.

(ii) Show that the GIT-quotient of ordered pairs of quadrics by the group $\mathrm{SL}(4)$ is isomorphic to the hypersurface of degree 6 in the weighted projective space $\mathbb{P}(1, 2, 2, 3, 3)$ given by the equation $t_1 t_2^2 - t_3 t_4 = 0$.

(iii Show that the GIT-quotient has a singular line and its general point corresponds to the orbit of the pair $V(\sum t_i^2), V((t_0^2 - t_1^2) + a(t_2^2 - t_3^2))$. ex:2.26

2.27 Let T_1 and T_2 be two conjugate triangles with respect to a conic C. Find the condition for C such that the vertices of T_1 and T_2 lie on a conic, or there exists a conic tangent to their sides. ex:2.27

2.28 Let C be a smooth conic and ℓ be a line in $\mathbb{P}^2$. The pole of ℓ with respect to C is called the center of C.

(i) Show that the center of the complex circle with respect to the line at infinity is the center of the circle.

(ii) Fix five lines in a general position. Show that the centers of conics tangent to any four lines with respect to the remaining line trace a line (called *Newton's line*). ex:2.28

2.29 Show that the subvariety of $|\mathcal{O}_{\mathbb{P}^n}(2)|$ of quadrics in $\mathbb{P}^n$ that touch a fixed linear subspace of codimension $k < n$ is a hypersurface of degree $n - k$. Identify it with a kth polar hyperface of the discriminant hypersurface.

2.30 Show that the intersection of the discriminant hypersurface of quadrics in $\mathbb{P}^n$ with its general first polar hypersurface is a codimension two subvariety of degree $\frac{1}{2}n(n+1)$ taken with multiplicity two. semplequadric

2.31 Show that the tangent hyperplane H to the discriminant hypersurface of quadrics in $\mathbb{P}^n$ at its nonsingular point is touching the subvariety of quadrics of rank one along the space of quadrics of rank one in H.

2.32 Show that the intersection number $e_i\mu_1^{a_1}\mu_2^{a_2}\rho^{a_3}, a_1+a_2+a_3=8$, is equal to 2^{a_i} times the number of quadric cones satisfying the condition $\mu_1^{a_1}\mu_2^{a_2}\rho^{a_3}$.

ex:2.29

2.33 Show that the intersection number $e_1e_2\mu_1^{a_1}\mu_2^{a_2}\rho^{a_3}, a_1+a_2+a_3=7$, is equal to 2^{a_2} times the number of reducible quadrics satisfying the condition $\mu_1^{a_1}\mu_2^{a_2}\rho^{a_3}$. is equal to 2^{a_i} times the number of quadric cones satisfying the condition $\mu_1^{a_1}\mu_2^{a_2}\rho^{a_3}$.

ex:2.30

2.34 Consider a space conic on $\mathbb{P}^n$ as a closed point $\mathsf{Q}_{\mathcal{R}}$, where $\mathcal{R}$ is in the universal quotient bundle over the Grassmannian $G_2(\mathbb{P}^n)$ of planes in $\mathbb{P}^n$.

- Show that $\mathsf{CQ}_{\mathcal{R}}$ is naturally isomorphic to the boundary divisor E_{n-2} in the space CQ_n of complete quadrics in $\mathbb{P}^n$.
- Using the intersection theory on CQ_3 show that the number of conics in $\mathbb{P}^3$ which meet a given planes, intersect b given lines, and whose spanning planes pass through a given set of $8-a-b$ points is finite, and equal to the number $2^{8-a-b}\mu_1^a\mu_2^b\rho^{8-a-b}$.
- Use the intersection theory on $\mathsf{CQ}_{\mathcal{R}}$ to confirm the previous exercise.
- Compute the number of conics in $\mathbb{P}^3$ that are tangent to 8 given quadrics in a general position.
- Compute the degree of the variety of planes in $\mathbb{P}^3$ that intersect 6 general lines at 6 points lying on a conic.

ex:2.31

2.35 Show that the condition α that a quadric surface contains a given line is equal to

$$\alpha = \frac{1}{4}(2\mu_2^2 - 3(\mu_1+\rho)\mu_2^2 + 3(\mu_1^2 + mu_3^2)\mu_2 + 2\mu_1\mu_2\rho - 2(\mu_1^3+\rho^3).$$

ex:2.32

2.36 Consider the variety $C(T)$ of plane conics that admit a given triangle T of lines as its self-conjugate triangle. Show that $C(T)$ is a surface in the space of conics, and its proper transform in the variety CQ of complete conics is isomorphic to the blow-up of the plane at three points.

ex:2.34

Historical Notes

There is a significant number of books dealing with the analytic geometry of conics. The most comprehensive source for the history of the subject is Coolidge's book [167]. Many facts and results about real conics treated synthetically can be found in textbooks in projective geometry. Coxeter's small book [174] is one of the best.

The theory of polarity for conics goes back to Poncelet [601]. Polar triangles and tetrahedra of a conic and a quadric surface were already studied by P. Serret [706]. In particular, he introduced the notions of self-conjugate triangles, quadrangles and pentagons. These objects were later intensively studied by T. Reye [614], [619] and R. Sturm [736, Bd. 3]. The subject of their study was called the *Polarraum*, i.e., a pair consisting of a projective space together with a nonsingular quadric.

Pascal's Theorem was discovered by B. Pascal in 1639 when he was 16 years old [576] but not published until 1779 [576]. It was independently rediscovered by C. MacLaurin in 1720 [501]. A large number of results about the geometry and combinatorics of 60 Pascal lines assigned to six points on a conic have been discovered by J. Steiner, J. Kirkman, A. Cayley, G. Salmon, L. Cremona and others. A good survey of these results can be found in Note 1 in Baker's book [29], v.2, and Notes in Salmon's book [651]. We will return to this in Chapter 9.

Poncelet's Closure Theorem, which is the second part of Darboux's Theorem 2.2.2 was first discoverd by J. Poncelet himself [601]. We refer the reader to the excellent account of the history of the Poncelet-related conics in [67]. A good elementary discussion of Poncelet's Theorem and its applications can be found in Flatto's book [305]. Other elementary and non-elementary treatments of the Poncelet properties and their generalizations can be found in [34], [35], [156], [158], [358], [359].

The relationship between Poncelet curves and vector bundles is discussed in [764], [545], [765], [780]. The Schwarzenberger bundles were introduced in [675]. We followed the definition given in [237]. The papers [532] and [400], [401] discuss the compactification of the variety of conjugate triangles. The latter two papers of N. Hitchin also discuss an interesting connection with Painleve equations.

The notion of the apolarity of conics is due to T. Reye [617]. However, J. Rosanes [636] used this notion earlier under the name conjugate conics. In the same paper, he also studied the representation of a conic as a sum of four squares of linear forms. The condition (2.11) for conjugate conics was first discovered by O. Hesse in [389]. He also proved that this property is poristic. The condition for Poncelet relation given in terms of invariants of a pair of conics (Theorem 2.3.14) was first discovered by A. Cayley [106], [111].

The theory of invariants of two conics and two quadric surfaces was first developed by G. Salmon (see [651], [653], vol. 1). The complete system of invariants, covariants, and contravariants of a pair of conics was given by J. Grace and A. Young [351]. P. Gordan has given a complete system of 580 invariants, covariants and contravariants of a pair of quadric surfaces [348]. Later H.W. Turnbull was able to reduce it to 123 elements [767]. In a series of papers of J. Todd, one can find further simplifications and more geometric interpretations of the system of combinants of two quadric surfaces [759], [760]. A good exposition of the theory of invariants can be found in Sommerville's and Todd's books [717], [761]. The latter book contains many examples and exercises, some of which were borrowed here.

Chasles' Theorem 2.3.4 about the covariant quadric was proven by him in [125] and reproved later by N. Ferrers [300] . A special case was known earlier to E. Bobillier [62]. Chasles' generalization of Pascal's Theorem to quadric surfaces can be found in [129]. Baker's book [29], v. 3, gives a good exposition of polar properties of quadric surfaces.

The proof of Theorem 2.3.21 is due to J. Coolidge [167], Chapter VI, §3. The result was known to G. von Staudt [720] (see [167], p. 66) and can also be found in Salmon's book on conics [651], p. 345. Although Salmon writes in the footnote on p. 345 that "I believe that I was the first to direct the attention to the importance of this conic in the theory of two conics", this conic was already known to Ph. La Hire [476] (see [167], p. 44). In Sommerville's book [716], the Salmon conic goes under the name *harmonic conic-locus* of two conics.

An excellent historical discussion of the enumerative geometry of conics can be found in [450]. We will give only a brief account of the main actors in this historical drama. J. Steiner was the first to attempt to compute the number of conics touching five fixed conics. However, the problem of finding the number of circles tangent to three fixed circles goes back to antiquity by the name *Apollonius's problem*. Steiner correctly proved that the variety of conics tangent to a fixed conic is a hypersurface of degree six, and wrongly concluded from this that the answer to the enumerative problem is 6^5 [722, p. 188]. We explained the reason of his mistake. This explanation of Steiner's mistake is due to L. Cremona [184]. In 1861, M. Chasles introduced two characteristic numbers for enumerative geometry of conics and showed that one characteristic number introduced earlier by J. de Jonquères is not enough [128]. He was the first to give the correct answer 3264 of the number of conics touching five fixed conics. The first modern justification of the number was given by F. Severi using his residual intersection theory [707, Footnote, p. 116]. The name "Vollständiger Kegelschnitt" is due to B, van der Waerden [794, p. 647]. The four classes μ_1, μ_2, e_1, e_2 in $\mathrm{Pic}(\mathsf{CQ})$ were introduced by Charles as the loci of conics passing through a point, tangent to a line, to be a double line with two foci, to be a line-pair, respectively. The relations between (μ_1, μ_2) and (e_1, e_2) was also obtained by Chasles. The numbers $\mu_1^a \mu_2^{5-a}$ were first computed by H. Zeuthen [816]. The multiple conditions we discussed in Remark 2.4.38 were first introduced by Charles, who used the formula $2d = \mu_1\mu_2$ to express the condition that a conic is tangent to two fixed lines at fixed points on the lines. Cremona was the first to express any multiple conditions in term of $a\rho + b\sigma + c\tau$, where $a, b, c \in \mathbb{Q}$ [184].

G. Halphen [371] was the first to discover that not all enumerative problems of conics can be expressed in terms of the intersection theory on the variety of complete conics.

The characteristic numbers $a\mu_1 + b\mu_2 + c\rho$ for enumerative geometry of quadrics were first introduced in 1966, in an unpublished work by Zeuthen (see [819]). A more complete theory was developed later by H. Schubert [665]. Schubert computed correctly all the intersection numbers $\mu^a\nu^b\rho^c$; they can be found in his book [667, pp. 104–105]. The book goes beyond the enumerative geometry of quadrics; for example, one can find there the determination of the number $5,819,539,783,680$ of twisted cubics tangent to 12 quadrics in $\mathbb{P}^3$. Note that the methods of Schubert were based on the degenerations and the Charles' Principle of Correspondence cannot be rigorously justified by modern techniques, although they lead to correct results. The rigorous methods based on the intersection theory on appropriate parameter spaces had to wait almost a hundred years.

The modern theory of the intersection theory on the space of complete quadrics was first developed in a series of papers by J. Semple [699], [700], [703] for conics and quadric surfaces, A. Alguneid [4]in dimension 4, and J. Tyrrell [769] for quadrics of arbitrary dimension. A good exposition of a modern enumerative geometry of quadric surfaces based on the intersection theory can be found in the book of Semple and Roth [701, Chapter XI]. The numbers $\mu^a\nu^b\nu^c$ computed first by Schubert can also be found in the book.

A good survey of the intersection theory on the varieties of complete quadrics is [477]. We refer to the work of C. De Concini and C. Procesi [208], [206] for theory of generalized varieties of complete quadrics as wonderful compactifications of certain homogeneous spaces of algebraic groups. the varieties.

3

Plane Cubics

Ch3

3.1 Equations

S:3.1

3.1.1 Elliptic Curves

SS:3.1.1

There are many excellent expositions of the theory of elliptic curves from their many aspects: analytical, algebraic and arithmetical (a short survey can be found in Hartshorne's book [379], Chapter IV). We will be brief here.

Let X be a nonsingular projective curve of genus one. By Riemann-Roch, for any divisor D of degree $d \geq 1$, we have $\dim H^0(X, \mathcal{O}_X(D)) = d$. If $d > 2$, the complete linear system $|D|$ defines an isomorphism $X \to C$, where C is a curve of degree d in $\mathbb{P}^{d-1}$ (called an *elliptic normal curve* of degree d). If $d = 2$, the map is of degree 2 onto $\mathbb{P}^1$. The divisor classes of degree 0 are parameterized by the Jacobian variety $\mathrm{Jac}(X)$ isomorphic to X. Fixing a point x_0 on X, the group law on $\mathrm{Jac}(X)$ transfers to a group law on X by assigning to a divisor class $\mathfrak{d}$ of degree 0 the divisor class $\mathfrak{d} + x_0$ of degree 1 represented by a unique point on X. The group law becomes

$$x \oplus y = z \in |x + y - x_0|. \tag{3.1}$$ grouplaw

The group of translation automorphisms of X acts transitively on the set $\mathrm{Pic}^d(X)$ of divisor classes of degree d. This implies that two elliptic normal curves are isomorphic if and only if they are projectively equivalent. In the case $d = 2$, this implies that two curves are isomorphic if and only if the two sets of four branch points of the double cover are projectively equivalent.

In this chapter, we will be mainly interested in the case $d = 3$. The image of X is a nonsingular plane cubic curve. There are two well-known normal forms for its equation. The first one is the *Weierstrass form* and the second one is the *Hesse form*. We will deal with the Hesse form in the next subsection. Let us start with the Weierstrass form.

By Proposition 1.1.17, $C = V(f)$ has an inflection point p_0. Without loss of generality, we may assume that $p_0 = [0,0,1]$ and the inflection tangent line at this point has the equation $t_0 = 0$. The projection from p_0 is the double cover $C \to \mathbb{P}^1$. It has four ramification points, the intersection points of C with the first polar. There are four tangent lines to C containing p_0. One of them is $V(t_0)$. The first polar $V(\frac{\partial f}{\partial t_2})$ of the point p_0 is a singular conic that intersects C at the tangency points of the four tangents, we immediately obtain that it consists of the line $V(t_0)$ and a line $V(t_2 + at_1 + bt_0)$ not passing through the point p_0. Changing the coordinates, we may assume that the line is equal to $V(t_2)$. Now, the equation of C takes the form

$$t_0 t_2^2 + \alpha t_1^3 + \beta t_1^2 t_0 + \gamma t_1 t_0^2 + \delta t_0^3 = 0,$$

where $\alpha \neq 0$. Replacing t_1 with $t_1 + \frac{\beta}{3\alpha} t_0$, and scaling the coordinates, we may assume that $\alpha = 1$ and $\beta = 0$. This gives us the *Weierstrass equation* of a nonsingular cubic:

$$t_0 t_2^2 + t_1^3 + a t_1 t_0^2 + b t_0^3 = 0. \tag{3.2}$$ weierstrass

It is easy to see that C is nonsingular if and only if the polynomial $x^3 + ax + b$ has no multiple roots, or, equivalently, its discriminant $\Delta = 4a^3 + 27b^2$ is not equal to zero.

Two Weierstrass equations define isomorphic elliptic curves if and only if there exists a projective transformation transforming one equation to another. It is easy to see that it happens if and only if $(\alpha', \beta') = (\lambda^3\alpha, \lambda^2\beta)$ for some nonzero constant λ. This can be expressed in terms of the *absolute invariant*

$$j = 2^6 3^3 \frac{4a^3}{4a^3 + 27b^2}. \tag{3.3}$$ absinv

Two elliptic curves are isomorphic if and only if their absolute invariants are equal. ∗

The projection $[t_0, t_1, t_2] \mapsto [t_0, t_1]$ exhibits C as a double cover of $\mathbb{P}^1$. Its ramification points are the intersection points of C and its polar conic $V(t_0 t_2)$. The cover has four branch points $[1, \lambda], [0, 1]$, where $\lambda^3 + a\lambda + b = 0$. The corresponding points $[1, \lambda, 0]$, and $[0, 0, 1]$ on C are the ramification points. If we choose $p_0 = [0, 0, 1]$ to be the zero point in the group law on C, then $2p \sim 2p_0$ for any ramification point p implies that p is a 2-torsion point. Any 2-torsion point is obtained in this way.

It follows from the computation above that any nonsingular plane cubic $V(f)$ is projectively isomorphic to the plane cubic $V(t_2^2 t_0 + t_1^3 + a t_1 t_0^2 + b t_0^3)$. The functions $S : f \mapsto a/27, T : f \mapsto 4b/27$ can be extended to the *Aronhold*

∗The coefficient $1728 = 2^6 3^3$ is needed to make this work in characteristic 2 and 3, otherwise j would not be defined for example when $a = 1, b = 0$ in characteristic 2.

invariants S and T of degrees 4 and 6 of a ternary cubic form. The explicit expressions of S and T in terms of the coefficients of f are rather long and can be found in many places (e.g. [242], [652]).

Fixing an order on the set of branch points, and replacing them by a projectively equivalent set, we may assume that the cubic polynomial $x^3 + ax + b$ is equal to $-x(x-1)(x-\lambda)$. This gives an affine equation of C

$$y^2 = x(x-1)(x-\lambda),$$

called the *Legendre equation.*

The number λ is equal to the cross ratio $R(q_1q_2; q_3q_4)$ of the four ordered branch points $(q_1, q_2, q_3, q_4) = (0, \lambda, 1, \infty)$. The absolute invariant (3.3) is expressed in terms of λ to give the following formula:

$$j = 2^8 \frac{(\lambda^2 - \lambda + 1)^3}{\lambda^2(\lambda - 1)^2}. \tag{3.4}$$

legendre

Remark 3.1.1. For any binary form $g(t_0, t_1)$ of degree 4 without multiple zeros, the equation

$$t_2^2 + g(t_0, t_1) = 0 \tag{3.5}$$

wew

defines an elliptic curve X in the weighted projective plane $\mathbb{P}(1, 1, 2)$. The four zeros of g are the branch points of the projection $X \to \mathbb{P}^1$ to the first two coordinates. So, every elliptic curve can be given by such an equation. The coefficients a, b in the Weierstrass equation are expressed in terms of the invariants S and T of binary quartics from Example 1.5.2. We have $a = -4S, b = -4T$. In particular.

$$j = \frac{27S(g)^3}{S(g)^3 - 27T(g)^2}.$$

Definition 3.1.2. *A nonsingular plane cubic $V(f)$ with Weierstrass Equation (3.2) is called* harmonic *(resp.* equianharmonic*) if $b = 0$ (resp. $a = 0$).*

We leave it to the reader to prove the following:

harmoniccubic **Theorem 3.1.3.** *Let $C = V(f)$ be a nonsingular plane cubic and c be any point on C. The following conditions are equivalent.*

(i) C is a harmonic (resp. equianharmonic cubic).
(ii) The absolute invariant $j = 1728$ (resp. $j = 0$).
(iii) The set of cross ratios of four roots of the polynomial $t_0(t_1^3 + at_1t_0^2 + bt_0^3)$ is equal to $\{-1, 2, \frac{1}{2}\}$ (resp. consists of two primitive cube roots of -1).
(iv) The group of automorphisms of C leaving the point c invariant is a cyclic group of order 4 (resp. 6).

Note that C is a harmonic cubic if and only if the invariant T of degree on the space of binary quartic forms (1.87) vanishes on the binary form g in Equation (3.5). A quartic binary form on which T vanishes is called a *harmonic binary quartic*. We know that a binary form g is harmonic if and only if it admits an apolar binary quadratic form. One can check that this form is nondegenerate if and only if g has no multiple zeros. In this case it can be written as a sum of two powers of linear forms $l_1^4 + l_2^4$. This exhibits an obvious symmetry of order 4. Changing coordinates we can reduce the form to $t_0^4 - t_1^4 = (t_0^2 + t_1^2)(t_0^2 - t_1^2)$. The pairs of zeros of the factors are harmonically conjugate pairs of points. This explains the name harmonic cubic.

Theorem 3.1.3 gives a geometric interpretation for the vanishing of the quadratic invariant S (1.87) on the space of binary quartics. It vanishes if and only if there exists a projective transformation of order 3 leaving the zeros of a binary forms invariant.

Another useful model of an elliptic curve is an elliptic normal quartic curve C in $\mathbb{P}^3$. There are two types of nondegenerate quartic curves in $\mathbb{P}^3$ that differ by the dimension of the linear system of quadrics containing the curve. In terminology of classical algebraic geometry, a space quartic curve is of the *first species* if the dimension is equal to 1, quartics of the *second species* are those which lie on a unique quadric. Elliptic curves are nonsingular quartics of the first species. The proof is rather standard (see, for example, [375]). By Lemma 8.6.1 from Chapter 8, we can write C as the intersection of two simultaneously diagonalized quadrics

$$Q_1 = V(\sum_{i=0}^{3} t_i^2), \quad Q_2 = V(\sum_{i=0}^{3} a_i t_i^2).$$

The pencil $\lambda Q_1 + \mu Q_2$ contains exactly four singular members corresponding to the parameters $[-a_i, 1], i = 0, 1, 2, 3$. The curve C is isomorphic to the double cover of $\mathbb{P}^1$ branched over these four points. This can be seen in many ways. Later, we will present one of them, a special case of Weil's Theorem on the intersection of two quadrics (the same proof can be found in Harris's book [375], Proposition 22.38). Changing a basis in the pencil of quadrics containing C, we can reduce the equations of C to the form

$$t_0^2 + t_1^2 + t_2^2 = t_1^2 + \lambda t_2^2 + t_3^2 = 0. \tag{3.6}$$

The absolute invariant of E is expressed via formula (3.4).

3.1.2 The Hesse equation

SS:3.1.2

Classical geometers rarely used Weierstrass equations. They preferred *Hesse's canonical equation* of a cubic curve:

$$t_0^3 + t_1^3 + t_2^3 + 6\alpha t_0 t_1 t_2 = 0. \tag{3.7}$$

hesse

Let us see that any nonsingular cubic can be reduced to this form by a linear change of variables.

Since any tangent line at an inflection point intersects the curve with multiplicity 3, applying (flexorder 1.24), we obtain that the curve has exactly 9 inflection points. Using the group law on an elliptic cubic curve with an inflection point $\mathfrak{o}$ as the zero, we can interpret any inflection point as a 3-torsion point. This, of course, agrees with the fact that the group $X[3]$ of 3-torsion points on an elliptic curve X is isomorphic to $(\mathbb{Z}/3\mathbb{Z})^2$.

Let H be a subgroup of order 3 of X. Since the elements from this group add up to zero, we see that the corresponding three inflection points p, q, r satisfy $p + q + r \sim 3\mathfrak{o}$. It is easy to see that the rational function on C with the divisor $p + q + r - 3\mathfrak{o}$ can be obtained as the restriction of the rational function $m(t_0, t_1, t_2)/l_0(t_0, t_1, t_2)$, where $V(m)$ defines the line containing the points p, q, r and $V(l_0)$ is the tangent to C at the point $\mathfrak{o}$. There are three cosets with respect to each subgroup H. Since the sum of elements in each coset is again equal to zero, we get 12 lines, each containing three inflection points. Conversely, if a line contains three inflection points, the sum of these points is zero, and it is easy to see that the three points form a coset with respect to some subgroup H. Each element of $(\mathbb{Z}/3\mathbb{Z})^3$ is contained in four cosets (it is enough to check this for the zero element).

A triangle containing the inflection points is called an *inflection triangle*. There are four inflection triangles and the union of their sides is the set of 12 lines from above. The configuration of 12 lines and 9 points, each line contains 3 points, and each point lies on four lines is the famous *Hesse arrangement of lines* $(12_3, 9_4)$.

Consider the polar conic of an inflection point. It splits into the union of the tangent line at the point and another line, called the *harmonic polar line* of the inflection point.

Lemma 3.1.4. *Let x be a point on an nonsingular cubic C. Any line ℓ passing through x intersects C at points y, z which are harmonically conjugate to the pair x, w, where w is the intersection point of the line and the conic polar $P_x(C)$.*

In Subsection SS:7.2.2 7.2.2 we will prove a more general statement where x is a point of multiplicity $d-2$ on an irreducible curve of degree d.

Proposition 3.1.5. *Let a, b, c be three collinear inflection points. The harmonic polar lines of three inflection points on a line ℓ intersect at the opposite vertex of the inflection triangle containing ℓ.*

Proof Let Δ be the inflection triangle with side ℓ containing the points a, b, c. Consider the three lines ℓ_i through a which join a with one of the inflection point x_i on the side of Δ. Let z_i be the other inflection point on ℓ_i (lying on the other side). By the previous Lemma, the harmonic polar line intersects each ℓ_i at a point y_i such that the cross ratio $R(ay_i; t_i z_i)$ is constant. This implies that the harmonic polar line is the line in the pencil of lines through the vertex that, together with the two sides and the line passing through a, make the same cross ratio in the pencil. Since the same is true for harmonic polar lines of the points b and c, we get the assertion. □

It follows from the previous proposition that the nine harmonic polar lines intersect by three at 12 edges of the inflection triangles, and each vertex belongs to four lines. This defines the *dual Hesse arrangement of lines* $(9_4, 12_3)$. It is combinatorially isomorphic to the arrangement of lines in the dual plane which is defined from the Hesse line arrangement via duality.

Now, it is easy to reduce a nonsingular cubic curve $C = V(f)$ to the Hesse canonical form. Choose coordinates such that one of the inflection triangles is the coordinate triangle. Let q be one of its vertices, say $q = [1,0,0]$, and x be an inflection point on the opposite line $V(t_0)$. Then, $P_x(C)$ is the union of the tangent to C at x and the harmonic polar of x. Since the latter passes through q, we have $P_{q^2x}(C) = P_{xq^2}(C) = 0$. Thus, the polar line $P_{q^2}(C)$ intersects the line $V(t_0)$ at three points. This can happen only if $P_{q^2}(C) = V(t_0)$. Hence, $V(\frac{\partial^2 f}{\partial t_0^2}) = V(t_0)$ and f has no terms $t_0^2 t_1, t_0^2 t_2$. We can write

$$f = at_0^3 + bt_1^3 + ct_2^2 + dt_0t_1t_2.$$

Since C is nonsingular, it is immediately checked that the coefficients a, b, c are not equal to zero. After scaling the coordinates, we arrive at the Hesse canonical form.

It is easy to check, by taking partials, that the condition that the curve given by the Hesse canonical form is nonsingular is

$$1 + 8\alpha^3 \neq 0. \tag{3.8}$$ discrm

By reducing the Hesse equation to a Weierstrass forms one can express the

Aronhold invariants S, T and the absolute invariant j in terms of the parameter α in (hesse 3.7):

$$\mathsf{S} = \alpha - \alpha^4, \tag{3.9}$$

$$\mathsf{T} = 1 - 20\alpha^3 - 8\alpha^6, \tag{3.10}$$

$$j = 2^{12}.3^3 \frac{(\alpha - \alpha^4)^3}{(1 + 8\alpha^3)^3}. \tag{3.11}$$

3.1.3 The Hesse pencil

Since the cubic C and its four inflection triangles pass through the same set of nine points, the inflection points of C, they belong to a pencil of cubic curves. This pencil is called the *Hesse pencil*. It is spanned by C and one of the inflection triangles, say the coordinate triangle. Thus, the Hesse pencil is defined by the equation

$$\lambda(t_0^3 + t_1^3 + t_2^3) + \mu t_0 t_1 t_2 = 0. \tag{3.12}$$ hessepencil

Its base points are

$$\begin{aligned} &[0, 1, -1], \quad [0, 1, -\epsilon], \quad [0, 1, -\epsilon^2], \\ &[1, 0, -1], \quad [1, 0, -\epsilon^2], \quad [1, 0, -\epsilon], \\ &[1, -1, 0], \quad [1, -\epsilon, 0], \quad [1, -\epsilon^2, 0], \end{aligned} \tag{3.13}$$ points

where $\epsilon = e^{2\pi i/3}$. They are the nine inflection points of any nonsingular member of the pencil. The singular members of the pencil correspond to the values of the parameters

$$(\lambda, \mu) = (0, 1),\ (1, -3),\ (1, -3\epsilon),\ (1, -3\epsilon^2).$$

The last three values correspond to the three values of α for which the Hesse equation defines a singular curve.

Any triple of lines containing the nine base points belongs to the pencil and forms its singular member. Here, they are:

$$\begin{gathered} V(t_0), \quad V(t_1), \quad V(t_2), \\ V(t_0 + t_1 + t_2),\ V(t_0 + \epsilon t_1 + \epsilon^2 t_2),\ V(t_0 + \epsilon^2 t_1 + \epsilon t_2), \\ V(t_0 + \epsilon t_1 + t_2),\ V(t_0 + \epsilon^2 t_1 + \epsilon^2 t_2),\ V(t_0 + t_1 + \epsilon t_2), \\ V(t_0 + \epsilon^2 t_1 + t_2),\ V(t_0 + \epsilon t_1 + \epsilon t_2),\ V(t_0 + t_1 + \epsilon^2 t_2). \end{gathered} \tag{3.14}$$ lines1

We leave it to a suspicious reader to check that

$$(t_0+t_1+t_2)(t_0+\epsilon t_1+\epsilon^2 t_2)(t_0+\epsilon^2 t_1+\epsilon t_2)=t_0^3+t_1^3+t_2^3-3t_0t_1t_2,$$

$$(t_0+\epsilon t_1+t_2)(t_0+\epsilon^2 t_1+\epsilon^2 t_2)(t_0+t_1+\epsilon t_2)=t_0^3+t_1^3+t_2^3-3\epsilon t_0t_1t_2,$$

$$(t_0+\epsilon^2 t_1+t_2)(t_0+\epsilon t_1+\epsilon t_2)(t_0+t_1+\epsilon^2 t_2)=t_0^3+t_1^3+t_2^3-3\epsilon^2 t_0t_1t_2.$$

The 12 lines (3.14) and nine inflection points (3.13) form the Hesse configuration corresponding to any nonsingular member of the pencil.

Choose $[0,1,-1]$ to be the zero point in the group law on C. Then, we can define an isomorphism of groups $\phi:(\mathbb{Z}/3\mathbb{Z})^2\to X[3]$ by sending $[1,0]$ to $[0,1,-\epsilon]$, $[0,1]$ to $[1,0,-1]$. The points of the first row in (3.13) are the subgroup H generated by $\phi([1,0])$. The points of the second row are the coset of H containing $\phi([0,1])$.

Remark 3.1.6. Note that, varying α in $\mathbb{P}^1\setminus\{-\frac{1}{2},-\frac{\epsilon}{2},-\frac{\epsilon^2}{2},\infty\}$, we obtain a family of elliptic curves X_α defined by Equation (3.7) with a fixed isomorphism $\phi_\alpha:(\mathbb{Z}/3\mathbb{Z})^2\to X_\alpha[3]$. After blowing up the 9 base points, we obtain a rational surface $S(3)$

$$f:S(3)\to\mathbb{P}^1 \qquad (3.15)$$ modfam

defined by the rational map $\mathbb{P}^2\dashrightarrow\mathbb{P}^1$, $[t_0,t_1,t_2]\mapsto[t_0t_1t_2,t_0^3+t_1^3+t_2^3]$. The fiber of f over a point $(a,b)\in\mathbb{P}^2$ is isomorphic to the member of the Hesse pencil corresponding to $(\lambda,\mu)=(-b,a)$. It is known that (3.15) is a *modular family* of elliptic curves with level 3, i.e. the universal object for the fine moduli space of pairs (X,ϕ), where X is an elliptic curve and $\phi:(\mathbb{Z}/3\mathbb{Z})^2\to X[3]$ is an isomorphism of groups. There is a canonical isomorphism $\mathbb{P}^1\cong Y$, where Y is the modular curve of level 3, i.e. a nonsingular compactification of the quotient of the upper half-plane $\mathcal{H}=\{a+bi\in\mathbb{C}:b>0\}$ by the group

$$\Gamma(3)=\{A=\begin{pmatrix}a&b\\c&d\end{pmatrix}\in\mathrm{SL}(2,\mathbb{Z}):A\equiv I_3\mod 3\},$$

which acts on $\mathcal{H}$ by Möbius transformations $z\mapsto\frac{az+b}{cz+d}$. The boundary of $H/\Gamma(3)$ in Y consists of four points (the cusps). They correspond to the singular members of the Hesse pencil.

3.1.4 The Hesse group

SS:3.1.4

The *Hesse group* G_{216} is the group of projective transformations that preserve the Hesse pencil of cubic curves. First, we see the obvious symmetries generated by the transformations

$$\tau : [t_0, t_1, t_2] \mapsto [t_0, \epsilon_3 t_1, \epsilon_3^2 t_2],$$
$$\sigma : [t_0, t_1, t_2] \mapsto [t_2, t_0, t_1].$$

They define a projective representation of the group $(\mathbb{Z}/3\mathbb{Z})^2$.

If we fix the group law by taking the origin to be $[0, 1, -1]$, then τ induces on each nonsingular fiber the translation automorphism by the point $[0, 1, -\epsilon]$ and σ is the translation by the point $[1, 0, -1]$.

Theorem 3.1.7. *The Hesse group G_{216} is a group of order 216 isomorphic to the semi-direct product*

$$(\mathbb{Z}/3\mathbb{Z})^2 \rtimes \mathrm{SL}(2, \mathbb{F}_3),$$

where the action of $\mathrm{SL}(2, \mathbb{F}_3)$ *on* $(\mathbb{Z}/3\mathbb{Z})^2$ *is the natural linear representation.*

Proof Let $g \in G_{216}$. It transforms a member of the Hesse pencil to another member. This defines a homomorphism $G_{216} \to \mathrm{Aut}(\mathbb{P}^1)$. An element of the kernel K leaves each member of the pencil invariant. In particular, it leaves invariant the curve $V(t_0t_1t_2)$. The group of automorphisms of this curve is generated by homotheties $[t_0, t_1, t_2] \mapsto [t_0, at_1, bt_2]$ and permutation of coordinates. Suppose σ induces a homothety. Since it also leaves invariant the curve $V(t_0^3 + t_1^3 + t_2^3)$, we must have $1 = a^3 = b^3$. To leave invariant a general member we also need that $a^3 = b^3 = bc$. This implies that g belongs to the subgroup generated by the transformation σ. An even permutation of coordinates belongs to a subgroup generated by the transformation τ. The odd permutation $\sigma_0 : [t_0, t_1, t_2] \mapsto [t_0, t_2, t_1]$ acts on the group of 3-torsion points of each nonsingular fiber as the negation automorphism $x \mapsto -x$. Thus, we see that

$$K \cong (\mathbb{Z}/3\mathbb{Z})^2 \rtimes \langle \sigma_0 \rangle.$$

Now, let I be the image of the group G_{216} in $\mathrm{Aut}(\mathbb{P}^1)$. It acts by permuting the four singular members of the pencil and thus leaves the set of zeros of the binary form

$$\Delta = (8t_1^3 + t_0^3)t_0$$

invariant. It follows from the invariant theory that this implies that H is a subgroup of $\mathfrak{A}_4$. We claim that $H = \mathfrak{A}_4$. Consider the projective transformations given by the matrices

$$\sigma_1 = \begin{pmatrix} 1 & 1 & 1 \\ 1 & \varepsilon & \varepsilon^2 \\ 1 & \varepsilon^2 & \varepsilon \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 1 & \varepsilon & \varepsilon \\ \varepsilon^2 & \varepsilon & \varepsilon^2 \\ \varepsilon^2 & \varepsilon^2 & \varepsilon \end{pmatrix}.$$

The transformations $\sigma_0, \sigma_1, \sigma_2$ generate a subgroup isomorphic to the quaternion group Q_8 with center generated by σ_0. The transformation

$$\sigma_3 : [t_0, t_1, t_2] \mapsto [\varepsilon t_0, t_2, t_1]$$

satisfies $\sigma_3^3 = \sigma_0$. It acts by sending a curve C_α from (3.7) to $C_{\varepsilon\alpha}$. It is easy to see that the transformations $\sigma_1, \sigma_2, \sigma_3, \tau$ generate the group isomorphic to $\mathrm{SL}(2, \mathbb{F}_3)$. Its center is (σ_0) and the quotient by the center is isomorphic to $\mathfrak{A}_4$. In other words, this group is the binary tetrahedral group. Note that the whole group can be generated by transformations $\sigma, \tau, \sigma_0, \sigma_1$. □

Recall that a linear operator $\sigma \in \mathrm{GL}(E)$ of a complex vector space E of dimension $n+1$ is called a *complex reflection* if it is of finite order and the rank of $\sigma - \mathrm{id}_E$ is equal to 1. The kernel of $\sigma - \mathrm{id}_E$ is a hyperplane in E, called the *reflection hyperplane* of σ. It is invariant with respect to σ and its stabilizer subgroup is a cyclic group. A *complex reflection group* is a finite subgroup G of $\mathrm{GL}(E)$ generated by complex reflections. One can choose a unitary inner product on E such that any complex reflection σ from E can be written in the form

$$s_{v,\eta} : x \mapsto x + (\eta - 1)(x, v)v,$$

where v is a vector of norm 1 perpendicular to the reflection hyperplane H_v of σ, and η is a nontrivial root of unity of order equal to the order of σ.

Recall the basic facts about complex reflection groups (see, for example, [719]):

- The algebra of invariants $S(E)^G \cong \mathbb{C}[t_0, \ldots, t_n]^G$ is freely generated by $n+1$ invariant polynomials $f_0, \ldots, f_n$ (geometrically, $E/G \cong \mathbb{C}^{n+1}$).
- The product of degrees d_i of the polynomials $f_0, \ldots, f_n$ is equal to the order of G.
- The number of complex reflections in G is equal to $\sum(d_i - 1)$.

All complex reflection groups were classified by G. Shephard and J. Todd [710]. There are five conjugacy classes of complex reflection subgroups of $\mathrm{GL}(3, \mathbb{C})$. Among them is the group G isomorphic to a central extension of degree 3 of the Hesse group. It is generated by complex reflections $s_{v,\eta}$ of order 3, where the reflection line H_v is one of the 12 lines (3.14) in $\mathbb{P}^2$ and v is the unit normal vector (a, b, c) of the line $V(at_0 + bt_1 + ct_2)$. Note that each reflection $s_{v,\eta}$ leaves invariant the hyperplanes with a normal vector orthogonal to v. For example, $s_{(1,0,0),\varepsilon}$ leaves invariant the line $V(t_0)$. This implies that each of the 12 complex reflections leave the Hesse pencil invariant. Thus, the image of G in $\mathrm{PGL}(3, \mathbb{C})$ is contained in the Hesse group. It follows from the classification of complex reflection groups (or could be checked directly, see Springer [719]) that it is equal to the Hesse group, and the subgroup of scalar matrices from G is a cyclic group of order 3.

Each of the 12 reflection lines defines two complex reflections. This gives 24 complex reflections in G. This number coincides with the number of elements of order 3 in the Hesse group, so, there are no more complex reflections in G. Let $d_1 \leq d_2 \leq d_3$ be the degrees of the invariants generating the algebra of invariants of G. We have $d_1 + d_2 + d_3 = 27, d_1d_2d_3 = 648$. This easily gives $d_1 = 6, d_2 = 9, d_3 = 12$. There are obvious reducible curves of degrees 9 and 12 in $\mathbb{P}^2$ invariant with respect to G. The curve of degree 9 is the union of the polar harmonic lines. Each line intersects a nonsingular member of the pencil at nontrivial 2-torsion points with respect to the group law defined by the corresponding inflection point. The equation of the union of nine harmonic polar lines is

$$f_9 = (t_0^3 - t_1^3)(t_0^3 - t_2^3)(t_1^3 - t_2^3) = 0. \tag{3.16}$$ inv9

The curve of degree 12 is the union of the 12 lines lines1 (3.14). Its equation is

$$f_{12} = t_0t_1t_2[27t_0^3t_1^3t_2^3 - (t_0^3 + t_1^3 + t_2^3)^3] = 0. \tag{3.17}$$ inv12

A polynomial defining an invariant curve is a *relative invariant* of G (it is an invariant with respect to the group $G' = G \cap \mathrm{SL}(3, \mathbb{C})$). One checks that the polynomial f_9 is indeed an invariant, but the polynomial f_{12} is only a relative invariant. So, there exists another curve of degree 12 whose equation defines an invariant of degree 12. Recall that the Hesse group acts on the base of the Hesse pencil via the action of the tetrahedron group $\mathfrak{A}_4$. It has three orbits with stabilizers of order 2,3 and 3. The first orbit consists of six points; the fibers over these points are harmonic cubics. The second orbit consists of four points such that the fibers over these points are equianharmonic cubics. The third orbit consists of four points corresponding to singular members of the pencil. It is not difficult to check that the product of the equations of the equianharmonic cubics defines an invariant of degree 12. Its equation is

$$f_{12}' = (t_0^3 + t_1^3 + t_2^3)[(t_0^3 + t_1^3 + t_2^3)^3 + 216t_0^3t_1^3t_2^3] = 0. \tag{3.18}$$ inv122

An invariant of degree 6 is

$$f_6 = 7(t_0^6 + t_1^6 + t_2^6) - 6(t_0^3 + t_2^3 + t_3^3)^2. \tag{3.19}$$ inv6

The product of the equations defining 6 harmonic cubics is an invariant of degree 18

$$f_{18} = (t_0^3 + t_1^3 + t_2^3)^6 - 540t_0^3t_1^3t_2^3(t_0^3 + t_1^3 + t_2^3)^3 - 5832t_0^6t_1^6t_2^6 = 0. \tag{3.20}$$ inv18

S:3.2

3.2 Polars of a Plane Cubic

SS:3.2.1

3.2.1 The Hessian of a plane cubic

Let $X = V(f)$ be a cubic hypersurface in $\mathbb{P}^n$. We know that the Hessian $\mathrm{He}(X)$ is the locus of points $a \in \mathbb{P}^n$ such that the polar quadric $P_a(X)$ is singular. Also, we know that, for any $a \in \mathrm{He}(X)$,

$$\mathrm{Sing}(P_a(X)) = \{b \in \mathbb{P}^2 : D_b(D_a(f)) = 0\}.$$

Since $P_b(P_a(X)) = P_a(P_b(X))$ we obtain that $b \in \mathrm{He}(X)$.

Theorem 3.2.1. *The Hessian* $\mathrm{He}(X)$ *of a cubic hypersurface X contains the Steinerian* $\mathrm{St}(X)$. *If* $\mathrm{He}(X) \neq \mathbb{P}^n$, *then*

$$\mathrm{He}(X) = \mathrm{St}(X).$$

For the last assertion, one only needs to compare the degrees of the hypersurfaces. They are equal to $n + 3$.

In particular, the rational map, if defined,

$$\mathrm{st}_X^{-1} : \mathrm{St}(X) \to \mathrm{He}(X), a \mapsto \mathrm{Sing}(P_a(X)) \tag{3.21}$$ stcub

is a birational automorphism of the Hessian hypersurface. We have noticed this already in Chapter 1.

Proposition 3.2.2. *Assume X has only isolated singularities. Then,* $\mathrm{He}(X) = \mathbb{P}^n$ *if and only if X is a cone over a cubic hypersurface in* $\mathbb{P}^{n-1}$.

Proof Let $W = \{(a, b) \in \mathbb{P}^n \times \mathbb{P}^n : P_{a,b^2}(X) = 0\}$. For each $a \in \mathbb{P}^n$, the fiber of the first projection over the point a is equal to the first polar $P_a(X)$. For any $b \in \mathbb{P}^n$, the fiber of the second projection over the point b is equal to the second polar $P_{b^2}(X) = V(\sum \partial_i f(b)t_i)$. Let $U = \mathbb{P}^n \setminus \mathrm{Sing}(X)$. For any $b \in U$, the fiber of the second projection is a hyperplane in $\mathbb{P}^n$. This shows that $p_2^{-1}(U)$ is nonsingular. The restriction of the first projection to U is a morphism of nonsingular varieties. The general fiber of this morphism is a regular scheme over the general point of $\mathbb{P}^n$. Since we are in characteristic 0, it is a smooth scheme. Thus, there exists an open subset $W \subset \mathbb{P}^n$ such that $p_1^{-1}(W) \cap U$ is nonsingular. If $\mathrm{He}(X) = 0$, all polar quadrics $P_a(X)$ are singular, and a general polar must have singularities inside of $p_2^{-1}(\mathrm{Sing}(X))$. This means that $p_1(p_2^{-1}(\mathrm{Sing}(X))) = \mathbb{P}^n$. For any $x \in \mathrm{Sing}(X)$, all polar quadrics contain x and either all of them are singular at x or there exists an open subset $U_x \subset \mathbb{P}^n$ such that all quadrics $P_a(X)$ are nonsingular at x for $a \in U_x$. Suppose that, for any $x \in \mathrm{Sing}(X)$, there exists a polar quadric that is nonsingular at x. Since the number of isolated singular points is finite, there will be an open set of points $a \in \mathbb{P}^n$ such that the fiber $p_1^{-1}(a)$ is nonsingular in $p_2^{-1}(\mathrm{Sing}(X))$. This is a contradiction. Thus, there exists a point $c \in \mathrm{Sing}(X)$ such that all polar quadrics are singular at x, therefore, c is a common solution of the systems of linear equations $\mathrm{He}(f_3)(a) \cdot X = 0, a \in \mathbb{P}^n$. Thus, the first partials of f_3 are linearly dependent. Now, we apply Proposition 1.1.6 to obtain that X is a cone. □

Remark 3.2.3. The example of a cubic hypersurface in $\mathbb{P}^4$ from Remark 1.1.16 shows that the assumption on singular points of X cannot be weakened. The singular locus of the cubic hypersurface is the plane $t_0 = t_1 = 0$.

Consider a plane cubic $C = V(f)$ with equation in the Hesse canonical form (3.7). The partials of $\frac{1}{3}f$ are

$$t_0^2 + 2\alpha t_1 t_2, \quad t_1^2 + 2\alpha t_0 t_2, \quad t_2^2 + 2\alpha t_0 t_1. \tag{3.22}$$

partialscubic

Thus, the Hessian of C has the following equation:

$$\mathrm{He}(C) = \begin{vmatrix} t_0 & \alpha t_2 & \alpha t_1 \\ \alpha t_2 & t_1 & \alpha t_0 \\ \alpha t_1 & \alpha t_0 & t_2 \end{vmatrix} = (1 + 2\alpha^3) t_0 t_1 t_2 - \alpha^2 (t_0^3 + t_1^3 + t_2^3). \tag{3.23}$$

In particular, the Hessian of the member of the Hesse pencil corresponding to the parameter $(\lambda, \mu) = (1, 6\alpha),\ \alpha \neq 0$, is equal to

$$t_0^3 + t_1^3 + t_2^3 - \frac{1 + 2\alpha^3}{\alpha^2} t_0 t_1 t_2 = 0, \tag{3.24}$$

hessianeq

or, if $(\lambda, \mu) = (1, 0)$ or $(0, 1)$, then the Hessian is equal to $V(t_0 t_1 t_2)$.

Lsing **Lemma 3.2.4.** *Let C be a nonsingular cubic in a Hesse's canonical form. The following assertions are equivalent:*

(i) $\dim \mathrm{Sing}(P_a(C)) > 0$*;*
(ii) $a \in \mathrm{Sing}(\mathrm{He}(C))$*;*
(iii) $\mathrm{He}(C)$ *is the union of three nonconcurrent lines;*
(iv) *C is isomorphic to the Fermat cubic* $V(t_0^3 + t_1^3 + t_1^3)$*;*
(v) $\mathrm{He}(C)$ *is a singular cubic;*
(vi) *C is an equianharmonic cubic;*
(vii) $\alpha(\alpha^3 - 1) = 0$.

Proof Use the Hesse equation for a cubic and for its Hessian. We see that $\mathrm{He}(C)$ is singular if and only if either $\alpha = 0$ or $1 + 8(-\frac{1+2\alpha^3}{6\alpha^2})^3 = 0$. Obviously, $\alpha = 1$ is a solution of the second equation. Other solutions are ϵ, ϵ^2. This corresponds to $\mathrm{He}(C)$, where C is of the form $V(t_0^3+t_1^3+t_1^3)$, or is given by the equation

$$t_0^3+t_1^3+t_2^3+6\epsilon^i t_0t_1t_2 = (\epsilon^i t_0+\epsilon t_1+t_2)^3+(t_0+\epsilon^i t_1+t_2)^3$$
$$+(t_0+t_1+\epsilon^i t_2)^3 = 0,$$

where $i = 1, 2$, or

$$t_0^3+t_1^3+t_2^3+6t_0t_1t_2 = (t_0+t_1+t_2)^3+(t_0+\epsilon t_1+\epsilon^2 t_2)^3$$
$$+(t_0+\epsilon^2 t_1+\epsilon t_2)^3 = 0.$$

This computation proves the equivalence of (iii), (iv), (v), and (vii).

Assume (i) holds. Then, the rank of the Hessian matrix is equal to 1. It is easy to see that the first two rows are proportional if and only if $\alpha(\alpha^3-1)=0$. Thus, (i) is equivalent to (vii), and hence to (iii), (iv), (v) and (vii). The point a is one of the three intersection points of the lines such that the cubic is equal to the sum of the cubes of linear forms defining these lines. Direct computation shows that (ii) holds. Thus, (i) implies (ii).

Assume (ii) holds. Again the previous computations show that $\alpha(\alpha^3-1)=0$ and the Hessian curve is the union of three lines. Now, (i) is directly verified.

The equivalence of (iv) and (vi) follows from Theorem 3.1.3 since the transformation $[t_0,t_1,t_2] \to [t_1,t_0,e^{2\pi i/3}t_2]$ generates a cyclic group of order 6 of automorphisms of C leaving the point $[1,-1,0]$ fixed. □

involution **Corollary 3.2.5.** *Assume that $C = V(f)$ is not projectively isomorphic to the Fermat cubic. Then, the Hessian cubic is nonsingular, and the map $a \mapsto$* $\mathrm{Sing}(P_a(C))$ *is an involution on* $\mathrm{He}(C)$ *without fixed points.*

Proof The only unproved assertion is that the involution does not have fixed points. A fixed point a has the property that $D_a(D_a(f)) = D_{a^2}(f) = 0$. It follows from Theorem 1.1.5 that this implies that $a \in \mathrm{Sing}(C)$. □

lambdamap *Remark* 3.2.6. Consider the Hesse pencil of cubics with parameters $(\lambda,\mu) = (\alpha_0, 6\alpha_1)$

$$C_{(\alpha_0,\alpha)} = V(\alpha_0(t_0^3+t_1^3+t_2^3)+6\alpha_1 t_0t_1t_2).$$

Taking the Hessian of each curve from the pencil we get the pencil

$$H_{(\alpha_0,\alpha)} = V(\alpha_0 t_0^3+t_1^3+t_2^3+6\alpha_1 t_0t_1t_2).$$

The map $C_{(\alpha_0,\alpha)} \to H_{(\alpha_0,\alpha)}$ defines a regular map

$$\mathfrak{h} : \mathbb{P}^1 \to \mathbb{P}^1, \quad [\alpha_0,\alpha_1] \mapsto [t_0,t_1] = [-\alpha_0\alpha_1^2, \alpha_0^3+2\alpha_1^3]. \qquad (3.25)$$ lamdamap

This map is of degree 3. For a general value of the inhomogeneous parameter $\lambda = t_1/t_0$, the pre-image consists of three points with inhomogeneous coordinate $\alpha = \alpha_1/\alpha_0$ satisfyfing the cubic equation

$$6\lambda\alpha^3 - 2\alpha^2 + 1 = 0. \tag{3.26}$$

We know that the points $[\alpha_0, \alpha_1] = [0, 1], [1, -\frac{1}{2}], [1, -\frac{\epsilon}{2}], [1, -\frac{\epsilon^2}{2}]$ correspond to singular members of the λ-pencil. These are the branch points of the map $\mathfrak{h}$. Over each branch point we have two points in the pre-image. The points

$$(\alpha_0, \alpha_1) = [1, 0], [1, 1], [1, \epsilon], [1, \epsilon^2]$$

are the ramification points corresponding to equianharmonic cubics. A nonramication point in the pre-image corresponds to a singular member.

Let $C_\alpha = C_{(1,\alpha)}$. If we fix a group law on a $H_\alpha = \mathrm{He}(C_\alpha)$, we will be able to identify the involution described in Corollary 3.2.5 with the translation automorphism by a nontrivial 2-torsion point η. Given a nonsingular cubic curve H together with a fixed-point-free involution τ, there exists a unique nonsingular cubic C_α such that $H = H_\alpha$ and the involution τ is the involution described in the corollary. Thus, the three roots of Equation (3.26) can be identified with 3 nontrivial torsion points on H_α. We refer the reader to Exercise 3.2 for a reconstruction of C_α from the pair (H_α, η).

Recall that the Cayleyan curve of a plane cubic C is the locus of lines $\overline{pq}$ in the dual plane such that $a \in \mathrm{He}(C)$ and b is the singular point of $P_a(C)$. Each such line intersects $\mathrm{He}(C)$ at three points a, b, c. The following gives the geometric meaning of the third intersection point.

Proposition 3.2.7. *Let c be the third intersection point of a line $\ell \in \mathrm{Cay}(C)$ and $\mathrm{He}(C)$. Then, ℓ is a component of the polar $P_d(C)$ whose singular point is c. The point d is the intersection point of the tangents of $\mathrm{He}(C)$ at the points a and b.*

Proof From the general theory of linear systems of quadrics, applied to the net of polar conics of C, we know that ℓ is a Reye line, i.e. it is contained in some polar conic $P_d(C)$ (see Subsection 1.1.7). The point d must belong to $\mathrm{He}(C)$ and its singular point c belongs to ℓ. Thus, c is the third intersection point of ℓ with C.

It remains for to prove the last assertion. Choose a group law on the curve $\mathrm{He}(C)$ by fixing an inflection point as the zero point. We know that the Steinerian involution is defined by the translation $x \mapsto x \oplus \eta$, where η is a fixed 2-torsion point. Thus, $b = a \oplus \eta$. It follows from the definition of the group law on a nonsingular cubic that the tangents $\mathbb{T}_a(\mathrm{He}(C))$ and $\mathbb{T}_b(\mathrm{He}(C))$ intersect at a point d on $\mathrm{He}(C)$. We have $d \oplus 2a = 0$, hence $d = -2a$. Since a, b, c lie on a line, we get $c = -a - b$ in the group law. After subtracting, we get $d - c = b - a = \eta$. Thus, the points x and c are an orbit of the Steinerian involution. This shows that c is the singular point of $P_d(C)$. By Proposition 1.2.5, $P_d(C)$ contains the points a, b. Thus, $\overline{ab}$ is a component of $P_d(C)$. □

It follows from the Proposition from above that the Cayleyan curve of a nonsingular cubic C parameterizes the line components of singular polar conics of C. It is also isomorphic to the quotient of $\mathrm{He}(C)$ by the Steinerian involution from Corollary 3.2.5. Since this involution does not have fixed points, the quotient map $\mathrm{He}(C) \to \mathrm{Cay}(C)$ is an unramified cover of degree two. In particular, $\mathrm{Cay}(C)$ is a nonsingular curve of genus one.

Let us find the equation of the Cayleyan curve. A line ℓ belongs to $\mathrm{Cay}(X)$ if and only if the restriction of the linear system of polar conics of X to ℓ is of dimension 1. This translates into the condition that the restriction of the partials of X to ℓ is a linearly dependent set of three binary forms. So, write ℓ in the parametric form as the image of the map $\mathbb{P}^1 \to \mathbb{P}^2$ given by $[u, v] \mapsto [a_0u + b_0v, a_1u + b_1v, a_2u + b_2v]$. The condition of the linear dependence is given by

$$\det\begin{pmatrix} a_0^2 + 2\alpha a_1a_2 & 2a_0b_0 + 2\alpha(a_1b_2 + a_2b_1) & b_0^2 + 2\alpha b_1b_2 \\ a_1^2 + 2\alpha a_0a_2 & 2a_1b_1 + 2\alpha(a_0b_2 + a_2b_0) & b_1^2 + 2\alpha b_0b_2 \\ a_2^2 + 2\alpha a_0a_1 & 2a_2b_2 + 2\alpha(a_0b_1 + a_1b_0) & b_2^2 + 2\alpha b_0b_1 \end{pmatrix} = 0.$$

The coordinates of ℓ in the dual plane are

$$[u_0, u_1, u_2] = [a_1b_2 - a_2b_1, a_2b_0 - a_0b_2, a_0b_1 - a_1b_0].$$

Computing the determinant, we find that the equation of $\mathrm{Cay}(X)$ in the coordinates u_0, u_1, u_2 is

$$u_0^3 + u_1^3 + u_2^3 + 6\alpha' u_0u_1u_2 = 0, \tag{3.27}$$

where $\alpha' = (1 - 4\alpha^3)/6\alpha$. Note that this agrees with the degree of the Cayleyan curve found in Proposition 1.1.26. Using formula (3.9) for the absolute invariant of the curve, this can be translated into an explicit relationship between the absolute invariant of an elliptic curve C and the isogenous elliptic curve $C/(\tau_e)$, where τ_e is the translation automorphism by a nontrivial 2-torsion point e.

Remark 3.2.8. The iterations of the map $\mathfrak{h}$ from (3.2.6) was studied in [406] and [101]. It is an interesting example of a complex dynamic in one variable. The critical points of $\mathfrak{h}$ are the four equianharmonic cubics in the pencil and its critical values are the four triangles. Since equianharmonic cubics are mapped under $\mathfrak{h}$ to the critical values, the map $\mathfrak{h}$ is a critically finite map in the sense of Thurston.

SS:3.2.3

3.2.2 The dual curve

Write the equation of a general line in the form $t_2 = u_0t_0 + u_1t_1$ and plug in Equation (3.7). The corresponding cubic equation has a multiple root if and only if the line is a tangent. We have

$$(u_0t_0 + u_1t_1)^3 + t_0^3 + t_1^3 + 6\alpha t_0t_1(u_0t_0 + u_1t_1)$$

$$= (u_0^3 + 1)t_0^3 + (u_1^3 + 1)t_1^3 + (3u_0^2u_1 + 6\alpha u_0)t_0^2t_1 + (3u_0u_1^2 + 6\alpha u_1)t_0t_1^2 = 0.$$

The condition that there is a multiple root is that the discriminant of the homogeneous cubic form in t_0, t_1 is equal to zero. Using the formula (1.86) for the discriminant of a cubic form, after plugging in, we obtain

$$(3u_0^2u_1 + 6\alpha u_0)^2(3u_0u_1^2 + 6\alpha u_1)^2 + 18(3u_0^2u_1 + 6\alpha u_0)(3u_0u_1^2 + 6\alpha u_1)(u_0^3 + 1)(u_1^3 + 1)$$

$$-4(u_0^3 + 1)(3u_0u_1^2 + 6\alpha u_1)^3 - 4(u_1^3 + 1)(3u_1u_0^2 + 6\alpha u_0)^3 - 27(u_0^3 + 1)^2(u_1^3 + 1)^2$$

$$= -27 + 864u_0^3u_1^3\alpha^3 + 648u_0^2u_1^2\alpha - 648\alpha^2u_0u_1^4 - 648\alpha^2u_0^4u_1 + 648\alpha^2u_0u_1$$

$$+1296\alpha^4u_0^2u_1^2 - 27u_1^6 - 27u_0^6 + 54u_0^3u_1^3 - 864u_1^3\alpha^3 - 864u_0^3\alpha^3 - 54u_1^3 - 54u_0^3 = 0.$$

It remains for us to homogenize the equation and divide by (-27) to obtain the equation of the dual curve

$$\begin{aligned} &u_0^6 + u_1^6 + u_2^6 - (2 + 32\alpha^3)(u_0^3u_1^3 + u_0^3u_2^3 + u_2^3u_1^3) \\ &-24\alpha^2u_0u_1u_2(u_0^3 + u_1^3 + u_2^3) - (24\alpha + 48\alpha^4)u_0^2u_1^2u_2^2 = 0. \end{aligned} \tag{3.28}$$

eqdualcubic

According to the Plücker formula (1.51), the dual curve of a nonsingular plane cubic has nine cusps. They correspond to the inflection tangents of the original curve. The inflection points are given in (3.12). Computing the equations of the tangents, we find the following singular points of the dual curve:

$$[-2m, 1, 1],\ [1, -2\alpha, 1],\ [1, 1, -2\alpha],\ [-2\alpha\varepsilon, \varepsilon^2, 1],\ [-2\alpha\varepsilon, 1, \varepsilon^2],$$

$$[\varepsilon^2, -2\alpha\varepsilon, 1],\ [1, -2\alpha\varepsilon, \varepsilon^2],\ [1, \varepsilon^2, -2\alpha\varepsilon],\ [\varepsilon^2, 1, -2\alpha].$$

The tangent of C at an inflection point a is a component of the polar conic $P_a(C)$, hence connects a to the singular point of the polar conic. This implies that the tangent line belongs to the Cayleyan curve $\mathrm{Cay}(C)$, hence the Cayleyan curve contains the singular points of the dual cubic. The pencil of plane curves of degree 6 spanned by the dual cubic $C^\vee$ and the Cayleyan cubic taken with multiplicity 2 is an example of an *Halphen pencil* of index 2 of curves of degree 6 with nine double base points (see Exercises to Chapter 7).

3.2.3 Polar s-gons

Since, for any three general points in $\mathbb{P}^2$, there exists a plane cubic singular at these points (the union of three lines), a general ternary cubic form does not admit polar triangles. Of course this is easy to see by counting constants.

By Lemma Lsing 3.2.4, a nonsingular cubic admits a polar triangle if and only if it is an equianharmonic cubic. Its polar triangle is unique. Its sides are the three first polars of C which are double lines.

cubtri **Proposition 3.2.9.** *A plane cubic admits a polar triangle if and only if either it is a Fermat cubic or it is equal to the union of three distinct concurrent lines.*

Proof Suppose $C = V(l_1^3+l_2^3+l_3^3)$. Without loss of generality, we may assume that l_1 is not proportional to l_2. Thus, after a linear change of coordinates, $C = V(t_0^3 + t_1^3 + l^3)$. If $l(t_0, t_1, t_2)$ does not depend on t_2, the curve C is the union of three distinct concurrent lines. Otherwise, we can change coordinates to assume that $l = t_2$ and get a Fermat cubic. □

By counting constants, a general cubic admits a *polar quadrangle*. It is clear that a polar quadrangle $\{[l_1], \dots, [l_4]\}$ is nondegenerate if and only if the linear system of conics in the dual plane through the points $[l_i]$ is an irreducible pencil (i.e. a linear system of dimension 1 whose general member is irreducible). This allows us to define a *nondegenerate generalized polar quadrangle* of C as a generalized quadrangle Z of C such that $|\mathcal{I}_Z(2)|$ is an irreducible pencil.

Let $g(t_0, t_1)$ be a binary form of degree 3. Its polar 3-hedron is the divisor of zeros of its apolar form of degree 3. Thus,

$$\mathrm{VSP}(g, 3) \cong |\mathrm{AP}_3(g)|^\vee \cong \mathbb{P}^2. \tag{3.29}$$ binap

This implies that any ternary cubic form $f = t_2^3 + g(t_0, t_1)$ admits degenerate polar quadrangles.

Also, if $C = V(g(t_0, t_1))$ is the union of three concurrent lines then any four distinct nonzero linear forms l_1, l_2, l_3, l_4 form a degenerate quadrangle of C. In fact, using the Van der Monde determinant, we obtain that the cubes $l_1^3, l_2^3, l_3^3, l_4^3$ form a basis in the space of binary cubic forms. So, the variety of sums of four powers of C is isomorphic to the variety of four distinct points in $\mathbb{P}^1$. Its closure $\mathrm{VSP}(C, 4)$ in the Hilbert scheme $\mathrm{Hilb}^4(\mathbb{P}^2)$ is isomorphic to $(\mathbb{P}^1)^{(4)} \cong \mathbb{P}^4$.

gener1 **Lemma 3.2.10.** *C admits a degenerate polar quadrangle if and only if it is one of the following curves:*

(i) an equianharmonic cubic;
(ii) a cuspidal cubic;
(iii) the union of three concurrent lines (not necessarily distinct).

Proof We only have to prove the converse. Suppose

$$f = l_1^3 + l_2^3 + l_3^3 + l_4^3,$$

where l_1, l_2, l_3 vanish at a common point a which we identify with a vector in E. We have

$$\frac{1}{3}D_a(f) = l_1(a)l_1^2 + l_2(a)l_2^2 + l_3(a)l_3^2 + l_4(a)l_4^2 = l_4(a)l_4^2.$$

This shows that the first polar $P_a(V(f))$ is either the whole $\mathbb{P}^2$ or the double line $2\ell = V(l_4^2)$. In the first case C is the union of three concurrent lines. Assume that the second case occurs. We can choose coordinates such that $a = [0,0,1]$ and $\ell = V(t_2)$. Write

$$f = g_0t_2^3 + g_1t_2^2 + g_2t_2 + g_3,$$

where g_k are homogeneous forms of degree k in variables t_0, t_1. Then, $D_a(f) = \partial_2 f = 3t_2^2g_0 + 2t_2g_1 + g_2$. This can be proportional to t_2^2 only if $g_1 = g_2 = 0, g_0 \neq 0$. Thus, $V(f) = V(g_0t_2^3 + g_3(t_0, t_1))$. If g_3 has no multiple linear factors, we get an equianharmonic cubic. If g_3 has a linear factor with multiplicity 2, we get a cuspidal cubic. Finally, if g_3 is a cube of a linear form, we reduce the latter to the form t_1^3 and get three concurrent lines.

□

aronhold *Remark* 3.2.11. We know that all equianharmonic cubics are projectively equivalent to the Fermat cubic. The orbit of the Fermat cubic $V(t_0^3 + t_1^3 + t_2^3)$ is somorphic to the homogeneous space $\mathrm{PSL}(3)/G$, where $G = (\mathbb{Z}/3\mathbb{Z})^2 \rtimes \mathfrak{S}_3$. Its closure in $|S^3(E^\vee)|$ is a hypersurface F and consists of curves listed in the assertion of the previous Lemma and also reducible cubics equal to the unions of irreducible conics with its tangent lines. The explicit equation of the hypersurface F is given by the *Aronhold invariant S* of degree 4 in the coefficients of the cubic equation. A nice expression for the invariant S in terms of a pfaffian of a skew-symmetric matrix was given by G. Ottaviani Ottaviani [564].

gener2 **Lemma 3.2.12.** *The following properties are equivalent:*

(i) $\mathrm{AP}_1(f) \neq \{0\}$*;*
(ii) $\dim \mathrm{AP}_2(f) > 2$*;*
(iii) $V(f)$ *is equal to the union of three concurrent lines.*

Proof By the apolarity duality,

$$(A_f)_1 \times (A_f)_2 \to (A_f)_3 \cong \mathbb{C},$$

we have

$$\dim(A_f)_1 = 3 - \dim \mathrm{AP}_1(f) = \dim(A_f)_2 = 6 - \dim \mathrm{AP}_2(f).$$

Thus, $\dim \mathrm{AP}_2(f) = 3 + \dim \mathrm{AP}_1(f)$. This proves the equivalence of (i) and (ii). By definition, $\mathrm{AP}_1(f) \neq \{0\}$ if and only if $D_\psi(f) = 0$ for some nonzero linear operator $\psi = \sum a_i \partial_i$. After a linear change of variables, we may assume that $\psi = \partial_0$, and then $\partial_0(f) = 0$ if and only if C does not depend on t_0, i.e. C is the union of three concurrent lines. □

Lemma 3.2.13. *Let Z be a generalized polar quadrangle of f. Then, $|\mathcal{I}_Z(2)|$ is a pencil of conics in $|E^\vee|$ contained in the linear system $|\mathrm{AP}_2(f)|$. If Z is nondegenerate, then the pencil has no fixed component. Conversely, let Z be a 0-dimensional cycle of length 4 in $|E|$. Assume that $|\mathcal{I}_Z(2)|$ is an irreducible pencil contained in $|\mathrm{AP}_2(f)|$. Then, Z is a nondegenerate generalized polar quadrangle of f.*

Proof The first assertion follows from the definition of nondegeneracy and Proposition 1.3.12. Let us prove the converse. Let $V(\lambda q_1 + \mu q_2)$ be the pencil of conics $|\mathcal{I}_Z(2)|$. Since $\mathrm{AP}(f)$ is an ideal, the linear system L of cubics of the form $V(q_1 l_1 + q_2 l_2)$, where l_1, l_2 are linear forms, is contained in $|\mathrm{AP}_3(f)|$. Obviously, it is contained in $|\mathcal{I}_Z(3)|$. Since $|\mathcal{I}_Z(2)|$ has no fixed part we may choose q_1 and q_2 with no common factors. Then, the map $E^\vee \oplus E^\vee \to I_Z(3)$ defined by $(l_1, l_2) \to q_1 l_1 + q_2 l_2$ is injective, hence $\dim L = 5$. Assume $\dim |\mathcal{I}_Z(3)| \geq 6$. Choose three points in general position on an irreducible member C of $|\mathcal{I}_Z(2)|$ and three non-collinear points outside C. Then, find a cubic K from $|\mathcal{I}_Z(3)|$ which passes through these points. Then, K intersects C with total multiplicity $4 + 3 = 7$, hence contains C. The other component of K must be a line passing through three non-collinear points. This contradiction shows that $\dim |\mathcal{I}_Z(3)| = 5$ and we have $L = |\mathcal{I}_Z(3)|$. Thus, $|\mathcal{I}_Z(3)| \subset |\mathrm{AP}_3(f)|$ and, by Proposition 1.3.12, Z is a generalized polar quadrangle of C.

□

Note that not every point in $\mathrm{Hilb}^4(\mathbb{P}^2)$ can be realized as a generalized quadrangle of a ternary cubic. Each point in the Hilbert scheme $\mathrm{Hilb}^4(\mathbb{P}^2)$ is the union of subschemes supported at one point. Let us recall analytic classification of closed subschemes $V(I)$ of length $h \leq 4$ supported at one point (see [72]).

- $h = 1$: $I = (x, y)$;
- $h = 2$: $I = (x, y^2)$;
- $h = 3$: $I = (x, y^3), (x^2, xy, y^2)$;
- $h = 4$: $I = (x, y^4), (x^2, y^2), (x^2, xy, y^3)$.

The subschemes Z of length 4 that cannot be realized as the base scheme of a pencil of conics, are those which contain a subscheme analytically isomorphic to one of the following schemes $V(x, y^3)$, $V(x, y^4)$, $V(x^2, xy, y^2)$, or $V(x^2, xy, y^3)$.

Theorem 3.2.14. *Assume that C is neither an equianharmonic cubic, nor a cuspidal cubic, nor the union of three concurrent lines. Then,*

$$\mathrm{VSP}(f, 4) \cong |\mathrm{AP}_2(f)|^\vee \cong \mathbb{P}^2.$$

If C is nonsingular, the complement of $\Delta = \mathrm{VSP}(f, 4) \setminus \mathrm{VSP}(f, 4)^o$ is a curve of degree 6 isomorphic to the dual of a nonsingular cubic curve. If C is a nodal cubic, then Δ is the union of a quartic curve isomorphic to the dual quartic of C and two lines. If C is the union of a nonsingular conic and a line intersecting it transversally, Δ is the union of a conic and two lines. If C is the union of a conic and its tangent line, then $\Delta = \mathrm{VSP}(f, 4)$.

Proof We will start with the case when C is nonsingular. We know that its equation can be reduced to the Hesse canonical form (3.7). The space of apolar quadratic forms is spanned by $\alpha u_0u_1 - u_2^2,\ \alpha u_1u_2 - u_0^2,\ \alpha u_0u_2 - u_1^2$. It is equal to the net of polar conics of the curve C' in the dual plane given by the equation

$$u_0^3 + u_1^3 + u_2^3 - 6\alpha u_0u_1u_2 = 0, \quad \alpha(\alpha^3 - 1) \neq 0. \tag{3.30}$$

The net $|\mathrm{AP}_2(f)|$ is base-point-free. Its discriminant curve is a nonsingular cubic, the Hessian curve of the curve C'. The generalized quadrangles are parameterized by the dual curve $\mathrm{He}(C')^\vee$. All pencils are irreducible, so there are no degenerate generalized quadrangles. Generalized quadrangles correspond to tangent lines of the discriminant cubic. So,

$$\mathrm{VSP}(f, 4) = |\mathrm{AP}_2(f)|^\vee, \tag{3.31}$$

and $\mathrm{VSP}(f, 4) \setminus \mathrm{VSP}(f, 4)^o = \mathrm{He}(C')^\vee$.

Next, assume that $C = V(t_2^2t_0 + t_1^3 + t_1^2t_0)$ is an irreducible nodal cubic.

The space of apolar quadratic forms is spanned by $u_0^2, u_1u_2, u_2^2 - u_1^2 + 3u_0u_1$. The net $|\mathrm{AP}_2(f)|$ is base-point-free. Its discriminant curve is an irreducible nodal cubic D. So, all pencils are irreducible, and (3.31) holds. Generalized quadrangles are parameterized by the union of the dual quartic curve $D^\vee$ and the pencil of lines through the double point.

Next, assume that $C = V(t_0^3 + t_0t_1t_2)$ is the union of an irreducible conic and a line which intersects the conic transversally.

The space of apolar quadratic forms is spanned by $u_1^2, u_2^2, 6u_1u_2 - u_0^2$. The net $|\mathrm{AP}_2(f)|$ is base-point-free. It is easy to see that its discriminant curve is the union of a conic and a line intersecting the conic transversally. The line component defines the pencil generated by $V(u_1^2)$ and $V(u_2^2)$. It has no fixed part but its members are singular. So, all generalized quadrangles are nondegenerate and (3.31) holds. The locus of generalized quadrangles consists of a conic and two lines.

Next, assume that $V(f) = V(t_0t_1t_2)$ is the union of three nonconcurrent lines.

The net $|\mathrm{AP}_2(f)|$ of apolar conics is generated by $V(u_0^2), V(u_1^2), V(u_2^2)$. It is base-point-free. The discriminant curve is the union of three nonconcurrent lines representing pencils of singular conics which have no fixed component. Thus, any pencil not containing a singular point of the discriminant curve defines a nondegenerate polar quadrangle. A pencil containing a singular point defines a nondegenerate generalized polar quadrangle. Again (3.31) holds and $\mathrm{VSP}(f,4) \setminus \mathrm{VSP}(f,4)^o$ consists of three nonconcurrent lines.

Finally, let $C = V(t_0(t_0t_1 + t_2^2))$ be the union of an irreducible conic and its tangent line. We check that $\mathrm{AP}_2(f)$ is spanned by $u_1^2, u_1u_2, u_2^2 - u_0u_1$. The discriminant curve is a triple line. It corresponds to the pencil $V(\lambda u_1^2 + \mu u_1u_2)$ of singular conics with the fixed component $V(u_1)$. There are no polar quadrangles. Consider the subscheme Z of degree 4 in the affine open set $u_0 \neq 0$ defined by the ideal supported at the point $[1,0,0]$ with ideal at this point generated by $(u_1/u_0)^2, u_1u_2/u_0^2$, and $(u_2/u_0)^2$. The linear system $|\mathcal{I}_Z(3)|$ is of dimension 5 and consists of cubics of the form $V(u_0u_1(au_1 + bu_2) + g_3(u_1,u_2))$. One easily computes $\mathrm{AP}_3(f)$. It is generated by the polynomial $u_0u_2^2 - u_0^2u_1$ and all monomials except $u_0^2u_1$ and $u_0u_2^2$. We see that $|\mathcal{I}_Z(3)| \subset |\mathrm{AP}_3(f)|$. Thus, Z is a degenerate generalized polar quadrangle of C and (3.31) holds.

□

Remark 3.2.15. We already know the variety $\mathrm{VSP}(f,4)$ in the case when C is the union of concurrent lines. In the remaining cases, which we excluded, the variety $\mathrm{VSP}(f,4)$ is a reducible surface. Its description is too involved to discuss it here. For example, if C is an equianharmonic cubic, it consists of four irreducible components. Three components are isomorphic to $\mathbb{P}^2$. They are disjoint and each contains an open dense subset parametrizing degenerate polar quadrangles. The fourth component contains an open subset of base schemes of irreducible pencils of apolar conics. It is isomorphic to the blow-up of $|\mathrm{AP}_2|^\vee$ at three points corresponding to reducible pencils. Each of the first three components intersects the fourth component along one of the three exceptional curves.

S:3.3

3.3 Projective Generation of Cubic Curves

SS:3.3.1

3.3.1 Projective generation

Suppose we have m different r-dimensional linear systems $|L_i|$ of hypersurfaces of degrees d_i in $\mathbb{P}^n$. Choose projective isomorphisms $\phi_i : \mathbb{P}^r \to |L_i|$ and consider the variety

$$Z = \{(\lambda, x) \in \mathbb{P}^r \times \mathbb{P}^n : x \in \phi_1(\lambda) \cap \ldots \cap \phi_m(\lambda)\}. \tag{3.32}$$ inc

The expected dimension of a general fiber of the first projection $\mathrm{pr}_1 : Z \to \mathbb{P}^r$ is equal to $n - m$. Assume

- Z is irreducible of dimension $r + n - m$;
- the second projection $\mathrm{pr}_2 : Z \to \mathbb{P}^n$ is of finite degree k on its image X.

Under these assumptions, X is an irreducible subvariety of dimension $r+n-m$.

pg **Proposition 3.3.1.**

$$\deg X = s_r(d_1, \ldots, d_m)/k,$$

where s_r is the r-th elementary symmetric function in m variables.

Proof It is immediate that Z is a complete intersection in $\mathbb{P}^r \times \mathbb{P}^n$ of m divisors of type $(1, d_i)$. Let Π be a general linear subspace in $\mathbb{P}^n$ of codimension $n-m+r$. We use the intersection theory from [315] (Fulton). Let $\bar{h}_1$ and $\bar{h}_2$ be the natural generators of $H^2(\mathbb{P}^r \times \mathbb{P}^n, \mathbb{Z})$ equal to the pre-images of the cohomology classes h_1, h_2 of a hyperplane in $\mathbb{P}^r$ and $\mathbb{P}^n$, respectively. We have $(\mathrm{pr}_2)_*([Z]) = k[X]$. By the projection formula,

$$(\mathrm{pr}_2)_*([Z]) = (\mathrm{pr}_2)_*(\prod_{j=1}^{m}(\bar{h}_1 + d_j\bar{h}_2)) = (\mathrm{pr}_2)_*(\sum_{j=1}^{m} s_j(d_1, \ldots, d_m)\bar{h}_1^j\bar{h}_2^{m-j})$$

$$= \sum_{j=1}^{m} s_j(d_1, \ldots, d_m)h_2^{m-j}(\mathrm{pr}_2)_*(\bar{h}_1^j) = s_r(d_1, \ldots, d_m)h_2^{m-r}.$$

Intersecting with h_2^{n-m+r}, we obtain that $k \deg X = s_r(d_1, \ldots, d_m)$.

□

Since through a general point in $\mathbb{P}^n$ passes a unique member of a pencil, $k = 1$ if $r = 1$.

The following example is *Steiner's construction* of rational normal curves of degree n in $\mathbb{P}^n$. We have already used it in the case of conics, referring the reader for the details to [360] (GH).

ex:3.3.2 *Example* 3.3.2. Let $r = 1, m = n$ and $d_1 = \ldots = d_n = 1$. Let $p_1, \ldots, p_n$ be linearly independent points in $\mathbb{P}^n$ and let $\mathcal{P}_i$ be the pencil of hyperplanes passing through the codimension 2 subspace spanned by all points except p_i. Choose a linear isomorphism $\phi_i : \mathbb{P}^1 \to \mathcal{P}_i$ such that the common hyperplane H spanned by all the points corresponds to different parameters $\lambda \in \mathbb{P}^1$.

Let $H_i(\lambda) = \phi_i(\lambda)$. A line contained in the intersection $H_1(\lambda) \cap \ldots \cap H_n(\lambda)$ meets H, and hence H meets each $H_i(\lambda)$. If H is different from each $H_i(\lambda)$, this implies that the base loci of the pencils $\mathcal{P}_i$ meet. However this contradicts the assumption that the points p_i are linearly independent. If $H = H_i(\lambda)$ for some i, then $H \cap H_j(\lambda)$ is equal to the base locus of $\mathcal{P}_j$. Thus, the intersection$H_1(\lambda) \cap \ldots \cap H_n(\lambda)$ consists of the point p_i. This shows that, under the first projection $\mathrm{pr}_1 : Z \to \mathbb{P}^1$, the incidence variety (3.32) is isomorphic to $\mathbb{P}^1$. In particular, all the assumptions on the pencils $\mathcal{P}_i$ are satisfied with $k = 1$. Thus, the image of Z in $\mathbb{P}^n$ is a rational curve R_n of degree n. If $\phi_i(\lambda) = H$, then the previous argument shows that $p_i \in R_n$. Thus, all points $p_1, \ldots, p_n$ lie on R_n. Since all rational curves of degree n in $\mathbb{P}^n$ are projectively equivalent, we obtain that any such curve can be projectively generated by n pencils of hyperplanes.

More generally, let $\mathcal{P}_1, \ldots, \mathcal{P}_n$ be n pencils of hyperplanes. Since a projective isomorphism $\phi_i : \mathbb{P}^1 \to \mathcal{P}_i$ is uniquely determined by the images of three different points, we may assume that $\phi_i(\lambda) = V(\lambda_0 l_i + \lambda_1 m_i)$ for some linear forms l_i, m_i. Then, the intersection of the hyperplanes $\phi_1(\lambda) \cap \ldots \cap \phi_n(\lambda)$ consists of one point if and only if the system of n linear equations with $n + 1$ unknowns

$$\lambda_0 l_1 + \lambda_1 m_1 = \ldots = \lambda_0 l_n + \lambda_1 m_n = 0$$

has a 1-dimensional space of solutions. Under some genericity assumption on the choice of the pencils, we may always assume it. This shows that the rational curve R_n is projectively generated by the pencils, and its equations are expressed by the condition that

$$\mathrm{rank} \begin{pmatrix} l_0 & l_1 & \ldots & l_n \\ m_0 & m_1 & \ldots & m_n \end{pmatrix} \leq 1.$$

Observe that the maximal minors of the matrix define quadrics in $\mathbb{P}^n$ of rank ≤ 4.

Example 3.3.3. Take two pencils $\mathcal{P}_i$ of planes in $\mathbb{P}^3$ through skew lines ℓ_i. Choose a linear isomorphism $\phi : \mathbb{P}^1 \to \mathcal{P}_i$. Then, the union of the lines $\phi_1(\lambda) \cap \phi_2(\lambda)$ is equal to a quadric surface in $\mathbb{P}^3$ containing the lines ℓ_1, ℓ_2.

3.3.2 Projective generation of a plane cubic

We consider a special case of the previous construction where $n = 2, r = 1$ and $m = 2$. By Proposition 3.3.1, X is a curve of degree $d_1 + d_2$. Assume that the base locus of the pencil $\mathcal{P}_i$ consists of d_i^2 distinct points and the two base loci have no points in common. It is clear that the union of the base loci is the set of $d_1^2 + d_2^2$ points on X.

Take a pencil of lines $\mathcal{P}_1$ and a pencil of conics $\mathcal{P}_2$. We obtain a cubic curve C containing the base point of the pencil of lines and four base points of the pencil of conics. The pencil $\mathcal{P}_2$ cuts out on C a g_2^1. We will use the following.

Lemma 3.3.4. *For any g_2^1 on an irreducible reduced plane cubic curve, the lines spanned by the divisor from g_2^1 intersect at one point on the curve.*

Proof The standard exact sequence

$$0 \to \mathcal{O}_{\mathbb{P}^2}(-2) \to \mathcal{O}_{\mathbb{P}^2}(1) \to \mathcal{O}_C(1) \to 0$$

gives an isomorphism $H^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(1)) \cong H^0(C, \mathcal{O}_C(1))$. It shows that the pencil g_2^1 is cut out by a pencil of lines. Its base point is the point whose existence is asserted in the Lemma. □

The point of intersection of lines spanned by the divisors from a g_2^1 was called by Sylvester the *coresidual point* of C (see [652], p. 134).

Let C be a nonsingular plane cubic. Pick up four points on C, no three of them lying on a line. Consider the pencil of conics through these points. Let q be the coresidual point of the g_2^1 on C defined by the pencil. Then, the pencil of lines through q and the pencil of conics projectively generate C.

Note that the first projection $\mathrm{pr}_1 : Z \to \mathbb{P}^1$ is a degree 2 cover defined by the g_2^1 cut out by the pencil of conics. It has four branch points corresponding to lines $\phi_1(\lambda)$ which touch the conic $\phi_2(\lambda)$.

There is another way to projectively generate a cubic curve. This time, we take three nets of lines with fixed isomorphisms ϕ_i to $\mathbb{P}^2$. Explicitly, if $\lambda = [\lambda_0, \lambda_1, \lambda_2] \in \mathbb{P}^2$ and $\phi_i(\lambda) = V(a_0^{(i)} t_0 + a_1^{(i)} t_1 + a_2^{(i)} t_2)$, where $a_j^{(i)}$ are linear forms in $\lambda_0, \lambda_1, \lambda_2$, then C is given by the equation

$$\det \begin{pmatrix} a_0^{(1)} & a_1^{(1)} & a_2^{(1)} \\ a_0^{(2)} & a_1^{(2)} & a_2^{(2)} \\ a_0^{(3)} & a_1^{(3)} & a_2^{(3)} \end{pmatrix} = 0.$$

This is an example of a determinantal equation of a plane curve which we will study in detail in the next Chapter.

3.4 Invariant Theory of Plane Cubics

S:3.4

3.4.1 Mixed concomitants

SS:3.4.1

The classical invariant theory dealt with objects more general that invariants of homogeneous forms. Let E, as usual, denote a complex vector space of dimension $n+1$. Recall that the main object of study in the invariant theory is a *mixed combinant*, an element Φ of the tensor product

$$\bigotimes_{i=1}^{r} S^{m_i}(S^{d_i}(E^\vee))^\vee \otimes \bigotimes_{i=1}^{k} S^{p_i}(E^\vee) \otimes \bigotimes_{i=1}^{s} S^{q_i}(E)$$

which is invariant with respect to the natural linear representation of $\mathrm{SL}(E)$ on the tensor product. We will be dealing here only with the cases when $r = 1, k, s \leq 1$. If $k = s = 0$, Φ is an *invariant* of degree m_1 on the space $S^d(E^\vee)$. If $k = 1, s = 0$, then Φ is a *covariant* of degree m and order p. If $k = 0, s = 1$, then Φ is a *contravariant* of degree m and class q. If $k = s = 1$, then Φ is a *mixed concomitant* of degree m, order p and class q.

Choosing a basis $u_0, \ldots, u_n$ in E, and the dual basis $t_0, \ldots, t_n$ in $E^\vee$, one can write an invariant $\Phi \in S^m(S^d(E^\vee))^\vee = S^m(S^d(E))$ as a homogeneous polynomial of degree m in coefficients of a general polynomial of degree d in $u_0, \ldots, u_n$ which are expressed as monomials of degree d in $u_0, \ldots, u_n$. Via polarization, we can consider it as a multihomogeneous function of degree $(d, \ldots, d)$ on $(E^*)^m$. *Symbolically*, it is written as a product of w sequences $(i_1 \ldots i_n)$ of numbers from $\{1, \ldots, m\}$ such that each number appears d times. The relation

$$(n+1)w = md$$

must hold. In particular, there are no invariants if $n+1$ does not divide md. The number w is called the *weight* of the invariant. When we apply a linear transformation, it is multiplied by the w-th power of the determinant.

A covariant $\Phi \in S^m(S^d(E^\vee))^\vee \otimes S^p(E^\vee)$ can be written as a polynomial of degree m in coefficients of a general polynomial of degree d and of degree p in coordinates $t_0, \ldots, t_n$. Via polarization, it can be considered as a multihomogeneous function of degree $(d, \ldots, d, p)$ on $(E^\vee)^m \times E$. Symbolically, it can be written as a product of w expressions $(j_0 \ldots j_n)$ and p expressions $(i)_x$, where each number from $\{1, \ldots, m\}$ appears d times. We must have

$$(n+1)w + pn = md.$$

A contravariant $\Phi \in S^m(S^d(E^\vee))^\vee \otimes S^q(E)$ can be written as a polynomial of degree m in coefficients of a general polynomial of degree d and of degree q in $u_0, \ldots, u_n$. Via polarization, it can be considered as a multihomogeneous function of degree $(d, \ldots, d, q)$ on $(E^\vee)^m \times E^\vee$. Symbolically, it can be written as a product of w expressions $(j_0 \ldots j_n)$ and q expressions $(i_1 \ldots i_n)_u$. We have

$$(n+1)w + qn = md.$$

A mixed concomitant $\Phi \in S^m(S^d(E^\vee))^\vee \otimes S^p(E^\vee) \otimes S^q(E)$ can be written as a polynomial of degree m in coefficients of a general polynomial of degree d, of degree p in $t_0, \ldots, t_n$, and of degree q in $u_0, \ldots, u_n$. Via polarization, it can be considered as a multihomogeneous function of degree $(d, \ldots, d, p, q)$ on $(E^\vee)^m \times E \times E^\vee$. Symbolically, it can be written as a product of w expressions $(j_0, \ldots, j_n)$, p expressions $(i)_x$ and q expressions $(i_1, \ldots, i_n)_\xi$, where each number from $\{1, \ldots, m\}$ appears d times. We have

$$(n+1)w + (a+b)n = md.$$

Note that instead of numbers $1, \ldots, m$ classics often employed m letters $a, b, c, \ldots$.

For example, we have met already the Aronhold invariants S and T of degrees 4 and 6 of a ternary cubic form. Their symbolic expressions are

$$\mathsf{S} = (123)(124)(134)(234) = (abc)(abd)(acd)(bcd),$$
$$\mathsf{T} = (123)(124)(135)(256)(456)^2 = (abc)(abd)(ace)(bef)(def)^2.$$

SS:3.4.2

3.4.2 Clebsch's transfer principle

This principle allows one to relate invariants of polynomials in n variables to contravariants and covariants of polynomials in $n+1$ variables.

Start from an invariant Φ of degree m on the space $S^d((\mathbb{C}^n)^\vee)$ of homogeneous polynomials of degree d. We will "transfer it" to a contravariant $\tilde{\Phi}$ on the space of polynomials of degree d in $n+1$ variables. First, we fix a volume form ω on E. A basis in a hyperplane $U \subset E$ defines a linear isomorphism $\mathbb{C}^n \to U$. We call a basis admissible if the pull-back of the volume form under this linear map is equal to the standard volume form $e_1 \wedge \ldots \wedge e_n$. For any $\alpha \in E^\vee$, choose an admissible basis $(v_1^\alpha, \ldots, v_n^\alpha)$ in $\mathrm{Ker}(\alpha)$. For any $(l_1, \ldots, l_m) \in (E^\vee)^m$, we obtain n vectors in $\mathbb{C}^n$, the columns of the matrix

$$A = \begin{pmatrix} l_1(v_1^\alpha) & \ldots & l_m(v_1^\alpha) \\ \vdots & \vdots & \vdots \\ l_1(v_n^\alpha) & \ldots & l_m(v_n^\alpha) \end{pmatrix}.$$

The value of Φ on this set of vectors can be expressed as a linear combination of the product of maximal minors $|A_I|$, where each column occurs d times. It is easy to see that each minor $A_{i_1\ldots i_n}$ is equal to the value of $l_{i_1} \wedge \ldots \wedge l_{i_n} \in \bigwedge^n E^\vee$ on $v^u_{i_1} \wedge \ldots \wedge v^u_{i_n}$ under the canonical pairing

$$\bigwedge^n E^\vee \times \bigwedge^n E \to \mathbb{C}.$$

Our choice of a volume from on E allows us to identify $\bigwedge^n E$ with $E^\vee$. Thus, any minor can be considered as multilinear function on $(E^\vee)^m \times E^\vee$ and its value does not depend on the choice of an admissible basis in $\mathrm{Ker}(u)$. Symbolically, $(i_1 \ldots i_n)$ becomes the bracket expression $(i_1 \ldots i_n)_u$. This shows that the invariant Φ, by restricting to the subspaces $\mathrm{Ker}(\alpha)$, defines a contravariant $\tilde{\Phi}$ on $S^d(E^\vee)$ of degree m and class $q = md/n$.

Example 3.4.1. Let Φ be the discriminant of a quadratic form in n variables. It is an invariant of degree $m = n$ on the space of quadratic forms. Its symbolic notation is $(12\ldots n)^2$. Its transfer to $\mathbb{P}^n$ is a contravariant $\tilde{\Phi}$ of degree n and class $2n/n = 2$. Its symbolic notation is $(12\ldots n)^2_u$. Considered as a map $\tilde{\Phi} : S^2(E)^\vee \to S^2(E)$, the value of $\tilde{\Phi}(q)$ on $u \in E^\vee$ is the discriminant of the quadratic form obtained from restriction of q to $\mathrm{Ker}(u)$. It is equal to zero if and only if the hyperplane $V(u)$ is tangent to the quadric $V(q)$. Thus, $V(\tilde{\Phi}(q))$ is the dual quadric $V(q)^\vee$.

Example 3.4.2. Consider the quadratic invariant S on the space of binary forms of even degree $d = 2k$ with symbolic expression $(12)^{2k}$. We write a general binary form $f \in S^d(U)$ of degree d symbolically,

$$f = (\xi_0 t_0 + \xi_1 t_1)^{2k} = (\eta_0 t_0 + \eta_1 t_1)^{2k},$$

where (ξ_0, ξ_1) and (η_0, η_1) are two copies of a basis in U and (t_0, t_1) is its dual basis. Then, the coefficients of f are equal to $\binom{d}{j} a_j$, where $a_j = \xi_0^j \xi_1^{2k-j} = \eta_0^j \eta_1^{2k-j}$. Thus, S is equal to

$$\begin{aligned}
(\xi_0\eta_1 - \xi_1\eta_0)^{2k} &= \sum_{j=0}^{2k} (-1)^j \tbinom{2k}{j} (\xi_0\eta_1)^j (\xi_1\eta_0)^{2k-j} \\
&= \sum_{j=0}^{2k} (-1)^j \tbinom{2k}{j} (\xi_0^j \xi_1^{2k-j})(\eta_1 \eta_0^{2k-j}) = \sum_{j=0}^{2k} (-1)^j \tbinom{d}{j} a_j a_{2k-j} \qquad (3.33)\\
&= 2\Big(\sum_{j=0}^{k} (-1)^j \tbinom{2k}{j} a_j a_{2k-j} + \tfrac{1}{2}\tbinom{2k}{k} a_k^2\Big).
\end{aligned}$$

We have already encountered this invariant in the case $d = 3$ (see Subsection binary 1.60).

The transfer of S is the contravariant of degree 2 and class d with symbolic expression $(abu)^d$. For example, when $d = 4$, its value on a quartic ternary form f is a quartic form in the dual space which vanishes on lines which cut out on $V(f)$ a harmonic set of 4 points. The transfer T of the invariant of degree 3 on the space of quartic binary forms defines a contravariant of class 6. Its value on a quartic ternary form is a ternary form of degree 6 in the dual space which vanishes on the set of lines which cut out in $V(f)$ an equianharmonic set of 4 points.

One can also define Clebsch's transfer of covariants of degree m and order p, keeping the factors i_x in the symbolic expression. The result of the transfer is a mixed concomitant of degree m, order p and class md/n.

SS:3.4.3

3.4.3 Invariants of plane cubics

Since this material is somewhat outside of the topic of the book, we state some of the facts without proof, referring to classical sources for the invariant theory (e.g. [150], t. 2, [652]).

We know that the ring of invariants of ternary cubic forms is generated by the Aronhold invariants S and T. Let us look for covariants and contravariants. As we know from Subsection 1.60, any invariant of binary form of degree 3 is a power of the discriminant invariant of order 4, and the algebra of covariants is generated over the ring of invariants by the identical covariant $\mathsf{U} : f \mapsto f$, the Hessian covariant H of order 2 with symbolic expression $(ab)a_xb_x$, and the covariant $\mathsf{J} = \mathrm{Jac}(f, \mathsf{H})$ of degree 3 and order 6 with symbolic expression $(ab)^2(ca)b_xc_x^2$. Clebsch's transfer of the discriminant is a contravariant F of degree 4 and class 6. Its symbolic expression is $(abu)^2(cdu)^2(acu)(bdu)$. Its value on a general ternary cubic form is the form defining the dual cubic curve. Clebsch's transfer of H is a mixed concomitant Θ of degree 2, order 2 and class 2. Its symbolic expression is $(abu)^2a_xb_x$. Explicitly, up to a constant factor,

$$\Theta = \det\begin{pmatrix} f_{00} & f_{01} & f_{02} & u_0 \\ f_{10} & f_{11} & f_{12} & u_1 \\ f_{20} & f_{21} & f_{22} & u_2 \\ u_0 & u_1 & u_2 & 0 \end{pmatrix}, \tag{3.34}$$

where $f_{ij} = \frac{\partial^2 f}{\partial t_i \partial t_j}$.

The equation $\Theta(f, x, u) = 0$, for fixed x, is the equation of the dual of the polar conic $P_x(V(f))$. The equation $\Theta(f, x, u) = 0$, for fixed u, is the equation of the locus of points x such that the first polar $P_x(V(f))$ is tangent to the line $V(u)$. It is called the *poloconic* of the line $V(u)$. Other description of the poloconic can be found in Exercise 3.3.

The Clebsch transfer of J is a mixed concomitant Q of degree 3, order 3 and class 3. Its symbolic expression is $(abu)^2(cau)c_x^2 b_x$. The equation $\mathsf{Q}(f, x, u) = 0$, for fixed u, is the equation of the cubic curve such that second polars $P_{x^2}(V(f))$ of its points intersect $V(u)$ at a point conjugate to x with respect to the poloconic of $V(u)$. A similar contravariant is defined by the condition that it vanishes on the set of pairs (x, u) such that the line $V(u)$ belongs to the Salmon envelope conic of the polars of x with respect to the curve and its Hessian curve.

An obvious covariant of degree 3 and order 3 is the Hessian determinant $\mathsf{H} = \det \mathrm{He}(f)$. Its symbolic expression is $(abc)^2 a_x b_x c_x$. Another covariant G is defined by the condition that it vanishes on the locus of points x such that the Salmon conic of the polar of x with respect to the curve and its Hessian curve passes through x. It is of degree 8 and order 6. Its equation is the following bordered determinant

$$\begin{pmatrix} f_{00} & f_{01} & f_{02} & h_0 \\ f_{10} & f_{11} & f_{12} & h_1 \\ f_{20} & f_{21} & f_{22} & h_2 \\ h_0 & h_1 & h_2 & 0 \end{pmatrix},$$

where $f_{ij} = \frac{\partial^2 f}{\partial t_i \partial t_j}$, $h_i = \frac{\partial \mathsf{H}(f)}{\partial t_i}$ (see [107],[150], t. 2, p. 313). The algebra of covariants is generated by $\mathsf{U}, \mathsf{H}, \mathsf{G}$ and the *Brioschi covariant* [74] $J(f, \mathsf{H}, \mathsf{G})$ whose value on the cubic (3.7) is equal to

$$(1 + 8\alpha^3)(t_1^3 - t_2^3)(t_2^3 - t_0^3)(t_0^3 - t_1^3).$$

Comparing this formula with (3.16), we find that it vanishes on the union of 9 harmonic polars of the curve. The square of the Hermite covariant is a polynomial in $\mathsf{U}, \mathsf{H}, \mathbf{G}$.

The Cayleyan of a plane cubic defines a contravariant P of degree 3 and class 3. Its symbolic expression is $(abc)(abu)(acu)(bcu)$. Its value on the curve in the Hesse form is given in (3.27). There is also a contravariant Q of degree 5 and class 3. In analogy with the form of the word Hessian, A. Cayley gave them the names the *Pippian* and the *Quippian* [108]. If $C = V(f)$ is given in the Hesse form (3.7), then

$$\mathsf{Q}(f) = V((1 - 10\alpha^3)(u_0^2 + u_1^3 + u_2^3) - 6\alpha^2(5 + 4\alpha^3)u_0u_1u_2).$$

The full formula can be found in Cayley's paper [107]). He also gives the formula

$$\mathsf{H}(6a\mathsf{P} + b\mathsf{Q}) = (-2Ta^3 + 48S^2a2b + 18TSab^2 + (T^3 + 16S^2)b^3\mathsf{P}$$

$$+(8\mathsf{S}a^3 + 3\mathsf{T}a^2b - 24\mathsf{S}^2ab^2 - \mathsf{T}\mathsf{S}^2b^3)\mathsf{Q},$$

where the product of a covariant and a contravariant is considered as the composition of the corresponding equivariant maps.

According to A. Clebsch, $\mathsf{Q}(f)$ vanishes on the locus of lines whose poloconics with respect to the Cayleyan of C are apolar to their poloconics with respect to C. Also, according to W. Milne and D. Taylor, $\mathsf{Q}(f)$ is the locus of lines which intersect C at three points such that the polar line of the Hessian curve $\mathsf{H}(f)$ with respect two of the points is tangent to $\mathsf{H}(f)$ at the third point (see [384]). This is similar to the property of the Pippian which is the set of lines which intersect C at three points such that the polar line with respect to two of the points is tangent to C at the third point. The algebra of contravariants is generated by $\mathsf{F}, \mathsf{P}, \mathsf{Q}$ and the *Hermite contravariant* Hermite [386]. Its value on the cubic in the Hesse form is equal to

$$(1 + 8\alpha^3)(u_1^3 - u_2^3)(u_2^3 - u_0^3)(u_0^3 - u_1^3).$$

It vanishes on the union of nine lines corresponding to the inflection points of the curve. The square of the Hermite contravariant is a polynomial in $\mathsf{F}, \mathsf{P}, \mathsf{Q}$.

E:2

Exercises

ex:3.1

3.1 Find the Hessian form of a nonsingular cubic given by the Weierstrass equation.

3.2 Let $H = \mathrm{He}(C)$ be the Hessian cubic of a nonsingular plane cubic curve C that is not an equianharmonic cubic. Let $\tau : H \to H$ be the Steinerian automorphism of H that assigns to $a \in H$ the unique singular point of $P_a(C)$.

(i) Let $\tilde{H} = \{(a, \ell) \in H \times (\mathbb{P}^2)^\vee : \ell \subset P_a(C)\}$. Show that the projection $p_1 : \tilde{H} \to H$ is an unramified double cover.

(ii) Show that $\tilde{H}$ is isomorphic to the Caylean curve $\mathrm{Cay}(C)$.

ex:3.2

3.3 Let $C = V(f) \subset \mathbb{P}^2$ be a nonsingular cubic.

(i) Show that the set $K(\ell)$ of second polars of C with respect to points on a fixed line ℓ is the dual conic of the poloconic of C with respect to ℓ.

(ii) Show that $K(\ell)$ is equal to the set of poles of ℓ with respect to polar conics $P_x(C)$, where $x \in \ell$.

(iii) What happens to the conic $K(\ell)$ when the line ℓ is tangent to C?

(iv) Show that the set of lines ℓ such that $K(\ell)$ is tangent to ℓ is the dual curve of C.

(v) Let $\ell = V(a_0t_0 + a_1t_1 + a_2t_2)$. Show that $K(\ell)$ can be given by the equation

$$g(a,t) = \det\begin{pmatrix} 0 & a_0 & a_1 & a_2 \\ a_0 & \frac{\partial^2 f}{\partial t_0^2} & \frac{\partial^2 f}{\partial t_0 \partial t_1} & \frac{\partial^2 f}{\partial t_0 \partial t_2} \\ a_1 & \frac{\partial^2 f}{\partial t_1 \partial t_0} & \frac{\partial^2 f}{\partial t_1^2} & \frac{\partial^2 f}{\partial t_1 \partial t_2} \\ a_2 & \frac{\partial^2 f}{\partial t_2 \partial t_0} & \frac{\partial^2 f}{\partial t_2 \partial t_1} & \frac{\partial^2 f}{\partial t_2^2} \end{pmatrix} = 0.$$

(vi) Show that the dual curve $C^\vee$ of C can be given by the equation (the *Schläfli equation*)

$$\det\begin{pmatrix} 0 & \xi_0 & \xi_1 & \xi_2 \\ \xi_0 & \frac{\partial^2 g(\xi,t)}{\partial t_0^2}(\xi) & \frac{\partial^2 g(\xi,t)}{\partial t_0 \partial t_1}(\xi) & \frac{\partial^2 g(\xi,t)}{\partial t_0 \partial t_2}(\xi) \\ \xi_1 & \frac{\partial^2 g(\xi,t)}{\partial t_1 \partial t_0}(\xi) & \frac{\partial^2 g(\xi,t)}{\partial t_1^2}(\xi) & \frac{\partial^2 g(\xi,t)}{\partial t_1 \partial t_2}(\xi) \\ \xi_2 & \frac{\partial^2 g(\xi,t)}{\partial t_2 \partial t_0}(\xi) & \frac{\partial^2 g(\xi,t)}{\partial t_2 \partial t_1}(\xi) & \frac{\partial^2 g(\xi,t)}{\partial t_2^2}(\xi) \end{pmatrix}.$$

ex:3.3

3.4 Let $C \subset \mathbb{P}^{d-1}$ be an elliptic curve embedded by the linear system $|\mathcal{O}_C(dp_0)|$, where p_0 is a point in C. Assume $d = p$ is prime.

(i) Show that the image of any p-torsion point is an osculating point of C, i.e., a point such that there exists a hyperplane (an *osculating hyperplane*) which intersects the curve only at this point.

(ii) Show that there is a bijective correspondence between the sets of cosets of $(\mathbb{Z}/p\mathbb{Z})^2$ with respect to subgroups of order p and hyperplanes in $\mathbb{P}^{p-1}$ which cut out in C the set of p osculating points.

(iii) Show that the set of p-torsion points and the set of osculating hyperplanes define a $(p^2_{p+1}, p(p+1)_p)$-configuration of p^2 points and $p(p+1)$ hyperplanes (i.e., each point is contained in $p+1$ hyperplanes and each hyperplane contains p points).

(iv) Find a projective representation of the group $(\mathbb{Z}/p\mathbb{Z})^2$ in $\mathbb{P}^{p-1}$ such that each osculating hyperplane is invariant with respect to some cyclic subgroup of order p of $(\mathbb{Z}/p\mathbb{Z})^2$.

ex:3.4

3.5 A point on a nonsingular cubic is called a *sextactic point* if there exists an irreducible conic intersecting the cubic at this point with multiplicity 6. Show that there are 27 sextactic points.

ex:3.5

3.6 The pencil of lines through a point on a nonsingular cubic curve C contains four tangent lines. Show that the twelve contact points of three pencils with collinear base points on C lie on 16 lines forming a configuration $(12_4, 16_3)$ (the *Hesse-Salmon configuration*).

ex:3.6

3.7 Show that the cross ratio of the four tangent lines of a nonsingular plane cubic curve that pass through a point on the curve does not depend on the point.

3.8 Prove that the second polar of a nonsingular cubic C with respect to the point a on the Hessian $\mathrm{He}(C)$ is equal to the tangent line $\mathbb{T}_b(\mathrm{He}(C))$, where b is the singular point of the polar conic $P_a(C)$.

ex:3.8

3.9 Let a, b be two points on the Hessian curve $\mathrm{He}(C)$ forming an orbit with the respect to the Steinerian involution. Show that the line $\overline{ab}$ is tangent to the dual of the Caylean curve $\mathrm{Cay}(C)$ at some point d. Let c be the third intersection point of $\mathrm{He}(C)$ with the line $\overline{ab}$. Show that the pairs (a, b) and (c, d) are harmonically conjugate.

ex:3.9

3.10 Show that from each point a on the $\mathrm{He}(C)$ one can pass three tangent lines to the dual curve of $\mathrm{Cay}(C)$. Let b be the singular point of $P_a(C)$. Show that the set of the three tangent lines consists of the line $\overline{ab}$ and the components of the reducible polar conic $P_b(C)$.

ex:3.10

3.11 Let $C = V(\sum_{0\le i\le j\le k\le 2} a_{ijk}t_it_jt_k)$. Show that the Cayleyan curve $\mathrm{Cay}(C)$ can be given by the equation

$$\det\begin{pmatrix} a_{000} & a_{001} & a_{002} & \xi_0 & 0 & 0 \\ a_{110} & a_{111} & a_{112} & 0 & \xi_1 & 0 \\ a_{220} & a_{221} & a_{222} & 0 & 0 & \xi_2 \\ 2a_{120} & 2a_{121} & 2a_{122} & 0 & \xi_2 & \xi_1 \\ 2a_{200} & 2a_{201} & 2a_{202} & \xi_2 & 0 & \xi_0 \\ 2a_{010} & 2a_{011} & 2a_{012} & \xi_1 & \xi_0 & 0 \end{pmatrix} = 0$$

ClebschLindemann [150], p. 245. ex:3.11

3.12 Show that any general net of conics is equal to the net of polars of some cubic curve. Show that the curve parameterizing the irreducible components of singular members of the net coincides with the Cayleyan curve of the cubic (it is called the *Hermite curve*. of the net) ex:3.12

3.13 Show that the group of projective transformations leaving a nonsingular plane cubic invariant is a finite group of order 18, 36 or 54. Determine these groups. ex:3.13

3.14 Find all ternary cubics C such that $\mathrm{VSP}(C,4)^o = \emptyset$. ex:3.14

3.15 Show that a plane cubic curve belongs to the closure of the Fermat locus if and only if it admits a first polar equal to a double line or the whole space. ex:3.15

3.16 Show that any plane cubic curve can be projectively generated by three pencils of lines. ex:3.16

3.17 Given a nonsingular conic K and a nonsingular cubic C, show that the set of points x such that $P_x(C)$ is inscribed in a self-polar triangle of K is a conic. ex:3.17

3.18 A complete quadrilateral is inscribed in a nonsingular plane cubic. Show that the tangent lines at the two opposite vertices intersect at a point on the curve. Also, show that the three points obtained in this way from the three pairs of opposite vertices are collinear. ex:3.18

3.19 Let $\mathfrak{o}$ be a point in the plane outside of a nonsingular plane cubic C. Consider the six tangents to C from the point $\mathfrak{o}$. Show that there exists a conic passing through the six points on C which lie on the tangents but not equal to the tangency points. It is called the *satellite conic* of C CremonaIntr [182]. Show that this conic is tangent to the polar conic $P_{\mathfrak{o}}(C)$ at the points where it intersects the polar line $P_{\mathfrak{o}^2}(C)$. ex:3.19

3.20 Show that two general plane cubic curves C_1 and C_2 admit a common polar pentagon if and only if the planes of apolar conics $|\mathrm{AP}_2(C_1)|$ and $|\mathrm{AP}_2(C_2)|$ intersect. ex:3.20

3.21 Let C be a nonsingular cubic and K be its apolar cubic in the dual plane. Prove that, for any point on C, there exists a conic passing through this point such that the remaining five intersection points with C form a polar pentagon of K Schlesinger1 [660]. ex:3.21

3.22 Let p, q be two distinct points on a nonsingular plane cubic curve. Starting from an arbitrary point p_1 find the third intersection point q_1 of the line $\overline{pp_1}$ with C, then define p_2 as the third intersection point of the line $\overline{qq_1}$ with C, and continue in this way to define a sequence of points $p_1, q_1, p_2, q_2, \ldots, q_k, p_{k+1}$ on C. Show that $p_{k+1} = p_1$ if and only if $p - q$ is a k-torsion point in the group law on C defined by a choice of some inflection point as the zero point. The obtained polygon $(p_1, q_1, \ldots, q_k, p_1)$ is called the *Steiner polygon* inscribed in C. ex:3.22

3.23 Show that the polar conic $P_x(C)$ of a point x on a nonsingular plane cubic curve C cuts out on C the divisor $2x + a + b + c + d$ such that the intersection points $\overline{ab} \cap \overline{cd}$, $\overline{ac} \cap \overline{bd}$ and $\overline{ad} \cap \overline{bc}$ lie on C. ex:3.23

3.24 Show that any intersection point of a nonsingular cubic C and its Hessian curve is a sextactic point on the latter. ex:3.24

3.25 Fix three pairs (p_i, q_i) of points in the plane in general position. Show that the closure of the locus of points x such that the three pairs of lines $\overline{xp_i}, \overline{xq_i}$ are members of a g_2^1 in the pencil of lines through x is a plane cubic.

3.26 Fix three points p_1, p_2, p_3 in the plane and three lines ℓ_1, ℓ_2, ℓ_3 in general position. Show that the set of points x such that the intersection points of $\overline{xp_i}$ with ℓ_i are collinear is a plane cubic curve [354].

Historical Notes

The theory of plane cubic curves originates from the works of I. Newton [549] and his student C. MacLaurin [502]. Newton was the first to classify real cubic curves, and he also introduced the Weierstrass equation. Much later, K. Weierstrass showed that the equation can be parameterized by elliptic functions, the Weierstrass functions $\wp(z)$ and $\wp(z)'$. The parameterization of a cubic curve by elliptic functions was widely used for defining a group law on the cubic. We refer to [656] for the history of the group law on a cubic curve. Many geometric results on cubic curves follow simply from the group law and were first discovered without using it. For example, the fact that the line joining two inflection points contains the third inflection point was discovered by MacLaurin much earlier before the group law was discovered. The book of Clebsch and Lindemann [150] contains many applications of the group law to the geometry of cubic curves.

The Hesse pencil was introduced and studied by O. Hesse [387],[388]. The pencil was also known as the *syzygetic pencil* (see [150]). It was widely used as a canonical form for a nonsingular cubic curve. More facts about the Hesse pencil and its connection to other constructions in modern algebraic geometry can be found in [22].

The Cayleyan curve first appeared in Cayley's paper [102]. The Schläfli equation of the dual curve from the Exercises was given by L. Schläfli in [657]. Its modern proof can be found in [325].

The polar polygons of plane cubics were first studied by F. London [492]. London proves that the set of polar 4-gons of a general cubic curve are base points of apolar pencils of conics in the dual plane. A modern treatment of some of these results is given in [235] (see also [613] for related results). A beautiful paper by G. Halphen [372] discusses the geometry of torsion points on plane cubic curves.

Poloconics of a cubic curve are studied extensively in Durège's book [265]. The term belongs to L. Cremona [182] (conic polar in Salmon's terminology). O. Schlessinger proved in [660] that any polar pentagon of a nonsingular cubic curve can be inscribed in an apolar cubic curve.

The projective generation of a cubic curve by a pencil and a pencil of conics was first given by M. Chasles. Other geometric ways to generate a plane cubic are discussed in Durège's book [265]. Steiner polygons inscribed in a plane cubic were introduced by J. Steiner in [721]. His claim that their existence is an example of a porism was given without proof. The proof was later supplied by A. Clebsch [142].

The invariants S and T of a cubic ternary form were first introduced by Aronhold [19]. G. Salmon gave the explicit formulas for them in [652]. The basic covariants and contravariants of plane cubics were given by A. Cayley [107]. He also introduced 34 basic concomitants [122]. They were later studied in detail by A. Clebsch and P. Gordan [147]. The fact that they generate the algebra of concomitants was first proved by P. Gordan and M. Noether [346], and by S. Gundelfinger [368]. A simple proof for the completeness of the set of basic covariants was given by L. Dickson [224]. One can find an exposition on the theory of invariants of ternary cubics in classical books on the invariant theory [351], [286].

Cremona's paper [182] is a fundamental source of the rich geometry of plane curves, and in particular, cubic curves. Other good sources for the classical geometry of cubic curves are books by Clebsch and Lindemann [150], t. 2, by H. Durège [265], by G. Salmon [652], by H. White [807] and by H. Schroter [664].

4

Determinantal Equations

Ch4

In this chapter, we will study varieties in $\mathbb{P}^n$ defined by minors of a matrix whose entries are homogeneous polynomials on projective coordinates.

4.1 Determinantal Varieties

S:4.1

4.1.1 Determinantal representation

SS:4.1.1

Let U and V be vector spaces over a field $\Bbbk$ of dimensions k and m. An element $\sigma \in U \otimes V$ can be viewed as a linear map $\sigma : U^\vee \to V$, or as a bilinear form on $U^\vee \otimes V^\vee$. Under the natural isomorphism $\tau : U \otimes V \to V \otimes U$, the map $\sigma : U^\vee \to V$ changes to the transpose map ${}^t\sigma : V^\vee \to U$. In the case $U = V$, the fixed points of τ in $U \otimes U$ correspond to quadratic forms on $U^\vee$ which we studied in Section S:2.4 2.4.

Let $(U \otimes V)_r$ be the subsvariety of tensors σ (considered as linear maps) of rank $\le r$. We denote by $|U \otimes V|$ the projective space $\mathbb{P}(U^\vee \otimes V^\vee)$ and denote by $|U \otimes V|_r$ the images of $(U \otimes V)_r$ in $|U \otimes V|$. The varieties $|U \otimes V|_r$ are closed subvarieties of the projective space $|U \otimes V|$, called the *universal determinant varieties*.

We set

$$ {}^lN(\sigma) = \mathrm{Ker}(\sigma) \subset U^\vee, \quad {}^rN(\sigma)) = \mathrm{Ker}({}^t\sigma) \subset V^\vee $$

the left (resp. the right) kernel of σ. We have

$$ {}^lN(\sigma)^\perp = (U^\vee/{}^lN(\sigma))^\vee \subset U = \mathrm{Im}({}^t\sigma), $$

$$ {}^rN(\sigma)^\perp = (V^\vee/{}^rN(\sigma))^\vee \subset V = \mathrm{Im}(\sigma). $$

Let

$$ \mathsf{u}_{UV} : (U^\vee)_{|U\otimes V|} \to (U^\vee)_{|U\otimes V|}(1) $$

be the universal linear map similar to the one we considered for quadratic forms. We can consider it as a section of the locally free sheaf $(U \otimes V)_{|U\otimes V|}(1)$. For any scheme X over $\Bbbk$, an invertible sheaf $\mathcal{L}$ on X, and a tensor $t \in (U\otimes V)_X \otimes \mathcal{L}$, there exists a unique morphism $f : X \to |U \otimes V|$ such that $f^*(\mathsf{u}_{UV}) = t$ and $f^*\mathcal{O}_{|U\otimes V|}(1) = \mathcal{L}$.

Let

$$\wedge^i \mathsf{u}_{UV} : \bigwedge^i U^\vee_{|U\otimes V|} \to \bigwedge^i V_{|U\otimes V|}(i)$$

be the ith exterior power of u_{UV}. We may consider it as an element of $\bigwedge^i(U \otimes V)_{|U\otimes V|}(i) = \bigwedge^i U_{|U\otimes V|} \otimes \bigwedge^i V)_{|U\otimes V|}(i)$. Then,

$$|U \otimes V|_r = Z(\wedge^{r+1}\mathsf{u}_{UV})$$

is the scheme of zeros of the section $\wedge^i \mathsf{u}_{UV}$ of $\bigwedge^{r+1}$.

Let

$$|U \otimes V|_r^\circ = \{[\sigma] \in |U \otimes V|_r : \mathrm{rank}(\sigma) = r\}.$$

This is an open subset of $|U \otimes V|_r$. Assume $k \le m$. Then $|U \otimes V|_k^\circ$ coincides with the largest open subset of $|U \otimes V|$ such that $\mathrm{Coker}(\mathsf{u}_{UV})$ is locally free sheaf of rank $m - k$ on this subset. Since $U_{|U\otimes V|}$ is a locally free sheaf, it does not have torsion subsheaves, hence, we have an exact sequence on $|U \otimes V|$

$$0 \to U_{|U\otimes V|}(-1)^\vee \overset{\mathsf{u}_{UV}}{\to} V_{|U\otimes V|} \to \mathcal{T}_{UV} \to 0, \tag{4.1}$$ Texseq

which is locally split over $|U \otimes V|_k^\circ$. Applying the functor $\mathcal{H}om$, and twisting by $\mathcal{O}(-1)$, we obtain the exact sequence

$$0 \to \mathcal{T}_{UV}^\vee(-1) \to V^\vee_{|U\otimes V|}(-1) \to U_{|U\otimes V|} \to \mathcal{E}xt^1(\mathcal{T}_{UV}, \mathcal{O}_{|U\otimes V|})(-1) \to 0$$

The first exact sequence defines a projective resolution of $\mathcal{T}_{|U\otimes V|}$ of length 2. For any point $[\sigma] \in |U \otimes V|$, we have $\mathrm{depth}((\mathcal{T}_{|U\otimes V|})_{[\sigma]}) \le \dim |U \otimes V| - 1 = km - 1$. If $k < m$, then $\mathrm{depth}((\mathcal{T}_{|U\otimes V|})_{[\sigma]}) \ge 2$. By definition, $\mathcal{T}$ is a *reflexive sheaf* on $|U \otimes V|$. . Equivalently, the canonical homomorphism $\mathcal{T}_{UV} \to j_*j^*\mathcal{T}_{|U\otimes V|}$ is an isomorphism, where j is an open embedding with complement of codimension ≥ 2.

If $k = m$, $\mathcal{T}_{|U\otimes V|}$ is supported on a proper closed subset $|U \otimes V|_{k-1}$, hence, $\mathcal{T}^\vee_{|U\otimes V|}(-1) = \{0\}$, and $\mathcal{T}_{VU} = \mathcal{E}xt^1(\mathcal{T}_{|U\otimes V|}, \mathcal{O}_{|U\otimes V|})(-1)$ is a reflexive sheaf. It is equal to the pull-back of $\mathcal{T}_{UV}$ under the transpose isomorphism $|V \otimes U| \to |U \otimes V|$.

The stratification (Z_c) considered in Subsection ~~2.4.2~~ SS:2.4.2 identifies Z_c with $|U \otimes V|^\circ_{m-r}$ and its closure $\overline{Z}_c$ with $|U \otimes V|_{m-r}$. Thus, $|U \otimes V|$ is stratified by closed subsvarieties

$$|U\otimes V| = |U\otimes V|_s \supset \cdots \supset |U\otimes V|_r \supset \cdots \supset |U\otimes V|_1 \supset \emptyset,$$

where $s = \min\{k, m\}$.

Let $\mathcal{T}_{UV}(r)$ be the restriction of $\mathcal{T}_{UV}$ to $'|U\otimes V|_r$. It is a locally free sheaf of rank $m-r$, and $|U\otimes V|_r'$ is the largest subset satisfying this property.

The surjection $V_{|U\otimes V|_r'} \to \mathcal{T}_{UV}(m-r)(-1)$ defines a morphism

$$\mathfrak{r}_{UV}(r) : |U\otimes V|_r' \to G(m-r, V^\vee) \cong G(r, V), \quad \sigma \mapsto \mathbb{P}(\mathrm{Coker}(\sigma)) = |^{m-r}N(\sigma)|$$

such that $\mathcal{T}_{UV}(r) = \mathfrak{r}_{UV}(r)^*\mathcal{K}(V)_r^\vee$, where $\mathcal{K}(V)_r$ is the universal subsheaf over the Grassmannian $G(r, V)$. Dually, we get the maps

$$\mathfrak{l}_{UV}(r) : |U\otimes V|_r' \to G(r, U) \cong G(k-r, U^\vee), \quad \sigma \mapsto |^l N(\sigma)|$$

such that ${}^t\mathcal{T}_{UV}(r) := \tau^*\mathcal{T}_{VU}(r) \cong \mathfrak{l}_{UV}(r)^*(\mathcal{R}_r)$, where $\mathcal{R}(U)_r$ is the universal quotient sheaf on $G(r, U)$.

Choosing a basis $(e_1, \ldots, e_k)$ in U and a basis $(e_1', \ldots, e_m')$ in V, we can identify the linear space $U^\vee \otimes V$ with the linear space of matrices $\mathrm{Mat}_{m,k}$ of size $m\times k$ with entries in $\Bbbk$, and the projective space $|U\otimes V|$ with $|\mathrm{Mat}_{m,k}|$. The subvariety $|U\otimes V|_r$ is identified with the variety of nonzero matrices (up to scalar multiple) of rank $\le r$.

The following theorem is an analog of Corollary cor:one 2.4.17 in the symmetric case (see ACGH [13, Chapter II]).

multdet **Theorem 4.1.1.** *Let* $\mathrm{Mat}_r(m,k)(r) \subset \mathbb{A}^{k\times m}$ *be the variety of matrices of rank* $\le r$. *Then,*

- $\mathrm{Mat}_r(m,k)$ *is an irreducible Cohen-Macaulay variety of codimension* $(m-r)(k-r)$*;*
- $\mathrm{Sing}(\mathrm{Mat}_r(m,k)) = \mathrm{Mat}_{r-1}(m,k)$*;*
- *the multiplicity of* $\mathrm{Mat}_r(m,k)$ *at a point* A *of rank* $s \le r$ *is equal to*

$$\mathrm{mult}_A \mathrm{Mat}_r(m,k) = \prod_{j=0}^{k-r-1} \frac{(m-s+j)!j!}{(r-s+j)!(m-r+j)!},$$

  in particular,
- *the degree of* $\mathrm{Mat}_{m,k}(r)$ *is equal to*

$$\deg \mathrm{Mat}_r(m,k) = \mathrm{mult}_0 \mathrm{Mat}_r(m,k) = \prod_{j=0}^{k-r-1} \frac{(m+j)!j!}{(r+j)!(k-r+j)!}.$$

Note that the formula for the degree of $\mathrm{Mat}_r(m,k)$ is originally due to C. Segre SegreDet [693].

segrelin *Example* 4.1.2. The last piece of the stratification $|U \otimes V|_1$ coincides with $|U \otimes V|_1'$. It is isomorphic to $|U| \times |V| \cong \mathbb{P}^{k-1} \times \mathbb{P}^{m-1}$ embedded in $|U \otimes V| \cong \mathbb{P}^{km-1}$ via the Segre map. Its degree is equal to the degree of the Segre variety equal to $\binom{k+m-2}{k-1}$ [375, Example 18.15]. It agrees with the formula for the degree of $\mathrm{Mat}_1(m, k)$.

tangentspace **Proposition 4.1.3.** *Let $\mathbb{T}_{[\sigma]}(|U \otimes V|)$ be the embedded tangent space of $|U \otimes V|_r'$ at a point $\sigma \in |U \otimes V|_r'$. Then*

$$\mathbb{T}_{[\sigma]}(|U \otimes V|) = |\{\phi : U^\vee \to V : \phi(\mathrm{Ker}(\sigma)) \subset \sigma(U^\vee)\}|$$

$$= |\{\phi \in U \otimes V : \phi(u^* \otimes v^*) = 0, \forall u^* \in {}^lN(\sigma), v^* \in {}^rN(\sigma)\}|.$$

Proof Let

$$\widetilde{|U \otimes V|}_r = \{([\sigma], x) \in |U \otimes V| \times |V| : x \in |\sigma(U^\vee)|\}.$$

The second projection to $|V|$ exhibits $\widetilde{|U \otimes V|}$ as a projective vector bundle of relative dimension $km - (k-r)(m-r)$. This implies that $\widetilde{|U \otimes V|}_r$ is a smooth variety of dimension $\dim |U \otimes V|_r = km - (k-r)(m-r)$. The first projection to $|U \otimes V|$ is a proper map which is an isomorphism over $|U \otimes V|_r'$. It is a resolution of singularities

$$\pi_r : \widetilde{|U \otimes V|}_r \to |U \otimes V|_r.$$

It identifies the embedded tangent space $\mathbb{T}_{[\sigma]}(|U \otimes V|_r)$ at a point $[\sigma] \in |U \otimes V|_r'$ with the projective space of maps $\phi : U^\vee \to V$ such that $\phi(\mathrm{Ker}(\sigma)) \subset \sigma(U^\vee)$. If we view σ as a bilinear form on $U^\vee \otimes V^\vee$, then the tangent space consists of bilinear forms $\tau \in U \otimes V$ such that $\tau(u^* \otimes v^*) = 0$ for all $u^* \in {}^lN(\sigma), v^* \in {}^rN(\sigma)$. □

Remark 4.1.4. There is a notion of a *complete collineation* similar to the notion of a complete quadric. A complete collineation is a point in the variety equal to the closure of the graph of the map

$$|U \otimes V|_s' \to |\bigwedge^2(U \otimes V)| \times \cdots \times |\bigwedge^s(U \otimes V)|, \quad [\sigma] \mapsto ([\wedge^2\sigma], \ldots, [\wedge^s\sigma]),$$

where $s = \min\{k, m\}$. We refer the interested reader to [477], [769], [779] for exposition of this theory.

normaldet **Proposition 4.1.5.** *Let $\mathcal{N}(r)$ be the normal sheaf of $|U \otimes V|_r'$ in $|U \otimes V|$. Then*

$$\mathcal{N}(r) \cong \mathcal{K}_r(U)^\vee \otimes \mathcal{R}_r(U)(1),$$

where $\mathcal{R}_r(U)$ is the universal quotient sheaf over $G(r, U)$ and $\mathcal{R}_r(U)$ is the universal subsheaf over $G(r, V)$.

Let E be a linear space of dimension $n+1$. It follows from the universal property of the map u_{UV} that there is a bijective correspondence between regular maps $\phi : |E| \to |U \otimes V|$ with $\phi^*\mathcal{O}_{|U\otimes V|}(1) \cong \mathcal{O}_{|E|}(l)$ and a homomorphisms of locally sheaves

$$\tilde{\phi} := \phi^*(\mathsf{u}_{UV}) : U^\vee_{|E|}(-l) \to V_{|E|},$$

or, equivalently, a section of $(U \otimes V)_{|E|}(l)$. The composition of ϕ with the transpose isomorphism defines the transpose determinantal representation ${}^t\phi : |E| \to |V \otimes U|$.

We set

$$D_r(\phi) = \phi^{-1}(|U \otimes V|_r) = \{x \in |E| : \mathrm{rank}(\tilde{\phi}(x)) \le r\}.$$

Clearly

$$D_r({}^t\phi) = \phi^{-1}(\tau^{-1}(|V \otimes U|_r).$$

Definition 4.1.6. *A* determinantal variety *of type* $(|k, m|_r, n)_l$ *in* $|E| \cong \mathbb{P}^n$ *is the pre-image of* $|U \otimes V|_r$ *under some regular map* $\phi : |E| \to |U \otimes V|$ *such that* $\phi^*\mathcal{O}_{|U\otimes V|}(1) \cong \mathcal{O}_{|E|}(l)$.

We will skip the subscript l if $l = 1$, i.e., we write $(|k, m|_r, n)$ if $l = 1$. In this case, we say that the determinantal representation is *linear*. We also abbreviate $(|k, k|_r, n)_l = [k_r, n]_l$.

We say that a projective subvariety $X \subset |E|$ admits a *determinantal representation* of type $(|k, m|_r, n)_l)$ if there exists $\phi : |E| \to |U \otimes V|$ such that $X = D_r(\phi)$.

We say that a determinantal representation of $X \subset |E|$ of type $([U, V]_r, n)_d$ is *proper* if, for any $r' \le r$

$$\mathrm{codim}(X \cap \phi^{-1}(|U \otimes V|_{r'}), X) = \mathrm{codim}(|U \otimes V|_{r'}, |U \otimes V|_r) = (r-r')(k+m-r-r').$$

In particular, this implies that $D_r(\phi)$ is a Cohen-Macaulay variety of codimension $(m-r)(k-r)$ in $|E|$ Fulton [315, Theorem 14.4].

We also say that $\phi : |E| \to |U \otimes V|$ is *transversal* if

$$\mathrm{Sing}(\phi^{-1}(|U \otimes V|_r)) = \phi^{-1}(|U \otimes V|_{r-1}), \quad r < \min\{m, k\}.$$

The following proposition follows immediately from Proposition tangentspace 4.1.3.

nonsing **Proposition 4.1.7.** *Assume* ϕ *is proper. A point* $[x] \in D_r(\phi) \setminus D_{r-1}(\phi)$ *is nonsingular if and only if*

$$\dim\{x \in |E| : \phi(x)(\mathrm{Ker}(\phi(x)) \otimes \mathrm{Ker}({}^t\phi(x))) = 0\} = n + 1 - (m-r)(k-r).$$

For example, suppose $n = m = d$ and $r = d-1$. Let $[x] \in D_{d-1}(\phi) \setminus D_{d-2}(\phi)$. Then, $\mathrm{Ker}(\phi(x))$ and $\mathrm{Ker}({}^t\phi(x))$ are 1-dimensional subspaces. Let u^*, v^* be their respective bases. Then, $[x]$ is a nonsingular point on $D_{d-1}(\phi)$ if and only if the tensor $u^* \otimes v^*$ is not contained in the kernel of the map ${}^t\phi : U^\vee \otimes V^\vee \to E^\vee$.

Example 4.1.8. The variety $|U \otimes V|_1$ is isomorphic to the Segre variety $\mathbb{P}^1 \to \mathbb{P}^{n-1} \hookrightarrow \mathbb{P}^{2n-1}$. Let $|E|$ be embedded in $\mathbb{P}^{2n-1}$ as a linear subspace defined by equation $t_{01} - t_{12} = 0, \ldots, t_{0n} - t_{1n-1} = 0$, where $t_{ij} = x_i y_j, i = 0, 1, j = 0, \ldots, n$ are projective coordinates in $\mathbb{P}^{2n-1}$. We use the coordinates $(t_0, \ldots, t_n) = (t_{01}, \ldots, t_{0n})$ in $|E|$. Then $|E| \cap |U \otimes V|_1$ is given by equations expressing the condition

$$\mathrm{rank}\begin{pmatrix} t_0 & t_1 & \ldots & t_{n-1} \\ t_1 & t_2 & \ldots & t_n \end{pmatrix} = 1.$$

This gives a determinantal representation of type $(|2, n|_1, n)_1$ for a normal rational curve (or, a Veronese curve) $\mathsf{v}_n(\mathbb{P}^1)$ in $\mathbb{P}^n$ from Subsection SS:3.3.1 3.3.1.

More generally, in Example ex:1.4.1 1.4.1, we considered the catalecticant matrix $\mathrm{Cat}_k(f)$ of size $(k+1) \times (d-k+1)$. Assuming $s \le k \le d-k$, we showed that the determinant variety of type $(|k+1, d-k+1|_r, d)_1$ defined by the condition

$$\mathrm{rank}(\mathrm{Cat}_k(f)) \le r$$

coincides with the secant variety $\mathrm{Sec}_r(R_d)$ of the Veronese curve $\mathsf{V}_1^d \subset \mathbb{P}^d$.

scroll *Example* 4.1.9. This example is a generalization of the previous example. Let $(a_1, \ldots, a_k)$ be non-negative integers with $a_1 + \cdots + a_k = n - k + 1$. Let $L_1, \ldots, L_k$ be linear subspaces of E of dimensions $a_i + 1$ such that $E = L_1 \oplus \cdots \oplus L_k$. Fix a rational normal curve R_i of degree a_i in each $|L_i| \cong \mathbb{P}^{a_i}$ and an isomorphism $\phi_i : \mathbb{P}^1 \to R_i$ (we agree that R_i is a point p_i if $a_i = 0$). Let $X = X_{a_1,\ldots,a_k}$ be the *join* of $R_1, \ldots, R_k$, that is, the smallest closed subvariety of $|E|$ that contains all $k-1$-dimensional subspaces $\langle \phi_1(t), \ldots, \phi_k(t) \rangle, t \in \mathbb{P}^1$. In other words, $X_{a_1,\ldots,a_k}$ is the projection to $|E|$ of the closure of the graph of the rational map $\mathbb{P}^1 \dashrightarrow G(k, E), t \mapsto \langle \phi_1(t), \ldots, \phi_k(t) \rangle$. Yet another description of $X_{a_1,\ldots,a_k}$ is as the image of the projective bundle over $\mathbb{P}^1$ $\mathbb{P}(\mathcal{E})$, where $\mathcal{E} = \mathcal{O}_{\mathbb{P}^1}(a_1) \oplus \cdots \mathcal{O}_{\mathbb{P}^1}(a_k)$ to $\mathbb{P}(\mathcal{O}_{\mathbb{P}(\mathcal{E})}(1))$ under the morphism ϕ given by the complete linear system $|\mathcal{O}_{\mathbb{P}(\mathcal{E})}(1)|$. Each surjection $\mathcal{E} \to \mathcal{O}_{\mathbb{P}^1}$ defines an embedding $\mathbb{P}^1 \to \mathbb{P}(\mathcal{E})$ whose composition with ϕ is equal to ϕ_i. Let $\eta = c_1(\mathcal{O}_{\mathbb{P}(\mathcal{E})}(1))$. Since $c_1(\mathcal{E}) = a_1 + \cdots + a_k$ and $c_i(\mathcal{E}) = 0, i > 0$, formula cohprojbundle (2.57) gives

$$\eta^k = \deg(X_{a_1,\ldots,a_k}) = h^{k-1} p^*(c_1(\mathcal{E})) = a_1 + \cdots + a_k = n - k + 1. \quad (4.2)$$ scolldegree

Of course, this formula can be also obtained by an elementary argument. If $a_1 = \cdots a_s = 0$, the variety $X_{a_1,\ldots,a_k}$ is a cone over $X_{a_{s+1},\ldots,a_k}$ with vertex

$\langle p_1, \ldots, p_s \rangle \cong \mathbb{P}^{s-1}$ The variety $X_{a_1,\ldots,a_k}$ is an example of a *scroll* which we will encounter often in the sequel. It is a determinantal variety of type $(|2, n-k+1|_1, n)_1$. We assume that $X_{a_1,\ldots,a_k}$ is not a cone, it is obvious that a cone over a determinantal variety is a determinantal variety of similar type. Then $X_{a_1,\ldots,a_k}$ is defined by the matrix

$$A = \begin{pmatrix} x_0^{(1)} & \ldots & x_{a_1-1}^{(1)} & \ldots & x_0^{(k)} & \ldots & x_{a_k-1}^{(k)} \\ x_1^{(1)} & \ldots & x_{a_1}^{(1)} & \ldots & x_1^{(k)} & \ldots & x_{a_k}^{(k)} \end{pmatrix},$$

where $(x_0^i, \ldots, x_{a_i}^i)$ are projective coordinates in $|L_i|$. The proof easily follows from the previous example and is left to the reader. Note that $\dim X_{a_1,\ldots,a_k} = k = \dim \mathrm{Mat}_1(2, n-k+1) - 1$, hence, the determinant representation is proper and $X_{a_1,\ldots,a_k}$ are Cohen-Macaulay variety.

Remark 4.1.10. Determinantal varieties are examples of *projectively generated* varieties. Let us elaborate.

We consider the map $\phi : |E| \to |U \otimes V|$ as coming from a linear map $E \to U \otimes V$. It can be considered as a tensor in $E^\vee \otimes U \otimes V$, or, as a linear map $\phi : U^\vee \to E^\vee \otimes V$. Let $A = (a_{ij})$ be a $m \times k$ matrix, where $a_{ij} \in E^\vee$ are linear forms in variables $t_0, \ldots, t_n$. Choose coordinates $(u_1, \ldots, u_k) \in U^\vee$ in U, and a basis $e_1^V, \ldots, e_m^V$ in V. Then ϕ is given by

$$\phi(u_j) = \sum_{i=1}^{m} a_{ij} e_i^V, \quad j = 1, \ldots, k.$$

Thus, each $u = \sum_{j=1}^{k} \alpha_j u_j$ defines m hyperplanes

$$H_i(u) = V(\sum_{j=1}^{k} a_{ij}\alpha_j), \quad i = 1, \ldots, m.$$

Each hyperplane vanishes on the subspace $B_i = V(a_{i1}, \ldots, a_{ik})$. In classical terminology, they belong to the *star* $]B_i[$ of B_i. They form a linear subspace $B_j^\perp$ of the dual space $|E^\vee|$.

Consider the subvariety of $\mathbb{P}^n$

$$X = \{x \in |E| : x \in H_1(u) \cap \ldots \cap H_k(u), \text{ for some } u \in |U^\vee|\}.$$

It is clear that

$$X = \{x \in |E| : \mathrm{rank} A(x) < k\}.$$

If $m < k$, we have $X = \mathbb{P}^n$, so we assume that $k \leq m$. If not, we replace A with its transpose matrix. In this way, we obtain a proper subvariety X of $\mathbb{P}^n$, a hypersurface, if $m = k$, with linear determinantal representation $X = \det A$. For any $x \in X$, let

$$^{l}N(x) := \{u \in |U^{\vee}| : x \in H_1(u) \cap \ldots \cap H_k(u)\}.$$

Then,

$$X_r' = \{x \in X : \dim {}^{l}N(x) \geq k - r\}, \quad r \leq k-1,$$

is the determinantal subvariety of $\mathbb{P}^n$ of type $(|k, m|_r, n)_1$ given by the condition $\mathrm{rank} A(x) \leq r$.

We will be mostly concerned with the determinantal representations of type $(d, d)_{d-1}, n)_l$ for a hypersurface $X = V(f)$ in $|E| = \mathbb{P}^n$ of degree ld. In this case, f is equal to the determinant of a square matrix of size $d \times d$ whose entries are homogeneous polynomials in coordinates in $|E|$ of degree l.

Note that a proper determinantal representation of $X \subset \mathbb{P}^n$ of type $(|k, m|_r, n]_l$ defines two coherent sheaves $\mathcal{T}_\phi := \phi^*\mathcal{T}_{UV}$ and ${}^t\mathcal{T}_\phi := \phi^*({}^t\mathcal{T}_{VU})$ on $|E| = \mathbb{P}^n$. Assume that $k \leq m$, We have an exact sequence:

$$\begin{gathered} 0 \to U_{|E|}^{\vee}(-l) \to V_{|E|} \to \mathcal{T}_\phi \to 0, \\ \mathcal{T}_\phi^{\vee} \to V_{|E|}^{\vee}(-l) \to U_{|E|} \to {}^t\mathcal{T}_\phi \to 0, \end{gathered} \tag{4.3}$$

resolvent1

Passing to the duals in the first exact sequence, we get

$${}^t\mathcal{T}_\phi \cong \mathcal{E}xt^1_{\mathcal{O}_{|E|}}(\mathcal{T}_\phi), \mathcal{O}_{|E|})(-l) \cong \mathcal{E}xt^1_{\mathcal{O}_{|E|}}(\mathcal{T}_\phi), \mathcal{O}_{|E|}(-l)). \tag{4.4}$$

firstext

Assume $k < m$, then $\mathrm{codim}(|U \otimes V|_{k-1}, |U \otimes V|) = m - k + 1 \geq 2$. For any coherent sheaf $\mathcal{F}$ on $|E|$, we have

$$\mathrm{depth}(\mathcal{F}_x) + \mathrm{proj.dim}(\mathcal{F}_x) = n \tag{4.5}$$

auslander

This implies that $\mathrm{depth}((\mathcal{T}_\phi)_x) = n - 1$. If $n \geq 3$, $\mathcal{T}_\phi$ is torsion-free, and, for any $x \in |E|$, $\mathrm{depth}((\mathcal{T}_\phi))_x \geq 2$. This shows that $\mathcal{T}_\phi$ is a *reflexive sheaf* on $|E|$ [379, Proposition 1.3]. Recall that a coherent sheaf $\mathcal{F}$ on a normal variety X is called *reflexive* if, for any open embedding $j : X \setminus Z$, where $\mathrm{codim}(Z, X) \geq 2$, the canonical homomorphism $\mathcal{F} \to j_*j^*\mathcal{F}$ is an isomorphism. A reflexive sheaf is free outside a closed subset of codimension ≥ 3.

Hartshorne

Restricting the exact sequence to $X = D_r(\phi)$, we obtain that X is equal to $\phi^{-1}(|U \otimes V|_r)$. We assume that ϕ is proper, so that $\mathrm{codim}(D_r(\phi), \mathbb{P}^n) = km - (m-r)(n-r)$. The surjections $V_X \to \mathcal{T}_\phi(r)$ and $U \to {}^t\mathcal{T}_\phi(r)$ define morphisms

$$\mathfrak{l} : X \to G(m, V), \quad \mathfrak{r} : X' \to G(k, U)$$

such that $\mathfrak{l}^*\mathcal{R}(V) \cong \mathcal{T}_\phi(r)$ and $\mathfrak{r}^*\mathcal{R}(U) \cong {}^t\mathcal{T}_\phi(r)$.

generalization *Remark* 4.1.11. We can follow the exposition in Section S:2.4 2.4 and replace the linear spaces U and V with locally free sheaves $\mathcal{U}$ and $\mathcal{V}$ on a Cohen-Macaulay scheme T over $\Bbbk$. This leads to the projective bundle $|\mathcal{U} \otimes_{O_T} \mathcal{V}|$ on T, and morphism $\phi : |\mathcal{E}| \to |\mathcal{U} \otimes_{O_T} \mathcal{V}|$. We are not pursuing this, and leave it to the reader.

4.1.2 Determinantal hypersurfaces and aCM-sheaves

SS:4.1.2

Let $\mathcal{F}$ be a coherent sheaf on $\mathbb{P}^n$ and

$$\Gamma_*(\mathcal{F}) = \bigoplus_{k=0}^{\infty} H^0(\mathbb{P}^n, \mathcal{F}(k)).$$

It is a graded module over the graded ring

$$S = \Gamma_*(O_{\mathbb{P}^n}) = \bigoplus_{k=0}^{\infty} H^0(\mathbb{P}^n, O_{\mathbb{P}^n}(k)) \cong \mathbb{C}[t_0, \dots, t_n].$$

We say that $\mathcal{F}$ is an *arithmetically Cohen-Macaulay sheaf* (*aCM sheaf*, for brevity) if $M = \Gamma_*(\mathcal{F})$ is a graded Cohen-Macaulay module over S. Recall that this means that every localization of M is a *Cohen-Macaulay module*, i.e. its depth is equal to its dimension. Let us identify M with the coherent sheaf on Spec A. The associated sheaf $\tilde{M}$ on Proj S is isomorphic to $\mathcal{F}$. Let $U = \text{Spec } S \setminus \mathfrak{m}_0$, where $\mathfrak{m}_0 = (t_0, \dots, t_n)$ is the irrelevant maximal ideal of the graded ring S. Since the projection $U \to \text{Proj } S = \mathbb{P}^n$ is a smooth morphism, the localizations of M at every maximal ideal different from $\mathfrak{m}$ are Cohen-Macaulay modules if and only if

- $\mathcal{F}_x$ is a Cohen-Macaulay module over $O_{\mathbb{P}^n,x}$ for all $x \in \mathbb{P}^n$.

The condition that the localization of $M = \Gamma_*(\mathcal{F})$ at $\mathfrak{m}_0$ is Cohen-Macaulay is satisfied if and only if the local cohomology $H^i_{\mathfrak{m}_0}(M)$ vanish for all i with $0 \le i < \dim M$. We have $H^i(U, M) = \oplus_{k\in\mathbb{Z}} H^i(\mathbb{P}^n, \tilde{M}(k))$. The exact sequence of local cohomology gives an exact sequence

$$0 \to H^0_{\mathfrak{m}}(M) \to M \to H^0(U, M) \to H^1_{\mathfrak{m}}(M) \to 0,$$

and isomorphisms

$$H^{i+1}_{\mathfrak{m}}(M) \cong H^i(U, M), \quad i > 0.$$

In the case $M = \Gamma_*(\mathcal{F})$, the map $M \to H^0(U, M) = \Gamma_*(\tilde{M})$ is an isomorphism, hence, $H^0_{\mathfrak{m}}(M) = H^1_{\mathfrak{m}}(M) = 0$. Since the canonical homomorphism $\widetilde{\Gamma_*}(\mathcal{F}) \to \mathcal{F}$ is bijective, the conditions $H^i_{\mathfrak{m}}(M) = 0, i > 1$, become equivalent to the conditions

- $H^i(\mathbb{P}^n, \mathcal{F}(k)) = 0, \quad 1 \le i < \dim \operatorname{Supp}(\mathcal{F}),\ k \in \mathbb{Z}.$

A subvariety $X \subset \mathbb{P}^n$ (as always, reduced and connected) is said to be *arithmetically Cohen-Macaulay* is its structure sheaf $\mathcal{O}_X$ is an aCM sheaf. It follows from above that X is arithmetically Cohen-Macaulay if and only if

(i) X is Cohen-Macaulay;
(ii) the restriction maps $r_n : H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^k}(k)) \to H^0(X, \mathcal{O}_X(k))$ are bijective for all $k \ge 0$;
(iii) $H^i(X, \mathcal{O}_X(k)) = 0, i > 0$ and all k.

An embedding $X \hookrightarrow \mathbb{P}^n$ is called *linearly normal* of condition (ii) is satified . Moreover, if, additionally, X is a normal variety, we say that the embedding is *projectively normal*. In this case, the projective coordinate ring of X is normal, and the converse is also true.

Let $\phi : |E| \to |U \otimes V|$ be a proper determinantal representation of type $([m, m]_{m-1}, n)_l$. Let $X = D_{m-1}(\phi)$. It is a hypersurface of degree $d = lm$. We apply the previous discussion to the sheaves

$$\mathcal{L} := \mathcal{T}_{UV}(\phi), \quad \mathcal{M} := {}^t\mathcal{T}_{UV}(\phi),$$

We have the exact sequences

$$\begin{aligned} 0 &\to U^\vee_{|E|}(-l) \to V_{|E|} \to \mathcal{L} \to 0, \\ 0 &\to V^\vee_{|E|}(-l) \to U_{|E|} \to \mathcal{M} \to 0. \end{aligned} \tag{4.6}$$

mainrxseq

It follows that proj.dim $\mathcal{F}_x = 1$ for all $x \in X = \operatorname{Supp}(\mathcal{F})$. This implies that depth $\mathcal{F}_x = n - 1$ for all $x \in X$. In particular, X is hypersurface in $\mathbb{P}^n$ and the stalks of $\mathcal{F}_x$ are Cohen-Macaulay modules over $\mathcal{O}_{\mathbb{P}^n,x}$.

Proposition 4.1.12. *Assume $n > 1$.*

(i) $H^i(\mathbb{P}^n, \mathcal{L}(j)) = H^i(\mathbb{P}^n, \mathcal{M}(j)) = 0, 1 \le i \le n-1, j \in \mathbb{Z}.$
(ii) $H^0(\mathbb{P}^n, \mathcal{L}(j)) = H^0(\mathbb{P}^n, \mathcal{M}(j)) = 0, j < 0.$

Proof The first assertion follows from the cohomological characterization of aCM-sheaves. The second assertion follows from twisting exact sequence (4.6) by $\mathcal{O}_{\mathbb{P}^n}(j)$ and applying H^0. □

Consider the restriction of $\mathcal{L}$ (or $\mathcal{M}$) to X. It is a Cohen-Macaulay sheaf of rank one on X which is an invertible sheaf on $X' = X \setminus \phi^{-1}(|U \otimes V|_{m-2})$. A Cohen-Macaulay sheaf of rank 1 is defined by a Weil divisor on X, not necessarily a Cartier divisor. Recall the definitions. Let X be a Noetherian integral scheme of dimension ≥ 1 and $X^{(1)}$ be its set of points of codimension 1 (i.e. points $x \in X$ with $\dim \mathcal{O}_{X,x} = 1$). We assume that X is regular in codimension 1, i.e. all local rings of points from $X^{(1)}$ are regular. In this case, we can define *Weil divisors* on X as elements of the free abelian group $\mathrm{WDiv}(X) = \mathbb{Z}^{X^{(1)}}$ and also define linear equivalence of Weil divisors and the group $\mathrm{Cl}(X)$ of linear equivalence classes of Weil divisors (see Hartshorne [379, Chapter II, §6]).

We identify a point $x \in X^{(1)}$ with its closure E in X. We call it an *irreducible divisor*. Any irreducible reduced closed subscheme E of codimension 1 is an irreducible divisor, the closure of its generic point.

For any Weil divisor D, let $\mathcal{O}_X(D)$ be the sheaf whose section on an open affine subset U consists of functions from the quotient field $Q(\mathcal{O}(U))$ such that $\mathrm{div}(\Phi) + D \geq 0$.

It follows from the definition that $\mathcal{O}_X(D)$ is torsion-free and, for any open subset $j : U \hookrightarrow X$ which contains all points of codimension 1, the canonical homomorphism of sheaves

$$\mathcal{O}_X(D) \to j_* j^* \mathcal{O}_X(D) \tag{4.7}$$

reflexive

is an isomorphism. These two conditions characterize *reflexive sheaves* on any normal integral scheme X. It follows from the theory of local cohomology that the latter condition is equivalent to the condition that, for any point $x \in X$ with $\dim \mathcal{O}_{X,x} \geq 2$, the depth of the $\mathcal{O}_{X,x}$-module $\mathcal{F}_x$ is greater than or equal to 2. By equivalent definition, a reflexive sheaf $\mathcal{F}$ is a coherent sheaf such that the canonical homomorphism $\mathcal{F} \to (\mathcal{F}^\vee)^\vee$ is an isomorphism. The sheaves $\mathcal{O}_X(D)$ are reflexive sheaves of rank one. Conversely, a reflexive sheaf $\mathcal{F}$ of rank 1 on a normal integral scheme is isomorphic to $\mathcal{O}_X(D)$ for some Weil divisor D. In fact, we restrict $\mathcal{F}$ to some open subset $j : U \hookrightarrow X$ with the complement of codimension ≥ 2 such that $j^*\mathcal{F}$ is locally free of rank 1. Thus, it corresponds to a Cartier divisor on U. Taking the closure of the corresponding Weil divisor in X, we get a Weil divisor D on X and it is clear that $\mathcal{F} = j_* j^* \mathcal{F} \cong \mathcal{O}_X(D)$. In particular, we see that any reflexive sheaf of rank 1 on a regular scheme is invertible. It is not true for reflexive sheaves of rank > 1. They are locally free outside of a closed subset of codimension ≥ 3 (see HartshorneRef [380]).

Reflexive sheaves of rank 1 form a group with respect to the operation

$$\mathcal{L} \cdot \mathcal{G} = ((\mathcal{L} \otimes \mathcal{G})^\vee)^\vee, \ \mathcal{L}^{-1} = \mathcal{L}^\vee.$$

For any reflexive sheaf $\mathcal{L}$ and an integer n we set

$$\mathcal{L}^{[n]} = ((\mathcal{L}^{\otimes n})^\vee)^\vee.$$

One checks that

$$\mathcal{O}_X(D + D') = \mathcal{O}_X(D) \cdot \mathcal{O}_X(D')$$

and the map $D \mapsto \mathcal{O}_X(D)$ defines an isomorphism from the group $\mathrm{Cl}(X)$ to the group of isomorphism classes of reflexive sheaves of rank 1.

As we observed in (4.4),

$$\mathcal{M} = {}^t\mathcal{T}(\phi) \cong \mathcal{E}xt^1_{\mathcal{O}_{\mathbb{P}^n}}(\mathcal{L}, \mathcal{O}_{\mathbb{P}^n}(-l)). \tag{4.8}$$ sheafG

In the following, we use some standard facts from the Grothendieck-Serre duality theory (see [365]). We have

$$\begin{aligned}
&\mathcal{E}xt^1_{\mathcal{O}_{\mathbb{P}^n}}(\mathcal{L}, \mathcal{O}_{\mathbb{P}^n}(-l)) \cong \mathcal{H}om_{\mathcal{O}_X}(\mathcal{L}, \mathcal{E}xt^1_{\mathcal{O}_{\mathbb{P}^n}}(\mathcal{O}_X, \mathcal{O}_{\mathbb{P}^n}(-l)))\\
&\cong \mathcal{H}om_{\mathcal{O}_X}(\mathcal{L}, \mathcal{E}xt^1_{\mathcal{O}_{\mathbb{P}^n}}(\mathcal{O}_X, \omega_{\mathbb{P}^n}))(n-l+1) \cong \mathcal{H}om_{\mathcal{O}_X}(\mathcal{L}, \omega_X)(n-l+1)\\
&\cong \mathcal{H}om_{\mathcal{O}_X}(\mathcal{L}, \mathcal{O}_X(d-n-1))(n-l+1) \cong \mathcal{L}^\vee(d-l),
\end{aligned} \tag{4.9}$$

where $\mathcal{L}^\vee = \mathcal{H}om_{\mathcal{O}_X}(\mathcal{L}, \mathcal{O}_X)$. Thus, (4.8) becomes

$$\mathcal{M} \cong \mathcal{L}^\vee(d-l). \tag{4.10}$$ sheafGG

normalsheafdet **Proposition 4.1.13.**

$$\mathcal{L} \cdot \mathcal{M} \cong \mathcal{O}_X(d-l).$$

Proof Since the normal sheaf $\mathcal{N}_{X/\mathbb{P}^n}$ is isomorphic to $\mathcal{O}_X(d)$, applying Proposition (4.1.5), we obtain the asserted isomorphism over the open subset X' of X with the complement of codimension 2. Writing $\mathcal{L} = \mathcal{O}_X(D)$ and $\mathcal{M} = \mathcal{O}_X(D')$ we obtain that $\mathcal{O}_X(D+D') = \mathcal{O}_X(d-l)$ on X', and hence, on the whole X. □

We have seen how a determinantal representation of a hypersurface in $\mathbb{P}^n$ leads to an aCM sheaf on $\mathbb{P}^n$. Now, let us see the reverse construction. Let $\mathcal{F}$ be an aCM-sheaf on $\mathbb{P}^n$ supported on a reduced and normal hypersurface X. We assume that $\mathcal{F}$ is of rank one on X. It is a reflexive sheaf on X. So, we can write it as $\mathcal{F} = \mathcal{O}_X(D)$ for some Weil divisor D on X. Let

$$\mathcal{G} = \mathcal{E}xt^1_{\mathcal{O}_{\mathbb{P}^n}}(\mathcal{L}, \mathcal{O}_{\mathbb{P}^n}(-l)) \cong \mathcal{F}^\vee(d-l) \cong \mathcal{O}_X(-D)(d-l). \tag{4.11}$$ dualF

Since $M = \Gamma_*(\mathcal{F})$ is a Cohen-Macaulay module over $S = \Gamma_*(\mathcal{O}_{\mathbb{P}^n})$ of depth $n-1$, its projective dimension is equal to 1. Since any graded projective module over the polynomial ring is isomorphic to the direct sum of free modules of rank 1, we obtain a resolution

$$0 \to \bigoplus_{i=1}^m S[-b_i] \to \bigoplus_{i=1}^m S[-a_i] \to \Gamma_*(\mathcal{F}) \to 0,$$

for some sequences of integers (a_i) and (b_i). Passing to the associated sheaves on the projective space, it gives a projective resolution of $\mathcal{F}$:

$$0 \to \bigoplus_{i=1}^{m} \mathcal{O}_{\mathbb{P}^n}(-b_i) \xrightarrow{\phi} \bigoplus_{i=1}^{m} \mathcal{O}_{\mathbb{P}^n}(-a_i) \to \mathcal{F} \to 0. \qquad (4.12)$$ res11

The homomorphism of sheaves ϕ is given by a square matrix A of size m. Its ij-th entry is a polynomial of degree $b_j - a_i$. The support X of $\mathcal{F}$ is equal to $V(\det A)_{\mathrm{red}}$. The degree of $Y = V(\det A)$ is equal to

$$d = (b_1 + \cdots + b_m) - (a_1 + \cdots + a_m). \qquad (4.13)$$ res2

We assume that the resolution is minimal, i.e. $b_j > a_i$ for all i, j. This can always be achieved by dropping the isomorphic summands in the first and the second module. The case we considered before is a special case when $\mathcal{F}$ is an aCM sheaf with

$$a_1 = \ldots = a_m = 0, \quad b_1 = \ldots = b_m = l. \qquad (4.14)$$ res3

In this case, A is a matrix of homogeneous forms of degree $l = m/d$.

P4.2.2 **Proposition 4.1.14.** *Let $\mathcal{F}$ be an aCM sheaf on $\mathbb{P}^n$ supported on a reduced hypersurface X and let (4.12) be its projective resolution. Then, (4.14) holds if and only if*

$$H^0(X, \mathcal{F}(-1)) = H^0(X, \mathcal{G}(-1)) = 0, \qquad (4.15)$$ van1

where $\mathcal{G}$ is its dual sheaf defined in (4.11).

Proof Taking global sections in the exact sequence (4.12), we immediately get that all a_i are non-negative. Taking the dual to the second exact sequence, we get an exact sequence

$$0 \to \bigoplus_{i=1}^{m} \mathcal{O}_{\mathbb{P}^n}(a_i - l) \xrightarrow{\phi} \bigoplus_{i=1}^{m} \mathcal{O}_{\mathbb{P}^n}(b_i - l) \to \mathcal{G} \to 0. \qquad (4.16)$$ res33

Thus, the condition $H^0(\mathbb{P}^n, \mathcal{G}(-1)) = 0$ implies that $b_i \le l$. Since $b_i > a_j \ge 0$, and $d = ml = \sum_{i=1}^m (b_i - a_i)$, we get that all b_i are equal to l and all a_j are equal to zero. □

Note that, by duality on X, we get

$$H^0(X, \mathcal{G}(-1)) = H^{n-1}(X, \mathcal{G}^\vee(1), \omega_X)$$

$$= H^{n-1}(X, \mathcal{F}(l - d + 1)(d - n - 1)) = H^{n-1}(X, \mathcal{F}(l - n)).$$

dixon **Theorem 4.1.15.** *Let $\mathcal{L} = \mathcal{F}$, $\mathcal{M} = \mathcal{G}$, where $\mathcal{F}, \mathcal{G}$ be as in Proposition 4.1.14. Assume, additionally, that*

$$\mathcal{L} \otimes \mathcal{M} \cong \mathcal{O}_X(d-l).$$

Then X admits a determinantal representation $\phi : |E| \to |U \otimes V|$ of type $(|d, d|_{d-1}, n)_l$ with $\mathcal{L} = \mathcal{T}_\phi$, $\mathcal{M} = {}^t\mathcal{T}_\phi$, $U = H^0(X, \mathcal{L})$, $V = H^0(X, \mathcal{M})$.

Proof By Proposition 4.1.14, $\mathcal{L}$ admits a projective resolutions (4.12) with $a_i = 0$ and $b_i = l$. It allows us to identify $\mathcal{O}_{|E|}^{\oplus m}$ with $U_{|E|}$, where $U = H^0(X, \mathcal{L})$. We also have a similar resolution for $\mathcal{M}$ which is obtained from the first resolution by taking $\mathcal{H}om(-, \mathcal{O}_{|E|})$ and tensoring with $\mathcal{O}_{|E|}(-l)$. This identifies $\mathcal{O}_{|E|}(-l)^{\oplus m}$ with $V_{|E|}$, where $V = H^0(X, \mathcal{M})$. Let $m = \dim U = \dim V$.

The surjections

$$V_{|E|} \cong \mathcal{O}_{\mathbb{P}^n}^{\oplus m} \to \mathcal{L}, \quad U_{|E|} \cong \mathcal{O}_{\mathbb{P}^n}^{\oplus m} \to \mathcal{M}$$

define the morphisms

$$\mathfrak{l} : \mathrm{Proj}(\mathbf{S}(\mathcal{L})) \to \mathbb{P}(V), \quad \mathfrak{r} : \mathrm{Proj}(\mathbf{S}(\mathcal{M})) \to \mathbb{P}(U).$$

If $n = 2$, $X = X'$, $\mathcal{L}$ and $\mathcal{M}$ are invertible sheaves on X, hence, $X = \mathbb{P}(\mathcal{L})) = \mathbb{P}(\mathcal{M})$. Otherwise $X' \cong \mathrm{Proj}(\mathbf{S}(\mathcal{L})) \cong \mathrm{Proj}(\mathbf{S}(\mathcal{M}))$, and we obtain rational maps

$$\mathfrak{l} : X \dashrightarrow \mathbb{P}(U), \quad \mathfrak{r} : X \dashrightarrow \mathbb{P}(V).$$

with the set of indeterminacy points of codimension ≥ 2.

Let

$$\psi : X \to \mathbb{P}(U \otimes V) \to \mathbb{P}(U \otimes V) \tag{4.17}$$ adjcurves

to be the composition of $(\mathfrak{l}, \mathfrak{r})$ and the Segre map s_2. It is given by the complete linear system $|\mathcal{L} \otimes \mathcal{M}| = |\mathcal{O}_X(d-1)|$ on X. The exact sequence

$$0 \to \mathcal{O}_{|E|}(-l) \to \mathcal{O}_{|E|}(d-l) \to \mathcal{O}_X(d-l)) \to 0$$

together with vanishing of $H^1(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(-l))$ shows that the map ψ is equal to the restriction of a map $\Psi : |E| \to \mathbb{P}(U \otimes V)$.

We can view Ψ as defined by a tensor in $S^{d-l}(E^\vee) \otimes U^\vee \otimes V^\vee$. In coordinates, it is a $m \times m$ matrix $A(t)$ with entries from the space of homogeneous polynomials of degree $d-l$. Since $\Psi|_X = \psi$, for any point $x \in X$, $A(x) = \mathfrak{l}(x) \cdot {}^t\mathfrak{r}(x)$ is of rank 1.

Let M be a 2×2 submatrix of $A(t)$. Since $\det M(x) = 0$ for $x \in X$, we have $f \mid \det M$. Consider a 3×3 submatrix N of $A(t)$. We have $\det \mathrm{adj}(N) = \det(N)^2$. Since the entries of $\mathrm{adj}(N)$ are determinants of 2×2 submatrices, we see that $f^3 \mid \det(N)^2$. Since X is irreducible, this immediately implies that $f^2 \mid \det(N)$.

Continuing in this way, we obtain that f^{m-2} divides all cofactors of the matrix A. Thus $B = f^{2-m}\mathrm{adj}(A)$ is a matrix with entries in $S^l(E^\vee)$. It defines a linear map $S^l(E) \to U \otimes V$. The associated polynomial map

$$\phi : |E| \to |U \otimes V|$$

is a determinantal representation of X of type $(|m, m|, n)_l$. We immediately check that $\mathcal{L} = \mathcal{T}_\phi, \mathcal{M} = {}^t\mathcal{T}_\phi$.

□

hom *Remark* 4.1.16. In the proof we encounter a birational map

$$\mathbb{P}(U \otimes V) \to \mathbb{P}(U^\vee \otimes V^\vee) = |U \otimes V|.$$

It is similar to the adjugate birational map (2.73) defined for orthogonal matrices. It follows from Example 1.1.4 that this map is given by the polars of the determinantal hypersurface $|U \otimes V|_{d-1}$. In fact, if $A = (t_{ij})$ is a matrix with independent variables as entries, then $\frac{\partial \det(A)}{\partial t_{ij}} = M_{ij}$, where M_{ij} is the ij-th cofactor of the matrix A. The map Adj is a birational map since $\mathrm{Adj}(A) = A_{-1}\det(A)$ and the map $A \to A_{-1}$ is obviously invertible. So, the determinantal equation is an example of a homogeneous polynomial such that the corresponding polar map is a birational map. Such a polynomial is called a *homaloidal polynomial* (see [240]).

SS:4.1.3 4.1.3 Symmetric and skew-symmetric aCM sheaves

Let $\mathcal{F}, \mathcal{G}$ be coherent sheaves on a subscheme T of $\mathbb{P}^n$. Suppose there is a homomorphism of coherent sheaves on X

$$\alpha : \mathcal{F} \to \mathcal{G}^\vee(N) \tag{4.18}$$

for some integer N. Passing to duals, we get a homomorphism $(\mathcal{G}^\vee)^\vee(-N) \to \mathcal{F}^\vee$. After twisting by N, we get a homomorphism $(\mathcal{G}^\vee)^\vee \to \mathcal{F}(N)$. Composing it with the natural homomorphism $\mathcal{G} \to (\mathcal{G}^\vee)^\vee$, we get a homomorphism

$${}^t\alpha : \mathcal{G} \to \mathcal{F}^\vee(N),$$

which we call the *transpose* of α.

We take $\mathcal{F} = \mathcal{L}, \mathcal{G} = \mathcal{M}$ defined by a determinant hypersurface $X \subset \mathbb{P}^n$. Applying Proposition 4.1.13, we obtain that the homomorphisms $\alpha : \mathcal{L} \to \mathcal{M}^\vee(d-l)$ and ${}^t\alpha : \mathcal{M} \to \mathcal{L}(d-l)$ are bijective over X'. Since $\mathcal{L}$ and $\mathcal{M}$ are reflexive sheaves, applying j_*, where $j : X' \hookrightarrow X$, we get isomorphisms on X $\mathcal{L} \to \mathcal{M}(d-l)$ and its transpose ${}^t\alpha : \mathcal{M} \to \mathcal{L}(d-l)$ on X.

Suppose $\mathcal{F} = \mathcal{G}$, A coherent sheaf $\mathcal{F}$, together with a homomorphism

α) as above, is called a ϵ-*symmetric* if α is an isomorphism and ${}^t\alpha = \epsilon\alpha$, where $\epsilon = \pm 1$. We say that $(\mathcal{F}, \alpha)$ is symmetric if $\epsilon = 1$ and skew-symmetric otherwise.

Our first observation is that the conditions $\mathcal{F} \to ((\mathcal{F})^\vee)^\vee$ is an isomorphism is a necessary condition for the existence of α such that $(\mathcal{F}, \alpha)$ is ϵ-symmetric. In particular, $\mathcal{F}$ must be a reflexive sheaf.

Suppose $\mathcal{L}$ is ϵ-symmetric. Then $\mathcal{L} \cong \mathcal{L}^\vee(d-l) \cong \mathcal{M}(d-l)$. This implies that

$$\mathcal{L} \cong \mathcal{M}.$$

We refer for the proof of the following result to Casnati [93] and BeauvilleDet [51, Theorem B].

T4.2.4 **Theorem 4.1.17.** *Let $(\mathcal{F}, \alpha)$ be an ϵ-symmetric aCM sheaf. Assume that $X_s = X$. Then, it admits a resolution of the form* (resl1 4.12), *where*

$$(a_1, \ldots, a_m) = (b_1 + N - d, \ldots, b_m + N - d),$$

and the map ϕ is defined by a symmetric matrix if $\epsilon = 1$ and a skew-symmetric matrix if $\epsilon = -1$.

beau **Corollary 4.1.18.** *Suppose $(\mathcal{F}, \alpha)$ is a symmetric sheaf with $N = d - l$ satisfying the vanishing conditions from* (van1 4.15). *Then, $\mathcal{F}$ admits a projective resolution*

$$0 \to U^\vee_{\mathbb{P}^n}(-1) \xrightarrow{\phi} U_{\mathbb{P}^n} \to \mathcal{F} \to 0,$$

where $U = H^0(\mathbb{P}^n, \mathcal{F})$ and ϕ is defined by a symmetric matrix A with linear entries. In particular, $\mathcal{F} \cong \mathcal{L} = \mathcal{T}(\phi)$ for some determinantal representation of $X = \mathrm{Supp}(\mathcal{F})$.

Note that, if $\mathcal{L}$ is skew-symmetric, the matrix A is skew-symmetric, hence, ϕ is not proper. We have $D_{m-1}(\phi) = \mathbb{P}^n$ if m is odd, and $D_{m-1}(\phi) = D_{m-2}(\phi)$ if m is even. We will noted discuss the skew-symmetric determinantal representation, called *pfaffian* representations, however we refer to the interested reader to BeauvilleDet [51, Theorem B] for the discussion of pfaffian surfaces. Apparently, pfaffian representations were not studied in classical literature

Suppose ϕ is a symmetric determinantal representation with $\mathcal{L} \cong \mathcal{M}$. Suppose $n + 1 - l = 2t$ is even. Twisting the isomorphism $\mathcal{L} \to \mathcal{M} = \mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \omega_X)(n + 1 - l)$ by $-t$, we obtain an isomorphism

$$\mathcal{L}(-t) \to \mathcal{H}om_{\mathcal{O}_X}(\mathcal{L}(-t), \omega_X).$$

def:thetachar **Definition 4.1.19.** *A rank 1 torsion-free coherent sheaf θ on a reduced variety Y with canonical sheaf ω_Y is called a* theta characteristic *if there exists an isomorphism*

$$\alpha : \theta \to \mathcal{H}om_{\mathcal{O}_Y}(\theta, \omega_Y).$$

Note that in the case when a theta characteristic θ is an invertible sheaf, we obtain

$$\theta^{\otimes 2} \cong \omega_Y.$$

In particular, if ω_X is an invertible sheaf and $\mathcal{L} = \mathcal{O}_X(D)$, then D is a $\mathbb{Q}$-Cartier divisor.

Since α and ${}^t\alpha$ differ by an automorphism of θ, and any automorphism of a rank 1 torsion-free sheaf is defined by a nonzero scalar multiplication, we can always choose an isomorphism α defining a structure of a symmetric sheaf on θ.

Let X be a reduced normal hypersurface of degree $d = lm$ in $\mathbb{P}^n$ and let θ be a theta characteristic on X. Assume $n + l - 1 = 2t$ is even. Then, $\mathcal{L} = \theta(t)$ satisfies $\mathcal{L}(t) \cong \mathcal{L}(t)^\vee(d - l)$, and hence, has a structure of a symmetric sheaf with $N = d - l$. Assume also that θ, considered as a coherent sheaf on $\mathbb{P}^n$, is an aCM sheaf satisfying the assumptions in Proposition P4.2.2 ~~4.1.14~~. Then, we obtain that $\mathcal{L}$ admits a resolution

$$0 \to \mathcal{O}_{\mathbb{P}^n}(-l)^{\oplus m} \to \mathcal{O}_{\mathbb{P}^n}^{\oplus m} \to \mathcal{L} \to 0,$$

defined by a symmetric matrix with homogeneous forms of degree l as its entries. The vanishing conditions from Proposition P4.2.2 ~~4.1.14~~ translate into one condition:

$$H^0(X, \theta(t - 1)) = 0. \qquad (4.19)$$ nv

4.1.4 Contact hypersurfaces

SS:4.1.4

Let X be a hypersurface of degree d in $|E| = \mathbb{P}^n$ given by the determinantal representation $\phi : |E| \to |U \times V|$ such that $\phi^*\mathcal{O}_{|U \otimes V|}(1) \cong \mathcal{O}_{|E|}(l)$. We have $\dim U = V = k = d/l$.

Let $(\mathfrak{l}, \mathfrak{r}) : X \dashrightarrow \mathbb{P}(U) \times \mathbb{P}(V) \subset \mathbb{P}(U \otimes V)$, be the rational map given by the left and the right kernel maps given by the aCM-sheaves $\mathcal{L}$ and $\mathcal{M}$ on X.

They define a bilinear map

$$r : |\mathcal{L}| \times |\mathcal{M}| = |U| \times |V| \to |\mathcal{L} \otimes \mathcal{M}| \cong |\mathcal{O}_X(d - l)|, \ (D_1, D_2) \mapsto \langle D_1, D_2 \rangle, \qquad (4.20)$$ ress1

where $\langle D_1, D_2 \rangle$ is the unique hypersurface of degree $d - l$ that cuts out the divisor $D_1 + D_2$ on X. Consider the incidence variety

$$F = \{(x, D_1, D_2) \in |E| \times |U| \times |V| : x \in \langle D_1, D_2 \rangle\}.$$

It is a hypersurface in $|E| \times |U| \times |V|$ of type $(d - l, 1, 1)$.

Proposition 4.1.20. *The incidence variety F is given by the bordered determinant*

$$\det\begin{pmatrix} a_{1k} & \dots & a_{1k} & v_0\\ a_{21} & \dots & a_{2k} & v_1\\ \vdots & \vdots & \vdots & \vdots\\ a_{k1} & \dots & a_{kk} & v_{k-1}\\ u_0 & \dots & u_{k-1} & 0\end{pmatrix} = 0. \tag{4.21}$$ borddet

Proof The *bordered determinant*(4.21) is equal to $(-1)^k \sum A_{ij}u_iv_j$, where A_{ij} is the (ij)-entry of the adjugate matrix $\mathrm{adj}(A)$. For a general point $x \in X$, the rank of the adjugate matrix $\mathrm{adj}(A(x))$ is equal to one. Thus, the bordered determinant defines a bilinear form of rank one in the space $U^\vee \otimes V^\vee$ of bilinear forms on $U \times V$. We can write it in the form $(\sum a_iv_i)(\sum b_ju_j)$, where $\mathfrak{l}(x) = [a_0, \dots, a_{d-1}], \mathfrak{r}(x) = [b_0, \dots, b_{d-1}]$. The hyperplane $V(\sum a_iv_i)$ (resp. $V(\sum b_iu_i)$) in $|U|$ (resp.$|V|$) defines a divisor $D_1 \in |\mathcal{L}|$ (resp. $|\mathcal{M}|$) such that $x \in \langle D_1, D_2\rangle$. This checks the assertion. □

Next, we use the following determinantal identity due to O. Hesse [390].

L4.1.7 **Lemma 4.1.21.** *Let $M = (c_{ij})$ be a square matrix of size k. Let*

$$D(A; u, v) := \begin{vmatrix} c_{11} & a_{12} & \dots & c_{1k} & u_0\\ c_{21} & a_{22} & \dots & c_{2k} & u_2\\ \vdots & \vdots & \vdots & \vdots & \vdots\\ c_{k1} & a_{k2} & \dots & c_{kk} & u_{k-1}\\ v_0 & v_2 & \dots & v_{k-1} & 0\end{vmatrix}.$$

Then,

$$D(A; u, u)D(A; v, v) - D(A; u, v)D(A, v, u) = P\det(A), \tag{4.22}$$ hesbord

where $P = P(c_{11}, \dots, c_{kk}; u_0, \dots, u_{k-1}; v_0, \dots, v_{k-1})$ is a polynomial of degree k in variables a_{ij} and of degree 2 in variables u_i and v_j.

Proof Consider $D(A; u, v)$ as a bilinear function in u, v satisfying $D(A; u, v) = D({}^tA; v, u)$. We have $D(A; u, v) = -\sum A_{ij}u_iv_j$, where A_{ij} is the (ij)-entry of $\mathrm{adj}(A)$. This gives

$$\begin{aligned} &D(A; u, u)D(A; v, v) - D(A; u, v)D(A; v, u)\\ &= (\sum A_{ij}u_iu_j)(\sum A_{ij}v_iv_j) - (\sum A_{ij}u_iv_j)(\sum A_{ji}u_iv_j)\\ &= \sum u_au_bv_cv_d(A_{ab}A_{dc} - A_{ac}A_{db}).\end{aligned}$$

Observe that $A_{ab}A_{dc} - A_{ac}A_{db}$ is equal to a 2×2-minor of $\mathrm{adj}(A)$. Thus, if $\det A = 0$, all these minors are equal to zero, and the left-hand side in (4.22) is equal to zero. This shows that $\det A$, considered as a polynomial in variables a_{ij}, divides the left-hand side of (4.22). Comparing the degrees of the expression in the variables a_{ij}, u_i, v_j, we get the assertion about the polynomial P. □

Let us see the geometric meaning of the lemma. For a fixed value ξ of the coordinates u, the hypersurface $T_\xi = V(D(A;\xi,\xi))$ intersects the hypersurface $X = V(\det A)$ along the sum of two divisors $V(D(A;\xi,\eta)) \in |\mathcal{L}|$ and $V(D(A;\eta,\xi) \in |\mathcal{M}|)$.

We specialize to the case when the determinantal representation is symmetric, i.e., $\mathcal{L} = \mathcal{M}$. Then, $U = V$, and (4.22) becomes

$$D(A;u,v)^2 - D(A;u,u)D(A;v,v) = P \det A. \tag{4.23}$$

bordsym

This time the curve $T_\xi = V(D(A;\xi,\xi))$ cuts out in X the divisor $2D_\eta$, where $D_\eta \in |\mathcal{L}|$, i.e., it touches X along D_η.

The algebraic family of hypersurfaces $V(D(u,u))$ of degree $d-l$ is a family of contact hypersurfaces of X. Each divisor from this family is touching X along a divisor of degree $\frac{1}{2}k(k-1)l$. projective transformation of U, the number of such families of contact curves is equal to the number of non-effective even theta characteristics on the curve C.

On many occasions in the sequel, we will be discussing *quadratic pencils* of hypersurfaces of degree l in $\mathbb{P}^n$. This is a special case where $k = 2$ and $d = 2k$. Its universal family is a hypersurface in $\mathbb{P}^1 \times \mathbb{P}^n$ of bidegree $(2, 2k)$ given by equation

$$A(t_0,\ldots,t_n)u_0^2 + 2B(t_0,\ldots,t_n)u_0u_1 + C(t_0,\ldots,t_n)u_1^2 = 0,$$

where A, B, and C are homogenous forms of degree l. A general point in $\mathbb{P}^n$ is contained in two members of the family. The hypersurface $D = V(B^2 - AC)$ is the *discriminant hypersurface* of the quadratic family. It is the branch divisor of the projection to the first factor. It is equal to the closure of the set of points that are contained in only one member of the family. The members of the family are contact hypersurfaces to D.

We assume that (A, B, C) are coprime. The closed subset $V(A, B, C)$ of $\mathbb{P}^n$ consists of *base points* of the quadratic pencil. If $n > 3$, it is always non-empty. Taking the partials, we see that the discriminant hypersurface has singular points at the base points.

Finally, let us give one more interesting application of bordered symmetric determinants. Consider the coefficient $P = P(t_0,\ldots,t_n; u_0,\ldots,u_{k-1}; u_0,\ldots,u_{k-1})$ in (4.22). Setting $[u] = [v]$, we obtain $P = 0$. This implies that P can be expressed as a polynomial of degree 2 in Plücker coordinates of lines in $\mathbb{P}^{k-1}$. Thus, $P = 0$ represents a family of *quadratic line complexes* in $G_1(\mathbb{P}^{k-1})$. parameterized by points in $\mathbb{P}^n$.

linecomplex **Proposition 4.1.22.** *Let $\phi : |E| \to |S^2(U)^\vee|$ define a symmetric linear determinant representation of a hypersurface X in $|E|$. of a hypersurface X. For any $x \in |E|$ the quadratic line complex $V(P(u, v; x))$ consists of lines in $\mathbb{P}(U)$ such that the dual subspace of codimension two in $|U|$ is tangent to the quadric $Q_x = \phi(x)$.*

Proof Note that the dual assertion is that the line is tangent to the dual quadric $Q_x^\vee$. The equation of the dual quadric is given by $D(A(x); u, u) = 0$. A line spanned by the points $[\xi] = [\xi_0, \dots, \xi_{d-1}]$ and $[\eta] = [\eta_0, \dots, \eta_{d-1}]$ is tangent to this quadric if and only if the restriction of this quadric to the line is given by a singular binary form in coordinates on the line. The discriminant of this quadratic form is $D(A(x); \xi, \xi)D(A(x); \eta, \eta) - D(A(x); \xi, \eta)^2$. We assume that the point x is a general point in the plane, in particular, it does not belong to X. Thus, this expression vanishes if and only if $P(\xi, \eta) = 0$. □

Note that we have already encountered these tangential quadratic line complexes in Example exa:2.3.11 2.3.11.

S:4.2

4.2 Plane Curves

In this section we will consider linear determininantal representations of plane curves. We will discuss some examples of determinantal representations of higher degree for plane curves later in Subsection 6.2.1 6.2.1.

SS:4.2.1

4.2.1 Nonsingular plane curves

Let us first consider a linear determinantal representation $\phi : |E| = \mathbb{P}^2 \to |U \otimes V|$ of a nonsingular plane curve $C = V(f) \subset \mathbb{P}^2$ of degree d. Since C is nonsingular, ϕ is proper and the sheaves $\mathcal{L}$ and $\mathcal{M}$ are invertible sheaves on C

L3 **Theorem 4.2.1.** *Let $g = \frac{1}{2}(d-1)(d-2)$ be the genus of the curve C. Then*

(i) $H^0(C, \mathcal{L}) \cong U$, $H^0(C, \mathcal{M}) \cong V$;
(ii) $H^0(C, \mathcal{L}(-1)) = H^0(C, \mathcal{M}(-1)) = \{0\}$;
(iii) $\mathcal{L} \otimes \mathcal{M} \cong \mathcal{O}_C(d-1)$;
(iv) $\mathcal{L} \cong \mathcal{M}^\vee(d-1)$;
(v) $H^1(C, \mathcal{L}(j)) = H^1(C, \mathcal{M}(j)) = \{0\}, j = -1, 0$;

(vi) $\deg(\mathcal{L}) = \deg(\mathcal{M}) = \frac{1}{2}d(d-1) = g - 1 + d.$

Proof Assertions (i)–(iv) were proven in a more general situation in Subsection 4.1.2. By adjunction, $\omega_C = \mathcal{O}_C(d-3)$. Thus $\deg(\mathcal{L}(-1)) = \deg(\mathcal{M})(-1) = g-1$. By Riemann-Roch, $H^0(C, \mathcal{L}(-1) \cong H^1(C, \mathcal{L}(-1)) = 0$. Similarly, we prove that $H^1(C, \mathcal{L}(-1)) = 0$. The vanishing of $H^1(C, \mathcal{L})$ and $H^1(C, \mathcal{M})$ also follow from Riemann-Roch and (i).

The last equality follows from the fact that the transpose map $\tau : |U \otimes V| \to |V \otimes U|$ is a projective isomorphism, and ϕ is a projective map, hence, the degree of $\mathcal{T}_{UV}(d-1)$ is equal to the degree of ${}^t\mathcal{T}_{UV}(d-1)$, hence, $\deg(\mathcal{L}) = \deg(\mathcal{M}) = \frac{1}{2}d(d-1)$. □

It follows from Theorem 4.1.15 that, conversely, a pair of invertible sheaves $\mathcal{L}$ and $\mathcal{M}$ satisfying (i)– (iv) defines a linear determinantal representation ϕ of C with $\mathcal{L} = \mathcal{T}_\phi, \mathcal{M} = {}^t\mathcal{T}_\phi$.

Remark 4.2.2. It follows that a linear determinantal representation of C is determined by an invertible sheaf $\mathcal{L}$ such that $\mathcal{L}(-1) \in \mathrm{Pic}^{g-1}(C) \setminus \Theta$, where Θ is hypersurface in $\mathrm{Pic}^{g-1}(C)$ of effective divisors of degree $g-1$. The involution $D \mapsto K_C - D$ of $\mathrm{Pic}^{g-1}(C)$ replaces $\mathcal{L}(-1)$ with $\mathcal{M}(-1)$. A symmetric matrix A defining the determinant representation is obtained after we choose a basis in $H^0(C, \mathcal{L})$ and $H^0(C, \mathcal{M})$. Thus, we obtain ththe orbit space of the group $G = \mathrm{GL}(d) \times \mathrm{GL}(d)$ acting on the set of square matrices of size $d \times d$ with entries in $E^\vee$ by left and right multiplication is isomorphic to $\mathrm{Pic}^{g-1}(C) \setminus \Theta$.

Let us now consider the symmetric case $\mathcal{L} = \mathcal{M}$. Specializing Theorem 4.2.1, we get

- $\mathcal{L}(-1)^{\otimes 2} \cong \mathcal{O}_C(d-3) = \omega_C$;
- $\deg(\mathcal{L}) = \frac{1}{2}d(d-1)$;
- $H^0(C, \mathcal{L}(-1)) = \{0\}$.

Recall that the first property implies that $\theta = \mathcal{L}(-1)$ is a theta characteristic on C that was defined in Subsection 4.1.3. The third property says that θ is non-effective. So, we obtain that a symmetric linear determinantal representation is defined by a non-vanishing theta characteristic θ on C and the map

$$\mathfrak{l} = \mathfrak{r} : C \to \mathbb{P}^d = |U|$$

is defined by the complete linear system $|\mathcal{L}| = |\theta(1)|$.

We can also specialize to the symmetric case the definition of the map (4.17). It is a map

$$\psi : C \to \mathbb{P}(S^2(U^\vee)) = |S^2 U|.$$

In coordinates, the map is given by

$$\psi(x) = \tilde{\mathfrak{l}}(x) \cdot {}^t\tilde{\mathfrak{l}}(x),$$

where $\tilde{\mathfrak{l}}(x)$ is the column of projective coordinates of the point $\mathfrak{l}(x)$. It is clear that the image of the map ψ is contained in the variety $\mathsf{Q}_{U^\vee}(1)$ of rank one quadrics in $\mathbb{P}(U) = |U^\vee|$. It follows from the proof of Theorem 4.1.15 that there exists a linear map $\phi : \mathbb{P}^2 \to |S^2(U^\vee)|$ such that its composition with the rational map defined by taking the adjugate matrix is equal, after restriction to C, to the map ψ. The image of ϕ is a net N of quadrics in $|U|$. The image $\phi(C)$ is the locus of singular quadrics in N. For each point $x \in C$, we denote the corresponding quadric by Q_x. The regular map $\mathfrak{l}$ is defined by assigning to a point $x \in C$ the singular point of the quadric Q_x. The image X of C in $|U|$ is a curve of degree equal to $\deg \mathcal{L} = \frac{1}{2}d(d-1)$.

dix2 **Proposition 4.2.3.** *The restriction map*

$$r : H^0(|U|, \mathcal{O}_{|U|}(2)) \to H^0(X, \mathcal{O}_X(2))$$

is bijective. Under the isomorphism

$$H^0(X, \mathcal{O}_X(2)) \cong H^0(C, \mathcal{L}^{\otimes 2}) \cong H^0(C, \mathcal{O}_C(d-1)),$$

the space of quadrics in $|U|$ is identified with the space of plane curves of degree $d-1$. The net of quadrics N is identified with the linear system of first polars of the curve C.

Proof Reversing the proof of property (iii) from Lemma 4.2.1, we see that the image of C under the map $\psi : C \to \mathbb{P}(U \otimes V)$ spans the space. In our case, this implies that the image of C under the map $C \to |S^2(U^\vee)|$ spans the space of quadrics in the dual space. If the image of C in $\mathbb{P}(U)$ were contained in a quadric Q, then Q would be apolar to all quadrics in the dual space, a contradiction. Thus, the restriction map r is injective. Since the spaces have the same dimension, it must be surjective.

The composition of the map $i : \mathbb{P}^2 \to |\mathcal{O}_{|U|}(2)|, x \mapsto Q_x$, and the isomorphism $|\mathcal{O}_{|U|}(2)| \cong |\mathcal{O}_{\mathbb{P}^2}(d-1)|$ is a map $s : \mathbb{P}^2 \to |\mathcal{O}_{\mathbb{P}^2}(d-1)|$. A similar map s' is given by the first polars $x \mapsto P_x(C)$. We have to show that the two maps coincide. Recall that $P_x(C) \cap C = \{c \in C : x \in \mathbb{T}_c(C)\}$. In the next lemma, we will show that the quadrics $Q_x, x \in \mathbb{T}_c(C)$, form the line in N of quadrics passing through the singular point of Q_c equal to $\mathfrak{r}(c)$. This shows that the quadric $Q_{\mathfrak{r}(x)}$ cuts out in $\mathfrak{l}(C)$ the divisor $\mathfrak{r}(P_x(C) \cap C)$. Thus, the curves $s(x)$ and $s'(x)$ of degree $d-1$ cut out the same divisor on C, and hence, they coincide. □

Lemma 4.2.4. *Let $W \subset S^d(U^\vee)$ be a linear subspace, and $|W|^{\mathrm{s}}$ be the locus of singular hypersurfaces. Assume $x \in |W|^{\mathrm{s}}$ is a nonsingular point of $|W|^{\mathrm{s}}$. Then, the corresponding hypersurface has a unique ordinary double point y and the embedded tangent space $\mathbb{T}_x(|W|^{\mathrm{s}})$ is equal to the hyperplane of hypersurfaces containing y.*

Proof Assume $W = S^d(V^\vee)$. Then, $|W|^{\mathrm{s}}$ coincides with the discriminant hypersurface $\mathsf{D}_d(|U|)$ of singular degree d hypersurfaces in $|U|$. If $|W|$ is a proper subspace, then $|W|^{\mathrm{s}} = |W| \cap \mathsf{D}_d(|U|)$. Since $x \in |W|^{\mathrm{s}}$ is a nonsingular point and the intersection is transversal. It follows from Example ex:dfscrim 1.2.3 that $\mathbb{T}_x(|W|^{\mathrm{s}}) = \mathbb{T}_x(\mathsf{D}_d(|U|)) \cap |W|$. This proves the assertion.

□

We see that a pair (C, θ), where C is a plane irreducible curve and θ is a non-effective even theta characteristic on C defines a net N of quadrics in $\mathbb{P}(H^0(C, \theta(1))) \cong \mathbb{P}^{d-1}$ such that $C = \mathsf{N}^{\mathrm{s}}$. Conversely, let N be a net of quadrics in $\mathbb{P}^{d-1} = |V|$. We know from Corollary cor:one 2.4.17 that the singular locus of the discriminant hypersurface $\mathsf{D}_2(d-1)$ of quadrics in $\mathbb{P}^{d-1}$ is of codimension 3 in $\mathbb{P}^d$. Thus, a general net N intersects $\mathsf{D}_2(d-1)$ transversally along a nonsingular curve C of degree d. This gives a representation of C as a symmetric determinant, and hence, defines an invertible sheaf $\mathcal{L}$ and a non-effective even theta characteristic θ. It is easy to see that $\mathcal{L}$ is equal to the pull-back of the invertible sheaf $\mathcal{V}_1$ on $\mathsf{D}_2(d-1)$ from Subsection SS:2.4.2 2.4.2. This gives a dominant rational map of varieties of dimension $(d^2+3d-16)/2$

$$G(3, S^2(U^\vee))/\mathrm{PGL}(U) \dashrightarrow |\mathcal{O}_{\mathbb{P}^2}(d)|/\mathrm{PGL}(3). \qquad (4.24)$$ eq4.8

The degree of this map is equal to the number of non-effective even theta characteristics on a general curve of degree d. In the next chapter, we will see that the number of non-vanishing theta characteristics on a general nonsingular curve of genus g is equal to $2^{g-1}(2^g+1)$, where $g = (d-1)(d-2)/2$ is the genus of the curve.

ex:d=2 *Example* 4.2.5. Take $d = 2$. Then, there is only one isomorphism class of $\mathcal{L}$ with $\deg \mathcal{L} = 1$. Since $\deg \mathcal{L}(-1) = -1$, $h^0(C, \mathcal{L}(-1)) = 0$, and so, $\mathcal{L} \cong \mathcal{M}$, and C admits a unique equivalence class of determinantal representations which can be chosen to be symmetric. For example, if $C = V(t_0t_1 - t_2^2)$, we can choose

$$A = \begin{pmatrix} t_0 & t_2 \\ t_2 & t_1 \end{pmatrix}.$$

We have $\mathbb{P}(U) \cong \mathbb{P}^1$, and $\mathfrak{r} = \mathfrak{l}$ maps C isomorphically to $\mathbb{P}^1$. There is only one family of contact curves of degree one. It is the system of tangents to C. It is parameterized by the conic in the dual plane, the dual conic of C. Thus, there is a natural identification of the dual plane with $\mathbb{P}(S^2U)$.

d=3 *Example* 4.2.6. Take $d = 3$. Then, $\mathrm{Pic}^{g-1}(C) = \mathrm{Pic}^0(C)$ and $\Theta = \mathrm{Pic}^0(C) \setminus \{\mathcal{O}_C\}$. Thus, the equivalence classes of determinantal representations are parameterized by the curve itself minus one point. There are three systems of contact conics. Let T be a contact conic cutting out a divisor $2(p_1+p_2+p_3)$. If we fix a group law on C defined by an inflection point $\mathfrak{o}$, then the points p_i add up to a nonzero 2-torsion point ϵ. We have $p_1 + p_2 + p_3 \sim 2\mathfrak{o} + \epsilon$. This implies that $\mathcal{L} \cong \mathcal{O}_C(2\mathfrak{o} + \epsilon)$. The contact conic that cuts out the divisor $2(2\mathfrak{o} + \epsilon)$ is equal to the union of the inflection tangent line at $\mathfrak{o}$ and the tangent line at ϵ (which passes through $\mathfrak{o}$). We know that each nonsingular curve can be written as the Hessian curve in three essentially different ways. This gives the three ways to write C as a symmetric determinant and also explicitly write the three algebraic systems of contact conics.

Let $(\mathcal{L}, \mathcal{M})$ define a determinantal representation of C. Let $\mathfrak{l} : C \hookrightarrow \mathbb{P}(U)$ be the reembedding of C in $\mathbb{P}(U)$ given by the linear system $|\mathcal{L}|$. For any $D_0 \in |\mathcal{M}|$, there exists $D \in |\mathcal{L}|$ such that D_0+D is cut out by a conic. Thus, we can identify the linear system $|\mathcal{L}|$ with the linear system of conics through D_0. This linear system defines a birational map $\sigma : \mathbb{P}^2 \dashrightarrow \mathbb{P}(U)$ with indeterminacy points in D_0. The map $\mathfrak{l} : C \to \mathbb{P}(U)$ coincides with the restriction of σ to C.

Consider the map

$$(\mathfrak{l}, \mathfrak{r}) : C \to \mathbb{P}(U) \times \mathbb{P}(V) \cong \mathbb{P}^2 \times \mathbb{P}^2.$$

We claim that

- The image of $(\mathfrak{l}, \mathfrak{r})$ is a complete intersection of three hyperplane sections in the Segre embedding of the product.

Let us prove it. Consider the restriction map (4.20)

$$U \times V = H^0(\mathbb{P}(U) \times \mathbb{P}(V), \mathcal{O}_{\mathbb{P}(U)}(1) \boxtimes \mathcal{O}_{\mathbb{P}(V)}(1)) \to H^0(X, \mathcal{O}_X(1)),$$

where X is the image of C in $\mathbb{P}(U \otimes V)$ under the composition of the map $(\mathfrak{l}, \mathfrak{r})$ and the Segre map. Here, we identify the spaces $H^0(C, \mathcal{L} \otimes \mathcal{M})$ and $H^0(X, \mathcal{O}_X(1))$. Since the map (4.20) is surjective, and its target space is of dimension 6, the kernel is of dimension 3. So, the image X of C in $\mathbb{P}(U) \times \mathbb{P}(V)$ is contained in the complete intersection of three hypersurfaces of type $(1, 1)$. By the adjunction formula, the intersection is a curve X' of arithmetic genus 1. Choose coordinates (u_0, u_1, u_2) in $U^\vee$ and coordinates (v_0, v_1, v_2) in V to be able to write the three hypersurfaces by equations

$$\sum_{0 \le i,j \le 2} a_{ij}^{(k)} u_i v_j = 0, \quad k = 1, 2, 3.$$

The projection of X to the first factor is equal to the locus of points $[u_0, u_1, u_2]$ such that the system

$$\sum_{i,j=0}^{2} a_{ij}^{(k)} u_i v_j = \sum_{j=0}^{2} \Big(\sum_{i=0}^{2} a_{ij}^{(k)} u_i \Big) v_j = 0, \quad k = 1, 2, 3,$$

has a nontrivial solution (v_0, v_1, v_2). The condition for this is

$$\begin{vmatrix} \sum_{i=0}^{2} a_{i0}^{(1)} u_i & \sum_{i=0}^{2} a_{i1}^{(1)} u_i & \sum_{i=0}^{2} a_{i2}^{(1)} u_i \\ \sum_{i=0}^{2} a_{i0}^{(2)} u_i & \sum_{i=0}^{2} a_{i1}^{(2)} u_i & \sum_{i=0}^{2} a_{i2}^{(2)} u_i \\ \sum_{i=0}^{2} a_{i0}^{(3)} u_i & \sum_{i=0}^{2} a_{i1}^{(3)} u_i & \sum_{i=0}^{2} a_{i2}^{(3)} u_i \end{vmatrix} = 0. \tag{4.25}$$

newdet

This checks that the projection of X' to the factor $\mathbb{P}(U)$ is a cubic curve, the same as the projection of X. Repeating the argument, replacing the first factor with the second one, we obtain that the projections of X' and X to each factor coincide. This implies that $X = X'$.

Recall that a linear determinantal representation of C is defined by a linear map $\phi : E \to U \otimes V$. Let us show that its image is the kernel of the restriction map. We identify its target space $H^0(X, \mathcal{O}_X(1))$ with $H^0(C, \mathcal{O}_C(d-1)) = H^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(d-1))$. In coordinates, the map ϕ is defined by $[x] \mapsto \sum a_{ij}(x) u_j \otimes v_i$, where $C = V(\det(a_{ij}))$. The restriction map is defined by the map $u_i \otimes v_j \mapsto \bar{A}_{ij}$, where A_{ij} is a (ij)-cofactor of the adjugate matrix of (a_{ij}) and the bar means the restriction to C. The composition is given by

$$x \mapsto \sum a_{ji} A_{ij} = \det(a_{ij}) \quad \text{restricted to } C.$$

Since the restriction of the determinant to C is zero, we see that E can be identified with the linear system of hyperplane sections of $\mathbb{P}(U) \times \mathbb{P}(V)$ defining the curve $(\mathfrak{l}, \mathfrak{r})(C)$.

Note that the determinant (4.25) gives a determinantal representation of the plane cubic C reembedded in the plane by the linear system $|\mathcal{L}|$. It is given by a linear map $U^\vee \to E^\vee \otimes V$ obtained from the tensor $\tau \in E^\vee \otimes U \otimes V$ which defines the linear map $\phi : E \to U \otimes V$.

SS:4.2.2

4.2.2 The moduli space

Let us consider the moduli space of pairs (C, A), where C is a nonsingular plane curve of degree d and A is a matrix of linear forms such that $C = V(\det A)$. To make everything coordinate-free and match our previous notations, we let

$\mathbb{P}^2 = |E|$ and consider A as a linear map $\phi : E \to U \otimes V = \mathrm{Hom}(U^\vee, V)$. Our equivalence relation on such pairs is defined by the natural action of the group $\mathrm{GL}(U) \times \mathrm{GL}(V)$ on $U \otimes V$. The composition of ϕ with the determinant map $U \otimes V \to \mathrm{Hom}(\bigwedge^d U^\vee, \bigwedge^d V) \cong \mathbb{C}$ is an element of $S^d(E^\vee)$. It corresponds to the determinant of the matrix A. Under the action of $(g, h) \in \mathrm{GL}(U) \times \mathrm{GL}(V)$, it is multiplied by $\det g \det h$, and hence. represents a projective invariant of the action. Consider ϕ as an element of $E^\vee \otimes U \otimes V$, and let

$$\det : E^\vee \otimes U \otimes V / \mathrm{GL}(U) \times \mathrm{GL}(V) \to |S^d E^\vee|$$

be the map of the set of orbits defined by taking the determinant. We consider this map as a map of sets since there is a serious issue here as to whether the orbit space exists as an algebraic variety. However, we are interested only in the restriction of the determinant map on the open subset $(E^\vee \otimes U \otimes V)^o$ defining nonsingular determinantal curves. One can show that the quotient of this subset is an algebraic variety.

We know that the fiber of the map det over a nonsingular curve C is bijective to $\mathrm{Pic}^{g-1}(C) \setminus \Theta$. Let be the universal family of nonsingular plane curves of degree d (and genus g). It defines a family

$$\tilde{\pi} : \mathcal{P}ic_d^{g-1} \to |S^d(E^\vee)|$$

whose fiber over a curve C is isomorphic to $\mathrm{Pic}^{g-1}(C)$. It is the relative Picard scheme of π. It comes with a divisor $\mathcal{T}$ such that its intersection with $\tilde{\pi}^{-1}(C)$ is equal to the divisor Θ. It follows from the previous sections that there is an isomorphism of algebraic varieties

$$(E^\vee \otimes U \otimes V)^o / \mathrm{GL}(U) \times \mathrm{GL}(V) \cong \mathcal{P}ic_d^{g-1} \setminus \mathcal{T}.$$

This shows that the relative Picard scheme $\mathcal{P}ic_d^{g-1}$ is a unirational variety. An easy computation shows that its dimension is equal to $d^2 + 1$.

It is a very difficult question to decide whether the variety $\mathcal{P}ic_d^{g-1}$ is rational. It is obviously rational if $d = 2$. It is known that it is rational for $d = 3$ and $d = 4$ [306]. Let us sketch a beautiful proof of the rationality in the case $d = 3$ due to M. Van den Bergh [782].

Theorem 4.2.7. *Assume $d = 3$. Then, $\mathcal{P}ic_3^0$ is a rational variety.*

Proof A point of $\mathcal{P}ic^0$ is a pair $(C, \mathcal{L})$, where C is a nonsingular plane cubic and $\mathcal{L}$ is the isomorphism class of an invertible sheaf of degree 0. Let D be a divisor of degree zero such that $\mathcal{O}_C(D) \cong \mathcal{L}$. Choose a line ℓ and let $H = \ell \cap C = p_1 + p_2 + p_3$. Let $p_i + D \sim q_i, i = 1, 2, 3$, where q_i is a point. Since $p_i - q_i \sim p_j - q_j$, we have $p_i + q_j \sim p_j + q_i$. This shows that the lines $\langle p_i, q_j \rangle$

and $\langle p_j, q_i\rangle$ intersect at the same point r_{ij} on C. Since, $p_i + q_j + r_{ij} \sim H$, it is immediately checked that

$$p_1 + p_2 + p_3 + q_1 + q_2 + q_3 + r_{12} + r_{23} + r_{13} \sim 3H.$$

This easily implies that there is a cubic curve that intersects C at the nine points. Together with C, they generate a pencil of cubics with the nine points as the set of its base points. Let $X = \ell^3 \times (\mathbb{P}^2)^3/\mathfrak{S}_3$, where $\mathfrak{S}_3$ acts by

$$\sigma : \big((p_1, p_2, p_3), (q_1, q_2, q_3)\big) = \big((p_{\sigma(1)}, p_{\sigma(2)}, p_{\sigma(3)}), (q_{\sigma(1)}, q_{\sigma(2)}, q_{\sigma(3)})\big).$$

The variety X is easily seen to be rational. The projection to $\ell^3/\mathfrak{S}_3 \cong \mathbb{P}^3$ defines a birational isomorphism between the product of $\mathbb{P}^3$ and $(\mathbb{P}^2)^3$. For each $x = (\mathcal{P}, \mathcal{Q}) \in X$, let $c(x)$ be the pencil of cubics through the points $p_1, p_2, p_3, q_1, q_2, q_3$ and the points $r_{ij} = \langle p_i, q_j\rangle$, where $(ij) = (12), (23), (13)$. Consider the set U' of pairs $(x, C), C \in c(x)$. The projection $(u, C) \mapsto u$ has fibres isomorphic to $\mathbb{P}^1$. Thus, the field of rational functions on X' is isomorphic to the field of rational functions on a conic over the field $\mathbb{C}(X)$. But this conic has a rational point. It is defined by fixing a point in $\mathbb{P}^2$ and choosing a member of the pencil passing through this point. Thus, the conic is isomorphic to $\mathbb{P}^1$ and $\mathbb{C}(X')$ is a purely transcendental extension of $\mathbb{C}(X)$. Now, we define a birational map from $\mathcal{P}ic_3^0$ to X'. Each $(C, \mathcal{L})$ defines a point of U' by ordering the set $\ell \cap C$, then defining q_1, q_2, q_3 as above. The member of the corresponding pencil through p_i's, q_i's, and r_{ij}'s is the curve C. Conversely, a point $(x, C) \in X'$ defines a point $(C, \mathcal{L})$ in $\mathcal{P}ic_3^0$. We define $\mathcal{L}$ to be the invertible sheaf corresponding to the divisor $q_1 + q_2 + q_3$. It is easy to see that these maps are inverse to each other. □

Remark 4.2.8. If we choose a basis in each space E, U, V, then a map $\phi : E \to \mathrm{Hom}(U, V)$ is determined by three matrices $A_i = \phi(e_i)$. Our moduli space becomes the space of triples (A_1, A_2, A_3) of $d \times d$ matrices up to the action of the group $G = \mathrm{GL}(d) \times \mathrm{GL}(d)$ simultaneously by left and right multiplication

$$(\sigma_1, \sigma_2) \cdot (A_1, A_2, A_3) = (\sigma_1 A_1 \sigma_2^{-1}, \sigma_1 A_2 \sigma_2^{-1}, \sigma_1 A_3 \sigma_2^{-1}).$$

Consider an open subset of maps ϕ such that A_1 is an invertible matrix. Taking $(\sigma_1, \sigma_2) = (1, A_1^{-1})$, we may assume that $A_1 = I_d$ is the identity matrix. The stabilizer subgroup of (I_d, A_2, A_3) is the subgroup of (σ_1, σ_2) such that $\sigma_1\sigma_2 = 1$. Thus, our orbit space is equal to the orbit space of pairs of matrices (A, B) up to simultaneous conjugation. The rationality of this space is a notoriously very difficult problem.

SS:4.2.3

4.2.3 Singular plane curves

Assume $n = 2$, and let C be a reduced irreducible curve of degree d. Let $\mathcal{F}$ be a coherent torsion-free sheaf on C. Since $\dim C = 1$, $\mathcal{F}$ is an aCM-sheaf. Also, the cohomological condition for an aCM sheaf are vacuous, hence, $\mathcal{F}$ is an aCM sheaf. In general, a Cohen-Macaulay module M over a local Noetherian ring R admits a dualizing R-module D, and

$$\text{depth } M + \max\{q : \text{Ext}^q_R(M, D) \neq 0\} = \dim R$$

(see [281]). In our case, the global dualizing sheaf is

$$\omega_C = \omega_{\mathbb{P}^2}(C) \cong \mathcal{O}_C(d-3),$$

the previous equality implies that $\mathcal{E}xt^q_{\mathcal{O}_C}(\mathcal{F}, \omega_C) = 0, \quad q > 0$, and

$$\mathcal{F} \to D(\mathcal{F}) := \mathcal{H}om_{\mathcal{O}_C}(\mathcal{F}, \omega_C) \cong \mathcal{F}^\vee \otimes \omega_C$$

is the duality, i.e. $\mathcal{F} \to D(D(\mathcal{F}))$ is an isomorphism.

If $\mathcal{F}$ satisfies the conditions from Proposition 4.1.14

$$H^0(C, \mathcal{F}(-1)) = H^1(C, D(\mathcal{F})(-1)) = 0, \tag{4.26}$$ condd

we obtain a determinantal representation $C = V(\det A)$ with linear entries. For a general point x on C, the corank of the matrix $A(x)$ is equal to the rank of $\mathcal{F}$. We shall assume that

$$\text{rank } \mathcal{F} = 1.$$

In this case, $\mathcal{F}$ is isomorphic to a subsheaf of the constant sheaf of rational functions on C. It follows from the resolution of $\mathcal{F}$ that

$$\chi(\mathcal{F}(-1)) = 0, \quad \chi(\mathcal{F}) = d.$$

Thus,

$$\deg \mathcal{F}(-1) := \chi(\mathcal{F}(-1)) + p_a(C) - 1 = p_a(C) - 1.$$

Also,

$$\deg \mathcal{F} = \deg D(\mathcal{F}) = d + p_a(C) - 1 = d(d-1)/2.$$

Suppose x is a singular point of C. Then, either $\text{rank}A(x) < d - 1$, or the image of the map $\phi : \mathbb{P}^2 \to |U \times V|_{d-1}$ is tangent to $|U \times V|_{d-1}$ at a point $\phi(x) \notin |U \times V|_{d-2}$. The sheaf $\mathcal{F}$ is not invertible at x only in the former case.

It is known that the isomorphism classes of rank 1 torsion-free sheaves of fixed degree d on an irreducible reduced algebraic curve C admit a moduli space which is a projective variety that contains an irreducible component which compactifies the generalized Jacobian variety $\text{Jac}^d(C)$ of C (see [10],

[609]). In the case of plane curves (and, by [609], only in this case), the moduli space is irreducible. Its dimension is equal to $p_a(C)$. We denote the moduli space by $\overline{\mathrm{Jac}}^d(C)$.

Let us describe in more detail rank 1 torsion-free sheaves $\mathcal{F}$ on C.

Let $p : \bar{C} \to C$ be the normalization morphism. Its main invariant is the *conductor ideal* $\mathfrak{c}$, the annihilator ideal of the sheaf $p_*\mathcal{O}_{\bar{C}}$. Obviously, it can be considered as an ideal sheaf in $\bar{C}$ equal to $p^{-1}(\mathfrak{c})$ (the image of $p^*(\mathfrak{c})$ in $\mathcal{O}_{\bar{C}}$ under the multiplication map, or, equivalently, $p^*(\mathfrak{c})$/torsion). For any $x \in C$, $\mathfrak{c}_x$ is the conductor ideal of the normalization $\bar{R}$ of the ring $R = \mathcal{O}_{C,x}$ equal to $\prod_{y\to x} \mathcal{O}_{\bar{C},y}$. Let

$$\delta_x = \text{length } \bar{R}/R.$$

Since, in our case, R is a Gorenstein local ring, we have

$$\dim_{\mathbb{C}} \bar{R}/\mathfrak{c}_x = 2\dim_{\mathbb{C}} \bar{R}/\mathfrak{c}_x = 2\delta$$

(see [705], Chapter 4, n.11).

Suppose R is isomorphic to the localization of $\mathbb{C}[[u,v]]/(f(u,v))$ at the origin. One can compute δ_x, using the following *Jung-Milnor formula* (see [435], [520], §10).

$$\deg \mathfrak{c}_x = \dim_{\mathbb{C}} R/J_f + r_x - 1, \tag{4.27}$$ jungmilnor

where J_f is the ideal generated by partial derivatives of f, and r_x is the number of analytic branches of C at the point x.

Let F be the cokernel of the canonical injection of sheaves $\mathcal{O}_C \to p_*(\mathcal{O}_{\bar{C}})$. Applying cohomology to the exact sequence

$$0 \to \mathcal{O}_C \to p_*\mathcal{O}_{\bar{C}} \to \mathsf{F} \to 0, \tag{4.28}$$ conseq

we obtain the *genus formula*

$$\chi(p_*(\mathcal{O}_{\bar{C}})) = \chi(\mathcal{O}_{\bar{C}}) = \chi(\mathcal{O}_C) + \sum_{x\in C} \delta_x. \tag{4.29}$$ forchar

Consider the sheaf of algebras $\mathcal{E}nd(\mathcal{F}) = \mathcal{H}om_{\mathcal{O}_C}(\mathcal{F},\mathcal{F})$. Since $\mathcal{E}nd(\mathcal{F})$ embeds into $\mathcal{E}nd(\mathcal{F}_\eta)$, where η is a generic point of C, and the latter is isomorphic to the field of rational functions on C, we see that $\mathcal{E}nd(\mathcal{F})$ is a coherent $\mathcal{O}_C$-algebra. It is finitely generated as a $\mathcal{O}_C$-module, and hence, it is finite and birational over C. We set $C' = \operatorname{Spec} \mathcal{E}nd(\mathcal{F})$ and let

$$\pi = \pi_{\mathcal{F}} : C' \to C$$

be the canonical projection. The normalization map $\bar{C} \to C$ factors through the map π. For this reason, π is called the *partial normalization* of C. Note that

$C' = C$ if $\mathcal{F}$ is an invertible sheaf. The algebra $\mathcal{E}nd(\mathcal{F})$ acts naturally on $\mathcal{F}$ equipping it with a structure of an $\mathcal{O}_{C'}$-module, which we denote by $\mathcal{F}'$. We have

$$\mathcal{F} \cong \pi_*\mathcal{F}'.$$

Recall that for any finite morphism $f : X \to Y$ of Noetherian schemes there is a functor $f^!$ from the category of $\mathcal{O}_Y$-modules to the category of $\mathcal{O}_X$-modules defined by

$$f^!\mathcal{M} = \mathcal{H}om_{\mathcal{O}_Y}(f_*\mathcal{O}_X, \mathcal{M}),$$

considered as a $\mathcal{O}_X$-module. The functor $f^!$ is the right adjoint of the functor f_* (recall that f^* is the left adjoint functor of f_*), i.e.

$$f_*\mathcal{H}om_{\mathcal{O}_X}(\mathcal{N}, f^!\mathcal{M}) \cong \mathcal{H}om_{\mathcal{O}_Y}(f_*\mathcal{N}, \mathcal{M}), \tag{4.30}$$

as bi-functors in $\mathcal{M}, \mathcal{N}$. If X and Y admit dualizing sheaves, we also have

$$f^!\omega_Y \cong \omega_X$$

(see [379, Chapter III, Exercises 6.10 and 7.2]).

Applying this to our map $\pi : C' \to C$, and taking $\mathcal{N} = \mathcal{O}_{C'}$, we obtain

$$\mathcal{F} \cong \pi_*\pi^!\mathcal{F}.$$

It is known that, for any torsion-free sheaves $\mathcal{A}$ and $\mathcal{B}$ on C', a morphism $\pi_*\mathcal{A} \to \pi_*\mathcal{B}$ is $\pi_*\mathcal{O}_{C'}$-linear (see, for example, [50], Lemma 3.1). This implies that the natural homomorphism

$$\mathrm{Hom}_{C'}(\mathcal{A}, \mathcal{B}) \cong \mathrm{Hom}_{C'}(\pi_*\mathcal{A}, \pi_*\mathcal{B}) \tag{4.31}$$

is bijective. This gives

$$\mathcal{F}' \cong \pi^!\mathcal{F}.$$

For any $\mathcal{F}' \in \overline{\mathrm{Jac}}^d(C')$,

$$\chi(\mathcal{F}') = d' + \chi(C')$$

(in fact, this equality is the definition of the degree d' of $\mathcal{F}'$, see [539])

$$d = \deg \pi_*\mathcal{F}' = \chi(\pi_*\mathcal{F}') - \chi(\mathcal{O}_C)$$

$$= \chi(\mathcal{F}') - \chi(\mathcal{O}_C) = d' + \chi(\mathcal{O}_{C'}) - \chi(\mathcal{O}_{C'}).$$

Definition 4.2.9. *The collection of $\mathcal{O}_{C,x}$-modules $\mathcal{F}_x, x \in \mathrm{Sing}(C)$, is called the* local type *of $\mathcal{F}$ ([587]). The* global invariant *is the isomorphism class of $\mathcal{E}nd_{\mathcal{O}_C}(\mathcal{F})$.*

It follows from Lemma 1.7 in [587] that the global type of $\mathcal{F}$ determines the isomorphism class of $\mathcal{F}$, up to tensoring with an invertible sheaf. Also, it is proven in the same lemma that the global type depends only on the collection of local types.

Lemma 4.2.10. *The global types of $\mathcal{F}$ and $D(\mathcal{F})$ are isomorphic, and*

$$\pi^! D(\mathcal{F}) \cong D(\pi^! \mathcal{F}).$$

Proof The first assertion follows from the fact that the dualizing functor is an equivalence of the categories. Taking $\mathcal{M} = \omega_C$ in (4.30), we obtain that $\pi_*(D(\pi^!\mathcal{F})) \cong D(\mathcal{F})$. The second assertion follows from (4.31). □

In fact, by Lemma 3.1 from [50], the map

$$\pi_* : \overline{\mathrm{Jac}}^{d'}(C') \to \overline{\mathrm{Jac}}^{d}(C)$$

is a closed embedding of projective varieties.

It follows from the duality that $\chi(\mathcal{F}) = -\chi(D(\mathcal{F}))$. Thus, the functor $\mathcal{F} \to D(\mathcal{F})$ defines an involution $D_{C'}$ on $\overline{\mathrm{Jac}}^{p_a(C')-1}(C')$ and an involution D_C on $\overline{\mathrm{Jac}}^{p_a(C)-1}(C)$. By Lemma 4.2.10, the morphism π_* commutes with the involutions.

Let us describe the isomorphism classes of the local types of $\mathcal{F}$. Let $\tilde{\mathcal{F}} = p^{-1}(\mathcal{F}) = p^*(\mathcal{F})/\text{torsion}$. This is an invertible sheaf on $\bar{C}$. The canonical map $\mathcal{F} \to p_*(p^*\mathcal{F})$ defines the exact sequence

$$0 \to \mathcal{F} \to p_*\tilde{\mathcal{F}} \to \mathsf{T} \to 0, \tag{4.32}$$

where T is a torsion sheaf whose support is contained in the set of singular points of C.

The immediate corollary of this is the following.

Lemma 4.2.11. *For any $x \in C$,*

$$\dim_{\mathbb{C}} \mathcal{F}(x) = \mathrm{mult}_x C,$$

where $\mathcal{F}(x)$ denotes the fiber of the sheaf $\mathcal{F}$ and $\mathrm{mult}_x C$ *denotes the multiplicity of the point x on C.*

Proof Since the cokernel of $\mathcal{F} \to p_*\tilde{\mathcal{F}}$ is a torsion sheaf, we have

$$\dim_{\mathbb{C}} \mathcal{F}(x) = \dim_{\mathbb{C}} \tilde{\mathcal{F}}(x) = \dim_{\mathbb{C}} p_*(\mathcal{O}_{\bar{C}})(x). \tag{4.33}$$

It is clear that the dimension of the fiber of a coherent sheaf is equal to the dimension of the fiber over the closed point of the formal completion of $\mathcal{F}_x$. Let R (resp. $\bar{R}$) denote the formal completion of $\mathcal{O}_{C,x}$ (resp. its normalization). We know that $\bar{R} = \prod_{y\to x} \bar{R}_y$, where $\bar{R}_y \cong \mathbb{C}[[t]]$. Let (u, v) be local parameters in R generating the maximal ideal $\mathfrak{m}$ of R. One can choose the latter isomorphism in such a way that the composition of the map $R \to \bar{R}$ with the projection map $\bar{R} \to \bar{R}_i$ is given by

$$(u, v) \mapsto (t_i^{m_i}, \sum_{j=m_i}^{\infty} a_j t^j),$$

where m_j is the multiplicity of the analytic branch of the curve C corresponding to the point y over x. It follows that

$$\dim_{\mathbb{C}} \bar{R}/\mathfrak{m} = \dim_{\mathbb{C}} \prod_{i=1}^{r_x} \mathbb{C}[[t]]/(t^{m_i}) = \sum_{i=1}^{r_x} m_i = \mathrm{mult}_x C.$$

Thus, the last dimension in (4.33) is equal to the multiplicity, and we are done. □

Corollary 4.2.12. *Suppose $\mathcal{F}$ satisfies (4.26), and hence, defines a linear determinantal representation $C = V(\det A)$. Then,*

$$d - \mathrm{rank}\ A(x) = \mathrm{mult}_x C.$$

We denote by $\delta_x(\mathcal{F})$ the length of $\mathcal{T}_x$. The length $\delta_x(\mathcal{F})$ of $\mathcal{T}_x$ is the local invariant of the $\mathcal{O}_{C,x}$-module $\mathcal{F}_x$ (see [357]). Let M be a rank 1 torsion-free module over $R = \mathcal{O}_{C,x}$ and $\bar{M} = M \otimes \bar{R}$/torsion. Let Q be the fraction field of R. Since $M \otimes_R Q \cong Q$, one can find a fractional ideal isomorphic to M. It is known that the isomorphism class of M can be represented by a fractional ideal J with local invariant $\delta(M) = \dim \bar{M}/M$ contained in $\bar{R}$ and containing the ideal $\mathfrak{c}(R)$, where $\mathfrak{c}(M)$ is the conductor ideal of R. This implies that local types of $\mathcal{F}$ at x with $\delta_x(\mathcal{F}) = \delta$ are parameterized by the fixed locus of the group R^* acting on the Grassmann variety $G(\delta, \bar{R}/\mathfrak{c}_x) \cong G(\delta, 2\delta)$ (see [357], Remark 1.4, [609], Theorem 2.3 (d)). The dimension of the fixed locus is equal to δ_x. Thus, local types with fixed local invariant δ are parameterized by a projective variety of dimension δ.

Example 4.2.13. Let C' be the proper transform of C under the blow-up of the plane at a singular point $x \in C$ of multiplicity m_x. Since it lies on a nonsingular surface, C' is a Gorenstein curve. The projection $\pi : C' \to C$ is a partial normalization. Let $\mathcal{F} = \pi_* \mathcal{O}_{C'}$. Then, $\mathfrak{m}_{C,x}^{m_x}$ contains the conductor $\mathfrak{c}_x$ and $\mathfrak{c}(\mathcal{F}_x) = \mathfrak{m}_{C,x}^{m_x - 1}$, hence, $\delta_x(\mathcal{F}) = m_x - 1$ (see [609], p. 219).

Let $\mathcal{F}$ define a linear determinantal representation $C = V(\det A)$. We know that $D(\mathcal{F})$ defines the linear representation corresponding to the transpose matrix ${}^t A$. The case when $\mathcal{F} \cong D(\mathcal{F})$ corresponds to the symmetric matrix A. We assume that rank $\mathcal{F} = 1$, i.e. $\mathcal{F}$ is a theta characteristic θ on C.

By duality, the degree of a theta characteritic θ is equal to $p_a(C) - 1$ and $\chi(\theta) = 0$. We know that each theta characteristic θ is isomorphic to $\pi_*\theta'$, where θ' is a theta characteristic on the partial normalization of C defined by θ. Since, locally, $\mathcal{E}nd(\theta') \cong \mathcal{O}_{C'}$, we obtain that θ' is an invertible sheaf on C'.

Let $\mathrm{Jac}(X)[2]$ denote the 2-torsion subgroup of the group $\mathrm{Jac}(X)$ of isomorphism classes of invertible sheaves on a curve X. Via tensor product it acts on the set $\mathrm{TChar}(C)$ of theta characteristics on C. The pull-back map p^* defines an exact sequence

$$0 \to G \to \mathrm{Jac}(C) \to \mathrm{Jac}(\bar{C}) \to 0. \tag{4.34}$$ expic

The group $\mathrm{Jac}(\bar{C})$ is the group of points on the Jacobian variety of $\bar{C}$, an abelian variety of dimension equal to the genus g of $\bar{C}$. The group $G \cong \mathcal{O}^*_{\bar{C}}/\mathcal{O}^*_C$ has a structure of a commutative group, isomorphic to the product of additive and multiplicative groups of $\mathbb{C}$. Its dimension is equal to $\delta = \sum_x \delta_x$. It follows from the exact sequence that

$$\mathrm{Jac}(C)[2] \cong (\mathbb{Z}/2\mathbb{Z})^{2g+b}, \tag{4.35}$$

where b is equal to the dimension of the multiplicative part of G. It is easy to see that

$$b = \#p^{-1}(\mathrm{Sing}(C)) - \#\mathrm{Sing}(C) = \sum_x (r_x - 1). \tag{4.36}$$ k

Proposition 4.2.14. *The group* $\mathrm{Jac}(C)[2]$ *acts transitively on the set of theta characteristics with fixed global type. The order of the stabilizer subgroup of a theta characteristic* θ *is equal to the order of the* 2*-torsion subgroup of the kernel of* $\pi^* : \mathrm{Jac}(C) \to \mathrm{Jac}(C')$.

Proof Let $\theta, \theta' \in \mathrm{TChar}(C)$ with the isomorphic global type. Since two sheaves with isomorphic global type differ by an invertible sheaf, we have $\theta' \cong \theta \otimes \mathcal{L}$ for some invertible sheaf $\mathcal{L}$. This implies

$$\theta' \otimes \mathcal{L} \cong \theta'^{\vee} \otimes \omega_C \cong \theta^{\vee} \otimes \mathcal{L}^{-1} \otimes \omega_C \cong \theta \otimes \mathcal{L}^{-1} \cong \theta \otimes \mathcal{L}.$$

By Lemma 2.1 from BeauvilleCounting [50], $\pi^*\mathcal{F} \cong \pi^*\mathcal{F} \otimes \mathcal{L}$ for some $\mathcal{L} \in \mathrm{Jac}(C)$ if and only if $\pi^*\mathcal{L} \cong \mathcal{O}_{C'}$. This gives $\pi^*\mathcal{L}^2 \cong \mathcal{O}_{C'}$, and hence, $\pi^*(\mathcal{L}) \in \mathrm{Jac}(C')[2]$. It follows from exact sequence expic (4.34) (where C is replaced with C') that $\mathrm{Jac}(C')$ is a divisible group, hence, the homomorphism $p^* : \mathrm{Jac}(C)[2] \to \mathrm{Jac}(C')[2]$ is surjective. This implies that there exists $\mathcal{M} \in \mathrm{Jac}(C)[2]$ such that $\pi^*(\mathcal{L} \otimes \mathcal{M}) \cong \mathcal{O}_{C'}$. Thus, we obtain

$$\theta' \otimes \mathcal{M} \cong \theta \otimes \mathcal{L} \otimes \mathcal{M} \cong \theta.$$

This proves the first assertion. The second assertion follows from the loc. cit. Lemma. □

Corollary 4.2.15. *The number of theta characteristics of global type defined by a partial normalization* $\pi : C' \to C$ *is equal to* $2^{2g+b-b'}$*, where* $b' = \#\pi^{-1}(\mathrm{Sing}(C)) - \#\mathrm{Sing}(C)$.

Recall that a theta characteristic θ defines a symmetric determinantal representation of C if and only if it satisfies $h^0(\theta) = 0$. So, we would like to know how many such theta characteritics exist. A weaker condition is that $h^0(\theta)$ is even. In this case, the theta characteristic is called *even*, and the remaining ones are called *odd*. The complete answer on the number of even theta characteristics on a plane curve C is not known. In the case when $\theta \in \mathrm{Jac}(C)$, the answer, in terms of some local invariants of singularities, can be found in HarrisTheta [374] (see also Libgober [486] for a topological description of the local invariants). The complete answer is known in the case when C has *simple* (or *ADE*) singularities.

simplesing1 **Definition 4.2.16.** *A singular point* $x \in C$ *is called a* simple singularity *if its local ring is formally isomorphic to the local ring of the singularity at the origin of one of the following plane affine curves*

$$\begin{aligned} a_k &: x^2 + y^{k+1} = 0, \quad k \geq 1, \\ d_k &: x^2y + y^{k-1} = 0, \quad k \geq 4 \\ e_6 &: x^3 + y^4 = 0, \\ e_7 &: x^3 + xy^3 = 0, \\ e_8 &: x^3 + y^5 = 0. \end{aligned}$$

According to GreuelKnorrer [356], a simple singularity is characterized by the property that there are only finitely many isomorphism classes of indecomposable torsion-free modules over its local ring. This implies that the set $\mathrm{TChar}(C)$ is finite if C is a plane curve with only simple singularities.

The number of even theta characteristics on an irreducible reduced plane curve C with only simple singularities is given in the following Theorem from Piontkowski [587].

Theorem 4.2.17. *The number of invertible even theta characteristics on C is*

2^{2g+b-1}	*if C has an* A_{4s+1}, D_{4s+2}*, or* E_7 *singularity,*
$2^{g+b-1}(2^g+1)$	*if C has no singularities as above, and has an even number of types* $A_{8s+2}, A_{8s+3}, A_{8s+4}, D_{8s+3}, D_{8s+4}, D_{8s+5}, E_6$,
$2^{g+b-1}(2^g-1)$	*otherwise.*

The number of non-invertible even theta characteristics on a curve with simple singularities depends on their known local types. An algorithm to compute them is given in Piontkowski [587].

E4.2.2 *Example* 4.2.18. Let C be a plane irreducible cubic curve. Suppose it has an ordinary node. This is a simple singularity of type A_1. We have $\mathrm{Jac}(C) \cong \mathbb{C}^*$ and $\mathrm{Jac}(C)[2] \cong \mathbb{Z}/2\mathbb{Z}$. The only partial normalization is the normalization map. There is one invertible theta characteristic θ_1 with $h^0(\theta_1) = 0$ and one non-invertible theta characteristic $\theta_2 \cong p_*\mathcal{O}_{\bar{C}}(-1)$ with $h^0(\theta_2) = 0$. It is isomorphic to the conductor ideal sheaf on C. Thus, there are two isomorphism classes of symmetric determinant representations for C. Without loss of generality we may assume that $C = V(t_0t_2^2 + t_1^3 + t_0t_1^2)$. The theta characteristic θ_1 defines the symmetric determinantal representation

$$t_0t_2^2 + t_1^3 + t_0t_1^2 = \det\begin{pmatrix} 0 & t_2 & t_1 \\ t_2 & -t_0 - t_1 & 0 \\ t_1 & 0 & -t_0 \end{pmatrix}.$$

Observe that rank $A(x) = 2$ for all points $x \in C$. The theta characteristic θ_2 defines the symmetric determinantal representation

$$t_0t_2^2 + t_1^3 + t_0t_1^2 = \det\begin{pmatrix} -t_0 & 0 & -t_1 \\ 0 & -t_1 & -t_2 \\ -t_1 & -t_2 & t_1 \end{pmatrix}.$$

The rank of $A(x)$ is equal to 1 for the singular point $x = [1, 0, 0]$ and equals 2 for other points on C.

Assume now that C is a cuspidal cubic with equation $V(t_0t_2^2 + t_1^3)$. There are no invertible theta characteristics and there is only one non-invertible. It is isomorphic to the conductor ideal sheaf on C. It defines the symmetric linear determinantal representation

$$t_0t_2^2 + t_1^3 = \det\begin{pmatrix} 0 & -t_2 & -t_1 \\ -t_2 & -t_1 & 0 \\ -t_1 & 0 & -t_0 \end{pmatrix}.$$

Remark 4.2.19. We restricted ourselves with irreducible curves. The case of reducible nodal curves was studied in CataneseTheta [100].

S:4.3

4.3 Linear Determinantal Representations of Surfaces

SS:4.3.1

4.3.1 Normal surfaces in $\mathbb{P}^3$

Let S be a normal surface of degree d in $\mathbb{P}^3$. We are looking for an aCM sheaf $\mathcal{L}$ on $\mathbb{P}^3$ with scheme-theoretical support equal to S. We also require that $\mathcal{L}$ is of rank 1 and satisfies the additional assumption (vanl 4.15)

$$H^0(\mathbb{P}^3, \mathcal{L}(-1)) = H^2(\mathbb{P}^3, \mathcal{L}(-2)) = 0. \tag{4.37}$$ acm0

Every such $\mathcal{L}$ will define a linear determinantal representation of S defined by the resolution (mainrxseq 4.6) of $\mathcal{L}$ such that rank $A(x) = d - 1$ for a general point on S.

Since the exact sequence (mainrxseq 4.6) shows that $\mathcal{F}$ is generated by its global sections, we see that $\mathcal{F} \cong \mathcal{O}_S(C)$ for some effective Weil divisor C. By taking a general section of $\mathcal{F}$ and applying the Bertini theorem, we may assume that C is an integral curve, nonsingular outside $\mathrm{Sing}(S)$.

Recall that, as an aCM sheaf, $\mathcal{L}$ satisfies the cohomological condition

$$H^1(\mathbb{P}^3, \mathcal{L}(j)) = 0, \ j \in \mathbb{Z}. \tag{4.38}$$ acm1

Let s be a nonzero section of $\mathcal{L}$ whose zero subscheme is an integral curve such that $\mathcal{L} \cong \mathcal{O}_S(C)$. The dual of the map $\mathcal{O}_S \xrightarrow{s} \mathcal{L}$ defines an exact sequence

$$0 \to \mathcal{L}^\vee(j) \to \mathcal{O}_S(j) \to \mathcal{O}_C(j) \to 0. \tag{4.39}$$ acm5

By Serre's Duality,

$$H^1(S, \mathcal{L}^\vee(j)) \cong H^1(S, \mathcal{L}(-j) \otimes \omega_S) \cong H^1(S, \mathcal{L}(d-4-j)) = 0.$$

Applying cohomology, we obtain that the restriction map

$$H^0(S, \mathcal{O}_S(j)) \to H^0(C, \mathcal{O}_C(j)) \tag{4.40}$$

is surjective for all $j \in \mathbb{Z}$. Recall that, by definition, this means that C is *projectively normal* in $\mathbb{P}^3$. Conversely, if C is projectively normal, we obtain (acm1 4.38).

Before we state the next theorem we have to remind ourselves of some facts about the intersection theory on a normal singular surface (see MumfordTopology [538]).

Let $\sigma : S' \to S$ be a resolution of singularities that we always assume to be minimal. Let $\mathcal{E} = \sum_{i \in I} E_i$ be its reduced exceptional locus. For any curve C on S we denote by $\sigma^{-1}(C)$ the proper transform of C and define

$$\sigma^*(C) := \pi^{-1}(C) + \sum_{i \in I} n_i E_i,$$

where n_i are rational numbers uniquely determined by the system of linear equations

$$0 = \sigma^*(C) \cdot E_i = \cdot\pi^{-1}(C) \cdot E_j + \sum_{i \in I} n_i E_i \cdot E_j = 0, \ j \in I.$$

Now, we define the intersection number $C \cdot C'$ of two curves S by

$$C \cdot C' := \sigma^*(C) \cdot \sigma^*(C').$$

This can be extended by linearity to all Weil divisors on S. It coincides with the usual intersection product on the subgroup of Cartier divisors. Also, it depends only on the equivalence classes of the divisors.

Recall that S admits a dualizing sheaf ω_S. It is a reflexive sheaf of rank 1, hence, determines the linear equivalence class of a Weil divisors denoted by K_S (the *canonical class* of S). It is a Cartier divisor class if and only if S is Gorenstein (as it will be in our case when S is a hypersurface). We have

$$K_{S'} = \sigma^*(K_S) + \Delta,$$

where $\Delta = \sum_{i \in I} a_i E_i$ is the *discrepancy divisor*. The rational numbers a_i are uniquely determined from linear equations

$$K_{S'} \cdot R_j = \sum_{i \in I} a_i E_i \cdot E_j, \ j \in I.$$

For any reduced irreducible curve C on S define

$$A_S(C) := -\tfrac{1}{2}(\sigma^*(C) - \sigma^{-1}(C))^2 + \tfrac{1}{2}\sigma^{-1}(C) \cdot \Delta - \delta,$$

where $\delta = h^0(p^* \mathcal{O}_{\bar{C}}/\mathcal{O}_C)$ is our familiar invariant of the normalization of C. The following results can be found in Blache [60].

blache **Proposition 4.3.1.** *For any reduced curve C on S and a Weil divisor D let $\mathcal{O}_C(D)$ be the cokernel of the natural injective map $\mathcal{O}_S(D - C) \to \mathcal{O}(D)$ extending the similar map on $S \setminus \mathrm{Sing}(S)$. Then,*

(i) $C \mapsto A_S(C)$ extends to a homomorphism $\mathrm{WDiv}(S)/\mathrm{Div}(S) \to \mathbb{Q}$ which is independent of a resolution;

(ii) $\chi(\mathcal{O}_C(D)) = \chi(\mathcal{O}_C) + C \cdot D - 2A_S(C)$;

(iii) $-2\chi(\mathcal{O}_C) = C^2 + C \cdot K_S - 2A_S(C)$.

Example 4.3.2. Assume that S has only ordinary double points. Then, a minimal resolution $\sigma : S' \to S$ has the properties that $\Delta = 0$ and $\mathcal{E} = E_1 + \cdots + E_k$, where k is the number of singular points and each E_i is a smooth rational curves with $E_i \cdot K_{S'} = 0$ (see more about this in Chapter 8). Let $\sigma^{-1}(C) \cdot E_i = m_i$. Then, easy computations show that

$$\sigma^*(C) = \sigma^{-1}(C) + \tfrac{1}{2}\sum_{i=1}^{n} m_i E_i,$$
$$C^2 = \sigma^{-1}(C)^2 + \tfrac{1}{2}\sum_{i=1}^{n} m_i^2,$$
$$C \cdot K_S = \sigma^{-1}(C) \cdot K_{S'},$$
$$A_S(C) = \frac{1}{4}\sum_{i=1}^{k} m_i^2 - \delta.$$

Now, we are ready to state and to prove the following theorem.

Theorem 4.3.3. *Let $\mathcal{L}$ be an aCM sheaf of rank 1. Then, $\mathcal{L}$ defines a linear determinantal representation of S if and only if $\mathcal{L} \cong \mathcal{O}_S(C)$ for some projectively normal integral curve C with*

$$\deg C = \tfrac{1}{2}d(d-1), \quad p_a(C) = \frac{1}{6}(d-2)(d-3)(2d+1).$$

Proof Suppose $\mathcal{L}$ defines a linear determinantal representation of S. Then, it is an aCM sheaf isomorphic to $\mathcal{O}_S(C)$ for some integral projectively normal curve C, and satisfies conditions (4.37) and (4.38).

We have

$$\chi(\mathcal{L}(-1)) = h^0(\mathcal{L}(-1)) - h^1(\mathcal{L}(-1)) + h^2(\mathcal{L}(-1)).$$

By (4.37) and (4.38), the right-hand side is equal to $h^2(\mathcal{L}(-1))$. Let H be a general plane section of S and

$$0 \to \mathcal{O}_S(-H) \to \mathcal{O}_S \to \mathcal{O}_H \to 0 \tag{4.41}$$

be the tautological exact sequence defining the ideal sheaf of H. Tensoring with $\mathcal{L}(-1)$, we obtain an exact sequence

$$0 \to \mathcal{L}(-2) \to \mathcal{L}(-1) \to \mathcal{L}(-1) \otimes \mathcal{O}_H \to 0.$$

It shows that the condition $h^2(\mathcal{L}(-2)) = 0$ from (4.37) implies $h^2(\mathcal{L}(-1)) = 0$, hence

$$\chi(\mathcal{L}(-1)) = 0. \tag{4.42}$$

Similar computation shows that

$$\chi(\mathcal{L}(-2)) = 0. \tag{4.43}$$

Tensoring the exact sequence (4.41) with $\mathcal{O}_S(C-H)$, we obtain an exact sequence

$$0 \to \mathcal{L}(-2) \to \mathcal{L}(-1) \to \mathcal{O}_H(C-H) \to 0.$$

Applying the Riemann-Roch Theorem to the sheaf $\mathcal{O}_H(C-H)$ on H, we get

$$\deg \mathcal{O}_H(C-H) = \deg C - d = \chi(\mathcal{O}_H(C-H)) - \chi(\mathcal{O}_H)$$

$$= \chi(\mathcal{L}(-1)) - \chi(\mathcal{L}(-2)) - \chi(\mathcal{O}_H) = -\chi(\mathcal{O}_H).$$

This gives

$$\deg C = d - \chi(\mathcal{O}_H) = d - 1 + \tfrac{1}{2}(d-1)(d-2) = \tfrac{1}{2}d(d-1),$$

as asserted.

Applying Proposition 4.3.1 (ii), we get,

$$\chi(\mathcal{O}_C) = -C\cdot C + C\cdot H + \chi(\mathcal{O}_C(C-H)) + 2A_S(C)$$

$$= \deg C - C^2 + \chi(\mathcal{O}_C(C-H)) + 2A_S(C).$$

By Proposition 4.3.1 (iii),

$$C^2 = -C\cdot K_S - 2\chi(\mathcal{O}_C) + 2A_S(C) = -(d-4)\deg C - 2\chi(\mathcal{O}_C) + 2A_S(C),$$

hence

$$-\chi(\mathcal{O}_C) = (d-3)\deg C + \chi(\mathcal{O}_C(C-H)).$$

The exact sequence

$$0 \to \mathcal{O}_S(-H) \to \mathcal{O}_S(C-H) \to \mathcal{O}_C(C-H) \to 0$$

gives

$$\chi(\mathcal{O}_C(C-H)) = \chi(\mathcal{L}(-1)) - \chi(\mathcal{O}_S(-1)) = -\chi(\mathcal{O}_S(-1)).$$

Easy computations of the cohomology of projective space gives

$$\chi(\mathcal{O}_S(-1)) = \binom{d}{3}.$$

Combining all together, we obtain

$$p_a(C) = 1 - \chi(\mathcal{O}_C) = 1 + \tfrac{1}{2}d(d-1)(d-3) - \frac{1}{6}d(d-1)(d-2)$$

$$= \frac{1}{6}(d-2)(d-3)(2d+1),$$

as asserted. We leave it to the reader to reverse the arguments and prove the converse. □

Example 4.3.4. We will study the case of cubic surfaces in more detail in Chapter 9. Let us consider the case of quartic surfaces. Assume first that S is nonsingular. Then, $\mathcal{L} \cong \mathcal{O}_S(C)$, where C is a projectively normal smooth curve of degree 6 and genus 3. The projective normality is equivalent to the condition that C is not hyperelliptic (Exercise 4.10). We also have $h^0(\mathcal{O}_X(C))$ = 4. According to Noether's theorem, the Picard group of a general surface of degree ≥ 4 is generated by a plane section. Since a plane section of a quartic surface is of degree 4, we see that a general quartic surface does not admit a determinantal equation. The condition that X contains a curve C as in above imposes one algebraic condition on the coefficients of a quartic surface (one condition on the moduli of quartic surfaces).

Suppose now that S contains such a curve. By (4.8), the transpose determinantal representation $C = \det {}^tA$ is defined by the sheaf $\mathcal{G} \cong \mathcal{F}^\vee(3) \cong \mathcal{O}_S(3H - C)$, where H is a plane section of S. We have two maps $\mathfrak{l} : S \to \mathbb{P}^3, \mathfrak{r} : S \to \mathbb{P}^3$ defined by the complete linear systems $|C|$ and $|3H - C|$. Since $C^2 = -C \cdot K_S - 2\chi(\mathcal{O}_C) = 4$, the images are quartic surfaces. We will see later, in Chapter 7, that the two images are related by a Cremona transformation from $|U^\vee| = |C|^\vee$ to $|V^\vee| = |3H - C|^\vee$.

We will find examples with singular surface S in the next Subsection.

SS:4.3.2

4.3.2 Symmetroid surfaces

These are surfaces in $\mathbb{P}^3$ which admit a linear determinantal representation $S = V(\det A)$ with a symmetric matrix A. The name was coined by A. Cayley.

As we know from Subsection 4.1.2, the determinantal representation is given by an aCM sheaf $\mathcal{L}$ satisfying

$$\mathcal{L} \cong \mathcal{L}^\vee(d-1). \tag{4.44}$$

For example, if S is a smooth surface of degree d, we have $\mathcal{L} \cong \mathcal{O}_S(C)$ and we must have $C \sim (d-1)H - C$, where H is a plane section. Thus, numerically, $C = \frac{1}{2}(d-1)H$, and we obtain $C^2 = \frac{1}{4}d(d-1)^2, C \cdot K_S = \frac{1}{2}d(d-1)(d-4)$, and $p_a(C) = 1 + \frac{3}{8}d(d-1)(d-3)$. It is easy to see that it disagrees with the formula for $p_a(C)$ for any $d > 1$. A more obvious reason why a smooth surface cannot be a symmetroid is the following. The codimension of the locus of quadrics in $\mathbb{P}^d$ of corank ≥ 2 is equal to 3. Thus, each three-dimensional linear system of quadrics intersects this locus, and hence, at some point $x \in S$, we must have rank $A(x) \leq d-2$. Since our sheaf $\mathcal{F}$ is an invertible sheaf, this is impossible.

So, we have to look for singular surfaces. Let us state the known analog of Theorem 4.1.1 in the symmetric case.

We know from Remark 2.4.19 that

$$\deg \mathsf{Q}_U(2) = \frac{1}{6}d(d^2-1). \tag{4.45}$$

Thus, we expect that a general cubic symmetroid has four ordinary points, a general quartic symmetroid has ten ordinary nodes, and a general quintic symmetroid has twenty ordinary nodes.

Note that a symmetroid surface of degree d is the Jacobian hypersurface of the web of quadrics equal to the image of the map $\phi : \mathbb{P}^3 \to \mathsf{Q}_{d-1}$ defined by the determinantal representation. We identify $|E|$ with a web W of quadrics in $\mathbb{P}(U)$. The surface S is the discriminant hypersurface $\mathsf{D}(|E|)$ of W. The left kernel map $\mathfrak{l} : S \dashrightarrow \mathbb{P}^{d-1}$ given by $|\mathcal{O}_S(C)|$ maps S onto the Jacobian surface $\mathrm{Jac}(|E|)$ in $\mathbb{P}(U)$. $|E|$ is a regular web of quadrics if $|E|$ intersects the discriminant hypersurface of quadrics in $\mathbb{P}(U)$ transversally. In this case, we have the expected number of singular points on S, and all of them are ordinary nodes. The surface S admits a minimal resolution $\sigma : \tilde{S} := \tilde{\mathsf{D}}(|E|) \to S$. The map $\mathfrak{l} = \tilde{\mathfrak{l}} \circ \sigma^{-1}$, where $\tilde{\mathfrak{l}} : \tilde{S} \to \mathrm{Jac}(|E|)$. The map is given by the linear system $|\sigma^{-1}(C)|$. The Jacobian surface is a smooth surface of degree equal to $\sigma^{-1}(C)^2$.

Proposition 4.3.5. *Let S' be the Jacobian surface of $|E|$, the image of S under the right kernel map $\mathfrak{r}$. Assume that $|E|$ is a regular web of quadrics. Then,* $\mathrm{Pic}(S')$ *contains two divisor classes η, h such that $h^2 = d, \eta^2 = \binom{d}{3}$, and*

$$2\eta = (d-1)h - \sum_{i=1}^{k} E_i,$$

where E_i are exceptional curves of the resolution $\sigma : \tilde{S} \to S$.

Proof We identify S' with the resolution $\tilde{S}$ by means of the map $\tilde{\mathfrak{r}}$. We take $h = \sigma^*(\mathcal{O}_{|E|}(1))$ and η to be $\tilde{\mathfrak{r}}^*(\mathcal{O}_{S'}(1))$. We follow the proof of Proposition 4.2.3 to show that, under the restriction $|\mathcal{O}_{\mathbb{P}(U)}(2)| \to |\mathcal{O}_{S'}(2)|$, the web of quadrics $|E|$ in $|\mathcal{O}_{\mathbb{P}(U)}(2)|$ is identified with the linear system of polars of S. This is a linear subsystem in $|\mathcal{O}_S(d-1)|$. Its pre-image in $\tilde{S}$ is contained in the linear system $|(d-1)h - \sum_{i=1}^k E_i|$. It is clear that $h^2 = d$. It follows from Proposition 2.4.19 that $4\eta^2 = (d-1)^2 d - 2\binom{d+1}{3}$. This easily gives the asserted value of η^2. □

Corollary 4.3.6.

$$\deg S' = \eta^2 = \tbinom{d}{3}.$$

Using the adjunction formula, we find

$$2p_a(\eta) - 2 = \eta_S^2 + \eta \cdot K_{S'} = \eta^2 + \tfrac{1}{2}d(d-1)(d-4) = \tbinom{d}{3} + \tfrac{1}{2}d(d-1)(d-4)$$

$$= \frac{1}{3}d(d-1)(2d-7).$$

This agrees with the formula for $p_a(C)$ in Theorem arnaud 4.3.3.

It follows from the proposition that the theta characteristic θ defining the symmetric determinantal representation of S is isomorphic to $\mathcal{O}_S(C)$, where $C = \sigma_*(D)$ for $D \in |\eta|$. We have $\mathcal{O}_S(C)^{\otimes 2} \cong \mathcal{O}_S(d-1)$ outside $\mathrm{Sing}(S)$. This gives $\theta^{[2]} \cong \mathcal{O}_S(d-1)$.

evennodes *Remark* 4.3.7. Suppose d is odd. Let

$$\xi := \tfrac{1}{2}(d-1)h - \eta.$$

Then, $\sum_{i=1}^k E_i \sim 2\xi$. If d is even, we let

$$\xi := \tfrac{1}{2}dh - \eta.$$

Then, $h + \sum_{i=1}^k E_i \sim 2\xi$. So, the set of nodes is even in the former case and weakly even in the latter case (see CataneseBabbage [98]). The standard construction gives a double cover of S' ramified only over nodes if the set is even and over the union of nodes and a member of $|h|$ if the set is weakly even. We will discuss even and weakly sets of nodes on quartic surfaces in Chapter 12.

The bordered determinant formula (borddet 4.21) for the family of contact curves extends to the case of surfaces. It defines a $(d-1)$-dimensional family of contact surfaces of degree $d-1$. The proper transform of a contact curve in S' belongs to the linear system $|\eta|$.

quarticsymmetroid *Example* 4.3.8. We will consider the case $d = 3$ later. Assume $d = 4$ and the determinantal representation is transversal, i.e. S has the expected number 10 of nodes. Let S' be its minimal resolution. The linear system η consists of curves of genus 3 and degree 6. It maps S' isomorphically onto a quartic surface in $\mathbb{P}^3$, the Jacobian surface of the web of quadrics defined by the determinantal representation. The family of contact surfaces is a 3-dimensional family of cubic surfaces passing through the nodes of S and touching the surface along some curve of genus 3 and degree 6 passing through the nodes. The double cover corresponding to the divisor class ξ is a regular surface of general type with $p_g = 1$ and $c_1^2 = 2$.

Consider the linear system $|2h - E_1|$ on S'. Since $(h-E_1)^2 = 2$, it defines a degree 2 map onto $\mathbb{P}^2$. Since $(2h - E_i)\cdot E_j = 0, i > 10$, the curves $E_i, i \neq 1$, are blown down to points. The curve R_1 is mapped to a conic K on the plane. One can show that the branch curve of the cover is the union of two cubic curves and the conic K is tangent to both of the curves at each intersection point. Conversely, the double cover of the plane branched along the union of two cubics, which both everywhere are tangent to a nonsingular conic, is isomorphic to a quartic symmetroid (see [172]). We refer to Chapter 1, where we discussed the Reye varieties associated to n-dimensional linear systems of quadrics in $\mathbb{P}^n$. In the case of the quartic symmetroid parameterizing singular quadrics in a web of quadrics in $\mathbb{P}^3$, the Reye variety is an Enriques surface.

Assume $d = 5$ and S has expected number 20 of nodes. The linear system η consists of curves of genus 11 and degree 10. It maps S' isomorphically onto a surface of degree 10 in $\mathbb{P}^4$, the Jacobian surface of the web of quadrics defined by the determinantal representation. The family of contact surfaces is a 4-dimensional family of quartic surfaces passing through the nodes of S and touching the surface along some curve of genus 11 and degree 10 passing through the nodes. The double cover X of S branched over the nodes is a regular surface of general type with $p_g = 4$ and $c_1^2 = 10$. It is easy to see that the canonical linear system on X is the pre-image of the canonical linear system on S. This gives an example of a surface of general type such that the canonical linear system maps the surface onto a canonically embedded normal surface, a counter-example to Babbage's conjecture (see [98]).

Exercises

4.1 Find explicitly all equivalence classes of linear determinantal representations of a nodal or a cuspidal cubic.

4.2 Show that a general binary form admits a unique equivalence class of symmetric determinantal representations.

4.3 The following problems lead to a symmetric determinantal expression of a plane rational curve [467].

(i) Show that, for any two degree d binary forms $p(u_0, u_1)$ and $q(u_0, u_1)$, there exists a unique $d \times d$ symmetric matrix $B(p, q) = (b_{ij})$ whose entries are bilinear functions of the coefficients of p and q such that

$$\begin{vmatrix} p(u_0,u_1) & p(v_0,v_1) \\ q(u_0,u_1) & q(v_0,v_1) \end{vmatrix} = (u_0v_1 - u_1v_0) \sum b_{ij} u_0^i u^{d-j} v_0^j v_1^{d-j}.$$

(ii) Show that the determinant of $B(p, q)$ (the *bezoutiant* of p, q) vanishes if and only if the two binary forms have a common zero.

(iii) Let p_0, p_1, p_2 be three binary forms of degree d without common zeros and C be the image of the map

$$\mathbb{P}^1 \to \mathbb{P}^2,\ [u_0, u_1] \mapsto [p_0(u_0, u_1), p_1(u_0, u_1), p_2(u_0, u_1)].$$

Show that C is given by the equation $f(t_0, t_1, t_2) = |B(t_0p_1 - t_1p_0, t_0p_2 - t_2p_0)| = 0$.

(iv) Prove that $f = |t_0B(p_1, p_2) - t_1B(t_0, t_2) - t_2B(t_0, t_1)|$ and any symmetric determinantal equation of C is equivalent to this.

4.4 Let $C = V(f)$ be a nonsingular plane cubic, p_1, p_2, p_3 be three non-collinear points. Let (A_0, A_1, A_2) define a quadratic Cremona transformation with fundamental points p_1, p_2, p_3. Let q_1, q_2, q_3 be another set of three points such that the six points $p_1, p_2, p_3, q_1, q_2, q_3$ are cut out by a conic. Let (B_0, B_1, B_2) define a quadratic Cremona transformation with fundamental points q_1, q_2, q_3. Show that

$$F^{-3} \det \mathrm{adj} \begin{pmatrix} A_0B_0 & A_0B_1 & A_0B_2 \\ A_1B_0 & A_1B_1 & A_1B_2 \\ A_2B_0 & A_2B_1 & A_2B_2 \end{pmatrix}$$

ex:4.4 is a determinantal equation of C.

ex:4.5 4.5 Find determinantal equations for a nonsingular quadric surface in $\mathbb{P}^3$.

4.6 Let $E \subset \mathrm{Mat}_d$ be a linear subspace of dimension 3 of the space of $d \times d$ matrices. Show that the locus of points $x \in \mathbb{P}^{d-1}$ such that there exists $A \in E$ such that $Ax = 0$ is defined by $\binom{d}{3}$ equations of degree 3. In particular, for any determinantal equation of a curve C, the images of C under the maps $\mathfrak{r} : \mathbb{P}^2 \to \mathbb{P}^{d-1}$ and ex:4.6 $\mathfrak{l} : \mathbb{P}^2 \to \mathbb{P}^{d-1}$ are defined by such a system of equations.

4.7 Show that the variety of nets of quadrics in $\mathbb{P}^n$ whose discriminant curve is ex:4.7 singular is reducible.

4.8 Let $C = V(\det A)$ be a linear determinantal representation of a plane curve C of degree d defiined by a rank 1 torsion-free sheaf $\mathcal{F}$ of global type $\pi : C' \to C$. Show that the rational map $\mathfrak{l} : C \to \mathbb{P}^{d-1}, x \mapsto |N(A(x))|$ extends to a regular ex:4.8 map $C' \to \mathbb{P}^{d-1}$.

4.9 Let C be a non-hyperelliptic curve of genus 3 and degree 6 in $\mathbb{P}^3$.

(i) Show that the homogeneous ideal of C in $\mathbb{P}^3$ is generated by four cubic polynomials f_0, f_1, f_2, f_3.

(ii) Show that the equation of any quartic surface containing C can be written in the form $\sum l_i f_i = 0$, where l_i are linear forms.

(iii) Show that (f_0, f_1, f_2, f_3) define a birational map f from $\mathbb{P}^3$ to $\mathbb{P}^3$. The ex:4.9 image of any quartic containing C is another quartic surface.

4.10 Show that a curve of degree 6 and genus 3 in $\mathbb{P}^3$ is projectively normal if and ex:4.10 only if it is not hyperelliptic.

4.11 Let C be a nonsingular plane curve of degree d and $\mathcal{L}_0 \in \mathrm{Pic}^{g-1}(C)$ with $h^0(\mathcal{L}_0) \neq 0$. Show that the image of C under the map given by the complete ex:4.11 linear system $\mathcal{L}_0(1)$ is a singular curve.

4.12 Let θ be a theta characteristic on a nonsingular plane curve of degree d with $h^0(\theta) = 1$. Show that the corresponding aCM sheaf on $\mathbb{P}^2$ defines an equation of C expressed as the determinant of a symmetric $(d-1) \times (d-1)$ matrix $(a_{ij}(t))$, where $a_{ij}(t)$ are of degree 1 for $1 \le i, j \le d-3$, $a_{1j}(t)$ are of degree 2, and ex:4.12 $a_{d-1d-1}(t)$ is of degree 3 BeauvilleDet [51].

4.13 Let $S = V(\det A)$ be a linear determinantal representation of a nonsingular quartic surface in $\mathbb{P}^3$. Show that the four 3×3 minors of the matrix B obtained from A by deleting one row define the equations of a projectively normal curve of degree ex:4.13 6 and genus 3 lying on S.

4.14 Show that any quartic surfaces containing a line and a rational normal cubic not ex:4.14 intersecting the line admits a determinantal representation.

4.15 Show that the Hessian hypersurface of a general cubic hypersurface in $\mathbb{P}^4$ is hypersurface of degree 5 whose singular locus is a curve of degree 20. Show that ex:4.15 its general hyperplane section is a quintic symmetroid surface.

4.16 Let C be a curve of degree $N(d) = d(d-1)/2$ and arithmetic genus $G(d) = \frac{1}{6}(d-2)(d-3)(2d+1)$ on a smooth surface of degree d in $\mathbb{P}^3$. Show that the linear system $|\mathcal{O}_S(-C)(d)|$ consists of curves of degree $N(d+1)$ and arithmetic genus $G(d+1)$.

4.17 Let S be a general symmetroid quintic surface in $\mathbb{P}^3$ and $|L|$ be the linear system of projectively normal curves of degree 10 and genus 11 that defines a symmetric linear determinantal representation of S. Let S' be the image of S under the rational map $\Phi : \mathbb{P}^3 \to \mathbb{P}^d = |\mathcal{O}_C|^\vee$ and W be the web of quadrics defining the linear determinantal representation of S. Consider the rational map $T : \mathbb{P}^4 \dashrightarrow \mathbb{P}^4$ defined by sending a point $x \in \mathbb{P}^4$ to the intersection of polar hyperplanes $P_x(Q), Q \in W$. Prove the following assertions (see [757].

(i) The indeterminacy locus of T is equal to S'.
(ii) The image of a general hyperplane H is a quartic hypersurface X_H.
(iii) The intersection $X_H \cap X_{H'} = S' \cup F$, F is a surface of degree 6.
(iv) Each 4-secant line of C contained in H (there are 20 of them) is blown down under T to 20 nodes of X_H.

4.18 Let $p_1, \ldots, p_5$ be five points in $\mathbb{P}^3$ in general linear position. Prove the following assertions (see [758]).

(i) Show that one can choose a point q_{ij} on the line $\overline{p_i p_j}$ such that the lines $\overline{p_1 q_{34}}, \overline{p_2 q_{45}}, \overline{p_3 q_{25}}, \overline{p_4 q_{12}}, \overline{p_5 q_{23}}$ form a closed space pentagon.
(ii) Show that the union of five lines $\overline{p_i p_j}$ and five lines defined in (i) is a curve of arithmetic genus 11.
(ii) Show that the linear system of quartic surfaces containing the 10 lines maps $\mathbb{P}^3$ to a quartic hypersurface in $\mathbb{P}^4$ with 45 nodes (the *Burhardt quartic threefold*).

4.19 Show that the equivalence classes of determinantal representations of a plane curve C of degree $2k$ with quadratic forms as entries correspond to aCM sheaves on C satisfying $h^0(\mathcal{F}(-1)) = 0$ and $\mathcal{F}(-\frac{1}{2}d-2)^\vee \cong \mathcal{F}(-\frac{1}{2}(d-2))$.

4.20 Show that the union of d different hyperplanes in $\mathbb{P}^n$ always admits a unique equivalence class of symmetric linear determinant representations.

4.21 Show that the secant varieties of a Veronese variety V_n^d are determinantal varieties.

Historical Notes

Apparently, O. Hesse was the first to state clearly the problem of representation of the equation of a hypersurface as a symmetric determinant of linear forms [390]. He did it for plane curves of order 4 [391]. He also showed that it can be done in 36 different ways corresponding to 36 families of contact cubics. For cubic curves, the representation follows from the fact that any cubic curve can be written in three ways as the Hessian curve of another cubic curve. This fact was also proven by Hesse [387, p. 89].

The fact that a general plane curve of degree d can be defined by the determinant of a symmetric $d \times d$ matrix with entries as homogeneous linear forms was first proved by A. Dixon [226]. Dixon's result was reproved later by J. Grace and A. Young [351]. Modern expositions of Dixon's theory were given by A. Beauville [46] and A. Tyurin [752] [753].

W. Barth was the first to introduce a non-invertible theta characteristics on a singular plane curve [32]. It was studied for nodal plane curves by A. Beauville [46] and F. Catanese [100], and for arbitrary singular curves of degree ≤ 4, by C.T.C. Wall [797].

It was proved by L. Dickson [223] that any plane curve can be written as the determinant of a not necessarily symmetric matrix with linear homogeneous forms as its entries. The relationship between linear determinantal representations of an irreducible plane curve of degree d and line bundles of degree $d(d-1)/2$ was first established in [166]. This was later elaborated by V. Vinnikov [791]. A deep connection between linear determinantal representations of real curves and the theory of colligations for pairs of commuting operators in a Hilbert space was discovered by M. Lifšic [490] and his school (see [491]).

The theory of linear determinantal representation for cubic surfaces was developed by L. Cremona [186]. Dickson proves in [223] that a general homogeneous form of degree $d > 2$ in r variables cannot be represented as a linear determinant unless $r = 3$ or $r = 4, d \leq 3$. The fact that a determinantal representation of quartic surfaces is possible only if the surface contains a projectively normal curve of genus 3 and degree 6 goes back to F. Schur [670]. However, it was A. Coble who was the first to understand the reason: by Noether's theorem, the Picard group of a general surface of degree ≥ 4 is generated by a plane section [159], p. 39. The case of quartic surfaces was studied in detail in a series of papers of T. Room [632]. Quartic symmetroid surfaces were first studied by A. Cayley [119]. They appear frequently in algebraic geometry. Coble's paper [157] studies (in a disguised form) the group of birational automorphisms of such surfaces. There is a close relationship between quartic symmetroids and Enriques surfaces (see [172], [259], Chapter 7] M. Artin and D. Mumford [25] used quartic symmetroids in their celebrated construction of counter-examples to the Lüroth Problem. A modern theory of symmetroid surfaces can be found in papers of A. Beauville [51] and F. Catanese [98].

We refer to [51] for a comprehensive survey of the modern theory of determinantal representations of hypersurfaces based on the theory of aCM sheaves. In this paper, numerous special examples of determinantal representations are found.

In classical algebraic geometry, a determinantal representation was considered as a special case of a projective generation of subvarieties in a projective space. It seems that the geometric theory of determinantal varieties started from the work of H. Grassmann in 1856 [355], where he considers the projective generation of a cubic surface by three collinear nets of planes. Grassmann's construction was greatly generalized in a series of papers by T. Reye [620]. In the last paper of the series, he studies curves of degree 10 and genus 11, which lead to linear determinantal representation of quintic surfaces.

Algebraic theory of determinantal varieties started from the work of F. S. Macaulay [500], where the fact that the loci of rank $\leq r$ square matrices are Cohen-Macaulay varieties can be found. The classical account of the theory of determinantal varieties is T. Room's monograph [633]. A modern treatment of determinantal varieties can be found in several books [13], [315], [360]. The book by W. Bruns and U. Vetter [77] gives a rather complete account of the recent development of the algebraic theory of determinantal ideals. The formula for the dimensions and the degrees of determinantal varieties in the general case of $m \times n$ matrices and also symmetric matrices goes back to C. Segre [693] and G. Giambelli [331], [332].

5
Theta Characteristics

Ch5

S:5.1

5.1 Odd and Even Theta Characteristics

We have already defined a theta characteristic in Chapter 4 (see Definition 4.1.19). In section 4.2 we discussed theta characteristics on singular curves. Here, we will study theta characteristics on a nonsingular projective curve in more detail.

S:5.1.1

5.1.1 Theta characteristics and quadratic forms over a field of characteristic 2

Let C be a nonsingular irreducible curve and θ be a theta characteristic on C. We can consider it either as an invertible sheaf or a divisor on C. By definition, $\theta^{\otimes 2} \cong \omega_C$, or $2D \sim K_C$. It follows from the definition that two theta characteristics, considered as divisor classes of degree $g-1$, differ by a 2-torsion divisor class. Since the 2-torsion subgroup $\mathrm{Jac}(C)[2]$ is isomorphic to $(\mathbb{Z}/2\mathbb{Z})^{2g}$, there are 2^{2g} theta characteristics. However, in general, there is no canonical identification between the set $\mathrm{TChar}(C)$ of theta characteristics on C and the set $\mathrm{Jac}(C)[2]$. One can say only that $\mathrm{TChar}(C)$ is an affine space over the vector space of $\mathrm{Jac}(C)[2] \cong \mathbb{F}_2^{2g}$.

There is one more structure on $\mathrm{TChar}(C)$ besides being an affine space over $\mathrm{Jac}(C)[2]$. Recall that the subgroup of 2-torsion points $\mathrm{Jac}(C)[2]$ is equipped with a natural symmetric bilinear form over $\mathbb{F}_2$, called the *Weil pairing*. It is defined as follows (see [13], Appendix B). Let ϵ, ϵ' be two 2-torsion divisor classes. Choose their representatives D, D' with disjoint supports. Write $\mathrm{div}(\phi) = 2D, \mathrm{div}(\phi') = 2D'$ for some rational functions ϕ and ϕ'. Then, $\frac{\phi(D')}{\phi'(D)} = \pm 1$. Here, for any rational function ϕ defined at points x_i, $\phi(\sum_i x_i) =$

$\prod_i \phi(x_i)$. Now, we set

$$\langle \epsilon, \epsilon' \rangle = \begin{cases} 1 & \text{if } \phi(D')/\phi'(D) = -1, \\ 0 & \text{otherwise.} \end{cases}$$

Note that the Weil pairing is a symplectic form, i.e. satisfies $\langle \epsilon, \epsilon \rangle = 0$. One can show that it is a nondegenerate symplectic form (see [541]).

For any $\vartheta \in \mathrm{TChar}(C)$, define the function

$$q_\vartheta : \mathrm{Jac}(C)[2] \to \mathbb{F}_2,\ \epsilon \mapsto h^0(\vartheta + \epsilon) + h^0(\vartheta).$$

The proof of the following Theorem can be found in [13], p. 290).

Theorem 5.1.1 (Riemann-Mumford Relation)**.** *The function q_ϑ is a quadratic form on* $\mathrm{Jac}(C)[2]$ *whose associated symmetric bilinear form is equal to the Weil pairing.*

Later we shall see that there are two types of quadratic forms associated with a fixed nondegenerate symplectic form: even and odd. They agree with our definition of an even and odd theta characteristic. The number of even (odd) theta characteristics is equal to $2^{g-1}(2^g+1)$ $(2^{g-1}(2^g-1))$.

An odd theta characteristic ϑ is obviously effective, i.e. $h^0(\vartheta) > 0$. If C is a canonical curve, then divisor $D \in |\vartheta|$ satisfies the property that $2D$ is cut out by a hyperplane H in the space $|K_C|^\vee$, where C is embedded. Such a hyperplane is called a *contact hyperplane*. It follows from above that a canonical curve either has $2^{g-1}(2^g-1)$ contact hyperplanes or infinitely many. The latter case happens if and only if there exists a theta characteristic ϑ with $h^0(\vartheta) > 1$. Such a theta characteristic is called a *vanishing theta characteristic*. An example of a vanishing odd theta characteristic is the divisor class of a line section of a plane quintic curve. An example of a vanishing even theta characteristic is the unique g_3^1 on a canonical curve of genus 4 lying on a singular quadric.

The geometric interpretation of an even theta characteristic is more subtle. In the previous chapter, we related theta characteristics, both even and odd, to determinantal representations of plane curves. The only known geometrical construction related to space curves that I know is the Scorza construction of a quartic hypersurface associated to a canonical curve and a non-effective theta characteristic. We will discuss this construction in Section 5.5.

Recall that a quadratic form on a vector space V over a field $\Bbbk$ is a map $q : V \to \Bbbk$ such that $q(av) = a^2 q(v)$ for any $a \in \Bbbk$ and any $v \in V$, and the map

$$b_q : V \times V \to \Bbbk, \quad (v, w) \mapsto q(v+w) - q(v) - q(w)$$

is bilinear (it is called the *polar bilinear form*). We have $b_q(v, v) = 2q(v)$ for any $v \in V$. In particular, q can be reconstructed from b_q if $\mathrm{char}(\Bbbk) \neq 2$. In the case when $\mathrm{char}(\Bbbk) = 2$, we get $b_q(v, v) \equiv 0$, hence b_q is a symplectic bilinear form. Two quadratic forms q, q' have the same polar bilinear form if and only if $q - q' = l$, where $l(v+w) = l(v) + l(w), l(av) = a^2 l(v)$ for any $v, w \in V, a \in \Bbbk$. If $\Bbbk$ is a finite field of characteristic 2, $\sqrt{l}$ is a linear form on V, and we obtain

$$b_q = b_{q'} \Longleftrightarrow q = q' + \ell^2 \tag{5.1}$$ nonuniq

for a unique linear form $\ell : V \to \Bbbk$.

Let $e_1, \ldots, e_n$ be a basis in V and $A = (a_{ij}) = (b_q(e_i, e_j))$ be the matrix of the bilinear form b_q. It is a symmetric matrix with zeros on the diagonal if $\mathrm{char}(\Bbbk) = 2$. It follows from the definition that

$$q(\sum_{i=1}^n x_i e_i) = \sum_{i=1}^n x_i^2 q(e_i) + \sum_{1 \le i < j \le n} x_i x_j a_{ij}.$$

The *rank* of a quadratic form is the rank of the matrix A of the polar bilinear form. A quadratic form is called *nondegenerate* if the rank is equal to $\dim V$. In a coordinate-free way, this is the rank of the linear map $V \to V^\vee$ defined by b_q. The kernel of this map is called the *radical* of b_q. The restriction of q to the radical is identically zero. The quadratic form q arises from a nondegenerate quadratic form on the quotient space. In the following, we assume that q is nondegenerate.

A subspace L of V is called *singular* if $q|L \equiv 0$. Each singular subspace is an *isotropic subspace* with respect to b_q, i.e. $b_q(v, w) = 0$ for any $v, w \in E$. The converse is true only if $\mathrm{char}(\Bbbk) \neq 2$.

Assume $\mathrm{char}(\Bbbk) = 2$. Since b_q is a nondegenerate symplectic form, $n = 2k$, and there exists a basis $e_1, \ldots, e_n$ in V such that the matrix of b_q is equal to

$$J_k = \begin{pmatrix} 0_k & I_k \\ I_k & 0_k \end{pmatrix}. \tag{5.2}$$ matrixJ

We call such a basis a *standard symplectic basis*. In this basis

$$q(\sum_{i=1}^n x_i e_i) = \sum_{i=1}^n x_i^2 q(e_i) + \sum_{i=1}^k x_i x_{i+k}.$$

Assume, additionally, that $\Bbbk^* = \Bbbk^{*2}$, i.e., each element in $\Bbbk$ is a square (e.g. $\Bbbk$ is a finite or algebraically closed field). Then, we can further reduce q to the form

$$q(\sum_{i=1}^{2k} x_i e_i) = (\sum_{i=1}^n \alpha_i x_i)^2 + \sum_{i=1}^k x_i x_{i+k}, \tag{5.3}$$ 222

where $q(e_i) = \alpha_i^2,\ i = 1, \ldots, n$. This makes (5.1) more explicit. Fix a nondegenerate symplectic form $\langle , \rangle : V \times V \to \Bbbk$. Each linear function on V is given by $\ell(v) = \langle v, \eta \rangle$ for a unique $\eta \in V$. By (5.1), two quadratic forms q, q' with the polar bilinear form equal to $\langle , \rangle$ satisfy

$$q(v) = q'(v) + \langle v, \eta \rangle^2$$

for a unique $\eta \in V$. Choose a standard symplectic basis. The quadratic form defined by

$$q_0\Big(\sum_{I=1}^{2k} x_i e_i\Big) = \sum_{i=1}^{k} x_i x_{i+k}$$

has the polar bilinear form equal to the standard symplectic form. Any other form with the same polar bilinear form is defined by

$$q(v) = q_0(v) + \langle v, \eta_q \rangle^2,$$

where

$$\eta_q = \sum_{i=1}^{2k} \sqrt{q(e_i)}\, e_i.$$

From now on, $\Bbbk = \mathbb{F}_2$, the field of two elements. In this case, $a^2 = a$ for any $a \in \mathbb{F}_2$. The formula (5.1) shows that the set $Q(V)$ of quadratic forms associated with the standard symplectic form is an affine space over V with addition $q + \eta, q \in Q(V), \eta \in V$, defined by

$$(q + \eta)(v) = q(v) + \langle v, \eta \rangle = q(v + \eta) + q(\eta). \tag{5.4}$$ translate

The number

$$\mathrm{Arf}(q) = \sum_{i=1}^{k} q(e_i) q(e_{i+k}) \tag{5.5}$$ arf

is called the *Arf invariant* of q. One can show that the Arc invariant is independent of the choice of a standard symplectic basis (see [361], Proposition 1.11). A quadratic form $q \in Q(V)$ is called *even* (resp. *odd*) if $\mathrm{Arf}(q) = 0$ (resp. $\mathrm{Arf}(q) = 1$).

If we choose a standard symplectic basis for b_q and write q in the form $q_0 + \eta_q$, then we obtain

$$\mathrm{Arf}(q) = \sum_{i=1}^{k} \alpha_i \alpha_{i+k} = q_0(\eta_q) = q(\eta_q). \tag{5.6}$$ arf2

In particular, if $q' = q + v = q_0 + \eta_q + v$,

$$\mathrm{Arf}(q') + \mathrm{Arf}(q) = q_0(\eta_q + v) + q_0(\eta_q) = q_0(v) + \langle v, \eta_q \rangle = q(v). \quad (5.7)$$ arf3

It follows from (5.6) [arf2] that the number of even (resp. odd) quadratic forms is equal to the cardinality of the set $q_0^{-1}(0)$ (resp. $q_0^{-1}(1)$). We have

$$|q_0^{-1}(0)| = 2^{k-1}(2^k + 1), \quad |q_0^{-1}(1)| = 2^{k-1}(2^k - 1). \quad (5.8)$$ even2

This is easy to prove by using induction on k.

Let $\mathrm{Sp}(V)$ be the group of linear automorphisms of the symplectic space V. If we choose a standard symplectic basis then

$$\mathrm{Sp}(V) \cong \mathrm{Sp}(2k, \mathbb{F}_2) = \{X \in \mathrm{GL}(2k)(\mathbb{F}_2) : {}^tX \cdot J_k \cdot X = J_k\}.$$

It is easy to see by induction on k that

$$|\mathrm{Sp}(2k, \mathbb{F}_2)| = 2^{k^2}(2^{2k} - 1)(2^{2k-2} - 1) \cdots (2^2 - 1). \quad (5.9)$$ grouporder

The group $\mathrm{Sp}(V)$ has two orbits in $Q(V)$, the set of even and the set of odd quadratic forms. An even quadratic form is equivalent to the form q_0 and an odd quadratic form is equivalent to the form

$$q_1 = q_0 + e_k + e_{2k},$$

where $(e_1, \ldots, e_{2k})$ is the standard symplectic basis. Explicitly,

$$q_1\Big(\sum_{i=1}^{2k} x_i e_i\Big) = \sum_{i=1}^{k} x_i x_{i+k} + x_k^2 + x_{2k}^2.$$

The stabilizer subgroup $\mathrm{Sp}(V)^+$ (resp. $\mathrm{Sp}(V)^-$) of an even quadratic form (resp. an odd quadratic form) is a subgroup of $\mathrm{Sp}(V)$ of index $2^{k-1}(2^k + 1)$ (resp. $2^{k-1}(2^k - 1)$). If $V = \mathbb{F}_2^{2k}$ with the symplectic form defined by the matrix J_k, then $\mathrm{Sp}(V)^+$ (resp. $\mathrm{Sp}(V)^-$) is denoted by $\mathrm{O}(2k, \mathbb{F}_2)^+$ (resp. $\mathrm{O}(2k, \mathbb{F}_2)^-$).

Let $\mathcal{X} \to S$ be a smooth projective morphism whose fiber X_s over a point $s \in S$ is a curve of genus $g > 0$ over the residue field $\kappa(s)$ of s. Let $\mathbf{Pic}^n_{\mathcal{X}/S} \to S$ be the *relative Picard scheme* of $\mathcal{X}/S$. It represents the sheaf in étale topology on S associated to the functor on the category of S-schemes defined by assigning to a S-scheme T the group $\mathrm{Pic}^d(X \times_S T)$ of isomorphism classes of invertible sheaves on $X \times_S T$ of relative degree n over T modulo tensor product with invertible sheaves coming from T. The S-scheme $\mathbf{Pic}^n_{\mathcal{X}/S} \to S$ is a smooth projective scheme over S. Its fiber over a point $s \in S$ is isomorphic to the Picard variety $\mathbf{Pic}^n_{\mathcal{X}_s/\kappa(s)}$ over the field $\kappa(s)$. The relative Picard scheme comes with a universal invertible sheaf $\mathcal{U}$ on $\mathcal{X} \times_S \mathbf{Pic}^n_{\mathcal{X}/S}$ (locally in étale topology). For any point $y \in \mathbf{Pic}^n_{\mathcal{X}/S}$ over a point $s \in S$, the restriction of $\mathcal{U}$ to the fiber of the second projection over y is an invertible sheaf $\mathcal{U}_y$ on $X_s \otimes_{\kappa(s)} \kappa(y)$ representing a point in $\mathrm{Pic}^n(\mathcal{X}_s \otimes \kappa(y))$ defined by y.

For any integer m, raising a relative invertible sheaf into m-th power defines a morphism

$$[m] : \mathbf{Pic}^n_{\mathcal{X}/S} \to \mathbf{Pic}^{mn}_{\mathcal{X}/S}.$$

Taking $n = 2g - 2$ and $m = 2$, the pre-image of the section defined by the relative canonical class $\omega_{\mathcal{X}/S}$ is a closed subscheme of $\mathbf{Pic}^{g-1}_{\mathcal{X}/S}$. It defines a finite cover

$$\mathcal{TC}_{\mathcal{X}/S} \to S$$

of degree 2^{2g}. The pull-back of $\mathcal{U}$ to $\mathcal{TC}_{\mathcal{X}/S}$ defines an invertible sheaf $\mathcal{T}$ over $\mathcal{P} = \mathcal{X} \times_S \mathcal{TC}_{\mathcal{X}/S}$ satisfying $\mathcal{T}^{\otimes 2} \cong \omega_{\mathcal{P}/\mathcal{TC}_{\mathcal{X}/S}}$. By a theorem of Mumford MumTheta [541], the parity of a theta characteristic is preserved in an algebraic family, thus the function $\mathcal{TC}_{\mathcal{X}/S} \to \mathbb{Z}/2\mathbb{Z}$ defined by $y \mapsto \dim H^0(U_y, \mathcal{T}_y) \mod 2$ is constant on each connected component of $\mathcal{TC}_{\mathcal{X}/S}$. Let $\mathcal{TC}^{\mathrm{ev}}_{\mathcal{X}/S}$ (resp. $\mathcal{TC}^{\mathrm{odd}}_{\mathcal{X}/S}$) be the closed subset of $\mathcal{TC}_{\mathcal{X}/S}$, where this function takes the value 0 (resp. 1). The projection $\mathcal{TC}^{\mathrm{ev}}_{\mathcal{X}/S} \to S$ (resp. $\mathcal{TC}^{\mathrm{odd}}_{\mathcal{X}/S} \to S$) is a finite cover of degree $2^{g-1}(2^g + 1)$ (resp. $2^{g-1}(2^g - 1)$).

It follows from above that $\mathcal{TC}_{\mathcal{X}/S}$ has at least two connected components.

Now, take $S = |\mathcal{O}_{\mathbb{P}^2}(d)|^{\mathrm{ns}}$ to be the space of nonsingular plane curves C of degree d and $\mathcal{X} \to |\mathcal{O}_{\mathbb{P}^2}(d)|^{\mathrm{ns}}$ be the universal family of curves defined by $\{(x, C) : x \in C\}$. We set

$$\mathcal{TC}_d = \mathcal{TC}_{\mathcal{X}/S},\ \mathcal{TC}_d^{\mathrm{ev/odd}} = \mathcal{TC}^{\mathrm{ev/odd}}_{\mathcal{X}/S}.$$

The proof of the following Proposition can be found in BeauvilleMon [47].

beauville2 **Proposition 5.1.2.** *If d is even or $d = 3$, $\mathcal{TC}_d$ consists of two irreducible components $\mathcal{TC}_d^{ev}$ and $\mathcal{TC}_d^{odd}$. If $d \equiv 1 \mod 4$, then $\mathcal{TC}_d^{ev}$ is irreducible but $\mathcal{TC}_d^{odd}$ has two irreducible components, one of which is the section of $\mathcal{TC}_d \to |\mathcal{O}_{\mathbb{P}^2}(d)|$ defined by $\mathcal{O}_{\mathbb{P}^2}((d-3)/2)$. If $d \equiv 3 \mod 4$, then $\mathcal{TC}_d^{odd}$ is irreducible but $\mathcal{TC}_d^{ev}$ has two irreducible components, one of which is the section of $\mathcal{TC}_d \to |\mathcal{O}_{\mathbb{P}^2}(d)|$ defined by $\mathcal{O}_{\mathbb{P}^2}((d-3)/2)$.*

Let $\mathcal{TC}_d^0$ be the open subset of $\mathcal{TC}_d^{\mathrm{ev}}$ corresponding to the pairs (C, ϑ) with $h^0(\vartheta) = 0$. It follows from the theory of symmetric determinantal representations of plane curves that $\mathcal{TC}_d^0/\mathrm{PGL}(3)$ is an irreducible variety covered by an open subset of a Grassmannian. Since the algebraic group $\mathrm{PGL}(3)$ is connected and acts freely on a Zariski open subset of $\mathcal{TC}_d^0$, we obtain that $\mathcal{TC}_d^0$ is irreducible. It follows from the previous Proposition that

$$\mathcal{TC}_d^0 = \mathcal{TC}_d^{\mathrm{ev}} \quad \text{if } d \not\equiv 3 \mod 4. \tag{5.10}$$

Note that there exist coarse moduli spaces $\mathcal{M}_g^{ev}$ and $\mathcal{M}_g^{\mathrm{odd}}$ of curves of genus g together with an even (odd) theta characteristic. We refer to [170] for the proof of the irreducibility of these varieties and for the construction of certain compactifications of these spaces. The recent paper of G. Farkas and A. Verra [297] established the birational properties of the varieties $\mathcal{M}_g^{ev}$.

5.2 Hyperelliptic curves

5.2.1 Equations of hyperelliptic curves

Let us first describe explicitly theta characteristics on hyperelliptic curves. Recall that a hyperelliptic curve of genus g is a nonsingular projective curve X of genus $g > 1$ admitting a degree 2 map $\varphi : C \to \mathbb{P}^1$. By Riemann–Hurwitz formula, there are $2g+2$ branch points $p_1, \ldots, p_{2g+2}$ in $\mathbb{P}^1$. Let $f_{2g+2}(t_0, t_1)$ be a binary form of degree $2g+2$ whose zeros are the branch points. The equation of C in the weighted projective plane $\mathbb{P}(1, 1, g+1)$ is

$$t_2^2 + f_{2g+2}(t_0, t_1) = 0. \tag{5.11}$$

Recall that a weighted projective space $\mathbb{P}(\mathbf{q}) = \mathbb{P}(q_0, \ldots, q_n)$ is defined as the quotient $\mathbb{C}^{n+1} \setminus \{0\}/\mathbb{C}^*$, where $\mathbb{C}^*$ acts by

$$t : [z_0, \ldots, z_n] \mapsto [t^{q_0} z_0, \ldots, t^{q_n} z_n].$$

A more general definition of $\mathbb{P}(\mathbf{q})$ which works over $\mathbb{Z}$ is

$$\mathbb{P}(\mathbf{q}) = \mathrm{Proj}\, \mathbb{Z}[T_0, \ldots, T_n],$$

where the grading is defined by setting $\deg T_i = q_i$. Here, $\mathbf{q} = (q_0, \ldots, q_n)$ are integers ≥ 1. We refer to [233] or [421] for the theory of weighted projective spaces and their subvarieties. Note that a hypersurface in $\mathbb{P}(\mathbf{q})$ is defined by a homogeneous polynomial where the unknowns are homogeneous of degree q_i. Thus, equation (5.11) defines a hypersurface of degree $2g+2$. Although, in general, $\mathbb{P}(\mathbf{q})$ is a singular variety, it admits a canonical sheaf

$$\omega_{\mathbb{P}(\mathbf{q})} = \mathcal{O}_{\mathbb{P}(\mathbf{q})}(-|\mathbf{q}|),$$

where $|\mathbf{q}| = q_0 + \cdots + q_n$. Here, the Serre sheaves are understood in the sense of the theory of projective spectrums of graded algebras. There is also the adjunction formula for a hypersurface $X \subset \mathbb{P}(\mathbf{q})$ of degree d

$$\omega_X = \mathcal{O}_X(d - |\mathbf{q}|). \tag{5.12}$$

In the case of a hyperelliptic curve, we have

$$\omega_C = \mathcal{O}_C(g-1).$$

The morphism $\varphi : C \to \mathbb{P}^1$ corresponds to the projection $[t_0, t_1, t_2] \mapsto [t_0, t_1]$ and we obtain that

$$\omega_C = \varphi^*\mathcal{O}_{\mathbb{P}^1}(g-1).$$

The weighted projective space $\mathbb{P}(1, 1, g+1)$ is isomorphic to the projective cone in $\mathbb{P}^{g+2}$ over the Veronese curve $v_{g+1}(\mathbb{P}^1) \subset \mathbb{P}^{g+1}$. The hyperelliptic curve is isomorphic to the intersection of this cone and a quadric hypersurface in $\mathbb{P}^{g+1}$ not passing through the vertex of the cone. The projection from the vertex to the Veronese curve is the double cover $\varphi : C \to \mathbb{P}^1$. The canonical linear system $|K_C|$ maps C to $\mathbb{P}^g$ with the image equal to the Veronese curve $v_{g-1}(\mathbb{P}^1)$.

A very useful birational model of a hyperelliptic curve is its plane model as a curve of degree $g+2$ with a singular point $\mathfrak{o}$ of multiplicity g. It is obtained as the image of a regular map $\phi : C \to \mathbb{P}^2$ given by the liner series $|g_2^1 + \mathfrak{a}|$, where $\mathfrak{a}$ is an effective divisor of degree g. The image of $\mathfrak{a}$ is the singular point $\mathfrak{o}$, and $\mathfrak{a}$ is equal the pre-image of $\mathfrak{o}$ under the normalization map. In particular, taking $\mathfrak{a}$ supported at one point, we obtain that every hyperelliptic curve of genus g admits plane model as a curve of degree $g+2$ with a simple singular points of type a_{g-1}.

SS:5.2.2

5.2.2 2-torsion points on a hyperelliptic curve

Let $c_1, \ldots, c_{2g+2}$ be the ramification points of the map φ. We assume that $\varphi(c_i) = p_i$. Obviously, $2c_i - 2c_j \sim 0$, hence the divisor class of $c_i - c_j$ is of order 2 in $\mathrm{Pic}(C)$. Also, for any subset I of the set $B_g = \{1, \ldots, 2g+2\}$,

$$\alpha_I = \sum_{i \in I} c_i - \#I c_{2g+2} = \sum_{i \in I}(c_i - c_{2g+2}) \in \mathrm{Pic}(C)[2].$$

Now, observe that

$$\alpha_{B_g} = \sum_{i \in B_g} c_i - (2g+2)c_{2g+2} = \mathrm{div}(\phi) \sim 0, \tag{5.13}$$ zero

where $\phi = t_2/(bt_0 - at_1)^{g+1}$ and $p_{2g+2} = [a, b]$ (we consider the fraction modulo (5.11) hypeq defining C). Thus,

$$c_i - c_j \sim 2c_i + \sum_{k \in B_g \setminus \{j\}} c_k - (2g+2)c_{2g+2} \sim \alpha_{B_g \setminus \{i,j\}}.$$

Adding to α_I the zero divisor $c_{2g+2} - c_{2g+2}$, we can always assume that $\#S$ is even. Also adding the principal divisor α_{B_g}, we obtain that $\alpha_I = \alpha_{\bar{I}}$, where $\bar{I}$ denotes $B_g \setminus I$.

Let $\mathbb{F}_2^{B_g} \cong \mathbb{F}_2^{2g+2}$ be the $\mathbb{F}_2$-vector space of functions $B_g \to \mathbb{F}_2$, or, equivalently, subsets of B_g. The sum is defined by the symmetric sum of subsets

$$I + J = I \cup J \setminus (I \cap J).$$

The subsets of even cardinality form a hyperplane. It contains the subsets $\emptyset$ and B_g as a subspace of dimension 1. Let E_g denote the quotient space. Elements of E_g are represented by subsets of even cardinality up to the complementary set (*bifid maps* in terminology of A. Cayley). We have

$$E_g \cong \mathbb{F}_2^{2g},$$

hence the correspondence $I \mapsto \alpha_I$ defines an isomorphism

$$E_g \cong \mathrm{Pic}(C)[2]. \tag{5.14}$$ weil*

Note that E_g carries a natural symmetric bilinear form

$$e : E_g \times E_g \to \mathbb{F}_2, \quad e(I, J) = \#I \cap J \mod 2. \tag{5.15}$$ sympform

This form is symplectic (i.e. $e(I, I) = 0$ for any I) and nondegenerate. The subsets

$$A_i = \{2i - 1, 2i\}, \quad B_i = \{2i, 2i + 1\}, \quad i = 1, \ldots, g, \tag{5.16}$$ sympbasis

form a standard symplectic basis.

Under isomorphism (5.14), this bilinear form corresponds to the Weil pairing on 2-torsion points of the Jacobian variety of C.

Remark 5.2.1. The symmetric group $\mathfrak{S}_{2g+2}$ acts on E_g via its action on B_g and preserves the symplectic form e. This defines a homomorphism

$$s_g : \mathfrak{S}_{2g+2} \to \mathrm{Sp}(2g, \mathbb{F}_2).$$

If $g = 1$, $\mathrm{Sp}(2, \mathbb{F}_2) \cong \mathfrak{S}_3$, and the homomorphism s_1 has the kernel isomorphic to the group $(\mathbb{Z}/2\mathbb{Z})^2$. If $g = 2$, the homomorphism s_2 is an isomorphism. If $g > 2$, the homomorphism s_g is injective but not surjective.

SS:5.2.3

5.2.3 Theta characteristics on a hyperelliptic curve

For any subset T of B_g set

$$\vartheta_T = \sum_{i \in T} c_i + (g - 1 - \#T c_{2g+2}) = \alpha_T + (g - 1)c_{2g+2}.$$

We have

$$2\vartheta_T \sim 2\alpha_T + (2g-2)c_{2g+2} \sim (2g-2)c_{2g+2}.$$

It follows from the proof of the Riemann–Hurwitz formula that

$$K_C = \varphi^*(K_{\mathbb{P}^1}) + \sum_{i\in B_g} c_i.$$

Choose a representative of $K_{\mathbb{P}^1}$ equal to $-2p_{2g+2}$ and use (5.13) to obtain

$$K_C \sim (2g-2)c_{2g+2}.$$

This shows that ϑ_T is a theta characteristic. Again adding and subtracting c_{2g+2} we may assume that $\#T \equiv g+1 \mod 2$. Since T and $\bar{T}$ define the same theta characteristic, we will consider the subsets up to taking the complementary set. We obtain a set Q_g which has a natural structure of an affine space over E_g, the addition is defined by

$$\vartheta_T + \alpha_I = \vartheta_{T+I}.$$

Thus, all theta characteristics are uniquely represented by the divisor classes ϑ_T, where $T \in Q_g$.

An example of an affine space over $V = \mathbb{F}_2^{2g}$ is the space of quadratic forms $q : \mathbb{F}_2^{2g} \to \mathbb{F}_2$ whose associated symmetric bilinear form b_q coincides with the standard symplectic form defined by (5.2). We identify V with its dual $V^\vee$ by means of b_0 and set $q + l = q + l^2$ for any $l \in V^\vee$.

For any $T \in Q_g$, we define the quadratic form q_T on E_g by

$$q_T(I) = \tfrac{1}{2}(\#(T+I) - \#T) = \#T \cap I + \tfrac{1}{2}\#I = \tfrac{1}{2}\#I + e(I,T) \mod 2.$$

We have (all equalities are modulo 2)

$$q_T(I+J) + q_T(I) + q_T(J)$$

$$= \tfrac{1}{2}(\#(I+J) + \#I + \#J) + e(I+J,T) + e(I,T) + e(J,T) = \#I \cap J.$$

Thus, each theta characteristic can be identified with an element of the space $Q_g = Q(E_g)$ of quadratic forms on E_g with polar form e.

Also notice that

$$(q_T + \alpha_I)(J) = q_T(J) + e(I,J) = \tfrac{1}{2}\#J + e(T,J) + e(I,J)$$

$$= \tfrac{1}{2}\#J + e(T+I,J) = q_{T+I}(J).$$

Lemma 5.2.2. *Let ϑ_T be a theta characteristic on a hyperelliptic curve C of genus g identified with a quadratic form on E_g. Then, the following properties are equivalent:*

(i) $\#T \equiv g+1 \mod 4$*;*

(ii) $h^0(\vartheta_T) \equiv 0 \mod 2$*;*

(iii) q_T *is even.*

Proof Without loss of generality, we may assume that p_{2g+2} is the point $(0,1)$ at infinity in $\mathbb{P}^1$. Then, the field of rational functions on C is generated by the functions $y = t_2/t_0$ and $x = t_1/t_0$. We have

$$\vartheta_T = \sum_{i \in T} c_i + (g-1-\#T)c_{2g+2} \sim (g-1+\#T)c_{2g+2} - \sum_{i \in T} c_i.$$

Any function ϕ from the space $L(\vartheta_T) = \{\phi : \mathrm{div}(\phi) + \vartheta_T \geq 0\}$ has a unique pole at c_{2g+2} of order $< 2g+1$. Since the function y has a pole of order $2g+1$ at c_{2g+2}, we see that $\phi = \varphi^*(p(x))$, where $p(x)$ is a polynomial of degree $\leq \frac{1}{2}(g-1+\#T)$ in x. Thus. $L(\vartheta_T)$ is isomorphic to the linear space of polynomials $p(x)$ of degree $\leq \frac{1}{2}(g-1+\#T)$ with zeros at $p_i, i \in T$. The dimension of this space is equal to $\frac{1}{2}(g+1-\#T)$. This proves the equivalence of (i) and (ii).

Let

$$U = \{1, 3, \ldots, 2g+1\} \subset B_g \tag{5.17}$$

be the subset of odd numbers in B_g. If we take the standard symplectic basis in E_g defined in (5.16), then we obtain that $q_U = q_0$ is the standard quadratic form associated to the standard symplectic basis. It follows from (5.6) that q_T is an even quadratic form if and only if $T = U + I$, where $q_U(I) = 0$. Let I consists of k even numbers and s odd numbers. Then, $q_U(I) = \#U \cap I + \frac{1}{2}\#I = m + \frac{1}{2}(k+m) = 0 \mod 2$. Thus $\#T = \#(U+S) = \#U + \#I - 2\#U \cap S = (g+1) + (k+m) - 2m = g+1+k-m$. Then, $m + \frac{1}{2}(k+m)$ is even, hence $3m+k \equiv 0 \mod 4$. This implies that $k-m \equiv 0 \mod 4$ and $\#T \equiv g+1 \mod 4$. Conversely, if $\#T \equiv g+1 \mod 4$, then $k-m \equiv 0 \mod 4$ and $q_U(I) = 0$. This proves the assertion. □

5.3 Theta Functions

5.3.1 Jacobian variety

Recall the definition of the Jacobian variety of a nonsingular projective curve C of genus g over $\mathbb{C}$. We consider C as a compact oriented 2-dimensional manifold of genus g. We view the linear space $H^0(C, K_C)$ as the space of holomorphic 1-forms on C. By integration over 1-dimensional cycles, we get a homomorphism of $\mathbb{Z}$-modules

$$\iota : H_1(C,\mathbb{Z}) \to H^0(C,K_C)^\vee, \quad \iota(\gamma)(\omega) = \int_\gamma \omega.$$

The image of this map is a lattice Λ of rank $2g$ in $H^0(C,K_C)^\vee$. The quotient by this lattice

$$\mathrm{Jac}(C) = H^0(C,K_C)^\vee/\Lambda$$

is a complex g-dimensional torus. It is called the *Jacobian variety* of C.

Recall that the cap product

$$\cap : H_1(C,\mathbb{Z}) \times H_1(C,\mathbb{Z}) \to H_2(C,\mathbb{Z}) \cong \mathbb{Z}$$

defines a nondegenerate symplectic form on the group $H_1(C,\mathbb{Z}) \cong \mathbb{Z}^{2g}$. Let $\alpha_1,\ldots,\alpha_g,\beta_1,\ldots,\beta_g$ be a standard symplectic basis. We choose a basis $\omega_1,\ldots,\omega_g$ of holomorphic 1-differentials on C such that

$$\int_{\alpha_i} \omega_j = \delta_{ij}. \tag{5.18}$$

normm

Let

$$\tau_{ij} = \int_{\beta_i} \omega_i.$$

The complex matrix $\tau = (\tau_{ij})$ is called the *period matrix*. The basis $\omega_1,\ldots,\omega_g$ identifies $H^0(C,K_C)^\vee$ with $\mathbb{C}^g$ and the period matrix identifies the lattice Λ with the lattice $\Lambda_\tau = [\tau\ I_g]\mathbb{Z}^{2g}$, where $[\tau\ I_g]$ denotes the block-matrix of size $g \times 2g$. The period matrix $\tau = \Re(\tau) + \sqrt{-1}\Im(\tau)$ satisfies

$$^t\tau = \tau, \quad \Im(\tau) > 0.$$

As is well-known (see [360]) this implies that $\mathrm{Jac}(C)$ is a projective algebraic group, i.e. an abelian variety. It is isomorphic to the Picard scheme $\mathbf{Pic}^0_{C/\mathbb{C}}$.

We consider any divisor $D = \sum n_x x$ on C as a 0-cycle on C. The divisors of degree 0 are boundaries, i.e. $D = \partial\gamma$ for some 1-chain β. By integrating over β we get a linear function on $H^0(C,K_C)$ whose coset modulo $\Lambda = \iota(H_1(C,\mathbb{Z}))$ does not depend on the choice of β. This defines a homomorphism of groups $p : \mathrm{Div}^0(C) \to \mathrm{Jac}(C)$. The *Abel-Jacobi Theorem* asserts that p is zero on principal divisors (Abel's part), and surjective (Jacobi's part). This defines an isomorphism of abelian groups

$$\mathfrak{a} : \mathrm{Pic}^0(C) \to \mathrm{Jac}(C) \tag{5.19}$$

which is called the *Abel-Jacobi map*. For any positive integer d let $\mathrm{Pic}^d(C)$

denote the set of divisor classes of degree d. The group $\mathrm{Pic}^0(C)$ acts simply transitively on $\mathrm{Pic}^d(C)$ via the addition of divisors. There is a canonical map

$$u_d : C^{(d)} \to \mathrm{Pic}^d(C),\ D \mapsto [D],$$

where we identify the symmetric product with the set of effective divisors of degree d. One can show that $\mathrm{Pic}^d(C)$ can be equipped with a structure of a projective algebraic variety (isomorphic to the Picard scheme $\mathbf{Pic}^d_{C/\mathbb{C}}$) such that the map u_d is a morphism of algebraic varieties. Its fibers are projective spaces, the complete linear systems corresponding to the divisor classes of degree d. The action of $\mathrm{Pic}^0(C) = \mathrm{Jac}(C)$ on $\mathrm{Pic}^d(C)$ is an algebraic action equipping $\mathrm{Pic}^d(C)$ with a structure of a torsor over the Jacobian variety.

Let

$$W^r_{g-1} = \{[D] \in \mathrm{Pic}^{g-1}(C) : h^0(D) \geq r+1\}.$$

In particular, W^0_{g-1} was denoted by Θ in Theorem dixon 4.1.15, where we showed that the invertible sheaves $\mathcal{L}_0 \in \mathrm{Pic}^{g-1}(C)$ defining a determinantal equation of a plane curve of genus g belong to the set $\mathrm{Pic}^{g-1}(C) \setminus W^0_{g-1}$. The fundamental property of the loci W^r_{g-1} is given by the following *Riemann-Kempf Theorem*.

RKT **Theorem 5.3.1.**

$$W^r_{g-1} = \{x \in W^0_{g-1} : \mathrm{mult}_x W^0_{g-1} \geq r+1\}.$$

Here, mult_x *denotes the multiplicity of a hypersurface at the point* x.

In particular, we get

$$W^1_{g-1} = \mathrm{Sing}(W^0_{g-1}).$$

From now on we will identify $\mathrm{Pic}^0(C)$ with the set of points on the Jacobian variety $\mathrm{Jac}(C)$ by means of the Abel-Jacobi map. For any theta characteristic ϑ the subset

$$\Theta = W^0_{g-1} - \vartheta \subset \mathrm{Jac}(C)$$

is a hypersurface in $\mathrm{Jac}(C)$. It has the property that

$$h^0(\Theta) = 1, \quad [-1]^*(\Theta) = \Theta, \qquad (5.20)$$ thetadiv

where $[m]$ is the multiplication by an integer m in the group variety $\mathrm{Jac}(C)$. Conversely, any divisor on $\mathrm{Jac}(C)$ satisfying these properties is equal to W^0_{g-1} translated by a theta characteristic. This follows from the fact that a divisor D on an abelian variety A satisfying $h^0(D) = 1$ defines a bijective map $A \to \mathrm{Pic}^0(A)$ by sending a point $x \in A$ to the divisor $t_x^*D - D$, where t_x is the translation map $a \mapsto a + x$ in the group variety, and $\mathrm{Pic}^0(A)$ is the group of divisor classes algebraically equivalent to zero. This fact implies that any two divisors satisfying properties (5.20) differ by translation by a 2-torsion point.

We call a divisor satisfying (5.20) a *symmetric theta divisor*. An abelian variety that contains such a divisor is called a *principally polarized abelian variety*.

Let $\Theta = W^0_{g-1} - \theta$ be a symmetric theta divisor on $\mathrm{Jac}(C)$. Applying Theorem 5.3.1 we obtain that, for any 2-torsion point $\epsilon \in \mathrm{Jac}(C)$, we have

$$\mathrm{mult}_\epsilon \Theta = h^0(\vartheta + \epsilon). \tag{5.21}$$ mult

In particular, $\epsilon \in \Theta$ if and only if $\theta + \epsilon$ is an effective theta characteristic. According to ϑ, the symmetric theta divisors are divided into two groups: even and odd theta divisors.

SS:5.3.2

5.3.2 Theta functions

The pre-image of Θ under the quotient map $\mathrm{Jac}(C) = H^0(C, K_C)^\vee/\Lambda$ is a hypersurface in the complex linear space $V = H^0(C, K_C)^*$ equal to the zero set of some holomorphic function $\phi : V \to \mathbb{C}$. This function ϕ is not invariant with respect to translations by Λ. However, it has the property that, for any $v \in V$ and any $\gamma \in \Lambda$,

$$\phi(v + \gamma) = e_\gamma(v)\phi(v), \tag{5.22}$$ th8

where e_γ is an invertible holomorphic function on V. A holomorphic function ϕ satisfying (5.22) is called a *theta function* with *theta factor* $\{e_\gamma\}$. The set of zeros of ϕ does not change if we replace ϕ with $\phi\alpha$, where α is an invertible holomorphic function on V. The function $e_\gamma(v)$ will change into the function $e_{\gamma'}(v) = e_\gamma(v)\phi(v+\gamma)\phi(v)^{-1}$. One can show that, after choosing an appropriate α, one may assume that

$$e_\gamma(v) = \exp(2\pi i(a_\gamma(v) + b_\gamma)),$$

where a_γ is a linear function and b_γ is a constant (see [540], Chapter 1, §1). We will assume that such a choice has been made.

It turns out that the theta function corresponding to a symmetric theta divisor Θ from (5.20) can be given in coordinates defined by a choice of a normalized basis (5.18) by the following expression

$$\theta\left[\begin{smallmatrix}\epsilon\\ \eta\end{smallmatrix}\right](\mathbf{z};\tau) = \sum_{r \in \mathbb{Z}^g} \exp \pi i\left[(\mathbf{r} + \tfrac{1}{2}\boldsymbol{\epsilon}) \cdot \tau \cdot (\mathbf{r} + \tfrac{1}{2}\boldsymbol{\epsilon}) + 2(\mathbf{z} + \tfrac{1}{2}\boldsymbol{\eta}) \cdot (\mathbf{r} + \tfrac{1}{2}\boldsymbol{\epsilon})\right], \tag{5.23}$$ tetcharacteristic

where $\boldsymbol{\epsilon}, \boldsymbol{\eta} \in \{0,1\}^g$ considered as column or raw vectors from $\mathbb{F}_2^g$. The function defined by this expression is called a *theta function with characteristic*. The theta factor $e_\lambda(z_1, \ldots, z_g)$ for such a function is given by the expression

$$e_\gamma(\mathbf{z}) = \exp -\pi i(\mathbf{m} \cdot \tau \cdot \mathbf{m} - 2\mathbf{z} \cdot \mathbf{m} - \boldsymbol{\epsilon} \cdot \mathbf{n} + \boldsymbol{\eta} \cdot \mathbf{m}),$$

where we write $\gamma = \tau \cdot \mathbf{m} + \mathbf{n}$ for some $\mathbf{m}, \mathbf{n} \in \mathbb{Z}^g$. One can check that

$$\theta \begin{bmatrix} \boldsymbol{\epsilon} \\ \boldsymbol{\eta} \end{bmatrix} (-\mathbf{z}; \tau) = \exp(\pi i \boldsymbol{\epsilon} \cdot \boldsymbol{\eta}) \theta \begin{bmatrix} \boldsymbol{\epsilon} \\ \boldsymbol{\eta} \end{bmatrix} (\mathbf{z}; \tau). \tag{5.24}$$

This shows that $\theta \begin{bmatrix} \boldsymbol{\epsilon} \\ \boldsymbol{\eta} \end{bmatrix} (-\mathbf{z}; \tau)$ is an odd (resp. even) function if and only if $\boldsymbol{\epsilon} \cdot \boldsymbol{\eta} = 1$ (resp. 0). In particular, $\theta \begin{bmatrix} \boldsymbol{\epsilon} \\ \boldsymbol{\eta} \end{bmatrix} (0; \tau) = 0$ if the function is odd. It follows from (5.21) that $\theta \begin{bmatrix} \boldsymbol{\epsilon} \\ \boldsymbol{\eta} \end{bmatrix} (0; \tau) = 0$ if θ is an odd theta characteristic or an effective even theta characteristic.

Taking $\boldsymbol{\epsilon}, \boldsymbol{\eta} = 0$, we obtain the *Riemann theta function*

$$\theta(\mathbf{z}; \tau) = \sum_{\mathbf{r} \in \mathbb{Z}^g} \exp \pi i(\mathbf{r} \cdot \tau \cdot \mathbf{r} + 2\mathbf{z} \cdot \mathbf{r}).$$

All other theta functions with characteristic are obtained from $\theta(\mathbf{z}; \tau)$ by a translate

$$\theta \begin{bmatrix} \boldsymbol{\epsilon} \\ \boldsymbol{\eta} \end{bmatrix} (\mathbf{z}; \tau) = \exp \pi i(\boldsymbol{\epsilon} \cdot \boldsymbol{\eta} + \boldsymbol{\epsilon} \cdot \tau \cdot \boldsymbol{\epsilon}) \theta(\mathbf{z} + \tfrac{1}{2}\tau \cdot \boldsymbol{\eta} + \tfrac{1}{2}\boldsymbol{\epsilon}; \tau).$$

In this way points on $\mathbb{C}^g$ of the form $\frac{1}{2}\tau \cdot \boldsymbol{\epsilon} + \frac{1}{2}\boldsymbol{\eta}$ are identified with elements of the 2-torsion group $\frac{1}{2}\Lambda/\Lambda$ of $\mathrm{Jac}(C)$. The theta divisor corresponding to the Riemann theta function is equal to W_{g-1}^0 translated by a certain theta characteristic κ called the *Riemann constant*. Of course, there is no distinguished theta characteristic; the definition of κ depends on the choice of a symplectic basis in $H_1(C, \mathbb{Z})$.

The multiplicity m of a point on a theta divisor $\Theta = W_{g-1}^0 - \vartheta$ is equal to the multiplicity of the corresponding theta function defined by vanishing partial derivatives up to order $m - 1$. Thus, the quadratic form defined by θ can be redefined in terms of the corresponding theta function as

$$q_\vartheta(\tfrac{1}{2}\tau \cdot \boldsymbol{\epsilon}' + \tfrac{1}{2}\boldsymbol{\eta}') = \mathrm{mult}_0 \theta \begin{bmatrix} \boldsymbol{\epsilon}+\boldsymbol{\epsilon}' \\ \boldsymbol{\eta}+\boldsymbol{\eta}' \end{bmatrix} (\mathbf{z}, \tau) + \mathrm{mult}_0 \theta \begin{bmatrix} \boldsymbol{\epsilon} \\ \boldsymbol{\eta} \end{bmatrix} (\mathbf{z}, \tau).$$

It follows from (5.24) that this number is equal to

$$\boldsymbol{\epsilon} \cdot \boldsymbol{\eta}' + \boldsymbol{\eta} \cdot \boldsymbol{\eta}' + \boldsymbol{\eta}' \cdot \boldsymbol{\eta}'. \tag{5.25}$$

A choice of a symplectic basis in $H_1(C, \mathbb{Z})$ defines a standard symplectic basis in $H_1(C, \mathbb{F}_2) \cong \frac{1}{2}\Lambda/\Lambda = \mathrm{Jac}(C)[2]$. Thus, we can identify 2-torsion points $\frac{1}{2}\tau \cdot \boldsymbol{\epsilon}' + \frac{1}{2}\boldsymbol{\eta}'$ with vectors $(\boldsymbol{\epsilon}', \boldsymbol{\eta}') \in \mathbb{F}_2^{2g}$. The quadratic form corresponding to the Riemann theta function is the standard one

$$q_0((\boldsymbol{\epsilon}', \boldsymbol{\eta}')) = \boldsymbol{\epsilon}' \cdot \boldsymbol{\eta}'.$$

The quadratic form corresponding to $\theta\left[\begin{smallmatrix}\boldsymbol{\epsilon}\\ \boldsymbol{\eta}\end{smallmatrix}\right](\mathbf{z};\tau)$ is given by (5.25). The Arf invariant of this quadratic form is equal to

$$\mathrm{Arf}(q_\vartheta) = \boldsymbol{\epsilon}\cdot\boldsymbol{\eta}.$$

SS:5.3.3

5.3.3 Hyperelliptic curves again

In this case we can compute the Riemann constant explicitly. Recall that we identify 2-torsion points with subsets of even cardinality of the set $B_g = \{1,\ldots,2g+2\}$ which we can identify with the set of ramification or branch points. Let us define a standard symplectic basis in C by choosing the 1-cycle α_i to be the path which goes from c_{2i-1} to c_{2i} along one sheet of the Riemann surface C and returns to c_{2i-1} along the other sheet. Similarly, we define the 1-cycle β_i by choosing the points c_{2i} and c_{2i+1}. Choose g holomorphic forms ω_j normalized by the condition (5.18). Let τ be the corresponding period matrix. Notice that each holomorphic 1-form changes sign when we switch the sheets. This gives

$$\begin{aligned}\tfrac{1}{2}\delta_{ij} = \tfrac{1}{2}\int_{\alpha_i}\omega_j &= \int_{c_{2i-1}}^{c_{2i}}\omega_j = \int_{c_{2i-1}}^{c_{2g+2}}\omega_j - \int_{c_{2i}}^{c_{2g+2}}\omega_j\\ &= \int_{c_{2i-1}}^{c_{2g+2}}\omega_j + \int_{c_{2i}}^{c_{2g+2}}\omega_j - 2\int_{c_{2i}}^{c_{2g+2}}\omega_j.\end{aligned}$$

Since

$$2\Big(\int_{c_{2i}}^{c_{2g+2}}\omega_1,\ldots,\int_{c_{2i}}^{c_{2g+2}}\omega_g\Big) = \mathfrak{a}(2c_{2i}-2c_{2g+2}) = 0,$$

we obtain

$$\iota(c_{2i-1}+c_{2i}-2c_{2g+2}) = \tfrac{1}{2}\mathbf{e}_i \mod \Lambda_\tau,$$

where, as usual, $\mathbf{e}_i$ denotes the i-th unit vector. Let A_i, B_i be defined as in (5.16). We obtain that

$$\mathfrak{a}(\alpha_{A_i}) = \tfrac{1}{2}\mathbf{e}_i \mod \Lambda_\tau.$$

Similarly, we find that

$$\mathfrak{a}(\alpha_{B_i}) = \tfrac{1}{2}\tau\cdot\mathbf{e}_i \mod \Lambda_\tau.$$

Now, we can match the set Q_g with the set of theta functions with characteristics. Recall that the set $U = \{1,3,\ldots,2g+1\}$ plays the role of the standard quadratic form. We have

$$q_U(A_i) = q_U(B_i) = 0, \quad i = 1,\ldots,g.$$

Comparing it with (5.25), we see that the theta function $\theta\left[\begin{smallmatrix}\boldsymbol{\epsilon}\\ \boldsymbol{\eta}\end{smallmatrix}\right](\mathbf{z};\tau)$ corresponding to ϑ_U must coincide with the function $\theta(\mathbf{z};\tau)$. This shows that

$$\iota_{c_{2g+2}}^{g-1}(\vartheta_U) = \iota_{c_{2g+2}}(\vartheta_U - k_{c_{2g+2}}) = 0.$$

Thus, the Riemann constant κ corresponds to the theta characteristic ϑ_U. This allows one to match theta characteristics with theta functions with theta characteristics.

Write any subset I of E_g in the form

$$I = \sum_{i=1}^{g} \epsilon_i A_i + \sum_{i=1}^{g} \eta_i B_i,$$

where $\boldsymbol{\epsilon} = (\epsilon_1, \ldots, \epsilon_g)$, $\boldsymbol{\eta} = (\eta_1, \ldots, \eta_g)$ are binary vectors. Then,

$$\vartheta_{U+I} \longleftrightarrow \theta\left[\begin{smallmatrix}\boldsymbol{\epsilon}\\ \boldsymbol{\eta}\end{smallmatrix}\right](\mathbf{z};\tau).$$

In particular,

$$\vartheta_{U+I} \in \mathrm{TChar}(C)^{\mathrm{ev}} \Longleftrightarrow \epsilon\cdot\eta = 0 \mod 2.$$

Example 5.3.2. We give the list of theta characteristics for a small genus. We also list 2-torsion points at which the corresponding theta function vanishes.

$g = 1$

3 even "thetas":

$$\vartheta_{12} = \theta\left[\begin{smallmatrix}1\\ 0\end{smallmatrix}\right] \quad (\alpha_{12}),$$

$$\vartheta_{13} = \theta\left[\begin{smallmatrix}0\\ 0\end{smallmatrix}\right] \quad (\alpha_{13}),$$

$$\vartheta_{14} = \theta\left[\begin{smallmatrix}0\\ 1\end{smallmatrix}\right] \quad (\alpha_{14}).$$

1 odd theta:

$$\vartheta_{\emptyset} = \theta\left[\begin{smallmatrix}1\\ 1\end{smallmatrix}\right] \quad (\alpha_{\emptyset}).$$

$g = 2$

10 even thetas:

$$\vartheta_{123} = \theta\left[\begin{smallmatrix}01\\ 10\end{smallmatrix}\right] \quad (\alpha_{12}, \alpha_{23}, \alpha_{13}, \alpha_{45}, \alpha_{46}, \alpha_{56}),$$

$$\vartheta_{124} = \theta\left[\begin{smallmatrix}00\\ 10\end{smallmatrix}\right] \quad (\alpha_{12}, \alpha_{24}, \alpha_{14}, \alpha_{35}, \alpha_{36}, \alpha_{56}),$$

$$\vartheta_{125} = \theta\left[\begin{smallmatrix}00\\ 11\end{smallmatrix}\right] \quad (\alpha_{12}, \alpha_{25}, \alpha_{15}, \alpha_{34}, \alpha_{36}, \alpha_{46}),$$

$$\vartheta_{126} = \theta\left[\begin{smallmatrix}11\\ 11\end{smallmatrix}\right] \quad (\alpha_{12}, \alpha_{16}, \alpha_{26}, \alpha_{34}, \alpha_{35}, \alpha_{45}),$$

$$\vartheta_{234} = \theta \begin{bmatrix} 10 \\ 01 \end{bmatrix} \quad (\alpha_{23}, \alpha_{34}, \alpha_{24}, \alpha_{15}, \alpha_{56}, \alpha_{16}),$$

$$\vartheta_{235} = \theta \begin{bmatrix} 10 \\ 00 \end{bmatrix} \quad (\alpha_{23}, \alpha_{25}, \alpha_{35}, \alpha_{14}, \alpha_{16}, \alpha_{46}),$$

$$\vartheta_{236} = \theta \begin{bmatrix} 01 \\ 00 \end{bmatrix} \quad (\alpha_{23}, \alpha_{26}, \alpha_{36}, \alpha_{14}, \alpha_{45}, \alpha_{15}),$$

$$\vartheta_{245} = \theta \begin{bmatrix} 11 \\ 00 \end{bmatrix} \quad (\alpha_{24}, \alpha_{25}, \alpha_{13}, \alpha_{45}, \alpha_{16}, \alpha_{36}),$$

$$\vartheta_{246} = \theta \begin{bmatrix} 00 \\ 00 \end{bmatrix} \quad (\alpha_{26}, \alpha_{24}, \alpha_{13}, \alpha_{35}, \alpha_{46}, \alpha_{15}),$$

$$\vartheta_{256} = \theta \begin{bmatrix} 00 \\ 01 \end{bmatrix} \quad (\alpha_{26}, \alpha_{25}, \alpha_{13}, \alpha_{14}, \alpha_{34}, \alpha_{56}).$$

6 odd thetas:

$$\vartheta_{1} = \theta \begin{bmatrix} 01 \\ 01 \end{bmatrix} \quad (\alpha_{\emptyset}, \alpha_{12}, \alpha_{13}, \alpha_{14}, \alpha_{15}, \alpha_{16}),$$

$$\vartheta_{2} = \theta \begin{bmatrix} 11 \\ 01 \end{bmatrix} \quad (\alpha_{\emptyset}, \alpha_{12}, \alpha_{23}, \alpha_{24}, \alpha_{25}, \alpha_{26}),$$

$$\vartheta_{3} = \theta \begin{bmatrix} 11 \\ 01 \end{bmatrix} \quad (\alpha_{\emptyset}, \alpha_{13}, \alpha_{23}, \alpha_{34}, \alpha_{35}, \alpha_{36}),$$

$$\vartheta_{4} = \theta \begin{bmatrix} 10 \\ 10 \end{bmatrix} \quad (\alpha_{\emptyset}, \alpha_{14}, \alpha_{24}, \alpha_{34}, \alpha_{45}, \alpha_{46}),$$

$$\vartheta_{5} = \theta \begin{bmatrix} 10 \\ 11 \end{bmatrix} \quad (\alpha_{\emptyset}, \alpha_{15}, \alpha_{35}, \alpha_{45}, \alpha_{25}, \alpha_{56}),$$

$$\vartheta_{6} = \theta \begin{bmatrix} 01 \\ 11 \end{bmatrix} \quad (\alpha_{\emptyset}, \alpha_{16}, \alpha_{26}, \alpha_{36}, \alpha_{46}, \alpha_{56}).$$

$g = 3$
36 even thetas $\vartheta_{\emptyset}, \vartheta_{ijkl}$,
28 odd thetas ϑ_{ij}.

$g = 4$
136 even thetas $\vartheta_{i}, \vartheta_{ijklm}$,
120 odd thetas ϑ_{ijk}.

S:5.4

5.4 Odd Theta Characteristics

SS:5.4.1

5.4.1 Syzygetic triads

We have already observed that effective theta characteristics on a canonical curve $C \subset \mathbb{P}^{g-1}$ correspond to *contact hyperplanes*, i.e. hyperplanes everywhere tangent to C. They are also called *bitangent hyperplanes* (not to be confused with hyperplanes tangent at ≥ 2 points).

An odd theta characteristic is effective and determines a contact hyperplane, a unique one if it is nonvanishing. In this section, we will study the configuration of contact hyperplanes to a canonical curve. Let us note here that a general canonical curve is determined uniquely by the configuration of its contact hyperplanes [89].

From now on, we fix a nondegenerate symplectic space (V, ω) of dimension $2g$ over $\mathbb{F}_2$. Let $Q(V)$ be the affine space of quadratic forms with associated symmetric bilinear form equal to ω. The Arf invariant divides $Q(V)$ into the union of two sets $Q(V)_+$ and $Q(V)_-$, of even or odd quadratic forms. Recall that $Q(V)_-$ is interpreted as the set of odd theta characteristics when $V = \mathrm{Pic}(C)$ and ω is the Weil pairing. For any $q \in Q(V)$ and $v \in V$, we have

$$q(v) = \mathrm{Arf}(q + v) + \mathrm{Arf}(q).$$

Thus, the function Arf is a symplectic analog of the function $h^0(\vartheta) \mod 2$ for theta characteristics.

The set $\tilde{V} = V \coprod Q(V)$ is equipped with a structure of a $\mathbb{Z}/2\mathbb{Z}$-graded vector space over $\mathbb{F}_2$. It combines the addition on V (the 0-th graded piece) and the structure of an affine space on $Q(V)$ (the 1-th graded piece) by setting $q+q' := v$, where $q' = q + v$. One can also extend the symplectic form on V to $\tilde{V}$ by setting

$$\omega(q, q') = q(q + q'), \quad \omega(q, v) = \omega(v, q) = q(v).$$

Definition 5.4.1. *A set of three elements q_1, q_2, q_3 in $Q(V)$ is called a* syzygetic triad *(resp. an* azygetic triad*) if*

$$\mathrm{Arf}(q_1) + \mathrm{Arf}(q_2) + \mathrm{Arf}(q_3) + \mathrm{Arf}(q_1 + q_2 + q_3) = 0 \ (\textit{resp.} \ = 1).$$

A subset of $k \geq 3$ elements in $Q(V)$ is called a azygetic set *if any subset of three elements is azygetic.*

Note that a syzygetic triad defines a set of four quadrics in $Q(V)$ that add up to zero. Such a tetrad of quadrics is called a *syzygetic tetrad*. Obviously, any subset of three elements in a syzygetic tetrad is a syzygetic triad.

Another observation is that three elements in $Q(V)_-$ form an azygetic triad if their sum is an element in $Q(V)_+$.

For any odd theta characteristic ϑ, any divisor $D_\eta \in |\vartheta|$ is of degree $g-1$. The condition that four odd theta characteristics ϑ_i form a syzygetic tetrad means that the sum of divisors D_{ϑ_i} are cut out by a quadric in $\mathbb{P}^{g-1}$. The converse is true if C does not have a vanishing even theta characteristic.

Let us now compute the number of syzygetic tetrads.

5.4.1 **Lemma 5.4.2.** *Let q_1, q_2, q_3 be a set of three elemenst in $Q(V)$. The following properties are equivalent:*

(i) q_1, q_2, q_3 *is a syzygetic triad;*
(ii) $q_1(q_2+q_3) = \mathrm{Arf}(q_2) + \mathrm{Arf}(q_3)$*;*
(iii) $\omega(q_1+q_2, q_1+q_3) = 0$.

Proof The equivalence of (i) and (ii) follows immediately from the identity

$$q_1(q_2+q_3) = \mathrm{Arf}(q_1) + \mathrm{Arf}(q_1+q_2+q_3).$$

We have

$$\omega(q_1+q_2, q_1+q_3) = q_1(q_1+q_3) + q_2(q_1+q_3)$$

$$= \mathrm{Arf}(q_1) + \mathrm{Arf}(q_3) + \mathrm{Arf}(q_2) + \mathrm{Arf}(q_1+q_2+q_3).$$

This shows the equivalence of (ii) and (iii). □

315 **Proposition 5.4.3.** *Let $q_1, q_2 \in Q(V)_-$. The number of ways in which the pair can be extended to a syzygetic triad of odd theta characteristics is equal to* $2(2^{g-1}+1)(2^{g-2}-1)$.

Proof Assume that q_1, q_2, q_3 is a syzygetic triad in $Q(V)_-$. By the previous lemma, $q_1(q_2+q_3) = 0$. Also, we have $q_2(q_2+q_3) = \mathrm{Arf}(q_3) + \mathrm{Arf}(q_2) = 0$. Thus, q_1 and q_2 vanish at $v_0 = q_2+q_3$. Conversely, assume $v \in V$ satisfies $q_1(v) = q_2(v) = 0$ and $v \neq q_1+q_2$ so that $q_3 = q_2+v \neq q_1, q_2$. We have $\mathrm{Arf}(q_3) = \mathrm{Arf}(q_2) + q_2(v) = 1$, hence $q_3 \in Q(V)_-$. Since $q_1(v) = q_1(q_2+q_3) = 0$, by the previous Lemma q_1, q_2, q_3 is a syzygetic triad.

Thus, the number of the ways in which we can extend q_1, q_2 to a syzygetic triad q_1, q_2, q_3 is equal to the cardinality of the set

$$Z = q_1^{-1}(0) \cap q_2^{-1}(0) \setminus \{0, v_0\},$$

where $v_0 = q_1+q_2$. It follows from arf2 (5.6) that $v \in Z$ satisfies $\omega(v, v_0) = q_2(v) + q_1(v) = 0$. Thus, any $v \in Z$ is a representative of a nonzero element in $W = v_0^\perp/v_0 \cong \mathbb{F}_2^{2g-2}$ on which q_1 and q_2 vanish. It is clear that q_1 and q_2 induce the same quadratic form q on W. It is an odd quadratic form. Indeed, we can choose a symplectic basis in V by taking as a first vector the vector v_0. Then, computing the Arf invariant of q_1 we see that it is equal to the Arf invariant of the quadratic form q. Thus, we get

$$\#Z = 2(\#Q(W)_- - 1) = 2(2^{g-2}(2^{g-1} - 1) - 1) = 2(2^{g-1} + 1)(2^{g-2} - 1).$$

□

number **Corollary 5.4.4.** *Let t_g be the the number of syzygetic tetrads of odd theta characteristics on a nonsingular curve of genus g. Then*

$$t_g = \frac{1}{3}2^{g-3}(2^{2g} - 1)(2^{2g-2} - 1)(2^{g-2} - 1).$$

Proof Let I be the set of triples (q_1, q_2, T), where $q_1, q_2 \in Q(V)_-$ and T is a syzygetic tetrad containing q_1, q_2. We count $\#I$ in two ways by projecting I to the set $\mathcal{P}$ of unordered pairs of distinct elements $Q(V)_-$ and to the set of syzygetic tetrads. Since each tetrad contains 6 pairs from the set $\mathcal{P}$, and each pair can be extended in $(2^{g-1} + 1)(2^{g-2} - 1)$ ways to a syzygetic tetrad, we get

$$\#I = (2^{g-1} + 1)(2^{g-2} - 1)\binom{2^{g-1}(2^g - 1)}{2} = 6t_g.$$

This gives

$$t_g = \frac{1}{3}2^{g-3}(2^{2g} - 1)(2^{2g-2} - 1)(2^{g-2} - 1).$$

□

Let V be a vector space with a symplectic or symmetric bilinear form. Recall that a linear subspace L is called isotropic if the restriction of the bilinear form to L is identically zero.

5.4.4 **Corollary 5.4.5.** *Let $\{q_1, q_2, q_3, q_4\}$ be a syzygetic tetrad in $Q(V)_-$. Then, $P = \{q_1 + q_i, \dots, q_4 + q_i\}$ is an isotropic -dimensional subspace in (V, ω) that does not depend on the choice of q_i.*

Proof It follows from Lemma 5.4.1 5.4.2 (iii) that P is an isotropic subspace. The equality $q_1 + \cdots + q_4 = 0$ gives

$$q_k + q_l = q_i + q_j, \tag{5.26}$$ eq6.1.6

where $\{i, j, k, l\} = \{1, 2, 3, 4\}$. This shows that the subspace P of V formed by the vectors $q_j + q_i$, $j = 1, \dots, 4$, is independent of the choice of i. One of its bases is the set $(q_1 + q_4, q_2 + q_4)$. □

SS:5.4.2

5.4.2 Steiner complexes

Let $\mathcal{P}$ be the set of unordered pairs of distinct elements in $Q(V)_-$. The addition map in $Q(V)_- \times Q(V)_- \to V$ defines a map

$$s : \mathcal{P} \to V \setminus \{0\}.$$

Definition 5.4.6. *The set of pairs from the same fiber $s^{-1}(v)$ of the map s is called a* Steiner complex. *It is denoted by* $\Sigma(v)$.

It follows from (5.26) [eq6.1.6] that any two pairs from a syzygetic tetrad belong to the same Steiner complex. Conversely, let $\{q_1, q_1'\}, \{q_2, q_2'\}$ be two pairs from $\Sigma(v)$. We have $(q_1 + q_1') + (q_2 + q_2') = v + v = 0$, showing that the tetrad (q_1, q_1', q_2, q_2') is syzygetic.

5.4.5 **Proposition 5.4.7.** *There are $2^{2g} - 1$ Steiner complexes. Each Steiner complex consists of $2^{g-1}(2^{g-1} - 1)$ pairs of elements from $Q(V)_-$. An an element $q \in Q(V)_-$ belongs to a Steiner complex $\Sigma(v)$ if and only if $q(v) = 0$.*

Proof Since $2^{2g} - 1 = \#(V \setminus \{0\})$, it suffices to show that the map $s : \mathcal{P} \to V \setminus \{0\}$ is surjective. The symplectic group $\mathrm{Sp}(V, \omega)$ acts transitively on $V \setminus \{0\}$ and on $\mathcal{P}$, and the map s is obviously equivariant. Thus, its image is a non-empty G-invariant subset of $V \setminus \{0\}$. It must coincide with the whole set.

By (5.7) [arf3], we have $q(v) = \mathrm{Arf}(q+v) + \mathrm{Arf}(q)$. If $q \in \Sigma(v)$, then $q+v \in Q(V)_-$, hence $\mathrm{Arf}(q + v) = \mathrm{Arf}(q) = 1$ and we get $q(v) = 0$. Conversely, if $q(v) = 0$ and $q \in \Sigma(v)$, we get $q + v \in Q(V)_-$ and hence $q \in \Sigma(v)$. This proves the last assertion. □

5.4.6 **Lemma 5.4.8.** *Let $\Sigma(v), \Sigma(v')$ be two Steiner complexes. Then,*

$$\#\Sigma(v) \cap \Sigma(v') = \begin{cases} 2^{g-1}(2^{g-2} - 1) & \text{if } \omega(v, v') = 0, \\ 2^{g-2}(2^{g-1} - 1) & \text{if } \omega(v, v') \neq 0. \end{cases}$$

Proof Let $q \in \Sigma(v) \cap \Sigma(v')$. Then, we have $q + q' = v, q + q'' = v'$ for some $q' \in \Sigma(v), q'' \in \Sigma(v')$. This implies that

$$q(v) = q(v') = 0. \tag{5.27}$$ er

Conversely, if these equalities hold, then $q + v, q + v' \in Q(V)_-$, $q, q' \in \Sigma(v)$, and $q, q'' \in \Sigma(v')$. Thus, we have reduced our problem to linear algebra. We want to show that the number of elements in $Q(V)_-$ that vanish at two nonzero vectors $v, v' \in V$ is equal to $2^{g-1}(2^{g-2} - 1)$ or $2^{g-2}(2^{g-1} - 1)$ depending on whether $\omega(v, v') = 0$ or 1. Let q be one such quadratic form. Suppose we have another q' with this property. Write $q' = q + v_0$ for some v_0. We have $q(v_0) = 0$

since q' is odd and

$$\omega(v_0, v) = \omega(v_0, v') = 0.$$

Let L be the plane spanned by v, v'. Assume $\omega(v, v') = 1$, then, we can include v, v' in a standard symplectic basis. Computing the Arf invariant, we find that the restriction of q to $L^\perp$ is an odd quadratic form. Thus, it has $2^{g-2}(2^{g-1}-1)$ zeros. Each zero gives us a solution for v_0. Assume $\omega(v, v') = 0$. Then, L is a singular plane for q since $q(v) = q(v') = q(v + v') = 0$. Consider $W = L^\perp/L \cong \mathbb{F}_2^{2g-4}$. The form q has $2^{g-3}(2^{g-2} - 1)$ zeros in W. Any representative v_0 of these zeros defines the quadratic form $q + v_0$ vanishing at v, v'. Any quadratic form we are looking for is obtained in this way. The number of such representatives is equal to $2^{g-1}(2^{g-2} - 1)$.

□

Definition 5.4.9. *Two Steiner complexes* $\Sigma(v)$ *and* $\Sigma(v')$ *are called* syzygetic *(resp.* azygetic*) if* $\omega(v, v') = 0$ *(resp.* $\omega(v, v') = 1$*).*

Theorem 5.4.10. *The union of three mutually syzygetic Steiner complexes* $\Sigma(v)$, $\Sigma(v')$ *and* $\Sigma(v + v')$ *is equal to* $Q(V)_-$.

Proof Since

$$\omega(v + v', v) = \omega(v + v', v') = 0,$$

we obtain that the Steiner complex $\Sigma(v + v')$ is syzygetic to $\Sigma(v)$ and $\Sigma(v')$. Suppose $q \in \Sigma(v) \cap \Sigma(v')$. Then, $q(v + v') = q(v) + q(v') + \omega(v, v') = 0$. This implies that $\Sigma(v) \cap \Sigma(v') \subset \Sigma(v + v')$ and hence $\Sigma(v), \Sigma(v'), \Sigma(v + v')$ share the same set of $2^{g-1}(2^{g-2} - 1)$ elements. This gives

$$\#\Sigma(v) \cup \Sigma(v') \cup \Sigma(v + v') = 6 \cdot 2^{g-2}(2^{g-1} - 1) - 2 \cdot 2^{g-1}(2^{g-2} - 1)$$

$$= 2^{g-1}(2^g - 1) = \#Q(V)_-.$$

□

Definition 5.4.11. *A set of three mutually syzygetic Steiner complexes is called a* syzygetic triad *of Steiner complexes. A set of three Steiner complexes corresponding to vectors forming a non-isotropic plane is called* azygetic triad *of Steiner complexes.*

Let $\Sigma(v_i), i = 1, 2, 3$ be an azygetic triad of Steiner complexes. Then,

$$\#\Sigma(v_1) \cap \Sigma(v_2) = 2^{g-2}(2^{g-1} - 1).$$

Each set $\Sigma(v_1) \setminus (\Sigma(v_1) \cap \Sigma(v_2))$ and $\Sigma(v_2) \setminus (\Sigma(v_1) \cap \Sigma(v_2))$ consists of

$2^{g-2}(2^{g-1}-1)$ elements. The union of these sets forms the Steiner complex $\Sigma(v_3)$. The number of azygetic triads of Steiner complexes is equal to $\frac{1}{3}2^{2g-2}(2^{2g}-1)$ (= the number of non-isotropic planes). We leave the proofs to the reader.

Let $\mathcal{S}_4(V)$ denote the set of syzygetic tetrads. By Corollary 5.4.5, each $T \in \mathcal{S}_4(V)$ defines an isotropic plane P_T in V. Let $\mathrm{Iso}_k(V)$ denote the set of k-dimensional isotropic subspaces in V.

P5.4.8 **Proposition 5.4.12.** *Let $\mathcal{S}_4(V)$ be the set of syzygetic tetrads. For each tetrad T, let P_T, denote the corresponding isotropic plane. The map*

$$\mathcal{S}_4(V) \to \mathrm{Iso}_2(V), \quad T \mapsto P_T,$$

is surjective. The fiber over a plane T consists of $2^{g-3}(2^{g-2}-1)$ tetrads forming a partition of the intersection of the Steiner complexes $\Sigma(v)$, where $v \in P \setminus \{0\}$.

Proof The surjectivity of this map is proved along the same lines as we proved Proposition 5.4.7. We use the fact that the symplectic group $\mathrm{Sp}(V, \omega)$ acts transitively on the set of isotropic subspaces of the same dimension. Let $T = \{q_1, \ldots, q_4\} \in \mathcal{S}_4(V)$. By definition, $P_T \setminus \{0\} = \{q_1+q_2, q_1+q_3, q_1+q_4\}$. Suppose we have another tetrad $T' = \{q_1', \ldots, q_4'\}$ with $P_T = P_{T'}$. Suppose $T \cap T' \neq \emptyset$. Without loss of generality, we may assume that $q_1' = q_1$. Then, after reindexing, we get $q_1 + q_i = q_1 + q_i'$, hence $q_i = q_i'$ and $T = T'$. Thus, the tetrads T with $P_T = P$ are disjoint. Obviously, any $q \in T$ belongs to the intersection of the Steiner complexes $\Sigma(v)$, $v \in P \setminus \{0\}$. It remains for us to apply Lemma 5.4.8. □

A closer look at the proof of Lemma 5.4.8 shows that the fiber over P can be identified with the set $Q(P^\perp/P)_-$.

Combining Proposition 5.4.12 with the computation of the number t_g of syzygetic tetrads, we obtain the number of isotropic planes in V:

$$\#\mathrm{Iso}_2(V) = \frac{1}{3}(2^{2g}-1)(2^{2g-2}-1). \tag{5.28}$$

Let $\mathrm{Iso}_2(v)$ be the set of isotropic planes containing a nonzero vector $v \in V$. The set $\mathrm{Iso}_2(v)$ is naturally identified with nonzero elements in the symplectic space $(v^\perp/v, \omega')$, where ω' is defined by the restriction of ω to $v^\perp$. We can transfer the symplectic form ω' to $\mathrm{Iso}_2(v)$. We obtain $\omega'(P, Q) = 0$ if and only if $P + Q$ is an isotropic 3-subspace.

Let us consider the set $\mathcal{S}_4(V, v) = \alpha^{-1}(\mathrm{Iso}_2(v))$. It consists of syzygetic tetrads that are invariant with respect to the translation by v. In particular, each tetrad from $\mathcal{S}_4(V, v)$ is contained in $\Sigma(v)$. We can identify the set $\mathcal{S}_4(V, v)$ with the set of cardinality 2 subsets of $\Sigma(v)/\langle v\rangle$.

There is a natural pairing on $\mathcal{S}_4(V, v)$ defined by

$$\langle T, T'\rangle = \tfrac{1}{2}\#T \cap T' \mod 2. \tag{5.29}$$ pairing2

5.4.9 **Proposition 5.4.13.** *For any $T, T' \in \mathcal{S}_4(V, v)$,*

$$\omega'(P_T, P_{T'}) = \langle T, T'\rangle.$$

Proof Let $X = \{\{T, T'\} \subset \mathcal{S}_4(V) : \alpha_v(T) \neq \alpha_v(T')\}$, $Y = \{\{P, P'\} \subset \mathrm{Iso}_2(v)\}$. We have a natural map $\tilde{\alpha}_v : X \to Y$ induced by α_v. The pairing ω' defines a function $\phi : Y \to \mathbb{F}_2$. The corresponding partition of Y consists of two orbits of the stabilizer group $G = \mathrm{Sp}(V, \omega)_v$ on Y. Suppose $\{T_1, T_2\}$ and $\{T_1', T_2'\}$ are mapped to the same subset $\{P, P'\}$. Without loss of generality, we may assume that T_1, T_1' are mapped to P. Thus

$$\begin{aligned}\langle T_1 + T_2', T_2 + T_1'\rangle &= \langle T_1, T_2\rangle + \langle T_1', T_2'\rangle + \langle T_1, T_1'\rangle + \langle T_2, T_2' \\ &= \langle T_1, T_2\rangle + \langle T_1', T_2'\rangle.\end{aligned}$$

This shows that the function $X \to \mathbb{F}_2$ defined by the pairing (5.29) is constant on fibers of $\tilde{\alpha}_v$. Thus, it defines a map $\phi' : Y \to \mathbb{F}_2$. Both functions are invariant with respect to the group G. This immediately implies that their two level sets either coincide or are switched. However, $\#\mathrm{Iso}_2(v) = 2^{2g-2} - 1$ and hence the cardinality of Y is equal to $(2^{2g-2} - 1)(2^{2g-3} - 1)$. Since this number is odd, the two orbits are of different cardinalities. Since the map $\tilde{\alpha}_v$ is G-equivariant, the level sets must coincide. □

SS:5.4.3

5.4.3 Fundamental sets

Suppose we have an ordered set S of $2g + 1$ vectors $(u_1, \ldots, u_{2g+1})$ satisfying $\omega(u_i, u_j) = 1$ unless $i = j$. It defines a standard symplectic basis by setting

$$v_i = u_1 + \cdots + u_{2i-2} + u_{2i-1},\ v_{i+g} = u_1 + \cdots + u_{2i-2} + u_{2i},\ i = 1, \ldots, g.$$

Conversely, we can solve the u_i's from the v_i's uniquely to reconstruct the set S from a standard symplectic basis.

Definition 5.4.14. *A set of* $2g + 1$ *vectors* $(u_1, \ldots, u_{2g+1})$ *with* $\omega(u_i, u_j) = 1$ *unless $i = j$ is called a* normal system *in* (V, ω).

We have established a bijective correspondence between normal systems and standard symplectic bases.

Recall that a symplectic form ω defines a nondegenerate null-system in V, i.e. a bijective linear map $f : V \to V^\vee$ such that $f(v)(v) = 0$ for all $v \in V$. Fix a basis $(e_1, \dots, e_{2g})$ in V and the dual basis $(t_1, \dots, t_{2g})$ in $V^\vee$ and consider vectors $u_i = e_1 + \cdots + e_{2g} - e_i, i = 1, \dots, 2g$ and $u_{2g+1} = e_1 + \cdots + e_{2g}$. Then, there exists a unique null-system $V \to V^\vee$ that sends u_i to t_i and u_{2g+1} to $t_{2g+1} = t_1 + \cdots + t_{2g}$. The vectors $u_1, \dots, u_{2g+1}$ form a normal system in the corresponding symplectic space.

Let $(u_1, \dots, u_{2g+1})$ be a normal system. We will identify nonzero vectors in V with points in the projective space $|V|$. Denote the points corresponding to the vectors u_i by ϵ_{i2g+2}. For any $i, j \neq 2g+2$, consider the line spanned by ϵ_{i2g+2} and ϵ_{j2g+2}. Let ϵ_{ij} be the third nonzero point in this line. Now, do the same with points ϵ_{ij} and ϵ_{kl} with the disjoint sets of indices. Denote this point by ϵ_{ijkl}. Note that the residual point on the line spanned by ϵ_{ij} and ϵ_{jk} is equal to ϵ_{ik}. Continuing in this way, we will be able to index all points in $|V|$ with subsets of even cardinality (up to complementary sets) of the set $B_g = \{1, \dots, 2g+2\}$. This notation will agree with the notation of 2-torsion divisor classes for hyperelliptic curves of genus g. For example, we have

$$\omega(p_I, p_J) = \#I \cap J \mod 2.$$

It is easy to compute the number of normal systems. It is equal to the number of standard symplectic bases in (V, ω). The group $\mathrm{Sp}(V, \omega)$ acts simply transitively on such bases, so their number is equal to

$$\#\mathrm{Sp}(2g, \mathbb{F}_2) = 2^{g^2}(2^{2g} - 1)(2^{2g-2} - 1) \cdots (2^2 - 1). \tag{5.30}$$

Now, we introduce the analog of a normal system for quadratic forms in $Q(V)$.

Definition 5.4.15. *A* fundamental set *in $Q(V)$ is an ordered azygetic set of $2g+2$ elements in $Q(V)$.*

The number $2g+2$ is the largest possible cardinality of a set in which any three elements are azygetic. This follows from the following immediate corollary of Lemma 5.4.2.

Lemma 5.4.16. *Let $B = (q_1, \dots, q_k)$ be an azygetic set. Then, the set $(q_1 + q_2, \dots, q_1 + q_k)$ is a normal system in the symplectic subspace of dimension $k-2$ spanned by these vectors.*

The Lemma shows that any fundamental set in $Q(V)$ defines a normal system in V, and hence a standard symplectic basis. Conversely, starting from a normal system $(u_1, \dots, u_{2g+1})$ and any $q \in Q(V)$ we can define a fundamental set

$(q_1, \ldots, q_{2g+2})$ by

$$q_1 = q, q_2 = q + u_1, \ldots, q_{2g+2} = q + u_{2g+1}.$$

Since the elements in a fundamental system add up to zero, the sam is true for a fundamental set.

Proposition 5.4.17. *There exists a fundamental set of* $2g + 2$ *elements with* k *odd quadratic forms such that*

$$k = \begin{cases} 0 & \mathit{if} g \equiv 0 \mod 4, \\ 1 & \mathit{if} g \equiv 1 \mod 4, \\ 2g+2 & \mathit{if} g \equiv 2 \mod 4, \\ 2g+1 & textif g \equiv 3 \mod 4. \end{cases}$$

Proof Let $(u_1, \ldots, u_{2g+1})$ be a normal system and $(t_1, \ldots, t_{2g+1})$ be its image under the map $V \to V^\vee$ defined by ω. Consider the quadratic form

$$q = \sum_{1 \le i < j \le 2g+1} t_i t_j.$$

It is immediately checked that

$$q(u_k) \equiv \binom{2g}{2} = g(2g-1) \equiv g \mod 4.$$

Passing to the associated symplectic basis, we can compute the Arf invariant of q to get

$$\mathrm{Arf}(q) = \begin{cases} 1 & \text{if } g \equiv 1 \mod 2 \\ 0 & \text{otherwise.} \end{cases}$$

This implies that

$$\mathrm{Arf}(q + t_k^2) = \mathrm{Arf}(q) + q(u_k) = \begin{cases} 0 & \text{if } g \equiv 0, 3 \mod 4, \\ 1 & \text{otherwise.} \end{cases}$$

Consider the fundamental set of quadrics $q, q + t_k^2, k = 1, \ldots, 2g + 1$. If $g \equiv 0 \mod 4$, the set consists of all even quadratic forms. If $g \equiv 1 \mod 4$, the quadratic form q is odd, all other quadratic forms are even. If $g \equiv 2 \mod 4$, all quadratic forms are odd. Finally, if $g \equiv 3 \mod 4$, then q is even, all other quadratic forms are odd. □

Definition 5.4.18. *A fundamental set with all or all but one quadratic forms are even or odd is called a* normal fundamental set.

One can show (see [154], p. 271) that any normal fundamental set is obtained as in the proof of the previous proposition.

Choose a normal fundamental set $(q_1, \ldots, q_{2g+2})$ such that the first $2g + 1$ quadrics are of the same type. Any quadratic form $q \in Q(V)$ can be written in the form

$$q_{2g+2} + \sum_{i \in I} t_i^2 = q + \sum_{i \in I} t_i^2,$$

where I is a subset of $[1, 2g+1] := \{1, \ldots, 2g+1\}$. We denote such a quadratic form by q_S, where $S = I \cup \{2g+2\}$ considered as a subset of $[1, 2g+2]$ modulo the complementary set. We can and will always assume that

$$\#S \equiv g + 1 \mod 2.$$

The quadratic form q_S can be characterized by the property that it vanishes on points p_{ij}, where $i \in S$ and $j \in \{1, \ldots, 2g + 2\}$.

The following properties can be checked.

Proposition 5.4.19. • *$q_S + q_T = \epsilon_{S+T}$;*
- *$q_S + \epsilon_I = q_{S+I}$;*
- *$q_S(\epsilon_T) = 0$ if and only if $\#S \cap T + \frac{1}{2}\#S \equiv 0 \mod 2$;*
- *$q_S \in Q(V)_+$ if and only if $\#S \equiv g + 1 \mod 4$.*

Again, we see that a choice of a fundamental set defines the notation of quadratic forms that agrees with the notation of theta characteristics for hyperelliptic curves.

Since fundamental sets are in a bijective correspondence with normal systems, their number is given by (5.30).

5.5 Scorza Correspondence

5.5.1 Correspondences on an algebraic curve

A *correspondence* of degree d between nonsingular curves C_1 and C_2 is a non-constant morphism T from C_1 to the d-th symmetric product $C_2^{(d)}$ of C_2.

$$\ell_T = \{(x, y) \in C_1 \times C_2 : y \in T(x)\}.$$

We have

$$T(x) = \ell_T \cap (\{x\} \times C_2), \tag{5.31}$$

where the intersection is scheme-theoretical.

One can extend the map (5.31) to any divisors on C_1 by setting $T(D) =$

$p_1^*(D) \cap \ell_T$. It is clear that a principal divisor goes to a principal divisor. Taking divisors of degree 0, we obtain a homomorphism of the Jacobian varieties

$$\phi_T : \mathrm{Jac}(C_1) \to \mathrm{Jac}(C_2).$$

The projection $\ell_T \to C_1$ is a finite map of degree d. Since T is not constant, the projection to C_2 is a finite map of degree d'. It defines a correspondence $C_2 \to C_1^{(d')}$ which is denoted by T^{-1} and is called the *inverse correspondence*. Its graph is equal to the image of T under the switch map $C_1 \times C_2 \to C_2 \times C_1$.

We will be dealing mostly with correspondences $T : C \to C^{(d)}$ and will identify T with its graph ℓ_T. If d is the degree of T and d' is the degree of T^{-1} we say that T is the correspondence of type (d, d'). A correspondence is *symmetric* if $T = T^{-1}$.∗ We assume that T does not contain the diagonal Δ of $C \times C$. A *united point* of correspondence is a common point with the diagonal. It comes with the multiplicity.

A correspondence $T : C \to C^{(d)}$ has *valence* v if the divisor class of $T(x)+vx$ does not depend on x.

valence **Proposition 5.5.1.** *The following properties are equivalent:*

(i) T has valence v;
(ii) the cohomology class $[T]$ in $H^2(C \times C, \mathbb{Z})$ is equal to

$$[T] = (d' + v)[\{x\} \times C] + (d + v)[C \times \{x\}] - v[\Delta],$$

where x is any point on C;
(iii) the homomorphism ϕ_T is equal to homomorphism $[-v] : \mathrm{Jac}(C) \to \mathrm{Jac}(C)$ of the multiplication by $-v$.

Proof (i) $\Rightarrow$ (ii). We know that there exists a divisor D on C such that the restriction $T + v\Delta - p_2^*(D)$ to any fiber of p_1 is linearly equivalent to zero. By the seesaw principle (MumfordAb [540] Chapter 2, Corollary 6), $T + v\Delta - p_2^*(D) \sim p_1^*(D')$ for some divisor D' on C. This implies that $[T] = \deg D'[\{x\} \times C] + \deg D[C \times \{x\}] - v[\Delta]$. Taking the intersections with a fiber of the projections, we find that $d' = \deg D' - v$ and $d = \deg D - v$.

(ii) $\Rightarrow$ (i) Let $p_1, p_2 : C \times C \to C$ be the projections. We use the well-known fact that the natural homomorphism of the Picard varieties

$$p_1^*(\mathrm{Pic}^0(C)) \oplus p_2^*(\mathrm{Pic}^0(C)) \to \mathrm{Pic}^0(C \times C)$$

is an isomorphism (see Hartshorne [379, Chapter III, Exercise 12.6]). Fix a point $x_0 \in C$

∗In classical terminology, a symmetric correspondence is an *involutory* correspondence and consider the divisor $T + \nu\Delta - (d' + \nu)(\{x_0\} \times C) - (d + \nu)(C \times \{x_0\})$. By assumption, it is algebraically equivalent to zero. Thus,

$$T + \nu\Delta \sim p_1^*(D_1) + p_2^*(D_2)$$

for some divisors D_1, D_2 on C. Thus, the divisor class $T(x) + \nu x$ is equal to the divisor class of the restriction of $p_2^*(D_2)$ to $\{x\} \times C$. Obviously, it is equal to the divisor class of D_2, hence is independent on x.

(i) ⇔ (iii) This follows from the definition of the homomorphism ϕ_T.

□

Note that for a general curve C of genus $g > 2$

$$\mathrm{End}(\mathrm{Jac}(C)) \cong \mathbb{Z}$$

(see [460]), so any correspondence has valence. An example of a correspondence without valence is the graph of an automorphism of order > 2 of C.

Observe that the proof of the Proposition shows that for a correspondence R with valence ν

$$T \sim p_1^*(D') + p_2^*(D) - \nu\Delta, \tag{5.32}$$ lineq

where D is the divisor class of $T(x)+\nu x$ and D' is the divisor class of $T^{-1}(x)+\nu x$. It follows from the Proposition that the correspondence T^{-1} has valence ν.

The next result is known as the *Cayley-Brill formula*.

cayley-brill **Corollary 5.5.2.** *Let T be a correspondence of type (a, b) on a nonsingular projective curve C of genus g. Assume that the valence of T is equal to ν. Then, the number of united points of T is equal to*

$$d + d' + 2\nu g.$$

This immediately follows from (5.32) and the formula $\Delta \cdot \Delta = 2 - 2g$.

Example 5.5.3. Let C be a nonsingular complete intersection of a nonsingular quadric Q and a cubic in $\mathbb{P}^3$. In other words, C is a canonical curve of genus 4 curve without vanishing even theta characteristic. For any point $x \in C$, the tangent plane $\mathbb{T}_x(Q)$ cuts out the divisor $2x + D_1 + D_2$, where $|x + D_1|$ and $|x+D_2|$ are the two g_3^1's on C defined by the two rulings of the quadric. Consider the correspondence T on $C \times C$ defined by $T(x) = D_1 + D_2$. This is a symmetric correspondence of type $(4, 4)$ with valence 2. Its 24 united points correspond to the ramification points of the two g_3^1's.

For any two correspondences T_1 and T_2 on C, one defines the *composition of correspondences* by considering $C\times C\times C$ with the projections $p_{ij} : C\times C\times C \to C \times C$ onto two factors and setting

$$T_1 \circ T_2 = (p_{13})_*\big(p_{12}^*(T_1) \cap p_{23}^*(T_2)\big).$$

Set-theoretically,

$$T_1 \circ T_2 = \{(x,y) \in C \times C : \exists z \in C : (x,z) \in T_1, (z,y) \in T_2\}.$$

Also $T_1 \circ T_2(x) = T_1(T_2(x))$. Note that, if $T_1 = T_2^{-1}$ and T_2 is of type (d,d'). we have $T_1(T_2(x)) - dx > 0$. Thus, the graph of $T_1 \circ T_2$ contains $d\Delta$. We modify the definition of the composition by setting $T_1 \diamond T_2 = T_1 \circ T_2 - s\Delta$, where s is the largest positive multiple of the diagonal component of $T_1 \circ T_2$.

Proposition 5.5.4. *Let $T_1 \circ T_2 = T_1 \diamond T_2 + s\Delta$. Suppose that T_i is of type (d_i, d_i') and valence ν_i. Then, $T_1 \diamond T_2$ is of type $(d_1d_2 - s, d_1'd_2' - s)$ and valence $-\nu_1\nu_2 + s$.*

Proof Applying Proposition 5.5.1, we can write

$$[T_1] = (d_1' + \nu_1)[\{x\} \times C] + (d_1 + \nu_1)[C \times \{x\}] - \nu_1[\Delta],$$

$$[T_2] = (d_2' + \nu_2)[\{x\} \times C] + (d_2 + \nu_2)[C \times \{x\}] - \nu_2[\Delta].$$

Easy computation with intersections gives

$$[T_1 \diamond T_2] = (d_1'd_2' - \nu_1\nu_2)[\{x\} \times C] + (d_1d_2 - \nu_1\nu_2)[C \times \{x\}] + (\nu_1\nu_2 - s)[\Delta]$$
$$= (d_1'd_2' - s + \nu)[\{x\} \times C] + (d_1d_2 - s + \nu)[C \times \{x\}] + \nu[\Delta],$$

where $\nu = -\nu_1\nu_2 + s$. This proves the assertion. □

lateral *Example* 5.5.5. In Baker's book [29], vol. 6, p. 11, the symmetric correspondence $T \diamond T^{-1}$ is called the *direct lateral correspondence*. If (r,s) is the type of T and γ is its valence, then it is easy to see that $T \circ T = T \diamond T^{-1} + s\Delta$, and we obtain that the type of $T \diamond T^{-1}$ is equal to $(s(r-1), s(r-1))$ and valence $s - \gamma^2$. This agrees with Baker's formula.

Here is one application of direct lateral correspondence. Consider a correspondence of valence 2 on a plane nonsingular curve C of degree d such that $T(x) = \mathbb{T}_c(C) \cap C - 2x$. In other words, $T(x)$ is equal to the set of the remaining $d-2$ intersection points of the tangent at x with C. For any point $y \in C$, the inverse correspondence assigns to y the divisor $P_y(C) - 2y$, where $P_y(C)$ is the first polar. A united point of $T \diamond T^{-1}$ is one of the two points at which a bitangent intersects the curve. We have $s = d(d-1) - 2, r = d-2, \nu = 2$. Applying the Cayley-Brill formula, we find that the number b of bitangents is expressed by the following formula

$$2b = 2(d(d-1)-2)(d-3) + (d-1)(d-2)(d(d-1)-6) = d(d-2)(d^2-9). \tag{5.33}$$ bit

As in the case of bitangents to the plane quartic, there exists a plane curve of degree $(d-2)(d^2-9)$ (a *bitangential curve* which cuts out on C the set of tangency points of bitangents (see [652, pp. 342=357]).

There are many other applications of the Cayley-Brill formula to enumerative geometry. Many of them go back to Cayley and can be found in Baker's book. Modern proofs of some of these formulas are available in the literature and we omit them.

Recall that a *k-secant line* of an irreducible space curve $C \subset \mathbb{P}^3$ of degree d is a line ℓ such that a general plane containing ℓ intersects C at $d-k$ points outside ℓ. Equivalently, the projection from ℓ defines a finite map $C \to \mathbb{P}^1$ of degree $d-k$.

The proof of the following formula can be found in [360, Chapter 2, §5].

Proposition 5.5.6. *Let C be a general space curve of genus g and degree d. Then, the number of 4-secant lines of C is given by the following formula:*

$$q = \frac{1}{12}(d-2)(d-3)^2(d-4) - \tfrac{1}{2}g(d^2-7d+13-g). \qquad (5.34)$$

There is a precise meaning of generality of a curve. We refer to loc. cit. or [482] for the explanation.

The set of trisecant lines is infinite and parameterized by a curve of degree

$$t = (d-2)\frac{(d-1)(d-3)-3g}{3}. \qquad (5.35)$$ trisecants

(see [482]).

SS:5.5.2

5.5.2 Scorza correspondence

Let C be a nonsingular projective curve of genus $g > 0$ and ϑ be a non-effective theta-characteristic on C.

Let

$$d_1 : C \times C \to \mathrm{Jac}(C),\ (x, y) \mapsto [x-y] \qquad (5.36)$$ difference

be the difference map. Let $\Theta = W^0_{g-1} - \vartheta$ be the symmetric theta divisor corresponding to ϑ. Define

$$R_\vartheta = d_1^{-1}(\Theta).$$

Set-theoretically,

$$(R_\vartheta)_{\mathrm{red}} = \{(x, y) \in C \times C : h^0(x+\vartheta-y) > 0\}.$$

Lemma 5.5.7. *R_ϑ is a symmetric correspondence of type (g, g), with the valence equal to -1 and without united points.*

Proof Since Θ is a symmetric theta divisor, the divisor $d_1^{-1}(\Theta)$ is invariant with respect to the switch of the factors of $X \times X$. This shows that R_ϑ is symmetric.

Fix a point x_0 and consider the map $i : C \to \mathrm{Jac}(C)$ defined by $i(x) = [x - x_0]$. It is known (see [59, Chapter 11, Corollary (2.2)]) that

$$\Theta \cdot \iota_*(C) = (C \times \{t_0\}) \cdot d_1^*(\Theta) = g.$$

This shows that R_ϑ is of type (g, g). Also it shows that $R_\vartheta(x_0) - x_0 + \vartheta \in W_{g-1}$. For any point $x \in C$, we have $h^0(\vartheta + x) = 1$ because ϑ is non-effective. Thus, $R_\vartheta(x)$ is the unique effective divisor linearly equivalent to $x + \vartheta$. By definition, the valence of R_ϑ is equal to -1. Applying the Cayley-Brill formula, we obtain that R_ϑ has no united points. □

Definition 5.5.8. *The correspondence R_ϑ is called the* Scorza correspondence.

Example 5.5.9. Assume $g = 1$ and fix a point on C equipping C with a structure of an elliptic curve. Then, ϑ is a nontrivial 2-torsion point. The Scorza correspondence R_ϑ is the graph of the translation automorphism defined by ϑ.

In general, R_ϑ could be neither reduced nor irreducible correspondence. However, for a general curve X of genus g everything is as expected.

Proposition 5.5.10. *Assume C is general in the sense that* $\mathrm{End}(\mathrm{Jac}(C)) \cong \mathbb{Z}$. *Then, R_ϑ is reduced and irreducible.*†

Proof The assumption that $\mathrm{End}(\mathrm{Jac}(C)) \cong \mathbb{Z}$ implies that any correspondence on $C \times C$ has valence. This implies that the Scorza correspondence is an irreducible curve and is reduced. In fact, it is easy to see that the valence of the sum of two correspondences is equal to the sum of valences. Since R_ϑ has no united points, it follows from the Cayley-Brill formula that the valence of each part must be negative. Since the valence of R_ϑ is equal to -1, we get a contradiction. □

It follows from (5.32) that the divisor class of R_ϑ is equal to

$$R_\vartheta \sim p_1^*(\vartheta) + p_2^*(\vartheta) + \Delta. \tag{5.37}$$

Since $K_{C \times C} = p_1^*(K_C) + p_2^*(K_C)$ and the restrictions of $p_1^*(\vartheta)$ and $p_2^*(\vartheta)$ to R_ϑ have the same degree, applying the adjunction formula and using that $\Delta \cap R = \emptyset$, we obtain

$$\omega_{R_\vartheta} \equiv 3p_1^*\omega_C. \tag{5.38}$$

†In a recent paper [297, Theorem 4.1] G. Farkas and A. Verra prove that, for a general pair (C, ϑ), the curve R_ϑ is smooth.

In particular, the arithmetic genus of R_ϑ is given by

$$p_a(R_\vartheta) = 3g(g-1)+1. \tag{5.39}$$

‡

Note that the curve R_ϑ is very special, for example, it admits a fixed-point free involution defined by the switching the factors of $X \times X$.

scorza2 **Proposition 5.5.11.** *Assume that C is not hyperelliptic. Let R be a symmetric correspondence on $C\times C$ of type (g,g), without united points and some valence. Then, there exists a unique non-effective theta characteristic ϑ on C such that $R = R_\vartheta$.*

Proof It follows from the Cayley-Brill formula that the valence ν of R is equal to -1. Thus, the divisor class of $R(x) - x$ does not depend on x. Since R has no united points, the divisor class $D = R(x) - x$ is not effective, i.e. $h^0(R(x) - x) = 0$. Consider the difference map $d_1 : C \times C \to \mathrm{Jac}(C)$. For any $(x, y) \in R$, the divisor $R(x) - y \sim D + x - y$ is effective and of degree $g - 1$. Thus, $d_1(R) + D \subset W^0_{g-1}$. Let $\sigma : X \times X \to X \times X$ be the switch of the factors. Then,

$$\phi(R) = d_1(\sigma(R)) = [-1](d_1(R)) \subset [-1](W^0_{g-1} - D) \subset W^0_{g-1} + D',$$

where $D' = K_C - D$. Since $R \cap \Delta = \emptyset$ and C is not hyperelliptic, the equality $d_1(x, y) = d_1(x', y')$ implies $(x, y) = (x', y')$. Thus, the difference map d_1 is injective on R. This gives

$$R = d_1^{-1}(W^0_{g-1} - D) = d_1^{-1}(W^0_{g-1} - D').$$

Restricting to $\{x\}\times C$ we see that the divisor classes D and D' are equal. Hence, D is a theta characteristic ϑ. By assumption, $h^0(R(x) - x) = h^0(\vartheta) = 0$, hence ϑ is non-effective. The uniqueness of ϑ follows from formula (5.37). □

Let $x, y \in R_\vartheta$. Then, the sum of two positive divisors $(R_\vartheta(x) - y) + (R_\vartheta(y) - x)$ is linearly equivalent to $x + \vartheta - y + y + \vartheta - x = 2\vartheta = K_C$. This defines a map

$$\gamma : R_\vartheta \to |K_C|,\ (x, y) \mapsto (R_\vartheta(x) - y) + (R_\vartheta(y) - x). \tag{5.40}$$

Recall from [360, p. 360], that the theta divisor Θ defines the *Gauss map*

$$\mathcal{G} : \Theta^0 \to |K_C|,$$

‡It is possible that the restrictions of $p_1^*(\vartheta)$ and $p_2^*(\vartheta)$ to R_ϑ are equal but we do not know how to prove it. The equality was claimed in the first edition of the book; I thank Emre Sertoz who spotted the gap.

where Θ^0 is the open subset of nonsingular points of Θ. It assigns to a point z the tangent space $T_z(\Theta)$ considered as a hyperplane in

$$T_z(\mathrm{Jac}(C)) \cong H^1(C, \mathcal{O}_C) \cong H^0(C, \mathcal{O}_C(K_C))^\vee.$$

More geometrically, $\mathcal{G}$ assigns to $D - \vartheta$ the linear span of the divisor D in the canonical space $|K_C|^\vee$ (see [13, p. 246]). Since the intersection of hyperplane $\gamma(x, y)$ with the canonical curve C contains the divisors $R(x) - y$ (and $R(y) - x$), and they do not move, we see that

$$\gamma = \mathcal{G} \circ d_1.$$

L5.4.5 **Lemma 5.5.12.**

$$\gamma^*(\mathcal{O}_{|K_C|}(1)) \cong \mathcal{O}_{R_\vartheta}(R_\vartheta) \cong p_1^*(K_C).$$

Proof The Gauss map $\mathcal{G}$ is given by the normal line bundle $\mathcal{O}_\Theta(\Theta)$. Thus, the map γ is given by the line bundle

$$d_1^*(\mathcal{O}_\Theta(\Theta)) = \mathcal{O}_{R_\vartheta}(d_1^*(\Theta)) \cong \mathcal{O}_{R_\vartheta}(R_\vartheta).$$

It remains for us to apply formula (5.37). □

The Gauss map is a finite map of degree $\binom{2g-2}{g-1}$. It factors through the map $\Theta^0 \to \Theta^0/(\iota)$, where ι is the negation involution on $\mathrm{Jac}(C)$. The map γ also factors through the involution of $X \times X$. Thus, the degree of the map $R_\vartheta \to \gamma(R_\vartheta)$ is equal to $2d(\vartheta)$, where $d(\vartheta)$ is some numerical invariant of the theta characteristic ϑ. We call it the *Scorza invariant*. Let

$$\ell(\vartheta) := \gamma(R_\vartheta).$$

We considered it as a curve embedded in $|K_C|$. Applying Lemma 5.5.12, we obtain the following.

Corollary 5.5.13.

$$\deg \ell(\vartheta) = \frac{g(g-1)}{d(\vartheta)}.$$

Remark 5.5.14. Let C be a canonical curve of genus g and R_ϑ be a Scorza correspondence on C. For any $x, y \in C$, consider the degree $2g$ divisor $D(x, y) = R_\vartheta(x) + R_\vartheta(y) \in |K_C + x + y|$. Since $|2K_C - (K_C + x + y)| = |K_C - x - y|$, we obtain that the linear system of quadrics through $D(x, y)$ is of dimension $\frac{1}{2}g(g+1) - 2g = \dim |\mathcal{O}_{\mathbb{P}^{g-1}}(2)| - 2g + 1$. This shows that the set $D(x, y)$ imposes one less condition on quadrics passing through this set. For example, if $g = 3$, we get that $D(x, y)$ is on a conic. If $g = 4$ it is the base set of a net of quadrics. We refer to [234] and [282] for projective geometry of sets imposing one less condition on quadrics (called *self-associated sets*).

5.5.3 Scorza quartic hypersurfaces

SS:S:5.5.3

The following construction due to G. Scorza needs some generality assumption on C.

Definition 5.5.15. *A pair* (C, ϑ) *is called* Scorza general *if the following properties are satisfied*

(i) R_ϑ *is a connected nonsingular curve;*
(ii) $d(\vartheta) = 1$*;*
(iii) $\ell(\vartheta)$ *is not contained in a quadric.*

We will see in the next chapter that a general canonical curve of genus 3 is Scorza general. For higher genus, this was proven in [742].

We continue to assume that C is non-hyperelliptic. Consider the canonical embedding $C \hookrightarrow |K_C|^\vee \cong \mathbb{P}^{g-1}$ and identify C with its image (the canonical model of C). For any $x \in C$, the divisor $R_\vartheta(x)$ consists of g points y_i. If all of them are distinct, we have g hyperplanes $\gamma(x, y_i) = \langle R_\vartheta(x) - y_i \rangle$, or, g points on the curve $\ell(\vartheta)$. More generally, we have a map $C \to C^{(g)}$ defined by the projection $p_1 : R_\vartheta \to C$. The composition of this map with the map $\gamma^{(g)} : C^{(g)} \to \ell(\vartheta)^{(g)}$ is a regular map $\phi : C \to \ell(\vartheta)^{(g)}$. Let $H \cap C = x_1 + \cdots + x_{2g-2}$ be a hyperplane section of C. Adding up the images of the points x_i under the map ϕ, we obtain $g(2g-2)$ points on $\ell(\vartheta)$.

scorzag

Proposition 5.5.16. *Let* $D = x_1 + \cdots + x_{2g-2}$ *be a canonical divisor on* C*. Assume* (C, ϑ) *is Scorza general. Then, the divisors*

$$\phi(D) = \sum_{i=1}^{2g-2} \phi(x_i), \quad D \in |K_C|,$$

span a linear system of divisors on $\ell(\vartheta)$ *which are cut out by quadrics.*

Proof First note that the degree of the divisor is equal to $2\deg \ell(\vartheta)$. Let $(x, y) \in R_\vartheta$ and $D_{x,y} = \gamma(x, y) = (R_\vartheta(x) - y) + (R_\vartheta(y) - x) \in |K_C|$. For any $x_i \in R_\vartheta(x) - y$, the divisor $\gamma(x, x_i)$ contains y. Similarly, for any $x_j \in R_\vartheta(y) - x$, the divisor $\gamma(y, x_j)$ contains x. This means that $\phi(D_{x,y})$ is cut out by the quadric $Q_{x,y}$ equal to the sum of two hyperplanes $\check{H}_x, \check{H}_y$ corresponding to the points $x, y \in C \subset |K_C|^\vee$ via the duality. The image of $|K_C|$ in $\ell(\vartheta)^{(g(2g-2))}$ spans a linear system L (since any map of a rational variety to $\mathrm{Jac}(\ell(\vartheta))$ is constant). Since $\ell(\vartheta)$ is not contained in a quadric, it generates $|K_C|$. This shows that all divisors in L are cut out by quadrics. The quadrics $Q_{x,y}$ span the space of quadrics in $|K_C|$ since otherwise there exists a quadric in $|K_C|^\vee$ apolar to all quadrics $Q_{x,y}$. This would imply that for a fixed $x \in C$, the divisor $R_\vartheta(x)$ lies in a hyperplane, the polar hyperplane of the quadric with respect to the point x. However, because ϑ is non-effective, $\langle R_\vartheta(x)\rangle$ spans $\mathbb{P}^{g-1}$. Thus, $\dim L \ge g(g+1)/2$, and, since no quadrics contains $\ell(\vartheta)$, L coincides with the linear system of divisors on $\ell(\vartheta)$ cut out by quadrics. □

Let $E = H^0(C,\omega_C)^\vee$. We can identify the space of quadrics in $|E|$ with $\mathbb{P}(S^2(E))$. Using the previous proposition, we obtain a map $|E^\vee| \to |S^2(E)|$. The restriction of this map to the curve $\ell(\vartheta)$ is given by the linear system $|\mathcal{O}_{\ell(\vartheta)}(2)|$. This shows that the map is given by quadratic polynomials, so defines a linear map

$$\alpha : S^2(E^\vee) \to S^2(E).$$

The proof of the proposition implies that this map is bijective.

T5.4.8 **Theorem 5.5.17.** *Assume (C,ϑ) is Scorza general. Then, there exists a unique quartic hypersurface $V(f)$ in $|E| = \mathbb{P}^{g-1}$ such that the inverse linear map α^{-1} is equal to the polarization map $\psi \mapsto D_\psi(f)$.*

Proof Consider $\alpha^{-1} : S^2(E) \to S^2(E^\vee)$ as a tensor $U \in S^2(E^\vee)\otimes S^2(E^\vee) \subset (E^\vee)^{\otimes 4}$ viewed as a 4-multilinear map $E^4 \to \mathbb{C}$. It is enough to show that U is totally symmetric. Then, α^{-1} is defined by the apolarity map associated to a quartic hypersurface. Fix a reduced divisor $R_\vartheta(x) = x_1 + \cdots + x_g$. Let H_i be the hyperplane in $|E|$ spanned by $R_\vartheta(x) - x_i$. Choose a basis $(t_1,\ldots,t_g)$ in $E^\vee$ such that $H_i = V(t_i)$. It follows from the proof of Proposition scorzag 5.5.16 that the quadratic map $\mathbb{P}(E^\vee) \to \mathbb{P}(S^2(E))$ assigns to the hyperplane H_i the quadric Q_{x,x_i} equal to the union of two hyperplanes associated to x and x_i via the duality. The corresponding linear map α satisfies

$$\alpha(t_j^2) = \xi_j(\sum_{i=1}^g b_i\xi_i),\ j = 1,\ldots,g, \tag{5.41}$$ w*

where $(\xi_1,\ldots,\xi_g)$ is the dual basis to $(t_1,\ldots,t_g)$, and $(b_1,\ldots,b_g)$ are the coordinates of the point x. This implies that

$$U(\xi_j,\sum_{i=1}^g b_i\xi_i,\xi_k,\xi_m) = \begin{cases} 1 & \text{if } j=k=m,\\ 0 & \text{otherwise}\end{cases} = U(\xi_k,\sum_{i=1}^g b_i\xi_i,\xi_j,\xi_m).$$

This shows that U is symmetric in the first and the third arguments when the second argument belongs to the curve $\ell(\vartheta)$. Since the curve $\ell(\vartheta)$ spans $\mathbb{P}(E^\vee)$, this is always true. It remains to use that U is symmetric in the first and the second arguments, as well as in the third and the fourth arguments. □

Definition 5.5.18. *Let (C,ϑ) be Scorza general pair consisting of a canonical curve of genus g and a non-effective theta characteristic ϑ. Then, the quartic* *hypersurface* $V(f)$ *is called the* Scorza quartic hypersurface *associated to* (C, ϑ).

We will study the Scorza quartic plane curves in the case $g = 3$. Very little is known about Scorza hypersurfaces for general canonical curves of genus > 3. We do not even know whether they are nonsingular. However, it follows from the construction that the hypersurface is given by a nondegenerate homogeneous form.

The Scorza correspondence has been recently extended to pairs (C, θ), where C is a hyperelliptic curve of genus 3 Grushevsky [366]. For arbitrary $g \geq 3$, the degenerations of the Scorza correspondences defined by (C, ϑ) when the pair degenerates in the compactification of the moduli space $\mathcal{M}_g^{\mathrm{ev}}$ was studied in FarkasIzadi [298],

SS:5.5.5

5.5.4 Contact hyperplanes of canonical curves

Let C be a nonsingular curve of genus $g > 0$. Fixing a point c_0 on C allows one to define an isomorphism of algebraic varieties $\mathrm{Pic}^d(C) \to \mathrm{Jac}(C)$, $[D] \mapsto [D - dc_0]$. Composing this map with the map $u_d : C^{(d)} \to \mathrm{Pic}^d(C)$ we obtain a map

$$u_d(c_0) : C^{(d)} \to \mathrm{Jac}(C). \tag{5.42}$$ ab

If no confusion arises, we drop c_0 from this notation. For $d = 1$, this map defines an embedding

$$u_1 : C \hookrightarrow \mathrm{Jac}(C).$$

For the simplicity of the notation, we will identify C with its image. For any $c \in C$, the tangent space of C at a point c is a one-dimensional subspace of the tangent space of $\mathrm{Jac}(C)$ at c. Using a translation automorphism, we can identify this space with the tangent space $T_0\mathrm{Jac}(C)$ at the zero point. Under the Abel-Jacobi map, the space of holomorphic one-forms on $\mathrm{Jac}(C)$ is identified with the space of holomorphic forms on C. Thus, we can identify $T_0\mathrm{Jac}(C)$ with the space $H^0(C, K_C)^\vee$. As a result, we obtain the *canonical map* of C

$$\varphi : C \to \mathbb{P}(H^0(C, K_C)^\vee) = |K_C|^\vee \cong \mathbb{P}^{g-1}.$$

If C is not hyperelliptic, the canonical map is an embedding.

We continue to identify $H^0(C, K_C)^\vee$ with $T_0\mathrm{Jac}(C)$. A symmetric odd theta divisor $\Theta = W_{g-1}^0 - \vartheta$ contains the origin of $\mathrm{Jac}(C)$. If $h^0(\vartheta) = 1$, the origin is a nonsingular point on Θ, and hence Θ defines a hyperplane in $T_0(\mathrm{Jac}(C))$, the tangent hyperplane $T_0\Theta$. Passing to the projectivization we have a hyperplane in $|K_C|^\vee$.

Proposition 5.5.19. *The hyperplane in $|K_C|^\vee$ defined by Θ is a contact hyperplane to the image $\varphi(C)$ under the canonical map.*

Proof Consider the difference map (5.36) $d_1 : C \times C \to \mathrm{Jac}(C)$. In the case when Θ is an even divisor, we proved in (5.37) that

$$d_1^*(\Theta) \sim p_1^*(\theta) + p_2^*(\theta) + \Delta. \tag{5.43}$$

Since two theta divisors are algebraically equivalent the same is true for an odd theta divisor. The only difference is that $d_1^*(\Theta)$ contains the diagonal Δ as the pre-image of 0. It follows from the definition of the map $u_1(c_0)$ that

$$u_1(c_0)(C) \cap \Theta = d_1^{-1}(\Theta) \cap p_1^{-1}(c_0) = c_0 + D_\vartheta,$$

where D_ϑ is the unique effective divisor linearly equivalent to ϑ. Let $\mathcal{G} : \Theta \to \mathbb{P}(T_0\mathrm{Jac}(C))$ be the Gauss map defined by translation of the tangent space at a nonsingular point of Θ to the origin. It follows from the proof of Torelli Theorem [13] that the Gauss map ramifies at any point where Θ meets $u_1(C)$. So, the image of the Gauss map intersects the canonical image with multiplicity ≥ 2 at each point. This proves the assertion. □

More explicitly, the equation of the contact hyperplane corresponding to Θ is given by the linear term of the Taylor expansion of the theta function $\theta\left[\begin{smallmatrix}\epsilon\\ \eta\end{smallmatrix}\right]$ corresponding to Θ. Note that the linear term is a linear function on $H^0(C, K_C)^\vee$, hence can be identified with a holomorphic differential

$$h_\Theta = \sum_{i=1}^{g} \frac{\partial\theta\left[\begin{smallmatrix}\epsilon\\ \eta\end{smallmatrix}\right](z,\tau)}{\partial z_i}(0)\omega_i,$$

where $(z_1, \ldots, z_g)$ are coordinates in $H^0(C, K_C)^\vee$ defined by a normalized basis $\omega_1, \ldots, \omega_g$ of $H^0(C, K_C)$. A nonzero section of $\mathcal{O}_{\mathrm{Jac}(C)}(\Theta)$ can be viewed as a holomorphic differential of order $\frac{1}{2}$. To make this more precise, i.e. describe how to get a square root of a holomorphic one-form, we use the following result (see [299, Proposition 2.2]).

Proposition 5.5.20. *Let Θ be a symmetric odd theta divisor defined by the theta function $\theta\left[\begin{smallmatrix}\epsilon\\ \eta\end{smallmatrix}\right]$. Then, for all $x, y \in C$,*

$$\theta\left[\begin{smallmatrix}\epsilon\\ \eta\end{smallmatrix}\right](d_1(x-y))^2 = h_\Theta(\varphi(x))h_\Theta(\varphi(y))E(x,y)^2,$$

where $E(x,y)$ is a certain section of $\mathcal{O}_{C\times C}(\Delta)$ (the prime-form*).*

An attentive reader should notice that the equality is not well-defined in many ways. First, the vector $\varphi(x)$ is defined only up to proportionality and the value of a section of a line bundle is also defined only up to proportionality. To make sense of this equality we pass to the universal cover of $\mathrm{Jac}(C)$ identified with $H^0(C, K_C)^\vee$ and to the universal cover U of $C \times C$ and extend the difference map and the map φ to the map of universal covers. Then, the prime-form is defined by a certain holomorphic function on U and everything makes sense. As the equality of the corresponding line bundles, the assertion trivially follows from (5.43).

Let

$$\mathfrak{r}\left[\begin{smallmatrix}\epsilon\\ \eta\end{smallmatrix}\right](x,y) = \frac{\theta\left[\begin{smallmatrix}\epsilon\\ \eta\end{smallmatrix}\right](d_1(x-y))}{E(x,y)}.$$

Since $E(x,y) = -E(y,x)$ and $\theta\left[\begin{smallmatrix}\epsilon\\ \eta\end{smallmatrix}\right]$ is an odd function, we have $\mathfrak{r}\left[\begin{smallmatrix}\epsilon\\ \eta\end{smallmatrix}\right](x,y) = \mathfrak{r}\left[\begin{smallmatrix}\epsilon\\ \eta\end{smallmatrix}\right](y,x)$ for any $x, y \in C \times C \setminus \Delta$. It satisfies

$$\mathfrak{r}\left[\begin{smallmatrix}\epsilon\\ \eta\end{smallmatrix}\right](x,y)^2 = h_\Theta(\varphi(x))h_\Theta(\varphi(y)). \tag{5.44}$$

Note that $E(x,y)$ satisfies $E(x,y) = -E(y,x)$, since $\theta\left[\begin{smallmatrix}\epsilon\\ \eta\end{smallmatrix}\right]$ is an odd function, we have $\mathfrak{r}\left[\begin{smallmatrix}\epsilon\\ \eta\end{smallmatrix}\right](x,y) = \mathfrak{r}\left[\begin{smallmatrix}\epsilon\\ \eta\end{smallmatrix}\right](y,x)$ for any $x, y \in C \times C \setminus \Delta$.

Now, let us fix a point $y = c_0$, so we can define the *root function* on C. It is a rational function on the universal cover of C defined by $\mathfrak{r}\left[\begin{smallmatrix}\epsilon\\ \eta\end{smallmatrix}\right](x, c_0)$.

Thus, every contact hyperplane of the canonical curve defines a root function.

Suppose we have two odd theta functions $\theta\left[\begin{smallmatrix}\epsilon\\ \eta\end{smallmatrix}\right], \theta\left[\begin{smallmatrix}\epsilon'\\ \eta'\end{smallmatrix}\right]$. Then, the ratio of the corresponding root functions is equal to $\theta\left[\begin{smallmatrix}\epsilon\\ \eta\end{smallmatrix}\right](d_1(x-c_0))/\theta\left[\begin{smallmatrix}\epsilon'\\ \eta'\end{smallmatrix}\right](d_1(x-c_0))$ and its square is a rational function on C, defined uniquely up to a constant factor depending on the choice of c_0. Its divisor is equal to the difference $2\vartheta - 2\vartheta'$. Thus, we can view the ratio as a section of $K_X^{\frac{1}{2}}$ with divisor $\theta - \theta'$. This section is not defined on C but on the double cover of C corresponding to the 2-torsion point $\vartheta - \vartheta'$. If we have two pairs $\vartheta_1, \vartheta', \vartheta_2, \vartheta_2'$ of odd theta characteristics satisfying $\vartheta_1 - \vartheta' = \vartheta_2 - \vartheta_2' = \epsilon$, i.e. forming a syzygetic tetrad, the product of the two ratios is a rational function on C with divisor $\vartheta_1 + \vartheta_2' - \vartheta_1' - \vartheta_2$. Following Riemann [625] and Weber [798], we denote this function by $(\vartheta_1\vartheta_1'/\vartheta_2\vartheta_2')^{1/2}$. By Riemann-Roch, $h^0(\vartheta_1 + \vartheta_2') = h^0(K_C + \epsilon) = g - 1$, and hence, any g pairs $(\vartheta_1, \vartheta_1'), \ldots, (\vartheta_g, \vartheta_g')$ of odd theta characteristics in a Steiner complex define g linearly independent functions $(\vartheta_1\vartheta_1'/\vartheta_g\vartheta_g')^{1/2}, \ldots, (\vartheta_{g-1}\vartheta_{g-1}'/\vartheta_g\vartheta_g')^{1/2}$. After scaling, and getting rid of squares by using (5.44), we obtain a polynomial in $h_{\Theta_1}(\varphi(x)), \ldots, h_{\Theta_g}(\varphi(x))$ vanishing on the canonical image of C.

Example 5.5.21. Let $g = 3$. We take three pairs of odd theta functions and get the equation

$$\sqrt{\vartheta_1\vartheta_1'} + \sqrt{\vartheta_2\vartheta_2'} + \sqrt{\vartheta_3\vartheta_3'} = 0. \tag{5.45}$$

After getting rid of square roots, we obtain a quartic equation of C

$$(lm + pq - rs)^2 - 4lmpq = 0, \tag{5.46}$$

where l, m, p, q, rs are the linear functions in z_1, z_2, z_3 defining the linear terms of the Taylor expansion at 0 of the odd theta functions corresponding to three pairs in a Steiner complex. The number of possible ways to write the equation of a plane quartic in this form is equal to $63 \cdot 20 = 1260$.

Remark 5.5.22. For any nonzero 2-torsion point, the linear system $|K_C + \epsilon|$ maps C to $\mathbb{P}^{g-2}$, the map is called the *Prym canonical map*. We have seen that the root functions $(\vartheta_1\vartheta_1'/\vartheta_2\vartheta_2')^{1/2}$ belong to $H^0(C, K_C + \epsilon)$ and can be used to define the Prym canonical map. For $g = 3$, the map is a degree 4 cover of $\mathbb{P}^1$ and we express the quartic equation of C as a degree 4 cover of $\mathbb{P}^1$.

Exercises

E:5

5.1 Let C be an irreducible plane curve of degree d with a $(d-2)$-multiple point. Show that its normalization is a hyperelliptic curve of genus $g = d-2$. Conversely, show that any hyperelliptic curve of genus g admits such a plane model. ex:5.1

5.2 Show that a nonsingular curve of genus 2 has a vanishing theta characteristic but a nonsingular curve of genus 3 has a vanishing theta characteristic if and only if it is a hyperelliptic curve. ex:5.2

5.3 Show that a nonsingular non-hyperelliptic curve of genus four has a vanishing theta characteristic if and only if its canonical model lies on a quadratic cone. ex:5.3

5.4 Find the number of vanishing theta characteristics on a hyperelliptic curve of genus g. ex:5.4

5.5 Show that a canonical curve of genus 5 has 10 vanishing even theta characteristics if and only if it is isomorphic to the intersection of three simultaneously diagonalized quadrics in $\mathbb{P}^4$. ex:5.5

5.6 Compute the number of syzygetic tetrads contained in a Steiner complex. ex:5.6

5.7 Show that the composition of two correspondences (defined as the composition of the multi-valued maps defined by the correspondences) with valences ν and ν' is a correspondence with valence $-\nu\nu'$. ex:5.7

5.8 Let $f : X \to \mathbb{P}^1$ be a non-constant rational function on a nonsingular projective curve X. Consider the fibered product $X \times_{\mathbb{P}^1} X$ as a correspondence on $X \times X$. Show that it has valence and compute the valence. Show that the Cayley-Brill formula is equivalent to the Riemann–Hurwitz formula. ex:5.8

5.9 Suppose that a nonsingular projective curve X admits a non-constant map to a curve of genus > 0. Show that there is a correspondence on X without valence. ex:5.9

5.10 Show that any correspondence on a nonsingular plane cubic has valence unless the cubic is harmonic or equianharmonic. ex:5.10

5.11 Describe all symmetric correspondences of type $(4, 4)$ with valence 1 on a canonical curve of genus 4. ex:5.11

5.12 Let R_ϑ be the Scorza correspondence on a curve C. Prove that a point $(x, y) \in R_\vartheta$ is singular if and only if x and y are ramification points of the projections $R_\vartheta \to C$. ex:5.12

5.13 Show that the Scorza map gives a birational isomorphism between the moduli space of canonical curves of genus 3 and the moduli space of its 36 : 1-cover defined by a choice of an even theta characteristic. Give examples of a similar birational isomorphism for curves of genus $g = 1, 2$. Are there any examples for $= 4$?

Historical Notes

It is too large a task to discuss the history of theta functions. However, we mention that the connection between odd theta functions with characteristics and bitangents to a quartic curves goes back to Riemann [625], [798]. There are numerous expositions of the theory of theta functions and Jacobian varieties (e.g., [13], [152], [541]). The theory of fundamental sets of theta characteristics goes back to A. Göpel and J. Rosenhein. . A good exposition can be found in Krazer's book [468]. As an abstract symplectic geometry over the field of two elements, it is discussed in [159], citeCobleTheta (see also a modern exposition in [644]).

The theory of correspondences on an algebraic curve originates from *Charles' Principle of Correspondence* [127]. It is a special case of the Cayley-Brill formula in the case $g = 0$. However, the formula was known and used earlier by E. de Jonquières [212], and later, but before Chasles, by L. Cremona in [182]. We refer to C. Segre [692] for the history of this discovery and the polemic between Chasles and de Jonquières on the priority of this discovery.

We have already encountered the application of Chasles' Principles to Poncelet polygons in Chapter 2. Cayley was the first who found this application [115]. Cayley was also the first to extend Chasles' Principle to a higher genus [115], although with incomplete proof. The first proof of the Cayley-Brill formula was given by A. Brill [73]. The notion of valence (die Werthigeit) was introduced by Brill. Hurwitz was the first to point out that only a general curve may admit a correspondence with valence [417]. Hurwitz also showed the existence of correspondences without valence. Baker's book [29, Vol. 6] is a good reference to many problems that can be solved by using the theory of correspondences. We refer to [715] for a fuller history of the theory of correspondences.

The number of bitangents to a plane curve was first computed by J. Plücker [597], [598]. The equations of bitangential curves were given by A. Cayley [109], G. Salmon [652] and O. Dersch [219].

The study of correspondences of type (g, g) with valence -1 was initiated by G. Scorza [678], [679]. In [680] Scorza gave a construction of a quartic hypersurface associated with a non-effective theta characteristic on a canonical curve of genus g. A modern exposition of Scorza's theory was first given in [235]. A survey of some new results about Scorza correspondences and the associated quartic hypersurfaces can be found in [821].

6

Plane Quartics

6.1 Bitangents

S:6.1

6.1.1 28 bitangents

S:6.1.1

A nonsingular plane quartic C is a non-hyperelliptic genus 3 curve embedded in $\mathbb{P}^2$ by its canonical linear system $|K_C|$. It has no vanishing theta characteristics, so the only effective theta characteristics are odd ones. The number of them is $28 = 2^2(2^3 - 1)$. Thus, C has exactly 28 contact lines, which, in this case, coincide with bitangents. Each bitangent is tangent to C at two points that may coincide. In the latter case, the bitangent is called a *inflection bitangent*.

We can apply the results from Section S:5.4 5.4 to the case $g = 3$. Let $V = \mathrm{Pic}(C)[2] \cong \mathbb{F}_2^6$ with the symplectic form ω defined by the Weil pairing. The elements of $Q(V)_-$, i.e., quadratic forms of odd type on V, will be identified with bitangents.

The union of four bitangents forming a syzygetic tetrad cuts out in C an effective divisor of degree 8. It is cut by some conic $V(q)$. There are $t_3 = 315$ syzygetic tetrads which are in a bijective correspondence with the set of isotropic planes in $\mathrm{Pic}(C)[2]$.

Since a syzygetic tetrad of bitangents and the conic $V(q)$ cuts out in C the same divisor, we obtain the following.

Proposition 6.1.1. *A choice of a syzygetic tetrad of bitangents $V(l_i), i = 1, \ldots, 4$, puts the equation of C in the form*

$$C = V(l_1 l_2 l_3 l_4 + q^2). \tag{6.1}$$

syzeg

Conversely, each such equation defines a syzygetic tetrad of bitangents. There are 315 ways to write f in this form.

There are 63 Steiner complexes of bitangents. Each complex consists of six pairs of bitangents ℓ_i, ℓ_i' such that the divisor class of $\ell_i \cap C - \ell_i' \cap C$ is a fixed nonzero two-torsion divisor class.

P6.1.2 **Proposition 6.1.2.** *Let $(l,m), (p,q), (r,s)$ be three pairs of linear forms defining three pairs of bitangents from a Steiner complex. Then, after scaling the forms, one can write the equation of C in the form*

$$4lmpq - (lm + pq - rs)^2 = 0, \tag{6.2}$$ squareroots

which is equivalent to the equation

$$\sqrt{lm} + \sqrt{pq} + \sqrt{rs} = 0 \tag{6.3}$$ squareroots2

after getting rid of square roots. Conversely, an equation of this form is defined by three pairs of bitangents from a Steiner complex. The number of ways in which the equation can be written in this form is equal to $1260 = \binom{6}{3} \cdot 63$.

Proof By (6.1), we can write

$$C = V(lmpq - a^2) = V(lmrs - b^2)$$

for some quadratic forms a, b. After subtracting the equations, we get

$$lm(pq - rs) = (a + b)(a - b).$$

If l divides $a + b$ and m divides $a - b$, then the quadric $V(a)$ passes through the point $l \cap m$. But this is impossible since no two bitangents intersect at a point on the quartic. Thus, we obtain that lm divides either $a + b$ or $a - b$. Without loss of generality, we get $lm = a + b, pq - rs = a - b$, and hence $a = \frac{1}{2}(lm + pq - rs)$. Therefore, we can define the quartic by the equation $4lmpq - (lm + pq - rs)^2 = 0$. Conversely, Equation (6.2) defines a syzygetic tetrad $V(l), V(m), V(p), V(q)$. By the symmetry of (6.3), we obtain two other syzygetic tetrads $V(l), V(m), V(r), V(s)$ and $V(p), V(q), V(r), V(s)$. Obviously, the pairs $(l,m), (p,q), (r,s)$ define the same 2-torsion divisor class, so they belong to a Steiner hexad. □

In the previous chapter, we found this equation by using theta functions (see (5.45)).

Remark 6.1.3. Consider the 4-dimensional algebraic torus

$$T = \{(z_1, z_2, z_3, z_4, z_5, z_6) \in (\mathbb{C}^*)^6 : z_1z_2 = z_3z_4 = z_5z_6\} \cong (\mathbb{C}^*)^4.$$

It acts on 6-tuples of linear forms $(l_1, \ldots, l_6) \in (\mathbb{C}^3)^6 \cong \mathbb{C}^{18}$ by scalar multiplication. The group $G = \mathbb{F}_2^3 \rtimes \mathfrak{S}_3$ of order 48 acts on the same space by permuting two forms in each pair $(l_i, l_{i+1}), i = 1, 3, 5$, and permuting the three pairs. This action commutes with the action of T and defines a linear action of the group $T \times G$ on $\mathbb{P}^{17} = \mathbb{C}^{18} \setminus \{0\}/\mathbb{C}^*$. The GIT-quotient $X = \mathbb{P}^{17}/(T \times G)$ is a projective variety of dimension 14. A rational map $X \dashrightarrow |\mathcal{O}_{\mathbb{P}^2}(4)|$ which assigns to a general orbit of $T \times G$ the quartic curve $V(\sqrt{l_l l_2} + \sqrt{l_3 l_4} + \sqrt{l_5 l_6})$ is a SL(3)-equivariant and of degree $48 \cdot 1260$. I do not know whether $X/\mathrm{SL}(3)$ is a rational variety; the orbit space $|\mathcal{O}_{\mathbb{P}^2}(4)|/\mathrm{SL}(3)$ is known to be a rational variety [444], [63].

We know that two Steiner complexes have either four or six common bitangents, depending on whether they are syzygetic or not. Each isotropic plane in $\mathrm{Pic}(C)[2]$ defines three Steiner complexes with common four bitangents. Two azygetic Steiner complexes have 6 common bitangents. The number of azygetic triads is equal to 336.

The projection from the intersection point of two bitangents defines a g^1_4 with two members of the form $2p + 2q$. It is possible that more than two bitangents are concurrent. However, we can prove the following.

Proposition 6.1.4. *No three bitangents forming an azygetic triad can intersect at one point.*

Proof Let $\vartheta_1, \vartheta_2, \vartheta_3$ be the corresponding odd theta characteristics. The 2-torsion divisor classes $\epsilon_{ij} = \vartheta_i - \vartheta_j$ form a non-isotropic plane. Let ϵ be a nonzero point in the orthogonal complement. Then, $q_{\eta_i}(\epsilon) + q_{\eta_j}(\epsilon) + \langle \eta_{ij}, \epsilon \rangle = 0$ implies that q_{η_i} take the same value at ϵ. We can always choose ϵ such that this value is equal to 0. Thus, the three bitangents belong to the same Steiner complex $\Sigma(\epsilon)$. Obviously, no two differ by ϵ, hence we can form 3 pairs from them. These pairs can be used to define the equation (6.2) of C. It follows from this equation that the intersection point of the three bitangents lies on C. But this is impossible because C is nonsingular. □

Remark 6.1.5. A natural question is whether the set of bitangents determines the quartic, i.e. whether two quartics with the same set of bitangents coincide. Surprisingly, it has not been answered by the ancients. Only recently it was proven that the answer is yes: [88] (for a general curve), [480] (for any nonsingular curve).

6.1.2 Aronhold sets

We know that in the case $g = 3$ a normal fundamental set of eight theta characteristics contains seven odd theta characteristics. The corresponding unordered set of seven bitangents is called an *Aronhold set*. It follows from (5.30) that the number of Aronhold sets is equal to $\#\mathrm{Sp}(6, \mathbb{F}_2)/7! = 288$.

A choice of an ordered Aronhold set defines a unique fundamental set that contains it. The eighth theta characteristic is equal to the sum of the characteristics from the Aronhold set. Thus, an Aronhold set can be defined as an azygetic set of seven bitangents.

A choice of an ordered Aronhold set allows one to index all 2-torsion divisor classes (resp. odd theta characteristics) by subsets of even cardinality (resp. of cardinality 2) of $\{1, \ldots, 8\}$, up to complementary set. Thus, we have 63 2-torsion classes $\epsilon_{ab}, \epsilon_{abcd}$ and 28 bitangents ℓ_{ij} corresponding to 28 odd theta characteristics ϑ_{ij}. The bitangents from the Aronhold set correspond to the subsets $(18, 28, \ldots, 78)$.

We also know that $\vartheta_A - \vartheta_B = \epsilon_{A+B}$. This implies, for example, that four bitangents $\ell_A, \ell_B, \ell_C, \ell_D$ form a syzygetic tetrad if and only if $A+B+C+D = 0$.

Following Cayley, we denote a pair of numbers from the set $\{1, \ldots, 8\}$ by a vertical line $|$. If two pairs have a common number we make them intersect. For example, we have the following.

- Pairs of bitangents: 210 of type $||$ and 168 of type $\vee$.
- Triads of bitangents:
  1. (syzygetic) 420 of type $\sqcup$, 840 azygetic of type $|||$;
  2. (azygetic) 56 of type $\triangle$, 1680 of type $\vee\ |$, and 280 of type $\backslash\!|\!/$.
- Tetrads of bitangents:
  1. (syzygetic) 105 azygetic of types $||||$, 210 of type $\square$;
  2. (asyzygetic) 560 of types $|\ \triangle$, 280 of type $\backslash\!\angle$, 1680 of type $|\backslash\!|\!/$, 2520 of type $\vee\vee$;
  3. (non syzygetic but containing a syzygetic triad) 2520 of type $||\ \vee$, 5040 of type $|\ \sqcup$, 3360 of type $\sqcup\!\!\ulcorner$, 840 of type $\triangle\!/$, 3360 of type $\backslash\!|\!\wedge\!\backslash$.

There are two types of Aronhold sets: $\ast$, $\backslash\!\angle\ \triangle$. They are represented by the sets $(12, 13, 14, 15, 16, 17, 18)$ and $(12, 13, 23, 45, 46, 47, 48)$. The number of the former type is 8, the number of the latter type is 280. Note that the different types correspond to orbits of the subgroup of $\mathrm{Sp}(6, \mathbb{F}_2)$ isomorphic to the permutation group $\mathfrak{S}_8$. For example, we have two orbits of $\mathfrak{S}_8$ on the set of Aronhold sets consisting of 8 and 280 elements.

L6.1.2 **Lemma 6.1.6.** *Three odd theta characteristics $\vartheta_1, \vartheta_2, \vartheta_3$ in a Steiner complex $\Sigma(\epsilon)$, no two of which differ by ϵ, are azygetic.*

Proof Let $\vartheta_i' = \vartheta_i + \epsilon, i = 1, 2, 3$. Then, $\{\vartheta_1, \vartheta_1', \vartheta_2, \vartheta_2'\}$ and $\{\vartheta_1, \vartheta_1', \vartheta_3, \vartheta_3'\}$ are syzygetic and have two common theta characteristics. By Proposition 5.4.13, the corresponding isotropic planes do not span an isotropic 3-space. Thus, $\langle \vartheta_1 - \vartheta_2, \vartheta_3 - \vartheta_1 \rangle = 1$, hence $\vartheta_1, \vartheta_2, \vartheta_3$ is an azygetic triad. □

The previous Lemma suggests a way to construct an Aronhold set from a Steiner set $\Sigma(\epsilon)$. Choose another Steiner set $\Sigma(\eta)$ azygetic to the first one. They intersect at six odd theta characteristics $\vartheta_1, \ldots, \vartheta_6$, no two of which differ by ϵ. Consider the set $\{\vartheta_1, \ldots, \vartheta_5, \vartheta_6 + \epsilon, \vartheta_6 + \eta\}$. We claim that this is an Aronhold set. By the previous Lemma all triads $\vartheta_i, \vartheta_j, \vartheta_k, i, j, k \leq 5$ are azygetic. Any triad $\vartheta_i, \vartheta_6 + \epsilon, \vartheta_6 + \eta, i \leq 5$, is azygetic too. In fact, $\vartheta_i((\vartheta_6 + \epsilon) + (\vartheta_6 + \eta)) = \vartheta_i(\epsilon + \eta) \neq 0$ since $\vartheta_i \notin \Sigma(\epsilon + \eta)$. So the assertion follows from Lemma 5.4.2. We leave it to the reader to check that remaining triads $\{\vartheta_i, \vartheta_j, \vartheta_6 + \epsilon\}, \{\vartheta_i, \vartheta_j, \vartheta_6 + \eta\}, i \leq 5$, are azygetic.

P6.1.22 **Proposition 6.1.7.** *Any six lines in an Aronhold set are contained in a unique Steiner complex.*

We use that the symplectic group $\mathrm{Sp}(6, \mathbb{F}_2)$ acts transitively on the set of Aronhold sets. So it is enough to check the assertion for one Aronhold set. Let it correspond to the index set $(12, 13, 14, 15, 16, 17, 18)$. It is enough to check that the first six are contained in a unique Steiner complex. By Proposition 5.4.7, it is enough to exhibit a 2-torsion divisor class ϵ_{ij} such that $\vartheta_{1k}(\epsilon_{ij}) = 0$ for $k = 2, 3, 4, 5, 6, 7$, and show its uniqueness. By Proposition 5.4.19, ϵ_{18} does the job.

Recall that a Steiner complex of theta characteristics on a genus 3 curve consists of six pairs of theta characteristics and the union of these pairs consists of 12 theta characteristics A subset of six of them will be called a *hexad*.

Corollary 6.1.8. *Any Steiner complex contains* 2^6 *azygetic hexads. Half of them are contained in another Steiner complex, necessarily azygetic to the first one. Any other hexad can be extended to a unique Aronhold set.*

Proof Let $\Sigma(\epsilon)$ be a Steiner complex consisting of six pairs of odd theta characteristics. Consider it as G-set, where $G = (\mathbb{Z}/2\mathbb{Z})^6$ whose elements, identified with subsets I of $[1, 6]$, act by switching elements in i-th pairs, $i \in I$. It is clear that G acts simply transitively on the set of azygetic sextuples in $\Sigma(\epsilon)$. For any azygetic complex $\Sigma(\eta)$, the intersection $\Sigma(\epsilon) \cap \Sigma(\eta)$ is an azygetic hexad. Note that two syzygetic complexes have only four bitangents in common. The number of such hexads is equal to $2^6 - 2^5 = 2^5$. Thus, the set of azygetic hexads contained in a unique Steiner complex is equal to $2^5 \cdot 63$. But this number is equal to the number $7 \cdot 288$ of subsets of cardinality 6 of Aronhold sets. By the previous Proposition, all such sets are contained in a unique Steiner complex. □

Let $(\vartheta_{18}, \ldots, \vartheta_{78})$ be an Aronhold set. By Proposition 6.1.7, the hexad $\vartheta_{28}, \ldots, \vartheta_{78}$ is contained in a unique Steiner complex $\Sigma(\epsilon)$. Let $\vartheta_{28}' = \vartheta_{28} +$

ϵ. By Proposition tete 5.4.19, the only 2-torsion point ϵ_{ij} at which all quadrics $\vartheta_{28}, \ldots, \vartheta_{78}$ vanish is the point ϵ_{18}. Thus, $\vartheta'_{28} = \vartheta_{28} + \epsilon_{18} = \vartheta_{12}$. This shows that the bitangent defined by ϑ'_{28} coincides with ϑ_{12}. Similarly, we see that the bitangents corresponding to $\vartheta_{i8} + \epsilon, i = 3, \ldots, 7$, coincide with the bitangents ϑ_{1i}.

SS:6.1.3

6.1.3 Riemann's equations for bitangents

Here, we show how to write equations of all bitangents knowing the equations of an Aronhold set of bitangents.

Let $\ell_1 = V(l_1), \ldots, \ell_7 = V(l_7)$ be an Aronhold set of bitangents of C. By Proposition arr 6.1.4, any three lines are not concurrent. We may assume that

$$\ell_1 = V(t_0), \ell_2 = V(t_1), \ell_3 = V(t_2), \ell_4 = V(t_0 + t_1 + t_2)$$

and the remaining ones are $\ell_{4+i} = V(a_{0i}t_0 + a_{1i}t_1 + a_{2i}t_2), i = 1, 2, 3.$

riemann **Theorem 6.1.9.** *There exist linear forms u_0, u_1, u_2 such that, after rescaling the linear forms,*

$$C = V(\sqrt{t_0u_0} + \sqrt{t_1u_1} + \sqrt{t_2u_2}).$$

The forms u_i can be found from equations

$$\begin{aligned} u_0 + u_1 + u_2 + t_0 + t_1 + t_2 &= 0, \\ \frac{u_0}{a_{01}} + \frac{u_1}{a_{11}} + \frac{u_2}{a_{21}} + k_1(a_{01}t_0 + a_{11}t_1 + a_{21}t_2) &= 0, \\ \frac{u_0}{a_{02}} + \frac{u_1}{a_{12}} + \frac{u_2}{a_{22}} + k_2(a_{02}t_0 + a_{12}t_1 + a_{22}t_2) &= 0, \\ \frac{u_0}{a_{03}} + \frac{u_1}{a_{13}} + \frac{u_2}{a_{23}} + k_3(a_{03}t_0 + a_{13}t_1 + a_{23}t_2) &= 0, \end{aligned}$$

where k_1, k_2, k_3 can be found from solving first linear equations:

$$\begin{pmatrix} \frac{1}{a_{01}} & \frac{1}{a_{02}} & \frac{1}{a_{03}} \\ \frac{1}{a_{11}} & \frac{1}{a_{12}} & \frac{1}{a_{13}} \\ \frac{1}{a_{21}} & \frac{1}{a_{22}} & \frac{1}{a_{23}} \end{pmatrix} \cdot \begin{pmatrix} \lambda_0 \\ \lambda_1 \\ \lambda_2 \end{pmatrix} = \begin{pmatrix} -1 \\ -1 \\ -1 \end{pmatrix},$$

and then solving the equations

$$\begin{pmatrix} \lambda_0 a_{01} & \lambda_1 a_{11} & \lambda_2 a_{21} \\ \lambda_0 a_{02} & \lambda_1 a_{12} & \lambda_2 a_{22} \\ \lambda_0 a_{03} & \lambda_1 a_{13} & \lambda_2 a_{23} \end{pmatrix} \cdot \begin{pmatrix} k_1 \\ k_2 \\ k_3 \end{pmatrix} = \begin{pmatrix} -1 \\ -1 \\ -1 \end{pmatrix}.$$

The equations of the remaining 21 bitangents are:

(1) $u_0 = 0,\ u_1 = 0,\ u_2 = 0,$

(2) $t_0 + t_1 + u_2 = 0,\ t_0 + t_2 + u_1 = 0,\ t_1 + t_2 + u_0 = 0,$

(3) $\frac{u_0}{a_{0i}} + k_i(a_{1i}t_1 + a_{2i}t_2) = 0,\ i = 1, 2, 3,$

(4) $\frac{u_1}{a_{1i}} + k_i(a_{0i}t_0 + a_{2i}t_2) = 0,\ i = 1, 2, 3,$

(5) $\frac{u_2}{a_{2i}} + k_i(a_{0i}t_0 + a_{1i}t_1) = 0,\ i = 1, 2, 3,$

(6) $\frac{t_0}{1-k_i a_{1i} a_{2i}} + \frac{t_1}{1-k_i a_{0i} a_{2i}} + \frac{t_2}{1-k_i a_{0i} a_{1i}} = 0,\ i = 1, 2, 3,$

(7) $\frac{u_0}{a_{0i}(1-k_i a_{1i} a_{2i})} + \frac{u_1}{a_{1i}(1-k_i a_{0i} a_{2i})} + \frac{u_2}{a_{2i}(1-k_i a_{0i} a_{1i})} = 0,\ i = 1, 2, 3.$

Proof By Proposition 6.1.7, six bitangents in our set of seven bitangents $\ell_1, \ldots, \ell_7$ are contained in a unique Steiner complex. Throwing away ℓ_1, ℓ_2, ℓ_3, we find three Steiner complexes partitioned in pairs

$$\begin{aligned} &(\ell_2, \xi_3),\ (\ell_3, \xi_2),\ (\ell_4, \xi_{41}),\ \ldots,\ (\ell_7, \xi_{71}), \\ &(\ell_3, \xi_1),\ (\ell_1, \xi_3),\ (\ell_4, \xi_{42}),\ \ldots,\ (\ell_7, \xi_{72}), \\ &(\ell_1, \xi_2),\ (\ell_2, \xi_1),\ (\ell_4, \xi_{43}),\ \ldots,\ (\ell_7, \xi_{73}). \end{aligned} \tag{6.4}$$

Since two Steiner complexes cannot contain more than six common bitangents, the bitangents $\xi_i = V(u_{i-1})$ and $\xi_{ij} = V(l_{ij})$ are all different and differ from $\ell_1, \ldots, \ell_7$. We continue to identify bitangents with odd theta characteristics, and the corresponding odd quadratic forms.

Now, we have

$$\ell_2 - \xi_3 = \ell_3 - \xi_2,\ \ell_3 - \xi_1 = \ell_1 - \xi_3,\ \ell_1 - \xi_2 = \ell_2 - \xi_1.$$

This implies that $\ell_1 - \xi_1 = \ell_2 - \xi_2 = \ell_3 - \xi_3$, i.e. the pairs (ℓ_1, ξ_1), (ℓ_2, ξ_2), and (ℓ_3, ξ_3) belong to the same Steiner complex Σ. One easily checks that

$$\langle \ell_1 - \xi_1, \ell_1 - \xi_2 \rangle = \langle \ell_2 - \xi_2, \ell_2 - \xi_3 \rangle = \langle \ell_3 - \xi_3, \ell_3 - \xi_1 \rangle = 0,$$

and hence Σ is syzygetic to the three complexes (6.4) and therefore it does not contain $\ell_i, i \geq 4$.

By Proposition 6.1.2 and its proof, we can write, after rescaling u_0, u_1, u_2,

$$C = V(4t_0t_1u_0u_1 - q_3^2) = V(4t_0t_2u_0u_2 - q_2^2) = V(4t_1t_2u_1u_2 - q_1^2), \tag{6.5}$$ 63

where

$$\begin{aligned} q_1 &= -t_0u_0 + t_1u_1 + t_2u_2, \\ q_2 &= t_0u_0 - t_1u_1 + t_2u_2, \\ q_3 &= t_0u_0 + t_1u_1 - t_2u_2. \end{aligned} \tag{6.6}$$

Next, we use the first Steiner complex from (6.4) to do the same by using the first three pairs. We obtain

$$C = V(4t_1u_2l_4l_{41} - q^2).$$

As in the proof of Proposition (6.1.2), we find that

$$q_1 - q = 2\lambda_1 t_1 u_2,\ q_1 + q = \frac{2(t_2u_2 - l_4 l_{41})}{\lambda_1}.$$

Hence

$$q_1 = \lambda_1 t_1 u_2 + \frac{t_2u_1 - l_4l_{41}}{\lambda_1} = -t_0u_0 + t_1u_1 + t_2u_3,$$

and we obtain

$$\begin{aligned} l_4l_{41} &= t_2u_1 - \lambda_1(-t_0u_0 + t_1u_1 + t_2u_3) + \lambda_1^2 t_1u_2, \\ l_4l_{42} &= t_1u_0 - \lambda_2(t_0u_0 - t_1u_1 + t_2u_3) + \lambda_2^2 t_2u_0, \\ l_4l_{43} &= t_0u_2 - \lambda_3(t_0u_0 + t_1u_1 - t_2u_3) + \lambda_3^2 t_0u_1. \end{aligned} \tag{6.7}$$

The last two equations give

$$l_4\Big(\frac{l_{42}}{\lambda_2} + \frac{l_{43}}{\lambda_3}\Big) = t_0\Big(-2u_0 + \lambda_3 u_1 + \frac{u_2}{\lambda_3}\Big) + u_0\Big(\lambda_2 t_2 + \frac{t_1}{\lambda_3}\Big). \tag{6.8}$$

The lines ℓ_4, ℓ_1, and ξ_1 belong to the third Steiner complex (6.4), and by Lemma 6.1.6 form an azygetic triad. By Proposition 6.1.4, they cannot be concurrent. This implies that the line $V(\lambda_2 t_2 + \frac{t_1}{\lambda_3})$ passes through the intersection point of the lines ξ_1 and ℓ_4. This gives a linear dependence between the linear functions $l_4 = a_0t_0 + a_1t_1 + a_2t_2, l_1 = t_0$ and $\lambda_2t_2 + \frac{t_1}{\lambda_3}$ (we can assume that $a_0 = a_1 = a_2 = 1$ but will do it later). This can happen only if

$$\lambda_2 = c_1a_2, \quad \frac{1}{\lambda_3} = c_1a_1,$$

for some constant c_1. Now, $\lambda_2 t_2 + \frac{1}{\lambda_3}t_1 = c_1(a_2t_2 + a_1t_1) = c_1(l_4 - a_0t_0)$, and we can rewrite (6.8) in the form

$$c_1 l_4\Big(\frac{l_{42}}{\lambda_2} + \frac{l_{43}}{\lambda_3} - c_1u_0\Big) = t_0\Big(-c_1(2 + a_0c_1)u_0 + \frac{u_1}{a_1} + \frac{u_2}{a_2}\Big).$$

This implies that

$$\frac{l_{42}}{\lambda_2} + \frac{l_{43}}{\lambda_3} = c_1u_0 + \frac{k_1}{c_1}t_0, \tag{6.9}$$

$$k_1l_4 = -c_1(2 + c_1a_0)u_0 + \frac{u_1}{a_1} + \frac{u_2}{a_2}, \tag{6.10}$$

for some constant k_1. Similarly, we get

$$k_2l_4 = -c_2(2 + c_2a_1)u_1 + \frac{u_0}{a_0} + \frac{u_2}{a_2},$$

$$k_3l_4 = -c_3(2 + c_3a_2)u_2 + \frac{u_1}{a_0} + \frac{u_2}{a_1}.$$

It is easy to see that this implies that

$$k_1 = k_2 = k_3 = k,\ c_1 = -a_0,\ c_2 = -a_1, c_3 = -a_2.$$

Equations (6.9)[64] and (6.10)[65] become

$$\frac{l_{42}}{\lambda_2} + \frac{l_{43}}{\lambda_3} = -a_0u_0 - \frac{k}{a_0}t_0, \tag{6.11}$$ 64'

$$kl_4 = \frac{u_0}{a_0} + \frac{u_1}{a_1} + \frac{u_2}{a_2}. \tag{6.12}$$ 65'

At this point, we can scale the coordinates to assume

$$a_1 = a_2 = a_2 = 1 = -k = 1,$$

and obtain our first equation

$$t_0 + t_1 + t_2 + u_0 + u_1 + u_2 = 0.$$

Replacing l_{41} with l_{51}, l_{61}, l_{71} and repeating the argument, we obtain the remaining three equations relating u_0, u_1, u_2 with t_0, t_1, t_2.

Let us find the constants k_1, k_2, k_3 for ℓ_5, ℓ_6, ℓ_7. We have found four linear equations relating six linear functions $t_0, t_1, t_2, u_0, u_1, u_2$. Since three of them form a basis in the space of linear functions, there must be one relation. We may assume that the first equation is a linear combination of the last three with some coefficients $\lambda_1, \lambda_2, \lambda_3$. This leads to the system of linear equations from the statement of the Theorem.

Finally, we have to find the equations of the 21 bitangents. The equations (6.5)[63] show that the lines ξ_1, ξ_2, ξ_3 are bitangents. Equation (6.11)[64'] and similar equations

$$\frac{l_{43}}{\lambda_3} + \frac{l_{41}}{\lambda_1} = -u_1 + t_1,$$

$$\frac{l_{41}}{\lambda_1} + \frac{l_{42}}{\lambda_2} = -u_2 + t_2,$$

after adding up, give

$$\frac{l_{41}}{\lambda_1} + \frac{l_{42}}{\lambda_2} + \frac{l_{43}}{\lambda_3} = t_0 + t_1 + t_2,$$

and then

$$\frac{l_{41}}{\lambda_1} = u_0 + t_1 + t_2,$$

$$\frac{l_{42}}{\lambda_1} = u_1 + t_0 + t_2,$$

$$\frac{l_{43}}{\lambda_1} = u_2 + t_0 + t_1.$$

This gives us three equations of type (2). Similarly, we get the expressions for l_{5i}, l_{6i}, l_{7i} which are the nine equations of types (3), (4), and (5).

Let us use the Aronhold set $(\ell_1, \ldots, \ell_7)$ to index bitangents by subsets (ij) of $\{1, \ldots, 8\}$. As we explained at the end of the previous section, we have

$$\xi_1 = \vartheta_{23}, \quad \xi_2 = \vartheta_{13}, \quad \xi_3 = \vartheta_{12},$$

$$\xi_{4k} = \vartheta_{k4}, \quad \xi_{5k} = \vartheta_{k5}, \quad \xi_{6k} = \vartheta_{k6}, \quad \xi_{7k} = \vartheta_{k7}, \quad k = 1, 2, 3.$$

The remaining bitangents are $\vartheta_{56}, \vartheta_{57}, \vartheta_{67}, \vartheta_{45}, \vartheta_{46}, \vartheta_{47}$. The first three look like $\vartheta_{23}, \vartheta_{13}, \vartheta_{12}$, they are of type $\triangle$. The second three look like $\vartheta_{5k}, \vartheta_{6k}, \vartheta_{7k}$, they are of type $\veebar$. To find the equations of triples of bitangents of type $\triangle$, we interchange the roles of the lines ℓ_1, ℓ_2, ℓ_3 with the lines ℓ_5, ℓ_6, ℓ_7. Our lines will be the new lines analogous to the lines ξ_1, ξ_2, ξ_3. Solving the system, we find their equations. To find the equations of the triple of bitangents of type $\veebar$, we delete ℓ_4 from the original Aronhold set, and consider the Steiner complex containing the remaining lines as we did in (6.4). The lines making the pairs with ℓ_5, ℓ_6, ℓ_7 will be our lines. We find their equations in the same manner as we found the equations for $\xi_{5k}, \xi_{6k}, \xi_{7k}$. □

Remark 6.1.10. The proof of the Theorem implies the following result, which can be found in [367]. Let (ℓ_1, ξ_1), be three pairs of bitangents from the same Steiner complex. Let (ℓ_4, ξ_4) be a fourth pair of bitangents from the Steiner complex given by pairs $(\ell_1, \xi_2), (\ell_2, \xi_1)$ as in (6.4) (where $\xi_4 = \xi_{43}$). Choose some linear forms l_i, m_i representing ℓ_i, ξ_i. Then, the equation of C can be given by

$$\big((l_4l_2l_3)(l_4m_2m_3)l_1m_1 + (l_1l_4l_3)(m_1l_4m_3)l_2m_2 - (l_1l_2l_4)(m_1m_2l_4)l_3m_3\big)^2$$

$$-4(l_4l_2l_3)(l_4m_2m_3)(l_1l_4l_3)(m_1l_4m_3)l_1m_1l_2m_2 = 0,$$

where the brackets denote the determinants of the matrix formed by the coefficients of the linear forms. In fact, this is Equation (6.5), where the determinants take care of scaling of the forms u_0, u_1, u_2 (use that, $V(l_4)$ can be taken to be $V(l_1 + l_2 + l_3)$ and we must keep the relation $l_1 + l_2 + l_3 + u_1 + u_2 + u_3 = 0$).

One can also find in loc.cit. paper of J. Guàrdia the expressions for l_i, m_i in terms of the period matrix of C.

Remark 6.1.11. We will see later in Subsection 6.3.3 that any seven lines in a general linear position can be realized as an Aronhold set for a plane quartic curve. Another way to see it can be found in [799, p. 447].

S:6.2
6.2 Determinant Equations of a Plane Quartic

6.2.1
6.2.1 Quadratic determinantal representations

Recall from Subsection SS:4.1.1 4.1.1 that a determinantal representation of type $(|k,k|_{k-1},2)_2$ of a nonsingular plane curve $C\subset |E|$ of degree $d=2k$ is defined by two exact sequences

$$0\to U^\vee_{|E|}(-2)\xrightarrow{\phi} V_{|E|}\to \mathcal{L}\to 0,$$
$$0\to V^\vee_{|E|}(-2)\xrightarrow{\phi} U_{|E|}\to \mathcal{M}\to 0, \qquad (6.13)$$ r1

where $\dim U=\dim V=k$ and $\mathcal{L},\mathcal{M}$ are invertible sheaves on C satisfying

$$\mathcal{L}\otimes\mathcal{M}\cong \mathcal{O}_C(d-2)\cong \omega_C(-1). \qquad (6.14)$$

The following is an analog of Theorem L3 4.2.1 whose proof we leave to the reader.

L3new **Theorem 6.2.1.** *Let $g=\frac{1}{2}(d-1)(d-2)$ be the genus of the curve C. Then*

(i) $H^0(C,\mathcal{L})\cong U,\ H^0(C,\mathcal{M})\cong V$*;*
(ii) $H^0(C,\mathcal{L}(-1))=H^0(C,\mathcal{M}(-1))=\{0\}$;
(iii) $H^1(C,\mathcal{L}(j))=H^1(C,\mathcal{M}(j))=\{0\},\ j=-1,0$*;*
(iv) $\deg(\mathcal{L})=\deg(\mathcal{M})=2k(k-1)$*;*
(v) $\deg(\mathcal{L}(1-k))=\deg(\mathcal{M}(1-k))=0$.

The maps

$$\mathfrak{l}:C\to\mathbb{P}(U),\quad \mathfrak{r}:C\to\mathbb{P}(V)$$

are given by the linear systems $|\mathcal{L}|$ and $|\mathcal{M}|$ of dimension $k-1=\frac{1}{2}(d-2)$ and degree $\frac{1}{2}d(d-2)$. If we write

$$\mathcal{L}(1-k)=\mathcal{O}_C(a)$$

for some divisor class of degree 0, then

$$|\mathcal{L}|=|(k-1)h+a,\quad |\mathcal{M}|=|(k-1)h-a,$$

where h is the divisor class of $\mathcal{O}_C(1)$.

The map

$$(\mathfrak{l},\mathfrak{r}):C\to\mathbb{P}(U)\times\mathbb{P}(V)\to\mathbb{P}(U\otimes V)=|U^\vee\otimes V^\vee|$$

is given the linear system $|\mathcal{L}\otimes\mathcal{M}|=|2(k-1)h|$. The restriction map to the image S of $(\mathfrak{l},\mathfrak{r})$

$$|\mathcal{O}_{\mathbb{P}(U)\times\mathbb{P}(V)}(1)|=|U\otimes V|\to|\mathcal{L}\otimes\mathcal{M}|=|2(k-1)h|, \qquad (6.15)$$ 6.18

defines a map

$$\mu : |(k-1)h+a|\times|(k-1)h-a| \to |\mathcal{O}_{\mathbb{P}^2}(2)|,\ (D_1,D_2)\mapsto \langle D_1,D_2\rangle = Q\cap C, \tag{6.16}$$ mapmu

where $\langle D_1, D_2\rangle$ is the unique curve of degree $2k-2$ that cuts out the divisor D_1+D_2 on C. Composed with the quadratic polarization linear map ϕ : $|S^2(E)| \to |U\otimes V|$ we get a map

$$\nu : |S^2(E)| \to |\mathcal{O}_{\mathbb{P}^2}(2k-2)| = |S^{2k-2}(E^\vee)|. \tag{6.17}$$ mapnu

A similar proof as used in the case of linear determinantal representations shows that this map coincides with the apolarity map corresponding to C.

For any $x\in C$, consider the tensor $\mathfrak{l}(x)\otimes\mathfrak{r}(x)$ as a hyperplane in $|U\otimes V|$. It intersects $|U|\times|V|$ at the subvariety of points whose image under the map μ vanishes at x. Choose a basis $(s_1,\ldots,s_k)$ in U and a basis $(s_1',\ldots,s_k')$ in V. The map ϕ is given by $\phi(x)=\sum a_{ij}s_i\otimes s_j$. It follows from above that the matrix $(\nu(s_i\otimes s_j))$ and the matrix $\mathrm{adj}((a_{ij}))$ coincide when restricted at C (up to a multiplicative factor). Since its entries are polynomials of degree less than $\deg C$, we see that they coincide for all x. This shows that the map ν can be written by the formula

$$\Big(\sum u_is_i,\sum v_js_j'\Big)\mapsto -\det\begin{pmatrix} a_{11}(t) & \ldots & a_{1k}(t) & v_1\\ \vdots & \vdots & \vdots & \vdots\\ a_{k1}(t) & \ldots & a_{kk}(t) & v_k\\ u_1 & \ldots & u_k & 0\end{pmatrix}. \tag{6.18}$$ bd2

Under the composition of the map, the zero set of the bordered determinant is a curve of degree $2k-2$. Consider the discriminant hypersurface $\mathsf{D}_{d-2}(2)$ of plane curves of degree $d-2=2k-2$. The pre-image of $\mathsf{D}_{d-2}(2)$ under the map (6.18) is a hypersurface X in $\mathbb{P}(U)\times\mathbb{P}(V)\cong\mathbb{P}^{k-1}\times\mathbb{P}^{k-1}$ given by a bihomogeneous equation of bidegree $(3(d-3)^2, 3(d-3)^2)$. Here, we use that $\deg\mathsf{D}_d(2)=3(d-1)^2$.

Now, it is time to specialize to the case $d=4$. In this case, the map $|\nu|$ is the map

$$|\nu| : |K_C+a|\times|K_C-a| \to |\mathcal{O}_{\mathbb{P}^2}(2)|.$$

In coordinates, the map ν is given by

$$(u_1s_1+u_2s_2, v_1s_1'+v_2s_2')\mapsto -u_0v_0a_{11}+u_0v_1a_{12}-u_1v_0a_{21}-u_1v_1a_{22}. \tag{6.19}$$ bd3

The map ϕ is given by

$$\phi(x)=\sum a_{ij}(x)s_i^*\otimes s_j'^*,$$

where $(s_1^*, s_2^*), (s_1'^*, s_2'^*)$ are the dual bases in $U^\vee$ and $V^\vee$. One can also explicitly see the maps $\mathfrak{l}$ and $\mathfrak{r}$:

$$\mathfrak{l}(x) = [-a_{21}(x), a_{11}(x)] = [-a_{22}(x), a_{12}(x)], \tag{6.20}$$

$$\mathfrak{r}(x) = [-a_{12}(x), a_{11}(x)] = [-a_{22}(x), a_{21}(x)]. \tag{6.21}$$

The intersection of the conics $V(a_{21}(t)) \cap V(a_{21}(t))$ lies on C, so $\mathfrak{l}$ is given by a pencil of conics with four base points $x_1, \ldots, x_4$ on C. The map $\mathfrak{r}$ is given by another pencil of conics whose base points $y_1, \ldots, y_4$, together with the base points $x_1, \ldots, x_4$, are cut out by a conic.

The hypersurface $X \subset \mathbb{P}(U) \times \mathbb{P}(V) \cong \mathbb{P}^1 \times \mathbb{P}^1$ is of bidegree $(3, 3)$. It is a curve of arithmetic genus 4. Its image under the Segre map is a canonical curve equal to the intersection of a nonsingular quadric and a cubic surface. The cubic surface is the pre-image of the determinantal cubic. It is a cubic symmetroid surface. We will discuss such cubic surfaces in Subsection 9.3.3. As we explained in Subsection 4.3.2, a cubic symmetroid surface admits a unique double cover ramified along the nodes. The restriction of this cover to X is an irreducible unramified cover $r : \tilde{X} \to X$. Let τ be the nontrivial 2-torsion divisor class on X corresponding to this cover (it is characterized by the property that $r^*(\tau) = 0$). The linear system $|K_X + \tau|$ maps X to $\mathbb{P}^2$. The image is a *Wirtinger plane sextic* with double points at the vertices of a complete quadrilateral. Conversely, we will explain in Chapter 9 that a cubic symmetroid surface with four nodes is isomorphic to the image of the plane under the linear system of cubics passing through the six vertices of a complete quadrilateral. This shows that any Wirtinger sextic is isomorphic to the intersection of a quadric and a cubic symmetroid surface. In this way, we see that any general curve of genus four is isomorphic to the curve X arising from a quadratic determinantal representation of a nonsingular plane quartic. We refer for this and for more of the geometry of Wirtinger sextics to [99].

The map (6.17) is just the apolarity map $\mathrm{ap}_2 : S^2(E) \to S^2(E^\vee)$ defined by the quartic C. It is bijective if the quartic C is nondegenerate. Under the composition $|E| \to |S^2(E)| \to |S^2(E^\vee)|$, the pre-image of the discriminant cubic hypersurface is the Hessian sextic of C.

Consider the hypersurface W of type $(1, 1, 2)$ in $|U| \times |V| \times |E|$ defined by the section of $\mathcal{O}_{\mathbb{P}(U)}(1) \boxtimes \mathcal{O}_{\mathbb{P}(V)}(1) \boxtimes \mathcal{O}_{\mathbb{P}(E)}(2)$ corresponding to the tensor defining the linear map $\phi : S^2(E) \to U \otimes V$. It is immediate that

$$W = \{(D_1, D_2, x) \in |K_C + a| \times |K_C - a| \times \mathbb{P}^2 : x \in \langle D_1, D_2 \rangle\}. \tag{6.22}$$

In coordinates, the equation of W is given by the bordered determinant (6.18).

Consider the projections

$$\mathrm{pr}_1 : W \to \mathbb{P}^1 \times \mathbb{P}^1, \quad \mathrm{pr}_2 : W \to \mathbb{P}^1. \tag{6.23}$$

The fibers of pr_1 are isomorphic (under pr_2) to conics.The discriminant curve is the curve X. The fiber of pr_2 over a point $x \in \mathbb{P}^2$ is isomorphic, under pr_1, to a curve on $\mathbb{P}^1 \times \mathbb{P}^1$ of degree $(1,1)$. In the Segre embedding, it is a conic. The discriminant curve is the curve C. Thus, W has two structures of a conic bundle. The two discriminant curves, X and C, come with the natural double cover parameterizing irreducible components of fibers. In the first case, it corresponds to the 2-torsion divisor class τ and X is a nontrivial unramified double cover. In the second case, the cover splits (since the factors of $\mathbb{P}^1 \times \mathbb{P}^1$ come with an order).

Remark 6.2.2. Recall that, for any unramified double cover of nonsingular curves $\pi : \tilde{S} \to S$, the Prym variety $\mathrm{Prym}(\tilde{S}/S)$ is defined to be the connected component of the identity of $\mathrm{Jac}(\tilde{S})/\,\pi^*\mathrm{Jac}(S)$.

$$\mathrm{Prym}(\tilde{X}/X) \cong \mathrm{Jac}(C).$$

This is a special case of the *trigonal construction* applied to trigonal curves (like ours X) discovered by S. Recillas [608] (see a survey of R. Donagi [260] about this and more general constructions of this sort). Note that, in general, the curve X could be singular even when C is not. However, the Prym variety is still defined.

Let $\mathcal{R}_g$ be the coarse moduli space of isomorphism classes of pairs $(S, \tilde{S})$, where S is a nonsingular curve of genus g and $\tilde{S} \to S$ is its unramified double cover. There is a *Prym map*

$$\mathfrak{p}_g : \mathcal{R}_g \to \mathcal{A}_{g-1}, \quad (S, \tilde{S}) \mapsto \mathrm{Prym}(\tilde{S}/S),$$

where $\mathcal{A}_{g-1}$ is the coarse moduli space of principally polarized abelian varieties of dimension $g-1$. In our case $g = 4$, the quadratic determinantal constructions allows us to describe the fiber over the Jacobian variety of a nonsingular canonical curve of genus 3. It is isomorphic to the *Kummer variety* $\mathrm{Kum}(C) = \mathrm{Jac}(C)/(\iota)$, where ι is the negation involution $a \mapsto -a$.

The map $\mathfrak{p}_g$ is known to be generically injective for $g \geq 7$ [310], a finite map of degree 27 for $g = 6$ [263], and dominant for $g \leq 5$ with fibers of dimension $3g - 3 - \frac{1}{2}g(g-1)$. We refer to [261] for the description of fibers.

The varieties $\mathcal{R}_g$ are known to be rational ([248] for $g = 2$, [248], [444] for $g = 3$, [99] for $g = 4$) and unirational for $g = 5$ [427], [788], $g = 6$ [262], [785] and $g = 7$ [788]). It is known to be of general type for $g > 13$ and $g \neq 15$ [296].

6.2.2 Symmetric quadratic determinants

SS:6.2.2

Assume that a quadratic determinantal representation of a nonsingular plane curve C of degree $d = 2k$ is symmetric. Then $\mathcal{L}(1-k) \cong \mathcal{M}(1-k)$ and $\mathcal{L}(1-k)^{\otimes 2} \cong \mathcal{O}_C$. Also, by Theorem 6.2.1, $H^0(C, \mathcal{L}(1-k)) = \{0\}$. Thus, we see that a quadratic determinantal representation of C correspond to non-trivial elements of $\mathrm{Jac}(C)[2]$. We also have

$$H^0(C, \mathcal{L}(-1) = H^1(C, \mathcal{L}(-1)) = 0.$$

We write $|\mathcal{L}(1-k)| = |\epsilon|$, where $\epsilon \in \mathrm{Jac}(C)[2] \setminus \{0\}$, hence $|\mathcal{L}| = |(k-1)h+\epsilon|$. In coordinates,

$$C := \det \begin{pmatrix} a_{11} & \dots & a_{1k} \\ \vdots & \vdots & \vdots \\ a_{k1} & \dots & a_{kk} \end{pmatrix} = 0,$$

where $a_{ij} = a_{ji}$ are homogeneous forms of degree 2. It comes with the maps,

$$\phi : |E| \to |S^2(U^\vee)|, \quad x \mapsto (a_{ij}(x)),$$

$$\mathfrak{l} : C \to \mathbb{P}(U) \cong \mathbb{P}^{k-1}, \quad x \mapsto |N(A(x))|.$$

It is given by the linear system $|(k-1)h + \epsilon|$. The map (6.15) becomes the restriction map of quadrics in $\mathbb{P}(U)$ to the image S of C under the map $\mathfrak{l}$

$$\nu : S^2(U) \to H^0(C, \mathcal{O}_C(2k-2)) = H^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(2k-2)).$$

The map (6.16) is the composition of the map $U \times U \to |S^2(U)|$ given by the complete linear system of quadrics in $|U|$ and the map ν. It factors through the symmetric square of $|U|$, and defines a map

$$|U|^{(2)} \to |\mathcal{O}_{\mathbb{P}^2}(2k-2)|. \tag{6.24}$$ nus

Recall that $|U|^{(2)}$ is isomorphic to the secant variety of $\mathsf{v}_2(|U|)$ in $|S^2(U)|$. The pre-image $X(\epsilon)$ of the determinantal hypersurface $\mathsf{D}_{2k-2}(2)$ of curves of degree $2k-2$ in $|S^2(U)|$ is a hypersurface of degree $3(d-3)^2$. Its intersection with $|U| \times |U|$, embedded by Segre, is a hypersurface of bidegree $(3(d-3)^2, 3(d-3)^2)$. It is invariant with respect to the switch involution of $|U| \times |U|$ and descends to a hypersurface in the quotient. Its pre-image under the Veronese map is a hypersurface $B(\epsilon)$ of degree $6(d-3)^2$ in $\mathbb{P}(U)$.

In coordinates, the multiplication map is given by the bordered determinant (6.18). Since A is symmetric, we have $D(A; u, v) = D(A; v, u)$, and the bordered determinantal identity (4.22) gives

$$D(A; u, v)^2 - D(A; u, u)D(A; v, v) = |A|P(t; u, v),$$

where $P(t;u,v)$ is of degree $2k-4$ in (t_0,t_1,t_2) and of bidegree $(2,2)$ in u,v. The curves $V(D(A;u,u))$ define a quadratic family of contact curves of degree $2k-2$. So, we have $2^{2g}-1$ of such families, where g is the genus of C.

Now, let us specialize to the case $k=2$. The determinantal equation of C corresponding to ϵ must be given by a symmetric quadratic determinant

$$\begin{vmatrix} a_{11} & a_{12} \\ a_{12} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{12}^2. \tag{6.25}$$

symdett3

Thus, we obtain the following.

thm:6.2.3 **Theorem 6.2.3.** *An equation of a nonsingular plane quartic can be written in the form*

$$\begin{vmatrix} a_1 & a_2 \\ a_2 & a_3 \end{vmatrix} = 0,$$

where a_1,a_2,a_3 are homogeneous forms of degree 2. The set of equivalence classes of such representations is in a bijective correspondence with the set of 63 nontrivial 2-torsion divisor classes in $\mathrm{Pic}(C)$.

The bordered determinant

$$D(A;u,u) = \begin{vmatrix} a_{11} & a_{12} & u_0 \\ a_{21} & a_{22} & u_1 \\ u_0 & u_1 & 0 \end{vmatrix} = -(a_{22}u_0^2 - 2a_{12}u_0u_1 + a_{11}u_1^2)$$

defines a family of *contact conics* of C. Each conic from the family touches C along a divisor from $|K_C+\epsilon|$.

Also identity (4.23) [bordsym] between the bordered determinants becomes in our case

$$\det\begin{pmatrix} D(A;u,u) & D(A;u,v) \\ D(A;u,v) & D(A;v,v) \end{pmatrix} = |A|P(u,v), \tag{6.26}$$

newid

where $P(u,v)$ is a bihomogeneous polynomial in u,v of bidegree $(2,2)$. Note that $P(u,v)$ is symmetric in u,v and $P(u,u)=0$. This shows that $P(u,v)$ can be written in the form

$$P(u,v) = (u_0v_1 - u_1v_0)(\alpha u_0v_0 + \beta(u_0v_1+u_1v_0) + \gamma u_1v_1),$$

where α,β,γ are some constants.

The variety $X(\epsilon)$ in $|U|\times|U| \cong \mathbb{P}^1\times\mathbb{P}^1$ is a curve of bidegree $(3,3)$. The difference from the general case of quadratic determinantal representations of C is that the curve $X(\epsilon)$ is defined by a symmetric bihomogeneous form. The symmetric product $|U|^{(2)}$ is isomorphic to $|S^2(U)| \cong \mathbb{P}^2$. The image of $X(\epsilon)$ in the plane is a curve $F(\epsilon)$ of degree 3. In intersects the Veronese curve

$|E| \hookrightarrow |S^2(U)|$ at 6 points. They are the images of the hypersurface $B(\epsilon) \subset |U|$ under the Veronese map $|E| \hookrightarrow |S^2(U)|$. So, we see another special property of $X(\epsilon)$. If it is nonsingular, it is a canonical *bielliptic curve* of genus four.

curve!of genus 4

One can easily compute the number of moduli of such curves. It is equal to six instead of nine for a general curve of genus 4. This agrees with our construction since we have six moduli for pairs (C, ϵ).

It follows from the definition that the curve $F(\epsilon)$ parameterizes unordered pairs D_1, D_2 of divisors $D \in |K_C + \epsilon|$ such that the conic $\langle D_1, D_2\rangle$ is equal to the union of two lines.

Let $\Pi(\epsilon)$ be the plane in $|\mathcal{O}_{\mathbb{P}^2}(2)|$ equal to the image of the map (6.24). It is a net of conics in $|E| = \mathbb{P}^2$. It is spanned by the contact conics to C. We can take for the basis of the net the conics

$$V(a_{11}) = \langle 2D_1\rangle,\ V(a_{12}) = \langle D_1, D_2\rangle,\ V(a_{22}) = \langle 2D_2\rangle,$$

where D_1, D_2 span $|K_C + \epsilon|$. In particular, we see that $\Pi(\epsilon)$ is base-point-free. Its discriminant curve is equal to the curve $F(\epsilon)$.

lemmma1

Proposition 6.2.4. *The cubic curve $F(\epsilon)$ is nonsingular if and only if the linear system $|K_C + \epsilon|$ does not contain a divisor of the form $2a + 2b$.*

Proof Let $\mathsf{D} = \mathsf{D}_2(2) \subset |\mathcal{O}_{\mathbb{P}^2}(2)|$ be the discriminant cubic. The plane section $\Pi(\epsilon) \cap \mathsf{D}_2(2)$ is singular if and only if $\Pi(\epsilon)$ contains a singular point of D represented by a double line, or if it is tangent to D at a nonsingular point. We proved in Example 1.2.3 that the tangent hypersurface of D at a nonsingular point represented by a reducible conic Q is equal to the space of conics passing through the singular point q of Q. If L is contained in the tangent hyperplane, then all conics from $\Pi(\epsilon)$ pass through q. But, as we saw earlier, the net of conics $\Pi(\epsilon)$ is base-point-free. This shows that $\Pi(\epsilon)$ intersects D transversally at each nonsingular point.

In particular, $F(\epsilon)$ is singular if and only if $\Pi(\epsilon)$ contains a double line. Assume that this happens. Then, we get two divisors $D_1, D_2 \in |K_C + \epsilon|$ such that $D_1 + D_2 = 2A$, where $A = a_1 + a_2 + a_3 + a_4$ is cut out by a line ℓ. Let $D_1 = p_1 + p_2 + p_3 + p_4, D_2 = q_1 + q_2 + q_3 + q_4$. Then, the equality of divisors (not the divisor classes)

$$p_1 + p_2 + p_3 + p_4 + q_1 + q_2 + q_3 + q_4 = 2(a_1 + a_2 + a_3 + a_4)$$

implies that either D_1 and D_2 share a point x, or $D_1 = 2p_1+2p_2, D_2 = 2q_1+2q_2$. The first case is impossible, since $|K_C + \epsilon - x|$ is of dimension 0. The second case happens if and only if $|K_C + \epsilon|$ contains a divisor $D_1 = 2a + 2b$. The converse is also true. For each such divisor the line $\overline{ab}$ defines a residual pair of points c, d such that $D_2 = 2c + 2d \in |K_C + \epsilon|$ and $\varphi(D_1, D_2)$ is a double line. $\square$

Remark 6.2.5. By analyzing possible covers of a plane cubic unramified outside the singular locus, one can check that $F(\epsilon)$ is either nonsingular or a nodal cubic, maybe reducible.

From now on, we assume that $F(\epsilon)$ is a nonsingular cubic. Since it parameterizes singular conics in the net $\Pi(\epsilon)$, it comes with a natural nontrivial two-torsion point ϵ. Recall that the corresponding unramified double cover of $F(\epsilon)$ is naturally isomorphic to the Cayleyan curve in the dual plane $\Pi(\epsilon)^\vee$ which parameterizes irreducible components of singular conics in the net.

steiner2 **Theorem 6.2.6.** *Let* $\Sigma(\epsilon) = \{(\ell_1, \ell_1'), \ldots, (\ell_6, \ell_6')\}$ *be a Steiner complex of twelve bitangents associated with the two-torsion divisor class* ϵ. *Each pair, considered as a divisor* $D_i = \ell_i + \ell_i' \in |K_C + \epsilon| = |U|$ *is mapped under the Veronese map* $|U| \to |S^2(U)|$ *to a point in* $F(\epsilon)$. *It belongs to the set* $B(\epsilon)$ *of six ramification points of the cover* $X(\epsilon) \to F(\epsilon)$. *The twelve bitangents, considered as points in the dual plane* $|S^2(U^\vee)|$, *lie on the cubic curve* $\tilde{F}(\epsilon)$.

Proof Let $(\vartheta_i, \vartheta_i')$ be a pair of odd theta characteristics corresponding to a pair (ℓ_i, ℓ_i') of bitangents from $\Sigma(\epsilon)$. They define a divisor $D = \vartheta_i + \vartheta_i' \in |K_C + \epsilon|$ such that D is the divisor of points of contact of a reducible contact conic, i.e., the union of two bitangents. This shows that $\vartheta_i, \vartheta_i' \in \tilde{F}(\epsilon)$. The point $(D, D) \in |K_C + \epsilon| \times |K_C + \epsilon|$ belongs to the diagonal in $|U| \times |U|$. These are the ramification points of the cover $X(\epsilon) \to F(\epsilon)$. They can be identified with the branch points of the cover $X(\epsilon) \to F(\epsilon)$. $\square$

So, we have a configuration of 63 cubic curves $\tilde{F}(\epsilon)$ in the plane $|S^2(U^\vee)|$ (beware that this plane is different from the plane $|E|$ containing C). Each contains twelve bitangents from a Steiner complex. Let S_1, S_2, S_3 be a syzygetic (resp. azygetic) triad of Steiner complexes. They define three cubic curves $\tilde{F}(\epsilon), \tilde{F}(\eta), \tilde{F}(\eta + \epsilon)$ with four (resp. six) common points.

Let us see what happens in the symmetric case with the two-way conic bundle $W \subset \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^2$ from (6.22) which we discussed in the previous subsection. First, its intersection with the product of the diagonal Δ of $\mathbb{P}^1 \times \mathbb{P}^1$ with $\mathbb{P}^2$ defines the universal family $\mathsf{U}(\epsilon)$ of the contact conics. It is isomorphic to a surface in $\mathbb{P}^1 \times \mathbb{P}^2$ of bidegree $(2, 2)$. The projection to $\mathbb{P}^2$ is a double cover branched along the quartic C. As we will see later, $\mathsf{U}(\epsilon)$ is isomorphic to a *del Pezzo surface of degree 2*. Its isomorphism class does not depend on ϵ. The projection $\mathsf{U}(\epsilon) \to \mathbb{P}^1$ is a conic bundle. It has six singular fibers that lie over six points at which the diagonal intersects the curve $X(\epsilon)$, i.e. the ramification points of the cover $X(\epsilon) \to F(\epsilon)$. The six branch points lie on a conic, the image of the diagonal Δ in $\mathbb{P}^2$. We will see later that a del Pezzo surface of degree 2 has 126 conic bundle structures; they are divided into 63 pairs which correspond to nonzero 2-torsion divisor classes on C.

The threefold W is invariant with respect to the involution of $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^2$ which switches the first two factors. The quotient $W^s = W/(\iota)$ is a hypersurface of bidegree $(2,2)$ in $(\mathbb{P}^1 \times \mathbb{P}^1)/(\iota) \times \mathbb{P}^2 \cong \mathbb{P}^2 \times \mathbb{P}^2$. The projection to the first factor is a conic bundle with the discriminant curve $B(\epsilon)$. The projection to the second factor is no longer a conic bundle. It is isomorphic to the pull-back of the universal family of lines $X(\epsilon) \to \mathbb{P}^2$ under the map of the base $\mathbb{P}^2 \to \mathbb{P}^2$ given by the net of conics $\Pi(\epsilon)$.

Remark 6.2.7. One can easily describe the Prym map $\mathfrak{p}_3 : \mathcal{R}_3 \to \mathcal{A}_2$ restricted to the open subset of canonical curves of genus three. A pair (C, η) defines an elliptic curve $F(\epsilon)$ and six branch points of the cover $X(\epsilon) \to F(\epsilon)$. The six points lie on the Veronese conic $|E| \hookrightarrow |S^2(U)|$. The cover $\tilde{C} \to C$ defined by ϵ is a curve of genus 5. The Prym variety $\mathrm{Prym}(\tilde{C}/C)$ is a principally polarized abelian variety of dimension two. One can show that $\mathrm{Prym}(\tilde{C}/C) \cong \mathrm{Jac}(H)$, where H is the hyperelliptic curve of genus 2 which is isomorphic to the branch cover of the Veronese conic ramified over $B(\epsilon)$ (see [480], [481]). Other description of the Prym map $\mathfrak{p}_3$ can be found in [786].

S:6.3

6.3 Even Theta Characteristics

SS:6.3.1

6.3.1 Contact cubics

Recall that each even theta characteristic ϑ on a nonsingular plane quartic curve C defines a three-dimensional family of contact cubics. The universal family of contact cubics is a hypersurface $\mathsf{W}_\vartheta \subset |E| \times \mathbb{P}(U) \cong \mathbb{P}^2 \times \mathbb{P}^3$ of bidegree $(2,3)$. If we choose coordinates (t_0, t_1, t_2) in $|E|$ and coordinates u_0, u_1, u_2, u_3 in $\mathbb{P}(U)$, then the equation of the family of contact cubics becomes

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} & a_{14} & u_0 \\ a_{21} & a_{22} & a_{23} & a_{24} & u_1 \\ a_{31} & a_{32} & a_{33} & a_{34} & u_2 \\ a_{41} & a_{42} & a_{43} & a_{44} & u_3 \\ u_0 & u_1 & u_2 & u_3 & 0 \end{vmatrix} = 0, \qquad (6.27)$$ borddet2

where (a_{ij}) is the symmetric matrix defining the net N_ϑ of quadrics defined by ϑ. The first projection $\mathsf{W}_\vartheta \to |E|$ is a quadric bundle with discriminant curve equal to C. Its fiber over a point $x \notin C$ is the dual of the quadric $Q_x = \phi(x)$. Its fiber over a point $x \in C$ is the double plane corresponding to the vertex of the quadric cone $\phi(x)$. Scheme-theoretically, the discriminant hypersurface of the quadric bundle is the curve C taken with multiplicity 3.

The second projection $\mathsf{W}_\vartheta \to \mathbb{P}^3$ is a fibration with fibers equal to contact cubics. Its discriminant surface D_ϑ is the pre-image of the discriminant hypersurface $D_3(2)$ of plane cubic curves in $|\mathcal{O}_{\mathbb{P}^2}(3)|$ under the map $\mathbb{P}^3 \to |\mathcal{O}_{\mathbb{P}^2}(3)|$ given by quadrics. This implies that D_ϑ is of degree 24 and its equation is of the form $F_8^3 + G_{12}^2 = 0$, where F_8 and G_{12} are homogeneous forms in $u_0, \ldots, u_3$ of the degrees indicated by the subscript.

Proposition 6.3.1. *The discriminant surface* D_ϑ *of the family of contact cubics is reducible and non-reduced. It consists of the union of 8 planes and a surface of degree 8 taken with multiplicity 2.*

Proof Let N_θ be the net of quadrics in $\mathbb{P}^3$ defined by ϑ. We know that the contact cubic $V(D(A;\xi,\xi))$ is isomorphic to the discriminant curve of the net of quadrics obtained by restricting N_ϑ to the plane H_ξ defined by the point ξ in the dual space. The contact cubic is singular if and only if the restricted net has either a base point or contains a conic of rank 1, i.e. a double line. The first case occurs if and only if the plane contains one of the base points of the net N_ϑ. There are eight of them (see the next subsection). This gives eight plane components of D_ϑ. The second case occurs if and only if the plane is tangent to a singular quadric in N_ϑ along a line. It is easy to compute the degree of the surface in $(\mathbb{P}^3)^\vee$ parameterizing such planes. Fix a general line ℓ in $\mathbb{P}^3$, the quadrics in N_ϑ which are tangent to ℓ are parameterized by a conic in N_ϑ. The conic intersects the discriminant curve C of N_ϑ at eight points. Thus, there are eight cones in N_ϑ that are tangent to ℓ. Let ℓ_i' be the line on the cone intersecting ℓ. Then, the plane spanned by the lines ℓ and ℓ' is tangent to the cone. Thus, we see that the degree of the surface parameterizing planes tangent to some cone in N_ϑ is of degree 8. The assertion about the multiplicity of the surface entering the discriminant is proved in Gizatullin [334], Theorem 7.2. □

Let F_ξ be a contact nodal cubic represented by a general point ξ in one of the eight plane components. It is tangent to C at six nonsingular points. On the other hand, a general point F_ξ on the other component of D_ϑ is a nodal cubic with a node at C.

We can see other singular contact cubics too. For example, 56 planes through three base points of the pencil N_ϑ correspond to the union of three asyzygetic bitangents. Another singular contact cubic is a *biscribed triangle*. It is the union of three lines such that C is tangent to the sides and also passes through the three vertices of the triangle. It is proved in [536] that the number of biscribed triangles in each of 36 families of contact cubics is equal to 8.

Remark 6.3.2. Note that each cubic curve F in the family of contact cubics comes with a distinguished 2-torsion point defined by the divisor class $\eta = d - 2h$, where $C \cap F = 2d$, and h is the intersection of F with a line. One can show that the 2-torsion point is nontrivial. The locus of zeros of the invariant surface $V(G_{12})$ of degree 12 parameterizes harmonic contact cubics F together with a nontrivial 2-torsion divisor class η. The group μ_4 of complex multiplications of $\mathrm{Jac}(F)$ acts on the set of 2-torsion divisor classes with two fixed points. If ϵ is invariant with respect to μ_4, then the Cayleyan curve of the cubic is also harmonic. Thus, the surface $V(G_{12})$ is reducible. One of its irreducible component describes the locus of harmonic contact cubics with harmonic Cayleyan. It is shown in [369](see a modern discussion of this surface in [334]) that the degree of this component is equal to 4. Thus, each pair (C, ϑ) defines a quartic surface Θ in $|K_C + \vartheta|$. It can be also described as the locus of planes Π in $|K_C + \vartheta|^\vee$ such that the restriction of N_ϑ to Π is a net of conics with harmonic discriminant curve and the Steinerian curve. The residual surface is of degree 8. It belongs to the pencil of octavic surfaces generated by $V(F_8)$ and 2Θ.

SS:6.3.2

6.3.2 Cayley octads

Let N_ϑ be the net of quadrics defined by the pair (C, ϑ) and Q_1, Q_2, Q_3 be its basis. The base locus of N_ϑ is the complete intersection of these quadrics. One expects that it consists of eight distinct points. Let us see that this is indeed true.

P6.3.1 **Proposition 6.3.3.** *The set of base points of the net of quadrics N_ϑ consists of eight distinct points, no three of which are collinear, and no four are coplanar.*

Proof If we have fewer than eight base points, then all nonsingular quadrics share the same tangent line at a base point. This implies that N_ϑ contains a quadric Q with a singular point at a base point. The computation of the tangent space of the discriminant hypersurface given in (1.46) shows that Q is a singular point of the discriminant curve C, a contradiction.

Suppose three points are on a line ℓ. This includes the case when two points coincide. This implies that ℓ is contained in all quadrics from $\mathcal{N}$. Take a point $x \in \ell$. For any quadric $Q \in \mathsf{N}_\vartheta$, the tangent plane of Q at x contains the line ℓ. Thus, the tangent planes form a pencil of planes through ℓ. Since N_ϑ is a net, there must be a quadric which is singular at x. Thus, each point of ℓ is a singular point of some quadric from N_ϑ. However, the set of singular points of quadrics from N_ϑ is equal to the nonsingular sextic S, the image of C under the map given by the linear system $|\vartheta(1)|$. This shows that no three points are collinear.

Suppose that four points lie in a plane Π. The restriction of N_ϑ to Π defines a linear system of conics through four points, no three of which are collinear. It is of dimension one. Thus, there exists a quadric in N_ϑ which contains Π. However, since C is nonsingular, all quadrics in N_ϑ are of corank ≤ 1. □

Definition 6.3.4. *A set of eight distinct points in* $\mathbb{P}^3$ *which is a complete intersection of three quadrics is called a* Cayley octad.

From now on we assume that a Cayley octad satisfies the properties from Proposition P6.3.1 6.3.3.

Let S be the sextic model of C defined by the linear system $|K_C + \vartheta|$.

T6.3.2 **Theorem 6.3.5.** *Let* $q_1, \ldots, q_8$ *be a Cayley octad. Each line* $\overline{q_i q_j}$ *intersects the sextic curve* S *at two points* $\varphi(p_i), \varphi(p_j)$. *The line* $\overline{p_i p_j}$ *is a bitangent of* C.

Proof The quadrics containing the line $\ell_{ij} = \overline{q_i q_j}$ form a pencil $\mathcal{P}$ in N_ϑ. Its base locus consists of the line ℓ_{ij} and a rational normal cubic curve R which intersects the line at two points (they could be equal). Note that the locus of singular quadrics in the net of quadrics containing R is a conic. Thus, the pencil $\mathcal{P}$ contains two (or one) singular quadrics with singular points at the intersection of R and ℓ_{ij}. In the net N_ϑ this pencil intersects the discriminant curve C at two points. Suppose one of these two points is an ordinary cusp. It is easy to check that the multiplicity of a zero of the discriminant polynomial of the pencil of quadrics is equal to the corank of the corresponding quadric. Since our pencil does not contain reducible quadrics, we see that this case does not occur. Hence, the pencil $\mathcal{P}$ in N_ϑ is a bitangent.

□

We can also see all even theta characteristics.

Theorem 6.3.6. *Let* $q_1, \ldots, q_8$ *be the Cayley octad associated to an even theta characteristic* ϑ. *Let* ϑ_{ij} *be the odd theta characteristic corresponding to the lines* $\overline{q_i q_j}$. *Then, any even theta characteristic different from* ϑ *can be represented by the divisor class*

$$\vartheta_{i,jkl} = \vartheta_{ij} + \vartheta_{ik} + \vartheta_{il} - K_C$$

for some distinct i, j, k, l.

Proof Suppose that $\vartheta_{i,jkl}$ is an odd theta characteristic ϑ_{mn}. Consider the plane π which contains the points q_i, q_j, q_k. It intersects S at six points corresponding to the theta characteristics $\vartheta_{ij}, \vartheta_{ik}, \vartheta_{jk}$. ince the planes cut out divisors from $|K_C + \vartheta|$, we obtain

$$\vartheta_{ij} + \vartheta_{ik} + \vartheta_{jk} \sim K_C + \vartheta.$$

This implies that

$$\vartheta_{jk} + \vartheta_{il} + \vartheta_{mn} \sim K_C + \vartheta.$$

Hence, the lines $\overline{q_j q_k}$ and $\overline{q_i q_l}$ lie in a plane π'. The intersection point of the lines $\overline{q_j q_k}$ and $\overline{q_i q_l}$ is a base point of two pencils in $\mathcal{N}$ and hence is a base point of $\mathcal{N}$. However, it does not belong to the Cayley octad. This contradiction proves the assertion. ◻

Remark 6.3.7. Note that

$$\vartheta_{i,jkl} = \vartheta_{j,ikl} = \vartheta_{k,ijl} = \vartheta_{l,ijk}.$$

Thus, $\vartheta_{i,jkl}$ depends only on the choice of a subset of four elements in $\{1, \ldots, 8\}$. Also it is easy to check that the complementary set defines the same theta characteristic. This gives $35 = \binom{8}{4}/2$ different even theta characteristics. Together with $\vartheta = \vartheta_\emptyset$, we obtain 36 even theta characteristics. Observe now that the notation ϑ_{ij} for odd thetas and $\vartheta_{i,jkl}, \vartheta_\emptyset$ agrees with the notation we used for theta characteristics on curves of genus 3. For example, any set $\vartheta_{18}, \ldots, \vartheta_{78}$ defines an Aronhold set. Or, a syzygetic tetrad corresponds to four chords forming a spatial quadrangle, for example, $\overline{p_1p_3}, \overline{p_2p_4}, \overline{p_2p_3}, \overline{p_1p_4}$.

Here, is another application of Cayley octads.

Proposition 6.3.8. *There are* 1008 *azygetic hexads of bitangents of C such that their* 12 *points of contact lie on a cubic.*

Proof Let ℓ_1, ℓ_2, ℓ_3 be an azygetic triad of bitangents. The corresponding odd theta characteristics add up to $K_C + \vartheta$, where ϑ is an even theta characteristic. Let O be the Cayley octad corresponding to the net of quadrics for which C is the Hessian curve and let $S \subset \mathbb{P}^3 = |K_C + \vartheta|^\vee$ be the corresponding sextic model of C. We know that the restriction map

$$|\mathcal{O}_{\mathbb{P}^3}(2)| \to |\mathcal{O}_S(2)| = |\mathcal{O}_C(3K_C)| = |\mathcal{O}_{\mathbb{P}^2}(3)|$$

is a bijection. We also know that the double planes in $|\mathcal{O}_{\mathbb{P}^3}(2)|$ are mapped to contact cubics corresponding to ϑ. The cubic curve $\ell_1 + \ell_2 + \ell_3$ is one of them. Using the interpretation of bitangents as chords of the Cayley octad given in Theorem 6.3.5, we see that the union of the three chords corresponding to ℓ_1, ℓ_2, ℓ_3 cut out on S six coplanar points.This means that the three chords span a plane in $\mathbb{P}^3$. Obviously, the chords must be of the form $\overline{q_iq_j}, \overline{q_iq_k}, \overline{q_jq_k}$, where $1 \le i < j < k \le 8$. The number of such triples is $\binom{8}{3} = 56$. Fixing such a triple of chords, we can find $\binom{5}{3} = 10$ triples disjoint from the fixed one. The sum of the six corresponding odd theta characteristics is equal to $3K$, and hence, the points of contact are on a cubic. We can also see it by using the determinantal identity (hesbord 4.22). Other types of azygetic hexads can be found by using the previous Remark. □

Altogether we find (see SalmonCurves [652]) the following possible types of such hexads.

- 280 of type $(12, 23, 31, 45, 56, 64)$;
- 168 of type $(12, 34, 35, 36, 37, 38)$;
- 560 of type $(12, 13, 14, 56, 57, 58)$.

Recall that the three types correspond to three orbits of the permutation group $\mathfrak{S}_8$ on the set of azygetic hexads whose points of contact are on a cubic. Note that not every azygetic hexad has this property. For example, a subset of an Aronhold set does not have this property.

For completeness sake, let us give the number of non-azygetic hexads whose points of contact are on a cubic. The number of them is equal 5040. Here, is the list.

- 840 of type $(12, 23, 13, 14, 45, 15)$;
- 1680 of type $(12, 23, 34, 45, 56, 16)$;
- 2520 of type $(12, 34, 35, 36, 67, 68)$.

SS:6.3.3

6.3.3 Seven points in the plane

Let $\mathcal{P} = \{p_1, \ldots, p_7\}$ be a set of seven distinct points in $\mathbb{P}^2$. We assume that
star the points satisfy the following conditions:

(∗) No three points are collinear, and no six lies on a conic.

Consider the linear system L of cubic curves through these points. The conditions on the points imply that L is of dimension 2 and each member of L is an irreducible cubic. A subpencil in L has two base points outside the base locus of L. The line spanned by these points (or the common tangent if these points coincide) is a point in the dual plane $\mathbb{P}(E)$. This allows us to identify the net L with the plane $\mathbb{P}^2 = |E|$ where the seven points lie. Nets of curves with this special property are *Laguerre nets* (see Theorem thm:laguerre 7.2.16).

post **Proposition 6.3.9.** *The rational map $f : |E| \dashrightarrow \mathbb{P}(E)$ given by the linear system L is of degree 2. It extends to a regular degree 2 finite map $\pi : X \to \mathbb{P}(E)$,* *where X is the blow-up of the set $\mathcal{P}$. The branch curve of ϕ is a nonsingular plane quartic C in $\mathbb{P}(E)$. The ramification curve R is the proper transform of a curve $B \subset |E|$ of degree 6 with double points at each p_i. Conversely, given a nonsingular plane quartic C, the double cover of $\mathbb{P}^2$ ramified over C is a nonsingular surface isomorphic to the blow-up of 7 points $p_1, \dots, p_7$ in the plane satisfying the condition above.*

The following diagram illustrates this proposition:

$$\begin{array}{ccc} & X & \\ {}^{\sigma}\swarrow & & \searrow^{\pi} \\ |E| \cong \mathbb{P}^2 & \overset{f}{\dashrightarrow} & \mathbb{P}^2 \cong \mathbb{P}(E) \end{array}$$

We postpone the proof of this Proposition until Section CAG-2:S:8.7 8.7. The surface X is a del Pezzo surface of degree 2 .

Following our previous notation, we denote the plane $L^\vee$ by $|E|$ for some vector space E of dimension 3. Thus, L can be identified with $\mathbb{P}(E)$. Let $\sigma : X \to \mathbb{P}^2$ be the blowing-up map. The curves $E_i = \sigma^{-1}(p_i)$ are exceptional curves of the first kind, (-1)-curves for short. We will often identify L with its proper transform in S equal to

$$|-K_X| = |3h - E_1 - \cdots - E_7|,$$

where $h = c_1(\sigma^*\mathcal{O}_{\mathbb{P}^2}(1))$ is the divisor class of the pre-image of a line in $\mathbb{P}^2$.

The pre-image of a line $\ell \subset |E|$ in $\mathbb{P}(E) = L$ is a nonsingular member of L if and only if ℓ intersects transversally C. In this case, it is a double cover of ℓ branched over $\ell \cap C$. The pre-image of a tangent line is a singular member, the singular points lie over the points of contact. Thus, the pre-image of a general tangent line is an irreducible cubic curve with a singular point at $\sigma(R)$. The pre-image of a bitangent is a member of $|-K_X|$ with two singular points (they may coincide if the bitangent is an inflection bitangent). It is easy to see that its image in the plane is either an irreducible cubic F_i with a double point at p_i or the union of a line $\overline{p_i p_j}$ and the conic K_{ij} passing through the point $p_k, k \neq i, j$. In this way we can account for all 28 = 7+21 bitangents. If we denote the bitangents corresponding to F_i by ℓ_{i8} and the bitangents corresponding to $\overline{p_i p_j} + K_{ij}$ by ℓ_{ij}, we can accommodate the notation of bitangents by subsets of cardinality 2 of $[1, 8]$.

The next proposition states that this notation agrees with the previous notation.

Proposition 6.3.10. *The images of the cubic curves F_i under the rational map*

$f : |E| \dashrightarrow \mathbb{P}(E)$ given by the linear system L of cubics through the seven points $p_1, \dots, p_7$ is an Aronhold set of bitangents to the branch curve C.

Proof Let $E_i + E_i'$ be the full transform of F_i under the blowing-up morphism $\sigma : X \to |E|$, where $\sigma(E_i) = E_i'$ and $\pi(E_i + E_i')$ is a bitangent ℓ_i of B. Let $E_i \cap E_i' = \{a_i, b_i\}$and $a_i', b_i' \in \mathbb{P}^2$ be their images under the map ϕ. Since $\pi^{-1}(\ell_i) = E_i + E_i'$, the ramification curve R of π passes through a_i, b_i. Suppose the tangency points $a_i' b_i'$ lie on a conic K. Let a, b be the residual pair of points in the intersection $K \cap B$. Then, $a + b \in |2K_B - \sum_{i=1}^3 (a_i' + b_i')|$ Since $2(a_i' + b_i') \in K_B|$, we obtain that $a + b \in |K_B|$, hence a, b are the tangency points of some bitangent ℓ of B. The pre-image of ℓ in X splits into two (-1)-curves $E + \gamma(E)$. We have

$$2 = 2\ell \cdot \ell_i = (E + E') \cdot (E_i + E_i') = E + \gamma(E)) \cdot (E_i + \gamma(E_i)) = 2(E \cdot E_i) + 2(E' \cdot E_i).$$

Replacing E with $\gamma(E)$, and reordering the set $\{E_1, \dots, E_7\}$, if needed, we may assume that $E \cdot E_i = 1, i = 1, \dots, k$ and $E \cdot E_i = 0, i > k$, where $k \leq 3$. But then $E = ae_0 - e_1 - \cdots - e_k$, and $E^2 = -1 = a^2 - k$. Since $0 < k \leq 3$, a^2 cannot be a square. this contradiction proves the lemma. □

Let $\ell' \in |h|$. Its image $\pi(\ell')$ in $|L|^\vee = |E|$ is a plane cubic G. The pre-image of G in X is the union of ℓ' and a curve ℓ'' in the linear system $3(3h - \sum E_i) - h| = |8h - 3\sum E_i|$. The curves ℓ' and ℓ'' intersect at six points. Since the cubic G splits in the cover π, it must touch the branch curve C at each intersection point with it. Thus, it is a contact cubic and hence the divisor $D = \phi(\ell' \cap \ell'')$ belongs to $|K_C + \vartheta|$ for some even theta characteristic ϑ. This shows that ℓ' cuts out in R the divisor from the linear system $|K_R + \vartheta|$. In other words, the inverse of the isomorphism $\pi|R : R \to C$ is given by a 2-dimensional linear system contained in $|K_C + \vartheta|$. The image B of R in the plane $|L|$ is a projection of a sextic model of C in $\mathbb{P}^3$ defined by the linear system $|K_C + \vartheta|$.

Let us record what we have found so far.

Proposition 6.3.11. *A choice of seven unordered points $p_1, \dots, p_7$ in the plane $L = |E|$ satisfying condition (*) from above defines a nonsingular plane quartic C in the dual plane $\mathbb{P}(E)$, and an even theta characteristic ϑ on C. The linear system $|L|$ of cubic curves through the seven points maps each its member with a double point at p_i to a bitangent ϑ_i of C. The seven bitangents $\vartheta_1, \dots, \vartheta_7$ form an Aronhold set of bitangents. The the map $\sigma : R \cong C \to B = \sigma(R)$ is given by $|K_C + \vartheta|$, where ϑ is an even theta characteristic. The set $(\vartheta_1, \dots, \vartheta_7, \vartheta)$ is a fundamental set of theta characteristics on C.*

We skip the proof of the last assertion and leave it to the reader.

Let us now see the reverse construction of a set of seven points defined by a pair (C, ϑ) as above.

Let N_ϑ be the linear system of quadrics in $|K_C + \vartheta| \cong \mathbb{P}^3$ defined by an even theta characteristic ϑ on C. Let $X \to \mathbb{P}^3$ be the blow-up of the Cayley octad $\mathrm{O} = \{q_1, \ldots, q_8\}$ of its base points. The linear system N_ϑ defines an elliptic fibration $f : X \to \mathsf{N}_\vartheta^\vee$. If we identify N_ϑ with $|E|$ by using the determinantal representation $\phi : |E| \to |\mathcal{O}_{\mathbb{P}^3}(2)|$, then N_ϑ can be identified with $\mathbb{P}(E)$. The images of fibers of f in $\mathbb{P}^3$ are quartic curves passing through O. The projection map from $\mathbb{P}^3$ from $q_8 \in \mathrm{O}$ is defined by a 2-dimensional linear subsystem H of $|K_C + \vartheta|$. The projections of quartic curves are cubic curves passing through the set $\mathcal{P} = \{p_1, \ldots, p_7\}$, where p_i is the projection of q_i. In this way we get a set of seven points that define (C, ϑ). The Aronhold set of bitangents ϑ_{i8} obtained from the Cayley octad corresponds to the Aronhold set $\vartheta_1, \ldots, \vartheta_7$ defined by the cubic curves F_i. They are the projections of the rational cubic curve R_i which together with the line $\overline{q_j q_8}$ form the base locus of a pencil of quadrics contained in N_ϑ.

So, we have proved the converse.

prop:6.3.11 **Proposition 6.3.12.** *A nonsingular plane quartic curve $C \subset |E|$ together with an even theta characteristic defines a unique Cayley octad $\mathrm{O} \subset |K_C + \vartheta| = \mathbb{P}^3$ such that the linear system of quadrics through* O *is the linear system of quadrics associated to (C, ϑ). The projection of* O *from one of its points to $\mathbb{P}^2$ plus a choice of an isomorphism $\mathbb{P}^2 \cong |E^\vee|$ defines a net of cubics through seven points $p_1, \ldots, p_7$. The blow-up of the seven points is a del Pezzo surface and its anti-canonical linear system defines a degree 2 finite map $X \to |E|$ branched over C. The ramification curve R of the map is the projection of the image of C under the linear system $|K_C + \vartheta|$.*

Note that in this way we account for all $288 = 8 \times 36$ Aronhold sets of seven bitangents. They are defined by a choice of an even theta characteristic and a choice of a point in the corresponding Cayley octad. We also obtain the following.

Corollary 6.3.13. *The moduli space U_2^7 of projective equivalence classes of unordered seven points in the plane is birationally isomorphic to the moduli space $\mathcal{M}_3^{\mathrm{ar}}$ of curves of genus 3 together with an Aronhold set of bitangents. It is (birationally) a* $8 : 1$*-cover of the moduli space $\mathcal{M}_3^{\mathrm{ev}}$ of curves of genus 3 together with an even theta characteristic. The latter space is birationally isomorphic to the moduli space of projective equivalence classes of Cayley octads.*

Remark 6.3.14. Both of the moduli spaces $\mathcal{M}_3^{\mathrm{ar}}$ and $\mathcal{M}_3^{\mathrm{ev}}$ are known to be rational varieties. The rationality of U_2^7 was proven by P. Katsylo in [443]. The rationality of $\mathcal{M}_3^{\mathrm{ev}}$ P. Katsylo [445]. It is also known, and much easier to prove, that the moduli space $\mathcal{M}^{\mathrm{odd}}$ of curves of genus three with an odd theta characteristic is rational [31].

rmk:dianode *Remark* 6.3.15. The elliptic fibration $f : X \to \mathbb{P}(E)$ defined by the linear system N_ϑ has 8 sections corresponding to the exceptional divisors over the points q_j. Its discriminant locus consists of lines in $|E|$ tangent to C, that is, the dual curve $C^\vee$ of C. If we fix one section, say the exceptional divisor over q_8, then all nonsingular fibers acquire a group structure. The closure of the locus of nontrivial 2-torsion points is a smooth surface W in X. Its image in $\mathbb{P}^3$ is a surface of degree 6 with triple points at $q_1, \ldots, q_8$, called the *Cayley dianodal surface* [119, Art. 42]. It is a determinantal surface of degree six equal to the Jacobian surface of the linear system of quartic surfaces with double points at $q_1, \ldots, q_7$. The linear system of quartics defines a map $X \to \mathbb{P}^6$ whose image is the cone over a Veronese surface in a hyperplane. The map is a double cover onto the image. The exceptional divisor over q_8 is mapped to the vertex of the cone. The surface W is the ramification locus of this map. Its image in $\mathbb{P}^6$ is the complete intersection of the cone and a cubic hypersurface. It is a surface of degree 12 with 28 nodes, the images of the lines $\overline{q_iq_j}$. The surface W is a minimal surface of general type with $p_g = 3$ and $K_W^2 = 3$. It is birationally isomorphic to the quotient of a symmetric theta divisor in $\mathrm{Jac}(C)$ modulo the involution $x \mapsto -x$. All of this is discussed in [159] and [234].

There is another similar elliptic fibration over $\mathbb{P}(E)$. Consider the universal family of the net L:

$$\mathsf{U} = \{(x, F) \in |E| \times L : x \in F\}.$$

The fiber of the first projection $\pi_1 : \mathsf{U} \to X$ over a point $x \in X$ can be identified, via the second projection, with the linear subsystem $L(x) \subset L$ of curves passing through the point x. If $x \notin \mathcal{P}$, $L(x)$ is a pencil, otherwise, it is the whole L. The second projection $\pi_2 : \mathsf{U} \to L$ is an elliptic fibration, its fiber over the point $\{F\}$ is isomorphic to F. It has seven regular sections

$$s_i : L \to \mathsf{U}, \quad F \mapsto (p_i, F).$$

There is another natural rational section $s_8 : L \to \mathsf{U}$ defined as follows. We know from 3.3.2 that any g_2^1 on a nonsingular cubic curve F is obtained by projection from the coresidual point $p \in F$ to a line. Take a curve $F \in L$ and restrict L to F. This defines a g_2^1 on F, and hence defines the coresidual point $\mathfrak{c}_F$. The section s_8 maps F to $\mathfrak{c}_8$. Although the images S_i of the first sections are disjoint in U, the image S_8 of s_8 intersects each S_j, $j \neq 8$, at the point (p_j, F_j)

(in this case the g^1_2 on F_i has a base point p_i, which has to be considered as the coresidual point of F_j). The universal family U is singular because the net N_ϑ has base points. The singular points are the intersection points of the sections S_j and S_8, $j \neq 8$. The variety X is a small resolution of the singular points. The exceptional curves are the proper transforms of lines $\overline{q_j q_8}$.

SS:6.3.4

6.3.4 The Clebsch covariant quartic

Here, we shall specialize the Scorza construction in the case of plane quartic curves. Consider the following symmetric correspondence on $\mathbb{P}^2$

$$R = \{(x, y) \in \mathbb{P}^2 \times \mathbb{P}^2 : \mathrm{rank} P_{yx}(C) = 1\}.$$

We know that a cubic curve has a polar quadric of rank 1 if and only if it lies in the closure of the projective equivalence class of the Fermat cubic. Equivalently, a cubic curve $G = V(g)$ has this property if and only if the Aronhold invariant S vanishes on g. In this case, we write $\mathsf{S}(G) = 0$.

Consider the projection of R to one of the factors. It is equal to

$$\mathfrak{C}(C) := \{x \in \mathbb{P}^2 : \mathsf{S}(P_x(C)) = 0\}.$$

By symmetry of polars, if $x \in \mathfrak{C}(C)$, then $R(x) \subset \mathfrak{C}(C)$. Thus, $S = \mathfrak{C}(C)$ comes with a symmetric correspondence

$$R_C := \{(x, y) \in S \times S : \mathrm{rank} P_{xy}(C) = 1\}.$$

Since the Aronhold invariant S is of degree 4 in coefficients of a ternary quartic, we obtain that $\mathfrak{C}(C)$ is a quartic curve or the whole $\mathbb{P}^2$. The case when $\mathfrak{C}(C) = \mathbb{P}^2$ happens, for example, when C is a Fermat quartic. For any point $x \in \mathbb{P}^2$ and any vertex y of the polar triangle of the Fermat cubic $P_x(C)$, we obtain $P_{yx}(C) = \mathbb{P}^2$.

The assignment $C \to \mathfrak{C}(C)$ lifts to a covariant

$$\mathfrak{C} : S^4(E^\vee) \to S^4(E^\vee)$$

which we call the *Scorza covariant* of quartics. We use the same notation for the associated rational map

$$\mathfrak{C} : |\mathcal{O}_{\mathbb{P}^2}(4)| \dashrightarrow |\mathcal{O}_{\mathbb{P}^2}(4)|.$$

sc *Example* 6.3.16. Assume that the equation of C is given in the form

$$at_0^4 + bt_1^4 + ct_2^4 + 6ft_1^2t_2^2 + 6gt_0^2t_2^2 + 6ht_0^2t_1^2 = 0.$$

Then, the explicit formula for the Aronhold invariant S (see SalmonCurves [652], p. 270) gives

$$\mathfrak{C}(C) := a't_0^4 + b't_1^4 + c't_2^4 + 6f't_1^2t_2^2 + 6g't_0^2t_2^2 + 6h't_0^2t_1^2 = 0,$$

where

$$a' = 6e^2h^2, \quad b' = 6h^2f^2, \quad c' = 6f^2g^2,$$
$$d' = bcgh - f(bg^2 + ch^2) - ghf^2,$$
$$e' = acfh - g(ch^2 + af^2) - fhg^2,$$
$$h' = abfg - h(af^2 + bg^2) - fgh^2.$$

For a general f the formula for $\mathfrak{C}$ is too long.

Consider the pencil of quartics defined by the equation

$$t_0^4 + t_1^4 + t_2^4 + 6\alpha(t_0^2t_1^2 + t_0^2t_2^2 + t_1^2t_2^2) = 0, \quad \alpha \neq 0. \tag{6.28}$$ pencil

Then, $\mathfrak{C}(C)$ is given by the equation

$$t_0^4 + t_1^4 + t_2^4 + 6\beta(t_0^2t_1^2 + t_0^2t_2^2 + t_1^2t_2^2) = 0,$$

where

$$6\beta\alpha^2 = 1 - 2\alpha - \alpha^2.$$

We find that $\mathfrak{C}(C) = C$ if and only if α satisfies the equation

$$6\alpha^3 + \alpha^2 + 2\alpha - 1 = 0.$$

One of the solutions is $\alpha = 1/3$; it gives a double conic. Two other solutions are $\alpha = \frac{1}{4}(-1 \pm \sqrt{-7})$. They give two curves isomorphic to the *Klein curve* $V(t_0^3t_1 + t_1^3t_2 + t_2^3t_0)$ with 168 automorphisms. We will discuss this curve later in this chapter.

We will be interested in the open subset of $|\mathcal{O}_{\mathbb{P}^2}(4)|$ where the map $\mathfrak{C}$ is defined and its values belong to the subset of nonsingular quartics.

P6.4.1 **Proposition 6.3.17.** *Suppose $\mathfrak{C}(C)$ is a nonsingular quartic. Then, C is either non-degenerate or has a unique irreducible apolar conic.*

Proof Suppose C does not satisfy the assumption. Then, C admits either a pencil of apolar conics or one reducible apolar conic. In any case, there is a reducible apolar conic. Hence, there exist two points x, y such that $P_{xy}(C) = \mathbb{P}^2$. This implies that $P_x(C)$ is a cone with triple point y. It follows from the explicit formula for the Aronhold invariant S that the curve $P_x(C)$ is a singular point in the closure of the variety of Fermat cubics. Thus, the image of the polar map $x \mapsto P_x(C)$ passes through the singular point. The pre-image of this point under the polar map is a singular point of C. □

Theorem 6.3.18. *Let $C = V(f)$ be a general plane quartic. Then, $S = \mathfrak{C}(C)$ is a nonsingular curve and there exists an even theta characteristic ϑ on S such* *that R_C coincides with the Scorza correspondence R_ϑ on S. Every nonsingular S together with an even theta characteristic is obtained in this way.*

Proof To show that $\mathfrak{C}(C)$ is nonsingular for a general quartic, it suffices to give one example when it happens. The Klein curve from Example 6.3.16 will do.

Let S be a nonsingular quartic and R_ϑ be the Scorza correspondence on S defined by a theta characteristic ϑ. It defines the Scorza quartic C. It follows immediately from (5.41) in the proof of Theorem 5.5.17 that for any point $(x, y) \in R_\vartheta$ the second polar $P_{x,y}(C)$ is a double line (in notation in the proof of the loc. cit. Theorem, $(x, y) = (x, x_i)$ and $V(t_i^2)$ is the double line). This shows that $P_x(C)$ is a Fermat cubic, and hence $\mathfrak{C}(C) = S$. Thus, we obtain that the Clebsch covariant $\mathfrak{C}$ is a dominant map whose image contains nonsingular quartics. Moreover, it inverts the Scorza rational map which assigns to (S, ϑ) the Scorza quartic. Thus, a general quartic curve C is realized as the Scorza quartic for some (S, ϑ), the correspondence R_C coincides with R_ϑ and $S = \mathfrak{C}(C)$. □

Suppose C is plane quartic with nonsingular $S = \mathfrak{C}(C)$. Suppose $R_C = R_\theta$ for some even theta characteristic on S. Let C' be the Scorza quartic assigned to (S, ϑ). Then, for any $x \in S$, $P_x(C) = P_x(C')$. Since S spans $\mathbb{P}^2$, this implies that $C = C'$. The generality condition in order that $R_C = R_\vartheta$ happens can be made more precise.

Proposition 6.3.19. *Suppose $S = \mathfrak{C}(C)$ satisfies the following conditions*

- *S is nonsingular;*
- *the Hessian of C is irreducible;*
- *S does not admit nonconstant maps to curves of genus 1 or 2.*

Then, $R_C = R_\theta$ for some even theta characteristic θ and C is the Scorza quartic associated to (S, θ).

Proof It suffices to show that R_C is a Scorza correspondence on S. Obviously, R_C is symmetric. As we saw in the proof of Proposition 6.3.17, the first condition shows that no polar $P_x(C), x \in S$, is the union of three concurrent lines. The second condition implies that the Steinerian of C is irreducible and hence does not contain S. This shows that, for any general point $x \in S$, the first polar $P_x(C)$ is projectively equivalent to a Fermat cubic. This implies that R_C is of type $(3, 3)$. Since C is nonsingular, $P_{x^2}(C)$ is never a double line or $\mathbb{P}^2$. Thus, R_C has no united points.

By Proposition 5.5.11, it remains for us to show that R_C has valence -1. Take a general point $x \in S$. The divisor $R_C(x)$ consists of the three vertices of its unique polar triangle. For any $y \in R_C(x)$, the side $\lambda = V(l)$ opposite to y

is defined by $P_y(P_x(C)) = P_x(P_y(C)) = V(l^2)$. It is a common side of the polar triangles of $P_x(C)$ and $P_y(C)$. We have $\ell \cap S = y_1 + y_2 + x_1 + x_2$, where $R_C(x) = \{y, y_1, y_2\}$ and $R_C(y) = \{x, x_1, x_2\}$. This gives

$$y_1 + y_2 + x_1 + x_2 = (R_C(x) - x) + (R_C(y) - y) \in |K_S|.$$

Consider the map $\alpha : S \to \mathrm{Pic}^2(S)$ given by $x \to [R(x) - x]$. Assume R_C has no valence, i.e., the map α is not constant. If we replace in the previous formula y with y_1 or y_2, we obtain that $\alpha(y) = \alpha(y_1) = \alpha(y_2) = K_S - \alpha(x)$. Thus, $\alpha : S \to \alpha(S) = S'$ is a map of degree ≥ 3. It defines a finite map of degree ≥ 3 from S to the normalization $\tilde{S}'$ of S'. Since a rational curve does not admit non-constant maps to an abelian variety, we obtain that $\tilde{S}'$ is of positive genus. By assumption, this is impossible. Hence, R_C has valence $v = -1$.

□

Let $|\mathcal{O}_{\mathbb{P}^2}(4)|^{\mathrm{snd}}$ be the open subset of plane quartics C such that $\mathfrak{C}(C)$ is a nonsingular quartic and the correspondence R_C is a Scorza correspondence R_ϑ. The Clebsch covariant defines a regular map

$$\tilde{\mathfrak{C}} : |\mathcal{O}_{\mathbb{P}^2}(4)|^{\mathrm{snd}} \to \mathcal{T}C_4^{\mathrm{ev}}, \quad C \mapsto (\mathfrak{C}(C), R_C). \tag{6.29}$$

tildecleb

By Proposition 5.1.2 the variety $\mathcal{T}C_4^{\mathrm{ev}}$ is an irreducible cover of degree 36 of the variety $|\mathcal{O}_{\mathbb{P}^2}(4)|$ of nonsingular quartics. By Proposition 5.1.2 the variety $\mathcal{T}C_4^{\mathrm{ev}}$ is an irreducible cover of degree 36 of the variety $|\mathcal{O}_{\mathbb{P}^2}(4)|$ of nonsingular quartics. The Scorza map defines a rational section of $\tilde{\mathfrak{C}}$. Since both the source and the target of the map are irreducible varieties of the same dimension, this implies that (6.29) is a birational isomorphism.

Passing to the quotients by PGL(3), we obtain the following:

scorzatheorem

Theorem 6.3.20. *Let $\mathcal{M}_3^{\mathrm{ev}}$ be the moduli space of curves of genus three together with an even theta characteristic. The birational map $S : |\mathcal{O}_{\mathbb{P}^2}(4)| \to \mathcal{T}C_4$ has the inverse defined by assigning to a pair (C, ϑ) the Scorza quartic. It induces a birational isomorphism*

$$\mathcal{M}_3 \cong \mathcal{M}_3^{\mathrm{ev}}.$$

The composition of this map with the forgetting map $\mathcal{M}_3^{\mathrm{ev}} \to \mathcal{M}_3$ is a rational self-map of $\mathcal{M}_3$ of degree 36.

Remark 6.3.21. The corollary generalizes to genus 3 the fact that the map from the space of plane cubics $|\mathcal{O}_{\mathbb{P}^2}(3)|$ to itself defined by the Hessian is a birational map to the cover $|\mathcal{O}_{\mathbb{P}^2}(3)|^{\mathrm{ev}}$, formed by pairs (X, ϵ), where ϵ is a nontrivial 2-torsion divisor class (an even characteristic in this case). Note that the Hessian covariant is defined similarly to the Clebsch invariant. We compose the polarization map $V \times S^3(E^\vee) \to S^2(E^\vee)$ with the discriminant invariant $S^2(E^\vee) \to \mathbb{C}$.

6.3.5 Clebsch and Lüroth quartics

SS:6.3.5

Since five general points in the dual plane lie on a singular quartic (a double conic), a general quartic does not admit a polar pentagon, although the count of constants suggests that this is possible. This remarkable fact was first discovered by J. Lüroth in 1868. Suppose a quartic C admits a polar pentagon $\{[l_1], \ldots, [l_5]\}$ (or the *polar pentalateral* $V(l_1), \ldots, V(l_5)$). Let $Q = V(q)$ be a conic in $\mathbb{P}(E)$ passing through the points $[l_1], \ldots, [l_5]$. Then, $q \in \mathrm{AP}_2(f)$. The space $\mathrm{AP}_2(f) \neq \{0\}$ if and only if $\det \mathrm{Cat}_2(f) = 0$. Thus, the set of quartics admitting a polar pentagon is the locus of zeros of the catalecticant invariant on the space $\mathbb{P}(S^4(E^\vee))$. It is a polynomial of degree 6 in the coefficients of a ternary form of degree 4.

Definition 6.3.22. *A plane quartic admitting a polar pentagon is called a* Clebsch quartic.

ndeg1

Lemma 6.3.23. *Let $C = V(f)$ be a Clebsch quartic. The following properties are equivalent.*

(i) C admits polar pentagon $\{[l_1], \ldots, [l_5]\}$ such that $l_1^2, \ldots, l_5^2 \in S^2(E^\vee)$ are linearly independent;
(ii) $\dim \mathrm{AP}_2(f) = 1$*;*
(iii) for any polar pentagon $\{[l_1], \ldots, [l_5]\}$ of C, $l_1^2, \ldots, l_5^2$ are linearly independent;
(iv) for any polar pentagon $\{[l_1], \ldots, [l_5]\}$ of C, no four of the points $[l_i]$ are collinear.

Proof (i) ⇒ (ii) For any $\psi \in \mathrm{AP}_2(f)$, we have

$$0 = D_\psi(f) = \sum D_\psi(l_i^2) l_i^2.$$

Since l_i^2 are linearly independent, this implies $D_\psi(l_i^2) = 0, i = 1, \ldots, 5$. This means that $V(\psi)$ is a conic passing through the points $[l_1], \ldots, [l_5]$. Five points in the plane determine unique conic unless four of the points are collinear. It is easy to see that in this case the quadratic forms $l_1^2, \ldots, l_5^2$ are linearly dependent. Thus, $\dim \mathrm{AP}_2(f) = 1$.

(ii) ⇒ (ii) Suppose $\{[l_1], \ldots, [l_5]\}$ is a polar pentagon of C with linearly dependent $l_1^2, \ldots, l_5^2$. Then, there exist two linearly independent functions ψ_1, ψ_2 in $S^2(E^\vee)^\vee = S^2(E)$ vanishing at $l_1^2, \ldots, l_5^2$. They are apolar to f, contradicting the assumption.

(iii) $\Rightarrow$ (iv) Suppose $\{[l_1], \ldots, [l_4]\}$ are collinear. Then, we can choose coordinates to write $l_1 = t_0, l_1 = t_1, l_3 = at_0+bt_1, l_4 = ct_0+dt_1$. Taking squares, we see that the five l_i^2 are linear combinations of four forms $t_0^2, t_1^2, t_0t_1, l_5$. This contradicts the assumption.

(iv) $\Rightarrow$ (i) Let $\{[l_1], \ldots, l_5]\}$ be a polar pentagon with no four collinear points. It is easy to see that it implies that we can choose four of the points such that no three among them are collinear. Now, change coordinates to assume that the corresponding quadratic forms are $t_0^2, t_1^2, t_2^2, a(t_0+t_1+t_2)^2$. Suppose $l_1^2, \ldots, l_5^2$ are linearly dependent. Then, we can write

$$l_5^2 = \alpha_1 t_0^2 + \alpha_2 t_1^2 + \alpha_3 t_2^2 + \alpha_4 (t_0+t_1+t_2)^2.$$

If two of the coefficients α_i are not zero, then the quadratic form in the right-hand side is of rank ≥ 2. The quadratic form in the left-hand side is of rank 1. Thus, three of the coefficients are zero, but the two of the points $[l_i]$ coincide. This contradiction proves the implication. □

Definition 6.3.24. *A Clebsch quartic is called* weakly nondegenerate *if it satisfies one of the equivalent conditions from the previous Lemma. It is called nondegenerate if the unique polar conic is irreducible.*

This terminology is somewhat confusing since a quartic was earlier called nondegenerate if it does not admit an apolar conic. I hope the reader can live with this.

It follows immediately from the definition that each polar pentalateral of a nondegenerate Clebsch quartic consists of five sides, no three of which pass through a point (a *complete pentalateral*). Considered as a polygon in the dual plane, this means that no three vertices are collinear. On the other hand, the polar pentalateral of a weakly nondegenerate Clebsch quartic may contain one or two triple points.

Let $C = V(\sum l_i^4)$ be a Clebsch quartic. If x lies in the intersection of two sides $V(l_i)$ and $V(l_j)$ of the polar pentalateral, then

$$P_x(C) = V(\sum_{k \neq i,j} l_k(x) l_k^3),$$

hence it lies in the closure of the locus of Fermat cubics. This means that the point x belongs to the quartic $\mathfrak{C}(C)$. When C is a general Clebsch quartic, $\mathfrak{C}(C)$ passes through each of 10 vertices of the polar complete pentalateral. In other words, $\mathfrak{C}(C)$ is a *Darboux plane curve* of degree 4 in sense of the definition below.

Let $\ell_1, \ldots, \ell_N$ be a set of N distinct lines in the planes, the union of which is called a *N-lateral*, or an *arrangement of lines*. A point of intersection x_{ij}

of two of the lines ℓ_i and ℓ_j is called a *vertex* of the N-lateral. The number of lines intersecting at a vertex is called the *multiplicity* of the vertex. An N-lateral with all vertices of multiplicity 2 is called a *complete N-lateral* (or a *general arrangement*). Considered as a divisor in the plane, it is a normal crossing divisor. The dual configuration of an N-lateral (the dual arrangement) consists of a set of N points corresponding to the lines and a set of lines corresponding to points. The number of points lying on a line is equal to the multiplicity of the line considered as a vertex in the original N-lateral.

Let $\mathcal{J}$ be the ideal sheaf of functions vanishing at each vertex x_{ij} with multiplicity $\geq \nu_{ij} - 1$, where ν_{ij} is the multiplicity of x_{ij}. A nonzero section of $\mathcal{J}(k)$ defines a plane curve of degree k that has singularities at each x_{ij} of multiplicity $\geq \nu_{ij} - 1$.

barth **Lemma 6.3.25.** *Let $\mathcal{A} = \{\ell_1, \ldots, \ell_N\}$ be an N-lateral. Then*

$$h^0(\mathbb{P}^2, \mathcal{J}(N-1)) = N.$$

Proof Let ℓ be a general line in the plane. It defines an exact sequence

$$0 \to \mathcal{J}(N-2) \to \mathcal{J}(N-1) \to \mathcal{J}(N-1) \otimes \mathcal{O}_\ell \to 0.$$

Since the divisor of zeros of a section of $\mathcal{J}(N-2)$ contains the divisor $\ell_i \cap (\sum_{j\neq i} \ell_j)$ of degree $N-1$, it must be the whole ℓ_i. Thus, $h^0(\mathcal{J}(N-2)) = 0$. Since $\mathcal{J}(N-1) \otimes \mathcal{O}_\ell \cong \mathcal{O}_{\mathbb{P}^1}(N-1)$, we have $h^0(\mathcal{J}(N-1) \otimes \mathcal{O}_\ell) = N$. This shows that $h^0(\mathcal{J}(N-1)) \leq N$. On the other hand, we can find N linear independent sections by taking the products f_j of linear forms defining $\ell_i, j \neq i$. This proves the equality. □

Definition 6.3.26. *A* Darboux curve *of degree $N-1$ is a plane curve defined by a nonzero section of the sheaf $\mathcal{J}(N-1)$ for some N-lateral of lines in the plane. A Darboux curve of degree 4 is called a* Lüroth quartic curve.

Obviously, any conic (even a singular one) is a Darboux curve. The same is true for cubic curves. The first case where a Darboux curve must be a special curve is the case $N = 5$.

It follows from the proof of Lemma 6.3.25 that a Darboux curve can be given by an equation

$$\sum_{i=1}^{N} \prod_{j\neq i} l_j = \prod_{i=1}^{N} l_i (\sum_{i=1}^{N} \frac{1}{l_i}) = 0 \qquad (6.30)$$ darboux

where $\ell_i = V(l_i)$.

From now on, we will be dealing with the case $N = 5$, i.e with Lüroth quartics. The details for the next computation can be found in the original paper by Lüroth [499], p. 46.

Lemma 6.3.27. *Let $C = V(\sum l_i^2)$ be a Clebsch quartic in $\mathbb{P}^2 = |E|$. Choose a volume form on E to identify $l_i \wedge l_j \wedge l_k$ with a number $|l_i l_j l_k|$. Then,*

$$\mathfrak{C}(C) = V(\sum_{s=1}^{5} k_s \prod_{i \neq s} l_i),$$

where

$$k_s = \prod_{i<j<k, r \notin \{i,j,k\}} |l_i l_j l_k|.$$

Proof This follows from the known symbolic expression of the Aronhold invariant

$$\mathsf{S} = (abc)(abd)(acd)(bcd).$$

If we polarize $4D_a(f) = \sum l_i(a) l_i^3$, we obtain a tensor equal to the tensor $\sum l_i(a) l_i \otimes l_i \otimes l_i \in (E^\vee)^{\otimes 3}$. The value of S is equal to the sum of the determinants $l_i(a) l_j(a) l_k(a) |l_i l_j l_k|$. When $[a]$ runs $\mathbb{P}^2$, we get the formula from the assertion of the Lemma. □

Looking at the coefficients $k_1, \ldots, k_5$, we observe that

- $k_1, \ldots, k_5 \neq 0$ if and only if C is nondegenerate;
- two of the coefficients $k_1, \ldots, k_5$ are equal to zero if and only if C is weakly degenerate and the polar pentalateral of C has one triple point;
- three of the coefficients $k_1, \ldots, k_5$ are equal to zero if and only if C is weakly nondegenerate and the polar pentalateral of C has two triple points;
- $\mathfrak{C}(C) = \mathbb{P}^2$ if the polar pentalateral has a point of multiplicity 4.

It follows from this observation, that a Lüroth quartic of the form $\mathfrak{C}(C)$ is always reducible if C admits a degenerate polar pentalateral. Since $\mathfrak{C}(C)$ does not depend on a choice of a polar pentalateral, we also see that all polar pentalaterals of a weakly nondegenerate Clebsch quartic are complete pentalaterals (in the limit they become generalized polar 5-hedra).

Thus, we see that, for any Clebsch quartic C, the quartic $\mathfrak{C}(C)$ is a Lüroth quartic. One can prove that any Lüroth quartic is obtained in this way from a unique Clebsch quartic (see [235]).

Let $C = V(f)$ be a nondegenerate Clebsch quartic. Consider the map

$$c : \mathrm{VSP}(f,5)^o \to |\mathcal{O}_{\mathbb{P}^2}(2)| \tag{6.31}$$

defined by assigning to $\{\ell_1,\dots,\ell_5\} \in VSP(f,5)^o$ the unique conic passing through these points in the dual plane. This conic is nonsingular and is apolar to C. The fibers of this map are polar pentagons of f inscribed in the apolar conic. We know that the closure of the set of Clebsch quartics is defined by one polynomial in coefficients of quartic, the catalecticant invariant. Thus, the varierty of Clebsch quartics is of dimension 13.

Let $\mathcal{E}^5$ be the variety of 5-tuples of distinct nonzero linear forms on E. Consider the map $\mathcal{E}^5 \to |\mathcal{O}_{\mathbb{P}^2}(4)|$ defined by $(l_1,\dots,l_5) \mapsto V(l_1^4+\cdots+l_5^4)$. The image of this map is the hypersurface of Clebsch quartics. A general fiber must be of dimension $15-13=2$. However, scaling the l_i by the same factor, defines the same quartic. Thus, the dimension of the space of all polar pentagons of a general Clebsch quartic is equal to 1. Over an open subset of the hypersurface of Clebsch quartics, the fibers of c are irreducible curves.

old **Proposition 6.3.28.** *Let $C = V(f)$ be a nondegenerate Clebsch quartic and Q be its apolar conic. Consider any polar pentagon of C as a positive divisor of degree 5 on Q. Then,* $\mathrm{VSP}(f,5)^o$ *is an open non-empty subset of a g_5^1 on Q.*

Proof Consider the correspondence

$$X = \{(x,\{\ell_1,\dots,\ell_5\}) \in Q \times \mathrm{VSP}(f,5)^o : x = [l_i] \text{ for some } i = 1,\dots,5\}.$$

Let us look at the fibers of the projection to Q. Suppose we have two polar pentagons of f with the same side $[l]$. We can write

$$f - l^4 = l_1^4 + \cdots + l_4^4,$$

$$f - \lambda l^4 = m_1^4 + \cdots + m_4^4.$$

For any $\psi \in S^2(E)$ such that $\psi(l_i) = 0, i = 1,\dots,4$, we get $D_\psi(f) = 12\psi(l)l^2$. Similarly, for any $\psi' \in S^2(E)$ such that $\psi'(m_i) = 0, i = 1,\dots,4$, we get $D_{\psi'}(f) = 12\lambda\psi'(l)l^2$. This implies that $V(\psi(l)\psi' - \psi'(l)\psi)$ is an apolar conic to C. Since C is a general Clebsch quartic, there is only one apolar conic. The set of $V(\psi)$'s is a pencil with base points $V(l_i)$, the set of $V(\psi')$ is a pencil with base points $V(l_i)$. This gives a contradiction unless the two pencils coincide. But then their base points coincide and the two pentagons are equal. This shows that the projection to Q is a one-to-one map. In particular, X is an irreducible curve.

Now, it is easy to finish the proof. The set of degree 5 positive divisors on $Q \cong \mathbb{P}^1$ is the projective space $|\mathcal{O}_{\mathbb{P}^1}(5)|$. The closure $\mathcal{P}$ of our curve of polar pentagons lies in this space. All divisors containing one fixed point in their support form a hyperplane. Thus, the polar pentagons containing one common side $[l]$ correspond to a hyperplane section of $\mathcal{P}$. Since we know that there is only one such pentagon and we take $[l]$ in an open Zariski subset of Q, we see that the curve is of degree 1, i.e. a line. So our curve is contained in a 1-dimensional linear system of divisors of degree 5. □

Remark 6.3.29. The previous Proposition shows why Lüroth quartics are special among Darboux curves. By Lemma 6.3.25, the variety of pairs consisting of an N-lateral and a curve of degree $N-1$ circumscribing it is of dimension $3N-1$. This shows that the dimension of the variety of Darboux curves of degree $N-1$ is equal to $3N-1-k$, where k is the dimension of the variety of N-laterals inscribed in a general Darboux curve. We can construct a Darboux curve by considering an analog of a Clebsch curve, namely a curve C admitting a polar N-gon. Counting constants shows that the expected dimension of the locus of such curves is equal to $3N-1-m$, where m is the dimension of the variety of polar N-gons of C. Clearly every such C defines a Darboux curve as the locus of $x \in \mathbb{P}^2$ such that $P_x(C)$ admits a polar $(N-2)$-gon. The equation of a general Darboux curve shows that it is obtained in this way from a generalized Clebsch curve. In the case $N=5$, we have $k=m=1$. However, already for $N=6$, the variety of Darboux quintics is known to be of dimension 17, i.e. $k=0$ [32]. This shows that there are only finitely many N-laterals that a general Darboux curve of degree 5 could circumscribe.

Suppose C is an irreducible Lüroth quartic. Then, it comes from a Clebsch quartic C' if and only if it circumscribes a complete pentalateral and C' is a nondegenerate Clebsch quartic. For example, an irreducible singular Lüroth quartic circumscribing a pentalateral with a triple point does not belong to the image of the Clebsch covariant. In any case, a Darboux curve of degree $N-1$ given by Equation (6.30), in particular, a Lüroth quartic, admits a natural symmetric linear determinantal representation:*

$$\det\begin{pmatrix} l_1+l_2 & l_1 & l_1 & \dots & l_1 \\ l_1 & l_1+l_3 & l_1 & \dots & l_1 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ l_1 & \dots & \dots & l_1 & l_1+l_N \end{pmatrix} = 0. \qquad (6.32)$$

It is clear that, if $l_1(x)=l_2(x)=l_3(x)=0$, the corank of the matrix at the point x is greater than 1. Thus, if the N-lateral is not a complete N-lateral, the theta characteristic defining the determinantal representation is not an invertible one. However, everything goes well if we assume that the Lüroth quartic comes from a nondegenerate Clebsch quartic. Before we state and prove the next Theorem, we have to recall some facts about cubic surfaces which we will

*This was communicated to me by B. van Geemen, but also can be found in Room's book [633], p. 178.

prove and discuss later in Chapter 9. A cubic surface K always admits a polar pentahedron, maybe a generalized one. Suppose that K is general enough so that it admits a polar pentahedron $V(L_1), \ldots, V(L_5)$ such that no four of the forms L_i are linearly dependent. In this case K is called a *Sylvester nondegenerate cubic* and the polar pentahedron is unique. If we write $K = V(L_1^3 + \cdots + L_5^3)$, then the Hessian surface of K can be written by the equation

$$\sum_{i=1}^{5} \prod_{i \neq j} L_i(z) = 0. \qquad (6.33)$$ le'

Obviously, a general plane section of the Hessian surface is isomorphic to a Lüroth quartic.

Theorem 6.3.30. *Let $\mathcal{N}$ be a net of quadrics in $\mathbb{P}^3$. The following properties are equivalent.*

(i) There exists a basis (Q_1, Q_2, Q_3) of $\mathcal{N}$ such that the quadrics Q_i can be written in the form

$$Q_j = V\Big(\sum_{i=1}^{5} a_{ij} L_i^2\Big), \quad j = 1, 2, 3, \qquad (6.34)$$ fram

where L_i are linear forms with any four of them being linearly independent.

(ii) There exists a Sylvester nondegenerate cubic surface K in $\mathbb{P}^3$ such that $\mathcal{N}$ is equal to a net of polar quadrics of K.

(iii) The discriminant curve C of $\mathcal{N}$ is a Lüroth quartic circumscribing a complete pentalateral $\{V(l_1), \ldots, V(l_5)\}$ and $\mathcal{N}$ corresponds to the symmetric determinantal representation (6.32) of C. darboux2

Proof (i) ⇒ (ii) Consider the Sylvester nondegenerate cubic surface K given by the *Sylverster equation*

$$K = V(L_1^3 + \cdots + L_5^3).$$

For any point $x = [v] \in \mathbb{P}^3$, the polar quadric $P_x(K)$ is given by the equation $V(\sum L_i(v) L_i^2)$. Let $A = (a_{ij})$ be the 5×3 matrix defining the equations of the three quadrics. Let

$$L_i = \sum_{j=0}^{3} b_{ij} z_j, \quad i = 1, \ldots, 5,$$

and let $B = (b_{ij})$ be the 5×4-matrix of the coefficients. By assumption, $\mathrm{rank} A = 4$. Thus, we can find a 4×3-matrix $C = (c_{ij})$ such that $B \cdot C = A$. If we take the points x_1, x_2, x_3 with coordinate vectors v_1, v_2, v_3 equal to the columns of the matrix C, then we obtain that $L_i(v_j) = a_{ij}$. This shows that $Q_i = P_{x_i}(K), i = 1, 2, 3$.

(ii) $\Rightarrow$ (i) Suppose we can find three non-collinear points $x_i = [v_i]$ and a Sylvester nondegenerate cubic surface K such that $Q_i = P_{x_i}(K), i = 1, 2, 3$. Writing K as a sum of 5 cubes of linear forms L_i, we obtain (i).

(i) $\Rightarrow$ (iii) Consider the five linear forms $l_i = a_{i1}t_0 + a_{i2}t_1 + a_{i3}t_2$. Our net of quadrics can be written in the form

$$Q(t_0, t_1, t_2) = V(\sum_{i=1}^{5} l_i(t_0, t_1, t_2)L_i(z_0, z_1, z_2, z_3)^2).$$

By scaling coordinates t_i and z_j, we may assume that the forms l_i and L_j satisfy

$$\begin{aligned} l_1 + l_2 + l_3 + l_4 + l_5 &= 0, \\ L_1 + L_2 + L_3 + L_4 + L_5 &= 0. \end{aligned} \tag{6.35}$$

The quadric $Q(a)$ is singular at a point x if and only if

$$\text{rank}\begin{pmatrix} l_1(a)L_1(x) & \dots & l_5(a)L_5(x) \\ 1 & \dots & 1 \end{pmatrix} = 1.$$

This is equivalent to that

$$l_1(a)L_1(x) = \dots = L_5(a)l_5(x). \tag{6.36}$$

forget

Taking into account (6.35), we obtain

$$\sum_{i=1}^{5} \frac{1}{l_i(a)} = 0,$$

or

$$\sum_{i=1}^{5} \prod_{i \neq j} l_i(a) = 0. \tag{6.37}$$

le

This shows that the discriminant curve is a Lüroth quartic given by the determinantal Equation (6.32).

(iii) $\Rightarrow$ (ii) Computing the determinant, we find the equation of C in the form (6.37). Then, we linearly embed $\mathbb{C}^3$ in $\mathbb{C}^4$ and find five linear forms L_i such that restriction of L_i to the image is equal to l_i. Since no four of the l_i are linearly dependent, no four of the L_i are linearly dependent. Thus, $K = V(\sum L_i^3)$ is a Sylvester nondegenerate cubic surface. This can be chosen in such a way that $\sum L_i = 0$ generates the space of linear relations between the forms. By definition, the image of C in $\mathbb{P}^3$ given by the forms l_i is the discriminant curve of the net of polars of K. □

Definition 6.3.31. *The even theta characteristic on a Lüroth curve defined by the determinantal representation* (5.5.11) *is called a* pentalateral *theta characteristic.*

By changing the pentalateral inscribed in a weakly nondegenerate Lüroth quartic C, we map $\mathbb{P}^1$ to the variety of nets of quadrics in $\mathbb{P}^3$ with the same discriminant curve C. Its image in the moduli space of nets of quadrics modulo projective transformations of $\mathbb{P}^3$ is irreducible. Since there are only finitely many projective equivalence classes of nets with the same discriminant curve, we obtain that the pentalateral theta characteristic does not depend on the choice of the pentalateral.

Suppose C is a nondegenerate Lüroth quartic equal to $\mathfrak{C}(C')$ for some Clebsch quartic C'. It is natural to guess that the determinantal representation of C given by determinant (6.32) corresponds to the pentalateral theta characteristic defined by the Scorza correspondence $R_{C'}$ on C. The guess is correct. We refer for the proof to [235], Theorem 7.4.1.

Remark 6.3.32. Since the locus of Clebsch quartics is a hypersurface (of degree 6) in the space of all quartics, the locus of Lüroth quartics is also a hypersurface. Its degree is equal to 54 ([528]). One can find modern proofs of this fact in [483], [756], and [565]. We also refer to Basement's paper [39] that discusses many aspects of the theory of Lüroth quartics, some of this was revised in [565] and [566]. For example, in the second paper, G. Ottaviani and E. Sernesi study the locus of singular Lüroth quartics and prove that it consists of two irreducible components. One of them is contained in the image of the Clebsch covariant. The other component is equal to the locus of Lüroth quartics circumscribing a pentalateral with a double point.

Note that the degree of the locus of three quadrics (Q_1, Q_2, Q_3) with discriminant curve isomorphic to a Lüroth quartic is equal to $4 \cdot 54 = 216$. It consists of one component of degree 6, the zero set of the Toeplitz invariant, and the other component of degree 210. The component of degree 6 corresponds to a choice of a pentagonal theta characteristic, the other component corresponds to other 35 theta characteristics, for which the monodromy is irreducible.

6.3.6 A Fano model of $\mathrm{VSP}(f, 6)$

Recall that a nondegenerate ternary quartic $f \in S^4(E^\vee)$ is one of the special cases from Theorem 1.3.20 where Corollary 1.4.13 applies. So, the variety $\mathrm{VSP}(f, 6)^o$ embeds in the Grassmann variety $G(3, \mathrm{AP}_3(f)^\vee) \cong G(3, 7)$. The image is contained in the subvariety $G(3, \mathrm{AP}_3(f))_\sigma$ of isotropic subspaces of the skew-symmetric linear map $\sigma : \Lambda^2 E \to \bigwedge^2 \mathrm{AP}_3(f)$. Choosing a basis in

E and identifying $\Lambda^2 E$ with $E^\vee$, we can view this map as a skew-symmetric 7×7-matrix M whose entries are linear functions on E. Let $L \subset \mathrm{AP}_3(f)^\vee$ be an isotropic subspace of σ. In appropriate coordinates (t_0, t_1, t_2), we can write M in the block-form

$$M = \begin{pmatrix} B & A \\ -{}^t A & 0 \end{pmatrix},$$

where B is a square skew-symmetric 4×4 matrix and A is a 4×3 matrix. The maximal minors of the matrix A generate an ideal in $\mathbb{C}[t_0, t_1, t_2]$ defining a closed 0-dimensional subscheme Z of length 6. This defines the map

$$G(3, \mathrm{AP}_3(f))_\sigma \to \mathrm{VSP}(f, 6)$$

which is the inverse of the map $\mathrm{VSP}(f, 6)^o \to G(3, \mathrm{AP}_3(f))_\sigma$ (see [607]).

The following Theorem is originally due to S. Mukai [535] and was reproved by a different method by K. Ranestad and F.-O. Schreyer [607], [663].

Theorem 6.3.33. *Let $f \in S^4(E^\vee)$ be a nondegenerate quartic form in three variables. Then, the map $VSP(f, 6)^o \to G(3, \mathrm{AP}_3(f)^\vee)_\sigma$ extends to an isomorphism*

$$\mu : \mathrm{VSP}(f, 6) \to G(3, \mathrm{AP}_3(f)^\vee)_\sigma.$$

If f is a general quartic, the variety $G(3, \mathrm{AP}_3(f)^\vee)_\sigma$ is a smooth threefold. Its canonical class is equal to $-H$, where H is a hyperplane section in the Plücker embedding of the Grassmannian.

Recall that a *Fano variety* of dimension n is a projective variety X with ample $-K_X$. If X is smooth, and $\mathrm{Pic}(X) \cong \mathbb{Z}$ and $-K_X = mH$, where H is an ample generator of the Picard group, then X is said to be of *index* m. The *degree* of X is the self-intersection number H^n . The number $g = \frac{1}{2}K_X^n + 1$ is called the *genus*.

In fact, in [534] S. Mukai announced a more precise result. The variety $\mathrm{VSP}(f, 6)$ is a Gorenstein Fano variety if f is not a Lüroth quartic and it is smooth, if $V(f)$ is nondegenerate and does not admit a complete quadrangle as its polar 6-side (a *complete quadrangle* is the union of six lines joining two out of four general points in the plane).

Remark 6.3.34. A Fano variety V_{22} of degree 22 (genus 12) and index 1 was omitted in the original classification of Fano varieties with the Picard number 1 due to Gino Fano. It was discovered by V. Iskovskikh. It has the same Betti numbers as the $\mathbb{P}^3$. It was proven by S. Mukai that every such variety is isomorphic to $\mathrm{VSP}(f, 6)$ for a unique quartic for $F = V(f)$. He also makes the relation between $\mathrm{VSP}(f, 6)$ and the corresponding V_{22} very explicit.

The Clebsch quartic curve $C = \mathfrak{C}(F)$ of F can be reconstructed from V_{22} as the Hilbert scheme of lines on V_{22} (in the anti-canonical embedding). The Scorza correspondence defining the corresponding even theta characteristic is the incidence relation of lines. The quartic is embedded in the plane of conics on V_{22}. In this way Mukai gets another proof of Corollary 6.3.20. Also he shows that through each point on V_{22} passes 6 conics taken with multiplicities. In the dual plane they correspond to a generalized polar hexagon of f (see [535], [536]).

By the same method, Ranestad and Schreyer extended the previous result to all exceptional cases listed in Subsection 1.4.3, where $n = 2$. We have

Theorem 6.3.35. *Let f be a general ternary form of degree $2k$. Then,*

- $k = 1$*:* $\mathrm{VSP}(f, 3) \cong G(2, 5)_\sigma$ *is isomorphic to a Fano variety of degree 5 and index 2;*
- $k = 2$*:* $\mathrm{VSP}(f, 6) \cong G(3, 7)_\sigma$ *is isomorphic to the Fano variety V_{22} of degree 22 and index 1;*
- $k = 3$*:* $\mathrm{VSP}(f, 10) \cong G(4, 9)_\sigma$ *is isomorphic to a K3 surface of degree* 38 *in* $\mathbb{P}^{20}$*;*
- $k = 4$*:* $\mathrm{VSP}(f, 15) \cong G(5, 11)_\sigma$ *is a set of 16 points.*

In the two remaining cases $(n, k) = (1, k)$ and $(n, k) = (3, 2)$, the variety $\mathrm{VSP}(f, k+1)$ is isomorphic to $\mathbb{P}^1$ (see 1.60) in the first case and, in the second case, At present, the birational type of $\mathrm{VSP}(f, 10)$ is unknown.

Let $C = V(f)$ be a nonsingular plane quartic and θ is an even theta characteristic on C. Let N_ϑ be the corresponding net of quadrics in $\mathbb{P}(H^0(C, \theta(1)))$. Let $\mathsf{N}_\vartheta^\perp$ be the apolar linear system of quadrics in the dual projective space $\check{\mathbb{P}}^3$. Its dimension is equal to 6. We say that a rational normal cubic R in $\check{\mathbb{P}}^3$ is associated to N_ϑ if the net of quadrics $|\mathcal{J}_R(2)|$ vanishing on R is contained in $\mathsf{N}_\vartheta^\perp$. In [663] F.-O. Schreyer constructs a linear map $\alpha : \bigwedge^2 \mathsf{N}_\vartheta^\perp \to \mathsf{N}_\vartheta$ and shows that the nets of quadrics defining the associated rational normal curves is parameterized by the subvariety $G(3, \mathsf{N}_\vartheta^\perp)_\alpha$ of isotropic subspaces of α. This reminds us of the construction of $G(3, \mathrm{AP}_3(f)^\vee)_\sigma$. In fact, consider the transpose map ${}^t\alpha : \mathsf{N}_\vartheta^\vee \to \bigwedge^2 (E^\perp)^\vee$ and pass to the third symmetric power to get a linear map

$$S^3(\mathsf{N}_\vartheta^\vee) \to S^3 \bigwedge^2 (\mathsf{N}_\vartheta^\perp)^\vee \to \bigwedge^6 (\mathsf{N}_\vartheta^\perp)^\vee \to \mathsf{N}_\vartheta^\perp.$$

Its kernel can be identified with $\mathrm{AP}_3(g)$, where $V(g)$ is the quartic, and $V(f)$

is its Scorza quartic. This gives another proof of the Scorza birational isomorphism between $\mathcal{M}_3$ and $\mathcal{M}_3^{\mathrm{ev}}$. A similar construction was announced earlier by S. Mukai [534].

6.4 Invariant Theory of Plane Quartics

Let $\mathsf{I}(d)$ denote the space of SL(3)-invariants of degree d in the linear action of SL(3) on the space of quartic ternary forms. We have already encountered an invariant I_6 of degree 6, the catalecticant invariant (invariant B in Salmon's notation from [652, Art 292]). It vanishes on the space of Clebsch quartics. Another familiar invariant is the discriminant invariant I_{27} of degree 27. We know some of the covariants of plane quartics. These are the Hessian invariant He of degree 3 and order 6, and the Clebsch covariant $\mathfrak{C}_4$ of degree 3 and order 4. It vanishes on the set of Lüroth quartics. Recall that the latter assigns to a general quartic the closure of the locus of points whose polar is equianharmonic cubic. There is a similar covariant $\mathfrak{C}_6$ of degree 4 and order 6 that assigns to a general quartic curve the closure of the locus of points whose polar cubics are harmonic. The Steinerian covariant of degree 12 and order 12 is a linear combination of $\mathfrak{C}_4^3$ and $\mathfrak{C}_6^2$. The dual analogs of the covariants $\mathfrak{C}_4$ and $\mathfrak{C}_6$ are the *equianharmonic* contravariant Φ_4 of class 4 and degree 2 and the *harmonic* contravariant Φ_6 of class 6 and degree 3.

Let us introduce the generating function

$$P(T) = \sum_{d=0}^{\infty} \dim_{\mathbb{C}} \mathsf{I}(d) T^d.$$

It has been computed by T. Shioda [712], and the answer is

$$P(T) = \frac{N(T)}{\prod_{i=1}^{6}(1-T^{3i})(1-T^{27})}, \tag{6.38}$$

where

$$N(T) = 1 + T^9 + T^{12} + T^{15} + 2T^{18} + 3T^{21} + 2T^{24} + 3T^{27} + 4T^{30} + 3T^{33} + 4T^{36}$$
$$+4T^{39} + 3T^{42} + 4T^{45} + 3T^{48} + 2T^{51} + 3T^{54} + 2T^{57} + T^{60} + T^{63} + T^{66} + T^{75}.$$

It was proven by J. Dixmier [225] that the algebra of invariants is finite over the free subalgebra generated by seven invariants of degrees 3, 6, 9, 12, 15, 18, 27. Shioda conjectured that one needs six more invariants of degrees 9, 12, 15, 18, 21, 21 to generate the whole algebra of invariants. This was proved by T. Ohno (unpublished but see [287]).

It follows that the invariant I_3 (the invariant A in Salmon's notation) is of the lowest possible degree, and it is a unique (up to proportionality) invariant of degree 3. The next lowest degree is 6 and the linear space of invariants of degree 6 is spanned by I_3^2 and I_6. The invariant I_3 has symbolic expression $(123)^4$. Its value on a quartic ternary form $F(x,y,z)$ is equal to the value of $F(x,y,z)$ on the Clebsch covariant of $\mathfrak{S}(F)$. So, it vanishes on the set of plane quartics apolar to the equianharmonic covariant Φ_4 (contravariant σ in Salmon's notation).

If we employ Salmon's notation for an equation for a ternary form of degree four

$$\begin{aligned} &ax^4+by^4+cz^4+6fy^2z^2+6gx^2z^2+6hx^2y^2+12lx^2yz12my^2xz \\ &+12nz^2xy+4a_2x^3y+4a_3x^2z+4b_1xy^3+4b_3y^3z+4c_1xz^3+4c_2yz^3, \end{aligned} \tag{6.39}$$

then

$$\begin{aligned} I_3 &= abc+3(af^2+bg^2+ch^2)-4(ab_3c_2+ba_3c_1+ca_2b_1)+6fgh \\ &+12(fl^2+gm^2+hn^2)-12lmn-12(a_2nf+a_3mf+b_1ng+b_3lg \\ &+c_1mn+c_2lh)+12(lb_1c_1+ma_2c_2+na_3b_3)+4a_2b_3c_1+a_3b_1c_2). \end{aligned} \tag{6.40}$$

One can generate a new invariant by using the polarity pairing between covariants and contravariants of the same order. The obtained invariant, if not zero, is of degree equal to the sum of degrees of the covariant and the contravariant. For example, $(\Phi_4(C), \mathfrak{C}_4(C))$ or $(\Phi_6(C), \mathrm{He}(C))$ give invariants of degree 6. It follows from (6.38) that all invariants of degree 6 are linear combinations of A_3^2 and A_6. However, $(\Phi_6(C), \mathfrak{C}_6(C))$ is a new invariant of degree 9. Taking here the Hessian covariant instead of $\mathfrak{C}_6(C)$, one obtains an invariant of degree 6.

There is another contravariant Ω of class 4 but of degree 5. It vanishes on the set of lines ℓ such that the unique anti-polar conic of ℓ contains ℓ (see [235], p. 274). The contravariant $A_3\Phi_4$ is of the same degree and order, but the two contravariants are different.

We can also generate new covariants and contravariants by taking the polar pairing at already known covariants and contravariants. For example, one gets a covariant conic σ of degree 5 by operating $\Phi_4(C)$ on $\mathrm{He}(C)$. Or we may operate C on $\Phi_6(C)$ to get a contravariant conic of degree 4.

Applying known invariants to covariants or contravariants gets a new invariant. However, they are of large degrees. For example, taking the discriminant of the Hessian, we get an invariant of degree 215. However, it is reducible, and contains a component of degree 48 representing an invariant that vanishes on the set of quartics which admit a polar conic of rank 1 [748]. There are other known geometrically meaningful invariants of large degree. For example, the Lüroth invariant of degree 54 vanishing on the locus of Lüroth quartics and the *Salmon invariant* of degree 60 vanishing on the locus of quartics with an inflection bitangent (see [162], [602]). It is a special case of the *undulation invariant* of plane curves of degree d that vanished on curves admitting a line touching it at some point with multiplicity ≥ 4. Its degree was computed by Cayley and Salmon in 1852, and it is equal to $6(d-3)(3d-2)$.

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The GIT-quotient of $|\mathcal{O}_{\mathbb{P}^2}(4)|$ by SL(3) and other compactifications of the moduli space of plane quartic curves were studied recently from different aspects. Unfortunately, it is too extensive a topic to discuss here. We refer to [21], [23],[381], [465], [493], [494].

6.5 Automorphisms of Plane Quartic Curves

S:6.5

6.5.1 Automorphisms of finite order

SS:6.5.1

Since an automorphism of a nonsingular plane quartic curve C leaves the canonical class K_C invariant, it is defined by a projective transformation. We first describe all possible cyclic groups of automorphisms of C.

finorder

Lemma 6.5.1. *Let σ be an automorphism of order $n > 1$ of a nonsingular plane quartic $C = V(f)$. Then, one can choose coordinates in such a way that a generator of the cyclic group (σ) is represented by the diagonal matrix*

$$\mathrm{diag}[1, \zeta_n^a, \zeta_n^b], \quad 0 \leq a < b < n,$$

where ζ_n is a primitive n-th root of unity, and f is given in the following list.

(i) $(n=2), (a,b) = (0,1)$,

$$t_2^4 + t_2^2 g_2(t_0,t_1) + g_4(t_0,t_1);$$

(ii) $(n=3), (a,b) = (0,1)$,

$$t_2^3 g_1(t_0,t_1) + g_4(t_0,t_1);$$

(iii) $(n=3), (a,b) = (1,2)$,

$$t_0^4 + \alpha t_0^2 t_1 t_2 + t_0 t_1^3 + t_0 t_2^3 + \beta t_1^2 t_2^2;$$

(iv) $(n=4), (a,b) = (0,1)$,

$$t_2^4 + g_4(t_0,t_1);$$

(v) $(n=4), (a,b) = (1,2)$,

$$t_0^4 + t_1^4 + t_2^4 + \alpha t_0^2 t_2^2 + \beta t_0 t_1^2 t_2;$$

(vi) $(n=6), (a,b)=(2,3)$,

$$t_0^4+t_2^4+\alpha t_0^2t_2^2+t_0t_1^3;$$

(vii) $(n=7), (a,b)=(1,3)$,

$$t_0^3t_2+t_2^3t_1+t_0t_1^3;$$

(viii) $(n=8), (a,b)=(3,7)$,

$$t_0^4+t_1^3t_2+t_1t_2^3;$$

(ix) $(n=9), (a,b)=(2,3)$,

$$t_0^4+t_0t_2^3+t_1^3t_2;$$

(x) $(n=12), (a,b)=(3,4)$,

$$t_0^4+t_1^4+t_0t_2^3.$$

Here, the subscripts in the polynomials g_i indicate their degree.

Proof Let us first choose coordinates such that σ acts by the formula

$$\sigma:[x_0,x_1,x_2]\mapsto[x_0,\zeta_n^ax_1,\zeta_n^bx_2],$$

where $a\le b<n$. If $a=b$, we can scale the coordinates by ζ^{-a}, and then permute the coordinates to reduce the action to the case, where $0\le a<b$.

We will often use that f is of degree ≥ 3 in each variable. This follows from the assumption that f is nonsingular. A form f is invariant with respect to the action if all monomials entering f with nonzero coefficients are eigenvectors of the action of σ on the space of quartic ternary forms. We denote by p_1,p_2,p_3 the points $[1,0,0],[0,1,0],[0,0,1]$.

Case 1: $a=0$.

Write f in the form:

$$f=\alpha t_2^4+t_2^3g_1(t_0,t_1)+t_2^2g_2(t_0,t_1)+t_2g_3(t_0,t_1)+g_4(t_0,t_1). \tag{6.41}$$ quar

Assume $\alpha\ne0$. Since $g_4\ne0$, if $\alpha\ne0$, we must have $4b\equiv0\mod n$. This implies that $n=2$ or 4. In the first case $g_1=g_3=0$, and we get case (i). If $n=4$, we must have $g_1=g_2=g_3=0$, and we get case (iv).

If $\alpha=0$, then $3b=0\mod n$. This implies that $n=3$ and $g_2=g_3=0$. This gives case (ii).

Case 2: $a\ne0$.

The condition $a<b<n$ implies that $n>2$.

Case 2a: The points p_1,p_2,p_3 lie on C.

This implies that no monomial t_i^4 enters f. We can write f in the form

$$f = t_0^3 a_1(t_1,t_2) + t_1^3 b_1(t_0,t_2) + t_2^3 c_1(t_0,t_1)$$

$$+ t_0^2 a_2(t_1,t_2) + t_1^2 b_2(t_0,t_2) + t_2^2 c_2(t_0,t_1),$$

where a_i, b_i, c_i are homogeneous forms of degree i. If one of them is zero, then we are in Case 1 with $\alpha = 0$. Assume that all of them are not zeros. Since f is invariant, it is clear that no t_i enters two different coefficients a_1, b_1, c_1. Without loss of generality, we may assume that

$$f = t_0^3 t_2 + t_2^3 t_1 + t_1^3 t_0 + t_0^2 a_2(t_2,t_3) + t_1^2 b_2(t_0,t_2) + t_2^2 c_2(t_0,t_1).$$

Now, we have $b \equiv a + 3b \equiv 3a \mod n$. This easily implies $7a \equiv 0 \mod n$ and $7b \equiv 0 \mod n$. Since $n|g.c.m(a,b)$, this gives $n = 7$, and $(a,b) = (1,3)$. By checking the eigenvalues of other monomials, we verify that no other monomials enter f. This is case (vii).

Case 2b: Two of the points p_1, p_2, p_3 lie on the curve.

After scaling and permuting the coordinates, we may assume that the point $p_1 = [1,0,0]$ does not lie on C. Then, we can write

$$f = t_0^4 + t_0^2 g_2(t_1,t_2) + t_0 g_3(t_1,t_2) + g_4(t_1,t_2),$$

where t_1^4, t_2^4 do not enter g_4.

Without loss of generality, we may assume that $t_1^3 t_2$ enters g_4. This gives $3a + b \equiv 0 \mod n$. Suppose $t_1 t_2^3$ enters g_4. Then, $a + 3b \equiv 0 \mod n$, and $8a \equiv n, 8b \equiv 0 \mod n$. As in the previous case, this easily implies that $n = 8$. This gives case (viii). If $t_1 t_2^3$ does not enter in g_4, then t_2^3 enters g_3. This gives $3b \equiv 0 \mod n$. Together with $3a + b \equiv 0 \mod n$, this gives $n = 3$ and $(a,b) = (1,2)$, or $n = 9$ and $(a,b) = (2,3)$. These are cases (iii) and (ix).

Case 2c: Only one point p_i lies on the curve.

Again we may assume that p_1, p_3 do not lie on the curve. Then, we can write

$$f = t_0^4 + t_1^4 + t_0^2 g_2(t_1,t_2) + t_0 g_3(t_1,t_2) + g_4(t_1,t_2),$$

where t_1^4, t_2^4 do not enter g_4. This immediately gives $4a \equiv 0 \mod n$. We know that either t_2^3 enters g_3, or $t_1 t_2^3$ enters g_4. In the first case, $3b \equiv 0 \mod n$ and together with $4a \equiv 0 \mod n$, we get $n = 12$ and $(a,b) = (3,4)$. Looking at the eigenvalues of other monomials, this easily leads to case (x). If $t_2^3 t_1$ enters g_4, we get $3b + a \equiv 0 \mod n$. Together with $4a \equiv 0 \mod n$, this gives $12b \equiv 0 \mod 12$. Hence, $n = 12$ or $n = 6$. If $n = 12$, we get $a = b = 3$, this has been considered before. If $n = 6$, we get $a = 3, b = 1$. This leads to the equation $t_0^4 + t_1^4 + \alpha t_0^2 t_1^2 + t_1 t_2^3 = 0$. After permutation of coordinates $(t_0, t_1, t_2) \mapsto (t_2, t_0, t_1)$, we arrive at case (vi).

Case 2d: None of the reference points lies on the curve.

In this case we may assume that

$$f = t_0^4 + t_1^4 + t_2^4 + t_0^2 g_2(t_1, t_2) + t_0 g_3(t_1, t_2) + t_1 t_2(\alpha t_1^2 + \beta t_2^2 + \gamma t_1 t_2).$$

Obviously, $4a = 4b = 0 \mod n$. If $n = 2$, we are in case (i). If $n = 4$, we get $(a, b) = (1, 2), (1, 3)$, or $(2, 3)$. Permuting $(t_0, t_1, t_2) \mapsto (t_2, t_0, t_1)$, and multiplying the coordinates by ζ_4^2, we reduce the case $(1, 2)$ to the case $(2, 3)$. The case $(1, 3)$ is also reduced to the case $(1, 2)$ by multiplying coordinates by ζ_4 and then permuting them. Thus, we may assume that $(a, b) = (1, 2)$. Checking the eigenvalues of the monomials entering f, we arrive at case (v). □

SS:6.5.2

6.5.2 Automorphism groups

We employ the notation from [165]: a cyclic group of order n is denoted by n, the semi-direct product $A \rtimes B$ is denoted by $A : B$, a central extension of a group A with kernel B is denoted by $B.A$. We denote by $L_n(q)$ the group $\mathrm{PSL}(n, \mathbb{F}_q)$.

Theorem 6.5.2. *The following Table is the list of all possible groups of automorphisms of a nonsingular plane quartic.*

Before we prove the theorem, let us comment on the parameters of the equations. First of all, their number is equal to the dimension of the moduli space of curves with the given automorphism group. The equations containing parameters may acquire additional symmetry for special values of parameters. Thus, in Type IV, one has to assume that $a \neq \frac{3}{2}(-1 \pm \sqrt{-7})$, otherwise the curve becomes isomorphic to the Klein curve (see [309], vol. 2, p. 209, or [627]). In Type V, the special values are $a = 0, \pm 2\sqrt{-3}, \pm 6$. If $a = 0$, we get the Fermat quartic, if $a = \pm 6$, we again get Type II (use the identity

$$x^4 + y^4 = \frac{1}{8}((x + y)^4 + (x - y)^4 + 6(x + y)^2(x - y)^2).$$

If $a = \pm 2\sqrt{-3}$, we get Type III (the identity

$$x^4 + y^4 + ax^2y^2 = \frac{e^{-\pi i/3}}{4}((x + iy)^4 + (x - iy)^4 + a(x + iy)^2(x - iy)^2)$$

exhibits an additional automorphism of order 3). In Type VII, we have to assume $b \neq 0$, otherwise the curve is of Type V. In Type VIII, $a \neq 0$, otherwise the curve is of type III. In Type IX, $a \neq 0$, otherwise the curve acquires an

Type	Order	Structure	Equation
I	168	$L_2(7)$	$t_0^3t_2 + t_0t_1^3 + t_1t_2^3$
II	96	$4^2 : \mathfrak{S}_3$	$t_0^4 + t_1^4 + t_2^4$
III	48	$4.\mathfrak{A}_4$	$t_0^4 + t_1^4 + 2\sqrt{-3}t_1^2t_2^2 + t_2^4$
IV	24	$\mathfrak{S}_4$	$t_0^4 + t_1^4 + t_2^4 + a(t_0^2t_1^2 + t_0^2t_2^2 + t_1^2t_2^2)$
V	16	4.2^2	$t_0^4 + t_1^4 + at_1^2t_2^2 + t_2^4$
VI	9	9	$t_0^4 + t_0t_2^3 + t_1^3t_2$
VII	8	D_8	$t_0^4 + t_1^4 + t_2^4 + at_0^2t_2^2 + bt_1^2t_0t_2$
VIII	6	6	$t_0^4 + t_2^4 + t_0t_1^3 + at_0^2t_2^2$
IX	6	$\mathfrak{S}_3$	$t_0^4 + t_0(t_1^3 + t_2^3) + at_0^2t_1t_2 + bt_1^2t_2^2$
X	4	2^2	$t_0^4 + t_1^4 + t_2^4 + at_0^2t_2^2 + bt_0^2t_1^2 + ct_1^2t_2^2$
XI	3	3	$t_2^3t_1 + t_0(t_1^3 + at_1^2t_0 + bt_1t_0^2 + ct_0^3)$
XII	2	2	$t_2^4 + t_2^2g_2(t_0,t_1) + t_0^4 + at_0^2t_1^2 + t_1^4$

autoquartic

Table 6.1 *Automorphisms of plane quartics*

automorphism of order 4. In Type X, all the coefficients a, b, c are different. We leave the cases XI and XII to the reader.

Proof Suppose G contains an element of order $n \geq 6$. Applying Lemma finorder 6.5.1, we obtain that C is isomorphic to a quartic of Type VIII ($n = 6$), I ($n = 7$), II ($n = 8$), VI ($n = 9$), and III ($n = 12$). Here, we use that, in case $n = 8$ (resp. $n = 12$), the binary form $t_1^3t_2 + t_1t_2^3$ (resp. $t_1^4 + t_0t_2^3$) can be reduced to the binary forms $t_1^4 + t_2^4$ (resp. $t_1^4 + 2\sqrt{-3}t_1^2t + t_2^4$) by a linear change of variables. It corresponds to a harmonic (resp. equianharmonic) elliptic curve.

Assume $n = 8$. Then, C is a Fermat quartic. Obviously, G contains a subgroup $G' = 4^2 : \mathfrak{S}_3$ of order 96. If it is a proper subgroup, then the order of G is greater than 168. By *Hurwitz's Theorem*, the automorphism group of a nonsingular curve of genus g is of order $\leq 84(g-1)$ (see Hartshorne [379, Chapter V, Exercise 2.5]). This shows that $G \cong 4^2 : \mathfrak{S}_3$, as in Type II.

Assume $n = 7$. Then, the curve is projectively isomorphic to the *Klein curve*, which we will discuss in the next subsection. We will show that its automorphism group is isomorphic to $L_2(7)$. This deals with Type I.

Now, we see that G may contain only Sylow 2-subgroups or 3-subgroups.

Case 1: G contains a 2-group.

First of all, the order $N = 2^m$ of G is less than or equal to 16. Indeed, by above, we may assume that G does not contain cyclic subgroups of order 2^a

with $a > 2$. By Riemann-Hurwitz formula

$$4 = N(2g' - 2) + N \sum (1 - \frac{1}{e_i}).$$

If $N = 2^m, m > 4$, then the right-hand side is divisible by 8.

So $N = 2^m, m \leq 4$. As is well-known, and is easy to prove, the center Z of G is not trivial. Pick up an element σ of order 2 in the center and consider the quotient $C \to C/(\sigma) = C'$. Since any projective automorphism of order 2 fixes a line ℓ pointwise, g has a fixed point on C. By Riemann-Hurwitz formula, C' is a curve of genus one, and the cover is ramified at four points. By choosing the coordinates such that $\sigma = \mathrm{diag}[-1, 1, 1]$, the equation of C becomes

$$t_2^4 + t_2^2 g_2(t_0, t_1) + g_4(t_0, t_1) = 0. \tag{6.42}$$

If $G = (\sigma)$, we get Type XII. Suppose $G = 2^2$ and τ is another generator. After a linear change of variables t_1, t_2, we may assume that τ acts as $[t_0, t_1, t_2] \mapsto [t_0, t_1, -t_2]$. This implies that g_2 does not contain the monomial $t_1 t_2$ and g_4 does not contain the monomials $t_1^3 t_2, t_1 t_2^3$. This leads to Type X.

If $G = (\tau) \cong \mathbb{Z}/4\mathbb{Z}$, there are two cases to consider corresponding to items (iv) and (v) in Lemma 6.5.1. In the first case, we may assume that $\tau : [t_0, t_1, t_2] \mapsto [t_0, t_1, it_2]$. This forces $g_2 = 0$. It is easy to see that any binary quartic without multiple zeros can be reduced to the form $t_1^4 + at_1^2 t_2^2 + t_2^4$. Now, we see that the automorphism group of the curve

$$V(t_0^4 + t_1^4 + at_1^2 t_2^2 + t_2^4), \ a \neq 0,$$

contains a subgroup generated by the transformations

$$g_1 : [t_0, t_1, t_2] \mapsto [it_0, t_1, t_2],$$
$$g_2 : [t_0, t_1, t_2] \mapsto [t_0, it_1, -it_2],$$
$$g_3 : [t_0, t_1, t_2] \mapsto [it_0, it_2, it_1].$$

The element g_1 generates the center, and the quotient is isomorphic to $2^2 := (\mathbb{Z}/2\mathbb{Z})^2$. We denote this group by 4.2^2. It is one of nine non-isomorphic non-abelian groups of order 16. Another way to represent this group is $D_8 : 2$. The dihedral subgroup D_8 is generated by g_2 and $g_1 g_3$. If $a = 0$, it is the Fermat curve of Type II.

In the second case, we may assume that $\tau : [t_0, t_1, t_2] \mapsto [t_0, it_1, -t_2]$. In this case, we can reduce the equation to the form (v) from Lemma 6.5.1. It is easy to see that G contains the dihedral group D_8. If there is nothing else, we get Type VII. There are two isomorphism classes of group of order 16 that contain D_8. They are $D_8 \times 2$ or 4.2^2 from above. In the former case, the group contains a subgroup isomorphic to $2^3 := (\mathbb{Z}/2\mathbb{Z})^3$. This group does not embed in PGL(3). In the latter case, the center is of order four, hence commutes with τ but does not equal to (τ). The equation shows that this is possible only if the coefficient $b = 0$, this leads to Type V.

Case 2: G contains a Sylow 3-subgroup.

Let Q be a Sylow 3-subgroup of G. Assume Q contains a subgroup Q' isomorphic to 3^2. By Riemann-Hurwitz formula, the quotient of C by a cyclic group of order 3 is either an elliptic curve or a rational curve. In the former case, the quotient map has two simple ramification points, in the latter case, it has five simple ramification points. In any case, the second generator of Q' fixes one of the ramification points. However, the stabilizer subgroup of any point on a nonsingular curve is a cyclic group. This contradiction shows that Q must be cyclic of order 3 or 9.

Case 2a: Q is of order 9.

If $Q = G$, we are getting Type VI. Thus, we may assume that G contains a Sylow 2-subgroup P of some order $2^m, m \leq 4$. By Sylow's Theorem, the number s_3 of Sylow 3-subgroups is equal to $1 + 3k$ and it divides 2^m. T his gives $s_3 = 1, 4, 16$. If $m = 1$, the subgroup Q is normal. The cover $C \to C/Q$ is ramified at five points with ramification indices $(9, 9, 3)$. If $Q \neq G$, then P contains a subgroup isomorphic to $9 : 2$. It does not contain elements of order 6. An element of order 2 in this group must fix one of the five ramification points. It generates a stabilizer subgroup of order 6 or 18. Both cases lead to a contradiction.

Suppose Q is not a normal subgroup. The number n_3 of Sylow 3-subgroups is equal to 4 if $m = 2, 3$, or 16 if $m = 4$. Consider the action of G on the set of 28 bitangents. It follows from the normal form of an automorphism of order 9 in Lemma 6.5.1 that Q fixes a bitangent. Thus, the cardinality of each orbit of G on the set of bitangents divides 2^m and the number of orbits is equal to 4 or 16. It is easy to see that this is impossible.

Case 2a: Q is of order 3.

If P contains an element of order 4 of type (v), then, by the analysis from Case 1, we infer that G contains D_8. If $P \cong D_8$, by Sylow's Theorem, the index of the normalizer $N_G(P)$ is equal to the number s_2 of Sylow 2-subgroups. This shows that $s_2 = 1$, and hence, P is a normal subgroup of G. An element of order 4 in P must commute with an element of order 3. Thus, G contains an element of order 12, hence the equation can be reduced to the Fermat equation of Type II. Thus, P must be of order 16. This leads to Type III.

So, we may assume that P does not contain an element of order 4 of type (v). If it does, then it must have the equation of Type V with $a = 0$. This leads again to the Fermat curve.

Finally, we arrive at the case when P has no elements of order 4. Then, P is an abelian group $(\mathbb{Z}/2\mathbb{Z})^m$, where $m \le 2$ (the group 2^3 does not embed in $\mathrm{Aut}(\mathbb{P}^2)$. If $m = 0$, we get Type XI; if $m = 1$, we get Type IX, if $m = 2$, we get Type IV.

□

SS:6.5.3

6.5.3 The Klein quartic

Recall that a quartic curve admitting an automorphism of order 7 is projectively equivalent to the quartic

$$C = V(t_0t_1^3 + t_1t_2^3 + t_0^3t_2). \tag{6.43}$$ klein1

The automorphism S of order 7 acts by the formula

$$S : [t_0, t_1, t_2] \mapsto [\epsilon t_0, \epsilon^2 t_1, \epsilon^4 t_2], \quad \epsilon = e^{2\pi i/7},$$

where we scaled the action to represent the transformation by a matrix from $\mathrm{SL}(3)$.

As promised, we will show that the group of automorphisms of such a quartic is isomorphic to the simple group $L_2(7)$ of order 168. By Hurwitz's Theorem, the order of this group is the largest possible for curves of genus 3.

Observe that the equation (6.43) has a symmetry given by a cyclic permutation U of the coordinates. It is easy to check that

$$USU^{-1} = S^4, \tag{6.44}$$ rel1

so that the subgroup generated by S, U is a group of order 21 isomorphic to the semi-direct product $7 : 3$.

By a direct computation, one checks that the following unimodular matrix defines an automorphism T of C of order two:

$$\frac{i}{\sqrt{7}}\begin{pmatrix} \epsilon - \epsilon^6 & \epsilon^2 - \epsilon^5 & \epsilon^4 - \epsilon^3 \\ \epsilon^2 - \epsilon^5 & \epsilon^4 - \epsilon^3 & \epsilon - \epsilon^6 \\ \epsilon^4 - \epsilon^3 & \epsilon - \epsilon^6 & \epsilon^2 - \epsilon^5 \end{pmatrix}. \tag{6.45}$$ fricke

We have

$$TUT^{-1} = U^2. \tag{6.46}$$ rel2

This shows that the subgroup generated by U, T is the dihedral group of order 6. One checks that the 49 products S^aTS^b are all distinct. In particular, the cyclic subgroup (S) is not normal in the group G generated by S, T, U. Since the order of G is divisible by $2 \cdot 3 \cdot 7 = 42$, we see that $\#G = 42, 84, 126$ or 168.

It follows from Sylow's Theorem that the subgroup (S) must be normal in the first three cases, and hence, $\#G = 168$, and by Hurwitz's Theorem

$$\mathrm{Aut}(C) = G = \langle S, U, T \rangle.$$

One checks that $V = (TS)^{-1}$ satisfies $V^3 = 1$ and the group has the presentation

$$G = \langle S, T, V : S^7 = V^3 = T^2 = STV = 1 \rangle.$$

Proposition 6.5.3. *The group* $\mathrm{Aut}(C)$ *is a simple group* G_{168} *of order* 168.

Proof Suppose H is a nontrivial normal subgroup of G. Assume that its order is divisible by 7. Since its Sylow 7-subgroup cannot be normal in H, we see that H contains all Sylow 7-subgroups of G. By Sylow's Theorem, their number is equal to 8. This shows that $\#H = 56$ or 84. In the first case, H contains a Sylow 2-subgroup of order 8. Since H is normal, all its conjugates are in H, and, in particular, $T \in H$. The quotient group G/H is of order 3. It follows from (6.46) that the coset of U must be trivial. Since 3 does not divide 56, we get a contradiction. In the second case, H contains S, T, U and hence coincides with G. So, we have shown that H cannot contain an element of order 7. Suppose it contains an element of order 3. Since all such elements are conjugate, H contains U. It follows from (6.44) that the coset of S in G/H is trivial, and hence, $S \in H$, contradicting the assumption. It remains to consider the case when H is a 2-subgroup. Then, $\#G/H = 2^a \cdot 3 \cdot 7$, with $a \leq 2$. It follows from Sylow's Theorem that the image of the Sylow 7-subgroup in G/H is normal. Thus, its pre-image in G is normal. This contradiction finishes the proof that G is simple. □

Remark 6.5.4. One can show that

$$G_{168} \cong \mathrm{PSL}(2, \mathbb{F}_7) \cong \mathrm{PSL}(3, \mathbb{F}_2).$$

The first isomorphism has a natural construction via the theory of automorphic functions. The Klein curve is isomorphic to a compactification of the modular curve $X(7)$, corresponding to the principal congruence subgroup of full level 7. The second isomorphism has a natural construction via considering a model of the Klein curve over a finite field of two elements (see [284]). We can see an explicit action of G on 28 bitangents via the geometry of the projective line $\mathbb{P}^1(\mathbb{F}_7)$ (see [176], [431]).

The group $\mathrm{Aut}(C)$ acts on the set of 36 even theta characteristics with orbits of cardinality $1, 7, 7, 21$ (see [235]. The unique invariant even theta characteristic

θ gives rise to a unique G-invariant in $\mathbb{P}^3 = \mathbb{P}(V)$, where $V = H^0(C, \theta(1))$. Using the character table, one can decompose the linear representation $S^2(V)$ into the direct sum of the 3-dimensional representation $E = H^0(C, \mathcal{O}_C(1))^\vee$ and a 7-dimensional irreducible linear representation. The linear map $E \to S^2(V)$ defines the unique invariant net of quadrics. This gives another proof of the uniqueness of an invariant theta characteristic. The corresponding representation of C as a symmetric determinant is due to F. Klein [455] (see also [272]). We have

$$\det\begin{pmatrix} -t_0 & 0 & 0 & -t_2 \\ 0 & t_2 & 0 & -t_2 \\ 0 & 0 & t_2 & -t_0 \\ t_2 & -t_2 & -t_0 & 0 \end{pmatrix} = t_0^3 t_2 + t_2^3 t_0 + t_1^3 t_0. \tag{6.47}$$

The group $\mathrm{Aut}(C)$ has 3 orbits on C with nontrivial stabilizers of orders $2, 3, 7$. They are of cardinality $84, 56$, and 24, respectively.

The orbit of cardinality 24 consists of inflection points of C. They are the vertices of the eight triangles with inflection tangents as their sides. These are eight contact cubics corresponding to the unique invariant theta characteristic. The eight inflection triangles coincide with eight biscribed triangles. The group acts on the eight triangles with stabilizer subgroup of order 21. In fact, the coordinate triangle is one of the eight triangles. The subgroup generated by S and U leaves it invariant. The element T of order 2 sends the coordinate triangle to the triangle with sides whose coordinates are the rows of the matrix (6.45). In fact, this is how the element T was found (see [455] or [309], vol. 2, p. 199).

We know that the inflection points are the intersection points of C and its Hessian given by the equation

$$\mathrm{He}(f) = 5t_0^2 t_2^2 t_2^2 - t_0 t_2^5 - t_0^5 t_1 - t_1 t_1^5 = 0.$$

So, the orbit of 24 points is cut out by the Hessian.

The orbit of cardinality 56 consists of the tangency points of 28 bitangents of C. An example of an element of order 3 is a cyclic permutation of coordinates. It has two fixed points $[1, \eta_3, \eta_3^2]$ and $[1, \eta_3^2, \eta_3]$ on C. They lie on the bitangent with equation

$$4t_0 + (3\eta_3^2 + 1)t_1 + (3\eta_3 + 1)t_2 = 0.$$

Define a polynomial of degree 14 by

$$\Psi = \det \begin{pmatrix} \frac{\partial^2 f}{\partial t_0^2} & \frac{\partial^2 f}{\partial t_0 t_1} & \frac{\partial^2 f}{\partial t_0 t_2} & \frac{\partial f}{\partial t_0} \\ \frac{\partial^2 f}{\partial t_1 t_0} & \frac{\partial^2 f}{\partial t_1^2} & \frac{\partial^2 f}{\partial t_1 t_2} & \frac{\partial f}{\partial t_1} \\ \frac{\partial^2 f}{\partial t_2 t_0} & \frac{\partial^2 f}{\partial t_2 t_1} & \frac{\partial^2 f}{\partial t_2^2} & \frac{\partial f}{\partial t_2} \\ \frac{\partial f}{\partial t_0} & \frac{\partial f}{\partial t_1} & \frac{\partial f}{\partial t_2} & 0 \end{pmatrix}.$$

One checks that it is invariant with respect to G_{168} and does not contain f as a factor. Hence, it cuts out in $V(f)$ a G-invariant positive divisor of degree 56. It must consist of a G_{168}-orbit of cardinality 56.

One can compute it explicitly (see WeberAlgebra [799], p. 524) to find that

$$\Psi = t_0^{14} + t_1^{14} + t_2^{14} - 34 t_0 t_1 t_2 (t_0^{10} t_1 + \cdots) - 250 t_0 t_1 t_2 (t_0^3 t_2^8 + \cdots) +$$

$$375 t_0^2 t_1^2 t_2^2 (t_0^6 t_1^2 + \cdots) + 18 (t_0^7 t_2^7 + \cdots) - 126 t_0^3 t_2^3 t_1^3 (t_0^3 t_2^2 + \cdots).$$

Here, the dots mean monomials obtained from the first one by permutation of variables.

The orbit of cardinality 84 is equal to the union of 21 sets, each consisting of four intersection points of C with the line of fixed points of a transformation of order two. An example of such a point is

$$[(\epsilon^4 - \epsilon^3)(\epsilon - \epsilon^6)\epsilon^4, (\epsilon^2 - \epsilon^5)(\epsilon - \epsilon^6)\epsilon, (\epsilon^4 - \epsilon^3)(\epsilon^2 - \epsilon^5)\epsilon^2].$$

The product ξ of the equations defining the 21 lines defines a curve of degree 21 which coincides with the curve $V(J(f, H, \Psi))$, where $J(f, H, \Psi)$ is the Jacobian determinant of f, the Hesse polynomial, and Ψ. It is a G_{168}-invariant polynomial of degree 21. Its explicit expression was given by P. Gordan in Gordan2 [347], p. 372:

$$\Xi = t_0^{21} + t_1^{21} + t_2^{21} - 7 t_0 t_1 t_2 (t_0^{17} t_1 + \cdots) + 217 t_0 t_1 t_2 (t_0^3 t_2^{15} + \cdots) -$$

$$308 t_0^2 t_1^2 t_2^2 (t_0^{13} t_1^2 + \cdots) - 57 (t_0^{14} t_1^7 + \cdots) - 289 (t_0^7 t_2^{14} + \cdots) +$$

$$4018 t_0^3 t_1^3 t_2^3 (t_0^2 t_2^{10} + \cdots) + 637 t_0^3 t_2^3 t_1^3 (t_0^9 t_2^3 + \cdots) +$$

$$1638 t_0 t_1 t_2 (t_0^{10} t_2^8 + \cdots) - 6279 t_0^2 t_1^2 t_2^2 (t_0^6 t_2^9 + \cdots) +$$

$$7007 t_0^5 t_1^5 t_2^5 (t_0 t_2^5 + \cdots) - 10010 t_0^4 t_1^4 t_2^4 (t_0^5 t_2^4 + \cdots) + 3432 t_0^7 t_1^7 t_2^7.$$

The group G_{168} admits a central extension $2.L_2(7) \cong \mathrm{SL}(2, \mathbb{F}_7)$. It has a linear representation in $\mathbb{C}^3$ where it acts as a complex reflection group. The algebra of invariants is generated by the polynomial f defining the Klein curve, the Hesse polynomial H, and the polynomials Ψ. The polynomial Ξ is a skew invariant, it is not invariant but its square is. We have (see [347], [309, Vol. 2, p. 208])

$$\Xi^2 = \Phi^3 - 88f^2H\Psi^2 + 16(63fH^4\Psi + 68f^4H^2\Psi - 16f^7\Psi$$
$$+108H^7 - 3752f^3H^5 + 1376f^6H^3 - 128f^9H). \tag{6.48}$$

(note that there is some discrepancy of signs in the formulas of Gordan and Fricke).

We have already mentioned that the Scorza quartic of the Klein quartic C coincides with C. The corresponding even theta characteristic is the unique invariant even theta characteristic θ. One can find all quartic curves X such that its Scorza quartic is equal to C (see [132], [235]).

The group G acts on the set of 63 Steiner complexes, or, equivalently, on the set of nontrivial two-torsion divisor classes of the Jacobian of the curve. There is one orbit of length 28, an orbit of length 21, and two orbits of length 7. Also, the group G_{168} acts on Aronhold sets with orbits of length $8, 168, 56$ and 56 [431]. In particular, there is no invariant set of seven points in the plane that defines C.

The variety $\mathrm{VSP}(C,6)$ is a Fano threefold V_{22} admitting G_{168} as its group of automorphisms. It is studied in [513].

Exercises

6.1 Show that two syzygetic tetrads of bitangents cannot have two common bitangents.

6.2 Let $C_t = V(tf+q^2)$ be a family of plane quartics over $\mathbb{C}$ depending on a parameter t. Assume that $V(f)$ is nonsingular and $V(f)$ and $V(q)$ intersect transversally at eight points $p_1,\ldots,p_8$. Show that C_t is nonsingular for all t in some open neighborhood of 0 in the usual topology and the limit of 28 bitangents when $t\to 0$ is equal to the set of 28 lines $\overline{p_ip_j}$.

6.3 Show that the locus of nonsingular quartics that admit an inflection bitangent is a hypersurface in the space of all nonsingular quartics.

6.4 Consider the Fermat quartic $V(t_0^4+t_1^4+t_2^4)$. Find all bitangents and all Steiner complexes. Show that it admits 12 inflection bitangents.

6.5 Show that a Fermat quartic has 12 inflection bitangents and this number is maximal possible [475].

6.6 Let $S=\{(\ell_1,\ell_1'),\ldots,(\ell_6,\ell_6')\}$ be a Steiner complex of 12 bitangents. Prove that the six intersection points $\ell_i\cap\ell_i'$ lie on a conic and all $\binom{28}{2}=378$ intersection points of bitangents lie on 63 conics.

6.7 Show that the pencil of conics passing through the four points of contact of two bitangents contains five members each passing through the points of contact of a pair of bitangents.

6.8 Show that a choice of $\epsilon \in \mathrm{Jac}(C)[2] \setminus \{0\}$ defines a conic Q and a cubic B such that C is equal to the locus of points x such that the polar $P_x(B)$ is touching Q.

6.9 Let $C = V(a_{11}a_{22} - a_{12}^2)$ be a representation of a nonsingular quartic C as a symmetric quadratic determinant corresponding to a choice of a 2-torsion divisor class ϵ. Let $\tilde{C}$ be the unramified double cover of C corresponding to ϵ. Show that $\tilde{C}$ is isomorphic to a canonical curve of genus 5 given by the equations

$$a_{11}(t_0,t_1,t_2) - t_3^2 = a_{12}(t_0,t_1,t_2) - t_3t_4 = a_{22}(t_0,t_1,t_2) - t_4^2 = 0$$

in $\mathbb{P}^4$.

6.10 Show that the moduli space of bielliptic curves of genus 4 is birationally isomorphic to the moduli space of isomorphism classes of genus three curves together with a nonzero 2-torsion divisor class.

6.11 A plane quartic $C = V(f)$ is called a *Caporali quartic* if $\mathrm{VSP}(f,4)^o \neq \emptyset$.

(i) Show that the C admits a pencil of apolar conics.
(ii) Show that the Clebsch covariant quartic $\mathfrak{C}(C)$ is equal to the union of four lines.
(iii) Show that any Caporali quartic is projectively isomorphic to the curve

$$at_0(t_1^3 - t_2^3) + bt_1(t_2^3 - t_1^3) + ct_2(t_0^3 - t_1^3) = 0$$

([84]).

6.12 Let q be a nondegenerate quadratic form in three variables. Show that $\mathrm{VSP}(q^2,6)^o$ is a homogeneous space for the group $\mathrm{PSL}(2,\mathbb{C})$.

6.13 Show that the locus of lines $\ell = V(l)$ such that the anti-polar conic of l^2 with respect to a quartic curve $V(f)$ is reducible is a plane curve of degree 6 in the dual plane.

6.14 Classify automorphism groups of irreducible singular plane quartics.

6.15 For each nonsingular plane quartic curve C with automorphism group G describe the ramification scheme of the cover $C \to C/G$.

6.16 Let C be the Klein quartic. For any subgroup H of $\mathrm{Aut}(C)$ determine the genus of H and the ramification scheme of the cover $C \to C/H$.

6.17 Show that a smooth plane quartic admits an automorphism of order 2 if and only if among its 28 bitangents four form a syzygetic set of bitangents intersecting at one point.

6.18 Show that the set of polar conics $P_{x^2}(C)$ of a plane quartic C, where x belongs to a fixed line, form a family of contact conics of another plane quartic C'.

6.19 Show that the description of bitangents via the Cayley octad can be stated in the following way. Let $C = \det A$ be the symmetric determinantal representation of C with the Cayley octad O. Let P be the 8×4-matrix with columns equal to the coordinates of the points in O. The matrix $M = {}^tPAP$ is a symmetric 8×8-matrix, and its entries are the equations of the bitangents (the *bitangent matrix*, see [592]).

6.20 Show that the bitangents participating in each principal 4×4-minor of the bitangent matrix from the previous exercise is a syzygetic tetrad, and the minor itself defines the equation of the form (6.1).

6.21 Let C and K be a general conic and a general cubic. Show that the set of points a such that $P_a(C)$ is tangent to $P_a(K)$ is a Lüroth quartic. Show that the set of polar lines $P_a(C)$ which coincide with polar lines $P_a(K)$ is equal to an Aronhold set of seven bitangents of the Lüroth quartic ([39]).

6.22 Prove *Sonya Kowalevskaya's Theorem*: a smooth quartic curve admits a bielliptic involution if and only if four of its bitangents intersect at one point [466].

6.23 Using the previous theorem, show that the set of 28 bitangents of the Klein quartic contains 21 subsets of four concurrent bitangents and each bitangent has 3 concurrency points.

6.24 Let $v_3 : |E| \to |S^3(E^\vee)|$ be the Veronese embedding corresponding to the apolarity map $\mathrm{ap}^1_f : E \to S^3(E^\vee)$ for a general plane quartic $V(f) \subset |E|$. Show that the variety $\mathrm{VSP}(f, 6)$ is isomorphic to the variety of 6-secant planes of the projection of the Veronese surface $v_3(|E|)$ to $|S^3(E^\vee)/\mathrm{ap}^1_f(E)| \cong \mathbb{P}^6$ ([513]).

6.25 Find a symmetric determinant expression for the Fermat quartic $V(t_0^4 + t_1^4 + t_2^4)$.

Historical Notes

The fact that a general plane quartic curve has 28 bitangents was first proved in 1850 by C. Jacobi [428] although the number was apparently known to J. Poncelet. The proof used Plücker formulas and did not apply to any nonsingular curve. Using contact cubics, O. Hesse extended this result to arbitrary nonsingular quartics [390].

The first systematic study of the configuration of bitangents began by O. Hesse [390],[391] and J. Steiner [725]. Steiner's paper does not contain proofs. They considered azygetic and syzygetic sets, and Steinercomplexes of bitangents, although the terminology was introduced later by Frobenius [312]. Hesse's approach used the relationship between bitangents and Cayley octads. The notion of a Steiner group of bitangents was introduced by A. Cayley in [116]. Weber [798] changed it to a Steiner complex in order not to be confused with the terminology in group theory.

The fact that the equation of a nonsingular quartic could be brought to the form (6.1) was first noticed by J. Plücker [598]. Equation (6.2), arising from a Steiner complex, appears first in Hesse's paper [391], §9. The determinantal identity for bordered determinants (6.27) appears in [390]. The number of hexads of bitangents with points of contact on a cubic curve was first computed by O. Hesse [390] and by G. Salmon [652].

The equation of a quartic as a quadratic determinant appeared first in Plücker [595, p. 228], and in Hesse [391], §10, [392]. Both knew that it could be done in 63 different ways. Hesse also proves that the 12 lines of a Steiner complex, considered as points in the dual plane, lie on a cubic. More details appear in Roth's paper [638] and later, in Coble's book [159].

S. Arinhold was the first to discover the relationship between bitangents of a plane quartic and seven points in the dual projective plane [20]. The fact that Hesse's construction and Aronhold' construction are equivalent via the projection from one point of a Cayley octad was first noticed by A. Dixon [228].

The relation of bitangents to theta functions with odd characteristics goes back to B. Riemann [625] and H. Weber [798] and was developed later by A. Clebsch [143] and G. Frobenius [312], [314]. In particular, Frobenius had found a relationship between the sets of seven points or Cayley octads with theta functions of genus 3. Coble's book [159] has a nice exposition of Frobenius's work. The equations of bitangents presented in Theorem 6.1.9 were first found by Riemann, with more details explained by H. Weber. A modern treatment of the theory of theta functions in genus three can be found in many papers. We refer to [321], [337], and the references therein.

The theory of covariants and contravariants of plane quartics was initiated by A. Clebsch in his fundamental paper about plane quartic curves [140]. In this paper, he introduces his covariant quartic $\mathfrak{C}(C)$ and the catalecticant invariant. He showed that the catalecticant vanishes if and only if the curve admits an apolar conic. Much later, G. Scorza [677] proved that the rational map S on the space of quartics is of degree 36 and related this number with the number of even theta characteristics. The interpretation of the apolar conic of a Clebsch quartic as the parameter space of inscribed pentagons was given by G. Lüroth [499]. In this paper (the first issue of Mathematische Annalen), he introduced the quartics that now bear his name. Darboux curves were first introduced by G. Darboux in [203]. They got a modern incarnation in a paper of W. Barth [32], where it was shown that the curve of jumping lines of a rank two vector bundle with trivial determinant is a Darboux curve. The modern exposition of works of F. Morley [528] and H. Bateman [39] on the geometry of Lüroth quartics can be found in papers of G. Ottaviani and E. Sernesi [563], [565], [566].

The groups of automorphisms of nonsingular plane quartic curves were classified by S. Kantor [438] and A. Wiman [808]. The first two curves from our table were studied earlier by F. Klein [455] and W. Dyck [268]. Of course, the Klein curve is the most famous and often appears in modern literature (see, for example, [714]).

The classical literature about plane quartics is enormous. We refer to Ciani's paper [133] for a survey of classical results, as well as to his contributions to the study of plane quartics, which are assembled in [135]. Other surveys can be found in [577] and [293].

7

Cremona Transformations

S:7.1

7.1 Homaloidal Linear Systems

SS:7.1.1

7.1.1 Linear systems and their base schemes

Recall that a *rational map* $f : X \dashrightarrow Y$ of algebraic varieties over a field $\Bbbk$ is a regular map defined on a dense open Zariski subset $U \subset X$. The largest such set, which f can be extended to as a regular map, is denoted by $\mathrm{dom}(f)$. A point $x \notin \mathrm{dom}(f)$ is called an *indeterminacy point*. Two rational maps are considered to be equivalent if their restrictions to an open dense subset coincide. A rational map is called *dominant* if $f : \mathrm{dom}(f) \to Y$ is a dominant regular map, i.e. the image is dense in Y. Algebraic varieties form a category with morphisms taken to be equivalence classes of dominant rational maps.

From now on, we restrict ourselves to rational maps of irreducible varieties over $\mathbb{C}$. We use f_{d} to denote the restriction of f to $\mathrm{dom}(f)$, or to any open subset of $\mathrm{dom}(f)$. A dominant map $f_{\mathrm{d}} : \mathrm{dom}(f) \to Y$ defines a homomorphism of the fields of rational functions $f^* : R(Y) \to R(X)$. Conversely, any homomorphism $R(Y) \to R(X)$ arises from a unique equivalence class of dominant rational maps $X \dashrightarrow Y$. If f^* makes $R(X)$ a finite extension of $R(Y)$, the degree of the extension is the *degree* of f. A rational map of degree one is called a *birational map*. It can also be defined as an invertible rational map.

We will further assume that X is a smooth projective variety. It follows that the complement of $\mathrm{dom}(f)$ is of codimension ≥ 2. Let $|V'| \subset |\mathcal{L}'|$ be a linear system of Cartier divisors on Y, then the pre-image of any divisor $D' \in |V'|$ under f_d is a Cartier divisor on $\mathrm{dom}(f)$. It can be uniquely extended to a Cartier divisor D on X. Also, $f_d^*\mathcal{L}'$ can be extended uniquely to an an invertible sheaf $\mathcal{L}$ on X such that $D \in |\mathcal{L}|$. The linear map $f^* : V' \to H^0(X, \mathcal{L})$ is injective and its image is a linear subspace $V \subset H^0(X, \mathcal{L})$. The linear system $|V| \subset |\mathcal{L}$

obtained in this way is called the *proper transform* of $|V'|$ and denoted by $f^{-1}(|V|)$.

Any rational map $f : X \dashrightarrow Y$ is defined by a linear system. Namely, we embed Y in a projective space $\mathbb{P}^r$ by a complete linear system $|V'| := |H^0(Y, \mathcal{L}')|$. Its divisors are hyperplane sections of Y. Then, f is defined by $f^{-1}(|V'|)$.

The rational map f is given in the usual way. Evaluating sections of V at a point, we get a map $\mathrm{dom}(f) \to \mathbb{P}(V)$ and,, by restriction, the map $\mathrm{dom}(f) \to \mathbb{P}(V')$, which factors through the map $Y \hookrightarrow \mathbb{P}(V')$. A choice of a basis $(s_0, \ldots, s_r)$ in V and a basis in V' defines a rational map $f : X \dashrightarrow Y \subset \mathbb{P}^r$. It is given by the formula

$$x \mapsto [s_0(x), \ldots, s_r(x)].$$

Let $\mathcal{L}$ be a line bundle and $V \subset H^0(X, \mathcal{L})$. Consider the natural evaluation map of sheaves

$$\mathrm{ev} : V \otimes \mathcal{O}_X \to \mathcal{L}$$

defined by restricting global sections to stalks of $\mathcal{L}$. Tensoring by $\mathcal{L}^{-1}$, we get a map

$$\mathrm{ev} : V \otimes \mathcal{L}^{-1} \to \mathcal{O}_X$$

whose image is a sheaf of ideals in $\mathcal{O}_X$. This sheaf of ideals is denoted $\mathfrak{b}(|V|)$ and is called the *base ideal* of the linear system $|V|$. The closed subscheme $\mathrm{Bs}(|V|)$ of X defined by this ideal is called the *base scheme* of $|V|$. The reduced scheme is called the *base locus*. In classical terminology, the base locus is the *F-locus*; its points are called *fundamental points*. We have

$$\mathrm{Bs}(|V|) = \cap_{D \in |V|} D = D_0 \cap \ldots \cap D_r \text{ (scheme-theoretically)},$$

where $D_0, \ldots, D_r$ are the divisors of sections forming a basis of V. The largest positive divisor F contained in all divisors from $|V|$ (equivalently, in the divisors $D_0, \ldots, D_r$) is called the *fixed component* of $|V|$. The linear system without fixed component is sometimes called *irreducible*. Each irreducible component of its base scheme is of codimension ≥ 2.

If $F = \mathrm{div}(s_0)$ for some $s_0 \in H^0(X, \mathcal{O}_X(F))$, then the multiplication by s_0 defines an injective map $\mathcal{L}(-F) \to \mathcal{L}$. The associated linear map $H^0(X, \mathcal{L}(-F)) \to H^0(X, \mathcal{L})$ defines an isomorphism from a subspace W of $H^0(X, \mathcal{L}(-F))$ onto V. The linear system $|W| \subset |\mathcal{L}(-F)|$ is irreducible and defines a rational map $f' : X \dashrightarrow \mathbb{P}(W) \cong \mathbb{P}(V)$.

A linear system is called *base-point-free*, or simply *free* if its base scheme is empty, i.e., $\mathfrak{b}(|V|) \cong \mathcal{O}_X$. The proper transform of such a system under a rational map is an irreducible linear system. In particular, the linear system $|V|$ defining a rational map $X \dashrightarrow Y$ as described in above, is always irreducible.

Here, are some simple properties of the base scheme of a linear system.

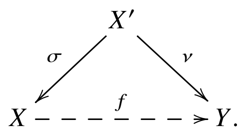
(i) $|V| \subset |\mathcal{L} \otimes \mathfrak{b}(|V|)| := |H^0(X, \mathfrak{b}(|V|) \otimes \mathcal{L})|$.

(ii) Let $\phi : X' \to X$ be a regular map, and $V' = \phi^*(V) \subset H^0(X', \phi^*\mathcal{L})$. Then, $\phi^{-1}(\mathfrak{b}(|V|)) = \mathfrak{b}(f^{-1}(|V|))$. Recall that, for any ideal sheaf $\mathfrak{a} \subset \mathcal{O}_X$, its inverse image $\phi^{-1}(\mathfrak{a})$ is defined to be the image of $\phi^*(\mathfrak{a}) = \mathfrak{a} \otimes_{\mathcal{O}_X} \mathcal{O}_{X'}$ in $\mathcal{O}_{X'}$ under the canonical multiplication map.

(iii) If $\mathfrak{b}(|V|)$ is an invertible ideal (i.e. isomorphic to $\mathcal{O}_X(-F)$ for some effective divisor F), then $\mathrm{dom}(f) = X$ and f is defined by the linear system $|\mathcal{L}(-F)|$.

(iv) If $\mathrm{dom}(f) = X$, then $\mathfrak{b}(|V|)$ is an invertible sheaf and $\mathrm{Bs}(|V|) = \emptyset$.

Definition 7.1.1. *A* resolution of indeterminacy *of a rational map* $f : X \dashrightarrow Y$ *of projective varieties is a pair of regular projective morphisms* $\pi : X' \to X$ *and* $\sigma : X' \to Y$ *such that* $f = \sigma \circ \pi^{-1}$ *and* π *is an isomorphism over* $\mathrm{dom}(f)$*:*


(7.1)

We say that a resolution is smooth (normal) if X' is smooth (normal).

We have encountered resolutions of indeterminacy in Section 2.4. For example, the space CQ of complete quadrics is a resolution of indeterminacy of the adjugate birational map Adj.

We denote by

$$\sigma : \mathrm{Bl}_Z(X) = \mathrm{Proj} \bigoplus_{k=0}^{\infty} \mathfrak{a}^k \to X$$

the *blow-up* of Z (see Subsection 2.4.1).

An ideal sheaf $\mathfrak{a} \subset \mathcal{O}_X$ is called a *contracted ideal* for a morphism $f : Y \to X$ satisfying $f_*\mathcal{O}_Y = \mathcal{O}_X$ if $\mathfrak{a} = f_*(\mathcal{I})$ for some sheaf of ideals $\mathcal{I}$ on Y [489, II:Definition (6.1)] If X is normal, an ideal $\mathfrak{a}$ is said to be *complete* if and only if it is contracted for every proper birational morphism $f : Y \to X$. A local definition is that an ideal I in a commutative a normal ring A is complete if is *integrally closed*, i.e.,

$$I = \bar{I} := \{x \in A : x^n + a_1x^{n-1} + \cdots + a_n = 0 \text{ for some } a_k \in I^k\}.$$

It follows from the universality property of the blow-up that $\mathfrak{a}$ is complete if and only $\mathfrak{a}$ is contracted for σ_+, where $\sigma^+ : \mathrm{Bl}_{V(\mathfrak{a})}(X)^+ \to \mathrm{Bl}_{V(\mathfrak{a})}(X) \to X$ is the composition of σ with the normalization map and E^+ is its exceptional divisor. In fact, for any Ideal $\mathfrak{a} \subset \mathcal{O}_X$, its *integral closure* $\bar{\mathfrak{a}}$ coincides with

$(\sigma^+)_+(\mathcal{O}_{V(\mathfrak{a})}(-E^+)$. If $E^+ = \sum r_i E_i$, where E_i are irreducible Weil divisors, then

$$f \in \bar{\mathfrak{a}} \Longleftrightarrow \mathrm{ord}_{E_i}(f) \geq r_i \quad \text{for all } i. \tag{7.2}$$

valuecrit

We have $\mathsf{Bl}_{V(\mathfrak{a})}(X)^+ = \mathsf{Bl}_{V(\mathfrak{a})}(X)$ if and only $\mathfrak{a}^m$ is integrally closed for all $m \geq 0$. If X is nonsingular, and $\dim X = 2$, then $m = 1$ suffices [815, Appendix 5],[489, II, Theorem 7.1]

Example 7.1.2. Let $\mathfrak{a} = (u^2, v^3) \subset \Bbbk[u, v]$, then $(uv^2)^2 + 0 \cdot uv^2 - u^2v^4 = 0$, since $u^2v^4 \in \mathfrak{a}^2$, $uv^2 \in \bar{\mathfrak{a}}$. The blow-up $\mathsf{Bl}_{V(\mathfrak{a})}(\mathbb{A}^2) \cong V(u^2t_0 - v^2t_1)$. It is not normal along the exceptional curve $E = V(u, v)$. On the other hand, the ideal (u, v^n) is integrally closed for any n.

Let $\bar{\mathfrak{a}} = \mathfrak{a}$ and $\mathfrak{a} = \mathfrak{q}_1 \cap \ldots \cap \mathfrak{q}_k$ as the intersection of primary sheaves of ideals. We assume that there are no inclusions among $\mathfrak{q}_i$'s. Then, each $\mathfrak{q}_i$ is integrally closed, and

$$\mathfrak{q}_i = \sigma_* \mathcal{O}_{\mathsf{Bl}_Z(X)^+}(-r_i E_i). \tag{7.3}$$

riEi

It follows that $\mathfrak{a}$ is integrally closed if and only if its primary components are integrally closed.

Remark 7.1.3. Let $Z = V(\mathcal{I}_Z)$ be an irreducible reduced subvariety of a nonsingular quasi-projective variety X over $\mathbb{C}$ with $\mathfrak{a} = \mathcal{I}_Z \subset \mathcal{O}_X$. The *m-symbolic power* of Z is the closed subscheme $Z^{\langle m \rangle} = V(\mathfrak{a}^{\langle m \rangle})$ of $mZ := V(\mathcal{I}_Z^m)$ such that $(Z^{\langle m \rangle})_{\mathrm{red}} = Z$. The sheaf of ideals $\mathfrak{a}^{\langle m \rangle}$ is called the mth symbolic power of $\mathfrak{a}$, For example, take $X = \mathrm{Spec}(\mathbb{C}[x, y, z])$ and $Z = V(xy, yz, xz)$. Then, $V(xyz)_{\mathrm{red}} = Z$ but $xyz \notin (xy, xz, yz)^2$. According to the theorem of Zariski and Nagata ,

$$\mathfrak{a}^{\langle m \rangle} = \{f \in \mathcal{O}_X : \mathrm{ord}_x(f) \geq m \text{ for all } x \in Z\}$$

(it is enough here to consider only the generic points of irreducible components of Z). It follows that

$$\mathfrak{a}^{\langle m \rangle} = \cap_{x \in Z} \mathfrak{m}_{x,Z}^m.$$

Another characterization of $\mathfrak{a}^{\langle m \rangle}$ in an affine variety X is

$$\mathfrak{a}^{\langle m \rangle} = \{f \in \mathcal{O}(X) : Df \in \mathfrak{a} \text{ for all differential operators } D \text{ on } X \text{ of order} < m\}.$$

It implies that $\mathfrak{a}^{\langle m \rangle} = \mathfrak{a}^m$ if Z is nonsingular, for example, Z is a closed point.

According to [279, Theorem A], for all $m \geq 0$,

$$\mathfrak{a}^{\langle em \rangle} \subset \mathfrak{a}^m,$$

where e is the codimension of Z.

For any irreducible component $Z_i = V(\mathfrak{a}_i)$ of $\mathrm{Bs}(|V|)_{\mathrm{red}}$ one defines the *multiplicity* of $|V|$ (or $\mathrm{Bs}(|V|)$, or $\mathfrak{b}(|V|)$) to be the largest m such that

$$\mathfrak{b}(|V|) \subset \mathfrak{a}_i^{\langle m \rangle}$$

[479, I:Definition 9.3.6]. It coincides with the multiplicity $\mathrm{mult}_{V(\mathfrak{a}_i)}(D)$ of a general divisor $D \in |V|$ along $V(\mathfrak{a}_i)$, i.e., the multiplicity of D at the generic point of $V(\mathfrak{a}_i)$. We use the notation $\mathrm{mult}_{Z_i}(|V|)$ ($\mathrm{mult}_{\mathfrak{a}_i}(\mathfrak{b}(|V|)$, $\mathrm{mult}_{Z_i}(\mathfrak{b}(|V|))$.

If $\mathfrak{b}(|V|)$ is integrally closed, and $\mathfrak{b}(|V|) = \sigma_*(\mathcal{O}_X(-\sum r_i E_i))$ as in (7.3), then $\mathrm{mult}_{\mathfrak{a}_i}(\mathfrak{b}(|V|)) \le r_i$. In example 7.1.22, the multiplicity is equal to one, but $r_1 = 2$.

Example 7.1.4. Let Y be a closed reduced subscheme of a smooth scheme X and $\mathfrak{a}$ be its sheaf of ideals. Let mY denote the closed subscheme of X defined by the ideal $\mathfrak{a}^m$.

For any graded algebra $A = \oplus_{i=0}^{\infty}$ generated by elements of degree one, the projective spectrum $\mathrm{Proj}(A)$ is isomorphic to the projective spectrum of the grades subalgebra $A^{(m)} := \oplus_{i=0} A_i^{(m)}$, where $A_i^{(m)} = A_{ei}$. This shows that there exists an isomorphism $v_m : \mathsf{Bl}_{mY}(X) \cong \mathsf{Bl}_Y(X)$. Under this isomorphic, the pre-image of the exceptional divisor of $\mathsf{Bl}_{mY}(X)$ is equal to mE, where E is the exceptional divisor of $\mathsf{Bl}_Y(X)$. t follows that

$$\sigma_* \mathcal{O}_{\mathsf{Bl}_Y(X)}(-mE) = \mathfrak{a}^m.$$

It is clear that $\mathrm{mult}_{\mathfrak{a}}(\mathfrak{a}^m) = m$.

Proposition 7.1.5. *Let $f : X \dashrightarrow Y$ be a rational map of irreducible varieties defined by a linear system $|V|$ with the base scheme $Z = V(\mathfrak{b}(|V|))$. Let $\sigma : \mathsf{Bl}_Z(X) \to X$ be the blow-up of Z. Then, there exists a unique regular map $\nu : \mathsf{Bl}_Z(X) \to Y$ such that (σ, ν) is a resolution of indeterminacy of f. For any resolution of indeterminacy (σ', ν') of f, there exists a unique morphism $\alpha : X' \to \mathsf{Bl}_Z(X)$ such that $\sigma' = \sigma \circ \alpha$, $\nu' = \nu \circ \alpha$,.*

Proof By properties (ii) and (iii) from above, the linear system $\sigma^{-1}(|V|) = |\sigma^*(\mathcal{L}) \otimes \sigma^{-1}(\mathfrak{b})|$ defines a regular map $\sigma : \mathsf{Bl}_Z(X) \to Y$. It follows from the definition of maps defined by linear systems that $f = \sigma \circ \nu^{-1}$. For any resolution, (X', σ', ν') of f, the base scheme of the inverse transform $\sigma^{-1}(|V|)$ on X' is equal to $\sigma^{-1}(\mathfrak{b})$. The morphism σ' is defined by the linear system $\sigma'^{-1}(|V|)$ and hence its base sheaf is invertible. This implies that σ' factors through the blow-up of Z. □

Note that we also obtain that the exceptional divisor of σ' is equal to the pre-image of the exceptional divisor of the blow-up of $\mathrm{Bs}(|V|)$.

Any diagram (7.1) defines the rational map $f = \nu \circ \sigma^{-1}$. So, if $\mathfrak{b}(|V|)$ is not integrally closed, we replace $X' = \mathrm{Bl}_{V(\mathfrak{b}(|V|))}(X)$ with its normalization X'^{+} and replace f with f^{+} as above. Obviously, f and f^{+} coincide on $\mathrm{dom}(f)$, hence $f = f^{+}$ as rational maps. Now, we can change the linear system to assume that its base ideal is integrally closed.

thm1 **Theorem 7.1.6.** *Assume that $f : X \dashrightarrow Y$ is a birational map of normal projective varieties and f is given by a linear system $|V| \subset |\mathcal{L}|$ equal to the inverse transform of a very ample complete linear system $|\mathcal{L}'|$ on Y. Let (X', σ, ν) be a resolution of indeterminacy of f and let E be the exceptional divisor of σ. Then, the canonical map*

$$V \to H^0(X', \sigma^*\mathcal{L}(-E))$$

is an isomorphism.

Proof Set $\mathfrak{b} = \mathfrak{b}(|V|)$. We have natural maps

$$V \to H^0(X, \mathcal{L} \otimes \mathfrak{b}) \to H^0(X', \sigma^*\mathcal{L} \otimes \sigma^{-1}(\mathfrak{b})) \xrightarrow{\cong} H^0(X', (\sigma^*\mathcal{L})(-E))$$

$$\xrightarrow{\cong} H^0(X', \sigma^*\mathcal{L}') \xrightarrow{\cong} H^0(Y, \sigma_*\sigma^*\mathcal{L}') \xrightarrow{\cong} H^0(Y, \mathcal{L}' \otimes \sigma_*\mathcal{O}_{X'}) \xrightarrow{\cong} H^0(Y, \mathcal{L}').$$

Here, we used the Main Zariski Theorem that asserts that $\sigma_*\mathcal{O}_{X'} \cong \mathcal{O}_Y$ because σ is a birational morphism and Y is normal [379, Chapter III, Corollary 11.4]. By definition of the linear system defining f, the composition of all these maps is a bijection. Since each map here is injective, we obtain that all the maps are bijective. One of the compositions is our map $V \to H^0(X', \sigma^*\mathcal{L}(-E))$, hence it is bijective. □

cor:7.1.4 **Corollary 7.1.7.** *Assume, additionally, that the resolution of indeterminacy (X, σ, ν) is normal. Then, the natural maps*

$$V \to H^0(X, \mathcal{L} \otimes \mathfrak{b}(|V|)) \to H^0(X', \sigma^*(\mathcal{L})(-E)) \to H^0(X, \mathcal{L} \otimes \overline{\mathfrak{b}(|V|)})$$

are bijective.

We apply Theorem 7.1.6 to the case when $f : \mathbb{P}^n \dashrightarrow \mathbb{P}^n$ is a birational map, a *Cremona transformation*. In this case, $\mathcal{L} = \mathcal{O}_{\mathbb{P}^n}(d)$ for some $d \geq 1$, called the (algebraic) *degree* of the Cremona transformation f. We take $|\mathcal{L}'| = |\mathcal{O}_{\mathbb{P}^n}(1)|$. The linear system $|V| = |\mathfrak{b}(|V|)(d)|$ defining a Cremona transformation is called a *homaloidal linear system*. In classical literature, members of a homaloidal linear system are called *homaloids*. More generally, a *k-homaloid* is a proper transform of a k-dimensional linear subspace in the target space.

P1.2.2 **Proposition 7.1.8.**

$$H^1(\mathbb{P}^n, \mathcal{L} \otimes \overline{\mathfrak{b}(|V|)}) = 0.$$

Proof Let (X, σ, ν) be the resolution of indeterminacy of f defined by the normalization of the blow-up of $\mathrm{Bs}(|V|)$. Let E be the exceptional divisor of $\sigma : X \to \mathbb{P}^n$. We know that $\sigma_*(\sigma^*\mathcal{L}(-E)) = \mathcal{L} \otimes \overline{\mathfrak{b}(|V|)}$ and $\nu^*\mathcal{L}(-E) \cong \nu^*\mathcal{O}_{\mathbb{P}^n}(1)$. The low degree exact sequence defined by the Leray spectral sequence, together with the projection formula, gives an exact sequence

$$0 \to H^1(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(1)) \to H^1(X, \nu^*\mathcal{O}_{\mathbb{P}^n}(1)) \to H^0(\mathbb{P}^n, R^1\nu_*\mathcal{O}_{X'} \otimes \mathcal{O}_{\mathbb{P}^n}(1)). \tag{7.4}$$

exa1

Let $\pi : X' \to X$ be a resolution of singularities of X. Then, we have the spectral sequence

$$E_2^{pq} = R^p\sigma_*(R^q\pi_*\mathcal{O}_{X'}) \Rightarrow R^{p+q}(\sigma \circ \pi)_*\mathcal{O}_{X'}.$$

It gives the exact sequence

$$0 \to R^1\pi_*(\pi_*\mathcal{O}_{X'}) \to R^1(\sigma \circ \pi)_*\mathcal{O}_{X'} \to \pi_*R^1\pi_*\mathcal{O}_{X'}.$$

Since X is normal, $\pi_*\mathcal{O}_{X'} = \mathcal{O}_X$. Since the composition $\sigma \circ \pi : X' \to \mathbb{P}^n$ is a birational morphism of nonsingular varieties, $R^1(\sigma \circ \pi)_*\mathcal{O}_{X'} = 0$. This shows that $R^1\sigma_*(\pi_*\mathcal{O}_{X'}) = 0$. Together with vanishing of $H^1(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(1))$, (7.4) implies that

$$H^1(X, \sigma^*\mathcal{L}(-E)) = 0.$$

It remains to use that the canonical map

$$H^1(\mathbb{P}^n, \mathcal{L} \otimes \overline{\mathfrak{b}(|V|)}) \cong H^1(\mathbb{P}^n, \sigma_*(\pi^*\mathcal{L}(-E)) \to H^1(X, \sigma^*\mathcal{L}(-E))$$

is injective (use the Čech cohomology, or the Leray spectral sequence). □

Using the exact sequence,

$$0 \to \overline{\mathfrak{b}(|V|)} \to \mathcal{O}_{\mathbb{P}^n} \to \mathcal{O}_{\mathbb{P}^n}/\overline{\mathfrak{b}(|V|} \to 0,$$

and tensoring it with $\mathcal{O}_{\mathbb{P}^n}(d)$, we obtain the following result, classically known as the *Postulation formula*.

postulation **Corollary 7.1.9.** *Let $|V|$ be a homaloidal linear system. Then*

$$h^0(\mathcal{O}_{V(\overline{\mathfrak{b}(|V|)})}(d)) = \binom{n+d}{d} - n - 1.$$

Let

$$\mathfrak{b}(|V|) = \bigoplus_{k=0}^{\infty} \mathfrak{b}(|V|)_k,$$

where $\mathfrak{b}(|V|)_k = H^0(\mathbb{P}^n, \mathfrak{b}(|V|)(k))$, be the grading of the base scheme ideal. We have

$$\mathfrak{b}(|V|)_d =\cong H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(1)) \cong \mathbb{C}^{n+1}. \tag{7.5}$$ post1

Also, we have

$$\mathfrak{b}(|V|)_k = 0,\ k < d. \tag{7.6}$$ post

Indeed, otherwise $|\mathfrak{b}(|V|)(d)|$ contains $|\mathcal{O}_{\mathbb{P}^n}(d-k)| + |\mathfrak{b}(|V|)(k)|$ and its dimension is strictly larger than $n+1$ if $k < d-1$, or it has fixed component if $k = d-1$.

The base locus $\mathrm{Bs}(|V|)$ could be very complicated, e.g., it could be non-reduced, contain embedded components, and possess other bad properties. The next proposition shows that not any closed subscheme of codimension ≥ 2 can be realized as the base scheme of a homaloidal linear syste,

completeintersection **Proposition 7.1.10.** $\mathrm{Bs}(|V|)$ *is not a complete intersection.*

Proof If $\mathrm{Bs}(|V|)$ is a complete intersection, then $\mathrm{Bs}(|V|)$ is equidimensional of codimension $c \leq n$ and its homogeneous ideal $\mathfrak{b}(|V|)$ is generated by forms $G_1 \ldots G_c$. By (7.5), we must have $\dim \mathfrak{b}(|V|)_d = n+1$. If G_i has degree $d_i < d$ then

$$G_i H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(d - d_i)) \subset V.$$

Then, for dimension reasons, $d - d_i = 1$. But then $V(G_i)$ is a fixed component of $\mathrm{Bs}(|V|)$, a contradiction. Hence, $d_i \geq d, i = 1, \ldots, c$. Since $c \leq n$ it follows that $\dim \mathfrak{b}(|V|)_d \leq c \leq n$, contradicting (7.5). □

Since $X' = \mathsf{Bl}_{V(\mathfrak{b}(|V|))}(X)$ is often singular, we will need to use the intersection theory on X' to compute the degree of the rational map, one often passes to a resolution of singularities of X'.

Definition 7.1.11. *Let* $|V| \subset |\mathcal{L}|$ *be a linear system on a nonsingular variety* X *without fixed components. A* log resolution *of* $|V|$ *is a projective birational morphism* $\mu : Y \to X$ *such that*

- $\mu^{-1}(\mathfrak{b}(|V|)) :== \mathfrak{b}(|V|) \cdot \mathcal{O}_Y = \mathcal{O}_Y(-F)$ *for some effective divisor* F.
- *Let* D *be a general member of* $|V|$ *and* $\mathrm{Exc}(\mu)$ *be the exceptional divisor of* μ*, then* $\mu^*(D) + \mathrm{Exc}(\mu)$ *is a divisor with simple normal crossings.*

Recall that a *simple normal crossing* divisor (sic-divisor, for short) is an effective divisor whose reduced irreducible components $D_i, i \in I$, are smooth, and, for any subset $J \subset I$, the intersection $\cap_{j \in J} D_i$ is the disjoint union of smooth subvarieties of codimension $|J|$.

Note that the first condition says that μ is a log resolution of the Ideal $\mathfrak{b}(|V|$ [479, II:Definition 9.1.12]

Since we are working over $\mathbb{C}$ a theorem of Hironaka implies that a log resolution always exists and can be obtained as a composition

$$Y = Y_N \xrightarrow{\mu_N} Y_{N-1} \xrightarrow{\mu_{N-1}} \dots \xrightarrow{\mu_2} Y_1 \xrightarrow{\mu_1} Y_0 = X, \tag{7.7}$$

where each morphism $\mu_i : Y_i \to Y_{i-1}$ is the blowing up of a smooth closed subscheme B_i of Y_{i-1}, which we can always assume to be of codimension ≥ 2.

For any $N \geq a > b \geq 1$, we set

$$\mu_{a,b} = \mu_a \circ \dots \circ \mu_{b+1} : Y_a \to Y_b, \quad \mu_{a,a} = \mu_a,$$

and $\mu = \mu_{N,0} : Y \to Y_0$.

Let $\mathcal{H} = |V|$ and $\mathcal{H}_i = \mu_{i1}^{-1}(\mathcal{H})$ be the proper transform of the linear system $\mathcal{H}$ in Y_i. Then B_i is contained in $\mathrm{Bs}(\mathcal{H}_i)$

Let $E_i = \nu_i^{-1}(B_i)$ be the exceptional divisor of $\nu_i : Y_{i+1} \to Y_i$. It is isomorphic to $\mathbb{P}(\mathcal{N}^\vee_{B_i/X_i})$. We denote by F_i its proper transform in Y. The linear system $\nu^{-1}(|V|)$ coincides with $|\mu^*(\mathcal{L})(-F)|$, where

$$F = \sum_{i \in I} m_i F_i,$$

and

$$m_i = \min_{D \in \mathcal{H}_i} \mathrm{mult}_{\eta_i}(D).$$

The first property of a log resolution and the universality property of the blow-up shows that there is a birational morphism

$$\nu : Y \to \mathsf{Bl}_{\mathrm{Bs}(\mathcal{H})}(X)^+.$$

It is a resolution of singularities of $\mathsf{Bl}_{\mathrm{Bs}(\mathcal{H})}(X)^+$, and also, a log resolution of the exceptional divisor $\mathrm{Exc}(\sigma)$ of $\mathsf{Bl}_{\mathrm{Bs}(\mathcal{H})}(X)^+ \to X$.

Let $\mathrm{Exc}(\mu) = \sum F_i$, we denote by $S(\mu)$ the simplicial complex on the set of irreducible components of all possible non-empty intersections $F_J = \cap_{i \in J} F_i, J \subset I$, with simplices equal to finite linearly ordered subsets of the set. The extended simplicial complex is obtained from $\tilde{S}(\mu)$ by adding a general member of $\mu^{-1}(|V|)$ and its intersections with all sets F_J. By a theorem of V. Danilov [198, Proposition 4], the topological realization of $\tilde{S}(\mu)$ is connected and simply-connected. Moreover, applying the main theorem from [198] and [479, Example 9.1.16], we obtain that the homotopy type of $\tilde{S}(\mu)$ does not depend on a log resolution.

The structure of a simplicial complex on $S(\mu)$ defines a partial order on the set of subvarieties B_i by writing $B_i > B_j$ if $\mu_{i,j}(E_i) \subset B_j$.

If, additionally, $\mu_{i,j} : Y_i \to Y_j$ maps B_i dominantly onto B_j, we say that B_i is *infinitely near* to B_j of order $k = i - j$ and write $B_i \succ_k B_j$ or simply $B_i \succ B_j$ if we do not want to specify the order.

enriquediagram *Example* 7.1.12. Assume X is a surface. Then all B_i are closed points $y_i \in Y_i$. All E_i's are (-1)-curves. Let $y_{i_1} \succ_1 y_{i_2} \succ_1 \cdots \succ_1 y_{i_k}$ be the longest sequence of infinitely near points. Then $\mu_{i_k,0} : Y_k \to X$ is an isomorphism in an open neighborhood of y_{y_k}, and we can identify it with a point in X. We say that y_{i_k} is a proper base point of $\mathcal{H}$. The curve $F_{i_1} = \mu_{N,i_1}^{-1}(E_{i_1})$. It is a (-1)-curve on Y. The simplicial complex $S(\mu)$ is a tree, and its longest paths correspond to (-1)-components of $\mathrm{Exc}(\mu)$. A log resolution is called minimal, if does not factor non-trivially through another log resolution. This is equivalent to that the morphism $\nu : Y \to \mathsf{Bl}^{+}_{\mathrm{Bs}(|V|)}(X)$ is a minimal resolution of the normal surface $\mathsf{Bl}^{+}_{\mathrm{Bs}(|V|)}(X)$. It is known that a minimal resolution of a normal surface is unique. This shows that a minimal resolution of $|V|$ is unique. Note that this is not true in higher dimension.

The next theorem is known as the *Noether-Fano inequality*.

noetherformula1 **Theorem 7.1.13.** *Let $\mu : Y = Y_N \to \cdots \to Y_0 = \mathbb{P}^n$ be a log resolution of a homaloidal linear system $\mathcal{H}$, and $\mu^{-1}(\mathcal{H}) = |dH - \sum_{i=1}^N m_i F_i)|$, where $\mu_i : Y_i \to Y_{i-1}$ is the blow-up with center B_i of codimension δ_i in Y_{i-1} and F_i be the proper transform in f of the exceptional divisor E_i. Then there exists i such that*

$$m_i > \frac{d(\delta_i - 1)}{n+1}.$$

Proof We know from Proposition canblowup 2.4.4 that $K_{Y_i} = \mu_i^*(K_{Y_{i-1}}) + \delta_i[E_i]$. By induction, we get

$$K_Y = \mu^*(K_{\mathbb{P}^n}) + \sum(\delta_i - 1)F_i = (-n-1)H + \sum(\delta_i - 1)F_i. \tag{7.8}$$

Let $\nu : Y \to \mathbb{P}^n$ be the second projection of the log resolution to the target space $\mathbb{P}^n$. Since $\nu_*(\mu^{-1}(\mathcal{H})) = |\mathcal{O}_{\mathbb{P}^n}(1)|$, and $\nu_*(K_Y) = K_{\mathbb{P}^n}$, for any positive rational number $t > \frac{1}{n+1}$,

$$|\mu^{-1}(\mathcal{H}) + tK_Y| = \emptyset.$$

Here, by definition, $\mu^{-1}(\mathcal{H}) + tK_Y = b\mathcal{H} + aK_Y$ if $t = a/b$. Now, we use that

$$\mu^{-1}(\mathcal{H}) + tK_Y = (dH - \sum_{i=1}^N m_i F_i) + t((-n-1)H + \sum_{i=1}^N (\delta_i - 1)F_i)$$

$$= (d - t(n+1)H - \sum_{i=1}^N (m_i - t(\delta_i - 1))F_i.$$

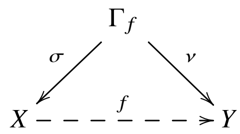
Taking $\frac{1}{n+1} < t \leq \frac{d}{n+1}$, we obtain that there exists at least one i such that $m_i - t(\delta_i - 1) \geq 0$. Hence,

$$m_i \geq t\delta_i > \frac{d(\delta_i - 1)}{n+1},$$

as asserted. □

SS:7.1.2 7.1.2 F-locus and P-locus

We define the *graph* Γ_f of a rational map $f : X \dashrightarrow Y$ as the closure Γ_f in $X \times Y$ of the graph $\Gamma_{f_{\mathrm{d}}}$ of $f_{\mathrm{d}} : \mathrm{dom}(f) \to Y$. Clearly, the graph, together with its projections to X and Y, defines a resolution of indeterminacy of the rational map f.



By the universal property of the graph, for any resolution (X', σ', ν') of f, the map $(\sigma', \nu') : X' \to X \times Y$ factors through the closed embedding $\Gamma_f \hookrightarrow X \times Y$. Thus, the first projection $\Gamma_f \to X$ has the universal property for morphisms which invert $\mathfrak{b}(|V|)$. Hence, it is isomorphic to the blow-up scheme $\mathsf{Bl}_X(\mathfrak{b}((|V|))$.

Suppose that $X = \mathbb{P}^n, Y = \mathbb{P}^n$ and f is given by a linear system of hypersurfaces of degree d. In coordinates,

$$f : (x_0, \ldots, x_n) \mapsto (F_0(x_0, \ldots, x_n), \ldots, F_n(x_0, \ldots, x_n)).$$

Then, Γ_f is defined by 2×2-minors of the matrix

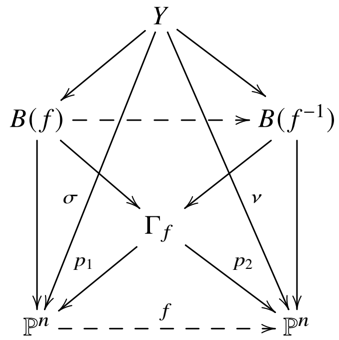
$$\begin{pmatrix} F_0(x) & F_1(x) & \ldots & F_n(x) \\ y_0 & y_1 & \ldots & y_n \end{pmatrix}, \tag{7.9}$$ graph

It is a subvariety of $\mathbb{P}^n \times \mathbb{P}^n$ given $\binom{n}{2}$ of equations of bidegree $(d, 1)$.

The *F-locus* of a Cremona transformation f is the reduced base locus of the linear system defining f. Its points are called the *fundamental points* or *indeterminacy points* (F-points, in classical terminology).

The *P-locus* of f is the union of irreducible hypersurfaces that are blown down to subvarieties of codimension ≥ 2. One can make this more precise and also give it a scheme-theoretical structure.

Let (X, σ', ν') be a normal resolution of indeterminacy of a birational map $f : \mathbb{P}^n \dashrightarrow \mathbb{P}^n$. We know that σ and ν) factors through the blow-ups $B(f)$ and $B(f^{-1})$ of the base ideals of f and f^{-1}. We assume that they are integrally closed, and hence, $B(f)$ and $B(f^{-1})$ are normal. We may find a log resolution X' of the exceptional divisor of $B(f) \to \mathbb{P}^n$ and a log resolution X'' of the exceptional divisor of $B(f^{-1})$. By Lazarsfeld [479, I:Example 9.1.6], we can find a common log resolution $Y \to B(f)$ and $Y \to B(f^{-1})$ to get the following commutative diagram:



We assume that (Y, σ, ν) is a minimal resolution satisfying the properties above. Let the base ideals of the homaloida systems defining f and f^{-1} be equal to $\sigma_* \mathcal{O}_Y(-\sum_{i\in I} r_i E_i)$ and $\nu_* \mathcal{O}_Y(-\sum_{j\in J} m_j F_j)$. Let J' be the largest subset of J such that the proper transform of F_j, $j \in J'$, in Y is not equal to the proper transform of some E_i in X. The image of the divisor $\sum_{j\in J'} F_j$ under σ is the *P-locus* of f. We can define the scheme-theoretical P-locus as the image of $\sum_{j\in J'} m_k F_j$. The image of any irreducible component of the P-locus is blown down under f (after we restrict ourselves to $\mathrm{dom}(f)$) to an irreducible component of the base locus of f^{-1}.

Let f be given by homogeneous polynomials $(F_0, \dots, F_n)$ and let $\tilde{f} : \mathbb{C}^{n+1} \to \mathbb{C}^{n+1}$ be the holomorphic map defined by the same polynomials. Then, the P-locus is the image in $\mathbb{P}^n$ of the locus of critical points of $\tilde{f}$. It is equal to the set of zeros of the determinant of the Jacobian matrix of $\tilde{f}$

$$J = \Big(\frac{\partial F_i}{\partial t_j}\Big)_{i,j=0,\dots,n}.$$

So, we expect that the P-locus is a hypersurface of degree $(n+1)(d-1)$. The component $P_j = \sigma(F_J)$ enters with the multiplicity

$$\mathrm{mult}_{P_j} = m_j(n - 1 - \dim(\nu(F_j))) \qquad (7.10)$$ hudsonformula

Hudson [414]. This often allows one to compute the degree of the map defined in geometric terms without an explicit formula.

standard *Example* 7.1.14. Consider the *standard quadratic transformation* given by

$$T_{\mathrm{st}} : [t_0, t_1, t_2] \mapsto [t_1t_2, t_0t_2, t_0t_1]. \tag{7.11}$$ st

It has three fundamental points $p_1 = [1,0,0], p_2 = [0,1,0], p_3 = [0,0,1]$. The P-locus is the union of three coordinate lines $V(t_i)$. The Jacobian matrix is

$$J = \begin{pmatrix} 0 & t_2 & t_1 \\ t_2 & 0 & t_0 \\ t_1 & t_0 & 0 \end{pmatrix}.$$

Its determinant is equal to $2t_0t_1t_2$. We may take $X = \mathrm{Bl}_{\mathbb{P}^2}(\{p_1, p_2, p_3\})$ as a smooth resolution of T_{st} (see Figure F7.1 7.1).

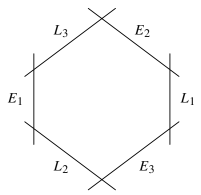


Figure 7.1 A resolution of T_{st} F7.1

Let E_1, E_2, E_3 be the exceptional divisors over the fundamental points p_1, p_2, p_3, and let $L_i, i = 1, 2, 3$, be the proper transforms of the coordinate lines $V(t_0), V(t_1), V(t_2)$, respectively. Then, the morphism $\sigma : X \to \mathbb{P}^2$ blows down L_1, L_2, L_3 to the points p_1, p_2, p_3, respectively. Note that $T_{\mathrm{st}}^{-1} = T_{\mathrm{st}}$, so there is no surprise here. Recall that the blow-up of a closed subscheme is defined uniquely only up to an isomorphism. The isomorphism τ between the blow-ups of the base scheme of T_{st} and T_{st}^{-1} that sends E_i to L_i is a lift of the Cremona transformation T_{st}^{-1}. The surface X is a *del Pezzo surface* of degree 6, a toric Fano variety of dimension 2. We will study such surfaces in Chapter 8. The complement of the open torus orbit is the hexagon of lines $E_1, E_2, E_3, L_1, L_2, L_3$ intersecting each other as in the picture. We call them lines because they become lines in the embedding $X \hookrightarrow \mathbb{P}^6$ given by the anti-canonical linear system. The automorphism τ of the surface is the extension of the inversion automorphism $z \to z^{-1}$ of the open torus orbit to the whole surface. It defines the symmetry of the hexagon which exchanges its opposite sides.

Now, let us consider the first *degenerate* standard quadratic transformation given by

$$T'_{\rm st} : [t_0, t_1, t_2] \mapsto [t_2^2, t_0t_1, t_0t_2]. \tag{7.12}$$ st'

It has two fundamental points $p_1 = [0, 1, 0]$ and $p_2 = [1, 0, 0]$. Blowing up the first point, we obtain a base point $p'_1 \succ p_1$. After blowing up p'_1 and p_2, we obtain birational morphism $\pi : X \to \mathbb{P}^2$ such that $T'_{\rm st}$ lifts to a biregular morphism $X \to \mathbb{P}^2$. The following figure 7.3 gives a picture of the exceptional curves of π and the proper transforms L'_1, L'_2 of the lines $L_1 = V(t_0)$ and $L_2 = V(t_2)$.

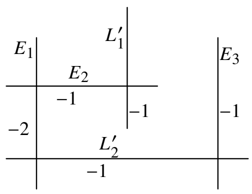


Figure 7.2 A resolution of $T'_{\rm st}$ F7.2

Here, E_1 is the exceptional curve of the first blow-up of p_1, and E_2 (resp. E_3) is the exceptional curve over p'_1 (resp. p_2).

The P-locus consists of the line $V(t_0)$ blown down to the point p_1 and the line $V(t_2)$ blown down to the point p_2.

The Jacobian matrix is

$$J = \begin{pmatrix} 0 & 0 & 2t_2 \\ t_1 & t_0 & 0 \\ t_2 & 0 & t_0 \end{pmatrix}.$$

Its determinant is equal to $-2t_0t_2^2$. The line $V(t_2)$ enters with multiplicity 2. In fact, the exceptional divisor of the second projection $\nu : Y \to \mathbb{P}^2$ is equal to $2L'_2 + E_1$.

The base scheme of $T'_{\rm st}$ is smooth at p_2 and locally isomorphic to $V(y^2, x)$ at the point p_1, where $y = t_2/t_1, x = t_0/t_1$. The blow-up of the base scheme is singular over p_1 with the singular point p'_2 corresponding to the tangent direction $t_0 = 0$. The singular point is locally isomorphic to the singularity of the surface $V(uv + w^2) \subset \mathbb{C}^3$ (a singularity of type A_1, see Example 1.2.8). It is obtained by blowing down the curve E_1 on X.

Finally, we can consider the *second degenerate standard quadratic transfor-*

mation given by the formula

$$T''_{\mathrm{st}} : [t_0, t_1, t_2] \mapsto [t_2^2 - t_0 t_1, t_1^2, t_1 t_2]. \tag{7.13}$$ st”

Its unique base point is $p_1 = [1, 0, 0]$. In affine coordinates $x = t_1/t_0, y = t_2/t_0$, the base ideal is $(x^2, xy, y^2 - x) = (xy, x - y^2)$. The blow-up of this ideal is isomorphic to $V((x-y^2)u + xyv)$ in $\mathbb{A}^2 \times \mathbb{P}^1$. It has a singular point $(0, 0, [0, 1])$ locally isomorphic to the surface singularity $x(u + y) - y^2 u = 0$. Replacing (x, y) with $(u^2 - x, y - u)$, we obtain that the singularity is locally isomorphic to the singularity $xy + z^3 = 0$ of type A_2 (see Example 1.2.8).

A smooth resolution of the transformation is obtained by blowing up infinitely near points $p_3 \succ p_2 \succ p_1$ corresponding to the direction $t_1 = 0$, followed by the direction of the proper transform of the conic $V(t_2^2 - t_0 t_1)$. The exceptional divisor of the morphism $\pi : X \to \mathbb{P}^2$ consists of three curves E_1, E_2, E_3, where E_1 (resp. E_2) is the proper transform of the exceptional curve of the first (resp. the second) blow-up, and E_3 is the exceptional curve over p_3. Blowing down $E_1 + E_2$, we obtain the blow-up of the base scheme of the transformation.

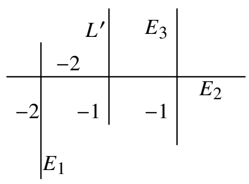


Figure 7.3 A resolution of T''_{st} F7.3

Here, L' denotes the proper transform of the line $L = V(t_1)$. The Jacobian matrix of T''_{st} is equal to

$$\begin{pmatrix} -t_1 & -t_0 & 2t_2 \\ 0 & 2t_1 & 0 \\ 0 & t_2 & t_1 \end{pmatrix}.$$

Its determinant is equal to $-2t_1^3$. So, the P-locus consists of one line $V(t_1)$ taken with multiplicity 3. In fact, the exceptiona divisor of $\nu : Y \to \mathbb{P}^2$ is equal to $3L_1' + 2E_3 + E_1$.

Let us look at F-locus.

def:contact **Definition 7.1.15.** *An isolated base point of a homaloidal linear system $\mathcal{H}$ is called a* point of s-contact *if there exists an open non-empty subset $U \subset \mathcal{H}$ such that, for all $D \in U$, a local equation at x of D in some affine coordinates*

$z_1, \dots, z_n$ is of the form:

$$z_1 + f_2(z_1, \dots, z_n) + \cdots + f_{s-1}(z_1, \dots, z_n) + \text{higher terms},$$

where f_k are homogeneous forms of degree k. We say that a nonsingular base point of a general D is a point of 0-contact.

It is easy to see that a point of s-contact imposes

$$\sum_{k=1}^{s-1} \binom{n-2+k}{k} = \binom{n-2+s}{n-2} \tag{7.14}$$ contact

condition in the postulation formula.

For example, the base point p_1 of T'_{st} (resp. T''_{st}) is a point of 1-contact (resp. 2-contact).

SS:7.1.3

7.1.3 The multi-degree of a Cremona transformation

Let $\Gamma_f \subset \mathbb{P}^n \times \mathbb{P}^n$ be the graph of a Cremona transformation $f : \mathbb{P}^n \dashrightarrow \mathbb{P}^n$. In the usual way, the graph Γ_f defines the linear maps of cohomology

$$f_k^* : H^{2k}(\mathbb{P}^n, \mathbb{Z}) \to H^{2k}(\mathbb{P}^n, \mathbb{Z}), \quad \gamma \mapsto (\mathrm{pr}_1)_*([\Gamma_f] \cap (\mathrm{pr}_2)^*(\gamma)),$$

where $\mathrm{pr}_i : \mathbb{P}^n \times \mathbb{P}^n \to \mathbb{P}^n$ are the projection maps. Since $H^{2k}(\mathbb{P}^n, \mathbb{Z}) \cong \mathbb{Z}$, these maps are defined by some numbers d_k, the vector $(d_0, \dots, d_n)$ is called the *multidegree* of f. In more details, we write the cohomology class $[\Gamma_f]$ in $H^*(\mathbb{P}^n \times \mathbb{P}^n, \mathbb{Z})$ as

$$[\Gamma_f] = \sum_{k=0}^{n} d_k h_1^k h_2^{n-k},$$

where $h_1 = \mathrm{pr}_1^* c_1(\mathcal{O}_{\mathbb{P}^n}(1))$ and $h_2 = \mathrm{pr}_2^* c_1(\mathcal{O}_{\mathbb{P}^n}(1))$. Then,

$$f_k^*(h^k) = (\mathrm{pr}_1)_*([\Gamma_f]) \cdot h_2^k = (\mathrm{pr}_1)_*(d_k h_1^k) = d_k h^k.$$

The map f is of finite degree if and only if all d_i are positive.

The number $d_0 = 1$ since $h_1^n \cdot [\Gamma_f] = 1$. The number d_n is equal to the degree of the map. If f is birational, it is equal to one. The number d_k is equal to the degree of the proper transform under f of a general linear subspace of codimension k. In particular, $d_1 = d$ is the algebraic degree of f.

If f is birational, we can invert f and obtain

$$\Gamma_{f^{-1}} = \tilde{\Gamma}_f,$$

where $\tilde{\Gamma_f}$ is the image of Γ_f under the self-map of $\mathbb{P}^n \times \mathbb{P}^n$ that switches the factors.

In the case when f is a birational map, we have $d_0 = d_n = 1$. We shorten the definition by saying that the multi-degree of a Cremona transformation is equal to $(d_1, \dots, d_{n-1})$. In particular, we see that $(d_n, d_{r-1}, \dots, d_0)$ is the multi-degree of f^{-1}. In particular, we see that $d_k = \deg(f(L_k)$, where L_k is a general linear subspace of codimension k.

The next result, due to L. Cremona, gives some restrictions on the multi-degree of a Cremona transformation.

Proposition 7.1.16 (Cremona's inequalities)**.** *For any $n \geq i, j \geq 0$,*

$$1 \leq d_{i+j} \leq d_i d_j, \quad d_{n-i-j} \leq d_{n-i} d_{n-j}.$$

Proof It is enough to prove the first inequality. The second one follows from the first one by considering the inverse transformation. Write a general linear subspace L_{i+j} of codimension $i + j$ as the intersection of a general linear subspace L_i of codimension i and a general linear subspace L_j of codimension j. Then, $f^{-1}(L_{i+j})$ is an irreducible component of the intersection $f^{-1}(L_i) \cap f^{-1}(L_j)$. By Bezout's Theorem,

$$d_{i+j} = \deg f^{-1}(L_{i+j}) \leq \deg f^{-1}(L_i) \deg f^{-1}(L_j) = d_i d_j.$$

□

Remark 7.1.17. There are more conditions on the multi-degree which follow from the irreducibility of Γ_f. For example, by using the *Hodge type inequality* (see [479, Corollary 1.6.3]), we get the inequality

$$d_i^2 \geq d_{i-1} d_{i+1}. \tag{7.15}$$

For example, if $n = 3$, the only nontrivial inequality following from the Cremona inequalities is $d_0 d_2 = d_2 \leq d_1^2$, and this is the same as the Hodge-type inequality. However, if $n = 4$, we get additional inequalities. For example, $(1, 2, 3, 5, 1)$ satisfies the Cremona inequalities but does not satisfy the Hodge-type inequality.

The following are the natural questions related to the classification of possible multi-degrees of Cremona transformations.

- Let $(1, d_1, \dots, d_{n-1}, 1)$ be a sequence of integers satisfying the Cremona inequalities and the Hodge-type inequalities: Does there exist an irreducible reduced close subvariety ℓ of $\mathbb{P}^n \times \mathbb{P}^n$ with $[\ell] = \sum d_k h_1^k h_2^{n-k}$?
- What are the components of the Hilbert scheme of this class containing an integral scheme?

Note that any irreducible reduced closed subvariety of $\mathbb{P}^n \times \mathbb{P}^n$ with multi-degree $(1, d_1, \ldots, d_{n-1}, 1)$ is realized as the graph of a Cremona transformation.

The multi-degree $(d_1, \ldots, d_{n-1})$ of a Cremona transformation can be computed using the theory of Segre classes $s(Z, X)$ which we already discussed in Subsection SS:2.4.2 2.4.2. There, we were mostly dealing with smooth varieties Z and X over a field $\Bbbk$. Here, we will be using Segre classes where Z is the base scheme of a Cremona transformation. In general, it is not smooth and even not necessarily reduced.

Let $|V|$ be a linear system of hypersurfaces of degree d in $\mathbb{P}^n$ that defines a rational map $f : \mathbb{P}^n \to \mathbb{P}^n$ of degree $d_n > 0$. We assume that $\mathfrak{b}(|V|)$ is normal, and let $Z = V(\mathfrak{b})$ be the closed subscheme of $\mathbb{P}^n$ defined by $\mathfrak{b}(|V|)$. We denote by $\mathcal{I}_Z(d)$ the sheaf $\mathcal{O}_{\mathbb{P}^n}(d) \otimes \bar{\mathfrak{b}}$.

Let $(\mathsf{Bl}_Z(\mathbb{P}^n), \sigma, \nu)$ be the resolution of indeterminacy of f. We know that X coincides with the graph Γ_f of f, and, by Corollary cor:7.1.4 7.1.7,

$$V = H^0(\mathbb{P}^n, \mathcal{I}_Z^d) = H^0(\mathbb{P}^n, \sigma_*\sigma^*\mathcal{O}_{\mathbb{P}^n}(d)(-E))$$

and

$$\sigma^*\mathcal{O}_{\mathbb{P}^n}(d)(-E) = \sigma^*\mathcal{O}_{\mathbb{P}^n}(1),$$

where E is the exceptional divisor of σ. Since σ is a birational morphism, the degree m of f is equal to the degree of ν. The latter is equal to the degree of the pre-image of a general point in the target. This gives

$$d_n = \nu^*(\sigma^*(h_2^n) = \nu^*(\sigma^*(h_2))^n = (d\sigma^*(h_1) - [E])^n,$$

where h_1 (resp. h_2) is the class of a hyperplane in the source (resp. target) $\mathbb{P}^n$. Abusing the notation, we identify h_1 and h_2 with their pull-backs in $\mathbb{P}^n \times \mathbb{P}^n$.

To compute d_n we need to compute all self-intersections $[E]^i$. This is where we use the Segre classes of $s(Z, \mathbb{P}^n)$.

Let $i : Z \hookrightarrow \mathbb{P}^n$ and $i_* : A(Z) \to A(\mathbb{P}^n)$. We write

$$i_*s_r(Z, \mathbb{P}^n) = s(Z, \mathbb{P}^n)_r h_1^{n-r},$$

and call $s(Z, \mathbb{P}^n)_r$ the rth *Segre number* of Z.

We keep the assumption that $\mathfrak{b}(|V|)$ is normal.

muldegree **Proposition 7.1.18.** *Let $(d_0, d_1, \ldots, d_n)$ be the multi-degree of a rational map of finite degree d_n. Then,*

$$d_k = d^k - \sum_{i=2}^{k} d^{k-i}\binom{k}{i} s(Z, \mathbb{P}^n)_{n-i}. \qquad (7.16)$$ multdegree

Proof Let $H = \sigma^*(h_1)$. Then

$$d_k = \pi_*[dH - E]^k \cdot h_1^{n-k} = \sum_{i=0}^{k}(-1)^i d^{k-i}\tbinom{k}{i}\pi_*([H]^{k-i}\cdot[E]^i))\cdot h_1^{n-k}$$

$$= \sum_{i=0}^{k}(-1)^i d^{k-i}\tbinom{k}{i}h_1^{k-i}\cdot\pi_*([E]^i)\cdot h_1^{n-k} = \sum_{i=0}^{k}(-1)^i d^{k-i}\tbinom{k}{i}\cdot\pi_*([E]^i)\cdot h_1^{n-i}$$

$$= d^k + \sum_{i=1}^{k}(-1)^i d^{k-i}\tbinom{k}{i}\cdot\pi_*([E]^i)\cdot h_1^{n-i} = d^k - \sum_{i=1}^{k}d^{k-i}\tbinom{k}{i}s(Z,\mathbb{P}^n)_{n-i}.$$

Finally, we use that the homaloidal linear system has no fixed components, thus codim$(\mathrm{Bs}(|V|, \mathbb{P}^n) \geq 2$, and hence $s(Z, \mathbb{P}^n)_{n-i} = 0$ for $i = 1$. □

postulation2 **Corollary 7.1.19.** *The linear system $|V|$ is homaloidal if and only if*

1. $d^n - \sum_{i=2}^{n} d^{n-i}\binom{n}{i}s(Z,\mathbb{P}^n)_{n-i} = 1$.
2. $h^0(\mathcal{O}_Z(d)) = \binom{n+d}{d} - n - 1$.

The second condition follows from Corollary 7.1.9.

cor:zerodimensional **Corollary 7.1.20.** *Assume that the base schemes of Cremona transformations f and f^{-1} are 0-dimensional. Then $n = 2$.*

Proof Since $s(Z, \mathbb{P}^n)_i = 0, i > 0$, we have $d_k = d^k, k \neq 1, n$ and the multi-degree of f is $(d, \ldots, d^{n-1})$. The multi-degree of f^{-1} is $(d^{n-1}, \ldots, d)$. Clearly, this is possible only if $n = 2$. □

steinequartics2 *Example* 7.1.21. Let $p_1, \ldots, p_4$ be four general points in $\mathbb{P}^3$ and Π is a plane containing p_4. Consider the linear system $|V| \subset |\mathcal{O}_{\mathbb{P}^3}(2)|$ of quadrics that contain the four points and tangent to Π at p_4. It is easy to see that the dimension of $|V|$ is equal to 3. Let $f : \mathbb{P}^3 \dashrightarrow \mathbb{P}^3$ be the rational map defined by $|V|$ and a basis $(u_0, u_1, u_2, u_3) = t_0(t_1 + t_2 + t_3), t_1t_2, t_1t_3, t_2t_3)$ in V. We compute its Jacobian matrix and find that the P-locus of f is equal to the union of four planes $V(t_1), V(t_2), V(t_3), V(t_1 + t_2 + t_3)$. The map f blows down the first three planes to the cross of coordinate lines $V(u_1u_2, u_1u_3, u_2u_3)$ in the target space, and blows down the last plane to the intersection of the lines. Since the multi-degree of f is equal to $(2, 4)$, the multidegree of f^{-1} is equal to $(4, 2)$. The inverse transformation f^{-1} is given by Steiner quartics from Subsection 2.1.1. They are the images of planes under the map f. The F-locus of f^{-1} consists of the cross of the coordinate lines. The base scheme of f^{-1} is equal to the $V(t_1t_2, t_1t_3, t_2t_3, t_1t_2t_3)$. It is the integral closure of the ideal (t_1t_2, t_1t_3, t_2t_3) defining the F-locus.

The Segre numbers are computed in terms of intersection theory on the blow-up of the base scheme. However, it is often singular, and the computation is very difficult. We can choose a log resolution $\sigma : X \to \mathbb{P}^n$ of $\mathrm{Bs}(|V|)$ and use Proposition pullbackintersection 2.4.5 to do the computations on a smooth variety. Recall that

$$|V| = |\sigma^*(\mathcal{O}_{\mathbb{P}^n}(d) \otimes \mathcal{O}_X(-\sum m_i F_i)|, \quad (7.17)$$ loghomaloid

where all F_i are smooth divisors and they intersect transversally.

ex:infinitelynear1 *Example* 7.1.22. Let $n = 2$ and (t_0, t_1, t_2) be projective coordinates. Let $Z = V(t_1^2, t_2)$. Then $X = \mathrm{Bl}_Z(\mathbb{P}^2)$ is the hypersurface in $\mathbb{P}^2 \times \mathbb{P}^1$ given by equations $t_1^2 u_0 - t_2 u_1 = 0$. Computing the partials, we find that the point $([1, 0, 0], [1, 0])$ is an ordinary double point of the surface X. The exceptional divisor E is equal to $V(t_1^2, t_0) = 2E_1$, where $E_1 = V(t_1, t_0)$. is a reduced divisor isomorphic to $\mathbb{P}^1$. We know that

$$\sigma(Z, \mathbb{P}^2)_0 = -(\sigma_E)_*(2E_1^2) = e_{(u^2, v)},$$

where we use the affine coordinates $u = t_1/t_0, v = t_2/t_0$ and denote by $e_{(u^2,v)}$ the multiplicity of the primary ideal $I = (u^2, v)$ in $\mathbb{C}[u, v]$. Since (u^2, v) is generated by a regular sequence, $e_{(u^2,v)} = l(\mathbb{C}[u, v]/I) = 2$. Thus, $E^2 = -2$. The proper transform of E in X' is a (-1)-curve E_1', and the exceptional divisor of $X' \to X \to \mathbb{P}^2$ is the Cartier divisor $2E_1' + E_2$. We immediately check that $(2E_1 + E_2) = -2$. Thus, $E^2 = -2$ agrees with the computation of the Segre class. Note that $E = 2E_1$ but E_1' is not a Cartier divisor. So, we cannot say that $e^2 = 4E_1^2$. It is true if E_1^2 is computed by using the intersection theory of Q-Cartier divisors according to which $E_1^2 = -1/2$.

Note that Z is equal to the base scheme of the homaloidal linear system of conics defining the standard Cremona transformation T_{st} considered in the previous subsection. The line $V(t_2)$ is a part of its F-locus.

In a similar fashion, we take $Z = V(t_1^k, t_2)$ to obtain that $E = nE_1$ and $E^2 = -n$. The blow-up $X = \mathrm{Bl}_Z(\mathbb{P}^2)$ has a singularity of type A_{n-1} (see Example ex:1.2.8 1.2.8). The pre-image of E in a minimal resolution $X' \to X$ is a divisor $E' = nE_1' + (n-1)E_2 + \cdots + E_n$, where E_1' is the proper transform of E_1. It is a (-1)-curve, all other curves E_i are (-2)-curves. We have $E^2 = E'^2 = -n$.

S:7.2

7.2 Planar Cremona Transformations

In this section, we will discuss Cremona transformations of the projective plane.

7.2.1 Exceptional configurations

SS:7.2.1

Let

$$\mu : Y = Y_N \xrightarrow{\mu_N} Y_{N-1} \xrightarrow{\mu_{N-1}} \ldots \xrightarrow{\mu_2} Y_1 \xrightarrow{\mu_1} Y_0 = X \qquad (7.18)$$ decom

be a log resolution (7.7) of a rational map $f : X \to X'$ of algebraic surfaces. Every birational morphism $Y \to X$ of smooth algebraic surfaces can be factored in this way. Here, each map $\mu_k : Y_k \to Y_{k-1}$ is the blow-up of a closed point $x_k \in Y_{k-1}$ with the exceptional (-1)-curve E_k. The effective divisor

$$\mathcal{E}_k = \mu_{N,0}^{-1}(E_k)$$

is called the *exceptional configuration*.

Let $|V| \subset |\mathcal{L}|$ be a linear system without fixed components that defines a rational map $f : X \to X'$. We know that its pre-image in the log resolution is equal to the linear system

$$|\mu^*(\mathcal{L}) - \sum_{i=1} m_i\mathcal{E}_i|,$$

Here, $m_i = \mathrm{mult}_{x_i}\mu_{k,0}^{-1}(D)$, where D is a general member of $|V|$ and $\mu_{k-1,1}^{-1}(D)$ its proper transform on X_k. We have

$$\mu_*\mathcal{O}_Y(-\sum_{i=1}^{n} m_i\mathcal{E}_i) = \mathfrak{b}(|V|).$$

The base ideal $\mathfrak{b}(|V|)$ is an integrally closed ideal and there is a birational morphism $Y \to \mathsf{Bl}_{\mathrm{Bs}(|V|)}(X)$ of normal surfaces.

Consider the category $\mathcal{B}_X$ of birational morphisms $\pi : X' \to X$ of nonsingular projective surfaces. Recall that a morphism from $(X' \xrightarrow{\pi'} X)$ to $(X'' \xrightarrow{\pi''} X)$ in this category is a regular map $\phi : X' \to X''$ such that $\pi'' \circ \phi = \pi'$.

bubblespace **Definition 7.2.1.** *The* bubble space X^{bb} *of a nonsingular surface X is the factor set*

$$X^{\mathrm{bb}} = \Big(\bigcup_{(X' \xrightarrow{\pi'} X) \in \mathcal{B}_X} X'\Big)/R,$$

where R is the following equivalence relation: $x' \in X'$ is equivalent to $x'' \in X''$ if the rational map $\pi''^{-1} \circ \pi' : X' \dashrightarrow X''$ maps isomorphically an open neighborhood of x' to an open neighborhood of x''.

It is clear that, for any $\pi : X' \to X$ from $\mathcal{B}_X$, we have an injective map $i_{X'} : X' \to X^{\mathrm{bb}}$. Let us identify points of X' with their images. If $\phi : X'' \to X'$ is a morphism in $\mathcal{B}_X$ which is isomorphic in $\mathcal{B}_{X'}$ to the blow-up of a point $x' \in X'$, any point $x'' \in \phi^{-1}(x')$ is called a point *infinitely near* x' of the first order. This is denoted by $x'' \succ_1 x'$. By induction, one defines an infinitely near point of order k, denoted by $x'' \succ_k x'$. This puts a partial order on X^{bb}, where $x > y$ if x is infinitely near to y. When we do not specify the order of an infinitely near point we write $x' \succ x$.

A log resolution of a rational map $f : X \to X'$ is an object of the category $\mathcal{B}_X$. The images of the base points $x_i \in X_{i-1}$ of the blowing up $\mu_i : X_i \to X_{i-1}$ are points in the bubble space. The order on the set of points x_i defined for a log resolution of any rational map in Subsection 7.1.1 agrees with the previous definition.

We say that a point $x \in X^{\mathrm{bb}}$ is of height k, if $x \succ_k x_0$ for some $x_0 \in X$. This defines the *height function* on the bubble space

$$\mathrm{ht} : X^{\mathrm{bb}} \to \mathbb{N}.$$

Clearly, $X = \mathrm{ht}^{-1}(0)$. Points of height zero are called *proper points* of the bubble space. They will be identified with points in X. They are minimal points with respect to the partial order on X^{bb}.

Definition 7.2.2. *A* bubble cycle *is an element* $\eta = \sum m(x)x$ *of* $\mathbb{Z}^{X^{\mathrm{bb}}}$ *satisfying the following properties:*

(i) $m(x) \geq 0$ *for any* $x \in X^{\mathrm{bb}}$*;*
(ii) $\sum_{x' \succ x} m_{x'} \leq m_x$.

We denote the subgroup of bubble cycles by $\mathcal{Z}_+(X^{\mathrm{bb}})$.

Recall that elements of $\mathbb{Z}^{X^{\mathrm{bb}}}$ are integer valued functions on X^{bb} with finite support. They added up as functions with values in $\mathbb{Z}$. We write elements of $\mathbb{Z}^{X^{\mathrm{bb}}}$ as finite linear combinations $\sum m(x)x$, where $x \in X^{\mathrm{bb}}$ and $m(x) \in \mathbb{Z}$ (similar to divisors on curves). Here, $m(x)$ is the value of the corresponding function at x.

Clearly, any bubble cycle η can be written in a unique way as a sum of bubble cycles Z_k such that the support of η_k is contained in $\mathrm{ht}^{-1}(k)$.

Let $\eta = \sum m_x x$ be a bubble cycle. We order the points from the support of η such that $x_i \succ x_j$ implies $j < i$. We refer to such an order as an *admissible order*. We write $\eta = \sum_{i=1}^N m_i x_i$. Then, we represent x_1 by a point on X and define $\pi_1 : X_1 \to X$ to be the blow-up of X with center at x_1. Then, x_2 can be represented by a point on X_1 as either infinitely near of order one to x_1 or as a point equivalent to a point on X. We blow up x_2. Continuing in this way, we get a sequence of birational morphisms as in (7.18) $\mu_\eta : Y_\eta \to Y$. Clearly, the bubble cycle η is equal to the bubble cycle $\sum_{i=1}^N m_i x_i$.

For any invertible sheaf $\mathcal{L}$ and a bubble cycle $\eta \in \mathcal{Z}_+(X^{\mathrm{bb}})$ we define the linear system

$$|\mathcal{L}-\eta| := |\mu^*\mathcal{L}\otimes\mathcal{O}_{Y_\eta}(-\sum_{i=1}^{N}\mathcal{E}_i|,$$

where $\mathcal{E}_i$ are exceptional configurations of teh rational map $\mu : Y_\eta \to X$. We have

$$\mu_*(\mu^*\mathcal{L}\otimes\mathcal{O}_{Y_\eta}(-\sum_{i=1}^{N}\mathcal{E}_i)) = \mathcal{L}\otimes\mathfrak{b}$$

where $\mathfrak{a} = \mu_*(\mathcal{O}_\eta(-\sum_{i=1}^N\mathcal{E}_i)$ is the complete base ideal of the linear system $|\mathcal{L}\otimes\mathfrak{a}|$.

The image of the partially ordered set of base points $x_1,\ldots,x_N$ of a log resolution of a rational map $f : X \to X'$ defines a bubble cycle $\eta\in\mathbb{Z}^{X^{\mathrm{bb}}}$. It is called the *fundamental bubble cycle* of the rational map. The morphism $\mu : Y_\eta \to X'$ from (decom 7.18) is given by the linear system $|\pi^*\mathcal{L}-\sum_{i=1}^N m_i\mathcal{E}_i|$, together with the birational morphism $\mu : Y_\eta \to X$ is resolution of indeterminacy of f.

We can describe a bubble cycle by a weighted oriented graph, called the *Enriques diagram*, by assigning to each point from its support a vertex, and joining two vertices by an oriented edge if one of the points is infinitely near another point of the first order. The arrow points to the vertex of lower height. We weight each vertex by the corresponding multiplicity.

The Enriques diagram of the fundamental bubble cycle of a rational map is equal to the simplicial complex of the log resolution (with weights deleted) which we defined in a more general situation.

Example 7.2.3. Suppose $\eta = \sum m_ix_i$, where all points x_i are proper. Then, the integrally closed ideal corresponding to η is equal to the product of the ideal sheaves $\mathfrak{m}_{x_i}^{m_i}$.

Let x_i be a proper point of η and $\eta_i\subset\eta$ the maximal part of η that is supported in the of infinitely near points to x_i from η. Then, $\mathfrak{a}_i = \mu_(\mathcal{O}_{Y_\eta}(-\eta_i)$ is the $\mathfrak{m}_{x_i,X}$-primary component of $\mathfrak{a} = \mu_\eta(\mathcal{O}_{Y_\eta}-\eta)$. If all $m_i = 1$ in η_i, then the ideal $\mathfrak{a}_i = (u, v^k)$, where (u,v) are local parameters at $x_i\in X$.

From now on, we use the intersection theory on a smooth projective surface and use the notation $D\cdot D'$ for the intersection of the divisor classes $[D]\cdot[D']$.

L711 **Lemma 7.2.4.** *Let $\pi : Y \to X$ be a birational morphism of nonsingular surfaces and let $\mathcal{E}_i, i = 1,\ldots,N$, be its exceptional configurations. Then*

$$\mathcal{E}_i\cdot\mathcal{E}_j = -\delta_{ij},$$

$$\mathcal{E}_i\cdot K_Y = -1.$$

Proof This follows from the standard properties of the intersection theory on surfaces. For any morphism of nonsingular projective surfaces $\phi : X' \to X$ and two divisors D, D' on X, we have

$$\phi^*(D) \cdot \phi^*(D') = \deg(\phi) D \cdot D'. \tag{7.19}$$ prop2

Also, if C is a curve such that $\phi(C)$ is a point, we have

$$C \cdot \phi^*(D) = 0. \tag{7.20}$$ prop22

Applying (prop2 7.19), we have

$$\mathcal{E}_i^2 = E_i^2 = -1.$$

Assume $i < j$. Applying (prop22 7.20) by taking $C = E_j$ and $D = E_i$, we obtain

$$0 = E_j \cdot \mu_{j,i}^*(E_i) = \pi_{Nj}^*(E_j) \cdot \pi_{Ni}^*(E_i) = \mathcal{E}_j \cdot \mathcal{E}_i.$$

This proves the first assertion.

To prove the second assertion, we use Proposition canblowup 2.4.4. It gives

$$K_Y = \mu^*(K_X) + \sum_{i=1} \mathcal{E}_i,$$

and hence,

$$K_Y \cdot \mathcal{E}_j = \Big(\sum_{i=1}^N \mathcal{E}_i\Big) \cdot E_j = \mathcal{E}_j^2 = -1.$$

□

Let $\mathfrak{b} = \sum_{i=1}^N m_i x_i$ be the fundamental bubble cycle of a Cremona transformation $f : \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$ and

$$\begin{array}{ccc} & X = Y_\eta & \\ {}^{\sigma}\swarrow & & \searrow^{\nu} \\ \mathbb{P}^2 & \dashrightarrow & \mathbb{P}^2 \end{array}$$

be its log resolution defined by η. The homaloidal linear system $|V|$ defining f is equal to $|dh - \eta|$. We identify η with a divisor $D_\eta = \sum_{i=1}^N m_i \mathcal{E}_i$ on X and identify the linear system $|dh - \eta|$ with the linear system $|dH - D_\eta|$ on X, where $H = \mu_\eta^* h$.

The vector $(d; m_1, \ldots, m_N)$ is called the *characteristic vector* of the homaloidal net, or, of a Cremona transformation defined by this net.

Obviously, $|V|$ is a homaloidal linear system if and only if

$$(dH - D_\eta)^2 = d^2 - \sum_{i=1} m_i^2 = 1. \tag{7.21}$$ virt1

Since

$$K_Y^2 = (-3H + \mathcal{E}_1 + \cdots + \mathcal{E}_N)^2 = 9 - n, \tag{7.22}$$

the birational map $\nu : X \to \mathbb{P}^2$ admits a factorization defining a log resolution of the inverse map f^{-1} with the same number of blow-ups.

Since a general member D of $|dH - \sum_{i=1}^N m_i\mathcal{E}_i|$ admits a birational morphism onto a line in the target $\mathbb{P}^2$, it must be a smooth rational curve. By the adjunction formula,

$$K_X \cdot D_\eta = -3d + \sum_{i=1}^N m_i = -D_\epsilon^2 = 2 = -3. \tag{7.23}$$ virt2

Subtracting (7.23) from (7.21), and applying Riemann-Roch and the Postulation formula from Corollary 7.1.9, we find

$$h^0(\mathcal{O}_{\mathbb{P}^2}/\mathfrak{b}(|\bar{V}|)) = \frac{1}{2}\sum_{i=1}^N m_i(m_i+1). \tag{7.24}$$ hoskin

The equalities (7.21), (7.23), (7.24) and Noether's inequality (7.1.13)

$$\max\{m_i\} > \frac{d}{3} \tag{7.25}$$ noetherineq1

allows one to list the possible characteristic vectors of Cremona transformations of low degree. We refer to [414, Table 1] for the list of characteristic numbers $(d, m_1, \ldots, m_N)$ with $d \le 16$.

poncelet *Example* 7.2.5. A Cremona transformation with $d = 2$ is called a *quadratic Cremona transformation*. We will study quadratic transformation in any $\mathbb{P}^n$ later in the chapter. A planar quadratic Cremona transformation is given by conics passing simply through three points including infinitely near points. We discussed such transformations in Subsection 7.1.2, where we chose a basis of the linear system that defines the Cremona involution T_{st} or its degenerations corresponding to $x_2 \succ x_1$ or $x_3 \succ x_2 \succ x_1$. It follows that any quadratic Cremona transformation is obtained by a composition of one of three standard Cremona involutions with a projective automorphism.

Following Poncelet, one gives the following geometric construction of a planar quadratic transformation. Let C_1 and C_2 be two conics intersecting at four distinct points. For each general point x in the plane let $f(x)$ be the intersection of the polar lines $P_x(C_1)$ and $P_x(C_2)$. Let us see that this defines an involutorial quadratic transformation with fundamental points equal to the singular points of three reducible conics in the pencil generated by C_1 and C_2. For any two different members C, C' of the pencil, $P_x(C_1) \cap P_x(C_2) = P_x(C) \cap P_x(C')$. Taking C to be a reducible conic and x to be its singular point, we obtain that

f is not defined at x. Since the pencil contains three reducible members, we obtain that f has three base points, hence f is given by a homaloidal net and hence is a birational map. Obviously, $x \in P_{f(x)}(C_1) \cap P_{f(x)}(C_2)$, hence f is an involution.

Of course, not every vector $(d; m_1, \ldots, m_N)$ satisfying equalities (7.21) and (7.23) is realized as the characteristic vector of a homaloidal net. There are other necessary conditions for a vector to be realized as the characteristic $(d; m_1, \ldots, m_N)$ for a homaloidal net. For example, if m_1, m_2 correspond to points of largest multiplicity, a line through the points should intersect a general member of the net non-negatively. This gives the inequality

$$d \geq m_1 + m_2.$$

Next, we take a conic through five points with maximal multiplicities. We get

$$2d \geq m_1 + \cdots + m_5.$$

Then, we take cubics through nine points, quartics through 14 points, and so on. The first case that can be ruled out in this way is $(5; 3, 3, 1, 1, 1, 1, 1)$. It satisfies the equalities (7.21) and (7.23) but does not satisfy the condition $m \geq m_1 + m_2$.

SS:7.2.2

7.2.2 de Jonquières transformations and hyperelliptic curves

Let $\mathcal{H} = |dh - \sum_{i=1}^N m_i x_i|$ be a homaloidal linear system, where we may assume that $m_1 \geq m_2 \cdots \geq m_N$. Obviously, x_1 must be a proper base point. Since $\mathcal{H}$ has no fixed components $\frac{d}{3} < m_1 \leq d - 1$. A Cremona transformation J_d defined by a homaloidal linear system with $m_1 = d - 1$ is called a *de Jonquières transformation*. We will give later other characterizations of such transformations and its extension to any space $\mathbb{P}^n$.

Lemma 7.2.6. *The characteristic vector of a de Jonquièyes transformation is* $(d, d-1, 1^{2d-2}) := (d, d-1, \underbrace{1, \ldots, 1}_{2d-2})$.

Proof Using (7.21) and (7.23), we get $d^2 - (d-1)^2 - \sum_{i=1}^N m_i^2 = 1 = 2d - 1 - \sum_{i=2}^N m_i^2 = 1$ and $3d - (d-1) - \sum_{i=2}^N m_i = 2d + 1 \sum_{i=2}^N m_i = 3$. Subtracting the first equality from the second one, we obtain $2 + \sum_{i=2}^N m_i(m_i - 1) = 2$. This could happen only if $m_2 = \cdots = m_N = 1$. Substituting these numbers in the first equality we get $N = 2d - 2$. □

It will be convenient to re-denote the base points:

$$(x_1, \ldots, x_{2d-1}) = (\mathfrak{o}, x_1, \ldots, x_{2d-2}).$$

A general member of the homaloidal linear system with characteristic vector $(d; d-1, 1\ \dots, 1)$ is a *monoidal curve*, i.e., a reduced curve of some degree n with a singular point of multiplicity $n-1$. Choosing projective coordinates such that the singular point $\mathfrak{o}$ is $[1,0,0]$ we write the equation of such a curve C in the form

$$t_0 b_{d-1}(t_1,t_2) + b_d(t_1,t_2) = 0, \tag{7.26}$$

where $b_s(t_1,t_2)$ is a binary form of degree indicated by the subscript. The number of conditions for passing simply through $2d-2$ points is equal to $2d-2$. Since the coefficients depend on $d+d+1 = 2d+1$ parameters, we see that a de Jonquièyes transformation can be given by curves from above with general b_{d-1} and b_d.

The lines $V(t_1)$ and $V(t_2)$ contain the point $\mathfrak{o}$. The restriction of the linear system to each of this line defines a linear series of degree one. Since the dimension of $\mathcal{H}$ is equal to two, there are members of $\mathcal{H}$ that contain these lines. This show that we can find a basis of $\mathcal{H}$ in the form

$$\begin{aligned} t_0' &= t_0 b_{d-1}(t_1,t_2) + b_d(t_1,t_2), \\ t_1' &= t_1(t_0 a_{d-2}(t_1,t_2) + a_{d-1}(t_1,t_2)), \\ t_2' &= t_2(t_0 a_{d-2}(t_1,t_2) + a_{d-1}(t_1,t_2)). \end{aligned} \tag{7.27}$$

Since $\mathcal{H}$ has no fixed components, the polynomials $F_d = t_0 b_{d-1} + b_d$ and $Q_{d-1} = t_0 a_{d-2} + a_{d-1}$ are coprime and $b_{d-1}a_{d-1} \neq b_d a_{d-2}$.

In affine coordinates $x = t_1/t_2$, $y = t_0/t_2$, the transformation is given by

$$(x', y') = \Big(x, \frac{y b_{d-1}'(x) + b_d'(x)}{y a_{d-2}'(x) + a_{d-1}'(x)}\Big). \tag{7.28}$$

afdej

Let us consider the closure of fixed points of J_d in $\mathrm{dom}(J_d)$. It is given by the affine equation

$$y b_{d-1}'(x) + b_d'(x) = y(y a_{d-2}'(x) + a_{d-1}'(x)).$$

Going back to our projective coordinates, the equation becomes

$$t_0^2 a_{d-2}(t_1,t_2) + t_0(a_{d-1}(t_1,t_2) - b_{d-1}(t_1,t_2)) - b_d(t_1,t_2) = 0. \tag{7.29}$$

hypp

This is a plane curve Γ of degree d with the point $\mathfrak{o} = [1,0,0]$ of multiplicity $d-2$. In Subsection 5.2.1 we saw that any hyperelliptic curve of genus $g = d-2$ admits such a birational model. The pencil of lines through $\mathfrak{o}$ defines the unique linear series g_2^1 on the curve. So, if Γ has no other singularities, it must be a rational curve if $d = 2$, an elliptic curve if $d = 3$, and a hyperelliptic curve of genus $d-2$ if $d \geq 4$.

The pencil generated by $V(t_1F_{d-1})$ and $V(t_2F_{d-1})$ is a part of our linear system $\mathcal{H}$. It shows that in an appropriate basis of the target $\mathbb{P}^2$, the Cremona transformation J_d blows down Γ to the point $\mathfrak{o}$. Hence Γ is a part of the P-locus. Other irreducible components of the P-locus are the lines $\langle \mathfrak{o}, x_i \rangle$. The total degree is equal to $3d - 3$ equal to the degree of the Jacobian of three polynomials of degree d. So, there is nothing else in the P-locus.

Let us find simple base points of $\mathcal{H}$. Let $x = [\alpha, \beta, \gamma]$ be a base point different from $\mathfrak{o}$. Then, either β or γ is not zero. Hence

$$\alpha b_{d-1}(\beta, \gamma) + b_d(\beta, \gamma) = \alpha a_{d-2}(\beta, \gamma) + a_{d-1}(\beta, \gamma) = 0.$$

If $\alpha \neq 0$ this happens if and only if

$$(b_{d-1} a_{d-1} - b_d a_{d-2})(\beta, \gamma) = 0.$$

If $\alpha = 0$, then the condition is $b_d(\beta, \gamma) = a_{d-1}(\beta, \gamma) = 0$, hence the point still satisfies the previous equation. Under some generality condition, this gives $2d - 2$ base points $x_1, \ldots, x_{2d-2}$. In general, some points come with multiplicities k_i which account for infinitely near points of level k_i. We see that $\{\mathfrak{o}, x_1, \ldots, x_{2d-2}\} = H_d \cap \Gamma$, where $\mathfrak{o}$ is the intersection point of multiplicity $(d-1)(d-2)$.

Note that a general member D of the homaloidal linear system intersects the line $\langle \mathfrak{o}, x_i \rangle$ with multiplicity $d - 1$ at $\mathfrak{o}$ and multiplicity one at x_i. This implies that each line belongs to the P-locus of J_d. Also, D intersects the curve Γ at $\mathfrak{o}$ with multiplicity $(d-1)^2$ and at the points x_i with multiplicity one. Since $D \cdot \ell = d(d-1) = (d-1)^2 + 2d - 2$, this implies that ℓ belongs to the P-locus two. The degree of the Jacobian is equal to $3(d-1) = d - 1 + 2d - 2$, thus there is nothing more in the P-locus.

Let us see when J_d is an involution. The affine equation shows that this happens if and only if the trace of the matrix $\begin{pmatrix} b'_{d-1} & b'_d \\ a'_{d-2} & a'_{d-1} \end{pmatrix}$ is equal to 0. Thus, the condition is

$$a_{d-1}(t_1, t_2) + b_{d-1}(t_1, t_2) = 0. \tag{7.30}$$ dejinv

In this case the hyperelliptic curve has the equation

$$t_0^2 a_{d-2}(t_1, t_2) + 2t_0 a_{d-1}(t_1, t_2) - b_d(t_1, t_2) = 0. \tag{7.31}$$ hypp2

The curve Γ coincides with the first polar $P_{\mathfrak{o}}(H_d)$ of H_d. The fundamental points x_i are the ramification points of the projection of H_m from the point $\mathfrak{o}$. It is clear that the curve H_d is nonsingular if and only if we have $2d - 2$ distinct simple fundamental points.

Example 7.2.7. Assume $d = 2$. Then a de Jonquières transformation is a quadratic transformation with base points $\mathfrak{o}, x_1, x_2$. The involution J_2 is one of the three standard quadratic involutions. The degenerate cases are $x_i \succ \mathfrak{o}, x_2 \succ x_1 \succ \mathfrak{o}$, and $x_2 \succ x_1$. The curve H_2 is a fixed conic passing through x_1, x_2 but not through $\mathfrak{o}$. The curve Γ is the line of the conic that contains x_1, x_2. It is the polar line of the conic if J_2 is an involution.

If $d = 3$, then H_3 is a cubic with a simple point at $\mathfrak{o}$, and Γ is a conic that passes through x_1, x_2, x_3, x_4 and tangent to the cubic at $\mathfrak{o}$. If J_d is an involution, the conic is the first polar of the cubic with pole at $\mathfrak{o}$.

Remark 7.2.8. Recall that a pair $\{a, b\}$ of distinct points in $\mathbb{P}^1$ defines an involutions $\sigma_{a,b}$ uniquely determined by the property that $\sigma_{a,b}(a) = a$ and $\sigma_{a,b}(b) = b$. It sends a point x to the unique point x' such that $x + x', 2a, 2b$ belong to the same linear series g_2^1 of degree 2 of $\mathbb{P}^1$. In other words, the pairs $\{a, b\}$ and $\{x, x'\}$ are harmonically conjugate (see Exercise 2.5). Another involution $\sigma_{a,b;p}$ associated with the pair $\{a, b\}$ requires fixing one point p on $\mathbb{P}^1$. It is uniquely determined by the property that $\sigma_{a,b;p}(a) = b, \sigma_{a,b}(b) = a, \sigma_{a,b}(p) = p$.

Any involution of $\mathbb{P}^1$ coincides with either $\sigma_{a,b}$ or $\sigma_{a,b;p}$ for some a, b, p. It is easy to check that the involutions $\sigma_{a,b}$ and $\sigma_{a,b;p}$ commute.

Using this simple observation, we see that H_d defines two Cremona I_1 and I_2 involutions of the plane. The involution I_1 sends a general point x to the point $\sigma_{a,b}(x)$, where $\{a, b\}$ are the residual intersection points of the line $\langle \mathfrak{o}, x\rangle$ with H_d. We leave it to the reader to check that it coincides with the de Jonquières involution J_d.

The second involution is the composition of I_1 with the lift to $\mathbb{P}^2$ of the hyperelliptic involution of H_d.

Obviously, a de Jonquières transformation leaves invariant the pencil of lines through the point $\mathfrak{o}$. This property characterizes such transformations.

characterization

Proposition 7.2.9. *Let f be a planar Cremona transformation. Suppose there exists a point $\mathfrak{o} \in \mathbb{P}^2$ such that the image of a general member of the pencil of lines through $\mathfrak{o}$ belongs to the same pencil. Then f is a de Jonquières transformation.*

Proof The homaloidal linear system of curves of degree d defining f restricts to a general line of the pencil as the linear system of degree $d - k$, where k is the number of the fundamental points of f on the line with total multiplicities k. Since the image of the line is a line in the same pencil, we must get $k = d - 1$. Since a general line in the pencil does not apps through the fundamental points except $\mathfrak{o}$, we obtain that $\mathfrak{o}$ is a fundamental point of multiplicity $d - 1$. This shows that f is a de Jonquiéres transformation. □

7.2.3 Nets of isologues and fixed points

SS:7.2.3

Let $f : \mathbb{P}^n \dashrightarrow \mathbb{P}^n$ be a Cremona transformation of degree d. Let $p \in \mathbb{P}^n$. Consider the locus of points $C_f(p)$ such that $x, f(x), p$ are collinear. This locus is called the *isologue* of p, the point p is called its *center*. In terms of equations, if f is given by polynomials $(f_0(t), \dots, f_n(t))$ of degree d and $p = [a_0, a_1, a_2]$, then $C_f(p)$ is given by equation

$$\operatorname{rank}\begin{pmatrix} a_0 & a_1 & \dots & a_n \\ t_0 & t_1 & \dots & t_n \\ f_0(t) & f_1(t) & \dots & f_n(t) \end{pmatrix} \le 2. \tag{7.32}$$ isol1

It follows immediately that, for general p, $C_f(p)$ is a curve of degree $d + 1$. One more observation is that

$$C_T(p) = C_{f^{-1}}(p). \tag{7.33}$$

Any $x \in \mathrm{dom}(f)$ determines a line $\langle x, f(x) \rangle$, and hence, we get a rational map

$$\iota_f : \mathbb{P}^n \dashrightarrow G_1(\mathbb{P}^n) \tag{7.34}$$ isologuemap

to the Grassmannian of lines in $\mathbb{P}^n$. The map is defined on the complement of the F-locus of f and the set of fixed points of f in $\mathrm{dom}(f)$. It is given by the minors of the submarix (7.32) of the last two rows.

The image of this map is closed subvariety $\mathfrak{G}(f)$ of $G_1(\mathbb{P}^n)$ of expected dimension n. The dimension can drop to $n - 1$ if, for a general point $x \in \mathbb{P}^3$, the image of a general point $y \in \langle x, f(x) \rangle$ is contained in contained in $\langle x, f(x) \rangle$. We will call $\mathfrak{G}(f)$ the *complex of lines* associated with f. If $n = 3$, $\mathfrak{G}(f)$ is a hypersurface in $G_1(\mathbb{P}^3)$, in classical terminology, it is a line complex. For any n, a line complex is a hypersurface in $G_1(\mathbb{P}^n)$ (see Section 10.2). So, for $n > 3$, the terminology is somewhat confusing, but we urge the reader to live with this. It is even more confusing when the dimensions $\mathfrak{G}(f)$ is equal to $n - 1$. In this case, f is called an *Arguesian transformation*. It is clear that f is Arguesian if and only if the linear system of ideologues is of dimension $n - 1$ instead of expected dimension n.

Note that $\mathfrak{G}(f) = \mathfrak{G}(f^{-1})$, the maps ι_T is equal to the composition $\iota_T \circ T$.

From now on, we assume that $n = 2$ and, for any point p, $C_f(p) \neq \mathbb{P}^2$. Then, $C_f(p)$ is a curve of degree $d + 1$. It passes through the fundamental points of f (because the last row in the determinant is identically zero for such point) and it passes through the *fixed points* of f, i.e., points $x \in \mathrm{dom}(f)$ such that $f(x) = x$ (because the last two rows are proportional). Also, $C_f(p)$ contains its center p (because the first two rows are proportional).

The next lemma is due to K. Doehleman [231].

karl **Lemma 7.2.10.** *The linear system of isologue curves $C_f(p)$ is a net, unless f is a de Jonquière transformation. In the latter case, the linear system contains the fixed part of degree d and its mobile part is a pencil of lines.*

Proof Suppose that the linear system of ideologues is a pencil. Let x be a general point in $\mathbb{P}^2$. Then x belongs to the pencil of ideologues $C_f(p)$, where $p \in \langle x, f(x)\rangle$. Since the whole linear system is a pencil, we see there is a point $p \in \mathbb{P}^2$ such that the points $x, f(x), p$ are collinear for all x in an open Zariski subset of $\mathbb{P}^2$. Equivalently, f leaves invariant a pencil of lines. By Proposition 7.2.9, f must be a de Jonquières transformation. Let d be the algebraic degree of f. We know that a de Jonquière transformation has a curve of degree d of fixed points. Thus, the moving part of the linear system of ideologues is a pencil of lines. □

Obviously, a one-dimensional irreducible component of the set of fixed points is a fixed component of the *net of isologues* $(C_f(p))_{p\in\mathbb{P}^2}$.

projgen *Remark* 7.2.11. It follows from the definition that the isologue curve $C_f(p)$ is *projectively generated* by the pencil of lines ℓ through p and the pencil of curves $f^{-1}(\ell)$. Recall that given two pencils $\mathcal{P}$ and $\mathcal{P}'$ of plane curves of degree d_1 and d_2 and a projective isomorphism $\alpha : \mathcal{P} \to \mathcal{P}'$, the union of points $Q \cap \alpha(Q), Q \in \mathcal{P}$, is a plane curve C. Assuming that the pencils have no common base points, C is a plane curve of degree $d_1 + d_2$. To see this, we take a general line ℓ and restrict $\mathcal{P}$ and $\mathcal{P}'$ to it. We obtain two linear series g^1_d and $g^1_{d'}$ on ℓ. The intersection $C \cap \ell$ consists of points common to divisors from g^1_d and $g^1_{d'}$. The number of such points is equal to the intersection of the diagonal of $\mathbb{P}^1 \times \mathbb{P}^1$ with a curve of bidegree (d, d'), hence it is equal to $d + d'$. It follows from the definition that C contains the base points of both pencils.

so **Proposition 7.2.12.** *Assume that f has no infinitely near fundamental points. Then, the multiplicity of a general isologue curve at a fundamental point x of multiplicity m is equal to m.*

Proof Let u, v be local affine parameters at x. For each homogeneous polynomial $\phi(t_0, t_1, t_2)$ vanishing at x with multiplicity $\geq m$, let $[\phi]_k := [\phi]_k(u, v)$ be the degree k homogeneous term in the Taylor expansion at x. If $V(f)$ is a general member of the homaloidal net, then $[f]_k = 0$ for $k < m$ and $[f_m] \neq 0$. Let B_m be the linear space of binary forms of degree m in variables u, v. Consider the linear map $\alpha : \mathbb{C}^3 \to B_m$ defined by

$$(a, b, c) \mapsto [(bt_2 - ct_1)f_0(t) + (ct_0 - at_2)f_1(t) + (at_1 - bt_0)f_2(t)]_m.$$

The map is the composition of the linear map $\mathbb{C}^3 \to \mathbb{C}^3$ defined by $(a, b, c) \mapsto ([bt_2 - ct_1]_0, [ct_0 - at_2]_0, [at_1 - bt_0]_0)$ and the linear map $\mathbb{C}^3 \to B_m$ defined by $(a, b, c) \mapsto [af_0 + bf_1 + cf_2]_m$. The rank of the first map is equal to 2, the kernel is generated by $[t_0]_0, [t_1]_0, [t_2]_0)$. Since no infinitely near point is a base point of the homaloidal net, the rank of the second map is greater than or equal to 2. This implies that the map α is not the zero map. Hence, there exists an isologue curve of multiplicity equal to m. □

cool *Remark* 7.2.13. Coolidge claims in [169, p. 460] that the assertion is true even in the case of infinitely near points. By a direct computation, one checks that the multiplicity of isologue curves of the degenerate standard Cremona transformation (7.13) at the unique base point is equal to 2.

numfixed **Corollary 7.2.14.** *Assume that the homaloidal net has no infinitely near base points and the net of isologues has no fixed component. Then, the number of fixed points of f is equal to $d + 2$.*

Proof Take two general points p, q in the plane. In particular, we may assume that the line $\ell = \langle p, q\rangle$ does not pass through the base points of the homaloidal net and the fixed points. Also, $p \notin C_f(q)$ and $q \notin C_f(p)$. Consider a point x in the intersection $C_f(p) \cap C_\phi(q)$. Assume that it is neither a base point nor a fixed point. Then, $p, q \in \langle x, f(x)\rangle$, hence $x, f(x), p, q$ lie on ℓ. Conversely, if $x \in \ell \cap C_f(p)$ and $x \neq p$, then the points $x, f(x), p$ are collinear and, since $q \in \ell$, we get that $x, f(x), q$ are collinear. This implies that $x \in C_f(q)$ and shows that the set of base points of the pencil of isologue curves $C_f(p), p \in \ell$, consists of base points of the homaloidal net, fixed points, and d points on ℓ (counted with multiplicities). The base points of the homaloidal net contribute $\sum_{i=1}^N m_i^2$ to the intersection. Applying Proposition 7.2.12, we obtain that fixed points contribute $d + 2 = (d+1)^2 - d - \sum_{i=1}^N m_i^2$ to the intersection. □

Note that the transformation from Remark 7.2.13 has no fixed points.

laguerre *Remark* 7.2.15. One can confirm the previous corollary by using the intersection theory on the graph Γ_f of f. Since there are no infinitely near base points, it is smooth and isomorphic to the blow-up of the base ideal. Its class in $A(\mathbb{P}^2 \times \mathbb{P}^2)$ is equal to $h_1^2 + dh_1h_2 + h_2^2$. Intersecting it with the class of the diagonal $[\Delta] = h_1^2 + h_1h_2 + h_2^2$, we obtain the number $d + 2$. In fact, one can use the argument for another proof of the corollary if we assume (that follows from the corollary) that no point in the intersection $\ell \cap \Delta$ lies on the exceptional curves of the projections.

The net of isologue curves without fixed component is a special case of a *Laguerre net*. It is defined by one of the following three equivalent properties.

thm:laguerre **Theorem 7.2.16.** *Let $|V|$ be an irreducible net of plane curves of degree d. The following properties are equivalent.*

(i) There exists a basis f_0, f_1, f_2 of V such that

$$t_0 f_0(t) + t_1 f_1(t) + t_2 f_2(t) = 0. \tag{7.35}$$ lag

(ii) For any basis f_0, f_1, f_2 of V, there exist three linearly independent linear forms l_1, l_2, l_3 such that

$$l_0 f_0 + l_1 f_1 + l_2 f_2 = 0.$$

(iii) There exists a basis f_0, f_1, f_2 of V such that

$$f_0 = t_1 g_2 - t_2 g_1, \quad f_1 = t_2 g_0 - t_0 g_2, \quad f_2 = t_0 g_1 - t_1 g_0,$$

where g_0, g_1, g_2 are homogeneous forms of degree $d-1$.

(iv) The base locus of a general pencil in $|V|$ is the union of the base locus of $|V|$ and a set of $d-1$ collinear points.

Proof The equivalence of the fist two properties is obvious. Obviously, (iii) implies (i). Suppose (i) holds. The Koszul complex in the ring of polynomials $S = \mathbb{C}[t_0, t_1, t_2]$ is an exact sequence

$$0 \to S \xrightarrow{\alpha} S^3 \xrightarrow{\beta} S^3 \xrightarrow{\gamma} S \to S/(t_0, t_1, t_2) \to 0,$$

where α is defined by $a \mapsto a(t_0, t_1, t_2)$. The map β is defined by the matrix

$$\begin{pmatrix} 0 & -t_2 & t_1 \\ t_2 & 0 & -t_0 \\ -t_1 & t_0 & 0 \end{pmatrix},$$

and the map γ is defined by $(a, b, c) \mapsto at_0 + bt_1 + ct_2$ (see Eisenbud [281], 17.2). Property (i) says that (f_0, f_1, f_2) belongs to the kernel of γ. Thus, it belongs to the image of β, and hence (iii) holds.

(i) $\Rightarrow$ (iv) Take two general curves $C_\lambda = V(\lambda_0 f_0 + \lambda_1 f_1 + \lambda_2 f_2)$ and $C_\mu = V(\mu_0 f_0 + \mu_1 f_1 + \mu_2 f_2)$ from the net. They intersect with multiplicity ≥ 2 at a point x if and only if x belongs to the Jacobian curve of the net. This shows that the set of pencils which intersect non-transversally outside the base locus is a proper closed subset of $\mathbb{P}(V)$. So, we may assume that $C(\mu)$ and $C(\nu)$ intersect transversally outside the base locus of the net. Let $p = [a]$ belong to $C_\lambda \cap C_\mu$ but does not belong to the base locus of $|V|$. Then, $(f_0(a), f_1(a), f_2(a))$ is a nontrivial solution of the system of linear equations with the matrix of coefficients equal to

$$\begin{pmatrix} \lambda_0 & \lambda_1 & \lambda_2 \\ \mu_0 & \mu_1 & \mu_2 \\ a_0 & a_1 & a_2 \end{pmatrix}.$$

This implies that the line spanned by the points $\lambda = [\lambda_0, \lambda_1, \lambda_2]$ and $\mu = [\mu_0, \mu_1, \mu_2]$ contains the point p. Thus, all base points of the pencil different from the base points of the net are collinear. Conversely, suppose a non-base point $[a] \neq \lambda, \mu$ lies on a line $\langle \lambda, \mu \rangle$ and belongs to the curve C_λ. Then, $(f_0(a), f_1(a), f_2(a))$ is a nontrivial solution of

$$\lambda_0 t_0 + \lambda_1 t_1 + \lambda_2 t_2 = 0, \ a_0 t_0 + a_1 t_1 + a_2 t_2 = 0,$$

and hence satisfies the third equation $\mu_0 t_0 + \mu_1 t_1 + \mu_2 t_2 = 0$. This shows that $a \in C_\lambda \cap C_\mu$. Thus, we see that the intersection $C_\lambda \cap C_\mu$ consists of $d - 1$ non-base points.

(iv) ⇒ (ii) We follow the proof from CoolidgeCurves [169, p. 423]. Let $V(f_0), V(f_1)$ be two general members intersecting at $d-1$ points on a line $V(l)$ not passing through the base points. Let p_i be the residual point on $V(f_i)$. Choose a general line $V(l_0)$ passing through p_2 and a general line $V(l_1)$ passing through p_1. Then, $V(l_0 f_0)$ and $V(l_1 f_1)$ contain the same set of $d+1$ points on the line $V(l)$, hence we can write

$$l_0 f_0 + c l_1 f_1 = l f_2, \tag{7.36}$$ 11

for some polynomial f_2 of degree d and some constant c. For any base point q of the net, we have $l_0(q) f_0(q) + c l_1(q) f_1(q) = l(q) f_2(q)$. Since $l(q) \neq 0$ and $f_0(q) = f_1(q) = 0$, we obtain that $f_2(q) = 0$. Thus, the curve $V(f_2)$ passes through each base point and hence belongs to the net $|V|$. This shows that f_0, f_1 and f_2 define a basis of $|V|$ satisfying property (ii). □

Corollary 7.2.17. *Let* $\mathfrak{b}$ *be the base ideal of a Laguerre net of curves of degree* d*. Then,* $h^0(\mathcal{O}_{\mathbb{P}^2}/\mathfrak{b}) = d^2 - d + 1$.

Proof It is clear that, any base-point, $\mathfrak{b}$ is generated by two general members of the net. By Bezout's Theorem $h^0(\mathcal{O}_{\mathbb{P}^2}/\mathfrak{b}) = d^2 - (d-1)$. □

Example 7.2.18. Take an irreducible net of cubic curves with seven base points. Then, it is a Laguerre net since two residual intersection points of any two general members are on a line. Thus, it is generated by the minors of the matrix

$$\begin{pmatrix} t_0 & t_1 & t_2 \\ g_0 & g_1 & g_2 \end{pmatrix},$$

where g_0, g_1, g_2 are quadratic forms. Recall that the linear system of cubics with seven base points satisfying the condition (*) from Section [SS:6.3.3] 6.3.3 defines a nonsingular quartic curve. It is known that the quartic curve is a Lüroth quartic if and only if there exists a cubic curve $V(f)$ such that $g_i = \frac{\partial f}{\partial t_i}$ (see [Bateman] [39], [OttavianiSernesi] [565]).

SS:7.2.4

7.2.4 Characteristic matrices

Consider a log resolution ([hironaka] 7.1) of a Cremona transformation f of degree d

$$\begin{array}{ccc} & X & \\ \pi \swarrow & & \searrow \sigma \\ \mathbb{P}^2 & \overset{f}{\dashrightarrow} & \mathbb{P}^2. \end{array}$$

Obviously, it gives a resolution of the inverse transformation f^{-1}. The roles of π and σ are interchanged. Let

$$\sigma : X = Y_N \xrightarrow{\sigma_N} Y_{N-1} \xrightarrow{\sigma_{N-1}} \dots \xrightarrow{\sigma_2} X_1 \xrightarrow{\sigma_1} X_0 = \mathbb{P}^2 \tag{7.37}$$

Since $K_X^2 = 9 - N$, the length of the sequence of blow-ups for σ is the same as one for π. Since the degree of f and f^{-1} coincide, the homaloidal net defining f^{-1} is $|dh - \xi|$, where ξ is the fundamental bubble cycle of f^{-1}. Let $\mathcal{E}'_1, \dots, \mathcal{E}'_N$ be the corresponding exceptional configurations.

Note that it could happen that all exceptional configurations of π are irreducible (i.e. no infinitely points are used to define π) but some of the exceptional configurations of σ are reducible (see Exercise [ex:7.star] 7.8).

A rational surface X admitting a birationa morphism $f : X \to \mathbb{P}^2$ is called a *basic rational surface*. An example of a rational surface which is not basic is a minimal ruled surface $\mathbf{F}_n, n \neq 1$. We refer to [DKEII] [259, Chapter 9] for the description of non-basic rational surfaces. A choice of the blrational morphism $f : X \to \mathbb{P}^2$ defines a basis $(e_0, e_1, \dots, e_N)$ in $\mathrm{Pic}(X)$, where $e_0 = f^*(h)$ is the pull-back of the divisor class of a line in the plane, and $e_i = [\mathcal{E}_i]$ are the divisor classes of the exceptional configurations. We call such a basis a *geometric basis* defined by the blowing down structure of X.

The divisor class of any irreducible curve C on X can be expressed in terms of a geometric basis by

$$[C] = de_0 - m_1 e_1 - \dots - m_N e_N.$$

The curve C is equal to the proper transform under f of an irreducible plane curve of degree d passing through the points p_i with multiplicity m_i. We will often identify the linear system $|de_0 - m_1 e_1 - \dots - m_N|$ on X with the linear system of plane curves of degree d passing through the points p_i with multiplicity $\geq m_i$.

We will often employ the classical notation

$$C^d(m_1, \ldots, m_N) \tag{7.38}$$

classicalnotation

for a plane curve of degree d with m_i-multiple points at $p_1, \ldots, p_N$.

Definition 7.2.19. *An* ordered resolution *of a Cremona transformation is the diagram* (hironaka 7.1) *together with an order of a sequence of the exceptional curves for σ and π (equivalently, a choice of an admissible order on the bubble cycles defining π and σ).*

Any ordered resolution of f defines two bases in $\mathrm{Pic}(X)$. The first basis is

$$\underline{e}' : e'_0 = \sigma^*(h),\ e'_1 = [\mathcal{E}'_1], \ldots, e'_N = [\mathcal{E}'_N].$$

The second basis is

$$\underline{e} : e_0 = \pi^*(h),\ e_1 = [\mathcal{E}_1], \ldots, e_N = [\mathcal{E}_N].$$

Here, as always, h denotes the class of a line in the plane.

We will always assume that the resolution of σ is minimal. This is equivalent to the property that $e'_j \neq e_i$ for any i, j. If $e'_j = e_i$, then the exceptional configurations $\mathcal{E}_i$ and $\mathcal{E}'_j$ are equal. We can change the admissible orders on the bubble cycles defining the maps π and σ to assume that $i = j = n - b$, where b is the number of irreducible components in $\mathcal{E}_i$, the exceptional divisor of $\pi_{N-i} : X \to X_i$ is equal to $\mathcal{E}_i$ and the exceptional divisor of $\sigma_{Ni} : X \to Y_i$ is equal to $\mathcal{E}'_i$. By the universal property of the blow-up, there exists an isomorphism $\phi : X_i \to Y_i$ such that $\phi \circ \pi_{Ni} = \sigma_{Nj}$. Thus, we can replace X with X_i and define a new resolution $\pi_{i0} : X_i \to \mathbb{P}^2, \sigma_{i0} \circ \phi : X_i \to \mathbb{P}^2$ of f. The old resolution factors through the new one.

Write

$$e'_0 = de_0 - \sum_{i=1}^{N} m_i e_i, \quad e'_j = d_j e_0 - \sum_{i=1}^{N} m_{ij} e_i,\ j > 0.$$

By the minimality property, we may assume that $d, d_1, \ldots, d_N > 0$. The matrix

$$A = \begin{pmatrix} d & d_1 & \ldots & d_N \\ -m_1 & -m_{11} & \ldots & -m_{1N} \\ \vdots & \vdots & \vdots & \vdots \\ -m_N & -m_{N1} & \ldots & -m_{NN} \end{pmatrix} \tag{7.39}$$

charmat

is called the *characteristic matrix* of f with respect to an ordered resolution. It is the matrix of change of basis from $\underline{e}$ to $\underline{e}'$.

Here $(d; m_1, \ldots, m_N)$ is the characteristic of f. In other columns, the vectors $(d_j, m_{1j}, \ldots, m_{Nj})$ describe the divisor classes of the exceptional configurations $\mathcal{E}'_j$ of σ. The image of $\mathcal{E}'_j$ in $\mathbb{P}^2$ is a curve in the linear system $|d_j h - \sum_{i=1}^N m_{ij} x_i|$. Its degree is equal to d_j. It may not be irreducible or reduced. Let E be a unique (-1)-component of the exceptional configuration $\mathcal{E}'_j$. It corresponds to a minimal point in the bubble cycle η' infinitely near x_j of order equal to the number of irreducible components of $\mathcal{E}'_j$ minus one. By the minimality assumption, the image $\pi(E)$ is an irreducible curve, and the image $\pi(\mathcal{E}_j)$ contains $\pi(E)$ with multiplicity equal to b_j.

The image of $\mathcal{E}_j$ under the map π is called a *total principal curve* of f. Its degree is equal to d_j. The reduced union of total principal curves is equal to the P-locus of f.

The characteristic matrix defines a homomorphism of free abelian groups

$$\phi_A : \mathbb{Z}^{1+N} \to \mathbb{Z}^{1+N}.$$

We equip $\mathbb{Z}^{1+N}$ with the standard hyperbolic inner product, where the norm-square v^2 of a vector $v = (a_0, a_1, \ldots, a_N)$ is defined by

$$v^2 = a_0^2 - a_1^2 - \cdots - a_N^2. \tag{7.40}$$

lattice1

Recall that a *quadratic lattice* (or just *lattice* if no confusion arises with other uses of this word) is a free abelian group equipped with an integral valued quadratic form. The group $\mathbb{Z}^{1+N}$ equipped with integral form (7.40) is an example of an odd unimodular quadratic lattice. It is customarily denoted by $\mathrm{I}^{1,N}$. We will discuss quadratic lattices in Chapter 8. Since both bases $\underline{e}$ and $\underline{e'}$ are orthonormal with respect to the inner product, we obtain that the characteristic matrix is orthogonal, i.e. belongs to the group $\mathrm{O}(\mathrm{I}^{1,N}) \subset \mathrm{O}(1,N)$, where $O(1,N)$ is the real orthogonal group of the hyperbolic space $\mathbb{R}^{1,N}$ with the hyperbolic norm-square defined by the quadratic form $x_0^2 - x_1^2 - \cdots - x_n^2$.

Recall that the orthogonal group $\mathrm{O}(1,N)$ consists of $(N+1)\times(N+1)$ matrices M such that

$$M^{-1} = J_{N+1}{}^t M J_{N+1}, \tag{7.41}$$

orth

where J_{N+1} is the diagonal matrix $\mathrm{diag}[1,-1,\ldots,-1]$.

In particular, the characteristic matrix A^{-1} of f^{-1} satisfies

$$A^{-1} = J^t A J = \begin{pmatrix} d & m_1 & \dots & m_N \\ -d_1 & -m_{11} & \dots & -m_{N1} \\ \vdots & \vdots & \vdots & \vdots \\ -d_N & -m_{1N} & \dots & -m_{NN} \end{pmatrix}. \tag{7.42}$$ inv3

It follows that the vector $(d; d_1, \dots, d_N)$ is equal to the characteristic vector of f^{-1}. Also, (7.23) implies that $d_1 + \dots + d_N = d - 3$. This shows that the sum of the degrees of a total principal curve of f is equal to the degree of the Jacobian J of the polynomials defining f. This explains the multiplicities of irreducible components of $V(J)$. They are larger than one when not all fundamental points are proper.

Let $f : X' \to X$ be a rational map of irreducible varieties. For any closed irreducible subvariety Z of X' with $X' \cap \mathrm{dom}(f) \neq \emptyset$, we denote by $f(Z)$ the closure of the image of $Z \cap \mathrm{dom}(f)$ under f.

transform **Proposition 7.2.20.** *Let $f : \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$ be a Cremona transformation with fundamental points $x_1, \dots, x_N$ and fundamental points $y_1, \dots, y_N$ of f^{-1}. Let A be the characteristic matrix of f. Let C be an irreducible curve on $\mathbb{P}^2$ of degree n which passes through the points y_i with multiplicities n_i. Let n' be the degree of $f(C)$ and let n_i' be the multiplicity of $f(C)$ at x_i. Then the vector $v = (n', -n_1', \dots, -n_N')$ is equal to $A^{-1} \cdot v$, where $v = (n, -n_1, \dots, -n_N)$.*

Proof Let (X, σ, ν) be a minimal resolution of f. The divisor class of the proper inverse transform $\pi^{-1}(C)$ in X is equal to $v = ne_0 - \sum n_i e_i$. If we rewrite it in terms of the basis $(e_0', e_1', \dots, e_N')$ we obtain that it is equal to $v' = n' e_0 - \sum n_i' e_i$, where $v' = Av$. Now, the image of $\pi^{-1}(C)$ under σ coincides with $\phi(C)$. By definition of the curves $\mathcal{E}_i$, the curve $\phi^{-1}(C)$ is a curve of degree n' passing through the fundamental points y_i of f^{-1} with multiplicities n_i'. □

Let C be a total principal curve of f and $ce_0 - \sum_{i=1}^N c_i e_i$ be the class of $\pi^{-1}(C)$. Let $v = (c, -c_1, \dots, -c_N)$. Since $f(C)$ is a point, $A \cdot v = -e_j'$ for some j.

Example 7.2.21. The following matrix is a characteristic matrix of the standard quadratic transformation T_{st} or its degenerations $T_{\mathrm{st}}, T_{\mathrm{st}}''$.

$$A = \begin{pmatrix} 2 & 1 & 1 & 1 \\ -1 & 0 & -1 & -1 \\ -1 & -1 & 0 & -1 \\ -1 & -1 & -1 & 0 \end{pmatrix}. \tag{7.43}$$ standardmatrix

This follows from Example 7.1.14.

The following is a characteristic matrix of a de Jonquières transformation

$$A = \begin{pmatrix} m & m-1 & 1 & \dots & 1 \\ -m+1 & -m+2 & -1 & \dots & -1 \\ -1 & -1 & -1 & \dots & 0 \\ -1 & -1 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ -1 & -1 & 0 & \dots & 0 \\ -1 & -1 & 0 & \dots & -1 \end{pmatrix}. \tag{7.44}$$

Observe that the canonical class K_X is an element of $\mathrm{Pic}(X)$ which can be written in both bases as

$$K_X = -3e_0 + \sum_{i=1}^{N} e_i = -3e_0' + \sum_{i=1}^{n} e_i'.$$

This shows that the matrix A considered as an orthogonal transformation of $I^{1,N}$ leaves the vector

$$\mathbf{k}_N = -3\mathbf{e}_0 + \mathbf{e}_1 + \cdots + \mathbf{e}_N = (-3, 1, \dots, 1)$$

invariant. Here, $\mathbf{e}_i$ denotes the unit vector in $\mathbb{Z}^{1+N}$ with $(i+1)$-th coordinate equal to 1 and other coordinates equal to zero.

The matrix A defines an orthogonal transformation of the orthogonal complement $(\mathbb{Z}\mathbf{k}_N)^{\perp}$.

Lemma 7.2.22. *The following vectors form a basis of* $(\mathbb{Z}\mathbf{k}_N)^{\perp}$.

$$N \geq 3 : \alpha_1 = \mathbf{e}_0 - \mathbf{e}_1 - \mathbf{e}_2 - \mathbf{e}_3, \quad \alpha_i = \mathbf{e}_{i-1} - \mathbf{e}_i,\ i = 2, \dots, N,$$
$$N = 2 : \alpha_1 = \mathbf{e}_0 - 3\mathbf{e}_1, \quad \alpha_2 = \mathbf{e}_1 - \mathbf{e}_2,$$
$$N = 1 : \alpha_1 = \mathbf{e}_0 - 3\mathbf{e}_1.$$

Proof Obviously, the vectors α_i are orthogonal to the vector $\mathbf{k}_N$. Suppose a vector $v = (a_0, a_1, \dots, a_N) \in (\mathbb{Z}\mathbf{k}_N)^{\perp}$. Thus, $3a_0 + \sum_{i=1}^{N} a_i = 0$, hence $-a_N = 3a_0 + \sum_{i=1}^{N-1} a_i$. Assume $N \geq 3$. We can write

$$v = a_0(\mathbf{e}_0 - \mathbf{e}_1 - \mathbf{e}_2 - \mathbf{e}_3) + (a_0 + a_1)(\mathbf{e}_1 - \mathbf{e}_2) + (2a_0 + a_1 + a_2)(\mathbf{e}_2 - \mathbf{e}_3)$$
$$+ \sum_{i=3}^{N-1} (3a_0 + a_1 + \cdots + a_i)(\mathbf{e}_i - \mathbf{e}_{i+1}).$$

If $N = 2$, we write $v = a_0(\mathbf{e}_0 - 3\mathbf{e}_1) + (3a_0 + a_1)(\mathbf{e}_1 - \mathbf{e}_2)$. If $N = 1$, $v = a_0(\mathbf{e}_0 - 3\mathbf{e}_1)$. □

It is easy to compute the matrix $Q_N = (a_{ij})$ of the restriction of the inner product to $(\mathbb{Z}\mathbf{k}_N)^\perp$ with respect to the basis (α_0, α_{N-1}). We have

$$(-8), \quad \text{if } N = 1, \quad \begin{pmatrix} -8 & 3 \\ 3 & -2 \end{pmatrix}, \quad \text{if } N = 2.$$

If $N \geq 3$, we have

$$a_{ij} = \begin{cases} -2 & \text{if } i = j, \\ 1 & \text{if } |i - j| = 1 \text{ and } i, j \geq 1, \\ 1 & \text{if } i = 0, j = 3, \\ 0 & \text{otherwise.} \end{cases}$$

For $N \geq 3$, the matrix $A + 2I_N$ is the incidence matrix of the graph from Figure 7.5 (the *Coxeter-Dynkin diagram* of type $T_{2,3,N-3}$).

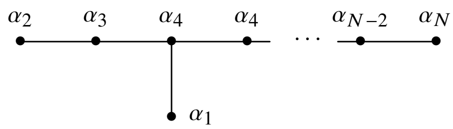


Figure 7.4 Coxeter-Dynkin diagram of type $T_{2,3,N-3}$ Coxdiag

For $3 \leq N \leq 8$ this is the Coxeter-Dynkin diagram of the root system of the semi-simple Lie algebra $\mathfrak{sl}_3 \oplus \mathfrak{sl}_2$ of type $A_2 + A_1$ if $N = 3$, of $\mathfrak{sl}_5$ of type $\mathfrak{A}_4$ if $N = 4$, of $\mathfrak{so}_{10}$ of type D_5 if $N = 5$ and of the exceptional simple Lie algebra of type E_N if $N = 6, 7, 8$.

We have

$$\mathbf{k}_N^2 = 9 - N.$$

This shows that the matrix Q_N is negative definite if $N < 9$, semi-negative definite with 1-dimensional null-space for $N = 9$, and of signature $(1, N - 1)$ for $N \geq 10$. By a direct computation one checks that its determinant is equal to $N - 9$.

Proposition 7.2.23. *Assume $N \leq 8$. There are only finitely many possible characteristic matrices. In particular, there are only finitely many possible characteristics of a homaloidal net with ≤ 8 base points.*

Proof Let

$$G = \{M \in \mathrm{GL}(N) : {}^tMQ_NM = Q_N\}.$$

Since Q_N is negative definite for $N \leq 8$, the group G is isomorphic to the orthogonal group $\mathrm{O}(N)$. The latter group is a compact Lie group. A characteristic matrix belongs to the subgroup $\mathrm{O}(Q_N) = G \cap \mathrm{GL}(N, \mathbb{Z})$. Since the latter is discrete, it must be finite. □

There are further properties of characteristic matrices for which we refer to [2] for the modern proofs. The most important of these is the following *Clebsch Theorem*.

Theorem 7.2.24. *Let A be the characteristic matrix. There exists a bijection $\beta : \mathbb{N} \to \mathbb{N}$ such that for any set I of columns with $d_i = n, i \in I$, there exists a set of rows J with $\#I = \#J$ such that $\mu_j = \beta(a), j \in J$.*

Note that subtracting two columns (or rows) with the same first entry and taking the inner product square, we easily get that they differ only at two entries by ± 1. This implies a certain symmetry of the matrix if one reorders the columns and rows according to Clebsch's Theorem. We refer for the details to [2].

SS:7.2.5

7.2.5 The Weyl groups

Let $\mathsf{E}_N = (\mathbb{Z}\mathbf{k}_N)^\perp \cong \mathbb{Z}^N$ equipped with the quadratic form obtained by the restriction of the inner product in $\mathsf{I}^{1,N}$. Assume $N \geq 3$. For any vector $\alpha \in \mathsf{E}_N$ with $\alpha^2 = -2$, we define the following element in $\mathrm{O}(E_N)$:

$$r_\alpha : v \mapsto v + (v, \alpha)\alpha.$$

It is called the *reflection* with respect to α. It acts identically on the orthogonal complement to α, and maps α to $-\alpha$.

Definition 7.2.25. *The subgroup $W(\mathsf{E}_N)$ of $\mathrm{O}(\mathsf{E}_N)$ generated by reflections r_{α_i} is called the* Weyl group *of E_N.*

The following proposition is stated without proof. It follows from the theory of groups generated by reflections (see, for example, [251], 4.3).

weylgroup

Proposition 7.2.26. *The Weyl group $W(\mathsf{E}_N)$ is of infinite index in $\mathrm{O}(\mathsf{E}_N)$ for $N > 10$. For $N \leq 10$,*

$$\mathrm{O}(\mathsf{E}_N) = W(\mathsf{E}_N) \rtimes (\tau),$$

where $\tau^2 = 1$ and $\tau = 1$ if $N = 7, 8$, $\tau = -1$ if $N = 9, 10$ and τ is induced by the symmetry of the Coxeter-Dynkin diagram for $N = 4, 5, 6$.

Note that any reflection can be extended to an orthogonal transformation of the lattice $\mathsf{I}^{1,N}$ (use the same formula). The subgroup generated by reflections $r_{\alpha_i}, i \neq 1$, acts as the permutation group $\mathfrak{S}_N$ of the vectors $\mathbf{e}_1, \ldots, \mathbf{e}_N$.

noetherineq **Lemma 7.2.27.** *(Noether's inequality) Let $v = (d, m_1, \ldots, m_N)$. Assume $d > 0, m_1 \geq \ldots \geq m_N > 0$, and*

(i) $\sum_{i=1}^{n} m_i^2 = d^2 + a$;
(ii) $\sum_{i=1}^{N} m_i = 3d - 2 + a$,

where $a \in \{-1, 0, 1\}$. Then

$$m_1 + m_2 + m_3 \geq d.$$

Proof We have

$$m_1^2 + \cdots + m_N^2 = d^2 - 1, \quad m_1 + \cdots + m_N = 3d - 3.$$

Multiplying equality (ii) by m_3 and subtracting it from equality (i), we obtain

$$m_1(m_1 - m_3) + m_2(m_2 - m_3) - \sum_{i \geq 4} m_i(m_3 - m_i) = d^2 + a - 3m_3(d - \tfrac{2-a}{3}).$$

We can rewrite the previous equality in the form

$$(d - \tfrac{2-a}{3})(m_1 + m_2 + m_3 - d - \tfrac{2-a}{3}) = (m_1 - m_3)(d - \tfrac{2-a}{3} - m_1) +$$

$$(m_2 - m_3)(d - \tfrac{2-a}{3} - m_2) + \sum_{i \geq 4} m_i(m_3 - m_i) + a + (\tfrac{2-a}{3})^2.$$

Note that $\frac{2-a}{3} < 1 \leq d$ unless $a = -1$ when $\frac{2-a}{3} = 1$. In any case, (i) and (ii) give that $d - \frac{2-a}{3} - m_i > 0$. Thus, all summands on the right-hand side are positive. In the left-hand side, the factor $d - \frac{2-a}{3}$ is positive unless $d = 1, a = -1$. In the latter case, all $m_i = 0$ contradicting our assumption that $m_N > 0$. Therefore, we obtain $m_1 + m_2 + m_3 > d + \frac{2-a}{3}$. Since $\frac{2-a}{3} = -\frac{1}{3}$ if it is not positive, this implies $m_1 + m_2 + m_3 > d$. □

The following corollary agrees with the more general Noether-Fano inequality noetherformula1 7.1.13.

fano2 **Corollary 7.2.28.**

$$m_1 > d/3.$$

We can apply Noether's Lemma to the case when $v = (d, m_1, \ldots, m_N)$ is the characteristic vector of a homaloidal net or when $d\mathbf{e}_0 - \sum m_i \mathbf{e}_i$ is the class of an exceptional configuration.

Definition 7.2.29. *Let $v = d\mathbf{e}_0 - \sum_{i=1}^{N} m_i \mathbf{e}_i \in \mathrm{I}^{1,N}$. We say that v is of* homaloidal type *(resp.*conic bundle type*,* exceptional type*) if it satisfies conditions (i) and (ii) from the above with $a = -1$ (resp. $a = 0$, resp. $a = 1$). We say that v is of* proper *homaloidal (exceptional type) if there exists a Cremona transformation whose characteristic matrix has v as the first (resp. second column).*

extype1 **Lemma 7.2.30.** *Let $v = d\mathbf{e}_0 - \sum_{i=1}^n m_i\mathbf{e}_i$ belong to the $W(\mathsf{E}_N)$-orbit of $\mathbf{e}_1$. Then $d \geq 0$. Let $\eta = \sum_{i=1}^N x_i$ be a bubble cycle and $\alpha_\eta : \mathsf{I}^{1,N} \to \mathrm{Pic}(Y_\eta)$ be an isomorphism of lattices defined by choosing some admissible order of η. Then $\alpha_\eta(v)$ is an effective divisor.*

Proof The assertion is true for $v = \mathbf{e}_1$. In fact, $\alpha_\eta(v)$ is the divisor class of the first exceptional configuration $\mathcal{E}_1$. Let $w = s_k \circ \cdots \circ s_1 \in W(\mathsf{E}_N)$ be written as the product of simple reflections with minimal possible k. One can show that k is uniquely defined by w. It is called the *length* of w. Let $v = w(\mathbf{e}_1) = (d', m_1', \ldots, m_N')$. We prove the assertion by using induction on the length of w. The assertion is obvious if $k = 1$ since $v' = \mathbf{e}_0 - \mathbf{e}_i - \mathbf{e}_j$ or differs from v by a permutation of the m_i's. Suppose the assertion is true for all w of length $\leq k$. Let w has length $k+1$. Without loss of generality, we may assume that s_{k+1} is the reflection with respect to some root $\mathbf{e}_0 - \mathbf{e}_1 - \mathbf{e}_2 - \mathbf{e}_3$. Then $d' = 2d - m_1 - m_2 - m_3 < 0$ implies $4d^2 < (m_1+m_2+m_3)^2 \leq 3(m_1^2+m_2^2+m_3^2)$, hence $d^2 - m_1^2 - m_2^2 - m_3^2 < -\frac{d^2}{3}$. If $d \geq 2$, this contradicts condition (i) of the exceptional type. If $d = 1$, we check the assertion directly by listing all exceptional types.

To prove the second assertion, we use the Riemann-Roch Theorem applied to the divisor class $\alpha_\eta(v)$. We have $\alpha_\eta(v)^2 = -1, \alpha_\eta(v) \cdot K_{Y_\eta} = -1$, hence $h^0(\alpha_\eta(v)) + h^0(K_{Y_\eta} - \alpha_\eta(v)) \geq 1$. Assume $h^0(K_{Y_\eta} - \alpha_\eta(v)) > 0$. Intersecting $K_Y - \alpha_\eta(v)$ with $e_0 = \alpha_\eta(\mathbf{e}_0)$, we obtain a negative number. However, the divisor class e_0 is nef on Y_η. This shows that $h^0(\alpha_\eta(v)) > 0$ and we are done. □

L7.4.10 **Lemma 7.2.31.** *Let v be a proper homaloidal type. Then it belongs to the $W(\mathsf{E}_N)$-orbit of the vector $\mathbf{e}_0$.*

Proof Let $v = d\mathbf{e}_0 - \sum_{i=1}^N m_i\mathbf{e}_i$ be a proper homaloidal type and η be the corresponding homaloidal bubble cycle. Let $w \in W(\mathsf{E}_N)$ and $\mathbf{v}' = w(v) = d'\mathbf{e}_0 - \sum_{i=1}^N m_i'\mathbf{e}_i$. We have $m_i' = \mathbf{e}_i \cdot v' = w^{-1}(\mathbf{e}_i) \cdot \mathbf{v}$. Since $w^{-1}(\mathbf{e}_i)$ represents an effective divisor on Y_η and v is the characteristic vector of the corresponding homaloidal net, we obtain $w^{-1}(\mathbf{e}_i) \cdot v \geq 0$, hence $m_i \geq 0$.

Obviously, $m_i \geq 0$. We may assume that $v \neq \mathbf{e}_0$, i.e. the homaloidal net has at least three base points. Applying Noether's inequality (noetherineq 7.2.27), we find m_i, m_j, m_k such that $m_i + m_j + m_k > d$. We choose the maximal possible such m_i, m_j, m_k. After reordering, we may assume that $m_1 \geq m_2 \geq m_3 \geq \ldots \geq m_N$. Note that this preserves the properness of the homaloidal type since the new order on η is still admissible. Applying the reflection s with respect to the vector $\mathbf{e}_0 - \mathbf{e}_1 - \mathbf{e}_2 - \mathbf{e}_3$, we obtain a new homaloidal type $v' = d'\mathbf{e}_0 - \sum_{i=1}^N m_i'\mathbf{e}_i$ with $d' = 2d - m_1 - m_2 - m_3 < d$. As we saw above, each $m_i \geq 0$. So, we can apply Noether's inequality again until we get $w \in W(\mathsf{E}_N)$ such that the number of nonzero coefficients m_i' of $\mathbf{v}' = w(v)$ is at most 2 (i.e. we cannot apply Noether's inequality anymore). A straightforward computation shows that such vector must be equal to $\mathbf{e}_0$. □

history *Remark* 7.2.32. Observe that the characteristic matrix of a quadratic transformation with fundamental points x_1, x_2, x_3 is the matrix of the reflection s_{α_1} with respect to the vector $\alpha_1 = \mathbf{e}_0 - \mathbf{e}_1 - \mathbf{e}_2 - \mathbf{e}_3$. So, the previous proposition seems to suggest that, by applying a sequence of quadratic transformation, we obtain a Cremona transformation with characteristic vector $(1, 0, \ldots, 0)$. It must be a projective transformation. In other words, any Cremona transformation is the composition of quadratic and projective transformations. This is the content of Noether's Factorization Theorem, which we will prove later in this section. The original proof by Noether was along these lines, where he wrongly presumed that one can always perform a standard quadratic transformation with fundamental points equal to the highest multiplicities, say m_1, m_2, m_3. The problem here is that the three points x_1, x_2, x_3 may not represent the fundamental points of a standard Cremona transformation when one of the following cases happens
gaps for the three fundamental points x_1, x_2, x_3 of highest multiplicities:

(i) $x_2 \succ x_1, x_3 \succ x_1$;
(ii) the base ideal in an affine neighborhood of x_1 is equal to (u^2, v^3) (cuspidal case).

charweyl **Theorem 7.2.33.** *Let A be a characteristic matrix of a homaloidal net. Then A belongs to the Weyl group $W(\mathsf{E}_N)$.*

Proof Let $A_1 = (d, -m_1, \ldots, -m_N)$ be the first column of A. Applying the previous lemma, we obtain $w \in W(\mathsf{E}_N)$, identified with a $(N+1) \times (N+1)$-matrix, such that the $w \cdot A_1 = \mathbf{e}_0$. Thus, the matrix $A' = w \cdot A$ has the first column equal to the vector $(1, 0, \ldots, 0)$. Since A' is an orthogonal matrix (with respect to the hyperbolic inner product), it must be the direct sum of the unit matrix I_1 of size one and an orthogonal matrix O of size $n - 1$. Since O has integer entries, it is equal to the product of a permutation matrix P and the diagonal matrix with ± 1 at the diagonal. Since $A \cdot \mathbf{k}_N = \mathbf{k}_N$ and ${}^t w \cdot \mathbf{k}_N = \mathbf{k}_N$, this easily implies that O is the identity matrix I_N. Thus, $w \cdot A = I_{N+1}$ and $A \in W(\mathsf{E}_N)$. □

homtype **Proposition 7.2.34.** *Every vector v in the $W(\mathsf{E}_N)$-orbit of $\mathbf{e}_0$ is a proper homaloidal type.*

Proof Let $v = w(\mathbf{e}_0)$ for some $w \in W(\mathsf{E}_N)$. Write w as the composition of simple reflections $s_k \circ \cdots \circ s_1$. Choose an open subset U of $(\mathbb{P}^2)^N$ such that an ordered set of points $(x_1, \ldots, x_N) \in U$ satisfies the following conditions:

(i) $x_i \neq x_j$ for $i \neq j$;

(ii) if $s_1 = s_{\mathbf{e}_0 - \mathbf{e}_i - \mathbf{e}_j - \mathbf{e}_k}$, then x_i, x_j, x_k are not collinear;

(iii) let f be the involutive quadratic transformation with the fundamental points x_i, x_j, x_k and let $(y_1, \ldots, y_N)$ be the set of points with $y_i = x_i, y_j = x_j, y_k = x_k$ and $y_h = f(x_h)$ for $h \neq i, j, k$. Then, $(y_1, \ldots, y_N)$ satisfies conditions (i) and (ii) for s_1 is replaced with s_2. Next, do it again by taking s_3 and so on.

It is easy to see that in this way U is a non-empty Zariski open subset of $(\mathbb{P}^2)^N$ such that $w(\mathbf{e}_0)$ represents the characteristic vector of a homaloidal net. □

excptype2 **Corollary 7.2.35.** *Every vector v in the $W(\mathsf{E}_N)$-orbit of $\mathbf{e}_1$ can be realized as a proper exceptional type.*

Proof Let $v = w(\mathbf{e}_1)$ for some $w \in W(\mathsf{E}_N)$. and η be a bubble cycle realizing the homaloidal type $w(\mathbf{e}_0)$ and f be the corresponding Cremona transformation with characteristic matrix A. Then, v is its second column, and hence corresponds to the first exceptional configuration $\mathcal{E}'_1$ for ϕ^{-1}. □

7.2.6 Symmetric Cremona transformations

SS:7.2.6

Assume that the characteristic vector $(d; m_1, \ldots, m_N)$ is of the form $(d; m, \ldots, m)$. In this case, the Cremona transformation is called *symmetric*. We have

$$d^2 - Nm^2 = 1, \quad 3d - Nm = 3.$$

Multiplying the second equality by m and subtracting from the first one, we obtain $d^2 - 3dm = 1 - 3m$. This gives $(d-1)(d+1) = 3m(d-1)$. The case $d = 1$ corresponds to a projective transformation. Assume $d > 1$. Then, we get $d = 3m - 1$, and hence, $3(3m-1) - Nm = 3$. Finally, we obtain

$$(9 - N)m = 6, \quad d = 3m - 1.$$

This gives us four possible cases:

(1) $m = 1, N = 3, d = 2$;
(2) $m = 2, N = 6, d = 5$;
(3) $m = 3, N = 7, d = 8$;
(4) $m = 6, N = 8, d = 17$.

The first case is obviously realized by a quadratic transformation with three fundamental points.

The second case is realized by the linear system of plane curves of degree 5 with six double points. Its characteristic matrix is the following:

$$\begin{pmatrix} 5 & 2 & 2 & 2 & 2 & 2 & 2 \\ -2 & 0 & -1 & -1 & -1 & -1 & -1 \\ -2 & -1 & 0 & -1 & -1 & -1 & -1 \\ -2 & -1 & -1 & 0 & -1 & -1 & -1 \\ -2 & -1 & -1 & -1 & 0 & -1 & -1 \\ -2 & -1 & -1 & -1 & -1 & 0 & -1 \\ -2 & -1 & -1 & -1 & -1 & -1 & 0 \end{pmatrix}. \tag{7.45}$$

ass5

We have to impose some conditions on the points x_i that guarantee that the linear system does not have fixed components. Since the linear system $|5H - D_\eta|$ is complete on a log resolution X, a fixed component belongs to a linear system $|d'H - D_{\eta'}|$, where $d' < 5$ and $h^0(d'H - D_{\eta'}) = 1$. We leave it to the reader to check that the conditions are

- $|h - x_i - x_j - x_k| = \emptyset$, i.e., no three points are on a line;
- $|2h - x_1 - \cdots - x_6| = \emptyset$, i.e., the six points are not on a conic;

Assume that all base points are proper points in the plane. Then, the P-locus of the transformation consists of six conics, each passing through five of the six base points. The same is true if there are points in the bubble cycle of height ≥ 3.

The third case of symmetric Cremona transformations is realized by a *Geiser involution*. We consider an irreducible net $\mathcal{N}$ of cubic curves through seven points $x_1, \ldots, x_7$ in the plane. The existence of such a net puts some conditions on the seven points. For example, no four points must be collinear, and no seven points lie on a conic. We leave it to the reader to check that these conditions are sufficient that such a net exists. Now, consider the transformation γ that assigns to a general point x in the plane the base point of the pencil of cubics from the net which pass through x. If points $x_1, \ldots, x_7$ satisfy condition $(*)$ from Subsection 6.3.3, then the net of cubics defines a rational map of degree 2 to the plane with a nonsingular quartic curve as the branch curve. The Geiser involution G is the rational deck transformation of this cover. Under weaker conditions on the seven points, the same is true. The only difference is that the branch curve may acquire simple singularities.

Let us confirm that the degree of the transformation γ is equal to 8. The image of a general line ℓ under the map given by $\mathcal{N}$ is a cubic curve L. Its pre-image is a curve of degree 9 passing through the points x_i with multiplicity 3. Thus, the union $\ell + L$ is invariant under f, hence $f(\ell) = L$. Since $f = f^{-1}$, this shows that the degree of f is equal to 8. It also shows that the homaloidal linear system consists of curves of degree 8 passing through the base points with multiplicities ≥ 3. In other words, the homaloidal linear system is equal to $|8h - 3\eta|$, where $\eta = x_1 + \cdots + x_7$. The P-locus of the Geiser involution consists of cubic curves passing through the base points with a node at one of the points. The total degree is equal to $7 \times 3 = 21$ that agrees with the degrees of the jacobian. The following is the characteristic matrix of the transformation:

$$\begin{pmatrix} 8 & 3 & 3 & \dots & 3 & 3 \\ -3 & -2 & -1 & \dots & -1 & -1 \\ -3 & -1 & -2 & \dots & -1 & -1 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \\ -3 & -1 & -1 & \dots & -1 & -2 \end{pmatrix} \tag{7.46}$$

If one composes γ with a projective transformation we obtain a transformation with the same characteristic matrix but not necessarily involutorial. Also, the bubble cycle η may not consist of only proper points, as soon as we continue to require that the linear system $|3h - \eta|$ has no fixed components. All admissible η's will be classified in Section 8.7 in Volume 2.

The last case is realized by a *Bertini involution*. We consider an irreducible pencil of cubic curves through a general set of 8 points $x_1, \ldots, x_8$. Let q be its ninth base point (it could be infinitely near one of the points x_i). For any general point x in the plane, let $F(x)$ be the member of the pencil containing x. Let q' be the intersection point of the tangent line at q with $F(x)$ and $\beta(x)$ be the residual point in the intersection of $F(x)$ with the line $\langle x, q' \rangle$. The transformation $x \longrightarrow \beta(x)$ is the Bertini involution. If we take q as the origin in the group law on a nonsingular cubic $F(x)$, then $\beta(x) = -x$.

Consider the web $\mathcal{N}$ of curves of degree 6 and genus 2 whose general member passes through each point x_i with multiplicity 2. The restriction of $\mathcal{N}$ to any $F(x)$ is a pencil with fixed part $2x_1 + \cdots + 2x_8$ and a moving part g_2^1. One of the members of this g_2^1 is the divisor $2q$ cut out by $2F(x'), x \neq x'$. As we have seen in Subsection 6.3.3, the members of this pencil are cut out by lines through the coresidual point on $F(x)$. This point must coincide with the point q. Thus, members of the g_2^1 are divisors $x + \beta(x)$. We will see in Section 8.8 that the web $\mathcal{N}$ defines a degree 2 rational map $f : \mathbb{P}^2 \dashrightarrow Q \subset \mathbb{P}^3$, where Q is a singular irreducible quadric in $\mathbb{P}^3$. The image of q is the vertex of the cone. The images of the curves $F(x)$ are lines on Q. Consider a general line ℓ in the plane. It is mapped to a curve of degree 6 on Q not passing through the vertex of Q. A curve on Q not passing through the vertex is always cut out by a cubic surface. In our case the curve $f(\ell)$ is cut out by a cubic surface. The pre-image of this curve is a curve of degree 18 passing through the points x_i with multiplicities

6. As in the case of the Geiser involution, this shows that $\beta(\ell)$ is a curve of degree 17 with 6-tuple points $x_1, \dots, x_8$. Thus, the homaloidal linear system defining the Bertini involution is equal to $|17h - 6\eta|$, where $\eta = x_1 + \cdots + x_8$. The P-locus of the Bertini involution consists of plane sextics with a triple point at one of the base points and double points at the remaining base points. The characteristic matrix of the Bertini involution (or its composition with a. projective transformation) is the following:

$$\begin{pmatrix} 17 & 6 & 6 & \dots & 6 & 6 \\ -6 & -3 & -2 & \dots & -2 & -2 \\ -6 & -2 & -3 & \dots & -2 & -2 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \\ -6 & -2 & -2 & \dots & -2 & -3 \end{pmatrix} \tag{7.47}$$

Again, we may consider η not necessarily consisting of proper points. All admissible η's will be classified in Section 8.7.

7.3 Noether's Factorization Theorem

7.3.1 Elementary transformations

First, let us recall the definition of a minimal rational ruled surface $\mathbf{F}_n$ (often called, in modern literature, a *Hirzebruch surface*). If $n = 0$ this is the surface $\mathbb{P}^1 \times \mathbb{P}^1$. If $n = 1$ it is isomorphic to the blow-up of one point in $\mathbb{P}^2$ with the ruling $\pi : \mathbf{F}_1 \to \mathbb{P}^1$ defined by the pencil of lines through the point. If $n > 1$, we consider the cone in $\mathbb{P}^{n+1}$ over a Veronese curve $\mathsf{V}_n^1 \subset \mathbb{P}^n$, i.e. we identify $\mathbb{P}^{n-1}$ with a hyperplane in $\mathbb{P}^n$ and consider the union of lines joining a fixed point p_0 not on the hyperplane with all points in V_n^1. The surface $\mathbf{F}_n$ is a minimal resolution of the vertex p_0 of the cone. The exceptional curve of the resolution is a smooth rational curve E_n with $E_n^2 = -n$. The projection from the vertex of the cone extends to a morphism

$$p : \mathbf{F}_n \to \mathbb{P}^1,$$

which defines a ruling. The curve E_n is its section called the *exceptional section*. In the case $n = 1$, the exceptional curve E_1 of the blow-up $\mathbf{F}_1 \to \mathbb{P}^2$ is also a section of the corresponding ruling $p : \mathbf{F}_1 \to \mathbb{P}^1$. It is also called the exceptional section.

We will use the general facts about projective bundles discussed in [379, Chapter V, §2] or recalled in Subsection 2.4.1.

prop **Proposition 7.3.1.** *Let $\pi : X \to \mathbb{P}^1$ be a morphism of a nonsingular surface such that all fibers over a non-empty open subset of $\mathbb{P}^1$ are isomorphic to $\mathbb{P}^1$. Then there exists a unique birational morphism $f : X \to \mathbf{F}_n$ such that $\pi = p \circ f$.*

Proof Let F be a smooth fiber. It follows from the assumption that $F \cong \mathbb{P}^1$. By the adjunction formula, $K_X \cdot F = -2 - F^2 = -2$. Thus, for any fiber F, we get $K_X \cdot F' = -2$. Writing F as a sum of its irreducible components, we find an irreducible component R with $K_X \cdot R < 0$. If F is irreducible, then $F^2 = 0$ and $K_X \cdot F < 0$ implies that $F \cong \mathbb{P}^1$, hence F is a smooth fiber. If F is reducible, $R^2 < 0$ and $R \cdot K_X < 0$ implies that $R^2 = R \cdot K_X = -1$, hence R is a (-1)-curve. Let $f' : X \to X'$ be the birational morphism that blows down R to a nonsingular point of X'. Replacing X with X' and repeating the argument, we find a birational morphism $f : X \to S$ over $\mathbb{P}^1$ such that all fibers $\pi' : S \to \mathbb{P}^1$ are smooth. The generic fiber S_η is smooth projective curve of genus $g = 0$ over the field K of rational functions on $\mathbb{P}^1$ isomorphic to $\mathbb{C}(t)$.

By Tsen's Theorem, Shafarevich [708, Chapter 1, 6.2, Corollary 4] $S_\eta \cong \mathbb{P}^1_\eta$. The closure of a rational point on S_η defines a section $\sigma : \mathbb{P}^1 \to S$ whose image is a smooth rational curve C on S such that $S \cdot F = 1$. It is clear that $\mathrm{Pic}(S) \cong \mathbb{Z}^2$ and we can choose a basis formed by the divisor class $\mathfrak{f}$ of a fiber F and the divisor class $[C]$ of a section. Let m be the largest positive number such that $E = C - mF$ is effective. We have $e = E^2 = C^2 - 2m$ and $E \cdot F = 1$. Each irreducble component of C' must contain a fiber. By maximality of m, E' is irreducible curve isomorphic to $\mathbb{P}^1$. Write $K_S = a\mathfrak{f} + b\mathfrak{e}$, where $\mathfrak{e} = [C']$. Intersecting with $\mathfrak{f}$ and $\mathfrak{e}$, and applying the adjunction formula, we get $b = -2$ and $a = e^2 - 2$. If $e^2 > 0$, $(E - F)^2 = e^2 - 2 > -2$ and $(E - F) \cdot K_S = e^2$. By Riemann-Roch, $h^0(E - F) > 0$, contradicting our choice of m. Thus $e \leq 0$. Setting $n = -e$, we get

$$K_X = (-2 - n)\mathfrak{f} - 2\mathfrak{e}, \quad \mathfrak{e}^2 = -n \leq 0. \tag{7.48}$$ canruled2

Consider the linear system $|n\mathfrak{f} + \mathfrak{e}|$. We have

$$(n\mathfrak{f} + \mathfrak{e})^2 = n, \ (n\mathfrak{f} + \mathfrak{e}) \cdot ((-2 - n)\mathfrak{f} - 2\mathfrak{e}) = -2 - n.$$

By Riemann-Roch, $\dim |n\mathfrak{f} + \mathfrak{e}| \geq n + 1$. The linear system $|n\mathfrak{f} + \mathfrak{e}|$ has no base points because it contains the linear system $|n\mathfrak{f}|$ with no base points. Thus it defines a regular map $\mathbb{P}(\mathcal{E}) \to \mathbb{P}^n$. Since $(n\mathfrak{f} + \mathfrak{e}) \cdot \mathfrak{e} = 0$, it blows down the section E to a point p. Since$(n\mathfrak{f} + \mathfrak{e}) \cdot \mathfrak{f} = a$, it maps fibres to lines passing through p. The degree of the image is $(n\mathfrak{f} + \mathfrak{e})^2 = n$. Thus, the image of the map is a surface of degree n equal to the union of lines through a point. It must be a cone over the Veronese curve V_n^1 if $n > 1$ and $\mathbb{P}^2$ if $n = 1$. The map is its minimal resolution of singularities. This proves the assertion in this case.

Assume $n = 0$. We leave it to the reader to check that the linear system $|\mathfrak{f} + \mathfrak{e}|$ maps X isomorphically to a quadric surface in $\mathbb{P}^3$. □

A surface X from the proposition is called a *raional ruled surface*. A surface isomorphic to the surface $\mathbf{F}_n$ is called a *minimal rational ruled surface*. The section E with $E^2 = -n \leq 0$ is called the *exceptional section*. If $n < 0$, it is a unique section with negative self-intersection. This immediately follows from the formula (canruled2 7.48) and the adjunction formula. In particular, we see that it coincides with the exceptional section of $\mathbf{F}_N$ defined in the first paragraph of this subsection. If $n = 0$, $\mathbf{F}_0 \cong \mathbb{P}^1 \times \mathbb{P}^1$, and the section with minimal self-intersections move in a base-point-free pencil defining one of the projections to $\mathbb{P}^1$.

hirzebruchchar **Proposition 7.3.2.** *Let $X = \mathbf{F}_n$ be a minimal ruled surface not isomorphic to $\mathbb{P}^2$. Then,*

$$X \cong \mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-n)).$$

Proof First, let us see that the projective bundle $\mathbb{P}(\mathcal{E})$, where $\mathcal{E} = \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-n)$ is indeed isomorphic $\mathbf{F}_n$. Let $\mathfrak{f}$ be the divisor class of a fiber, and $\mathfrak{e}$ be the divisor class of the image E of the unique section $s_0 : \mathbb{P}^1 \to \mathbb{P}(\mathcal{E})$ defined by a surjection $\mathcal{E} \to \mathcal{O}_{\mathbb{P}^1}(-n)$. By Proposition sernesi 2.4.2 (in this case, one can refer to Hartshorne [379, Chapter V, §2, Proposition 2.6]), $s_0^*(\mathcal{N}_{E/X}) \cong \mathcal{O}_{\mathbb{P}^1}(-n)$. This implies that $\mathfrak{e}^2 = -n$. Assume $n > 0$. Consider the linear system $|n\mathfrak{f} + \mathfrak{e}|$. We have

$$(n\mathfrak{f} + \mathfrak{e})^2 = n, \ (n\mathfrak{f} + \mathfrak{e}) \cdot ((-2 - n)\mathfrak{f} - 2\mathfrak{e}) = -2 - n.$$

By Riemann-Roch, $\dim |n\mathfrak{f} + \mathfrak{e}| \geq n + 1$. The linear system $|n\mathfrak{f} + \mathfrak{e}|$ has no base points because it contains the linear system $|n\mathfrak{f}|$ with no base points. Thus, it defines a regular map $\mathbb{P}(\mathcal{E}) \to \mathbb{P}^n$. Since $(n\mathfrak{f} + \mathfrak{e}) \cdot \mathfrak{e} = 0$, it blows down the section E to a point p. Since$(n\mathfrak{f} + \mathfrak{e}) \cdot \mathfrak{f} = a$, it maps fibers to lines passing through p. The degree of the image is $(n\mathfrak{f} + \mathfrak{e})^2 = n$. Thus, the image of the map is a surface of degree n equal to the union of lines through a point. It must be a cone over the Veronese curve V_n^1 if $n > 1$ and $\mathbb{P}^2$ if $n = 1$. The map is its minimal resolution of singularities. This proves the assertion in this case.

Assume $n = 0$. We leave it to the reader to check that the linear system $|\mathfrak{f} + \mathfrak{e}|$ maps X isomorphically to a quadric surface in $\mathbb{P}^3$. □

Let $\pi : \mathbf{F}_n \to \mathbb{P}^1$ be a ruling of $\mathbf{F}_n$ (the unique one if $n \neq 0$). Let $x \in \mathbf{F}_n$ and F_x be the fiber of the ruling containing x. If we blow up x, the proper transform $\bar{F}_x$ of F_x is an exceptional curve of the first kind. We can blow it down to obtain a nonsingular surface X. The projection π induces a morphism $\pi' : X \to \mathbb{P}^1$ with any fiber isomorphic to $\mathbb{P}^1$. Let S_0 be the exceptional section or any section with the self-intersection 0 if $n = 0$ (such a section is of course equal to a fiber of the second ruling of $\mathbf{F}_0$). Assume that $x \notin S_0$. The proper transform $\bar{S}_0$ of S_0 on the blow-up has the self-intersection equal to $-n$, and its image in X has the self-intersection equal to $-n+1$. Applying Proposition 7.3.1, we obtain that $X \cong \mathbf{F}_{n-1}$. This defines a birational map

$$\mathrm{elm}_x : \mathbf{F}_n \dashrightarrow \mathbf{F}_{n-1}.$$

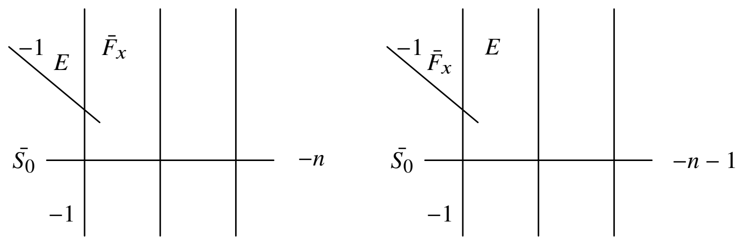


Figure 7.5 Elementary transformation

Here, in Figure 7.4, on the left, we blow down $\bar{F}_x$ to obtain $\mathbf{F}_{n-1}$, and, on the right, we blow down $\bar{F}_x$ to obtain $\mathbf{F}_{n+1}$.

Assume that $x \in E_n$. Then, the proper inverse transform of S_0 on the blow-up has self-intersection $-n-1$ and its image in X has the self-intersection equal to $-n-1$. Applying Proposition 7.3.2, we obtain that $X \cong \mathbf{F}_{n+1}$. This defines a birational map

$$\mathrm{elm}_x : \mathbf{F}_n \dashrightarrow \mathbf{F}_{n+1}.$$

A birational map elm_x is called an *elementary transformation*.

Remark 7.3.3. Let $\mathcal{E}$ be a locally free sheaf over a nonsingular curve B. As we explained in Subsection 2.4.1, a point $x \in \mathbb{P}(\mathcal{E})$ is defined by a surjection $\mathcal{E}(x) \to \kappa(x)$, where $\kappa(x)$ is considered as the structure sheaf of the closed point x. Composing this surjection with the natural surjection $\mathcal{E} \to \mathcal{E}(x)$, we get a surjective morphism of sheaves $\phi_x : \mathcal{E} \to \kappa(x)$. Its kernel $\mathrm{Ker}(\phi_x)$ is a subsheaf of $\mathcal{E}$ which has no torsion. Since the base is a regular 1-dimensional scheme, the sheaf $\mathcal{E}' = \mathrm{Ker}(\phi_x)$ is locally free. Thus, we have defined an operation on locally free sheaves. It is also called an elementary transformation.

Consider the special case when $B = \mathbb{P}^1$ and $\mathcal{E} = \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-n)$. We have an exact sequence

$$0 \to \mathcal{E}' \to \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-n) \xrightarrow{\phi_x} \kappa_x \to 0.$$

The point x belongs to the exceptional section S_0 if and only if ϕ_x factors through $\mathcal{O}_{\mathbb{P}^1}(-n) \to \kappa_x$. Then, $\mathcal{E}' \cong \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-n-1)$ and $\mathbb{P}(\mathcal{E}') \cong \mathbf{F}_{n+1}$.

The inclusion of sheaves $\mathcal{E}' \subset \mathcal{E}$ gives rise to a rational map $\mathbb{P}(\mathcal{E}) \dashrightarrow \mathcal{P}(\mathcal{E}')$ which coincides with elm_x. If $x \notin S_0$, then ϕ_x factors through $\mathcal{O}_{\mathbb{P}^1}$, and we obtain $\mathcal{E}' \cong \mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-n)$. In this case $\mathbb{P}(\mathcal{E}') \cong \mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-n+1)) \cong \mathbf{F}_{n-1}$ and again, the inclusion $\mathcal{E}' \subset \mathcal{E}$ defines a rational map $\mathbb{P}(\mathcal{E}) \dashrightarrow \mathbb{P}(\mathcal{E}')$ which coincides with elm_x. We refer for this sheaf-theoretical interpretation of elementary transformation to [378]. A more general definition applied to projective bundles over any algebraic variety can be found in [33], [754].

Let $x, y \in \mathbf{F}_n$. Assume that $x \in S_0$, $y \notin S_0$ and $\pi(x) \neq \pi(y)$. Then, the composition

$$e_{x,y} = \mathrm{elm}_y \circ \mathrm{elm}_x : \mathbf{F}_n \dashrightarrow \mathbf{F}_n$$

is a birational automorphism of $\mathbf{F}_n$. Here, we identify the point y with its image in $\mathrm{elm}_x(\mathbf{F}_n)$. If $n = 0$, we have to fix one of the two structures of a projective bundle on $\mathbf{F}_0$. Similarly, we get a birational automorphism $e_{y,x} = \mathrm{elm}_y \circ \mathrm{elm}_x$ of $\mathbf{F}_n$. We can also extend this definition to the case when $y >_1 x$, where y does not correspond to the tangent direction defined by the fiber passing through x or the exceptional section (or any section with self-intersection 0). We blow up x, then y, and then blow down the proper transform of the fiber through x and the proper inverse transform of the exceptional curve blown up from x.

SS:7.3.2

7.3.2 Birational automorphisms of $\mathbb{P}^1 \times \mathbb{P}^1$

Let X be a rational variety and let $\phi : X \dashrightarrow \mathbb{P}^n$ be a birational isomorphism. It defines a homomorphism of the groups of birational automorphisms

$$\mathrm{Bir}(\mathbb{P}^n) \to \mathrm{Bir}(X), \quad f \mapsto \phi^{-1} \circ f \circ \phi$$

with the inverse

$$\mathrm{Bir}(X) \to \mathrm{Bir}(\mathbb{P}^n), \quad g \mapsto \phi \circ g \circ \phi^{-1}.$$

Here, we realize this simple observation by taking $X = \mathbb{P}^1 \times \mathbb{P}^1$, identified with a nonsingular quadric Q in $\mathbb{P}^3$. We identify $\mathbb{P}^2$ with a plane in $\mathbb{P}^3$ and take $\phi : Q \dashrightarrow \mathbb{P}^2$ to be the projection map p_{x_0} from a point x_0. Let a, b be the images of the two lines on Q containing the point x_0. The inverse map ϕ^{-1} is given by the linear system $|2h - q_1 - q_2|$ of conics through the points q_1, q_2, and a choice of an appropriate basis in the linear system. Let

$$\Phi_{x_0} : \mathrm{Bir}(Q) \to \mathrm{Bir}(\mathbb{P}^2)$$

be the corresponding isomorphism of groups.

A birational automorphism of $\mathbb{P}^1 \times \mathbb{P}^1$ is given by a linear system $|mh_1 + kh_2 - \eta|$, where h_1, h_2 are the divisor classes of fibers of the projection maps

$\mathrm{pr}_i : \mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^1$, and η is a bubble cycle on Q. If we fix coordinates $(u_0, u_1), (v_0, v_1)$ on each factor of $\mathbb{P}^1 \times \mathbb{P}^1$, then a birational automorphism of the product is given by four bihomogeneous polynomials R_0, R_1, R_0', R_1' of bidegree (m, k):

$$([a_0, a_1], [b_0, b_1]) \mapsto ([R_1(a, b), R_2(a, b], [R_0'(a, b), R_1'(a, b)]).$$

Explicitly, let us use an isomorphism

$$\mathbb{P}^1 \times \mathbb{P}^1 \to Q, \quad ([a_0, a_1], [b_0, b_1]) \mapsto [a_0 b_0, a_0 b_1, a_1 b_0, a_1 b_1],$$

where $Q = V(z_0 z_3 - z_1 z_2)$. Take $x_0 = [0, 0, 0, 1]$. The projection map p_{x_0} is given by $[z_0, z_1, z_2, z_3] \mapsto [z_0, z_1, z_2]$. The inverse map $p_{x_0}^{-1}$ can be given by the formulas

$$[t_0, t_1, t_2] \mapsto [t_0^2, t_0 t_1, t_0 t_2, t_1 t_2].$$

It is not defined at the points $q_1 = [0, 1, 0]$ and $q_2 = [0, 0, 1]$.

If g is given by R_0, R_1, R_0', R_1', then $\Phi_{x_0}(g)$ is given by the formula

$$[z_0, z_1, z_2] \mapsto [R_0(a, b) R_0'(a, b), R_0(a, b) R_1'(a, b), R_1(a, b) R_0'(a, b)],$$

where $[z_0, z_1, z_2] = [a_0 b_0, a_0 b_1, a_1 b_0]$ for some $[a_0, b_0], [b_0, b_1] \in \mathbb{P}^1$.

If $f : \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$ is given by the polynomials P_0, P_1, P_2, then $\Phi_{x_0}^{-1}(f)$ is given by the formula

$$[z_0, z_1, z_2, z_3] \mapsto [P_0(z')^2, P_0(z') P_1(z'), P_0(z') P_2(z'), P_1(z') P_2(z')], \tag{7.49}$$ expl

where $P_i(z') = P_i(z_0, z_1, z_2)$.

Let $\mathrm{Aut}(Q) \subset \mathrm{Bir}(Q)$ be the subgroup of biregular automorphisms of Q. It contains a subgroup $\mathrm{Aut}(Q)^o$ of index 2 that leaves invariant each family of lines on Q. By acting on each factor of the product $\mathbb{P}^1 \times \mathbb{P}^1$, it becomes isomorphic to the product $\mathrm{PGL}(2) \times \mathrm{PGL}(2)$.

lll **Lemma 7.3.4.** *Let $\sigma \in \mathrm{Aut}(Q)^o$. If $\sigma(x_0) \neq x_0$, then $\Phi_{x_0}(\sigma)$ is a quadratic transformation with fundamental points $a, b, p_{x_0}(\sigma^{-1}(x_0))$. If $\sigma(x_0) = x_0$, then $\Phi_{x_0}(\sigma)$ is a projective transformation.*

Proof If $x = \sigma(x_0) \neq x_0$, then the F-locus of $f = \Phi_{x_0}(\sigma)$ consists of three points q_1, q_2 and $p_{x_0}(x)$. It follows from (7.21) virt1, that it must be a quadratic transformation. If $\sigma(x_0) = x_0$, then the map f is not defined only at q_1 and q_2. The rational map $\phi : \mathbb{P}^2 \dashrightarrow Q$ can be resolved by blowing up the two points q_1, q_2 followed by blowing down the proper transform of the line $\langle q_1, q_2 \rangle$. It is clear that it does not have infinitely near fundamental points. Since any non-projective planar Cremona transformation has at least three fundamental points, we obtain that the map f extends to an automorphism of $\mathbb{P}^2$. □

Remark 7.3.5. The image $\Phi_{x_0}(\mathrm{Aut}(Q)$ consists of quadratic or projective transformations which leave invariant the linear system of conics through two points q_1, q_2. These are complex conics discussed in Subsection 2.2.3. Over reals, when we deal with real conics through the ideal points in the line at infinity, the group $\Phi_{x_0}(\mathrm{Aut}(Q))$ is known as the *Inversive group* in dimension 2 (see MorleyBook [529]).

The subgroup $\Phi_{x_0}(\mathrm{Aut}(Q))$ of $\mathrm{Cr}(2) = \mathrm{Bir}(\mathbb{P}^2)$ is an example of a linear algebraic subgroup of the Cremona group $\mathrm{Cr}(2)$. All such subgroups in $\mathrm{Cr}(2)$ were classified by F. Enriques Enriques [291]. In particular, he showed that any linear algebraic subgroup of rank 2 in $\mathrm{Cr}(2)$ is contained in a subgroup isomorphic to $\mathrm{Aut}(\mathbf{F}_n)$ for some n. There is a generalization of this result to the group $\mathrm{Cr}(n) = \mathrm{Bir}(\mathbb{P}^n)$ (see Demazure2 [218]). Instead of minimal ruled surfaces one considers smooth toric varieties of dimension n.

Take two points x, y in Q which do not lie on a line and consider the birational transformation $e_{x,y} := \mathrm{elm}_x \circ \mathrm{elm}_y$ defined in the previous Subsection. Recall that to define $e_{x,y}$, we have to fix one of the two structures of a projective bundle on Q. We do not exclude the case when there is only one proper point among x and y, say $y > x$. It is easy to see that the linear system defining the transformation $e_{x,y}$ is equal to $|2h_1 + h_2 - x - y|$, where h_1 is the class of a fiber of the projective bundle structure $\mathrm{pr} : Q \to \mathbb{P}^1$.

112 **Proposition 7.3.6.** *$\Phi_{x_0}(e_{x,y})$ is a product of quadratic transformations. Moreover, if $x_0 \in \{x, y\}$, then $\Phi_{x_0}(t_{x,y})$ is a quadratic transformation. Otherwise, $\Phi_{x_0}(t_{x,y})$ is the product of two quadratic transformations.*

Proof Let $\tau : X \to Q$ be the blow-up of the bubble cycle $x + y$. It factors into the composition of the blow-up $\tau_1 : Q_x \to Q$ of x and the blow-up $\tau_2 : Q' \to Q_x$ of y. Suppose $x_0 \in \{x, y\}$. Without loss of generality, we may assume that $x_0 = x$. The composition of rational maps $\pi = p_{x_0} \circ \tau : Q' \dashrightarrow \mathbb{P}^2$ is a regular map. Let $\alpha : X \to Q$ be the blowing-down of the proper transforms of the fiber ℓ_x (resp. ℓ_y) of $\mathrm{pr} : Q \to \mathbb{P}^1$ containing x (resp. y). The composition $\sigma = p_{x_0} \circ \alpha : Q' \to Q \dashrightarrow \mathbb{P}^2$ is also a regular map. The two morphisms $\sigma, \pi : X \to \mathbb{P}^2$ define a resolution of the birational map $\Phi_{x_0}(e_{x,y})$. It is immediate that this resolution coincides with a resolution of a quadratic transformation with fundamental points $q_1, q_2, p_{x_0}(y)$. Note that, if $y > x$, then $p_{x_0}(y) > q_2$, where the line ℓ_y is blown down to q_2 under the map $Q'_x \to \mathbb{P}^2$.

If $x_0 \neq x, y$, we compose $e_{x,y}$ with an automorphism g of Q such that $\sigma(x_0) = x$. Then,

$$\Phi_{x_0}(e_{x,y} \circ g) = \Phi_{x_0}(e_{x_0, g^{-1}(y)}) = \Phi_{x_0}(e_{x,y}) \circ \Phi_{x_0}(g).$$

By Lemma 111 7.3.4, $\Phi_{x_0}(g)$ is a quadratic transformation. By the previous case,

$\Phi_{x_0}(e_{x_0,\sigma^{-1}(y)})$ is a quadratic transformation. Also, the inverse of a quadratic transformation is a quadratic transformation. Thus, $\Phi_{x_0}(e_{x,y})$ is a product of two quadratic transformations.

□

Proposition 7.3.7. *Let $f : \mathbf{F}_n \dashrightarrow \mathbf{F}_m$ be a birational map. Assume that f commutes with the projections of the minimal ruled surfaces to $\mathbb{P}^1$. Then, f is a composition of biregular maps and elementary transformations.*

Proof Let (X, σ, π) be a resolution of indeterminacy of f. The morphism π (resp. σ) is the blowing up of an admissible ordered bubble cycle $\eta = (x_1, \dots, x_N)$ (resp. $\xi = (y_1, \dots, y_N)$). Let $p_1 : \mathbf{F}_n \to \mathbb{P}^1$ and $p_2 : \mathbf{F}_m \to \mathbb{P}^1$ be the structure morphisms of the projective bundles. The two composition $p_1 \circ \pi$ and $p_2 \circ \sigma$ coincide and define a map

$$\phi : X \to \mathbb{P}^1.$$

Let $a_1, \dots, a_k$ be points in $\mathbb{P}^1$ such that $F_i = \phi^{-1}(a_i) = \pi^*(p_1^{-1}(a_i))$ is a reducible curve. We have $\pi_*(F_i) = p_1^{-1}(a_i)$ and $\sigma_*(F_i) = p_2^{-1}(a_i)$. Let E_i be the unique component of R_i which is mapped onto $p_1^{-1}(a_i)$ and E_i' be the unique component of F_i which is mapped surjectively to $p_2^{-1}(a_i)$. The pre-images in X of the maximal points in η and ξ (with respect to the admissible order) are (-1)-curves $E_1, \dots, E_k$ and $E_1', \dots, E_{k'}'$. Let E be a (-1)-curve component of F_i that is different from $E_1, \dots, E_k$ and $E_1', \dots, E_{k'}'$. We can reorder the order of the blow-ups to assume that $\pi(E) = x_N$ and $\sigma(E) = y_N$. Let $\pi_N : X \to X_{N-1}$ be the blow-up of x_N and $\sigma_N : X \to Y_{N-1}$ be the blow-up of y_N. Since π_N and σ_N blow down the same curve, there exists an isomorphism $\phi : X_{N-1} \cong Y_{N-1}$. Thus, we can replace the resolution (X, σ, π) with

$$(X_{N-1}, \pi_1 \circ \ldots \circ \pi_{N-1}, \sigma_1 \circ \ldots \circ \sigma_{N-1} \circ \phi).$$

Continuing in this way, we may assume that x_N and y_N are the only maximal points of π and σ such that $p_1(x_N) = p_2(y_N) = a_i$. Let $E = \pi^{-1}(x_N)$ and $E' = \sigma^{-1}(y_N)$. Let $R \neq E'$ be a component of $\phi^{-1}(a_i)$ which intersects E. Let $x = \pi(R)$. Since $x_N \succ x$, and no other points is infinitely near x, we get $R^2 = -2$. Blowing down E, we get that the image of R has self-intersection -1. Continuing in this way, we get two possibilities:

(1)

$$F_i = E_i + E_i', \quad E_i^2 = E_i'^2 = -1, E_i \cdot E_i' = 1,$$

(2)

$$F_i = E_i + R_1 + \cdots + R_k + E_i', \quad E_i^2 = E_i'^2 = -1,$$

$$R_i^2 = -1, E_i \cdot R_1 = \ldots = R_i \cdot R_{i+1} = R_k \cdot E_i' = 1,$$

and all other intersections are equal to zero.

In the first case, $f = \mathrm{elm}_{x_N}$. In the second case, let $g : X \to X'$ be the blow-down of E_i, let $x = \pi(R_1 \cap E_i)$. Then, $f = f' \circ \mathrm{elm}_x$, where f' satisfies the assumption of the proposition. Continuing in this way, we write f as the composition of elementary transformations. □

Let J be a de Jonquières transformation of degree m with fundamental points $\mathfrak{o}, x_1, \ldots, x_{2m-2}$. We use the notation from Subsection 7.3.6. Let $\pi : X \to \mathbb{P}^2$ be the blow-up of the base points. We factor π as the composition of the blow-up $\pi_1 : X_1 \to X_0 = \mathbb{P}^2$ of the point $\mathfrak{o}$ and the blow-ups $\pi_i : X_{i+1} \to X_i$ of the points x_i. Let $p : X_1 \to \mathbb{P}^1$ be the map given by the pencil of lines through the point $\mathfrak{o}$. The composition $\phi : X \to X_1 \to \mathbb{P}^1$ is a *conic bundle*. This means that its general fiber is isomorphic to $\mathbb{P}^1$ and it has $2m-2$ singular fibers F_i over the points a_i corresponding to the lines $\ell_i = \overline{\mathfrak{o}x_i}$. Each singular fiber is equal to the union of two (-1)-curves $F_i' + F_i''$ intersecting transversally at one point x_i'. The curve F_i' is the proper transform of the line ℓ_i, and the curve F_i'' is the proper transform of the exceptional curve E_i of the blow-up $X_{i+1} \to X_i, i \geq 1$. The proper transform E of the exceptional curve of $X_1 \to X_0$ is a section of the conic bundle $\phi : X \to \mathbb{P}^1$. It intersects the components F_i'. The proper transform $\overline{\ell}$ of the curve ℓ is another section. It intersects the components F_i''. Moreover, it intersects E at $2m-2$ points $z_1, \ldots, z_{2m-2}$ corresponding to the common branches of ℓ and the proper transform H_m' of the hyperelliptic curve H_m at the point $\mathfrak{o}$. The curve H_m' is a 2-section of the conic bundle (i.e. the restriction of the map ϕ to H_m' is of degree 2).

Recall that the curve ℓ and the lines ℓ_i form the P-locus of J. Let $\sigma : X \to \mathbb{P}^2$ be the blow-down of the curves $F_i', \ldots, F_{2m-2}'$ and $\overline{\ell}$. The morphisms $\sigma, \pi : X \to \mathbb{P}^2$ define a resolution of the transformation J. We may assume that σ is the composition of the blow-downs $X \to Y_{2m-3} \to \ldots \to Y_1 \to Y_0 = \mathbb{P}^2$, where $Y_1 \to Y_0$ is the blow-down of the image of $\overline{\ell}$ under the composition $X \to \ldots \to Y_1$, and $Y_2 \to Y_1$ is the blow-down of the image of F_1' in Y_2.

The surfaces X_1 and Y_1 are isomorphic to $\mathbf{F}_1$. The morphisms $X \to X_1$ and $X \to Y_1$ define a resolution of the birational map $T' : \mathbf{F}_1 \dashrightarrow \mathbf{F}_1$ equal to the composition of $2m-2$ elementary transformations

$$\mathbf{F}_1 \overset{\mathrm{elm}_{x_2'}}{\dashrightarrow} \mathbf{F}_0 \dashrightarrow \mathbf{F}_1 \dashrightarrow \ldots \dashrightarrow \mathbf{F}_0 \overset{\mathrm{elm}_{x_{2m-2}'}}{\dashrightarrow} \mathbf{F}_1.$$

If we take x_0 to be the image of ℓ_1 under $\mathrm{elm}_{x_2'}$, and use it to define the isomorphism $\Phi_{x_0} : \mathrm{Bir}(\mathbf{F}_0) \to \mathrm{Bir}(\mathbb{P}^2)$, then we obtain that $f = \Phi_{x_0}(T')$, where T' is the composition of transformations $e_{x'_i, x'_{i+1}} \in \mathrm{Bir}(\mathbf{F}_0)$, where $i = 3, 5, \ldots, 2m-3$. Applying Proposition 7.3.6 (112), we obtain the following.

Theorem 7.3.8. *A de Jonquières transformation is equal to a composition of quadratic transformations.*

SS:7.3.3

7.3.3 Noether-Fano-Iskovskikh inequality

First, we generalize Corollary 7.2.28 (fano2) to birational maps of any rational surfaces. The same idea works even for higher-dimensional varieties. Let $f : S \dashrightarrow S'$ be a birational map of surfaces. Let $\pi : X \to S, \sigma : X \to S'$ be its resolution. Let $\mathcal{H}'$ be a linear system on X' without base points. For any $H' \in \mathcal{H}', H \in \mathcal{H}$,

$$\sigma^*(H') \sim \pi^*(H) - \sum_i m_i \mathcal{E}_i,$$

where $\mathcal{E}_i$ are the exceptional configurations of the map π. Since $\mathcal{H}'$ has no base points, $\sigma^*(\mathcal{H}')$ has no base points. Thus, any divisor $\sigma^*(H')$ intersects non-negatively any curve on X. In particular,

$$\sigma^*(H') \cdot \mathcal{E}_i = -m_i \mathcal{E}_i^2 = m_i \geq 0. \tag{7.50}$$ pos

This can be interpreted by saying that $f^{-1}(H')$ belongs to the linear system $|H - \eta|$, where $\eta = \sum m_i x_i$ is the bubble cycle on S defined by π.

NF **Theorem 7.3.9.** *(Noether-Fano-Iskovskikh inequality) Assume that there exists some integer $m_0 \geq 0$ such that $|H' + mK_{S'}| = \emptyset$ for $m \geq m_0$. For any $m \geq m_0$ such that $|H + mK_S| \neq \emptyset$, there exists i such that*

$$m_i > m.$$

Moreover, we may assume that x_i is a proper point in S.

Proof We know that $K_X = \pi^*(K_S) + \sum_i \mathcal{E}_i$. Thus, we have the equality in $\mathrm{Pic}(X)$

$$\sigma^*(H') + mK_X = (\pi^*(H + mK_S)) + \sum (m - m_i)\mathcal{E}_i.$$

Applying σ_* to the left-hand side we get the divisor class $H' + mK_{S'}$ which, by assumption, cannot be effective. Since $|\pi^*(H + mK_S)| \neq \emptyset$, applying σ_* to the right-hand side, we get the sum of an effective divisor and the image of the divisor $\sum_i (m - m_i)\mathcal{E}_i$. If all $m - m_i$ are non-negative, it is also an effective divisor, and we get a contradiction. Thus, there exists i such that $m - m_i < 0$.

The last assertion follows from the fact that $m_i \geq m_j$ if $x_j > x_i$. □

Example 7.3.10. Assume $S = S' = \mathbb{P}^2$, $[H] = dh$ and $[H'] = h$. We have $|H + K_{S'}| = |-2h| = \emptyset$. Thus, we can take $m_0 = 1$. If $d \geq 3$, we have for any $1 \leq a \leq d/3$, $|H' + aK_S| = |(d-3a)h| \neq \emptyset$. This gives $m_i > d/3$ for some i. This is Corollary 7.2.28.

nex *Example* 7.3.11. Let $S = \mathbf{F}_n$ and $S' = \mathbf{F}_r$ be the minimal rational ruled surfaces. Let $\mathcal{H}' = |\mathfrak{f}'|$, where $\mathfrak{f}'$ is the divisor class of a fiber of the fixed projective bundle structure on S'. The linear system $|\mathfrak{f}'|$ is a pencil without base points. So, we can write $\sigma^*(\mathcal{H}') = |\pi^*(a\mathfrak{f} + b\mathfrak{e}) - \eta|$ for some bubble cycle, where $\mathfrak{f}, \mathfrak{e}$ are the divisor classes of a fiber and the exceptional section on S. Here, (X, π, σ) is a resolution of f. Thus, $\mathcal{H} \subset |a\mathfrak{f} + b\mathfrak{e}|$.

By (10.49),

$$K_S = -(2+n)\mathfrak{f} - 2\mathfrak{e}, \quad K_{S'} = -(2+r)\mathfrak{f}' - 2\mathfrak{e}'. \tag{7.51}$$ canclassruled

Thus, $|H' + K_{S'}| = |(-1-n)\mathfrak{f} - 2\mathfrak{e}| = \emptyset$. We take $m_0 = 1$. We have

$$|a\mathfrak{f} + b\mathfrak{e} + mK_S| = |(a - m(2+n))\mathfrak{f} + (b - 2m)\mathfrak{e}|.$$

Assume that

$$1 < b \leq \frac{2a}{2+n}.$$

If $m = [b/2]$, then $m \geq m_0$ and both coefficients $a - m(2+n)$ and $b - 2m$ are non-negative. Thus, we can apply Theorem 7.3.9 to find an index i such that $m_i > m \geq b/2$.

In the special case, when $n = 0$, i.e. $S = \mathbb{P}^1 \times \mathbb{P}^1$, the inequality $b \leq a$ implies that there exists i such that $m_i > b/2$.

A similar argument also can be applied to the case $S = \mathbb{P}^2, S' = \mathbf{F}_r$. In this case, $\mathcal{H} = |ah|$ and $|h + mK_S| = |(a-3m)h|$. Thus, we can take $m = [a/3]$ and find i such that $m_i > a/3$.

7.3.4 Noether's Factorization Theorem

SS:7.3.4 We shall prove the following.

birr **Theorem 7.3.12.** *The group* $\mathrm{Bir}(\mathbf{F}_0)$ *is generated by biregular automorphisms and a birational automorphism* $e_{x,y}$ *for some pair of points* x, y.

Applying Proposition 7.3.6, we obtain the following Noether's Factorization Theorem.

Corollary 7.3.13. *The group* $\mathrm{Bir}(\mathbb{P}^2)$ *is generated by projective automorphisms and quadratic transformations.*

Let us prove Theorem 7.3.12. Let $f : \mathbf{F}_n \dashrightarrow \mathbf{F}_m$ be a birational map. Let

$$\mathrm{Pic}(\mathbf{F}_n) = \mathbb{Z}\mathfrak{f} + \mathbb{Z}\mathfrak{e}, \quad \mathrm{Pic}(\mathbf{F}_m) = \mathbb{Z}\mathfrak{f}' + \mathbb{Z}\mathfrak{e}',$$

where we use the notation from the previous Subsection. We have two bases in $\mathrm{Pic}(X)$

$$\underline{e} : \pi^*(\mathfrak{f}),\ \pi^*(\mathfrak{e}),\ e_i = [\mathcal{E}_i],\ i = 1, \ldots, N,$$

$$\underline{e'} : \pi^*(\mathfrak{f}'),\ \pi^*(\mathfrak{e}'),\ e_i' = [\mathcal{E}_i'],\ i = 1, \ldots, N.$$

For simplicity of notation, let us identify $\mathfrak{f}, \mathfrak{e}, \mathfrak{f}', \mathfrak{e}'$ with their inverse transforms in $\mathrm{Pic}(X)$. Similar to the case of birational maps of projective plane, we can use an ordered resolution (X, π, σ) of f to define its characteristic matrix A.

Lemma 7.3.14. *Let f be a quadratic transformation with two (resp. one) proper base points. Then f is equal to the composition of two (resp. four or less) quadratic transformations with proper base points.*

Proof Composing the transformation f with a projective transformation, we may assume that f is either T_{st}' or $T = T_{\mathrm{st}}''$ (see Example 7.1.14). In the first case, we compose f with the quadratic transformation T' with fundamental points $[1,0,0], [0,1,0], [1,0,1]$ given by the formula:

$$[t_0', t_1', t_2'] = [t_1t_2, t_1(t_0 - t_2), t_2(t_0 - t_2)].$$

The composition $T' \circ T_{\mathrm{st}}'$ is given by the formula

$$[t_0', t_1', t_2'] = [t_0^2t_1t_2, t_0t_2^2(t_2 - t_0), t_0t_1t_2(t_2 - t_0)] = [t_0t_1, t_2(t_2 - t_0), t_1(t_2 - t_0)].$$

It is a quadratic transformation with three fundamental points $[0,1,0]$, $[1,0,0]$, and $[1,0,1]$.

In the second case, we let T' be the quadratic transformation

$$[t_0', t_1', t_2'] = [t_0t_1, t_1t_2, t_2^2]$$

with two proper fundamental points $[1,0,0], [0,1,0]$. The composition $T' \circ T_{\mathrm{st}}''$ is given by

$$[t_0', t_1', t_2'] = [t_1^2(t_2^2 - t_0t_1), t_1^2t_2^2, t_1^3t_2] = [t_2^2 - t_0t_1, t_1t_2, t_2^2].$$

It is a quadratic transformation with two proper base points. By the above, T' and $T' \circ T$ are equal to the composition of two quadratic transformations with three proper points. Thus, f is a composition of four, or less, quadratic transformations with three proper base points. □

lall **Lemma 7.3.15.** *Let $f : \mathbf{F}_0 \dashrightarrow \mathbf{F}_0$ be a birational automorphism equal to a composition of elementary transformations. Then, f is equal to a composition of biregular automorphisms of $\mathbf{F}_0$ and a transformation $e_{x,y}$ for a fixed pair of points x, y, where y is not infinitely near x.*

Proof It follows from Proposition 7.3.6 and the previous lemma that $e_{x,y}$, where $y >_1 x$, can be written as a composition of two transformations of type $e_{x',y'}$ with no infinitely near points. Now, notice that the transformations $e_{x,y}$ and $e_{x',y'}$ for different pairs of points differ by an automorphism of $\mathbf{F}_0$ which sends x to x' and y to y'. Suppose we have a composition f of elementary transformations

$$\mathbf{F}_0 \overset{\mathrm{elm}_{x_1}}{\dashrightarrow} \mathbf{F}_1 \overset{\mathrm{elm}_{x_2}}{\dashrightarrow} \dots \overset{\mathrm{elm}_{x_{k-1}}}{\dashrightarrow} \mathbf{F}_1 \overset{\mathrm{elm}_{x_k}}{\dashrightarrow} \mathbf{F}_0.$$

If no $\mathbf{F}_0$ occurs among the surfaces $\mathbf{F}_n$ here, then f is a composition of even number k of elementary transformations preserving the projections to $\mathbb{P}^1$. It is clear that not all points x_i are images of points in $\mathbf{F}_0$ lying on the same exceptional section as x_1. Let x_i be such a point (maybe infinitely near x_1). Then, we compose f with e_{x_i,x_1} to obtain a birational map $T' : \mathbf{F}_0 \dashrightarrow \mathbf{F}_0$ which is a composition of $k-2$ elementary transformations. Continuing in this way, we write f as a composition of transformations $e_{x',y'}$.

If $\mathbf{F}_1 \overset{\mathrm{elm}_{x_{i-1}}}{\dashrightarrow} \mathbf{F}_0 \overset{\mathrm{elm}_{x_i}}{\dashrightarrow} \mathbf{F}_1$ occurs, then elm_{x_i} may be defined with respect to another projection to $\mathbb{P}^1$. Then, we write this as a composition of the automorphism τ of $\mathbb{P}^1 \times \mathbb{P}^1$ which switches the factors and the elementary transformation with respect to the first projection. Then we repeat this if such $(\mathbf{F}_0, \mathrm{elm}_{x_j})$ occurs again. □

Let $f : \mathbf{F}_0 \dashrightarrow \mathbf{F}_0$ be a birational transformation. Assume the image of $|\mathfrak{f}|$ is equal to $|a\mathfrak{f} + b\mathfrak{e} - \sum m_x x|$. Applying the automorphism τ, if needed, we may assume that $b \le a$. Thus, by using Example 7.3.11, we can find a point x with $m_x > b/2$. Composing f with elm_x, we obtain that the image of $|\mathfrak{f}|$ in $\mathbf{F}_1$ is the linear system $|a'\mathfrak{f}' + b\mathfrak{e}' - m_{x'}x' - \sum_{y \ne x'} m_y y|$, where $m_{x'} = b - m_x < m_x$. Continuing in this way, we get a map $T' : \mathbf{F}_0 \dashrightarrow \mathbf{F}_q$ such that the image of $|\mathfrak{f}|$ is the linear system $|a'\mathfrak{f}' + b\mathfrak{e}' - \sum m_x x|$, where all $m_x \le b/2$. If $b = 1$, we get all $m_i = 0$. Thus, T' is everywhere defined and hence $q = 0$. The assertion of the Theorem is verified.

Assume $b \ge 2$. Since all $m_i \le b/2$, we must have, by Example 7.3.11,

$$b > \frac{2a'}{2+q}.$$

Since the linear system $|a'f' + bs'|$ has no fixed components, we get

$$(a'\mathfrak{f}' + b\mathfrak{e}') \cdot \mathfrak{e}' = a' - bq \geq 0.$$

Thus, $q \leq a'/b < (2+q)/2$, and hence $q \leq 1$. If $q = 0$, and we get $b > a'$. Applying τ, we will decrease b and will start our algorithm again until we either arrive at the case $b = 1$, and we are done, or arrive at the case $q = 1$, and $b > 2a'/3$ and all $m_{x'} \leq b/2$.

Let $\pi : \mathbf{F}_1 \to \mathbb{P}^2$ be the blowing down of the exceptional section to a point q. Then, the image of a fiber $|\mathfrak{f}|$ on $\mathbf{F}_1$ under π is equal to $|h - q|$. Hence the image of our linear system in $\mathbb{P}^2$ is equal to $|a'h - (a'-b)q - \sum_{p \neq q} m'_p p|$. Obviously, we may assume that $a' \geq b$; hence, the coefficient at q is non-negative. Since $b > 2a'/3$, we get $a' - b < a'/3$. By Example 7.3.11, there exists a point $p \neq q$ such that $m'_p > a'/3$. Let $\pi(x) = p$ and $\mathcal{E}_1$ be the exceptional curve corresponding to x and s be the exceptional section in $\mathbf{F}_1$. If $x \in S$, the divisor class $\mathfrak{e} - e_1$ is effective and is represented by the proper inverse transform of s in the blow-up of x. Then,

$$(a'\mathfrak{f} + b\mathfrak{e} - m'_x e_1 - \sum_{i>1} m'_i e_i) \cdot (\mathfrak{e} - e_1) \leq a' - b - m'_x < 0.$$

This is impossible because the linear system $|a'\mathfrak{f} + b\mathfrak{e} - m_x x - \sum_{y \neq x} y|$ on $\mathbf{F}_1$ has no fixed part. Thus, x does not lie on the exceptional section. If we apply elm_x, we arrive at $\mathbf{F}_0$ and may assume that the new coefficient at f' is equal to $a' - m'_x$. Since $m'_x > a'/3$ and $a' < 3b/2$, we see that $a' - m'_x < b$. Now, we apply the switch automorphism τ to decrease b. Continuing in this way, we obtain that f is equal to a product of elementary transformations and automorphisms of $\mathbf{F}_0$. We finish the proof of Theorem 7.3.12 by applying Lemma 7.3.15.

Applying Lemma 7.3.4, Proposition 7.3.6, and Lemma 7.3.15, we obtain the following.

Corollary 7.3.16. *The group* $\mathrm{Cr}(2)$ *of Cremona transformations of* $\mathbb{P}^2$ *is generated by projective automorphisms and the standard Cremona transformation* T_{st}.

Remark 7.3.17. It is known that for $n > 2$, the *Cremona groups* $\mathrm{Cr}(n) := \mathrm{Bir}(\mathbb{P}^n)$ cannot be generated by the subgroup of projective transformations and a countable set of other transformations. For $n = 3$, this is a classical result of Hilda Hudson [414]. A modern, and different, proof for $n \geq 3$ can be found in [571].

S:7.4

7.4 Smooth Homaloidal Linear Systems

SS:7.4.1

7.4.1 Definition and examples

Definition 7.4.1. *A homaloidal linear system* $\mathcal{H}$ *is called* smooth *if the reduced base scheme* Z_{red} *is smooth, and, each* $\mathfrak{a}_i$*-primary component* $\mathfrak{b}_i$ *of* $\mathfrak{b}(|V|)$ *coincides with some power* $\mathfrak{b}_i^{m_i}$.

In particular, any isolated point of multiplicity one is not a point of contact.

Remark 7.4.2. We know from Proposition completeintersection 7.1.10 that the base scheme of a Cremona transformation cannot be a complete intersection. Let $X \subset \mathbb{P}^n$ be a smooth irreducible non-degenerate subvariety of $\mathbb{P}^n$. Recall that *Hartshorne's conjecture* says that X is a complete intersection as soon as $\dim X > \frac{2n}{3}$. So, assuming that this conjecture is true, we obtain that

$$\dim \mathrm{Bs}(|V|) \le \frac{2n}{3},$$

for any smooth homaloidal linear system.

In many examples $\mathrm{Bs}(|V|)_{\mathrm{red}}$ is smooth but $\mathrm{Bs}(|V|) \ne \mathrm{Bs}(|V|)_{\mathrm{red}}$.

An additional quite strong condition is that $\mathrm{Bs}(|V|)_{\mathrm{red}}$ is smooth, integral and moreover, f admits a resolution (σ, ν), where π is the blow-up of $\mathrm{Bs}(|V|)_{\mathrm{red}}$. Such a situation has been studied by B. Crauder and S.Katz Crauder [179]. In particular, they show that, assuming Hartshorne's conjecture,

$$\mathrm{Bs}(|V|) = \mathrm{Bs}(|V|)_{\mathrm{red}}.$$

They also show that, if Hartshorne's conjecture holds and $n \ge 7$, then $d \le 4$.

We denote a smooth homaloidal linear system by

$$\mathcal{H} = |dh - \sum m_i Z_i| \tag{7.52}$$ smoothhomaloid

where Z_i are irreducible components of the F-locus and m_i is the multiplicity of $\mathrm{Bs}(|V|)$ at Z_i,

It follows from the definition that the birational map defined by a smooth homaloidal linear system admits a smooth resolution isomorphic to the blow-up of the reduced base scheme. The exceptional divisor is equal to $\sum m_i E_i$, where E_i is the exceptional divisor of the blow-up of Z_i. This implies

$$\begin{aligned} d_n &= 1 = d^n + (-1)^n \sum m_i^n E_i^n = d^n - \sum m_i^n s(Z_i, \mathbb{P}^n)_0, \\ d_k &= d^k + (-1)^k \sum m_i^k E_i^k) h_1^{n-k} = d^k - \sum m_i^k s(Z_i, \mathbb{P}^n)_{n-k}, \quad k = 0, \ldots, n-1. \end{aligned} \tag{7.53}$$ smooth degree

To compute the dimension of $\mathcal{H}$, we use the exact sequence

$$0 \to \mathcal{I}_{Z_i}^{m_i - 1} / \mathcal{I}_{Z_i}^{m_i} \to \mathcal{O}_{m_i Z_i} \to \mathcal{O}_{(m_i - 1) Z_i} \to 0$$

It allows to compute $\chi(\mathcal{O}_{m_iZ_i})$ by induction. We have $\mathcal{I}_{Z_i}^{m_i-1}/\mathcal{I}_{Z_i}^{m_i} \cong S^{m_i-1}(\mathcal{N}_{Z_i/\mathbb{P}^n}^\vee)$, so we need to twist by $\mathcal{O}_{\mathbb{P}^n}(d)$ and compute the Chern classes of $S^{m_i-1}(\mathcal{N}_{Z_i/\mathbb{P}^n}^\vee)(d)$ by the standard formulas, and apply Riemann-Roch.

The following theorem is due to Margherita Beloch Beloch [53, Parte Terza].

thm:beloch **Theorem 7.4.3.** *There are no smooth homaloidal linear systems in $\mathbb{P}^3$ with non-empty* 0*-dimensional F-locus.*

Proof Let $|dh - \sum_{i=1}^N m_ip_i|$ be a smooth homaloidal linear system with F-locus equal to a set of points $p_1,\ldots,p_N$. We may assume that $m_1 \ge \cdots \ge m_N$. By Noether's inequality,

$$m_1 > \frac{n-1}{n+1}d.$$

So, we get $m_1 > \frac{1}{2}d$.

Assume first that $d = 2,3$. If $d = 2$, we get $8 = \sum m_i^3 + 1$, and hence, the only solution $N = 7, m_i = 1$. But the dimension of this linear system is equal to one. We leave it to the reader to analyze the case $d = 3$ in a similar manner.

Let $X \to \mathbb{P}^3$ be a log resolution of the base scheme of $|V|$ and S' be the proper transform of a general member S of $|V|$.

Since $d \ge 4$, we can find t such that $4t \le d < 4t+4$. Noether's inequality implies $m_1 > 2t$. Assume $d = 4t$ and $m_1 = 2t+2s$. Then, $|S' + (t-s)K_X| = \emptyset$ implies $m_2 > 2(t-s)$. The line joining p_1 and p_2 intersects S with multiplicity $> (2t+2s)+(2t-s) = 4t = d$. This contradicts our assumption that $|V|$ has no fundamental curves.

Assume $d = 4t$ and $m_1 = 2t+2s-1$. Then, $|S'+(t-s)K_X| = \emptyset$ implies $m_2 \ge 2(t-s)+1$, and hence, $m_2 = 2(t-s)+1$. There is no contradiction in this case, we only deduce that there exists a line intersecting S' with multiplicity d. Let r be the number of points of multiplicity equal to m_2. The $(t-s)$-adjoint surface $S_{t-s} \in |S+(t-s)K_{\mathbb{P}^3}|$ is of degree $4s$ and has the point p_1 of multiplicity $4s-1$ and r simple points. Since it is empty, counting the dimension of $|S'+(t-s)K_X|$, we get

$$(\binom{4s+3}{3}-1)-\binom{4s+1}{3}-r<0.$$

This implies $r \ge (4s+1)^2$. Now, we count $\dim|V|$, and get

$$4 = \dim V \le \binom{4t+3}{3} - \binom{2t+2s+1}{3} - (4s+1)^2(\binom{2t-2s+3}{3}$$

$$= \frac{1}{6}(-128a^3s^2 - 64a^3s - 384a^2s^2 + 48a^3 - 48a^2s - 256as^2 + 72a^2 + 16as + 24a),$$

where $a = t - s > 0, s > 0$. It is immediate to see that the last sum is negative, and we get a contradiction.

Other cases with $d = 4t+1, 4t+2, 4t+3$ are treated in a similar manner, and we leave it to the reader to finish the proof. □

E2.3.2 *Example* 7.4.4. It is possible that $\mathrm{Bs}(f)$ is 0-dimensional but the homaloidal linear system is not smooth. There are hidden one-dimensional infinitely near components. For example, consider the linear system of quadrics in $\mathbb{P}^3$ passing through four non-collinear points and tangent to a fixed plane containing one of the points. Choosing coordinates, we may assume that the points are $p_1 = [1,0,0,0]$, $p_2 = [0,1,0,0]$, $p_3 = [0,0,1,0]$, $p_4 = [0,0,0,1]$, and the tangent plane at the point p_4 has equation $t_0+t_1+t_2 = 0$. After we blow up the first three points, we obtain that the inverse image of the linear system has the base locus equal to a line in the exceptional divisor E_4 over the point p_4. If we blow up this line, we resolve the indeterminacy of the birational map. The exceptional divisor consists of E_1, E_2, E_3, E_4', E_5, where E_4' is the proper transform of E_4 isomorphic to $\mathbb{P}^2$ and E_5 is the exceptional divisor over the line isomorphic to the minimal ruled surface $\mathbf{F}_2$. The divisors E_4' and E_5 intersect along a curve C which is the exceptional section in E_5 and a line in $\mathbb{P}^2$.

The base ideal in a neighborhood of the point p_4 is isomorphic to the ideal $(xy, yz, xz, x+y+z)$. After we make the change of variables $x \to x+y+z$, it becomes isomorphic to the ideal (x, y^2, yz, z^2). It is easy to see that the blow-up scheme is isomorphic to the projective cone over the blow-up of the maximal ideal (y, z). It has a singular point locally isomorphic to the cone over the Veronese surface. Its exceptional divisor is isomorphic to the quadratic cone. The birational morphism from the resolution above to the blow-up of the base scheme is the contraction of the divisor E_4' to the singular point of the blow-up.

So, the homaloidal linear system can be written in the form $|2H - p_1 - p_2 - p_3 - Z_4|$, where $p_4 = (Z_4)_{\mathrm{red}}$, and Z_4 is locally given by a primary ideal (x, y^2, yz, z^2).

Finally, note that the topological realization of the simplicial complex $S(\mu)$ of the log resolution of the rational map from the previous example is homeomorphic to the two-dimensional sphere [198, §4].

beloch2 *Remark* 7.4.5. All Cremona transformations in $\mathbb{P}^3$ with 0-dimensional F-locus were classified by Beloch [53, p. 64]. Besides the transformation with four base points from Example 7.4.4, there are two more transformations. The first one is given by the linear system of cubic surfaces with a double points at p_1, simple points at p_2, p_3, p_4, and one of the points is a point of 4-contact (in sense of

Definition def:contact 7.1.15).* The second one is given by the linear system of quartics with an ordinary triple base point and a point of 6-contact.

Note that the two transformations with $d = 2, 3$ were known to Cremona.

In the following examples, we will use the following lemma that follows immediately from the Euler exact sequence.

lem:7.4.6 **Lemma 7.4.6.** *Let S be a smooth irreducible r-dimensional subvariety of $\mathbb{P}^n$ and $c(\Theta_S)$ be the total Chern class of S, Then,*

$$s(S, \mathbb{P}^n) = \frac{c(\Theta_S)}{(1 + h_S)^{n+1}} = c(\Theta_S)(\sum_{i=0}^{n-r}(-1)^i \binom{n+i}{n} h_S^i),$$

where $h_S = c_1(O_S(1)) = i^*h$.

For any $0 \le t \le r$,

$$s_{r-t}(S, \mathbb{P}^n) = \sum_{i=0}^{t}(-1)^{t-i} c_i(\Theta_S) \binom{n+t-i}{n} h_S^{t-i}$$

and, setting $\deg(c_i(\Theta_S)) = c_i(\Theta_S) \cdot h^{r-i}$, we obtain

$$s(S, \mathbb{P}^n)_{r-t} = \sum_{\alpha=0}^{t}(-1)^{t-i} \binom{n+t-i}{n} \deg(c_i(\Theta_S)). \qquad (7.54)$$ substitute1

This allows us to compute the multidegree of the rational map. We have

$$d^k = d^k - \sum_{i=1}^{N}\Big(\sum_{j=\delta_i}^{k} d^{k-j} \tbinom{k}{j} m_i^j s(Z_i, \mathbb{P}^n)_{n-j}\Big), \qquad (7.55)$$ multsmooth

where $\delta_i = \operatorname{codim}(Z_i, \mathbb{P}^n)$.

Using (substitute1 7.54), we obtain the following:

smoothmultidegree **Proposition 7.4.7.** *Let f be a Cremona transformation given by a smooth homaloidal linear system (smoothhomaloid 7.52). Let $\delta_i = n - \dim(Z_i)$. Then,*

$$d_k = d^k - \sum_{i=1}^{N}\Big[\Big(\sum_{t=0}^{k-\delta_i} d^{k-\delta_i-t} \tbinom{k}{\delta_i+t} m_i^{\delta_i+t}\Big)\Big(\sum_{j=0}^{t}(-1)^{t-j} \tbinom{n+t-j}{n} c_j(\Theta_{Z_i})\Big)\Big]. \qquad (7.56)$$

In particular,

$$d_n = d^n - \sum_{i=1}^{N}\Big[\Big(\sum_{t=0}^{n-\delta_i} d^{n-\delta_i-t} \tbinom{n}{\delta_i+t} m_i^{\delta_i+t}\Big)\Big(\sum_{j=0}^{t}(-1)^{t-j} \tbinom{n+t-j}{n} c_j(\Theta_{Z_i})\Big)\Big]. \qquad (7.57)$$

*In Beloch's definition this is a point of contact of order 3.

and

$$d_k = d^k, \quad \textit{if } k < \max\{\delta_i\}.$$

ex:quadrocubic *Example* 7.4.8. Assume that the F-locus consists of N_1 smooth curves $C_1, \ldots, C_{N_1}$ of genus g_i and N_2 isolated points $x_{N_1+1}, \ldots, x_N$. Then, we get

$$d_n = d^n + \sum_{i=1}^{N_1} (m_i^n(n+1) - dnm_i^{n-1}) \deg(C_i) + m_i^n(2g_i - 2) - \sum_{i=N_1+1}^{N} m_i^n = 1,$$

$$d_{n-1} = d^{n-1} - \sum_{i=1}^{N_1} m_i^{n-1} \deg(C_i), \quad d_k = d^k, \quad k \geq 2.$$

Let us specialize.

Take $n = 3$. We have

$$1 = d^3 + \sum_{i=1}^{N_1} [(4m_i^3 - 3dm_i^2) \deg C_i + m_i^3(2g_i - 2)] - \sum_{i=N_1+1}^{N} m_i^3. \tag{7.58}$$ F2.8

Take $d = 2$. We have

$$1 = 8 + \sum_{i=1}^{N_1} [(4m_i^3 - 6m_i^2) \deg C_i + m_i^3(2g_i - 2)] - \sum_{i=N_1+1}^{N} m_i^3. \tag{7.59}$$ F2.9

Since each isolated base point imposes at least one condition on quadrics, their number is at most 5, and we get

$$-2 \geq \sum_{i=N_1+1}^{N} m_i^3 - 7 = \sum_{i=1}^{N_1} [(4m_i^3 - 6m_i^2) \deg C_i + m_i^3(2g_i - 2)]$$

$$\geq \sum_{i=1}^{N_1} m_i^2 [(2m_i - 3)(\deg(C_i) - 1) + m_i - 3).$$

The only possible cases here are: (a) $N_1 = 1, N_2 = 3, m_1 = \ldots = m_4 = 1, \deg(C_1) = 1$ and (b) $N_1 = 1, N_2 = 1, m_1 = m_2 = 1, \deg(C_1) = 2$. In the first case, $d_2 = 4 - 1 = 3$, so the multidegree is $(2, 3)$ and the inverse is a transformation of bidegree $(3, 2)$. This is an example of a *quadro-cubic* Cremona transformation. The P-locus is equal to the union of four planes, three are spanned by the line and one of the points, and the fourth is spanned by the three points. They are all blown down to four lines, one enters with multiplicity 2, and other three intersect it.

In the second case, $d_2 = 2$, we get a transformation of bidegree $(2, 2)$. We will study transformations of multidegree $(2, \ldots, 2)$ in the next section.

Take $d = 3$. We leave it to the reader to verify that there is only one transformation whose F-locus consists of one double line and six isolated points. The inverse transformation is of order $3^2 - 4 = 5$.

7.4.2 Special Cremona transformations

SS:7.4.2

Following [278] we call a Cremona transformation *special* if its base scheme Z is irreducible and smooth. Since a special transformation admits a log resolution with Picard number 2, the P-locus is also irreducible and coincides with the d_1-secant variety of the base scheme. Its degree is equal to $d_1 d_{n-1} - 1$ [278, Proposition 2.3]. However, it is not necessarily smooth (but always generically reduced). It follows from Noether's inequality that the codimension δ of the base locus satisfies the inequality:

$$d_1 < \frac{n+1}{\delta - 1} \tag{7.60}$$

noethersmooth

Applying Proposition 7.4.7, we obtain

$$d_k = d^k - \Big[\Big(\sum_{t=0}^{k-\delta} d^{k-\delta-t}\tbinom{k}{\delta+t}\Big)\Big(\sum_{j=0}^{t}(-1)^{t-j}\tbinom{n+t-j}{n}c_j(\Theta_Z)\Big)\Big]. \tag{7.61}$$

We start with the case $\delta = n - 1$, that is, $Z = C$ is a smooth curve of genus g and some degree c.

Proposition 7.4.9. *There are two possible types of special Cremona transformations with one-dimensional base scheme C:*

1. $n = 3, g = 3, \deg(C) = 6, d_1 = 3$;
2. $n = 4, g = 1, \deg(C) = 5, d_1 = 2$.

Proof Applying 7.1.13, we get $d_1 < \frac{n+1}{n-2}$. This gives $d_1 \le 3$, and, if $d_1 = 3$, $n \ge 4$. If $d_1 = 2$, the 1-secant variety must be a hypersurface, hence $n = 3$. If $d_1 = 3$, then the 2-secant variety must be a hypersurface, hence $n = 4$.

Assume $d_1 = 3, n = 3$. It follows from Example 7.4.8 that $1 = 27 - 5c + (2g - 2)$. By Postulation, $3 = \dim|\mathcal{O}_{\mathbb{P}^3}(3)| - (3c + 1 - g)$. This gives, $5c + 2(1-g) = 26$ and $3c + (1-g) = 16$. The only solution $c = \deg(C) = 6$ and $g = 3$. This transformation is an example of a bilinear Cremona transformation. We will study them later.

Assume $d_1 = 2, n = 4$. As above, we get $1 = 16 - 3c + 2 - 2g$ and $4 = \dim|\mathcal{O}_{\mathbb{P}^4}(2)| - (2c + 1 - g)$. This gives $3c + 2(g-1) = 15$ and $10 = 2c + (1-g)$, and hence, $c = 5, g = 1$. Since a general plane does not intersect C, $d_2 = 4$. The restriction of the linear system to a general hyperplane H is given by the linear system of quadrics with 5 isolated base points. The image $f(H)$ is a cubic hypersurface. Thus, $d_3 = 3$. We obtain an example of a quadro-cubic transformation in $\mathbb{P}^4$ discussed in [701, Chapter VIII, §5.2]. □

The proposition is proven in [179, Theorem 2.2]. In the same paper, Theorem 3.3 gives the classification in the case $\dim Z = 2$.

crauderkatz **Theorem 7.4.10.** *A Cremona transformation with 2-dimensional smooth connected base scheme Z is one of the following:*

(i) $n = 4, d = 3$, *Z is an elliptic scroll of degree 5, the base scheme of the inverse of the quadro-cubic transformation from above;*
(ii) $n = 4, d = 4$, *Z is a determinantal variety of degree 10 given by* 4×4-*minors of a* 4×5-*matrix of linear forms (a bilinear transformation, see later);*
(iii) $n = 5, d = 2$, *Z is a Veronese surface;*
(iv) $n = 6, d = 2$, *Z is an elliptic scroll of degree 7;*
(v) $n = 6, d = 2$, *Z is a surface of degree eight, the image of the projective plane under a rational map given by the linear system of quartics through eight points.*

We already encountered the case of a Veronese surface in Section 2.4. The transformation is the adjugate involution Adj_3 for the space of conics in $\mathbb{P}^2$. Noether's inequality gives $d_1 \le 2$ if $n \ge 5$. The 1-secant variety of a surface is of dimension 5 unless Z is a Veronese surface in $\mathbb{P}^5$. This gives case (iv) or $n = 6$. If $n = 4$, we can get $d_1 = 3$ or 4.

There is no classification for higher-dimensional Z. However, we have the following nice results of L. Ein and N. Shepherd-Barron [278].

Recall that a *Severi-Zak variety* is a closed smooth subvariety Z of $\mathbb{P}^n$ of dimension $\frac{1}{3}(2n-4)$ such that the secant variety is a proper subvariety of $\mathbb{P}^{n+1}$. All such varieties are classified by F. Zak (see [478]). The list is as follows:

(i) Z is a Veronese surface in $\mathbb{P}^5$;
(ii) Z is the Grassmann variety $G_1(\mathbb{P}^5)$ embedded in the Plücker space $\mathbb{P}^{14}$;
(iii) Z is the Severi variety $\mathsf{s}_2(\mathbb{P}^2 \times \mathbb{P}^2) \subset \mathbb{P}^8$;
(iv) Z is the E_6-variety, a 16-dimensional homogeneous variety in $\mathbb{P}^{26}$.

In all these cases the secant variety of the Severi variety Z is a cubic hypersurface X with the singular locus equal to Z. A theorem of Ein and Shepherd-Barron asserts that a simple Cremona transformation $f : \mathbb{P}^n \dashrightarrow \mathbb{P}^n$ with $d_1 = d_{n-2} = 2$ (a *quadro-quadratic transformation*) has the base scheme equal to one of the four Severi-Zak varieties. Another result from the same paper gives a classification of special Cremona transformations with codimension two F-locus. If $n \leq 4$ such transformations are covered by Theorem 7.4.10. the paper of Crauder and Katz. A new transformation could be only in $\mathbb{P}^5$ and it is a transformation of degree $d_1 = 5$ with the base scheme equal to a determinantal variety of type $(|5,6|_5,4)_1$. It is given by maximal minors of the (5×6)-matrix defining the determinantal variety.

Let G be a semi-simple complex algebraic group and $\rho : G \to \mathrm{GL}(E)$ be its linear representation. Assume that it is *pre-homogenous*, i.e., it acts transitively outside an irreducible hypersurface (equivalently, the algebra of invariants is generated by one invariant P). The classification of irreducible pre-homogeneous representations is known [655], [713]. Let X be the image of this hypersurface in the projective space $|E|$.

Recall that a polynomial $F(t_0,\ldots,t_n)$ is called *homaloidal* if its partial derivatives define a Cremona transformation. An example of a homaloidal polynomial is the determinant of a square matrix with indeterminant entries, orthogonal or general. It defines the adjugate involution $A \mapsto \mathrm{adj}(A)$.

The following theorem is proved in loc. cit. paper of Ein and Shepherd-Barron.

einshepherdbarron **Theorem 7.4.11.** *Assume that the Hessian of P is not identically zero. Then, either*

1. *P is homaloidal and defines a Cremona involution.*
2. *$n = 2k+1$, and the transformation defined by*

$$\left(\frac{\partial P}{\partial t_{k+1}},\ldots,\frac{\partial P}{\partial t_{2k+1}},-\frac{\partial P}{\partial t_0},\ldots,-\frac{\partial P}{\partial t_k}\right)$$

is a Cremona involution.

An example of pre-homogenous transformation is given a Severi-Zak variety. The unique invariant is of degree three and $V(P)$ is equal to the secant variety of the Severi-Zak variety. The group and the representations (G,E) are the following:

1. $(\mathrm{SL}(3), S^2(\mathbb{C}^3))$;
2. $(\mathrm{SL}(6), \bigwedge^2(\mathbb{C}^6))$;
3. $(\mathrm{SL}(3)\times\mathrm{SL}(3), \mathbb{C}^3\otimes\mathbb{C}^3)$;
4. $(E_6, \mathbb{C}^{27})$.

S:7.5

7.5 Quadratic Cremona Transformations

These are transformations of algebraic degree $d_1 = 2$. The complete classification of quadratic transformations in arbitrary $\mathbb{P}^n$ is not known. However, there are partial results which we will discuss in this section.

SS:7.5.1

7.5.1 Elementary quadratic transformations and complex spheres

These are transformation of multidegree $(2, \ldots, 2)$. In fact, it follows from the Cremona inequalities, that a Cremona transformation is an elementary quadratic transformation if and only if $d_1 = d_2$.

Let us show that any vector $(2, \ldots, 2)$ is realized as the multi-degree of a Cremona transformation. For $n = 2$, we can take the homaloidal linear system of conics through three non-collinear points. We can view a pair of the fundamental points as a 0-dimensional quadric in the line spanned by these points. This admits an immediate generalization to higher-dimensional spaces.

Consider the linear system of quadrics in $\mathbb{P}^n$ containing a fixed smooth quadric Q_0 of dimension $n - 2$. It maps $\mathbb{P}^n$ to a quadric Q in $\mathbb{P}^{n+1}$. We may choose coordinates such that

$$Q_0 = V(t_0, \sum_{i=1} t_i^2),$$

so that the hyperplane $H = V(z_0)$ is the linear span of Q_0. Then, the linear system is spanned by the quadrics $V(\sum t_i^2), V(t_0 t_i), i = 0, \ldots, n$. A general quadric in the linear system has an equation

$$a_{n+1}(\sum_{i=1}^n t_i^2) - 2\sum_{i=1}^n a_i t_0 t_i = 0. \tag{7.62}$$

complexsphere0

If $a_{n+1} \neq 0$, we may assume that $a_{n+1} = 1$, and rewrite it in the form

$$\sum_{i=1}^n (t_i - a_i t_0)^2 - (a_0 + \sum_{i=0}^n a_i^2) t_0^2 = 0.$$

If $a_0 \neq 0$, we divide by a_0^2 and get the equation

$$a_0^2 (\sum_{i=1}^n (\frac{t_i}{t_0} - \frac{a_i}{a_0})^2 = 2a_0 a_{n+1} + \sum_{i=1}^n a_i^2. \tag{7.63}$$

n-sphere

It can be viewed as the equation of a sphere in the affine space $\mathbb{A}^n$ with center at the point $(\frac{a_1}{a_0}, \ldots, \frac{a_n}{a_0})$ and radius-square $R^2 = 2a_0 a_{n+1} + \sum_{i=1}^n a_i^2$. Of course, all the coefficients here are complex numbers, so the radius could be equal to zero. We say that the quadric given by equation (n-sphere 7.63) is a *complex sphere*. . In Subsection SS:2.2.3 2.2.3 we discussed complex circles.

If we take $(2t_0^2, -2t_0t_1, \ldots, -2t_0t_n, \sum_{i=1}^n t_i^2)$ as a basis of the linear system of quadrics, we obtain that the image of the rational map

$$g : \mathbb{P}^n \dashrightarrow \mathbb{P}^{n+1}, \quad (t_0, \ldots, t_n) \mapsto (2t_0^2, -2t_0t_1, \ldots, -2t_0t_n, \sum_{i=1}^n t_i^2), \qquad (7.64)$$ qqt

defined by the linear system is the quadric

$$Q : 2x_0x_{n+1} + \sum_{i=1} x_i^2 = 0.$$

Its polar hyperplane with pole at $(a_0, \ldots, a_{n+1})$ is given by equation

$$a_0x_{n+1} + a_{n+1}x_0 - \sum_{i=1}^n a_ix_i = 0$$

Its pre-image in $\mathbb{P}^n$ is the quadric (complexsphere0 7.62). We see that the complex spheres with radius zero correspond to poles lying on the quadric Q in $\mathbb{P}^{n+1}$.

This simple observation is the cornerstone of the spherical geometry. By passing to real points, it allows one to translate all assertions about n-dimensional spheres in $\mathbb{R}^n$ into equivalent assertions about points in $\mathbb{P}^{n+1}(\mathbb{R})$.

Over $\mathbb{C}$, it gives an isomorphism between the projective orthogonal group of the quadric Q isomorphic to $\mathrm{PO}(n+2)$ and the *Inversion group* $\mathrm{Inv}(n+1)$ of Cremona transformations of $\mathbb{P}^n$ that transforms spheres to spheres, maybe degenerate.

inversion *Example* 7.5.1. The first historical example of a Cremona transformation is the *inversion* map. Recall the inversion transformation from the plane geometry. Given a circle of radius R, a point $x \in \mathbb{R}^2$ with distance r from the center of the circle is mapped to the point on the same ray at the distance R/r (as in the picture below).

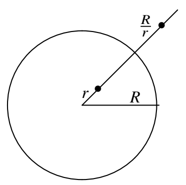


Figure 7.6 Inversion transformation in the plane

In the affine plane $\mathbb{C}^2$, the transformation is given by the formula

$$(x, y) \mapsto \Big(\frac{Rx}{x^2 + y^2}, \frac{Ry}{x^2 + y^2}\Big).$$

In projective coordinates, the transformation is given by the formula

$$(t_0,t_1,t_2) \mapsto (t_1^2+t_2^2, Rt_1t_0, Rt_2t_0).$$

We see that this transformation corresponds to a projective transformation of $\mathbb{P}^3$ given by a formula (in coordinates as above):

$$[x_0,x_1,x_2,x_3] \mapsto [x_1, -\tfrac{1}{2}x_1, -\tfrac{1}{2}x_2, x_0].$$

Let $f:\mathbb{P}^n \dashrightarrow \mathbb{P}^n$ be the Cremona transformation of $\mathbb{P}^n$ equal to the composition of the map $g:\mathbb{P}^n \dashrightarrow \mathbb{P}^{n+1}$ from above and the projection map $\mathrm{pr}_q$from a point $q \in Q$. It is the identity map only if $q = [0,\ldots,0,1] \in Q$. The F-locus of f is equal to the indeterminacy locus of the rational map g that coincides with the quadric $Q_0 = V(t_0, \sum_{i=1}^n t_i^2)$. The P-locus is equal to the hyperplane $V(t_0)$. It is blown down to the point $q_0 = [0,\ldots,0,1]$.

The homaloidal linear system is smooth. The multi-degree of f is equal to $(2,\ldots,2)$. This can be seen geometrically as follows.

For a general linear subspace L of codimension $k > 0$, its pre-image under the projection map $\mathrm{pr}_q : Q \dashrightarrow \mathbb{P}^n$ is the intersection of Q with the subspace $L' = \langle L, q\rangle$ spanned by L and q. It is a quadric in L'. Since we can assume that the point q_0 does not belong to L', the projection of this quadric from the point q is a quadric in the projection of L' from the same point. Thus, $d_k = 2$. This shows that the multidegree of the transformation is equal to $(2,\ldots,2)$. We can confirm this by using Lemma lem:7.4.6 7.4.6. We use the embedding $i : Q_0 \hookrightarrow H \cong \mathbb{P}^{n-1}$ to get

$$c(\Theta_{Q_0}) = c(i^*\Theta_H)c(\mathcal{N}_{Q_0/H})^{-1} = (1+h_0)^n(1+2h_0)^{-1},$$

where $h_0 = i^*(c_1(\mathcal{O}_{\mathbb{P}^n}(1))$. Now, we apply Lemma lem:7.4.6 7.4.6 to get

$$s(Q_0,\mathbb{P}^n) = (1+2h_0)^{-1}(1+h_0)^{-1} = \sum_{t=0}^{n-1}(2^t-1)(-1)^t h_0^t.$$

Using (multsmooth 7.55), we easily find

$$d_k = 2^k - 2^k(\sum_{i=1}(-1)^i\binom{k}{i} + 2\sum_{i=1}^{k} 2^{k-1}(-1)^i\binom{k}{i} = 2^k + 2^k 2(1-2^k) = 2, k \neq n.$$

If $k = n$, we have to subtract 1 because there is one isolated simple base point.

Note that in case $n = 2$, the quadric Q_0 is the set of two distinct points a, b in a fixed line ℓ. If we choose the center q of projection $Q \dashrightarrow \mathbb{P}^2$ such that $\mathrm{pr}_q(q_0) \notin \ell$, we get a quadratic transformation of $\mathbb{P}^2$ with three fundamental points. If $\mathrm{pr}_q(q_0) \in \ell$ but different from a or b, we get the first degenerate quadratic transformation with fundamental points $x_2 \succ x_1, x_3$. Finally, if $\mathrm{pr}_{q_0} \in \{a,b\}$ we get a quadratic transformation with fundamental points $x_3 \succ x_2 \succ x_1$.

Let us show that any Cremona transformation of multidegree $(2, \dots, 2)$ is a degeneration of the the transformation $f : \mathbb{P}^n \to Q \dashrightarrow \mathbb{P}^n$ as above.

Proposition 7.5.2. *The F-locus of any Cremona transformation f of multidegree $(2, 2, \dots, 2, 2)$ is a codimension two subvariety of $\mathbb{P}^n$ of degree 2.*

Proof The assertion is obvious if $n = 2$. Assume that $n \geq 3$. The image of a general plane Π in the source $\mathbb{P}^n$ is of degree 2. It is equal to the projection of the Veronese surface to $\mathbb{P}^n$ and its degree is equal to 2. This is possible only if Π contains two base points. Thus, the codimension 2 part of $\mathrm{Bs}(f)$ is a quadric Q_0. Let H be the hyperplane spanned by the quadric. The restriction of the map to H is given by quadrics containing Q_0, and the image of H is a quadric because f^{-1} is of multidegree $(2, \dots, 2)$. Without loss of generality, we may assume that $H = V(t_0)$. Let f be given by a formula

$$[t_0, \dots, t_n] \mapsto [f_0(t_0, \dots, t_n), \dots, f_n(t_0, \dots, t_n)],$$

where f_i are quadratic forms. Obviously, H belongs to the P-locus. We may assume that it is blown down to a point $[0, \dots, 0, 1]$. Substituting $x_0 = 0$, we obtain that $f_i = t_0 l_i(t_0, t_1, \dots, t_n), i = 0, \dots, n-1$, where l_i are linear forms. Another change of coordinates allows us to assume that $l_0 = t_1, \dots, l_{n-1} = t_n$. Since Q_0 belongs to the F-locus, its equation must be $f_n(t_1, \dots, t_n) = 0$. It is immediate to see that our transformation is equal to the composition of the rational map to a quadric $f_n(x_0, \dots, x_{n-1}) - x_0 x_{n+1} = 0$ and the projection from a point on this quadric. This is a degeneration of our transformation, where the quadric was nonsingular. □

There are different kinds of the transformations of multidegree $(2, \dots, 2)$ dependent on the degeneration of the quadric in the base locus and the position of the isolated base point. In the case $n = 3$, all such degenerate transformations were classified in PanVust [573].

SS:7.5.2

7.5.2 Non-elementary quadratic transformations

We start with non-elementary quadratic Cremona transformations in $\mathbb{P}^3$. Here, we use *Cremona's classifying method* for Cremona transformations in $\mathbb{P}^3$. Let $f : \mathbb{P}^3 \dashrightarrow \mathbb{P}^3$ be a Cremona transformation. Let V be a general member of the homaloidal linear system $\mathcal{H}$. Then, the restriction of f to Φ defines a birational map $f : V \dashrightarrow H$ from V to a plane in the target space $\mathbb{P}^3$. The restriction of $\mathcal{H}$ to V is a linear system $\mathcal{H}_V$ whose fixed components are one-dimensional irreducible components of the F-locus. Since V is rational we find a rational map $\sigma : \mathbb{P}^2 \dashrightarrow V$ such that the pre-image of $\mathcal{H}_V$ is a homoloidal linear system in $\mathbb{P}^2$ that defines Cremona transformation $f \circ \sigma : \mathbb{P}^2 \dashrightarrow V \to H \cong \mathbb{P}^2$. The degree of the plane homoloidal linear system is less than or equal to d_2.

We employ this method here, by taking $n = 3$ and $(d_1, d_2) = (2, d_2)$. It follows from Cremona's inequalities that $d_2 \in \{2, 3, 4\}$. Since we are interested in non-elementary quadratic transformations, we exclude the case $d_2 = 2$. The transformation $V : f^{-1} \circ \sigma$ is now a planar transformation of degree d_2.

Assume $d_2 = 3$. Then, V is a quadric surface. Assume that it is smooth and choose $\sigma : \mathbb{P}^2 \dashrightarrow V$ be the inverse of the projection from a point $p_0 \in V$. The map σ has two fundamental points a_1, a_2 and the line spanned by these points is blown down to p_0. Since $d_2 = 3$, we are looking for a homaloidal linear system of curves of degree ≤ 3. If the degree is 3, then its characteristic vector is $(3; 2, 1, 1, 1, 1)$ with base points $(c_0, a_1, a_2, b_1, b_2)$. The image of this linear system in V consists of twisted cubics, the residual to the line ℓ in the complete intersection of V with another member of $\mathcal{H}$. The line ℓ contains p_0 and it is blown down under σ^{-1} to the point a_1. Another line on V containing p_0 is blown down to a_2. The isolated fundamental points of f are the points $a_2, \sigma(b_1), \sigma(b_2)$. We encountered this transformation in Example 7.4.8.

We have found a general transformation of bidegree $(2, 3)$. There are 10 types of degenerate transformations (see [573]). one of the isolated fundamental points may lie on the fundamental line ℓ. The quadrics from $\mathcal{H}$ are tangent to ℓ at this point.

Assume $d_2 = 4$. Then, we are looking for a homaloidal linear system in the plane whose image in the quadric V is a curve of degree 4. In this case, the Φ-locus consists of isolated points. Again, we encountered this kind of a Cremona transformation in Example 7.4.4. We refer to [573] for the description of five possible degenerations of this transformation.

In the rest of the subsection, we will study *quadro-quadratic transformations*, i.e. transformations of multidegree $(2, d_2, \ldots, d_{n-2}, 2)$.

We have seen already examples of non-elementary quadratic Cremona transformation of multidegree $(2, d_1, \ldots, d_{n-2}, 2)$. These were special transformations with the base scheme equal to one of the four Severi varieties. The P-locus of these transformations are cubic hypersurfaces equal to the secant varieties of the base scheme. Their equations are the determinants of a general 3×3-Hermitian matrix with entries in one of the four complexified Hurwitz's *composition algebras* $K \otimes \mathbb{C}$, where $K = \mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O}$. The vector space of such Hermitian matrices admits a structure of a *Jordan algebra* of rank 3, a commutative unitary finite-dimensional algebra $\mathbb{J}$ over a field $\Bbbk$ (which we take equal to be $\mathbb{C}$)) satisfying the Jordan relation $x^2(xy) = x(x^2y)$. The *rank* of a Jordan algebra is the dimension of the subalgebra generated by a general element x and the unity e.

We will be interested in Jordan algebras of rank 3. Elements of such an algebra satisfy a cubic equation

$$x^3 - Tr(x)x^2 + S(x)x - N(x)e = 0, \tag{7.65}$$ jordan1

where $\mathrm{Tr}(x), S(x)$, and $N(x) \in \Bbbk$.Writing x as a linear combination of a basis of R, we find that the coefficients are linear, quadratic, and cubic polynomials in coefficients. An element x is invertible if and only if $N(x) \neq 0$, the inverse is given by $\frac{1}{N(x)}x^*$, where $x^* = x^2 - \mathrm{Tr}(x)x + S(x)e$. We have

$$(x^*)^* = N(x).$$

The easiest example of a Jordan algebra $\mathbb{J}$ is obtained from any associative finite-dimensional algebra with an involution $x \mapsto x'$ by changing its multiplication rule $x \bullet y := \frac{1}{2}(xy' + x'y)$. For example, we can take any algebra of square matrices of size n with the transpose involution. The rank of the associated Jordan algebra is equal to n

The left-hand side of (jordan1 7.65) is the characteristic polynomial. The algebra of Hermitian 3 × 3-matrices over octonions is not obtained in this way, it is an example of an *exceptional Jordan algebra*.

The application of Jordan algebras to Cremona transformations is explained by the fact that the transformation

$$f_{\mathbb{J}} : |\mathbb{J}| \dashrightarrow |\mathbb{J}|, \quad [x] \mapsto [x^*]$$

is a quadratic Cremona involution in $|\mathbb{J}| \cong \mathbb{P}^n$, where $n+1 = \dim_{\mathbb{C}} \mathbb{J}$. Composing it with any projective automorphism, we obtain examples of quadro-quadratic Cremona transformations, not necessarily involutions.

For example, the Jordan algebras corresponding to Hurwitz's composition algebras give us the quadro-quadric Cremoma transformations with smooth base scheme from Theorem einshepherdbarron 7.4.11.

Here is another example.

Example 7.5.3. A Jordan algebra of rank 1 is the field $\mathbb{C}$, a Jordan algebra of rank 2 is isomorphic to the algebras $\mathbb{J}(q) = \mathbb{C} \times V$, where (V, q) is a quadratic space with the quadratic form $q : V \to \mathbb{C}$. The multiplication law is defined by

$$(\lambda, \mathbf{v}) \bullet (\lambda', \mathbf{v}') = (\lambda\lambda' + b_q(\mathbf{v}, \mathbf{v}'), \lambda\mathbf{v}' + \lambda'\mathbf{v}),$$

where $b_q(x, y) = \frac{1}{2}(q(x + y) - q(x) - q(y))$ is the polar symmetric bilinear form of q. Its unit element is $e = (1, \mathbf{0})$

We check

$$x^2 = (\lambda, \mathbf{v})^2 = (\lambda^2 + q(\mathbf{v}), 2\lambda\mathbf{v}) = 2\lambda(\lambda, \mathbf{v}) + (q(\mathbf{v}) - \lambda^2)e = f(x)x - S(x)e,$$

where $f(x) = 2\lambda, S(x) = \lambda^2 - q(\mathbf{v})$. In particular, $x^* = x - f(x)e = (-\lambda, \mathbf{v})$ satisfies $x \bullet x^* = -S(x)e$, and $x^{-1} = -\frac{x^*}{S(x)}$.

Now, consider the direct product of algebras $\mathbb{J} = \mathbb{C} \times \mathbb{J}(q)$ with the unit $\bar{e} = (1, e)$. Let $X = (\alpha, x)$, then

$$X^{-1} = (\alpha^{-1}, (\lambda^2 - q(\mathbf{v}))^{-1}(-\lambda, \mathbf{v})).$$

Let us fix coordinates in $\mathbb{J}$ such that $X = (x_0, -x_1, x_2, \ldots, x_n)$ and $q = -\sum_{i=2}^{r} x_i^2$. Then,

$$X^{-1} = (x_0^{-1}, (\sum_{i=1}^{n} x_i^2)^{-1}(x_1, x_2, \ldots, x_n)),$$

Considered as a rational map $\mathbb{P}^n \dashrightarrow \mathbb{P}^n$, it is given by quadratic polynomials

$$f(x) = (\sum_{i=1}^{r} x_i^2, x_0 x_1, \ldots, x_n).$$

When $r = n$, we recognize an elementary quadratic transformation. If $r < n$, it is its degeneration. We check that the cubic polynomial $N(x)$ in this case is equal to $x_0(x_1^2 + \cdots + x_r^2)$. Not all degenerations of an elementary quadratic transformation are obtained in this way. Other degenerations correspond to a possible position of isolated base points (see [573] and [589]).

Let us see now that any quadro-quadratic transformation is defined by some Jordan algebra of rank 3. This is a marvelous result that one has to take with a grain of salt because the classification of Jordan algebras of arbitrary dimension in unknown. On the other hand, as we will see, a geometric construction of a quadro-quadratic transformation leads to the construction of a new Jordan algebra.

Let $f : \mathbb{P}^n \dashrightarrow \mathbb{P}^n$ be a quadro-quadratic Cremona transformation given by quadratic polynomials:

$$[t_0, \ldots, t_n] \mapsto [f_0, \ldots, t_n), \ldots, f_n(t_0, \ldots, t_n)].$$

Composing it with the of $g = f^{-1}$

$$g : [t_0, \ldots, t_n] \mapsto [g_0(t_0, \ldots, t_n), \ldots, g_n(t_0, \ldots, t_n)],$$

we obtain $g_i(f_0, \ldots, f_n) = t_i N(t_0, \ldots, t_n))$, where N is a cubic polynomial. It is clear that $V(N)_{\mathrm{red}}$ is equal to the P-locus of f.

Consider f and g as holomorphic maps $f, g : \mathbb{C}^{n+1} \to \mathbb{C}^{n+1}$ and $N(x)$ as a holomorphic map $\mathbb{C}^{n+1} \to \mathbb{C}^{n+1}$ given by the diagonal matrix $N(t)I_{n+1}$.

The composition $j = N^{-1} \circ f$ îs given by rational homogenous functions $b_i(t) = f_i(t)/N(t)$ of degree -1. The jacobian matrix $dj_f(x)$ of this map is a square matrix of size $n+1$ with entries

$$A_{ij} = \frac{\partial b_i}{\partial t_j} = N^{-1}\frac{\partial f_i}{\partial t_j} + f_i\frac{\partial N^{-1}}{\partial t_j}$$

which are rational homogenous functions of degree -2. The inverse dj_f^{-1} is given by homogenous rational functions of degree 2. It defines a rational map

$$P_f : \mathbb{C}^{n+1} \to \mathrm{Mat}_{n+1}(\mathbb{C}) = \mathrm{End}(\mathbb{C}^{n+1}.$$

Let

$$B_f(x,y) = P_f(x+y) - P(x) - P(y) : \mathbb{C}^{n+1} \times \mathrm{Mat}_{n+1}(\mathbb{C})$$

be the associated symmetric rational bilinear form with values in $\mathrm{End}(E)$.In fact, it extends to a polynomial bilinear form. Let $e \in \mathbb{C}^{n+1}$ be such that $N(x) \neq 0$ and $j_f(x)$ is invertible. Replacing f by $P_f(e) \circ f$, if needed, we may assume that $P_f(e) = \mathrm{id}_E$. We define a multiplication law on E by setting:

$$x \bullet y = \tfrac{1}{2}P_f(x,y)(e). \tag{7.66}$$

It follows from PirioRussoXJC [591, Theorem 3.4] (the proof relies on McCrimmon [509, Theorem 4.4 and Remark 4.5]) that this multiplication law makes E into a Jordan algebra. If $j_f(x)$ is defined, one checks that $j_f(x) = x^{-1}$, hence $x^* = f(x)$ that implies that E is a Jordan algebra of rank 3.

Since, all quadro-quadratic transformation f in $\mathbb{P}^n, n \leq 3$ are elementary or their degenerations, we will concentrate on the case $n > 3$. The multidegree of f is equal to $(2, d_1, \ldots, d_{n-2}, 2)$. The Cremona inequalities give $d_1^2 \geq 2d_2, d_2^2 \geq d_1d_3, \ldots, d_{n-2}^2 \geq 2d_{n-3}$

brunoverra *Example* 7.5.4. Assume $n = 4$. The only possible multidegrees are $(2,3,2)$ or $(2,4,2)$. Assume $d_2 = 3$. Then, the image of a general plane is a cubic surface in $\mathbb{P}^4$. It must span $\mathbb{P}^4$, and hence it is a cubic ruled surface. We will see in Subsection CAG-2:SS:9.2.1 9.2.1 that it is equal to the image of the minimal ruled surface $\mathbf{F}_1$ by a map given by the linear system $|2\mathfrak{f} + \mathfrak{e}|$. This shows that $f : \mathbb{P}^2 \dashrightarrow \mathbb{P}^4$ is given by the linear system of conics through a fixed point, and hence, one of the irreducible components of the F-locus must be a plane. Other irreducible components must be of dimension ≤ 1.

Now, let H be a general hyperplane in the source $\mathbb{P}^4$. Then, $f(H) = Q$ is a quadric in the target $\mathbb{P}^4$. Composing the map f with the projection from a point on Q, we get a birational map $P^3 \cong H \dashrightarrow \mathbb{P}^3$. Its multidegree is equal to $(2,3)$. We know from Subsection SS:7.4.1 7.4.1 that the transformation must be from Example ex:quadrocubic 7.4.8. Its F-locus is the union of a line and three isolated points. We already saw the line, it is the intersection of H with the F-locus of f. Since one of the base points of $f : H \to Q$ comes from the projection $Q \dashrightarrow \mathbb{P}^3$, we need to add only two lines to the F-locus. Let us count the dimensions. Since the plane in the base locus imposes six conditions and a line three conditions, the lines must be skew lines intersecting the plane.

Let us now check that, indeed, we have a homaloidal linear system. We take a log resolution of f equal to the composition ν of the blow-up $\nu_1 : Y_1 \to \mathbb{P}^4$ along the lines ℓ_1, ℓ_2 followed by the blow-up $\nu_2 : Y_2 \to Y_1$ of the proper transform $\bar{\Pi}$ of the plane Π. Let $E_1 + E_2$ be the exceptional divisor of ν_1. Since $\mathcal{N}_{\ell_i/\mathbb{P}^4} \cong \mathcal{O}_{\ell_i}(1)^{\oplus 3}$, $E_i^4 = -s(\ell_i, \mathbb{P}^4)_0 = 3$. We have $\mathcal{N}_{\bar{\Pi}/Y_1} = \mathcal{O}_{\bar{\Pi}}(\ell)^{\oplus 2}$, where ℓ is the proper transform of the line joining the intersection points $\ell_i \cap \Pi$. Let E be the exceptional divisor of ν_2, we get $E^4 = -s(\bar{\Pi}, Y_1)_0 = -3\ell^2 = 3$. The class of the proper transforms $\bar{E}_i$ of E_i in $A(Y_2)$ is equal to the class $\nu_2^* E_i$, hence, $\bar{E}_i^4 = 3$. We compute other intersections and find $(2H - E - \bar{E}_1 - \bar{E}_2)^4 = 2^4 + 24H^2E^2 + E^4 + \bar{E}_1^4 + \bar{E}_2^4 = 16 - 24 + 9 = 1$. Thus, our linear system is homoloidal.

Assume $d_2 = 4$. Similar arguments show that the restriction of the map to a general plane is given by a linear system of conics and its image is a quartic surface. This implies that the surface is a projection of a Veronese surface, and the F-locus is of dimension ≤ 1. Take a general hyperplane, its image under f is a quadric with two base points, and composing with the projection to $\mathbb{P}^3$, we get a birational map $\mathbb{P}^3 \dashrightarrow \mathbb{P}^3$ with isolated base points. Now, we apply Beloch's classification of such transformations from Remark 7.4.5. There is only one quadratic transformation that was already discussed in Example 7.4.4. Its F-locus consists of 4 points and the quadrics are tangent to a fixed plane at one of the points. It follows that the F-locus of f is a curve of degree 3. We refer to [76], where it is shown that the F-locus is equal to the union of a conic C and a line L intersecting at one point such that all quadrics from the homaloidal linear system are tangent to a hyperplane H tangent to C.

fano *Example* 7.5.5. An easy example of a quadro-quadratic transformation in $\mathbb{P}^5$ is the adjugation involution Adj_2 on space of conics which we considered in Section 2.4. Its base scheme is a Veronese surface. Since the image of a general plane is a surface of degree 4, the multi-degree of the involution is equal to $(2, 4, 4, 2)$.

The next example of a quadro-quadratic transformation in $\mathbb{P}^5$ is more interesting. Consider three planes pairwise intersecting at one point. We can choose coordinates to assume that the planes are $V(t_1, t_2, t_3), V(t_0, t_2, t_4), V(t_0, t_1, t_5)$. Consider the linear system $|V| \subset |\mathcal{O}_{\mathbb{P}^3}(2)|$ of quadrics with the base locus equal to the union of the planes. The dimension of this linear system is equal to 5 and one can find a basis formed by the monomials such that the transformation is given by the formula:

$$f : [t_0, \dots, t_5] \mapsto [-t_1t_2, -t_0t_2, -t_0t_1, t_0t_3, t_1t_4, t_2t_5]. \tag{7.67}$$ basisgiz

In this basis, the transformation f is a Cremona involution. One can find a log resolution of the rational transformation f given by the linear system as the composition of the blow-up $\nu_1 : Y_1 \to \mathbb{P}^3$ with center at the set of three intersection points, followed by the blow-up $\nu_2 : Y_2 \to Y_1$ of the proper transforms of the planes. The proper transform $\bar{\Pi}_i$ of each plane in the first blow-up is isomorphic to the minimal ruled surface $\mathbf{F}_1$. It has the normal sheaf isomorphic to $\mathcal{O}_{\bar{\Pi}_i}(\mathfrak{f}_i)^{\oplus 3}$, where $\mathfrak{f}_i$ is the divisor class of a ruling. We have $s(\bar{\Pi}_i, Y_1) = (1 + \mathfrak{f}_i)^{-3}$. The proper transform of $|V|$ in y_1 is equal to $|2h_1 - E_1 - E_2 - E_3|$, where h_1 is the divisor class of the pre-image of a hyperplane in X_1 and $E_1 + E_2 + E_3$ is the exceptional divisor. Let $Z = \sum \bar{\Pi}$, we get $s(Z, Y_1)_1 = -3\sum \mathfrak{f}_i, s(Z, Y_1)_2 = Z$. This easily gives $d_5 = 2^5 - 3 + 30 = 1$. Thus, $|V|$ is homoloidal. Since f is an involution, the multidegree is equal to $(2, a, a, 2)$. Here a is equal to the degree of the image of a 3-dimensional subspace of $\mathbb{P}^5$. The map is given by a web of quadrics through three points, and hence, the degree is equal to 5.

It was observed by Gizatullin [336] that the Cremona involution of multidegree $(2, 5, 5, 2)$ can be viewed as the action of the standard planar quadratic Cremona transformation on the 5-dimenensional space of conics. Applying Noether's factorization theorem, we obtain that the subgroup of the group $\mathrm{Cr}(5)$ of Cremona transformations of $\mathbb{P}^5$ generated by f and the group of projective transformations is isomorphic to $\mathrm{Cr}(2)$.

Let us look at the corresponding Jordan algebra of dimension 6. Straightforward computation show that, we can take $e = (1, 1, 1, 0, 0, 0)$ and obtain

$$x \bullet_f y = \tfrac{1}{2}(2x_0y_0, 2x_1y_1, 2x_2y_2,$$

$$x_3(y_1 + y_2) + y_3(x_1 + x_2), x_4(y_0 + y_2) + y_4(x_0 + x_2), x_5(y_0 + y_1) + y_5(x_0 + x_1)).$$

We verify that

$$(t_0, t_1, t_2, t_3, t_4, t_5) \bullet_f f(t_0, t_1, t_2, t_3, t_4, t_5) = -t_0t_1t_2 \cdot e,$$

confirming that $f(t) = t^*$ with $N = -t_0t_1t_2$.

The following generalization of the transformation f can be found in [589, 2.2.1]. Let $A = A_1 \oplus A_2 \oplus A_3$ be the direct sum of linear spaces of dimensions n_1, n_2, n_3. Define

$$T_A : (t_0, x_1, x_2, \mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3) \mapsto (t_1t_2, t_0t_2, t_0t_2, t_0\mathbf{a}_1, t_1\mathbf{a}_2, t_2\mathbf{a}_3),$$

where $\mathbf{a}_i \in A_i$. Our transformation f is the special case with $n_1 = n_2 = n_3 = 1$. One check that the cubic N is equal to $t_0t_1t_2$.

Remark 7.5.6. The analog of a twisted cubic for a Jordan algebra $\mathbb{J}$ of dimension r is the map

$$|\mathbb{J}| \to |\mathbb{C} \oplus \mathbb{J} \oplus \mathbb{J} \oplus \mathbb{C}, \quad [x] \mapsto [1, x, x^*, N(x)].$$

The image of this map is an irreducible subvariety of $X \subset \mathbb{P}^{2r+1}$. It satisfies the property that any three general points lie on a twisted cubic [591], [643]. In is proved in loc. cit. that any variety with this property arises from some Jordan algebra of rank 3.

For example, taking $\mathbb{J}$ to be one of the Jordan algebras arising from Hurwitz's composition algebras, we get:

- $\mathbb{R}$: $X = \mathrm{LG}_3(\mathbb{C}^6) \subset \mathbb{P}^{13}$, 6-dimensional Lagrangian Grassmannian,
- $\mathbb{C}$: $X = G(3, 6) \subset \mathbb{P}^{19}$, the 9-dimensional Grassmannian,
- $\mathbb{H}$: $X = OG_6(\mathbb{C}^{12}) \subset \mathbb{P}^{31}$, 15-dimensional orthogonal Grassmannian,
- $\mathbb{O}$: X is the E_7-variety in $\mathbb{P}^{55}$ of dimension 27.

All these homogenous varieties are isomorphic to minimal closed orbits of a simple algebraic group G acting in the projetivization of its irreducible linear representation:

1. $(\mathrm{Sp}(6), \mathbb{C}^{14})$,
2. $(\mathrm{SL}(6), \bigwedge^3 \mathbb{C}^6)$,
3. $(\mathrm{SO}(6)$, the spin representation $\mathbb{C}^{2^5})$,
4. $(E_7, \mathbb{C}^{56})$.

The last variety is defined to be the minimal closed orbit of the exceptional simple group G of type E_7 acting in the projectivization of its minuscule 56-dimensional irreducible linear representation. In all these examples, $X = \mathrm{Sing}(F_4)$, where F_4 is a certain quartic hypersurface. In the last case, it is the famous *Cartan quartic* whose equation is given by a generator of the algebra of invariants of G in $\mathbb{C}^{56}$. This should be compared with the corresponding classification of quadro-quadratic Cremona tranformations with smooth connected base schemes given in Theorem 7.4.11. In this case, X coincide with the singular locus of a certain cubic hypersurface. In the last case of the classification, the cubic is the famous Cartan cubic hypersurface (see Remark 9.1.15).

A *Holy Grail Problem* is to find the family of 3-dimensional Calabi-Yau manifolds Y such that the cohomology $H^2(Y,\mathbb{C}) \cong \mathbb{J}$ realize the representation (G,V) from Theorem 7.4.11 and the cohomology $H^3(Y^m,\mathbb{C}) \cong \mathbb{C}\oplus\mathbb{J}\oplus\mathbb{J}\oplus\mathbb{C}$ realizes the representation (G,V) from above. The direct sum decomposition is the Hodge decomposition. The cubic form on $\mathbb{J}$ should correspond to the Yukawa coupling in $H^{2,1}(Y^m,\mathbb{C}) = \mathbb{J}$ and the cubic cup-product on $H^2(Y,\mathbb{C})$.

7.6 Bilinear Cremona Transformations

S:7.6

Here, we encounter again aCM sheaves that we used in Chapter 4.

Definition 7.6.1. *A closed subscheme Z of $\mathbb{P}^n$ of pure dimension r is called* arithmetically Cohen-Macaulay *(aCM for short) if its ideal sheaf $\mathcal{J}_Z$ is an aCM sheaf.*

Assume that $\operatorname{codim} Z = 2$. Then, as in Chapter 4, we obtain a locally free resolution

$$0 \to \bigoplus_{i=1}^{m} \mathcal{O}_{\mathbb{P}^n}(-a_i) \to \bigoplus_{j=1}^{m+1} \mathcal{O}_{\mathbb{P}^n}(-b_j) \to \mathcal{J}_Z \to 0 \qquad (7.68)$$ res1

for some sequences of integers (a_i) and (b_j).

The numbers (a_i) and (b_j) are determined from the Hilbert polynomials of Z.

We will consider a special case of resolution of the form (4.16) which we used in the theory of linear determinantal representations of hypersurfaces:

$$0 \to U^\vee_{\mathbb{P}^n}(-n-1) \to V_{\mathbb{P}^n}(-n) \to \mathcal{J}_Z \to 0, \qquad (7.69)$$ exseq1

where U,V are linear spaces of dimensions n and $n+1$, respectively. By twisting the exact sequence, and taking cohomology, we obtain natural isomorhisms

$$U \cong H^{n-1}(\mathbb{P}^n,\mathcal{J}_Z), \quad V \cong H^0(\mathbb{P}^n,\mathcal{J}_Z(n)).$$

The resolution of $\mathcal{J}_Z$ allows one to compute the Hilbert polynomial of the subscheme Z. We get

$$\chi(\mathcal{O}_Z(k)) = \chi(\mathcal{O}_{\mathbb{P}^n}(k)) - \chi(\mathcal{J}_Z(k)) = \binom{n+k}{n} - \binom{k}{n} - n\binom{k-1}{n-1}. \qquad (7.70)$$ hilb1

It also defines an isomorphism between Z and the determinantal variety given by the linear map

$$\phi : E \to U \otimes V, \qquad (7.71)$$ newphi

where $\mathbb{P}^n = |E|$. In coordinates, it is given by $n \times (n+1)$ matrix A with linear functions on E as its entries. The maximal minors of A generate the homogeneous ideal of Z. Let

$$\mathfrak{r} : E \to V^\vee$$

be the right kernel maps. It defines the rational maps of projective spaces

$$|\mathfrak{r}| : |E| \dashrightarrow \mathbb{P}(V).$$

hilb *Remark* 7.6.2. The Hilbert scheme of an aCM subscheme Z of $\mathbb{P}^n$ admitting a resolution (7.69) is isomorphic to an open subset of the projective space of $(n+1) \times n$ matrices $A(t)$ of linear forms such that the rank of $A(t)$ is equal to n for an open non-empty subset of $\mathbb{P}^n$. It is a connected smooth variety of dimension $n(n^2-1)$ (see [581] or [285]).

P3.4.2 **Theorem 7.6.3.** *The map $T_\phi = |\mathfrak{r}| : |E| \dashrightarrow \mathbb{P}(V)$ is a birational map with base scheme Z. Its multi-degree is equal to $(d_k) = (\binom{n}{k})$.*

Proof In coordinates, the map $|\mathfrak{r}|$ is defined by $n \times n$ minors of the matrix A. The subscheme Z is given scheme-theoretically by these minors. In particular, we already see that the degree of the map is equal to n. Let us view the linear map ϕ as an element of the tensor product $E^\vee \otimes U \otimes V$. Consider it as a linear map

$$\psi : E \otimes V^\vee \to U. \qquad (7.72)$$ psi

It may be considered as a collection of n bilinear forms on $E \otimes V^\vee$. It is immediate that $v^* = \mathfrak{r}(e)$ for some $v^* \in V^\vee$ and $e \in E$ if and only if $\psi(e \otimes v^*) = 0$. This relation is symmetric, so $v^* = \mathfrak{r}(e)$ if and only if $e = \mathfrak{r}'(v^*)$, where $\mathfrak{r}' : V^\vee \to E$ is the right kernel map for the linear map $\phi' : V^\vee \to U \otimes E^\vee$ defined by applying to the tensor ϕ the isomorphism $E^\vee \otimes U \otimes V \to V^\vee \otimes U \otimes E$. Thus, the map $T_{\phi'} = \mathfrak{r}'$ defines the inverse of T_ϕ.

In coordinates, if we choose a basis $e_0, \ldots, e_n$ in E, a basis $u_1, \ldots, u_n$ in U and a basis $v_0, \ldots, v_n$ in V, then the linear map ϕ can be written as a tensor

$$\phi = a_{ij}^k := \sum_{0 \le k, j \le n; 1 \le i \le n}^{n} a_{ij}^k t_k \otimes u_i \otimes v_j.$$

The matrix A is equal to $t_0 A_0 + \cdots + t_n A_n$, where $A_k = (a_{ij}^k)$. The bilinear map ψ is given by n square matrices $X_i = (a_{ij}^k)$ of size $n+1$, where k is the index for the columns, and j is the index for the rows. The graph of the Cremona map

$|\mathfrak{r}|$ is given by n bilinear equations in $|E| \times \mathbb{P}(V)$

$$\sum_{j,k=0}^{n} t_k v_j a_{ij}^k = 0, \quad i = 1, \ldots, n. \tag{7.73}$$

These equations define the graph of the transformation T_ϕ. Also, note that the matrix B defining the linear map $\phi' : V^\vee \to U \otimes E^\vee$ is equal to $v_0 B_0 + \cdots + v_n B_n$, where $B_j = (a_{ij}^k)$. Here, k is now the row index, and i is the column index.

It is easy to compute the cohomology class of the graph (7.73) of T_ϕ. It is equal to

$$(h_1 + h_2)^n = \sum_{k=0}^{n} \binom{n}{k} h_1^i h_2^{n-i}.$$

□

We can also see another determinantal variety, this time defined by the transpose of (7.72)

$${}^t\psi : U^\vee \to E^\vee \otimes V. \tag{7.74}$$

Let $\mathsf{D}_k \subset \mathbb{P}(U)$ be the pre-image of the determinantal variety of bilinear forms on $E \otimes V^\vee$ (or linear maps $E^\vee \to V$) of rank $\le k$. We have regular kernel maps

$$\mathfrak{l}_\psi : \mathsf{D}_n \setminus \mathsf{D}_{n-1} \to |E|, \quad \mathfrak{r}_\psi : \mathsf{D}_n \setminus \mathsf{D}_{n-1} \to \mathbb{P}(V).$$

By definition, the image of the first map is equal to the base scheme Z of the rational map $|\mathfrak{r}|$ considered in the previous theorem. The image of the second map is of course the base scheme of the inverse map. In particular, we see that the base schemes of T_ϕ and T_ϕ^{-1} are birationally isomorphic to the variety D_n.

Note the special case where $E = V^\vee$ and the image of ${}^t\psi$ is contained in the space of symmetric bilinear maps $E \times V^\vee \to \mathbb{C}$. In this case,

$$T_\phi = T_\phi^{-1}.$$

The bilinear map is given by a $(n-1)$-dimensional linear system $|L| \subset |\mathcal{O}_{\mathbb{P}^n}(2)|$ of quadrics in $\mathbb{P}^n$. It assigns to a general point $x \in \mathbb{P}^n$, the intersection $\cap_{Q \in |L|} P_x(Q)$ of polars of quadrics with the pile at x. It is clear that the base scheme of f coincides with the base scheme of $|L|$ in this case. If $n = 2$, this is a quadratic involution conjugate to the standard Cremona involution T_{st}.

Example 7.6.4. Consider the *standard Cremona transformation* of degree n in $\mathbb{P}^n$ given by

$$T_{\mathrm{st}} : [t_0, \ldots, t_n] \mapsto [\frac{t_0 \cdots t_n}{t_0}, \ldots, \frac{t_0 \cdots t_n}{t_n}]. \tag{7.75}$$

In affine coordinates, $z_i = t_i/t_0$, it is given by the formula

$$(z_1, \ldots, z_n) \mapsto (z_1^{-1}, \ldots, z_n^{-1}).$$

The transformation T_{st} is an analog of the standard quadratic transformation of the plane in higher dimension.

The base ideal of T_{st} is generated by $t_1 \cdots t_n, \ldots, t_0 \cdots t_{n-1}$. It is equal to the ideal generated by the maximal minors of the $n \times n$ matrix

$$A(t) = \begin{pmatrix} t_0 & 0 & \cdots & 0 \\ 0 & t_1 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & t_{n-1} \\ -t_n & -t_n & \ldots & -t_n \end{pmatrix}.$$

The $(n-1)$-dimensional linear system of quadrics is spanned by the quadrics $V(t_k^2 - t_n^2), k = 0, \ldots, n-1$.

The base scheme of T_{st} is equal to the union of the coordinate subspaces of codimension 2.

It follows from the proof of Theorem 7.6.3 that the graph of T_{st} is isomorphic to the closed subvariety X of $\mathbb{P}^n \times \mathbb{P}^n$ given by n bilinear equations

$$t_i y_i - t_n y_n = 0, \quad i = 0, \ldots, n-1.$$

It is isomorphic to the blow-up of the union of coordinate subspaces of codimension 2. The action of the torus $(\mathbb{C}^*)^{n+1}$ on $\mathbb{P}^n$ (by scaling the coordinates) extends to a biregular action on X. In the case $n = 2$, the toric surface X is a del Pezzo surface of degree 6 isomorphic to the blow-up of 3 points in the plane, no three of which are collinear. In $n > 2$ the variety is singular.

Example 7.6.5. Let $\alpha : U^\vee \to E^\vee \otimes V$ be a linear determinantal representation of a nonsingular plane quartic $C \subset \mathbb{P}(U) \cong \mathbb{P}^2$ given by the linear system $|K_C + a|$. The image Z of C in $|E|$ under the right kernel map $\mathfrak{r}$ is a curve Z of degree 6 and genus 3. Let $\phi : E \to U \otimes V$ be the linear map obtained from the tensor $\phi \in U \otimes E^\vee \otimes V$. Then, the bilinear Cremona transformation $|E| \to \mathbb{P}(V)$ defined by this map is given by cubic polynomials generating the ideal of Z. Note that Z is an aCM subscheme of $|E| \cong \mathbb{P}^3$. Its Hilbert polynomial is $6t - 2$ in agreement with (7.70). Conversely, any irreducible and reduced curve of degree 6 and arithmetic genus 3 not lying on a quadric is arithmetically Cohen-Macaulay and admits a resolution of type (7.69) (see [285, p. 430]). Assume Z is arithmetically Cohen-Macaulay. The bilinear Cremona transformation defined by such a curve is classically known as a *cubo-cubic transformations* (see [701]).

In fact, an example of a standard Cremona transformation in $\mathbb{P}^3$ shows that one can often drop the assumption that Z is an integral curve. In this example, Z is the union of 6 coordinate lines, and is a curve of degree 6 and of arithmetic genus 3, and it does not lie on a quadric. Another example of this sort is when Z is the union of 4 skew lines and two lines intersecting them. There are examples when Z is not reduced, e.g. with the reduced scheme equal to a rational normal curve. I do not know whether any closed subscheme Z of degree 6 (in the sense that $[Z] = 6[\text{line}]$) with $h^0(\mathcal{O}_Z) = 1, h^1(\mathcal{O}_Z) = 3$, and not lying on a quadric surface, admits a resolution of type (7.69).

Assume Z is a smooth curve and let us describe the P-locus of the corresponding Cremona transformation. Obviously, any line intersecting Z at three distinct points (a *trisecant line*) must be blown down to a point (otherwise a general cubic in the linear system intersects the line at more than 3 points). Consider the surface $\mathrm{Tri}(Z)$ of Z, the closure in $\mathbb{P}^3$ of the union of lines intersecting Z at three points. Note that no line intersects Z at > 3 points because the ideal of Z is generated by cubic surfaces. Consider the linear system of cubics through Z. If all of them are singular, by Bertini's Theorem, there will be a common singular point at the base locus, i.e. at Z. But this easily implies that Z is singular, contradicting our assumption. Choose a nonsingular cubic surface S containing Z. By the adjunction formula, we have $Z^2 = -K_S \cdot Z + \deg K_Z = 6 + 4 = 10$. Take another cubic S' containing Z. The intersection $S \cap S'$ is a curve of degree 9, the residual curve A is of degree 3 and $Z + A \sim -3K_S$ easily gives $Z \cdot A = 18 - 10 = 8$. Note that the curves A are the proper transforms of lines under the Cremona transformation. They are rational curves of degree 3. We know that the base scheme of the inverse transformation f^{-1} is a curve of degree 6 isomorphic to Z. Replacing f with f^{-1}, we obtain that the image of a general line ℓ under f is a rational curve of degree 3 intersecting Z' at eight points. These points are the images of eight trisecants intersecting ℓ. This implies that the degree of the trisecant surface $\mathrm{Tri}(Z)$ is equal to 8. Since the degree of the determinant of the Jacobian matrix of a transformation of degree 3 is equal to 8, we see that there is nothing else in the P-locus.

The linear system of planes containing a trisecant line ℓ cuts out on Z a linear series of degree 6 with moving part of degree 3. It is easy to see, by using Riemann-Roch, that any g^1_3 on a curve of genus 3 must be of the form $|K_Z - x|$ for a unique point $x \in Z$. Conversely, for any point $x \in Z$, the linear system $|\mathcal{O}_Z(1) - K_Z + x|$ is of dimension 0 and of degree 3 (here we use that $|\mathcal{O}_Z(1)| = |K_Z + \mathfrak{a}|$, where $\mathfrak{a}$ is not effective divisor class of degree 2). Thus, it defines a trisecant line (maybe a tangent line at some point). This shows that the curve R parameterizing trisecant lines is isomorphic to Z. This agrees with the fact that R must be isomorphic to the base curve of the inverse transformation. The Cremona transformation can be resolved by blowing up the curve Z and then blowing down the proper transform of the surface $\mathrm{Tri}(Z)$. The exceptional divisor is isomorphic to the minimal ruled surface with the base curve equal to Z. It is the universal family of lines parameterized by Z. Its image in the target $\mathbb{P}^3$ is the surface $\mathrm{Tri}(Z')$, where Z' is the base locus of the inverse transformation (the same curve, only re-embedded by the linear system $|K_Z + a'|$, where $a' \in |K_Z - a|$).

arguesiancubic *Example* 7.6.6. Here, we will discuss an example of a non-reduced ACM curve of degree 6 and arithmetic genus 3.

Fix an isomorphism $\nu : R_3 \to \mathbb{P}^1$ and let $\mathcal{L}_n = \nu^*(\mathcal{O}_{\mathbb{P}^1}(n))$. Since $h^0(\mathcal{N}_{R_3/\mathbb{P}^3}) = 12$ (=the number of projective parameters for twisted cubics in $\mathbb{P}^3$) and $c_1(\mathcal{N}_{R_3/\mathbb{P}^3}) = 10$, we easily find that

$$\mathcal{N}_{R_3/\mathbb{P}^3} \cong \mathcal{L}_5^{\oplus 2}. \tag{7.76}$$ normaltwistedcubic

Choose a surjection $u : \mathcal{L}_{-5}^{\oplus 2} \to \mathcal{L}_{-4}$. Let $\bar{u} : \mathcal{I}_{R_3} \to \mathcal{I}_{R_3}/\mathcal{I}_{R_3}^2 \to \mathcal{L}_{-4}$ be the composition of u with the surjection $\mathcal{I}_{R_3} \to \mathcal{I}_{R_3}/\mathcal{I}_{R_3}^2$. It kernel defines a closed subscheme Z of $\mathbb{P}^3$ with $Z_{\mathrm{red}} = R_3$. By definition, we have an exact sequences

$$0 \to \mathcal{I}_Z \to \mathcal{I}_{R_3} \xrightarrow{\bar{u}} \mathcal{L}_{-4} \to 0,$$

and

$$0 \to \mathcal{L}_{-4} \to \mathcal{O}_Z \to \mathcal{O}_{R_3} \to 0.$$

Tensoring by $\mathcal{O}_{\mathbb{P}^3}(k)$, and taking cohomology, we obtain $\chi(\mathcal{O}_Z(k)) = \chi(\mathcal{O}_{R_3}(k)) + \chi(\mathcal{O}_{\mathbb{P}^1}(3k-4)) = 6k-2$. Thus, $p_a(Z) = 3$ and $\deg(Z) = 6$. We also find that $h^0(\mathcal{O}_Z(k)) = 6k-2, k \geq 0$, hence Z is an aCM-subscheme. We have $h^0(\mathcal{I}_Z(3)) = h^0(\mathcal{O}_{\mathbb{P}^3}(3) - h^0(\mathcal{O}_Z(3)) = 20 - 16 = 4$, and, as in the previous example, we check that $|\mathcal{I}_Z(3)|$ is a homaloidal linear system. One can find a log resolution of the base scheme Z by considering the composition of the blow-up R_3 followed by the blow-up of the section C of the exceptional divisor $E = \mathbb{P}(\mathcal{I}_{R_3}/\mathcal{I}_{R_3}^2)$ defined by the surjection u. We may consider C as a curve infinitely near to R_3. We refer to [574] (PanRusso) for more examples of Cremona transformation with the non-reduced base scheme obtained in a similar way.

Let $\sigma : X = \mathsf{Bl}_{R_3}(\mathbb{P}^3) \to \mathbb{P}^3$. The linear system $|I_{R_3}(2)|$ lifts to X and defines a $\mathbb{P}^1$-bundle structure $\pi : X \to \mathbb{P}^2$ on X. The pre-image of a point in $\mathbb{P}^2 = \mathbb{P}(H^0(R_3, I_{R_3}(2)))$ is a secant line of R_3, the residual to R_3 in the corresponding pencil of quadrics. The restriction of π to E_1 is the second ruling of E, the first one is defined by σ_{E_1}. The exceptional divisor E_1 of X is isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$. Let $\mathfrak{f}_1$ be the divisor class of the fiber of the projection $\sigma_E : E \to R_3$ and $\mathfrak{f}_2$ is the divisor class of the other ruling. If we write

$E_1 = \mathbb{P}(\mathcal{L}_{-5}^{\oplus 2})$, then $c_1(\mathcal{O}(1)) = -5\mathfrak{f}_1 + \mathfrak{f}_2$. The restriction of $2H - E_1$ to E_1 is linearly equivalent on E_1 to $6\mathfrak{f}_1 + (-5\mathfrak{f}_1 + \mathfrak{f}_2) = \mathfrak{f}_1 + \mathfrak{f}_2$. This shows that the map π restricted to E_1 is a double cover $\pi_E : E \to \mathbb{P}^2$. The pre-image of a point under π_E are the two ends of the corresponding secant of R_3.

Using Proposition 2.4.2, we find $u^*\mathcal{N}_{C/X} \cong \mathcal{L}_2$. Thus C is a section of $\sigma_{E_1} : E_1 \to R_3$ with self-intersection equal to 2. It implies that the divisor class of C on E is equal to $\mathfrak{f}_1 + \mathfrak{f}_2$. So, a choice of the surjection u defining Z corresponds to a choice of a divisor of bidegree $(1, 1)$ on E_1. The image of C under $\pi_E : C \to \mathbb{P}^2$ is isomorphic to the projection of a conic on a quadric from a point outside of the quadric. It is a line ℓ_C in the plane or the branch conic. In the latter case, the conic is cut out by the polar of the quadric with the pole equal to the center of the projection. The line ℓ_C defines a quadric Q_Z from $|\mathcal{I}_{R_3}(2)|$ contradicting the property of Z to be an aCM-subscheme. Thus, the image of C is a conic. Its pre-image in $\mathbb{P}^3$ is the developable quartic of R_3 swept by tangent lines to R_3 (see Subsection 10.4.4 and Remark 10.4.24). The surface is the P-locus of the Cremona transformation f. It is a substitute of the octic ruled surface in the case of a smooth aCM-curve Z of genus 3 and degree 6. The lift of f to the log resolution $\mathsf{Bl}_C(X) \to \mathbb{P}^3$ blows down the proper transform of E_1 to twisted cubic in the target $\mathbb{P}^3$ and maps E_2 to its developable quartic surface.

The cubic-cubic Cremona transformation, under an appropriate choices of bases in the source and the target $\mathbb{P}^3$ coule be made into an ibvolution. It assigns to a geenral point x in $\mathbb{P}^3$ the point $f(x)$ such that the pair $\{x, f(x)\}$ is harmonically conjugate to the pair of the end-points of the unique secant of R_3 containing x (see [288]).

Remark 7.6.7. Let Z be a closed aCM subscheme of codimension two in $\mathbb{P}^5$ defined by a resolution

$$0 \to \mathcal{O}_{\mathbb{P}^5}(-4)^3 \to \mathcal{O}_{\mathbb{P}^5}(-3)^4 \to \mathcal{I}_Z \to 0.$$

It is a determinantal variety in $\mathbb{P}^5$ with the right kernel map $\mathfrak{r} : Z \to \mathbb{P}^2$ isomorphic to a projective bundle $\mathbb{P}(\mathcal{E})$, where $\mathcal{E}$ is a rank 2 bundle on $\mathbb{P}^2$ with $c_1(\mathcal{E}) = 0$ and $c_2(\mathcal{E}) = 6$ (see [560], [562]). Thus, Z is a scroll in $\mathbb{P}^5$, called a *Bordiga scroll*. A general hyperplane section of Z is a surface S of degree 6 in $\mathbb{P}^4$ with ideal sheaf defined by a resolution

$$0 \to \mathcal{O}_{\mathbb{P}^4}(-4)^3 \to \mathcal{O}_{\mathbb{P}^4}(-3)^4 \to \mathcal{I}_S \to 0.$$

It is a determinantal surface in $\mathbb{P}^4$ with the right kernel map $\mathfrak{r} : S \to \mathbb{P}^2$ isomorphic to the blow-up of 10 points in $\mathbb{P}^2$. The embedding of S in $\mathbb{P}^4$ is given by the linear system of quartic curves passing through the ten points. The surface S is of degree 6, classically known as a *Bordiga surface* [65]. Finally, a general hyperplane section of S is a sextic of genus 3 in $\mathbb{P}^3$ discussed in Example 7.6.5.

7.7 Involutions in the Cremona group

The Cremona group $\mathrm{Cr}_{\Bbbk}(n)$ is the group of Cremona transformation of $\mathbb{P}^n_{\Bbbk}$. Equivalently, it is the group of automorphisms of the field $\Bbbk(x_1, \ldots, x)$ of rational functions with coefficients in $\Bbbk$ that leaves constants unchanged. In this section, we will give only a brief discussion about the structure of the group $\mathrm{Cr}(n) = \mathrm{Cr}_{\mathbb{C}}(n)$, the subject is too enormous to include in the present book. We restrict ourselves only to discussion of elements of order 2 and give some references to the literature at the end of the chapter.

7.7.1 Planary involutions

We have already encountered examples of Cremona involutions in Subsections 7.2.2 (de Jonquières involutions) and 7.2.6 (Geiser and Betini involutions).

Before we show that any involution is conjugate to one of those, we have to introduce some general approach to the classification of finite subgroups of $\mathrm{Cr}(2)$. We refer for details and references to [252].

Lemma 7.7.1. *Let G be a finite subgroup of* $\mathrm{Cr}(2)$. *There exists a birational map $\phi : S \to \mathbb{P}^2$ of smooth projective surfaces such that $\phi^{-1} \circ G \circ \phi \subset \mathrm{Aut}(S)$.*

Proof Let $U = \cap_{g \in G}\mathrm{dom}(g)$. It is an open subset of $\mathbb{P}^2$ on which G acts. Let $U' = U/G$ be the orbit space. It is a normal algebraic surface. Choose any normal projective completion X' of U'. Let S' be the normalization of X' in the field of rational functions of U. This is a normal projective surface on which G acts by biregular transformations. It remains to define S to be a G-invariant resolution of singularities (see [489]).

□

We say that G lifts biregularly to S (or regularizes on S).

The lemma shows that one can classify the conjugacy classes of finite subgroups G of $\mathrm{Cr}(2)$ by classifying all possible finite groups of automorphisms of smooth rational surfaces. Given such a surface S and $G \subset \mathrm{Aut}(S)$, we choose a birational map $\phi : S \to \mathbb{P}^2$ and obtain a finite subgroup $\phi \circ G \circ \phi^{-1}$ of $\mathrm{Cr}(2)$. Its conjugacy class does not depend on a choice of ϕ. The lemma shows that any finite subgroup of $\mathrm{Cr}(2)$ is obtained in this way.

The next crucial step is to choose the pair (S, G) minimal in the category of all G-equivariant birational morphisms $f : (S, G) \to (S', G)$. The following theorem, due to Yu. I. Manin, that clarifies the classical result of S. Kantor Kantor [438], describes all minimal G-surfaces. Its more modern proof is based on Mori's theory of minimal models.

minimal **Theorem 7.7.2.** *Let (S, G) be a minimal G-surface. Then,* $\mathrm{Pic}(S)^G \cong \mathbb{Z}$ *or* $\mathbb{Z}^{\oplus 2}$. *In the first case, S is a del Pezzo surface of degree $d = K_S^2 \neq 7, 8$. In the second case, S is a conic bundle, i.e. there exists a morphism $\phi : S \to \mathbb{P}^1$ such that its general fiber is isomorphic to $\mathbb{P}^1$ and any singular fiber is the union of two (-1)-curves intersecting transversally at one point.*

Let $f \in \mathrm{Cr}(2)$ with $f^2 = \mathrm{id}$. Assume we are in the first case. Let σ be a biregular lift of f to some $\sigma \in \mathrm{Aut}(S)$. If $S = \mathbb{P}^2$, then σ is conjugate to the transformation $(t_0, t_1, t_2) \mapsto (-t_0, t_1, t_2)$. In affine coordinates, it acts as $(x, y) \mapsto (-x, -y)$. Since the involutions $x \mapsto -x$ and $x \mapsto 1/x$ are conjugate in $\mathrm{Aut}(\mathbb{P}^1)$ (by means of the map $x \mapsto \frac{x-1}{x+1}$), we see that T_{st} is conjugate to a projective involution. Since T_{st} is an example of a de Jonquières transformation, we may assume that the minimal surface $S \neq \mathbb{P}^2$.

we will show in the next chapter that $1 \leq d = K_S^2 \leq 9$ and a del Pezzo surface of degree $d = 7$ is isomorphic to the blow-up of two points in $\mathbb{P}^2$. A surface of degree 8 is isomorphic to $\mathbf{F}_0$ or $\mathbf{F}_1$. Assume that S is a del Pezzo surface of degree d with $\mathrm{Pic}(S)^G \cong \mathbb{Z}$. If $d = 9$, then $S \cong \mathbb{P}^2$. If $d = 7$ or $S \cong \mathbf{F}_1$, S contains an invariant curve with negative self-intersection, hence $\mathrm{Pic}(S)^G$ is of rank ≥ 2. If $S \cong \mathbf{F}_0, \mathrm{Pic}(S)^G \cong \mathbb{Z}^2$.

Assume that S is a del Pezzo surface of degree $d \leq 6$ with $\mathrm{Pic}(S)^{(\sigma)} = \mathbb{Z}$. Since $K_S \in \mathrm{Pic}(S)^{(\sigma)}$, we obtain that σ acts as $-\mathrm{id}$ on the orthogonal complement $K_S^\perp$ of K_S. Since an element of the kernel of $\rho : \mathrm{Aut}(X) \to W(S)$ (non-trivial only in the case $d = 6$) acts identically on $\mathrm{Pic}(S)$, we may assume that $\rho(\sigma)$ is an element of order 2 in $W(S)$ that acts as the minus identity on $K_S^\perp \cong \mathsf{E}_{9-d}$. The only case where the Weyl group contains an element that acts as $-\mathrm{id}$ on the root lattice is the case $d = 2$ (resp. 1). We explained in Subsection SS:7.2.6 7.2.6 that the involution σ in this case is conjugate to a Geiser (resp. Bertini) involution.

Let us now assume now that $\mathrm{Pic}(S)^G \cong \mathbb{Z}^2$, i.e., S admits a structure of a conic bundle $\phi : S \to \mathbb{P}^1$. Let

$$\begin{array}{ccc} S & \xrightarrow{\mu} & \mathbf{F}_n \\ & {\scriptstyle\phi}\searrow \quad \swarrow{\scriptstyle\nu} & \\ & \mathbb{P}^1 & \end{array}$$

be the birational map of conic bundles onto its relative minimal model. We assume that n is chosen here as minimal as possible.

Suppose μ is an isomorphism $S \to \mathbf{F}_n$. Obviously, $n \neq 1$ since we can blow down equivariantly the exceptional section to obtain $\mathbb{P}^2$. If $n \geq 2$, then σ leaves invariant the exceptional section $\mathfrak{e}$. If σ acts identically on $\mathfrak{e}$, then it has one fixed point on each fiber not lying on $\mathfrak{e}$. Applying the elementary transformation e_x at any of the se points, we obtain a σ-equivariant birational morphism $S \dashrightarrow \mathbf{F}_{n-1}$. By minimality of n, we obtain that n must be equal to $zero$.

If $S = \mathbf{F}_0 \cong \mathbb{P}^1 \times \mathbb{P}^1$, then σ either belongs to the connected component of $\mathrm{Aut}(S)$ or it is conjugate to the involution that switches two factors of $\mathbb{P}^1 \times \mathbb{P}^1$. In the first case, σ is conjugate to either $(x, y) \mapsto (1/x, y)$ or $(x, y) \mapsto (1/x, 1/y)$. In the first case, it is conjugate to the projective involution $(x_0, x_1, x_2) \mapsto (-x_0, x_1, x_2)$. In the second case, it is conjugate to the standard Cremona T_{st}. so, the transformations are conjugate. If σ is the switch, it is conjugate to the standard to the projective involution $(x_0, x_1, x_2) \mapsto (x_0, x_2, x_1)$.

Therefore, we may assume that $\mu : S \to \mathbf{F}_n$ is not an isomorphism. Hence, the conic bundle has $k > 0$ singular fibers $F_i = R_i^+ + R_i^-$. Let E be a section of ϕ with minimal self-intersection $-m$. We assume that E intersects R_i^+ intersects E. Then, we blow down the components R_i^- and get a minimal ruled surface isomorphic to $\mathbf{F}_m$. By minimality assumption on n, we get $m \geq n$. On the other hand, the proper transform of the exceptional section $\mathfrak{e}$ of $\mathbf{F}_n$ has self-intersection $\leq -n$. This shows that $m = n$, and S is obtained by blowing up k points on $\mathbf{F}_n$ not lying on $\mathfrak{e}$.

The image $\sigma(R_i^\pm)$ of any irreducible component of a singular fiber, is an irreducible component of some fiber F_j. If $j \neq i$, the pair $R_i^\pm + \sigma(R_i^\pm)$ can be equivariantly blown down contradicting the minimality of (S, σ). Obviously, $\sigma(R_i^\pm) \neq R_i^\pm$. The remaining possibilities is that $\sigma(R_i^+) = R_i^-$ for all i.

Let $E' = \sigma(E)$, we may assume that E intersects each R_i^+ and E' intersects each R_i^-. Since that are sections they do not pass through We can find a basis of $\mathrm{Pic}(S)_\mathbb{Q}$ in the form $\mathbf{f}, K_S, R_1^+, \ldots, R_k^+$, where $\mathbf{f}$ is the divisor class of a fiber. Intersecting $E' - E$ with each element of the basis, we easily find $[E'] - [E'] = \frac{1}{2}k\mathbf{f} - \sum_{i=1}^k R_i^+$. This gives $(E' - E)^2 = -2n - 2E \cdot E' = -k$, hence that $k = 2k_0$ and

$$E \cdot E' = k_0 - n. \tag{7.77}$$ intformula1

Lemma 7.7.3. *An involution σ of S with $\mathrm{Pic}(S)^{(\sigma)} = \mathbb{Z}^2$ acts identically on the base of the conic bundle.*

Proof Suppose it does not act identically. Then σ leaves invariant exactly two fibers of ϕ and switches over fibers in pairs. As we saw above, the minimality assumption on (S, σ) implies that there are only two singular fibers, and they are invariant with respect to σ. The singular points of the two singular fibers are the fixed points of σ on S, Applying (7.77), we obtain that $E \cdot E' = 1 - n$. If $n = 0$, the point $E \cap E'$ is fixed by σ, but it cannot be a singular point of a fiber. Thus, $n = 1$, and we obtain two disjoint (-1)-curves on S switched by σ. This contradicts the minimality of (S, σ). □

Lemma 7.7.4. *The locus $S^{(\sigma)}$ of fixed points of σ is a smooth bisection C of μ with ramification points $x_1, \ldots, x_k$. In particular, $k = 2g + 2$.*

Proof Consider the action of σ on the generic fiber $F_\eta \cong \mathbb{P}^1_\eta$ of ϕ. The involution σ fixed two points on the geometric generic fiber $F_{\bar{\eta}}$. This shows that σ fixed one divisor class on F_η of degree 2 such that its closure C in S is fixed by σ pointwise. Since the set of a fixed points $S^{(\sigma)}$ of σ is smooth (this is true for any automorphism of finite order since we are working over $\mathbb{C}$), C is either a smooth irreducible curve or the sum $C_1 + C_2$ of two disjoint sections. Obviously, each C_i intersects one of the irreducible components of a singular fiber, and since they are switched C cannot be invariant. Thus, C is an irreducible curve of some genus g. Since the irreducible components of singular fibers are switched by σ, the curve C intersects each singular fiber F_i at the point $x_i = R_i^+ \cap R_i^-$. It intersects each nonsingular fiber F at two points or tangent at one point in F. Since no smooth σ-invariant curve can be tangent to C (its image in the quotient $S/(\sigma)$ locally splits in the cover at the intersection point with the branch divisor), we obtain that $S^{(\sigma)}$ consists of k points $x_1, \ldots, x_k$. Therefore, $k = 2g + 2$. Note that it confirms the Lefschetz fixed-point formula $4 - k = 2 + \mathrm{Trace}(\sigma : H^2(S, \mathbb{Q}) \to H^2(S, \mathbb{Q})) = e(S^g)$. □

It is now easy to finish the proof that σ is conjugate to a de Jonquières involution associated with hyperelliptic curve of genus g. Since $2g + 2 \geq 2(n - 1)$, we can blow-down $n - 1$ components in reducible fibers intersecting E and $n - 1$ components in other reducible fibers intersecting E' to obtain a birational morphism $S \to S'$, where the image of E and E' are (-1)-curves $\bar{E}$ and $\bar{E}'$ intersecting at $g + 1 - n$ points. Writing $[C] = a\mathbf{f} + b[E] + \sum_i a_i R_i^+$ with rational coefficients, we easily get

$$[C] = (k_0 + n)\mathbf{f} + 2E - \sum_i R_i^+,$$

where we used that $E \cdot C = E \cdot E' = k_0 - n$. This implies that $C^2 = k$, hence the image C' of C in S' has self-intersection $2k$. It intersect $\bar{E}$ at $k_0 - n + n - 1 = nk_0 - 1 = g$ points. Finally, let $\pi : S' \to \mathbb{P}^2$ be the blowing down of E and

$k - 2(n-1)$ images of $R^-_{2n-1}, \ldots, R^-_i$. The image of C' has self intersection $2k + g = 2(2g+2) + g^2 = (g+2)^2$. So, it is a plane curve of degree $g+2$ with a singular point (the image of $\bar{E}$) of multiplicity g.

It is equal to the proper transform of a plane hyperelliptic curve H_g of degree $g+2$ under the blow-up of its singular point $\mathfrak{o}$ and $2g+2$ Weierstrass points. Our involution is conjugate to the de Jonquières involution associated with H_g,.

Remark 7.7.5. One can show that the three types of conjugacy classes of involutions in Cr(2) are not conjugate to each other. In fact, it is easy to see that two involutions that contain a non-isomorphic non-rational irreducible components in their locus of fixed points are not conjugate.

One can use the minimal model theory and known automorphism groups of del Pezzo surface (see Chapters 8 and 9) to classify all conjugacy classes of finite subgroups of $C(2)$. The difficult part of this classification is distinguishing the conjugacy classes. We refer for this classification to [252].

SS:7.7.2

7.7.2 de Jonquiére involutions of $\mathbb{P}^n$

In this subsection, we will extend the definition of a planar de Jonquières involution to arbitrary $\mathbb{P}^n$.

Let X be a reduced irreducible hypersurface of degree m in $\mathbb{P}^n$ that contains a linear subspace of points of multiplicity $m-2$. Such a hypersurface is called *submonoidal* (a *monoidal hypersurface* is a hypersurface of degree m which contains a linear subspace of points multiplicity $m-1$). For example, every smooth hypersurface of degree ≤ 3 is submonoidal.

Let X be a submonoidal hypersurface with a singular point $\mathfrak{o}$ of multiplicity $m-2$. Let us choose the coordinates such that $\mathfrak{o} = [1, 0, \ldots, 0]$. Then, X is given by an equation

$$F_m = t_0^2 a_{m-2}(t_1, \ldots, t_n) + 2t_0 a_{d-1}(t_1, \ldots, t_n) + a_m(t_1, \ldots, t_n) = 0, \quad (7.78)$$ eqhyp

where the subscripts indicate the degrees of the homogeneous forms. For a general point $x \in X$, let us consider the intersection of the line $\ell_x = \overline{\mathfrak{o}x}$ with X. It contains $\mathfrak{o}$ with multiplicity $m-2$ and the residual intersection is a set of two points a, b in ℓ_x. Define $f(x)$ to be the point on ℓ_x such that the pairs $\{a, b\}$ and $\{x, f(x)\}$ are harmonically conjugate. We call it a *de Jonquières involution* (observe that $f = f^{-1}$).

Let us find an explicit formula for the de Jonquières involution, which we have defined. Let $x = [\alpha_0, \ldots, \alpha_n]$ and let $[u + v\alpha_0, v\alpha_1, \ldots, v\alpha_n]$ be the parametric equation of the line ℓ_x. Plugging in (7.78), we find

$$(u + v\alpha_0)^2 v^{m-2} a_{m-2}(\alpha_1, \ldots, \alpha_n) + 2(u + v\alpha_0)v^{m-1} a_{m-1}(\alpha_1, \ldots, \alpha_n)$$

$$+v^m a_m(\alpha_1, \dots, \alpha_n) = 0.$$

Canceling v^{m-2}, we see that the intersection points of the line ℓ_x with X are the two points corresponding to the zeros of the binary form $Au^2 + 2Buv + Cv^2$, where

$$(A, B, C) = (a_{m-2}(x), \alpha_0 a_{m-2}(x) + a_{m-1}(x), F_m(x)).$$

The points x and $f(x)$ correspond to the parameters satisfying the quadratic equation $A'u^2 + 2B'uv + C'v^2 = 0$, where $AA' + CC' - 2BB' = 0$. Since x corresponds to the parameters $[0, 1]$, we have $C' = 0$. Thus $f(x)$ corresponds to the parameters $[u, v] = [-C, B]$, and

$$f(x) = [-C + B\alpha_0, B\alpha_1, \dots, B\alpha_n].$$

Plugging in the expressions for C and B, we obtain the following formula for the transformation f

$$t_0' = -t_0 a_{m-1}(t_1, \dots, t_n) - a_m(t_1, \dots, t_n),$$
$$t_i' = t_i(a_{m-2}(t_1, \dots, t_n)t_0 + a_{m-1}(t_1, \dots, t_n)), \quad i = 1, \dots, n.$$

In affine coordinates $z_{i+1} = t_i/t_n,\ i = 0, \dots, n-1$, the formulas are

$$z_1' = -\frac{a_{m-1}(z_2, \dots, z_n)' z_1 + a_m(z_2, \dots, z_n)'}{a_{m-2}(z_2, \dots, z_n)' z_1 + a_{m-1}(z_2, \dots, z_n)'},$$
$$z_i' = z_i,\ i = 2, \dots, n.$$

A de Jonquières involution is an example of a *dilated Cremona transformation*. Starting from a Cremona transformation f in $\mathbb{P}^{n-1}$ we seek to extend it to a Cremona transformation in $\mathbb{P}^n$. More precisely, if $p_{\mathfrak{o}} : \mathbb{P}^n \dashrightarrow \mathbb{P}^{n-1}$ is a projection map from a point $\mathfrak{o}$, we want to find a Cremona transformation $\overline{T} : \mathbb{P}^n \dashrightarrow \mathbb{P}^{n-1}$ such that $p_{\mathfrak{o}} \circ \bar{T} = T \circ p_{\mathfrak{o}}$. Suppose that f is given by degree d homogeneous polynomials $(G_1, \dots, G_n)$. Composing with a projective transformation in $\mathbb{P}^n$, we may assume that $\mathfrak{o} = [1, 0, \dots, 0]$. Thus, the transformation $\overline{T}$ must be given by $(F_0, QG_1, \dots, QG_n)$, where Q and F_0 are coprime polynomials of degrees r and $d + r$. The following result can be found in PanGen [571].

ipan **Proposition 7.7.6.** *Let $(G_1, \dots, G_n)$ be homogeneous polynomials of degree d in $t_1, \dots, t_n$. Let $F_0 = t_0A_1 + A_2, Q = t_0B_1 + B_2$, where A_1, A_2, B_1, B_2 are homogeneous polynomials in $t_1, \dots, t_n$ of degrees $d + r - 1, d + r, r - 1, r$, respectively. Assume that F_0 and Q are coprime and $A_1B_2 \neq A_2B_1$. Then, the polynomials $(F_0, QG_1, \dots, QG_n)$ define a Cremona transformation of $\mathbb{P}^n$ if and only if $(G_1, \dots, G_n)$ define a Cremona transformation of $\mathbb{P}^{n-1}$.*

Proof Let $F'(z_1, \ldots, z_n)$ denote the dehomogenization of a homogeneous polynomial $F(t_0, \ldots, t_n)$ in the variable t_1. It is obvious that $(F_0, \ldots, F_n)$ defines a Cremona transformation if and only if:

$$\mathbb{C}(F_1/F_0, \ldots, F_n/F_0) := \mathbb{C}(F_1'/F_0', \ldots, F_n'/F_0') = \mathbb{C}(z_1, \ldots, z_n).$$

Consider the ratio $F_0/QG_1 = \frac{t_0A_1+A_2}{t_0GB_1+GB_2}$. Dehomogenizing with respect to t_1, we can write the ratio in the form $\frac{az_1+b}{cz_1+d}$, where $a, b, c, d \in \mathbb{C}(z_2, \ldots, z_n)$. By our assumption, $ad - bc \neq 0$. Then,

$$\mathbb{C}(F_1/F_0, \ldots, F_n/F_0) = \mathbb{C}(F_0/QG_1, G_2/G_1, \ldots, G_n/G_1)$$

$$= \mathbb{C}(G_2/G_1, \ldots, G_n/G_1)(F_0/QG_1) = \mathbb{C}(G_2/G_1, \ldots, G_n/G_1)(\frac{az_1 + b}{cz_1 + d}).$$

This field coincides with $\mathbb{C}(z_1, \ldots, z_n)$ if and only if $\mathbb{C}(G_2/G_1, \ldots, G_n/G_1)$ coincides with $\mathbb{C}(z_2, \ldots, z_n)$.

□

Taking $G_i = t_i, i = 1, \ldots, n$, and

$$F_0 = -t_0a_{m-1}(t_1, \ldots, t_n) - a_m(t_1, \ldots, t_n),$$
$$Q = a_{m-2}(t_1, \ldots, t_n)t_0 + a_{m-1}(t_1, \ldots, t_n),$$

we see that a de Jonquières involution is dilated from the identity transformation of $\mathbb{P}^{n-1}$. If we replace F_0 with $t_0b_{m-1}(t_1, \ldots, t_n) + b_m(t_1, \ldots, b_m)$, where b_{d-1}, b_m are any polynomials of indicated degrees such that F_0 and Q still satisfy the assumptions of Proposition 7.7.6, then we get a Cremona transformation, not necessarily involutive. In fact, one defines a general de Jonquières transformation as follows.

defdejonq **Definition 7.7.7.** *A Cremona transformation $f : \mathbb{P}^n \dashrightarrow \mathbb{P}^n$ is called a* de Jonquières transformation *if there exists a rational map $f : \mathbb{P}^n \dashrightarrow \mathbb{P}^k$ birationally isomorphic to the projection map* $\mathrm{pr}_2 : \mathbb{P}^{n-k} \times \mathbb{P}^k \to \mathbb{P}^k$ *and a Cremona transformation $T' : \mathbb{P}^k \dashrightarrow \mathbb{P}^k$ such that $f \circ T = T' \circ f$.*

In algebraic language, this definition is equivalent to f defining an automorphism Φ of the field of rational functions in $z_1, \ldots, z_n$ of the form

$$(z_1, \ldots, z_n) \mapsto (R_1, \ldots, R_k, R_{k+1}, \ldots, R_n),$$

where $R_1, \ldots, R_k$ are rational functions in variables $z_1, \ldots, z_k$ with coefficients in $\mathbb{C}$ and $R_{k+1}, \ldots, R_n$ are rational functions in variables $z_{k+1}, \ldots, z_n$ with coefficients in the field $\mathbb{C}(z_1, \ldots, z_k)$.

A de Jonquières transformation obtained by dilating the identity map of $\mathbb{P}^{n-1}$ is the special case when $k = n - 1$ and T' is the identity. It is easy to compute its multidegree. Take a general linear k-codimensional subspace L of $\mathbb{P}^n$. We can write L as the intersection of $k-1$ hyperplanes $H_i = V(l_i(t_1, \ldots, t_n))$ containing the point $\mathfrak{o}$ and one hyperplane $H_k = V(l_k(t_0, \ldots, t_n))$ which does not contain $\mathfrak{o}$. The pre-image of the first $k-1$ hyperplanes H_i are reducible hypersurfaces $D_i = V(t_i Q)$ of degree m. The pre-image of H_k is a hypersurface D_k of degree m. The intersection of the hypersurface $V(Q)$ with D_k is contained in the base scheme of f. Thus, the degree of the intersection $D_1 \cdots D_k$ outside the base locus is equal to m. This shows that the multi-degree of f is equal to $(m, \ldots, m)$. Note that the case $m = 2$ corresponds to quadratic transformations we studied in Subsection 7.2.1. In the notation from this Subsection, the point $\mathfrak{o}$ is the isolated base point, and the submonoidal hypersurface in this case is a quadric hypersurface Q such that the quadric component Q_0 of the base locus is equal to the intersection $Q \cap P_{\mathfrak{o}}(Q)$.

Remark 7.7.8. The following is Cremona's original construction of a space de Jonquièyes transformation [185]. Consider a rational curve R of bidegree $(1, m-2)$ on a nonsingular quadric Q in $\mathbb{P}^3$. Let ℓ be a line on Q which intersects R at $m-2$ distinct points. For each point x in the space, there exists a unique line joining a point on ℓ and on R. In fact, the plane spanned by x and ℓ intersects R at a unique point r outside $R \cap \ell$ and the line $\langle x, r\rangle$ intersects ℓ at a unique point s. Take two general planes Π and Π' and consider the following birational transformation $f : \Pi \dashrightarrow \Pi'$. Take a general point $p \in \Pi$, and find the unique line passing through p and intersecting R at a point r and intersecting ℓ. at a point s. The line intersects Π' at the point $f(p)$. For a general line ℓ in Π, the union of lines $\langle r, s\rangle, r \in R, s \in L$, is a ruled surface of degree m. Its intersection with Π' is a curve of degree m. This shows that the transformation f is of degree m. It has $2m-2$ simple base points. They are $m-1$ points in $\Pi' \cap R$ and $m-1$ points which are common to the line $\Pi \cap \Pi'$ and the $m-1$ lines joining the point $\ell \cap \Pi$ with the points in the intersection $\Pi \cap R$. Finally, the point $\ell \cap \Pi'$ is a base point of multiplicity $m-1$. Identifying Π and Π' by means of an isomorphism, we obtain a de Jonquières transformation.

SS:7.7.3

7.7.3 Geiser type involutions

The set of conjugacy classes of birational involutions σ of $\mathbb{P}^n, n \geq 3$, are divided into two classes: *rational* or *irrational*, dependent on whether the quotient $\mathbb{P}^n/(\sigma)$ is a rational variety or not. Obviously, any rational involution is conjugate to the deck transformation of a rational map $\mathbb{P}^n \dashrightarrow \mathbb{P}^n$ of degree 2. Of course, all planar involutions are rational.

An example of a unirational involution is provided by any non-rational but unirational n-dimensional variety X with the minimal degree of a dominant rational map $\mathbb{P}^n \dashrightarrow X$ equal to two, for example, a cubic threefold $X \subset \mathbb{P}^4$ [146, Appendix B].

Let f be a Cremona involution and $\mathfrak{G}(f) \subset G_1(\mathbb{P}^n)$ be the associated complex of lines in $\mathbb{P}^n$ (see Subsection 7.2.3). Let

$$\mathcal{U}_f = \{(x,p) \in \mathbb{P}^n \times \mathbb{P}^n : x \in C_f(p)\}$$

be the universal family of isologue curves of f. Then the fiber over a general point x of the first projection is the line $\langle x, f(x)\rangle \in \mathfrak{G}(f)$. We have the following commutative diagram

$$\begin{array}{ccccc} \mathcal{U}_f & \overset{\tilde{\iota}_T}{\dashrightarrow} & Z_{\mathfrak{G}(f)} & \xrightarrow{p_{\mathfrak{G}(f)}} & \mathbb{P}^n \\ \downarrow{\scriptstyle \mathrm{pr}_1} & & \downarrow{\scriptstyle q_{\mathfrak{G}(f)}} & & \\ \mathbb{P}^n & \overset{\iota_T}{\dashrightarrow} & \mathfrak{G}(f) & & \end{array} \tag{7.79}$$

Here, the composition of the maps in the upper row coincides with the second projection $\mathrm{pr}_2 : \mathcal{U}_T \to \mathbb{P}^n$.

The rational map

$$\iota_f : \mathbb{P}^n \dashrightarrow \mathfrak{G}(f), \quad x \mapsto \langle x, f(x)\rangle$$

factors through the orbits space $\mathbb{P}^n/(f)$ (considered as any birational model of the field of invariants $\mathbb{C}(z_1,\ldots,z_n)^{(f)}$) and defines a rational map

$$\bar{\iota}_f : \mathbb{P}^n/(f) \dashrightarrow \mathfrak{G}(f).$$

There are three possible scenarios.

1. $\bar{\iota}_f$ is a birational map;
2. $\bar{\iota}_f$ has one-dimensional fibers, i.e. f is an Arguesian involution.
3. $\bar{\iota}_f$ is of finite degree > 1.

In the first case, $\mathfrak{G}(f)$ is a n-dimensional subvariety of the Grassmannian $G_1(\mathbb{P}^n)$. For example, if $n = 3$, $\mathfrak{G}(f)$ is a complex of lines in $\mathbb{P}^3$. Its important invariant is the *degree* (see Section 10.2). It is equal to the degree of the curve of lines passing through a general point $x \in \mathbb{P}^n$ under the Plücker embedding of the Grassmannian.

In the second case, $\dim \mathfrak{G}(f) = n - 1$. If $n = 3$, it is a surface of lines in $\mathbb{P}^3$. Its order is the number of lines passing through a general point of $\mathbb{P}^3$.

It is clear that the complex of lines $\mathfrak{G}(f)$ of f is a unirational variety. In the first two cases, $\mathfrak{G}(f)$ is isomorphic to the quotient of the graph $\Gamma_f \subset \mathbb{P}^n \times \mathbb{P}^n$ of f by the involution switching the factors. It is a rational involution if and only if $\mathfrak{G}(f)$ is a rational variety. In the second case, $\bar{\iota}$ defines a birational map to the restriction $Z_{\mathfrak{G}(f)}$ of the universal line bundle $Z_{G_1(\mathbb{P}^n)} \to G_1(\mathbb{P}^n)$ to $\mathfrak{G}(f)$. It sends x to the point $(x, \langle x, f(x)\rangle) \in Z_{G_1(\mathbb{P}^n)}$. Again, f is a rational involution if and only if $\mathfrak{G}(f)$ is a rational variety. I do not know what happens in the third case, in fact, I do not know an example of Cremona involution in this case.

Let us give some examples of rational involutions of the first kind.

Example 7.7.9. Let f be a symmetric bilinear Cremona involution. Recall that its graph Γ_f is a complete intersection in $\mathbb{P}^n \times \mathbb{P}^n$ of n symmetric divisors of bidegree $(1, 1)$. The involution is defined by switching the factors. The Segre map embeds Γ_f onto a linear section of the Segre variety. The quotient embeds into $\mathbb{P}^{\frac{1}{2}n(n+1)}$ as a variety of degree $\frac{1}{2}\binom{2n}{n}$ [249, §5]. In the case $n = 2$, this is a cubic surface. For a general pencil of quadrics, it is isomorphic to the Cayley cubic surface. For larger n and general linear system of quadrics, the quotient has 2^n singular points, the images of 2^n base points of the linear system. This generalization of the Cayley cubic surface can be also described as the intersection of the determinant variety $\mathsf{Q}_n(2)$ of quadrics of rank 2 with the linear subspace defined by the condition that the diagonal elements of a general symmetric matrix are equal. The map to the Grassmannian is the map studied in Subsection 2.4.4 that assigns to the matrix the dual of its null-space. For $n = 3$, $\mathfrak{G}(f)$ is isomorphic to a 3-fold in $\mathbb{P}^6$ of degree 10 with 8 singular points, locally isomorphic to the cone over a Veronese surface.

Example 7.7.10. We have already encountered the following example of a rational involution of $\mathbb{P}^3$ in Remark 6.3.15. It can be considered as an analog of the planar Bertini involution, although it is known by the name a *Kantor involution* [157].

The linear system $|4h - 2\mathcal{P}|$ of quartic surfaces passing with multiplicity two through a general set $\mathcal{P} = \{p_1, \ldots, p_7\}$ of seven points in $\mathbb{P}^3$ defines a degree 2 map from $\mathbb{P}^3$ to the cone over the Veronese surface in $\mathbb{P}^6$. It lifts to an anti-canonical map $\mathsf{Bl}_{\mathcal{P}}(\mathbb{P}^3) \to \mathbb{P}(1, 1, 1, 2)$ of degree 2. The target of the map is isomorphic to the cone over the Veronese surface. The locus of fixed points of the involution is the Cayley dianode sextic surface with the set $\mathcal{P}$ of triple points and one isolated point p_8. The Kantor involution is the deck transformation of this map.

The net N of quadrics through $p_1, \ldots, p_7$ has the eighth base point p_8. It defines an elliptic fibration $\mathsf{Bl}_{p_1,\ldots,p_8}(\mathbb{P}^3) \to N^*$ with the section equal to the exceptional divisor over p_8. The transformation f is the negation involution on fibers of the fibration. The projection to $\mathbb{P}^2$ from p_8 defines a net of cubics through the projections of $\mathcal{P}$. For a general point $x \in \mathbb{P}^3$, let $C(x)$ be the unique quartic elliptic curve passing through $p_1, \ldots, p_8, x$. Then $f(x)$ is equal to the residual intersection point of $C(x)$ with the plane containing the line $\langle x, p_8 \rangle$ and tangent to $C(x)$ at p_8. For a general point $p \in \mathbb{P}^3$, the fiber of the projection $p_{\mathfrak{G}(f)} : Z_{\mathfrak{G}(f)} \to \mathbb{P}^3$ is a plane curve isomorphic to the isologue curve $C_\gamma(p')$ of the Geiser involution γ with center at the projection of p from the point p_8. So, the degree of $\mathfrak{G}(f)$ is equal to $8 + 1 = 9$. This implies that $\mathfrak{G}(f)$ is a complete intersection of $G_1(\mathbb{P}^3)$ with a hypersurface of degree 9 in the Plücker space $\mathbb{P}^5$. It is birationally isomorphic to the cone over the Veronese surface in $\mathbb{P}^6$.

The image of a general plane Π under the map given by the linear system $|4h - 2\mathcal{P}|$ projected to the Veronese surface in $\mathbb{P}^5$ with degree 4. It is cut by a quartic hypersurface in $\mathbb{P}^6$. Its pre-image in $\mathbb{P}^3$ is the union of Π and a surface of degree 15 passing through $p_1, \ldots, p_7$ with multiplicity 8. This shows that the degree of the Cantor involution is equal to 15. Its F-locus consists of the seven points and the inion of 21 chords $\langle p_i, p_j \rangle$.

The next example is another generalization of a planar Geiser involution.

Example 7.7.11. The starting point is an observation that the ideal sheaf $\mathcal{I}_Z$ of the set Z of seven points in $\mathbb{P}^2 = |E|$ in a general position is an aCM-sheaf defined by a resolution

$$0 \to \mathcal{O}_{\mathbb{P}^2}(-5) \oplus \mathcal{O}_{\mathbb{P}^2}(-4) \to \mathcal{O}_{\mathbb{P}^2}(-3)^3 \to \mathcal{I}_Z \to 0.$$

The resolution is defined by a matrix

$$A = \begin{pmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{pmatrix}$$

with polynomial entries a_i of degree 2 and b_i of degree 1. It defines a map $|E| \to |H^0(\mathbb{P}^2, \mathcal{I}_Z(3))^\vee|$ whose graph is a complete intersection of divisors of bidegree $(2, 1)$ and $(1, 1)$. Computing the multidegree, we find that the map is a rational of degree 2 and of algebraic degree 3. It coordinates, it is given by the maximal minors of the matrix A. We recognize the map from Proposition post 7.6. The deck transformation of this cover is a Geiser planar involution.

Now, we see how to extend this map to any $\mathbb{P}^n$. We consider a locally free resolution

$$0 \to \mathcal{O}_{\mathbb{P}^n}(-1)^{n-1} \oplus \mathcal{O}_{\mathbb{P}^n}(-2) \to \mathcal{O}_{\mathbb{P}^n}^{n+1} \to \mathcal{J}_Z(n+1) \to 0,$$

where Z is a codimension 2 closed aCM-subscheme in $\mathbb{P}^n$ with Hilbert polynomial

$$\chi(\mathcal{O}_Z(t)) = \chi(\mathcal{O}_{\mathbb{P}^n}(t)) - \chi(\mathcal{I}_Z(t)) = \chi(\mathcal{O}_{\mathbb{P}^n}(t)) - (n+1)\chi(\mathcal{O}_{\mathbb{P}^n}(t-n-1))$$
$$+ (n-1)\chi(\mathcal{O}_{\mathbb{P}^n}(t-n-2)) + \chi(\mathcal{O}_{\mathbb{P}^n}(t-n-3))$$
$$= \binom{t+n}{n} - (n+1)\binom{t-1}{n} + (n-1)\binom{t-2}{n} + \binom{t-3}{n}. \tag{7.80}$$

Again, we see that the resolution is defined by a matrix of size $n \times (n+1)$ with the linear entries in the first n rows and quadratic entries in the last row. The maximal minors of the matrix provide us with a rational map $\mathbb{P}^n \dashrightarrow \mathbb{P}^n$ of algebraic degree $n+1$. Its graph in $\mathbb{P}^n \times \mathbb{P}^n$ is a complete intersection of $n-1$ divisors of type $(1,1)$ and one divisor of type $(2,1)$. We compute $(h_1+h_2)^{n-1}(2h_1+h_2)$ to obtain that the multi-degree of this map is equal to $(1, d_1, \ldots, d_{n-1}, 2)$, where $d_k = 2\binom{n-1}{k-1} + \binom{n-1}{k}$. In particular, the degree of the map is equal to 2.

Take $n = 3$. Computing the Hilbert polynomial, we find that the base scheme Z is a smooth aCM curve of degree 11 and genus 14. For any smooth quartic S containing Z, the residual curve $C \in |\mathcal{O}_S(4) - Z|$ of quartics containing Z is a curve of genus 2 and degree 5.

We have $|\mathcal{O}_S(2) - C| = \{R\}$, where R is a rational normal cubic on S. The linear system $|C| = |\mathcal{O}_S(2) - R|$ defines a double cover $S \to \mathbb{P}^2$ with the branch curve of degree 6. Thus, the restriction of the double cover $\mathbb{P}^3 \dashrightarrow \mathbb{P}^3$ defined by $|\mathcal{I}_Z(4)|$ to S coincides with the double cover $S \to \mathbb{P}^2$ defined by the linear system $|C|$.

Let $\tilde{\phi} : X = \mathsf{Bl}_Z(\mathbb{P}^3) \dashrightarrow \mathbb{P}^3$ be the rational lift of ϕ to the blow-up of the branch curve Z. We have $K_X = -4H + E$, where H is the pre-image of the divisor class of a plane in $\mathbb{P}^3$, and E is the exceptional divisor of the blow-up. We see that the proper transform of the linear system $|\mathcal{I}_{\mathbb{P}^3}(4) - Z|$ to X coincides with the linear system $|-K_X|$. The variety X is a *weak Fano variety* of degree $(-K_X)^3 = 2$. This means that $-K_X$ is nef and big. It can be found in the list of weak Fano varieties obtained by blowing up a smooth curve in $\mathbb{P}^3$ (see [61] and [195]). Also, the map $\tilde{\phi}$ coincides with the anti-canonical map, in a complete analogy with the planar Geiser involution.

The exceptional divisor E of the blow-up is a $\mathbb{P}^1$-bundle over Z isomorphic to $\mathbb{P}(\mathcal{N}^\vee_{Z/\mathbb{P}^3})$. Applying Lemma (7.4.6), we find that $s(Z, \mathbb{P}^3) = [Z] - 70[pt]$ and compute $-K_X \cdot E^2 = -4H \cdot Z + E^3 = -44 + 70 = 26$.

It follows from the Cayley formula for the number of 4-secant lines of a curve of genus g and degree d

$$t_4 = \frac{(d-2)(d-3)^2(d-4)}{12} - \frac{(d^2-7d+13-g)g}{2}, \qquad (7.81)$$

foursecant

that Z has 35 4-lines [360, Chapter 2, §5]). They are blown down to ordinary double points of B.

The ramification divisor $\mathrm{Ram}(\tilde{\phi})$ belongs to $|-3K_X|$. Its image under the blowing down morphism $X \to \mathbb{P}^3$ belongs to $|3(4H-Z)|$. It is a surface of degree 12 that contains Z with multiplicity 3. The branch divisor B of $\tilde{\phi}$ is a surface of degree 6. So, X is a birational model of a *double sextic solid*. Note that the double cover of $\mathbb{P}^3$ branched along a general surface of degree 6 is a non-rational variety. In our case, it is a rational variety.

The image of a general plane Π under the map ϕ is given by the linear system $|L| = |\mathcal{O}_\Pi(4) - Z\cap\Pi|$. It is a rational surface V_5 of degree $16-11=5$, a projection of a Bordiga sextic surface from a point on it. Since each 4-secant of Z intersects Π, the image of Π contains the thirty-five singular points of the branch sextic surface.

We have $\tilde{\phi}^{-1}(\phi(\Pi)) \in |5(4H-E)|$, hence $f(\Pi) \in |19h-5Z|$. Thus, the degree of f is equal to 19. The base locus of f is equal to the union of the curve Z taken with multiplicity 5 and 35 4-secants.

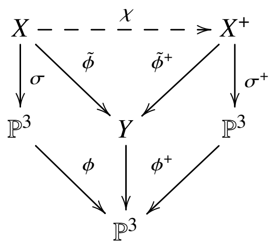
Remark 7.7.12. Similar to the case $(g,d) = (3,6)$, there are two irreducible components in the Hilbert scheme of smooth curves of genus 14 and degree 11. The other component parameterizes non-aCM curves lying on a quadric as curves of bidegree $(3,8)$ [61, Proposition 2.7].

Remark 7.7.13. There are other examples of a smooth curve Z of genus g and degree d such that $X = \mathsf{Bl}_Z(\mathbb{P}^3)$ is a weak Fano 3-fold such that $-K_X^3 = 2$, so that the anti-canonical map is of degree 2. In each case, the deck transformation gives an example of a Cremona involution. The values (d,g) satisfying these properties are $(8,2), (9,6), (10,10)$, and our case $(11,14)$. The anti-canonical map is given by the linear system $|\mathcal{O}_{\mathbb{P}^3}(4) - Z|$ of quartic surfaces passing through Z. For a general quartic Y in this linear system $|\mathcal{O}_Y(4) - Z|$ is a linear system of genus 2 curves. It follows, as above, that the branch divisor of the anti-canonical map is a surface of degree six. It follows from (7.81) that it has, respectively, $31, 30, 31$ singular points. The homaloidal linear systems defining the corresponding Cremona involutions are $|\mathcal{O}_{\mathbb{P}^3}(32) - 8Z - L|, |\mathcal{O}_{\mathbb{P}^3}(28) - 7Z - L|, |\mathcal{O}_{\mathbb{P}^3}(24) - 6Z - L|$, where L denotes the union of 4-secants.

Remark 7.7.14. In the previous examples, we encountered quartic surfaces Y which contain a smooth curve C of genus 2 of degree $d = 16 - \deg(Z)$, where $Z \in |\mathcal{O}_Y(4) - C|$. The birational involution defined by $|C|$ is an example of a

Gizatullin involution of a quartic surface, a birational involution of a smooth quartic surface that can be lifted to a Cremona involution of $\mathbb{P}^3$. There are examples of biregular involutions of smooth quartic surface which cannot be lifted to a Cremona involution Oguiso [557].

Remark 7.7.15. We have the following commutative diagram of birational maps



Here $Y \to \mathbb{P}^3$ is the double cover of $\mathbb{P}^3$ branched along a sextic surface with 35 ordinary double points. Each of the birational morphisms $\tilde{\phi}$ and $\tilde{\phi}^+$ is a small contraction that represent two small resolutions of 35 ordinary double points on Y. The birational morphisms $\tilde{\phi}$ and $\tilde{\phi}^+$ are divisorial contractions. The exceptional divisors are ruled surfaces isomorphic to the exceptional divisors of σ and σ^+.

The birational map χ is an example of an *flop*. The birational morphism σ^+ is the blow-up of a curve $Z' \subset \mathbb{P}^3$ projectively isomorphic to Z. The birational map $\sigma^+ \circ \chi \circ \sigma$ îs an example of a *Sarkisov link*. Since, χ is an example of a *pseudo-isomorphism* of algebraic varieties, a birational isomorphism which is an isomorphism on the complement of a closed subset of codimension ≥ 2. It pull-back homomorphism $\chi^* : \mathrm{Pic}(X^+) \to \mathrm{Pic}(X)$ is bijective and sends K_{X^+} to K_X. The proper transform of the divisor E^+ under χ belongs to $|-18K_X - E|$ Cutrone [195].

Example 7.7.16. One can apply the dilation operation from Subsection 7.7.2 to planar Cremona involutions. In this way, we obtain examples of rational de Jonqui`eres involution and examples of dilated Geiser an Bertini involutions. It was shown by Coble that the dilated Geiser (resp. Bertini) involution is given by the linear system $|15H - 14p_9 - 6(p_1 + \cdots + p_8)|$ (resp. $|33H - 32p_9 - 6(p_1 + \cdots + p_8)|$, where $(p_1, \ldots, p_8, p_9$ (resp. $(p_1, \ldots, p_8, p_9)$ is the dilated set of 7 (resp. 8 points) in $\mathbb{P}^2$ CobleSym [157]. Both involutions leave invariant a Cayley quartic symmetroid $\mathsf{D}(W)$ associated with a general web of quadrics in $\mathbb{P}^3$ with the set of ten nodes containing the points p_i. The Kantor involution together with the dilated Geiser and Bertini involutions descend to biregular automorphisms of the Reye congruence lines associated with W and generate its group of automorphisms DKEII [259, 8.4, 8.5].

7.7.4

7.7.4 Arguesian involutions

Recall that a Cremona transformation f of $\mathbb{P}^n$ is called Arguesian if the complex of lines $\mathfrak{G}(f)$ is of dimension $n-1$. In this subsection, we will give some classical examples of Arguesian involutions.

transfT7 *Example* 7.7.17. The following is another generalization of the planar Geiser involution. Assume $n = 3$, and consider a rational map $\phi : \mathbb{P}^3 \dashrightarrow \mathbb{P}^3$ defined by the linear system $\mathcal{H} = |2H - p_1 - \ldots - p_6|$, where $p_1, \ldots, p_6$ are points in a general linear position. Take a general point $x \in \mathbb{P}^3$, then the quadrics from $\mathcal{H}$ passing through x form a net. The base locus of the net consists of 8 points, we have already seven base points $p_1, \ldots, p_6, x$. By definition, $f(x)$ is the eighth base point.

The involution is the deck transformation $T_7 : \mathbb{P}^3 \to \mathbb{P}^3 = \mathcal{H}^*$ defined by the web of quadrics $\mathcal{H}$. We will see later, in Subsection 12.3.1 that the branch divisor of the map is a Kummer quartic surface, and the ramification divisor is a Weddle surface. encountereed this example earlier A general plane is mapped to a quartic surface, a projection of a Veronese surface. The pre-image of this quartic surface consists of the plane and a surface of degree 7. This shows that the degree of the involution is equal to 7. It is given by the linear system $|7H - 4\mathcal{P}|$. It is an example of a *regular* Cremona transformation (see [159], [234]). Its F-locus is determined by a finite set of indeterminacy points. In this case, the lines $\ell_{ij} = \langle p_i, p_j \rangle$ belong to the F-locus because a general member of the homalidal linear system has points of multiplicity 4 at p_i. Also the unique twisted cubic R_3 containing the points p_i belongs to F-locus becaus $3 \cdot 7 < 4 \cdot 6$. It is similar to that we had for teh standard cubic transformation of $\mathbb{P}^3$ with the edges of the tetrahedron as its set of F-points. . The lift of T_7 to $\mathsf{Bl}_{\mathcal{P}}(\mathbb{P}^3)$ blows down the proper transform of ℓ_{ij} (resp. R_3) to the line ℓ_{ij} (resp. R_3) in the target $\mathbb{P}^3$.

The P-locus consists the six quadrics from the linear system which have a singular point at one of the points p_j. They are blown down to the points p_i.

Let ℓ be the secant line containing a point $x \in \mathbb{P}^3$ (ot tangent to R_3 if $x \in R_3$). It intersects the Weddle at two points x_1, x_2 on ℓ. The image $T(x)$ is the point on ℓ such that the pairs $\{x, T_7(x)\}$ and $\{x_1, x_2\}$ are harmonically conjugate. Indeed, there is a pencil in $\mathcal{H}$ that contains $R_3 + \ell$. The restriction of $\mathcal{H}$ to ℓ contains the intersection of the cones with vertics at x_1, x_2 to ℓ. It defines a g_2^1 on ℓ with the ramification points x_1, x_2. The net of quadrics in $\mathcal{H}$ that contain x, contains $T(x)$ as another base point. Thus, $\{x, T(x)\}$ is a member of g_2^1, and hence, the pairs $\{x, T_7(x)\}$ and $\{x_1, x_2\}$ are harmonically conjugate.

Remark 7.7.18. One can show that the Cremona group Cr(3) contains a subgroup isomorphic to an elementary abelian 2-group 2^5 of rank 5 that leaves the Weddle surface invariant. One of the non-trivial elements in this group is our transformation f [159, Chapter III, §36]. This gives an example of a group of birational automorphisms of a quartic surface of order 16 that consists of Gizatullin involutions.

The Coble representation $\mathrm{cr}_6 : W(D_6) \to \mathrm{Bir}(P_3^6)$ defined in [159], [234] has the kernel that coincides with the normal subgroup 2^5 of the Weyl group $W(D_6)$. Let

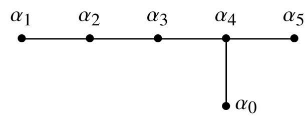


Figure 7.7 Coxeter-Dynkin diagram of type D_6 Coxdiagd6

The kernel is the smallest normal subgroup containing the product of simple reflections $\gamma = s_{\alpha_0}s_{\alpha_5}$. The image of this involution is a cubic transformation with fundamental point $p_1, \ldots, p_4$. It switches the points p_5 and p_6. The septic transformation T_7 is equal to the composition $\gamma(\sigma_1\gamma\sigma_1^{-1})(\sigma_2\gamma\sigma_2^{-1}$, where $\sigma_1 = (35)(56), \sigma_2 = (15)(26)$ belong to the subgroup $\mathfrak{S}_6 \subset W(D_6)$ generated by $\alpha_1, \ldots \alpha_5$. (see [159, p. 116].

transfT3 *Example* 7.7.19. Fix a rational normal cubic curve R_3 in $\mathbb{P}^3$ and consider the involution $x \mapsto T_3(x)$ such that the pairs $\{x, T_3(x)\}$ and $\{x_1, x_2\}$ are harmonically conjugate. This is similar to the previous example. However, in this case, T_3 is a cubo-cubic transformation. We encountered this transformation in Example 7.6.6. Its F-locus is R_3 and irs P-locus is the tangential quartic surface with the double curve R_3. The image of a general plane Π is a cubic surface with 3 double points in $\Pi \cap R_3$. So, the transformation is different from the standard cubic transformation given in (7.75), where the images of planes were 4-nodal cubic surfaces.

The image of a general line ℓ is a rational cubic curve R. Since ℓ intersects the tangential quartic surface at four points, it meets four tangents of R_3 which are blown down to four intersection points of R_3 and $T_3(R_3)$. Note that the line complexes associated to T_3 and T_7 are the same; the planes $R_3^{(2)} \cong \mathbb{P}^2$ embedded in $\mathbb{P}^5$ by the Plücker embedding. However, the maps of $\bar{\iota}$ are different.

The following two examples are taken from [338].

godeaux1 *Example* 7.7.20. Let $V(\lambda q_1 + \mu q_2)$ be a pencil of quadrics in $\mathbb{P}^n$. Fix a point p not belonging to the base locus B of the pencil, For a general point $x \in \mathbb{P}^3$, restrict the pencil to the line $\langle x, p\rangle$ and consider the quadric from the pencil containing x. It contains another point $f(x)$ on the same line. The transformation $x \to f(x)$ is an Arguesian involution. Let us find the formula for the transformation f.

Choose projective coordinates such that $p = [a_0, \ldots, a_n]$. A line $\langle x, p\rangle$ is given parametrically by $[sx_0+ta_0, \ldots, sx_n+ta_n]$. The point x is contained in a quadric $\lambda q_1(x)+\mu q_2(x)$, so that we may assume that $[\lambda, \mu] = [q_2(x), -q_1(x)]$. The second intersection point on the line $\langle x, p\rangle$ is given by the linear equation

$$q_2(x)q_1(sx_0 + ta_0, \ldots, sx_n + ta_n) - q_1(x)q_2(sx_0 + ta_0, \ldots, sx_n + ta_n)$$

$$= [q_2(x)b_1(x, p) - q_1(x)b_2(x, p)]s + [q_2(x)q_1(p) - q_1(x)q_2(p)]t = 0.$$

Denote the coefficients at s and t by $A(x, p)$ and $B(x, p)$, we obtain

$$f(x) = [B(x, p)x + A(x, p)p]$$

The degree of f is equal to 3. Note that, if the line $\langle x, p\rangle$ intersects the base locus B of the pencil, then $A(x, p)$ and $B(X, p)$ are proportional, and hence, f blows contracts this line. The F-locus is equal to $B \cup \{p\}$ and the P-locus is equal to the cone over B with vertex at p. It is contracted to the curve B. The complex of lines $\mathfrak{G}(f)$ is an α-plane of lines through p, a congruence of order one and class 0.

godeaux2 *Example* 7.7.21. This time, we take the congruence S of lines intersecting two skew lines ℓ and ℓ'. Fix a general pencil of quadrics $\mathcal{P}$ and consider the following involution f of $\mathbb{P}^3$. Let Q be the unique quadric from $\mathcal{P}$ that contains a general point $x \in \mathbb{P}^3$ and let ℓ_x be the unique line $\ell_x \in S$ passing through x. Then, by definition, $Q \cap \ell_x = \{x, f(x)\}$. Obviously, $f(\ell_x) = \ell_x$, so f is an Arguesian involution,.

It is clear that the base curve B of $\mathcal{P}$ is a fundamental curve. For a general point $x \in \ell$ there exists a unique $Q \in \mathcal{P}$ containing x. It intersects the plane $\langle x, \ell'\rangle$ along a conic. All lines in this plane passing through x belong to S, hence the image of the ruling of the exceptional divisor of $\mathrm{Bl}_\ell(\mathbb{P}^3)$ over x is mapped to a conic. This shows that the lines ℓ and ℓ' enter in the base scheme with multiplicity 2.

We will explain in Example CAG-2:Type1 11.4.1 in Volume II that the surface of secant lines of B is a congruence of lines of degree 8. Since all lines intersecting ℓ_i is a hyperplane section of $G_1(\mathbb{P}^3)$, we obtain that there are exactly eight secant lines $\ell_1, \ldots, \ell_8$ of B that intersect ℓ and ℓ'. Let Q_i be the unique quadric in $\mathcal{P}$ that contains ℓ_i. Any line on Q_i intersecting ℓ_i is blown down to the point of its intersection with ℓ_i. This shows that Q_i belongs to the P-locus, and also shows that ℓ_i enters with multiplicity one in the base scheme. Similarly, we see that B

enters with multiplicity one. Applying (7.10), we get $d(d-1) = 2\cdot4+4+8 = 20$, hence $d = 5$.

The P-locus consists of the eight quadrics Q_i and the ruled surface of degree 4 of rays from S that intersect B. The degrees add up to $20 = 4\cdot(d-1)$.

Let Π be a general plane, the map $V = \Pi \to f(\Pi)$ is given by the linear system of curves of degree 5 with two two base points of multiplicity 2 and 12 simple base points. Since f is an involution V is also equal to the pre-image of a general plane under f. It contains two double lines, the quartic curve B, and eight secant lines ℓ_i. The curve B is the image of the unique plane quartic with two double points $\Pi\cap\ell, \Pi\cap\ell'$, and eight simple points $\Pi\cap\ell_i$. The surface has also eight conics, the images of the lines joining a double base point with one of the points in $\Pi\cap B$.

A closed subvariety X_n of $\mathbb{P}^{2n+1}$ of dimension n is called a *subvariety with one apparent double point* (*OADP subvariety*, for short) if a general point in $\mathbb{P}^n$ lies on a unique secant line of X. A twisted cubic is the simplest example of an OADP variety. We will see in Section 8.5 that a nonsingular del Pezzo surface of degree 5 anti-canonically embedded in $\mathbb{P}^5$ is an OADP subvariety of dimension 2.

An OADP subvariety X of $\mathbb{P}^n$ defines a Cremona involution of $\mathbb{P}^n$ in a way similar to the definition of a de Jonquières involution. For a general point $x\in\mathbb{P}^n$ we find a unique secant line of X intersecting X at two points (a,b), and then we define the unique $f(x)$ such that the pair $\{x, f(x)\}$ is harmonically conjugate to $\{a,b\}$. A Cremona involution obtained in such a way is called an *OADP-involution*. The complex of lines $\mathfrak{G}(f)$ of an OADP-involution is birationally isomorphic to the symmetric product $X^{(2)}$.

An infinite series of examples of OADP subvarietes was given by D. Babbage [27] and W. Edge [276]. They are now called the *Edge varieties*. The Edge varieties are of two kinds. The first kind is a general divisor $E_{n,2n+1}$ of bidegree $(1,2)$ in $\mathbb{P}^1\times\mathbb{P}^n$ embedded by Segre in $\mathbb{P}^{2n+1}$. Its degree is equal to $2n+1$. For example, when $n = 1$, we obtain a twisted cubic in $\mathbb{P}^3$. If $n = 2$, we obtain a del Pezzo surface in $\mathbb{P}^5$. The second type is a general divisor of bidegree $(0,2)$ in $\mathbb{P}^1\times\mathbb{P}^n$.

For example, when $n = 1$, we get the union of two skew lines. When $n = 2$, we get a quartic ruled surface $S_{2,5}$ in $\mathbb{P}^5$ isomorphic to $\mathbb{P}^1\times\mathbb{P}^1$ embedded by the linear system of divisors of bidegree $(1,2)$. A smooth OADP surface in $\mathbb{P}^5$ is either an Edge variety of dimension 2, or a scroll $S_{1,5}$ of degree 4 [642].

We refer to [137] and [11] for more information about OADP subvarieties.

An example of a reducible OADP curve is the curve $\Gamma = \ell + C_{n-1}$ from the description of congruences of lines of order one. We will give an example of the OADP-involution in this case in Chapter 11.

Exercises

7.1 Let $|L|$ be the linear system of quadrics through a rational normal curve of degree n in $\mathbb{P}^n$. Show that its dimension is equal to $\frac{1}{2}(n+1(n-2)$ and the image of the rational map given by the linear system is of degree $2^n - n^2 + n - 2$. If $n = 4$, show that the image is isomorphic to a smooth quadric in $\mathbb{P}^5$ that can be identified with the Plücker embedding of the Grassmannian $G(2,4)$.

7.2 Let $Z = \ell_1 \cup \ell_2$ is the union of two lines in $\mathbb{P}^3$ intersecting at a point p_0. Show that $i_*s(Z, \mathbb{P}^n) = 2h^2 - 6h^3$. Find the base scheme of the linear system of quadrics containing Z. Show that the linear system defines a birational map onto a quadric of corank one in $\mathbb{P}^4$ and the inverse map is the projection of the quadric from a nonsingular point.

7.3 Let Z be the union of three non-coplanar linesin $\mathbb{P}^3$ intersecting at one point. Show that $i_*s(Z, \mathbb{P}^3) = 3h^2 - 10h^3$. Describe the blow-up of Z, and find its log resolution.

7.4 A set of lines in $\mathbb{P}^3$ is called *homaloidal* if there exist some positive integers $d, m_1, \ldots, m_n$ such that the linear system $|dH - \sum_{i=1}^n \alpha_i \ell_i|$ is homaloidal. Show that the union of four skew lines in $\mathbb{P}^3$ and two lines intersecting them is a homaloidal set. Find its P-locus, as well as the base scheme and the P-locus of the inverse Cremona transformation.

7.5 Let x be an isolated base point of s-contact point of a homaloidal linear system. Show that x contributes $-s-1$ to the formula for the degree d_n of the Cremona transformation.

7.6 Consider a rational map defined by

$$[t_0, t_1, t_2] \mapsto [t_1t_2(t_0 - t_2)(t_0 - 2t_1), t_0t_2(t_1 - t_2)(t_0 - 2t_1), t_0t_1(t_1 - t_2)(t_0 - t_2)].$$

Show that it is a Cremona transformation and find the Enriques diagram of the corresponding bubble cycle. ex:7.7

7.7 Let C be a plane curve of degree d with a singular point p. Let $\pi : X \to \mathbb{P}^2$ be a sequence of blow-ups which resolves the singularity. Define the bubble cycle $\eta(C, p) = \sum m_i x_i$ as follows: $x_1 = p$ and $m_1 = \mathrm{mult}_p C$, $x_2, \ldots, x_k$ are infinitely near points to p of order one such that the proper transform C' of C in the blow-up $\mathsf{Bl}_p(\mathbb{P}^2)$ contains these points, $m_i = \mathrm{mult}_{x_i} C'$, $i = 2, \ldots, k$, and so on.

(i) Show that the arithmetic genus of the proper transform of C in X is equal to $\frac{1}{2}(d-1)(d-2) - \frac{1}{2}\sum_i m_i(m_i - 1)$.

(ii) Describe the Enriques diagram of $\eta(C, p)$, where $C = V(t_0^{b-a}t_1^a + t_2^b)$, $p = [1, 0, 0]$, and $a \leq b$ are positive integers. ex:7.8

7.8 Give an example of a planar Cremona transformation with no infinitely near base points such that the inverse transformation has infinitely near base points. ex:7.star

7.9 Show that two hyperelliptic plane curves H_m and H'_m of degree m and genus $m-2$ are birationally isomorphic if and only if there exists a de Jonquières transformation which transforms one curve to another. ex:7.9

7.10 Consider a set of five points $\mathfrak{o}, x_1, x_2, x_3, x_4$ such that the last three points are collinear. Consider a de Joinquières transformation of degree three with fundamental point $\mathfrak{o}$ of multiplicity 2 and simple fundamental points at the points x_i. ex:7.10 Show that one of five total principal curves is reducible. hypp

7.11 Let H_{g+2} be a hyperelliptic curve given by Equation (7.29). Consider the linear system of hyperelliptic curves $H_{q+2} = V(t_2^2 g_q(t_0,t_1) + 2t_2 g_{q+1}(t_0,t_1) + g_{q+2}(t_0,t_1))$ such that $f_g g_{q+2} - 2f_{g+1}g_{q+1} + f_{g+2}g_q = 0$. Show that

(i) the curves H_{q+2} exist if $q \geq (g-2)/2$;

(ii) the branch points of H_{g+2} belong to H_{q+2} and vice versa;

(iii) the curve H_{q+2} is invariant with respect to the de Jonquières involution IH_{g+2} defined by the curve H_{g+2} and the curve H_{g+2} is invariant with respect to the de Jonquières involution IH_{q+2} defined by the curve H_{q+2};

(iv) the involutions IH_{g+2} and IH_{q+2} commute with each other;

(v) the fixed locus of the composition $H_{g+2} \circ H_{q+2}$ is given by the equation

$$f_{g+q+3} = \det\begin{pmatrix} f_g & f_{g+1} & f_{g+2} \\ g_q & g_{q+1} & g_{q+2} \\ 1 & -t_2 & t_2^2 \end{pmatrix} = 0;$$

(vi) the de Jonquières transformations that leave the curve H_{g+2} invariant form a group. It contains an abelian subgroup of index 2 that consists of transformations which leave H_{g+2} fixed pointwise.

ex:7.11 ex:7.13

7.12 Find the automorphism group of the surface $\mathbf{F}_n$.

7.13 Let C be an irreducible plane curve of degree $d > 1$ passing through some points $x_1, \ldots, x_n$ with multiplicities $m_1 \geq \ldots \geq m_n$. Assume that its proper inverse transform under the blowing up the points $x_1, \ldots, x_n$ is a smooth rational curve ex:7.15 $\bar{C}$ with $\bar{C}^2 = -1$. Show that $m_1 + m_2 + m_3 > d$.

7.14 Let $(m, m_1, \ldots, m_n)$ be the characteristic vector of a Cremona transformation. ex:7.16 Show that the number of fundamental points with $m_i > m/3$ is less than 9.

7.15 Compute the characteristic matrix of the composition $T \circ T'$ of a de Jonquières transformation f with fundamental points $\mathfrak{o}, x_1, x_2, \ldots, x_{2d-2}$ and a quadratic ex:7.17 transformation T' with fundamental points $\mathfrak{o}, x_1, x_2$.

7.16 Let $\sigma : \mathbb{A}^2 \to \mathbb{A}^2$ be an automorphism of the affine plane given by a formula $(x,y) \to (x + P(y), y)$, where P is a polynomial of degree d in one variable. Consider σ as a Cremona transformation. Compute its characteristic matrix. In the case $d = 3$ write as a composition of projective transformations and quadratic ex:7.18 transformations.

7.17 Show that every Cremona transformation is a composition of the following maps ("links"):

(i) the switch involution $\tau : \mathbf{F}_0 \to \mathbf{F}_0$;

(ii) the blow-up $\sigma : \mathbf{F}_1 \to \mathbb{P}^2$;

(iii) the inverse $\sigma^{-1} : \mathbb{P}^2 \dashrightarrow \mathbf{F}_1$;

(iv) an elementary transformation $\mathrm{elm}_x : \mathbf{F}_q \dashrightarrow \mathbf{F}_{q\pm 1}$.

ex:7.19 Ex:7.22 ex:7.20

7.18 Show that any planar Cremona transformation is a composition of de Jonquières transformations and projective automorphisms.

7.19 Let $\mathcal{P}$ be a linear pencil of plane curves whose general member is a curve of geometric genus 1 and $f : \mathbb{P}^2 \dashrightarrow \mathbb{P}^1$ be a rational map it defines.

(i) Show that there exist birational morphisms $\pi : X \to \mathbb{P}^2, \phi : X \to \mathbb{P}^1$ with

$f = \phi \circ \pi^{-1}$ such that $\phi : X \to \mathbb{P}^1$ is a relatively minimal rational elliptic surface.

(ii) Use the formula for the canonical class of an elliptic surface to show that the divisor class of a fiber is equal to $-mK_X$ for some positive integer m.

(iii) Show that there exists a birational morphism $\sigma : X \to \mathbb{P}^2$ such that the image of the elliptic fibration is an *Halphen pencil* of index m, i.e. a linear pencil of curves of degree $3m$ with nine m-multiple base points (including infinitely near).

(iv) Conclude by deducing *Bertini's Theorem*, which states that any linear pencil of plane elliptic curves can be reduced by a plane Cremona transformation to an Halphen pencil.

ex:7.22

7.20 Describe a log resolution of the Cremona transformations of degree 3 and 4 from Remark 7.4.5. Find its simplicial complex and describe the inverse transformations of degrees 9 and 16.

7.21 Consider the Arguesian involution f from Example 7.6.6. Show that, for any general point $x \in \mathbb{P}^3$, the pair $\{x, f(x)\}$ is harmonically conjugate to the pair $\{p_1, p_2\}$, where p_1, p_2 are the point on R_3 lying on the unique secant of R_3 passing through x.

7.22 Let C be a smooth curve in $\mathbb{P}^n$ of genus g and degree $\deg(C)$ and let $\mathcal{I}_C \to \mathcal{I}_C/\mathcal{I}_Z^2 \to \mathcal{L}$ is the composition of surjections, where $\mathcal{L}$ be an invertible sheaf on C of degree a. Show the linear system $|\mathcal{I}_Z(d)|$ defined by the closed subscheme Z with the ideal sheaf $\mathcal{I}_Z = \mathrm{Ker}(\mathcal{I}_C \to \mathcal{L})$ is homaloidal if and only if

$$d^n + (4 - 4g - 2(n+1)\deg(C) + 2a) - 2dn\deg(C) = 1.$$

and $h^0(\mathcal{I}_Z(d)) = n + 1$. The Arguesian involution from the previous example corresponds to the case $n = 3, d = 3, g = 0, \deg(C) = 3, a = -4$. Find new examples of homaloidal linear systems $h^0(\mathcal{I}_Z(d))$.

7.23 A Cremona transformation f is called *monomial* if its restriction to the open subset $U \cong (\mathbb{C}^*)^n$ complementary to the union of the coordinate hyperplanes $V(t_i)$ is given by $(z_1, \ldots, z_n) \to (\mathbf{z}^{\mathbf{m}_1}, \ldots, \mathbf{z}^{\mathbf{m}_n})$, where $\mathbf{z}^{\mathbf{m}_i} = z_1^{m_{i1}} \cdots z_n^{m_{in}}$ are monomials in affine coordinates $z_i = t_i/t_0, i = 1, \ldots, n$.

(i) Let X_Σ be a toric variety defined by a fan Σ containing U as its dense torus orbit. Show that the matrix $M = (m_{ij})$ is invertible and defines a linear automorphism $g_M \in \mathrm{GL}(n, \mathbb{R}^n)$. It also defines an isomorphism of toric varieties $f_M : X_\Sigma \to X'_{\Sigma}$, where $\Sigma' = g_M(\Sigma)$. Show that its restriction to the open subset U coincides with the birational map given by the binomials $\mathbf{z}^{\mathbf{m}_i}$.

(ii) Let Π be a common subdivision of Σ and Σ'. Show that the projections $\sigma : X_\Pi \to X_\Sigma$ and $\nu : X_\Pi \to X_{\Sigma'}$ define a resolution of the birational transformation $f_M = \nu \circ \sigma^{-1}$.

(iii) In the case when Σ is defined by the vectors $\mathbf{e}_1, \ldots, \mathbf{e}_n, -(\mathbf{e}_1 + \cdots + \mathbf{e}_n)$, X_Σ and $X_{\Sigma'}$ can be identified with $\mathbb{P}^n$ and f_M defines a Cremona transformation. Show that the degree d_k is equal to the mixed volume $\mathrm{Vol}(\underbrace{P, \ldots, P}_{k}, \underbrace{g(P), \ldots, g(P)}_{n-k})$.

Confirm that $d_k = \binom{n}{k}$ if $M = -I_n$ and f_M is the standard Cremona transformation in $\mathbb{P}^n$.

(iv) Use the known generators of the group $\mathrm{GL}(n, \mathbb{Z})$ [177, 7.1] to show that the monomial Cremona transformations form a group generated by projective and quadratic transformations [343].

7.24 A Cremona transformation f of $\mathbb{P}^n$ is called *regularizable* if there exists a rational variety X, a birational morphism $\phi : X \to \mathbb{P}^n$, and an automorphism g of X such that $f = \phi \circ g \circ \phi^{-1}$. We showed that any f of finite order in $\mathrm{Cr}(n)$ is regularizable. Show that a general quadratic transformation in $\mathrm{Cr}(2)$ is not regularizable.

7.25 Describe the base schemes of the inverse of the transformations of bidegree $(2,4)$, $(3,9)$ and $(4,16)$ from Remark 7.4.5.

7.26 A Cremona transformation is called *monoidal* if its homaloidal linear system $\mathcal{H}$ consists of monoidal hypersurfaces (see Subsection 7.7.2). Assume $n = 3$ and $\mathcal{H}$ consists of surfaces of degree d with a line Γ of multiplicity $d - 1$.

(i) Suppose the base scheme of $\mathcal{H}$ consists of skew lines $\ell_1, \ldots, \ell_a$ intersecting Γ and general simple points $p_1, \ldots, p_b$ outside Γ. Show that $2a + b = 3(d - 1)$.
(ii) Show that the inverse transformation f^{-1} is also monoidal with d' satisfying $a + 2b = 3(d' - 1)$.
(iii) Find the P-locus of the Cremona transformation defined by $\mathcal{H}$.
(iv) Give examples of homaloidal linear systems $\mathcal{H}$.

7.27 Show that two planar Cremona transformations whose loci of fixed points are not birationally equivalent cannot be conjugate in the Cremona group $\mathrm{Cr}(2)$. Using this, give a finer description of conjugacy classes of planar Cremona involutions. of the corresponding Geiser involution.

7.28 Consider the Cremona transformation f from Example 7.7.21 defined by a pencil of quadrics in $\mathbb{P}^3$ and two skew lines ℓ_1 and ℓ_2.

(i) Show that there are 8 secant lines of the base curve C of the pencil of quadrics that intersect both ℓ_1 and ℓ_2.
(ii) Show that the base scheme of f consists of the lines ℓ_1 and ℓ_2 taken with multiplicity 2, the curve C and the eight secants.
(iii) Show that degree of f is equal to 5.

Historical Notes

The monograph of Hilda Hudson [414] remains the only monograph devoted exclusively to Cremona transformations of $\mathbb{P}^2$ and $\mathbb{P}^3$. Several books in algebraic geometry contain a chapter devoted to Cremona transformations. Thus, Coolidge's book gives a rather complete exposition of the theory of planar Cremona transformation and a book by Semple and Roth [701] discusses many Cremona transformations besides transformations of $\mathbb{P}^2$ and $\mathbb{P}^3$. Volume IV of Sturm's book [736] gives many examples of Cremona transformations of $\mathbb{P}^3$. There are also two small books by L. Godeaux [341], [342] that contain only some basic facts with a few examples.

A recent book by J. Deserti [222] is devoted to the Cremona group and its subgroup, the topic which we conscientiously omitted since it will lead us far afield.

The main source of references to work on Cremona transformations prior to 1927-1932 is [414] and [715].

A comprehensive history of the theory of Cremona transformations can be found in several sources [168], [414], and [715]. Here, we give only a brief sketch.

Two memoirs of L. Cremona [183] and [185] published in 1863 and 1864 initiated the general study of Cremona transformations. However, examples of birational transformations have been known since antiquity, for instance, the inversion transformation. The example of a quadratic transformation presented in Example 7.2.5 goes back to Poncelet [601], although the first idea of a general quadratic transformation must be credited to C. MacLaurin [501]. It was generally believed that all birational transformations must be quadratic, and much work was done in developing the general theory of quadratic transformations. The first transformation of arbitrary degree was constructed in 1859 by E. de Jonquières in [214], the de Jonquières transformations. His memoir remained unpublished until 1885 although he published an abstract of his work in 1864 [213]. In his first memoir [183], Cremona constructs a general de Jonquières transformation without reference to de Jonquières. We reproduced his construction in Section 7.2.3. Cremona gives credit to de Jonquières in his second paper. R. Sturm [731] first studies symmetric transformations of order five. Symmetric transformations of order 8 were first studied by C. Geiser [324] and of order 17 by E. Bertini [55]. In his second memoir, Cremona lays the foundation of the general theory of plane birational transformations. He introduces the notion of fundamental points and principal curves establishes the equalities (7.21) and (7.23), proves that the numbers of fundamental points of the transformation and its inverse coincide, principal curves are rational, and computes all possible characteristic vectors up to degree 10. The notion of a homaloidal linear system was introduced by Cremona later, first for space transformations in [189] and then for plane transformations in [190]. The word "homaloid" means flat and was used by J. Sylvester to mean a linear subspace of a projective space. More generally, it was applied by A. Cayley to rational curves and surfaces. Cremona also introduced the net of isologues and proved that the number of fixed points of a general transformation of degree d is equal to $d+2$. In the special case of de Jonquière transformations this was also done by de Jonquière in [214]. The notion of isologue curves belongs to him as well as the formula for the number of fixed points.

Hudson's book [414] discusses many special Cremona transformations in $\mathbb{P}^3$. In her words, the most interesting space transformation is the bilinear cubo-cubic transformation with the base curve of genus three and degree six. It was first constructed by L. Magnus in 1837 [503]. In modern times, bilinear transformations, under the name determinantal transformations, were studied by I. Pan [572], [575], and by G. Gonzales-Sprinberg [344].

The first major result in the theory of plane Cremona transformations after Cremona's work was Noether's Theorem. W. Clifford guessed the statement of this theorem in 1869 [153]. The original proof of M. Noether in [552] based on Noether's inequality contained a gap, which we explained in Remark 7.2.32. Independently, J. Rosanes found the same proof and made the same mistake [634]. In [554], Noether tried to correct his mistake, taking into account the presence of infinitely near fundamental points of highest multiplicities where one cannot apply a quadratic transformation. He took into account the case of infinitely near points with different tangent directions but overlooked the cuspidal case. The result was accepted for thirty years until in 1901 C. Segre pointed out that the cuspidal case was overlooked [684]. In the same year, G. Castelnuovo [97] gave a complete proof along the same lines as used in this chapter. In 1916, J. Alexander [5] raised objections to Castelnuovo's proof and gave a proof without using de Jonquières transformations [5]. This seems to be a still accepted proof. It is reproduced, for example, in [3].

The characteristic matrices of Cremona transformation were used by S. Kantor [438] and later by P. Du Val [266]. The latter clearly understood the connection to reflection groups. The description of proper homaloidal and exceptional types as orbits of the Weyl groups was essentially known to H. Hudson. There are numerous modern treatments; these started from M. Nagata [543] and culminated in the monograph of M. Alberich-Carramiñana [2]. A modern account of Clebsch's Theorem and its history can also be found there. Theorem 7.2.33 is usually attributed to Nagata, although it was known to S. Kantor and A. Coble.

The original proof of Bertini's Theorem on elliptic pencils discussed in Exercise 7.19 can be found in [55]. The Halphen pencils were studied by G. Halphen in [373]. A modern proof of Bertini's Theorem can be found in [232]. A survey of results about reducing other linear systems of plane curves by planar Cremona transformation to linear systems of curves of lower degrees can be found in [715] and in [342]. The formalism of bubble spaces originated from the classical notion of infinitely near points was first introduced by Yu. Manin [504].

The theory of decomposition of Cremona transformation via composition of elementary birational isomorphisms between minimal ruled surfaces has a vast generalization to higher dimensions under the name *Sarkisov program* (see [171]).

We intentionally omitted the discussion of finite subgroups of the Cremona group Cr(2); the modern account of this classification and its history can be found in [252].

The term Arguesian involution is in honor of a projective geometer J.L.A. Argues de Gratigny.

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