Classical Algebraic Geometry: a modern view.II

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Preface

The main purpose of the present treatise is to give an account of some of the topics in algebraic geometry, which, while having occupied the minds of many mathematicians in previous generations, have fallen out of fashion in modern times. Often in the history of mathematics, new ideas and techniques make the work of previous generations of researchers obsolete. This mainly refers to the foundations of the subject and the fundamental general theoretical facts used heavily in research. Even the greatest achievements of the past generations, which can be found, for example, in the work of F. Severi on algebraic cycles or in the work of O. Zariski in the theory of algebraic surfaces, have been greatly generalized and clarified so that they now remain only of historical interest. In contrast, the fact that a nonsingular cubic surface has 27 lines or a plane quartic has 28 bitangents cannot be improved and continues to fascinate modern geometers. One of the goals of this present work is then to save from oblivion the work of many mathematicians who discovered these classic tenets and many other beautiful results.

In writing this book the greatest challenge the author has faced was distilling the material down to what should be covered. The number of concrete facts, examples of special varieties, and beautiful geometric constructions that have accumulated during the classical period of the development of algebraic geometry is enormous, and what the reader is going to find in the book is only the tip of the iceberg; a work that is like a taste sampler of classical algebraic geometry. It avoids most of the material found in other modern books on subject, such as, for example, [10], where one can find many classical results on algebraic curves. Instead, it tries to assemble or, in other words, to create a compendium of material that either cannot be found, is too dispersed to be found easily, or is not treated adequately by contemporary research papers. On the other hand, while most of the material treated in the book exists in classical treatises in algebraic geometry, their somewhat archaic terminology, and what is by now completely forgotten background knowledge makes these books useful to but a handful of experts in classical literature. Lastly, one must admit that the author's personal taste also has much sway in the choice of material.

The reader should be warned that the book is by no means an introduction to algebraic geometry. Although some of the exposition can be followed with only a minimum background in algebraic geometry, for example, based on Shafarevich's book [708] it often relies on current cohomological techniques, such as those found in Hartshorne's book [379]. The idea was to reconstruct a result by using modern techniques but not necessarily its original proof. For one, the ingenious geometric constructions in those proofs were often beyond the author's abilities to follow them completely. Understandably, the price of this was often to replace a beautiful geometric argument with a dull cohomological one. For those looking for a less demanding sample of some of the topics covered in the book, the recent beautiful book [54] may be of great use.

No attempt has been made to give a complete bibliography. To give an idea of such an enormous task, one could mention that the report on the status of topics in algebraic geometry submitted to the National Research Council in Washington in 1928 [715] contains more than 500 items of bibliography by 130 different authors only on the subject of planar Cremona transformations (covered in one of the chapters of the present book.) Another example is the bibliography on cubic surfaces compiled by J. E. Hill [394] in 1896 which alone contains 205 titles. Meyer's article [515] cites around 130 papers published between 1896 and 1928. The title search in MathSciNet reveals more than 200 papers refereed since 1940, many of them published only in the past 20 years. How sad it is when one considers the impossibility of saving from oblivion so many names of researchers of the past who have contributed so much to our subject.

A word about exercises: some of them are easy and follow from the definitions, and some are hard; they are included to provide additional facts not covered in the main text. In this case, we sometimes indicate the sources for the statements and solutions.

I am very grateful to many people for their comments and corrections to many previous versions of the manuscript. I am especially thankful to Sergey Tikhomirov, whose help in the mathematical editing of the book was essential for getting rid of many mistakes in the previous versions. The author bears sole responsibility for all the errors still found in the book.

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8.1 First Properties

8.1.1 Surfaces of degree d in $\mathbb{P}^d$

Recall that a subvariety $X \subset \mathbb{P}^n$ is called *nondegenerate* if it is not contained in a proper linear subspace. All varieties we consider here are assumed to be reduced. Let $d = \deg(X)$. We have the following well-known (i.e., can be found in modern text-books, e.g. [360], [375]) result.

Theorem 8.1.1. *Let X be an irreducible nondegenerate subvariety of $\mathbb{P}^n$ of dimension k and degree d. Then, $d \geq n - k + 1$, and the equality holds only in one of the following cases:*

(i) X is a quadric hypersurface;
(ii) X is a Veronese surface V_2^4 in $\mathbb{P}^5$;
(iii) X is a cone over a Veronese surface V_2^4 in $\mathbb{P}^5$;
(iv) X is a rational normal scroll.

Recall that a *rational normal scroll* is defined as follows. Choose k disjoint linear subspaces $L_1, \ldots, L_k$ in $\mathbb{P}^n$ that together span the space. Let $a_i = \dim L_i$. We have $\sum_{i=1}^k a_i = n - k + 1$. Consider Veronese maps $\mathsf{v}_{a_i} : \mathbb{P}^1 \to L_i$ and define $S_{a_1,\ldots,a_k;n}$ to be the union of linear subspaces spanned by the points $\mathsf{v}_{a_1}(x), \ldots, \mathsf{v}_{a_k}(x)$, where $x \in \mathbb{P}^1$. It is clear that $\dim S_{a_1,\ldots,a_k;n} = k$ and it is easy to see that $\deg S_{a_1,\ldots,a_k;n} = a_1 + \cdots + a_k$. Here, we assume that $a_1 \leq a_2 \leq \ldots \leq a_k$.

Note that, in the special case when $k = 1$, the definition of a rational normal scroll coincides with the definition of a Veronese curve of degree n. . For this reason, a Veronese curve is often called a *rational normal curve* of degree n (in classical terminology, *rational norm curve*).

A rational normal scroll $S_{a_1,a_2,n}$ of dimension 2 with $a_1 = a, a_2 = n - 1 - a$

will be re-denoted by $S_{a,n}$. Its degree is $n-1$ and it lies in $\mathbb{P}^n$. For example, $S_{1,3}$ is a nonsingular quadric in $\mathbb{P}^3$ and $S_{0,3}$ is an irreducible quadric cone.

Corollary 8.1.2. *Let S be an irreducible nondegenerate surface of degree d in $\mathbb{P}^n$. Then, $d \geq n-1$ and the equality holds only in one of the following cases:*

(i) X is a nonsingular quadric in $\mathbb{P}^3$;
(ii) X is a quadric cone in $\mathbb{P}^3$;
(iii) X is a Veronese surface $v_2(\mathbb{P}^2)$ in $\mathbb{P}^5$;
(iv) X is a rational normal scroll $S_{a,n} \subset \mathbb{P}^n$.

The del Pezzo surfaces come next. Let X be an irreducible nondegenerate surface of degree d in $\mathbb{P}^d$. A general hyperplane section H of X is an irreducible curve of degree d. Let $p_a = h^1(X, \mathcal{O}_X)$ denote its arithmetic genus. There are two possibilities: $p_a = 0$ or $p_a = 1$. In fact, projecting to $\mathbb{P}^3$ from a general set of $d-3$ nonsingular points, we get an irreducible curve H' of degree 4 in $\mathbb{P}^3$. Taking nine general points in H', we find an irreducible quadric surface Q containing H'. If Q is singular, then its singular point lies outside H'. We assume that Q is nonsingular, the other case is considered similarly. Let $\mathfrak{f}_1$ and $\mathfrak{f}_2$ be the divisor classes of the two rulings generating $\mathrm{Pic}(Q)$. Then, $H' \in |a\mathfrak{f}_1 + b\mathfrak{f}_2|$ with $a, b \geq 0$ and $a + b = \deg H' = 4$. This gives $(a,b) = (3,1), (1,3)$, or $(2,2)$. In the first two cases, $p_a(H') = 0$. In the third case, $p_a(H') = 1$.

Proposition 8.1.3. *An irreducible nondegenerate surface X of degree d in $\mathbb{P}^d$ with hyperplane sections of arithmetic genus equal to 0 is isomorphic to a projection of a surface of degree d in $\mathbb{P}^{d+1}$.*

Proof Obviously, X is a rational surface. Assume that X is embedded in $\mathbb{P}^d$ by a complete linear system; otherwise, it is a projection from a surface of the same degree in $\mathbb{P}^{N+1}$. A birational map $f : \mathbb{P}^2 \dashrightarrow X$ is given by a linear system $|mh - \eta|$ for some bubble cycle $\eta = \sum m_i x_i$. We have

$$d = \deg X = m^2 - \sum_{i=1}^{N} m_i^2,$$

$$r = \dim |mh - \eta| \geq \tfrac{1}{2}\big(m(m+3) - \sum_{i=1}^{n} m_i(m_i+1)\big).$$

Since hyperplane sections of X are curves of arithmetic genus 0, we get

$$(m-1)(m-2) = \sum_{i=1}^{N} m_i(m_i - 1).$$

Combining all this together, we easily get

$$r \geq d + 1.$$

Since X is nondegenerate, $r = d + 1$. Thus, X is a surface of degree d in $\mathbb{P}^{d+1}$ and we get a contradiction. □

Recall that an irreducible reduced curve of arithmetic genus $p_a = 0$ is a nonsingular rational curve. It follows from the proposition that every surface X embedded in $\mathbb{P}^n$ by a complete linear system with rational hyperplane sections has degree $n + 1$. By Corollary 8.1.2, it must be either a scroll or a Veronese surface. For example, if we take $m = 4, N = 3, m_1 = m_2 = m_3 = 2$, we obtain a surface of degree 4 in $\mathbb{P}^5$. It is a Veronese surface in disguise. Indeed, if we compose the map with a quadratic transformation with fundamental points at x_1, x_2, x_3, we obtain that the image is given by the linear system of conics in the plane, so the image is a Veronese surface. On the hand, if we take $m = 3, N = 1, m_1 = 2$, we get a surface X of degree 5 in $\mathbb{P}^6$. The family of lines through the point x_1 is mapped to a ruling of lines on X, so X is a scroll.

Proposition 8.1.4. *Suppose X is a scroll of degree d in $\mathbb{P}^d$, $d > 3$, that is not a cone. Then, X is a projection of a scroll of degree d in $\mathbb{P}^{d+1}$.*

Proof Projecting a scroll from a point on the surface, we get a surface of degree d' in $\mathbb{P}^{d-1}$ satisfying

$$d = kd' + 1, \tag{8.1}$$

where k is the degree of the rational map defined by the projection. Since the image of the projection is a nondegenerate surface, we obtain $d' \geq d - 2$, the only solution is $k = 1$ and $d' = d - 1$. Continuing in this way, we arrive at a cubic surface in $\mathbb{P}^3$. By Proposition 8.1.6, it is a cone, hence it is a rational surface. We will see later, in Subsection 10.4.1 that a rational scroll is a projection of a normal rational scroll $S_{a,n}$ of degree $n - 1$ in $\mathbb{P}^n$. □

The classical definition of a del Pezzo surface is the following:

Definition 8.1.5. *A* del Pezzo surface *is a nondegenerate irreducible surface of degree d in $\mathbb{P}^d$ that is not a cone and not isomorphic to a projection of a surface of degree d in $\mathbb{P}^{d+1}$.*

According to the classical definition (see [701], 4.5.2), a subvariety X is called *normal subvariety* if it is not a projection of a subvariety of the same degree.

Recall that a closed nondegenerate subvariety X of degree d in $\mathbb{P}^n$ is called *linearly normal* if the restriction map

$$r : H^0(\mathbb{P}^n, \mathcal{O}_X(1)) \to H^0(X, \mathcal{O}_X(1)) \tag{8.2}$$

is bijective.

The relation between the two definitions is the following one.

Proposition 8.1.6. *Suppose X is a normal nondegenerate subvariety in $\mathbb{P}^n$. Then, X is linearly normal. Conversely, if X is linearly normal and normal (i.e. coincides with its normalization), then it is a normal subvariety.*

Proof It is clear that X is nondegenerate if and only if r is injective. If it is not surjective, linear system $|\mathcal{O}_X(1)|$ embeds X in $\mathbb{P}^m$ with $m > n$ with the image X' of the same degree, and X is a projection of X.

Conversely, suppose the restriction map r is surjective and X is a projection of X' of the same degree. The center of the projection does not belong to X', so the projection is a regular map $p : X' \to X$. We have $p^*\mathcal{O}_X(1) \cong \mathcal{O}_{X'}(1)$. By the projection formula $p_*p^*\mathcal{O}_{X'}(1) \cong \mathcal{O}_X(1) \otimes \pi_*\mathcal{O}_{X'}$. Since X is normal, $p_*\mathcal{O}_{X'} \cong \mathcal{O}_X$ (see [379, Chapter III, §11]). Thus, the canonical homomorphism

$$H^0(X, \mathcal{O}_X(1)) \to H^0(X', \mathcal{O}_{X'}(1)) \cong H^0(X, p_*p^*\mathcal{O}_X(1)) \tag{8.3}$$

is bijective. Since r is bijective,

$$\dim H^0(X, \mathcal{O}_X(1)) = \dim H^0(X', \mathcal{O}_{X'}(1)) = n + 1.$$

Since X' is nondegenerate, $\dim H^0(X', \mathcal{O}_{X'}(1)) \geq n + 2$. This contradiction proves the assertion. □

Let $S_d \subset \mathbb{P}^d$ be a del Pezzo surface. Assume $d \geq 4$. As in the proof of Proposition 8.1.4, we project S_d from a general subset of $d - 3$ nonsingular points to obtain a cubic surface S_3 in $\mathbb{P}^3$. Suppose S_3 is a cone over a cubic curve with vertex x_0. A general plane section of S_3 is the union of three concurrent lines. Its pre-image in S_4 is the union of four lines passing through the pre-image x_0' of x_0. This means that the point x_0' is a singular point of multiplicity 4 equal to the degree of S_4. Clearly, it must be a cone. Proceeding in this way back to S_d, we obtain that S_d is a cone, a possibility that we have excluded. Next, assume that S_3 is not a normal surface. We will see later that it must be a scroll. A general hyperplane section of S_4 passing through the center of the projection $S_4 \dashrightarrow S_3$ is a curve of degree 4 and arithmetic genus 1. Its image in S_3 is a curve of degree 3 and arithmetic genus 1. So, it is not a line. The pre-image of a general line on S_3 must be a line on S_4. Thus, S_4 is a scroll.

Going back to S_d, we obtain that S_d is a scroll. This has been also excluded. Thus, we obtain that a general projection of S_d from a set of $d-3$ nonsingular points is a normal cubic surface.

Let us derive immediate corollaries of this.

Proposition 8.1.7. *The degree d of a del Pezzo surface S_d is less than or equal to* 9.

Proof We follow the original argument of del Pezzo. Let $S_d \dashrightarrow S_{d-1}$ be the projection from a general point $p_1 \in S_d$. It extends to a regular map $S'_d \to S_{d-1}$, where S'_d is the blow-up of p_1. The image of the exceptional curve E_1 of the blow-up is a line ℓ_1 in S'_d. Let $S_{d-1} \to S_{d-2}$ be the projection from a general point in S_{d-1}. We may assume that the projection map $S_d \dashrightarrow S_{d-1}$ is an isomorphism over p_2 and that p_2 does not lie on ℓ_1. Continuing in this way, we arrive at a normal cubic surface S_3, and the images of lines ℓ_1, and so on, will be a set of disjoint lines on S_3. We will see later that a normal cubic surface does not have more than six skew lines. This shows that $d \leq 9$. □

Proposition 8.1.8. *A del Pezzo surface S_d is a normal surface (i.e. coincides with its normalization in the field of rational functions).*

Proof We follow the same projection procedure as in the previous proof. The assertion is true for $d = 3$. The map $S'_4 \to S_3$ is birational map onto a normal surface. Since we may assume that the center p of the projection $S_4 \dashrightarrow S_3$ does not lie on a line, the map is finite and of degree 1. Since S_3 is normal, it must be an isomorphism. In fact, the local ring A of a point $x \in S'_4$ is integral over the local ring A' of its image x' and both rings have the same fraction field Q. Thus, the integral closure of A in Q is contained in the integral closure of A' equal to A'. This shows that A coincides with A'. Thus, we see that S_4 is a normal surface. Continuing in this way, we get that $S_5, \ldots, S_d$ are normal surfaces. □

8.1.2 Rational double points

Here, we recall without proof some facts about rational double points (RDP) singularities which we will often use later. The proofs can be found in many sources, for example, [24], [611], [586].

Recall that we say that a variety X has *rational singularities* if there exists a resolution of singularities $\pi : Y \to X$ such that $R^i\pi_*\mathcal{O}_Y = 0, i > 0$. One can show that, if there exists one resolution with this property, any resolution of singularities satisfies this property. Also, one can give a local definition of a rational singularity $x \in X$ by requiring that the stalk $(R^i\pi_*\mathcal{O}_Y)_x$ vanishes for $i > 0$. Note that a nonsingular point is, by definition, a rational singularity.

We will be interested in rational singularities of normal algebraic surfaces. Let $\pi : Y \to X$ be a resolution of singularities. We can always choose it to be minimal in the sense that it does not factor nontrivially through another resolution of singularities. This is equivalent to the property that the fibers of π do not contain (-1)-curves. A minimal resolution always exists and is unique, up to isomorphism. A curve in the fiber $\pi^{-1}(x)$ is called an *exceptional curve*.

Let $Z = \sum n_i E_i$, where $n_i \geq 0$ and E_i are irreducible components of $\pi^{-1}(x)$, called *exceptional components*. We say that Z is a *fundamental cycle* if $Z \cdot E_i \leq 0$ for all E_i and Z is minimal (in terms of order on the set of effective divisors) with this property. A fundamental cycle always exists and is unique.

Proposition 8.1.9. *The following properties are equivalent:*

(i) x is a rational singularity;
(ii) the canonical maps $\pi^* : H^i(X, \mathcal{O}_X) \to H^i(Y, \mathcal{O}_Y)$ are bijective;
(iii) for every curve (not necessarily reduced) Z supported in $\pi^{-1}(x)$, one has $H^1(Z, \mathcal{O}_Z) = 0$;
(iv) for every curve Z supported in $\pi^{-1}(x)$, $p_a(Z) := 1 + \frac{1}{2}Z \cdot (Z + K_Y) \leq 0$.

Recall that the *multiplicity* of a point x on a variety X is the multiplicity of the maximal ideal $\mathfrak{m}_{X,x}$ defined in any textbook in commutative algebra. If X is a hypersurface, the multiplicity is equal to the degree of the first nonzero homogeneous part in the Taylor expansion of the affine equation of X at the point x.

If x is a rational surface singularity, then $-Z^2$ is equal to its multiplicity, and $-Z^2+1$ is equal to the embedding dimension of x (the dimension of $\mathfrak{m}_{X,x}/\mathfrak{m}_{X,x}^2$) [24, Corollary 6]. It follows that a rational double point is locally isomorphic to a hypersurface singularity, and hence, it is a Gorenstein singularity. The converse is also true, a rational Gorenstein surface singularity has multiplicity 2.

Suppose now that x is a rational double point of a normal surface X. Then, each exceptional component E satisfies $H^1(E, \mathcal{O}_E) = 0$. This implies that $E \cong \mathbb{P}^1$. Since the resolution is minimal, $\mathsf{E}^2 \leq -2$. By the adjunction formula, $E^2 + E \cdot K_Y = -2$ implies $E \cdot K_Y \geq 0$. Let $Z = \sum n_i E_i$ be a fundamental cycle. Then, by (iii) from above,

$$0 = 2 + Z^2 \leq -Z \cdot K_Y = -\sum n_i (E_i \cdot K_Y).$$

This gives $E_i \cdot K_Y = 0$ for every E_i. By the adjunction formula, $E_i^2 = -2$.

Let K_X be a canonical divisor on X. This is a Weyl divisor, the closure of a canonical divisor on the open subset of nonsingular points. Let $\pi^*(K_X)$ be its pre-image on Y. We can write

$$K_Y = \pi^*(K_X) + \Delta,$$

where Δ is a divisor supported in $\pi^{-1}(x)$. Suppose x is a Gorenstein singularity. This means that ω_X is locally free at x, i.e., one can choose a representative of K_X which is a Cartier divisor in an open neighborhood of x. Thus, we can choose a representative of $\pi^*(K_X)$ which is disjoint from $\pi^{-1}(x)$. For any exceptional component E_i, we have

$$0 = K_Y \cdot E_i = E_i \cdot \pi^*(K_X) + E_i \cdot \Delta = E_i \cdot \Delta.$$

It is known that the *intersection matrix* $(E_i \cdot E_j)$ of exceptional components is negative definite [538]. This implies that $\Delta = 0$.

To sum up, we have the following.

Proposition 8.1.10. *Let $\pi : Y \to X$ be a minimal resolution of a rational double point x on a normal surface X. Then, each exceptional component of π is a (-2)-curve and $K_Y = \pi^*(K_X)$.*

8.1.3 A blow-up model of a del Pezzo surface

Let us show that a del Pezzo surface satisfies the following properties that we will take for a more general definition of a del Pezzo surface.

Theorem 8.1.11. *Let S be a del Pezzo surface of degree d in $\mathbb{P}^d$. Then, all its singularities are rational double points and ω_S^{-1} is an ample invertible sheaf.*

Proof The assertion is true if $d = 3$. It follows from the proof of Proposition 8.1.8 that S is isomorphic to the blow-up of a cubic surface at $d - 3$ nonsingular points. Thus, the singularities of S are isomorphic to singularities of a cubic surface which are RDP. In particular, the canonical sheaf ω_S of S is an invertible sheaf.

Let C be a general hyperplane section. It defines an exact sequence

$$0 \to \mathcal{O}_S \to \mathcal{O}_S(1) \to \mathcal{O}_C(1) \to 0.$$

Tensoring by ω_S, and applying the adjunction formula for C, we obtain an exact sequence

$$0 \to \omega_S \to \omega_S(1) \to \omega_C \to 0.$$

Applying Serre's duality and Proposition 8.1.9, we obtain

$$H^1(S, \omega_S) \cong H^1(S, \mathcal{O}_S) = 0.$$

Since C is an elliptic curve, $\omega_C \cong \mathcal{O}_C$. The exact sequence implies that $H^0(S, \omega_S(1)) \neq 0$. Let D be an effective divisor defined by a nonzero section of $\omega_S(1)$. By the adjunction formula, its restriction to a general hyperplane section is zero. Thus, D is zero. This shows that $\omega_S(1) \cong \mathcal{O}_S$, hence $\omega_S \cong \mathcal{O}_S(-1)$. In particular, $\omega_S^{-1} \cong \mathcal{O}_S(1)$ is ample (in fact, very ample). □

Definition 8.1.12. *A normal algebraic surface S is called a* del Pezzo surface *if its canonical sheaf ω_S is invertible, ω_S^{-1} is ample and all singularities are rational double points.*

By the previous Theorem and by Propositions 8.1.8, a del Pezzo surface of degree d in $\mathbb{P}^d$ is a del Pezzo surface in this new definition. Note that one takes a more general definition of a del Pezzo surface without assuming the normality property (see [612]). However, we will not pursue this.

Let $\pi : X \to S$ be a minimal resolution of singularities of a del Pezzo surface. Our goal is to show that X is a rational surface isomorphic either to a minimal rational surface $\mathbf{F}_0$, or $\mathbf{F}_2$, or is obtained from $\mathbb{P}^2$ by blowing up a bubble cycle of length ≤ 8.

Lemma 8.1.13. *Any irreducible reduced curve C on X with negative self-intersection is either a (-1)-curve or (-2)-curve.*

Proof By adjunction,

$$C^2 + C \cdot K_S = \deg \omega_C = 2 \dim H^1(C, \mathcal{O}_C) - 2 \geq -2.$$

By Proposition 8.1.10, the assertion is true if C is an exceptional curve of the resolution of singularities $\pi : X \to S$. Suppose $\pi(C) = C'$ is a curve. Since $-K_S$ is ample, there exists some $m > 0$ such that $|-mK_S|$ defines an isomorphism of S onto a surface S' in $\mathbb{P}^n$. Thus, $|-mK_X|$ defines a morphism $X \to S'$ which is an isomorphism outside the exceptional divisor of π. Taking a general section in $\mathbb{P}^n$, we obtain that $-mK_X \cdot C > 0$. By the adjunction formula, the only possibility is $C^2 = -1$, and $H^1(C, \mathcal{O}_C) = 0$. □

Recall that a divisor class D on a nonsingular surface X is called *nef* if $D \cdot C \geq 0$ for any curve C on X. It is called *big* if $D^2 > 0$. It follows from the proof of the previous Lemma that $-K_X$ is nef and big.

Lemma 8.1.14. *Let X be a minimal resolution of a del Pezzo surface S. Then*

$$H^i(X, \mathcal{O}_X) = 0, \ i \neq 0.$$

Proof Since S has rational double points, by Proposition 8.1.10, the sheaf ω_S is an invertible ample sheaf and

$$\omega_X \cong \pi^*(\omega_S). \tag{8.4}$$

Since, $\omega_S \cong \mathcal{O}_S(-A)$ for some ample divisor A, we have $\omega_X \cong \mathcal{O}_X(-A')$, where $A' = \pi^*(A)$ is nef and big. We write $0 = K_X + A$ and apply Ramanujam's Vanishing Theorem ([604], [479], vol. I, Theorem 4.3.1): for any nef and big divisor D on a nonsingular projective variety X

$$H^i(X, \mathcal{O}_X(K_X + D)) = 0, \ i > 0.$$

□

Theorem 8.1.15. *Let X be a minimal resolution of a del Pezzo surface. Then, either $X \cong \mathbf{F}_0$, or $X \cong \mathbf{F}_2$, or X is obtained from $\mathbb{P}^2$ by blowing up $N \leq 8$ points in the bubble space.*

Proof Let $f : X \to X'$ be a morphism onto a minimal model of X. Since $-K_X$ is nef and big, $K_{X'} = f_*(K_X)$ is not nef but big. It follows from the classification of algebraic surfaces that X' is a minimal ruled surface. Assume X' is not a rational surface. By Lemma 8.1.14, $H^1(X', \mathcal{O}_{X'}) = 0$. If $p : X' \to B$ is a ruling of X', we must have $B \cong \mathbb{P}^1$ (use that the projection $p : X' \to B$ satisfies $p_*\mathcal{O}_{X'} \cong \mathcal{O}_B$ and this defines a canonical injective map $H^1(B, \mathcal{O}_B) \to H^1(X', \mathcal{O}_{X'})$). Thus, $X' = \mathbf{F}_n$ or $\mathbb{P}^2$. Assume $X' = \mathbf{F}_n$. If $n > 2$, the proper transform in X of the exceptional section of X' has self-intersection $-r \leq -n < -2$. This contradicts Lemma 8.1.13. Thus, $n \leq 2$. If $n = 1$, then composing the map $X' = \mathbf{F}_1 \to \mathbb{P}^2$, we obtain a birational morphism $X \to X' \to \mathbb{P}^2$, so the assertion is verified.

Assume $n = 2$, and the birational morphism $f : X \to X' = \mathbf{F}_2$ is not an isomorphism. Then, it is an isomorphism over the exceptional section (otherwise we get a curve on X with self-intersection < -2). Thus, it factors through a birational morphism $f : X \to Y \to \mathbf{F}_2$, where Y is the blow-up of a point $y \in \mathbf{F}_2$ not on the exceptional section. Let $Y \to Y'$ be the blow-down morphism of the proper transform of a fiber of the ruling of $\mathbf{F}_2$ passing through the point y. Then, Y' is isomorphic to $\mathbf{F}_1$, and the composition $X \to X' \to Y \to Y' \to \mathbb{P}^2$ is a birational morphism to $\mathbb{P}^2$.

Assume $n = 0$ and $f : X \to \mathbf{F}_2$ is not an isomorphism. Again, we factor f as the composition $X \to Y \to \mathbf{F}_0$, where $Y \to \mathbf{F}_0$ is the blow-up of a point $y \in \mathbf{F}_0$. Blowing down the proper transforms of the lines through y, we get a morphism $Y \to \mathbb{P}^2$ and the composition $X \to Y \to \mathbb{P}^2$.

The last assertion follows from the known behavior of the canonical class under a blow-up. If $\pi : S \to \mathbb{P}^2$ is a birational morphism which is a composition of N blow-ups, then

$$K_X^2 = K_{\mathbb{P}^2}^2 - N = 9 - N. \tag{8.5}$$

Since $K_X^2 > 0$, we obtain $N < 9$.

□

Definition 8.1.16. *The number* $d = K_X^2$ *is called the* degree *of a del Pezzo surface.*

It is easy to see that it does not depend on a minimal resolution of S. Note that this definition agrees with the definition of the degree of a del Pezzo surface $S \subset \mathbb{P}^d$ in its classical definition. Indeed, let H be a hyperplane section of S, the intersection theory of Cartier divisors show that

$$d = H^2 = \pi^*(H)^2 = p^*(-K_S)^2 = (-K_X)^2 = K_X^2.$$

Suppose S is a nonsingular del Pezzo surface. Since $K_{\mathbf{F}_2}$ is not ample, we obtain the following.

Corollary 8.1.17. *Assume that* S *is a nonsingular del Pezzo surface. Then,* $S \cong \mathbf{F}_0$ *or is obtained by blowing-up of a bubble cycle in* $\mathbb{P}^2$ *of* ≤ 8 *points.*

Definition 8.1.18. *A* weak del Pezzo surface *is a nonsingular surface* S *with* $-K_S$ *nef and big.*

So, we see that a minimal resolution of a singular del Pezzo surface is a weak del Pezzo surface. The proof of Theorem 8.1.15 shows that a weak del Pezzo surface is isomorphic to $\mathbf{F}_0, \mathbf{F}_2$ or to the blow-up of a bubble cycle on $\mathbb{P}^2$ that consists of ≤ 8 points.

Remark 8.1.19. Recall that a Fano variety is a nonsingular projective variety X with $-K_X$ ample. A quasi-Fano variety is a nonsingular variety with $-K_X$ big and nef. Thus, a nonsingular del Pezzo surface is a Fano variety of dimension 2, and a weak del Pezzo surface is a quasi-Fano variety of dimension 2.

Definition 8.1.20. *A* blowing down structure *on a weak del Pezzo surface* S *is a composition of birational morphisms*

$$\pi : S = S_N \xrightarrow{\pi_N} S_{N-1} \xrightarrow{\pi_{N-1}} \dots \xrightarrow{\pi_2} S_1 \xrightarrow{\pi_1} \mathbb{P}^2,$$

where each $\pi : S_i \to S_{i-1}$ *is the blow-up a point* x_i *in the bubble space of* $\mathbb{P}^2$.

A blowing-down structure of a weak del Pezzo surface defines a geometric basis $(e_0, e_1, \dots, e_N)$ in $\text{Pic}(S)$ (see Subsection 7.2.4). A choice of a geometric basis defines an isomorphism of free abelian groups

$$\phi : \mathbb{Z}^{N+1} \to \text{Pic}(S) \quad \text{such that } \phi(k_N) = K_S,$$

where $k_N = -3\mathbf{e}_0 + \mathbf{e}_1 + \cdots + \mathbf{e}_N$. The class e_0 is the full pre-image of the class h of a line in the plane, and the classes e_i are the divisor classes of the exceptional configurations $\mathcal{E}_i$. We call such an isomorphism a *geometric marking*.

Definition 8.1.21. *A pair (S,ϕ), where S is a weak del Pezzo surface and ϕ is a marking (resp. geometric marking) $\mathbb{Z}^{N+1} \to \mathrm{Pic}(S)$ is called a* marked *(resp.* geometrically marked*) weak del Pezzo surface.*

The bubble cycle η appearing in a blowing-up model of a weak del Pezzo surface must satisfy some restrictive conditions. Let us find them.

Lemma 8.1.22. *Let X be a nonsingular projective surface with $H^1(X,\mathcal{O}_X)=0$. Let C be an irreducible curve on X such that $|-K_X-C|\neq\emptyset$ and $C\notin|-K_X|$. Then, $C\cong\mathbb{P}^1$.*

Proof We have $-K_X\sim C+D$ for some nonzero effective divisor D, and hence $K_X+C\sim -D\nsim 0$. This shows that $|K_X+C|=\emptyset$. By Riemann-Roch,

$$0=h^0(\mathcal{O}_X(K_X+C))=\tfrac{1}{2}((K_X+C)^2-(K_X+C)\cdot K_X)+1$$

$$-h^1(\mathcal{O}_X)+h^2(\mathcal{O}_X)\geq 1+\tfrac{1}{2}(C^2+K_X\cdot C)=h^1(\mathcal{O}_C).$$

Thus, $H^1(C,\mathcal{O}_C)=0$, and, as we noted earlier, this implies that $C\cong\mathbb{P}^1$. □

Proposition 8.1.23. *Let S be a weak del Pezzo surface.*

(i) Let $f:S\to\bar{S}$ be a blowing down of a (-1)-curve E. Then, $\bar{S}$ is a weak del Pezzo surface.
(ii) Let $\pi:S'\to S$ be the blowing-up with center at a point x not lying on any (-2)-curve. Assume $K_S^2>1$. Then, S' is a weak del Pezzo surface.

Proof (i) We have $K_S=f^*(K_{\bar{S}})+E$, and hence, for any curve C on $\bar{S}$, we have

$$K_{\bar{S}}\cdot C=f^*(K_{\bar{S}})\cdot f^*(C)=(K_S-E)\cdot f^*(C)=K_S\cdot f^*(C)\leq 0.$$

Also, $K_{\bar{S}}^2=K_S^2+1>0$. Thus, $\bar{S}$ is a weak del Pezzo surface.

(ii) Since $K_S^2>1$, we have $K_{S'}^2=K_S^2-1>0$. By Riemann-Roch,

$$\dim|-K_{S'}|\geq\tfrac{1}{2}((-K_{S'})^2-(-K_{S'}\cdot K_{S'}))=K_{S'}^2\geq 0.$$

Thus, $|-K_{S'}|\neq\emptyset$, and hence, any irreducible curve C with $-K_{S'}\cdot C<0$ must be a proper component of some divisor from $|-K_{S'}|$ (it cannot be linearly equivalent to $-K_{S'}$ because $(-K_{S'})^2>0$). Let $E=\pi^{-1}(x)$. We have $-K_{S'}\cdot E=1>0$. So, we may assume that $C\neq E$. Let $\bar{C}=f(C)$. We have

$$-K_{S'}\cdot C=\pi^*(-K_S)\cdot C-E\cdot C=-K_S\cdot\bar{C}-\mathrm{mult}_x(\bar{C}).$$

Since $f_*(K_{S'})=K_S$ and $C\neq E$, the curve $\bar{C}$ is a proper irreducible component of some divisor from $|-K_S|$. By Lemma 8.1.22, $\bar{C}\cong\mathbb{P}^1$. Thus, $\mathrm{mult}_x\bar{C}\leq 1$ and hence $0>-K_{S'}\cdot C\geq -K_S\cdot\bar{C}-1$. This gives $-K_S\cdot\bar{C}=0$ and $x\in\bar{C}$

and hence $\bar{C}$ is a (-2)-curve. Since x does not lie on any (-2)-curve we get a contradiction. □

Corollary 8.1.24. *Let $\eta = \sum_{i=1}^r x_i$ be a bubble cycle on $\mathbb{P}^2$ and S_η be its blow-up. Then, S_η is a weak del Pezzo surface if and only if*

(i) $r \leq 8$;
(ii) the Enriques diagram of η is the disjoint union of chains;
(iii) $|\mathcal{O}_{\mathbb{P}^2}(1) - \eta'| = \emptyset$ for any $\eta' \subset \eta$ consisting of four points;
(iv) $|\mathcal{O}_{\mathbb{P}^2}(2) - \eta'| = \emptyset$ for any $\eta' \subset \eta$ consisting of seven points.

Proof The necessity of condition (i) is clear. We know that S does not contain curves with self-intersection < -2. In particular, any exceptional cycle $\mathcal{E}_i$ of the birational morphism $\pi : S \to \mathbb{P}^2$ contains only smooth rational curves E with $E^2 = -1$ or -2. This easily implies that the bubble points corresponding to each exceptional configuration $\mathcal{E}_i$ represent a totally ordered chain. This checks condition (ii).

Suppose (iii) does not hold. Let D be an effective divisor from the linear system $|\mathcal{O}_{\mathbb{P}^2}(1) - \eta'|$. We can change the admissible order on η to assume that $\eta' = x_1 + x_2 + x_3 + x_4$. Then, the divisor class of the proper transform of D in Y_η is equal to $e_0 - e_1 - e_2 - e_3 - e_4 - \sum_{i \geq 4} m_i e_i$. Its self-intersection is obviously ≤ -3.

Suppose (iv) does not hold. Let $D \in |\mathcal{O}_{\mathbb{P}^2}(2) - \eta'|$. Arguing as above, we find that the divisor class of the proper transform of D is equal to $2e_0 - \sum_{i=1}^7 e_i - \sum_{i \geq 7} m_i e_i$. Its self-intersection is again ≤ -3.

Let us prove the sufficiency. Let $\mathcal{E}_N = \pi_N^{-1}(x_N)$ be the last exceptional configuration of the blow-down $Y_\eta \to \mathbb{P}^2$. It is an irreducible (-1)-curve. Obviously, $\eta' = \eta - x_N$ satisfies conditions (i)-(iv). By induction, we may assume that $S' = S_{\eta'}$ is a weak del Pezzo surface. Applying Proposition 8.1.23, we have to show that x_N does not lie on any (-2)-curve on S'. Condition (ii) implies that it does not lie on any irreducible component of the exceptional configurations $\mathcal{E}_i, i \neq N$. We will show in the next section that any (-2)-curve on a week del Pezzo surface S' of degree ≤ 7 is either blown down to a point under the canonical map $S_{\eta'} \to \mathbb{P}^2$ or equal to the proper inverse transform of a line through three points, or a conic through five points. If x_N lies on the proper inverse transform of such a line (resp. a conic), then condition (iii) (resp. (iv)) is not satisfied. This proves the assertion. □

A set of bubble points satisfying conditions (i)-(iv) above is called a set of points in *almost general position*.

We say that the points are in *general position* if the following hold:

(i) all points are proper points;
(ii) no three points are on a line;
(iii) no six points on a conic;
(iv) no cubic passes through the points with one of the point being a singular point.

Proposition 8.1.25. *The blow-up of $N \leq 8$ points in $\mathbb{P}^2$ is a del Pezzo surface if and only if the points are in general position.*

8.2 The E_N-lattice

8.2.1 Quadratic lattices

A (quadratic) *lattice* is a free abelian group $M \cong \mathbb{Z}^r$ equipped with a symmetric bilinear form $M \times M \to \mathbb{Z}$. A relevant example of a lattice is the second cohomology group modulo torsion of a compact smooth 4-manifold (e.g., a nonsingular projective surface) with respect to the cup-product. Another relevant example is the Picard group modulo numerical equivalence of a nonsingular projective surface equipped with the intersection pairing.

The values of the symmetric bilinear form will be often denoted by (x, y) or $x \cdot y$. We write $x^2 = (x, x)$. The map $x \mapsto x^2$ is an integer-valued quadratic form on M. Conversely, such a quadratic form $q : M \to \mathbb{Z}$ defines a symmetric bilinear form by the formula $(x, y) = q(x+y)-q(x)-q(y)$. Note that $x^2 = 2q(x)$.

Let $M^\vee := \mathrm{Hom}_Z(M, \mathbb{Z})$ and

$$\iota_M : M \to M^\vee, \quad \iota_M(x)(y) = x \cdot y.$$

We say that M is *nondegenerate* if the homomorphism ι_M is injective. In this case the group

$$\mathrm{Disc}(M) = M^\vee/\iota_M(M)$$

is a finite abelian group. It is called the *discriminant group* of M. If we choose a basis to represent the symmetric bilinear form by a matrix A, then the order of $\mathrm{Disc}(M)$ is equal to $|\det(A)|$. The number $\mathrm{disc}(M) = \det(A)$ is called the *discriminant* of M. A different choice of a basis changes A to ${}^tCAC$ for some $C \in \mathrm{GL}(n, \mathbb{Z})$, so it does not change $\det(A)$. A lattice is called *unimodular* if $|\mathrm{disc}(M)| = 1$.

Tensoring M with reals, we get a real symmetric bilinear form on $M_\mathbb{R} \cong \mathbb{R}^r$. We can identify M with an abelian subgroup of the inner product space $\mathbb{R}^r$ generated by a basis in $\mathbb{R}^r$. The Sylvester signature (t_+, t_-, t_0) of the inner product space $M_\mathbb{R}$ is called the *signature* of M. We write (t_+, t_-) if $t_0 = 0$. For example, the signature of $H^2(X, \mathbb{Z})/\text{Torsion} \cong \mathbb{Z}^{b_2}$ for a nonsingular projective surface X is equal to $(2p_g + 1, b_2 - 2p_g - 1)$, where $p_g = \dim H^0(X, \mathcal{O}_X(K_X))$. This follows from the Hodge Theory (see [38, Chapter IV, §2]). The signature on the lattice of divisor classes modulo numerical equivalence $\mathrm{Num}(X) = \mathrm{Pic}(X)/_\equiv \cong \mathbb{Z}^\rho$ is equal to $(1, \rho - 1)$ (this is called the *Hodge Index Theorem*, see [379, Chapter V, Theorem 1.9]).

Let $N \subset M$ be a subgroup of M. The restriction of the bilinear form to N defines a structure of a lattice on N. We say that N together with this form is a *sublattice* of M. We say that N is of *finite index m* if M/N is a finite group of order m. Let

$$N^\perp = \{x \in M : x \cdot y = 0, \forall y \in N\}.$$

Note that $N \subset (N^\perp)^\perp$ and the equality takes place if and only if N is a *primitive sublattice* (i.e. M/N is torsion-free).

We will need the following facts.

Lemma 8.2.1. *Let M be a nondegenerate lattice and let N be its nondegenerate sublattice of finite index m. Then,*

$$|\mathrm{disc}(N)| = m^2 |\mathrm{disc}(M)|.$$

Proof Since N is of finite index in M, the restriction homomorphism $M^\vee \to N^\vee$ is injective. We will identify $M^\vee$ with its image in $N^\vee$. We will also identify M with its image $\iota_M(M)$ in $M^\vee$. Consider the chain of subgroups

$$N \subset M \subset M^\vee \subset N^\vee.$$

Choose a basis in M, a basis in N, and the dual bases in $M^\vee$ and $N^\vee$. The inclusion homomorphism $N \to M$ is given by a matrix A and the inclusion $N^\vee \to M^\vee$ is given by its transpose ${}^tA$. The order m of the quotient M/N is equal to $|\det(A)|$. The order of $N^\vee/M^\vee$ is equal to $|\det({}^tA)|$. They are equal. Now, the chain of lattices from above has the first and the last quotient of order equal to m, and the middle quotient is of order $|\mathrm{disc}(M)|$. The total quotient $N^\vee/N$ is of order $|\mathrm{disc}(N)|$. The assertion follows. □

Lemma 8.2.2. *Let M be a unimodular lattice and N be its nondegenerate primitive sublattice. Then,*

$$|\mathrm{disc}(N^\perp)| = |\mathrm{disc}(N)|.$$

Proof Consider the restriction homomorphism $r : M \to N^\vee$, where we identify M with $M^\vee$ by means of ι_M. Its kernel is equal to $N^\perp$. Composing r with the projection $N^\vee/\iota_N(N)$ we obtain an injective homomorphism

$$M/(N + N^\perp) \to N^\vee/\iota_N(N).$$

Notice that $N^\perp \cap N = \{0\}$ because N is a nondegenerate sublattice. Thus, $N^\perp + N = N^\perp \oplus N$ is of finite index i in M. Also, the sum is orthogonal, so that the matrix representing the symmetric bilinear form on $N \oplus N^\perp$ can be chosen to be a block matrix.

We denote the orthogonal direct sum of two lattices M_1 and M_2 by $M_1 \oplus M_2$. This shows that $\mathrm{disc}(N \oplus N^\perp) = \mathrm{disc}(N)\mathrm{disc}(N^\perp)$. Applying Lemma 8.2.1, we get

$$\#(M/N \oplus N^\perp) = \sqrt{|\mathrm{disc}(N^\perp)||\mathrm{disc}(N)|} \leq \#(N^\vee/N) = |\mathrm{disc}(N)|.$$

This gives $|\mathrm{disc}(N^\perp)| \leq |\mathrm{disc}(N)|$. Since $N = (N^\perp)^\perp$, exchanging the roles of N and $N^\perp$, we get the opposite inequality. □

Lemma 8.2.3. *Let N be a nondegenerate sublattice of a unimodular lattice M. Then,*

$$\iota_M(N^\perp) = \mathrm{Ann}(N) := \mathrm{Ker}(r : M^\vee \to N^\vee) \cong (M/N)^\vee.$$

Proof Under the isomorphism $\iota_M : M \to M^\vee$ the image of $N^\perp$ is equal to $\mathrm{Ann}(N)$. Since the functor $\mathrm{Hom}_{\mathbb{Z}}(-, \mathbb{Z})$ is left exact, applying it to the exact sequence

$$0 \to N \to M \to M/N \to 0,$$

we obtain an isomorphism $\mathrm{Ann}(N) \cong (M/N)^\vee$. □

A morphism of lattices $\sigma : M \to N$ is a homomorphism of abelian groups preserving the bilinear forms. If M is a nondegenerate lattice, then σ is necessarily injective. We say in this case that σ is an *embedding* of lattices. An embedding is called *primitive* if its image is a primitive sublattice. An invertible morphism of lattices is called an *isometry*. The group of isometries of a lattice M to itself is denoted by $\mathrm{O}(M)$ and is called the *orthogonal group* of M.

Let $M_{\mathbb{Q}} := M \otimes \mathbb{Q} \cong \mathbb{Q}^n$ with the symmetric bilinear form of M extended to a symmetric $\mathbb{Q}$-valued bilinear form on $M_{\mathbb{Q}}$. The group $M^\vee$ can be identified with the subgroup of $M_{\mathbb{Q}}$ consisting of vectors v such that $(v, m) \in \mathbb{Z}$ for any $m \in M$. Suppose that M is nondegenerate lattice. The finite group $\mathrm{Disc}(M)$ can be equipped with a quadratic form defined by

$$q(\bar{x}) = (x, x) \mod \mathbb{Z},$$

where $\bar{x}$ denotes a coset $x + \iota_M(M)$. If M is an *even lattice*, i.e. $m^2 \in 2\mathbb{Z}$ for all $m \in M$, then we take values modulo $2\mathbb{Z}$. The group of automorphisms of $\mathrm{Disc}(M)$ leaving the quadratic form invariant is denoted by $\mathrm{O}(\mathrm{Disc}(M))$.

The proof of the next lemma can be found in [550].

Lemma 8.2.4. *Let $M \subset N$ be a sublattice of finite index. Then, the inclusion $M \subset N \subset N^\vee \subset M^\vee$ defines the subgroup N/M in $\mathrm{Disc}(M) = M^\vee/M$ such that the restriction of the quadratic form of $\mathrm{Disc}(M)$ to it is equal to zero. Conversely, any such subgroup defines a lattice N containing M as a sublattice of finite index.*

The group $\mathrm{O}(M)$ acts naturally on the dual group $M^\vee$ preserving its bilinear form and leaving the subgroup $\iota_M(M)$ invariant. This defines a homomorphism of groups

$$\alpha_M : \mathrm{O}(M) \to \mathrm{O}(\mathrm{Disc}(M)).$$

Lemma 8.2.5. *Let N be a primitive sublattice in a nondegenerate lattice M. Then, an isometry $\sigma \in \mathrm{O}(N)$ extends to an isometry of M acting identically on $N^\perp$ if and only if $\sigma \in \mathrm{Ker}(\alpha_N)$.*

8.2.2 The E_N-lattice

Let $\mathsf{I}^{1,N} = \mathbb{Z}^{N+1}$ equipped with the symmetric bilinear form defined by the diagonal matrix $\mathrm{diag}(1, -1, \dots, -1)$ with respect to the standard basis

$$\mathbf{e}_0 = (1, 0, \dots, 0),\ \mathbf{e}_1 = (0, 1, 0, \dots, 0), \dots, \mathbf{e}_N = (0, \dots, 0, 1)$$

of $\mathbb{Z}^{N+1}$. Any basis defining the same matrix will be called an *orthonormal basis*. The lattice $\mathsf{I}^{1,N}$ is a unimodular lattice of signature $(1, N)$.

Consider the special vector in $\mathsf{I}^{1,N}$ defined by

$$\mathbf{k}_N = (-3, 1, \dots, 1) = -3\mathbf{e}_0 + \sum_{i=1}^{N} \mathbf{e}_i. \tag{8.6}$$

Definition 8.2.6. *A* E_N-lattice *is a quadrtatic lattice isomorphis to the sublattice of $\mathsf{I}^{1,N}$ given by*

$$\mathsf{E}_N = (\mathbb{Z}\mathbf{k}_N)^\perp.$$

Since $\mathbf{k}_N^2 = 9 - N$, it follows from Lemma 8.2.2, that E_N is a negative definite lattice for $N \leq 8$. Its discriminant group is a cyclic group of order $9 - N$. Its quadratic form is given by the value on its generator equal to $-\frac{1}{9-N} \mod \mathbb{Z}$ (or $2\mathbb{Z}$ if N is odd).

Lemma 8.2.7. *Assume $N \geq 3$. The following vectors form a basis of E_N*

$$\alpha_1 = \mathbf{e}_0 - \mathbf{e}_1 - \mathbf{e}_2 - \mathbf{e}_3,\ \alpha_i = \mathbf{e}_{i-1} - \mathbf{e}_i,\ i = 2, \dots, N.$$

The matrix of the symmetric bilinear form of E_N with respect to this basis is equal to

$$C_N = \begin{pmatrix} -2 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & \dots & 0 \\ 0 & -2 & 1 & 0 & 0 & 0 & 0 & 0 & \dots & 0 \\ 0 & 1 & -2 & 1 & 0 & 0 & 0 & 0 & \dots & 0 \\ 1 & 0 & 1 & -2 & 1 & 0 & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & 1 & -2 & 1 & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & 0 & 1 & -2 & 1 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & \dots & 0 & 0 & 0 & -2 & 1 \end{pmatrix} \tag{8.7}$$

Proof By inspection, each $\boldsymbol{\alpha}_i$ is orthogonal to $\mathbf{k}_N$. Suppose $(a_0, a_1, \dots, a_N)$ is orthogonal to $\mathbf{k}_N$. Then,

$$3a_0 + a_1 + \cdots + a_N = 0. \tag{8.8}$$

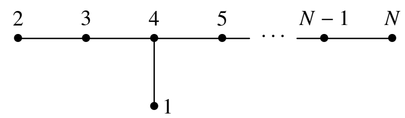
We can write this vector as follows:

$$(a_0, a_1, \dots, a_N) = a_0\boldsymbol{\alpha}_1 + (a_0 + a_1)\boldsymbol{\alpha}_2 + (2a_0 + a_1 + a_2)\boldsymbol{\alpha}_3$$

$$+(3a_0 + a_1 + a_2 + a_3)\boldsymbol{\alpha}_4 + \cdots + (3a_0 + a_1 + \cdots + a_{N-1})\boldsymbol{\alpha}_N.$$

We use here that (8.8) implies that the last coefficient is equal to $-a_N$. We leave the computation of the matrix to the reader. □

One can express the matrix C_N by means of the incidence matrix A_N of the following graph with N vertices.



$N \geq 4$

Figure 8.1 Coxeter-Dynkin diagram of type E_N

We have $C_N = -2I_N + A_N$.

8.2.3 Roots

A vector $\alpha \in \mathsf{E}_N$ is called a *root* if $\alpha^2 = -2$. A vector $(d, m_1, \dots, m_N) \in \mathrm{I}^{1,N}$ is a root if and only if

$$d^2 - \sum_{i=1}^{N} m_i^2 = -2, \quad 3d - \sum_{i=1}^{N} m_i = 0. \tag{8.9}$$

Using the inequality $(\sum_{i=1}^N m_i)^2 \leq N \sum_{i=1}^N m_i^2$, it is easy to find all solutions.

Proposition 8.2.8. *Let* $N \leq 8$ *and*

$$\alpha_{ij} = \mathbf{e}_i - \mathbf{e}_j, 1 \leq i < j \leq N,$$
$$\alpha_{ijk} = \mathbf{e}_0 - \mathbf{e}_i - \mathbf{e}_j - \mathbf{e}_k,\ 1 \leq i < j < k \leq N.$$

Any root in E_N *is equal to* $\pm\alpha$, *where* α *is one of the following vectors:*

N= 3 : $\alpha_{ij}, \alpha_{123}$. *Their number is* 8.
N= 4 :$\alpha_{ij}, \alpha_{ijk}$. *Their number is* 20.
N= 5 : $\alpha_{ij}, \alpha_{ijk}$. *Their number is* 40.
N= 6 :$\alpha_{ij}, \alpha_{ijk}, 2\mathbf{e}_0 - \mathbf{e}_1 - \cdots - \mathbf{e}_6$. *Their number is* 72.
N= 7 : $\alpha_{ij}, \alpha_{ijk}, 2\mathbf{e}_0 - \mathbf{e}_1 - \cdots - \mathbf{e}_7 - \mathbf{e}_i$. *Their number is* 126.
N= 8 :$\alpha_{ij}, \alpha_{ijk}, 2\mathbf{e}_0 - \mathbf{e}_1 - \cdots - \mathbf{e}_8 - \mathbf{e}_i - \mathbf{e}_j, 3\mathbf{e}_0 - \mathbf{e}_1 - \cdots - \mathbf{e}_8 - \mathbf{e}_i$. *Their number is* 240.

For $N \geq 9$, the number of roots is infinite. From now on, we assume

$$3 \leq N \leq 8.$$

An ordered set B of roots $\{\beta_1, \ldots, \beta_r\}$ is called a *root basis* if they are linearly independent over $\mathbb{Q}$ and

$$\beta_i \cdot \beta_j \geq 0.$$

A root basis is called *irreducible* if it is not equal to the union of non-empty subsets B_1 and B_2 such that $\beta_i \cdot \beta_j = 0$ if $\beta_i \in B_1$ and $\beta_j \in B_2$. The symmetric $r \times t$-matrix $C = (a_{ij})$, where $a_{ij} = \beta_i \cdot \beta_j$ is called the *Cartan matrix* of the root basis.

Definition 8.2.9. *A* Cartan matrix *is a symmetric integer matrix* (a_{ij}) *with* $a_{ii} = -2$ *and* $a_{ij} \geq 0$, *or such a matrix multiplied by* -1.

We will deal only with Cartan matrices C with $a_{ii} = -2$. The matrix $C + 2I$, where I is the identity matrix of the size equal to the size of C, can be taken as the incidence matrix of a non-oriented graph Γ_C with an ordered set of vertices in which we put the number $a_{ij} - 2$ at the edge corresponding to vertices i and j if this number is positive. The graph is called the *Coxeter-Dynkin diagram* of C. The Cartan matrix C_N for $N = 6, 7, 8$ has the corresponding graph pictured in Figure 8.2.

Cartan matrix is called *irreducible* if the graph Γ_C is connected.

If C is a negative definite irreducible Cartan matrix, then its Coxeter-Dynkin diagram is one of the types indicated in Figure 8.2 (see [70]). A lattice with

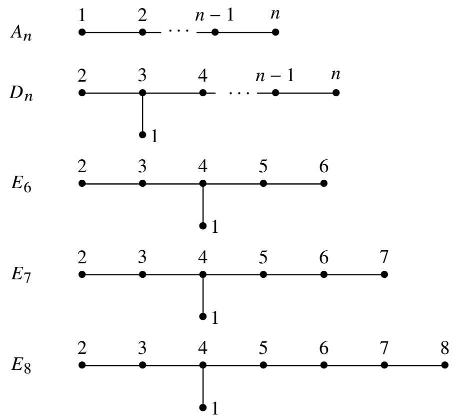


Figure 8.2 Coxeter-Dynkin diagrams of types A,D, E

quadratic form defined by a negative (positive) definite Cartan matrix is called a *root lattice*. Thus, the lattice $\mathsf{E}_N, N \leq 8$, is an example of a root lattice.

For $3 \leq n \leq 5$, we will use E_n to denote the Coxeter-Dynkin diagrams of types $A_2 + A_1 (N = 3)$, $A_4 (N = 4)$ and $D_5 (N = 5)$.

Example 8.2.10. We know that exceptional components E_i of a minimal resolution of a RDP are (-2)-curves. We have already used the fact that the intersection matrix $(E_i \cdot E_j)$ is negative definite. This implies that the intersection matrix is a Cartan matrix.

Proposition 8.2.11. *The Cartan matrix C of an irreducible root basis in E_N is equal to an irreducible Cartan matrix of type A_r, D_r, E_r with $r \leq N$.*

Definition 8.2.12. *A* canonical root basis *in E_N is a root basis with Cartan matrix* (8.7) *and the Coxeter-Dynkin diagram from Figure 8.1. Its element is called a* simple root.

An example of a canonical root basis is the basis $(\boldsymbol{\alpha}_1, \ldots, \boldsymbol{\alpha}_N)$.

Theorem 8.2.13. *Any canonical root basis is obtained from a unique orthonormal basis $(v_0, v_1, \ldots, v_n)$ in $\mathsf{I}^{1,N}$ such that $\mathbf{k}_N = -3v_0 + v_1 + \cdots + v_N$ by the formula*

$$\beta_1 = v_0 - v_1 - v_2 - v_3,\ \beta_i = v_{i-1} - v_i,\ i = 2, \ldots, N. \tag{8.10}$$

Proof Given a canonical root basis $(\beta_1, \ldots, \beta_N)$ we solve for v_i in the system of equations (8.10). We have

$$
\begin{aligned}
v_i &= v_N + \sum_{i=2}^{N} \beta_i, \; i = 1, \ldots, N-1, \\
v_0 &= \beta_1 + v_1 + v_2 + v_3 = \beta_1 + 3v_N + 3\sum_{i=4}^{N} \beta_i + 2\beta_3 + \beta_2, \\
-\mathbf{k}_N &= 3v_0 - v_1 - \cdots - v_N = 9v_N + 9\sum_{i=4}^{N} \beta_i + 6\beta_3 + 3\beta_2 \\
&- (v_N + \sum_{i=2}^{N} \beta_i) - (v_N + \sum_{i=3}^{N} \beta_i) - \cdots - (v_N + \beta_N) - v_N.
\end{aligned}
\tag{8.11}
$$

This gives

$$
v_N = -\frac{1}{9-N}(\mathbf{k}_N + 3\beta_1 + 2\beta_2 + 4\beta_3 + \sum_{i=3}^{N} (9-i)\beta_{i+1}).
$$

Taking the inner product of both sides with β_i, we find $(v_N, \beta_i) = 0, i = 1, \ldots, N-1$, and $(v_N, \beta_N) = 1$. Thus, all v_i belong to $(\mathbf{k}_N \perp \mathsf{E}_N)^\vee$. The discriminant group of this lattice is isomorphic to $(\mathbb{Z}/(9-N)\mathbb{Z})$ and the only isotropic subgroup of order $9-N$ is the diagonal subgroup. This shows that $\mathsf{E}_N^\vee$ is the only sublattice of $(\mathbf{k}_N \perp \mathsf{E}_N)^\vee$ of index $9-N$, hence $v_i \in \mathsf{E}_N^\vee$ for all i. It is immediately checked that $(v_0, v_1, \ldots, v_N)$ is an orthonormal basis and $\mathbf{k}_N = -3v_0 + v_1 + \cdots + v_N$. □

Corollary 8.2.14. *Let* $\mathrm{O}(\mathsf{I}^{1,N})_{\mathbf{k}_N}$ *be the stabilizer subgroup of* $\mathbf{k}_N$. *Then,* $\mathrm{O}(\mathsf{I}^{1,N})_{\mathbf{k}_N}$ *acts simply transitively on the set of canonical root bases in* E_N.

Each canonical root basis $\underline{\beta} = (\beta_1, \ldots, \beta_N)$ defines a partition of the set of roots $\mathcal{R}$

$$
\mathcal{R} = \mathcal{R}_+ \coprod \mathcal{R}_-,
$$

where $\mathcal{R}_+$ is the set of non-negative linear combinations of β_i. The roots from $\mathcal{R}_+$ ($\mathcal{R}_-$) are called *positive* (*negative*) roots with respect to the root basis $\underline{\beta}$. It is clear that $\mathcal{R}_- = \{-\alpha : \alpha \in \mathcal{R}_+\}$.

For any canonical root basis $\underline{\beta}$, the subset

$$
C_{\underline{\beta}} = \{x \in \mathsf{I}^{1,N} \otimes \mathbb{R} : (x, \beta_i) \geq 0\}
$$

is called a *Weyl chamber* with respect to $\underline{\beta}$. A subset of a Weyl chamber that consists of vectors such that $(v, \beta_i) = 0$ for some subset $I \subset \{1, \ldots, N\}$ is called a *face*. A face corresponding to the empty set is equal to the interior of the Weyl chamber. The face corresponding to the subset $\{1, \ldots, N\}$ is spanned by the vector $\mathbf{k}_N$.

For any root α, let

$$r_\alpha : \mathsf{I}^{1,N} \to \mathsf{I}^{1,N}, \quad \mathsf{v} \mapsto \mathsf{v} + (\mathsf{v}, \alpha)\alpha.$$

It is immediately checked that $r_\alpha \in \mathrm{O}(\mathsf{I}^{1,N})_{\mathbf{k}_N}$, $r_\alpha(\alpha) = -\alpha$ and $r_\alpha(v) = v$ if $(v, \alpha) = 0$. The isometry r_α is called the *reflection* in the root α. By linearity, r_α acts as an orthogonal transformation of the real inner product space $\mathbb{R}^{1,N} := \mathsf{I}^{1,N} \otimes \mathbb{R}$.

The following is a basic fact from the theory of finite reflection groups. We refer for the proof to numerous textbooks on this subject (e.g. [70], [437]).

Theorem 8.2.15. *Let C be a Weyl chamber defined by a canonical root basis $\underline{\beta}$. Let $W(\mathsf{E}_N)$ be the subgroup of $\mathrm{O}(\mathsf{E}_N)$ generated by reflections r_{β_i}. For any $x \in \mathbb{R}^{1,N}$, there exists $w \in W(\mathsf{E}_N)$ such that $w(x) \in C$. If $x, w(x) \in C$, then $x = w(x)$ and x belongs to a face of C. The union of Weyl chambers is equal to $\mathbb{R}^{1,N}$. Two Weyl chambers intersect only along a common face.*

Corollary 8.2.16. *The group $W(\mathsf{E}_N)$ acts simply transitively on canonical root bases, and Weyl chambers. It coincides with the group $\mathrm{O}(\mathsf{I}^{1,N})_{\mathbf{k}_N}$.*

The first assertion follows from the Theorem. The second assertion follows from Corollary 8.2.14 since $W(\mathsf{E}_N)$ is a subgroup of $\mathrm{O}(\mathsf{I}^{1,N})_{\mathbf{k}_N}$.

Corollary 8.2.17.

$$\mathrm{O}(\mathsf{E}_N) = W(\mathsf{E}_N) \times \langle\tau\rangle,$$

where τ is an isometry of E_N, which is realized by a permutation of roots in a canonical basis leaving invariant the Coxeter-Dynkin diagram. We have $\tau = 1$ for $N = 7, 8$ and $\tau^2 = 1$ for $N \neq 7, 8$.

Proof By Lemma 8.2.5, the image of the restriction homomorphism

$$\mathrm{O}(\mathsf{I}^{1,N})_{\mathbf{k}_N} \to \mathrm{O}(\mathsf{E}_N)$$

is equal to the kernel of the homomorphism $\alpha : \mathrm{O}(\mathsf{E}_N) \to \mathrm{O}(\mathrm{Disc}(\mathsf{E}_N))$. It is easy to compute $\mathrm{O}(\mathrm{Disc}(\mathsf{E}_N))$ and find that it is isomorphic to $\mathbb{Z}/\tau\mathbb{Z}$. Also, it can be checked that α is surjective and the image of the symmetry of the Coxeter-Dynkin diagram is the generator of $\mathrm{O}(\mathrm{Disc}(\mathsf{E}_N))$. It remains to apply the previous corollary. □

The definition of the group $W(\mathsf{E}_N)$ does not depend on the choice of a canonical basis and hence coincides with the definition of Weyl groups $W(\mathsf{E}_N)$ from Chapter 7. Note that Corollary 8.2.16 also implies that $W(\mathsf{E}_N)$ is generated by reflections r_α for all roots α in E_N. This is true for $N \le 10$ and is not true for $N \ge 11$.

Proposition 8.2.18. *If $N \ge 4$, the group $W(\mathsf{E}_N)$ acts transitively on the set of roots.*

Proof Let $(\beta_1, \dots, \beta_N)$ be a canonical basis from (8.10). Observe that the subgroup of $W(\mathsf{E}_N)$ generated by the reflections with respect to the roots $\beta_2, \dots, \beta_N$ is isomorphic to the permutation group $\mathfrak{S}_N$. It acts on the set $\{\mathbf{e}_1, \dots, \mathbf{e}_N\}$ by permuting its elements and leaves $\mathbf{e}_0$ invariant. This implies that $\mathfrak{S}_N$ acts on the roots $\alpha_{ij}, \alpha_{ijk}$, via its action on the set of subsets of $\{1, \dots, N\}$ of cardinality 2 and 3. Thus, it acts transitively on the set of roots α_{ij} and on the set of roots $\boldsymbol{\alpha}_{ijk}$. Similarly, we see that it acts transitively on the set of roots $2\mathbf{e}_0 - \mathbf{e}_{i_1} - \cdots - \mathbf{e}_{i_6}$ and $-\mathbf{k}_8 - \mathbf{e}_i$ if $N = 8$. Also, applying r_α to α, we get $-\alpha$. Now, the assertion follows from the following computation

$$\begin{aligned} r_{\beta_1}(-\mathbf{k}_8 - \mathbf{e}_8) &= 2\mathbf{e}_0 - \mathbf{e}_1 - \mathbf{e}_4 - \cdots - \mathbf{e}_8, \\ r_{\beta_1}(2\mathbf{e}_0 - \mathbf{e}_1 - \cdots - \mathbf{e}_6) &= \boldsymbol{\alpha}_{456}, \\ r_{\beta_1}(\boldsymbol{\alpha}_{124}) &= \boldsymbol{\alpha}_{34}. \end{aligned}$$

□

A sublattice R of E_N isomorphic to a root lattice is called a *root sublattice*. By definition, it has a root basis $(\beta_1, \dots, \beta_r)$ such that the matrix $(\beta_i \cdot \beta_j)$ is a Cartan matrix. Each such sublattice is isomorphic to the orthogonal sum of root lattices with irreducible Cartan matrices.

The types of root sublattices in the lattice E_N can be classified in terms of their root bases by the following procedure due to A. Borel and J. de Siebenthal [66] and, independently by E. Dynkin [269].

Let D be the Coxeter-Dynkin diagram. Consider the extended diagram by adding one more vertex which is connected to other edges as shown on the following *extended Coxeter-Dynkin diagrams*. Consider the following set of elementary operations over the diagrams D and their disconnected sums $D_1 + \cdots + D_k$. Extend one of the components D_i to get the extended diagram. Consider its subdiagram obtained by deleting subset of vertices. Now, all possible root bases are obtained by applying recursively the elementary operations to the initial Coxeter-Dynkin diagram of type E_N and all its descendants.

8.2.4 Fundamental weights

Let $\underline{\beta} = (\beta_1, \beta_2, \dots, \beta_N)$ be a canonical root basis (8.10) in E_N. Consider its dual basis $(\beta_1^*, \dots, \beta_N^*)$ in $\mathsf{E}_N^\vee \otimes \mathbb{Q}$. Its elements are called *fundamental*

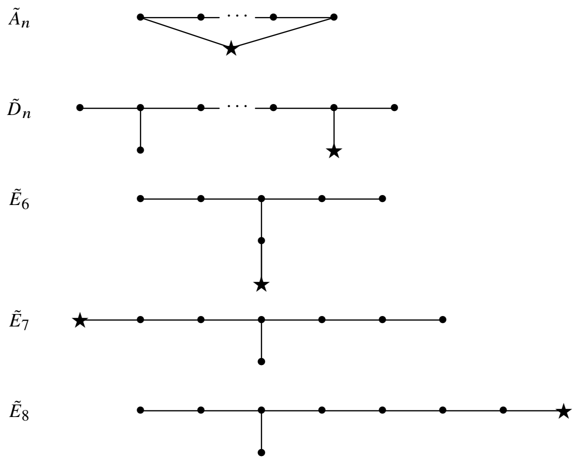


Figure 8.3 Extended Coxeter-Dynkin diagrams of types $\tilde{A}, \tilde{D}, \tilde{E}$

weights. We use the expressions for β_i from Theorem 8.2.13. Let us identify $\mathsf{E}_N^\vee$ with $(\mathbf{k}_N^\perp)^\vee = \mathsf{I}^{1,N}/\mathbb{Z}\mathbf{k}_N$. Then, we can take for representatives of β_j^* the following vectors from $\mathsf{I}^{1,N}$:

$$\begin{aligned}
\beta_1^* &= v_0,\\
\beta_2^* &= v_0 - v_1,\\
\beta_3^* &= 2v_0 - v_1 - v_2,\\
\beta_i^* &= v_i + \cdots + v_N,\ i = 4, \ldots, N.
\end{aligned}$$

Definition 8.2.19. *A vector in $\mathsf{I}^{1,N}$ is called an* exceptional vector *if it belongs to the $W(\mathsf{E}_N)$-orbit of β_N^*.*

Proposition 8.2.20. *A vector $v \in \mathsf{I}^{1,N}$ is exceptional if and only if $\mathbf{k}_N \cdot v = -1$ and $v^2 = -1$. The set of exceptional vectors is the following*

$N = 3, 4 : \mathbf{e}_i, \mathbf{e}_0 - \mathbf{e}_i - \mathbf{e}_j;$

$N = 5 : \mathbf{e}_i, \mathbf{e}_0 - \mathbf{e}_i - \mathbf{e}_j, 2\mathbf{e}_0 - \mathbf{e}_1 - \cdots - \mathbf{e}_5;$

$N = 6 : \mathbf{e}_i, \mathbf{e}_0 - \mathbf{e}_i - \mathbf{e}_j, 2\mathbf{e}_0 - \mathbf{e}_1 - \cdots - \mathbf{e}_6 + e_i;$

$N = 7 : \mathbf{e}_i, \mathbf{e}_0 - \mathbf{e}_i - \mathbf{e}_j, 2\mathbf{e}_0 - \mathbf{e}_1 - \cdots - \mathbf{e}_7 + \mathbf{e}_i + \mathbf{e}_j; -\mathbf{k}_7 - \mathbf{e}_i;$

$N = 8 : \mathbf{e}_i, \mathbf{e}_0 - \mathbf{e}_i - \mathbf{e}_j, 2\mathbf{e}_0 - \mathbf{e}_1 - \cdots - \mathbf{e}_8 + \mathbf{e}_i + \mathbf{e}_j + e_k; -\mathbf{k}_8 + \mathbf{e}_i - \mathbf{e}_j;$
$\quad -\mathbf{k}_8 + \mathbf{e}_0 - \mathbf{e}_i - \mathbf{e}_j - \mathbf{e}_k, -\mathbf{k}_8 + 2e_0 - \mathbf{e}_{i_1} - \cdots - \mathbf{e}_{i_6}, -2\mathbf{k}_8 - e_i.$

The number of exceptional vectors is given by Table 8.1.

N	3	4	5	6	7	8
#	6	10	16	27	56	240

Table 8.1 *Number of exceptional vectors*

Proof Similarly to the case of roots, we solve the equations

$$d^2 - \sum_{i=1}^{N} m_i^2 = -1, \ 3d - \sum_{i=1}^{N} m_i = 1.$$

First we immediately get the inequality $(3d-1)^2 \leq N(d^2+1)$ which gives $0 \leq d \leq 4$. If $d = 0$, the condition $\sum m_i^2 = d^2 + 1$ and $k_N \cdot v = -1$ gives the vectors $\mathbf{e}_i$. If $d = 1$, this gives the vectors $\mathbf{e}_0 - \mathbf{e}_i - \mathbf{e}_j$, and so on. Now, we use the idea of Noether's inequality (see Lemma 7.2.27) to show that all these vectors $(d, m_1, \ldots, m_N)$ belong to the same orbit of $W(\mathsf{E}_N)$. We apply permutations from $\mathfrak{S}_N$ to assume $m_1 \geq m_2 \geq m_3$, then use the reflection $r_{\alpha_{123}}$ to decrease d. □

Corollary 8.2.21. *The orders of the Weyl groups $W(\mathsf{E}_N)$ are given by Table 8.2.*

N	3	4	5	6	7	8
$\#W(\mathsf{E}_N)$	12	5!	$2^4 \cdot 5!$	$2^3 \cdot 3^2 \cdot 6!$	$2^6 \cdot 3^2 \cdot 7!$	$2^7 \cdot 3^3 \cdot 5 \cdot 8!$

Table 8.2 *Orders of the Weyl groups*

Proof Observe that the orthogonal complement of $\mathbf{e}_N$ in $\mathsf{I}^{1,N}$ is isomorphic to I^{N-1}. Since $\mathbf{e}_N^2 = -1$, by Lemma 8.2.5, the stabilizer subgroup of $\mathbf{e}_N$ in $\mathrm{O}(\mathsf{I}^{1,N})$ is equal to $\mathrm{O}(\mathsf{I}^{1,N-1})$. This implies that the stabilizer subgroup of $\mathbf{e}_N$ in $W(\mathsf{E}_N)$ is equal to $W(\mathsf{E}_{N-1})$. Obviously, $W(\mathsf{E}_3) \cong \mathfrak{S}_3 \times \mathfrak{S}_2$ and $W(\mathsf{E}_4) \cong \mathfrak{S}_5$. Thus,

$$\begin{aligned}
\#W(\mathsf{E}_5) &= 16 \cdot \#W(\mathsf{E}_4) = 2^4 \cdot 5!, \\
\#W(\mathsf{E}_6) &= 27 \cdot \#W(\mathsf{E}_5) = 2^3 \cdot 3^2 \cdot 6!, \\
\#W(\mathsf{E}_7) &= 56 \cdot \#W(\mathsf{E}_6) = 2^6 \cdot 3^2 \cdot 7!, \\
\#W(\mathsf{E}_8) &= 240 \cdot \#W(\mathsf{E}_7) = 2^7 \cdot 3^3 \cdot 5 \cdot 8!.
\end{aligned}$$

□

To describe automorphism groups of del Pezzo surfaces, we will use the known classification of conjugacy classes of elements in the Weyl groups of root systems of finite type given in Table 8.2.4. According to [91], they are indexed by certain graphs. In many cases, they coincide with the Coxeter-Dynkin diagrams of the orthogonal sum of root lattices.

Let R be a root lattice of rank l defined by a negative-definite Cartan matrix C. We say that it is irreducible if C is not the direct sum of Caratn matrices. Recall that any irreducible root basis is of type A_n, D_n, E_6, E_7, E_8 with the Coxeter-Dynkin diagram from Figure 8.3.

Assume that R is of rank l and let $(\alpha_1, \ldots, \alpha_l)$ be its root basis. The element $\mathfrak{h}_R = r_{\alpha_1} \cdots r_{\alpha_l}$ is called a Coxeter element of $W(R)$. Its conjugacy class does not depend on the order of the reflections in the product nor on the choice of a root basis. Its order is called the *Coxeter number* of R and it has exactly l eigenvalues different from 1 in its action in the complex linear space $R\otimes\mathbb{C} \cong \mathbb{C}^l$. If R is the orthogonal sum of irreducible root lattices, then a Coxeter element of R is the product of Coxeter elements of the irreducible summands.

An element $w \in W(R)$ is said to be of *Coxeter type* if it is conjugate to the product of Coxeter elements of a primitive sublattice of R isomorphic to a root lattice. We denote the conjugacy class of such an element by the type of the Coxeter-Dynkin diagram of the sublattice. However, one has to be warned that there could be different lattice embedding not conjugate with respect to $W(R)$. To find root sublattices of a root lattice, one uses the following the *Borel–de Siebenthal–Dynkin algorithm* that can be derived from [66] or [269].

Let R be an irreducible root lattice with Dynkin diagram Γ. Extend it to the affine Dynkin diagram $\tilde{\Gamma}$ by adding the maximal root $\alpha_{\max}$. Then, delete one vertex v different from the new one. The remaining diagram $\Gamma_v = \tilde{\Gamma} \setminus \{v\}$ is the Dynkin diagram of a root sublattice of M of the same rank, maybe equal to R. We repeat the process until no new sublattices are created in this way. For example, a subdiagram of type A_n does not create new sublattices of the same rank. All root sublattices of the same rank are obtained in this way.

In [91] R. Carter classifies the conjugacy classes of elements of the Weyl group $W(R)$ of a root lattice R. It turns out that not all conjugacy classes are of Coxeter type. They are described by certain graphs $D_n(a_i), E_n(a_i)$ that contain a cycle. Here, D_n, E_n means that the conjugacy class belongs to $W(\mathsf{D}_n)$ or $W(\mathsf{E}_n)$ and the eigenvalues of its representative different form 1.

The following table gives the conjugacy classes of elements defined by connected Carter graphs.

Graph	Order	Characteristic polynomial	Trace
A_k	$k+1$	$t^k + t^{k-1} + \cdots + 1$	-1
D_k	$2k-2$	$(t^{k-1}+1)(t+1)$	-1
$D_k(a_1)$	l.c.m$(2k-4,4)$	$(t^{k-2}+1)(t^2+1)$	0
$D_k(a_2)$	l.c.m$(2k-6,6)$	$(t^{k-3}+1)(t^3+1)$	0
$\vdots$	$\vdots$	$\vdots$	$\vdots$
$D_k(a_{\frac{k}{2}-1})$	even k	$(t^{\frac{k}{2}}+1)^2$	0
E_6	12	$(t^4-t^2+1)(t^2+t+1)$	-1
$E_6(a_1)$	9	t^6+t^3+1	0
$E_6(a_2)$	6	$(t^2-t+1)^2(t^2+t+1)$	1
E_7	18	$(t^6-t^3+1)(t+1)$	-1
$E_7(a_1)$	14	t^7+1	0
$E_7(a_2)$	12	$(t^4-t^2+1)(t^3+1)$	0
$E_7(a_3)$	30	$(t^5+1)(t^2-t+1)$	1
$E_7(a_4)$	6	$(t^2-t+1)^2(t^3+1)$	2
E_8	30	$t^8+t^7-t^5-t^4-t^3+t+1$	-1
$E_8(a_1)$	24	t^8-t^4+1	0
$E_8(a_2)$	20	$t^8-t^6+t^4-t^2+1$	0
$E_8(a_3)$	12	$(t^4-t^2+1)^2$	0
$E_8(a_4)$	18	$(t^6-t^3+1)(t^2-t+1)$	1
$E_8(a_5)$	15	$t^8-t^7+t^5-t^4+t^3-t+1$	1
$E_8(a_6)$	10	$(t^4-t^3+t^2-t+1)^2$	2
$E_8(a_7)$	12	$(t^4-t^2+1)(t^2-t+1)^2$	2
$E_8(a_8)$	6	$(t^2-t+1)^4$	4

Table 8.3 *Carter graphs and characteristic polynomials*

8.2.5 Gosset polytopes

Consider the real vector space $\mathbb{R}^{N,1} = \mathbb{R}^{N+1}$ with the inner product $\langle , \rangle$ defined by the quadratic form on $\mathsf{I}^{1,N}$ multiplied by -1. All exceptional vectors lie in the affine space $V_N = \{x \in \mathbb{R}^{N,1} : (\mathbf{k}_N, x) = 1\}$ and belong to the unit sphere S^N. Let Σ_N be the convex hull of the exceptional vectors. For any two vectors $w, w' \in \mathsf{S}^N$, the vector $w - w'$ belongs to the even quadratic lattice E_N, hence $2 \leq \langle w - w', w - w' \rangle = 2 - 2\langle w, w' \rangle$. This shows that the minimal distance $\langle w-w', w-w' \rangle^{1/2}$ between two vertices is equal to $\sqrt{2}$ and occurs only when the vectors w and w' are orthogonal. This implies that the edges of Σ_N correspond to pairs of orthogonal exceptional vectors. The difference of such vectors is a root $\alpha = w - w'$ such that $\langle \alpha, w \rangle = 1$. The reflections $s_\alpha : x \mapsto x - \langle x, \alpha \rangle \alpha$ sends w to w'. Thus, the reflection hyperplane $H_\alpha = \{x \in V_N : \langle x, \alpha \rangle = 0\}$ intersects the edge at the midpoint. It permutes two adjacent vertices. The Weyl group $W(\mathsf{E}_N)$ acts on Σ_N with the set of vertices forming one orbit. The edges coming out of a fixed vertex correspond to exceptional vectors orthogonal to the vertex. For example, if we take the vertex corresponding to the vector $\mathbf{e}_N$, then the edges correspond to exceptional vectors for the root system E_{N-1}. Thus, the vertex figure at each vertex (i.e. the convex hull of midpoints of edges coming from the vertex) is isomorphic to Σ_{N-1}. A convex polytope with isomorphic vertex figures is called a *semi-regular polytope* (a *regular polytope* satisfies the additional property that all facets are isomorphic).

The polytopes Σ_N are Gosset polytopes discovered by T. Gosset in 1900 [350]. Following Gosset, we denoted them by $(N-4)_{21}$. We refer to [175, p. 202], for the following facts about their combinatorics. Each polytope Σ_N has two $W(\mathsf{E}_N)$-orbits on the set of facets. One of them is represented by the convex hull of exceptional vectors $\mathbf{e}_1, \ldots, \mathbf{e}_N$ orthogonal to the vector e_0. It is a $(N-1)$-simplex α_{N-1}. The second one is represented by the convex hull of exceptional vectors orthogonal to $\mathbf{e}_0 - \mathbf{e}_1$. It is a cross-polytope β_{N-1} (a cross-polytope β_i is the bi-pyramide over β_{i-1} with β_2 being a square). The number of facets is equal to the index of the stabilizer group of $\mathbf{e}_0$ or $\mathbf{e}_0 - \mathbf{e}_1$ in the Weyl group. The rest of faces are obtained by induction on N. The number of k-faces in Σ_N is given in Table 8.3 (see [175, 11.8]).

k/N	3	4	5	6	7	8
0	6	10	16	27	56	240
1	$3\alpha + 6\alpha$	30	80	216	756	6720
2	$2\alpha + 3\beta$	$10\alpha + 20\alpha$	160	720	4032	60480
3		$5\alpha + 5\beta$	$40\alpha + 80\alpha$	1080	10080	241920
4			$16\alpha + 10\beta$	$432\alpha+$ 216α	12096	483840
5				$72\alpha_5+$ 27β	$2016\alpha+$ 4032α	483840
6					$576\alpha+$ 126β	$69120\alpha+$ 138240α
7						$17280\alpha+$ $+2160\beta$

Table 8.4 *Gosset polytopes*

The Weyl group $W(\mathsf{E}_N)$ acts transitively on the set of k-faces when $k \leq N-2$. Otherwise, there are two orbits; their cardinality can be found in the table. The dual (reciprocal) polytopes are not semi-regular anymore since the group of symmetries has two orbits on the set of vertices. One is represented by the vector $\mathbf{e}_0$ and another by $\mathbf{e}_0 - \mathbf{e}_1$.

8.2.6 (-1)-curves on del Pezzo surfaces

Let $\phi : \mathsf{I}^{1,N} \to \mathrm{Pic}(S)$ be a geometric marking of a weak del Pezzo surface S. The intersection form on $\mathrm{Pic}(S)$ equips it with the structure of a quadratic lattice. Since ϕ sends an orthonormal basis of $\mathsf{I}^{1,N}$ to an orthonormal basis of $\mathrm{Pic}(S)$, the isomorphism ϕ is an isomorphism of lattices. By definition of a geometric marking, $\phi(K_X) = \mathbf{k}_N$. Thus, the isometry of lattices ϕ restricts to an isometry

$$\phi : K_S^{\perp} \to \mathsf{E}_N. \tag{8.12}$$

We define the Weyl group $W(S)$ of S as the subgroup of the orthogonal group $\mathrm{O}(\mathrm{Pic}(S)$ that fixes K_X. The isometry ϕ defines an ismorphism

$$W(S) \to W(\mathsf{E}_N), \quad \sigma \mapsto \phi \circ \sigma^{-1}\phi. \tag{8.13}$$

The image of an exceptional vector is the divisor class E such that $E^2 = E \cdot K_S = -1$. By Riemann-Roch, E is an effective divisor class. Write it as a sum of irreducible components $E = R_1 + \cdots + R_k$. Intersecting with K_S, we obtain that there exists a unique component, say R_1 such that $R_1 \cdot K_S = -1$. For all other components, we have $R_i \cdot K_S = 0$. It follows from the adjunction formula that any such component is a (-2)-curve. So, if S is a nonsingular del Pezzo surface, the image of any exceptional divisor is a (-1)-curve on S, and we have a bijection between the set of exceptional vectors in E_N and (-1)-curves on S. If S is a weak del Pezzo surface, we use the following.

Lemma 8.2.22. *Let D be a divisor class with $D^2 = D \cdot K_S = -1$. Then, $D = E + R$, where R is a non-negative sum of (-2)-curves, and E is either a (-1)-curve or $K_S^2 = 1$ and $E \in |-K_S|$ and $E \cdot R = 0, R^2 = -2$. Moreover D is a (-1)-curve if and only if for each (-2)-curve R_i on S we have $D \cdot R_i \geq 0$.*

Proof Fix a geometric basis $(e_0, e_1, \ldots, e_N)$ in $\mathrm{Pic}(S)$. We know that $e_0^2 = 1, e_0 \cdot K_S = -3$. Hence, $((D \cdot e_0)K_S + 3D) \cdot e_0 = 0$, and hence,

$$\big((D \cdot e_0)K_S + 3D\big)^2 = -6D \cdot e_0 - 9 + (D \cdot e_0)^2 K_S^2 < 0.$$

Thus, $-6D \cdot e_0 - 9 < 0$ and hence $D \cdot e_0 > -9/6 > -2$. This shows that $(K_S - D) \cdot e_0 = -3 - D \cdot e_0 < 0$, and since e_0 is nef, we obtain that $|K_S - D| = \emptyset$.

Applying Riemann-Roch, we get $\dim |D| \geq 0$. Write an effective representative of D as a sum of irreducible components and use that $D \cdot (-K_S) = 1$. Since $-K_S$ is nef, there is only one component E entering with coefficient 1 and satisfying $E \cdot K_S = -1$, all other components are (-2)-curves. If $D \sim E$, then $D^2 = E^2 = -1$ and E is a (-1)-curve. Let $\pi : S' \to S$ be a birational morphism of a weak del Pezzo surface of degree 1 (obtained by blowing up $8-k$ points on S in general position not lying on E). We identify E with its pre-image in S'. Then, $(E + K_{S'}) \cdot K_{S'} = -1 + 1 = 0$, hence, by Hodge Index Theorem, either $S' = S$ and $E \in |-K_S|$, or

$$(E + K_{S'})^2 = E^2 + 2E \cdot K_{S'} + K_{S'}^2 = E^2 - 1 < 0.$$

Since $E \cdot K_S = -1$, E^2 is odd. Thus, the only possibility is $E^2 = -1$. If $E \in |-K_S|$, we have $E \cdot R_i = 0$ for any (-2)-curve R_i, hence $E \cdot R = 0, R^2 = -2$.

Assume $R \neq 0$. Since $-1 = E^2 + 2E \cdot R + R^2$ and $E^2 \leq 1$, $R^2 \leq -2$, we get $E \cdot R \geq 0$, where the equality take place only if $E^2 = 1$. In both cases we get

$$-1 = (E + R)^2 = (E + R) \cdot R + (E + R) \cdot E \geq (E + R) \cdot R.$$

Thus, if $D \neq E$, we get $D \cdot R_i < 0$ for some irreducible component of R. This proves the assertion. □

The number of (-1)-curves on a nonsingular del Pezzo surface is given in Table 8.1. It is also can be found in Table 8.4. It is the number of vertices of the Gosset polytope. Other faces give additional information about the combinatorics of the set of (-1)-curves. For example, the number of k-faces of type α is equal to the number of sets of $k+1$ non-intersecting (-1)-curves.

We can also see the geometric realization of the fundamental weights:

$$w_1 = \mathbf{e}_0,\ w_2 = \mathbf{e}_0 - \mathbf{e}_1,\ w_3 = 2\mathbf{e}_0 - \mathbf{e}_1 - \mathbf{e}_2,\ w_i = \mathbf{e}_1 + \cdots + \mathbf{e}_N,\ i = 4, \ldots, N.$$

The image of w_1 under a geometric marking represents the divisor class e_0. The image of w_2 represents $e_0 - e_1$. The image of w_3 is $2e_0 - e_1 - e_2$. Finally, the images of the remaining fundamental weights represent the classes of the sums of disjoint (-1)-curves.

Recall the usual attributes of the minimal model program. Let $\mathrm{Eff}(S)$ be the *effective cone* of a smooth projective surface S, i.e. the open subcone in $\mathrm{Pic}(S) \otimes \mathbb{R}$ spanned by effective divisor classes. Let $\overline{\mathrm{Eff}}(S)$ be its closure. The Cone Theorem [462] states that

$$\overline{\mathrm{Eff}}(S) = \overline{\mathrm{Eff}}(S)_{K_S \geq 0} + \sum_i \mathbb{R}[C_i],$$

where $\overline{\mathrm{Eff}}(S)_{K_S \geq 0} = \{x \in \overline{\mathrm{Eff}}(S) : x \cdot K_S \geq 0\}$ and $[C_i]$ are *extremal rays* spanned by classes of smooth rational curves C_i such that $-C_i \cdot K_X \leq 3$.

Recall that a subcone τ of a cone K is extremal if there exists a linear function ϕ such that $\phi(K) \geq 0$ and $\phi^{-1}(0) \cap K = \tau$. In the case when K is a polyhedral cone, an extremal subcone is a face of K.

Theorem 8.2.23. *Let S be a nonsingular del Pezzo surface of degree d. Then,*

$$\overline{\mathrm{Eff}}(S) = \sum_{i=1}^{k} \mathbb{R}[C_i],$$

where the set of curves C_i is equal to the set of (-1)-curves if $d \neq 8, 9$. If $d = 8$ and S is isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$, then $k = 2$, and the $[C_i]$'s are the classes of the two rulings on S. If $d = 8$ and $S \cong \mathbf{F}_1$, then $k = 2$ and $[C_1]$ is the class of the exceptional section, and $[C_2]$ is the class of a fiber. If $d = 9$, then $k = 1$ and $[C_1]$ is the class of a line.

Proof Since S is a del Pezzo surface, $\overline{\mathrm{Eff}}(S)_{K_S \geq 0} = \{0\}$, so it suffices to find the extremal rays. It is clear that $E \cdot K_S = -1$ implies that any (-1)-curve generates an extremal ray. Choose a geometric marking on S to identify $\mathrm{Pic}(S)$ with $\mathsf{I}^{1,N}$. Let C be a smooth rational curve such that $c = -C \cdot K_S \leq 3$. By the adjunction formula, $C^2 = -2 + c$. If $c = 1$, C is a (-1)-curve. If $c = 2$, applying Lemma 7.2.27, we follow the proof of Proposition 8.2.20 to obtain that all vectors with $v \in \mathsf{I}^{1,N}$ satisfying $v \cdot \mathbf{k}_N = -2$ and $(v, v) = 0$ belong to the same orbit of $W(\mathsf{E}_N)$. Thus, if $d < 8$, we may assume that $v = e_0 - e_1$, but then $v = (e_0 - e_1 - e_2) + e_2$ is equal to the sum of t wo exceptional vectors, hence $[C]$ is not extremal. If $c = 3$, then $C^2 = 1, C \cdot K_S = -3$. Again, we can apply Noether's inequality and the proof of Lemma 7.2.31 to obtain that all such vectors belong to the same orbit. Take $v = e_0$ and write $e_0 = (e_0 - e_1 + e_2) + e_1 + e_2$ to obtain that $[C]$ is not extremal if $d < 8$. We leave the cases $d = 8, 9$ to the reader. □

Corollary 8.2.24. *Assume $d < 8$. Let $\phi : \mathsf{I}^{1,N} \to \mathrm{Pic}(S)$ be a geometric marking of a nonsingular del Pezzo surface. Then, $\phi^{-1}(\overline{\mathrm{Eff}}(S))$ is equal to the Gosset polytope.*

Recall from [462] that any extremal face F of $\overline{\mathrm{Eff}}(S)$ defines a contraction morphism $\phi_F : S \to Z$. The two types of extremal faces of a Gosset polytope define two types of contraction morphisms: α_k-type and β_k-type. The contraction of the α_k-type blows down the set of disjoint (-1)-curves that are the vertices of the set. The contraction of the β_k-type defines a conic bundle structure on S. It is a morphism onto $\mathbb{P}^1$ with general fiber isomorphic to $\mathbb{P}^1$ and singular fibers equal to the union of two (-1)-curves intersecting transversally at one point. Thus, the number of facets of type β of the Gosset polytope is equal to the number of conic bundle structures on S.

Another attribute of the minimal model program is the *nef cone* $\mathrm{Nef}(S)$ in $\mathrm{Pic}(S) \otimes \mathbb{R}$ spanned by divisor classes D such that $D \cdot C \geq 0$ for any effective divisor class C. The nef cone is the dual of $\overline{\mathrm{Eff}}(S)$. Under a geometric marking it becomes isomorphic to the dual of the Gosset polytope. It has two types of vertices represented by the normal vectors to facets. One type is represented by the Weyl group orbit of the vector e_0 and another by the vector $e_0 - e_1$.

8.2.7 Effective roots

Let $\phi : \mathsf{I}^{1,N} \to \mathrm{Pic}(S)$ be a geometric marking of a weak del Pezzo surface of degree $d = 9 - N$. The image of a root $\alpha \in \mathsf{E}_N$ is a divisor class D such that $D^2 = -2$ and $D \cdot K_S = 0$. We say that α is an *effective root* if $\phi(\alpha)$ is an effective divisor class. An effective root representing a (-2)-curve will be called a *nodal root*. Let $\sum_{i \in I} n_i R_i$ be its effective representative. Since $-K_S$ is nef, we obtain that $R_i \cdot K_S = 0$. Since $K_S^2 > 0$, we also get $R_i^2 < 0$. Together with the adjunction formula, this implies that each R_i is a (-2)-curve. Since a (-2)-curve does not move, we will identify it with its divisor class.

Proposition 8.2.25. *Let S be a weak del Pezzo surface of degree $d = 9 - N$. The number r of (-2)-curves on S is less than or equal to N. The sublattice N_S of $\mathrm{Pic}(S)$ generated by (-2)-curves is a root lattice of rank r.*

Proof Since each nodal curve is contained in $K_S^\perp$ and $R_i \cdot R_j \geq 0$ for $i \neq j$, it suffices to prove that the set of (-2)-curves is linearly independent over $\mathbb{Q}$. Suppose that this is not true. Then, we can find two disjoint sets of curves $R_i, i \in I$, and $R_j, j \in J$, such that

$$\sum_{i \in I} n_i R_i \sim \sum_{j \in J} m_j R_j,$$

where n_i, m_j are some non-negative rational numbers. Taking the intersection of both sides with R_i, we obtain that

$$R_i \cdot \sum_{i \in I} n_i R_i = R_i \cdot \sum_{j \in J} m_j R_j \geq 0.$$

This implies that

$$(\sum_{i \in I} n_i R_i)^2 = \sum_{i \in I} n_i R_i \cdot \big(\sum_{i \in I} n_i R_i\big) \geq 0.$$

Since $(\mathbb{Z} K_S)^\perp$ is negative definite, this could happen only if $\sum_{i \in I} n_i R_i \sim 0$. Since all coefficients are non-negative, this happens only if all $n_i = 0$. For the same reason, each m_i is equal to 0. □

Let $\eta = x_1 + \cdots + x_N$ be the bubble cycle defined by the blowing down structure $S = S_N \to S_{N-1} \to \ldots S_1 \to S_0 = \mathbb{P}^2$ defining the geometric marking. It is clear that $\phi(\alpha_{ij}) = e_i - e_j$ is effective if and only if $x_i \succ_{i-j} x_j$. It is a nodal root if and only if $i = j + 1$.

A root α_{ijk} is effective if and only if there exists a line whose proper transform on the surfaces $S_{i-1}, S_{j-1}, S_{k-1}$ pass through the points x_i, x_j, x_k. It is a nodal root if and only if all roots $\alpha_{i',j',k'}$ with $x_{i'} > x_i, x_{j'} > x_j, x_{k'} > x_k$ are not effective.

The root $2\mathbf{e}_0 - \mathbf{e}_{i_1} - \cdots - \mathbf{e}_{i_6}$ is nodal if and only if its image in $\mathrm{Pic}(S)$ is the divisor class of the proper transform of an irreducible conic passing through the points $x_{i_1}, \ldots, x_{i_6}$.

The root $3\mathbf{e}_0 - \mathbf{e}_1 - \cdots - \mathbf{e}_8 - \mathbf{e}_i$ is nodal if and only if its image in $\mathrm{Pic}(S)$ is the divisor class of the proper transform of an irreducible cubic with double points at x_i and passing through the rest of the points.

Definition 8.2.26. *A* Dynkin curve *is a reduced connected curve R on a projective nonsingular surface X such that its irreducible components R_i are -2-curves and the matrix $(R_i \cdot R_j)$ is a Cartan matrix. The type of a Dynkin curve is the type of the corresponding root system.*

Under a geometric marking, a Dynkin curve on a weak del Pezzo surface S corresponds to an irreducible root base in the lattice E_N. We use the Borel-de Siebenthal-Dynkin procedure to determine all possible root bases in E_N.

Theorem 8.2.27. *Let R be a Dynkin curve on a projective nonsingular surface X. There is a birational morphism $f : X \to Y$, where Y is a normal surface satisfying the following properties:*

(i) $f(R)$ is a point;
(ii) the restriction of f to $X \setminus R$ is an isomorphism;
(iii) $f^*\omega_Y \cong \omega_X$.

Proof Let H be a very ample divisor on X. Since the intersection matrix of components of $R = \sum_{i=1}^n R_i$ has nonzero determinant, we can find rational numbers r_i such that

$$(\sum_{i=1}^n r_i R_i) \cdot R_j = -H \cdot R_j, \quad j = 1, \ldots, n.$$

It is known and that the entries of the inverse of a Cartan matrix are non-positive. Thus, all r_i's are non-negative numbers. Replacing H with some multiple mH, we may assume that all r_i are non-negative integers. Let $D = \sum r_i R_i$. Since

$H + D$ is an effective divisor and $(H + D) \cdot R_i = 0$ for each i, we have $\mathcal{O}_X(H + D) \otimes \mathcal{O}_{R_i} = \mathcal{O}_{R_i}$. Consider the standard exact sequence

$$0 \to \mathcal{O}_X(H) \to \mathcal{O}_X(H + D) \to \mathcal{O}_D \to 0.$$

Replacing H with mH, we may assume, by Serre's Duality, that $h^1(\mathcal{O}_X(H)) = 0$ and $\mathcal{O}_X(H)$ is generated by global sections. Let $s_0, \ldots, s_{N-1}$ be sections of $\mathcal{O}_X(H)$ which define an embedding of X in $\mathbb{P}^{N-1}$. Consider them as sections of $\mathcal{O}_X(H + D)$. Let s_N be a section of $\mathcal{O}_X(H + D)$ that maps to $1 \in H^0(X, \mathcal{O}_D)$. Consider the map $f' : X \to \mathbb{P}^N$ defined by the sections $(s_0, \ldots, s_N)$. Then, $f'(D) = [0, \ldots, 0, 1]$ and $f'|X \subset D$ is an embedding. So, we obtain a map $f : X \to \mathbb{P}^N$ satisfying properties (i) and (ii). Since X is normal, f' factors through a map $f : X \to Y$, where Y is normal. Let ω_Y be the canonical sheaf of Y (it is defined to be equal to the sheaf $j_*\omega_{Y \setminus f'(R)}$, where $j : Y \setminus f'(R) \to Y$ is the natural open embedding). We have

$$\omega_X = f^*\omega_Y \otimes \mathcal{O}_X(A)$$

for some divisor A. Since $K_X \cdot R_i = 0$ for each i, and $f^*\omega_Y \otimes \mathcal{O}_{R_i} = \mathcal{O}_{R_i}$ we get $A \cdot R_i = 0$. Since the intersection matrix of R is negative definite we obtain $A = 0$. □

Applying the projection formula and property (iii), we obtain

$$\omega_Y \cong f_*\omega_X.$$

Since f is a resolution of singularities and Y is a normal surface, and hence Cohen-Macaulay, this property is equivalent to that Y has rational singularities [462, Lemma 5.12]. For any canonical root basis $\beta_1, \ldots, \beta_N$ in a root system of type $\mathsf{E}_N, N \le 8$, there exists a positive root $\beta_{\max}$ satisfying the property $\beta_{\max} \cdot \beta_i \le 0, i = 1, \ldots, N$. For an irreducible root system, it is equal to the following vector

$$A_n : \beta_{\max} = \beta_1 + \cdots + \beta_n;$$
$$D_n : \beta_{\max} = \beta_1 + \beta_2 + 2\beta_3 + \cdots + 2\beta_{n-1} + \beta_n;$$
$$E_6 : \beta_{\max} = 2\beta_1 + \beta_2 + 2\beta_3 + 3\beta_4 + 2\beta_5 + \beta_6;$$
$$E_7 : \beta_{\max} = 2\beta_1 + 2\beta_2 + 3\beta_3 + 4\beta_4 + 3\beta_5 + 2\beta_6 + \beta_7;$$
$$E_8 : \beta_{\max} = 3\beta_1 + 2\beta_2 + 4\beta_3 + 6\beta_4 + 5\beta_5 + 4\beta_6 + 3\beta_7 + 2\beta_8.$$

In the root sublattice defined by a Dynkin curve it represents the fundamental cycle Z. Since $\beta_{\max}^2 = -2$, we see that there the singular point $f(R)$ admits a fundamental cycle Z with $Z^2 = -2$. Thus, $f(R)$ is a RDP. As we already observed in Example 8.2.10 the exceptional components of a RDP form a Dynkin curve.

An example of a RDP is the singularity of the orbit of the origin of the orbit space $V = \mathbb{C}^2/\ell$, where ℓ is a finite subgroup of $\mathrm{SL}(2)$. The orbit space is isomorphic to the affine spectrum of the algebra of invariant polynomials $A = \mathbb{C}[X,Y]^\ell$. It has been known since Felix Klein that the algebra A is generated by three elements u, v, w with one single basic relation $F(u,v,w) = 0$. The origin $(0,0,0)$ of the surface $V(F) \subset \mathbb{C}^3$ is a RDP with the Dynkin diagram of type A_n, D_n, E_n dependent on ℓ in the following way. A nontrivial cyclic group of order $n+1$ corresponds to type A_n, a binary dihedral group of order $4n, n \geq 2$, corresponds to type D_{n+2}, a binary tetrahedral group of order 24 corresponds to type E_6, a binary octahedron group of order 48 corresponds to type E_7, and binary icosahedral group of order 120 corresponds to type E_8. It is known that the local analytic isomorphism class of a RDP is determined by the Dynkin diagram (see [585]). This gives the following.

Theorem 8.2.28. *A RDP is locally analytically isomorphic to one of the following singularities*

$$\begin{aligned} A_n &: z^2 + x^2 + y^{n+1} = 0, \quad n \geq 1, \\ D_n &: z^2 + y(x^2 + y^{n-2}) = 0, \quad n \geq 4, \\ E_6 &: z^2 + x^3 + y^4 = 0, \\ E_7 &: z^2 + x^3 + xy^3 = 0, \\ E_8 &: z^2 + x^3 + y^5 = 0. \end{aligned} \tag{8.14}$$

The corresponding Dynkin curve is of respective type A_n, D_n, E_n.

Comparing this list with the list of simple singularities of plane curves from Definition 4.2.16, we find that a surface singularity is a RDP if and only if it is locally analytically isomorphic to a singularity at the origin of the double cover of $\mathbb{C}^2$ branched along a curve $F(x,y)$ with a simple singularity at the origin. The types match.

Remark 8.2.29. A RDP is often named an *ADE-singularity* for the reason clear from above. Also, it is often called a *Du Val singularity* in honor of P. Du Val who was the first to characterize them by property (iii) from Theorem 8.2.2. They are also called *Klein singularities* for the reason explained above.

8.2.8 Cremona isometries

Definition 8.2.30. *Let S be a weak del Pezzo surface. An orthogonal transformation σ of* $\mathrm{Pic}(S)$ *is called a* Cremona isometry *if $\sigma(K_S) = K_S$ and σ sends any effective class to an effective class. The group of Cremona isometries will be denoted by* $\mathrm{Cris}(S)$.

It is clear that Cris(S) is a subgroup of $W(S)$ defined in (8.13).

Lemma 8.2.31. *Let*

$$C^n = \{D \in \mathrm{Pic}(S) : D \cdot R \geq 0 \text{ for any } (-2)\text{-curve } R\}.$$

For any $D \in \mathrm{Pic}(S)$, there exists $w \in W(S)^n$ such that $w(D) \in C^n$. If $D \in C^n$ and $w(D) \in C^n$ for some $w \in W(S)^n$, then $w(D) = D$. In other words, C^n is a fundamental domain for the action of $W(S)^n$ in Pic(S).

Proof The set of (-2)-curves forms a root basis in the Picard lattice Pic(S) and $W(S)^n$ is its Weyl group. The set C^n is a chamber defined by the root basis. Now, the assertion follows from the theory of finite reflection groups, which we have already employed for a similar assertion in the case of a canonical root basis in E_N. □

Proposition 8.2.32. *An isometry σ of* Pic(S) *is a Cremona isometry if and only if it preserves the canonical class and sends a (-2)-curve to a (-2)-curve.*

Proof Clearly, any Cremona isometry sends the class of an irreducible curve to the class of an irreducible curve. Since it also preserves the intersection form, it sends a (-2)-curve to a (-2)-curve.

Let us prove the converse. Let D be an effective class in Pic(S) with $D^2 \geq 0$. Then, $-K_S \cdot D > 0$ and $(K_S - D) \cdot D < 0$. This gives $-K_S \cdot \sigma(D) > 0$, $\sigma(D)^2 \geq 0$. Since $(K_S - \sigma(D)) \cdot (-K_S) = -K_S^2 + \sigma(D) \cdot K_S < 0$, we have $|K_S - \sigma(D)| = \emptyset$. By Riemann-Roch, $|\sigma(D)| \neq \emptyset$.

So, it remains to show that σ sends any (-1)-curve E to an effective divisor class. By the previous Lemma, for any (-2)-curve R, we have $0 < E \cdot R = \sigma(E) \cdot \sigma(R)$. Since $\sigma(R)$ is a (-2)-curve, and any (-2) curve is obtained in this way, we see that $\sigma(E) \in C^n$. Hence, $\sigma(E)$ is a (-1)-curve. □

Corollary 8.2.33. *Let $\mathcal{R}$ be the set of effective roots of a marked del Pezzo surface (S, ϕ). Then, the group of Cremona isometries* Cris(S) *is isomorphic to the subgroup of the Weyl group of E_N that leaves the subset $\mathcal{R}$ invariant.*

Let $W(S)^n$ be the subgroup of $W(S)$ generated by reflections with respect to (-2)-curves. It acts on a marking $\varphi : \mathsf{I}^{1,N} \to \mathrm{Pic}(S)$ by composing on the left.

By Lemma 8.2.22, a divisor D with $D^2 = D \cdot K_S = -1$ belongs to C^n if and only if it is a (-1)-curve. This and the previous lemma imply the following.

Proposition 8.2.34. *Let $\phi : W(S) \to W(\mathsf{E}_N)$ be an isomorphism of groups defined by a geometric marking on S. There is a natural bijection*

$$(-1)\text{-curves on } S \longleftrightarrow W(S)^n \backslash \phi^{-1}(Exc_N),$$

where Exc_N is the set of exceptional vectors in $\mathsf{I}^{1,N}$.

Theorem 8.2.35. *For any marked weak del Pezzo surface (S, φ), there exists $w \in W(S)^n$ such that $(S, w \circ \varphi)$ is geometrically marked weak del Pezzo surface.*

Proof We use induction on $N = 9 - K_S^2$. Let $e_i = \phi(\mathbf{e}_i), i = 0, \ldots, N$. It follows from the proof of Lemma 8.2.22, that each e_i is an effective class. Assume e_N is the class of a (-1)-curve E. Let $\pi_N : S \to S_{N-1}$ be the blowing down of E. Then, $e_0, e_1, \ldots, e_{N-1}$ are equal to the pre-images of the divisor classes $e_0', e_1', \ldots, e_{N-1}'$ on S_{N-1} which define a marking of S_{N-1}. By induction, there exists an element $w \in W(S_{N-1})^n$ such that $w(e_0'), w(e_1'), \ldots, w(e_{N-1}')$ defines a geometric marking. Since $\pi_N(e_N)$ does not lie on any (-2)-curve (otherwise S is not a weak del Pezzo surface), we see that for any (-2)-curve R on S_{N-1}, $\pi_N^*(R)$ is a (-2)-curve on S. Thus, under the canonical isomorphism $\mathrm{Pic}(S) \cong \pi_N^*(\mathrm{Pic}(S_{N-1})) \perp \mathbb{Z}e_N$, we can identify $W(S_{N-1})^n$ with a subgroup of $W(S)^n$. Applying w to $(e_0, \ldots, e_{N-1})$, we get a geometric marking of S.

If e_N is not a (-1)-curve, then we apply an element $w \in W(S)^n$ such that $w(e_N) \in C^n$. By Lemma 8.2.22, $w(e_N)$ is a (-1)-curve. Now, we have a basis $w(e_0), \ldots, w(e_N)$ satisfying the previous assumption. □

Corollary 8.2.36. *There is a bijection from the set of geometric markings on S and the set of left cosets $W(S)/W(S)^n$.*

Proof The group $W(S)$ acts simply transitively on the set of markings. By Theorem 8.2.35, each orbit of $W(S)^n$ contains a unique geometric marking. □

Corollary 8.2.37. *The group* $\mathrm{Cris}(S)$ *acts on the set of geometric markings of S.*

Proof Let $(e_0, \ldots, e_N)$ defines a geometric marking, and $\sigma \in \mathrm{Cris}(S)$. Then, there exists $w \in W(S)^n$ such that $\omega(\sigma(e_0)), \ldots, \omega(\sigma(e_N))$ defines a geometric marking. By Proposition 8.2.32, $\sigma(e_N)$ is the divisor class of a (-1)-curve E, hence it belongs to C^n. By Lemma 8.2.31, we get $w(\sigma(e_N)) = \sigma(e_1)$. This shows that $w \in W^n(\bar{S})$, where $S \to \bar{S}$ is the blow-down $\sigma(E)$. Continuing in this way, we see that $w \in W(\mathbb{P}^2)^n = \{1\}$. Thus, $w = 1$ and we obtain that σ sends a geometric marking to a geometric marking. □

Let $\varphi : \mathsf{I}^{1,N} \to \mathrm{Pic}(S)$ and $\varphi' : \mathsf{I}^{1,N} \to \mathrm{Pic}(S)$ be two geometric markings corresponding to two blowing-down structures $\pi = \pi_1 \circ \ldots \circ \pi_N$ and $\pi' = \pi_1' \circ \ldots \circ \pi_N'$. Then, $T = \pi' \circ \pi^{-1}$ is a Cremona transformation of $\mathbb{P}^2$ and $w = \varphi \circ \varphi'^{-1} \in W(\mathsf{E}_N)$ is its characteristic matrix. Conversely, if T is a Cremona transformation with fundamental points $x_1, \ldots, x_N$ such that their blow-up is a weak del Pezzo surface S, a characteristic matrix of T defines a pair of geometric markings φ, φ' of S and an element $w \in W(\mathsf{E}_N)$ such that

$$\varphi = \varphi' \circ w.$$

Example 8.2.38. Let S be a nonsingular del Pezzo surface of degree 3 and let $\pi : S \to \mathbb{P}^2$ be the blow-up of six points. Let $(e_0, e_1, \ldots, e_6)$ be the geometric marking and $\alpha = 2e_0 - e_1 - \cdots - e_6$. The reflection $w = s_\alpha$ transforms the geometric marking $(e_0, e_1, \ldots, e_6)$ to the geometric marking $(e_0', e_1', \ldots, e_6')$, where $e_0' = 5e_0 - 2(e_1 + \cdots + e_6), e_i' = 2e_0 - (e_1 + \cdots + e_6) + e_i, i = 1, \ldots, 6$. The corresponding Cremona transformation is the symmetric involutorial transformation of degree 5 with characteristic matrix given in (7.45)

Let S be a weak del Pezzo surface of degree d and $\mathrm{Aut}(S)$ be its group of biregular automorphisms. By functoriality, $\mathrm{Aut}(S)$ acts on $\mathrm{Pic}(S)$ leaving the canonical class K_S invariant. Thus, $\mathrm{Aut}(S)$ acts on the lattice $K_X^\perp = (\mathbb{Z}K_S)^\perp$ preserving the intersection form. Let

$$\rho : \mathrm{Aut}(S) \to \mathrm{O}(K_X^\perp), \quad \sigma \mapsto \sigma^*,$$

be the corresponding homomorphism.

Proposition 8.2.39. *The image of ρ is contained in the group* $\mathrm{Cris}(S)$. *If S is a nonsingular del Pezzo surface, the kernel of ρ is trivial if $d \le 5$. If $d \ge 6$, then the kernel is a linear algebraic group of dimension* $2d - 10$.

Proof Clearly, any automorphism induces a Cremona isometry of $\mathrm{Pic}(S)$. We know that it is contained in the Weyl group. An element in the kernel does not change any geometric basis of $\mathrm{Pic}(S)$. Thus, it descends to an automorphism of $\mathbb{P}^2$ which fixes an ordered set of $k = 9 - d$ points in general linear position. If $k \ge 4$ it must be the identity transformation. Assume $k \le 3$. The assertion is obvious when $k = 0$.

If $k = 1$, the surface S is the blow-up of one point. Each automorphism leaves the unique exceptional curve invariant and acts trivially on the Picard group. The group $\mathrm{Aut}(S)$ is the subgroup of $\mathrm{Aut}(\mathbb{P}^2)$ fixing a point. It is a connected linear algebraic group of dimension six isomorphic to the semi-direct product $\mathbb{C}^2 \rtimes \mathrm{GL}(2)$.

If $k = 2$, the surface S is the blow-up of two distinct points p_1, p_2. Each automorphism leaves the proper inverse transform of the line $\langle p_1, p_2 \rangle$ invariant. It either leaves the exceptional curves E_1 and E_2 invariant or switches them. The kernel of the Weyl representation consists of elements that do not switch E_1 and E_2. It is isomorphic to the subgroup of $\mathrm{Aut}(\mathbb{P}^2)$ which fixes two points in $\mathbb{P}^2$ and is isomorphic to the group G of invertible matrices of the form

$$\begin{pmatrix} 1 & 0 & * \\ 0 & * & * \\ 0 & 0 & * \end{pmatrix}.$$

Its dimension is equal to 4. The image of the Weyl representation is a group of order 2. So $\mathrm{Aut}(S) = G \rtimes C_2$.

If $k = 3$, the surface S is the blow-up of three, non-collinear points. The kernel of the Weyl representation is isomorphic to the group of invertible diagonal 3×3 matrices modulo scalar matrices. It is isomorphic to the 2-dimension torus $(\mathbb{C}^*)^2$. □

Corollary 8.2.40. *Let S be a del Pezzo surface of degree $d \le 5$, then $\mathrm{Aut}(S)$ is isomorphic to a subgroup of the Weyl group $W(\mathsf{E}_{9-d})$.*

We will see later examples of automorphisms of weak del Pezzo surfaces of degree 1 or 2 which act trivially on $\mathrm{Pic}(S)$.

8.3 Anti-canonical Models

In this section, we will show that any weak del Pezzo surface of degree $d \ge 3$ is isomorphic to a minimal resolution of a del Pezzo surface of degree d in $\mathbb{P}^d$. In particular, any nonsingular del Pezzo surface of degree $d \ge 3$ is isomorphic to a nonsingular surface of degree d in $\mathbb{P}^d$.

8.3.1 Anti-canonical linear systems

Lemma 8.3.1. *Let S be a weak del Pezzo surface with $K_S^2 = d$. Then*

$$\dim H^0(S, \mathcal{O}_S(-rK_S)) = 1 + \tfrac{1}{2}r(r+1)d.$$

Proof By Ramanujam's Vanishing Theorem, which we already used, for any $r \ge 0$ and $i > 0$,

$$H^i(S, \mathcal{O}_S(-rK_S)) = H^i(S, \mathcal{O}_S(K_S + (-r-1)K_S)) = 0. \tag{8.15}$$

The Riemann-Roch Theorem gives

$$\dim H^0(S, \mathcal{O}_S(-rK_S)) = \tfrac{1}{2}(-rK_S - K_S)\cdot(-rK_S) + 1 = 1 + \tfrac{1}{2}r(r+1)d.$$

□

Theorem 8.3.2. *Let S be a weak del Pezzo surface of degree d and $\mathcal{R}$ be the union of (-2)-curves on S. Then, we have the following.*

(i) $|-K_S|$ has no fixed part.
(ii) If $d > 1$, then $|-K_S|$ has no base points.

(iii) If $d > 2$, $|-K_S|$ defines a regular map ϕ to $\mathbb{P}^d$ which is an isomorphism outside $\mathcal{R}$. The image surface $\bar{S}$ is a del Pezzo surface of degree d in $\mathbb{P}^d$. The image of each connected component of $\mathcal{R}$ is a RDP of $\phi(S)$.

(iv) If $d = 2$, $|-K_S|$ defines a regular map $\phi : S \to \mathbb{P}^2$. It factors as a birational morphism $f : S \to \bar{S}$ onto a normal surface and a finite map $\pi : \bar{S} \to \mathbb{P}^2$ of degree 2 branched along a curve B of degree 4. The image of each connected component of $\mathcal{N}$ is a RDP of $\bar{S}$. The curve B is either nonsingular or has only simple singularities.

(v) If $d = 1$, $|-2K_S|$ defines a regular map $\phi : S \to \mathbb{P}^3$. It factors as a birational morphism $f : S \to \bar{S}$ onto a normal surface and a finite map $\pi : \bar{S} \to Q \subset \mathbb{P}^3$ of degree 2, where Q is a quadric cone. The morphism π is branched along a curve B of degree 6 cut out on Q by a cubic surface. The image of each connected component of $\mathcal{N}$ under f is a RDP of $\bar{S}$. The curve B either nonsingular or has only simple singularities.

Proof The assertions are easily verified if $S = \mathbf{F}_0$ or $\mathbf{F}_2$. So we assume that S is obtained from $\mathbb{P}^2$ by blowing up $k = 9 - d$ points t_i.

(i) Assume there is a fixed part F of $|-K_S|$. Write $|-K_S| = F + |M|$, where $|M|$ is the mobile part. If $F^2 > 0$, by Riemann-Roch,

$$\dim |F| \geq \tfrac{1}{2}(F^2 - F \cdot K_S) \geq \tfrac{1}{2}(F^2) > 0,$$

and hence F moves. Thus, $F^2 \leq 0$. If $F^2 = 0$, we must also have $F \cdot K_S = 0$. Thus, $F = \sum n_i R_i$, where R_i are (-2)-curves. Hence, $[f] \in (\mathbb{Z}K_S)^{\perp}$, and hence, $F^2 \leq -2$ (the intersection form on $(\mathbb{Z}K_S)^{\perp}$ is negative definite and even). Thus, $F^2 \leq -2$ and

$$M^2 = (-K_S - F)^2 = K_S^2 + 2K_S \cdot F + F^2 \leq K_S^2 + F^2 \leq d - 2,$$
$$-K_S \cdot M = K_S^2 + K_S \cdot F \leq d.$$

Suppose $|M|$ is irreducible. Since $\dim |M| = \dim |-K_S| = d$, the linear system $|M|$ defines a rational map to $\mathbb{P}^d$ whose image is a nondegenerate irreducible surface of degree $\leq d - 3$ (strictly less if $|M|$ has base points). This contradicts Theorem 8.1.1.

Now, assume that $|M|$ is reducible, i.e., it defines a rational map to a nondegenerate curve $W \subset \mathbb{P}^d$ of some degree t. By Theorem 8.1.1, we have $t \geq d$. Since S is rational, W is a rational curve and the pre-image of a general hyperplane section is equal to the disjoint sum of t linearly equivalent curves. Thus, $M \sim tM_1$ and

$$d \geq -K_S \cdot M = -tK_S \cdot M_1 \geq d(-K_S \cdot M_1).$$

Since $-K_S \cdot M = 0$ implies $M^2 < 0$ and a curve with negative self-intersection does not move, this gives $-K_S \cdot M_1 = 1, d = t$. But then $M^2 = d^2 M_1^2 \leq d - 2$ gives a contradiction.

(ii) Assume $d > 1$. We have proved that $|-K_S|$ is irreducible. A general member of $|-K_S|$ is an irreducible curve C with $\omega_C = \mathcal{O}_C(C + K_S) = \mathcal{O}_C$. If C is smooth, then it is an elliptic curve and the linear system $|\mathcal{O}_C(C)|$ is of degree $d > 1$ and has no base points. The same is true for a singular irreducible curve of arithmetic genus 1. This is proved in the same way as in the case of a smooth curve. Consider the exact sequence

$$0 \to \mathcal{O}_S \to \mathcal{O}_S(C) \to \mathcal{O}_C(C) \to 0.$$

Applying the exact sequence of cohomology, we see that the restriction of the linear system $|C| = |-K_S|$ to C is surjective. Thus, we have an exact sequence of groups

$$0 \to H^0(S, \mathcal{O}_S) \to H^0(S, \mathcal{O}_S(C)) \to H^0(S, \mathcal{O}_C(C)) \to 0.$$

Since $|\mathcal{O}_C(C)|$ has no base points, we have a surjection

$$H^0(S, \mathcal{O}_C(C)) \otimes \mathcal{O}_C \to \mathcal{O}_C(C).$$

This easily implies that the homomorphism

$$H^0(S, \mathcal{O}_S(C)) \otimes \mathcal{O}_C \to \mathcal{O}_S(C)$$

is surjective. Hence, $|C| = |-K_S|$ has no base points.

(iii) Assume $d > 2$. Let $x, y \in S$ be two points outside $\mathcal{R}$. Let $f : S' \to S$ be the blowing up of x and y with exceptional curves E_x and E_y. By Proposition 8.1.23, S' is a weak del Pezzo surface of degree $d - 2$. We know that the linear system $|-K_{S'}|$ has no fixed components. Thus,

$$\dim |-K_S - x - y| = \dim |-K_{S'} - E_x - E_y| \geq 1.$$

This shows that $|-K_S|$ separates points. Also, the same is true if $y >_1 x$ and x does not belong to any (-1)-curve E on S or $x \in E$ and y does not correspond to the tangent direction defined by E. Since $-K_S \cdot E = 1$ and $x \in E$, the latter case does not happen.

Since $\phi : S \dashrightarrow \bar{S}$ is a birational map given by a complete linear system $|-K_S|$, its image is a nondegenerate surface of degree $d = (-K_S)^2$. Since $-K_S \cdot R = 0$ for any (-2)-curve, we see that ϕ blows down R to a point p. If $d = 3$, then $\bar{S}$ is a cubic surface with isolated singularities (the images of connected components of $\mathcal{N}$). It is well known that a hypersurface with no singularities in codimension 1 is a normal variety. Thus, $\bar{S}$ is a normal surface. If $d = 4$, then S is obtained by a blow-up one point on a weak del Pezzo surface S' of degree 3. This point does not lie on a (-2)-curve. Thus, $\bar{S}'$ is obtained from $\bar{S}$ by a linear projection from a nonsingular point. We have explained already in Proposition 8.1.8 that this implies that $\bar{S}$ is a normal surface.

The fact that singular points of $\bar{S}$ are RDP is proven in the same way as we have proved assertion (iii) of Theorem 8.2.27.

(iv) Assume $d = 2$. By (ii), the linear system $|-K_S|$ defines a regular map $\phi : S \to \mathbb{P}^2$. Since $K_S^2 = 2$, the map is of degree 2. Using Stein's factorization [379, Chapter III, Corollary 11.5], it factors through a birational morphism onto a normal surface $f : S \to \bar{S}$ and a finite degree 2 map $\pi : \bar{S} \to \mathbb{P}^2$. Also, we know that $f_*\mathcal{O}_S = \mathcal{O}_{\bar{S}}$. A standard Hurwitz-type formula gives

$$\omega_{\bar{S}} \cong \pi^*(\omega_{\mathbb{P}^2} \otimes \mathcal{L}), \tag{8.16}$$

where $s \in H^0(\mathbb{P}^2, \mathcal{L}^{\otimes 2})$ vanishes along the branch curve W of π. We have

$$\mathcal{O}_S(K_S) = \omega_S = (\pi \circ f)^*\mathcal{O}_{\mathbb{P}^2}(-1) = f^*(\pi^*\mathcal{O}_{\mathbb{P}^2}(-1)).$$

It follows from the proof of Theorem 8.2.27 (iii) that singular points of $\bar{S}$ are RDP. Thus, $f^*\omega_{\bar{S}} = \omega_S$, and hence

$$f^*\omega_{\bar{S}} \cong f^*(\pi^*\mathcal{O}_{\mathbb{P}^2}(-1)).$$

Applying f_* and using the projection formula and the fact that $f_*\mathcal{O}_X = \mathcal{O}_Y$, we get $\omega_{\bar{S}} \cong \pi^*\mathcal{O}_{\mathbb{P}^2}(-1)$. It follows from (8.16) that $\mathcal{L} \cong \mathcal{O}_{\mathbb{P}^2}(2)$ and hence $\deg W = 4$.

Proof of (v). Let $\pi : S \to \mathbb{P}^2$ be the blow-up of 8 points $x_1, \ldots, x_8$. Then, $|-K_S|$ is the proper inverse transform of the pencil $|3h - x_1 - \cdots - x_8|$ of plane cubics passing through the points $x_1, \ldots, x_8$. Let x_9 be the ninth intersection point of two cubics generating the pencil. The point $x_9' = \pi^{-1}(x_9)$ is the base point of $|-K_S|$. By Bertini's Theorem, all fibers except finitely many, are nonsingular curves (the assumption that the characteristic is zero is important here). Let F be a nonsingular member from $|-K_S|$. Consider the exact sequence

$$0 \to \mathcal{O}_S(-K_S) \to \mathcal{O}_S(-2K_S) \to \mathcal{O}_F(-2K_S) \to 0. \tag{8.17}$$

The linear system $|\mathcal{O}_F(-2K_S)|$ on F is of degree 2. It has no base points. We know from (8.15) that $H^1(S, \mathcal{O}_S(-K_S)) = 0$. Thus, the restriction map

$$H^0(S, \mathcal{O}_S(-2K_S)) \to H^0(F, \mathcal{O}_F(-2K_S))$$

is surjective. By the same argument as we used in the proof of (ii), we obtain that $|-2K_S|$ has no base points. By Lemma 8.3.1, $\dim|-2K_S| = 3$. Let $\phi : S \to \mathbb{P}^3$ be a regular map defined by $|-2K_S|$. Its restriction to any nonsingular member F of $|-K_S|$ is given by the linear system of degree 2 and hence is of degree 2.

Therefore, the map f is of degree $t > 1$. The image of ϕ is a surface of some degree k. Since $(-2K_S)^2 = 4 = kt$, we conclude that $k = t = 2$. Thus, the image of ϕ is a quadric surface Q in $\mathbb{P}^3$ and the images of $F \in |-K_S|$ is a line l_F on Q. Since all lines l_F intersect at the point $\phi(t_9')$, Q is a quadric cone with the vertex $\phi(t_9')$.

Let $S \xrightarrow{\pi} S' \xrightarrow{\phi'} Q$ be the Stein factorization. Note that a (-2)-curve R does not pass through the base point x_9' of $|-K_S|$ (because $-K_S \cdot R = 0$). Thus, $\pi(x_9')$ is a nonsingular point q' of S'. Its image in Q is the vertex q of Q. Since ϕ' is a finite map, the local ring $\mathcal{O}_{S',q'}$ is a finite algebra over $\mathcal{O}_{Q,q}$ of degree 2. After completion, we may assume that $\mathcal{O}_{S',q'} \cong \mathbb{C}[[u,v]]$. If $u \in \mathcal{O}_{Q,q}$, then v satisfies a monic equation $v^2 + av + b$ with coefficients in $\mathcal{O}_{Q,q}$, where, after changing v to $v + \frac{1}{2}a$, we may assume that $a = 0$. Then, $\mathcal{O}_{Q,q}$ is equal to the ring of invariants in $\mathbb{C}[[u,v]]$ under the automorphism $u \mapsto u, v \mapsto -v$ which as easy to see isomorphic to $\mathbb{C}[[u,v^2]]$. However, we know that q is a singular point so the ring $\mathcal{O}_{Q,q}$ is not regular. Thus, we may assume that $u^2 = a, v^2 = b$ and then $\mathcal{O}_{Q,q}$ is the ring of invariants for the action $(u,v) \mapsto (-u,-v)$. This action is free outside the maximal ideal (u,v). This shows that the finite map ϕ' is unramified in a neighborhood of q' with q' deleted. In particular, the branch curve Q of ϕ' does not pass through q. We leave it to the reader to repeat the argument from the proof of (iv) to show that the branch curve W of ϕ belongs to the linear system $|\mathcal{O}_Q(3)|$. □

Let X be a weak del Pezzo surface of degree $d \le 3$. The image of a (-1)-curve on X under the antticanonical map is a line on the anti-canonical model S of X in $\mathbb{P}^d$. Conversely, any line ℓ on a del Pezzo surface S of degree d in $\mathbb{P}^d$ is the image of a (-1)-curve E on its minimal resolution X. It passes through a singular point if and only if E intersects a component of a Dynkin curve blown down to this singular point. By Proposition 8.2.34, the set of lines on S is in a bijective correspondence with the set of orbits of exceptional vectors in the lattice $K_X^\perp \cong \mathsf{E}_{9-d}$ with respect to the Weyl group of the root sublattice of generated by (-2)-curves. This justifies calling a (-1)-curve on a weak del Pezzo surface a *line*.

8.3.2 Anti-canonical model

Let X be a normal projective algebraic variety and let D be a Cartier divisor on X. It defines the graded algebra

$$R(X,D) = \bigoplus_{r=0}^{\infty} H^0(S, \mathcal{O}_S(rD)),$$

which depends only (up to isomorphism) on the divisor class of D in $\mathrm{Pic}(X)$. Assume $R(X,D)$ is finitely generated, then $X_D = \mathrm{Proj}\ R(X,D)$ is a projective variety. If $s_0,\dots,s_n$ are homogeneous generators of $R(X,D)$ of degrees $q_0,\dots,q_n$ there is a canonical closed embedding into the weighted projective space

$$X_D \hookrightarrow \mathbb{P}(q_0,\dots,q_n).$$

Also, the evaluation homomorphism of sheaves of graded algebras

$$R(X,D) \otimes \mathcal{O}_X \to S(\mathcal{L}) = \bigoplus_{r=0}^{\infty} \mathcal{O}_S(rD)$$

defines a morphism

$$\varphi_{\mathrm{can}} : X = \mathrm{Proj}(S(\mathcal{L})) \to X_D.$$

For every $r > 0$, the inclusion of subalgebras

$$S(H^0(X,\mathcal{O}_X(rD))) \to R(X,D)$$

defines a rational map

$$\tau_r : X_D \dashrightarrow \mathbb{P}(H^0(X,\mathcal{O}_X(rD))).$$

The rational map $\phi_{|rD|} : X \dashrightarrow \mathbb{P}(H^0(X,\mathcal{O}_X(rD)))$ is given by the complete linear system $|rD|$ factors through φ

$$\phi_{|rd|} : X \overset{\varphi}{\dashrightarrow} X_D \overset{\tau_r}{\dashrightarrow} \mathbb{P}(H^0(X,\mathcal{O}_X(rD))).$$

A proof of the following proposition can be found in [205, 7.1].

Proposition 8.3.3. *Suppose $|rD|$ has no base points for some $r > 0$ and $D^{\dim X} > 0$. Then,*

(i) $R(X,D)$ *is a finitely generated algebra;*
(ii) X_D *is a normal variety;*
(iii) $\dim X_D = \max_{r>0} \dim \phi_{|rD|}(X)$*;*
(iv) if $\dim X_D = \dim X$*, then* φ *is a birational morphism.*

We apply this to the case when $X = S$ is a weak del Pezzo surface and $D = -K_S$. Applying the previous proposition, we easily obtain that

$$X_{-K_S} \cong \bar{S},$$

where we use the notation of Theorem 8.3.2. The variety $\bar{S}$ is called the *anti-canonical model* of S. If S is of degree $d > 2$, the map $\tau_1 : \bar{S} \to \mathbb{P}^d$ is a closed embedding, hence $R(S,-K_S)$ is generated by $d+1$ elements of order 1.

If $d = 2$, the map τ_1 is the double cover of $\mathbb{P}^2$. This shows that $R(S, -K_S)$ is generated by three elements s_0, s_1, s_2 of degree 1 and one element s_3 of degree 2 with a relation $s_3^2 + a_4(s_0, s_1, s_2) = 0$ for some homogeneous polynomial a_4 of degree 2. This shows that $\bar{S}$ is isomorphic to a hypersurface of degree 4 in $\mathbb{P}(1, 1, 1, 2)$ given by an equation

$$t_3^2 + a_4(t_0, t_1, t_2) = 0. \tag{8.18}$$

In the case $d = 1$, by Lemma 8.3.1 we obtain that

$$\dim R(S, -K_S)_1 = 2,\ \dim R(S, -K_S)_2 = 4,\ \dim R(S, -K_S)_3 = 7.$$

Let s_0, s_1 be generators of degree 1, let s_2 be an element of degree 2 that is not in $S^2(R(S, -K_S)_1)$ and let s_3 be an element of degree 3 that is not in the subspace generated by $s_0^3, s_0s_1^2, s_0^2s_1, s_1^3, s_2s_0, s_2s_1$. The subring $R(S, -K_S)'$ generated by s_0, s_1, s_2, s_3 is isomorphic to $\mathbb{C}[t_0, t_1, t_2, t_3]/(F(t_0, t_1, t_2, t_3))$, where

$$F = t_3^2 + t_2^3 + a_4(t_0, t_1)t_2 + a_6(t_0, t_1),$$

and $a_4(t_0, t_1)$ and $a_6(t_0, t_1)$ are binary forms of degrees 4 and 6. The projection $[t_0, t_1, t_2, t_3] \mapsto [t_0, t_1, t_2]$ is a double cover of the quadratic cone $Q \subset \mathbb{P}^3$ which is isomorphic to the weighted projective plane $\mathbb{P}(1, 1, 2)$. Applying Theorem 8.3.2, one can show that the rational map $\bar{S} \dashrightarrow \operatorname{Proj} R(S, -K_S)'$ is an isomorphism. This shows that the anti-canonical model $\bar{S}$ of a weak del Pezzo surface of degree one is isomorphic to a hypersurface $V(F)$ of degree 6 in $\mathbb{P}(1, 1, 2, 3)$.

Recall from Subsection 4.1.2 that a nondegenerate subvariety X of $\mathbb{P}^n$ is called projectively normal if X is normal and the natural restriction map

$$H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(m)) \to H^0(X, \mathcal{O}_X(m))$$

is surjective for all $m \geq 0$. This can be restated in terms of the vanishing of cohomology

$$H^1(\mathbb{P}^n, \mathcal{I}_X(m)) = 0,\ m > 0 \text{ (resp. } m = 1),$$

where $\mathcal{I}_X$ is the ideal sheaf of X. If X is a normal surface, this is equivalent to that the ideal sheaf $\mathcal{I}_X$ is an aCM sheaf.

Theorem 8.3.4. *Let S be a weak del Pezzo surface, then the anti-canonical ring $R(S, -K_S)$ is a normal Cohen-Macaulay ring. In particular, if $d \geq 3$, the anti-canonical model X of S of degree d in $\mathbb{P}^d$ is arithmetically Cohen-Macaulay and projectively normal.*

Proof For $d \leq 2$, this follows from the explicit description of the ring. It is quotient of a ring of polynomials by a principal ideal, and it has singularities in codimension ≥ 2. By Serre's criterion, it is a normal domain (see [281], 11.2).

It follows from (8.15) that $H^1(X, \mathcal{O}_X(k)) = 0, k \geq 0$, and by Serre's duality, $H^1(X, \mathcal{O}_X(k)) \cong H^1(X, \mathcal{O}_X(1-k)) = 0$ for all $k < 0$. Using our discussion of aCM sheaves in Subsection 4.1.2, we obtain that it remains to prove that a del Pezzo surface of degree d in $\mathbb{P}^d$ is projectively normal.

Let H be a general hyperplane. Tensoring the exact sequence

$$0 \to \mathcal{O}_{\mathbb{P}^n}(m-1) \to \mathcal{O}_{\mathbb{P}^n}(m) \to \mathcal{O}_H(m) \to 0$$

with $\mathcal{I}_X$ we get an exact sequence

$$0 \to \mathcal{I}_X(m-1) \to \mathcal{I}_X(m) \to \mathcal{I}_{H\cap X}(m) \to 0. \tag{8.19}$$

We know that a general hyperplane section $C = S \cap H$ is an elliptic curve of degree d in H which is a projectively normal curve in H. Thus,

$$H^1(C, \mathcal{I}_C(m)) = 0, \ m > 0.$$

We know that S is linearly normal surface in $\mathbb{P}^d$. This implies that

$$H^1(\mathbb{P}^d, \mathcal{I}_X(1)) = 0.$$

The exact sequence gives that $H^1(\mathbb{P}^d, \mathcal{I}_X(2)) = 0$. Continuing in this way, we get that $H^1(\mathbb{P}^d, \mathcal{I}_X(m)) = 0, m > 0$.

□

Corollary 8.3.5. *If $d \geq 4$, the anti-canonical ring of a weak del Pezzo surface is generated by $\frac{1}{2}d(d-3)$ elements of degree 2.*

Proof Let $S \subset \mathbb{P}^d$ be an anti-canonical model of a weak del Pezzo surface of degree $d \geq 4$. Since S is projectively normal,

$$\begin{aligned} \dim I_S(2) &= \dim H^0(\mathbb{P}^d, \mathbb{P}^d(2)) - \dim H^0(S, \mathcal{O}_S(-2K_S)) \\ &= \binom{d+2}{2} - 3d - 1 = \tfrac{1}{2}d(d-3). \end{aligned} \tag{8.20}$$

Let E be a general hyperplane section of S. It is an elliptic curve of degree d embedded in $\mathbb{P}^{d-1}$ by a complete linear system. It is known to be a normal embedding, and that E is a scheme-theoretical intersection of quadrics [415, Theorem IV.1.3]. This implies that

$$\dim I_E(2) = \binom{d+1}{2} = 2d = \tfrac{1}{2}d(d-3) = \dim I_S(2).$$

Let $J(S)_2$ be the ideal in $\oplus_{n\geq 0} H^0(\mathbb{P}^d, \mathcal{O}_{\mathbb{P}^d}(n)$ generated by $I_X(2)$.Then, $S \subset X = V(J(S)_2)$. For a general hyperplane H, the intersection $X \cap H$ contains $E = S \cap H$, and they are generated by the same linear space of quadrics. Thus, they coincide. Since H is general, $X = S$. □

One can also derive a projective resolution of an anti-canonical model $S = \mathrm{Proj}(R(S, -K_S)) \subset \mathbb{P}^d$ of a weak del Pezzo surface of degree $d \geq 4$. We use the following more general result [405, Theorem 1].

Theorem 8.3.6. *Let X be a an arithmetically Cohen-Macaulay subvariety of* $\mathbb{P}^n$ *of dimension* ≥ 1 *and* $\deg(X) = \mathrm{codim}(X) + 2$. *Then, the minimal free resolution of X is*

$$0 \to \mathcal{O}_{\mathbb{P}^n}(-p-2) \to \mathcal{O}_{\mathbb{P}^n}(-p)^{\oplus \alpha_{p-1}} \to \mathcal{O}_{\mathbb{P}^n}(-p+1)^{\oplus \alpha_{p-2}} \to \cdots$$
$$\cdots \to \mathcal{O}_{\mathbb{P}^n}(-2)^{\oplus \alpha_1} \to \mathcal{O}_{\mathbb{P}^n} \to \mathcal{O}_X \to 0,$$

where p is the projective dimension of the projective coordinate ring of X and $\alpha_i = i\binom{p+1}{i+1} - \binom{p}{i-1}$ *for* $1 \leq i \leq p-2$.

Since $\deg(S) = d$ and $\mathrm{codim}(S) = d - 2$, Theorem 8.3.4 allows us to apply this theorem.

Corollary 8.3.7. *We get the following resolvents for an anti-canonical model of a del Pezzo surface:*

$d = 4$:

$$0 \to \mathcal{O}_{\mathbb{P}^4}(-4) \to \mathcal{O}_{\mathbb{P}^4}(-2)^{\oplus 2} \to \mathcal{O}_{\mathbb{P}^4} \to \mathcal{O}_X \to 0.$$

$d = 5$:

$$0 \to \mathcal{O}_{\mathbb{P}^5}(-5) \to \mathcal{O}_{\mathbb{P}^5}(-3)^{\oplus 5} \to \mathcal{O}_{\mathbb{P}^5}(-2)^{\oplus 5} \to \mathcal{O}_{\mathbb{P}^5} \to \mathcal{O}_X \to 0.$$

$d = 6$:

$$0 \to \mathcal{O}_{\mathbb{P}^6}(-6) \to \mathcal{O}_{\mathbb{P}^6}(-4)^{\oplus 9} \to \mathcal{O}_{\mathbb{P}^6}(-3)^{\oplus 16} \to \mathcal{O}_{\mathbb{P}^6}(-2)^{\oplus 9} \to \mathcal{O}_{\mathbb{P}^6} \to \mathcal{O}_X \to 0.$$

$d = 7$:

$$0 \to \mathcal{O}_{\mathbb{P}^7}(-7) \to \mathcal{O}_{\mathbb{P}^7}(-5)^{\oplus 14} \to \mathcal{O}_{\mathbb{P}^7}(-4)^{\oplus 35}$$
$$\to \mathcal{O}_{\mathbb{P}^7}(-3)^{\oplus 35} \to \mathcal{O}_{\mathbb{P}^7}(-2)^{\oplus 14} \to \mathcal{O}_{\mathbb{P}^7} \to \mathcal{O}_X \to 0.$$

$d = 8$:

$$0 \to \mathcal{O}_{\mathbb{P}^8}(-8) \to \mathcal{O}_{\mathbb{P}^8}(-6)^{\oplus 20} \to \mathcal{O}_{\mathbb{P}^8}(-5)^{\oplus 64} \to \mathcal{O}_{\mathbb{P}^8}(-4)^{\oplus 90}$$
$$\to \mathcal{O}_{\mathbb{P}^8}(-3)^{\oplus 64} \to \mathcal{O}_{\mathbb{P}^8}(-2)^{\oplus 20} \to \mathcal{O}_{\mathbb{P}^8} \to \mathcal{O}_X \to 0.$$

$d = 9$:

$$0 \to \mathcal{O}_{\mathbb{P}^9}(-9) \to \mathcal{O}_{\mathbb{P}^9}(-7)^{\oplus 27} \to \mathcal{O}_{\mathbb{P}^9}(-6)^{\oplus 105} \to \mathcal{O}_{\mathbb{P}^9}(-5)^{\oplus 189}$$
$$\to \mathcal{O}_{\mathbb{P}^9}(-4)^{\oplus 189} \to \mathcal{O}_{\mathbb{P}^9}(-3)^{\oplus 105} \to \mathcal{O}_{\mathbb{P}^9}(-2)^{\oplus 27} \to \mathcal{O}_{\mathbb{P}^9} \to \mathcal{O}_X \to 0.$$

Remark 8.3.8. An irreducible subvariety X of $\mathbb{P}^n$ is said to satisfy *Property* (N_p) if it admits a projective resolvent

$$\cdots \to \oplus_j \mathcal{O}_{\mathbb{P}^n}(\alpha_{2,j}) \to \oplus_j \mathcal{O}_{\mathbb{P}^n}(\alpha_{1,j}) \to \mathcal{O}_{\mathbb{P}^n} \to \mathcal{O}_X \to 0$$

with $\alpha_{i,j} = i+1$ for $1 \le i \le p$ and all j [479, I:1.8.D]. Inspecting the projective resolvents of anti-canonical models of weak del Pezzo surfaces of degree $d \ge 4$, we find that they satisfy property (N_{d-3}).

8.4 Del Pezzo surfaces of degree ≥ 6

8.4.1 Del Pezzo surfaces of degree 7, 8, 9

A weak del Pezzo surface of degree 9 is isomorphic to $\mathbb{P}^2$. Its anti-canonical model is a Veronese surface V_3^2. It does not contain lines.

A weak del Pezzo surface of degree 8 is isomorphic to either $\mathbf{F}_0$, or $\mathbf{F}_1$, or $\mathbf{F}_2$. In the first two cases, it is a del Pezzo surface isomorphic to its anti-canonical model in $\mathbb{P}^8$. If $S \cong \mathbf{F}_0$, the anti-canonical model is a Veronese-Segre surface embedded in $\mathbb{P}^8$ by the complete linear system of divisors of type $(2,2)$. It does not contain lines.

If $S \cong \mathbf{F}_1$, the anti-canonical model is a surface of degree 8 in $\mathbb{P}^8$ embedded by the linear system $|3\mathfrak{f} + 2\mathfrak{e}|$, where $\mathfrak{f}$ is the divisor class of a fiber and $\mathfrak{e}$ is the divisor class of the exceptional section e. The image of e is the unique line ℓ on the surface. The images of the fibers of the ruling of $\mathbf{F}_1$ are conics. They define a structure of a smooth conic bundle on the surface with the section ℓ. If $S \cong \mathbf{F}_2$, the anti-canonical model is isomorphic to the quadratic cone Q embedded in $\mathbb{P}^8$ by the complete linear system $|\mathcal{O}_Q(2)|$. It does not contain lines.

A weak del Pezzo surface of degree 7 is isomorphic to the blow-up of two points x_1, x_2 in $\mathbb{P}^2$. If the points are proper, the anti-canonical model of S is a nonsingular surface that contains three lines representing the divisor classes $e_1, e_2, e_0 - e_1 - e_2$. If only one point is proper, then it has one singular point of type A_1 and contains two intersecting lines representing the classes e_1 and $e_0 - e_1 - e_2$. In both cases, the surface is isomorphic to a projection of the Veronese surface V_3^2 from a secant line of the surface. In the second case, the secant line is tangent to the Veronese surface.

The automorphism groups of a nonsingular del Pezzo surfaces of degree ≥ 7 were described in Subsection 8.2.8.

8.4.2 Del Pezzo surfaces of degree 6

A weak del Pezzo surface S of degree 6 is isomorphic to the blow-up of a bubble cycle $\eta = x_1 + x_2 + x_3$. Up to a change of an admissible order, we have the following possibilities:

(i,i') x_1, x_2, x_3 are three proper non-collinear (collinear) points;
(ii, ii') $x_2 \succ x_1, x_3$ are non-collinear (collinear) points;
(iii, iii') $x_3 \succ x_2 \succ x_1$ are non-collinear (collinear) points.

In cases (i), (ii) and (iii) the net of conics $|\mathcal{O}_{\mathbb{P}^2}(2) - \eta|$ is homaloidal and the surface S is isomorphic to a minimal resolution of the graph of the Cremona transformation T defined by this net. Since a quadratic Cremona transformation is a special case of a bilinear Cremona transformation, its graph is a complete intersection of two hypersurfaces of bidegree $(1,1)$ in $\mathbb{P}^2 \times \mathbb{P}^2$. Under the Segre map, the graph embeds in $\mathbb{P}^6$ and the composition of the maps

$$\Phi : S \to \ell_T \hookrightarrow \mathbb{P}^2 \times \mathbb{P}^2 \overset{s}{\hookrightarrow} \mathbb{P}^6,$$

is the map given by the anti-canonical linear system. Its image is a del Pezzo surface of degree 6 embedded in $\mathbb{P}^6$. It is a nonsingular surface in case (i) and it has one singular point of type A_1 in case (ii) and type A_2 in case (iii). The two maps $S \to \mathbb{P}^2$ are defined by the linear systems $|e_0|$ and $|2e_0 - e_1 - e_2 - e_3|$.

The set of (-1)-curves and (-2)-curves on a weak del Pezzo surface of types (i) (resp. (ii), resp. (iii)) is pictured in Figure 7.1 (resp. Figure 7.2, resp. Figure 7.3).

In the cases where the points x_1, x_2, x_3 are collinear, S has a unique map to $\mathbb{P}^2$ defined by the linear system $|e_0|$ and it is not related to Cremona transformations.

Surfaces of types (i), (ii), (ii'), (iii') are examples of *toric surfaces*. They contain an algebraic torus as its open Zariski set U, and the action of U on itself by translations extends to a biregular action of U on S. The complement of U is the union of orbits of dimension 0 and 1. It supports an anti-canonical divisor. For example, in case (i), the complement of U is the union of six lines on the surface.

The anti-canonical model of a weak toric del Pezzo surface is a toric del Pezzo surface of degree six in $\mathbb{P}^6$. It is nonsingular only in case (i).

The types of singular points and the number of lines on a del Pezzo surface of degree 6 are given in Table 8.5 below.

The secant variety of a nonsingular del Pezzo surface of degree 6 in $\mathbb{P}^6$ is of expected dimension five. In fact, projecting from a general point, we obtain a nonsingular surface of degree six in $\mathbb{P}^5$. It follows from the classification of Severi-Zak varieties from Subsection 7.4.2 that a surface in $\mathbb{P}^5$ with the secant

Bubble cycle	(i)	(ii)	(iii)	(i')	(ii')	(iii')
Singular points	$\emptyset$	A_1	A_2	A_1	$2A_1$	$A_1 + A_2$
Lines	6	4	2	3	2	1

Table 8.5 *Lines and singular points on a del Pezzo surface of degree 6*

variety of dimension four is a Veronese surface. More precisely, we have the following description of the secant variety.

Theorem 8.4.1. *Let S be a nonsingular del Pezzo surface of degree 6 in $\mathbb{P}^6$. Then, S is projectively equivalent to the subvariety given by equations expressing the rank condition*

$$\operatorname{rank}\begin{pmatrix} t_0 & t_1 & t_2 \\ t_3 & t_0 & t_4 \\ t_5 & t_6 & t_0 \end{pmatrix} \le 1.$$

The secant variety $\operatorname{Sec}(X)$ *is the cubic hypersurface defined by the determinant of this matrix.*

Proof We know that S is isomorphic to the intersection of the Segre variety $\mathsf{S}_{2,2} \cong \mathbb{P}^2 \times \mathbb{P}^2 \hookrightarrow \mathbb{P}^8$ by a linear subspace L of codimension 2. If we identify $\mathbb{P}^8$ with the projectivization of the space of 3×3-matrices, then the Segre variety $\mathsf{S}_{2,2}$ is the locus of matrices of rank 1, hence it is defined, even scheme-theoretically, by the 2×2-matrices. A secant line of S is contained in L and is a secant of $\mathsf{S}_{2,2}$. It represents a matrix equal to the sum of matrices of rank 1. Hence, each secant is contained in the determinantal cubic hypersurface. Thus, the secant variety of S is the intersection of the cubic by the linear subspace L, so it is a cubic hypersurface in $\mathbb{P}^6$.

Explicitly, we find the linear space L as follows. The map $S \to \mathbb{P}^2 \times \mathbb{P}^2$ is given by the map (π_1, π_2), where $\pi_i : S \to \mathbb{P}^2$ are given by the linear systems $|e_0|$ and $|2e_0 - e_1 - e_2 - e_3|$. Choose a basis z_0, z_1, z_2 in $|e_0|$ and a basis z_1z_2, z_0z_1, z_0z_1 in $|2e_0 - e_1 - e_2 - e_3|$ corresponding to the standard quadratic transformation T_{st}. Then, the graph of T_{st} is equal to the intersection of $\mathsf{S}_{2,2} \subset |\mathrm{Mat}_{3,3}|$ with equal diagonal entries $a_{11} = a_{22} = a_{33}$ corresponding to the relations $z_0(z_1z_2) = z_1(z_0z_2) = z_2(z_0z_1)$. This gives the equations from the assertion of the theorem. □

Let us describe the group of automorphisms of a nonsingular del Pezzo surface of degree six. The surface is obtained by blowing up three non-collinear points x_1, x_2, x_3. We may assume that their coordinates are $[1, 0, 0]$, $[0, 1, 0]$, $[0, 0, 1]$.

We know from Section 8.2.6 that the kernel of the linear representation ρ : $\mathrm{Aut}(S) \to \mathrm{O}(\mathrm{Pic}(S))$ is a 2-dimensional torus. The root system is of type $A_2 + A_1$, so the Weyl group is isomorphic to $2 \times \mathfrak{S}_3 \cong D_{12}$, where D_{12} is the dihedral group of order 12. Let us show that the image of the Weyl representation is the whole Weyl group.

We choose the standard generators s_1, s_2, s_3 of $W(S) \cong W(\mathsf{E}_3)$ defined by the reflections with respect to the roots $e_0 - e_1 - e_2, e_1 - e_2, e_2 - e_3$. The reflection s_1 acts as the standard quadratic transformation T_{st}, which is lifted to an automorphism of S. It acts on the hexagon of lines on S by switching the opposite sides. The reflection s_2 (resp. s_3) acts as a projective transformation that permutes the points x_1, x_2 and fixes x_3 (resp. permutes x_2 and x_3 and fixes x_1). The subgroup $\langle s_2, s_3 \rangle \cong D_6 \cong \mathfrak{S}_3$ acts on the hexagon of lines by natural embedding $D_6 \hookrightarrow \mathrm{O}(2)$.

We leave it to the reader to prove the following:

Theorem 8.4.2. *Let S be a del Pezzo surface of degree* 6. *Then,*

$$\mathrm{Aut}(S) \cong (\mathbb{C}^*)^2 \rtimes \mathfrak{S}_3 \times \mathfrak{S}_2.$$

If we represent the torus as the quotient group of $(\mathbb{C}^*)^3$ *by the diagonal subgroup* $\Delta \cong \mathbb{C}^*$, *then the subgroup* $\mathfrak{S}_3$ *acts by permutations of factors, and the cyclic subgroup* $\mathfrak{S}_2$ *acts by the inversion automorphism* $z \mapsto z^{-1}$.

Finally, we mention that the Gosset polytope $\Sigma_3 = -1_{21}$ corresponding to a nonsingular del Pezzo surface of degree 6 is an octahedron. This agrees with the isomorphism $W(\mathsf{E}_3) \cong D_{12}$. The surface has two blowing-down morphisms $S \to \mathbb{P}^2$ corresponding to two α-facets and three conic bundle structures corresponding to the pencils of lines through three points on the plane.

8.5 Del Pezzo Surfaces of Degree 5

8.5.1 Lines and singularities

A weak del Pezzo surface S of degree 5 is isomorphic to the blow-up of a bubble cycle $\eta = x_1 + x_2 + x_3 + x_4$. The only assumption on the cycle is that $|h - \eta| = \emptyset$. Let e_0, e_1, e_2, e_3, e_4 be a geometric basis defined by an admissible order of η. There are the following five possibilities:

(i) x_1, x_2, x_3, x_4 are proper points;

(ii) $x_2 \succ x_1, x_3, x_4$;

(iii) $x_3 \succ x_2 \succ x_1, x_4$;

(iv) $x_2 \succ x_1, x_4 \succ x_3$;

(v) $x_4 \succ x_3 \succ x_2 \succ x_1$.

There are the following root sublattices in a root lattice of type A_4:

$$A_1, A_1 + A_1, A_2, A_1 + A_2, A_3, A_4.$$

In case (i), S is a del Pezzo surface or has one Dynkin curve of type A_1 if three points are collinear.

In case (ii), we have three possibilities for Dynkin curves: A_1 if no three points are collinear, $A_1 + A_1$ if x_1, x_2, x_3 are collinear, A_2 if x_1, x_3, x_4 are collinear.

In case (iii), we have three possibilities: A_2 if no three points are collinear, A_3 if x_1, x_2, x_3 are collinear, $A_1 + A_2$ if x_1, x_2, x_4 are collinear.

In case (iv), we have two possibilities: $A_1 + A_1$ if no three points are collinear, $A_2 + A_1$ if x_2, x_3, x_4 or x_1, x_2, x_3 are collinear,

In case (v), we have two possibilities: A_3 if x_1, x_2, x_3 are not collinear, A_4 otherwise.

It can be checked that the cases with the same root bases are obtained from each other by a Cremona isometry. So, they lead to isomorphic surfaces.

Table 8.6 below gives the possibilities of lines and singular points on the anti-canonical model of a del Pezzo surface of degree 5 in $\mathbb{P}^5$.

Singular points	$\emptyset$	A_1	$2A_1$	A_2	$A_1 + A_2$	A_3	A_4
Lines	10	7	5	4	3	2	1

Table 8.6 *Lines and singular points on a del Pezzo surface of degree 5*

From now on, we will study nonsingular del Pezzo surfaces of degree 5. Since any set of four points in general position is projectively equivalent to the set of reference points $[1, 0, 0]$, $[0, 1, 0]$, $[0, 0, 1]$, $[1, 1, 1]$, we obtain that all nonsingular del Pezzo surfaces of degree 5 are isomorphic. A nonsingular del Pezzo surface of degree 5 has 10 lines. The union of them is a divisor in $|-2K_S|$.

The Gosset polytope $\Sigma_4 = 0_{21}$ has five facets of type α corresponding to contractions of five disjoint lines on S and five pencils of conics corresponding to the pencils of lines through a point in the plane and the pencil of conics through the four points.

The incidence graph of the set of 10 lines is the famous *Petersen graph*. The incidence graph of the set of 10 lines is the famous *Petersen graph*.

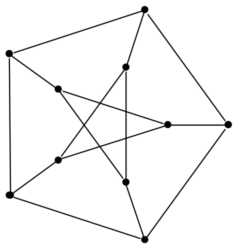


Figure 8.4 Petersen graph

8.5.2 Equations

In this subsection, we use some elementary properties of Grassmann varieties $G_k(\mathbb{P}^n) = G(k+1, n+1)$ of k-dimensional subspaces in $\mathbb{P}^n$ (equivalently, $(k+1)$-dimensional linear subspaces of $\mathbb{C}^{n+1}$). We refer to Chapter 10 for the proof of all properties we will use.

Proposition 8.5.1. *Let S be a nonsingular del Pezzo surface of degree five in $\mathbb{P}^5$. Then, S is isomorphic to a linear section of the Grassmann variety $G_1(\mathbb{P}^4)$ of lines in $\mathbb{P}^4$.*

Proof It is known that the degree of $G = G_1(\mathbb{P}^4)$ in the Plücker embedding is equal to 5 and $\dim G = 6$. It is also known that the canonical sheaf is equal to $\mathcal{O}_G(-5)$. By the adjunction formula, the intersection of G with a general linear subspace of codimension 4 is a nonsingular surface X with $\omega_X \cong \mathcal{O}_X(-1)$. This must be a del Pezzo surface of degree 5. Since all del Pezzo surfaces of degree 5 are isomorphic, the assertion follows. □

Corollary 8.5.2. *Let S be a nonsingular del Pezzo surface of degree 5 in $\mathbb{P}^5$. Then, its homogeneous ideal is generated by five linearly independent quadrics.*

Proof Since S is projectively normal, applying Lemma 8.3.1, we obtain that the linear system of quadrics containing S is of dimension 4. It is known that the homogeneous ideal of the Grassmannian $G(2, 5)$ is generated by five quadrics. In fact, the Grassmannian is defined by five pfaffians of principal 4×4 minors of a general skew-symmetric 5×5-matrix. So, restricting this linear system to the linear section of the Grassmannian, we obtain that the quadrics containing S define S scheme-theoretically. □

Let $\mathbb{P}^4 = |E|$ for some linear space E of dimension 5. For any line $\ell = |U|$ in $|E|$, the dual subspace $U^\perp$ in $E^\vee$ defines a plane $|U^\perp|$ in $\mathbb{P}(E)$. This gives a natural isomorphism between the Grassmannians $G_1(|E|)$ and $G_2(\mathbb{P}(E))$. Dually, we get an isomorphism $G_2(|E|) \cong G_1(\mathbb{P}(E))$.

Fix an isomorphism $\bigwedge^5 E \cong \mathbb{C}$, and consider the natural pairing

$$\bigwedge^2 E \times \bigwedge^3 E \to \bigwedge^5 E \cong \mathbb{C}$$

defined by the wedge product. It allows one to identify $(\bigwedge^2 E)^\vee = \bigwedge^2 E^\vee$ with $\bigwedge^3 E$. The corresponding identification of the projective spaces does not depend on the choice of an isomorphism $\bigwedge^5 E \cong \mathbb{C}$. A point $U \in G(2, E)$ is orthogonal to a point $V \in G(3, E)$ if and only if $|U| \cap |V| \neq \emptyset$. We know that a quintic del Pezzo surface S is contained in the base locus of a web W of hyperplanes in $|\bigwedge^2 E|$. The web of hyperplanes, considered as a 3-dimensional subspace of $|\bigwedge^2 E^\vee| \cong |\bigwedge^3 E|$ intersects $G_3(|E|)$ at 5 points $\Lambda_1, \ldots, \Lambda_5$. Thus, any point in S intersects $\Lambda_1, \ldots, \Lambda_5$.

Conversely, let $\Lambda_1, \ldots, \Lambda_5$ be the five planes in $|E|$ such that, considered as points in the space $|\bigwedge^3 E|$, they span a general 3-dimensional subspace W. Then, $W^\vee \cap G_2(|E|)$ is a general 5-dimensional subspace in $|\bigwedge^2 E|$ which cuts $G_2(|E|)$ along a quintic del Pezzo surface.

Let us record this.

Proposition 8.5.3. *A nonsingular del Pezzo quintic is isomorphic to the variety of lines in $\mathbb{P}^4$ that intersect five planes in $\mathbb{P}^4$ that span a general 3-dimensional subspace in the Plücker space $\mathbb{P}^9$. Via duality, it is also isomorphic to the variety of planes in $\mathbb{P}^4$ that intersect five lines in $\mathbb{P}^4$ that span a general 3-dimensional subspace of the Plücker space.*

Let S be a del Pezzo surface of degree 5 in $\mathbb{P}^5$. The linear system of cubics in $\mathbb{P}^5$ containing S has dimension 24. Let us see that any nonsingular cubic fourfold containing X is rational (the rationality or the irrationality of a general cubic fourfold is unknown at the moment).

Lemma 8.5.4. *Let S be a nonsingular del Pezzo surface S of degree 5 in $\mathbb{P}^5$. For any point x outside S there exists a unique secant line of S containing x.*

Proof It is known that $\mathrm{Sec}(X) = \mathbb{P}^5$ since any nondegenerate nonsingular surface in $\mathbb{P}^5$ with secant variety of dimension four is a Veronese surface. Let $a, b \in S$ such that $x \in \ell = \langle a, b\rangle$. Consider the projection $p_\ell : X \dashrightarrow \mathbb{P}^3$ from the line ℓ. Its image is a cubic surface S_3 isomorphic to the anti-canonical model of the blow-up of S at a, b. If $a = b$, the line ℓ is tangent to S, and one of the points is infinitely near the other. In this case the cubic surface is singular. The map $p_\ell : S \setminus \ell$ is an isomorphism outside a, b. Suppose x belongs to another secant $\ell' = \langle c, d\rangle$. Then, the projection of the plane $\langle \ell, \ell'\rangle$ spanned by ℓ and ℓ' is a point on the cubic surface whose pre-image contains c, d. This shows that p_ℓ is not an isomorphism outside $\ell \cap S$. This contradiction proves the assertion. □

Theorem 8.5.5. *Let F be an irreducible cubic fourfold containing a nonsingular del Pezzo surface S of degree 5 in $\mathbb{P}^5$. Then, F is a rational variety.*

Proof Consider the linear system $|\mathcal{I}_S(2)|$ of quadrics containing S. It defines a regular map $Y \to \mathbb{P}^4$, where Y is the blow-up of S. Its fibers are proper transforms of secants of X. This shows that the subvariety of $G_1(\mathbb{P}^5)$ parameterizing secants of S is isomorphic to $\mathbb{P}^4$. Let take a general point z in F. By the previous Lemma, there exists a unique secant of X passing through z. By Bezout's Theorem, no other point outside S lies on this secant. This gives a rational injective map $F \dashrightarrow \mathbb{P}^4$ defined outside S. Since a general secant intersects F at three points, with two of them on S, we see that the map is birational. □

Remark 8.5.6. According to a result of A. Beauville [51, Proposition 8.2], any smooth cubic fourfold containing S is a pfaffian cubic hypersurface, i.e., is given by the determinant of a skew-symmetric matrix with linear forms as its entries. Conversely, any pfaffian cubic fourfold contains a nondegenerate surface of degree 5, i.e. an anti-canonical weak del Pezzo surface or a scroll.

8.5.3 Automorphism group

Let us study automorphisms of a nonsingular del Pezzo surface of degree 5. Recall that the Weyl group $W(\mathsf{E}_4)$ is isomorphic to the Weyl group $W(A_4) \cong \mathfrak{S}_5$. By Proposition 8.2.39, we have a natural injective homomorphism

$$\rho : \mathrm{Aut}(S) \cong \mathfrak{S}_5.$$

Theorem 8.5.7. *Let S be a nonsingular del Pezzo surface of degree 5. Then,*

$$\mathrm{Aut}(S) \cong \mathfrak{S}_5.$$

Proof We may assume that S is isomorphic to the blow-up of the reference points $x_1 = [1,0,0], x_2 = [0,1,0], x_3 = [0,0,1]$ and $x_4 = [1,1,1]$. The group $\mathfrak{S}_5$ is generated by its subgroup isomorphic to $\mathfrak{S}_4$ and an element of order 5. The subgroup $\mathfrak{S}_4$ is realized by projective transformations permuting the points $x_1, \ldots, x_4$. The action is realized by the standard representation of $\mathfrak{S}_4$ in the hyperplane $z_1 + \cdots + z_4 = 0$ of $\mathbb{C}^4$ identified with $\mathbb{C}^3$ by the projection to the first three coordinates. An element of order 5 is realized by a quadratic transformation with fundamental points x_1, x_2, x_3 defined by the formula

$$T : [t_0, t_1, t_2] \mapsto [t_0(t_2 - t_1), t_2(t_0 - t_1), t_0 t_2]. \tag{8.21}$$

It maps the line $V(t_0)$ to the point x_2, the line $V(t_1)$ to the point x_4, the line $V(x_2)$ to the point x_1, the point x_4 to the point x_3. □

Note that the group of automorphisms acts on the Petersen graph of 10 lines and defines an isomorphism with the group of symmetries of the graph.

Let S be a del Pezzo surface of degree 5. The group $\mathrm{Aut}(S) \cong \mathfrak{S}_5$ acts linearly on the space $V = H^0(S, \mathcal{O}_S(-K_S)) \cong \mathbb{C}^6$. Let us compute the character of this representation. Choose the following basis in the space V:

$$(t_0^2t_1 - t_0t_1t_2, t_0^2t_2 - t_0t_1t_2, t_1^2t_0 - t_0t_1t_2, t_1^2t_2 - t_0t_1t_2, t_2^2t_0 - t_0t_1t_2, t_2^2t_1 - t_0t_1t_2). \quad (8.22)$$

Let $s_1 = (12), s_2 = (23), s_3 = (34), s_4 = (45)$ be the generators of $\mathfrak{S}_5$. It follows from the proof of Theorem 8.5.7 that s_1, s_2, s_3 generate the subgroup of $\mathrm{Aut}(S)$ which is realized by projective transformations permuting the points x_1, x_2, x_3, x_4, represented by the matrices

$$s_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \ s_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, s_3 = \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & -1 \end{pmatrix}.$$

The last generator s_4 is realized by the standard quadratic transformation T_{st}. Choose the following representatives of the conjugacy classes in $\mathfrak{S}_5$ different from the conjugacy class of the identity element id:

$$g_1 = (12), \ g_2 = (123) = s_2s_1, g_3 = (1234) = s_3s_2s_1,$$

$$g_4 = (12345) = s_4s_3s_2s_1, g_5 = (12)(34) = s_1s_3, g_6 = (123)(45) = s_3s_2s_1s_4.$$

The subgroup generated by s_1, s_2 acts by permuting the coordinates t_0, t_1, t_2. The generator s_3 acts as the projective transformation

$$(y_1, \ldots, y_6) \mapsto (y_1 - y_2 + y_5, -y_2 + y_5 - y_6, y_3 - y_4 + y_6, -y_4 - y_5 + y_6, -y_6, -y_5),$$

where $(y_1, \ldots, y_6)$ is the basis from (8.22). Finally, s_4 acts by

$$(y_1, y_2, y_3, y_4, y_5, y_6) \mapsto (y_6, y_4, y_5, y_2, y_3, y_1).$$

Simple computation gives the character vector of the representation

$$\chi = (\chi(1), \chi(g_1), \chi(g_2), \chi(g_3), \chi(g_4), \chi(g_5), \chi(g_6)) = (6, 0, 0, 0, 1, -2, 0).$$

Using the character table of $\mathfrak{S}_5$, we find that χ is the character of an irreducible 6-dimensional irreducible representation isomorphic to $V = \bigwedge^2 R_{\mathrm{st}}$, where R_{st} is the standard 4-dimensional irreducible linear representation of $\mathfrak{S}_5$ with character vector $(4, 2, 1, 0, -1, 0, -1)$ (see [316, p. 28]). The linear system $|-K_S|$ embeds S in $\mathbb{P}(V)$. Since V is isomorphic to its dual representation, we can identify $\mathbb{P}(V)$ with $|V|$.

We will see later in Chapter 10 that $G_1(\mathbb{P}^4)$, embedded in $\mathbb{P}^9$, is defined by five pfaffians of principal minors of a skew-symmetric 5×5-matrix (p_{ij}), where $p_{ji} = -p_{ij}, i < j$, are the Plücker coordinates. The group $\mathfrak{S}_5$ acts on $\mathbb{P}^9$ via its natural representation on $\bigwedge^2 W$, where W is an irreducible representation of $\mathfrak{S}_5$ with character $(5, 1, -1, -1, 0, 1, 1)$. The representation $\bigwedge^2 W$ decomposes into irreducible representation $V \oplus R'_{\mathrm{st}}$, where R'_{st} is the standard 4-dimensional representation of $\mathfrak{S}_5$ tensored with the sign representation U'.

Now, let us consider the linear representation of $\mathfrak{S}_5$ on the symmetric square $S^2(V^vee)$. Using the formula

$$\chi_{S^2(V)}(g) = \tfrac{1}{2}(\chi(g)^2 + \chi(g^2)),$$

we get

$$\chi_{S^2(V)} = (21, 3, 0, -1, 1, 5, 0).$$

Taking the inner product with the character of the trivial representation, we get 1. This shows that the subspace of invariant vectors $U = S^2(V)^{\mathfrak{S}_5}$ is one-dimensional. Similarly, we find that $\dim S^2(V)$ contains one copy of the one-dimensional sign representation U' of $\mathfrak{S}_5$. The equation of the union of ten lines, considered as an element of $S^2(V)$, is represented by the equation of the union of six lines $\langle x_i, x_j \rangle$, where $x_1, \ldots, x_4$ are the reference points. It is

$$F = t_0 t_1 t_2 (t_0 - t_1)(t_0 - t_2)(t_1 - t_2) = 0.$$

It is easy to check that F transforms under $\mathfrak{S}_5$ as the sign representation. It is less trivial, but straightforward, to find a generator of the vector space $S^2(V)^{\mathfrak{S}_5}$. It is equal to

$$G = 2\sum t_i^4 t_j^2 - 2\sum t_i^4 t_j t_k - 2\sum t_i^3 t_j^3 - \sum t_i^3 t_j^2 t_k + 6t_0^2 t_1^2 t_2^2. \qquad (8.23)$$

Its singular points are the reference points $x_1, \ldots, x_4$. In another coordinate system, the equation looks even better:

$$t_0^6 + t_1^6 + t_2^6 + (t_0^2 + t_1^2 + t_2^2)(t_0^4 + t_1^4 + t_2^4) - 12t_0^2 t_1^2 t_2^2 = 0.$$

(see [274]). The singular points here are the points

$$[1, -1, -1], \quad [-1, 1, -1], \quad [-1, -1, 1], \quad [1, 1, 1].$$

The equation $G = 0$ reveals obvious symmetry with respect to the group generated by the permutation of the coordinates corresponding to the generators s_1 and s_2. It is also obviously invariant with respect to the standard quadratic transformation T_{st} which we can write in the form $[t_0, t_1, t_2] \mapsto [1/t_0, 1/t_1, 1/t_2]$. Less obvious is the invariance with respect to the generator s_3.

The $\mathfrak{S}_5$-invariant plane sextic $W_6 = V(G)$ is called the *Wiman sextic*. Its proper transform on S is a smooth curve of genus six in $|-2K_S|$. All curves in the pencil of sextics spanned by $V(\lambda F + \mu G)$ (the *Wiman pencil*) are $\mathfrak{A}_5$-invariant. It contains two $\mathfrak{S}_5$-invariant members $V(F)$ and $V(G)$.

Remark 8.5.8. It is known that a del Pezzo surface of degree 5 is isomorphic to the GIT-quotient P_1^5 of the space $(\mathbb{P}^1)^5$ by the group $\mathrm{SL}(2)$ (see [234]). The group $\mathfrak{S}_5$ is realized naturally by the permutation of factors. The isomorphism is defined by assigning to any point x on the surface the five ordered points $(x_1, \ldots, x_4, x_5 = x)$, where $p_1, \ldots, p_4$ are the tangent directions of the conic in the plane passing through the points x_1, x_2, x_3, x_4, x. The isomorphism from P_1^5 onto a quintic surface in $\mathbb{P}^5$ is given by the linear system of bracket-functions $(ab)(cd)(ef)(hk)(lm)$, where $a, b, c, d, e, f, h, k, l, m$ belong to the set $\{1, 2, 3, 4, 5\}$ and each number from this set appears exactly 2 times.

Remark 8.5.9. Let S be a weak del Pezzo surface and D be a smooth divisor in $|-2K_S|$. The double cover X of S branched over D is a *K3 surface*. If we take S to be a nonsingular del Pezzo surface of degree 5 and D to be the proper transform of the Wiman sextic, we obtain a K3 surface with automorphism group containing the group $\mathfrak{S}_5 \times 2$. The cyclic factor here acts on the cover as the deck transformation. Consider the subgroup of $\mathfrak{S}_5 \times 2$ isomorphic to $\mathfrak{S}_5$ that consists of elements $(\sigma, \epsilon(\sigma))$, where $\epsilon : \mathfrak{S}_5 \to \{\pm 1\}$ is the sign representation. This subgroup acts on X symplectically, i.e. leaves a nonzero holomorphic 2-form on X invariant. The list of maximal groups of automorphisms of K3 surfaces which act symplectically was given by S. Mukai [533]. We find the group $\mathfrak{S}_5$ in this list (although the example in the paper is different).

In Section 11.3 we will discuss a realization of a del Pezzo surface of degree 5 as a congruence of lines of order two and class three.

8.6 Quartic del Pezzo Surfaces

Here, we study in more detail del Pezzo surfaces of degree 4. Their minimal resolutions of singularities are obtained by blowing up five points in $\mathbb{P}^2$ and hence vary in a two-dimensional family.

8.6.1 Equations

Lemma 8.6.1. *Let X be the complete intersection of two quadrics in $\mathbb{P}^n$. Then, X is nonsingular if and only if it is isomorphic to the variety*

$$\sum_{i=0}^{n} t_i^2 = \sum_{i=0}^{n} a_i t_i^2 = 0,$$

where the coefficients $a_0, \dots, a_n$ are all distinct.

Proof The pencil of quadrics has the discriminant hypersurface Δ defined by a binary form of degree $n+1$. If all quadrics are singular, then, by Bertini's theorem they share a singular point. This implies that X is a cone, and hence singular. Conversely, if X is a cone, then all quadrics in the pencil are singular. Suppose Δ consists of less than $n+1$ points. The description of the tangent space of the discriminant hypersurface of a linear system of quadrics (see Example 1.2.3) shows that a multiple point corresponds to either a quadric of corank ≥ 2 or to a quadric of corank 1 such that all quadrics in the pencil contain its singular point. In both cases, X contains a singular point of one of the quadrics in the pencil causing X to be singular. Conversely, if X has a singular point, all quadrics in the pencil are tangent at this point. One of them must be singular at this point causing Δ to have a multiple point.

So, we see that X is nonsingular if and only if the pencil contains exactly $n+1$ quadrics of corank one. It is a standard fact from linear algebra that, in this case, the quadrics can be simultaneously diagonalized (see, for example, [319] or [404, Volume 2, Chapter XIII]). Thus, we see that, after a linear change of coordinates, X can be given by equations from the assertion of the Lemma. If two coefficients a_i are equal, then the pencil contains a quadrics of corank ≥ 2, and hence Δ has a multiple point. □

Theorem 8.6.2. *Let S be a del Pezzo surface S of degree 4. Then, S is a complete intersection of two quadrics in $\mathbb{P}^4$. Moreover, if S is nonsingular, the equations of the quadrics can be reduced, after a linear change of variables, to the diagonal forms:*

$$\sum_{i=0}^{4} t_i^2 = \sum_{i=0}^{4} a_i t_i^2 = 0,$$

where $a_i \neq a_j$ for $i \neq j$.

Proof By Theorem 8.3.4, S is projectively normal in $\mathbb{P}^4$. This gives the exact sequence

$$0 \to H^0(\mathbb{P}^4, \mathcal{I}_S(2)) \to H^0(\mathbb{P}^4, \mathcal{O}_{\mathbb{P}^4}) \to H^0(S, \mathcal{O}_S(2)) \to 0.$$

By Lemma 8.3.1,

$$\dim H^0(S, \mathcal{O}_S(2)) = \dim H^0(S, \mathcal{O}_S(-2K_S)) = 13.$$

This implies that S is the base locus of a pencil of quadrics. Now, the assertion follows from the previous Lemma. □

Following the classical terminology, an anti-canonical model of a weak del Pezzo surface of degree 4 in $\mathbb{P}^4$ is called a *Segre quartic surface*.

One can say more about equations of singular del Pezzo quartics. Let Q be a pencil of quadrics in $\mathbb{P}^n$. We view it as a line in the space of symmetric matrices of size $n+1$ spanned by two matrices A, B. Assume that Q contains a nonsingular quadric, so that we can choose B to be a nonsingular matrix. Consider the λ-matrix $A + \lambda B$ and compute its elementary divisors. Let $\det(A + \lambda B) = 0$ have r distinct roots $\alpha_1, \ldots, \alpha_r$. For every root α_i we have elementary divisors of the matrix $A + \lambda B$

$$(\lambda - \alpha_i)^{e_i^{(1)}}, \ldots, (\lambda - \alpha_i)^{e_i^{(s_i)}}, \quad e_i^{(1)} \le \ldots \le e_i^{(s_i)}.$$

The *Segre symbol* of the pencil Q is the collection

$$[(e_1^{(1)} \ldots e_1^{(s_1)})(e_2^{(1)} \ldots e_2^{(s_2)}) \ldots (e_r^{(1)} \ldots e_r^{(s_r)})].$$

It is a standard result in linear algebra (see, the references in the proof of Lemma 8.6.1) that one can simultaneously reduce the pair of matrices (A, B) to the form (A', B') (i.e. there exists an invertible matrix C such that $CAC^t = A', CBC^t = B'$) such that the corresponding quadratic forms Q_1', Q_2' have the following form

$$Q_1' = \sum_{i=1}^{r} \sum_{j=1}^{s_i} p(\alpha_i, e_i^{(j)}), \quad Q_2' = \sum_{i=1}^{r} \sum_{j=1}^{s_i} q(e_i^{(j)}), \tag{8.24}$$

where

$$p(\alpha, e) = \alpha \sum_{i=1}^{e} t_i t_{e+1-i} + \sum_{i=1}^{e-1} t_{i+1} t_{e+1-i},$$

$$q(e) = \sum_{i=1}^{e} t_i t_{e+1-i}.$$

It is understood here that each $p(\alpha, e)$ and $q(e)$ are written in disjoint sets of variables. This implies the following.

Theorem 8.6.3. *Let X and X' be two complete intersections of quadrics and $\mathcal{P}, \mathcal{P}'$ be the corresponding pencils of quadrics. Assume that $\mathcal{P}$ and $\mathcal{P}'$ contains a nonsingular quadric. Let H and H' be the set of singular quadrics in $\mathcal{P}$ and $\mathcal{P}'$ considered as sets marked with the corresponding part of the Segre symbol. Then, X is projectively equivalent to X' if and only if the Segre symbols of $\mathcal{P}$ and $\mathcal{P}'$ coincide and there exists a projective isomorphism $\phi : \mathcal{P} \to \mathcal{P}'$ such that $\phi(H) = H'$ and the marking is preserved.*

Applying this to our case $n = 4$, we obtain the following possible Segre symbols:

$r = 5$: [11111];
$r = 4$: [(11)111], [2111];
$r = 3$: [(11)(11)1], [(11)21], [311], [221], [(12)11];
$r = 2$: [14], [(31)1], [3(11)], [32], [(12)2], [(12)(11)];
$r = 1$: [5], [(14)].

Here, r is the number of singular quadrics in the pencil. Note that the case $[(1, 1, 1, 1, 1)]$ leads to linearly dependent matrices A, B, so it is excluded for our purpose. Also, in cases [(111)11], [(1111)1], [(112)1], [(22)1], there is a reducible quadric in the pencil, so the base locus is a reducible. Finally, the cases [(23)], [(113)], [(122)], and [(1112)] correspond to cones over a quartic elliptic curve.

8.6.2 Cyclide quartics

Let S be a del Pezzo surface of degree four in $\mathbb{P}^4$. Let us project S to $\mathbb{P}^3$, first, from a nonsingular point $p \in S$. Assume that p does not lie on a line contained in S. Then, the image of the projection is a cubic surface V_3 in $\mathbb{P}^3$. Its singular points are the projections of the singular points of S, and they are of the same type as singular points on S. If p lies on a line, then the projection of the line is a new singular point. It is an ordinary double point if the line does not contain any singular points of S. Note that no three lines on S are coplanar since the pencil of hyperplanes through this line cuts out, residually, a pencil of lines on S. So, no point is a common point of 3 lines.

If the center of the projection is a singular point of S, then the projection of S is a quadric.

Now, let us assume that the center of the projection p does not lie on S. Let Q_p be the unique quadric from the pencil which contains p.

Theorem 8.6.4. *Assume that the quadric Q_p is nonsingular. Then, the projection X of S from p is a quartic surface in $\mathbb{P}^3$ which is singular along a nonsingular conic. Any irreducible quartic surface Q in $\mathbb{P}^3$ whose one-dimensional part of* Sing(Q) *consists of a nonsingular conic arises in this way from a Segre quartic surface S in $\mathbb{P}^4$. The surface S is nonsingular if and only if X is nonsingular outside the conic.*

Proof First of all, let us see that X is indeed a quartic surface. If not, the projection is a finite map of degree 2 onto a quadric. In this case, the pre-image of the quadric in $\mathbb{P}^4$ is a quadratic cone containing S with the vertex at the center of the projection. This is excluded by the assumption.

Let H be the tangent hyperplane of Q_p at p and $C = H \cap S$. The intersection $H \cap Q_p$ is an irreducible quadric in H with a singular point at p. The curve C lies on this quadric and is cut out by a quadric $Q' \cap H$ for some quadric $Q' \neq Q$ from the pencil. Thus, the projection from p defines a degree 2 map from C to a nonsingular conic K equal to the projection of the cone $H \cap Q_p$. It spans the plane in $\mathbb{P}^3$ equal to the projection of the hyperplane H. Since the projection defines a birational isomorphism from S to X that is not an isomorphism over the conic K, we see that X is singular along K. It is also nonsingular outside K (since we assume that S is nonsingular).

Conversely, let Q be a quartic surface such that the one-dimensional part of $\mathrm{Sing}(Q)$ is a nonsingular conic C. As we saw in Subsection 7.2.1, the linear system $|\mathcal{I}_C(2)|$ of quadrics through C maps $\mathbb{P}^3$ onto a quadric X_1 in $\mathbb{P}^4$. The pre-image of a quadric $X_2 \neq X_1$ under this rational map is a quartic surface X containing C as a double curve. The intersection $S = X_1 \cap X_2$ is a Segre quartic surface. The image of the plane Π containing C is a point p on X_1. The inverse map $S \dashrightarrow X$ is the projection from p. Since the rational map $\mathbb{P}^3 \dashrightarrow X_1$ is an isomorphism outside Π, the quartic Q is nonsingular outside C if and only if S is nonsingular.

Note that, if $\mathrm{Sing}(Q)$ contains a double line beside the double conic, Q is a ruled surface. The image of Q is still a complete intersection $X_1 \cap X_2$ of two quadrics, but it contains a double line. So, the surface is not normal, and hence it is not a del Pezzo surface.

□

In the classical literature, a quartic surface in $\mathbb{P}^3$ singular along a conic is called a *cyclide quartic surface*.

If we choose the equation of the conic C in the form $V(t_1^2 + t_2^2 + t_3^2, t_0)$, then formula (7.64) shows that the equation of the quartic cyclide can be written in the form $V(\sum a_{ij} z_i z_j)$, where $(z_0, z_1, z_2, z_3) = (t_1^2+t_2^2+t_3^2, t_0t_1, t_0t_2, t_0t_3)$. Since the quartic is irreducible, we may assume that $a_{00} \neq 0$, hence the equation of a cyclide surface can be reduced to the form

$$(t_1^2 + t_2^2 + t_3^2)^2 + t_0^2 g_2(t_0, t_1, t_2, t_3) = 0. \tag{8.25}$$

In particular, if $g_2 = t_0 g_1$, the conic becomes cuspidal double curve. Note that this can be generalized to any dimension. We obtain a quartic hypersurface

$$(\sum_{i=1}^{n} t_i^2)^2 + t_0^2 g_2(t_0, \dots, t_n) = 0$$

singular along the quadric $V(t_0) \cap V(\sum_{i=1}^n t_i^2)$. In dimension one, we obtain a quartic curve with two double points (a *cyclide curve*).

We can draw immediate corollaries of Theorem 8.6.4:

Corollary 8.6.5. *Assume that a cyclide quartic surface X is a general projection of a nonsingular quartic del Pezzo surface. Then,*

(i) *X contains 16 lines (they are the the projections of 16 lines on the quartic del Pezzo surface).*
(ii) $\mathrm{Sing}(X)$ *is a smooth conic C_0 that contains four pinch points, the branch points of the cover $C_0' \to C_0$, where C_0' is the pre-image of C_0 under the normalization map.*
(iii) *X is the image of $\mathbb{P}^2$ under a rational map defined by a general web of cubic curves with 5 base points.*

Next, we consider the projection of a nonsingular Segre surface from a nonsingular point p on a singular quadric Q from the pencil containing S. The tangent hyperplane H of Q at p intersects Q along the union of two planes. Thus, H intersects S along the union of two conics intersecting at two points. This is a degeneration of the previous case. The projection is a *degenerate cyclide surface*. It is isomorphic to the pre-image of a quadric in $\mathbb{P}^4$ under a map given by the linear system of quadrics in $\mathbb{P}^3$ containing the union of two coplanar lines (a degeneration of the conic K from above). Its equation can be reduced to the form

$$t_1^2 t_2^2 + t_0^2 g_2(t_0, t_1, t_2, t_3) = 0.$$

Finally, let us assume that the center of the projection is the singular point p of a cone Q from the pencil. In this case, the projection defines a degree 2 map $S \to \bar{Q}$, where $\bar{Q}$ is a nonsingular quadric in $\mathbb{P}^3$, the projection of Q. The branch locus of this map is a nonsingular quartic elliptic curve of bidegree $(2, 2)$. If we choose the diagonal equations of S as in Theorem 8.6.2, and take the point $p = [1, 0, 0, 0, 0]$, then Q is given by the equation

$$(a_2 - a_1)t_1^2 + (a_3 - a_1)t_2^2 + (a_3 - a_1)t_3^2 + (a_4 - a_1)t_4^2 = 0.$$

It is projected to the quadric with the same equations in coordinates $[t_1, \ldots, t_4]$ in $\mathbb{P}^3$. The branch curve is cut out by the quadric with the equation

$$t_1^2 + t_2^2 + t_3^2 + t_4^2 = 0.$$

A more general cyclide quartic surface is obtained by a projection of a singular quartic surface in $\mathbb{P}^3$. Such surfaces were classified by C. Segre [685].

Let us finish this subsection with one more property of cyclide quartic surfaces which we will need in Section 11.4. For any other quadric Q_t from the pencil, the projection defines a degree two map ramified over

Proposition 8.6.6. *Let Q be a cyclide quartic surface equal to the projection of a quartic del Pezzo surface $S \subset \mathbb{P}^4$ from a point p. Then there is a quadratic pencil of quadrics such that Q coincides with its discriminant surface. All members of the pencil are tangent to Q along a quartic curve.*

Proof Recall that we discussed quadratic pencils of quadrics in Subsection 4.1.4. Write the equation (8.25) in the form

$$(uq + vt_0^2)^2 + t_0^2(u^2 g_2 - 2uvq - v^2 t_0^2) = 0,$$

where $q = t_1^2 + t_2^2 + t_3^2$. We see that the quadratic pencil $V(u^2 g_2 - 2uvq + v^2 t_0^2)$ consists of quadrics that touch the surface along the intersection with the quadric $V(uq + vt_0^2)$. The discriminant surface of the quadratic pencil is equal to $V(q^2 + g_2 t_0^2)$, and hence coincides with Q. □

8.6.3 Lines and singularities

Let S be a quartic del Pezzo surface and X be its minimal resolution of singularities. The surface X is obtained by blowing up a bubble cycle $\eta = x_1 + \cdots + x_5$ of points in almost general position. Applying the procedure of Borel-De Sibenthal-Dynkin, we obtain the following list of types of root bases in the lattice $K_X^\perp \cong \mathsf{E}_5$:

$$D_5, A_3 + 2A_1, D_4, A_4, 4A_1, A_2 + 2A_1, A_3 + A_1, A_3, 3A_1, A_2 + A_1, A_2, 2A_1, A_1.$$

All of these types can be realized as the types of root bases defined by Dynkin curves.

$D_5 : x_5 \succ x_4 \succ x_3 \succ x_2 \succ x_1,$ and x_1, x_2, x_3 are collinear;

$A_3 + 2A_1 : x_3 \succ x_2 \succ x_1, x_5 \succ x_4,\ x_1, x_4, x_5$ and x_1, x_2, x_3 are collinear;

$D_4 : x_4 \succ x_3 \succ x_2 \succ x_1,$ and x_1, x_2, x_5 are collinear;

$A_4 : x_5 \succ x_4 \succ x_3 \succ x_2 \succ x_1;$

$4A_1 : x_2 \succ x_1, x_4 \succ x_3,\ x_1, x_2, x_5$ and x_3, x_4, x_5 are collinear;

$2A_1 + A_2 : x_3 \succ x_2 \succ x_1, x_5 \succ x_4$ and x_1, x_2, x_3 are collinear;

$A_1 + A_3 : x_3 \succ x_2 \succ x_1, x_5 \succ x_4,$ and x_1, x_4, x_5 are collinear;

$A_3 : x_4 \succ x_3 \succ x_2 \succ x_1,$ or $x_3 \succ x_2 \succ x_1$ and x_1, x_2, x_4 are collinear;

$A_1 + A_2 : x_3 \succ x_2 \succ x_1, x_5 \succ x_4;$

$3A_1 : x_2 \succ x_1,\ x_4, \succ x_3,$ and x_1, x_3, x_5 are collinear;

$A_2 : x_3 \succ x_2 \succ x_1$;

$2A_1 : x_2 \succ x_1, x_4 \succ x_3$, or x_1, x_2, x_3 and x_1, x_3, x_4 are collinear;

$A_1 : x_1, x_2, x_3$ are collinear.

This can also be stated in terms of equations indicated in the next table. The number of lines is also easy to find by looking at the blow-up model. We have the following table (see [750]).

$\emptyset$	A_1	$2A_1$	$2A_1$	A_2	$3A_1$	A_1+A_2	A_3
[11111]	[2111]	[221]	[(11)111]	[311]	[(11)21]	[32]	[41]
16	12	9	8	8	6	6	5
A_3	A_1+A_3	A_2+2A_1	$4A_1$	A_4	D_4	$2A_1+A_3$	D_5
[(21)11]	[(21)2]	[3(11)]	[(11)(11)1]	[5]	[(31)1]	[(21)(11)]	[(41)]
4	3	4	4	3	2	2	1

Table 8.7 *Lines and singularities on a weak del Pezzo surface of degree 4*

We refer to Segre's table [690] (one can find it also in [430, Art. 53]) that contains the classification of quartic cyclide surfaces with singular points outside the double conic.

The following is the figure of lines on a smooth Segre quartic surface:

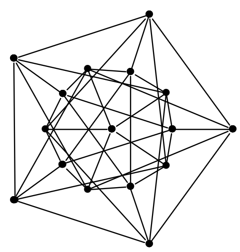


Figure 8.5 Lines on a nonsingular del Pezzo quartic surface

The Gosset polytope $\Sigma_5 = 1_{21}$ has 16 facets of type α and ten facets of type β. They correspond to contractions of 5 disjoint lines and pencils of conics arising from the pencils of lines through one of the five points in the plane and pencils of conics through four of the five points.

The list of singularities on a quartic del Pezzo surface gives the list of possible singular points of a quartic cyclide surface outside the double conic. Using this, we can compute the class of cyclide surfaces. Note that we cannot apply directly the Plücker-Teissier formula (1.2.7) from Section 1.2 since we have non-isolated singularities. We use a different proof that can also be applied to derive the Plücker-Teissier formula when all singular points of a surface are double rational points.

Proposition 8.6.7. *Let Q be a cyclide quartic surface obtained by projection of a quartic del Pezzo surface from a general point in $\mathbb{P}^4$ and let $x_1, \ldots, x_k$ be rational double points on Q with Milnor numbers $\mu_1, \ldots, \mu_k$. Let X^* be its dual surface. Then,*

$$\check{d} := \deg(X^*) = 12 - e(X, x).$$

Here $e(X,x) = \mu(X, x_i) + \mu(H(x) \cap x_i)$, where $H(x)$ is a general hyperplane passing through x_i.

Proof Choose a general line ℓ in $\mathbb{P}^3$ that intersects Q transversally at four points. Consider the pencil of plane sections of Q by planes H through ℓ. It defines a fibration $f : X = \mathsf{Bl}_{Q\cap\ell}(Q) \to \mathbb{P}^1$ whose general fiber F is isomorphic to a plane quartic curve with two nodes. The usual formula for the Euler-Poincaré characteristic of a fibered variety (in fact, any CW-complex) gives

$$e(X) = e(\mathbb{P}^1)e(F) + \sum_{t\in\mathbb{P}^1}(e(F_t) - e(F)) \tag{8.26}$$

(see, for example, [360, p.509]). Let us find special fibers that with non-zero $e(F_t) - e(F)$. A general fiber F is isomorphic to a 2-nodal plane quartic curve. It is easy to see, by passing to the normalization of F, that $e(F) = -2$. The line ℓ intersects the plane $\Pi = \langle C\rangle$ at one point. The planes that cut out Π along the line passing through the intersection point and tangent to C define two special fibers isomorphic to a plane quartic curve with a node and a cusps. We have $E(F_t) = -1$. Other special fibers are cut out by planes through the singular point of Q and there will be $\check{d}$ planes tangent to Q. They define special fibers isomorphic to plane quartic with three ordinary double points. We have $e(F_t) = -1$.

The projection $S \to Q$ defines an isomorphism $S \setminus F \cong Q \setminus C$, where F is a quartric elliptic curve that is mapped to C. This gives $e(Q) = e(Q \setminus C) = e(S \setminus F) = e(S)$. This gives $e(X) = 4 + e(Q) = 4 + e(S) + 2 = e(S) + 6$. So, collecting everything together, we find

$$\check{d} = e(X) - (-4 + 2 + \mu(Q \cap H(x_i), x_i) = 6 - \sum_{i=1}^{k}\mu(Q, x_i)$$

Now, the surface S has a minimal resolution of singularities isomorphic to a weak del Pezzo surface S' of degree 4. Each exceptional curve over a point x_i

contributes $\mu(S, x_i)$ to the second Betti number of S'. This gives

$$e(S) = e(S') - \sum_{i=1}^{k} \mu(x_i) = 12 - \sum_{i=1}^{k} (\mu(S, x_i) + \mu(H(x) \cap X)).$$

The possible Milnor numbers can be derived from Table 8.7. We can compute $\mu(H(x) \cap X, x)$ in each case, and find $\mu(H(x) \cap X, x) = 2$ if x_i is of type D_4, D_5 and $\mu(H(x) \cap X, x) = 1$ otherwise

□

Let us summarize our computation in the following Table 8.8.

$\emptyset$	A_1	$2A_1$	$2A_1$	A_2	$3A_1$	$A_1 + A_2$	A_3
12	10	8	8	9	6	7	8
A_3	$A_1 + A_3$	$A_2 + 2A_1$	$4A_1$	A_4	D_4	$2A_1 + A_3$	D_5
8	6	5	4	7	6	6	5

Table 8.8 *The class of a general cyclide quartic with isolated rational double points*

Example 8.6.8. A quartic del Pezzo surface $S \subset \mathbb{P}^4$ with singularities of type $4A_1$ or $2A_1 + A_3$ has a remarkable property, it admits a double cover ramified only at the singular points. We refer to [173] for more details about these quartic surfaces. The projections of these surfaces to $\mathbb{P}^3$ are cubic symmetroid surfaces which we will discuss in Subsection 9.3.3. The cover is the quadric surface $\mathbf{F}_0$ in the first case and the quadric cone Q in the second case. Assume that S has four ordinary nodes. The linear system $|I_S(2)|$ is spanned by two quadrics of corank 2 and, in appropriate projective coordinates, the equation of S is

$$x_0^2 + x_1x_2 = x_0^2 + x_3x_4 = 0.$$

The plane section $V(x_0)$ is equal to the union of a quadrangle of lines with vertices at the singular point. The projection of S from a nonsingular point of a quadric from $|I_S(2)|$ is a special quartic cyclide surface, classically known as a *Dupin quartic cyclide surface*. We will continue to discuss Dupin quartic cyclides later in Sections 11.3 and 12.1. Observe from Table 8.8 another remarkable property of a Dupin cyclide surface: it is projectively self-dual. Two concurrent sides of the quadrangle of lines on S span a plane. The projections of these planes in $\mathbb{P}^3$ define four planes $V(l_1), V(l_2), V(m_1), V(m_2)$ such that each line $V(l_1, l_2)$ and $V(m_1, m_2)$ contains a pair of nodes. The projection of the quadrangle of lines on S with vertices at the nodes are the lines $\ell_{ij} = V(l_i, m_j)$.

Each line intersects the double conic at one point. The equation of the Dupin cyclide can be written in the form

$$(t_0^2 + l_1 l_2 - m_1 m_2)^2 - 2t_0^2 l_1 l_2 = 0.$$

The double conic C is equal to $V(t_0, t_0^2 + l_1 l_2 - m_1 m_2)$. The quadratic pencil can be given in terms of the Hesse determinant

$$\det\begin{pmatrix} l_1 l_2 & t_0^2 + l_1 l_2 - m_1 m_2 & u \\ t_0^2 + l_1 l_2 - m_1 m_2 & t_0^2 & v \\ u & v & 0 \end{pmatrix} = 0.$$

The base points of the quadratic pencil satisfy $t_0 = l_1 l_2 = m_1 m_2 = 0$. These are the intersection points of the lines ℓ_{ij} with C. We can choose projective coordinates to write the equation in the form

$$((at_0+bt_1+ct_2+dt_3)^2+t_0t_1-t_2t_3)^2-4(at_0+bt_1+ct_2+dt_3)^2t_0t_1 = 0, \quad d(ab-cd) \neq 0. \tag{8.27}$$

The four nodes are the points $[0, 0, \alpha, 1], [1, \beta, 0, 0]$, where $(c\alpha + d)^2 - \alpha = 0, (a + \beta b)^2 - \beta = 0$. Later, in Subsection 11.4.3 we will give other representations of a Dupin cyclide quartic as the discriminant surface of a quadratic pencil.

8.6.4 Automorphisms

Let S be a del Pezzo surface of degree four. We know from Corollary 8.2.40 that the natural homomorphism

$$\rho : \mathrm{Aut}(S) \to W(S) \cong W(\mathsf{D}_5) \tag{8.28}$$

is injective.

Proposition 8.6.9.

$$W(D_5) \cong 2^4 \rtimes \mathfrak{S}_5,$$

where 2^k *denotes the elementary abelian group* $(\mathbb{Z}/2\mathbb{Z})^k$.

Proof This is a well-known fact from the theory of reflection groups. However, we give a geometric proof exhibiting the action of $W(D_5)$ on $\mathrm{Pic}(S)$. Fix a geometric basis $(e_0, \ldots, e_5)$ corresponding to an isomorphism S and the blow-up of five points $x_1, \ldots, x_5$ in general position. Consider five pairs of pencils of conics defined by the linear systems

$$L_i = |e_0 - e_i|, \quad L_i' = |2e_0 - \sum_{j=1}^{5} e_j + e_i|, \quad i = 1, \ldots, 5.$$

Let $\alpha_1, \dots, \alpha_5$ be the canonical root basis defined by the geometric basis and $r_i = r_{\alpha_i}$ be the corresponding reflections. Then, $r_2, \dots, r_5$ generate $\mathfrak{S}_5$ and act by permuting L_i's and L_i'. Consider the product $r_1 \circ r_5$. It is immediately checked that it switches L_4 with L_4' and L_5 with L_5' leaving L_i, L_i' invariant for $i = 1, 2, 3$. Similarly, a conjugate of $r_1 \circ r_5$ in $W(D_5)$ does the same for some other pair of the indices. The subgroup generated by the conjugates is isomorphic to 2^4. Its elements switch the L_i with L_i' in an even number of pairs of pencils. This defines a surjective homomorphism $W(D_5) \to \mathfrak{S}_5$ with a kernel containing 2^4. Comparing the orders of the groups we see that the kernel is 2^4 and we have an isomorphism of groups asserted in the Proposition. □

We know that the pencil of quadrics containing S has exactly five singular members Q_i of corank one. Each quadric Q_i is a cone over a nonsingular quadric in $\mathbb{P}^3$. It contains two rulings of planes containing the vertex of Q_i. Since $S = Q_i \cap Q$ for some nonsingular quadric Q, we see that S contains two pencils of conics $|C_i|$ and $|C_i'|$ such that $C_i \cap C_i' = 2$. In the blow-up model of S these are the pencils of conics $|L_i|$ and $L_i'|$ which we used in the proof of the previous Proposition. The group $W(S)$ acts on the set of pairs of pencils of conics and on the set of five singular quadrics Q_i. The subgroup 2^4 acts trivially on the set of singular quadrics.

Theorem 8.6.10. *Let S be a nonsingular del Pezzo surface of degree four. The image of the homomorphism*

$$\mathrm{Aut}(S) \to 2^4 \rtimes \mathfrak{S}_5$$

contains the normal subgroup 2^4. The quotient group is isomorphic to either a cyclic group C_n of order $n \in \{1, 2, 4\}$ or to the dihedral group D_6 or D_{10}.

Proof Consider the map

$$|L_i| \times |L_i'| \to |-K_S|, \quad (D, D') \mapsto D + D'.$$

Its image generates a 3-dimensional linear system contained in $|-K_S|$. This linear system defines the projection map $\psi_i : S \to \mathbb{P}^3$. Since $D_i \cdot D_i' = 2$ for $D_i \in L_i, D_i' \in L_i'$, the degree of the map is equal to 2. So the image of ψ is a quadric in $\mathbb{P}^3$. This shows that the center of the projection is the vertex of one of the five singular quadric cones in the pencil of quadrics containing S. The deck transformation $g_i, i = 1, \dots, 5$, of the cover is an automorphism and these five automorphisms generate a subgroup H of $\mathrm{Aut}(S)$ isomorphic to 2^4. One can come to the same conclusion by looking at the equations from Theorem 8.6.2 of S. The group of projective automorphisms generated by the transformations which switch t_i to $-t_i$ realizes the subgroup 2^4.

Let G be the quotient of $\mathrm{Aut}(S)$ by the subgroup 2^4. The group $\mathrm{Aut}(S)$ acts on the pencil $|\mathcal{I}_S(2)|$ of quadrics containing S leaving invariant the subset of five singular quadrics. The singular quadrics in the pencil correspond to points $[a_i, -1]$, where we use the equations (8.6.1) of S. Let us change the coordinates in $\mathbb{P}^4$ to assume that the singular quadrics correspond to points

$$[1,0], \quad [0,1], \quad [1,-1], \quad [-a,1], \quad [1,-b] \tag{8.29}$$

This changes the equations of S to

$$t_1^2 + t_2^2 + t_3^2 + bt_4^2 = t_0^2 + t_2^2 + at_3^2 + t_4^2 = 0.$$

Let ϵ_n be primitive roots of unity of degree $n = 3, 4, 5$. Consider the following matrices:

$$g_2 := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \; g_3 := \begin{pmatrix} \epsilon_3 & 0 \\ 0 & \epsilon_3^{-1} \end{pmatrix}, \; g_4 := \begin{pmatrix} \frac{1+\epsilon_4}{2} & \frac{1-\epsilon_4}{2} \\ \frac{1-\epsilon_4}{2} & \frac{1+\epsilon_4}{2} \end{pmatrix}, \; g_5 := \begin{pmatrix} -\epsilon_5 & -1 \\ 1 & 0 \end{pmatrix}.$$

Each matrix g_n is well-defined and has order n in PGL_2.

When $a = b$, the matrix g_2 leaves invariant the five points from (8.29). Thus when $a = b$, we have exhibited the group $G = \langle g_2 \rangle \cong 2$. If $a = b = \epsilon_3$, then $\mathfrak{S}_3 \cong \langle g_2, g_3 \rangle \subseteq G$. If $a = b = i$, then $4 \cong \langle g_4 \rangle \subseteq G$. If $a = b = \phi$, then $D_{10} \cong \langle g_2, g_5 \rangle \subseteq G$.

It remains to demonstrate that these are the only subgroups that occur. First, observe that there is no non-trivial automorphism of $\mathbb{P}^1$ which fixes three or more points. Thus, as a subgroup of $\mathfrak{S}_5$, the group G cannot contain any transpositions. Up to conjugacy, these are the only subgroups of $\mathfrak{S}_5$ containing no transpositions:

$$1,\ 2,\ 3,\ 4,\ 2^2,\ 5,\ \mathfrak{S}_3,\ D_{10},\ 5 \rtimes 4,\ \mathfrak{A}_5. \tag{8.30}$$

(Note there are other subgroups isomorphic to 2, 2^2 and $\mathfrak{S}_3$ which *do* have transpositions.)

This means that G contains elements only of orders 1, 2, 3, 4 and 5; and that an element of order 2 must act as a double transposition in $\mathfrak{S}_5$. Since cyclic subgroups of fixed order are all conjugate in PGL_2 (consider diagonalizations or Jordan canonical forms), we conclude that every non-trivial cyclic subgroup is generated by one of g_2, g_3, g_4, g_5 up to conjugacy.

Moreover, we claim that up to the choice of coordinates the five points can be put into the form (8.29) with one of the matrices g_2, g_3, g_4, g_5 given above. For order 2, there is only one fixed point which we may assume is -1; up to scaling, we may assume one of the orbits is $\{0, \infty\}$ and so the other must be $\{-a, -a^{-1}\}$ for some choice of a. For the other orders, we only have to show that the group and set of invariant points are unique up to conjugacy; that is, they can be put into form (8.29) then follows. For order 3, the group acts via $x \mapsto \epsilon_3 x$, so the two fixed points are 0 and ∞; scaling x commutes with the group action, so we may assume the non-trivial orbit contains 1 and so also ϵ_3, ϵ_3^2. For order 4, the group acts via $x \mapsto \epsilon_4 x$ fixing either 0 or ∞; scaling x commutes with the group action, and so we may assume the non-trivial orbit is $\{1, -1, \epsilon_4, -\epsilon_4\}$. The change of coordinates $x \to x^{-1}$ normalizes the group action and leaves invariant the non-trivial orbit $\{1, -1, \epsilon_4, -\epsilon_4\}$ and so we may assume the fixed point is 0. For order 5, the group acts diagonally and we may scale to ensure the orbit consists of the 5th fifth roots of unity.

Due to our choice of coordinates for the five points and g_2, g_3, g_4, g_5 above, we see that the presence of an element of order 3 forces G to contain $\mathfrak{S}_3$. It only remains to exclude the following possibilities: 2^2 and $\mathfrak{A}_5$, and $5 \rtimes 4$. The group 2^2 must act faithfully on the tangent space at the fixed point, which is impossible. The group $\mathfrak{A}_5$ contains a subgroup isomorphic to 2^2 and so it also does not occur. Finally, the group $5 \rtimes 4$ does not act on $\mathbb{P}^1$. □

It follows from the proof that a del Pezzo surface of degree four with group of automorphisms larger than 2^4 is projectively isomorphic to a surface of one the following types:

The following surfaces together realize possible groups $G = \mathrm{Aut}(S)/2^4$. We choose different projective coordinates that exhibit obvious symmetry of the surfaces.

$$C_2 : t_1^2 + t_2^2 + at_3^2 + t_4^2 = t_0^2 + t_2^2 + t_3^2 + at_4^2 = 0, a \neq 0, \quad a^3, a^4, a^5 \neq 1;$$

$$\mathfrak{S}_3 : t_0^2 + \epsilon_3 t_1^2 + \epsilon_3^2 t_2^2 + t_3^2 = t_0^2 + \epsilon_3^2 t_2^2 + \epsilon_3 t_3^2 + t_4^2 = 0,$$

$$C_4 : t_0^2 + t_1^2 + t_3^2 + t_4^2 = t_0^2 + \epsilon_4 t_2^2 - t_3^2 - \epsilon_4 t_4^2 = 0;$$

$$D_{10} : t_0^2 + \epsilon_5 t_1^2 + \epsilon_5^2 t_2^2 + \epsilon_5^3 t_3^2 + \epsilon_5^4 t_4^2 = \epsilon_5^4 t_0^2 + \epsilon_5^3 t_1^2 + \epsilon_5^2 t_2^2 + \epsilon_5 t_3^2 + t_4^2 = 0.$$

All surfaces admit an automorphism of order 2 defined by

$$[t_0, t_1, t_2, t_3, t_4] \mapsto [t_2, t_3, t_0, t_1, t_4].$$

The automorphism of order $n = 3, 4, 5]$ is defined by a cyclic permutation of coordinates

$$n = 3 : [x_0, x_1, x_2, x_3, x_4] \mapsto (x_1, x_2, x_0, \epsilon_3^2 x_3, \epsilon_3 x_4),$$

$$n = 4 : [x_0, x_1, x_2, x_3, x_4] \mapsto (x_1, x_2, x_3, x_0],$$

Order	Carter's notation	Trace	Char. Poly	Realizable
1	$\emptyset$	5	Φ_1^5	yes
2	$2A_1$	1	$\Phi_1^3\Phi_2^2$	yes
2	$4A_1$	-3	$\Phi_1\Phi_2^4$	yes
2	A_1	3	$\Phi_1^4\Phi_2$	
2	$3A_1$	-1	$\Phi_1^2\Phi_2^3$	
2	$2A_1$	1	$\Phi_1^3\Phi_2^2$	yes
3	A_2	2	$\Phi_1^3\Phi_3$	yes
4	$A_1 + A_3$	-1	$\Phi_1\Phi_2^2\Phi_4$	yes
4	A_3	1	$\Phi_1^2\Phi_2^1\Phi_4$	yes
4	$2A_1 + A_3$	-3	$\Phi_2^3\Phi_4$	
4	$D_4(a_1)$	1	$\Phi_1\Phi_4^2$	yes
5	A_4	0	$\Phi_1\Phi_5$	yes
6	$A_1 + A_2$	0	$\Phi_1^2\Phi_2\Phi_3$	
6	$2A_1 + A_2$	-2	$\Phi_1\Phi_2^2\Phi_3$	yes
6	D_4	0	$\Phi_1\Phi_2^2\Phi_6$	yes
8	D_5	-1	$\Phi_2\Phi_8$	yes
12	$D_5(a_1)$	0	$\Phi_2\Phi_4\Phi_6$	

Table 8.9 *Conjugacy classes in $W(\mathsf{D}_5)$.*

$$n = 5 \ : [x_0, x_1, x_2, x_3, x_4] \mapsto (x_1, x_2, x_3, x_4, x_0],$$

Let us consider the action (8.28) of $\mathrm{Aut}(S)$ on $\mathrm{Pic}(S)$. We fix a geometric basis to identify $W(S)$ with $W(\mathsf{D}_5)$. We shall find the conjugacy classes of the images of automorphisms of S in $W(\mathsf{D}_5)$.

We will use, here and later, the *Lefschetz fixed-point-formula* that states that in our case

$$2 + \mathrm{Trace}(\sigma^*|H^2(S, \mathbb{Q})) = 3 + \mathrm{Trace}(\sigma^*|K_S^\perp) = e(S^\gamma), \tag{8.31}$$

where $e(S^\sigma)$ denotes the Euler-Poincaré characteristic of the locus S^σ of fixed points of g.

In Subsection 8.2.4 we discussed the classification of conjugacy classes of elements of the Weyl group of a root basis. We use Table 8.2.4 and its notation to list the conjugacy classes of elements of $W(\mathsf{D}_5)$.

Here Φ_n denotes the cyclotomic polynomial whose roots are primitive nth roots of 1.

Let us check the last column. The subgroup 2^4 contains two conjugacy classes of elements of order 2. They represented by an involution that changes the signs at two or one coordinates. The first involution σ_1 fixes a hyperplane in the ambient space $\mathbb{P}^4$. Its fixed locus S^σ consists of a hyperplane section of S, which is a quartic elliptic curve. Since $e(S^\sigma) = 0$, the table shows that the conjugacy class is of type $2A_1$. The involution σ_2 of the second type acts in $\mathbb{P}^4$ fixing a plane and a line pointwise. It is immediate to see that S^g consists of four isolated points. This shows that σ belongs to the conjugacy class of type $2A_1$.

Note that there are two different lattice embeddings $\mathsf{A}_1^{\oplus 4} \hookrightarrow \mathsf{D}_5$. The conjugacy class of the involution σ corresponds to the embedding conjugate to the natural embedding of the sublattice spanned by simple roots $\boldsymbol{\alpha}_1$ and $\boldsymbol{\alpha}_2$ in the canonical basis Lemma 8.2.7. The sublattice spanned by simple roots $\boldsymbol{\alpha}_3, \boldsymbol{\alpha}_5$ represents the other conjugacy class. The involution that realizes this conjugacy class is the involution σ_3 of the surface with $a = b$ that switches the coordinates t_3, t_4 and t_0, t_1. Its set of fixed points in $\mathbb{P}^4$ is the union of the plane $V(t_0 - t_1, t_3 - t_4)$ and the line $V(t_2, t_0 + t_1, t_3 + t_4)$. They intersect S along the conic and two isolated points. We see that $e(S^g) = 4$ as in the case of the involution σ_2. It follows from Theorem 8.6.10 that any other involution is conjugate to one of the involutions $\sigma_1, \sigma_2, \sigma_3$.

If σ is of order 3, then it has 5 fixed points

$$[1, 1, 1, 0, 0], [1, \epsilon_3, \epsilon_3^2, 0, \pm\sqrt{-3}], [1, \epsilon_3^2, \epsilon_3, \pm\sqrt{-3}, 0].$$

So, the trace of σ^* is equal to 2. This confirms that σ is of type A_2.

If σ is of order 4, then it is either given by an isomorphic lift of the automorphism of order 4 from G, or it is one of the two non-conjugate lifts of an automorphism of order 2 from G. In the first case, it has four fixed points $[e^3, e^2, e, 1, 0]$, where $e^4 = 1$. This implies that $\mathrm{Trace}(\sigma^*) = 1$, hence σ is of type A_3 or $D_4(a_1)$. The square of the conjugacy class is of type $2A_1$ and the square of the second conjugacy class is of type $4A_1$. We know that it must be of type $2A_1$ (the same set of fixed point). Hence, σ is of type A_3. If $\sigma^2 \in 2^4$, then the conjugacy class of σ^2 is of type $4A_1$ or $2A_1$. In the first case, σ is of type $D_4(a_1)$, and, in the second case, it is of type $A_1 + A_3$.

Finally, if σ is of order 5. Then $e(S^\sigma)$ consists of 5 points $[1, e, e^2, e^3, e^4] me^5 = 1$. This determines the trace of σ^* and the conjugacy class that must be of type A_4.

There are two conjugacy classes of order 6. They are represented by the product of an element of order 3 from G and of of two non-conjugate of order 2 from the subgroup 2^4. Its cube must be a realizable conjugacy class of order 2. This excludes the conjugacy class $A_1 + A_2$ whose cube is of type A_1. The cube of the conjugacy class of type D_4 (resp. $2A_1 + A_2$ is of type $4A_1$ (resp. $2A_1$).

There is only one conjugacy class of order 8. It is realized by a non-isomorphic lift the cyclic subgroup of G generated by an element of order

4. Since the quotient group G does not contain elements of order 6, an unique possible conjugacy class of order 12 is not realizable.

Remark 8.6.11. In 1894 G. Humbert [412] discovered a plane sextic Γ with five ordinary cusps that has the automorphism group isomorphic to 2^4. Its proper transform on the blow-up of $\mathbb{P}^2$ at the five cusps is a nonsingular curve ℓ' of genus 5 on a del Pezzo quartic surface S. It is canonically embedded in $\mathbb{P}^5$. The curve ℓ' is cut out by a quadric $V(\sum a_i^2 t_i^2)$, where we consider S given by the equations from Theorem 8.6.2 (see [273]). The curve is tangent to all 16 lines on S. The double cover of S branched along this curve is a K3 surface isomorphic to a nonsingular model of a Kummer quartic surface. The following equation of ℓ was found by W. Edge [276]

$$9t_2^2(t_2^2 - t_0^2)(t_2^2 - t_1^2) + (t_1^2 + 3t_2^2 - 4t_0^2)(t_1 + 2t_0)^2(t_1^2 - t_0^2) = 0.$$

The curve has peculiar properties: the residual points of each line containing two cusps coincide, and the two contact points are on a line passing through a cusp; the residual points of the conic through the five cusps coincide and all cuspidal tangents pass through the contact point (see loc.cit.). The five maps $S \to \mathbb{P}^1$ defined by the pencils of conics, restricted to Γ', define five g_4^1's on Γ'. The quotient by the involution defined by the negation of one of the coordinates t_i is an elliptic curve. This makes the 5-dimensional Jacobian variety of Γ' isogenous to the product of five elliptic curves (this is how it was found by Humbert). The quotient of Γ' by the involution defined by the negation of two coordinates t_i is a curve of genus 3. It is isomorphic to the quartic curve with automorphism group isomorphic to 2^3.

By taking special del Pezzo surfaces with isomorphism groups $2^4 \rtimes D_6$ and $2^4 \rtimes D_{10}$ we obtain curves of genus 5 with automorphism groups of order 96 and 160 (see [273]).

Let $p_1, \ldots, p_6$ be six points in $\mathbb{P}^3$ in general linear position. A *Humbert curve* can be also defined as the locus of tangency points of lines passing through p_6 with rational normal cubics passing through $p_1, \ldots, p_5$ (see [29, Vol. 6, p. 24]). It is also characterized by the property that it has ten effective even theta characteristics (see [783]).

The double cover of S ramified over Γ' is a K3 surface isomorphic to a nonsingular model of a Kummer quartic surface with 16 nodes. The pre-images of the 16 lines on S split into 32 curves, the images of a subset of 16 of them on the Kummer surface are 16 nodes, and the images of the remaining 16 curves are the 16 conics cut by 16 tropes of the surface. The surface S admits a nonsingular model as a surface in the Grassmannian $G_1(\mathbb{P}^3)$ of degree 2 and class equal to 2. It is one of the irreducible components of the surface $\mathrm{Bit}(S)$ of bitangents of a Kummer quartic surface. We will discuss all of this in Section 11.3.

8.7 Del Pezzo surfaces of degree 2

8.7.1 Singularities

Let S be a weak del Pezzo surface of degree 2. Recall that the anti-canonical linear system defines a birational morphism $\phi' : S \to X$, where X is the anti-canonical model of S isomorphic to the double cover of $\mathbb{P}^2$ branched along a plane quartic curve C with at most simple singularities (see Subsection 6.3.3). We have already discussed nonsingular del Pezzo surfaces of degree 2 in Chapter 6, in particular, the geometry associated with seven points in the plane in general position. A nonsingular del Pezzo surface is isomorphic to the double cover of the projective plane ramified over a nonsingular plane quartic. It has 56 lines corresponding to 28 bitangents of the branch curve.

Let $\phi : S \to \mathbb{P}^2$ be the composition of ϕ and the double cover map $\sigma : X \to \mathbb{P}^2$. The restriction of ϕ to a (-1)-curve E is a map of degree $-K_S \cdot E = 1$. Its image in the plane is a line ℓ. The pre-image of ℓ is the union of E and a divisor $D \in |-K_S - E|$. Since $-K_S \cdot D = 1$, the divisor D is equal to $E' + R$, where E' is a (-1)-curve and R is the union of (-2)-curves. Also, we immediately find that $E \cdot D = 2, D^2 = -1$. There are three possible cases:

(i) $E \neq E',\ E \cdot E' = 2$;

(ii) $E \neq E', E \cdot E' = 1$;

(iii) $E \neq E', E = E'$.

In the first case, the image of E is a line ℓ tangent to C at two nonsingular points. The image of $D - E'$ is a singular point of C. By Bezout's Theorem, ℓ cannot pass through the singular point. Hence, $D = E'$ and ℓ is a bitangent of C.

In the second case, $E \cdot (D - E') = 1$. The line ℓ passes through the singular point $\phi(D - E')$ and it is tangent to C at a nonsingular point.

Finally, in the third case, ℓ is a component of C.

Of course, when S is a del Pezzo surface, the quartic C is nonsingular, and we have 56 lines paired into 28 pairs corresponding to 28 bitangents of C. Let $\pi : S \to \mathbb{P}^2$ be the blow-up of seven points $x_1, \ldots, x_7$ in general position. Then, 28 pairs of lines are the proper inverse transforms of the isolated pairs of curves:

21 pairs: a line through x_i, x_j and the conic through the complementary five points;

7 pairs: a cubic with a double point at x_i and passing through other points plus the exceptional curve $\pi^{-1}(x_i)$.

We use the procedure of Borel-de Siebenthal-Dynkin to compile the list of root bases in E_7. It is convenient first to compile the list of maximal (by inclusions) root bases of type A, D, E (see [437, §12]).

Type	rank $n-1$	rank n
A_n	$A_k + A_{n-k-1}$	
D_n	A_{n-1}, D_{n-1}	$D_k + D_{n-k}, k \geq 2$
E_6	D_5	$A_1 + A_5, A_2 + A_2 + A_2$
E_7	E_6	$A_1 + D_6, A_7, A_2 + A_5$
E_8		$D_8, A_1 + E_7, A_8, A_2 + E_6, A_4 + A_4$

Table 8.10 *Maximal root bases*

Here, $D_2 = A_1 + A_1$ and $D_3 = A_3$.

From this we easily find the following table of root bases in E_7. Note that

r	Types
7	$E_7, A_1 + D_6, A_7, 3A_1 + D_4, A_1 + 2A_3, A_5 + A_2, 7A_1$
6	$E_6, D_5 + A_1, D_6, A_6, A_1 + A_5, 3A_2, 2A_1 + D_4, 2A_3,$ $3A_1 + A_3, 6A_1, A_1 + A_2 + A_3, A_2 + A_4$
5	$D_5, A_5, A_1 + D_4, A_1 + A_4, A_1 + 2A_2, 2A_1 + A_3,$ $3A_1 + A_2, A_2 + A_3, 5A_1$
≤ 4	$D_4, A_{i_1} + \cdots + A_{i_k}, i_1 + \cdots + i_k \leq 4$

Table 8.11 *Root bases in the* E_7*-lattice*

there are two root bases of types $A_1 + A_5$, $A_2 + 2A_1$, $3A_1$, $A_1 + A_3$ and $4A_1$ which are not equivalent with respect to the Weyl group.

The simple singularities of plane quartics were classified by P. Du Val [267], Part III.

A_1: one node;

$2A_1$: two nodes;

A_2: one cusp;

$3A_1$: irreducible quartic with three nodes;

$3A_1$: a cubic and a line;

$A_1 + A_2$: one node and one cusp;

A_3: one tacnode (two infinitely near ordinary double points);

$4A_1$: a nodal cubic and a line;

$4A_1$: two conics intersecting at 4 points;

$2A_1 + A_2$: two nodes and one cusp;

$A_1 + A_3$: a node and a tacnode;

$A_1 + A_3$: cubic and a tangent line;

A_4: one rhamphoid cusp (two infinitely near cusps);

$2A_2$: two cusps;

D_4: an ordinary triple point;

$5A_1$: a conic and two lines;

$3A_1 + A_2$: a cuspidal cubic and a line;

$2A_1 + A_3$: two conics tangent at one point;

$2A_1 + A_3$: a nodal cubic and its tangent line;

$A_1 + A_4$: a rhamphoid cusp and a node;

$A_1 + 2A_2$: a cusp and two nodes;

$A_2 + A_3$: a cusp and a tacnode;

A_5: one oscnode (two infinitely near cusps);

A_5: a cubic and its inflection tangent;

D_5: nodal cubic and a line tangent at one branch;

$A_1 + D_4$: a nodal cubic and line through the node;

E_6: an irreducible quartic with one e_6-singularity;

D_6: triple point with one cuspidal branch;

$A_1 + A_5$: two conics intersecting at two points with multiplicities 3 and 1;

$A_1 + A_5$: a nodal cubic and its inflection tangent;

$6A_1$: four lines in general position;

$3A_2$: a three-cuspidal quartic;

$2A_1 + D_4$: two lines and conic through their intersection point;

$D_5 + A_1$: cuspidal cubic and a line through the cusp;

$2A_3$: two conics intersecting at two points with multiplicities 2;

$3A_1 + A_3$: a conic plus its tangent line plus another line;

$A_1 + A_2 + A_3$: cuspidal cubic and its tangent;

A_6: one oscular rhamphoid cusp (three infinitely near $x_1 > x_2 > x_1$ cusps);

$A_2 + A_4$: one rhamphoid cusp and a cusp;

E_7: cuspidal cubic and its cuspidal tangent;

$A_1 + D_6$: conic plus tangent line and another line through point of contact;

$D_4 + 3A_1$: four lines with three concurrent;

A_7: two irreducible conics intersecting at one point;

$A_5 + A_2$: cuspidal cubic and an inflection tangent;

$2A_3 + A_1$: conic and two tangent lines.

Note that all possible root bases are realized except $7A_1$ (this can be realized in characteristic 2). One can compute the number of lines but this is rather tedious. For example, in the case A_1, we have 44 lines and a one-nodal quartic C has 22 proper bitangents (i.e. lines with two nonsingular points of tangency) and six bitangents passing through the node.

The Gosset polytope $\Sigma_7 = 3_{21}$ has 576 facets of type α and 126 facets of type β. They correspond to contractions of seven disjoint (-1)-curves and pencils of conics arising from seven pencils of lines through one of the seven points in the plane, 35 pencils of conics through four points, 42 pencils of cubic curves through six points with a node at one of these points, 35 pencils of 3-nodal quartics through the seven points, and seven pencils of quintics through the seven points with six double points.

8.7.2 Automorphisms of del Pezzo surfaces of degree 2

Let S be a del Pezzo surface of degree 2. Choose a geometric basis S that defines an isomorphism $W(S) \cong W(\mathsf{E}_N)$. We fix a geometric basis and identify the two groups. By Corollary 8.2.40, the natural homomorphism

$$\rho : \mathrm{Aut}(S) \to W(E_7)$$

is injective.

We know that the Weyl group $W(\mathsf{E}_7)$ contains the center generated by an element w_0 of order 2. It acts as the minus identity on the root lattice E_7. We will show that it belongs to the image of ρ.

Let S be a weak del Pezzo surface and $p_1, \dots, p_7$ be the seven points in $\mathbb{P}^2$ that define our fixed geometric basis. In Subsection 7.2.6 we defined the planar Geiser Cremona involution with base points $p_1, \dots, p_7$. The involution is regularized on $S \to \mathbb{P}^2$. The anti-canonical map $\phi : S \to \mathbb{P}^2$ factors through a birational morphism $\nu : S \to S'$ that blows down all (-2)-curves on S and a finite degree two morphism $\phi' : S' \to \mathbb{P}^2$. Let γ' be the deck transformation of the cover ϕ'. It is a biregular automorphism of S' that coincides with the lift of the Geiser Cremona involution to S' as a birational involution. In fact, it lifts to a biregular automorphism of S. Since ν is a minimal resolution of singularities of S', this follows from the existence of an equivariant minimal resolution of singularities of surfaces [489] and the uniqueness of a minimal resolution of singularities of surfaces. The involution γ of S obtained in this way is called the Geiser involution of S. Note that we use the same notation γ as for the Geiser Cremona involution.

Proposition 8.7.1. *Let S be a del Pezzo surface of degree two. The image $\rho(\gamma)$ of the Geiser involutuion γ in $W(\mathsf{E}_7)$ does not depend on a choice of a geometric basis and coincides with the element w_0 of $W(\mathsf{E}_7)$.*

Proof Since the lift of the Geiser Cremona involution to S coincides with γ, we can use the characteristic matrix A from (7.46) for the matrix of γ in the same geometric basis. By inspection of the matrix, we see that $e_0 + A \cdot e_0 = -3K_S, e_i + A \cdot e_i = -K_X$. This implies that $v \cdot A = 0$ for any divisor class $v = a_0e_0 + a_1e_1 + \cdots + a_7e_7$ with $3a_0 = a_1 + \cdots = a_7$, we get $v + \cdot A \cdot v = 0$. Thus, $\rho(\gamma) = -\mathrm{id}_{\mathsf{E}_7}$. Since $w_0 = -\mathrm{id}_{E_7}$ generates the center of the Weyl group, this property does not depend on a choice of a geometric basis. □

Note that the element w_0 acts on the Gosset polytope 3_{21} as the reflection with respect to the center defined by the vector $\frac{1}{2}\mathbf{k}_7 = -\frac{1}{56}\sum v_i$, where v_i are the exceptional vectors. The 28 orbits on the set of vertices correspond to 28 bitangents of a nonsingular plane quartic.

The assertion of the proposition is not true anymore if S is weak del Pezzo surface. In fact, γ sends any (-2)-curve on S to a (-2)-curve, and cannot act as the minus identity on $K_S^\perp$.

For example, suppose that the branch curve has an ordinary triple point, i.e., it is a simple point of type d_4. Then the double cover $S' \to \mathbb{P}^2$ has a double rational point of type D_4. The surface S is a minimal resolution of the singular point and the Geiser involution of S acts identically on the central component of the exceptional curve and on the proper transform of the branch curve which is a smooth rational curve. We have $e(S^\gamma) = 4$ and $\mathrm{Trace}(\gamma^*|K_S^\perp) = 1$.

In a similar manner, one can go through the classification of possible simple singular points of a plane quartic and find the action of the Geiser involution on the exceptional curve of the minimal resolution of S'. We leave it to the reader to provek the following proposition.

Proposition 8.7.2. *The Geiser involution γ has no isolated fixed points. Its locus of fixed points is the disjoint union of smooth curves $W + R_1 + \cdots + R_k$, where $R_1, \ldots, R_k$ are among irreducible components of Dynkin curves. The curve W* *is the normalization of the branch curve of the double cover $\phi : S \to \mathbb{P}^2$. A Dynkin curve of type A_{2k} has no fixed components, a Dynkin curve of type A_{2k+1} has one fixed component equal to the central component. A Dynkin curve of type D_4, D_5, D_6, E_6, E_7 have fixed components marked by squares in their Coxeter-Dynkin diagrams given in Table 8.12 below.*

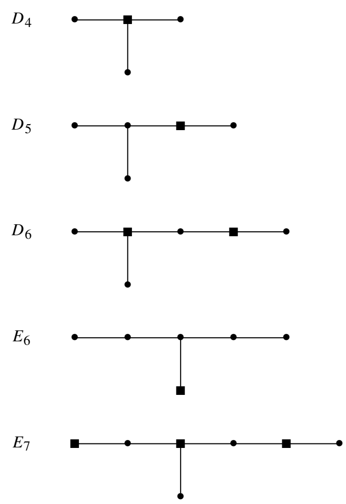


Table 8.12 *Fixed locus of the Geiser involution*

Let S be a del Pezzo surface of degree 2 and γ be its Geiser involution. The the anti-canonical map $\phi : S \to \mathbb{P}^2$ factors through the quotient map $S \to S/(\gamma)$ and an isomorphism $S/(\gamma) \to \mathbb{P}^2$. This gives an injective homomorphism

$$\bar{\rho} : \mathrm{Aut}(S)/(\gamma) \to W(S)/(\gamma) \to W(\mathsf{E}_7)/(w_0).$$

It is known that

$$W(\mathsf{E}_7) = W(\mathsf{E}_7)^+ \times (w_0),$$

where $W(\mathsf{E}_7)^+$ is a simple group isomorphic to the group $\mathrm{Sp}(6, \mathbb{F}_2)$. This isomorphism is obtained by considering the action of $W(\mathsf{E}_7)^+$ on the root lattice modulo 2. The group $\mathrm{Aut}(S)/(\gamma)$ acts naturally on the branch curve defining an injective homomorphism

$$\alpha : \mathrm{Aut}(S)/(\gamma) \to \mathrm{Aut}(B).$$

Let H be a subgroup of $W(E_7)^+$. Denote by H^+ a lift of H to an isomorphic subgroup of W^+. Any other isomorphic lift of H is defined by a nontrivial homomorphism $\alpha : H \to \langle w_0 \rangle \cong \mathbb{Z}/2\mathbb{Z}$. Elements of H are the products $h\alpha(h), h \in H^+$. We denote such a lift by H_α. Thus, all lifts are parametrized by the group $\mathrm{Hom}(H, \langle w_0 \rangle)$ and H^+ corresponds to the trivial homomorphism. Note that $wH_\alpha w^{-1} = (w'Hw'^{-1})_\alpha$, where w' is the image of $w \in W(\mathsf{E}_7)$ in $W(E_7)'$. In particular, two lifts of the same group are never conjugated. A lift is trivial if it contains w_0. In this case $H_\alpha = H \times (w_0)$.

Now we apply this to our geometric situation. The group $\mathrm{Aut}(S) \subset W(S)$ is the trivial lift of the subgroup $\mathrm{Aut}(B)$ of $W(S)^+ \cong W(\mathsf{E}_7)^+$. Since we know the automorphism groups of plane quartic curves, we find the list of possible automorphism groups of del Pezzo surfaces of degree two in Table 8.13 below.

Type	Order	Structure	Equation
I	336	$2 \times L_2(7)$	$t_3^2 + t_0^3 t_1 + t_1^3 t_2 + t_2^3 t_0$
II	192	$2 \times (4^2 : \mathfrak{S}_3)$	$t_3^2 + t_0^4 + t_1^4 + t_2^4$
III	96	$2 \times 4\mathfrak{A}_4$	$t_3^2 + t_2^4 + t_0^4 + \alpha t_0^2 t_1^2 + t_1^4$
IV	48	$2 \times \mathfrak{S}_4$	$t_3^2 + t_2^4 + t_1^4 + t_0^4 + a(t_0^2 t_1^2 + t_0^2 t_2^2 + t_1^2 t_2^2)$
V	32	2×4.2^2	$t_3^2 + t_2^4 + t_0^4 + a t_0^2 t_1^2 + t_1^4$
VI	18	18	$t_3^2 + t_0^4 + t_0 t_1^3 + t_1 t_2^3$
VII	16	$2 \times D_8$	$t_3^2 + t_2^4 + t_0^4 + t_1^4 + a t_0^2 t_1^2 + b t_2^2 t_0 t_1$
VIII	12	2×6	$t_3^2 + t_2^3 t_0 + t_0^4 + t_1^4 + a t_0^2 t_1^2$
IX	12	$2 \times \mathfrak{S}_3$	$t_3^2 + t_2^4 + a t_2^2 t_0 t_1 + t_2(t_0^3 + t_1^3) + b t_0^2 t_1^2$
X	8	2^3	$t_3^2 + t_2^4 + t_1^4 + t_0^4 + a t_2^2 t_0^2 + b t_1^2 t_2^2 + c t_0^2 t_1^2$
XI	6	6	$t_3^2 + t_2^3 t_0 + a_4(t_0, t_1)$
XII	4	2^2	$t_3^2 + t_2^4 + t_2^2 f_2(t_0, t_1) + a_4(t_0, t_1)$
XIII	2	2	$t_3^2 + a_4(t_0, t_1, t_2)$

Table 8.13 *Groups of automorphisms of del Pezzo surfaces of degree 2*

Remark 8.7.3. As we explained in Subsection 7.7.1 the classification of conjugacy classes of finite subgroupsn of the Cremona group Cr(2) includes the conjugacy classes of minimal finite groups F of the group $\mathrm{Aut}(S)$ of automorphisms of a del Pezzo surface S of degree 2. They are obtained as lifts of finite subgroups of $\mathrm{Aut}(B)$ that satisfy the property that $\mathrm{Pic}(S)^G \cong \mathbb{Z}$. The trivial lift is always minimal since it contains w_0 that acts on $K_S^\perp$ as the minus identity. To find the minimal non-trivial lifts of finite subgroups of $\mathrm{Aut}(B)$ is not an easy task, accomplished in [252].

Remark 8.7.4. The isomorphism of groups $W(\mathsf{E}_7) \cong \mathrm{Sp}(6, \mathbb{F}_2)$ has a geometric interpretation after we identify $H^1(B, \mathbb{F}_2)$ with $\mathbb{F}_2^6$ and equip it with a nondegenerate symplectic form $\xi : H^1(B, \mathbb{F}_2) \times H^1(B, \mathbb{F}_2) \to \mathbb{F}_2$ defined by the cap-product. There is a natural isomorphism $K_S^\perp/2K_S^\perp \to \mathrm{Jac}(B)[2] \cong H^1(B, \mathbb{F}_2)$ that defines an isomorphism of groups $W(S)/(\gamma) to \mathrm{Aut}(H^1(B, \mathbb{F}_2, \xi)$ [234, Chapter IX].

We leave it to a curious reader the task of classifying automorphism groups of weak del Pezzo surfaces. Notice that in the action of $\mathrm{Aut}(S)$ in the Picard group they correspond to certain subgroups of the group $\mathrm{Cris}(S)$. Also, the action is not necessarily faithful; for example, the Geiser involution acts trivially on $\mathrm{Pic}(S)$ in the case of a weak del Pezzo surface that is defined by the branch curve B with a simple singular point of type e_7.

8.8 Del Pezzo Surfaces of Degree 1

8.8.1 Singularities

Let S be a weak del Pezzo surface of degree one. It is isomorphic to the blow-up of a bubble cycle of eight points in an almost general position. The anti-canonical model X of S is a finite cover of degree 2 of a quadratic cone Q ramified over a curve B in the linear system $|\mathcal{O}_Q(3)|$. It is nonsingular or has simple singularities. The list of types of possible Dynkin curves is easy to compile. First, we observe that all diagrams listed for the case of the E_7-lattice are included in the list. They correspond to the natural inclusion of the set of simple roots of the lattice E_7 into the set of simple roots of the lattice E_8. Also, all the diagrams A_1+T, where T is from the list of possible Dynkin curves on a del Pezzo surface of degree 2 are also included. They correspond to the lattice embedding $\mathsf{E}_7 \oplus \mathsf{A}_1 \hookrightarrow \mathsf{E}_8$. We give only the new types.

r	Types
8	$E_8, A_8, D_8, 2A_4, A_1+A_2+A_5, A_3+D_5, 2D_4,$ $A_2+E_6, A_3+D_5, 4A_2$
7	$D_7, A_2+D_5, A_3+A_4, A_3+D_4$
6	A_2+D_4

Table 8.14 *Root bases in the* E_8*-lattice*

Note that there are two root bases for each of the types A_7, $2A_3$, A_1+A_5, $2A_1+A_3$, and $4A_1$, which are not equivalent with respect to the Weyl group.

The following result of P. Du Val [267] will be left without proof. Note that Du Val uses the following notation:

$$A_1 = [\ ],\ A_n = [3^{n-1}],\ n \geq 2,\ D_n = [3^{n-3,1,1}],\ n \geq 4,$$

$$E_6 = [3^{3,2,1}],\ E_7 = [3^{4,2,1}],\ E_8 = [3^{5,2,1}].$$

Theorem 8.8.1. *All types of root bases in* E_8 *can be realized by Dynkin curves except the cases* $7A_1, 8A_1, D_4 + 4A_1$.

In fact, Du Val describes explicitly the singularities of the branch sextic similarly to the case of weak del Pezzo surfaces of degree 2 (see also Table 8.10).

The number of lines on a del Pezzo surface of degree 1 is equal to 240. Note the coincidence with the number of roots. The reason is simple, for any root $\alpha \in \mathsf{E}_8$, the sum $-\mathbf{k}_8 + \alpha$ is an exceptional vector. The image of a line under the cover $\phi : S \to Q$ is a conic. The plane spanning the conic is a *tritangent plane*, i.e. a plane touching the branch sextic W at three points. There are 120 tritangent planes, each cut out a conic in Q which splits under the cover in the union of two lines intersecting at three points. Note that the effective divisor D of degree 3 on W such that $2D$ is cut out by a tritangent plane, is an odd theta characteristic on W. This gives another explanation of the number $120 = 2^3(2^4 - 1)$.

The Gosset polytope $\Sigma_8 = 4_{21}$ has 17280 facets of type α corresponding to contractions of sets of eight disjoint (-1)-curves, and 2160 facets of type β corresponding to conic bundle structures arising from the pencils of conics $|de_0 - m_1e_1 - \cdots - m_8|$ in the plane which we denote by $(d; m_1, \ldots, m_8)$:

- 8 of type $(1; 1, 0^7)$,
- 70 of type $(2; 1^4, 0^5)$,
- 168 of type $(3; 2, 1^5, 0^2)$,
- 280 of type $(4; 2^3, 1^4, 0)$,
- 8 of type $(4; 3, 1^7)$,
- 56 of type $(5; 2^6, 1, 0)$,
- 280 of type $(5; 3, 2^3, 1^4)$,
- 420 of type $(6; 3^2, 2^4, 1^2)$,
- 280 of type $(7; 3^4, 2^3, 1)$,
- 56 of type $(7; 4, 3, 2^6)$,
- 8 of type $(8; 3^7, 1)$,
- 280 of type $(8; 4, 3^4, 2^3)$,
- 168 of type $(9; 4^2, 3^5, 2)$,
- 70 of type $(10; 4^4, 3^4)$,
- 8 of type $(11; 4^7, 3)$,

Observe the symmetry $(d; m_1, \ldots, m_8) \mapsto -4\mathbf{k}_8 - (d; m_1, \ldots, m_8)$.

We know that the linear system $|-K_S|$ is an irreducible pencil with one base point x_0. Let $\tau : F \to S$ be its blow-up. The proper inverse transform of $|-K_S|$

in F is a base-point-free pencil of curves of arithmetic genus 1. It defines an elliptic fibration $\varphi : F \to \mathbb{P}^1$. The exceptional curve $E = \tau^{-1}(x_0)$ is a section of the fibration. Conversely, let $\varphi : F \to \mathbb{P}^1$ be an elliptic fibration on a rational surface F which admits a section E and is relative minimal in the sense that no fiber contains a (-1)-curve. It follows from the theory of elliptic surfaces that $-K_F$ is the divisor class of a fiber and E is a (-1)-curve. Blowing down E, we obtain a rational surface S with $K_S^2 = 1$. Since K_F is obviously nef, we obtain that K_S is nef, so S is a weak del Pezzo surface of degree 1.

Let $\varphi : F \to \mathbb{P}^1$ be a rational elliptic surface with a section E. The section E defines a rational point e on a generic fiber F_η, considered as a curve over the functional field K of the base of the fibration. It is a smooth curve of genus 1, so it admits a group law with the zero equal to the point e. It follows from the theory of relative minimal models of surfaces that any automorphism of F_η over K extends to a biregular automorphism of F over $\mathbb{P}^1$. In particular, the negation automorphism $x \to -x$ extends to an automorphism of F fixing the curve E. Its descent to the blowing down of E is the Bertini involution.

Let D be a Dynkin curve on S. The point x_0 cannot lie on D. In fact, otherwise the proper transform R' of a component of D that contains t_0 is a (-3)-curve on F. However, $-K_F$ is nef on F, hence $K_F \cdot R' \leq 0$ contradicting the adjunction formula. This implies that the pre-image $\tau^*(D)$ of D on F is a Dynkin curve contained in a fiber. The whole fiber is equal to the union of $\tau^*(D) + R$, where R is a (-2)-curve intersecting the zero section E. Kodaira's classification of fibers of elliptic fibrations shows that the intersection graph of the irreducible components of each reducible fiber is equal to one of the extended Coxeter-Dynkin diagrams.

The classification of Dynkin curves on a weak del Pezzo surfaces of degree 1 gives the classification of all possible collections of reducible fibers on a rational elliptic surface with a section. Conversely, the classification of possible reducible fibers of a rational elliptic surfaces gives the classification of possible Dynkin curves on a weak del Pezzo surface of degree one. The equation of the anti-canonical model in $\mathbb{P}(1, 1, 2, 3)$

$$t_3^2 + t_2^3 + a_4(t_0, t_1)t_2 + a_6(t_0, t_1) = 0. \tag{8.32}$$

After the dehomogenization $t = t_1/t_0, x = t_2/t_0^2, y = t_3/t_0^3$, we obtain the *Weierstrass equation* of the elliptic surface

$$y^2 + x^3 + a_4(t)x + a_6(t) = 0.$$

There is a classification of all possible singular fibers of rational elliptic surfaces in terms of the order of vanishing at a point on the base of the coefficients $a(t), b(t)$ and the discriminant $\Delta = 4a(t)^3 + 27b(t)^2$ (see, for example, [673]).

8.8.2 Automorphisms of del Pezzo surfaces of degree one

Let S be a weak del Pezzo surface of degree ne. Consider the degree 2 regular map $\phi : S \to Q$ defined by the linear system $|-2K_S|$. Let $S \to S' \to Q$ be its Stein factorization, where $\phi' : S' \to Q$ is a finite morphism of degree 2. Similarly to the case of weak del Pezzo surfaces of degree two, we define the *Bertini involution* β as the lift to $\mathrm{Aut}(S)$ of the deck transformation of the double cover ϕ'.

The proof of the following proposition is almost a word-by-word repetition of the proof of Proposition 8.7.1.

Proposition 8.8.2. *Let S be a del Pezzo surface of degree one. The image $\rho(\beta)$ of the Bertini involution β in $W(\mathsf{E}_8)$ does not depend on a choice of a geometric basis and coincides with the element w_0 of $W(\mathsf{E}_8)$ generating the center of $W(\mathsf{E}_8)$. It acts as the minus identity on the sublattice $K_S^\perp \cong \mathsf{E}_8$.*

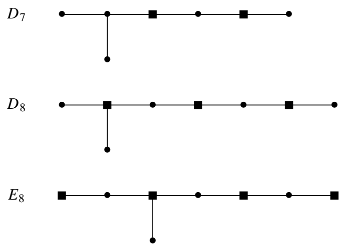
Here we use that w_0 acts as the minus identity on the lattice E_8. It acts on the Gosset polytope 4_{21} as the reflection with respect to the center defined by the vector $\mathbf{k}_8 = -\frac{1}{240}\sum v_i$, where v_i are the exceptional vectors. The 120 orbits on the set of vertices correspond to 120 tritangent planes of the branch curve of the Bertini involution.

As in the case of weak del Pezzo surfaces of degree two, the Bertinin involution does not act as the minus identity on $K_S^\perp$. Also, the difference between the two cases. The Bertini has always an isolated fixed point equal to the pre-image of the vertex of Q under the map ϕ.

The following proposition is the analog of Proposition 8.7.2.

Proposition 8.8.3. *The one-dimensional part of the locus of fixed points S^β of the Bertini involution β is the disjoint union of smooth curves $W + R_1 + \cdots + R_k$, where $R_1, \ldots, R_k$ are among irreducible components of Dynkin curves. The curve W is the normalization of the branch curve of the double cover $\phi : S \to Q$. A Dynkin curve of type A_{2k} has no fixed components, a Dynkin curve of type A_{2k+1} has one fixed component equal to the central component. A Dynkin curve of types D_4, D_7, D_8, E_8 have fixed components marked by square on their*

Coxeter-Dynkin diagrams. The fixed components of Dynkin curves of the other types are given in the diagrams from Proposition 8.7.2.



(8.33)

Assume that S is a del Pezzo surface. Then, the fixed locus of the Bertini involution is a smooth irreducible curve W of genus 4 isomorphic to the branch curve B of the cover and the base point of $|-K_S|$. It belongs to the linear system $|-3K_S|$ and hence its image in the plane is a curve of degree 9 with triple points at $x_1,\ldots,x_8$.

Suppose S is a del Pezzo surface of degree one. Then, the trace of β in $K_S^\perp$ is equal to -8. It follows from (8.31), where γ is replaced with β, that $e(S^\beta) = -5$. This agrees with the fact that $e(B) = -6$ and $e(\text{point}) = 1$.

We can also consider the Bertini involution as a planar Cremona involution of the plane. The characteristic matrix (7.47) coincides with the matrix of β^* in its action on $\mathrm{Pic}(S)$.

From now on, S is a del Pezzo surface of degree one. Let $\pi : X \to S$ be the blow-up of the isolated fixed point $\mathfrak{o}$ of the Bertini involution β. The group $\mathrm{Aut}(S)$ acts naturally on X and can be identified with the subgroup $\mathrm{Aut}(X)_0$ of $\mathrm{Aut}(X)$ that fixes the section $C_0 = \pi^{-1}(\mathfrak{o})$ invariant. The restriction of the image of β to the generic fiver X_η of f is the negation involution of the elliptic curve X_η with the zero element of the group law equal to the point of X_η corresponding to the section C_0. The group $\mathrm{Aut}(X)$ in its action on $\mathrm{Pic}(S)$ leaves invariant $-K_X|$ and hence preserves the elliptic fibration. Let

$$q : \mathrm{Aut}(X)_0 \to \mathrm{Aut}(\mathbb{P}^1) = \mathrm{Aut}(|-K_S|^*)$$

be the natural homomorphism of the base of the elliptic fibration. Its kernel is isomorphic to the group of automorphisms of the elliptic curve X_η preserving the group law. It contains β and the quotient group $\mathrm{Aut}(X)_0/(\beta)$ is a cyclic group of order 2 if the coefficient a_4 in (8.32) vanishes and order 3 if a_4 vanishes.

Let G be the image of $\mathrm{Aut}(X)_0$ in $\mathrm{Aut}(\mathbb{P}^1) \cong \mathrm{PGL}(2)$. Recall the well-known classification of finite subgroup Γ of $\mathrm{PGL}(2)$, called *polyhedral groups*. They are distinguished by the set of the special orbits, i.e., orbits of cardinality equal to $\#\Gamma/e_i$, where $e_i > 1$ is the order of the stabilizer subgroup of a point in the orbit.

(i) $\Gamma = C_n$, a cyclic group of order n, $(e_1, e_2) = (n, n)$;
(ii) $\Gamma = D_{2k}$, $n = 2k$, $(e_1, e_2, e_3) = (2, 2, k)$;
(iii) $\Gamma = \mathfrak{A}_4$, $n = 12$, $(e_1, e_2, e_3) = (2, 3, 3)$;
(iv) $\Gamma = \mathfrak{S}_4$, $n = 24$, $(e_1, e_2, e_3) = (2, 3, 4)$;
(v) $\Gamma = \mathfrak{A}_5$, $n = 60$, $(e_1, e_2, e_3) = (2, 3, 5)$.

Our group G must be one of these subgroups. Observe, that in its action on $\mathbb{P}^1$, the group G leaves invariant the zeroes of the binary forms $a_4(t_0, t_1)$ and $a_6(t_0, t_1)$.

Let $\overline{\Gamma}$ be a lift of Γ to a finite subgroup of $\mathrm{SL}(2)$. Since the kernel of the homomor[hism $\mathrm{SL}(2) \to \mathrm{PGL}(2)$ is the group of order 2 generated by the matrix $-I_2$, The kernel of $\overline{\Gamma} \to \Gamma$ is of order ≤ 2. It is easy to see that a non-cyclic group Γ cannot be isomorphically lifted to a finite subgroup of SL(2), and the same is true for cyclic groups of even order. The groups $\overline{\Gamma}$ are called *binary polyhedral groups*, binary dihedral in case (ii), binary tetrahedral in case (iii), binary octahedral in case (iv), and binary icosahedral in case (v). They are usually denoted by $\bar{\mathsf{D}}_{2n}, \bar{\mathsf{T}}, \bar{\mathsf{O}}, \bar{\mathsf{I}}$. They are of orders $4n, 24, 48, 120$, respectively.

The group $\mathrm{SL}(2)$ acts linearly on the symmetric algebra $S^2((\mathbb{C}^2)^\vee)$ preserving the grading. The binary forms a_4 and a_6 are relative invariants of $\tilde{G}$. This means that

$$g^*(a_4, a_6) = (\chi_1(g)a_4, \chi_2(a_6),$$

where $\chi_1, \chi_2 \in \mathrm{Hom}(\overline{\Gamma}, \mathbb{C}^*)$.

The set of relative invariants of $\overline{\Gamma}$ is a finitely generated graded algebra. Let us give the set of natural generators of this algebra, classically called *Gründformen*. We identify the group PGL(2) with the group of fractional-linear transformations $z \mapsto \frac{az+b}{cz+d}$. In the following, we give a representative of the conjugacy class of $\overline{\Gamma}$.

Case 1: Γ is a cyclic group of order n.

$\overline{\Gamma}$ is a cyclic group of order n generated by the matrix

$$g = \begin{pmatrix} \epsilon_n & 0 \\ 0 & \epsilon_n^{-1} \end{pmatrix}.$$

The group Γ is generated by the transformation $\bar{g} : z \mapsto \epsilon_n z$ if n is odd and $\bar{g} : z \mapsto \epsilon_n^2$ if n is even.

The exceptional orbits are $\{0\}$ and $\{\infty\}$. The Grundformen are

$$\Phi_1 = t_0, \quad \Phi_2 = t_1$$

with characters

$$\chi_1(g) = \epsilon_n, \quad \chi_2(g) = \epsilon_n^{-1}.$$

Case 2: $\Gamma = D_{2n}$.

The binary group $\overline{\Gamma}$ is generated by the matrices

$$g_1 = \begin{pmatrix} \epsilon_{2n} & 0 \\ 0 & \epsilon_{2n}^{-1} \end{pmatrix}, \quad g_2 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}.$$

Any exceptional orbit is the orbit of a fixed point of some element $g \in G$ different from $\pm I_2$. The fixed points of g_1 are $0, \infty$. Applying g_2, we see that they form one orbit of cardinality 2. The fixed points of g_2 are ± 1. Applying powers of g_1 we get two exceptional orbits. One is formed by nth roots of 1, the other one is formed by nth roots of -1. The Grundformen are

$$\Phi_1 = t_0^n + t_1^n, \quad \Phi_2 = t_0^n - t_1^n, \quad \Phi_3 = t_0 t_1. \tag{8.34}$$

The generators g_1 and g_2 act on the Grunforms with characters

$$\chi_1(g_1) = -1,\ \chi_1(g_2) = i^n,$$
$$\chi_2(g_1) = -1,\ \chi_2(g_2) = -i^n,$$
$$\chi_3(g_1) = 1,\ \chi_3(g_2) = -1.$$

Case 3: $\Gamma = \mathsf{T}$.

The group $\overline{\Gamma}$ is generators by the matrices

$$g_1 = \begin{pmatrix} \epsilon_4 & 0 \\ 0 & \epsilon_4^{-1} \end{pmatrix}, \quad g_2 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, \quad g_3 = \frac{1}{1-i}\begin{pmatrix} 1 & i \\ 1 & -i \end{pmatrix}$$

The fixed points of g_1 are $0, \infty$ permuted under g_2. Their stabilizer is of order 2. Thus orbit O_1 with $e_1 = 2$ consists of 6 points. Applying powers of g_3 to 0 and 1 we see that O_1 consists of the points $0, \infty, 1, -1, i, -i$. The fixed points of g_3 are $\frac{1-i}{2}(1 \pm \sqrt{3})$. Applying g_1 and g_2 to these points, we obtain two orbits of cardinality 4

$$O_2 = \{\pm\frac{1-i}{2}(1+\sqrt{3}), \pm\frac{1+i}{2}(1-\sqrt{3})\},\ O_3 = \{\pm\frac{1-i}{2}(1-\sqrt{3}), \pm\frac{1+i}{2}(1+\sqrt{3})\}.$$

The Grundformen are

$$\Phi_1 = t_0 t_1 (t_0^4 - t_1^4), \quad \Phi_2 = t_0^4 + 2\sqrt{-3}t_0^2 t_1^2 + t_1^4, \quad \Phi_3 = t_0^4 - 2\sqrt{-3}t_0^2 t_1^2 + t_1^4.$$

The characters are

$$\chi_1(g_1) = \chi_1(g_2) = \chi_1(g_3) = 1,$$
$$\chi_2(g_1) = \chi_2(g_2) = \chi_2(g_3) = \epsilon_3,$$
$$\chi_3(g_1) = \chi_3(g_2) = 1,\ \chi_3(g_3) = \epsilon_3^2.$$

Case 3: $\Gamma = \mathsf{O}$.

The binary group $\overline{\mathsf{O}}$ is generators by

$$g_1 = \begin{pmatrix} \epsilon_8 & 0 \\ 0 & \epsilon_8^{-1} \end{pmatrix}, \quad g_2 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, \quad g_3 = \frac{1}{1-i}\begin{pmatrix} 1 & i \\ 1 & -i \end{pmatrix}.$$

The fixed points with stabilizers of order 4 are $0, \infty, 1, -1, i, -i$. They form the first orbit O_1. The fixed points with stabilizers of order 3 form the second orbit of order eight. It consists of the union of the orbits O_2 and O_3 from the previous case.

The group $\overline{\Gamma}$ isomorphic to the permutation group $\mathfrak{S}_4$. It has two conjugacy classes of elements of order 2. One of them belongs to the conjugacy class of an element of order 4, say of g_1. Another one is a stabilizer of an exceptional orbit O_3 of cardinality 12. It corresponds to a transposition in S_4. It can be realized by the product $g_1g_2 : z \mapsto i/z$. Its fixed points are $\pm\epsilon_8$. The corresponding orbit O_3 consists of

$$\epsilon_8^k, \quad \frac{i\epsilon_8^k + i}{\epsilon_8^k - 1}, \quad \frac{i\epsilon_8^k - 1}{i\epsilon_8^k + 1}, \quad k = 1, 3, 5, 7.$$

Now, it is easy to list the Grundformen. They are

$$\Phi_1 = t_0t_1(t_0^4 - t_1^4),\ \Phi_2 = t_0^8 + 14t_0^4t_1^4 + t_1^8 = (t_0^4 + 2\sqrt{-3}t_0^2t_1^2 + t_1^4)(t_0^4 - 2\sqrt{-3}t_0^2t_1^2 + t_1^4),$$

$$\Phi_3 = (t_0^4 + t_1^4)((t_0^4 + t_1^4)^2 - 36t_0^4t_1^4).$$

The characters are

$$\chi_1(g_1) = -1,\ \chi_1(g_2) = \chi_1(g_3) = 1,$$
$$\chi_2(g_1) = \chi_2(g_2) = \chi_2(g_3) = 1,$$
$$\chi_3(g_1) = -1,\ \chi_3(g_2) = \chi(g_3) = 1.$$

Case 4: $\Gamma = \mathsf{I}$. The smallest orbit of Γ is of cardinality $60/5 = 12$. So, we see that this group cannot occur in our case. Nevertheles, for completeness sake we give the information about the Gründforms of . $\overline{\Gamma}$. The group is generated by

$$g_1 = \begin{pmatrix} \epsilon_{10} & 0 \\ 0 & \epsilon_{10}^{-1} \end{pmatrix}, \quad g_2 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, \quad g_3 = \frac{1}{\sqrt{5}}\begin{pmatrix} \epsilon_5 - \epsilon_5^4 & \epsilon_5^2 - \epsilon_5^3 \\ \epsilon_5^2 - \epsilon_5^3 & -\epsilon_5 + \epsilon_5^4 \end{pmatrix}.$$

Note that we can replace g_3 with a generator of order 3 equal to $g_1 g_3 g_1$. We already used that the orbit O_3 with stabilizer subgroups of $\bar{G}$ of order 5 are roots of the Grundform

$$\Phi_3 = t_0 t_1 (t_0^{10} + 11 t_0^5 t_1^5 - t_1^{10}).$$

Other Grundformen must be of degree 30 and 20. Recall that the Hessian determinant of a function in two variables is the determinant of the Hessian matrix of its partial derivatives of second order. The hessian of Φ_3 must be of degree 20. It is easy to see, using the chain rule, that the hessian of a relative invariant is a relative invariant. Thus the hessian of Φ_3 is a Grundform corresponding to the orbit O_2 with stabilizers of order 3. The direct computation gives

$$\Phi_2 = -(t_0^{20} + t_1^{20}) + 228(t_0^{15} t_1^5 - t_0^5 t_1^{15}) - 494 t_0^{10} t_1^{10}.$$

Next, we need a Grundform of degree 30. Recall that the jacobian of functions f, g in two variables is the determinant of the matrix whose first row are partial derivatives of the first order of f and the second row is the same for g. The jacobian of Φ_2, Φ_3 must be of degree 30, and it is easy to see that it is a relative invariant. This gives us a Grundform of degree 30

$$\Phi_1 = t_0^{30} + t_1^{30} + 522(t_0^{25} t_1^5 - t_0^5 t_1^{25}) - 10005(t_0^{20} t_1^{10} + t_0^{10} t_1^{20}).$$

Since $\bar{G} \cong A_5$ is a simple group and all Grundformen are of even degree, we see that the characters are trivial.

In the following, we use that a multiple root of a_6 is not a root of a_4 (otherwise the surface X is singular). We denote by ϵ_k a primitive k-th root of unity.

Case 1: $G = C_n$, Any monomial $t_0^i t_1^j$ is a relative invariant with $\chi(\sigma) = \epsilon_n^{i-j}$ if n is odd and $\chi(\sigma) = \epsilon_{2n}^{i-j}$ if n is even. In Table 8.15 below we list relative invariants which are not monomials.

Case 2: $\Gamma = D_{2n}$ is a dihedral group of order $n = 2k$. It is generated by two matrices

$$\sigma_1 = \begin{pmatrix} \epsilon_{2k} & 0 \\ 0 & \epsilon_{2n}^{-1} \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}.$$

Case 3: $\Gamma = \mathsf{A}_4$.

Up to the variable change $t_0 \to i t_0, t_1 \to t_1$, we have only one case

$$a_4 = t_0^4 + 2\sqrt{-3} t_0^2 t_1^2 + t_1^4, \ (\chi(\sigma_1), \chi(\sigma_2), \chi(\sigma_3)) = (1, 1, \epsilon_3), \tag{8.35}$$

$$a_6 = t_0 t_1 (t_0^4 - t_1^4), \ (\chi(\sigma_1), \chi(\sigma_2), \chi(\sigma_3)) = (1, 1, 1). \tag{8.36}$$

n	a_4	$\chi(\sigma)$	a_6	$\chi(\sigma)$
2	$at_0^4 + bt_0^2t_1^2 + ct_1^4$	1	$at_0^6 + t_0^2t_1^2(bt_0^2 + ct_1^2) + dt_1^6$	-1
	$t_0t_1(at_0^2 + bt_1^2)$	-1	$t_0t_1(at_0^4 + bt_0^2t_1^2 + ct_1^4)$	1
3	$t_0(at_0^3 + bt_1^3)$	ϵ_3	$at_0^6 + bt_0^3t_1^3 + ct_1^6$	1
	$t_1(at_0^3 + bt_1^3)$	ϵ_3^2	$t_0t_1^2(at_0^3 + bt_1^3)$	ϵ_3^2
			$t_0^2t_1(at_0^3 + bt_1^3)$	ϵ_3
4	$at_0^4 + bt_1^4$	-1	$t_0^2(at_0^4 + bt_1^4)$	-i
			$t_0t_1(at_0^4 + bt_1^4)$	-1
			$t_1^2(at_0^4 + bt_1^4)$	i
5			$t_0(at_0^5 + bt_1^5)$	ϵ_5
			$t_1(at_0^5 + bt_1^5)$	ϵ_5^4
6			$at_0^6 + bt_1^6$	-1

Table 8.15 *Relative invariants:* $\Gamma = C_n$

n	a_4	$\chi(\sigma_1)$	$\chi(\sigma_2)$	a_6	$\chi(\sigma_1)$	$\sigma_2)$
2	$a(t_0^4 + t_1^4) + bt_0^2t_1^2$	1	1	$t_0t_1\big(a(t_0^4 + t_1^4) + bt_0^2t_1^2\big)$	1	-1
	$t_0t_1(t_0^2 - t_1^2)$	-1	-1	$a(t_0^6 + t_1^6) + bt_0^2t_1^2(t_0^2 + t_1^2)$	-1	-1
	$t_0^4 - t_1^4$	1	-1	$a(t_0^6 - t_1^6) + bt_0^2t_1^2(t_0^2 - t_1^2)$	-1	1
3	$t_0^2t_1^2$	1	1	$t_0^6 + t_1^6 + at_0^3t_1^3$	1	-1
				$t_0^6 + t_1^6 + at_0^3t_1^3$	1	-1
4	$t_0^4 \pm t_1^4$	-1	±1	$t_0t_1(t_0^4 \pm t_1^4)$	-1	∓1
6	$t_0^2t_1^2$	1	1	$t_0^6 \pm t_1^6$	-1	∓1

Table 8.16 *Relative invariants:*$\Gamma = D_{2n}$

Case 4: $\Gamma = \mathfrak{S}_4$. It is generated by matrices

$$\sigma_1 = \begin{pmatrix} \epsilon_8 & 0 \\ 0 & \epsilon_8^{-1} \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \frac{1}{\sqrt{2}} \begin{pmatrix} \epsilon_8^{-1} & \epsilon_8^{-1} \\ \epsilon_8^5 & \epsilon_8 \end{pmatrix}.$$

There is only one, up to a change of variables, relative invariant of degree ≤ 6. It is

$$a_6 = t_0t_1(t_0^4 - t_1^4).$$

It is an invariant of $\bar{G}$. In this case $a_4 = 0$.

In the next theorem, we list all possible groups $G' = \mathrm{Aut}(S)/\langle\beta\rangle$ and their lifts G to subgroups of $\mathrm{Aut}(S)$. We extend the action of $\bar{G}$ on the coordinates t_0, t_1 to an action on the coordinates t_0, t_1, t_2. Note that not all combinations of (a_4, a_6) admit such an extension. For example, a common root of a_4 and a_6 must be a simple root of a_6 since otherwise the surface S is singular.

In the following list, the vector $\mathbf{c} = (c_0, c_1, c_2, c_3)$ will denote the transformation $[t_0, t_1, t_2, t_3] \mapsto [c_0t_0, c_1t_1, c_2t_2, c_3t_3]$. The Bertini transformation β corresponds to the vector $(1, 1, 1, -1)$.

1. *Cyclic groups G'*

(i) $G' = 2, G = \langle(1, -1, 1, 1), \beta\rangle \cong 2^2$,

$$a_4 = at_0^4 + bt_0^2t_1^2 + ct_1^4, \quad a_6 = dt_0^6 + et_0^4t_1^2 + ft_0^2t_1^4 + gt_1^6.$$

(ii) $G' = 2, G = \langle(1, -1, -1, i)\rangle$,

$$a_4 = at_0^4 + bt_0^2t_1^2 + ct_1^4, \quad a_6 = t_0t_1(dt_0^4 + et_0^2t_1^2 + ft_1^4).$$

(iii) $G' = 3, G = \langle(1, \epsilon_3, 1, -1)\rangle \cong 6$,

$$a_4 = t_0(at_0^3 + bt_1^3), \quad a_6 = at_0^6 + bt_0^3t_1^3 + ct_1^6.$$

(iv) $G' = 3, G = \langle(1, \epsilon_3, \epsilon_3, -1)\rangle$,

$$a_4 = t_0^2t_1^2, \quad at_0^6 + bt_0^3t_1^3 + ct_1^6.$$

(v) $G' = 3, G = 6, \mathbf{a} = (1, 1, \epsilon_3, -1)$,

$$a_4 = 0.$$

(vi) $G' = 4, G = \langle(i, 1, -1, i), \beta\rangle \cong 4 \times 2$,

$$a_4 = at_0^4 + bt_1^4, \quad a_6 = t_0^2(ct_0^4 + dt_1^4).$$

(vii) $G' = 4, G = \langle(i, 1, -i, -\epsilon_8)\rangle \cong 8$,

$$a_4 = at_0^2t_1^2, \quad a_6 = t_0t_1(ct_0^4 + dt_1^4),$$

(viii) $G' = 5, G = \langle(1, \epsilon_5, 1, -1)\rangle \cong 10$,

$$a_4 = at_0^4, \quad a_6 = t_0(bt_0^5 + t_1^5).$$

(ix) $G' = 6, G = \langle(1, \epsilon_6, 1, 1), \beta\rangle \cong 2 \times 6$.

$$a_4 = t_0^4, \quad a_6 = at_0^6 + bt_1^6.$$

(x) $G' = 6, G = \langle(\epsilon_6, 1, \epsilon_3^2, 1), \beta\rangle \cong 2 \times 6$,

$$a_4 = t_0^2t_1^2, \quad a_6 = at_0^6 + bt_1^6.$$

(xi) $G' = 6, G = \langle(-1, 1, \epsilon_3, 1), \beta\rangle \cong 2 \times 6$,

$$a_4 = 0, \quad a_6 = dt_0^6 + et_0^4t_1^2 + ft_0^2t_1^4 + gt_1^6,$$

(xii) $G' = 10, G = \langle(1, \epsilon_{10}, -1, i)\rangle \cong 20$,

$$a_4 = at_0^4, \quad a_6 = t_0t_1^5.$$

(xiii) $G' = 12, G = \langle(\epsilon_{12}, 1, \epsilon_3^2, -1), \beta\rangle \cong 2 \times 12$,

$$a_4 = at_0^4, \quad a_6 = t_1^6.$$

(xiv) $G' = 12, G = \langle(i, 1, \epsilon_{12}, \epsilon_8)\rangle \cong 24$,

$$a_4 = 0, \quad a_6 = t_0t_1(t_0^4 + bt_1^4).$$

(xv) $G' = 15, G = \langle(1, \epsilon_5, \epsilon_3, \epsilon_{30})\rangle \cong 30$,

$$a_4 = 0, \quad a_6 = t_0(t_0^5 + t_1^5).$$

2. *Dihedral groups*

(i) $G' = 2^2, G = D_8$,

$$a_4 = a(t_0^4 + t_1^4) + bt_0^2t_1^2, \quad a_6 = t_0t_1[c(t_0^4 + t_1^4) + dt_0^2t_1^2], a \neq 0, d \neq 0$$

$$G' = 3 \times 2^2, G = D_8, a = b = 0, c, d \neq 0,$$

$$\sigma_1 : [t_0, t_1, t_2, t_3] \mapsto [t_1, -t_0, t_2, it_3],$$
$$\sigma_2 : [t_0, t_1, t_2, t_3] \mapsto [t_1, t_0, t_2, t_3],$$

$$\sigma_1^4 = \sigma_2^2 = 1, \sigma_1^2 = \beta, \sigma_2\sigma_1\sigma_2^{-1} = \sigma_2^{-1}.$$

(ii) $G' = 2^2, G = 2.D_4$,

$$a_4 = a(t_0^4 + t_1^4) + bt_0^2t_1^2, \quad a_6 = t_0t_1(t_0^4 - t_1^4),$$

$$\sigma_1 : [t_0, t_1, t_2, t_3] \mapsto [t_0, -t_1, -t_2, it_3],$$
$$\sigma_2 : [t_0, t_1, t_2, t_3] \mapsto [t_1, t_0, -t_2, it_3],$$

$$\sigma_1^2 = \sigma^2 = (\sigma_1\sigma_2)^2 = \beta.$$

(iii) $G' = D_6, G = D_{12}$,

$$a_4 = at_0^2t_1^2, \quad a_6 = t_0^6 + t_1^6 + bt_0^3t_1^3,$$

$$\sigma_1 : [t_0, t_1, t_2, t_3] \mapsto [t_0, \epsilon_3t_1, \epsilon_3t_2, -t_3],$$
$$\sigma_2 : [t_0, t_1, t_2, t_3] \mapsto [t_1, t_0, t_2, t_3],$$

$$\sigma_1^3 = \beta, \sigma_2^2 = 1, \ \sigma_2\sigma_3\sigma_2^{-1} = \sigma_1^{-1}.$$

(v) $G' = D_8, G = D_{16}$,

$$a_4 = at_0^2t_1^2, \quad a_6 = t_0t_1(t_0^4 + t_1^4),$$

$$\sigma_1 : [t_0, t_1, t_2, t_3] \mapsto [\epsilon_8 t_0, \epsilon_8^{-1} t_1, -t_2, it_3],$$
$$\sigma_2 : [t_0, t_1, t_2, t_3] \mapsto [t_1, t_0, t_2, t_3],$$

$$\sigma_1^4 = \beta, \sigma_2^2 = 1, \ \sigma_2\sigma_1\sigma_2^{-1} = \sigma_1^{-1}.$$

(vi) $G' = D_{12}, G = 2.D_{12}$,

$$a_4 = at_0^2t_1^2, \quad a_6 = t_0^6 + t_1^6,$$

$$\sigma_1 : [t_0, t_1, t_2, t_3] \mapsto [t_0, \epsilon_6 t_1, \epsilon_3^2 t_2, t_3],$$
$$\sigma_2 : [t_0, t_1, t_2, t_3] \mapsto [t_1, t_0, t_2, t_3], \sigma_3 = \beta.$$

We have

$$\sigma_1^6 = \sigma_2^2 = \sigma_3^3 = 1, \ \sigma_2\sigma_1\sigma_2^{-1} = \sigma_1^{-1}\sigma_3.$$

3. *Other groups*

(i) $G' = \mathfrak{A}_4, G = 2.\mathfrak{A}_4$,

$$a_4 = t_0^4 + 2\sqrt{-3}t_0^2t_1^2 + t_2^4, \quad a_6 = t_0t_1(t_0^4 - t_1^4),$$

$$\sigma_1 = \begin{pmatrix} i & 0 & 0 & 0 \\ 0 & -i & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \ \sigma_2 = \begin{pmatrix} 0 & i & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \sigma_3 = \frac{1}{\sqrt{2}} \begin{pmatrix} \epsilon_8^{-1} & \epsilon_8^{-1} & 0 & 0 \\ \epsilon_8^5 & \epsilon_8 & 0 & 0 \\ 0 & 0 & \sqrt{2}\epsilon_3 & 0 \\ 0 & 0 & 0 & \sqrt{2} \end{pmatrix}.$$

(ii) $G' = 3 \times D_4, G = 3 \times D_8$,

$$a_4 = 0, \quad a_6 = t_0t_1(t_0^4 + at_0^2t_1^2 + t_1^4).$$

(iii) $G' = 3 \times D_6, G = 6.D_6 \cong 2 \times 3.D_6$,

$$a_4 = 0, \quad a_6 = t_0^6 + at_0^3t_1^3 + t_1^6.$$

It is generated by

$$\sigma_1 : [t_0, t_1, t_2, t_3] \mapsto [t_0, t_1, \epsilon_3 t_2, t_3],$$
$$\sigma_2 : [t_0, t_1, t_2, t_3] \mapsto [t_0, \epsilon_3 t_1, t_2, t_3],$$
$$\sigma_3 : [t_0, t_1, t_2, t_3] \mapsto [t_1, t_0, t_2, t_3].$$

They satisfy $\sigma_3 \cdot \sigma_2 \cdot \sigma_3^{-1} = \sigma_2^{-1}\sigma_1^4$.

(iv) $G' = 3 \times D_{12}, G = 6.D_{12}$,

$$a_4 = 0, \quad a_6 = t_0^6 + t_1^6.$$

It is generated by

$$\begin{aligned}
\sigma_1 &: [t_0, t_1, t_2, t_3] \mapsto [t_0, t_1, \epsilon_3 t_2, t_3],\\
\sigma_2 &: [t_0, t_1, t_2, t_3] \mapsto [t_0, \epsilon_6 t_1, t_2, t_3],\\
\sigma_3 &: [t_0, t_1, t_2, t_3] \mapsto [t_1, t_0, t_2, t_3].
\end{aligned}$$

We have $\sigma_3 \cdot \sigma_2 \cdot \sigma_3^{-1} = \sigma_2^{-1}\sigma_1$.

(v) $G' = 3 \times \mathfrak{S}_4, G = 3 \times 2.\mathfrak{S}_4$,

$$a_4 = 0, \quad a_6 = t_0 t_1 (t_0^4 - t_1^4),$$

$$\sigma_1 = \begin{pmatrix} \epsilon_8 & 0 & 0 & 0\\ 0 & \epsilon_8^{-1} & & 0\\ 0 & 0 & -1 & 0\\ 0 & 0 & 0 & i \end{pmatrix}, \ \sigma_2 = \begin{pmatrix} 0 & 1 & 0 & 0\\ 1 & 0 & 0 & 0\\ 0 & 0 & -1 & 0\\ 0 & 0 & 0 & i \end{pmatrix},$$

$$\sigma_3 = \frac{1}{\sqrt{2}} \begin{pmatrix} \epsilon_8^{-1} & \epsilon_8^{-1} & 0 & 0\\ \epsilon_8^5 & \epsilon_8 & 0 & 0\\ 0 & 0 & \sqrt{2} & 0\\ 0 & 0 & 0 & \sqrt{2} \end{pmatrix}, \quad \sigma_4 = \begin{pmatrix} 1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & \epsilon_3 & 0\\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Table 8.17 below gives a list of the full automorphism groups of del Pezzo surfaces of degree one.

Type	Order	Structure	a_4	a_6
I	144	$3 \times (\overline{T} : 2)$	0	$t_0t_1(t_0^4 - t_1^4)$
II	72	$3 \times 2D_{12}$	0	$t_0^6 + t_1^6$
III	36	$6 \times D_6$	0	$t_0^6 + at_0^3t_1^3 + t_1^6$
IV	30	30	0	$t_0(t_0^5 + t_1^5)$
V	24	$\overline{T}$	$a(t_0^4 + 2\sqrt{-3}t_0^2t_1^2 + t_1^4)$	$t_0t_1(t_0^4 - t_1^4)$
VI	24	$2D_{12}$	$at_0^2t_1^2$	$t_0^6 + t_1^6$
VII	24	2×12	t_0^4	t_1^6
VIII	24	$3 \times D_8$	0	$t_0t_1(t_0^4 + t_1^4 + at_0^2t_1^2)$
IX	20	20	t_0^4	$t_0t_1^5$
X	16	D_{16}	$at_0^2t_1^2$	$t_0t_1(t_0^4 + t_1^4)$
XI	12	D_{12}	$t_0^2t_1^2$	$t_0^6 + at_0^3t_1^3 + t_1^6$
XII	12	2×6	0	$g_3(t_0^2, t_1^2)$
XIII	12	2×6	t_0^4	$at_0^6 + t_1^6$
XIV	10	10	t_0^4	$t_0(at_0^5 + t_1^5)$
XV	8	Q_8	$t_0^4 + t_1^4 + at_0^2t_1^2$	$bt_0t_1(t_0^4 - t_1^4)$
XVI	8	2×4	$at_0^4 + t_1^4$	$t_0^2(bt_0^4 + ct_1^4)$
XVII	8	D_8	$t_0^4 + t_1^4 + at_0^2t_1^2$	$t_0t_1(b(t_0^4 + t_1^4) + ct_0^2t_1^2)$
XVIII	6	6	0	$a_6(t_0, t_1)$
XIX	6	6	$t_0(at_0^3 + bt_1^3)$	$ct_0^6 + dt_0^3t_1^3 + t_1^6$
XX	4	4	$g_2(t_0^2, t_1^2)$	$t_0t_1f_2(t_0^2, t_1^2)$
XXI	4	2^2	$g_2(t_0^2, t_1^2)$	$g_3(t_0^2, t_1^2)$
XXII	2	2	$a_4(t_0, t_1)$	$a_6(t_0, t_1)$

Table 8.17 *Groups of automorphisms of del Pezzo surfaces of degree 1*

The parameters here satisfy some conditions in order for the different types not to overlap.

Exercises

8.1 Show that a del Pezzo surface of degree 8 in $\mathbb{P}^8$ isomorphic to $\mathbf{F}_0$ is projectively isomorphic to the image of $\mathbb{P}^2$ defined by the linear system of plane quartic curves with two fixed double points.

8.2 Let S be a weak del Pezzo surface of degree 6. Show that its anti-canonical model is isomorphic to a hyperplane section of the Segre variety $\mathsf{s}_{1,1,1}(\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1)$ in $\mathbb{P}^7$.

8.3 Show that a general point in $\mathbb{P}^6$ is contained in three secants of a del Pezzo surface of degree six.

8.4 Prove that a del Pezzo surface of degree six in $\mathbb{P}^6$ has the property that all hyperplanes intersecting the surface along a curve with a singular point of multiplicity ≥ 3 have a common point in $\mathbb{P}^6$. According to [762] this distinguishes this surface among all other smooth projections of the Veronese surface $\mathsf{V}_3^2 \subset \mathbb{P}^9$ to $\mathbb{P}^6$ (see [781]).

8.5 Describe all weak del Pezzo surfaces which are toric varieties (i.e. contain an open Zariski subset isomorphic to the torus $(\mathbb{C}^*)^2$ such that each translation of the torus extends to an automorphism of the surface).

8.6 Show that a del Pezzo surface of degree 5 embeds into $\mathbb{P}^1 \times \mathbb{P}^2$ as a hypersurface of bidegree $(1, 2)$.

8.7 Show that a canonical curve of genus 6 in $\mathbb{P}^5$ lies on a unique del Pezzo quintic surface [41], [711].

8.8 Consider a nonsingular del Pezzo surface S of degree 5 in $\mathbb{P}^5$ as the variety of lines intersecting five planes spanning a 3-dimensional space in the Plücker space. Prove that the pencil of hyperplanes through each of the planes cuts out on S a pencil of conics.

8.9 Show that the Petersen graph of ten lines on a del Pezzo quintic surface contains 12 pentagons and each pentagon represents five lines contained in a hyperplane.

8.10 Show that the union of tangent planes to a nonsingular del Pezzo surface S of degree $d \geq 5$ in $\mathbb{P}^d$ not isomorphic to a quadric is a hypersurface of degree $4(d-3)$ which is singular along S with multiplicity 4 [275],[29], vol. 6, p.275.

8.11 Show that the quotient of a nonsingular quadric by an involution with four isolated fixed points is isomorphic to a quartic del Pezzo surface with four nodes.

8.12 Let S be a quartic del Pezzo surface obtained by blowing up five points in the plane. Show that there exists a projective isomorphism from the conic containing the five points and the pencil of quadrics whose base locus is an anti-canonical model of S such that the points are sent to singular quadrics.

8.13 Show that the Wiman pencil of four-nodal plane sextic curves contains two 10-nodal rational curves [274].

8.14 Show that the linear system of quadrics in $\mathbb{P}^3$ with $8 - d$ base points in general position maps $\mathbb{P}^3$ onto a 3-fold in $\mathbb{P}^{d+1}$ of degree d. Show that a del Pezzo surface of degree $d \leq 8$ in $\mathbb{P}^d$ is projectively equivalent to a hyperplane section of this threefold.

8.15 Show that the projection of a del Pezzo surface of degree d in $\mathbb{P}^d$ from a general point in the space is a surface of degree d in $\mathbb{P}^{d-1}$ with the double curve of degree $d(d-3)/2$.

8.16 Compute the number of (-1)-curves on a weak del Pezzo surfaces of degree one or two.

8.17 Let X be a Bordiga surface obtained by the blow-up of ten general points in the plane and embedded in $\mathbb{P}^4$ by the linear system of quartic curves passing through the ten points. Show that X is a OADP surface.

8.18 Let X be a rational elliptic surface. Show that any pair of two disjoint sections defines an involution on X whose fixed locus is a nonsingular curve of genus 3 and the quotient by the involution is isomorphic to the ruled surface $\mathbf{F}_1$.

Historical Notes

As the name suggests, P. del Pezzo was the first who laid the foundation of the theory. In his paper of 1887 [215], he proves that a non-ruled nondegenerate surface of degree d in $\mathbb{P}^d$ can be birationally projected to a cubic surface in $\mathbb{P}^3$

from $d-3$ general points on it. del Pezzo showed that the images of the tangent planes at the points are skew lines on the cubic surface and deduced from this that $d \leq 9$. He also gave a blow-up model of del Pezzo surfaces of degree $d \geq 3$, found the number of lines, and studied some singular surfaces. A realization of a del Pezzo surface of degree 5 as the variety of planes in $\mathbb{P}^4$ intersecting five planes is due to C. Segre [690]. He called the five planes the associated planes. The quartic cyclide surfaces in $\mathbb{P}^3$ with a nodal conic were first studied in 1864 by G. Darboux [199] and M. Moutard [530] and a year later by E. Kummer [472]. The detailed exposition of Darboux's work can be found in [201], [202]. Some special types of these surfaces were considered much earlier by Ch. Dupin [264]. Kummer was the first to observe the existence of five quadratic cones whose tangent planes cut out two conics on the surface (the *Kummer cones*). They correspond to the five singular quadrics in the pencil defining the corresponding quartic surface in $\mathbb{P}^4$. A. Clebsch finds a plane representation of a quartic cyclide by considering a web of cubics through five points in the plane [145]. He also finds in this way the configuration of 16 lines previously discovered by Darboux and proves that the Galois group of the equation for the 16 lines is isomorphic to $2^4 \rtimes \mathfrak{S}_5$. An 'epoch-making memoir' (see [701], p. 141) of C. Segre [685] finishes the classification of quartic cyclides by considering them as projections of a quartic surface in $\mathbb{P}^4$. Jessop's book [430] contains a good exposition of the theory of singular quartic surfaces, including quartic cyclide surfaces. At the same time, Segre classified the anti-canonical models of singular del Pezzo surfaces of degree 4 in terms of the pencil of quadrics they are defined by. The Segre symbol describing a pencil of quadratic forms was introduced earlier by A. Weiler [802]. The theory of canonical forms of pencils of quadrics was developed by K. Weierstrass [801] based an earlier work of J. Sylvester [738]. J. Steiner was probably the first who related seven points in the plane with curves of genus 3 by proving that the locus of singular points of the net of cubic curves is a plane sextic with nodes at the seven points [724]. A. Clebsch should be considered as a founder of the theory of del Pezzo surfaces of degree 2. In his memoir [148] on rational double planes he considers a special case of double planes branched along a plane quartic curve. He shows that the pre-images of lines are cubic curves passing through a fixed set of seven points. He identifies the branch curve with the Steiner sextic and relates the Aronhold set of seven bitangents with the seven base points. Although C. Geiser was the first to discover the involution defined by the double cover, he failed to see the double plane construction.

E. Bertini, in [55], while describing his birational involution of the plane, proves that the linear system of curves of degree 6 with eight double base points has the property that any curve from the linear system passing through a general point x must also pass through a unique point x' (which are in the Bertini involution). He mentions that the same result was proved independently by L. Cremona. This can be interpreted by saying that the linear system defines a rational map of degree 2 onto a quadric surface. Bertini also shows that the set of fixed points of the involution is a curve of degree 9 with triple points at the base points.

The classification of double singular points on algebraic surfaces in $\mathbb{P}^3$ started from the work of G. Salmon [645] who introduced the following notation C_2 for an ordinary node, B_k for *binode* (the tangent cone is the union of two different planes), which depend on how the intersection of the planes intersect the surface, and *unode* U_k with the tangent cone being a double plane. The indices here indicate the difference k between the degree of the dual surface and the dual of the nonsingular surface of the same degree. This nomenclature can be applied to surfaces in spaces of arbitrary dimension if the singularity is locally isomorphic to the singularities in above. For del Pezzo surfaces the defect k cannot exceed 8 and all corresponding singularities must be rational double points of types $A_1 = C_2$, $A_{k-1} = B_k$, $D_{k-2} = U_k, k = 6, 7$, $E_6 = U_8$. Much later, P. Du Val [267] characterized these singularities as ones that do not affect the conditions on adjunctions, the conditions that can be applied to any normal surface. He showed that each RDP is locally isomorphic to either a node C_2, or binode B_k, or unode U_k, or other unodes $U_8^* = E_6, U_9^* = E_7$ and $U_{10}^* = E_8$ (he renamed U_8 with U_8^*). A modern treatment of RDP singularities was given by M. Artin [24].

In the same series of papers, P. Du Val classifies all possible singularities of anti-canonical models of weak del Pezzo surfaces of any degree and relates them to Coxeter's classification of finite reflection groups. The relationship of this classification to the study of the singular fibers of a versal deformation of a simple elliptic singularities was found by J. Mérindol [514], H. Pinkham [584], [771], and E. Looijenga (unpublished).

In a fundamental paper by G. Timms [750] one can find a detailed study of the hierarchy of del Pezzo surfaces obtained by projections from a Veronese surface of degree 9. In this way, he finds all possible configurations of lines and singularities. Possible projections of a nonsingular del Pezzo surface from a point outside the surface were studied by H. Baker [29, Vol. 6, p. 275].

The Weyl group $W(\mathsf{E}_6)$ and $W(\mathsf{E}_7)$ as the Galois group of 27 lines on a nonsingular cubic surface and the group of 28 bitangents of a nonsingular plane quartic were first studied by C. Jordan [432]. These groups are discussed in many classical textbooks on algebra (e.g. [799, B. II], [80]). S. Kantor [438] realized the Weyl groups $W(\mathsf{E}_n)$ as groups of linear transformations preserving a quadratic form of signature $(1, n)$ and a linear form. This led him to describe the characteristic matrices of Cremona transformations with less than nine fundamental points. A. Coble extended this to any number N of fundamental points that gave the first occurrence of infinite Coxeter groups. In fact, Coble showed that the groups are reflections groups generated by the symmetric group $\mathfrak{S}_N$ and one reflection corresponding to the standard quadratic Cremona transformation [155, Part 2]. Earlier, W. Burnside described the same generation of the finite Weyl groups $W(E_6)$ and $W(E_7)$ [79].

We refer to [70] for the history of Weyl groups, reflection groups and root systems. These parallel directions of study of Weyl groups have been reconciled only recently.

The Gosset polytopes were discovered in 1900 by T. Gosset [350]. The notation n_{21} belongs to him. They were later rediscovered by E. Elte and H. S. M. Coxeter (see [178]) but only Coxeter realized that their groups of symmetries are reflection groups. The relationship between the Gosset polytopes n_{21} and curves on del Pezzo surfaces of degree $5 - n$ was found by Du Val [266]. In the case of $n = 2$, it goes back to [662]. The fundamental paper of Du Val is the origin of a modern approach to the study of del Pezzo surfaces by means of root systems of finite-dimensional Lie algebras [217], [504].

We refer to modern texts on del Pezzo surfaces [701], [504], [217], [463].

9

Cubic Surfaces

9.1 Lines on a Nonsingular Cubic Surface

9.1.1 The E_6-lattice

Let us study the lattice $\mathsf{I}^{1,6}$ and its sublattice E_6 in more detail.

Definition 9.1.1. *A* sixer *in* $\mathsf{I}^{1,6}$ *is a set of six mutually orthogonal exceptional vectors in* $\mathsf{I}^{1,6}$.

An example of a sixer is the set $\{\mathbf{e}_1, \dots, \mathbf{e}_6\}$.

Lemma 9.1.2. *Let* $\{v_1, \dots, v_6\}$ *be a sixer. Then, there exists a unique root* α *such that*

$$(v_i, \alpha) = 1, \quad i = 1, \dots, 6.$$

Moreover, $(w_1, \dots, w_6) = (r_\alpha(v_1), \dots, r_\alpha(v_6))$ *is a sixer satisfying*

$$(v_i, w_j) = 1 - \delta_{ij}.$$

The root associated to $(w_1, \dots, w_6)$ *is equal to* $-\alpha$.

Proof The uniqueness is obvious since $v_1, \dots, v_6$ are linearly independent, so no vector is orthogonal to all of them. Let

$$v_0 = \frac{1}{3}(-\mathbf{k}_6 + v_1 + \cdots + v_6) \in \mathbb{R}^{1,6}.$$

Let us show that $v_0 \in \mathsf{I}^{1,6}$. Since $\mathsf{I}^{1,6}$ is a unimodular lattice, it suffices to show that (v_0, v) is an integer for all $v \in \mathsf{I}^{1,6}$. Consider the sublattice N of $\mathsf{I}^{1,6}$ spanned by $v_1, \dots, v_6, \mathbf{k}_6$. We have $(v_0, v_i) = 0, i > 0$, and $(v_0, \mathbf{k}_6) = -3$. Thus, $(v_0, \mathsf{I}^{1,6}) \subset 3\mathbb{Z}$. By computing the discriminant of N, we find that it is equal to 9. By Lemma 8.2.1 N is a sublattice of index 3 of $\mathsf{I}^{1,6}$. Hence, for any $x \in \mathsf{I}^{1,6}$

we have $3x \in N$. This shows that

$$(v_0, x) = \frac{1}{3}(v_0, 3x) \in \mathbb{Z}.$$

Now, let us set

$$\alpha = 2v_0 - v_1 - \cdots - v_6. \tag{9.1}$$

We check that α is a root, and $(\alpha, v_i) = 1, i = 1, \ldots, 6$.

Since r_α preserves the symmetric bilinear form, $\{w_1, \ldots, w_6\}$ is a sixer. We have

$$(v_i, w_j) = \big(v_i, r_\alpha(v_j)\big) = \big(v_i, v_j + (v_j, \alpha)\alpha\big) = (v_i, v_j) + (v_i, \alpha)(v_j, \alpha)$$
$$= (v_i, v_j) + 1 = 1 - \delta_{ij}.$$

Finally, we check that

$$(r_\alpha(v_i), -\alpha) = \big(r_\alpha^2(v_i), -r_\alpha(\alpha)\big) = -(v_i, \alpha) = 1.$$

□

The two sixers with opposite associated roots form a *double-six* of exceptional vectors.

We recall the list of exceptional vectors in E_6 in terms of the standard orthonormal basis in $\mathsf{I}^{1,6}$.

$$\mathbf{a}_i = \mathbf{e}_i, \quad i = 1, \ldots, 6; \tag{9.2}$$
$$\mathbf{b}_i = 2\mathbf{e}_0 - \mathbf{e}_1 - \cdots - \mathsf{E}_6 + \mathbf{e}_i, \quad i = 1, \ldots, 6; \tag{9.3}$$
$$\mathbf{c}_{ij} = \mathbf{e}_0 - \mathbf{e}_i - \mathbf{e}_j, \quad 1 \le i < j \le 6. \tag{9.4}$$

Theorem 9.1.3. *The following is the list of 36 double-sixes with corresponding associated roots.*

1 of type D

$$\begin{matrix} \mathbf{a}_1 & \mathbf{a}_2 & \mathbf{a}_3 & \mathbf{a}_4 & \mathbf{a}_5 & \mathbf{a}_6 & & \alpha_{\max} \\ \mathbf{b}_1 & \mathbf{b}_2 & \mathbf{b}_3 & \mathbf{b}_4 & \mathbf{b}_5 & \mathbf{b}_6 & & -\alpha_{\max} \end{matrix},$$

15 of type D_{ij}

$$\begin{matrix} \mathbf{a}_i & \mathbf{b}_i & \mathbf{c}_{jk} & \mathbf{c}_{jl} & \mathbf{c}_{jm} & \mathbf{c}_{jn} & & \alpha_{ij} \\ \mathbf{a}_j & \mathbf{b}_j & \mathbf{c}_{ik} & \mathbf{c}_{il} & \mathbf{c}_{im} & \mathbf{c}_{in} & & -\alpha_{ij} \end{matrix},$$

20 of type D_{ijk}

$$\begin{matrix} \mathbf{a}_i & \mathbf{a}_j & \mathbf{a}_k & \mathbf{c}_{lm} & \mathbf{c}_{mn} & \mathbf{c}_{ln} & & \alpha_{ijk} \\ \mathbf{c}_{jk} & \mathbf{c}_{ik} & \mathbf{c}_{ij} & \mathbf{b}_n & \mathbf{b}_l & \mathbf{b}_m & & -\alpha_{ijk} \end{matrix}.$$

Here, $\boldsymbol{\alpha}_{\max}$ *is the maximal root of the root system* $\boldsymbol{\alpha}_1, \ldots, \boldsymbol{\alpha}_6$ *equal to* $2\mathbf{e}_0 -$

$\mathbf{e}_1 - \cdots - \mathsf{E}_6$. *The reflection with respect to the associated root interchanges the rows, preserving the order.*

Proof We have constructed a map from the set of sixers (resp. double-sixes) to the set of roots (resp. pairs of opposite roots). Let us show that no two sixers $\{v_1, \ldots, v_6\}$ and $\{w_1, \ldots, w_6\}$ can define the same root. Since $w_1, \ldots, w_6, \mathbf{k}_6$ span a sublattice of finite index in $\mathsf{I}^{1,6}$, we can write

$$v_i = \sum_{j=1}^{6} a_j w_j + a_0 \mathbf{k}_6 \tag{9.5}$$

with some $a_j \in \mathbb{Q}$. Assume that $v_i \neq w_j$ for all j. Taking the inner product of both sides with α, we get

$$1 = a_0 + \cdots + a_6. \tag{9.6}$$

Taking the inner product with $-\mathbf{k}_6$, we get $1 = a_1 + \cdots + a_6 - 3a_0$, hence $a_0 = 0$. Taking the inner product with w_j, we obtain $-a_j = (v_i, w_j)$. Applying Proposition 10.34, we get $a_j \leq -1$. This contradicts (9.6). Thus, each v_i is equal to some w_j.

The verification of the last assertion is straightforward. □

Proposition 9.1.4. *The group* $W(\mathsf{E}_6)$ *acts transitively on the sets of sixers and double-sixes. The stabilizer subgroup of a sixer (resp. double-six) is of order* $6!$ *(*$2 \cdot 6!$*).*

Proof We know that the Weyl group $W(\mathbf{E}_N)$ acts transitively on the set of roots and the number of sixers is equal to the number of roots. This shows that all sixers form one orbit. The stabilizer subgroup of the sixer $(\mathbf{a}_1, \ldots, \mathbf{a}_6)$ (and hence of a root) is the group $\mathfrak{S}_6$. The stabilizer of the double-six D is the subgroup $\langle \mathfrak{S}_6, s_{\alpha_0} \rangle$ of order 2.6!. □

One can check that two different double-sixes can share either four or six exceptional vectors. More precisely, we have

$$\#D \cap D_{ij} = 4, \; \#D \cap D_{ijk} = 6,$$

$$\#D_{ij} \cap D_{kl} = \begin{cases} 4 & \text{if } \#\{i,j\} \cap \{k,l\} = 0, \\ 6 & \text{otherwise;} \end{cases}$$

$$\#D_{ij} \cap D_{klm} = \begin{cases} 4 & \text{if } \#\{i,j\} \cap \{k,l,m\} = 0, 2, \\ 6 & \text{otherwise;} \end{cases}$$

$$\#D_{ijk} \cap D_{lmn} = \begin{cases} 4 & \text{if } \#\{i,j\} \cap \{k,l\} = 1, \\ 6 & \text{otherwise.} \end{cases}$$

A pair of double-sixes is called a *syzygetic duad* (resp. *azygetic duad*) if they have four (resp. six) exceptional vectors in common.

The next Lemma is an easy computation.

Lemma 9.1.5. *Two double-sixes with associated roots α, β form a syzygetic duad if and only if $(\alpha, \beta) \in 2\mathbb{Z}$.*

This can be interpreted as follows. Consider the vector space

$$V = \mathsf{E}_6/2\mathsf{E}_6 \cong \mathbb{F}_2^6 \tag{9.7}$$

equipped with the quadratic form

$$q(x + 2\mathsf{E}_6) = \tfrac{1}{2}(x, x) \mod 2.$$

Since E_6 is an even lattice, the definition makes sense. The associated symmetric bilinear form is the symplectic form

$$(x + 2\mathsf{E}_6, y + 2\mathsf{E}_6) = (x, y) \mod 2.$$

Each pair of opposite roots $\pm\alpha$ defines a vector v in V with $q(v) = 1$. It is easy to see that the quadratic form q has the Arf-invariant equal to 1 and hence vanishes on 28 vectors. The remaining 36 vectors correspond to 36 pairs of opposite roots or, equivalently, double-sixes.

Note that we have a natural homomorphism of groups

$$W(\mathsf{E}_6) \cong O(6, \mathbb{F}_2)^- \tag{9.8}$$

obtained from the action of $W(\mathsf{E}_6)$ on V. It is an isomorphism. This is checked by verifying that the automorphism $v \mapsto -v$ of the lattice E_6 does not belong to the Weyl group W and then comparing the known orders of the groups.

It follows from above that a syzygetic pair of double-sixes corresponds to orthogonal vectors v, w. Since $q(v + w) = q(v) + q(w) + (v, w) = 0$, we see that each nonzero vector in the isotropic plane s panned by v, w comes from a double-six.

A triple of pairwise syzygetic double-sixes is called a *syzygetic triad* of double-sixes. They span an isotropic plane. Similarly, we see that a pair of azygetic double-sixes spans a non-isotropic plane in V with three nonzero vectors corresponding to a triple of double-sixes which are pairwise azygetic. It is called an *azygetic triad* of double-sixes.

We say that three azygetic triads form a *Steiner complex of triads of double-sixes* if the corresponding planes in V are mutually orthogonal. It is easy to see that an azygetic triad contains 18 exceptional vectors and thus defines a set of nine exceptional vectors (the omitted ones). The set of 27 exceptional vectors omitted from three triads in a Steiner complex is equal to the set of 27 exceptional vectors in the lattice $I^{1,6}$. There are 40 Steiner complexes of triads:

10 of type

$$\ell_{ijk,lmn} = (D, D_{ijk}, D_{lmn}),\ (D_{ij}, D_{ik}, D_{jk}),\ (D_{lm}, D_{ln}, D_{mn}),$$

30 of type

$$\ell_{ij,kl,mn} = (D_{ij}, D_{ikl}, D_{jkl}),\ (D_{kl}, D_{kmn}, D_{lmn}),\ (D_{mn}, D_{mij}, D_{nij}).$$

Theorem 9.1.6. *The Weyl group $W(\mathsf{E}_6)$ acts transitively on the set of triads of azygetic double-sixes with stabilizer subgroup isomorphic to the group $\mathfrak{S}_3 \times (\mathfrak{S}_3 \wr \mathfrak{S}_2)$ of order 432. It also acts transitively on Steiner complexes of triads of double-sixes. A stabilizer subgroup is a maximal subgroup of $W(\mathsf{E}_6)$ of order* 1296 *isomorphic to the wreath product $\mathfrak{S}_3 \wr \mathfrak{S}_3$.*

Proof We know that a triad of azygetic double-sixes corresponds to a pair of roots (up to replacing the root with its negative) α, β with $(\alpha, \beta) = \pm 1$. This pair spans a root sublattice Q of E_6 of type A_2. Fix a root basis. Since the Weyl group acts transitively on the set of roots, we find $w \in W$ such that $w(\alpha) = \boldsymbol{\alpha}_{\max}$. Since $(w(\beta), \boldsymbol{\alpha}_{\max}) = (\beta, \alpha) = 1$, we see that $w(\beta) = \pm \boldsymbol{\alpha}_{ijk}$ for some i, j, k. Applying elements from $\mathfrak{S}_6$, we may assume that $w(\beta) = -\boldsymbol{\alpha}_{123}$. Obviously, the roots $\boldsymbol{\alpha}_{12}, \boldsymbol{\alpha}_{23}, \boldsymbol{\alpha}_{45}, \boldsymbol{\alpha}_{56}$ are orthogonal to $w(\alpha)$ and $w(\beta)$. These roots span a root sublattice of type $2A_2$. Thus, we obtain that the orthogonal complement of Q in E_6 contains a sublattice of type $2A_2 \perp A_2$. Since $|\mathrm{disc}(A_2)| = 3$, it follows easily from Lemma 8.2.1 that $Q^\perp$ is a root lattice of type $A_2 + A_2$ ($2A_2$, for short). Obviously, any automorphism Wthe two roots α, β invariant leaves invariant the sublattice Q and its orthogonal complement $Q^\perp$. Thus, the stabilizer contains a subgroup isomorphic to $W(A_2) \times W(A_2) \times W(A_2)$ and the permutation of order 2 which switches the two copies of A_2 in $Q^\perp$. Since $W(A_2) \cong \mathfrak{S}_3$ we obtain that a stabilizer subgroup contains a subgroup of order $2 \cdot 6^3 = 432$. Since its index is equal to 120, it must coincide with the stabilizer group.

It follows from above that a Steiner complex corresponds to a root sublattice of type $3A_2$ contained in E_6. The group $W(A_2) \wr \mathfrak{S}_3$ of order $3 \cdot 432$ is contained in the stabilizer. Since its index is equal to 40, it coincides with the stabilizer. □

Remark 9.1.7. The notions of syzygetic (azygetic) pairs, triads and a Steiner complex of triads of double-sixes is analogous to the notions of syzygetic (azygetic) pairs, triads, and a Steiner complex of bitangents of a plane quartic (see Chapter 6). In both cases we deal with a 6-dimensional quadratic space $\mathbb{F}_2^6$. However, they have different Arf invariants.

A triple v_1, v_2, v_3 of exceptional vectors is called a *tritangent trio* if

$$v_1 + v_2 + v_3 = -\mathbf{k}_6.$$

If we view exceptional vectors as cosets in $I^{1,6}/\mathbb{Z}\mathbf{k}_6$, this is equivalent to saying that the cosets add up to zero.

It is easy to list all tritangent trios.

Lemma 9.1.8. *There are 45 tritangent trios:*

30 of type

$$\mathbf{a}_i, \mathbf{b}_j, \mathbf{c}_{ij}, \quad i \neq j,$$

15 of type

$$\mathbf{c}_{ij}, \mathbf{c}_{kl}, \mathbf{c}_{mn}, \quad \{i, j\} \cup \{k, l\} \cup \{m, n\} = \{1, 2, 3, 4, 5, 6\}.$$

Theorem 9.1.9. *The Weyl group acts transitively on the set of tritangent trios.*

Proof We know that the permutation subgroup $\mathfrak{S}_6$ of the Weyl group acts on tritangent trios by permuting the indices. Thus, it acts transitively on the set of tritangent trios of the same type. Now, consider the reflection w with respect to the root α_{123}. We have

$$r_{\alpha_{123}}(\mathbf{a}_1) = \mathbf{e}_1 + \alpha_{123} = \mathbf{e}_0 - \mathbf{e}_3 - \mathbf{e}_4 = \mathbf{c}_{34},$$
$$r_{\alpha_{123}}(\mathbf{b}_2) = (2\mathbf{e}_0 - \mathbf{e}_1 - \mathbf{e}_3 - \mathbf{e}_4 - \mathbf{e}_5 - \mathsf{E}_6) - \alpha_{123} = \mathbf{e}_0 - \mathbf{e}_5 - \mathsf{E}_6 = \mathbf{c}_{56},$$
$$r_{\alpha_{123}}(\mathbf{c}_{12}) = \mathbf{e}_0 - \mathbf{e}_1 - \mathbf{e}_2 = \mathbf{c}_{12}.$$

Thus, $w(\mathbf{a}_1, \mathbf{b}_2, \mathbf{c}_{12}) = (\mathbf{c}_{34}, \mathbf{c}_{56}, \mathbf{c}_{12})$. This proves the assertion. □

Remark 9.1.10. The stabilizer subgroup of a tritangent trio is a maximal subgroup of $W(\mathsf{E}_6)$ of index 45 isomorphic to the Weyl group of the root system of type F_4.

Let $\Pi_1 = \{v_1, v_2, v_3\}$ and $\Pi_2 = \{w_1, w_2, w_3\}$ be two tritangent trios with no common elements. We have

$$(v_i, w_1 + w_2 + w_3) = -(v_i, \mathbf{k}_6) = 1$$

and, by Proposition 10.34, $(v_i, w_j) \geq 0$. This implies that there exists a unique j such that $(v_i, w_j) = 1$. After reordering, we may assume $j = i$. Let $u_i =$

$-k_6 - v_i - w_i$. Since $u_i^2 = -1, (u_i, k_6) = -1$, the vector u_i is an exceptional vector. Since

$$u_1 + u_2 + u_3 = \sum_{i=1}^{3}(-k_6 - v_i - w_i) = -3k_6 - \sum_{i=1}^{3} v_i - \sum_{i=1}^{3} w_i = -\mathbf{k}_6,$$

we get a new tritangent trio $\Pi_3 = (u_1, u_2, u_3)$. The union $\Pi_1 \cup \Pi_2 \cup \Pi_3$ contains nine lines $v_i, w_i, u_i, i = 1, 2, 3$. There is a unique triple of tritangent trios that consists of the same nine lines. It is formed by tritangent trios $\Pi_i' = (v_i, w_i, u_i), i = 1, 2, 3$. Any pair of triples of tritangent trios that consists of the same set of nine lines is obtained in this way. Such a pair of triples of tritangent trios is called a pair of *conjugate triads of tritangent trios*.

We can list all conjugate pairs of triads of tritangent trios:

$$(I)\ \begin{matrix} \mathbf{a}_i & \mathbf{b}_j & \mathbf{c}_{ij} \\ \mathbf{b}_k & \mathbf{c}_{jk} & \mathbf{a}_j \\ \mathbf{c}_{ik} & \mathbf{a}_k & \mathbf{b}_i \end{matrix}, \quad (II)\ \begin{matrix} \mathbf{c}_{ij} & \mathbf{c}_{kl} & \mathbf{c}_{mn} \\ \mathbf{c}_{ln} & \mathbf{c}_{im} & \mathbf{c}_{jk} \\ \mathbf{c}_{km} & \mathbf{c}_{jn} & \mathbf{c}_{il} \end{matrix}, \quad (III)\ \begin{matrix} \mathbf{a}_i & \mathbf{b}_j & \mathbf{c}_{ij} \\ \mathbf{b}_k & \mathbf{a}_l & \mathbf{c}_{kl} \\ \mathbf{c}_{ik} & \mathbf{c}_{jl} & \mathbf{c}_{mn} \end{matrix}. \quad (9.9)$$

Here, a triad is represented by the columns of the matrix and its conjugate triad by the rows of the same matrix. Altogether we have $20 + 10 + 90 = 120$ different conjugate pairs of triads.

There is a bijection from the set of pairs of conjugate triads to the set of azygetic triads of double-sixes. The 18 exceptional vectors contained in the union of the latter is the complementary set of the set of nine exceptional vectors defined by a triad in the pair. Here, is the explicit bijection.

$$\begin{matrix} \mathbf{a}_i & \mathbf{b}_j & \mathbf{c}_{ij} \\ \mathbf{b}_k & \mathbf{c}_{jk} & \mathbf{a}_j \\ \mathbf{c}_{ik} & \mathbf{a}_k & \mathbf{b}_i \end{matrix} \leftrightarrow D_{ij}, D_{ik}, D_{jk};$$

$$\begin{matrix} \mathbf{c}_{ij} & \mathbf{c}_{kl} & \mathbf{c}_{mn} \\ \mathbf{c}_{ln} & \mathbf{c}_{im} & \mathbf{c}_{jk} \\ \mathbf{c}_{km} & \mathbf{c}_{jn} & \mathbf{c}_{il} \end{matrix} \leftrightarrow D, D_{ikn}, D_{jlm};$$

$$\begin{matrix} \mathbf{a}_i & \mathbf{b}_j & \mathbf{c}_{ij} \\ \mathbf{b}_k & \mathbf{a}_l & \mathbf{c}_{kl} \\ \mathbf{c}_{ik} & \mathbf{c}_{jl} & \mathbf{c}_{mn} \end{matrix} \leftrightarrow D_{mn}, D_{jkm}, D_{jkn}.$$

Recall that the set of exceptional vectors omitted from each triad entering in a Steiner complex of triads of azygetic double-sixes is the set of 27 exceptional vectors. Thus, a Steiner complex defines three pairs of conjugate triads of tritangent trios which contains all 27 exceptional vectors. We have 40 such triples of conjugate pairs.

Theorem 9.1.11. *The Weyl group acts transitively on the set of 120 conjugate* *pairs of triads of tritangent trios. A stabilizer subgroup H is contained in the maximal subgroup of $W(\mathsf{E}_6)$ of index 40 realized as a stabilizer of a Steiner complex. The quotient group is a cyclic group of order 3.*

Proof This follows from the established bijection between pairs of conjugate triads and triads of azygetic double-sixes and Theorem 9.1.6. In fact it is easy to see directly the transitivity of the action. It is clear that the permutation subgroup $\mathfrak{S}_6$ acts transitively on the set of pairs of conjugate triads of the same type. Since the Weyl group acts transitively on the set of tritangent trios, we can send a tritangent trio $(\mathbf{c}_{ij}, \mathbf{c}_{kl}, \mathbf{c}_{mn})$ to a tritangent trio $(\mathbf{a}_i, \mathbf{b}_j, \mathbf{c}_{ij})$. By inspection, this sends a conjugate pair of type III to a pair of conjugate triads of type I. Also it sends a conjugate pair of type II to type I or III. Thus, all pairs are W-equivalent. □

Remark 9.1.12. Note that each monomial entering into the expression of the determinant of the matrix (9.9) expressing a conjugate pair of triads represents three orthogonal exceptional vectors. If we take only monomials corresponding to even (resp. odd) permutations we get a partition of the set of nine exceptional vectors into the union of three triples of orthogonal exceptional vectors such that each exceptional vector from one triple has a nonzero intersection with two exceptional vectors from any other triple.

9.1.2 Lines and tritangent planes

Let S be a nonsingular cubic surface in $\mathbb{P}^3$. Fix a geometric marking $\phi : I^{1,6} \to \mathrm{Pic}(S)$. We can transfer all the notions and the statements from the previous subsection to the Picard lattice $\mathrm{Pic}(S)$. The image of an exceptional vector is the divisor class of a line on S. So, we will identify exceptional vectors with lines on S. There are 27 lines. A tritangent trio of exceptional vectors defines a set of three coplanar lines. The plane containing them is called a *tritangent plane*. There are 45 tritangent planes.

Thus, we have 72 sixers of lines, 36 double-sixes, and 40 Steiner complexes of triads of double-sixes. If $e_0, e_1, \ldots, e_6$ define a geometric marking, then we can identify the divisor classes e_i with the exceptional curves of the blow-up $S \to \mathbb{P}^2$ of six points $x_1, \ldots, x_6$ in general position. They correspond to the exceptional vectors $\mathbf{a}_i$. We identify the proper transforms of the conic through the six points excluding the x_i with the exceptional vector $\mathbf{b}_i$. Finally, we identify the line through the points x_i and x_j with the exceptional vector $\mathbf{c}_{ij}$. Under the geometric marking, the Weyl group $W(\mathsf{E}_6)$ becomes isomorphic to the index two subgroup of the isometry group of $\mathrm{Pic}(S)$ leaving the canonical class invariant (see Corollary 8.2.17). It acts transitively on the set of lines, sixes, double-sixes, tritangent planes, and on the set of conjugate pairs of triples of tritangent planes.

An elementary geometric proof of the fact that any nonsingular cubic surface contains 27 lines can be found in [610]. The first proof of A. Cayley applies only to general nonsingular cubic surfaces. For completeness sake, let us reproduce the original proof of Cayley [104].

Theorem 9.1.13. *A general nonsingular cubic surface contains* 27 *lines and* 45 *tritangent planes.*

Proof First of all, let us show that any cubic surface contains a line. Consider the incidence variety

$$X = \{(S,\ell) \in |\mathcal{O}_{\mathbb{P}^3}(3)| \times G : \ell \subset S\}.$$

The assertion follows if we show that the first projection is surjective. It is easy to see that the fibers of the second projections are linear subspaces of codimension 4. Thus, $\dim X = 4 + 15 = 19 = \dim |\mathcal{O}_{\mathbb{P}^3}(3)|$. To show the surjectivity of the first projection, it is enough to find a cubic surface with only finitely many lines on it. Let us consider the surface S given by the equation

$$t_1t_2t_3 - t_0^3 = 0.$$

Suppose a line ℓ lies on S. Let $[a_0, a_1, a_2, a_3] \in \ell$. If $a_0 \neq 0$, then $a_i \neq 0, i \neq 0$. On the other hand, every line hits the planes $V(t_i)$. This shows that ℓ is contained in the plane $V(t_0)$. However, there are only three lines on S contained in this plane: $t_i = t_0 = 0, i = 1, 2, 3$. Therefore S contains only three lines. This proves the first assertion.

We already know that every cubic surface $S = V(f)$ has at least one line. Pick up such a line ℓ_0. Without loss of generality, we may assume that it is given by the equation:

$$t_2 = t_3 = 0.$$

Thus

$$f = t_2q_0(t_0, t_1, t_2, t_3) + t_3q_1(t_0, t_1, t_2, t_3) = 0, \tag{9.10}$$

where q_0 and q_1 are quadratic forms. The pencil of planes $\Pi_{\lambda,\mu} = V(\lambda t_2 - \mu t_3)$ through the line ℓ_0 cuts out a pencil of conics on S. The equation of the conic in the plane $\Pi_{\lambda,\mu}$ is

$$A_{00}(\lambda,\mu)t_0^2 + A_{11}(\lambda,\mu)t_1^2 + A_{22}(\lambda,\mu)t_2^2 +$$

$$2A_{01}(\lambda,\mu)t_0t_1 + 2A_{12}(\lambda,\mu)t_1t_2 + 2A_{02}(\lambda,\mu)t_0t_2 = 0,$$

where A_{00}, A_{11}, A_{01} are binary forms of degree 1, A_{02}, A_{12} are binary forms of degree 2 and A_{22} is a binary form of degree 3. The discriminant equation of this conic is equal to

$$\begin{vmatrix} A_{00} & A_{01} & A_{02} \\ A_{01} & A_{11} & A_{12} \\ A_{02} & A_{12} & A_{22} \end{vmatrix} = 0.$$

This is a homogeneous equation of degree five in variables λ, μ. Thus, we expect five roots of this equation which gives us five reducible conics. This is a tricky point because we do not know whether the equation has five distinct roots. First, we can exhibit a nonsingular cubic surface and a line on it and check that the equation indeed has five distinct roots. For example, let us consider the cubic surface

$$2t_0t_1t_2 + t_3(t_0^2 + t_1^2 + t_2^2 + t_3^2) = 0.$$

The equation becomes $\lambda(\lambda^4 - \mu^4) = 0$. It has five distinct roots. This implies that, for general nonsingular cubic surface, we have five reducible residual conics. Note that no conic is a double line since otherwise, the cubic surface is singular.

Thus, each solution of the quintic equation defines a tritangent plane Π_i of S consisting of three lines, one of them is ℓ_0. Thus, we found 11 lines on X: the line ℓ_0 and five pairs of lines ℓ_i, ℓ_i' lying in the plane Π_i. Pick up some plane, say Π_1. We have 3 lines ℓ_0, ℓ_1, ℓ_2 in Π_1. Replacing ℓ_0 by ℓ_1, and then by ℓ_2, and repeating the construction, we obtain four planes through ℓ_1 and four planes through ℓ_2 not containing ℓ_0 and each containing a pair of additional lines. Altogether we found $3 + 8 + 8 + 8 = 27$ lines on S. To see that all lines are accounted for, we observe that any line intersecting either ℓ_0, or ℓ_1, or ℓ_2 lies in one of the planes we have considered before. So, it has been accounted for. Now, let ℓ be any line. We find a plane Π through ℓ that contains three lines ℓ, ℓ' and ℓ'' on S. This plane intersects the plane containing $\ell,_0 \ell_1$, and ℓ_1' along a line. This line intersects S at some point on ℓ and on one of the lines ℓ_0, ℓ_1, ℓ_1'. Thus, ℓ intersects one of the lines ℓ_0, ℓ_1, ℓ_1' and has been accounted for.

It remains to count tritangent planes. Each line belongs to five tritangent planes, and each tritangent plane contains three lines. This easily shows that there are 45 tritangent planes. □

Remark 9.1.14. Reid's extension of Cayley's proof to any nonsingular surface uses some explicit computations. Instead, we may use that the number of singular conics in the pencil of conics residual to a line determines the topological Euler-Poincaré characteristic of the surface. Applying formula (8.26) for the Euler-Poincaré characteristic $e(S)$ of a fibered surface, we obtain $e(S) = 4 + s$, where s is the number of singular conics. Since any two nonsingular surfaces are homeomorphic (they are parameterized by an open subset of a projective space), we obtain that the number s is the same for all nonsingular surfaces. We know that $s = 5$ for the example above, hence $s = 5$ for all nonsingular surfaces. Also, we obtain $\chi(S) = 9$, which, of course, agrees with the fact that S is the blow-up of six points in the plane.

The closure of the effective cone $\overline{\mathrm{Eff}}(S)$ of a nonsingular cubic surface is isomorphic to the Gosset polytope $\Sigma_6 = 2_{21}$. It has 72 facets corresponding to sixes and 27 faces corresponding to conic bundles on S. In a geometric basis $(e_0, e_1, \ldots, e_6)$ they are expressed by the linear systems of types $|e_0 - e_1|, |2e_0 - e_1 - e_2 - e_3 - e_4|, |3e_0 - 2e_1 - e_2 - \cdots - e_6|$. The center of $\overline{\mathrm{Eff}}(S)$ is equal to $O = -\frac{1}{3}K_S = (e_1 + \cdots + e_{27})/27$, where $e_1, \ldots, e_{27}$ are the divisor classes of lines. A double-six represents two opposite facets whose centers lie on a line passing through O. In fact, if we consider the double-six $(e_i, e_i' = 2e_0 - e_1 - \cdots - e_6 + e_i), i = 1, \ldots, 6$, then

$$\frac{1}{12}(\sum_{i=1}^{6} e_i) + \frac{1}{12}\sum_{i=1}^{6} e_i' = -\frac{1}{3}K_S = O.$$

The line joining the opposite face is perpendicular to the facets. It is spanned by the root corresponding to the double-six. The three lines e_i, e_j, e_k in a tritangent plane add up to $-K_S$. This can be interpreted by saying that the center of the triangle with vertices e_i, e_j, e_k is equal to the center O. This easily implies that the three lines joining the center O with e_i, e_j, e_k are coplanar.

Remark 9.1.15. Let a_i, b_i, c_{ij} denote the set of 27 lines on a nonsingular cubic surface. Consider them as 27 unknowns. Let F be the cubic form in 27 variables equal to the sum of 45 monomials $a_i b_j c_{ij}, c_{ij} c_{kl} c_{mn}$ corresponding to tritangent planes. It was shown by E. Cartan that the group of projective automorphisms of the cubic hypersurface $V(F)$ in $\mathbb{P}^{26}$ is isomorphic to the simple complex Lie group of type E_6. We refer to [498] for integer models of this cubic.

9.1.3 Schur's quadrics

There are 36 double-sixes of lines on a nonsingular cubic surface S corresponding to 36 double-sixes of exceptional vectors in the lattice $I^{1,6}$. Let $((\ell_1, \ldots, \ell_6), (\ell_1', \ldots, \ell_6'))$ be one of them. Choose a geometric marking $\phi : I^{1,6} \to \mathrm{Pic}(S)$ such that $\phi(\mathbf{e}_i) = e_i = [\ell_i], i = 1, \ldots, 6$. The linear system $|e_0|$ defines a birational map $\pi_1 : S \to {}_1\mathbb{P}^2 = |e_0|^\vee$ which blows down the lines ℓ_i to points $x_1, \ldots, x_6$. The class of the line ℓ_i' is equal to $2e_0 - (e_1 + \cdots + e_6) + e_i$. Its image in the plane ${}_1\mathbb{P}^2$ is the conic C_i passing through all p_j except p_i.

Let $\phi' : I^{1,6} \to \mathrm{Pic}(S)$ be the geometric marking such that $\phi'(\mathbf{e}_i) = \ell'_i$. It is obtained from ϕ by composing ϕ with the reflection $s = s_{\alpha_{\max}} \in O(I^{1,6})$. We have

$$e'_0 = s(e_0) = e_0 + 2(2e_0 - e_1 - \cdots - e_6) = 5e_0 - 2e_1 - \cdots - 2e_6.$$

The linear system $|e'_0|$ defines a birational map $\pi' : S \to {}_2\mathbb{P}^2 = |e'_0|^\vee$ which blows down the lines ℓ'_i to points x'_i in ${}_2\mathbb{P}^2$. The Cremona transformation

$$T = \pi_2 \circ \pi_1^{-1} : {}_1\mathbb{P}^1 \dashrightarrow {}_2\mathbb{P}^2$$

is the symmetric Cremona transformation of degree 5. It is given by the homaloidal linear system $|\mathcal{I}^2_{x_1,\ldots,x_6}(5)|$. The P-locus of T consists of the union of the conics C_i. Note that the ordered sets of points $(x_1, \ldots, x_6)$ and $(x'_1, \ldots, x'_6)$ are not projectively equivalent.

Consider the map

$$_1\mathbb{P}^2 \times {}_2\mathbb{P}^2 = |e_0| \times |e'_0| \to |e_0 + e'_0| = |-2K_S| \cong |\mathcal{O}_{\mathbb{P}^3}(2)| \cong \mathbb{P}^9. \qquad (9.11)$$

It is isomorphic to the Segre map $\mathsf{s}_{2,2} : \mathbb{P}^2 \times \mathbb{P}^2 \to \mathbb{P}^8$, and its image is a hyperplane H in the space of quadrics in $\mathbb{P}^3$. Let Q be the unique quadric in the dual space of quadrics which is apolar to H.

The following beautiful result belongs to F. Schur [672].

Theorem 9.1.16 (F. Schur)**.** *The quadric Q is nonsingular. The polar of each line ℓ_i with respect to the dual quadric $Q^\vee$ is equal to ℓ'_i. The quadric $Q^\vee$ is uniquely determined by this property.*

Proof Let $(\ell_1, \ldots, \ell_6)$ and $(\ell'_1, \ldots, \ell'_6)$ form a double-six. We use the notations a_i, b_j, c_{ij} (resp. (a_i, b'_j, c'_{ij})) for lines defined by the geometric basis $(e_0, \ldots, e_6)$ (resp. $(e'_0, \ldots, e'_6)$). The divisor class of the sum of six lines a_i, a_j, c_{ij} and a'_i, b'_k, c'_{ik} is equal to

$$e_i + e_j + (e_0 - e_i - e_j) + 2e_0 - (e_1 + \cdots + e_6 - e_j) + 2e_0 - (e_1 + \cdots + e_6 - e_k) +$$

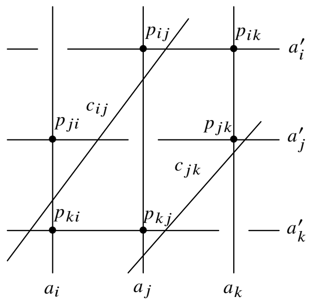
$$(e_0 - e_j - e_k) = 6e_0 - 2(e_1 + \cdots + e_6) = -2K_S.$$

The corresponding quadric Q_{ijk} cuts out six lines distributed into two triples of coplanar lines a_i, a'_j, c_{jk} and a_j, a'_k, c'_{jk}. Thus, Q_{ijk} consists of the union of two planes H_{ij} and H_{jk} (note that k here could be equal to i). This implies that, considered as points in the dual space, the polar plane $(H_{ij})^\perp_Q$ of H_{ij} with respect to Q contains H_{jk}. Let p_{ij} be the point in the original space $\mathbb{P}^3$ which corresponds to the hyperplane $(H_{ij})^\perp_Q$ in the dual projective space. Then,

$H_{ij} \subset (p_{ij})^{\perp}_{Q^\vee}$, or, $p_{ij} \in (H_{ij})^{\perp}_{Q^\vee}$. The inclusion $H_{ij} \in (H_{jk})^{\perp}_{Q}$ means that $p_{ij} \in H_{jk}$. Since $H_{bc} \in (H_{ab})^{\perp}_{Q}$ for any three indices a, b, c, we get

$$p_{ij} \in H_{jk} \cap H_{ji} \cap H_{ki} = \{a_j + a'_k + c_{jk}\} \cap \{a_j + a'_i + c_{ji}\} \cap \{a_k + a'_i + c_{ki}\}.$$

The point $a_j \cap a'_i$ belongs to the intersection. Since no three tritangent planes intersect along a line, we obtain that $p_{ij} = a_j \cap a'_i$, and, similarly, $p_{ji} = a_i \cap a'_j$. Now, we use that $p_{ji} \in (H_{ji})^{\perp}_{Q^\vee}$ and $p_{ij} \in (H_{ij})^{\perp}_{Q^\vee}$. Since $a_i \in H_{ij}, a'_i \subset H_{ji}$, we obtain that the points p_{ij} and p_{ji} are orthogonal with respect to $Q^\vee$. Similarly, we find that the pairs p_{ki}, p_{ik} and p_{jk}, p_{jk} are orthogonal. Since a_i contains p_{ji}, p_{ki}, and a'_i contains p_{ik}, p_{ij}, we see that the lines a_i and a'_i are orthogonal with respect to $Q^\vee$.



Let us show that Q is a nondegenerate quadric. Suppose Q is degenerate, then its set of singular points is a non-empty linear space L_0. Thus, for any subspace L of the dual space of $\mathbb{P}^3$, the polar subspace $L^{\perp}_{Q}$ contains L_0. Therefore, all the points p_{ij} lie in a proper subspace of $\mathbb{P}^3$. However, this is impossible since some of these points lie on a pair of skew lines and span $\mathbb{P}^3$. Thus, the dual quadric $Q^\vee$ is nonsingular, and the lines ℓ_i, ℓ'_i are orthogonal with respect to $Q^\vee$.

Let us prove the uniqueness of $Q^\vee$. Suppose we have two quadrics Q_1 and Q_2 such that $\ell'_i = (\ell_i)^{\perp}_{Q_i}, i = 1, \ldots, 6$. Let Q be a singular quadric in the pencil spanned by Q_1 and Q_2. Let K be its space of singular points. Then, K is orthogonal to each subspace of $\mathbb{P}^3$. Hence, it is contained in ℓ'_i and ℓ_i. Since these lines are skew, we get a contradiction.

□

Definition 9.1.17. *Let* $(\ell_1, \ldots, \ell_6), (\ell'_1, \ldots, \ell'_6)$ *be a double-six of lines on a nonsingular cubic surface* S. *The unique quadric* Q *such that* $(\ell_i)^{\perp}_{Q} = \ell'_i$ *is called the* Schur quadric *with respect to the double-six.*

Consider the bilinear map corresponding to the pairing (9.11)

$$H^0(S, \mathcal{O}_S(e_0)) \times H^0(S, \mathcal{O}_S(5e_0 - 2e_1 - \cdots - 2e_6))$$
$$\to H^0(S, \mathcal{O}_S(-2K_S)) = H^0(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(2)).$$

A choice of an equation of the dual of the Schur quadric defines a linear map $H^0(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(2)) \to \mathbb{C}$. Composing the pairing with this map, we obtain an isomorphism

$$H^0(S, \mathcal{O}_S(5e_0 - 2e_1 - \cdots - 2e_6)) \cong H^0(S, \mathcal{O}_S(e_0))^\vee.$$

This shows that the Schur quadric allows us to identify the plane ${}_1\mathbb{P}^2$ and ${}_2\mathbb{P}^2$ as the dual to each other. Under this identification, the linear system $|-2K_S - e_0|$ defines an involutive Cremona transformation $\mathbb{P}^2 \dashrightarrow \mathbb{P}^2$.

Fix six points $x_1, \ldots, x_6 \in \mathbb{P}^2$ in general positions. The linear system $|6h - 2x_1 - \cdots - 2x_6|$ is equal to the pre-image of the linear system of quadrics in $\mathbb{P}^3 = |3e_0 - x_1 - \cdots - x_6|^\vee$ under the map $\mathbb{P}^2 \dashrightarrow \mathbb{P}^3$ given by the linear system $|3h - x_1 - \cdots - x_6|$. The pre-image of the Schur quadric corresponding to the double-six $(e_1, \ldots, e_6), (e_1', \ldots, e_6')$ is a curve of degree 6 with double points at $x_1, \ldots, x_6$. It is called the *Schur sextic* associated with six points. Note that it is defined uniquely by the choice of six points. The proper transform of the Schur sextic under the blow-up of the points is a nonsingular curve of arithmetic genus 4. In the anti-canonical embedding, it is the intersection of the Schur quadric with the cubic surface.

Proposition 9.1.18. *The six double points of the Schur sextic are biflexes, i.e., the tangent line to each branch is tangent to the branch with multiplicity* ≥ 3.

Proof Let Q be the Schur quadric corresponding to the Schur sextic and ℓ_i be the lines on the cubic surface S corresponding to the points $x_1, \ldots, x_6$. Let $\ell_i \cap Q = \{a, b\}$ and $\ell_i' \cap Q = \{a', b'\}$. We know that

$$P_a(Q) \cap Q = \{x \in Q : a \in \mathbb{T}_x(Q)\}.$$

Since $\ell_i' = (\ell_i)_Q^\perp$, we have

$$\ell_i' \cap Q = (P_a(Q) \cap P_b(Q)) \cap Q = \{a', b'\}.$$

This implies that $a', b' \in \mathbb{T}_a(Q)$ and hence the lines $\langle a, a' \rangle, \langle a, b' \rangle$ span $\mathbb{T}_a(Q)$. The tangent plane $\mathbb{T}_a(Q)$ contains the line ℓ_i' and hence intersects the cubic surface S along ℓ_i' and some conic $K(a)$. We have

$$\mathbb{T}_a(K(a)) = \mathbb{T}_a(S) \cap \mathbb{T}_a(Q) = \mathbb{T}_a(Q \cap S).$$

Thus, the conic $K(a)$ and the curve $C = Q \cap S$ are tangent at the point a. Since the line ℓ_i' is equal to the proper transform of the conic C' in $\mathbb{P}^2$ passing through the points $x_j, j \neq i$, the conic $K(a)$ is the proper transform of some line ℓ in the plane passing through x_i. The point a corresponds to the tangent direction at x_i defined by a branch of the Schur sextic at x_i. The fact that $K(a)$ is tangent to C at a means that the line ℓ is tangent to the branch with multiplicity ≥ 3. Since the same is true, when we replace a with b, we obtain that x_i is a biflex of the Schur sextic. □

Remark 9.1.19. A biflex is locally given by an equation whose Taylor expansion looks like $xy + xy(ax + by) + f_4(x, y) + \cdots$. This shows that one has to impose five conditions to get a biflex. To get six biflexes for a curve of degree 6 one has to satisfy 30 linear equations. The space of homogeneous polynomials of degree 6 in three variables has dimension 28. So, one does not expect that such sextics exist.

Also observe that the set of quadrics Q such that $\ell_Q^\perp = \ell'$ for a fixed pair of skew lines (ℓ, ℓ') is a linear (projective) subspace of codimension 4 of the 9-dimensional space of quadrics. So, the existence of the Schur quadric is unexpected!

I do not know whether, for a given set of six points on $\mathbb{P}^2$ defining a nonsingular cubic surface, there exists a unique sextic with biflexes at these points. We refer to [238], where the Schur sextic is realized as the curve of jumping lines of the second kind of a rank 2 vector bundle on $\mathbb{P}^2$.

Example 9.1.20. Let S be the *Clebsch diagonal surface* given by two equations in $\mathbb{P}^4$:

$$\sum_{i=1}^{5} t_i = \sum_{i=1}^{5} t_i^3 = 0. \tag{9.12}$$

It exhibits an obvious symmetry defined by permutations of the coordinates. Let $a = \frac{1}{2}(1 + \sqrt{5}), a' = \frac{1}{2}(1 - \sqrt{5})$ be two roots of the equation $x^2 - x - 1 = 0$. One checks that the skew lines

$$\ell : t_1 + t_3 + at_2 = at_3 + t_2 + t_4 = at_2 + at_3 - t_5 = 0$$

and

$$\ell' : t_1 + t_2 + a't_4 = t_3 + a't_1 + t_4 = a't_1 + a't_4 - t_5 = 0$$

lie on S. Applying to each line even permutations we obtain a double-six. The Schur quadric is $\sum t_i^2 = \sum t_i = 0$.

Let $\pi_1 : S \to {}_1\mathbb{P}^2, \pi_2 : S \to {}_2\mathbb{P}^2$ be two birational maps defined by blowing down two sixes forming a double-six. We will see later in Subsection 9.3.2 that there exists a 3×3-matrix $A = (a_{ij})$ of linear forms such that $S = V(\det A)$. The map π_1 (resp. π_2) is given by the left (resp. right) kernel of A. In coordinates, it is given by a row (resp. column) of $\mathrm{adj}(A)$. The composition of the map $(\pi_1, \pi_2) : S \to {}_1\mathbb{P}^2 \times {}_2\mathbb{P}^2$ with the Segre map ${}_1\mathbb{P}^2 \times {}_2\mathbb{P}^2 \to \mathbb{P}^8$ is given by $x \mapsto [\mathrm{adj}(A)(x)]$. We immediately identify this map with the map (9.11). Thus, the entries of $\mathrm{adj}(A)$ define the quadrics in the image of this map. They are apolar to the dual of the Schur quadrics.

Let $x = [z_0, \ldots, z_3]$ be any point in $\mathbb{P}^3$. The polar quadric of S with center at x is given by the equation

$$\begin{vmatrix} Da_{11} & Da_{12} & Da_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} + \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ Da_{21} & Da_{22} & Da_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} + \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ Da_{31} & Da_{32} & Da_{33} \end{vmatrix} = 0,$$

where D is the linear differential operator $\sum z_i \frac{\partial}{\partial t_i}$. It is clear that the left-hand side this equation is a linear combination of the entries of $\mathrm{adj}(A)$. Thus, all polar quadrics of S are apolar to the duals of all 36 Schur quadrics. This proves the following.

Proposition 9.1.21. *The duals of the 36 Schur quadrics belong to the 5-dimensional projective space of quadrics apolar to the 3-dimensional linear system of polar quadrics of S.*

This result was first mentioned by H. Baker in [28], its proof appears in his book [29], Vol. 3, p. 187. In the notation of Theorem 9.1.3, let Q_α is the Schur quadric corresponding to the double-six defined by the root α (see Theorem 9.1.3). Any three of type $Q_{\alpha_{\max}}, Q_{\alpha_{123}}, Q_{\alpha_{456}}$ are linearly dependent. Among $Q_{\alpha_{ij}}$'s at most five are linearly independent ([631]).

Remark 9.1.22. We refer to [238] for the relationship between Schur quadrics and rank 2 vector bundles on $\mathbb{P}^2$ with odd first Chern class. The case of cubic surfaces corresponds to vector bundles with $c_1 = -1$ and $c_2 = 4$. For higher n values the Schur quadrics define some polarity relation for a configuration of $\binom{n+1}{2}$ lines and $(n-2)$-dimensional subspaces in $\mathbb{P}^n$ defined by a *White surface* X, the blow-up of a set Z of $\binom{n+1}{2}$ points in the plane which do not lie on a curve of degree $n-1$ and no n points among them are collinear [805]. The case $n = 3$ corresponds to cubic surfaces and the case $n = 4$ to Bordiga surfaces. The linear system $|\mathcal{I}_Z(n)|$ embeds X in $\mathbb{P}^n$. The images of the exceptional curves are lines, and the images of the curves through all points in Z except one (for each point, there is a unique such curve) spans a subspace of dimension $n-3$. The configuration generalizes a double-six on a cubic surface. The difference here is that, in the case $n > 3$, the polarity of the configuration exists only for a non-general White surface.

9.1.4 Eckardt points

A point of intersection of three lines in a tritangent plane is called an *Eckardt point*. As we will see later, the locus of nonsingular cubic surfaces with an Eckardt point is of codimension one in the moduli space of cubic surfaces.

Recall that the fixed locus of an automorphism τ of order 2 of $\mathbb{P}^n$ is equal to the union of two subspaces of dimensions k and $n-k-1$. The number k determines the conjugacy class of τ in the group $\mathrm{Aut}(\mathbb{P}^n) \cong \mathrm{PGL}(n+1)$. In the terminology of classical projective geometry, a projective automorphism with a hyperplane of fixed points is called a *homology*. A homology of order 2 was called a *harmonic homology*. The isolated fixed point is the *center* of the homology.

Proposition 9.1.23. *There is a bijective correspondence between the set of Eckardt points on a nonsingular cubic surface S and the set of harmonic homologies in $\mathbb{P}^3$ with center in S.*

Proof Let $x = \ell_1 \cap \ell_2 \cap \ell_3 \in S$ be an Eckardt point. Choose coordinates such that $x = [1,0,0,0]$ and the equation of the tritangent plane is $t_1 = 0$. The equation of S is

$$t_0^2 t_1 + 2t_0 g_2 + g_3 = 0, \tag{9.13}$$

where g_2, g_3 are homogeneous forms in t_1, t_2, t_3. The polar quadric $P_x(S)$ contains the three coplanar lines ℓ_i passing through one point. This implies that $P_x(S)$ is the union of two planes; one of them is $V(t_1)$. Since the equation of $P_x(S)$ is $t_0 t_1 + g_2 = 0$, we obtain that $g_2 = t_1 g_1(t_1, t_2, t_3)$. Making one more coordinate change $t_0 \to t_0 + g_1$, we reduce the equation to the form $t_0^2 t_1 + g_3'(t_1, t_2, t_3)$. The intersection $V(t_0) \cap S$ is isomorphic to the cubic curve $V(g_3)$. Now, we define the homology

$$\tau : [t_0, t_1, t_2, t_3] \mapsto [-t_0, t_1, t_2, t_3]. \tag{9.14}$$

Obviously, it leaves S invariant and has x as its isolated fixed point. The other component of the fixed locus is the cubic curve $V(t_0) \cap V(S)$.

Conversely, assume S admits a projective automorphism τ of order 2 with one isolated fixed point p on S. Choose projective coordinates such that τ is given by formula (9.14). Then, S can be given by Equation (9.13). The surface is invariant with respect to τ if and only if $g_2 = 0$. The plane $V(t_0)$ is the tritangent plane with Eckardt point $[1,0,0,0]$. □

It is clear that the automorphism τ is defined by the projection from the Eckardt point x. It extends to a biregular automorphism of the blow-up $\pi : S' \to S$ of the point x which fixes pointwisely the exceptional curve E of π.

The surface S' is a weak del Pezzo surface of degree 2. It has three disjoint (-2)-curves R_i equal to the proper transforms of the lines ℓ_i containing x. The projection map $S' \to \mathbb{P}^2$ is equal to the composition of the birational morphism $S' \to X$ which blows down the curves R_i and a finite map of degree 2 $X \to \mathbb{P}^2$. The surface X is an anti-canonical model of S' with three singular points of type A_1. The branch curve of $X \to \mathbb{P}^2$ is the union of a line and a nonsingular cubic intersecting the line transversally. The line is the image of the exceptional curve E.

Example 9.1.24. Consider a *cyclic cubic surface* S given by equation

$$f_3(t_0, t_1, t_2) + t_3^3 = 0,$$

where $C = V(f_3)$ is a nonsingular plane cubic in the plane with coordinates t_0, t_1, t_2. Let ℓ be an inflection tangent of C. We can choose coordinates such that $\ell = V(t_1)$ and the tangency point is $[1, 0, 0]$. The equation of S becomes

$$t_0^2 t_1 + t_0 t_1 g_1(t_1, t_2) + t_1 g_2(t_1, t_2) + t_2^3 + t_3^3 = 0.$$

The pre-image of the line ℓ under the projection map $[t_0, t_1, t_2, t_3] \mapsto [t_0, t_1, t_2]$ splits into the union of three lines with equation $t_1 = t_2^3 + t_3^3 = 0$. The point $[1, 0, 0, 0]$ is an Eckardt point. The surface contains nine Eckardt points because there are nine inflection points on a nonsingular plane cubic. Note that the corresponding nine tritangent planes contain all 27 lines.

Example 9.1.25. Consider a cubic surface given by equations

$$\sum_{i=0}^{4} a_i t_i^3 = \sum_{i=0}^{4} t_i = 0,$$

where $a_i \neq 0$. We will see later that a general cubic surface is projectively equivalent to such a surface. Assume $a_0 = a_1$. Then, the point $p = [1, -1, 0, 0, 0]$ is an Eckardt point. In fact, the tangent plane at this point is $t_0 + t_1 = t_2 + t_3 + t_4 = 0$. It cuts out the surface along the union of three lines intersecting at the point p. Similarly, we have an Eckardt point whenever $a_i = a_j$ for some $i \neq j$. Thus, we may have 1, 2, 3, 4, 6, or 10 Eckardt points dependent on whether we have just two equal coefficients, or two pairs of equal coefficients, or three equal coefficients, or a pair and a triple of equal coefficients, or four equal coefficients, or five equal coefficients. The other possibilities for the number of Eckardt points are nine, as in the previous example, or 18 when the surface is isomorphic to a Fermat cubic surface. We will prove later that no other case occurs.

Let us prove the following proposition, which we will use later.

Proposition 9.1.26. *Let x and y be two Eckardt points on S such that the line*

$\ell = \langle x, y \rangle$ is not contained in S. Then, ℓ intersects S in a third Eckardt point. Moreover, no three Eckardt points lie on a line contained in the surface.

Proof Let τ be the harmonic homology involution of S defined by the Eckardt point x. Then, ℓ intersects S at the point $z = \tau(y)$. The points y and z are on the line $\langle x, y \rangle$. If ℓ is not contained in S, then it is not contained in the polar quadric $P_x(S)$, and hence does not intersect the 1-dimensional component F of the fixed locus of τ. This shows that $y \neq z$. On the other hand, if ℓ is contained in S, then it is one of the three lines in the tritangent plane containing x. Suppose that we have two more Eckardt points on ℓ. Then, the plane that cuts out F intersects the corresponding tritangent planes of the two new Eckardt points. This implies that ℓ contains three fixed points of the involution τ, a contradiction. □

Proposition 9.1.27. *Let x_1, x_2, x_3 be three collinear Eckardt points. Then, the involutions τ_i corresponding to these points generate a subgroup of automorphisms isomorphic to $\mathfrak{S}_3$. If two Eckardt points x_1, x_2 lie on a line $\ell \subset S$, then the involutions commute, and the product fixes the line and the other line which contains the tangency points of three tritangent planes through ℓ.*

Proof Suppose three Eckardt points lie on a line ℓ. Obviously, each τ_i leaves the line $\ell = \langle x_1, x_2 \rangle$ invariant. Thus, the subgroup G generated by the three involutions leaves the line invariant and permutes the three Eckardt points. This defines a homomorphism $G \to \mathfrak{S}_3$ which is obviously surjective. Let g be a nontrivial element from the kernel. Then, it leaves three points fixed, and hence leaves all points on the line fixed. Without loss of generality, we may assume that $g = \tau_1\tau_2$ or $g = \tau_1\tau_2\tau_3$. Since τ_1 and τ_2, and $\tau_1\tau_2, \tau_3$ act differently on ℓ, we get $g = 1$.

Now, suppose that two Eckardt points lie on a line ℓ contained in the surface. Obviously, τ_i fixes both points x_1 and x_2. Since a finite automorphism group of $\mathbb{P}^1$ fixing two points is cyclic, the product $\tau = \tau_1\tau_2$ is of order 2; it fixes ℓ pointwise, and also fixes the line ℓ' equal to intersection of the planes of fixed points of τ_1, τ_2. This line intersects each tritangent plane through ℓ at some point. Hence, each such plane is invariant with respect to τ, and the tangency points of the remaining three tritangent planes lie on ℓ'. □

Let us project S from a point $x \in S$ that is not an Eckardt point. Suppose x does not lie on any line in S. Then, the blow-up S' of S at x is a del Pezzo surface of degree 2. The projection map lifts to a finite double cover of $\mathbb{P}^2$ branched along a nonsingular quartic curve B. The 27 lines, together with the exceptional curve E of the blow-up, map to the 28 bitangents of B. The image of a sixer of lines and the curve E is an Aronhold set of seven bitangents. This relationship between 27 lines on a cubic surface and 28 bitangents of a plane quartic was first discovered by C. Geiser [323] in 1860.

If x lies on one line, S' is a weak del Pezzo surface of degree 2 with one (-2)-curve R. The projection map lifts to a degree map $S' \to \mathbb{P}^2$ which factors through the blowing down map $S \to X$ of R and a finite map of degree 2 $X \to \mathbb{P}^2$ branched over a 1-nodal quartic curve. If x lies on two lines, then we have a degree 2 map $S' \to X \to \mathbb{P}^2$, where X has two A_1-singularities, and the branch curve of $X \to \mathbb{P}^2$ is a 2-nodal quartic.

9.2 Singularities of Cubic Surfaces

9.2.1 Non-normal cubic surfaces

Let X be an irreducible cubic surface in $\mathbb{P}^3$. Assume that X is not normal and is not a cone over a singular cubic curve. Then, its singular locus contains a one-dimensional part C of some degree d. Let m be the multiplicity of a general point of C. By Bertini's Theorem, a general plane section H of X is an irreducible plane cubic that contains d singular points of multiplicity m. Since an irreducible plane cubic curve has only one singular point of multiplicity 2, we obtain that the singular locus of X is a line.

Let us choose coordinates in such a way that C is given by the equations $t_0 = t_1 = 0$. Then, the equation of X must look like

$$l_0 t_0^2 + l_1 t_0 t_1 + l_2 t_1^2 = 0,$$

where $l_i, i = 0, 1, 2$, are linear forms in t_0, t_1, t_2. This shows that the left-hand side contains t_2 and t_3 only in degree 1. Thus, we can rewrite the equation in the form

$$t_2 f + t_3 g + h = 0, \tag{9.15}$$

where f, g, h are binary forms in t_0, t_1, the first two of degree 2, and the third one of degree 3.

Suppose f, g are proportional. Then, the equation can be rewritten in the form $(at_1 + bt_2)f + h = 0$, which shows that X is a cone. A pair of non-proportional binary quadratic forms f, g can be reduced to the form $t_0^2 + t_1^2, at_0^2 + bt_1^2$, or $t_0 t_1, at_0^2 + t_0 t_1$ (corresponding to the Segre symbol (2)). After making a linear change of variables t_2, t_3, we arrive at two possible equations

$$t_2 t_0^2 + t_3 t_1^2 + (at_0 + bt_1)t_0^2 + (ct_0 + dt_1)t_1 = 0,$$
$$t_2 t_0 t_1 + t_3 t_0^2 + (at_0 + bt_1)t_0^2 + (ct_0 + dt_1)t_1 = 0.$$

Replacing t_2 with $t_2' = t_2 + at_0 + bt_1$ and t_3 with $t_3' = t_3 + ct_0 + dt_1$, we obtain two canonical forms of non-normal cubic surfaces that are not cones.

The plane sections through the singular line of the surface define a structure of a scroll on the surface.

Theorem 9.2.1. *Let X be an irreducible non-normal cubic surface. Then, either X is a cone over an irreducible singular plane cubic, or it is projectively equivalent to one of the following cubic surfaces singular along a line:*

(i) $t_0^2 t_2 + t_1^2 t_3 = 0$;

(ii) $t_2 t_0 t_1 + t_3 t_0^2 + t_1^3 = 0$.

The two surfaces are not projectively isomorphic.

The last assertion follows from considering the normalization $\bar{X}$ of the surface X. In both cases it is a nonsingular surface, however in (i), the pre-image of the singular line is irreducible, but in the second case it is reducible.

We have already seen two cubic scrolls in $\mathbb{P}^3$ in Subsection 2.1.1. They are obtained as projections of the cubic scroll $S_{1,4}$ in $\mathbb{P}^4$ isomorphic to the rational minimal ruled surface $\mathbb{F}_1$ (a del Pezzo surface of degree 8). There are two possible centers of the projection: the center lies in the plane spanned by the image of the exceptional (-1)-curve and a line from the ruling, or it lies outside of this plane. Case (1) corresponds to the second possibility, and case (ii) to the first one.

9.2.2 Lines and singularities

From now on, we assume that S is a normal cubic surface that is not a cone. Thus, its singularities are rational double points, and S is a del Pezzo surface of degree 3.

Let X be a minimal resolution of singularities of S. All possible Dynkin curves on X can be easily found from the list of root bases in E_6.

$$
\begin{aligned}
&(r=6)\ \ E_6,\ A_6,\ D_4+A_2,\ \sum_{k=1}^{s} A_{i_k}, i_1+\cdots+i_s=6,\\
&(r=5)\ \ D_5,\ D_4+A_1,\ \sum_{k=1}^{s} A_{i_k}, i_1+\cdots+i_s=5,\\
&(r=4)\ \ D_4,\ \sum_{k=1}^{s} A_{i_k}, i_1+\cdots+i_s=4,\\
&(r=3)\ \ A_3,\ A_2+A_1,\ 3A_1,\\
&(r=2)\ \ A_2,\ A_1+A_1,\\
&(r=1)\ \ A_1.
\end{aligned}
$$

The following Lemma is easily verified, and we omit its proof.

Lemma 9.2.2. *Let* $x_0=(1,0,0,0)$ *be a singular point of* $S=V(f_3)$. *Write*

$$f_3=t_0g_2(t_1,t_2,t_3)+g_3(t_1,t_2,t_3),$$

where g_2,g_3 *are homogeneous polynomials of degrees 2 and 3, respectively. Let* $x=[a_0,a_1,a_2,a_3]\in S$. *If the line* $\langle x_0,x\rangle$ *is contained in S, then the point* $q=[a_1,a_2,a_3]$ *is a common point of the conic* $V(g_2)$ *and the cubic* $V(g_3)$. *If, moreover, x is a singular point of S, then the conic and the cubic intersect at q with multiplicity* >1.

Corollary 9.2.3. $V(f_3)$ *has at most four singular points. Moreover, if* $V(f_3)$ *has four singular points, then each point is of type* A_1.

Proof Let x_0 be a singular point which we may assume to be the point $[1,0,0,0]$ and apply Lemma 9.2.2. Suppose we have more than 4 singular points. The conic and the cubic will intersect at least in four singular points with multiplicity >1. Since they do not share an irreducible component (otherwise f_3 is reducible), this contradicts Bézout's Theorem. Suppose we have four singular points and x_0 is not of type A_1. Since x_0 is not an ordinary double point, the conic $V(g_2)$ is reducible. Then, the cubic $V(g_3)$ intersects it at three points with multiplicity >1 at each point. It is easy to see that this also contradicts Bézout's Theorem. □

Lemma 9.2.4. *The cases,* $A_{i_1}+\cdots+A_{i_k}, i_1+\cdots+i_k=6$, *except the cases* $3A_2, A_5+A_1$ *do not occur.*

Proof Assume $M=A_{i_1}+\cdots+A_{i_k}, i_1+\cdots+i_k=6$. Then, the discriminant d_M of the lattice M is equal to $(i_1+1)\cdots(i_k+1)$. By Lemma 8.2.1, $3|d_M$, one of the numbers, say $i_1 + 1$, is equal either to 3 or 6. If $i_1 + 1 = 6$, then $M = A_5 + A_1$. If $i_1 + 1 = 3$, then $(i_2 + 1) \dots (i_k + 1)$ must be a square, and $i_2 + \cdots + i_k = 4$. It is easy to see that the only possibilities are $i_2 = i_3 = 2$ and $i_2 = i_3 = i_4 = i_5 = 1$. The last possibility is excluded by applying Corollary 9.2.3. □

Lemma 9.2.5. *The cases $D_4 + A_1$ and $D_4 + A_2$ do not occur.*

Proof Let x_0 be a singular point of S of type D_4. Again, we assume that $x_0 = [1, 0, 0, 0]$ and apply Lemma 9.2.2. As we have already noted, the singularity of type D_4 is analytically (or formally) isomorphic to the singularity $z^2+xy(x+y) = 0$. This shows that the conic $V(g_2)$ is a double line ℓ. The plane $z = 0$ cuts out a germ of a curve with 3 different branches. Thus, there exists a plane section of $S = V(f_3)$ passing through x_0 which is a plane cubic with 3 different branches at x_0. Obviously, it must be the union of 3 lines with a common point at x_0. Now, the cubic $V(g_3)$ intersects the line ℓ at 3 points corresponding to the lines through x_0. Thus, S cannot have more singular points. □

Let us show that all remaining cases are realized. We will exhibit the corresponding del Pezzo surface as the blow-up of six bubble points $p_1, \dots, p_6$ in $\mathbb{P}^2$.

A_1: 6 proper points in $\mathbb{P}^2$ on an irreducible conic;

A_2: $p_3 \succ_1 p_1$;

$2A_1$: $p_2 \succ_1 p_1, p_4 \succ_1 p_3$;

A_3: $p_4 \succ_1 p_3 \succ_1 p_2 \succ_1 p_1$;

$A_2 + A_1$: $p_3 \succ_1 p_2 \succ_1 p_1, p_5 \succ_1 p_4$;

A_4: $p_5 \succ_1 p_4 \succ_1 p_3 \succ_1 p_2 \succ_1 p_1$;

$3A_1$: $p_2 \succ_1 p_1, p_4 \succ_1 p_3, p_6 \succ_1 p_5$;

$2A_2$: $p_3 \succ_1 p_2 \succ_1 p_1, p_6 \succ_1 p_5 \succ_1 p_4$;

$A_3 + A_1$: $p_4 \succ_1 p_3 \succ_1 p_2 \succ_1 p_1, p_6 \succ_1 p_5$;

A_5: $p_6 \succ_1 p_5 \succ_1 p_4 \succ_1 p_3 \succ_1 p_2 \succ_1 p_1$;

D_4: $p_2 \succ_1 p_1, p_4 \succ_1 p_3, p_6 \succ_1 p_5$ and p_1, p_3, p_5 are collinear;

$A_2 + 2A_1$: $p_3 \succ_1 p_2 \succ_1 p_1, p_5 \succ_1 p_4$, and $|h - p_1 - p_2 - p_3| \neq \emptyset$;

$A_4 + A_1$: $p_5 \succ_1 p_4 \succ_1 p_3 \succ_1 p_2 \succ_1 p_1$ and $|2h - p_1 - \cdots - p_6| \neq \emptyset$;

D_5: $p_5 \succ_1 p_4 \succ_1 p_3 \succ_1 p_2 \succ_1 p_1$ and $|h - p_1 - p_2 - p_6| \neq \emptyset$;

$4A_1$: $p_1, \dots, p_6$ are the intersection points of 4 lines in a general linear position;

$2A_2 + A_1$: $p_3 \succ_1 p_2 \succ_1 p_1, p_6 \succ_1 p_5 \succ_1 p_4$ and $|h - p_1 - p_2 - p_3| \neq \emptyset$;

$A_3 + 2A_1$: $p_4 \succ_1 p_3 \succ_1 p_2 \succ_1 p_1, p_6 \succ_1 p_5$ and $|h - p_1 - p_2 - p_3| \neq \emptyset$;

$A_5 + A_1$: $p_6 >_1 p_5 >_1 p_4 >_1 p_3 >_1 p_2 >_1 p_1$ and $|2h - p_1 - \cdots - p_6| \neq \emptyset$;

E_6: $p_6 >_1 p_5 >_1 p_4 >_1 p_3 >_1 p_2 >_1 p_1$ and $|h - p_1 - p_2 - p_3| \neq \emptyset$;

$3A_2$: $p_3 >_1 p_2 >_1 p_1, p_6 >_1 p_5 >_1 p_4, |h-p_1-p_2-p_3| \neq \emptyset, |h-p_4-p_5-p_6| \neq \emptyset$;

Projecting from a singular point and applying Lemma 9.2.2 we see that each singular cubic surface can be given by the following equations.

A_1: $V(t_0g_2(t_1,t_2,t_3)+g_3(t_1,t_2,t_3))$, where $V(g_2)$ is a nonsingular conic which intersects $V(g_3)$ transversally;

A_2: $V(t_0t_1t_2 + g_3(t_1,t_2,t_3))$,where $V(t_1t_2)$ intersects $V(g_3)$ transversally;

$2A_1$: $V(t_0g_2(t_1,t_2,t_3)+g_3(t_1,t_2,t_3))$, where $V(g_2)$ is a nonsingular conic which is simply tangent to $V(g_3)$ at one point;

A_3: $V(t_0t_1t_2+g_3(t_1,t_2,t_3))$, where $V(t_1t_2)$ intersects $V(g_3)$ at the point $[0,0,1]$ and at other 4 distinct points;

$A_2 + A_1$: $V(t_0t_1t_2 + g_3(t_1,t_2,t_3))$, where $V(g_3)$ is tangent to $V(t_2)$ at $[1,0,0]$;

A_4: $V(t_0t_1t_2 + g_3(t_1,t_2,t_3))$, where $V(g_3)$ is tangent to $V(t_1)$ at $[0,0,1]$;

$3A_1$: $V(t_0g_2(t_1,t_2,t_3)+g_3(t_1,t_2,t_3))$, where $V(g_2)$ is nonsingular and is tangent to $V(g_3)$ at 2 points;

$2A_2$: $V(t_0t_1t_2 + g_3(t_1,t_2,t_3))$, where $V(t_1)$ intersects $V(g_3)$ transversally and $V(t_2)$ is an inflection tangent to $V(g_3)$ at $[1,0,0]$;

$A_3 + A_1$: $V(t_0t_1t_2 + g_3(t_1,t_2,t_3))$, where $V(g_3)$ passes through $[0,0,1]$ and $V(t_1)$ is tangent to $V(g_3)$ at the point $[1,0,0]$;

A_5: $V(t_0t_1t_2 + g_3(t_1,t_2,t_3))$, where $V(t_1)$ is an inflection tangent of $V(g_3)$ at the point $[0,0,1]$;

D_4: $V(t_0t_1^2 + g_3(t_1,t_2,t_3))$, where $V(t_1)$ intersects transversally $V(g_3)$;

A_2+2A_1: $V(t_0t_1t_2+g_3(t_1,t_2,t_3))$, where $V(g_3)$ is tangent $V(t_1t_2)$ at two points not equal to $[0,0,1]$;

$A_4 + A_1$: $V(t_0t_1t_2 + g_3(t_1,t_2,t_3))$, where $V(g_3)$ is tangent to $V(t_1)$ at $[0,0,1]$ and is tangent to $V(t_2)$ at $[1,0,0]$;

D_5: $V(t_0t_1^2 + g_3(t_1,t_2,t_3))$, where $V(t_1)$ is tangent to $V(g_3)$ at $[0,0,1]$;

$4A_1$: $V(t_0g_2(t_1,t_2,t_3)+g_3(t_1,t_2,t_3))$, where $V(g_2)$ is nonsingular and is tangent to $V(g_3)$ at 3 points;

$2A_2 + A_1$: $V(t_0g_2(t_1,t_2,t_3) + g_3(t_1,t_2,t_3))$, where $V(g_2)$ is tangent to $V(g_3)$ at 2 points $[1,0,0]$ with multiplicity 3;

$A_3 + 2A_1$: $V(t_0t_1t_2 + g_3(t_1,t_2,t_3))$, where $V(g_3)$ passes through $[0,0,1]$ and is tangent to $V(t_1)$ and to $V(t_2)$ at one point not equal to $[0,0,1]$;

$A_5 + A_1$: $V(t_0t_1t_2 + g_3(t_1, t_2, t_3))$, where $V(t_1)$ is an inflection tangent of $V(g_3)$ at the point $[0, 0, 1]$ and $V(t_2)$ is tangent to $V(g_3)$;

E_6: $V(t_0t_1^2 + g_3(t_1, t_2, t_3))$, where $V(t_1)$ is an inflection tangent of $V(g_3)$.

$3A_2$: $V(t_0t_1t_2+g_3(t_1, t_2, t_3))$, where $V(t_1)$, $V(t_2)$ are inflection tangents of $V(g_3)$ at points different from $[0, 0, 1]$.

Remark 9.2.6. Applying a linear change of variables, one can simplify the equations. For example, in type XXI (see Table 9.1), we may assume that the inflection points are $[1, 0, 0]$ and $[0, 1, 0]$. Then, $g_3 = t_3^3 + t_1t_2L(t_1, t_2, t_3)$. Replacing t_0 with $t_0' = t_0 + L(t_1, t_2, t_3)$, we reduce the equation to the form

$$t_0t_1t_2 + t_3^3 = 0. \tag{9.16}$$

Another example is the E_6-singularity (type XX). We may assume that the inflection point is $[0, 0, 1]$. Then, $g_3 = t_2^3 + t_1g_2(t_1, t_2, t_3)$. The coefficient at t_3^2 is not equal to zero, otherwise the equation is reducible. After a linear change of variables we may assume that $g_2 = t_3^2 + at_1^2 + bt_1t_2 + ct_2^2$. Replacing t_0 with $t_0 + at_1 + bt_2$, we may assume that $a = b = 0$. After scaling the unknowns, we get the equation of the surface

$$t_0t_1^2 + t_1t_2^2 + t_1t_3^2 + t_2^3 = 0. \tag{9.17}$$

Table 9.1 below gives the classification of possible singularities of a cubic surface, the number of lines and the class of the surface.

Type	Singularity	Lines	Class	Type	Singularity	Lines	Class
I	$\emptyset$	27	12	XII	D_4	6	6
II	A_1	21	10	XIII	$A_2 + 2A_1$	8	5
III	A_2	15	9	XIV	$A_4 + A_1$	4	5
IV	$2A_1$	16	8	XV	D_5	3	5
V	A_3	10	8	XVI	$4A_1$	9	4
VI	$A_2 + A_1$	11	7	XVII	$2A_2 + A_1$	5	4
VII	A_4	6	7	XVIII	$A_3 + 2A_1$	5	4
VIII	$3A_1$	12	6	XIX	$A_5 + A_1$	2	4
IX	$2A_2$	7	6	XX	E_6	1	4
X	$A_3 + A_1$	7	6	XXI	$3A_2$	3	3
XI	A_5	3	6				

Table 9.1 *Singularities of cubic surfaces*

Note that the number of lines can be checked directly by using the equations. The map from $\mathbb{P}^2$ to S is given by the linear system of cubics generated by

$V(g_3), V(t_1g_2), V(t_2g_2), V(t_3g_2)$. The lines are images of lines or conics that intersect a general member of the linear system with multiplicity 1 outside base points. The class of the surface can be computed by applying the Plücker-Teissier formula from Theorem 1.2.7. We use that the Milnor number of an A_n, D_n, E_n singularity is equal to n, and the Milnor number of the singularity of a general plane section through the singular point is equal to 1 if type is A_n and 2 otherwise.

Example 9.2.7. The cubic surface with three singular points of type A_2 given by Equation (9.16) plays an important role in the Geometric Invariant Theory of cubic surfaces. It represents the unique isomorphism class of a strictly semistable point in the action of the group $\mathrm{SL}(4)$ in the space of cubic surfaces. Table 9.1 shows that it is the only normal cubic surface whose dual surface is also a cubic surface. By the Reciprocity Theorem $X \cong (X^\vee)^\vee$, the dual surface cannot be a cone or a scroll. Thus, the surface of type XXI is the only self-dual cubic surface.

Another interesting special case is the surface with four A_1-singularities. In notation above, let us choose coordinates such that the three tangency points of the conic $V(g_2)$ and the cubic $V(g_3)$ are $[1,0,0], [0,1,0], [0,0,1]$. After scaling the coordinates, we may assume that $g_2 = t_1t_2 + t_1t_3 + t_2t_3$. An example of a cubic tangent to the conic at the three points is the union of three tangent lines

$$h = (t_1 + t_2)(t_1 + t_3)(t_2 + t_3) = g_2(t_1 + t_2 + t_3) - t_1t_2t_3.$$

Any other cubic can be given by equation $lg_2 + h = 0$, where l is a linear form. Replacing t_0 with $t_0' = -(t_0 + l + t_1 + t_2 + t_3)$, we reduce the equation to the form

$$t_0(t_1t_2 + t_1t_3 + t_2t_3) + t_1t_2t_3 = t_0t_1t_2t_3\Big(\frac{1}{t_0} + \frac{1}{t_1} + \frac{1}{t_2} + \frac{1}{t_3}\Big) = 0. \tag{9.18}$$

A cubic surface with four nodes is called the *Cayley cubic surface*. As we see, all Cayley cubics are projectively equivalent. They admit $\mathfrak{S}_4$ as its group of automorphisms.

Let us find the dual surface of a Cayley surface. Table 9.1 shows that it must be a quartic surface. The equation of a tangent plane at a general point $[a, b, c, d]$ is

$$\frac{t_0}{a^2} + \frac{t_1}{b^2} + \frac{t_2}{c^2} + \frac{t_3}{d^2} = 0.$$

Thus, the dual surface is the image of S under the map

$$[t_0, t_1, t_2, t_3] \mapsto [\xi_0, \xi_1, \xi_2, \xi_3] = [1/t_0^2, 1/t_1^2, 1/t_2^2, 1/t_3^2].$$

Write $t_i = 1/\sqrt{\xi_i}$ and plug in Equation (9.18). We obtain the equation

$$(\sqrt{\xi_0\xi_1\xi_2\xi_3})^{-2}(\sqrt{\xi_0} + \sqrt{\xi_1} + \sqrt{\xi_2} + \sqrt{\xi_3}) = 0.$$

This shows that the equation of the dual quartic surface is obtained from the equation

$$\sqrt{\xi_0} + \sqrt{\xi_1} + \sqrt{\xi_2} + \sqrt{\xi_3} = 0$$

by getting rid of the irrationalities. We get the equation

$$(\sum_{i=0}^{3} \xi_i^2 - 2 \sum_{0\le i<j\le 3} \xi_i\xi_j)^2 - 64\xi_0\xi_1\xi_2\xi_3 = 0.$$

The surface has three singular lines $\xi_i + \xi_k = \xi_l + \xi_m = 0$. They meet at one point $[1, 1, 1, 1]$. The only quartic surface with this property is a Steiner quartic surface from Subsection 2.1.1. Thus, the dual of the Cayley cubic surface is a Steiner quartic surface.

9.3 Determinantal Equations of Cubic Surfaces

9.3.1 Cayley-Salmon equation

Let S' be a minimal resolution of singularities of a del Pezzo cubic surface S. Choose a geometric marking $\phi : I^{1,6} \to \mathrm{Pic}(S')$ and consider the image of one of 120 conjugate pairs of triples of tritangent trios from (9.9). Write them as a table:

$$\begin{matrix} e_{11} & e_{12} & e_{13} \\ e_{21} & e_{22} & e_{23}. \\ e_{31} & e_{32} & e_{33} \end{matrix} \qquad (9.19)$$

Suppose the divisor classes e_{ij} are the classes of (-1)-curves on S'. Then, their images in S are lines ℓ_{ij}. The lines defined by the i-th row (the j-th column) lie in a plane Λ_i (Λ'_j), a tritangent plane of S. The union of the planes Λ_i contains all nine lines ℓ_{ij}. The same is true for the planes Λ'_j. The pencil of cubic surfaces spanned by the cubics $\Lambda_1 + \Lambda_2 + \Lambda_3$ and $\Lambda'_1 + \Lambda'_2 + \Lambda'_3$ must contain the cubic S. This shows that we can choose the equations $l_i = 0$ of Λ_i and the equations $m_j = 0$ of Λ'_j such that S can be given by equation

$$l_1l_2l_3 + m_1m_2m_3 = \det\begin{pmatrix} l_1 & m_1 & 0 \\ 0 & l_2 & m_2 \\ -m_3 & 0 & l_3 \end{pmatrix} = 0. \qquad (9.20)$$

The equation of a cubic surface of the form (9.20), where the nine lines defined by the equations $l_i = m_j = 0$ are all different, is called *Cayley-Salmon equation*. Note that the lines ℓ_{ii} are skew (otherwise we have four lines in one plane). We say that two Cayley-Salmon equations are equivalent if they define the same unordered sets of three planes $V(l_i)$ and $V(m_j)$.

Suppose a cubic surface can be given by a Cayley-Salmon equation. Each plane $V(l_i)$ contains three different lines $\ell_{ij} = V(l_i) \cap V(m_j)$, $j = 1, 2, 3$, and hence, it is a tritangent plane. After reindexing, we may assume that the lines ℓ_{ii} are skew lines. Let e_{ij} be the divisor classes of the pre-images of the lines in S'. They form the image of a conjugate pair of tritangent trios under ome geometric marking of S'.

Theorem 9.3.1. *Let S be a normal cubic surface. The number of the equivalence classes of Cayley-Salmon equations for S is equal to 120 (type I), 10 (type II), 1 (type III, IV, VIII), and zero otherwise.*

Proof We know that the number of conjugate pairs of triads of tritangent trios of exceptional vectors is equal to 120. Thus, the number of conjugate triads of triples of tritangent planes on a nonsingular cubic surface is equal to 120.

Suppose S has one node. We take a blow-up model of S' as the blow-up of six proper points on an irreducible conic. We have ten tables of type II in (9.9) which give us ten pairs of conjugate triples of tritangent planes.

Suppose S has three nodes. We take the blow-up model corresponding to a bubble cycle $x_1 + \cdots + x_6$ with $x_4 \succ x_1, x_5 \succ x_2, x_6 \succ x_3$. The set of lines ℓ_{ij} are represented by the divisor classes of

$$e_0 - e_1 - e_4, e_0 - e_2 - e_5, e_0 - e_3 - e_6, e_k, 2e_0 - e_1 - \cdots - e_6 + e_k, k = 4, 5, 6.$$

It is easy to see that this is the only possibility.

We leave it to the reader to check the assertion in the remaining cases. □

Suppose a normal cubic surface S contains three skew lines ℓ_1, ℓ_2, ℓ_3. Consider the pencil of planes $\mathcal{P}_i$ through the line ℓ_i. For any general point $x \in \mathbb{P}^3$ one can choose a unique plane $\Pi_i \in \mathcal{P}_i$ containing the point x. This defines a rational map

$$f : \mathbb{P}^3 \dashrightarrow \mathcal{P}_1 \times \mathcal{P}_2 \times \mathcal{P}_3. \tag{9.21}$$

Suppose the intersection of the planes Π_i contains a line ℓ. Then, ℓ intersects ℓ_1, ℓ_2, ℓ_3, and hence either belongs to S or does not intersect S at a point outside the three lines. This shows that the restriction of the map f to S is a birational map onto its image X.

The composition of map (9.21) with the Segre map defines a rational map

$$f : \mathbb{P}^3 \dashrightarrow \mathcal{S} \subset \mathbb{P}^7,$$

where $\mathcal{S}$ is isomorphic to the Segre variety $\mathsf{s}(\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1)$. Since the pencils $\mathcal{P}_i$ generate the complete linear system $|V|$ of cubic surfaces containing the lines ℓ_i, the map f is given by $|V|$. Since our cubic S is a member of $|V|$, it is equal to the pre-image of a hyperplane section H of X. The Segre variety $\mathcal{S}$ is of degree 6, so H is isomorphic to a surface S_6 of degree 6 in $\mathbb{P}^6$ and the restriction of f to S is a birational map onto S_6. The hyperplane section H is defined by a divisor of tridegree $(1,1,1)$ on the Segre variety. This gives the following.

Theorem 9.3.2 (F. August)**.** *Any cubic surface containing three skew lines ℓ_1, ℓ_2, ℓ_3 can be generated by three pencils $\mathcal{P}_i$ of planes with base locus ℓ_i in the following sense. There exists a correspondence R of degree* $(1,1,1)$ *on $\mathcal{P}_1 \times \mathcal{P}_2 \times \mathcal{P}_3$ such that*

$$S = \{x \in \mathbb{P}^3 : x \in \Pi_1 \cap \Pi_2 \cap \Pi_3 \textit{ for some } (\Pi_1, \Pi_2, \Pi_3) \in R\}.$$

Note that a del Pezzo surface S_6 of degree 6 containing in the Segre variety $\mathsf{s}(\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1)$ has three different pencils of conics. So, it is either nonsingular, or has one node. In the first case, it is a toric surface which can be given by the equation

$$u_0 v_0 w_0 + u_1 v_1 w_1 = 0,$$

where $(u_0, u_1), (v_0, v_1), (w_0, w_1)$ are projective coordinates in each factor of $(\mathbb{P}^1)^3$. The equation can be considered as a linear equations in the space $\mathbb{P}^7$ with coordinates $u_i v_j w_k$. If S_6 is singular, it is not a toric surface. The corresponding weak del Pezzo surface is the blow-up of three collinear points. It contains only three lines.

Suppose S_6 is a nonsingular surface. Then, we can identify the coordinates (u_0, u_1) with the coordinates in the pencil $\mathcal{P}_1$ of planes through ℓ_1 and, similarly, for the other two pairs of coordinates. Thus, the cubic surface is equal to the set of solutions of the system of linear equations

$$\begin{aligned} u_0 l_1(t_0, t_1, t_2, t_3) &= u_1 m_1(t_0, t_1, t_2, t_3), \\ v_0 l_2(t_0, t_1, t_2, t_3) &= v_1 m_2(t_0, t_1, t_2, t_3), \\ w_0 l_3(t_0, t_1, t_2, t_3) &= w_1 m_3(t_0, t_1, t_2, t_3), \end{aligned}$$

where l_i, m_i are linear forms and $u_0 v_0 w_0 + u_1 v_1 w_1 = 0$. This immediately gives Cayley-Salmon equation of S. Conversely, the choice of Cayley-Salmon equation gives August's projective generation of S.

Remark 9.3.3. The rational map $f : \mathbb{P}^3 \dashrightarrow \mathcal{S} \subset \mathbb{P}^7$ is a birational map. To see this, we take a general line ℓ_4 in $\mathbb{P}^3$. The image of this line is a rational curve of degree 3 in X. The composition of f and the projection $\mathbb{P}^7 \dashrightarrow \mathbb{P}^3$ from the subspace spanned by $f(\ell_4)$ is a rational map $T : \mathbb{P}^3 \dashrightarrow \mathbb{P}^3$. It is given by the linear system of cubics passing through the lines $\ell_1, \ell_2, \ell_3, \ell_4$. Since four skew lines in $\mathbb{P}^3$ have two *transversals* (i.e. lines intersecting the four lines), the base locus of the linear system also contains the two transversal lines. The union of the six lines is a reducible projectively normal curve of arithmetic genus 3. The transformation T is a bilinear Cremona transformation. The P-locus of T consists of the union of four quadrics Q_k, each containing the lines $\ell_i, i \neq k$ (since it must be of degree 8, there is nothing else). In particular, the map f blows down the quadric Q_4 to a curve parameterizing the ruling of Q_4 that does not contain the lines ℓ_1, ℓ_2, ℓ_3. If S_6 is a nonsingular surface, we must blow down exactly three lines on S, hence the lines ℓ_1, ℓ_2, ℓ_3 have three transversals contained in S. Also, if S_6 is nonsingular, the singular points of S must lie on the lines ℓ_1, ℓ_2, ℓ_3. This can be checked in all cases where Cayley-Salmon equation applies.

Corollary 9.3.4. *Let S be a nonsingular cubic surface. Then, S is projectively equivalent to a surface*

$$V(t_0t_1t_2 + t_3(t_0 + t_1 + t_2 + t_3)l(t_0, \dots, t_3)).$$

A general S can be written in this form in exactly 120 ways (up to projective equivalence).

Proof We will prove later that a nonsingular cubic surface has at most 18 Eckardt points. Thus, we can choose the linear forms m_1, m_2, m_2 such that the lines $\ell_{1j}, \ell_{2j}, \ell_{3j}, j = 1, 2, 3$ do not intersect. This implies that the linear forms l_1, l_2, l_3, m_j are linearly independent. Similarly, we may assume that the linear forms $m_1, m_2, m_3, l_j, j = 1, 2, 3$ are linearly independent. Choose coordinates such that $l_1 = t_0, l_2 = t_1, l_3 = t_2, m_1 = t_3$. The equation of S can be written in the form $t_0t_1t_2 + t_3m_2m_3 = 0$. Let $m_2 = \sum a_it_i$. It follows from the previous assumption, that the coefficients a_i are all nonzero. After scaling the coordinates, we may assume that $m_2 = t_0 + t_1 + t_2 + t_3$ and we take $l = m_3$. □

9.3.2 Hilbert-Burch theorem

The Cayley-Salmon equation has a determinantal form and hence gives a determinantal representation of a cubic surface. Unfortunately, it applies to only a few of the 21 different types of cubics. By other methods, we will see that a determinantal representation exists for any normal cubic surface of a type different from XX. We will use the following result from commutative algebra (see [281]).

Theorem 9.3.5 (Hilbert-Burch)**.** *Let I be an ideal in a polynomial ring R such that* $\mathrm{depth}(I) = \mathrm{codim} I = 2$ *(thus R/I is a Cohen-Macaulay ring). Then, there exists a projective resolution*

$$0 \longrightarrow R^{n-1} \xrightarrow{\phi_2} R^n \xrightarrow{\phi_1} R \longrightarrow R/I \longrightarrow 0.$$

The i-th entry of the vector $(a_1, \ldots, a_n)$ defining ϕ_1 is equal to $(-1)^i c_i$, where c_i is the complementary minor obtained from the matrix A defining ϕ_2 by deleting its i-th row.

We apply this theorem to the case when $R = \mathbb{C}[t_0, t_1, t_2]$ and I is the homogeneous ideal of a closed 0-dimensional subscheme Z of $\mathbb{P}^2 = \mathrm{Proj}(R)$ generated by four linearly independent homogeneous polynomials of degree 3. Let $\mathcal{I}_Z$ be the ideal sheaf of Z. Then, $(I_Z)_m = H^0(\mathbb{P}^2, \mathcal{I}_Z(m))$. By assumption,

$$H^0(\mathbb{P}^2, \mathcal{I}_Z(2)) = 0. \tag{9.22}$$

Applying the Hilbert-Burch Theorem, we find a resolution of the graded ring R/I

$$0 \longrightarrow R(-4)^3 \xrightarrow{\phi_2} R(-3)^4 \xrightarrow{\phi_1} R \longrightarrow R/I \to 0,$$

where ϕ_2 is given by a 3×4-matrix A whose entries are linear forms in t_0, t_1, t_2. Passing to the projective spectrum, we get an exact sequence of sheaves

$$0 \longrightarrow U \otimes \mathcal{O}_{\mathbb{P}^2}(-4) \xrightarrow{\phi_2} V \otimes \mathcal{O}_{\mathbb{P}^2}(-3) \xrightarrow{\phi_1} \mathcal{I}_Z \longrightarrow 0,$$

where U, V are vector spaces of dimension 3 and 4. Twisting by $\mathcal{O}_{\mathbb{P}^2}(3)$, we get the exact sequence

$$0 \longrightarrow U \otimes \mathcal{O}_{\mathbb{P}^2}(-1) \xrightarrow{\tilde{\phi}_2} V \otimes \mathcal{O}_{\mathbb{P}^2} \xrightarrow{\tilde{\phi}_1} \mathcal{I}_Z(3) \longrightarrow 0. \tag{9.23}$$

Taking global sections, we obtain

$$V = H^0(\mathbb{P}^2, \mathcal{I}_Z(3)).$$

Twisting fact sequence (9.23) by $\mathcal{O}_{\mathbb{P}^2}(-2)$, and using the canonical trace isomorphism $H^2(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(-3)) \cong \mathbb{C}$, we obtain that

$$U = H^1(\mathbb{P}^2, \mathcal{I}_Z(1)).$$

The exact sequence

$$0 \to \mathcal{I}_Z(1) \to \mathcal{O}_{\mathbb{P}^2}(1) \to \mathcal{O}_Z \to 0$$

shows that

$$U \cong \mathrm{Coker}(H^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(1)) \to H^0(\mathcal{O}_Z)) \cong \mathrm{Coker}\big(\mathbb{C}^3 \to \mathbb{C}^{h^0(\mathcal{O}_Z)}\big).$$

Since $\dim U = 3$, we obtain that $h^0(\mathcal{O}_Z) = 6$. Thus, Z is a 0-cycle of length 6.

Now, we see that the homomorphism $\tilde{\phi}_2$ of vector bundles is defined by a linear map

$$\phi : E \to \mathrm{Hom}(U, V) = U^\vee \otimes V, \tag{9.24}$$

where $\mathbb{P}^2 = |E|$. We can identify the linear map ϕ with the tensor $\mathfrak{t} \in E^\vee \otimes U^\vee \otimes V$. Let us now view this tensor as a linear map

$$\psi : V^\vee \to \mathrm{Hom}(E, U^\vee) = E^\vee \otimes U^\vee. \tag{9.25}$$

The linear map (9.24) defines a rational map, the right kernel map,

$$f : |E| \to |V^\vee| = |\mathcal{I}_Z(3)|^\vee, \quad [v] \mapsto |\phi(v)(U)^\perp|.$$

It is given by the linear system $|\mathcal{I}_Z(3)|$. In coordinates, it is given by maximal minors of the matrix A defining ϕ_2. Thus, S is contained in the locus of $[\alpha]$ such that α belongs to the pre-image of the determinantal locus in $\mathrm{Hom}(E, U^\vee)$. It is a cubic hypersurface in the space $\mathrm{Hom}(E, U^\vee)$. Thus, the image of f is contained in a determinantal cubic surface S. Since the intersection scheme of two general members C_1, C_2 of the linear system $|\mathcal{I}_Z(3)|$ is equal to the 0-cycle Z of degree 6, the image of f is a cubic surface. This gives a determinantal representation of S.

Theorem 9.3.6. *Assume S is a normal cubic surface. Then, S admits k equivalence classes of linear determinantal representations, where k depends on type of S, and is given in Table 9.2 below.*

I	II	III	IV	V	VI	VII	VIII	IX	X	XI
72	50	24	34	60	64	52	66	60	58	42
XII	XIII	XIV	XV	XVI	XVII	XVIII	XIX	XX	XXI	
48	62	50	32	64	58	56	40	0	54	

Table 9.2 *Number of determinantal representations*

Proof Let S' be a minimal resolution of singularities of S. It follows from the previous construction that a blowing-down structure defined by a bubble cycle of six points not containing in a conic, gives a determinantal representation of S. Conversely, suppose $S \subset \mathbb{P}(V)$, and we have a linear map (9.25) for some

3-dimensional vector spaces E and U defining a determinantal representation of S. Then, ψ defines a map of vector bundles $U\otimes\mathcal{O}_{\mathbb{P}^2}(-4) \to V\otimes\mathcal{O}_{\mathbb{P}^2}(-3)$, and the cokernel of this map is the ideal sheaf of a 0-cycle of length 6. Its blow-up is isomorphic to S. Since S is normal, the ideal sheaf is integrally closed, and hence corresponds to a bubble cycle η whose blow-up is isomorphic to S'.

So, the number of equivalence classes of linear determinantal representations is equal to the number of nef linear systems $|e_0|$ on S' which define a birational morphism $S' \to \mathbb{P}^2$ isomorphic to the blow-up of bubble cycle η of six points not lying on a conic. It follows from the proof of Lemma 9.1.1 that there is a bijection between the set of vectors $v \in I^{1,6}$ with $v^2 = 1, v\cdot \mathbf{k}_6 = -3$ and the set of roots in E_6. The corresponding root α can be written in the form $\alpha = 2e-v_1-\cdots-v_6$, where $(v_1,\ldots,v_6)$ is a unique sixer of exceptional vectors. If we choose a geometric marking $\phi : I^{1,6} \to \mathrm{Pic}(S')$ defined by this sixer, then $\phi(\alpha)$ is an effective root if and only if the bubble cycle corresponding to this marking lies on a conic. Thus, the number of determinantal representations is equal to the number of non-effective roots in $K_{S'}^{\perp}$ modulo the action of the subgroup W_0 of $W(E_6)$ generated by the reflections in nodal roots. In other words, this is the number of roots in E_6 (equal to 72) minus the number of roots in the root sublattice defining the types of singularities on S. Now, we use the known number of roots in root lattices and get the result. Note the exceptional case of a surface with a E_6-singularity. Here, all roots are effective, so S does not admit a determinantal representation. □

Consider the left and right kernel maps for the linear map (9.25)

$$\mathfrak{l} : S \dashrightarrow |E|, \quad \mathfrak{r} : S \dashrightarrow |U|.$$

The composition of these maps with the resolution of singularities $S' \to S$ is the blowing-down map $\mathfrak{l}' : S' \to |E|$ and $\mathfrak{r}' : S' \to |U|$. When S is nonsingular, these are two maps defined by a double-six. The corresponding Cremona transformation $|E| \dashrightarrow |U|$ is given by the homaloidal linear system $|\mathcal{I}_Z^2(5)| = |5e_0-2(e_1-\cdots-e_6)|$. To identify the space U with $H^0(|E|,\mathcal{I}_Z^2(5))^\vee$, we consider the linear map

$$S^2(\psi) : S^2(V^\vee) \to S^2(E^\vee\otimes U^\vee) \to \bigwedge^2 E^\vee \otimes \bigwedge^2 U^\vee \cong E\otimes V,$$

where the last isomorphism depends on a choice of volume forms on $E^\vee$ and $U^\vee$. Dualizing, we get a linear pairing

$$E^\vee \otimes U^\vee \to S^2(V).$$

If we identify $H^0(S,\mathcal{O}_S(-2K_S))$ with S^2V, and $E^\vee$ with $H^0(S,\mathcal{O}_S(e_0))$, then $U^\vee$ can be identified with $H^0(S,\mathcal{O}_S(-2K_S-e_0))$. Note that we have also identified U with the cokernel of the map $r : E \to H^0(\mathbb{P}^2, \mathcal{O}_Z)$. Let us choose a basis in $E \cong \mathbb{C}^3$ and an order of points in Z, hence a basis in $H^0(\mathbb{P}^2, \mathcal{O}_Z) \cong \mathbb{C}^6$. The map $\mathbb{C}^6 \to U = \mathrm{Coker}(r)$ gives six vectors in U. The corresponding six points in $|U|$ is the bubble cycle defining the blowing-down structure $\mathfrak{r} : S \to |U|$. This is a special case of the construction of *associated* sets of points (see [234], [282], [755]).

Remark 9.3.7. We can also deduce Theorem 9.3.6 from the theory of determinantal equations from Chapter 4. Applying this theory, we obtain that S admits a determinantal equation with entries linear forms if it contains a projectively normal curve C such that

$$H^0(S, \mathcal{O}_S(C)(-1)) = H^2(S, \mathcal{O}_S(C)(-2)) = 0. \tag{9.26}$$

Moreover, the set of non-equivalent determinantal representations is equal to the set of divisor classes of such curves. Let $\pi : S' \to S$ be a minimal resolution of singularities and $C' = \pi^*(C)$. Since $\pi^*\mathcal{O}_S(-1) = \mathcal{O}_{S'}(K_{S'})$, the conditions (9.26) are equivalent to

$$H^0(S', \mathcal{O}_{S'}(C' + K_{S'})) = H^2(S', \mathcal{O}_{S'}(C' + 2K_{S'})) = 0. \tag{9.27}$$

Since C' is nef, $H^1(S', \mathcal{O}_{S'}(C' + K_{S'})) = 0$. Also

$$H^2(S', \mathcal{O}_{S'}(C' + K_X S')) = H^0(S', \mathcal{O}_{S'}(-C')) = 0.$$

By Riemann-Roch,

$$0 = \chi(\mathcal{O}_{S'}(C' + K_{S'})) = \tfrac{1}{2}((C' + K_{S'})^2 - (C' + K_{S'}) \cdot K_{S'}) + 1$$

$$= \tfrac{1}{2}(C'^2 + C' \cdot K_{S'}) + 1.$$

Thus, C' is a smooth rational curve, hence C is a smooth rational curve. It is known that a rational normal curve in $\mathbb{P}^n$ must be of degree n. Thus, $-K_{S'} \cdot C' = 3$, hence $C'^2 = 1$. The linear system $|C'|$ defines a birational map $\pi : S' \dashrightarrow \mathbb{P}^2$. Let $e_0 = [C'], e_1, \ldots, e_6$ be the corresponding geometric basis of $\mathrm{Pic}(S')$. The condition

$$0 = H^2(X, \mathcal{O}_{S'}(C' + 2K_{S'})) = H^0(S', \mathcal{O}_{S'}(-C' - K_{S'})) = 0$$

is equivalent to

$$|2e_0 - e_1 - \cdots - e_6| = \emptyset. \tag{9.28}$$

9.3.3 Cubic symmetroids

A *cubic symmetroid* is a hypersurface in $\mathbb{P}^n$ admitting a representation as a symmetric (3×3)-determinant whose entries are linear forms in $n+1$ variables. Here, we will be interested in cubic symmetroid surfaces. An example of a cubic symmetroid is the Cayley four-nodal cubic surface

$$t_0t_1t_2 + t_0t_1t_3 + t_0t_2t_3 + t_1t_2t_3 = \det\begin{pmatrix} t_0+t_3 & t_3 & t_3 \\ t_3 & t_1+t_3 & t_3 \\ t_3 & t_3 & t_2+t_3 \end{pmatrix},$$

which we have already encounter before. By choosing the singular points to be the reference points $[1,0,0,0]$, $[0,1,0,0]$, $[0,0,1,0]$, $[0,0,0,1]$, it is easy to see that cubic surfaces with 4 singularities of type A_1 are projectively isomorphic. Since the determinantal cubic hypersurface in $\mathbb{P}^5$ is singular along a surface, a nonsingular cubic surface does not admit a symmetric determinantal representation.

Lemma 9.3.8. *Let $L \subset |\mathcal{O}_{\mathbb{P}^2}(2)|$ be a pencil of conics. Then, it is projectively isomorphic to one of the following pencils:*

(i) $\lambda(t_0t_1 - t_0t_2) + \mu(t_1t_2 - t_0t_2) = 0$;
(ii) $\lambda(t_0t_1 + t_0t_2) + \mu t_1t_2 = 0$;
(iii) $\lambda(t_0t_1 + t_2^2) + \mu t_0t_2 = 0$;
(iv) $\lambda t_2^2 + \mu t_0t_1 = 0$;
(v) $\lambda t_0^2 + \mu(t_0t_2 + t_1^2) = 0$;
(vi) $\lambda t_0^2 + \mu t_1^2 = 0$;
(vii) $\lambda t_0t_1 + \mu t_0t_2 = 0$;
(viii) $\lambda t_0t_1 + \mu t_0^2 = 0$.

Proof The first five cases correspond to the Segre symbols $[1,1,1]$, $[(2)1]$, $[(3)]$, $[(11)1]$, $[(12)]$, respectively. For the future use, we chose different bases. The last three cases correspond to pencils of singular conics. □

Theorem 9.3.9. *A cubic symmetroid is a del Pezzo surface if and only if it is projectively isomorphic to one of the following determinantal surfaces:*

(i) $C_3 = V(t_0t_1t_2 + t_0t_1t_3 + t_0t_2t_3 + t_1t_2t_3)$ *with four RDP of type* A_1;
(ii) $C_3' = V(t_0t_1t_2 + t_1t_3^2 - t_2t_3^2)$ *with two RDP of type* A_1 *and one RDP of type* A_3;
(iii) $C_3'' = V(t_0t_1t_2 - t_3^2(t_0+t_2) - t_1t_2^2)$ *with one RDP of type* A_1 *and one RDP of type* A_5.

Proof Let $A = (l_{ij})$ be a symmetric 3×3 matrix with linear entries l_{ij} defining the equation of S. It can be written in the form $A(t) = t_0A_0 + t_1A_1 + t_2A_2 + t_3A_3$, where $A_i, i = 1, 2, 3, 4$, are symmetric 3×3 matrices. Let W be a linear system of conics spanned by the conics

$$C_i = [t_0, t_1, t_2] \cdot A \cdot \begin{pmatrix} t_0 \\ t_1 \\ t_2 \end{pmatrix} = 0.$$

Each web of conics is apolar to a unique pencil of conics. Using the previous lemma, we find the following possibilities. We list convenient bases in corresponding dual four-dimensional spaces of quadratic forms.

(i) $\xi_0^2,\ \xi_1^2,\ \xi_2^2,\ 2(\xi_0\xi_1 + \xi_1\xi_2 + \xi_0\xi_2)$;
(ii) $\xi_0^2,\ \xi_1^2,\ \xi_2^2,\ 2(\xi_0\xi_1 - \xi_0\xi_2)$;
(iii) $\xi_0^2,\ \xi_1^2,\ 2\xi_0\xi_1 - \xi_2^2,\ 2\xi_1\xi_2$;
(iv) $\xi_0^2,\ \xi_1^2,\ 2\xi_0\xi_2,\ 2(\xi_1\xi_2 - \xi_0\xi_1)$;
(v) $2\xi_0\xi_2 - \xi_1^2,\ \xi_2^2,\ 2\xi_0\xi_1,\ 2\xi_1\xi_2$;
(vi) $\xi_2^2,\ 2\xi_0\xi_1,\ 2\xi_1\xi_2,\ 2\xi_0\xi_2$;
(vii) $\xi_0^2,\ \xi_1^2,\ \xi_2^2,\ 2\xi_0\xi_1$;
(viii) $\xi_1^2,\ \xi_2^2,\ 2\xi_0\xi_2,\ 2\xi_1\xi_2$.

The corresponding determinantal varieties are the following.

(i)

$$\det\begin{pmatrix} t_0 & t_3 & t_3 \\ t_3 & t_1 & t_3 \\ t_3 & t_3 & t_2 \end{pmatrix} = t_0t_1t_2 + t_3^2(-t_0 - t_2 - t_1 + 2t_3) = 0.$$

It has four singular points $[1, 0, 0, 0]$, $(0, 1, 0, 0]$, $[0, 0, 1, 0]$, and $[1, 1, 1, 1]$. The surface is the Cayley four-nodal cubic.

(ii)

$$\det\begin{pmatrix} t_0 & t_3 & -t_3 \\ t_3 & t_1 & 0 \\ -t_3 & 0 & t_2 \end{pmatrix} = t_0t_1t_2 - t_1t_3^2 - t_2t_3^2 = 0.$$

It has two ordinary nodes $[0, 1, 0, 0]$, $[0, 0, 1, 0]$ and a RDP $[1, 0, 0, 0]$ of type A_3.

(iii)

$$\det\begin{pmatrix} t_0 & t_2 & 0 \\ t_2 & t_1 & t_3 \\ 0 & t_3 & -t_2 \end{pmatrix} = -t_0t_1t_2 - t_0t_3^2 + t_2^3 = 0.$$

The surface has an ordinary node at $[1,0,0,0]$ and a RDP of type A_5 at $[0,1,0]$.

(iv)

$$\det\begin{pmatrix} t_0 & -t_3 & t_2 \\ -t_3 & t_1 & t_3 \\ t_2 & t_3 & 0 \end{pmatrix} = t_3^2(-t_0-2t_2) - t_1t_2^2 = 0.$$

It has a double line $t_3 = t_2 = 0$.

(v)

$$\det\begin{pmatrix} 0 & t_2 & t_0 \\ t_2 & -t_0 & t_3 \\ t_0 & t_3 & t_1 \end{pmatrix} = -t_1t_2^2 + 2t_0t_2t_3 + t_0^3 = 0.$$

The surface has a double line $t_0 = t_2 = 0$.

(vi)

$$\det\begin{pmatrix} 0 & t_1 & t_3 \\ t_1 & 0 & t_2 \\ t_3 & t_2 & t_0 \end{pmatrix} = -t_0t_1^2 + 2t_1t_2t_3 = 0.$$

The surface is the union of a plane and a nonsingular quadric.

(vii)

$$\det\begin{pmatrix} t_0 & 0 & 0 \\ 0 & t_1 & t_3 \\ 0 & t_3 & t_2 \end{pmatrix} = t_0(t_1t_2 - t_3^2) = 0.$$

The surface is the union of a plane and a quadratic cone.

(viii)

$$\det\begin{pmatrix} 0 & 0 & t_2 \\ 0 & t_0 & t_3 \\ t_2 & t_3 & t_1 \end{pmatrix} = t_0t_2^2 = 0.$$

The surface is reducible.

□

Remark 9.3.10. Let S be a cone over a plane cubic curve C. We saw in Example 4.2.18 that any irreducible plane cubic curve admits a symmetric determinantal representation. This gives a symmetric determinantal representation of the cone over the cubic; however, it is not defined by a web of conics. In fact, we see from the list in above that no irreducible cone is given by a web of conics. I have not seen an a priori proof of this.

If S is irreducible non-normal surface, then S admits a symmetric determinantal representation. This corresponds to cases (iv) and (v) from the proof of the previous Theorem. Case (iv) (resp. (v)) gives a surface isomorphic to the surface from case (i) (resp. (ii)) of Theorem 9.3.9. We also see that a reducible cubic surface that is not a cone admits a symmetric determinantal representation only if it is the union of an irreducible nonsingular (singular) quadric and its tangent (non-tangent) plane. The plane is a tangent if the quadric is nonsingular; it intersects the quadric transversally.

Remark 9.3.11. The three symmetroid del Pezzo cubic surfaces S can be characterized among all del Pezzo cubics by the property that they admit a double cover $\pi : \bar{S} \to S$ ramified only over the singular points. They can be obtained by a projection of a quadric surface from Example 8.6.6.

9.4 Representation as a Sums of Cubes

9.4.1 Sylvester's pentahedron

Counting constants, we see that it is possible that a general homogeneous cubic form in four variables can be written as a sum of five cubes of linear forms in finitely many ways. Since there are no cubic surfaces singular at five general points, the theory of apolarity tells us that the count of constants gives a correct answer. The following result of J. Sylvester gives more.

Theorem 9.4.1. *A general homogeneous cubic form f in four variables can be written as a sum:*

$$f = l_1^3 + l_2^3 + l_3^3 + l_4^3 + l_5^3, \tag{9.29}$$

where l_i are linear forms in four variables, no two are proportional. The forms are defined uniquely, up to scaling by a cubic root of unity.

Proof The variety of cubic forms $f \in S^3(E^\vee)$ represented as a sum of five cubes lies in the image of the dominant map of 20-dimensional spaces $(E^\vee)^5 \to S^3(E^\vee)$. The subvariety of $(E^\vee)^5$ that consists of 5-tuples of linear forms containing four linearly dependent forms is a hypersurface. Thus, we may assume that in a representation of f as a sum of five cubes of linear forms, any set of four linear forms are linearly independent.

Suppose

$$f = \sum_{i=1}^{5} l_i^3 = \sum_{i=1}^{5} m_i^3.$$

Let x_i, y_i be the points in the dual space $(\mathbb{P}^3)^\vee$ corresponding to the hyperplanes $V(l_i), V(m_i)$. The first five and the last five are distinct points. Consider the linear system of quadrics in $(\mathbb{P}^3)^\vee$ which pass through the points $x_5, y_1, \ldots, y_5$. Its dimension is larger than or equal 3. Choose a web $|W|$ contained in this linear system. Applying the corresponding differential operator to f we find a linear relations between the linear forms l_1, l_2, l_3, l_4. Since we assumed that they are linearly independent, we obtain that all quadrics in the web contain $x_1, \ldots, x_4$. Thus, all quadrics in the web pass through x_i, y_j.

Suppose the union of the sets $X = \{x_1, \ldots, x_5\}$ and $Y = \{y_1, \ldots, y_5\}$ contains nine distinct points. Since three quadrics intersect at ≤ 8 points unless they contain a common curve, the web $|W|$ has a curve B in its base locus. Because an irreducible nondegenerate curve of degree 3 is not contained in the base locus of a web of quadrics, $\deg B \leq 2$. Suppose B contains a line ℓ_0. Since neither X nor Y is contained in a line, we can find a point x_i outside ℓ_0. Consider a plane Π spanned by ℓ_0 and x_i. The restriction of quadrics to Π is a pencil of conics with fixed line ℓ_0 and the base point x_i. This implies that $|W|$ contains a pencil of quadrics of the form $\Pi \cup \Pi'$, where Π' belongs to a pencil of planes containing a line ℓ passing through x_i. Since $X \cup Y$ is contained in the base locus of any pencil in $|W|$, we see that $X \cup Y \subset \Pi \cup \ell$. Now, we change the point x_i to some other point y_j not in Π. We find that $X \cup Y$ is contained in $\Pi' \cup \ell'$. Hence, the set is contained in $(\Pi \cup \ell) \cap (\Pi' \cup \ell')$. It is the union of ℓ_0 and a set Z consisting of either the line $\langle x_i, y_j \rangle$ or a set of ≤ 3 points. This implies that one of the sets X and Y has four points on ℓ_0. Then, X or Y spans a plane, a contradiction.

Suppose B is a conic. Then, we restrict $|W|$ to the plane Π it spans, and obtain that $|W|$ contains a net of quadrics of the form $\Pi \cup \Pi'$, where Π' is a net of planes. This implies that $X \cup Y$ is contained in $\Pi \cup Z$, where Z is either empty or consists of one point. Again this leads to contradiction with the assumption on linear independence of the points.

We may now assume that $m_5 = \lambda_5 \ell_5, m_4 = \lambda_4 l_4$, for some nonzero constants λ_4, λ_5, and get

$$\sum_{i=1}^{3} l_i^3 + (1 - \lambda_4^3) l_4^3 + (1 - \lambda_5^3) l_5^3 = \sum_{i=1}^{3} m_i^3.$$

Take the linear differential operator of the second order corresponding to the double plane containing the points y_1, y_2, y_3. It gives a linear relation between $l_1, \ldots, l_5$ which must be trivial. Since the points $y_1, y_2, y_3, y_4 = x_4$ and $y_1, y_2, y_3, y_5 = x_5$ are not coplanar, we obtain that $\lambda_4^3 = \lambda_5^3 = 1$. Taking the differential operator of the first order corresponding to the plane through y_1, y_2, y_3, we obtain a linear relation between the quadratic forms l_1^2, l_2^2, l_3^2. Since no two of l_1, l_2, l_3 are proportional, this is impossible. Thus, all the coefficients in the linear relation are equal to zero, hence $x_1, x_2, x_3, y_1, y_2, y_3$ are coplanar.

The linear system of quadrics through $y_1, \ldots, y_5$ is 4-dimensional. By an argument from above, each quadric in the linear system contains x_1, x_2, x_3 in its base locus. Since $x_1, x_2, x_3, y_1, y_2, y_3$ lie in a plane Π, and no three points x_i's or y_j's are collinear, the restriction of the linear system to the plane is a fixed conic containing the six points. This shows that the dimension of the linear system is less than or equal to 3. This contradiction shows that the sets $\{x_1, x_2, x_3\}, \{y_1, y_2, y_3\}$ have two points in common. Thus, we can write

$$l_1^3 + (1 - \lambda_2^3) l_2^3 + (1 - \lambda_3^3) l_3^3 = m_1^3.$$

The common point of the planes $V(l_1), V(l_2), V(l_3)$ lies on $V(m_1)$. After projecting from this point, we obtain that the equation of a triple line can be written as a sum of cubes of three linearly independent linear forms. This is obviously impossible. So, we get $\lambda_2^3 = \lambda_3^3 = 1$, hence $m_1 = \lambda_1 l_1$, where $\lambda_1^3 = 1$. So, all cubes λ_i^3 are equal to 1.

□

Corollary 9.4.2. *A general cubic surface is projectively isomorphic to a surface in $\mathbb{P}^4$ given by equations*

$$\sum_{i=0}^{4} a_i z_i^3 = \sum_{i=0}^{4} z_i = 0. \tag{9.30}$$

The coefficients $(a_0, \ldots, a_4)$ are determined uniquely up to permutation and a common scaling.

Proof Let $S = V(f)$ be a cubic surface given by Equation (9.30). Let $b_0 l_1 + \cdots + b_4 l_5 = 0$ be a unique, up to proportionality, linear relation. Consider the embedding of $\mathbb{P}^3$ into $\mathbb{P}^4$ given by the formula

$$[y_0, \ldots, y_4] = [l_1(t_0, \ldots, t_3), \ldots, l_5(t_0, \ldots, t_3)].$$

Then, the image of S is equal to the intersection of the cubic hypersurface $V(\sum y_i^3)$ with the hyperplane $V(\sum b_i y_i)$. Now, make the change of coordinates $z_i = b_i y_i$, if $b_i \neq 0$ and $z_i = y_i$ otherwise. In the new coordinates, we get Equation (9.30), where $a_i = b_i^3$. The Sylvester presentation is unique, up to permutation of the linear functions l_i, multiplication l_i by third roots of 1, and a common scaling. It is clear that the coefficients $(a_0, \ldots, a_4)$ are determined uniquely up to permutation and common scaling. □

We refer to equations (9.30) as *Sylvester equations* of a cubic surface.

Recall from Subsection 6.3.5, that a cubic surface $V(f)$ is called Sylvester nondegenerate if it admits Equation (9.29), where any four linear forms are linearly independent.

It is clear that in this case the coefficients $a_1, \ldots, a_5$ are all nonzero.

If four of the linear forms in (9.29) are linearly dependent, after a linear change of variables, we may assume that $l_1 = t_0, l_2 = t_1, l_3 = t_2, l_4 = t_3, l_5 = at_0 + bt_1 + ct_2$. The equation becomes

$$f = t_3^3 + g(t_0, t_1, t_2), \tag{9.31}$$

where g_3 is a ternary cubic form. We called such surfaces $V(f)$ cyclic.

Remark 9.4.3. Suppose a cubic surface $V(f)$ admits Sylvester equations. Then, any net of polar quadrics admits a common polar pentahedron. The condition that a net of quadrics admits a common polar pentahedron is given by the vanishing of the Toeplitz invariant Λ from (1.68). Using this fact, Toeplitz gave another proof of the existence of the Sylvester pentahedron for a general cubic surface [749].

9.4.2 The Hessian surface

Suppose S is given by the Sylvester equations (9.30). Let us find the equation of its Hessian surface $\mathrm{He}(S)$ of S. Recall that $\mathrm{He}(S)$ is the locus of points whose polar quadric is singular. For our surface S lying in the hyperplane plane $H = V(\sum z_i) \subset \mathbb{P}^4$, this means that this is the locus of points $z = [\alpha_0, \ldots, \alpha_4] \in H$ with $\sum \alpha_i = 0$ such that the polar quadric is tangent to H at some point. The equation of the polar quadric is $\sum \alpha_i a_i z_i^2 = 0$.

It is tangent to H if the point $[1, \ldots, 1]$ lies in the dual quadric $\sum \frac{1}{\alpha_i a_i} u_i^2 = 0$. Here, we omit the term with $a_i = 0$. Thus, the equation of the Hessian surface is

$$\sum_{i=0}^{4} \frac{1}{a_i z_i} = 0, \quad \sum z_i = 0,$$

where we have to reduce to the common denominator to get an equation of a quartic hypersurface. If all $a_i \neq 0$, we get the equation

$$z_0 \cdots z_5 \Big(\sum_{i=0}^{4} \frac{A_i}{z_i}\Big) = \sum_{i=0}^{4} z_i = 0, \tag{9.32}$$

where $A_i = (a_0 \cdots a_5)/a_i$. If some coefficients a_i are equal to zero, say $a_0 = \ldots = a_k = 0$, the Hessian surface $\mathrm{He}(S)$ becomes the union of planes $V(z_i) \cap V(\sum z_i), i = 0, \ldots, k$, and a surface of degree $3 - k$.

Assume that $S = V(f)$ is Sylvester nondegenerate, so the Hessian surface $\mathrm{He}(S)$ is irreducible. The ten lines

$$\ell_{ij} = V(z_i) \cap V(z_j) \cap V(\sum z_i)$$

are contained in $\mathrm{He}(S)$. The ten points

$$p_{ijk} = V(z_i) \cap V(z_j) \cap V(z_j) \cap V(\sum z_i)$$

are singular points of $\mathrm{He}(S)$.

The union of the planes $V(z_i) \cap V(\sum z_i)$ is called the *Sylvester pentahedron*, the lines ℓ_{ij} are its *edges*, the points p_{ijk} are its *vertices*.

Remark 9.4.4. Recall that the Hessian of any cubic hypersurface admits a birational automorphism σ which assigns to the polar quadric of corank 1 its singular point. Let X be a minimal nonsingular model of $\mathrm{He}(S)$. It is a K3 surface. The birational automorphism σ extends to a biregular automorphism of X. It exchanges the proper transforms of the edges with the exceptional curves of the resolution. One can show that for a general S, the automorphism of X has no fixed points, and hence the quotient is an Enriques surface.

We know that a cubic surface admitting a degenerate Sylvester equation must be a cyclic surface. Its Hessian is the union of a plane and the cone over a cubic curve. A cubic form may not admit a polar pentahedral, so its equation may not be written as a sum of powers of linear forms. For example, consider a cubic surface given by equation

$$t_0^3 + t_1^3 + t_2^3 + t_3^3 + 3t_3^2(at_0 + bt_1 + ct_2) = 0.$$

For a general choice of the coefficients, the surface is nonsingular and non-cyclic. Its Hessian surface has the equation

$$t_0t_1t_2t_3 + t_0t_1t_2(at_0 + bt_1 + ct_2) - t_3^2(a^2t_1t_2 + b^2t_0t_2 + c^2t_0t_1) = 0.$$

It is an irreducible surface with an ordinary node at $[0, 0, 0, 1]$ and singular points $[0, 0, 1, 0]$, $[0, 1, 0, 0]$, $[1, 0, 0, 0]$ of type A_3. So, we see that the surface cannot be Sylvester nondegenerate. The surface does not admit a polar pentahedral, it admits a generalized polar pentahedral in which two of the planes coincide. We refer to [626] and [204] for more examples of cubic surfaces with degenerate Hessian.

Proposition 9.4.5. *A cubic surface given by a nondegenerate Sylvester equation* (9.30) *is nonsingular if and only if, for all choices of signs,*

$$\sum_{i=0}^{4} \pm \frac{1}{\sqrt{a_i}} \neq 0. \tag{9.33}$$

Proof The surface is singular at a point $(z_0, \dots, z_4)$ if and only if

$$\operatorname{rank}\begin{pmatrix} a_0z_0^2 & a_1z_1^2 & a_2z_2^2 & a_3z_3^2 & a_4z_4^2 \\ 1 & 1 & 1 & 1 & 1 \end{pmatrix} = 1.$$

This gives $a_iz_i^2 = c, i = 0, \dots, 3$, for some $c \neq 0$. Thus, $z_i = \pm c/\sqrt{a_i}$ for some choice of signs, and the equation $\sum z_i = 0$ gives (9.33). Conversely, if (9.33) holds for some choice of signs, then $[\pm\frac{1}{\sqrt{a_0}}, \dots, \pm\frac{1}{\sqrt{a_4}}]$ satisfies $\sum z_i = 0$ and $\sum a_iz_i^3 = 0$. It also satisfies the equations $a_it_i^2 = a_jt_j^2$. Thus, it is a singular point. □

9.5 The Segre Cubic Primal

9.5.1 Cremona's hexahedral equations

The Sylvester theorem has the deficiency that it cannot be applied to any nonsingular cubic surface. The Cremona's hexahedral equations that we consider here work for any nonsingular cubic surface.

Theorem 9.5.1 (L. Cremona)**.** *Assume that a cubic surface S is not a cone and admits a Cayley-Salmon equation (e.g. S is a nonsingular surface). Then, S is isomorphic to a cubic surface in $\mathbb{P}^5$ given by the equations*

$$\sum_{i=0}^{5} t_i^3 = \sum_{i=0}^{5} t_i = \sum_{i=0}^{5} a_it_i = 0. \tag{9.34}$$

Proof Let $S = V(l_1l_2l_3 + m_1m_2m_3)$ be a Cayley-Salmon equation of S. Let us try to find some constants such that the linear forms, after scaling, add up to zero. Write

$$l_i' = \lambda_il_i, \quad m_i' = \mu_im_i, \ i = 1, 2, 3.$$

Since S is not a cone, four of the linear forms are linearly independent. After reordering the linear forms, we may assume that the linear forms l_1, l_2, l_3, m_1 are linearly independent. Let

$$m_2 = al_1 + bl_2 + cl_3 + dm_1, \ m_3 = a'l_1 + b'l_2 + c'l_3 + d'l_4.$$

The constants λ_i, μ_i must satisfy the following system of equations

$$\lambda_1 + a\mu_2 + a'\mu_3 = 0,$$
$$\lambda_2 + b\mu_2 + b'\mu_3 = 0,$$
$$\lambda_3 + c\mu_2 + c'\mu_3 = 0,$$
$$\mu_1 + d\mu_2 + d'\mu_3 = 0,$$
$$\lambda_1\lambda_2\lambda_3 + \mu_1\mu_2\mu_3 = 0.$$

The first four linear equations allow us to express linearly all unknowns in terms of μ_2, μ_3. Plugging in the last equation, we get a cubic equation in μ_2/μ_3. Solving it, we get a solution. Now, set

$$z_1 = l_2' + l_3' - l_1', \quad z_2 = l_3' + l_1' - l_2', \quad z_3 = l_1' + l_2' - l_3',$$

$$z_4 = \mu_2' + \mu_3' - \mu_1', \quad z_5 = \mu_3' + \mu_1' - \mu_2', \quad z_6 = \mu_1' + \mu_2' - \mu_3'.$$

One checks that these six linear forms satisfy the equations from the assertion of the Theorem. □

Equations (9.34) of a del Pezzo cubic surface are called *Cremona's hexahedral equations*. We will return to these equations in the next subsection, where we will show that they apply to any cubic surface obtained from the anti-canonical model of the blow-up a semi-stable set of points in $\mathbb{P}^2$.

Corollary 9.5.2 (T. Reye). *A general homogeneous cubic form f in four variables can be written as a sum of six cubes in ∞^4 different ways. In other words,*

$$\dim \mathrm{VSP}(f, 6)^o = 4.$$

Proof This follows from the proof of the previous theorem. Consider the map

$$(\mathbb{C}^4)^6 \to \mathbb{C}^{20}, \quad (l_1, \ldots, l_6) \mapsto l_1^3 + \cdots + l_6^3.$$

It is enough to show that it is dominant. We show that the image contains the open subset of nonsingular cubic surfaces. In fact, we can use a Cayley-Salmon equation $l_1l_2l_3 + m_1m_2m_3$ for $S = V(f)$ and apply the proof of the theorem to obtain that, up to a constant factor,

$$f = z_1^3 + z_2^3 + z_3^3 + z_4^3 + z_5^3 + z_6^3.$$

□

Suppose a nonsingular cubic surface S is given by equations (9.34). They allow us to locate 15 lines on S such that the remaining lines form a double-six. The equations of these lines in $\mathbb{P}^5$ are

$$z_i + z_j = 0,\ z_k + z_l = 0,\ z_m + z_n = 0,\ \sum_{i=1}^{6} a_i z_i = 0,$$

where $\{i, j, k, l, m, n\} = \{1, 2, 3, 4, 5, 6\}$. Let us denote the line given the above equations by $l_{ij,kl,mn}$.

Let us identify a pair a, b of distinct elements in $\{1, 2, 3, 4, 5, 6\}$ with a transposition (ab) in $\mathfrak{S}_6$. We have the product $(ij)(kl)(mn)$ of three commuting transpositions corresponding to each line $l_{ij,kl,mn}$. The group $\mathfrak{S}_6$ admits a unique (up to a composition with a conjugation) outer automorphism which sends each transposition to the product of three commuting transpositions. In this way, we can match lines $l_{ij,kl,mn}$ with exceptional vectors c_{ab} of the E_6-lattice. To do it explicitly, one groups together five products of three commuting transpositions in such a way that they do not contain a common transposition. Such a set is called a *total* and the triples (ij, kl, mn) are called *synthemes*. Here, is the set of six totals

$$\begin{aligned}
T_1 &= (12)(36)(45),\ (13)(24)(56),\ (14)(26)(35),\ (15)(23)(46),\ (16)(25)(34), \quad (9.35)\\
T_2 &= (12)(36)(45),\ (13)(25)(46),\ (14)(23)(56),\ (15)(26)(34),\ (16)(24)(35),\\
T_3 &= (12)(35)(46),\ (13)(24)(56),\ (14)(25)(36),\ (15)(26)(34),\ (16)(23)(45),\\
T_4 &= (12)(34)(56),\ (13)(25)(46),\ (14)(26)(35),\ (15)(24)(36),\ (16)(23)(45),\\
T_5 &= (12)(34)(56),\ (13)(26)(45),\ (14)(25)(36),\ (15)(23)(46),\ (16)(24)(35),\\
T_6 &= (12)(35)(46),\ (13)(26)(45),\ (14)(23)(56),\ (15)(24)(36),\ (16)(25)(34).
\end{aligned}$$

Two different totals T_a, T_b contain one common product $(ij)(kl)(mn)$. The correspondence $(a, b) \mapsto (ij)(kl)(mn)$ defines the outer automorphism

$$\alpha : \mathfrak{S}_6 \to \mathfrak{S}_6. \tag{9.36}$$

For example, $\alpha((12)) = (12)(36)(45)$ and $\alpha((23)) = (15)(26)(34)$.

After we matched the lines $l_{ij,kl,mn}$ with exceptional vectors c_{ab}, we check that this matching defines an isomorphism of the incidence subgraph of the lines with the subgraph of the incidence graph of 27 lines on a cubic surface whose vertices correspond to exceptional vectors c_{ab} .

Theorem 9.5.3. *Cremona's hexahedral equations of a nonsingular cubic surface S defines an ordered double-six of lines. Conversely, a choice of an ordered double-six defines uniquely Cremona hexahedral equations of S.*

Proof We have seen already the first assertion of the theorem. If two surfaces given by hexahedral equations define the same double-six, then they have in common 15 lines. Obviously, this is impossible. Thus, the number of different hexahedral equations of S is less than or equal to 36. Now, consider the identity

$$(z_1 + \cdots + z_6)\big((z_1 + z_2 + z_3)^2 + (z_4 + z_5 + z_6)^2 - (z_1 + z_2 + z_3)(z_4 + z_5 + z_6)\big)$$

$$= (z_1 + z_2 + z_3)^3 + (z_4 + z_5 + z_6)^3 = z_1^3 + \cdots + z_6^3$$

$$+3(z_2 + z_3)(z_1 + z_3)(z_1 + z_2) + 3(z_4 + z_5)(z_5 + z_6)(z_4 + z_6).$$

It shows that Cremona hexahedral equations define a Cayley-Salmon equation

$$(z_2 + z_3)(z_1 + z_3)(z_1 + z_2) + (z_4 + z_5)(z_5 + z_6)(z_4 + z_6) = 0,$$

where we have to eliminate one unknown with help of the equation $\sum a_i z_i = 0$. Applying permutations of $z_1, \ldots, z_6$, we get 10 Cayley-Salmon equations of S. Each set of nine lines formed by the corresponding conjugate pair of triads of tritangent planes are among the 15 lines determined by the hexahedral equation. It follows from the classification of the conjugate pairs that we have 10 such pairs of lines c_{ij}'s (type II). Thus, a choice of Cremona hexahedral equations defines exactly 10 Cayley-Salmon equations of S. Conversely, it follows from the proof of Theorem 9.5.1 that each Cayley-Salmon equation gives three Cremona hexahedral equations (unless the cubic equation has a multiple root). Since we have 120 Cayley-Salmon equations for S we get $36 = 360/10$ hexahedral equations for S. They match with 36 double-sixes. □

9.5.2 Invariants of six points in $\mathbb{P}^1$

Let $p_1, \ldots, p_m$ be a set of points in $\mathbb{P}^n$, where $m > n + 1$. For any ordered subset $(p_{i_1}, \ldots, p_{i_{n+1}})$ of $n + 1$ points, we denote by $(i_1 \ldots i_{n+1})$ the determinant of the matrix whose rows are projective coordinates of the points $(p_{i_1}, \ldots, p_{i_{n+1}})$ in this order. We consider $(i_1 \ldots i_{n+1})$ as a section of the invertible sheaf $\otimes_{j=1}^{n+1} p_{i_j}^* \mathcal{O}_{\mathbb{P}^n}(1)$ on $(\mathbb{P}^n)^m$. It is called a *bracket-function*. A monomial in bracket-functions such that each index $i \in \{1, \ldots, m\}$ occurs exactly d times defines a section of the invertible sheaf

$$\mathcal{L}_d = \bigotimes_{i=1}^{n} p_i^* \mathcal{O}_{\mathbb{P}^n}(d).$$

According to the Fundamental Theorem of Invariant Theory (see [242]) the subspace $(R_n^m)(d)$ of $H^0((\mathbb{P}^n)^m, \mathcal{L}_d)$ generated by such monomials is equal to the space of invariants $H^0((\mathbb{P}^n)^m, \mathcal{L}_d)^{\mathrm{SL}(n+1}$, where the group $\mathrm{SL}(n + 1)$ acts linearly on the space of sections via its diagonal action on $(\mathbb{P}^n)^m$. The graded ring

$$R_n^m = \bigoplus_{d=0}^{\infty}(R_n^m)(d) \tag{9.37}$$

is a finitely generated algebra. Its projective spectrum is isomorphic to the GIT-quotient

$$P_n^m := (\mathbb{P}^n)^m//\mathrm{SL}(n+1)$$

of $(\mathbb{P}^n)^m$ by $\mathrm{SL}(n+1)$. The complement U^{ss} of the set of common zeros of generators of the algebra R_n^m admits a regular map to P_n^m. The set U^{ss} does not depend on the choice of generators. Its points are called *semi-stable*. Let U^{s} be the largest open subset such that the fibers of the restriction map $U^{\mathrm{s}} \to P_n^m$ are orbits. Its points are called *stable*.

It follows from the Hilbert-Mumford numerical stability criterion that an ordered set of points $(p_1, \ldots, p_m)$ in $\mathbb{P}^1$ is semi-stable (resp. stable) if and only if at most $\frac{1}{2}m$ (resp. $< \frac{1}{2}m$) points coincide. We have already seen the definition of the bracket functions in the case $m = 4$. They define the cross ratio of four points

$$[p_1, p_2, p_3, p_4] = \frac{(12)(34)}{(13)(24)}.$$

The cross ratio can be viewed as a rational map $(\mathbb{P}^1)^4 \dashrightarrow \mathbb{P}^1$. It is defined on the open set U^{s} of points where no more than two coincide and it is an orbit space over the complement of three points $0, 1, \infty$.

In the case of points in $\mathbb{P}^2$ the condition of stability (semi-stability) is that at most $\frac{1}{3}m$ (resp. $< \frac{1}{3}m$) coincide and at most $\frac{2}{3}m$ (resp. $< \frac{2}{3}m$) points are on a line.

Proposition 9.5.4. *Let $\mathcal{P} = (p_1, \cdots, p_6)$ be an ordered set of distinct points in $\mathbb{P}^1$. The following conditions are equivalent.*

(i) There exists an involution of $\mathbb{P}^1$ such that the pairs $(p_1, p_2), (p_3, p_4)$, (p_5, p_6) are orbits of the involution.
(ii) The binary forms $g_i, i = 1, 2, 3$, with zeros $(p_1, p_2), (p_3, p_4), (p_5, p_6)$ are linearly dependent.
(iii) Let x_i be the image of p_i under a Veronese map $\mathbb{P}^1 \to \mathbb{P}^2$. Then, the lines $\langle x_1, x_2\rangle, \langle x_3, x_4\rangle, \langle x_5, x_6\rangle$ are concurrent.
(iv) The bracket-function $(14)(36)(25) - (16)(23)(54)$ vanishes at $\mathcal{P}$.

Proof (i) $\Leftrightarrow$ (ii) Let $f : \mathbb{P}^1 \to \mathbb{P}^1$ be the degree 2 map defined by the involution. Let f be given by $[t_0, t_1] \mapsto [g_1(t_0, t_1), g_2(t_0, t_1)]$, where g_1, g_2 are binary forms of degree 2. By choosing coordinates in the target space, we may assume that $f(p_1) = f(p_2) = 0, f(p_3) = f(p_4) = 1, f(p_5) = f(p_6) = \infty$, i.e. $g_1(p_1) = g_1(p_2) = 0, g_2(p_3) = g_2(p_4) = 0, (g_1 - g_2)(p_5) = (g_1 - g_2)(p_6) = 0$. Obviously, the binary forms $g_1, g_2, g_3 = g_1 - g_2$ are linearly dependent. Conversely, suppose g_1, g_2, g_3 are linearly dependent. By scaling, we may assume that $g_3 = g_1 - g_2$. We define the involution by $[t_0, t_1] \mapsto [g_1(t_0, t_1), g_2(t_0, t_1)]$.

(ii) $\Leftrightarrow$ (iii) Without loss of generality, we may assume that $p_i = [1, a_i]$ and $g_1 = t_1^2 - (a_1 + a_2)t_0t_1 + a_1a_2t_0^2, g_2 = t_1^2 - (a_3 + a_4)t_0t_1 + a_3a_4t_0^2, g_3 = t_1^2 - (a_5 + a_6)t_0t_1 + a_5a_6t_0^2$. The condition that the binary forms are linearly dependent is

$$D = \det\begin{pmatrix} 1 & a_1 + a_2 & a_1a_2 \\ 1 & a_3 + a_4 & a_3a_4 \\ 1 & a_5 + a_6 & a_5a_6 \end{pmatrix} = 0. \tag{9.38}$$

The image of p_i under the Veronese map $[t_0, t_1] \mapsto [t_0^2, t_0t_1, t_2^2]$ is the point $x_i = [1, a_i, a_i^2]$. The line $\langle x_i, x_j \rangle$ has equation

$$\det\begin{pmatrix} t_0 & t_1 & t_2 \\ 1 & a_i & a_i^2 \\ 1 & a_j & a_j^2 \end{pmatrix} = (a_j - a_i)(a_ia_jt_0 - (a_i + a_j)t_1 + t_2) = 0.$$

Obviously, the three lines are concurrent if and only if (9.38) is satisfied.

(iii) $\Leftrightarrow$ (iv) We have

$$\begin{pmatrix} 1 & a_1 + a_2 & a_1a_2 \\ 1 & a_3 + a_4 & a_3a_4 \\ 1 & a_5 + a_6 & a_5a_6 \end{pmatrix} \cdot \begin{pmatrix} a_1^2 & a_3^2 & a_5^2 \\ -a_1 & -a_3 & -a_5 \\ 1 & 1 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & (a_3 - a_1)(a_3 - a_2) & (a_5 - a_1)(a_5 - a_2) \\ (a_1 - a_3)(a_1 - a_4) & 0 & (a_5 - a_3)(a_5 - a_4) \\ (a_1 - a_5)(a_1 - a_6) & (a_3 - a_5)(a_3 - a_6) & 0 \end{pmatrix}.$$

Taking the determinant, we obtain

$$D(a_1 - a_3)(a_1 - a_5)(a_3 - a_5) =$$

$$= \det\begin{pmatrix} 0 & (a_3 - a_1)(a_3 - a_2) & (a_5 - a_1)(a_5 - a_2) \\ (a_1 - a_3)(a_1 - a_4) & 0 & (a_5 - a_3)(a_5 - a_4) \\ (a_1 - a_5)(a_1 - a_6) & (a_3 - a_5)(a_3 - a_6) & 0 \end{pmatrix} =$$

$$= (a_3 - a_5)(a_5 - a_1)(a_1 - a_3)[(a_1 - a_4)(a_3 - a_6)(a_5 - a_2) + (a_6 - a_1)(a_2 - a_3)(a_4 - a_5)].$$

Since the points are distinct, canceling by the product $(a_3 - a_5)(a_5 - a_1)(a_1 - a_3)$, we obtain

$$(a_1 - a_4)(a_3 - a_6)(a_5 - a_2) + (a_6 - a_1)(a_2 - a_3)(a_4 - a_5) =$$

$$= (14)(36)(25) - (16)(23)(54) = 0.$$

□

We let

$$[ij, kl, mn] := (il)(kn)(jm) - (jk)(lm)(ni). \tag{9.39}$$

For example, $[12, 34, 56] = (14)(36)(25) - (16)(23)(54)$. Note that determinant (9.38) does not change if we permute $(a_i, a_{i+1}), i = 1, 3, 5$. It also does not change if we apply an even permutation of the pairs, and changes the sign if we apply an odd permutation.

Let us identify the set $(1, 2, 3, 4, 5, 6)$ with points $(\infty, 0, 1, 2, 3, 4, 5)$ of the projective line $\mathbb{P}^1(\mathbb{F}_5)$. The group $\mathrm{PSL}(2, \mathbb{F}_5) \cong \mathfrak{A}_5$ acts on $\mathbb{P}^1(\mathbb{F}_5)$ via Moebius transformations $z \mapsto \frac{az+b}{cz+d}$. Let $u_0 = [\infty 0, 14, 23]$ and let $u_i, i = 1, \dots, 4$, be obtained from u_0 via the action of the transformation $z \mapsto z + i$. Let

$$U_1 := u_0 + u_1 + u_2 + u_3 + u_4$$

$$= \big([\infty 0, 14, 23] + [\infty 1, 20, 34] + [\infty 2, 31, 40] + [\infty 3, 42, 01] + [\infty 4, 03, 12]\big).$$

Obviously, U_1 is invariant under the subgroup of order 5 generated by the transformation $z \mapsto z + 1$. It is also invariant under the transformation $\tau : z \mapsto -1/z$. It is well known that $\mathfrak{A}_5$ is generated by these two transformations. The orbit of U_∞ under the group $\mathfrak{A}_6$ acting by permutations of $\infty, 0, \dots, 4$ consists of six functions $U_1, U_2, U_3, U_4, U_5, U_6$. We will rewrite them now returning to our old notation of indices by the set $(1, 2, 3, 4, 5, 6)$.

$$\begin{pmatrix} U_1 \\ U_2 \\ U_3 \\ U_4 \\ U_5 \\ U_6 \end{pmatrix} = \begin{pmatrix} 0 & [12,36,45] & [13,24,56] & [14,35,26] & [15,46,23] & [16,25,34] \\ & 0 & [15,26,34] & [13,46,25] & [16,35,24] & [14,23,56] \\ & & 0 & [16,23,45] & [14,25,36] & [12,35,46] \\ & & & 0 & [12,34,56] & [15,36,24] \\ & & & & 0 & [13,45,26] \\ & & & & & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}, \tag{9.40}$$

where the matrix is skew-symmetric. We immediately observe that

$$U_1 + U_2 + U_3 + U_4 + U_5 + U_6 = 0. \tag{9.41}$$

Next observe that the triples of pairs $[ij, kl, mn]$ in each row of the matrix constitute a total from (9.35). One easily computes the action of $\mathfrak{S}_6$ on U_i's. For example,

$$(12) : (U_1, U_2, U_3, U_4, U_5, U_6) \mapsto (-U_2, -U_1, -U_6, -U_5, -U_4, -U_3).$$

Its trace is equal to 1.

Recall that there are four isomorphism classes of irreducible 5-dimensional linear representations of the permutation group $\mathfrak{S}_6$. They differ by the trace of a transposition (ij).

If the trace is equal to 3, the representation is isomorphic to the standard representation V_{st} in the space

$$V = \{(z_1, \ldots, z_6) \in \mathbb{C}^6 : z_1 + \cdots + z_6 = 0\}.$$

It coincides with the action of the Weyl group $W(A_6)$ on the root lattice A_6. It corresponds to the partition $(5, 1)$ of 6.

If the trace is equal to -3, the representation is isomorphic to the tensor product of the standard representation and the 1-dimensional sign representation. It corresponds to the dual partition $(2, 1, 1, 1, 1)$.

If the trace is equal to 1, the representation is isomorphic to the composition of the outer automorphism $\alpha : \mathfrak{S}_6 \to \mathfrak{S}_6$ and the standard representation. It corresponds to the partition $(3, 3)$.

If the trace is equal to -1, the representation is isomorphic to the tensor product of the previous representation and the sign representation. It corresponds to the partition $(2, 2, 2)$.

So, our representation on the linear space $V = (R_1^6)(1)$ associated with the partition $(3, 3)$.

One checks that the involution $(12)(34)(56)$ acts as

$$(U_1, U_2, U_3, U_4, U_5, U_6) \mapsto (-U_1, -U_2, -U_3, -U_5, -U_4, -U_6). \tag{9.42}$$

Its trace is equal to -3. A well-known formula from the theory of linear representations

$$\dim V^G = \frac{1}{\#G} \sum_{g \in G} \mathrm{Trace}(g)$$

shows that the dimension of the invariant subspace for the element $(12)(34)(56)$ is equal to 1. It follows from (9.42) that the function $U_4 - U_5$ is invariant. On the other hand, we also know that the function $[12, 34, 56]$ is invariant too. This gives $U_4 - U_5 = c[12, 36, 54]$ for some scalar c. Evaluating these functions on a point set $(p_1, \ldots, p_6)$ with $p_1 = p_2, p_3 = p_6, p_4 = p_5$ we find that $c = 6$. Now, applying permutations we obtain:

$$U_1 - U_2 = 6[12, 36, 45],\ U_1 - U_3 = 6[13, 24, 56],\ U_1 - U_4 = 6[14, 35, 26], \quad (9.43)$$
$$U_1 - U_5 = 6[15, 46, 23],\ U_1 - U_6 = 6[16, 25, 34],\ U_2 - U_3 = 6[15, 26, 34],$$
$$U_2 - U_4 = 6[13, 46, 25],\ U_2 - U_5 = 6[16, 35, 24],\ U_2 - U_6 = 6[14, 23, 56],$$
$$U_3 - U_4 = 6[16, 45, 23],\ U_3 - U_5 = 6[14, 25, 36],\ U_3 - U_6 = 6[12, 46, 35],$$
$$U_4 - U_5 = 6[12, 43, 56],\ U_4 - U_6 = 6[15, 36, 24],\ U_5 - U_6 = 6[13, 45, 26].$$

Similarly, we find that $U_1 + U_2$ is the only anti-invariant function under σ and hence coincides with $c(12)(36)(45)$. After evaluating the functions at a point set $(p_1, \ldots, p_6)$ with $p_1 = p_3, p_2 = p_4, p_5 = p_6$ we find that $c = 4$. In this way, we get the relations:

$$U_1 + U_2 = 4(12)(36)(45),\ U_1 + U_3 = 4(13)(42)(56),\ U_1 + U_4 = 4(41)(53)(26), \quad (9.44)$$
$$U_1 + U_5 = 4(15)(46)(32),\ U_1 + U_6 = 4(16)(25)(34),\ U_2 + U_3 = 4(15)(26)(43),$$
$$U_2 + U_4 = 4(13)(46)(25),\ U_2 + U_5 = 4(16)(35)(42),\ U_2 + U_6 = 4(14)(23)(56),$$
$$U_3 + U_4 = 4(16)(54)(32),\ U_3 + U_5 = 4(14)(25)(63),\ U_3 + U_6 = 4(12)(46)(53),$$
$$U_4 + U_5 = 4(12)(34)(56),\ U_4 + U_6 = 4(15)(36)(24),\ U_5 + U_6 = 4(13)(45)(62).$$

Applying (9.41), we obtain

$$U_1 = (12)(36)(45) + (13)(42)(56) + (14)(35)(26) + (15)(46)(32) + (16)(25)(34), \quad (9.45)$$
$$U_2 = (12)(36)(45) + (13)(46)(25) + (14)(56)(23) + (15)(26)(43) + (16)(24)(53),$$
$$U_3 = (12)(53)(46) + (13)(42)(56) + (14)(52)(36) + (15)(26)(43) + (16)(23)(45),$$
$$U_4 = (12)(34)(56) + (13)(46)(25) + (14)(35)(26) + (15)(24)(36) + (16)(23)(45),$$
$$U_5 = (12)(34)(56) + (13)(54)(26) + (14)(52)(36) + (15)(46)(32) + (16)(24)(53),$$
$$U_6 = (12)(53)(46) + (13)(54)(26) + (14)(56)(23) + (15)(36)(24) + (16)(25)(34).$$

We see that our functions are in bijective correspondence with six totals from above. The functions $U_1, \ldots, U_6$ are known as the *Joubert functions*.

It is easy to see that the functions U_i do not vanish simultaneously on semistable point sets. Thus, they define a morphism

$$J : \mathrm{P}_1^6 \to \mathbb{P}^5.$$

Theorem 9.5.5. *The morphism J defined by the Joubert functions is an isomorphism onto the subvariety S_3 of $\mathbb{P}^5$ given by the equations*

$$\sum_{i=0}^{5} z_i = \sum_{i=0}^{5} z_i^3 = 0. \quad (9.46)$$

Proof It is known that the graded ring R_1^6 is generated by the following bracket-functions (*standard tableaux*):

$$(12)(34)(56),\ (12)(35)(46),\ (13)(24)(56),\ (13)(25)(46),\ (14)(25)(36)$$

(see [234]). The subspace of $R_1^6(1)$ generated by the Joubert functions is invariant with respect to $\mathfrak{S}_6$. Since $R_1^6(1)$ is an irreducible representation, this implies that the relation $\sum U_i = 0$ spans the linear relations between the Joubert functions. Consider the sum $\Sigma = \sum U_i^3$. Obviously, it is invariant with respect to $\mathfrak{A}_6$. One immediately checks that an odd permutation in $\mathfrak{S}_6$ transforms each sum Σ to $-\Sigma$. This implies that $\Sigma = 0$ whenever two points p_i and p_j coincide. Hence, Σ must be divisible by the product of 15 functions (ij). This product is of degree 5 in coordinates of each point but Σ is of degree 3. This implies that $\Sigma = 0$. Since the functions U_i generate the graded ring R_1^6, by definition of the space P_1^6, we obtain an isomorphism from P_1^6 to a closed subvariety of S_3. Since the latter is irreducible and of dimension equal to the dimension of P_1^6, we obtain the assertion of the theorem. □

9.5.3 Segre cubic primal and Cremona's hexahedral equations

The cubic threefold S_3 defined in Theorem 9.5.5 is called the *Segre cubic primal.* [1] We will often consider it as a hypersurface in $\mathbb{P}^4$.

It follows immediately by differentiating that the cubic hypersurface S_3 has 10 double points. They are the points $p = [1, 1, 1, -1, -1, -1]$ and others obtained by permuting the coordinates. A point p is given by the equations $z_i + z_j = 0, 1 \le i \le 3, 4 \le j \le 6$. By using (9.41) this implies that p is the image of a point set with $p_1 = p_4 = p_6$ or $p_2 = p_3 = p_5$. Thus, the singular points of the Segre cubic primal are the images of semi-stable, but not stable, point sets.

Also, S_3 has 15 planes

$$\Pi_{ij,kl,mn} : z_i + z_j = z_k + z_l = z_l + z_m = 0. \tag{9.47}$$

Let us see that they are the images of point sets with two points that coincide. Without loss of generality, we may assume that $z_1 + z_2 = z_3 + z_4 = z_5 + z_6 = 0$. Again from (9.41), we obtain that $(12)(36)(45), (16)(23)(45)$ and $(13)(26)(45)$ vanish. This happens if and only if $p_4 = p_5$.

We know that the locus of point sets $(q_1, \ldots, q_6)$ such that the pairs (q_i, q_j), (q_k, q_l), and (q_m, q_n) are orbits of an involution are defined by the equation $[ij, kl, mn] = 0$. By (9.43), we obtain that they are mapped to a hyperplane section of S_3 defined by the equation $z_a - z_b = 0$, where $\alpha((ab)) = (ij)(kl)(mn)$.

It follows from Cremona's hexahedral equations that a nonsingular cubic surface is isomorphic to a hyperplane section of the Segre cubic. In a Theorem below we will make it more precise. But first, we need some lemmas.

Lemma 9.5.6. *Let $x_1, \ldots, x_6$ be six points in $\mathbb{P}^2$. Let $\{1, \ldots, 6\} = \{i, j\} \cup$*

[1] According to classical terminology, a *primal* is a hypersurface in a projective space (see, for example, [701, p. 10].

$\{k,l\} \cup \{m,n\}$. *The condition that the lines* $\langle x_i, x_j\rangle, \langle x_k, x_l\rangle, \langle x_m, x_n\rangle$ *are concurrent is*

$$(ij,kl,mn) := (kli)(mnj) - (mni)(klj) = 0. \tag{9.48}$$

Proof The expression $(kli)(mnx) - (mni)(klx)$ can be considered as a linear function defining a line in $\mathbb{P}^2$. Plugging in $x = x_i$ we see that it passes through the point x_i. Also if x is the intersection point of the lines $\langle x_k, x_l\rangle$ and $\langle x_m, x_n\rangle$, then, writing the coordinates of x as a linear combination of the coordinates of $x_k x_l$, and of $x_m x_n$, we see that the line passes through the point x. Now, equation (9.48) expresses the condition that the point x_j lies on the line passing through x_i and the intersection point of the lines $\langle x_k, x_l\rangle$ and $\langle x_m, x_n\rangle$. This proves the assertion. □

The functions (ij,kl,mn) change the sign after permuting two numbers in one pair. They change the sign after permuting two pairs of numbers.

It is known (see [234]) that the space $R_2^6(1)$ is generated by bracket-functions $(ijk)(lmn)$. Its dimension is equal to 5 and it has a basis corresponding to standard tableaux

$$(123)(456),\ (124)(356),\ (125)(346),\ (134)(256),\ (135)(246).$$

The group $\mathfrak{S}_6$ acts linearly on this space via permuting the numbers $1,\ldots,6$.

Let

$$\begin{pmatrix}\bar U_1\\ \bar U_2\\ \bar U_3\\ \bar U_4\\ \bar U_5\\ \bar U_6\end{pmatrix} = \begin{pmatrix} 0 & (12,36,45) & (13,24,56) & (14,26,35) & (15,46,23) & (16,34,25)\\ & 0 & (15,26,34) & (13,25,46) & (16,35,24) & (14,56,23)\\ & & 0 & (16,45,23) & (14,36,25) & (12,46,35)\\ & & & 0 & (12,56,34) & (15,36,24)\\ & & & & 0 & (13,45,26)\\ & & & & & 0 \end{pmatrix}\begin{pmatrix}1\\1\\1\\1\\1\\1\end{pmatrix}.$$

Equations (9.43) extend to the functions $\bar U_i$.

Note that the transposition (12) acts on the functions $\bar U$ as

$$(\bar U_1,\bar U_2,\bar U_3,\bar U_4,\bar U_5,\bar U_6) \mapsto (\bar U_2,\bar U_1,\bar U_6,\bar U_5,\bar U_4,\bar U_3).$$

The trace is equal to -1. This shows that the representation $(R_2^6)(1)$ is different from the representation $(R_1^6)(1)$; it is associated to the partition $(2,2,2)$. One checks that the substitution $(12)(34)(56)$ acts by

$$(\bar U_1,\bar U_2,\bar U_3,\bar U_4,\bar U_5,\bar U_6) \mapsto (\bar U_1,\bar U_2,\bar U_3,\bar U_5,\bar U_4,\bar U_6).$$

The trace is equal to 3. This implies that the sign representation enters the representation of the cyclic group $\langle(12)(34)(56)\rangle$ on $(R_2^6)(1)$ with multiplicity 1. Thus, the space of anti-invariant elements is one-dimensional. It is spanned by $\bar U_4 - \bar U_5$. Since the function $(12)(34)(56)$ is anti-invariant, we obtain that $\bar U_4 - \bar U_5 = c(12)(34)(56)$. Again, as above, we check that $c = 6$. In this way, the equations (9.43) extend to the functions $\bar{U}_i$ with $[ij, kl.mn]$ replaced with $(ij)(kl)(mn)$.

Lemma 9.5.7. *We have the relation*

$$\bar{U}_1 + \bar{U}_2 + \bar{U}_3 = -6(146)(253) \tag{9.49}$$

and similar relations obtained from this one by permuting the set $(1, \ldots, 6)$.

Proof Adding up, we get

$$\bar{U}_1 + \bar{U}_2 + \bar{U}_3 = \big((14, 26, 35) + (14, 56, 23) + (14, 25, 36)\big) + \big((16, 34, 25)$$

$$+(16, 35, 24) + (16, 45, 23)\big) + \big((15, 46, 23) + (13, 25, 46) + (12, 46, 35)\big).$$

Next we obtain

$$(14, 26, 35) + (14, 56, 23) + (14, 25, 36) = (142)(536) - (146)(532) + (146)(523)$$

$$-(143)(526) + (142)(563) - (143)(562) = -2(146)(253),$$

$$(16, 34, 25) + (16, 35, 24) + (16, 45, 23) = (163)(524) - (164)(523) + (165)(243)$$

$$-(163)(245) + (164)(325) - (165)(324) = -2(146)(253),$$

$$(15, 46, 23) + (13, 25, 46) + (12, 46, 35) = (465)(312) - (462)(315) + (463)(152)$$

$$-(461)(153) + (465)(123) - (463)(125) = 2\big((465)(312) - (462)(315) + (463)(152)\big).$$

Now, we use the Plücker relation (10.4)

$$(ijk)(lmn) - (ijl)(kmn) + (ijm)(kln) - (ijn)(klm) = 0. \tag{9.50}$$

It gives

$$(465)(312) - (462)(315) + (463)(152) = -(146)(253).$$

Collecting all of this together, we get the assertion. □

Let $(p_1, \ldots, p_6)$ be a fixed ordered set of six points in $\mathbb{P}^2$. Consider the following homogeneous cubic polynomials in coordinates $x = (t_0, t_1, t_2)$ of a point in $\mathbb{P}^2$.

$$F_1 = (12x)(36x)(45x) + (13x)(42x)(56x) + (14x)(26x)(35x) + (15x)(46x)(32x) + (16x)(34x)(25x),$$
$$F_2 = (12x)(36x)(45x) + (13x)(25x)(46x) + (14x)(56x)(23x) + (15x)(26x)(43x) + (16x)(24x)(53x),$$
$$F_3 = (12x)(35x)(46x) + (13x)(42x)(56x) + (14x)(52x)(36x) + (15x)(26x)(43x) + (16x)(45x)(23x),$$
$$F_4 = (12x)(34x)(56x) + (13x)(46x)(25x) + (14x)(35x)(26x) + (15x)(36x)(24x) + (16x)(23x)(45x),$$
$$F_5 = (12x)(34x)(56x) + (13x)(54x)(26x) + (14x)(52x)(36x) + (15x)(46x)(32x) + (16x)(24x)(53x),$$
$$F_6 = (12x)(53x)(46x) + (13x)(54x)(26x) + (14x)(56x)(23x) + (15x)(36x)(24x) + (16x)(25x)(34x).$$

The next theorem shows that hyperplane sections of the Segre cubic primal are cubics surfaces together with a choice of their Cremona's hexahedral equations.

Theorem 9.5.8. *The rational map*

$$\Phi : \mathbb{P}^2 \dashrightarrow \mathbb{P}^5,\ x \mapsto [(F_1(x), \ldots, F_6(x)]$$

has the image given by the equations

$$z_1^3 + z_2^3 + z_3^3 + z_4^3 + z_5^3 + z_6^3 = 0, \tag{9.51}$$

$$z_1 + z_2 + z_3 + z_4 + z_5 + z_6 = 0,$$

$$a_1 z_1 + a_2 z_2 + a_3 z_3 + a_4 z_4 + a_5 z_5 + a_6 z_6 = 0,$$

where $(a_1, \ldots, a_6)$ *are the values of* $(\bar{U}_1, \ldots, \bar{U}_6)$ *at the point set* $(p_1, \ldots, p_6)$*. They satisfy* $a_1 + \cdots + a_6 = 0$*.*

Proof Take $x = (1, 0, 0)$, then each determinant (ijx) is equal to the determinant (ij) for the projection of $p_1, \ldots, p_6$ to $\mathbb{P}^1$. Since all the bracket-functions are invariant with respect to SL(3) we see that any (ijx) is the bracket-function for the projection of the points to $\mathbb{P}^1$ with center at x. This shows that the relations for the functions U_i imply similar relations for the polynomials F_i. This is an example of Clebsch's transfer principle, which we discussed in Subsection 3.4.2. Let us find the additional relation of the form $\sum_{i=0}^{5} a_i z_i = 0$. Consider the cubic curve

$$C = a_1 F_1(x) + \cdots + a_6 F_6(x) = 0,$$

where $a_1, \ldots, a_6$ are as in the assertion of the theorem. We have already noted that (ij, kl, mn) are transformed by $\mathfrak{S}_6$ in the same way as $(ij)(kl)(mn)$ up to the sign representation. Thus, the expression $\sum_i a_i F_i(x)$ is transformed to itself under an even permutation and is transformed to $-\sum_i a_i F_i(x)$ under an odd permutation. Thus, the equation of the cubic curve is invariant with respect to the order of the points $p_1, \ldots, p_6$. Obviously, C vanishes at the points p_i. Suppose we prove that C vanishes at the intersection point of the lines $\langle p_1, p_2 \rangle$ and $\langle p_3, p_4 \rangle$, then, by symmetry, it vanishes at the intersection points of all possible pairs of lines, and hence contains five points on each line. Since C is of degree 3 this implies that C vanishes on 15 lines; hence, C is identically zero and we are done.

So, let us prove that the polynomial C vanishes at the point $p = \langle p_1, p_2 \rangle \cap \langle p_3, p_4 \rangle$. Recall from analytic geometry (or multi-linear algebra) that p can be represented by the vector $(v_1 \times v_2) \times (v_3 \times v_4) = (v_1 \wedge v_2 \wedge v_3)v_4 - (v_1 \wedge v_2 \wedge v_4)v_3 = (123)v_4 - (124)v_3$. Thus, the value of (ijx) at p is equal to

$$(ijp) = (123)(ij4) - (124)(ij3) = (12)(ij)(34). \tag{9.52}$$

Applying Clebsch's transfer principle to (9.44), we obtain

$$F_1(x) + F_2(x) = 4(12x)(36x)(45x), \;\; F_4(x) + F_5(x) = 4(12x)(34x)(56x),$$

$$F_1(x) + F_6(x) = 4(16x)(25x)(34x), \;\; F_3(x) + F_6(x) = 4(12x)(53x)(46x),$$

$$F_2(x) + F_3(x) = (15x)(26x)(43x).$$

This implies that $F_1 + F_2, \; F_4 + F_5, \; F_1 + F_6, \; F_3 + F_6, \; F_2 + F_3$ all vanish at p. Thus, the value of C at p is equal to

$$(a_4 - a_5)F_4(p) + (a_2 + a_6 - a_1 - a_3)F_6(p)$$

$$= (a_4 - a_5)(F_4(p) + F_6(p)) + (a_2 + a_6 + a_5 - a_1 - a_3 - a_4)F_6(p)$$

$$= (a_4 - a_5)(F_4(p) + F_6(p)) + (a_2 + a_5 + a_6)(F_1(p) + F_3(p)).$$

Here, we used that $a_1 + \cdots + a_6 = 0$ and $F_1(p) + F_3(p) + 2F_6(p) = 0$. By Lemma 9.5.7,

$$a_4 - a_5 = (a_4 + a_1 + a_2) - (a_5 + a_1 + a_2) = 6(125)(436) - 6(126)(435) = 6(12, 43, 56).$$

$$a_2 + a_5 + a_6 = 6(346)(125).$$

By using (9.44) and (9.52), we get

$$F_4(p) + F_6(p) = (51p)(42p)(36p) = (42p)(12, 34, 15)(12, 36, 34),$$

$$F_1(p) + F_3(p) = (13p)(42p)(56p) = (42p)(12, 56, 34)(12, 13, 34).$$

Collecting this together, we obtain that the value of $\frac{1}{6}C$ at p is equal to

$$(12, 43, 56)(42p)[(12, 34, 15)(12, 36, 34) + (125)(436)(12, 13, 34)).$$

It remains to check that

$$(12, 34, 15)(12, 36, 34) + (125)(436)(12, 13, 34)$$

$$= (125)(314)(123)(364) + (125)(463)(123)(134) = 0.$$

□

Recall that the Segre cubic contains 15 planes defined by equations (9.47) $\Pi_{ij,kl,mn}$, where $\{i,j\}\cup\{k,l\}\cup\{m,n\}=[1,6]$. The intersection of this plane with the hyperplane $H:\sum a_iz_i=0$ is the union of three lines on the cubic surface. In this way we see 15 lines. Each hyperplane $H_{ij}:z_i+z_j=0$ cuts out the Segre cubic S_3 along the union of three planes $\Pi_{ij,kl,mn}$, where the union of $\{k,l\}$ and $\{m,n\}$ is equal to $[1,6]\setminus\{i,j\}$. The hyperplane H intersects $H_{ij}\cap\mathsf{S}_3$ along the union of three lines. Thus, we see 15 tritangent planes and 15 lines forming a configuration (15_3). This is a subconfiguration of the configuration $(27_5,45_3)$ of 27 lines and 45 tritangent planes on a nonsingular cubic surface.

The Segre cubic is characterized by the property that it has 10 nodes.

Theorem 9.5.9. *Let S be a normal cubic hypersurface in $\mathbb{P}^4$ with 10 ordinary double nodes. Then, S is isomorphic to the Segre cubic primal.*

Proof Choose projective coordinates such that one of the singular points is the point $[1,0,0,0,0]$. The equation of S can be written in the form

$$t_0A(t_1,\dots,t_4)+B(t_1,\dots,t_4)=0.$$

By taking the partials, we obtain that the degree 6 curve $C=V(A,B)$ in $\mathbb{P}^3$ has nine singular points. Since $[1,0,0,0,0]$ is an ordinary double point, the quadratic form A is nondegenerate. Thus, the curve C is a curve of bidegree $(3,3)$ on a nonsingular quadric $V(A)$. It is a curve of arithmetic genus with nine singular points. It is easy to see that this is possible only if C is the union of six lines, two triples of lines from each of the two rulings. Since $\mathrm{Aut}(\mathbb{P}^1)$ acts transitively on the set of ordered triple of points, we can fix the curve C. Two cubics $V(B)$ and $V(B')$ cut out the same curve C on $V(A)$ if and only if $B'-B=AL$, where L is a linear form. Replacing t_0 by t_0+L, we can fix B. □

It follows from the proof that no cubic hypersurface in $\mathbb{P}^4$ has more than ten ordinary double points. Thus, the Segre cubic primal can be characterized, up to projective equivalence, by the property that it has maximal number of ordinary double points.

Proposition 9.5.10. *Let $p_1,\dots,p_5$ be points in $\mathbb{P}^3$ in general linear position. The linear system of quadrics through these points defines a rational map $\mathbb{P}^3\dashrightarrow\mathbb{P}^4$ whose image is isomorphic to the Segre cubic primal.*

Proof It is clear that the dimension of the linear system is equal to 4. To compute the degree of the image, we have to compute the number of intersection points of three general quadrics from the linear system and subtract the number of base points. Three general quadrics intersect at eight points, subtracting five, we get three. So, the image of the rational map is a cubic hypersurface

S in $\mathbb{P}^4$. For each line $\ell_{ij} = \langle p_i, p_j \rangle$, the general member of the linear system intersects ℓ_{ij} only at the points p_i, p_j. This implies that the image of the line in $\mathbb{P}^4$ is a point. It is easy to see that no other line in $\mathbb{P}^4$, except the ten lines ℓ_{ij}, is blown down to a point. This implies that the image of ℓ_{ij} is an isolated singular point of S. Let $Y \to \mathbb{P}^3$ be the blow-up of the points $p_1, \ldots, p_5$. The composition $f : Y \to \mathbb{P}^3 \dashrightarrow \mathbb{P}^4$ defines a regular birational map from Y to S. It is a small resolution of S in the sense that the pre-images of the singular points are not divisors but curves. Let $Y' \to Y$ be the blow-up of the proper transforms of the lines ℓ_{ij} in Y. The normal bundle of a line ℓ n $\mathbb{P}^3$ is isomorphic to $\mathcal{O}_\ell(1) \oplus \mathcal{O}_\ell(1)$. It follows from some elementary facts of the intersection theory (see [315], Appendix B.6) that the normal bundle of the proper transform $\bar{\ell}_{ij}$ of ℓ_{ij} is isomorphic to $\mathcal{O}_{\bar{\ell}_{ij}}(-1) \oplus \mathcal{O}_{\bar{\ell}_{ij}}(-1)$. This implies that the pre-image of $\bar{\ell}_{ij}$ in Y' is isomorphic to the product $\mathbb{P}^1 \times \mathbb{P}^1$. Thus, the composition $Y' \to Y \to S$ is a resolution of singularities with the exceptional divisor over each singular point of S isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$. It is well known that it implies that each singular point of S is an ordinary double point of S. Applying Theorem 9.5.9, we obtain that S is isomorphic to the Segre cubic primal. □

Remark 9.5.11. According to [302], the Segre cubic primal admits 1024 small resolutions in the category of complex manifolds. By the action of $\mathfrak{S}_6$ they are divided into 13 isomorphism classes. Six of the classes give projective resolutions.

9.5.4 Castenuovo-Richmond quartic threefold

The coefficients $(a_1, \ldots, a_6)$ in Theorem 9.5.8 can be viewed as elements of the 5-dimensional linear space $V = (R_2^6)(1)$. Since the functions $\bar{U}_i$ add up to zero, $a_1 + \cdots + a_6 = 0$. They map the moduli space P_2^6 of ordered sets of six points in $\mathbb{P}^2$ to the hyperplane $V(\sum t_i)$ in $\mathbb{P}^5$. We know that the action of $\mathfrak{S}_6$ on P_2^6 defines an irreducible representation of type $(2, 2, 2)$ on V and the functions $\bar{U}_i$ are transformed according to the same representation. It is known that the algebra R_2^6 is generated by the space $(R_2^6)(1)$ and one element Υ from $(R_2^6)(2)$ [159, §40], [234, Chapter I]. We have

$$\Upsilon = (123)(145)(246)(356) - (124)(135)(236)(456). \tag{9.53}$$

If we replace 6 with x and consider this as an equation of a conic in $\mathbb{P}^2$, we observe that the expression vanishes when $x = p_1, p_2, p_3, p_4, p_5$. Thus, the conic passes through the points $p_1, p_2, p_3, p_4, p_5, x$. So, the function Υ vanishes on the set of points $(p_1, \ldots, p_6)$ lying on a conic. This is a hypersurface X in

P_2^6. One shows that Υ^2 is a polynomial of degree 4 in generators of $(R_2^6)(1)$. This implies that the image of X is a quartic hypersurface in $\mathbb{P}(V)$. Since the map $X \to \mathbb{P}(V)$ is $\mathfrak{S}_6$-equivariant, the image of X can be given by a $\mathfrak{S}_6$-invariant polynomial in t_i. Since the representation V is self-dual, and is obtained from the standard representation of $\mathfrak{S}_6$ on V by composing with the outer automorphism, the invariant functions are symmetric polynomials. So, the equation of the image of X is equal to

$$s_2^2 + \lambda s_4 = 0,$$

where $s_k = \sum_{i=0}^5 t_i^k$. The coefficient λ can be found from the fact that the hypersurface X is singular at the locus of strictly semi-stable points represented by points sets $p_i = p_j$ and the remaining four points are collinear. The locus consists of 15 lines. A simple computation shows that the only symmetric quartic with this property is the quartic $V(s_2^2 - 4s_4)$ (see [322], Theorem 4.1).

The quartic threefold CR_4 in $\mathbb{P}^5$ given by the equations

$$\sum_{i=0}^{5} t_i = 0, \quad (\sum_{i=0}^{5} t_i^2)^2 - 4\sum_{i=0}^{5} t_i^4 = 0 \tag{9.54}$$

will be called the *Castelnuovo-Richmond quartic*.

Corollary 9.5.12. *The variety* P_2^6 *is isomorphic to the double cover of* $\mathbb{P}^4$ *ramified over the Castelnuovo-Richmond quartic. It can be given by the equations*

$$t_5^2 + (\sum_{i=0}^{5} t_i^2)^2 - 4\sum_{i=0}^{5} t_i^4 = 0, \quad \sum_{i=0}^{5} t_i = 0. \tag{9.55}$$

in $\mathbb{P}(1,1,1,1,1,2)$.

The involution $(t_0,\ldots,t_6) \mapsto (t_0,\ldots,t_5,-t_6)$ is the *association involution*. Applying it to the projective equivalence class of a general point set $(p_1,\ldots,p_6)$ we obtain the projective equivalence class of a set $(q_1,\ldots,q_6)$ such that the blow-ups of the two sets are isomorphic cubic surfaces, and the two geometric markings are defined by a double-six. We refer for all of this to [234].

Consider the projective dual variety $(\mathsf{S}_3)^\vee$ of the Segre cubic primal. Since S_3 has ten ordinary nodes, the Plücker-Teissier formula shows that $(\mathsf{S}_3)^\vee$ is a quartic hypersurface. The duals of the hyperplanes H_{ij} define 15 points in the dual $\mathbb{P}^4$. The duals of the planes $\Pi_{ij,kl,mn}$ are 15 lines. They are singular lines of CR_4. The 15 lines and 15 points form a configuration (15_3) in the dual space.

Proposition 9.5.13. *The dual variety of the Segre cubic primal is isomorphic* *to the Castelnuovo-Richmond quartic hypersurface:*

$$\mathsf{CR}_4 \cong (\mathsf{S}_3)^\vee.$$

Proof We may assume that S_3 is given by the equation $\sum_{i=0}^4 t_i^3 - (\sum_{i=0}^4 t_i)^3 = 0$ in $\mathbb{P}^4$, and the group $\mathfrak{S}_6$ acts by letting its subgroup $\mathfrak{S}_5$ permute $t_0, \ldots, t_4$ and sending the transposition (56) to the transformation $t_i \mapsto t_i, i \le 4, t_4 \mapsto -L$, where $L = t_0 + \cdots + t_4$. The polar map is given by polynomials $F_i = t_i^2 - L^2, i = 0, \ldots, 4$. After a linear change of the coordinates y_i in the target space

$$y_i' = y_i - \frac{1}{3}(y_0 + y_1 + y_2 + y_3 + y4), \quad i = 0, \ldots, 4,$$

we obtain that the linear representation of $\mathfrak{S}_6$ on the target space is isomorphic to the representation on the t_i's. Thus, the dual hypersurface is isomorphic to a quartic threefold in $\mathbb{P}^5$ given by the equations

$$\sum_{i=0}^5 y_i = 0, \quad s_2^2 + \lambda s_4 = 0,$$

where $s_k = \sum_{i=0}^5 y_i^k$. Under the polar map, the 15 planes in $\mathfrak{S}_3$ are mapped to 15 singular lines on the dual variety. A straightforward computation shows that this implies that the parameter λ is equal to -4 (see [322], Theorem 4.1). □

9.6 Moduli Spaces of Cubic Surfaces

9.6.1 Projective invariants of cubic surfaces

The methods of the Geometric Invariant Theory (GIT) allow one to construct the moduli space of nonsingular cubic surfaces $\mathcal{M}_{\mathrm{cub}}$ as an open subset of the GIT-quotient

$$\mathbb{P}(S^3((\mathbb{C}^4))//\mathrm{SL}(4) = \mathrm{Proj} \bigoplus_{d=0}^{\infty} S^d(S^3((\mathbb{C}^4)^\vee)^\vee)^{\mathrm{SL}(4)}. \tag{9.56}$$

The analysis of stability shows that, except for one point, the points of this variety represent the orbits of cubic surfaces with ordinary double points. The exceptional point corresponds to the isomorphism class of a unique cubic surface with three A_2-singularities. It is isomorphic to the surface $V(xyz + w^3)$. So, the GIT-quotient is a natural compactification $\overline{\mathcal{M}}_{\mathrm{cub}}$ of the moduli space $\mathcal{M}_{\mathrm{cub}}$. The computations from the classical invariant theory due to G. Salmon [650], [654] and A. Clebsch [139] (see a modern exposition in [416]) show that the graded ring of invariants is generated by elements I_d of degrees $d =$

8, 16, 24, 32, 40 and 100 (a modern proof of completeness can be found in [52]).

The first four basic invariants are invariants with respect to the group G of invertible matrices with the determinant equal to ± 1. This explains why their degrees are divisible by 8 (see [242]). The last invariant is what the classics called a skew invariant, it is not an invariant of G but an invariant of $\mathrm{SL}(4)$. There is one basic relation expressing I_{100}^2 as a polynomial in the remaining invariants. The graded subalgebra generated by elements of degree divisible by 8 is freely generated by the first five invariants. Since the projective spectrum of this subalgebra is isomorphic to the projective spectrum of the whole algebra, we obtain an isomorphism

$$\overline{\mathcal{M}}_{\mathrm{cub}} \cong \mathbb{P}(8, 16, 24, 32, 40) \cong \mathbb{P}(1, 2, 3, 4, 5). \tag{9.57}$$

This, of course, implies that the moduli space of cubic surfaces is a rational variety.

The discriminant Δ of a homogeneous cubic form in four variables is expressed in terms of the basic invariants by the formula

$$\Delta = (I_8^2 - 64 I_{16})^2 - 2^{14}(I_{32} + 2^{-3} I_8 I_{24}) \tag{9.58}$$

(the exponent -3 is missing in Salmon's formula, and also, the coefficient at I_{32} was wrong, it is corrected in [204]).

We may restrict the invariants to the open Zariski subset of Sylvester nondegenerate cubic surfaces, It allows one to identify the first four basic invariants with symmetric functions of the coefficients of the Sylvester equations. Salmon's computations give

$$I_8 = \sigma_4^2 - 4\sigma_3\sigma_5, \quad I_{16} = \sigma_1\sigma_5^3, \quad I_{24} = \sigma_4\sigma_5^4, \quad I_{32} = \sigma_2\sigma_5^6, \quad I_{40} = \sigma_5^8, \tag{9.59}$$

where σ_i are elementary symmetric polynomials. Evaluating Δ from above, we obtain a symmetric polynomial of degree 8 obtained from (9.33) by eliminating the irrationality.

The invariant I_{40} restricts to $(a_0a_1a_2a_3a_4)^8$. It does not vanish on the set of Sylvester nondegenerate cubic surfaces. Its locus of zeros is the closure of the locus of Sylvester-degenerate nonsingular cubic surfaces.

The skew-invariant I_{100} is given by the equation

$$I_{100} = (a_0a_1a_2a_3a_4)^{19} \det \begin{pmatrix} a_0 & a_1 & a_2 & a_3 & a_4 \\ a_0^{-1} & a_1^{-1} & a_2^{-1} & a_3^{-1} & a_4^{-1} \\ a_0^2 & a_1^2 & a_2^2 & a_3^2 & a_4^2 \\ a_0^3 & a_1^3 & a_2^3 & a_3^3 & a_4^3 \\ 1 & 1 & 1 & 1 & 1 \end{pmatrix}.$$

It vanishes on the closure of the locus of nonsingular surfaces with an Eckardt point. Observe that it vanishes if $a_i = a_j$ and that agrees with Example 9.1.25.

Following [204] we can interpret (9.59) as a rational map

$$\mathbb{P}(\mathbb{C}^4)/\mathfrak{S}_5 \cong \mathbb{P}(1,2,3,4,5) \dashrightarrow \overline{\mathcal{M}}_{\mathrm{cub}} \cong \mathbb{P}(1,2,3,4,5).$$

We have

$$\sigma_1 = \frac{I_{16}}{\sigma_5^3}, \quad \sigma_2 = \frac{I_{32}}{\sigma_5^6}, \quad \sigma_3 = \frac{I_{24}^2 - I_8 I_{40}}{\sigma_5^9}, \quad \sigma_4 = \frac{I_{24}I_{40}}{\sigma_5^{12}}, \quad \sigma_5 = \frac{I_{40}^2}{\sigma_5^{15}}.$$

This gives the inverse rational map

$$\overline{\mathcal{M}}_{\mathrm{cub}} \dashrightarrow \mathbb{P}(\mathbb{C}^4)/\mathfrak{S}_5.$$

The map is not defined at the set of points where all the invariants I_{8d} vanish except I_8. It is shown in [204], Theorem 6.1 that the set of such points is the closure of the orbit of the Fermat cubic surface.

A cubic surface in $\mathbb{P}^3$ can be given as a hyperplane section of a cubic threefold in $\mathbb{P}^4 = |W|$. In this way, the theory of projective invariants of cubic surfaces becomes equivalent to the theory of projective invariants of PGL(5) in the space $S^3(W^\vee) \times W^\vee$. The Cremona hexahedral equations of a cubic surface represent a subvariety of this representation isomorphic to $\mathbb{C}^6$. Clebsch's transfer principle (for a modern explanation see [416]) allows one to express projective invariants of GL(4) as polynomial functions on $\mathbb{C}^6$. The degree of an invariant polynomial of degree m is equal to its weight $3m/4$. In particular, the basic polynomials $I_8, \ldots, I_{100}$ become polynomials $J_6, J_{12}, J_{18}, J_{24}, J_{30}, J_{75}$ in $(a_1, \ldots, a_6)$ of degrees indicated in the subscript. The first five polynomials are symmetric polynomials in $a_1, \ldots, a_6$, the last one is a skew-symmetric polynomial. For example,

$$J_6 = 24(4\sigma_2^3 - 3\sigma_3^2 - 16\sigma_2\sigma_4 + 12\sigma_6)$$

(see [155] Part III, p. 336, and [718]).

The skew-invariant J_{75} defining the locus of cubic surfaces with an Eckardt points is reducible. It contains as a factor of degree 15 the discriminant $\prod_{i<j}(a_i - a_j)$ of the polynomial $(X - a_1)\cdots(X - a_6)$. The remaining factor of degree 60 is equal to the product of 30 polynomials of the form

$$T_{1256;3} = (126)(356)(134)(253) - (136)(256)(123)(354), \tag{9.60}$$

where we use Lemma 9.5.7 to express the product of two brackets as a function $a_i + a_j + a_k$. The vanishing of $T_{1256;3}$ expresses the condition that the conic through the points p_1, p_2, p_3, p_5, p_6 is touched at p_3 by the line $\langle p_3, p_4\rangle$ (equivalently, the tritangent plane defined by the lines $e_3, 2e_0 - \sum e_i, e_0 - e_3 - e_4$ has an Eckardt point). Together with 15 polynomials $\bar{U}_i - \bar{U}_j$, this accounts for 45 hypersurfaces defining the locus of cubic surfaces with an Eckardt point. Note that the formulas for $\bar{U}_i - \bar{U}_j$ and $T_{1256;3}$ allow one to compute the number of Eckardt points on a surface given by Cremona's hexahedral equations. For example, if we have one pair of equal coefficients a_i, we have an Eckardt point on the surface. However, it is not a necessary condition, because an Eckardt point may arise from the vanishing of a function of type $T_{1256;3}$. For example, a cyclic cubic surface has nine Eckardt points, and they cannot be found only from the equalities of the coefficients a_i.

We can also find the expression of the discriminant invariant Δ (9.58) in terms of the coefficients $a_0, \ldots, a_5$.

We know that the quartic symmetric polynomials $\sigma_2^2 - 4\sigma_4$ in $a_1, \ldots, a_6$ equal to the squares of the function Υ from (9.53) representing points sets on a conic. Thus, we see that the discriminant invariant in $(a_0, \ldots, a_5)$, being of of degree 24, must be a scalar multiple of the product of powers of $(\sigma_2 - 4\sigma_4)$ and powers of $(a_i + a_j + a_k)$, $1 \le i < j < k \le 5$ representing points sets with three collinear points. The only way to make a symmetric polynomial of degree 24 in this way is to take all factors in the first power. We also use that σ_1 vanishes on $(a_1, \ldots, a_6)$. The computer computation gives the following expression in terms of the elementary symmetric polynomials:

$$\Delta = (\sigma_2^2 - 4\sigma_4)(\sigma_3^4\sigma_4^2 - 2\sigma_2\sigma_3^3\sigma_4\sigma_6 + \sigma_2^2\sigma_3^2\sigma_5^2 + 2\sigma_3^2\sigma_4\sigma_5^2 - 2\sigma_2\sigma_3\sigma_5^4 + 2\sigma_2\sigma_3\sigma_4\sigma_6 -$$

$$8\sigma_3^2\sigma_4^2\sigma_6 - 2\sigma_2^3\sigma_3\sigma_5\sigma_6 + 8\sigma_2\sigma_3\sigma_4\sigma_5\sigma_5\sigma_6 + 2\sigma_2^2\sigma_5^2\sigma_6 + \sigma_2^4\sigma_6^2 - 8\sigma_2^2\sigma_4\sigma_6^2 + 16\sigma_4^2\sigma_6^2).$$

9.6.2 Moduli of cubic surfaces with some additional structure

One can also describe the moduli spaces of cubic surfaces equipped with some additional structure. For example, we know that a choice of an ordered sixer of lines on a cubic surface S is equivalent to putting the equation of S in a Cremona's hexahedral form. The moduli space of smooth cubic surfaces together with an ordered sixer of lines is isomorphic to an open subspace of the moduli space P_2^6 which we discussed in subsection 9.5.1. It is equal to the open subset of P_2^6 equal to the image in the GIT-quotient of the open set of sixtuples of points $(p_1, \ldots, p_6)$ such that the their blow-up is a nonsingular cubic surface. We denote this open subset by $\mathcal{M}_{\mathrm{cub}}^{\mathrm{gm}}$ and call it the moduli space of geometrically marked cubic surfaces. We already mentioned that the algebra R_2^6 of invariants on the space $(\mathbb{P}^2)^6$ is generated by the symmetric algebra of the linear subspace $(R_2^6)(1) \cong \mathbb{C}^5$ of invariants $t_0, \ldots, t_4$ of degree and one element of degree two Υ from (9.53). It is known that the square of Υ is a polynomial F_4 of degree four in $t_0, \ldots, t_4$ with $V(F_4)$ projectively equivalent to the Castelnuovo-Richmond quartic. More precisely, denoting Υ by t_5, we get

$$y_5^2 + F_4(t_0, \ldots, t_4) = 0, \tag{9.61}$$

where we rewrite equations (9.55) by eliminating y_5 and replacing y_i with t_i. The involution defined by changing the sign of t_5 corresponds to the association involution that switches the sixers in the double-sixer of lines. The equation (9.61) gives a biregular model of the moduli space P_2^6 as the double cover of $\mathbb{P}^4$ ramified over the Castelnuovo-Richmond quartic. In modern terminology, it is known as the *Coble variety*.

In fact, the explicit identity between bracket-functions gives the relation between Υ and invariants from R_2^6 in the form

$$\Upsilon^2 - (-t_2t_3 + t_1t_4 + t_0t_1 + t_0t_4 - t_0t_2 - t_0t_3 - t_0^2)^2 - 4t_0t_0t_1t_4(-t_0 + t_1 - t_2 - t_3 + t_4) = 0. \tag{9.62}$$

The change of variables

$$(x_0, \ldots, x_5) = (t_0, t_1, t_4, -t_0 - t_2, -t_0 - t_3)$$

leads to the equation of the Castelnuovo-Richmond quartic in the form

$$(x_0x_1 + x_0x_2 - x_2x_4)^2 - 4x_0x_1x_2(x_0 + x_1 + x_2 + x_4) = 0.$$

This equation can be found in [423], where it is shown that the quartic hypersurface defined by this equation is isomorphic to a compactification of the moduli space of principally polarized abelian surface with a 2-level structure. For this reason, in modern literature, the Castelenuovo-Richmond quartic goes under the name *Igusa quartic*.

The blow-up of a set of six points in general position is isomorphic to a cubic surface S. Fixing an order defines a geometric basis of $\mathrm{Pic}(S)$. The Weyl group $W(\mathsf{E}_6)$ acts transitively on geometric bases, and the birational quotient of P_2^6 by the action of $W(\mathsf{E}_6)$ is isomorphic to $\overline{\mathcal{M}}_{\mathrm{cub}}$. The forgetful map

$$\mathrm{P}_2^6 \dashrightarrow \overline{\mathcal{M}}_{\mathrm{cub}} \tag{9.63}$$

is of degree equal to $\#W(\mathsf{E}_6)$. The action of the subgroup $\mathfrak{S}_6$ of the Weyl group is easy to describe. It is a regular action on P_2^6 via permuting the points. In the model of P_2^6 given by equation (9.55), the action is achieved by permuting the coordinates $t_0, \ldots, t_5$ according to the representation of type $(2, 2, 2)$. The quotient is isomorphic to the double cover

$$(\mathsf{P}_2^6)/\mathfrak{S}_6 \to \mathbb{P}^4/\mathfrak{S}_6 \cong \mathbb{P}(2,3,4,5,6).$$

It is ramified over the image of the hypersurface $V(\Upsilon) \subset \mathsf{P}_2^6$ parameterizing points sets on a conic. The branch locus is the image of the Castelnuovo-Richmond quartic CR_4 in the quotient. It is isomorphic to $\mathbb{P}(2,3,5,6)$. In the cubic surface interpretation, the ramification locus is birationally isomorphic to cubic surfaces with a node. This shows that the moduli space of singular cubic surfaces is birationally isomorphic to $\mathbb{P}(2,3,5,6)$, and hence, it is a rational variety.

The quotient $(\mathsf{P}_2^6)/\mathfrak{S}_6$ can be viewed as a birational model of the moduli space of cubic surfaces together with a choice of a double-six. The previous isomorphism shows that this moduli space is rational. In [44], by a beautiful geometric construction, the authors deduce from this fact the rationality of the moduli space of curves C of genus 2 equipped with a cyclic subgroup of $\mathrm{Jac}(C)[3]$. Note that the rationality of the moduli space of cubic surfaces together with an unordered sixer of lines is unknown.

The functions $\bar{U}_i$, taken as generators of the space $(R_2^6)(1)$, allow one to identify some special loci in P_2^6 with ones in $\overline{\mathcal{M}}_{\mathrm{cub}}$. For example, we know from (9.43) that $\bar{U}_1 - \bar{U}_2 = 0$ represents the locus of points sets $(p_1,\ldots,p_6)$ such that the lines $\langle p_1,p_2\rangle, \langle p_3,p_6\rangle, \langle p_4,p_5\rangle$ are collinear. This corresponds to a cubic surface with an Eckardt point. Changing the order of points, this gives 15 hypersurfaces in P_2^6 permuted by $\mathfrak{S}_6$. Another example is a hypersurface $V(\bar{U}_1 + \bar{U}_2 + \bar{U}_3)$. According to Lemma 9.5.7, it corresponds to the locus of points set $(p_1,\ldots,p_6)$, where the points p_1,p_4,p_6 or p_2,p_3,p_5 are collinear. They are permuted by $\mathfrak{S}_6$ and give 20 hypersurfaces in P_2^6. The image of these hypersurfaces under the map (9.63) is contained in the locus of singular surfaces.

The Coble variety P_2^6 is a singular variety on which the Weyl group $W(\mathsf{E}_6)$ acts only birationally. A smooth birational model of P_2^6 on which $W(\mathsf{E}_6)$ acts regularly was constructed by I. Naruki [548]. It admits a birational morphism to P_2^6 which is a resolution of singularities of P_2^6.

Another additional structure on a cubic surface is the fixing of a line on it. Any geometrically marked subic surface $(S,\ell_1,\ldots,\ell_6)$ defined a cubic surface marked with one line, namely (S,ℓ_6). Since the stabilizer subgroup of $W(\mathsf{E}_6)$ of an exceptional vector is equal to $W(\mathsf{D}_5)$, we can define the moduli space $\mathcal{M}_{\mathrm{cub}}^{\mathrm{line}}$ of *one line marked cubic surfaces* as the image of $\mathcal{M}_{\mathrm{cub}}^{\mathrm{gm}}$ under the quotient map

$$\mathsf{P}_2^6 \to \mathsf{P}_2^6/W(\mathsf{D}_5).$$

The quotient $\overline{\mathcal{M}}_{\mathrm{cub}}^{\mathrm{line}} = \mathrm{P}_2^6/W(\mathsf{D}_5)$ is a natural compactification of $\mathcal{M}_{\mathrm{cub}}^{\mathrm{line}}$.

Fixing a line ℓ on a cubic surface, we obtain the pencil of residual conics. Choose the projective coordinates x_0, x_1, x_2, x_3 such that $\ell = V(x_0, x_1)$. Let $\Pi_{s,t}$ be a plane $V(tx_0 - sx_1)$ containing the line. Write the equation F in the form

$$A_3(x_0,x_1) + B_1(x_2,x_3)x_0^2 + 2B_2(x_2,x_3)x_0x_1$$

$$+B_3(x_2,x_3)x_1^2 + C_1(x_2,x_3)x_0 + C_2(x_2,x_3)x_1 = 0.$$

The equation of the residual conic is

$$x_1^2A_3(s,t) + (s^2B_1(x_2,x_3) + 2stB_2(x_2,x_3)$$

$$+t^2B_3(x_2,x_3))x_1 + sC_1(x_2,x_3) + tC_2(x_2,x_3) = 0.$$

One can rewrite the equation of the conic in the form

$$(x_1,x_2,x_3)\cdot\begin{pmatrix} a_{11}(s,t) & a_{12}(s,t) & a_{13}(s,t)\\ a_{12}(s,t) & a_{22}(s,t) & a_{23}(s,t)\\ a_{13}(s,t) & a_{23}(s,t) & a_{33}(s,t)\end{pmatrix}\cdot\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix}, \tag{9.64}$$

where $a_{11}(s,t)$ is a binary form of degree 3, $a_{12}(s,t)$ and $a_{13}(s,t)$ are binary forms of degree 2, and other entries are linear forms. We see that the determinant D_3 of the matrix is a polynomial of degree 5 and its zeros are correspond to tritangent planes. The minor formed by the last two rows and two columns is a polynomial D_2 of degree two. It corresponds to conics tangent to the line ℓ. This defines a map from the moduli space

$$\overline{\mathcal{M}}_{\mathrm{cub}}^{\mathrm{line}} \to \big(\mathbb{P}(S^5(\mathbb{C}^2)) \times \mathbb{P}(S^2(\mathbb{C}^2))\big)/\!/\mathrm{SL}(2),$$

where the GIT-quotient is taken with respect to the linearization defined by the invertible sheaf $p_1^*\mathcal{O}(1) \otimes p_2^*\mathcal{O}(1)$. It is proven in [247, Theorem 3.6] that the map is an isomorphism on the open subset parameterizing nodal cubic surfaces.

Let ℓ' be another line on S disjoint from ℓ. By acting with the groups $W(\mathsf{D}_5)$, we can fix it without changing ℓ. A smooth cubic surface with two disjoint lines is isomorphic to $S' = \mathsf{Bl}_\Sigma(\mathbf{F}_0)$, where Σ is a set of five points, with no two lying on the same ruling and not all lying on a conic. The map $S \to S'$ is given by the linear system $|-2K_S - \ell - \ell'|$. It maps the lines to rational normal cubics B_1 and B_2 intersecting at five points from Σ. We may assume that B_1 is of bidegree $(2,1)$ and B_2 is of bidegree $(1,2)$. The divisor class $B = B_1 + B_2$ is divisible by 3, and hence, there exists a cyclic cover $X \to S'$ of degree 3 branched along B. The usual Hurwitz-type formula gives $\omega_X \cong \mathcal{O}_X$. The surface X' is singular over the singular points of B, a minimal resolution of singularities X'

is a smooth K3 surface with five Dynkin curves of type A_2 over the singular points of X. In [247], the authors prove that the isomorphism class of the K3 surface X' does not depend on a choice of the lines ℓ and ℓ'. The assigning to S the isomorphism class of the K3 surface and using the theory of periods of K3 surfaces, one establishes an isomorphism and defines an isomorphism

$$\overline{\mathcal{M}}_{\mathrm{cub}} \cong \overline{\Gamma\backslash\mathbb{B}_4},$$

from a compactification of the moduli space of cubic surfaces to the quotient of the four-dimensional complex ball by a certain discrete group of automorphisms. A similar uniformization theorem was proved earlier in [8] by assigning to $S = V(F(x_0, x_1, x_2, x_3))$ the cubic hypersurface $Y = V(x_4^3 + F(x_0, x_1, x_2, x_3)$ and using the fact that the intermediate Jacobian variety $\mathrm{Jac}(Y)$. It is a principally polarized abelian variety of dimension five admitting complex multiplication by Eisenstein integers. A compactification of the moduli space of such abelian varieties is a complex ball quotient by the group Γ.

Another interesting additional structure on a cubic surface S is a choice of an anti-canonical divisor D that allows to view the pair (S, D) as a log-surface, and applies the theory of stable pairs [6], [370], [461]. to construct the moduli space of such pairs.

Remark 9.6.1. The story goes on. The group $W(\mathsf{E}_6)$ acts birationally on the space P_2^6 by changing the markings and Coble describes in [155], Part III, rational invariants of this action. He also defines a linear system of degree 10 of elements of degree three in R_2^6 which gives a $W(\mathsf{E}_6)$-equivariant embedding of a certain blow-up of P_2^6 in $\mathbb{P}^9$ corresponding to some irreducible 10-dimensional linear representation of the Weyl group. For a modern treatment of this construction we refer to [164] and [320]. Other $W(\mathsf{E}_6)$-equivariant birational models of P_2^6 were given in [548] and [370].

9.7 Automorphisms of Cubic Surfaces

In this section, we compute the groups of automorphisms of smooth cubic surfaces.

9.7.1 Cyclic groups of automorphisms

Let $W = W(\mathsf{E}_6)$ be the Weyl group of a simple root system of type E_6. In subsection 8.2.4 we discussed the classification of conjugacy classes of elements in the finite Weyl group $W(\mathsf{E}_n)$. Table 8.2.4 contains the list of conjugacy classes and their characterictic polynomials in Carter's notation. Here, we specialize to the case $W(\mathsf{E}_6)$. In this case, the classification can also be found in [165], or [504]. We give the dictionary between the notations used in these sources.

	Carter	Atlas	Manin	order	#Cent	Trace	Char
x	$\emptyset$	1A	c_1	1	51840	6	Φ_1^6
x	$4A_1$	2A	c_3	2	1152	-2	$\Phi_2^4\Phi_1^2$
x	$2A_1$	2B	c_2	2	192	2	$\Phi_2^2\Phi_1^4$
	A_1	2C	c_{16}	2	1440	4	$\Phi_2\Phi_1^5$
	$3A_1$	2D	c_{17}	2	96	0	$\Phi_2^3\Phi_1^3$
x	$3A_2$	3A	c_{11}	3	648	-3	Φ_3^3
x	A_2	3C	c_6	3	216	3	$\Phi_3\Phi_1^4$
x	$2A_2$	3D	c_9	3	108	0	$\Phi_3^2\Phi_1^2$
x	$D_4(a_1)$	4A	c_4	4	96	2	$\Phi_4^2\Phi_1^2$
x	A_1+A_3	4B	c_5	4	16	0	$\Phi_2^2\Phi_4\Phi_1^2$
	$2A_1+A_3$	4C	c_{19}	4	96	-2	$\Phi_2^3\Phi_4\Phi_1$
	A_3	4D	c_{18}	4	32	2	$\Phi_2\Phi_4\Phi_1^3$
x	A_4	5A	c_{15}	5	10	1	$\Phi_5\Phi_1^2$
x	$E_6(a_2)$	6A	c_{12}	6	72	1	$\Phi_3\Phi_6^2$
x	D_4	6C	c_{21}	6	36	1	$\Phi_2^2\Phi_6\Phi_1^2$
x	A_1+A_5	6E	c_{10}	6	36	-2	$\Phi_2^2\Phi_6\Phi_3$
x	$2A_1+A_2$	6F	c_8	6	24	-1	$\Phi_2^2\Phi_3\Phi_1^2$
	A_1+A_2	6G	c_7	6	36	1	$\Phi_2\Phi_3\Phi_1^3$
	A_1+2A_2	6H	c_{22}	6	36	-2	$\Phi_2\Phi_3^2\Phi_1$
	A_5	6I	c_{23}	6	12	0	$\Phi_2\Phi_3\Phi_6\Phi_1$
x	D_5	8A	c_{20}	8	8	0	$\Phi_2^2\Phi_8\Phi_1$
x	$E_6(a_1)$	9A	c_{14}	9	9	0	Φ_9
	A_1+A_4	10A	c_{25}	10	10	-1	$\Phi_2\Phi_5\Phi_1$
x	E_6	12A	c_{13}	12	12	-1	$\Phi_3\Phi_{12}$
	$D_5(a_1)$	12C	c_{24}	12	12	1	$\Phi_2\Phi_6\Phi_4\Phi_1$

Table 9.3 *Conjugacy classes in* $W(\mathsf{E}_6)$

Lemma 9.7.1. *The conjugacy classes* $A_1, 3A_1, 2A_1 + A_2, A_3, A_1 + A_2, A_1 + 2A_2, A_5, A_1 + A_4$, *and* $D_5(a_1)$ *are not realizable by an automorphism of a nonsingular cubic surface.*

Proof Each of the conjugacy classes $A_1, 3A_1, 2A_1+A_2, A_3, A_1+A_2, A_1+2A_2$ is conjugate to an element from the r subgroup $W(\mathsf{E}_6)_{e_6}$ of $W(\mathsf{E})6)$ isomorphic to $W(\mathsf{D}_5)$ that fixes the fundamental weight represented by the exceptional vector e_6. In the realization of this element as an automorphism σ of S, we can choose a geometric basis sich that σ fixes the exceptional curve representing

e_6. Blowing it down, we obtain an equivariant morphism to a del Pezzo surface S' of degree four. This shows that σ is a lift of an automorphism of S' realizing one of these conjugacy classes. It follows from Table 8.9 that such conjugacy classes are not realizable.

Looking at the characteristic polynomials, we check that the third power of the conjugacy class of type A_5 is of type $3A_1$, the fifth power of an element of type $A_1 + A_4$ is of type A_1, and the third power of an element of type $D_5(a_1)$ is of type $2A_1 + A_2$. Since these powers are realizable, we see that the last three conjugacy classes are not realizable too. □

The next theorem shows that all other conjugacy classes are realizable.

Theorem 9.7.2. *Let S be a nonsingular cubic surface admitting a non-trivial automorphism σ of order n. Then S is equivariantly isomorphic to one of the following surfaces $V(F)$ with*

$$\sigma = [t_0, t_1, t_2, t-3] \mapsto [t_0, \epsilon_n^a t_1, \epsilon_n^b t_2, \epsilon_n^c t_3], \quad \epsilon_n = e^{2\pi i/n}. \tag{9.65}$$

order	Type	(a, b, c)	equation	fixed locus
2	$4A_1$	$(0,0,1)$	$t_3^2 L(t_0, t_1, t_2) + t_0^3 + t_1^3 + t_2^3 + \alpha t_0 t_1 t_2$	cubic+1 pt
2	$2A_1$	$(0,1,1)$	$t_0^3 + t_0(t_1^2 + t_2^2 + t_3^2) + t_1(\alpha t_2^2 + \beta t_3^2)$	line+3pts
3	$3A_2$	$(0,0,1)$	$t_0^3 + t_1^3 + t_2^3 + t_3^3 + \alpha t_0 t_1 t_2$	C
3	A_2	$(0,1,1)$	$t_0^3 + t_1^3 + t_2^3 + t_3^3$	6pts
3	$2A_2$	$(0,1,2)$	$t_0^3 + t_1^3 + t_2 t_3(t_0 + \alpha t_1) + t_2^3 + t_3^3$	3pts
4	$D_4(a_1)$	$(0,2,1)$	$t_2 t_3^2 + t_0 t_2^2 + t_1(t_1 + t_0)(t_1 + \alpha t_0)$	5pts
4	$A_1 + A_3$	$(2,1,3)$	$t_0 t_1 t_2 + t_0(t_0^2 + t_1^2) + t_1(t_3^2 + t_2^2)$	3pts
5	A_4	$(1,2,3)$	$t_0(t_0 t_2 + t_1^2) + t_3(t_2^2 + t_1 t_3)$	4pts
6	$E_6(a_2)$	$(0,3,2)$	$t_0^3 + t_1^3 + t_3^3 + t_2^2(\alpha t_0 + t_1)$	4pts
6	D_4	$(0,2,5)$	$f_3(t_0, t_1) + t_2 t_3^2 + t_2^3$	4pts
6	$A_1 + A_5$	$(4,2,1)$	$t_3^2 t_1 + t_0^3 + t_1^3 + t_2^3 + \alpha t_0 t_1 t_2$	1pt
6	$2A_1 + A_2$	$(4,1,3)$	$t_0^3 + \alpha t_0 t_3^2 + t_1 t_2^2 + t_1^3$	2pts
8	D_5	$(4,3,2)$	$t_0^3 + t_0 t_1^2 + t_1 t_3^2 + t_2^2 t_3$	3pts
9	$E_6(a_1)$	$(4,1,7)$	$t_0^3 + t_3^2 t_1 + t_1^2 t_2 + t_2^2 t_3$	3pts
12	E_6	$(4,1,10)$	$t_0^3 + t_1^3 + (t_1 t_3 + t_2^2) t_3$	2pts

Table 9.4 *Cyclic groups of automorphisms of a cubic surface*

Proof Assume that $n = 2$. Then σ extended to a projective automorphism of $\mathbb{P}^3$. Its set $F(\sigma)$ of fixed points is a plane and a point or two skew lines. In the first case, we may assume that σ acts with $(a, b, c) = (0, 0, 0, 1)$. The quation must be of the form $t_3^2 f_1(t_0,t_1,t_2,t_3) + f_3(t_0,t_1,t_2) = 0$. The surface is nonsingular if and only if the cubic curve $V(f_3)$ is nonsingular. It remains to reduce it to the Hesse form.

In the second case, σ acts with $(a,b,c) = (0,1,1)$. We know that S^σ contains one of the lines in $F(\sigma)$. We may assume that this line is $V(t_0,t_1)$. Thus, each monomial entering in the equation of S must contain t_0 or t_1 and t_2, t_3 enters with exponent 2 or the monomial contains t_2t_3. This gives an equation of the form

$$t_0q_1(t_2,t_3) + t_1q_2(t_2,t_3) + g_3(t_0,t_1) = 0$$

where q_1, q_2 are quadratic forms, and g_3 is a binary form of degree 3. Since S is nonsingular, g_3 does not have multiple roots, so it can be reduced by a linear change of variables to the form $t_0^3 + t_0t_2^2$. The equation becomes of the form

$$t_0^3 + t_0a(t_1,t_2,t_3) + t_1b(t_2,t_3) = 0.$$

Since S is nonsingular, $V(t_1,t_2,t_3)$ and $V(b(t_2,t_3)$ are nonsingular. Thus, we can reduce a_2 to the form $t_1^2+t_2^2+t_3^2$. By a linear transformation of t_2,t_2 leaving $t_2^2+t-3^2$ unchanged, we can reduce $b(t_2,t_3)$ to the form $\alpha t_2(t_3+\beta t_2)$. We see that the new equation from the table

$$t_0^3 + t_0(t_1^2+t_2^2+t_3^2) + t_1(\alpha t_2^2 + \beta t_3^2) = 0$$

has 2^2-symmetry generated by involutions of type $2A_1$ and $4A_1$.

- σ is of type $3A_2$.

In this case, $e(S^\sigma) = 0$, so S^σ is a plane section of S. We may assume that the equation of the plane is $x_3 = 0$, and σ acts identically on the plane. Thus, the equation must be

$$t_3^3 + g_3(t_0,t_1,t_2) = 0.$$

The surface is a cyclic cubic surface. Since $V(g_3)$ must be nonsingular, we can reduce it to a Hesse form to obtain the equation

$$t_3^3 + t_0^3 + t_1^3 + t_2^3 + at_0t_1t_2 = 0. \tag{9.66}$$

- σ is of type A_2.

In this case, $e(S^\sigma) = 6$, so $F(\sigma)$ consists of two skew lines, and S^σ is the intersection of the lines with S. We may assume that the lines are $V(t_0,t_1)$ and $V(t_1,t_2)$ and σ acts identically on the first line and the action is with $(a,b,c) = (0,0,1,1)$. The equation must be of the form $f_3(t_0,t_1) + g_3(t_1,t_2) = 0$. Since S is nonsingular, the binary cubic forms have no multiple zeros, and we can reduce the equation to the asserted form.

- σ is of type $2A_2$.

In this case, $e(S^\sigma) = 3$ and $F(\sigma)$ is again the union of a line and two points. It follows that S^σ is the intersection of the line with S. We may assume that the action is given by $(a, b, c) = (0, 1, 2)$ and the isolated fixed points in $F(\sigma)$ are $[0, 0, 1, 0]$ and $[0, 0, 0, 1]$. Also, the line in $F(\sigma$ is $V(t_2, t_3)$. This implies that the monomials t_2^3 and t_3^3 enter in the equation. Other monomials containing t_2 or t_3 must contain t_2t_3. Thus, S can be given by the equation

$$t_2^3 + t_3^3 + t_2t_3L(t_0, t_1) + g_3(t_0, t_1) = 0.$$

To make it different from the previous case, we have to assume that $L \neq 0$. By a further change of coordinates, we can reduce it to the equation

$$t_2^3 + t_3^3 + t_2t_3(t_0 + at_1) + t_0^3 + t_1^3 = 0. \tag{9.67}$$

- σ is of type $D_4(a_1)$.

Note that σ^2 belongs to the conjugacy class $4A_1$. In this case, $e(S^\sigma) = 5$, and S^σ consists of five points. Again, $F(\sigma)$ consists of a line and two points, but the points lie on S. The square of the conjugacy class $D_4(a_1)$ is of type $4A_1$. This implies that we may assume that σ acts with $(a, b, c) = (0, 2, 1)$. We may assume that the line from $F(\sigma)$ is $V(t_2, t_3)$ and the other fixed points are $[0, 0, 1, 0]$ and $[0, 0, 0, 1]$. Thus, any monomial containing t_2 or t_3 must be one of the forms $t_2t_3^2, t_2^2t_0, t_2^2t_1$. This leads to an equation

$$t_2t_3^2 + t_0t_2^2 + t_1t_2^2 + f_3(t_0, t_1) = 0. \tag{9.68}$$

Since S is nonsingular, the coefficient of f_3 at t_1^3 is not zero. A linear change $t_1 \mapsto at_0 + bt_0$ gives the equation from the table.

- σ is of type $A_1 + A_3$.

In this case, $e(S^\sigma) = 3$, and σ^2 must be of type $2A_1$. Analyzing possible actions of σ on S^{σ^2} we find that S^σ consists of three points, two lie on the line from $F(\sigma)$ and one is an isolated fixed point of σ^2. The linear action of σ has four distinct eigenvalues. So, we may assume that σ acts with $(a, b, c) = (2, 1, 3)$ and the fixed point from $F(\sigma)$ not lying in S is $[1, 0, 0, 0]$. Other fixed points are $[0, 1, 0, 0]$, $[0, 0, 1, 0]$, and $[0, 0, 0, 1]$. The monomial containing t_0 must be $t_0^3, t_0t_1t_2, t_0t_1^2$. After scaling the coordinates, we get the equation [2]

$$\lambda t_0t_2t_3 + t_1(t_2^2 + t_3^2) + t_0^3 + t_0t_1^2 = 0. \tag{9.69}$$

If we choose the new coordinates to transform t_2t_3 to $t_2^2 + t_3^2$, after rescaling, we get the asserted equation.

[2] This corrects the mistake in [252], where the equation defines a singular surface.

- σ is of type A_4.

In this case, $e(S^\sigma) = 4$, so S^σ has four isolated points or consists of a line ℓ and a point x_0. In the latter case, a general line joining the pointx_0 with a point on the line must be tangent to S at x_0. Thus, the plane spanned by the line and the point x_0 is a tritangent plane. There are five tritangent planes containing ℓ, and σ has one fixed plane among them. Hence, σ fixes all other planes. So, S^σ consists of four points. Since σ acts on an invariant line with two fixed points, we obtain that no three points are collinear. Thus, we may assume that the fixed points are $[1, 0, 0, 0]$, $[0, 1, 0, 0]$, $[0, 0, 1, 0]$, $[0, 0, 0, 1]$ and the linear action of σ has four distinct eigenvalues. It follows that we may assume that σ acts with $(a, b, c) = (1, 2, 3)$. In the linear action of σ in The linear space $S^3((\mathbb{C}^4)^\vee)$ of a homogeneous polynomial of degree 3 decomposes into the direct sum of eigensubspaces of the linear action of σ with eigenvalues $\epsilon_5 i$. One checks that the polynomials from the eigenspaces V_λ with the eigenvalue $\epsilon_5^i, i \neq 2$ define singular surfaces. The invariant monomials from the subspace $V_{\epsilon_5^2}$ are $t_0^2 t_2, t_0 t_1^2$, and $t_2^2 t_3, t_1 t_3^2$. After scaling the coordinates, we get the equation from the table.

- σ is of type $E_6(a_2)$.

Similarly to the previous case, we prove that $S^\sigma)$ cannot contain a line and hence consists of three points spanning $\mathbb{P}^3$. Note that σ^3 is of type $4A_1$ and σ^2 is of type $3A_2$. So, we can choose coordinates to assume that σ acts with $(a, b, c) = (0, 3, 2)$ and the equation is in the form (9.66)

$$t_3^3 + g_3(t_0, t_1, t_2) = 0.$$

The only way to make it invariant is to assume that $g_3 = t_2^2(at_0+bt_1)+h_3(t_0, t_1)$. Reducing h_3 to the sum of cubics, and scaling t_2, we get the equation from the table.

- σ is of type D_4.

In this case, σ^3 is of type $4A_1$ and σ^2 is of type A_2. We may assume that σ acts with $(a, b, c) = (0, 2, 5)$. The equation contains invariant monomials of degree 3 in t_0, t_1 and the monomials $t_2 t_3^2, t_2^3$. This leads to the equation from the table.

- σ is of type $A_1 + A_5$.

In this case, $e(S^\sigma) = 1$, and hence, S^σ consists of one point. It follows that $F(\sigma)$ consists of four points, hence the linear action of σ has four distinct eigenvalues. We check that σ^3 is of type $4A_1$ and σ^2 is of type $2A_2$. This implies that we may assume that σ acts with $(a, b, c) = (4, 2, 1)$. The invariant monomials are $t_0^3, t_0t_1t_2, t_1t_3^2, t_1^3, t_2^3$. This leads to the equation from the table.

- σ is of type $2A_1 + A_2$.

As in case $E_6(a_2)$, we show that S^σ does not contain a line. Hence it consists of two points. We check that σ^2 is of type A_2 and σ^3 is of type $2A_1$. This shows that we may assume that σ acts with $(a, b, c) = (4, 1, 3)$. So, again the linear action has four distinct eigenvalues. The invariant monomials are $t_0^3, t_0t_3^2, t_1t_2^2$ and t_1^3. After scaling, we get the equation from the table.

- σ is of type D_5.

The trace is equal to 0, so S^σ consists of three points. This is an element of type $D_4(a_1)$ of order 4 that acts with $(a, b, c) = (0, 2, 1)$. It follows that the three fixed points are not on a line, hence σ acts linearly with four distinct eigenvalues. We may assume that σ acts with $(a, b, c) = (4, 3, 2)$. Computing invariant monomials, we find that, after scaling the coordinates, we get the equation from the table.

- σ is of type $E_6(a_1)$.

The cube of σ is an element of type $3A_1$. So, the surface S is a cyclic surface $V(t_3^3 + f_3(t_0, t_1, t_2))$. To get an automorphism of order 9 we f_3 to admit a linear automorphism such that f is an eigenvector with eigenvalue ϵ_3. It is easy to see that f_3 can be reduced to the form $t_0^2t_1 + t_1^2t_2 + t_2^2t_0$, and we get the equation from the table.

- σ is of type E_6.

The cube of σ is an element of type $E_6(a_2)$. It acts linearly with $(a, b, c) = (2, 1, 3)$, The square of σ is an element of type $E_6(a_2)$ that acts with $(a, b, c) = (0, 3, 2)$. This easily implies that we may assume that σ acts with $(a, b, c) = (4, 1, 10)$. The invariant monomials are $t_0^3, t_1^3, t_2^2t_3, t_1t_3^2$, So, after scaling we get the equation from the table.

□

9.7.2 Maximal subgroups of $W(\mathsf{E}_6)$

We will need some known information about the structure of the Weyl group of type E_6.

Theorem 9.7.3. *Let H be a maximal subgroup of $W(\mathsf{E}_6)$. Then, one of the following cases occurs:*

(i) $H \cong 2^4 : \mathfrak{S}_5$ *of order* $2^4 \cdot 5!$ *and index 27;*
(ii) $H \cong \mathfrak{S}_6 \times 2$ *of order* $2 \cdot 6!$ *and index 36;*
(iii) $H \cong 3_+^{1+2} : 2\mathfrak{S}_4$ *of order* 1296 *and index 40;*
(iv) $H \cong \mathfrak{S}_3 \wr \mathfrak{S}_3 \cong 3^3 : (\mathfrak{S}_4 \times 2)$ *of order* 1296 *and index 40;*
(v) $H \cong \big(2.(\mathfrak{A}_4 \times \mathfrak{A}_4).2\big).2$ *of order* 1152 *and index 45.*

Here, we use the notations from the Atlas [165], where $\mathbb{Z}/n\mathbb{Z} = n$, semi-direct products: $H \ltimes G = H : G$, 3_+^{1+2} denotes the group of order 3^3 of exponent p, and $A.B$ denote a group with normal subgroup isomorphic to A and quotient isomorphic to B.

Let us identify the group $W(\mathsf{E}_6)$ with the Weyl group of the lattice $K_S^\perp$ defined by a nonsingular cubic surface S. We recognize a maximal subgroup from (i) as the stabilizer subgroup of a line on S.

A maximal subgroup H of type (ii) is the stabilizer subgroup of a double-six. Its subgroup isomorphic to $\mathfrak{S}_6$ permutes lines in one of the sixes.

I do not know a geometric interpretation of a maximal subgroup of type (iii).

By Theorem 9.1.6, a maximal subgroup of type (iv) is isomorphic to the stabilizer subgroup of a Steiner complex of triads of double-sixes. It also coincides with a stabilizer subgroup of the root sublattice of type $A_2 + A_2 + A_2$. There is another interpretation of this subgroup in terms of a compactification of the moduli space of cubic surfaces (see [548]).

A maximal subgroup of type (v) is the stabilizer subgroup of a tritangent plane.

Proposition 9.7.4. *$W(\mathsf{E}_6)$ contains a unique normal subgroup $W(\mathsf{E}_6)'$. It is a simple group and its index is equal to 2.*

Proof Choose a root basis $(\alpha_1, \dots, \alpha_6)$ in the root lattice E_6. Let $s_0, \dots, s_5$ be the corresponding simple reflections. Each element $w \in W(\mathsf{E}_6)$ can be written as a product of the simple reflections. Let $\ell(w)$ be the minimal length of the word needed to write w as such a product. For example, $\ell(1) = 0, \ell(s_i) = 1$. One shows that the function $\ell : W(\mathsf{E}_6) \to \mathbb{Z}/2\mathbb{Z}$, $w \mapsto \ell(w) \mod 2$ is a homomorphism of groups. Its kernel $W(\mathsf{E}_6)'$ is a subgroup of index 2. The restriction of the function ℓ to the subgroup $H \cong \mathfrak{S}_6$ generated by the reflections $s_1, \dots, s_5$ is the sign function. Suppose K is a normal subgroup of $W(\mathsf{E}_6)'$. Then, $K \cap H$ is either trivial or equal to the alternating subgroup $\mathfrak{A}_6$ of index 2. It remains to use the fact that $H \times (r)$ is a maximal subgroup of $W(\mathsf{E}_6)$ and s is a reflection that does not belong to $W(\mathsf{E}_6)'$. □

Remark 9.7.5. Recall that we have an isomorphism (9.8) of groups

$$W(\mathsf{E}_6) \cong \mathrm{O}(6, \mathbb{F}_2)^-.$$

The subgroup $W(\mathsf{E}_6)'$ is isomorphic to the commutator subgroup of $\mathrm{O}(6,\mathbb{F}_2)^-$.

Let us mention other realizations of the Weyl group $W(\mathsf{E}_6)$.

Proposition 9.7.6.

$$W(\mathsf{E}_6)' \cong \mathrm{SU}_4(2),$$

where $\mathrm{SU}_4(2)$ *is the group of linear transformations with determinant 1 of* $\mathbb{F}_4^4$ *preserving a nondegenerate Hermitian product with respect to the Frobenius automorphism of* $\mathbb{F}_4$.

Proof Let $\mathbf{F} : x \mapsto x^2$ be the Frobenius automorphism of $\mathbb{F}_4$. We view the expression

$$\sum_{i=0}^{3} t_i^3 = \sum_{i=0}^{3} t_i \mathbf{F}(t_i)$$

as a nondegenerate Hermitian form in $\mathbb{F}_4^4$. Thus, $\mathrm{SU}_4(2)$ is isomorphic to the subgroup of the automorphism group of the cubic surface S defined by the equation

$$t_0^3 + t_1^3 + t_2^3 + t_3^3 = 0$$

over the field $\bar{\mathbb{F}}_2$. The Weyl representation (which is defined for nonsingular cubic surfaces over fields of arbitrary characteristic) of $\mathrm{Aut}(S)$ defines a homomorphism $\mathrm{SU}_4(2) \to W(\mathsf{E}_6)$. The group $\mathrm{SU}_4(2)$ is known to be simple and of order equal to $\frac{1}{2}|W(\mathsf{E}_6)|$. This defines an isomorphism $\mathrm{SU}_4(2) \cong W(\mathsf{E}_6)'$. □

Proposition 9.7.7.

$$W(\mathsf{E}_6) \cong \mathrm{SO}(5,\mathbb{F}_3), \quad W(\mathsf{E}_6)' \cong \mathrm{SO}(5,\mathbb{F}_3)^+,$$

where $\mathrm{SO}(5,\mathbb{F}_3)^+$ *is the subgroup of elements of spinor norm 1.*

Proof Let $V = \mathsf{E}_6/3\mathsf{E}_6$. Since the discriminant of the lattice E_6 is equal to 3, the symmetric bilinear form defined by

$$\langle v + 3\mathsf{E}_6, w + 3\mathsf{E}_6 \rangle = -(v, w) \mod 3$$

has non-trivial radical. It has a 1-dimensional radical spanned by the vector

$$v_0 = 2\boldsymbol{\alpha}_1 + \boldsymbol{\alpha}_1 + 2\boldsymbol{\alpha}_4 + \boldsymbol{\alpha}_5 \mod 3\mathsf{E}_6.$$

The quadratic form $q(v) = (v, v) \mod 3$ defines a nondegenerate quadratic form on $\bar{V} = V/\mathbb{F}_3 v_0 \cong \mathbb{F}_3^5$. We have a natural injective homomorphism $W(\mathsf{E}_6) \to \mathrm{O}(5,\mathbb{F}_2)$. Comparing the orders, we find that the image is a subgroup of index 2. It must coincide with $\mathrm{SO}(5,\mathbb{F}_3)$. Its unique normal subgroup of index 2 is $\mathrm{SO}(5,\mathbb{F}_3)^+$. □

Remark 9.7.8. Let E be a vector space of odd dimension $2k+1$ over a finite field $\mathbb{F}_q$ equipped with a nondegenerate symmetric bilinear form. An element $v \in E$ is called a *plus vector* (resp. *minus vector*) if (v, v) is a square in $\mathbb{F}_q^*$ (resp. is not a square $\in \mathbb{F}_q^*$). The orthogonal group $\mathrm{O}(E)$ has three orbits in $|E|$: the set of isotropic lines, the set of lines spanned by plus vectors, and the set of lines spanned by minus vectors. The isotropic subgroup of a non-isotropic vector v is isomorphic to the orthogonal group of the subspace $v^\perp$. The restriction of the quadratic form to $v^\perp$ is of Witt index k if v is a plus vector and of Witt index $k-1$ if v is a minus vector. Thus, the stabilizer group is isomorphic to $\mathrm{O}(2k, \mathbb{F}_q)^\pm$. In our case, when $k = 2$ and $q = 3$, we obtain that the minus vectors correspond to cosets of roots in $V = \mathsf{E}_6/3\mathsf{E}_6$, hence the stabilizer of a minus vector is isomorphic to the stabilizer of a double-six, i.e. a maximal subgroup of $W(\mathsf{E}_6)$ of index 36. The stabilizer subgroup of a plus vector is a group of index 45 and isomorphic to the stabilizer of a tritangent plane. The stabilizer of an isotropic plane is a maximal subgroup of type (iii), and the stabilizer subgroup of an isotropic line is a maximal subgroup of type (iv).

9.7.3 Groups of automorphisms

Now, we are ready to classify all possible subgroups of automorphisms of a nonsingular cubic surface.

In Table9.5 below we use the notation $\mathcal{H}_3(3)$ for the *Heisenberg group* of unipotent 3×3-matrices with entries in $\mathbb{F}_3$.

Theorem 9.7.9. *The following is the list of all possible groups of automorphisms of nonsingular cubic surfaces.*

Here, in the third row, α is a root of the equation $8x^6 + 20x^3 - 1 = 0$, and, in the next row, $a \neq \alpha$ and also $a \neq a^4$, otherwise the surface is of Type II. Similar restrictions must be made for other parameters. There are also conditions for the surface to be nonsingular.

Proof Let S be a nonsingular cubic surface.

- Suppose $\mathrm{Aut}(S)$ contains an element from the conjugacy class A_2.

Table 9.4 shows that S is isomorphic to the Fermat cubic $V(t_0^3 + t_1^3 + t_2^3 + t_3^3)$. Obviously, its automorphism group contains a subgroup G isomorphic to $3^3 : \mathfrak{S}_4$. To see that it coincides with this group, we use that G is a subgroup of index 2 of a maximal subgroup H of type (iv). As we noted before, the group H is the stabilizer subgroup of a root lattice $A_2 + A_2 + A_2$. It contains an element represented by a reflection in one copy of the lattice and the identity on other

Type	Order	Structure	equation	Eckardt
I	648	$3^3 : \mathfrak{S}_4$	$t_0^3 + t_1^3 + t_2^3 + t_3^3$	18
II	120	$\mathfrak{S}_5$	$t_0^2t_1 + t_1^2t_2 + t_2^2t_3 + t_3^2t_0$	10
III	108	$\mathcal{H}_3(3) : 4$	$t_0^3 + t_1^3 + t_2^3 + t_3^3 + 6at_1t_2t_3$	9
IV	54	$\mathcal{H}_3(3) : 2$	$t_0^3 + t_1^3 + t_2^3 + t_3^3 + 6at_1t_2t_3$	9
V	24	$\mathfrak{S}_4$	$t_0^3 + t_0(t_1^2 + t_2^2 + t_3^2) + at_1t_2t_3$	6
VI	12	$\mathfrak{S}_3 \times 2$	$t_2^3 + t_3^3 + at_2t_3(t_0 + t_1) + t_0^3 + t_1^3$	4
VII	8	8	$t_3^2t_2 + t_2^2t_1 + t_0^3 + t_0t_1^2$	1
VIII	6	$\mathfrak{S}_3$	$t_2^3 + t_3^3 + at_2t_3(t_0 + bt_1) + t_0^3 + t_1^3$	3
IX	4	4	$t_3^2t_2 + t_2^2t_1 + t_0^3 + t_0t_1^2 + at_1^3$	1
X	4	2^2	$t_0^3 + t_0(t_1^2 + t_2^2 + t_3^2) + t_1(at_2^2 + bt_3^2)$	2
XI	2	2	$t_0^2(t_1 + bt_2 + at_3) + t_1^3 + t_2^3 + t_3^3 + at_1t_2t_3$	1

Table 9.5 *Groups of automorphisms of cubic surfaces*

copies. This element has a trace equal to 4, so belongs to the conjugacy class A_1. It is not realized by an automorphism of a nonsingular cubic surface. This gives Type I from the table.

- Suppose $\mathrm{Aut}(S)$ contains an element of order 5.

Table 9.4 shows that S is isomorphic to the surface

$$t_0^2t_1 + t_1^2t_2 + t_2^2t_3 + t_3^2t_0 = 0. \tag{9.70}$$

Consider the embedding of S in $\mathbb{P}^4$ given by the linear functions

$$\begin{aligned} z_0 &= t_0 + t_1 + t_2 + t_3, \\ z_1 &= \epsilon t_0 + \epsilon^3 t_1 + \epsilon^4 t_2 + \epsilon^2 t_3, \\ z_2 &= \epsilon^2 t_0 + \epsilon t_1 + \epsilon^3 t_2 + \epsilon^4 t_3, \\ z_3 &= \epsilon^3 t_0 + \epsilon^4 t_1 + \epsilon^2 t_2 + \epsilon t_3, \\ z_4 &= \epsilon^4 t_0 + \epsilon^2 t_1 + \epsilon t_2 + \epsilon^3 t_3, \end{aligned} \tag{9.71}$$

where $\epsilon^5 = 1$. One checks that $\sum_{i=0}^4 z_i = 0$ and (9.70) implies that also $\sum_{i=0}^4 z_i^3 = 0$. This shows that S is isomorphic to the following surface in $\mathbb{P}^4$:

$$\sum_{i=0}^4 z_i^3 = \sum_{i=0}^4 z_i = 0. \tag{9.72}$$

These equations exhibit a subgroup G of automorphisms of S isomorphic to $\mathfrak{S}_5$.

Assume that G is a proper subgroup of $\mathrm{Aut}(S)$. Note that the only maximal subgroup of $W(\mathsf{E}_6)$ that contains a subgroup isomorphic to $\mathfrak{S}_5$ is a subgroup H of type (i) or (ii). If H is of type (i), then $\mathrm{Aut}(S)$ contains one of the involutions from the subgroup 2^4. The group H is isomorphic to the Weyl group $W(\mathsf{D}_5)$. We encountered it as the Weyl group of a del Pezzo surface of degree 4. It follows from the proof of Proposition 8.6.9 that nontrivial elements of the subgroup 2^4 are conjugate to the composition of reflection $s_{\alpha_1} \circ s_{\alpha_5}$. Its trace is equal to 1. Thus, this element belongs to the conjugacy class $3A_1$ that is not realized by an automorphism. If H is of type (ii), then G is contained in $\mathfrak{S}_6$ or contains an element that commutes with G. It is immediately seen that the surface does not admit an involution that commutes with all elements in G. Since $\mathfrak{S}_5$ is a maximal subgroup of $\mathfrak{S}_6$, in the first case, we obtain that $\mathrm{Aut}(S)$ contains a subgroup isomorphic to $\mathfrak{S}_6$. However, a cyclic permutation g of order 6 acts on E_6 by cyclically permuting vectors $\mathbf{e}_1, \ldots, \mathbf{e}_6$ and leaving $\mathbf{e}_0$ invariant. Its trace is equal to 1. This shows that g belongs to the conjugacy class $A_1 + 2A_1$ and is not realized by an automorphism. This gives us type II from Table 9.5.

- Suppose $\mathrm{Aut}(S)$ contains an element of type $3A_2$.

From Table 9.4, we infer that S is a cyclic surface which is projectively isomorphic to the surface go $S = V(t_0^3 + t_1^3 + t_2^3 + t_3^3 + at_1t_2t_3)$. Obviously, it contains a group of automorphisms G isomorphic to $3.(3^2 : 2)$. The central element of order 3 is realized by the matrix $\mathrm{diag}[1, 1, 1, \epsilon_3]$. The quotient group is isomorphic to a group of projective automorphisms of the plane cubic $C = S \cap V(t_3)$. In the group law, the group is generated by translations by points of order 3 and the inversion automorphism. For special parameter a we get more automorphisms corresponding to a harmonic or an equianharmonic cubic. Let us see that there is nothing else in $\mathrm{Aut}(S)$. An equianharmonic cubic is projectively isomorphic to the Fermat cubic, so it will give Type I. The remaining two cases will give us surfaces of types III and IV.

The subgroup 3.3^2 is isomorphic to the Heisenberg group $\mathcal{H}_3(3)$ of upper-triangular 3×3 matrices with entries in $\mathbb{F}_3$ with 1 at the diagonal. In the notation of the Atlas, it is group 3_+^{1+2}. We see that it is contained in the only maximal subgroup which is of type (iii). The element generating the center of 3_+^{1+2} is a central element in the maximal subgroup. Thus, any extra automorphism commutes with the central element, and hence descends to an automorphism of the cubic curve C. This proves that $G = \mathrm{Aut}(S)$.

- Suppose $\mathrm{Aut}(S)$ contains an element of type D_5.

Consulting Table 9.4, we infer that S is isomorphic to the surface of type VII. The only maximal subgroup of $W(\mathsf{E}_6)$, which contains an element of order 8, is a subgroup H of order 1152. As we know it stabilizes a tritangent plane. In our case, the tritangent plane is $t_2 = 0$. It has the Eckardt point $x = [0,0,0,1]$. Thus, $G = \mathrm{Aut}(S)$ is a subgroup of the linear tangent space T_xS. If any element of G acts identically on the set of lines in the tritangent plane, then it acts identically on the projectivized tangent space, and hence G is a cyclic group. Obviously this implies that G is of order 8. Assume that there is an element τ which permutes cyclically the lines. Let G' be the subgroup generated by σ and τ. Obviously, $\tau^3 = \sigma^k$. Since G does not contain elements of order 24, we may assume that $k = 2$ or 4. Obviously, τ normalizes $\langle\sigma\rangle$ since otherwise we have two distinct cyclic groups of order 8 acting on a line with a common fixed point. It is easy to see that this is impossible. Since $\mathrm{Aut}(\mathbb{Z}/8\mathbb{Z})) \cong (\mathbb{Z}/2\mathbb{Z})^2$ this implies that σ and τ commute. Thus, $\sigma\tau$ is of order 24, which is impossible. This shows that $\mathrm{Aut}(S) \cong \mathbb{Z}/8\mathbb{Z}$.

By taking powers of elements of order 9 and 12, we obtain surfaces with automorphism groups which we have already classified. So, we may assume that $\mathrm{Aut}(S)$ does not contain elements of order 5, 8, 9, and 12. In a similar manner, we may assume that any element of order 3 belongs to the conjugacy class $2A_2$, and an element of order 6 belongs to the conjugacy class of type $E_6(a_1)$ or $A_1 + A_5$.

- Suppose $\mathrm{Aut}(S)$ contains an element of type $2A_2$.

Assume $\mathrm{Aut}(S)$ contains an element σ from conjugacy class $3D$. Then, the surface is isomorphic to $V(t_2^3 + t_3^3 + t_2t_3(t_0 + at_1) + t_0^3 + t_1^3)$. We assume that $a \neq 0$. Otherwise, the surface is a cyclic surface and admits an automorphism of type $3A_2$. This has been already taken care of. Let τ be an involution which exchanges the coordinates t_2 and t_3. The subgroup H generated by σ and τ is isomorphic to $\mathfrak{S}_3$. The involution τ is of type $4A_1$, it is a harmonic homology. Thus, the three involutions in H define three Eckardt points x_1, x_2, x_3. They are on the line $\ell = V(t_0) \cap V(t_1)$. The group H acts faithfully on the set of the three Eckardt points.

By Proposition 9.1.27, a triple of collinear Eckardt points defines a subgroup of $\mathrm{Aut}(S)$ isomorphic to $\mathfrak{S}_3$. If the triples are disjoint, then the subgroups do not have a common involution, hence they intersect only at the identity. Otherwise, they have one common involution.

Suppose we have an automorphism $g \notin H$. If $gHg^{-1} = H$, then, replacing g with the product with some involution in H, we may assume that g commutes with σ. This shows that we can simultaneously diagonalize the matrices representing g and σ. It is immediately checked from the equation of the surface that this is possible only if $a = 1$ and g is the transformation that switches t_0 and t_1. So, if $gHg^{-1} \neq H$, we obtain that $\mathrm{Aut}(S)$ is isomorphic to $\mathfrak{S}_3$ or $2 \times \mathfrak{S}_3$. This gives types VI and VIII.

Let us assume that $H' = gHg^{-1} \neq H$. Then, H' is the subgroup defined by the three Eckardt points y_i on the line $\ell' = g(\ell)$. Since each of the involutions corresponding to the points x_i commutes with at most one involution corresponding to the points y_i, we obtain that one of the lines $\langle x_i, y_j \rangle$ contains the third Eckardt point and defines a subgroup of $\mathrm{Aut}(S)$ isomorphic to $\mathfrak{S}_3$ which has one common involution with H. Replacing H' with this subgroup, we may assume that the lines ℓ and ℓ' intersect at $x_1 = y_1$ and, hence span a plane Π. Each of the pairs of lines $(\langle x_i, y_2 \rangle, \langle x_i, y_3 \rangle), i = 2, 3$, contains at most one line contained in S. Applying Proposition 9.1.27, we either get a complete quadrilateral in Π with six Eckardt points as its vertices and its three diagonals lying on S or there are more than nine Eckardt points on Π. Note that a plane section of S not containing a line on S intersects the 27 lines at 27 points, an Eckardt point is counted with multiplicity 3. This shows that an irreducible plane section of S contains ≤ 9 Eckardt points. If it contains a line with two Eckardt points on it, then the number is, at most, seven. This eliminates the second possibility. It follows from the structure of $W(E_6)$ that the first possibility gives that the four subgroups isomorphic to $\mathfrak{S}_3$ defined by the sides of the quadrilateral generate a subgroup G of $\mathrm{Aut}(S)$ isomorphic to $\mathfrak{S}_4$. The list of maximal subgroups of $W(E_6)$ shows that either $G = \mathrm{Aut}(S)$, or $\mathrm{Aut}(S) \cong \mathfrak{S}_5$ and hence S is the Clebsch diagonal surface given by equation (9.70).

- Suppose $\mathrm{Aut}(S)$ contains an element of type $2A_1$.

We proved in Theorem 9.7.2 that the equation of the surface can be reduced to the equation of a cubic surface of type XI from the table. Its automorphism group contains the subgroup generated by involutions of different types. For general parameters a, b, the automorphism group cannot be larger than 2^2. In fact, the existence of an additional automorphism will reduce the number of parameters.

- Suppose $\mathrm{Aut}(S)$ contains an element of order 4.

If σ belongs to the conjugacy class $A_1 + A_3$, then Table 9.4 shows that $\mathrm{Aut}(S)$ contains an additional automorphism of type $2A_2$. This leads to a surface of type V with $\mathrm{Aut}(S) \cong \mathfrak{S}_4$. If σ is of type $D_4(a_1)$, then the equation of the surface is (9.68). This is a cubic surface of type IX with a cyclic group of automorphisms of order 4. Here, we have to assume that the surface is not isomorphic to the surface of type VII. It follows from the proof of the next Corollary that in all previous cases, except type VII, the automorphism group is generated by involutions of type $4A_1$. Thus, our surface cannot be reduced to one of the previous cases.

Finally, it remains for us to consider the following case.

- $\mathrm{Aut}(S)$ contains only involutions of type $4A_1$, i.e. harmonic homologies.

Suppose we have two such involutions. They define two Eckardt points x_1 and x_2. By Proposition 9.1.27, if the line $\langle x_1, x_2\rangle$ is contained in S, then the involutions commute. If the line does not belong to S, then the two involutions generate $\mathfrak{S}_3$ that contains an element of order 3. Suppose we have a third involution defining a third Eckardt point x_3. Then, we have a tritangent plane formed by the lines $\langle x_i, x_j\rangle$. Obviously, it coincides with each tritangent plane corresponding to the Eckardt points x_i. This contradiction shows that we can have at most two commuting involutions. This gives the last two cases of our theorem. The condition that there is only one involution of type $2A$ is that the line $V(t_1+t_2+at_3)$ does not pass through an inflection point of the plane curve $V(t_0)$. □

The next corollary can be checked case by case, and its proof is left as an exercise.

Corollary 9.7.10. *Let* $\mathrm{Aut}(S)^o$ *be the subgroup of* $\mathrm{Aut}(S)$ *generated by involutions of type* $2A$*. Then,* $\mathrm{Aut}(S)^o$ *is a normal subgroup of* $\mathrm{Aut}(S)$ *such that the quotient group is either trivial or a cyclic group of order 2 or 4. The order 4 could occur only for the surface of type VII. The order 2 occurs only for surfaces of type IX.*

Finally, we explain the last column of Table 9.5. We already noticed that the Fermat surface has 18 Eckardt points. A harmonic involution of a surface of type II corresponds to a transposition in $\mathfrak{S}_5$. Their number is equal to 10. The surfaces of types III and IV are cyclic surfaces, we have already explained that they have nine Eckardt points. This can be also confirmed by looking at the structure of the group. A surface of type VI has four Eckardt points. They correspond to four harmonic symmetries. Three of them come from the subgroup $\mathfrak{S}_3$ and the fourth one corresponds to the central involution. Of course, we can see it in the equation. The fourth Eckardt point is $[1,-1,1,-1]$. Surfaces of type VII and IX have one involution of type 2A. Surfaces of type X have two and surfaces of type XI have only one.

9.7.4 The Clebsch diagonal cubic

We have already defined the Clebsch diagonal surface in Example 9.1.20 as a nonsingular cubic surface given by equations

$$t_0^3 + \cdots + t_3^3 - (t_0 + t_1 + t_2 + t_3)^3 = 0.$$

In the proof of Theorem 9.7.9 we found an explicit isomorphism to the surface in $\mathbb{P}^3$ with equation

$$t_0^2 t_1 + t_1^2 t_2 + t_2^2 t_3 + t_3^2 t_0 = 0.$$

The Sylvester pentahedron of the surface is $V(t_0 t_1 t_2 t_3 (t_0 + t_1 + t_2 + t_3))$. Its ten vertices are the Eckardt points. Each edge is a line going through three Eckardt points.

Each face of the pentahedron intersects the tetrahedron formed by the other four faces along three diagonals, they are lines on S (this explains the name of the surface). In this way, we get 15 lines, the $\mathfrak{S}_5$-orbit of the line

$$t_0 = t_1 + t_2 = t_3 + t_4 = 0.$$

The remaining 12 lines form a double-six. Their equations are as follows. Let η be a primitive 5-th root of unity. Let $\sigma = (a_1, \ldots, a_4)$ be a permutation of $\{1, 2, 3, 4\}$. Each line ℓ_σ spanned by the points $[1, \eta^{a_1}, \ldots, \eta^{a_4}]$ and $[1, \eta^{-a_1}, \ldots, \eta^{-a_4}]$ belongs to the surface. This gives $12 = 4!/2$ different lines. Here, is one of the ordered double-sixes formed by the twelve lines

$$(\ell_{1234}, \ell_{1243}, \ell_{1324}, \ell_{1342}, \ell_{1432}, \ell_{1423}),\ (\ell_{2413}, \ell_{2431}, \ell_{3412}, \ell_{3421}, \ell_{2312}, \ell_{2321}). \tag{9.73}$$

The Schur quadric Q corresponding to this double-six is the quadric

$$t_0^2 + \cdots + t_4^2 = 0, \quad t_0 + \cdots + t_4 = 0.$$

For example, the polar line of ℓ_{1234} is the line given by equations

$$\sum_{i=0}^{4} \eta^i t_i = \sum_{i=0}^{4} \eta^{-i} t_i = \sum_{i=0}^{4} t_i = 0$$

and, as is easy to see, it coincides with the line ℓ_{2413}. The Schur quadric intersects ℓ_{ijkl} at two points $[1, \eta^i, \eta^j, \eta^k, \eta^l]$ and $[1, \eta^{-i}, \eta^{-j}, \eta^{-k}, \eta^{-l}]$.

The group $\mathfrak{S}_5$ (as well as its subgroup $\mathfrak{S}_4$) acts transitively on the double-six. The group $\mathfrak{A}_5$ stabilizes a sixer.

The intersection $Q \cap S$ is the *Bring curve* of genus 4 given by the equations

$$t_0^3 + \cdots + t_4^3 = t_0^2 + \cdots + t_4^2 = t_0 + \cdots + t_4 = 0.$$

Its automorphism group is isomorphic to $\mathfrak{S}_5$. The image of this curve under the map $\pi_1 : S \to \mathbb{P}^2$ which blows down the first half $(\ell_1, \dots, \ell_6)$ of the double-six (9.73) is the Schur sextic with nodes at the points $p_i = \pi(\ell_i)$.

Consider the stereographic projection from the 2-dimensional sphere S^2 : $\{(a,b,c) \in \mathbb{R}^3 : a^2+b^2+c^2 = 1\}$ to the Riemann sphere $(a,b,c) \mapsto z = \frac{a+ib}{1-c}$. A rotation around the axis $\mathbb{R}(a,b,c)$ about the angle 2ϕ corresponds to the Möbius transformation

$$z \mapsto \frac{(\alpha+i\beta)z-(\gamma-\delta i)}{(\gamma+\delta i)z+(\alpha-\beta i)},$$

where $\alpha = \cos\phi, \beta = a\sin\phi, \gamma = b\sin\phi, \delta = c\sin\phi$. The icosahedron group $\mathfrak{A}_5$ acting by rotation symmetries of an icosahedron inscribed in S^2 defines an embedding of $\mathfrak{A}_5$ in the group $\mathrm{PGL}(3)$. One can choose the latter embedding as a subgroup generated by the following transformations S, U, T of orders $5, 2, 2$ (with $S \circ T$ of order 3) (represented by the Möbius transformations

$$S : z \mapsto \eta^2 z, \quad U : z \mapsto -z^{-1}, \quad T : z \mapsto \frac{(\eta-\eta^4)z+\eta^2-\eta^3}{(\eta^2-\eta^3)z+\eta^4-\eta}.$$

The orbit of the north pole of the sphere under the corresponding group of rotations is an icosahedron. It is known that the icosahedron group has three exceptional orbits in $\mathbb{P}^1$ with stabilizers of orders $5, 3, 2$. They are the sets of zeros of the homogeneous polynomials

$$\Phi_{12} = z_0 z_1 (z_1^{10} + 11 z_0^5 z_1^5 - z_0^{10}),$$
$$\Phi_{20} = -(z_0^{20} + z_1^{20}) + 228(z_1^{15} z_0^5 - z_1^5 z_0^{15}) - 494 z_1^{10} z_0^{10},$$
$$\Phi_{30} = z_0^{30} + z_1^{30} + 522(z_1^{25} z_0^5 - z_1^5 z_0^{25}) - 10005(z_1^{20} z_0^{10} + z_1^{10} z_0^{20}).$$

The isomorphism $\mathrm{SU}(2)/\pm 1 \to \mathrm{SO}(3)$ defines a 3-dimensional complex linear representation of $\mathfrak{A}_5$ which embeds $\mathfrak{A}_5$ in $\mathrm{PGL}(3)$. In an appropriate coordinate system, it leaves the conic $K = V(t_0^2 + t_1 t_2)$ invariant. The group $\mathfrak{A}_5$ acts in the plane in such a way that the Veronese map

$$[z_0, z_1] \mapsto [-z_0 z_1, z_0^2, -z_1^2] \tag{9.74}$$

is equivariant. The six lines

$$V(t_1), \quad V(t_2), \quad V(t_0 + \eta^i t_1 + \eta^{-i} t_2), \quad i = 0,1,2,3,4, \tag{9.75}$$

cut out on K the set

$$V(\Phi_{12}) = \{0, \infty, \eta^i(\eta+\eta^{-1}), \eta^i(\eta^2+\eta^{-2})\}, \quad i = 0, \dots, 4.$$

The poles of the six lines with respect to the conic is the set of six points

$$[1,0,0], \ \ [1, 2\eta^i, 2\eta^{-i}], \ i = 0,1,2,3,4.$$

They are called the *fundamental set of points*.

The image of the rational map $\mathbb{P}^2 \dashrightarrow \mathbb{P}^4$ defined by the five cubics $(F_0, F_1, F_2, F_3, F_4)$

$$F_i = \eta^i(4t_0^2t_2 - t_1t_2^2) + \eta^{2i}(-2t_0t_2^2 + t_1^3) + \eta^{3i}(2t_0t_1^2 - t_2^3) + \eta^{4i}(-4t_0^2t_1 + t_1^2t_2),$$

is the Clebsch diagonal cubic given by equations (9.12) (see [457], II, 1, §5).

The equation of the Schur sextic (also called the *Klein sextic* in this case) is

$$B = F_0^2 + \cdots + F_4^2 = 10(4t_0^2t_2 - t_1t_2^2)(-4t_0^2t_1 + t_1^2t_2) + 10(-2t_0t_2^2 + t_1^3)(2t_0t_1^2 - t_2^3)$$

$$= -20(8t_0^4t_1t_2 - 2t_0^2t_1^2t_2^2 + t_1^3t_2^3 - t_0t_1^5 - t_0t_2^5) = 0.$$

The 12 intersection points of the sextic with the conic K are the images of the 12 roots of Φ_{12} under the Veronese map (9.74). The images of 30 roots of Φ_{30} are the intersection points of K with the union of 15 lines joining pairwise the six fundamental points. Let D be the product of the linear forms defining these lines

$$\eta^\nu t_1 - \eta^{-i}t_2,\ (1+\sqrt{5})t_0 + \eta^i t_1 + \eta^{-i}t_2,\ (1-\sqrt{5})t_0 + \eta^{-i}t_1 + \eta^i t_2.$$

The images of these lines under the map given by the polynomials F_i's are the 15 diagonals of the Clebsch cubic. The images of 20 roots of Φ_{20} are cut out by an invariant curve of degree 10 given by equation

$$C = G_0^2 + G_1G_2$$

where

$$G_0 = -8t_0^3t_1t_2 + 6t_0t_1^2t_2^2 - t_1^5 - t_2^5,$$
$$G_1 = 16t_0^3t_2^2 - 8t_0^2t_1^3 - 4t_0t_1t_2^3 + 2t_1^4t_2,$$
$$G_2 = 16t_0^2t_1^3 - 8t_0^2t_2^3 - 4t_0t_1^3t_2 + 2t_1t_2^4$$

are quintic polynomials which define the $\mathfrak{A}_5$-equivariant symmetric Cremona transformation of degree 5. The curve $V(C)$ (with the source and the target identified via the duality defined by the conic K) is equal to the image of the conic K. The curve $V(C)$ is a rational curve which has each fundamental point as its singular points of multiplicity 4 with two ordinary cuspidal branches.

The four polynomials of degrees 2, 6, 10 and 15

$$A = t_0^2 + t_1t_2, \quad B, \quad C, \quad D$$

generate the algebra $\mathbb{C}[t_0, t_1, t_2]^{\mathfrak{A}_5}$ of invariant polynomials. The relation between the fundamental invariants is

$$D^2 = -1728B^5 + C^3 + 720ACB^3 - 80A^2C^2B + 64A^3(5B^2 - AC)^2$$

(see [457], II, 4, §3). The even part of the graded ring $\mathbb{C}[t_0, t_1, t_2]^{\mathfrak{A}_5}$ is freely generated by polynomials A, B, C of degrees $2, 6, 10$, so that

$$\mathbb{P}^2/\mathfrak{A}_5 \cong \mathrm{Proj}\, \mathbb{C}[t_0, t_1, t_2]^{\mathfrak{A}_5} \cong \mathbb{P}(1, 3, 5).$$

Remark 9.7.11. The Clebsch diagonal surface and the Bring curve of genus 4 play a role in the theory of modular forms. Thus, the Bring curve is isomorphic to the modular curve $\overline{\mathcal{H}/\ell}$, where $\ell = \ell_0(2) \cap \ell(5)$. It is also realized as the curve of fixed points of the Bertini involution on the del Pezzo surface of degree 1 obtained from the elliptic modular surface $S(5)$ of level 5 by blowing down the zero section [78], [546]. The blow-up of the Clebsch diagonal surface at its 10 Eckardt points is isomorphic to a minimal resolution of the *Hilbert modular surface* $\overline{\mathcal{H} \times \mathcal{H}/\Gamma}$, where Γ is the 2-level principal congruence subgroup of the Hilbert modular group associated to the real field $\mathbb{Q}(\sqrt{5})$ [398]. The curve C of degree 10 is isomorphic to the image of the diagonal in $\mathcal{H} \times \mathcal{H}$ under the involution switching the factors [399].

Remark 9.7.12. The pencil of curves of degree 6 $\lambda A^3 + \mu B = 0$ has remarkable properties, studied by R. Winger [809]. It has 12 base points, each point is an inflection point for all members of the pencil. The curves share common tangents at these points. They are the six lines (9.75). These lines count for 12 inflection tangents and 24 bitangents of each curve. The pencil contains three singular fibers: the curve $V(B)$, the union of the six lines, and a rational curve W with ten nodes forming an orbit of $\mathfrak{A}_5$ with stabilizer subgroup isomorphic to $\mathfrak{S}_3$. The union of the lines corresponds to the parameter $[\lambda, \mu] = [1, -1]$. The rational sextic corresponds to the parameter $[\lambda, \mu] = [32, 27]$. Other remarkable members of the pencil correspond to the parameter $[1 + \alpha, -\alpha]$, where $\alpha = (-9 \pm 3\sqrt{-15})/20$. These are the nonsingular *Valentiner sextics* with automorphism group isomorphic to $\mathfrak{A}_6$.

Let $\mathrm{PSO}(3) \cong \mathrm{SO}(3)$ be the group of projective automorphisms leaving invariant the conic $K = V(t_0^2 + t_1 t_2)$. Via the Veronese map, it is isomorphic to $\mathrm{PSL}(2)$. We have described explicitly the embedding $\iota : \mathfrak{A}_5 \hookrightarrow \mathrm{SO}(3)$. There are two non-isomorphic 3-dimensional irreducible representations of $\mathfrak{A}_5$ dual to each other. Note that the transformations S and S^{-1} are not conjugate in $\mathfrak{A}_5$, so that the dual representations are not isomorphic. In our representation, the trace of S is equal to $1 + \eta + \eta^{-1} = 1 + 2\cos 2\pi/5 = (1 + \sqrt{5})/2$ and, in the dual representation, the trace of S is equal to $1 + \eta^2 + \eta^{-2} = 2\cos 4\pi/5 = (1 - \sqrt{5})/2$. The polar lines of the fundamental set of six points define the fundamental set in the dual representation. Thus, each subgroup of $\mathrm{SO}(3)$ isomorphic to $\mathfrak{A}_5$ defines two sets of fundamental points, one in each of the two dual planes. We call them *icosahedral sets* of six points. The group $\mathrm{SO}(3)$ acts by conjugation on the set of subgroups isomorphic to $\mathfrak{A}_5$, with two conjugacy classes. This shows that the set of dual pairs of fundamental sets is parameterized by the homogeneous space $\mathrm{SO}(3)/\mathfrak{A}_5$.

The six fundamental lines (9.75) form a polar hexagon of the double conic $V(A^2)$ as the following identity shows (see [535], p. 261):

$$30(t_0^2 + t_1t_2)^2 = 25t_0^4 + \sum(t_0 + \eta^i t_1 + \eta^{-i} t_2)^4.$$

This shows that an icosahedral set in the dual plane is a polar hexagon of A^2. Hence, $\mathrm{VSP}(A^2, 6)$ contains a subvariety isomorphic to the homogeneous space $\mathrm{SO}(3)/\mathfrak{A}_5$. As we have explained in Subsection 1.4.4, this variety embeds into the Grassmannian $G(3, \mathcal{H}^3)$, where $\mathcal{H}^3$ is the 7-dimensional linear space of cubic harmonic polynomial with respect to the quadratic form $q = t_0^2 + t_1t_2$. Its closure is the subvariety of $G(3, \mathcal{H}^3)_\sigma$ of $G(3, \mathcal{H}^3)$ of subspaces isotropic to the Mukai skew forms σ_{ω, A^2}. It is a smooth irreducible Fano variety of genus 12 (see [535]). A compactification of the homogeneous space $\mathrm{SO}(3)/\mathfrak{A}_5$ isomorphic to $G(3, \mathcal{H}^3)_\sigma$ was constructed earlier by S. Mukai and H. Umemura in [532]. It is isomorphic to the closure of the orbit of $\mathrm{SL}(2)$ acting on the projective space of binary forms of degree 12.

Recall that the dual of the 4-dimensional space of cubic polynomials vanishing at the polar hexagon is a 3-dimensional subspace of $\mathcal{H}^3$ which is isotropic with respect to Mukai's skew forms. It follows from Theorem 6.3.33 that the variety $\mathrm{VSP}(A^2, 6)$ is the closure of $\mathrm{SO}(3)/\mathfrak{A}_5$ and isomorphic to a Fano variety of genus 12.

Observe that the cubic polynomials F_v are harmonic with respect to the Laplace operator corresponding to the dual quadratic form $q^\vee = -\frac{1}{4}\xi_0^2 + \xi_1\xi_2$. Thus, each fundamental set in the plane defines a four-dimensional subspace of the space $'\mathcal{H}^3$ of harmonic cubic polynomials with respect to $q^\vee$. This space is dual to the 3–dimensional subspace in $\mathcal{H}^3$ defined by the dual fundamental set with respect to the polarity pairing $\mathcal{H}^3 \times' \mathcal{H}^3 \to \mathbb{C}$. Note that the intersection of two four-dimensional subspaces in the 7-dimensional space $'\mathcal{H}^3$ of cubic polynomials is nonzero. Thus, for each of the two fundamental sets, there is a harmonic polynomial vanishing at both sets. One can show that the set of harmonic cubic curves vanishing at infinitely many fundamental sets is parameterized by a surface in the dual projective space $\mathbb{P}^3$ which is isomorphic to the Clebsch diagonal cubic surface under the map given by the Schur quadric (see [402]).

We refer to [512] for more of the beautiful geometry associated to the Bring curve and the Clebsch diagonal cubic surface.

Exercises

9.1 Let T_{st} be the standard Cremona transformation, considered as a biregular automorphism σ of a nonsingular del Pezzo surface S of degree six. Show that the orbit space $S/\langle\sigma\rangle$ is isomorphic to a Cayley four-nodal cubic surface.

9.2 Show that a cubic surface can be obtained as the blow-up of five points on $\mathbb{P}^1\times\mathbb{P}^1$. Find the conditions on the five points such that the blow-up is isomorphic to a nonsingular cubic surface. Show that each pair of skew lines on a cubic surface is intersected by five skew lines which can be blown down to 5 points on a nonsingular quadric.

9.3 Compute the number of m-tuples of skew lines on a nonsingular surface for $m = 2, 3, 4, 5$.

9.4 Suppose a quadric intersects a cubic surface along the union of three conics. Show that the three planes defined by the conics pass through three lines in a tritangent plane.

9.5 Let Γ and Γ' be two rational normal cubics in $\mathbb{P}^3$ containing a common point p. For a general plane Π through p let $\Pi\cap\Gamma = \{p, p_1, p_2\}, \Pi\cap\Gamma' = \{p, p_1', p_2'\}$ and $f(p) = \langle p_1, p_1\rangle \cap \langle p_1', p_2'\rangle$. Consider the set of planes through p as a hyperplane H in the dual space $(\mathbb{P}^3)^\vee$. Show that the image of the rational map $H \dashrightarrow \mathbb{P}^3, \Pi \mapsto f(p)$ is a nonsingular cubic surface and every such cubic surface can be obtained in this way.

9.6 Show that the linear system of quadrics in $\mathbb{P}^3$ spanned by quadrics which contain a degree 3 rational curve on a nonsingular cubic surface S can be spanned by the quadrics defined by the minors of a matrix defining a determinantal representation of S.

9.7 Show that a cubic surface with three nodes is isomorphic to a surface $V(w^3 + w(xy + yz + xz) + \lambda xyz)$. Show that the surface admits an Eckardt point if and only if $\lambda = \pm\sqrt{-2}$.

9.8 Let ℓ be a line on a del Pezzo cubic surface and E be its proper transform on the corresponding weak del Pezzo surface X. Let $\mathcal{N}$ be the sublattice of $\mathrm{Pic}(X)$ spanned by irreducible components of exceptional divisors of $\pi : X \to S$. Define the multiplicity of ℓ by

$$m(\ell) = \frac{\#\{\sigma \in \mathrm{O}(\mathrm{Pic}(X)) : \sigma(E) - E \in \mathcal{N}\}}{\#\{\sigma \in \mathrm{O}(\mathrm{Pic}(X)) : \sigma(E) = E\}}.$$

Show that the sum of the multiplicities is always equal to 27.

9.9 Show that the 24 points of intersection of a Schur quadric with the corresponding double-six lie on the Hessian of the surface ([29], vol. 3, p. 211).

9.10 Consider a Cayley-Salmon equation $l_1l_2l_3 - l_1'l_2'l_3' = 0$ of a nonsingular cubic surface.

(i) Show that the six linear polynomials l_i, l_i' satisfy the following linear equations

$$\sum_{j=1}^{3} a_{ij}l_j = \sum_{j=1}^{3} a_{ij}'l_j' = 0,\ i = 1, 2, 3,$$

where

$$\sum_{i=1}^{3} a_{ij} = 0,\ j = 1, 2, 3, \quad a_{i1}a_{i2}a_{i3} = a_{i1}'a_{i2}'a_{i3}',\ i = 1, 2, 3.$$

(ii) Show that for each $i = 1, 2, 3$ the nine planes

$$a_{ij}l_i - a'_{ij}l'_j = 0, \; i, j = 1, 2, 3$$

contain 18 lines common to three planes. The 18 lines obtained in this way form three double-sixes associated to the pair of conjugate triads defined by the Cayley-Salmon equation.

(iii) Show that the Schur quadrics defined by the three double-sixes can be defined by the equations

$$\sum_{j=1}^{3} a_{2j}a_{3j}l_j^2 - \sum_{j=1}^{3} a_{2j}a_{3j}l'^2_j = 0,$$

$$\sum_{j=1}^{3} a_{1j}a_{3j}l_j^2 - \sum_{j=1}^{3} a_{1j}a_{3j}l'^2_j = 0,$$

$$\sum_{j=1}^{3} a_{1j}a_{2j}l_j^2 - \sum_{j=1}^{3} a_{1j}a_{2j}l'^2_j = 0$$

([229]).

9.11 Prove the following theorem of Schläfli: given five skew lines in $\mathbb{P}^3$ and a line intersecting them all, there exists a unique cubic surface that contains a double-six including the seven lines ([230]).

9.12 For each type of a cubic surface with nontrivial group of automorphisms, find its Cremona hexahedral equations.

9.13 Show that the pull-back of a bracket-function (ijk) under the Veronese map is equal to $(ij)(jk)(ik)$.

9.14 Let S be a weak del Pezzo surface and R be a Dynkin curve on S. Show that S admits a double cover ramified only over R if and only if the sum of irreducible components in R is divisible by 2 in the Picard group. Using this, classify all del Pezzo surfaces which admit a double cover ramified only over singular points.

9.15 Show that the Segre cubic primal is isomorphic to a tangent hyperplane section of the cubic fourfold with nine lines given by the equation $xyz - uvw = 0$ (the *Perazzo primal* [579], [30]).

9.16 Consider the following *Cayley's family of cubic surfaces* in $\mathbb{P}^3$ with parameters l, m, n, k.

$$w[x^2 + y^2 + z^2 + w^2 + (mn + \frac{1}{mn})yz + (ln + \frac{1}{ln})xz + (lm + \frac{1}{lm})xy$$

$$+(l + \frac{1}{l})xw + (m + \frac{1}{m})yw + (n + \frac{1}{n})zw] + kxyz = 0.$$

Find the equations of 45 tritangent planes whose equations depend rationally on the parameters l, m, n, k.

9.17 Show that the polar quadric of a nonsingular cubic surface with respect to an Eckardt point is equal to the union of two planes.

9.18 Show that the equation of the dual of a nonsingular cubic surface can be written in the form $A^3 + B^2 = 0$, where A and B are homogeneous forms of degree 4 and 6, respectively. Show that the dual surface has 27 double lines and a curve of degree 24 of singularities of type A_2.

9.19 Show that any normal cubic surface can be given as the image of a plane under a Cremona transformation of $\mathbb{P}^3$ of degree three.

9.20 Show that a general cubic surface can be projectively generated by three nets of planes.

9.21 Show that the Eckardt points are singular points of the parabolic curve of a nonsingular cubic surface.

9.22 Show that each line on a nonsingular cubic surface intersects the parabolic curve with multiplicity two.

9.23 Find an $\mathfrak{A}_5$-invariant determinantal representation of the Clebsch diagonal cubic.

9.24 Use the Hilbert-Burch Theorem to show that any White surface (see Remark 9.1.22) is isomorphic to a determinantal surface W in $\mathbb{P}^n$ of degree $\binom{n}{2}$.

9.25 Let $\mathcal{P}$ be a pencil of quadrics in $\mathbb{P}^3$ with the nonsingular locus. For a general point $x \in \mathbb{P}^3$, consider the polar planes $P_x(Q)$ of quadrics $Q \in \mathcal{P}$.

(i) Show that the union of conics $P_x(Q) \cap Q$ is a nonsingular cubic surface S.

(ii) Show that the line $\ell = \cap_{Q \in \mathcal{P}} P_x(Q)$ is contained in S.

(iii) Let $x \in Q_0$ for a unique $Q_0 \in \mathcal{P}$ and $Q_1, \ldots, Q_4$ be the four singular quadrics in the pencil. Show that the five planes $P_x(Q_i), i = 0, \ldots, 4$, are the five tritangent planes of S that contain ℓ.

(iv) Show that any nonsingular cubic surfaces arises in this way.

(see [730, p. 16], where the cubic surface is called *die Pampolare* of the point x).

Historical Notes

Good sources for the historical references here are [384], [515], and [577]. According to [515], the study of cubic surfaces originates from the work of J. Plücker [593] on the intersection of quadrics and cubics and L. Magnus [503] on maps of a plane by a linear system of cubics.

However, it is customary to think that the theory of cubic surfaces starts from Cayley's and Salmon's discovery of 27 lines on a nonsingular cubic surface [104], [646] (see the history of discovery in [653], n. 529a, p. 183). Salmon's proof was based on his computation of the degree of the dual surface [645] and Cayley's proof uses the count of tritangent planes through a line, the proof we gave here. It is reproduced in many modern discussions of cubic surfaces (e.g. [610]). The number of tritangent planes was computed by [646] and Cayley [104]. Cayley gives an explicit four-dimensional family of cubic surfaces with a fixed tritangent plane (see Exercise 9.17). In 1851 J. Sylvester claimed, without proof, that a general cubic surface can be written uniquely as a sum of five cubes of linear forms [740]. This fact was proven ten years later by A. Clebsch [141]. In 1854, L. Schläfli discovered 36 double-sixes on a nonsingular cubic surface. This and other results about cubic surfaces were published later in [658]. In 1855, H. Grassmann proved that three collinear nets of planes generate a cubic surface [355]. The fact that a general cubic surface can be obtained in this way (this implies a linear determinantal representation of the surface) has a long history. In 1862, F. August proved that a general cubic surface can be generated by three pencils of planes [26]. L. Cremona deduces from this that a general cubic surface admits Grassmann's generation [186]. In 1904, R. Sturm pointed out that Cremona's proof had a gap. The gap was fixed by C. Segre in [694]. In the same paper, Segre proves that any normal cubic surface that does not contain a singularity of type E_6 has a linear determinantal representation. In 1956, J. Steiner introduced the Steiner systems of lines [726]. This gives 120 essentially different Cayley-Salmon equations of a nonsingular cubic surface. The existence of which was first shown by Cayley [104] and Salmon [646].

Cubic surfaces with a double line were classified in 1862 by A. Cayley [112] and, via a geometric approach, by Cremona [180]. In 1863, Schläfli [657] classified singular cubic surfaces with isolated singularities, although most of these surfaces were already known to G. Salmon [646]. The old notations for A_k-singularities are C_2 for A_1 (conic-node), B_{k+1} (biplanar nodes) for $A_k, k, > 1$ and U_{k+1} (uniplanar node) for D_k. The subscript indicates the decrease of the class of the surface. In [118] Cayley gives a combinatorial description of the sets of lines and tritangent planes on singular surfaces. He also gives the equations of the dual surfaces. Even before the discovery of 27 lines, in a paper of 1844 [102], Cayley studied a four-nodal cubic surface that we now call a Segre cubic surface. He finds its equation and realizases the suarfe as the image of the plane under the map given by the linear system of cubic curves passing through the vertices of a complete quadrilateral. Schläfli and later F. Klein [454] and L. Cremona [186] also studied the reality of singular points and lines. Benjamino Segre's book [683] on cubic surfaces treats real cubic surfaces with special detail.

In 1866, A. Clebsch proved that a general cubic surface can be obtained as the image of a birational map from the projective plane given by cubics through six points [144]. Using this, he showed that Schläfli's notation a_i, b_i, c_{ij} for 27 lines correspond to the images of the exceptional curves, conics through five points, and lines through two points. This important result was independently proven by Cremona in his memoir [186] of 1868, which got him the prize (shared with R. Sturm) offered by R. Steiner through the Royal Academy of Sciences of Berlin in 1864 and awarded in 1866. Some of the results from this memoir are discussed from a modern point of view in [246]. Many results from Cremona's memoir were independently proved by R. Sturm [730], and many of them were announced by Steiner (who did not provide the proofs). In particular, Cremona proved the result, anticipated in the work of Magnus, that any cubic surface can be obtained as the image of a plane under the cubocubic birational transformation of $\mathbb{P}^3$. Both of the memoirs contain a lengthy discussion of Steiner systems of tritangent planes. We refer to [246] for a historical discussion of Cremona's work on cubic surfaces.

Cremona's hexahedral equations were introduced by L. Cremona in [194]. Although known to T. Reye [616] (in geometric form, no equations can be found in his paper), Cremona was the first to prove that the equations are determined by a choice of a double-six. The Joubert functions were introduced by P. Joubert in [434]. Richmond was the first to use these functions to give a parameterization of the Segre cubic primal S_3 (Theorem 9.5.5) [623]. It later appears in the work of Coble [155], who proved the completeness of the system of Joubert invariants of six ordered points on $\mathbb{P}^1$. The Segre cubic primal arose in the work of C. Segre on cubic threefolds with singular points [686]. Its realization as the GIT-quotient space of ordered sets of six points in $\mathbb{P}^1$ is due to Coble. The dual quartic hypersurface was first studied by G. Castelnuovo [96] and later, in greaat detail, by H. Richmond [623]. It was called the Castelnuovo quartic by E. Ciani [134]. The relation of the Cremona-Richmond quartic hypersurface Cr_4 to the theory of invariant of 6 ordered points in $\mathbb{P}^2$ (Theorem 9.5.8) is due to Coble [155]. The parameterization of S_3 by quadrics in $\mathbb{P}^3$ through five reference points (Proposition 9.5.10) is due to Richmond (see also [29, Vol. IV, Chapter 5] and [30], where one can find many facts about the Segre cubic and the Castelnuovo-Richmond quartic). We will return to the geometry of the Segre cubic primal in the next chapter.

F. Eckardt gives a complete classification of cubic surfaces with Eckardt points (called Ovalpoints in [654]) in terms of their Hessian surface [270]. He also considers singular surfaces. A modern account of this work can be found [204]. The Clebsch Diagonalfläche with 10 Eckardt points was first studied by A. Clebsch in [149]. It has an important role in Klein's investigation of the Galois group of a quintic equation [457].

The classification of possible groups of automorphisms of nonsingular cubic surfaces was initiated by S. Kantor [438]. Some of the mistakes in his classification were later corrected by A. Wiman [808]. However, Wiman also made a small mistake in his claim that, in case VII, the group is a dihedral group of order 12. Segre's book [683] contains several mistakes, for example, he missed case VII. The first complete, purely algebraical, classification was given in 1997 by T. Hosoh [409]. Apparently he was not aware of Wiman's paper.

In 1897, J. Hutchinson showed in [418] that the Hessian surface of a nonsingular cubic surface could be isomorphic to the Kummer surface of the Jacobian of a genus 2 curve. This happens if the invariant $I_8I_{24} + 8I_{32}$ vanishes [637]. The group of birational automorphisms of the Hessian of a cubic surface was described only recently [241].

The relationship of the Gosset polynomial 2_{21} to 27 lines on a cubic surface was first discovered in 1910 by P. Schoute [662] (see [757]). The Weyl group $W(\mathsf{E}_6)$ as the Galois group of 27 lines was first studied by C. Jordan [432].

Together with the group of 28 bitangents of a plane quartic isomorphic to $W(\mathbf{E}_7)$, it is discussed in many classical text-books in algebra (e.g. [799], B. II). S. Kantor [438] realized the Weyl group $W(\mathbf{E}_n), n \leq 8$, as a group of linear transformations preserving a quadratic form of signature $(1, n)$ and a linear form. A. Coble [155], Part II, was the first who showed that the group is generated by the permutations group and one additional involution. Apparently, independently of Coble, this fact was rediscovered by P. Du Val [266]. We refer to [70] for the history of Weyl groups, reflection groups, and root systems. Note that the realization of the Weyl group as a reflection group in the theory of Lie algebras was obtained by H. Weyl in 1928, ten years later after Coble's work.

As we have already mentioned in the previous chapter, the Gosset polytopes were discovered in 1900 by T. Gosset [350]. The notation n_{21} belongs to him. They were rediscovered later by E. Elte and H. S. M. Coxeter (see [178]), but only Coxeter realized that their groups of symmetries are reflection groups. The relationship between the Gosset polytopes n_{21} and curves on del Pezzo surfaces of degree $5 - n$ was found by Du Val [266]. This fundamental paper is the origin of a modern approach to the study of del Pezzo surfaces by means of root systems of finite-dimensional Lie algebras [217], [504].

Volume 3 of Baker's book [29] contains a lot of information about the geometry of cubic surfaces. Yu. Manin's book [504] is a good source on cubic surfaces with emphasis on the case of a non-algebraically closed base field. It has been used as one of the main sources in the study of the arithmetic of del Pezzo surfaces.

10
Line Geometry

10.1 Grassmannians of Lines

10.1.1 Generalities about Grassmannians

In subsection 2.4.1, we introduced the notion of a Grassmannian bundle $G(k,\mathcal{E})$, where $\mathcal{E}$ is a locally free sheaf of rank $n+1$ on a scheme S. Here, we specialize and assume that $\mathcal{E}$ is a linear space E of dimension $n+1$ over $\mathbb{C}$ (mots of what follows works over any field $\Bbbk$). For brevity of the notation, we denote $G(k,E)$ by $\mathbb{G}$. The canonical exact sequence (2.33) becomes

$$0 \to \mathcal{S}_{\mathbb{G}} \to E_{\mathbb{G}}^{\vee} \to \mathcal{Q}_{\mathbb{G}} \to 0, \tag{10.1}$$

where we returned to the standard notations for the universal subbundle $\mathcal{S}_{\mathbb{G}}$ and the universal quotient bundle $\mathcal{Q}_{\mathbb{G}}$. The surjective map $E_{\mathbb{G}}^{\vee} \to \mathcal{Q}_{\mathbb{G}}$ defines a closed embedding $\mathbb{P}(\mathcal{Q}_{\mathbb{G}}) \hookrightarrow \mathbb{P}(E_{\mathbb{G}}^{\vee}) = \mathbb{G}\times|E|$. Passing to the closed fibers, we see that the fibers of the projective bundle $\mathbb{P}(\mathcal{Q}_{\mathbb{G}})$ are subspaces of dimension k in E, or dually, quotient spaces of $E^{\vee}$ of dimension $n+1-k$. The dual exact sequence

$$0 \to \mathcal{Q}_{\mathbb{G}}^{\vee} \to E_{\mathbb{G}} \to \mathcal{S}_{\mathbb{G}}^{\vee} \to 0, \tag{10.2}$$

shows that $\mathbb{P}(\mathcal{S}_{\mathbb{G}}^{\vee})$ is a projective subbundle of $\mathbb{P}(E_{\mathbb{G}}) = \mathbb{P}(E)\times\mathbb{G}$. Its fibers are projective subspaces of dimension $k-1$ of $\mathbb{P}(E)$, or, linear k-dimensional subspaces of $E^{\vee}$. If we choose a basis in E, the notations become $G(m,n+1)$ or $G_{m-1}(\mathbb{P}^n)$.

The surjection $E_{\mathbb{G}}^{\vee} \to \mathcal{Q}_{\mathbb{G}}$ defines a surjection $\bigwedge^m E_{\mathbb{G}}^{\vee} \to \bigwedge^k \mathcal{Q}_{\mathbb{G}}$. Since the latter sheaf is an invertible sheaf, we obtain the Plücker embedding

$$\mathbb{G} \hookrightarrow |\bigwedge^m E|. \tag{10.3}$$

.

A choice of a basis in E and a choice of a basis $(v_1,\dots,v_m)$ of $L\in G(m,E)$ defines a matrix A_L of size $m\times(n+1)$ and rank m whose i-th row consists of coordinates of the vector v_i. Two such matrices A and B define the same linear subspace if and only if there exists a matrix $C\in\mathrm{GL}(m)$ such that $CA=B$. In this way, $G(m,E)$ can be viewed as the orbit space of the action of $\mathrm{GL}(m)$ on the open subset of $\mathrm{Mat}_m(m,n+1)$ of rank m matrices. By the First Fundamental Theorem of Invariant Theory, the orbit space is isomorphic to the projective spectrum of the subring of the polynomial ring in $\binom{n+1}{m}$ variables $X_{ij}, 1\le i\le m, 1\le j\le n+1$, generated by the maximal minors of the matrix $X=(X_{ij})$. A choice of an order on the set of maximal minors (we will always use the lexicographic order) defines an embedding of the orbit space in $\mathbb{P}^{\binom{n+1}{m}-1}$. It is isomorphic to the Plücker embedding. In coordinates $(t_0,\dots,t_n)\in(E^\vee)^{n+1}$, the maximal minors $X_{i_1\dots i_m}$ can be identified with m-vectors $p_{i_1\dots i_m}=t_{i_1}\wedge\dots\wedge t_{i_m}\in\bigwedge^m E^\vee=(\bigwedge^m E)^\vee$. Considered as coordinates in the vector space $\bigwedge^m E$, they are called the *Plücker coordinates*.

The maximal minors $X_{i_1\dots i_k}$ satisfy the *Plücker equations*

$$\sum_{k=1}^{m+1}(-1)^k p_{i_1,\dots,i_{m-1},j_k}p_{j_1,\dots,j_{k-1},j_{k+1},\dots,j_{m+1}}=0, \tag{10.4}$$

where $(i_1,\dots,i_{m-1})$ and $(j_1,\dots,j_{m+1})$ are two strictly increasing subsets of $[1,n+1]$. These relations are easily obtained by considering the left-hand-side expression as an alternating $(m+1)$-multilinear function on $\mathbb{C}^m$. It is known that these equations define $G(m,n+1)$ scheme-theoretically in $\mathbb{P}^{\binom{n+1}{m}-1}$ (see, for example, [404], vol. 2).

The open subset $D(p_I)\cap G(m,n+1)$ is isomorphic to the affine space $\mathbb{A}^{m(n-m)}$. The isomorphism is defined by assigning to a matrix A defining L, the point (z_J), where $z_J=|A_J|/|A_I|$ taken in some fixed order. This shows that $G(m,n+1)$ is a smooth rational variety of dimension $m(n+1-m)$.

Let

$$p_{\mathbb{G}}:Z_{\mathbb{G}}:=\mathbb{P}(Q_{\mathbb{G}})\to|E|,\quad q_{\mathbb{G}}:Z_{\mathbb{G}}=\mathbb{P}(Q_{\mathbb{G}})\to\mathbb{G} \tag{10.5}$$

denote the compositions of the closed embedding $\mathbb{P}(Q_{\mathbb{G}})\hookrightarrow|E|\times\mathbb{G}$ with the projections to $|E|$ and $\mathbb{G}$.

The fiber of the projection $p_{\mathbb{G}}$ over a point $x=[v]\in|E|$ can be canonically identified with $G(m-1,E/\mathbb{C}v)$. Recall that the quotient spaces $E/\mathbb{C}v, v\in E$, are the fibers of the quotient sheaf $E\otimes\mathcal{O}_{|E|}/\mathcal{O}_{|E|}(-1)$ which is isomorphic to the twisted tangent sheaf $\mathcal{T}_{|E|}(-1)$ via the Euler exact sequence

$$0\to\mathcal{O}_{|E|}\to\mathcal{O}_{|E|}(1)\otimes E\to\mathcal{T}_{|E|}\to 0.$$

The projection map $p_{\mathbb{G}}$ is the *Grassmann bundle* $G(m-1, \mathcal{T}_{|E|}(-1))$. In particular, if $m = 2$,

$$Z_{\mathbb{G}} = |\mathcal{T}_{|E|}(-1)| = \mathbb{P}(\Omega^1_{|E|}(1)). \tag{10.6}$$

Let us compute the canonical sheaf $\omega_{\mathbb{G}}$ of $\mathbb{G}$.

Lemma 10.1.1. *Let $\mathcal{T}_{\mathbb{G}}$ be the tangent bundle of $\mathbb{G}$. There is a natural isomorphism of sheaves*

$$\mathcal{T}_{\mathbb{G}} \cong \mathcal{S}_{\mathbb{G}} \otimes \mathcal{Q}_{\mathbb{G}}^{\vee},$$
$$\omega_{\mathbb{G}} \cong \mathcal{O}_{\mathbb{G}}(-n-1),$$

where $\mathcal{O}_{\mathbb{G}}(1)$ is taken with respect to the Plücker embedding.

Proof Let us trivialize $\mathcal{Q}_{\mathbb{G}}$ over an open subset U to assume that $\mathcal{Q}_{\mathbb{G}}^{\vee} = L \otimes \mathcal{O}_U$, where L is a linear space of dimension k. Then, U is isomorphic to the quotient of an open subset of $\mathrm{Hom}(L, E)$ by $\mathrm{GL}(L)$. The tangent bundle of U becomes isomorphic to

$$\mathrm{Hom}(L, E)/\mathrm{End}(L, L) \cong L^{\vee} \otimes E/L^{\vee} \otimes L \cong L^{\vee} \otimes (E/L).$$

These isomorphisms can be glued together to define a global isomorphism $\mathcal{T}_{\mathbb{G}} \cong \mathcal{S}_{\mathbb{G}} \otimes \mathcal{Q}_{\mathbb{G}}^{\vee}$.

Since $\bigwedge^k E \to \bigwedge^k Q_{\mathbb{G}}$ defines the Plücker embedding, we have

$$c_1(\mathcal{Q}_{\mathbb{G}}) = c_1(\mathcal{O}_{\mathbb{G}}(1)).$$

Now, the second isomorphism follows from a well-known formula for the first Chern class of the tensor product of vector bundles (see [379, Appendix A]). □

Since $\mathsf{Z}_{\mathbb{G}}$ is a projective bundle over $\mathbb{G}$, we can apply formula (2.29) for the canonical sheaf of a projective bundle to obtain

$$\omega_{Z_{\mathbb{G}}/\mathbb{G}} \cong q_{\mathbb{G}}^*(\bigwedge^k \mathcal{Q}_{\mathbb{G}}) \otimes p_{\mathbb{G}}^*\mathcal{O}_{|E|}(-k) \cong q_{\mathbb{G}}^*\mathcal{O}_{\mathbb{G}}(1) \otimes p_{\mathbb{G}}^*\mathcal{O}_{|E|}(-m),$$
$$\omega_{Z_{\mathbb{G}}} \cong \omega_{Z_{\mathbb{G}}/\mathbb{G}} \otimes q_{\mathbb{G}}^*(\omega_{\mathbb{G}}) \cong q_{\mathbb{G}}^*\mathcal{O}_{\mathbb{G}}(-n) \otimes p_{\mathbb{G}}^*\mathcal{O}_{|E|}(-k),$$

The general linear group $\mathrm{GL}(E)$ admits a natural linear representation on $\bigwedge^k E$. It defines a natural action of the group $\mathrm{PGL}(E)$ on the Grassmannian $\mathbb{G} = G(k, E)$. It follows from the computation of $\omega_{\mathbb{G}}$ that $\mathbb{G}$ is a Fano variety. In particular, any automorphism of $\mathbb{G}$ is defined by a projective automorphism of $|\bigwedge^m E|$ after we use the Plücker embedding of $\mathbb{G}$.

We refer for the proof of the following Theorem to [375, Theorem 10.19].

Theorem 10.1.2. *The homomorphism*

$$\mathrm{PGL}(E) \to \mathrm{Aut}(G(m, E))$$

is an isomorphism if $\dim E = n$ *is even and an isomorphism onto a subgroup of index* 2 *if* n *is odd.*

10.1.2 Schubert varieties

Let us recall some facts about the cohomology ring $H^*(\mathbb{G}, \mathbb{Z})$ of $\mathbb{G} = G_r(\mathbb{P}^n)$ (see [315], Chapter 14).

Fix a flag

$$A_0 \subset A_1 \subset \ldots \subset A_r \subset \mathbb{P}^n$$

of subspaces of dimension $0 \le a_0 < a_1 < \cdots < a_r \le n - r$, and define the *Schubert variety*

$$\Omega(A_0, A_1, \ldots, A_r) = \{\Pi \in \mathbb{G} : \dim \Pi \cap A_i \ge i, \ i = 0, \ldots, r\}.$$

This is a closed subvariety of $\mathbb{G}$ of dimension $\sum_{i=0}^{r}(a_i - i)$. Its homology class $[\Omega(A_0, A_1, \ldots, A_r)]$ in $H_*(\mathbb{G}, \mathbb{Z})$ depends only on $a_0, \ldots, a_r$. It is called a *Schubert cycle* and is denoted by $(a_0, \ldots, a_r)$. Let $a_0 = n - r - d, a_i = n - r + i, i = 1, \ldots, r$. The varieties

$$\Omega(A_0) := \Omega(A_0, \ldots, A_r) = \{\Pi \in \mathbb{G} : \Pi \cap A_0 \neq \emptyset\}$$

are called the *special Schubert varieties*. Their codimension is equal to d. Their cohomology classes are denoted by $\sigma_d, d = \dim A_0$.

Under the Poincaré duality $H_*(\mathbb{G}, \mathbb{Z}) \to H^*(\mathbb{G}, \mathbb{Z})$, the cycles $(a_0, \ldots, a_r)$ are mapped to *Schubert classes* $\sigma_{\lambda_0,\ldots,\lambda_r}$ defined in terms of the Chern classes

$$\sigma_s = c_s(\mathcal{S}_{\mathbb{G}}^{\vee}) \in H^{2s}(\mathbb{G}, \mathbb{Z}), \ s = 1, \ldots, n - r,$$

by the *determinantal formula*

$$\sigma_{\lambda_0,\ldots,\lambda_r} = \det(\sigma_{\lambda_i + j - i})_{0 \le i, j \le r},$$

where $\lambda_i = n - r + i - a_i, \ i = 0, \ldots, r$. The cohomology classes σ_s are dual to the cohomology classes of special Schubert varieties $\Omega(A_0)$, where $\dim A_0 = n - r - s$.

The tautological exact sequence (10.1) shows that

$$1 = (\sum c_s(\mathcal{Q}_{\mathbb{G}}))(\sum c_s(\mathcal{S}_{\mathbb{G}})).$$

In particular,

$$\sigma_1 = c_1(\mathcal{S}_{\mathbb{G}}^{\vee}) = c_1(\mathcal{Q}_{\mathbb{G}}) = c_1(\mathcal{O}_{\mathbb{G}}(1)),$$
$$\sigma_2 = c_2(\mathcal{S}_{\mathbb{G}}^{\vee}) = \sigma_1^2 - c_2(\mathcal{Q}_{\mathbb{G}}).$$

Remark 10.1.3. Our notation is dual to the notation from [315] since our definitions of the projective space $\mathbb{P}(E)$ are dual. Thus, our $\mathbb{Q}_{\mathbb{G}}$ is Fultons's $S^{\vee}$ and our $\mathbb{S}_{\mathbb{G}}$ is Fulton's $Q^{\vee}$. So, Fulton's interpretation of $c_s(Q)$ as the classes of special Schubert varieties from [315, 14.7] must be applied to the dual Grassmannian $G(k, E^{\vee})$.

A proof of the following result can be found in [315, 14.7] or [404, Vol. 2].

Proposition 10.1.4. *The cohomology ring $H^*(\mathbb{G}, \mathbb{Z})$ is generated by the special Schubert classes σ_s. The Schubert cycles $(a_0, \ldots, a_r)$ with $\sum_{i=0}^r (a_i - i) = d$ freely generate $H_{2d}(\mathbb{G}, \mathbb{Z})$. The Schubert classes $\{\lambda_0, \ldots, \lambda_r\}$ with $d = \sum_{i=0}^r \lambda_i$ freely generate $H^{2d}(\mathbb{G}, \mathbb{Z})$. In particular,*

$$\mathrm{Pic}(\mathbb{G}) \cong H^2(\mathbb{G}, \mathbb{Z}) = \mathbb{Z}\sigma_1.$$

It follows from the proposition above that $H^*(\mathbb{G}, \mathbb{Z})$ is isomorphic to the Chow ring $A^*(\mathbb{G})$ of algebraic cycles on $\mathbb{G}$. Under the Poincaré Duality $\gamma \mapsto \alpha_\gamma$, the intersection form on cycles $\langle \gamma, \mu \rangle$ is defined by

$$\langle \gamma, \mu \rangle = \int_\mu \alpha_\gamma = \int_G \alpha_\gamma \wedge \alpha_\mu := \alpha_\gamma \cdot \alpha_\mu.$$

The intersection form on $A^*(\mathbb{G})$ is calculated by using the *Pieri's formulas*

$$\sigma_{\lambda_0, \ldots, \lambda_r} \cdot \sigma_s = \sum \sigma_{\mu_0, \ldots, \mu_r}, \tag{10.7}$$

where the sum is taken over all $\{\mu\}$ such that

$$n - r \geq \mu_0 \geq \lambda_0 \geq \mu_1 \geq \lambda_1 \geq \ldots \geq \mu_r \geq \lambda_r \geq \mu_r \geq \lambda_r$$

and $\sum \mu_i = s + \sum \lambda_i$.

Another useful formulas that can be found in [315, Examples 14.7.4 and 14.7.11] are the following:

$$(a_0, \ldots, a_r) \cdot (b_0, \ldots, b_r) = \begin{cases} 1 & \text{if } a_i + b_{r-i} = n, 0 \leq i \leq r, \\ 0 & \text{otherwise,} \end{cases} \tag{10.8}$$

where $(a_0, \ldots, a_r)$ and $(b_0, \ldots, b_r)$ are of complementary dimension.

$$(a_0, \ldots, a_r) \cdot \sigma_1^k = k! \sum \left| \left(\frac{1}{(a_i - b_j)!}\right)_{0 \leq i, j \leq r} \right| (b_0, \ldots, b_r), \tag{10.9}$$

where the sum is taken over all $(b_0, \dots, b_r)$ with $\sum b_i - \sum a_i = k$, and the entry $\frac{1}{(a_i - b_j)}$ in the matrix is equal zero if $a_i < b_j$.

For example, taking $k = (n-r)(r+1) = \dim G_r(\mathbb{P}^n)$, we get the following formula computing the degree of $G_r(\mathbb{P}^n)$

$$\deg G_r(\mathbb{P}^n) = \frac{1!2!\dots r! \dim G!}{(n-r)!(n-r+1)!\dots n!}. \tag{10.10}$$

Here, are some special cases. We set, for $\sigma_{\lambda_0,\lambda_r} \neq \sigma_{0,\dots,0}$,

$$\sigma_{\lambda_1,\dots,\lambda_k} := \sigma_{\lambda_1,\dots,\lambda_k,0,\dots,0},$$
$$\sigma_0 = \sigma_{0,\dots,0}.$$

Then

$$\sigma_1^2 = \sigma_2 + \sigma_{1,1},$$
$$\sigma_1 \cdot \sigma_2 = \sigma_3 + \sigma_{2,1},$$
$$\sigma_1 \cdot \sigma_{1,1} = \sigma_{2,1}.$$

Example 10.1.5. Let us look at the Grassmannian $G_1(\mathbb{P}^3) = G(2,4)$ of lines in $\mathbb{P}^3$. The Plücker equations are reduced to one quadratic relation

$$p_{12}p_{34} - p_{13}p_{24} + p_{14}p_{23} = 0. \tag{10.11}$$

This is a nonsingular quadric in $\mathbb{P}^5$, often called the *Klein quadric*. The Schubert class of codimension 1 is represented by the special Schubert variety $\Omega(\ell)$ of lines intersecting a given line ℓ. We have two codimension 2 Schubert cycles σ_2 and $\sigma_{1,1}$ represented by the Schubert varieties $\Omega(x)$ of lines containing a given point x and $\Omega(\Pi)$ of lines containing in a given plane Π. Each of these varieties is isomorphic to $\mathbb{P}^2$. In classical terminology, $\Omega(x)$ is an *α-plane* and $\Omega(\Pi)$ is a *β-plane*. We have a 1-dimensional Schubert cycle $\sigma_{2,1}$ represented by the Schubert variety $\Omega(x, \Pi)$ of lines in a plane Π containing a given point $x \in \Pi$. It is isomorphic to $\mathbb{P}^1$. Thus

$$A^*(G(2,4)) = \mathbb{Z}[G] \oplus \mathbb{Z}\sigma_1 \oplus (\mathbb{Z}\sigma_2 + \mathbb{Z}\sigma_{1,1}) \oplus \mathbb{Z}\sigma_{2,1} \oplus \mathbb{Z}[\text{point}].$$

Tthe two Schubert classes of codimension two represent two different rulings of the Klein quadric by planes.

We have

$$\sigma_2 \cdot \sigma_{1,1} = 0, \ \sigma_2^2 = 1, \sigma_{1,1}^2 = 1. \tag{10.12}$$

Write $\sigma_1^2 = a\sigma_2 + b\sigma_{1,1}$. Intersecting both sides with σ_2 and $\sigma_{1,1}$, we obtain $a = b = 1$ confirming Pieri's formula (10.7). Squaring σ_1^2, we obtain $\deg G = \sigma_1^4 = 2$, confirming the fact that $G(2,4)$ is a quadric in $\mathbb{P}^5$.

Theorem 10.1.2 applied to $G_1(\mathbb{P}^3)$ is easy to explain. The Klein quadric is a smooth quadric in $\mathbb{P}^5$, and hence admits two rulings by planes, α-planes and β-planes. The group of automorphism of $G_1(\mathbb{P}^3)$ has an index two subgroup that preserves each ruling. Since any α-plane is of the form $\Omega(x), x \in \mathbb{P}^3$, this subgroup acts in $\mathbb{P}^3$ proving the surjectivity of the action of $\mathrm{PGL}(4)$ on $G_1(\mathbb{P}^3)$ on this subgroup. One can interpret the whole group $\mathrm{Aut}(G_1(\mathbb{P}^3)$ as the group $\Sigma\mathrm{PGL}(4)$ of collineations and correlations of $\mathbb{P}^3$ discussed in Subsection 1.1.2

An irreducible surface S in $G_1(\mathbb{P}^3)$ is called a *congruence of lines*. Its cohomology class $[S]$ is equal to $m\sigma_2 + n\sigma_{1,1}$. The number m (resp. n) is classically known as the *order* of S (resp. *class*). It is equal to the number of lines in S passing through a general point in $\mathbb{P}^3$ (resp. contained in a general plane). The sum $m + n$ is equal to $\sigma_1 \cdot [S]$ and hence coincides with the degree of S in $\mathbb{P}^5$. We will study congruences of lines in $\mathbb{P}^3$ in great detail in Chapter 11.

Example 10.1.6. Let $\mathbb{G} = G_2(\mathbb{P}^4)$. We have the following Schubert varieties in $\mathbb{G} = G_2(\mathbb{P}^4) = G(3,5)$.

dim	σ_λ	(a_0, a_1, a_2)	Schubert variety
6	σ_0	$(2,3,4)$	$G_2(\mathbb{P}^4)$
5	σ_1	$(1,3,4)$	$\{\Pi : \Pi \cap \ell_0 \neq \emptyset\}$
4	σ_2	$(0,3,4)$	$\{\Pi : x_0 \in \Pi\}$
4	$\sigma_{1,1}$	$(1,2,4)$	$\{\Pi : \dim \Pi \cap \Pi_0 \geq 1\}$
3	$\sigma_{1,1,1}$	$(1,2,3)$	$\{\Pi : \Pi \subset H_0\}$
3	$\sigma_{2,1}$	$(0,2,4)$	$\{\Pi : x_0 \in \Pi, \Pi \cap \Pi_0 \neq \emptyset, x_0 \in \Pi_0\}$
2	$\sigma_{2,2}$	$(0,1,4)$	$\{\Pi : \ell_0 \subset \Pi\}$
2	$\sigma_{2,1,1}$	$(0,2,3)$	$\{\Pi : x_0 \subset \Pi \subset H_0\}$
1	$\sigma_{2,2,1}$	$(0,1,3)$	$\{\Pi : \ell_0 \subset \Pi \subset H_0\}$
0	$\sigma_{2,2,2}$	$(0,1,2)$	$\Pi : \Pi = \Pi_0$

Table 10.1 *Schubert classes in* $G_2(\mathbb{P}^4)$

We have

$$\begin{aligned}
\sigma_1^2 &= \sigma_2 + \sigma_{1,1}, \\
\sigma_1^3 &= \sigma_3 + 2\sigma_{2,1}, \\
\sigma_1^4 &= 2\sigma_{2,2} + 3\sigma_{2,1,1}, \\
\sigma_1^5 &= 5\sigma_{2,2,1}, \\
\sigma_1^6 &= 5 = \dim G_2(\mathbb{P}^4),
\end{aligned}$$

One can also check that the basis $(\sigma_2, \sigma_{1,1})$ of $H^2(\mathbb{G}, \mathbb{Z})$ is dual to the basis $(\sigma_{2,1,1}, \sigma_{2,2})$ of $H^4(\mathbb{G}, \mathbb{Z})$, and the basis $(\sigma_3, \sigma_{2,1})$ of $H^3(\mathbb{G}, \mathbb{Z})$ is orthogonal.

Of course, using the duality between lines an planes, we obtain the similar computations for the Grassmannian of lines $G_1(\mathbb{P}^4)$ in $\mathbb{P}^4$.

dim	σ_λ	(a_0, a_1)	Schubert variety	dual cycle
6	σ_0	$(3,4)$	$G_1(\mathbb{P}^4)$	σ_0
5	σ_1	$(2,4)$	$\{\ell : \ell \cap \Pi_0 \neq \emptyset\}$	σ_1
4	σ_2	$(1,4)$	$\{\ell : \ell \cap \ell_0 \neq \emptyset\}$	$\sigma_{1,1}$
4	$\sigma_{1,1}$	$(0,3)$	$\{\ell : \ell \subset H_0\}$	σ_2
3	$\sigma_{2,1}$	$(0,3)$	$\{\ell : \ell \subset H_0, \ell \cap \ell_0 \neq \emptyset, \ell_0 \subset H_0\}$	$\sigma_{2,1}$
3	σ_3	$(0,4)$	$\{\ell : x_0 \in \ell\}$	$\sigma_{1,1,1}$
2	$\sigma_{2,2}$	$(1,2)$	$\{\ell : \ell \subset \Pi_0\}$	$\sigma_{2,2}$
2	$\sigma_{3,1}$	$(0,3)$	$\{\ell : x_0 \in \ell \subset H_0\}$	$\sigma_{2,2,1}$
1	$\sigma_{3,2}$	$(0,2)$	$\{\ell_0 \subset \Pi_0, x_0 \subset \ell\}$	$\sigma_{2,2,1}$
0	$\sigma_{3,3}$	$(0,1)$	$\{\ell\}$	$\sigma_{2,2,2}$

Table 10.2 *Schubert classes in* $G_1(\mathbb{P}^4)$

Here, the dual cycle means the corresponding cycle in $H^*(G_2(\mathbb{P}^4), \mathbb{Z})$ under the duality isomorphism $\phi : G_1(\mathbb{P}^4) \to G_2(\check{\mathbb{P}}^4)$.

Let S be an irreducible surface in $\mathbb{G} = G_2(\mathbb{P}^4)$ and $[S]$ be its cohomology class. We have

$$[S] = m\sigma_{2,2} + n\sigma_{2,1,1}.$$

Let $p_S : Z_S \to \mathbb{P}^4$ be the restriction of the projection $p_\mathbb{G} : Z_\mathbb{G} \to \mathbb{P}^4$ to $q_\mathbb{G}^{-1}(S)$. Since $\dim Z_S = 4$, we expect that the map p_S is of finite degree. The projection $q(p_S^{-1}(x)) \subset S$ over a general point $x \in \mathbb{P}^4$ consists of the subvariety planes in S passing through x. Its cohomology class is equal to $[S] \cdot \sigma_2 = (m\sigma_{2,2} + n\sigma_{2,1,1}) \cdot \sigma_2 = m$. Similarly the projection of $p_S^{-1}(\Pi)$ of a general plane Π in $\mathbb{P}^4$ is equal to the set of planes in S intersecting Π along a line. Its cohomology class is equal to $[S] = \sigma_1^4 = 2\sigma_{2,2} + 3\sigma_{2,1,1}$. So, $(m, n) = (2, 3)$ in this case, and the projection $p_S : Z_S \to \mathbb{P}^4$ is a degree 2 map.

We refer to [130, Proposition 2.4.5], where it is proved that the branch divisor of the cover coincides with the Castelnuovo-Richmond quartic hypersurface CR_4. If we consider the dual Grassmannian $G_1(\check{P}^4)$ and take $\check{S}$ to be the image of S, then the similar projection $Z_{\check{S}} \to \mathbb{P}^4$ of the threefold $q^{-1}(\check{S}) \subset Z_{\mathbb{G}^*}$ has the image isomorphic to the Segre cubic primal S_3. [441].

Example 10.1.7. The middle-dimensional part $H^{n-1}(G_1(\mathbb{P}^n)$ is generated by σ_{n-1} and $\sigma_{n-2,1}$. The corresponding Schubert varieties are the set of lines passing through a fixed point or contained in a hyperplane, respectively. An irreducible $(n-1)$-dimensional subvariety S of $G_1(\mathbb{P}^n)$ is called a *congruence of lines* . Its cohomology class $[S] = m\sigma_{n-1} + n\sigma_{n-2,1}$. The number m (resp. n) is called the *order* (resp. *class*) of the congruence. it is equal to the number of lines passing through a fixed point in $\mathbb{P}^n$ (resp. the number of lines contained in a general hyperplane).

However, be aware that some authors define congruence as an irreducible surface or a codimension two subvariety in $G_1(\mathbb{P}^n)$ (see, for example, [701]). Their cohomology class is also determined by two integers (m, n) such that $[S] = m\sigma_{n-1,n-3} + n\sigma_{n-2,n-2}$ in the case of surfaces and $[S] = m\sigma_2 + n\sigma_{1,1}$ in the case of codimension two subvarieties. In the case of surfaces, the number m (resp. n) is equal to the number of lines in S that are contained in a 3-dimensional subspace of $\mathbb{P}^n$ and pass through a fixed point in this subspace (resp. lines from S contained in a fixed plane). In the case of codimension two subvarieties, the number m is equal to the number of lines in S intersecting a fixed codimension 3 subspace (resp. contained in a fixed hyperplane). One can also call m (resp. n) to be the order (resp. class) of S.

There are numerous applications of Schubert calculus to enumerative problems in algebraic geometry. Let us prove the following nice result, which can be found in many classical textbooks (first proven by L. Cremona [181]).

Proposition 10.1.8. *The number of common secant lines of two general rational normal curves in $\mathbb{P}^3$ is equal to 10.*

Proof Consider the congruence of lines formed by secants of a rational normal cubic curve in $\mathbb{P}^3$. Through a general point in $\mathbb{P}^3$ passes one secant. In a general plane lie three secants. Thus, the order of the congruence is equal to 1 and the class is equal to 3. Applying (10.12), we obtain that the two congruences intersect at 10 points. □

Remark 10.1.9. Let R_1 and R_2 be two general rational normal cubic curves in $\mathbb{P}^3$ and let $\mathcal{N}_i$ be the net of quadrics through R_i. The linear system $\mathcal{W}$ of quadrics in the dual space that is apolar to the linear system $\mathcal{N}$ spanned by $\mathcal{N}_1$ and $\mathcal{N}_2$ is of dimension 3. The Steinerian quartic surface defined by this linear system contains 10 lines, the singular lines of 10 reducible quadrics from $\mathcal{W}$. The dual of these lines are the 10 common secants of R_1 and R_2 (see [618], [516], [159]). Also observe that the 5-dimensional linear system $\mathcal{N}$ maps R_i to a curve C_i of degree 6 spanning the plane Π_i in $\mathcal{N}^\vee$ apolar to the plane $\mathcal{N}_j$. The 10 pairs of intersection points of C_i with the ten common secants correspond to the branches of the ten singular points of C_i.

10.1.3 Secant varieties of Grassmannians of lines

From now on, we will restrict ourselves with the Grassmannian of lines in $\mathbb{P}^n = |E|$. Via contraction, one can identify $\bigwedge^2 E$ with the space of linear maps $u : E^\vee \to E$ such that the transpose map ${}^t u$ is equal to $-u$. Explicitly,

$$v \wedge w(l) = l(v)w - l(w)v.$$

The rank of u is the rank of the map. Since ${}^t u = -u$, the rank takes even values. The Grassmann variety $G(2, E)$ is the set of points $[u]$, where u is a map of rank 2.

After fixing a basis in E, we can identify $\bigwedge^2 E$ with the space of skew-symmetric matrices $A = (p_{ij})$ of size $(n+1) \times (n+1)$. The Grassmann variety $G(2, E)$ is the locus of rank 2 matrices, up to proportionality. The entries $p_{ij}, i < j$, are the Plücker coordinates. In particular, $G(2, E)$ is the zero set of the 4×4 pfaffians of A. In fact, each of the Plücker equations is given by the 4×4 pfaffian of the matrix (p_{ij})

$$p_{ij}p_{kl} - p_{ik}p_{jl} + p_{il}p_{jk} = \mathrm{Pf}\begin{pmatrix} 0 & p_{ij} & p_{ik} & p_{il} \\ -p_{ij} & 0 & p_{jk} & p_{jl} \\ -p_{ik} & -p_{jk} & 0 & p_{kl} \\ -p_{il} & -p_{jl} & -p_{kl} & 0 \end{pmatrix}.$$

Another way to look at $G(2, E)$ is to use the decomposition

$$E \otimes E \cong S^2(E) \oplus \bigwedge^2 E.$$

It identifies $G(2, E)$ with the projection of the Segre variety $\mathsf{s}_2(|E| \times |E|) \subset |E \otimes E|$ to $|\bigwedge^2 E|$ from the subspace $|S^2(E)|$.

The formula (10.10) for the degree of the Grassmannian gives in our special case

$$\deg G_1(\mathbb{P}^n) = \frac{(2n-2)!}{(n-1)!n!}. \tag{10.13}$$

One can also compute the degrees of Schubert varieties

$$\deg \Omega(a_0, a_1) = \frac{(a_0 + a_1 - 1)!}{a_0!a_1!}(a_1 - a_0). \tag{10.14}$$

Lemma 10.1.10. *The rank of $u \in \bigwedge^2 E$ is equal to the smallest number k such that ω can be written as a sum $u_1 + \cdots + u_k$ of 2-vectors u_i of rank 2.*

Proof It suffices to show that, for any u of rank $2k \geq 4$, there exists a 2-vector u_1 of rank 2 such that $u - u_1$ is of rank $\leq 2k - 2$. Let R be the kernel of u and $l_0 \notin R$. Choose $v_0 \in E$ such that, for any $l \in R$, $l(v_0) = 0$ and $l_0(v_0) = 1$. By skew-symmetry of u, for any $l, m \in E^\vee$, $m(u(x)) = -l(u(m))$. Consider the difference $u' = u - v_0 \wedge u(l_0)$. For any $l \in R$, we have

$$u'(l) = u(l) - l(v_0)u(l_0) + l(u(l_0))v_0 = l(u(l_0))v_0 = -l_0(u(l))v_0 = 0.$$

This shows that $R \subset \mathrm{Ker}(u)$. Moreover, we have

$$u'(l_0) = u(l_0) + l_0(u(l_0))v_0 - l_0(v_0)u(l_0) = u(l_0) - u(l_0) = 0.$$

This implies that $\mathrm{Ker}(u')$ is strictly larger than $\mathrm{Ker}(u)$. ◻

This gives the following.

Proposition 10.1.11. *The variety*

$$\mathbb{G}_k =: \{[u] \in |\bigwedge^2 E| : \ u \text{ has rank} \leq 2k+2\}$$

is equal to the k-secant variety $\mathrm{Sec}_k(\mathbb{G})$ of $\mathbb{G}$.

Let $t = [\frac{n-3}{2}]$, then t is the maximal number k such that $\mathrm{Sec}_k(\mathbb{G}) \neq |\bigwedge^2 E|$. So the Plücker space is stratified by the rank of its points and the strata are the following:

$$|E| \setminus \mathbb{G}_t, \ \mathbb{G}_t \setminus \mathbb{G}_{t-1}, \ \ldots, \mathbb{G}_1 \setminus \mathbb{G}, \ \mathbb{G}. \tag{10.15}$$

It follows from the previous remarks that $\mathbb{G}_k \setminus \mathbb{G}_{k-1}$ is the orbit of a matrix of rank $2k+2$ and size $(n+1) \times (n+1)$ under the action of $\mathrm{GL}(n+1)$. Therefore,

$$\dim \mathbb{G}_k = \dim \mathrm{GL}(n)/H_k,$$

where H_k is the stabilizer of a skew-symmetric matrix of rank $2k+2$. An easy computation gives the following.

Proposition 10.1.12. *Let $0 \leq k \leq t$, then*

$$d_k = \dim \mathbb{G}_k = (k+1)(2n-2k-1) - 1. \tag{10.16}$$

Let $X \subset \mathbb{P}^r$ be a reduced and nondegenerate closed suvariety. The *k-th defect* of X can be defined as

$$\delta_k(X) = \min\big((k+1)\dim X + k, r\big) - \dim \mathrm{Sec}_k(X),$$

which is the difference between the expected dimension of the k-secant variety of X and the effective one. We say that X is *k-defective* if $\mathrm{Sec}_k(X)$ is a proper subvariety and $\delta_k(X) > 0$.

Example 10.1.13. Let $n = 2t + 3$, then $\mathbb{G}_t \subset |\bigwedge^2 E|$ is the pfaffian hypersurface of degree $t + 2$ in $|\bigwedge^2 E|$ parameterizing singular skew-symmetric matrices (a_{ij}) of size $2t+4$. The expected dimension of $\mathbb{G}_t$ is equal to $4t^2 + 8t + 5$, that is larger than $\dim |\bigwedge^2 E| = \binom{2t+4}{2} - 1$. Thus, $d_t(\mathbb{G}) = \dim \mathbb{G}_t + 1$ and $\delta_t(\mathbb{G}) = 1$.

In the special case $n = 5$, the variety $G_1(\mathbb{P}^5)$ is one of the four Severi-Zak varieties.

Using Schubert varieties one can describe the projective tangent space of $\mathbb{G}_k$ at a given point $p = [u] \notin \mathbb{G}_{k-1}$. Let $K = \mathrm{Ker}(u) \subset E^\vee$. Since the rank of u is equal to $2k + 2$, the dual subspace $K^\perp \subset E$ defines a linear subspace

$$\Lambda_p = |K^\perp|$$

of $|E|$ of dimension $2k + 1$. Let $\Omega(\Lambda_p)$ be the corresponding special Schubert variety and let $\langle \Omega(\Lambda_p) \rangle$ be its linear span in the Plücker space.

Proposition 10.1.14.

$$\mathbb{T}_p(\mathbb{G}_k) = \langle \Omega(\Lambda_p) \rangle.$$

Proof Since $\mathbb{G}_k \setminus \mathbb{G}_{k-1}$ is a homogeneous space for $\mathrm{GL}(n + 1)$, we may assume that the point p is represented by a 2-vector $u = \sum_{i=0}^{k} e_{2i+1} \wedge e_{2i+2}$, where $(e_1, \ldots, e_n)$ is a basis in E. The corresponding subspace $K^\perp$ is spanned by $e_1, \ldots, e_{2k+2}$. A line ℓ intersects Λ_p if and only if it can be represented by a bivector $v \wedge w$, where $v \in K^\perp$. Thus, $W = \langle \Omega(\Lambda_p) \rangle$ is the span of points $[e_i \wedge e_j]$, where either i or j is less than or equal to $2k + 2$. In other words, W is given by vanishing of $\binom{n-2k-1}{2}$ Plücker coordinates p_{ab}, where $a, b > 2k + 2$. It is easy to see that this agrees with the formula for $\dim \mathrm{Sec}_k(\mathbb{G})$. So, it is enough to show that W is contained in the tangent space. We know that the equations of $\mathrm{Sec}_k(\mathbb{G})$ are given by the pfaffians of size $4k + 4$. Recall the formula for the pfaffians from Exercise 2.1 in Chapter 2:

$$\mathrm{Pf}(A) = \sum_{S \in \mathcal{S}} \pm \prod_{(ij) \in S} a_{ij},$$

where S is a set of pairs $(i_1, j_1), \ldots, (i_{2k+2}, j_{2k+2})$ such that $1 \le i_s < j_s \le 4k + 4$, $s = 1, \ldots, 2k + 2$, $\{i_1, \ldots, i_{2k+2}, j_1, \ldots, j_{2k+2}\} = \{1, \ldots, 4k + 4\}$. Consider the Jacobian matrix of $\mathbb{G}_k$ at the point p. Each equation of $\mathbb{G}_k$ is obtained by a choice of a subset I of $\{1, \ldots, n\}$ of cardinality $4k + 4$ and writing the pfaffian of the submatrix of (p_{ij}) formed by the columns and rows with indices in I. The corresponding row of the Jacobian matrix is obtained by taking the partials of this equation with respect to all p_{ij} evaluated at the point p. If $a, b \le 2k + 2$, then one of the factors in the product $\prod_{(ij) \in S} p_{ij}$ corresponds to a pair (i, j), where $i, j > 2k + 2$. When we differentiate with respect to p_{ab} its value at p is equal to zero. Therefore, the corresponding entry in the Jacobian matrix is equal to zero. So, all nonzero entries in a row of the Jacobian matrix correspond to the coordinates of vectors from W that are equal to zero. Thus, W is contained in the space of solutions. □

Taking $k = 0$, we obtain the following.

Corollary 10.1.15. *For any $\ell \in \mathbb{G}$,*

$$\mathbb{T}_\ell(\mathbb{G}) = \langle \Omega(\ell) \rangle.$$

Let Λ be any subspace of $\mathbb{P}^n$ of dimension $2k + 1$ and

$$P_\Lambda = \{p \in |\bigwedge^2 E| : \Lambda = \Lambda_p\}.$$

This is the projectivization of the linear space of skew-symmetric matrices of rank $2k+2$ with the given nullspace of dimension $2k+2$. An easy computation using formula (10.16) shows that $\dim P_\Lambda = (2k + 1)(k + 1) - 1$.

Let

$$\gamma_k : \mathbb{G}_k \setminus \mathbb{G}_{k-1} \to G(d_k + 1, \bigwedge^2 E), \quad d_k = \dim \mathbb{G}_k,$$

be the *Gauss map* which assigns to a point its embedded tangent space. Applying Proposition 10.1.14, we obtain the following:

Corollary 10.1.16.

$$\gamma_k^{-1}(\langle \Omega(\Lambda) \rangle) = P_\Lambda.$$

In particular, any hyperplane in the Plücker space containing $\Omega(\Lambda)$ *is tangent to* $\mathrm{Sec}_k(\mathbb{G})$ *along the subvariety* P_Λ *of dimension* $(2k + 1)(k + 1) - 1$.

Example 10.1.17. Let $\mathbb{G} = G(2, 6)$. We have already observed that the secant variety $\mathbb{G}_1$ is a cubic hypersurface X in $\mathbb{P}^{14}$ defined by the pfaffian of the 6×6 skew-symmetric matrix whose entries are Plücker coordinates p_{ij} . The Gauss map is the restriction to X of the polar map $\mathbb{P}^{14} \dashrightarrow \check{\mathbb{P}}^{14}$ given by the partials of the cubic. The singular locus of X is equal to $G(2, 6)$, it is defined by the polars of X. The polar map is a Cremona transformation in $\mathbb{P}^{14}$. This is one of the examples of Cremona transformations defined by Severi-Zak varieties (see Subsection 7.4.2).

Let X be a subvariety of $\mathbb{G}$, and Z_X be the pre-image of X under the projection $q : \mathsf{Z}_\mathbb{G} \to \mathbb{G}$. The image of Z_X in $\mathbb{P}^n$ is the union of lines $\ell \in X$. We will need the description of its set of nonsingular points.

Proposition 10.1.18. *The projection $p_X : \mathsf{Z}_X \to \mathbb{P}^n$ is smooth at (x, ℓ) if and only if*

$$\dim_\ell \Omega(x) \cap \mathbb{T}_\ell(X) = \dim_{(x,\ell)} p_X^{-1}(x).$$

Proof Let $(x, \ell) \in \mathsf{Z}_X$ and let F be the fiber of $p_X : \mathsf{Z}_X \to \mathbb{P}^n$ passing through the point (x, ℓ) identified with the subset $\Omega(x) \cap X$ under the projection $q_{\mathbb{G}} : \mathsf{Z}_X \to \mathbb{G}$. Then,

$$\mathbb{T}_{x,\ell}(F) = \mathbb{T}_\ell(\Omega(x)) \cap \mathbb{T}_\ell(X) = \Omega(x) \cap \mathbb{T}_\ell(X). \tag{10.17}$$

This proves the assertion. □

Corollary 10.1.19. *Let $Y = p_X(\mathsf{Z}_X) \subset \mathbb{P}^n$ be the union of lines $\ell \in X$. Assume X is nonsingular and $p_X^{-1}(x)$ is a finite set. Suppose $\dim_\ell \Omega(x) \cap \mathbb{T}_\ell(X) = 0$ for some $\ell \in X$ containing x. Then, x is a nonsingular point of Y.*

10.2 Linear Line Complexes

10.2.1 Linear line complexes and apolarity

An irreducible and reduced codimension one subvariety of $\mathfrak{C} \subset \mathbb{G} = G_1(\mathbb{P}^n)$ is called a *line complex*. Since we know that $\mathrm{Pic}(\mathbb{G})$ is generated by $\mathcal{O}_{\mathbb{G}}(1)$ we see that $\mathfrak{C} \in |\mathcal{O}_{\mathbb{G}}(d)|$ for some $d \geq 1$. The number d is called the *degree* of the line complex.

An example of a line complex $\mathfrak{C}$ of degree d in $G_1(\mathbb{P}^n)$ is the *Chow form* of a subvariety $X \subset \mathbb{P}^n$ of codimension 2 (see [325]). It parameterizes lines that have non-empty intersection with X. Its degree is equal to the degree of X. When X is linear, this is of course the special Schubert variety $\Omega(X)$.

A *linear line complex* is a line complex of degree 1, that is a hyperplane section $\mathfrak{C} = H \cap \mathbb{G}$ of $\mathbb{G}$. If no confusion arises we will sometimes identify $\mathfrak{C}$ with the corresponding hyperplane H. A linear line complex is called *special* if it is equal to the special Schubert variety $\Omega(\Pi)$, where Π is a subspace of codimension $n - 2$. The corresponding hyperplane is tangent to the Grassmannian at any point ℓ such that $\ell \subset \Pi$. In particular, when $n = 3$, the special linear line complex is isomorphic to a quadric cone.

For any $\omega \in (\bigwedge^2 E)^\vee = \bigwedge^2 E^\vee$, let $\mathfrak{C}_\omega$ denote the linear line complex defined by the hyperplane $V(\omega)$. In coordinates, if $\omega = \sum a_{ij} p_{ij}$, the linear line complex $\mathfrak{C}_\omega$ is given by adding to the Plücker equations the additional equation

$$\sum_{0 \leq i < j \leq n} a_{ij} p_{ij} = 0.$$

For example, the line complex $V(p_{ij})$ parameterizes the lines intersecting the coordinate $(n-2)$-plane $t_k = 0, k \neq i, j$, in $\mathbb{P}^n$.

Remark 10.2.1. It follows from the Euler exact sequence that there is a natural isomorphism

$$H^0(|E|, \Omega^1_{|E|}(2)) \cong \mathrm{Ker}\big(E^\vee \otimes E^\vee \to S^2(E^\vee)\big) \cong \bigwedge^2 E^\vee. \tag{10.18}$$

Also we know from (10.6) that the incidence variety $\mathsf{Z}_{\mathbb{G}}$ is isomorphic to the projective bundle $\mathbb{P}(\Omega^1_{|E|}(1)) \cong \mathbb{P}(\Omega^1_{|E|}(2))$. Thus, a linear line complex can be viewed as a divisor in the linear system $|\mathcal{O}_{\mathsf{Z}_{\mathbb{G}}}(1)|$, where $p_*\mathcal{O}_{\mathsf{Z}_{\mathbb{G}}}(1) \cong \Omega^1_{|E|}(2)$. The fiber of $\mathsf{Z}_{\mathbb{G}}$ over a point $x \in |E|$ is isomorphic to the projectivized tangent space $\mathbb{P}(\Omega^1_{|E|}(x)) \cong |T_x(|E|)|$.

Choose local coordinates $z_1, \ldots, z_n$ in $|E|$ defining the basis $(\frac{\partial}{\partial z_1}, \ldots, \frac{\partial}{\partial z_n})$ in tangent spaces, then, for any nonzero $\omega \in \bigwedge^2 E^\vee$, the line complex $\mathfrak{C}_\omega \in \bigwedge^2 E^\vee$ is locally given by an expression

$$\sum_{i=1}^n A_i(z_1, \ldots, z_n) dz_i = 0.$$

This equation is called the *Pfaff partial differential equation*. More generally, any line complex of degree d can be considered as the zero set of a section of $\mathcal{O}_{\mathsf{Z}_{\mathbb{G}}}(d)$ and can be locally defined by the *Monge's partial differential equation*

$$\sum_{i_1+\cdots+i_n=d}^n A_{i_1,\ldots,i_n} dz_1^{i_1} \ldots dz_n^{i_n} = 0.$$

We refer to S. Lie's book [487] for the connection between the theory of Monge equations and line complexes.

The projective equivalence classes of linear line complexes coincide with the orbits of $\mathrm{GL}(E)$ acting naturally on $|\bigwedge^2 E^\vee|$. The $\mathrm{GL}(E)$-orbit of a linear line complex $\mathfrak{C}_\omega$ is uniquely determined by the *rank* $2k$ of ω. We will identify ω with the associated linear map $E \to E^\vee$. Let

$$\mathsf{S}_\omega = |\mathrm{Ker}(\omega)|. \tag{10.19}$$

It is called the *center* of a linear line complex $\mathfrak{C}_\omega$. We have encountered it in Chapter 2. This is a linear subspace of $|E|$ of dimension $n-2k$, where $2k$ is the rank of ω.

Proposition 10.2.2. *Let $\mathfrak{C}_\omega$ be a linear line complex and S_ω be its center. Then, the Schubert variety $\Omega(\mathsf{S}_\omega)$ is contained in $\mathfrak{C}_\omega$ and*

$$G_1(\mathsf{S}_\omega) = \mathrm{Sing}(\mathfrak{C}_\omega).$$

Proof Since $\mathrm{GL}(E)$ acts transitively on the set of linear line complexes of equal rank, we may assume that $\omega = \sum_{i=1}^k e_i^* \wedge e_{k+i}^*$, where $e_1^*, \ldots, e_n^*$ is a basis of $E^\vee$ dual to a basis $e_1, \ldots, e_n$ of E. The linear space $\mathrm{Ker}(\omega)$ is spanned by $e_i, i > 2k$. A line ℓ intersects S_ω if and only if it can be represented by a bivector $v \wedge w \in \bigwedge^2 E$, where $[v] \in \mathsf{S}_\omega$. Therefore, the linear span of the Schubert variety $\Omega(\mathsf{S}_\omega)$ is generated by bivectors $e_i \wedge e_j$, where $i < 2k$. It is obvious that it is contained in the hyperplane $V(\omega) = \langle \mathfrak{C}_\omega \rangle \subset |\bigwedge^2 E|$. This checks the first assertion.

It follows from Corollary 10.1.15 that

$$\ell \in \mathrm{Sing}(\mathfrak{C}_\omega) \Longleftrightarrow \mathbb{T}_\ell(\mathbb{G}) \subset V(\omega) \Longleftrightarrow \Omega(\ell) \subset \mathfrak{C}_\omega.$$

Suppose $\Omega(\ell) \subset \mathfrak{C}_\omega$ but ℓ does not belong to S_ω. We can find a point in ℓ represented by a vector $v = \sum a_i e_i$, where $a_i \neq 0$ for some $i \leq 2k$. Then, the line represented by a bivector $v \wedge e_{k+i}$ intersects ℓ but does not belong to $\mathfrak{C}_\omega$ (since $\omega(v \wedge e_{k+i}) = a_i \neq 0$). Thus, $\Omega(\ell) \subset \mathfrak{C}_\omega$ implies $\ell \subset \mathsf{S}_\omega$. Conversely, this inclusion implies $\Omega(\ell) \subset \Omega(\mathsf{S}_\omega) \subset \mathfrak{C}_\omega$. This proves the second assertion. □

It follows from the proposition that any linear line complex is singular unless its rank is equal to $2[\frac{n+1}{2}]$, maximal possible. Thus, the set of hyperplanes in the Plücker space that are tangent to $\mathbb{G}$ can be identified with the set of linear line complexes of rank $\leq 2[\frac{n-3}{2}]$. Consider $G(2, E^\vee)$ in its Plücker embedding in $\mathbb{P}(\bigwedge^2 E)$. Exchanging the roles of E and $E^\vee$, we obtain the following beautiful result.

Corollary 10.2.3. *Let $t = [\frac{n-3}{2}]$, then $\mathrm{Sec}_t(\mathbb{G})$ is equal to the dual variety of the Grassmannian $G(2, E^\vee)$ in $\mathbb{P}(\bigwedge^2 E)$.*

When $n = 4, 5$ we obtain that $G(2, E)$ is dual to $G(2, E^\vee)$. When $n = 6$ we obtain that the dual of $G(2, E^\vee)$ is equal to $\mathrm{Sec}_1(G(2, E))$. This agrees with Example 10.1.17.

For any linear subspace L of E, let

$$L_\omega = \omega(L)^\perp = \{w \in E : \omega(v, w) = 0, \forall v \in L\}.$$

For any subspace $\Lambda = |L| \subset |E|$, let

$$i_\omega(\Lambda) = |L_\omega|.$$

It is clear that $[v \wedge w] \in \mathbb{G}$ belongs to $\mathfrak{C}_\omega$ if and only if $\omega(v, w) = 0$. Thus,

$$\mathfrak{C}_\omega = \{\ell \in \mathbb{G} : \ell \subset i_\omega(\ell)\}. \tag{10.20}$$

Clearly $i_\omega(\Lambda)$ contains the center $\mathsf{S}_\omega = |\mathrm{Ker}(\alpha_\omega)|$ of $\mathfrak{C}_\omega$. Its dimension is equal to $n - \dim \Lambda + \dim \Lambda \cap \mathsf{S}_\omega$.

Since ω is skew-symmetric, for any point $x \in |E|$,

$$x \in i_\omega(x).$$

When ω is nonsingular, we obtain a bijective correspondence between points and hyperplanes classically known as a *null-system*.

In the special case when $n = 3$ and $\mathsf{S}_\omega = \emptyset$, this gives the *polar duality* between points and planes. The plane $\Pi(x)$ corresponding to a point x is called the *null-plane* of x. The point t_Π corresponding to a plane Π is called the *null-point* of Π. Note that $x \in \Pi(x)$ and $x_\Pi \in \Pi$. Also in this case the lines ℓ and $i_\omega(\ell)$ are called *polar lines*. We also have a correspondence between lines in $\mathbb{P}^3$

$$i_\omega : G_1(\mathbb{P}^3) \to G_1(\mathbb{P}^3), \quad \ell \mapsto i_\omega(\ell).$$

Note that the lines ℓ and $i_\omega(\ell)$ are always skew or coincide. The set of fixed points of i_ω on $G_1(\mathbb{P}^3)$ is equal to $\mathfrak{C}_\omega$. Since $\mathfrak{C}_\omega$ is nonsingular, the pole c of $V(\omega)$ with respect to the Klein quadric $\mathbb{G}$ does not belong to $\mathbb{G}$. It is easy to see that i_ω is the deck transformation of the projection of $\mathbb{G}$ in $\mathbb{P}^5$ from the point c. Thus,

$$\mathbb{G}/(i_\omega) \cong \mathbb{P}^4.$$

The hyperplane $\langle \mathfrak{C}_\omega \rangle$ is the polar hyperplane $P_c(\mathbb{G})$. The ramification divisor of the projection $\mathbb{G} \to \mathbb{P}^4$ is the linear complex $\mathfrak{C}_\omega = P_c(\mathbb{G}) \cap \mathbb{G}$. The branch divisor is a quadric in $\mathbb{P}^4$.

If $\mathfrak{C}_\omega$ is singular, then it coincides with the Schuber variety $\Omega(\ell)$, where $\ell = \mathsf{S}_\omega$. For any $\ell \neq \mathsf{S}_\omega$, we have $i_\omega(\ell) = \mathsf{S}_\omega$ and $i_\omega(\mathsf{S}_\omega) = \mathbb{P}^3$.

Proposition 10.2.4. *Let $\mathfrak{C}_\omega$ be a nonsingular linear line complex in $G_1(\mathbb{P}^n)$. Let ℓ be a line in $\mathbb{P}^n$. Then, any line $\ell' \in \mathfrak{C}_\omega$ intersecting ℓ also intersects $i_\omega(\ell)$. The intersection of $\mathfrak{C}_\omega$ with the special Schubert variety $\Omega(l)$ consists of lines intersecting the line ℓ and the codimension 2 subspace $i_\omega(\ell)$.*

Proof Let $x = \ell \cap \ell'$. Since $x \in \ell'$, we have $\ell' \subset i_\omega(\ell') \subset i_\omega(x)$. Since $x \in \ell$, we have $i_\omega(\ell) \subset i_\omega(x)$. Thus, $i_\omega(x)$ contains ℓ' and $i_\omega(\ell)$. Since $\mathfrak{C}_\omega$ is nonsingular, $\dim i_\omega(x) = n - 1$, hence the line ℓ' intersects the $(n-2)$-plane $i_\omega(\ell)$.

Conversely, suppose ℓ' intersects ℓ at a point x and intersects $i_\omega(\ell)$ at a point x'. Then, $x, x' \in i_\omega(\ell')$ and hence $\ell' = \overline{xx'} \subset i_\omega(\ell')$. Thus, ℓ' belongs to $\mathfrak{C}_\omega$. ◻

Definition 10.2.5. *A linear line complex $\mathfrak{C}_\omega$ in $|\bigwedge^2 E|$ is called* apolar *to a linear line complex $\mathfrak{C}_{\omega^*}$ in $|\bigwedge^2 E^\vee|$ if $\omega^*(\omega) = 0$.*

In the case $n = 3$, we can identify $|\bigwedge^2 E|$ with $|\bigwedge^2 E^\vee|$ by using the polarity defined by the Klein quadric. Thus, we can speak about apolar linear line complexes in $\mathbb{P}^3$. In Plücker coordinates, this gives the relation

$$a_{12}b_{34} + a_{34}b_{12} - a_{13}b_{24} - a_{24}b_{13} + a_{14}b_{23} + a_{23}b_{14} = 0. \tag{10.21}$$

Lemma 10.2.6. *Let $\mathfrak{C}_\omega$ and $\mathfrak{C}_{\omega'}$ be two nonsingular linear line complexes in $\mathbb{P}^3$. Then, $\mathfrak{C}_\omega$ and $\mathfrak{C}_{\omega'}$ are apolar to each other if and only if $g = \omega^{-1} \circ \omega' \in \mathrm{GL}(E)$ satisfies $g^2 = 1$.*

Proof Take two skew lines ℓ, ℓ' in the intersection $\mathfrak{C}_\omega \cap \mathfrak{C}_{\omega'}$. Choose coordinates in E such that ℓ and ℓ' are two opposite edges of the coordinate tetrahedron $V(t_0t_1t_2t_3)$, say $\ell : t_0 = t_2 = 0$, and $\ell' : t_1 = t_3 = 0$. Then, the linear line complexes have the following equations in Plücker coordinates

$$\mathfrak{C}_\omega : ap_{12} + bp_{34} = 0; \quad \mathfrak{C}_{\omega'} : cp_{12} + dp_{34} = 0.$$

The condition that $\mathfrak{C}_\omega$ and $\mathfrak{C}_{\omega'}$ are apolar is $ad + bc = 0$. The linear maps $\omega, \omega' : E \to E^\vee$ are give by the matrices

$$A = \begin{pmatrix} 0 & a & 0 & 0 \\ -a & 0 & 0 & 0 \\ 0 & 0 & 0 & b \\ 0 & 0 & -b & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & c & 0 & 0 \\ -c & 0 & 0 & 0 \\ 0 & 0 & 0 & d \\ 0 & 0 & -d & 0 \end{pmatrix}.$$

This gives

$$A^{-1}B = \begin{pmatrix} c/a & 0 & 0 & 0 \\ 0 & c/a & 0 & 0 \\ 0 & 0 & d/b & 0 \\ 0 & 0 & 0 & d/b \end{pmatrix} = \frac{a}{c}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}.$$

This shows that $(A^{-1}B)^2$ defines the identical transformation of $|E|$. It is easy to see that conversely, this implies that $ad + bc = 0$. □

In particular, a pair of nonsingular apolar linear line complexes defines an involution of $|E|$. Any pair of linear line complexes defines a projective transformation of $|E|$ as follows. Take a point x, define its null-plane $\Pi(x)$ with respect to ω, and then take its null-point y with respect to ω'. For apolar line complexes, we must get an involution. That is, the null-plane of y with respect to ω must coincide with the null-plane of x with respect to ω'.

Since any set of nonsingular mutually apolar linear line complexes is linearly independent, we see that the maximal number of mutually apolar linear line complexes is equal to 6. If we choose these line complexes as coordinates z_i in $\bigwedge^2 E$, we will be able to write the equation of the Klein quadric in the form

$$Q = \sum_{i=0}^{5} z_i^2.$$

Since each pair of apolar linear line complexes defines an involution in $|\bigwedge^2 E|$, we obtain 15 involutions. They generate an elementary abelian group $(\mathbb{Z}/2\mathbb{Z})^4$ of projective transformations in $\mathbb{P}^3$. The action of this group arises from a linear representation in $\mathbb{C}^4$ of the non-abelian group $\mathcal{H}_2$ (a *Heisenberg group*) given by a central extension

$$1 \to \mu_2 \to \mathcal{H}_2 \to (\mathbb{Z}/2\mathbb{Z})^4 \to 1.$$

We denote the subgroup of PGL(3) generated by the 15 involutions defined by six mutually apolar line complexes by $\mathcal{H}_2'$.

An example of six mutually apolar linear line complexes is the set

$$(p_{12} + p_{34}, i(p_{34} - p_{12}), p_{13} - p_{24}, -i(p_{24} + p_{13}), p_{14} + p_{23}, i(p_{23} - p_{14})),$$

where $i = \sqrt{-1}$. These coordinates in the Plücker space are called the *Klein coordinates*.

A set of six mutually apolar linear line complexes defines a symmetric (16_6)-*configuration* of points and planes. It is called the *Kummer configuration*. It is formed by 16 points and 16 planes in $\mathbb{P}^3$ such that each point is a null-point of six planes, each with respect to one of the six line complexes. Also each plane is a null-plane of six points with respect to one of the six line complexes. To construct such configuration one can start from any point $p_1 = [a_0, a_1, a_2, a_3] \in \mathbb{P}^3$ such that no coordinate is equal to zero. Assume that our six apolar line complexes correspond to Klein coordinates. The first line complex is $p_{12} + p_{34} = e_1^* \wedge e_2^* + e_3^* \wedge e_4^*$. It transforms the point p_1 to the plane $-a_1t_0 + a_0t_1 + a_3t_2 - a_2t_3 = 0$. Taking other line complexes we get five more null-planes:

$$\begin{aligned}
a_1t_0 - a_0t_1 + a_3t_2 - a_2t_3 &= 0,\\
a_2t_0 - a_3t_1 - a_0t_2 + a_1t_3 &= 0,\\
a_2t_0 + a_3t_1 - a_0t_2 - a_1t_3 &= 0,\\
a_3t_0 + a_2t_1 - a_1t_2 - a_0t_3 &= 0,\\
-a_3t_0 + a_2t_1 - a_1t_2 + a_0t_3 &= 0.
\end{aligned}$$

Next we take the orbit of p_1 with respect to the Heisenberg group $\mathcal{H}_2$. It consists of 16 points. Computing the null-planes of each point, we find altogether 16

$[a_0, a_1, a_2, a_3]$	$[a_1, a_0, a_3, a_2]$	$[a_0, -a_1, a_2, -a_3]$	$[a_1, -a_0, a_3, -a_2]$
$[a_2, a_3, a_0, a_1]$	$[a_3, a_2, a_1, a_0]$	$[a_2, -a_3, a_0, -a_1]$	$[a_3, -a_2, a_1, -a_0]$
$[a_0, a_1, -a_2, -a_3]$	$[a_1, a_0, -a_3, -a_2]$	$[a_0, -a_1, -a_2, a_3]$	$[a_1, -a_0, -a_3, a_2]$
$[a_2, a_3, -a_0, -a_1]$	$[a_3, a_2, -a_1, -a_0]$	$[a_2, -a_3, -a_0, a_1]$	$[a_3, -a_2, -a_1, a_0]$

Figure 10.1 Kummer configuration of 16 points in $\mathbb{P}^3$

planes forming with the 16 points a (16_6)-configurations. The following table gives the coordinates of the 16 points.

A point $(\alpha, \beta, \gamma, \delta)$ in this table is contained in six planes $at_0+bt_1+ct_2+dt_3 = 0$, where (a, b, c, d) is one of the following:

$$(\delta, -\gamma, \beta, -\alpha),\ (\delta, \gamma, -\beta, -\alpha),\ (\gamma, \delta, -\alpha, -\beta),$$

$$(-\gamma, \delta, \alpha, -\beta),\ (-\beta, \alpha, \delta, -\gamma),\ (\beta, -\alpha, \delta, -\gamma).$$

Dually, a plane $\alpha t_0 + \beta t_1 + \gamma t_2 + \delta t_3 = 0$ contains six points $[a, b, c, d]$, where (a, b, c, d) is as above.

One checks directly that the six null-points of each of the 16 planes of the configuration lie on a conic. So, this gives us a configuration of 16 conics in $\mathbb{P}^3$, each containing six points of the configuration. Also, observe that any two conics intersect at 2 points.

There is a nice symbolic way to exhibit the (16_6)-configuration. After we fix an order on a set of six mutually apolar linear line complexes, we will be able to identify the group $\mathcal{H}_2'$ with the group E_2 defined by 2-element subsets of the set $\{1, 2, 3, 4, 5, 6\}$ (see Subsection 5.2.2). A subset of two elements $\{i, j\}$ corresponds to the involution defined by a pair of apolar line complexes. We take the ordered set of apolar linear line complexes defined by the Klein coordinates. First we match the orbit of the point $[a_0, a_1, a_2, a_3]$ from the table from above with the left-hand side of the following table. To find the six planes that contain a point from the (ij)-th spot we look at the same spot in the right-hand side of the following table. Take the involutions in the i-th row and j-th column but not at the (ij)-spot. These involutions are matched with the planes containing the point. As always we identify a plane $a_0t_0 + a_1t_1 + a_2t_2 + a_3t_3$ with the point $[a_0, a_1, a_2, a_3]$. For example, the point $\emptyset$ is contained in six planes $(15), (13), (26), (46), (24), (35)$. Conversely, take a plane corresponding to the (ij)-th spot in the right-hand side of the table. The point contained in this plane can be found in the same row and the same column in the left-hand side of the table excluding the (ij)-th spot. For example, the plane $\emptyset$ contains the points $(45), (34), (35), (16), (12), (26)$.

$$\begin{array}{cccccccc} \emptyset & (45) & (34) & (35) & (14) & (15) & (13) & (26) \\ (16) & (23) & (25) & (24) & (46) & (56) & (36) & (12) \\ (12) & (36) & (56) & (46) & (24) & (25) & (23) & (16) \\ (26) & (13) & (15) & (14) & (35) & (34) & (45) & \emptyset \end{array}.$$

Another way to remember the rule of the incidence is as follows. A point corresponding to an involution (ab) is contained in a plane corresponding to an involution (cd) if and only if

$$(ab) + (cd) + (24) \in \{\emptyset, (16), (26), (36), (46), (56)\}.$$

Consider a regular map $\mathbb{P}^3 \to \mathbb{P}^4$ defined by the polynomials

$$t_0^4 + t_1^4 + t_2^4 + t_3^4,\ t_0^2t_3^2 + t_1^2t_2^2,\ t_1^2t_3^2 + t_0^2t_2^2,\ t_2^2t_3^2 + t_0^2t_1^2,\ t_0t_1t_2t_3.$$

Observe that this map is invariant with respect to the action of the Heisenberg group $\mathcal{H}_2$. So, it defines a regular map

$$\Phi : \mathbb{P}^3/\mathcal{H}_2' \to \mathbb{P}^4.$$

Proposition 10.2.7. *The map Φ defines an isomorphism*

$$\mathbb{P}^3/\mathcal{H}_2 \cong X,$$

where X is a quartic hypersurface in $\mathbb{P}^4$ given by the equation

$$z_0^2z_4^2 - z_0z_1z_2z_3 + z_1^2z_2^2 + z_2^2z_3^2 + z_1^2z_3^2 - 4z_4^2(z_1^2 + z_2^2 + z_3^2) + z_4^4 = 0. \tag{10.22}$$

Proof Since the map is given by five polynomials of degree 4, the degree of the map times the degree of the image must be equal to 4^3. We know that its degree must be a multiple of 16, this implies that either the image is $\mathbb{P}^3$ or a quartic hypersurface. Since the polynomials are linearly independent the first case is impossible. A direct computation gives the equation of the image. □

Note that the fixed-point set of each nontrivial element of the Heisenberg group $\mathcal{H}_2$ consists of two skew lines. For example, the involution

$$(12) : [a_0, a_1, a_2, a_3] \mapsto [a_0, a_1, -a_2, -a_3]$$

fixes pointwise the lines $t_0 = t_1 = 0$, and $t_2 = t_3 = 0$. Each line has a stabilizer subgroup of index 2. Thus, the images of the 30 lines form the set of 15 double lines on X. The stabilizer subgroup acts on the line as the dihedral group D_4. It has six points with a nontrivial stabilizer of order 2. Altogether we have $30 \times 6 = 180$ such points which form 15 orbits. These orbits and the double lines form a (15_3)-configuration. The local equation of X at one of these orbits is $v^2 + xyz = 0$.

We will prove later in Subsection 10.4.3 that the orbit space $X = \mathbb{P}^3/\mathcal{H}_2$ is isomorphic to the Castelnuovo-Richmond quartic.

10.2.2 Six lines

We know that any five lines in $\mathbb{P}^3$, considered as points in the Plücker $\mathbb{P}^5$, are contained in a linear line complex. In fact, in a unique linear line complex if the lines are linearly independent. A set of six lines is contained in a linear line complex only if they are linearly dependent. The 6×6-matrix of its Plücker coordinates must have a nonzero determinant. An example of six dependent lines is the set of lines intersecting a given line ℓ. They are contained in the special line complex which coincides with the Schubert variety $\Omega(\ell)$. We will give a geometric characterization of a set of six linearly dependent lines that contains a subset of five linearly independent lines.

Lemma 10.2.8. *Let $\sigma : \mathbb{P}^1 \to \mathbb{P}^1$ be an involution. Then, the graph of σ is an irreducible curve $\ell_\sigma \subset \mathbb{P}^1 \times \mathbb{P}^1$ of bidegree $(1, 1)$ such that $\iota(\ell_\sigma) = \ell_\sigma$, where ι is the automorphism $(x, y) \mapsto (y, x)$. Conversely, any curve on $\mathbb{P}^1 \times \mathbb{P}^1$ with these properties is equal to the graph of some involution.*

Proof This is easy and left to the reader. □

Corollary 10.2.9. *Let σ, τ be two different involutions of $\mathbb{P}^1$. Then, there exists a unique common orbit $\{x, y\}$ with respect to σ and τ.*

We will need the following result of M. Chasles.

Theorem 10.2.10 (M. Chasles)**.** *Let Q be a nondegenerate quadric in $\mathbb{P}^3$ and let σ be an automorphism of order 2 of Q which is the identity on one of the rulings. Then, the set of lines in $\mathbb{P}^3$, which are either contained in this ruling or intersect an orbit of lines in the second ruling form a linear complex. Conversely, any linear line complex is obtained in this way from some pair (Q, σ).*

Proof Consider the set X of lines defined as in the first assertion of the Theorem. Take a general plane Π and a point $x \in \Pi$. Consider the Schubert variety $\Omega(x, \Pi)$. It is a line in the Plücker space. The plane intersects Q along a conic C. Each line from $\Omega(x, \Pi)$ intersects C at two points. This defines an involution on C. Each line from the second ruling intersects C at one point. Hence, σ defines another involution on C. By Corollary 10.2.9 there is a unique common orbit. Thus, there is a unique line from $\Omega(x, \Pi)$ which belongs to X. Thus, X is a linear line complex.

Let ℓ_1, ℓ_2, ℓ_3 be any three skew lines in a line complex $X = \mathfrak{C}_\omega$. Let Q be a quadric containing these lines. It is obviously nonsingular. The lines belong to some ruling of Q. Take any line ℓ from the other ruling. Its polar line $\ell' = i_\omega(\ell)$ intersects ℓ_1, ℓ_2, ℓ_3 (because it is skew to ℓ or coincides with it). Hence, ℓ' lies on Q. Now, we have an involution on the second ruling defined by the polarity with respect to X. If $m \in X$ is not contained in the first ruling, then m intersects a line ℓ from the second ruling. By Proposition 10.2.4, it also intersects ℓ'. This is the description of X from the assertion of the Theorem.

□

Remark 10.2.11. Let C be the curve in $G(2,4)$ parameterizing lines in a ruling of a nonsingular quadric Q. Take a general line ℓ in $\mathbb{P}^3$. Then, $\Omega(\ell)$ contains two lines from each ruling, the ones which pass through the points $Q \cap \ell$. This implies that C is a conic in the Plücker embedding. A linear line complex X either intersects each conic at two points and contains two or one line from the ruling, or contains C and hence contains all lines from the ruling.

Lemma 10.2.12. *Let ℓ be a line intersecting a nonsingular quadric Q in $\mathbb{P}^3$ at two different points x, y. Let $\mathbb{T}_x(Q) \cap Q = \ell_1 \cup \ell_2$ and $\mathbb{T}_y(Q) \cap Q = \ell_1' \cup \ell_2'$, where ℓ_1, ℓ_1' and ℓ_2, ℓ_2' belong to the same ruling. Then, the polar line $\ell_Q^\perp$ intersects Q at the points $x' = \ell_1 \cap \ell_2'$ and $y' = \ell_2 \cap \ell_1'$.*

Proof Each line on Q is self-polar to itself. Thus, $P_x(Q)$ is the tangent plane $\mathbb{T}_x(Q)$ and, similarly, $P_y(Q) = \mathbb{T}_y(Q)$. This shows that $\ell_Q^\perp = \mathbb{T}_x(Q) \cap \mathbb{T}_y(Q) = \overline{x'y'}$. □

Lemma 10.2.13. *let $\ell_1, \ell_2, \ell_3, \ell_4$ be four skew lines in $\mathbb{P}^3$. Suppose not all of them are contained in a quadric. Then, there are exactly two lines which intersect all of them. These lines may coincide.*

Proof This is, of course, well known. It can be checked by using the Schubert calculus since $\sigma_1^4 = \# \cap_{i=1}^4 \Omega(\ell_i) = 2$. A better geometric proof can be given as follows. Let Q be the quadric containing the first three lines. Then, ℓ_4 intersects Q at two points p, q which may coincide. The lines through these points belonging to the ruling not containing ℓ_1, ℓ_2, ℓ_3 intersect $\ell_1, \ldots, \ell_4$. Conversely, any line intersecting $\ell_1, \ldots, \ell_4$ is contained in this ruling (because it intersects Q at three points) and passes through the points $\ell_4 \cap Q$.

□

Theorem 10.2.14. *Let $(\ell_1, \ldots, \ell_6)$ be a set of six lines and let $(\ell_1', \ldots, \ell_6')$ be the set of polar lines with respect to some nonsingular quadric Q. Assume that the first five lines are linearly independent in the Plücker space. Then,*

$(\ell_1,\dots,\ell_6)$ belong to a nonsingular linear line complex if and only if there exists a projective transformation T such that $T(\ell_i)=\ell_i'$. This condition does not depend on the choice of Q.

Proof First let us check that this condition does not depend on a choice of Q. For each line ℓ let $\ell_Q^\perp$ denote the polar line with respect to Q. Suppose $A(\ell)=\ell_Q^\perp$ for some projective transformation A. Let Q' be another nonsingular quadric. We have to show that $\ell_{Q'}^\perp = B(\ell)$ for some other projective transformation B depending only on A but not on ℓ. Let us identify E with $\mathbb{C}^{n+1}$ and a quadric Q with a nonsingular symmetric matrix. Then, $A(\ell)=\ell_Q^\perp$ means that $xQAy=0$ for any vectors x,y in ℓ. We have to find a matrix B such that $xQ'By=0$. We have

$$xQAy = xQ'(Q'^{-1}QA)y = xQ'By,$$

where $B=Q'^{-1}QA$. This checks the claim.

Suppose the set $(\ell_1,\dots,\ell_6)$ is projectively equivalent to $(\ell_1',\dots,\ell_6')$, where ℓ_i' are polar lines with respect to some quadric Q. Replacing Q with a quadric containing the first three lines ℓ_1,ℓ_2,ℓ_3, we may assume that $\ell_i'=\ell_i, i=1,2,3$. We identify Q with $\mathbb{P}^1\times\mathbb{P}^1$. If $\ell_j\cap Q=(a_j,b_j),(a_j',b_j')$ for $j=4,5,6$, then, by Lemma 10.2.12, $\ell_j'\cap Q=(a_j,b_j'),(a_j',b_j)$. Suppose $\ell_i'=A(\ell_i)$. Then, A fixes three lines in the first ruling hence sending Q to itself. It is also identical on the first ruling. It acts on the second ruling by switching the coordinates $(b_i,b_j'), j=4,5,6$. Thus, A^2 has three fixed points on $\mathbb{P}^1$, hence A^2 is the identity. This shows that $A=\sigma$ as in Chasles' Theorem 10.2.10. Hence, the lines $\ell_i,\ell_i',\ i=1,\dots,6$, belong to the linear complex.

Conversely, assume $\ell_1,\dots,\ell_6$ belong to a nonsingular linear line complex $X=\mathfrak{C}_\omega$. Applying Lemma 10.2.13, we find two lines ℓ,ℓ' which intersect $\ell_1,\ell_2,\ell_3,\ell_4$ (two *transversals*). By Proposition 10.2.4, the polar line $i_\omega(\ell)$ intersects $\ell_1,\ell_2,\ell_3,\ell_4$. Hence, it must coincide with either ℓ or ℓ'. The first case is impossible. In fact, if $\ell=\ell'$, then $\ell\in X$. The pencil of lines through $\ell\cap\ell_1$ in the plane $\langle\ell,\ell_1\rangle$ spanned by ℓ,ℓ_1 is contained in X. Similarly, the line $\Omega(\ell\cap\ell_2,\langle\ell,\ell_2\rangle)$ is contained in X. Let Π be the plane of lines spanned by these two lines in $\mathbb{G}$. It is contained in X. Thus, Π cuts out in $\mathbb{G}$ a pair of lines. Thus, X is singular at the point of intersections of these two lines. This is a contradiction.

Thus, we see that ℓ,ℓ' is a pair of polar lines. Now, the pair of transversals $\tau,\tau'=i_\omega(\tau)$ of $\ell_1,\ell_2,\ell_3,\ell_5$ is also a pair of polar lines. Consider the quadric Q spanned by ℓ_1,ℓ_2,ℓ_3. The four transversals are the four lines from the second ruling of Q. We can always find an involution σ on Q which preserves the first ruling and such that $\sigma(\ell)=\ell',\sigma(\tau)=\tau'$. Consider the linear line complex

X' defined by the pair (Q, σ). Since $\ell_1, \ldots, \ell_5$ belong to X, and any line complex is determined by five linearly independent lines, we have the equality $X = X'$. Thus, ℓ_6 intersects Q at a pair of lines in the second ruling, which is in the involution σ. But σ is defined by the polarity with respect to X (since $\ell_1, \ell_2, \ell_3 \in H$ and the two involutions share two orbits corresponding to the pairs $(\ell, \ell'), (\tau, \tau')$). This implies $(\ell_1, \ldots, \ell_6) = \sigma(\ell_1', \ldots, \ell_6')$, where $\ell_i' = \ell_i^{\perp}$. □

Corollary 10.2.15. *Let $\ell_1, \ldots, \ell_6$ be six skew lines on a nonsingular cubic surface S. Then, they are linearly independent in the Plücker space.*

Proof We first check that any five lines among the six lines are linearly independent. Assume that $\ell_1, \ldots, \ell_5$ are linearly dependent. Then, one of them, say ℓ_5, lies in the span of $\ell_1, \ell_2, \ell_3, \ell_4$. Let $\{\ell_1', \ldots, \ell_6'\}$ be the set of six skew lines which together with $(\ell_1, \ldots, \ell_6)$ form a double-six. Then, $\ell_1, \ell_2, \ell_3, \ell_4$ lie in the linear line complex $\Omega(\ell_5')$, and hence, ℓ_5 lies in it too. However, this is impossible because ℓ_5 is skew to ℓ_5'.

We know that there exists the unique quadric Q such that ℓ_i' are polar to Q with respect to Q (the Schur quadric). But $(\ell_1', \ldots, \ell_6')$ is not projectively equivalent to $(\ell_1, \ldots, \ell_6)$. Otherwise, S and its image S' under the projective transformation T will have six common skew lines. It will also have common transversals of each subset of four. Thus, the degree of the intersection curve is larger than 9. This shows that the cubic surfaces S and S' coincide and T is an automorphism of S. Its action on $\mathrm{Pic}(S)$ is a reflection with respect to the root corresponding to the double-six. It follows from Section 9.5 that S does not admit such an automorphism. □

Remark 10.2.16. The group $\mathrm{SL}(4)$ acts diagonally on the Cartesian product $\mathbb{G}^6$. Consider the sheaf $\mathcal{L}$ on $\mathbb{G}^6$ defined as the tensor product of the sheaves $p_i^*\mathcal{O}_{\mathbb{G}}(1)$, where $p_i : \mathbb{G}^6 \to \mathbb{G}$ is the i-th projection. The group $\mathrm{SL}(4)$ acts naturally in the space of global sections of $\mathcal{L}$ and its tensor powers. Let

$$R = \bigoplus_{i=0}^{\infty} H^0(\mathbb{G}^6, \mathcal{L}^i)^{\mathrm{SL}(4)}.$$

This is a graded algebra of finite type and its projective spectrum $\mathrm{Proj}(R)$ is the GIT-quotient $\mathbb{G}^6//\mathrm{SL}(4)$. The variety $\mathbb{G}^6$ has an open invariant Zariski subset U which is mapped to $\mathbb{G}^6//\mathrm{SL}(4)$ with fibers equal to $\mathrm{SL}(4)$-orbits. This implies that $\mathbb{G}^6//\mathrm{SL}(4)$ is an irreducible variety of dimension 9. Given six ordered general lines in $\mathbb{P}^3$, their Plücker coordinates make a 6×6-matrix. Its determinant can be considered as a section from the first graded piece R_1 of R. The locus of zeros of this section is a closed subvariety of $\mathbb{G}^6$ whose general point is a six-tuple of lines contained in a linear line complex. The image of this locus in $\mathbb{G}^6//\mathrm{SL}(4)$ is a hypersurface F. Now, the polar duality of lines by means of a nondegenerate quadric defines an involution on $\mathbb{G}^6$. Since it does not depend on the choice of a quadric up to projective equivalence, the involution descends to an involution of $\mathbb{G}^6//\mathrm{SL}(4)$. The set of fixed points of this involution is the hypersurface F. One can show that the quotient by the duality involution is an open subset of a certain explicitly described 9-dimensional toric variety X (see [242, 11.3]).

Finally, observe that a nonsingular cubic surface together with a choice of its geometric marking defines a double-six, which is an orbit of the duality involution in $\mathbb{G}^6//\mathrm{SL}(4)$ and hence a unique point in X that does not belong to the branch locus of the double cover $\mathbb{G}^6//\mathrm{SL}(4) \to X$. Thus, we see that the four-dimensional moduli space of geometrically marked nonsingular cubic surfaces embeds in the 9-dimensional toric variety X.

10.2.3 Linear systems of linear line complexes

Let $W \subset \bigwedge^2 E^\vee$ be a linear subspace of dimension $r+1$. After projectivization and restriction to $G(2,E) \cong G_1(\mathbb{P}^n)$, it defines an r-dimensional linear system $|W|$ of linear line complexes. Let

$$\mathfrak{C}_W = \cap_{\omega \in W} \mathfrak{C}_\omega \subset G(2,E)$$

be the base scheme of $|W|$. It is a subvariety of $G(2,E)$ of dimension $2n-3-r$. Its canonical class is given by the formula

$$\omega_{\mathfrak{C}_W} \cong \mathcal{O}_{\mathfrak{C}_W}(r-n). \tag{10.23}$$

In particular, if smooth, it is a Fano variety if $r < n$, a Calabi-Yau variety if $r = n$, and a variety of general type if $r > n$.

We also define the *center variety* S_W

$$\mathsf{S}_W = \bigcup_{\omega \in W} \mathsf{S}_\omega.$$

It is also called the *singular variety* of W.

For any $x = [v] \in \mathsf{S}_W$, there exists $\omega \in W$ such that $\omega(v,v') = 0$ for all $v' \in E$, or, equivalently, the line $\ell = \overline{xy}$ is contained in $\mathfrak{C}_\omega$ for all y. This implies that the codimension of $\Omega(x) \cap \mathfrak{C}_W$ in $\Omega(x)$ is $\le r$, less than the expected number $r+1$. Conversely, since $\Omega(x)$ is irreducible, if the codimension of the intersection $\le r$, then $\Omega(x)$ must be contained in some $\mathfrak{C}_\omega$, and hence $x \in \mathsf{S}_\omega$. Thus, we have proved the following.

Proposition 10.2.17.

$$\mathsf{S}_W = \{x \in |E| : \dim \Omega(x) \cap \mathfrak{C}_W \geq n - r - 1\}$$

$$= \{x \in |E| : \Omega(x) \subset \mathfrak{C}_\omega \textit{ for some } \omega \in W\}.$$

For any linear subspace Λ in $|E|$ we can define the *polar subspace* with respect to $|W|$ by

$$i_W(\Lambda) = \bigcap_{\omega \in W} i_\omega(\Lambda).$$

Since $x \in i_\omega(x)$ for any linear line complex $\mathfrak{C}_\omega$, we obtain that, for any $x \in |E|$,

$$x \in i_W(x).$$

It is easy to see that

$$\dim i_W(x) = n - r + \dim |\{\omega \in W : x \in \mathsf{S}_\omega\}|. \tag{10.24}$$

Now, we are ready to give examples.

Example 10.2.18. A pencil $|W|$ of linear line complexes in $\mathbb{P}^3 = |E|$ is defined by a line in the Plücker space $\mathbb{P}^5 = |\bigwedge^2 E^\vee|$ which intersects the Klein quadric $G(2, E^\vee)$ at two points or one point with multiplicity 2. The intersection points correspond to special linear line complexes intersecting a given line. Thus, the base locus of a general pencil of linear line complexes consists of lines intersecting two skew lines. It is a nonsingular congruence of lines in $G_1(\mathbb{P}^3)$ of order and degree equal to 1. It is isomorphic to a nonsingular quadric in $\mathbb{P}^3$. It may degenerate into the union of an α-plane and a β-plane if the two lines are coplanar, or to a singular quadric if the two lines coincide.

Example 10.2.19. Assume $r = 1$, $n = 2k$ and $|W|$ does not intersect the set of linear line complexes with corank > 1 (the variety of such linear line complexes is of codimension 3 in $|\bigwedge^2 E^\vee|$). Then, we have a map $|W| \cong \mathbb{P}^1 \to \mathbb{P}^{2k}$ which assigns to $[\omega] \in |W|$ the center S_ω of $\mathfrak{C}_\omega$. The map is given by the pfaffians of the principal minors of a skew-symmetric matrix of size $(n+1) \times (n+1)$, so the center variety S_W of $|W|$ is a rational curve R_k of degree k in $\mathbb{P}^{2k}$. By Proposition 10.2.17 any secant line of R_k is contained in $\mathfrak{C}_W$. For example, taking $n = 4$, we obtain that the center variety is a conic in a plane contained in $\mathfrak{C}_W$.

Now, assume that $r = 2$. We obtain that S_W is a projection of the Veronese surface V_k^2 and the variety of trisecant lines of the surface is contained in $\mathfrak{C}_W$. We have seen it already in the case $k = 2$ (see Subsection 2.1.3).

Example 10.2.20. Let $r = 3$ and $n = 4$ so we have a web $|W|$ of linear line complexes in $\mathbb{P}^9 = |\bigwedge^2 E^\vee|$. We assume that $|W|$ is general enough so that it intersects the Grassmann variety $\mathbb{G}^* = G(2, E^\vee)$ in a finite set of points. We know that the degree of $G(2,5)$ is equal to five, thus, $|W|$ intersects G^* at five points. Consider the rational map $|W| = \mathbb{P}^3 \dashrightarrow \mathsf{S}_W \subset \mathbb{P}^4$ which assigns to $[\omega] \in |W|$ the center of $\mathfrak{C}_\omega$. As in the previous examples, the map is given by pfaffians of skew-symmetric matrices of size 4×4. They all vanish at the set of five points $p_1, \ldots, p_5$. The pre-image of a general line in $\mathbb{P}^4$ is equal to the residual set of intersections of three quadrics and hence consists of three points. Thus, the map is a birational map onto a cubic hypersurface. Any line joining two of the five points is blown down to a singular point of the cubic hypersurface. Thus, the cubic is isomorphic to the Segre cubic primal. Observe now that $\mathfrak{C}_W$ is a del Pezzo surface of degree 5 and the singular variety of $|W|$ is equal to the projection of the incidence variety $\{(x, \ell) \in \mathbb{P}^4 \times \mathfrak{C}_W : x \in \ell\}$ to $\mathbb{P}^4$. It coincides with the center variety S_W.

One can see the center variety S_W of $|W|$ as the degeneracy locus of a map of rank 3 vector bundles over $|E|$. First, we identify $H^0(|E|, \Omega^1_{|E|}(2))$ with $\bigwedge^2 E^\vee$. To do this, we use the dual Euler exact sequence twisted by $\mathcal{O}_{|E|}(2)$

$$0 \to \Omega^1_{|E|}(2) \to E^\vee \otimes \mathcal{O}_{|E|}(1) \to \mathcal{O}_{|E|}(2) \to 0. \qquad (10.25)$$

Passing to the global sections, we obtain an isomorphism

$$H^0(|E|, \Omega^1_{|E|}(2)) \cong \mathrm{Ker}(E^\vee \otimes E^\vee \to S^2(E)^\vee) \cong \bigwedge^2 E^\vee.$$

The composition of the inclusion map $W \to \bigwedge^2 E^\vee$ and the evaluation map $\bigwedge^2 E^\vee \to \Omega^1_{|E|}(2)$ defines a morphism of vector bundles

$$\sigma : W \otimes \mathcal{O}_{|E|} \to \Omega^1_{|E|}(2).$$

The degeneracy locus of this morphism consists of points $x \in |E|$ such that the composition of $E \otimes \mathcal{O}_{|E|}(-1) \to \mathcal{T}^1_{|E|}(-2)$ and the dual map $\sigma^\vee : \mathcal{T}^1_{|E|}(-2) \to W^\vee \otimes \mathcal{O}_{|E|}$ is not of full rank at x. For any $x = [v] \in |E|$, the map of fibers $\phi(x)$ sends a vector v' to the linear function on W defined by $\omega \mapsto \omega(v, v')$. This linear function is equal to zero if and only if the line $\langle [v], [v'] \rangle$ intersects $\mathfrak{C}_W$. Applying Proposition 10.2.17, we obtain that the degeneracy locus of point $x = [v]$ for which the rank of $\phi(x)$ is smaller than $r + 1$ must be equal to S_W.

If we choose coordinates and take a basis of W defined by $r + 1$ skew-symmetric bilinear forms $\omega_k = \sum a^{(k)}_{ij} dt_i \wedge dt_j$, then the matrix is

$$\begin{pmatrix} \sum_{s=0}^n a^{(1)}_{1,s} t_s & \ldots & \sum_{s=0}^n a^{(1)}_{n,s} t_s \\ \vdots & \vdots & \vdots \\ \sum_{s=0}^n a^{(r+1)}_{1,s} t_s & \ldots & \sum_{s=0}^n a^{(r+1)}_{n,s} t_s \end{pmatrix},$$

where $a_{ij}^k = -a_{ji}^k$.

The expected dimension of the degeneracy locus is equal to $n - r$. Assume that this is the case. It follows from [315, Example 14.3.2] that

$$\deg \mathsf{S}_W = \deg c_{n-r}(\Omega^1_{|E|}(2)) = \sum_{i=0}^{n-r} (-1)^i \binom{n-i}{r}. \tag{10.26}$$

Example 10.2.21. Assume $n+1 = 2k$. If $\omega \in W$ is nondegenerate, then $\mathsf{S}_\omega = \emptyset$. Otherwise, $\dim \mathsf{S}_\omega \geq 1$. Thus, the varieties S_W are ruled by linear subspaces. For a general W of dimension $1 < r < n$, the dimensions of these subspaces is equal to 1 and each point in S_W is contained in a unique line S_ω. In other words, S_W is a scroll with one-dimensional generators parameterized by the subvariety B of $|W|$ of degenerate ω's. Thus, B is equal to the intersection of $|W|$ with a pfaffian hypersurface of degree k in $|\Lambda^2 E^\vee|$. The scrolls S_W are called *Palatini scrolls*. If $n = 3$, the only Palatini scroll is a quadric in $\mathbb{P}^3$ and B is a conic. In $\mathbb{P}^5$ we get a 3-dimensional Palatini scroll of degree 7 defined by a web $|W|$ of linear line complexes. The family of generators B is a cubic surface in $|W|$. We refer to [561] for the study of this scroll. There is also a Palatini ruled surface of degree 6 defined by a net of linear line complexes. Its generators are parameterized by a plane cubic curve. If we take W with $\dim W = 5$, we get a quartic hypersurface in $\mathbb{P}^5$.

10.3 Quadratic Line Complex

10.3.1 Generalities

Recall that a quadratic line complex $\mathfrak{C}$ is the intersection of the Grassmannian $\mathbb{G} = G(2, E) \subset |\bigwedge^2 E|$ with a quadric hypersurface Q. We assume that this intersection is irreducible and reduced. Since $\omega_\mathbb{G} \cong \mathcal{O}_\mathbb{G}(-n-1)$, by the adjunction formula

$$\omega_\mathfrak{C} \cong \mathcal{O}_\mathfrak{C}(1-n).$$

If $\mathfrak{C}$ is nonsingular, i.e., the intersection is transversal, we obtain that $\mathfrak{C}$ is a Fano variety of index $n - 1$.

Consider the incidence variety $\mathsf{Z}_\mathfrak{C}$ together with its natural projections $p_\mathfrak{C} : \mathsf{Z}_\mathfrak{C} \to \mathbb{P}^n$ and $q_\mathfrak{C} : \mathsf{Z}_\mathfrak{C} \to K$. For each point $x \in \mathbb{P}^n$, the fiber of $p_\mathfrak{C}$ is isomorphic to the intersection of the Schubert variety $\Omega(x)$ with Q. We know that $\Omega(x)$ is isomorphic to $\mathbb{P}^{n-1}$ embedded in $|\bigwedge^2 E|$ as a linear subspace. Thus, the fiber is isomorphic to a quadric in $\mathbb{P}^{n-1}$. This shows that $\mathfrak{C}$ admits a structure of a *quadric bundle*, i.e., a fibration with fibers isomorphic to a quadric hypersurface. The important invariant of a quadric bundle is its *discriminant locus*. This is the set of points of the base of the fibration over which the fiber is a singular quadric or the whole space. In our case, we have the following classical definition.

Definition 10.3.1. *The* singular variety Δ *of a quadratic line complex is the set of points* $x \in \mathbb{P}^n$ *such that* $\Omega(x) \cap Q$ *is a singular quadric in* $\Omega(x) = \mathbb{P}^{n-1}$ *or* $\Omega(x) \subset Q$.

We will need the following fact from linear algebra.

Lemma 10.3.2. *Let* $A = (a_{ij}), B = (b_{ij})$ *be two matrices of sizes* $k \times m$ *and* $m \times k$ *with* $k \le m$. *Let* $|A_I|, |B_I|, I = (i_1, \dots, i_k), 1 \le i_1 < \dots < i_k \le m$, *be maximal minors of A and B. For any* $m \times m$*-matrix* $G = (g_{ij})$,

$$|A \cdot G \cdot B| = \sum_{I,J} g_{IJ} |A_I||B_J|,$$

where $g_{IJ} = g_{i_1 j_1} \cdots g_{i_k j_k}$.

Proof Consider the product of the following block matrices

$$\begin{pmatrix} A \cdot B & A \\ 0_{mk} & I_m \end{pmatrix} \cdot \begin{pmatrix} I_k & 0_{km} \\ -B & I_m \end{pmatrix} = \begin{pmatrix} 0_{kk} & A \\ -B & I_m \end{pmatrix}, \tag{10.27}$$

where 0_{ab} is the zero matrix of size $a \times b$ and I_a is the identity matrix of size $a \times a$. The determinant of the first matrix is equal to $|A \cdot B|$, and the determinant of the second matrix is equal to 1. Applying the Laplace formula, we find that the determinant of the product is equal to $\sum |A_I||B_I|$. Now, we apply (10.27), replacing A with $A \cdot G$. Write a j-th column of $A \cdot G$ as the sum $\sum_{i=1}^m g_{ij} A_i$, where A_i are the columns of A. Then,

$$|(A \cdot G)_{j_1,\dots,j_k}| = \sum_{1 \le i_1 < \dots < i_k \le m} g_{i_1 j_1} g_{i_2 j_2} \cdots g_{i_k j_k} |A_{i_1,\dots,i_k}|.$$

This proves the assertion. □

Suppose we have a bilinear form $b : E \times E \to \mathbb{K}$ on a vector space E over a field $\mathbb{K}$. Let $G = \big(b(e_i, e_j)\big)$ be the matrix of the bilinear form with respect to a basis $e_1, \dots, e_m$. Let L be a subspace of E with basis $f_1, \dots, f_k$. Then, the matrix $G_L = \big(b(f_i, f_j)\big)$ is equal to the product ${}^t A \cdot G \cdot A$, where the $f_j = \sum a_{ij} e_i$. It follows from the previous lemma that $|G_L| = \sum_{I,J} g_{IJ} |A_I||A_J|$. If we extend b to $\bigwedge^k E$ by the formula

$$b(v_1, \dots, v_k; w_1, \dots, w_k) = \det\big(b(v_i, w_j)\big),$$

then the previous formula gives an explicit expression for $b(f_1 \wedge \dots \wedge f_k, f_1 \wedge$

$\ldots \wedge f_k)$. If $E = \mathbb{R}^n$ and we take b to be the Euclidean inner-product, we get the well-known formula for the area of the parallelogram spanned by vectors $f_1, \ldots, f_k$ in terms of the sum of squares of maximal minors of the matrix with columns equal to f_j. If $m = 3$ this is the formula for the length of the cross-product of two vectors.

Proposition 10.3.3. Δ *is a hypersurface of degree* $2(n-1)$.

Proof Consider the map

$$i : |E| \to G(n, \bigwedge^2 E),\ x \mapsto \Omega(x). \tag{10.28}$$

If $x = [v_0]$, the linear subspace of $\bigwedge^2 E$ corresponding to $\Omega(x)$ is the image of E in $\bigwedge^2 E$ under the map $v \mapsto v \wedge v_0$. This is an n-dimensional subspace $\Lambda(x)$ of $\bigwedge^2 E$. Hence, it defines a point in the Grassmann variety $G(n, \bigwedge^2 E)$. If we write $v_0 = \sum_{i=0}^n a_i e_i$, where we assume that $a_n \neq 0$, then $\Lambda(x)$ is spanned by the vectors $e_i \wedge v_0 = \sum_{j \neq i} a_j e_i \wedge e_j, i = 0, \ldots, n$. Thus, the rows of the matrix of Plücker coordinates of the basis are linear functions in coordinates of v_0. Its maximal minors are polynomials of order $n+1$. Observe now that each (in)-th column contains a_n in the i-th row and has zero elsewhere. This easily implies that all maximal minors are divisible by a_n. Thus, the Plücker coordinates of $\Lambda(x)$ are polynomials of degree $n-1$ in coordinates of v_0. We see now that the map i is given by a linear system of divisors of degree $n-1$. Fix a quadric Q in $|\bigwedge^2 E|$ that does not vanish on $\mathbb{G}$. For any $n-1$-dimensional linear subspace L of $|\bigwedge^2 E|$, the intersection of Q with L is either a quadric or the whole L. Let us consider the locus D of L's such that this intersection is a singular quadric. We claim that this is a hypersurface of degree 2.

Let $b : E \times E$ be a nondegenerate symmetric bilinear form on a vector space E of dimension $n+1$. The restriction of b to a linear subspace $L \subset E$ with a basis $(f_1, \ldots, f_k)$ is a degenerate bilinear form if and only if the determinant of the matrix $\big(b(f_i, f_j)\big)$ is equal to zero. If we write $f_i = \sum a_{ij} e_j$ in terms of a basis in E, we see that this condition is polynomial of degree $2k$ in coefficients a_{ij}. By the previous lemma, this polynomial can be written as a quadratic polynomial in maximal minors of the matrix (a_{ij}). Applying this to our situation, we interpret the maximal minors as the Plücker coordinates of L and obtain that D is a quadric hypersurface.

It remains to use the fact that $\Delta = i^{-1}(D)$, where i is given by polynomials of degree $n-1$. $\square$

Let

$$\Delta_k = \{x \in \Delta : \operatorname{corank} Q \cap \Omega(x) \geq k\}.$$

These are closed subvarieties of Δ_k.

Let

$$\tilde{\Delta} = \{(x, \ell) \in \mathsf{Z}_{\mathfrak{C}} : \text{rank } d_{p_{\mathfrak{C}}}(x, \ell) < n\}. \tag{10.29}$$

In other words, $\tilde{\Delta}$ is the locus of points in $\mathsf{Z}_{\mathfrak{C}}$ where the projection $p_{\mathfrak{C}} : \mathsf{Z}_{\mathfrak{C}} \to \mathbb{P}^n$ is not smooth. This set admits a structure of a closed subscheme of $Z_{\mathfrak{C}}$ defined locally by vanishing of the maximal minors of the Jacobian matrix of the map $p_{\mathfrak{C}}$. Globally, we have the standard exact sequence of the sheaves of differentials

$$0 \to p_{\mathfrak{C}}^* \Omega^1_{\mathbb{P}^n} \xrightarrow{\delta} \Omega_{\mathsf{Z}_{\mathfrak{C}}} \to \Omega^1_{\mathsf{Z}_{\mathfrak{C}}/\mathbb{P}^n} \to 0, \tag{10.30}$$

and the support of $\tilde{\Delta}$ is equal to the set of points where $\Omega^1_{\mathsf{Z}_{\mathfrak{C}}/\mathbb{P}^n}$ is not locally free. Locally the map δ is given by a matrix of size $n \times (2n-2)$. Thus, $\tilde{\Delta}$ is given locally by $n \times n$ minors of this matrix and is of dimension n.

Tensoring (10.30) with the residue field $\kappa(p)$ at a point $p = (x, \ell) \in \mathsf{Z}_{\mathfrak{C}}$, we see that $\tilde{\Delta}$ is equal to the degeneracy locus of points where the map $\delta_p : (p_{\mathfrak{C}}^* \Omega^1_{\mathbb{P}^n})_p \to (\Omega^1_{\mathsf{Z}_{\mathfrak{C}}})_p$ is not injective. Using the *Thom-Porteous formula* (see [315, 14.4]), we can express the class of $\tilde{\Delta}$ in $H^*(Z_{\mathfrak{C}}, \mathbb{Z})$.

Definition 10.3.4. *Let $\mathfrak{C}$ be a line complex of degree d in $G_1(\mathbb{P}^n)$. A line ℓ in $\mathfrak{C}$ is called* singular *if ℓ is a singular point of the intersection $\Omega(x) \cap \mathfrak{C}$ for some $x \in \mathbb{P}^n$ or any point on $\Omega(x)$ if $\Omega(x) \subset \mathfrak{C}$. The locus $S(\mathfrak{C})$ of singular lines is called the* singular variety *of $\mathfrak{C}$.*

Proposition 10.3.5. *Assume $n+1 = 2k$ and $\mathfrak{C}$ is nonsingular. Then, the singular variety $S(\mathfrak{C})$ of $\mathfrak{C}$ is equal to the intersection of $\mathfrak{C}$ with a hypersurface of degree $k(d-1)$.*

Proof Let ℓ be a singular line of $\mathfrak{C} = \mathbb{G} \cap X$, where X is a hypersurface of degree d. We have $\Omega(x) \subset \mathbb{T}_\ell(\mathfrak{C}) = \mathbb{T}_\ell(\mathbb{G}) \cap \mathbb{T}_\ell(X)$. Thus, $\Omega(x) \subset \mathbb{T}_\ell(X) \cap \mathbb{G}$. By Proposition 10.2.4, the linear line complex $\mathbb{T}_\ell(X) \cap \mathbb{G}$ consists of lines intersecting a line and its polar $(n-2)$-plane unless it is singular. Since $\Omega(x)$ is not contained in the Schubert variety of lines intersecting a codimension 2 linear subspace, we obtain that $\mathbb{T}_\ell(X) \cap \mathbb{G}$ is singular. This shows that the singular variety $S(\mathfrak{C})$ of $\mathfrak{C}$ consists of lines in $\mathfrak{C}$ such that $\mathbb{T}_\ell(X)$ coincides with a tangent hyperplane of $\mathbb{G}$. In other words,

$$S(\mathfrak{C}) = \gamma^{-1}(\mathbb{G}^\vee), \tag{10.31}$$

where $\gamma : \mathfrak{C} \to (\mathbb{P}^n)^\vee$ is the restriction of the Gauss map $X \to (\mathbb{P}^n)^\vee$ to $\mathfrak{C}$. Since $\mathfrak{C}$ is nonsingular, X is nonsingular at any point of $X \cap \mathbb{G}$ and, hence, γ is well-defined. It remains to use that γ is given by polynomials of degree $d-1$, the partials of X. □

Let $n = 3$ and let $\mathfrak{C}$ be a line complex defined by a hypersurface $X = V(\Phi)$ of degree d in the Plücker space. The equation of the singular surface $S(\mathfrak{C})$ in Plücker coordinates is easy to find. Let $\Phi_{ij} = \frac{\partial \Phi}{\partial p_{ij}}(l)$, where $[l] = \ell$. The tangent hyperplane to X at the point ℓ is given by the equation

$$\sum_{1 \le i < j \le 4} \Phi_{ij}(l) p_{ij} = 0.$$

Since the dual quadric $\mathbb{G}^\vee$ is given by the same equation as $\mathbb{G}$, we obtain the equation of $S(\mathfrak{C})$ in $\mathfrak{C}$:

$$\Phi_{12}\Phi_{34} - \Phi_{13}\Phi_{24} + \Phi_{14}\Phi_{23} = 0.$$

10.3.2 Intersection of two quadrics

Let Q_1, Q_2 be two quadrics in $\mathbb{P}^n$ and $X = Q_1 \cap Q_2$. We assume that X is nonsingular. It follows from the proof of Theorem 8.6.2 that this is equivalent to the condition that the pencil $\mathcal{P}$ of quadrics spanned by Q_1, Q_2 has exactly $n+1$ singular quadrics of corank one. This set can be identified with a set of $n+1$ points $p_1, \ldots, p_{n+1}$ in $\mathbb{P}^1 \cong \mathcal{P}$.

If $n = 2g+1$, we get the associated nonsingular hyperelliptic curve C of genus g, the double cover of $\mathbb{P}^1$ branched at $p_1, \ldots, p_{2g+2}$.

The variety X is of degree 4 in $\mathbb{P}^n, n \ge 3$, of dimension $n-2$. Its canonical class is equal to $-(n-3)h$, where h is the class of a hyperplane section. If $n = 4$, it is a quartic del Pezzo surface.

Theorem 10.3.6 (A. Weil)**.** *Assume $n = 2g+1$. Let $F(X)$ be the variety of $g-1$-dimensional linear subspaces contained in X. Then, $F(X)$ is isomorphic to the Jacobian variety of the curve C and also to the intermediate Jacobian of X.*

Proof We will restrict ourselves only to the case $g = 2$, leaving the general case to the reader. For each $\ell \in F(X)$, consider the projection map $p_\ell : X' = X \setminus \ell \to \mathbb{P}^3$. For any point $x \in X$ not on ℓ, the fiber over $p_\ell(x)$ is equal to the intersection of the plane $\ell_x = \langle \ell, x \rangle$ with X'. The intersection of this plane with a quadric Q from the pencil $\mathcal{P}$ is a conic containing ℓ and another line ℓ'. If we take two nonsingular generators of $\mathcal{P}$, we find that the fiber is the intersection of two lines or the whole $\ell' \in F(X)$ intersecting ℓ. In the latter case, all points on $\ell' \setminus \ell$ belong to the same fiber. Since all quadrics from the pencil intersect the plane $\langle \ell, \ell' \rangle$ along the same conic $\ell \cup \ell'$, there exists a unique quadric $Q_{\ell'}$ from the pencil which contains $\langle \ell, \ell' \rangle$. The plane belongs to one of the two rulings of planes on $Q_{\ell'}$ (or a unique family if the quadric is singular). Note that each quadric from the pencil contains at most one plane in each ruling which contains ℓ (two members of the same ruling intersect along a subspace of even codimension). Thus, we can identify the following sets:

- pairs (Q, r), where $Q \in \mathcal{P}$, r is a ruling of planes in Q,
- $B = \{\ell' \in F(X) : \ell \cap \ell' \neq \emptyset\}$.

If we identify $\mathbb{P}^3$ with the set of planes in $\mathbb{P}^5$ containing ℓ, the latter set becomes a subset of $\mathbb{P}^3$. Let D be the union of ℓ''s from B. The projection map p_ℓ maps D to B with fibers isomorphic to $\ell' \setminus \ell \cap \ell'$.

Extending p_ℓ to a morphism $f : \bar{X} \to \mathbb{P}^3$, where $\bar{X}$ is the blow-up of X with center at ℓ, we obtain that f is an isomorphism outside B and that the fibers over points in B are isomorphic to $\mathbb{P}^1$. Observe that $\bar{X}$ is contained in the blow-up $\bar{\mathbb{P}}^5$ of $\mathbb{P}^5$ along ℓ. The projection f is the restriction of the projection $\bar{\mathbb{P}}^5 \to \mathbb{P}^3$ which is a projective bundle of relative dimension 2. The crucial observation now is that B is isomorphic to our hyperelliptic curve C. In fact, consider the incidence variety

$$\mathcal{X} = \{(Q, \pi) \in \mathcal{P} \times G_2(\mathbb{P}^5) : \pi \subset Q\}.$$

Its projection to $\mathcal{P}$ has fiber over Q isomorphic to the rulings of planes in Q. It consists of two connected components outside of the set of singular quadrics and one connected component over the set of singular quadrics. Taking the Stein factorization, we get a double cover of $\mathcal{P} = \mathbb{P}^1$ branched along six points. It is isomorphic to C.

Now, the projection map p_ℓ maps each line ℓ' intersecting ℓ to a point in $\mathbb{P}^3$. We will identify the set of these points with the curve B. A general plane in $\mathbb{P}^3$ intersects B at $d = \deg B$ points. The pre-image of the plane under the projection $p_\ell : X \dashrightarrow \mathbb{P}^3$ is isomorphic to the complete intersection of two quadrics in $\mathbb{P}^4$. It is a del Pezzo surface of degree four, hence it is obtained by blowing up five points in $\mathbb{P}^2$. Thus, $d = 5$. An easy argument using Riemann-Roch shows that B lies on a unique quadric $\mathcal{Q} \subset \mathbb{P}^3$. Its pre-image under the projection $\bar{X} \to \mathbb{P}^3$ is the exceptional divisor E of the blow-up $\bar{X} \to X$. We have $\mathcal{N}_\ell/\mathbb{P}^4 \cong \mathcal{O}_\ell(2) \oplus \mathcal{O}_\ell(2)$ and $\mathcal{N}_{X/\mathbb{P}^4} \cong \mathcal{O}_X(2) \oplus \mathcal{O}_X(2)$. The exact sequence of normal bundles (2.42) shows that the normal bundle of ℓ in X is trivial, so $E \cong \mathbb{P}^1 \times \mathbb{P}^1$, and hence, $\mathcal{Q}$ is a nonsingular quadric. Thus, (X, ℓ) defines a biregular model $B \subset \mathbb{P}^3$ of C such that B is of degree five and lies on a unique nonsingular quadric $\mathcal{Q}$. One can show that the latter condition is equivalent to the condition that the invertible sheaf $\mathcal{O}_B(1) \otimes \omega_B^{-2}$ is not effective. It is easy to see that B is of bidegree $(2, 3)$.

Let us construct an isomorphism between $\mathrm{Jac}(C)$ and $F(X)$. Recall that $\mathrm{Jac}(C)$ is birationally isomorphic to the symmetric square $C^{(2)}$ of the curve

C. The canonical map $C^{(2)} \to \mathrm{Pic}^2(C)$ defined by $x + y \mapsto [x + y]$ is an isomorphism over the complement of one point represented by the canonical class of C. Its fiber over K_C is the linear system $|K_C|$. Also, note that $\mathrm{Pic}^2(C)$ is canonically identified with $\mathrm{Jac}(C)$ by sending a divisor class ξ of degree 2 to the class $\xi - K_C$.

Each line ℓ' skew to ℓ is projected to a secant line of B. In fact, $\langle \ell, \ell' \rangle \cap X$ is a quartic curve in the plane $\langle \ell, \ell' \rangle \cong \mathbb{P}^3$ that contains two skew line components. The residual part is the union of two skew lines m, m' intersecting both ℓ and ℓ'. Thus, ℓ' is projected to the secant line joining two points on C, which are the projections of the lines m, m'. If $m = m'$, then ℓ' is projected to a tangent line of B. Thus, the open subset of lines in X skew to ℓ is mapped bijectively to an open subset of $C^{(2)}$ represented by "honest" secants of C, i.e. secants which are not 3-secants. Each line $\ell' \in F(X) \setminus \{\ell\}$ intersecting ℓ is projected to a point b of B. The line f of the ruling of Q intersecting B with multiplicity three and passing through a point $b \in B$ defines a positive divisor D of degree 2 such that $f \cap B = b + D$. The divisor class $[D] \in \mathrm{Pic}^2(C)$ is assigned to ℓ'. So we see that each trisecant line of B (they are necessarily lie on Q) defines three lines passing through the same point of ℓ. By taking a section of X by a hyperplane tangent to X at a point $x \in X$, we see that x is contained in four lines (taken with some multiplicity). Finally, the line ℓ itself corresponds to K_C. This establishes an isomorphism between $\mathrm{Pic}^2(C)$ and $F(X)$.

□

Note that we have proved that X is a rational variety by constructing an explicit rational map from X to $\mathbb{P}^3$. This map becomes a regular map after we blow up a line ℓ on X. The image of the exceptional divisor is a quadric. This map blows down the union of lines on X that intersect ℓ to a genus 2 curve C of degree 5 lying on the quadric. The inverse map $\mathbb{P}^3 \dashrightarrow X \subset \mathbb{P}^5$ is given by the linear system of cubic hypersurfaces through the curve C. It becomes a regular map after we blow up C. Since any trisecant line of C defined by one of the rulings of the quadric blows down to a point, the image of the proper transform of the quadric is the line ℓ on X. The exceptional divisor is mapped to the union of lines on X intersecting ℓ.

10.3.3 Kummer surfaces

We consider the case $n = 3$. The quadratic line complex $\mathfrak{C}$ is the intersection of two quadrics $\mathbb{G} \cap Q$. We shall assume that $\mathfrak{C}$ is nonsingular. Let C be the associated hyperelliptic curve of genus 2.

First, let us look at the singular surface Δ of $\mathfrak{C}$. By Proposition 10.3.3, it is a quartic surface. For any point $x \in \Delta$, the conic $C_x = \mathfrak{C} \cap \Omega(x)$ is the union of two lines. A line in $\mathbb{G}$ is always equal to a one-dimensional Schubert variety. In fact, $\mathbb{G}$ is a nonsingular quadric of dimension 4, and hence contains two 3-dimensional families of planes. These are the families realized by the Schubert planes $\Omega(x)$ and $\Omega(\Pi)$. Hence, a line must be a pencil in one of these planes, which shows that $C_x = \Omega(x, \Pi_1) \cup \Omega(x, \Pi_2)$ for some planes Π_1, Π_2 in $\mathbb{P}^3$. Any line in $\mathfrak{C}$ is equal to some $\Omega(x, A)$ and hence is equal to an irreducible component of the conic C_x. Thus, we see that any line in $\mathfrak{C}$ is realized as an irreducible component of a conic $C_x, x \in \mathfrak{C}$. It follows from Theorem 10.3.6 that the variety of lines $F(\mathfrak{C})$ in $\mathfrak{C}$ is isomorphic to the Jacobian variety of C.

Proposition 10.3.7. *The variety $F(\mathfrak{C})$ of lines in $\mathfrak{C}$ is a double cover of the quartic surface Δ. The cover ramifies over the set Δ_1 of points such that the conic $C_x = p_{\mathfrak{C}}^{-1}(x)$ is a double line.*

Let $x \in \Delta$ and $C_x = \Omega(x, \Pi_1) \cup \Omega(x, \Pi_2)$. A singular point of C_x is called a *singular line* of $\mathfrak{C}$. If $x \notin \Delta_1$, then C_x has only one singular point equal to $\Omega(x, \Pi_1) \cap \Omega(x, \Pi_2)$. Otherwise, it has the whole line of them.

Let $S = S(\mathfrak{C})$ be the singular surface of $\mathfrak{C}$. By Proposition 10.3.5, S is a complete intersection of three quadrics.

By the adjunction formula, we obtain $\omega_S \cong O_S$. The assertion that S is nonsingular follows from its explicit equations (10.32) given below. Thus, S is a K3 surface of degree 8.

Theorem 10.3.8. *The set of pairs (x, ℓ), where ℓ is a singular line containing x is isomorphic to the variety $\tilde{\Delta} \subset Z_{\mathfrak{C}}$, the locus of points where the morphism $p_{\mathfrak{C}} : Z_{\mathfrak{C}} \to \mathbb{P}^3$ is not smooth. It is a nonsingular surface with the trivial canonical class. The projection $p_{\mathfrak{C}} : \tilde{\Delta} \to \Delta$ is a resolution of singularities. The projection $q_{\mathfrak{C}} : \tilde{\Delta} \to S$ is an isomorphism. The surface S is equal to $\mathfrak{C} \cap Q$, where Q is a quadric in $\mathbb{P}^5$.*

Proof The first assertion is obvious since the fibers of $p_{\mathfrak{C}} : Z_{\mathfrak{C}} \to \mathbb{P}^3$ are isomorphic to the conics C_x. To see that $q_{\mathfrak{C}}$ is one-to-one we have to check that a singular line ℓ cannot be a singular point of two different fibers C_x and C_y. The planes $\Omega(x)$ and $\Omega(y)$ intersect at one point $\ell = \overline{xy}$ and hence span $\mathbb{P}^4$. If Q is tangent to both planes at the same point ℓ, the two planes are contained in $\mathbb{T}_\ell(Q) \cap \mathbb{T}_\ell(G)$, hence $\mathfrak{C} = Q \cap G$ is singular at ℓ. This contradicts our assumption on $\mathfrak{C}$. Thus, the projection $\tilde{\Delta} \to S$ is one-to-one. Since the fibers of $q_{\mathfrak{C}} : Z_{\mathfrak{C}} \to \mathfrak{C}$ are projective lines, this easily implies that the restriction of $q_{\mathfrak{C}}$ to $\tilde{\Delta}$ is an isomorphism onto S.

□

Theorem 10.3.9. *The set Δ_1 consists of 16 points, and each point is an ordinary double point of the singular surface Δ.*

Proof Let $A = F(\mathfrak{C})$ be the variety of lines in $\mathfrak{C}$. We know that it is a double cover of Δ ramified over the set Δ_1. Since Δ is isomorphic to S outside Δ_1, we see that A admits an involution with a finite set F of isolated fixed points such that the quotient is birationally isomorphic to a K3 surface. The open set $A \setminus F$ is an unramified double cover of the complement of $s = \#F$ projective lines in the K3 surface S. For any variety Z we denote by $e_c(Z)$ the topological Euler-Poincaré characteristic with compact support. By the additivity property of e_c, we get $e_c(A-S) = e(A) - s = 2(e_s(S) - 2s) = 48 - 4s$. Thus, $e(A) = 48 - 3s$. Since $A \cong \mathrm{Jac}(C)$, we have $e(A) = 0$. This gives $s = 16$. Thus, Δ has 16 singular points. Each point is resolved by a (-2)-curve on S. This implies that each singular point is a rational double point of type A_1, i.e. an ordinary double point. □

Definition 10.3.10. *For any abelian variety A of dimension g the quotient of A by the involution $a \mapsto -a$ is denoted by* $\mathrm{Kum}(A)$ *and is called the* Kummer variety *of A.*

Note that $\mathrm{Kum}(A)$ has 2^{2g} singular point locally isomorphic to the cone over the Veronese variety V_g^{g-1}. In the case $g = 2$ we have 16 ordinary double points. It is easy to see that any involution with this property must coincide with the negation involution (look at its action in the tangent space, and use that A is a complex torus). This gives the following.

Corollary 10.3.11. *The singular surface Δ of $\mathfrak{C}$ is isomorphic to the Kummer surface of the Jacobian variety of the hyperelliptic curve C of genus 2.*

The Kummer variety of the Jacobian variety of a nonsingular curve is called a *jacobian Kummer variety*.

Proposition 10.3.12. *The surface S contains two sets of 16 disjoint lines.*

Proof The first set is formed by the lines $q_{\mathfrak{C}}(p_{\mathfrak{C}}^{-1}(z_i))$, where $z_1, \ldots, z_{16}$ are the singular points of the singular surface. The other set comes from the dual picture. We can consider the dual incidence variety

$$\check{\mathsf{Z}}_{\mathfrak{C}} = \{(\Pi, \ell) \in (\mathbb{P}^3)^\vee \times \mathfrak{C} : \ell \subset \Pi\}.$$

The fibers of the projection to $(\mathbb{P}^3)^\vee$ are conics. Again we define the singular surface $\check{\Delta}$ as the locus of planes such that the fiber is the union of lines. A line in the fiber is a pencil of lines in the plane. These pencils form the set of lines in $\mathfrak{C}$. The lines are common to two pencils if lines are singular lines of $\mathfrak{C}$. Thus, we see that the surface S can be defined in two ways by using the incidence $Z_{\mathfrak{C}}$ or $\check{Z}_{\mathfrak{C}}$. As before, we prove that $\check{\Delta}$ is the quotient of the abelian surface A and is isomorphic to the Kummer surface of C. The lines in S corresponding to singular points of $\check{\Delta}$ is the second set of 16 lines. □

Choosing six mutually apolar linear line complexes, we write the equation of the Klein quadric as a sum of squares. The condition of nondegeneracy allows one to reduce the quadric Q to the diagonal form in these coordinates. Thus, the equation of the quadratic line complex can be written in the form

$$\sum_{i=0}^{5} t_i^2 = \sum_{i=0}^{5} a_i t_i^2 = 0. \tag{10.32}$$

Since $\mathfrak{C}$ is nonsingular, $a_i \neq a_j$, $i \neq j$. The parameters in the pencil corresponding to six singular quadrics are $(t_0, t_1) = (-a_0, 1), i = 0, \ldots, 5$. Thus, the hyperelliptic curve C has the equation

$$t_2^2 = (t_1 + a_0 t_0) \cdots (t_1 + a_5 t_0),$$

which has to be considered as an equation of degree 6 in the weighted plane $\mathbb{P}(1, 1, 3)$.

To find the equation of the singular surface S of $\mathfrak{C}$, we apply (10.31). The dual of the quadric $V(\sum a_i t_i^2)$ is the quadric $V(\sum a_i^{-1} u_i^2)$. Its pre-image under the Gauss map defined by the quadric $V(\sum t_i^2)$ is the quadric $V(\sum a_i^{-1} x_i^2)$. After scaling $t_i \mapsto a_i t_i$, we obtain that the surface S, a nonsingular model, of the Kummer surface, is given by the equations

$$\sum_{i=0}^{5} t_i^2 = \sum_{i=0}^{5} a_i t_i^2 = \sum_{i=0}^{5} a_i^2 t_i^2 = 0. \tag{10.33}$$

We know that the surface given by above equations contains 32 lines. Consider six lines ℓ_i in $\mathbb{P}^2$ given by the equations

$$X_0 + a_i X_1 + a_i^2 X_2 = 0, \quad i = 0, \ldots, 5. \tag{10.34}$$

Since the points $(1, a_i, a_i^2)$ lie on the conic $X_0 X_2 - X_1^2 = 0$, the lines ℓ_i are tangent to the conic.

Lemma 10.3.13. *Let $X \subset \mathbb{P}^{2k-1}$ be a variety given by complete intersection of k quadrics*

$$q_i = \sum_{j=0}^{2k-1} a_{ij} t_j^2 = 0, \quad i = 1, \ldots, k.$$

Consider the group G of projective transformations of $\mathbb{P}^{2k-1}$ that consists of transformations

$$[t_0, \ldots, t_{2k-1}] \mapsto [\epsilon_0 t_0, \ldots, \epsilon_{2k-1} t_{2k-1}],$$

where $\epsilon_i = \pm 1$ and $\epsilon_0 \cdots \epsilon_{2k-1} = 1$. Then, X/G is isomorphic to the double cover of $\mathbb{P}^{k-1}$ branched along the union of $2k$ hyperplanes with equations explicitly given below in (10.35).

Proof Let $R = \mathbb{C}[t_0, \ldots, t_{2k-1}]/(q_1, \ldots, q_k)$ be the ring of projective coordinates of X. Then, the subring of invariants R^G is generated by the cosets of $t_0^2, \ldots, t_{2k-1}^2$ and $t_0 \cdots t_{2k-1}$. Since $(t_0 \cdots t_{2k-1})^2 = t_0^2 \cdots t_{2k-1}^2$, we obtain that

$$R^G \cong \mathbb{C}[t_0, \ldots, t_{2k-1}, t]/I,$$

where I is generated by

$$\sum_{j=0}^{2k-1} a_{ij} t_j, \ i = 1, \ldots, k, \quad t^2 - t_0 \cdots t_{2k-1}.$$

Let $A = (a_{ij})$ be the matrix of the coefficients a_{ij}. Its rank is equal to k. Choose new coordinates t_i' in $\mathbb{C}^{2k}$ such that $t_{i+k-1}' = \sum_{j=0}^{2k-1} a_{ij} t_j, \ i = 1, \ldots, k$. Write

$$t_i = \sum_{j=0}^{k-1} b_{ij} t_j' \mod (t_k', \ldots, t_{2k-1}'), \ i = 0, \ldots, 2k-1.$$

Then,

$$X/G \cong \operatorname{Proj} R^G \cong \operatorname{Proj}(\mathbb{C}[t_0', \ldots, t_{k-1}', t])/\Big(t^2 - \prod_{i=0}^{2k-1} \sum_{j=0}^{k-1} b_{ij} t_j']\Big).$$

Thus, X/G is isomorphic to the double cover of $\mathbb{P}^{k-1}$ branched along the hyperplanes

$$\sum_{j=0}^{k-1} b_{ij} z_j = 0, \ j = 0, \ldots, 2k-1. \tag{10.35}$$

□

Corollary 10.3.14. *Suppose the set of $2k$ points*

$$[a_{00}, \ldots, a_{k0}], \ldots, [a_{0\ 2k-1}, \ldots, a_{k\ 2k-1}]$$

in $\mathbb{P}^{k-1}$ is projectively equivalent to an ordered set of points on a Veronese curve of degree $k-1$. Then, X/G is isomorphic to the double cover of $\mathbb{P}^{k-1}$ branched along the hyperplanes

$$a_{0j} z_0 + \cdots + a_{k-1j} z_{k-1} = 0, \quad i = 0, \ldots, 2k-1.$$

Proof Choose coordinates such that the matrix $A = (a_{ij})$ has the form

$$A = \begin{pmatrix} 1 & 1 & \dots & 1 \\ \alpha_1 & \alpha_2 & \dots & \alpha_{2k} \\ \vdots & \vdots & \vdots\vdots & \\ \alpha_1^{k-1} & \alpha_2^{k-1} & \dots & \alpha_{2k}^{k-1} \end{pmatrix}.$$

Let

$$D_j = \prod_{1 \le i < j \le k} (\alpha_j - \alpha_i)$$

and

$$f(x) = (x - \alpha_1) \cdots (x - \alpha_k) = a_0 + a_1 x + \cdots + a_k x^k,$$

$$f_j(x) = \frac{f(x)}{D_j(x - \alpha_j)} = a_{0j} + a_{1j}x + \cdots + a_{k-1j}x^{k-1}, \ j = 1, \dots, k.$$

We have

$$B = \begin{pmatrix} 1 & 1 & \dots & 1 \\ \alpha_1 & \alpha_2 & \dots & \alpha_k \\ \vdots & \vdots & \vdots\vdots & \\ \alpha_1^{k-1} & \alpha_2^{k-1} & \dots & \alpha_k^{k-1} \end{pmatrix}^{-1} = \begin{pmatrix} a_{01} & a_{11} & \dots & a_{k-11} \\ a_{02} & a_{12} & \dots & a_{k-12} \\ \vdots & \vdots & \vdots & \\ a_{0k} & a_{1k} & \dots & a_{k-1k} \end{pmatrix}.$$

Multiplying A by B on the left we obtain

$$B \cdot A = \begin{pmatrix} 1 & 0 & 0 & \dots & 0 & f_1(\alpha_{k+1}) & \dots & f_1(\alpha_{2k}) \\ 0 & 1 & 0 & \dots & 0 & f_2(\alpha_{k+1}) & \dots & f_2(\alpha_{2k}) \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 1 & f_k(\alpha_{k+1}) & \dots & f_k(\alpha_{2k}) \end{pmatrix}$$

$$= \begin{pmatrix} f_1(\alpha_1) & \dots & f_1(\alpha_k) & f_1(\alpha_{k+1}) & \dots & f_1(\alpha_{2k}) \\ f_2(\alpha_1) & \dots & f_2(\alpha_k) & f_2(\alpha_{k+1}) & \dots & f_2(\alpha_{2k}) \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ f_k(\alpha_1) & \dots & f_k(\alpha_k) & f_k(\alpha_{k+1}) & \dots & f_k(\alpha_{2k}) \end{pmatrix}.$$

The polynomials $f_1(x), \dots, f_k(x)$ form a basis in the space of polynomials of degree $\le k - 1$. We see that the columns of the matrix $B \cdot A$ can be taken as the projective coordinates of the images of points $[1, \alpha_1], \dots, [1, \alpha_{2k}] \in \mathbb{P}^1$ under a Veronese map. Under the projective transformation defined by the matrix B, the ordered set of columns of matrix A is projectively equivalent to the set of points defined by the column of the matrix $B \cdot A$. Write the matrix $B \cdot A$ in the block-form $[I_k\ C]$. Then, the null-space of this matrix is the columns space of the matrix $[-C\ I_k]$. It defines the same set of points up to a permutation. □

The following lemma is due to A. Verra.

Lemma 10.3.15. *Let X be the base locus of a linear system $\mathcal{N}$ of quadrics of dimension $k-1$ in $\mathbb{P}^{2k-1}$. Suppose that*

- *$\mathcal{N}$ contains a nonsingular quadric;*
- *X contains a linear subspace Λ of dimension $k-2$;*
- *X is not covered by lines intersecting Λ.*

Then, X is birationally isomorphic to the double cover of $\mathcal{N}$ branched over the discriminant hypersurface of $\mathcal{N}$.

Proof Let Λ be a linear subspace of dimension $k-2$ contained in X. Take a general point $x \in X$ and consider the span $\Pi = \langle \Lambda, x \rangle$. By our assumption, x is not contained in any line. The restriction of the linear system $\mathcal{N}$ to Π is a linear system of quadrics in $\Pi \cong \mathbb{P}^{k-1}$ containing Λ and x in its base locus. The residual components of these quadrics are hyperplanes in Π containing x. The base locus of this linear system of hyperplanes consists only of x, because otherwise x will be contained in a line on X intersecting Λ. Our assumption excludes this. Thus, the dimension of the restriction of $\mathcal{N}$ to Π is equal to $k-2$. This implies that there exists a unique quadric in $\mathcal{N}$ containing Π. This defines a rational map $X \dashrightarrow \mathcal{N}$. A general member of $\mathcal{N}$ is a nonsingular quadric in $\mathbb{P}^{2k-1}$. It contains two rulings of $(k-1)$-planes. Our $(k-1)$-plane Π belongs to one of the rulings. The choice of a ruling to which Π belongs defines a rational map to the double cover $Y \to \mathcal{N}$ branched along the discriminant variety of $\mathcal{N}$ parameterizing singular quadrics. The latter is constructed by considering the second projection of the incidence variety

$$\{(\Pi, Q) \in G_k(\mathbb{P}^{2k-1}) \times \mathcal{N} : \Pi \in \mathcal{N}\}$$

and applying the Stein factorization. Now, we construct the inverse rational map $Y \dashrightarrow X$ as follows. Take a nonsingular quadric $Q \in \mathcal{N}$ and choose a ruling of $(k-1)$-planes in Q. If $Q = V(q)$, then $\Pi = |L|$, where L is an isotropic k-dimensional linear subspace of the quadratic form q, hence it can be extended to a unique maximal isotropic subspace of q in any of the two families of such subspaces. Thus, Λ is contained in a unique $(k-1)$-plane Π from the chosen ruling. The restriction of $\mathcal{N}$ to Π is a linear system of quadrics of dimension $k-2$ with Λ contained in the base locus. The free part of the linear system is a linear system of hyperplanes through a fixed point x. This point belongs to all quadrics in $\mathcal{N}$, hence belongs to X. So. this point is taken to be the value of our map at the pair Q plus a ruling. □

Applying this lemma to the case when the linear system of quadrics consists of diagonal quadrics, we obtain that the discriminant hypersurface in $\mathcal{N}$ is the union of hyperplanes

$$\sum_{i=0}^{k} a_{ij} t_i = 0, \ j = 0, \ldots, 2k+1.$$

This shows that in the case when the hyperplanes, considered as points in the dual space, lie on a Veronese curve, the base locus X of $\mathcal{N}$ is birationally isomorphic to the quotient X/G.

This applies to our situation and gives the following.

Theorem 10.3.16. *The surface S given by* (10.33) *is birationally isomorphic to the double cover of $\mathbb{P}^2$ branched along the six lines $\ell_i = V(z_0 + a_i z_1 + a_i^2 z_2)$. It is also birationally isomorphic to the quotient S/G, where G consists of involutions $[t_0, \ldots, t_5] \mapsto [\epsilon_0 t_0, \ldots, \epsilon_5 t_5]$ with $\epsilon_0 \cdots \epsilon_5 = 1$.*

Remark 10.3.17. In Chapter 12, we will study quartic surfaces with m ordinary nodes and show that such a surface is obtained as the double cover of the plane branched along a nodal curve of degree 6 admitting a contact conic. The Kummer surface is the special case when $m = 16$. The branch curve must have 15 nodes, and this is possible only if it is the union of 6 lines intersecting by pairs in 15 points.

Theorem 10.3.18. *A Kummer surface is projectively isomorphic to a quartic surface in $\mathbb{P}^3$ with equation*

$$A(x^4 + y^4 + z^4 + w^4) + 2B(x^2y^2 + z^2w^2) \tag{10.36}$$
$$+2C(x^2z^2 + y^2w^2) + 2D(x^2w^2 + z^2y^2) + 4Exyzw = 0,$$

where

$$A(A^2 + E^2 - B^2 - C^2 - D^2) + 2BCD = 0. \tag{10.37}$$

The cubic hypersurface defined by the equation in above is isomorphic to the Segre cubic primal.

Proof Choosing apolar linear line complexes, we transform the Klein quadric to the form $t_1^2 + \ldots + t_6^2 = 0$. Consider the Heisenberg group with nonzero elements defined by involutions associated to a pair of apolar linear line complexes. The Heisenber group is induced by transformations of $\mathbb{P}^3$ listed in Subsection

10.2.1. In these coordinates, the equation of the Kummer surface must be invariant with respect to these transformations. It is immediately checked that this implies that the equation must be as in (10.36). It remains to check the conditions on the coefficients. We know that a Kummer surface contains singular points. Taking the partials, we find

$$Ax^3 + x(By^2 + Cz^2 + Dw^2) + Eyzw = 0,$$

$$Ay^3 + y(Bx^2 + Cw^2 + Dz^2) + Exzw = 0,$$

$$Az^3 + z(Bw^2 + Cx^2 + Dy^2) + Exyw = 0,$$

$$Aw^3 + w(Bz^2 + Cy^2 + Dx^2) + Exyz = 0.$$

Multiplying the first equation by y and the second equation by x, and adding up the two equations, we obtain

$$(A + B)(x^2 + y^2) + (C + D)(z^2 + w^2) = \alpha \frac{x^2 + y^2}{x^2y^2}, \tag{10.38}$$

where $\alpha = -Exyzw$. Similarly, we get

$$(C + D)(x^2 + y^2) + (A + B)(z^2 + w^2) = \alpha \frac{z^2 + w^2}{z^2w^2}. \tag{10.39}$$

Dividing the first equation by $x^2 + y^2$, the second equation by $z^2 + w^2$, and adding up the results, we obtain

$$2(A + B) + (C + D)\left(\frac{z^2 + w^2}{x^2 + y^2} + \frac{z^2 + w^2}{x^2 + y^2}\right) = \alpha\left(\frac{1}{x^2y^2} + \frac{1}{z^2w^2}\right). \tag{10.40}$$

Multiplying both sides of (10.38) and (10.39), and dividing both sides by $(x^2 + y^2)(z^2 + w^2)$, we obtain

$$(A + B)^2 + (C + D)^2 + (A + B)(C + D)\left(\frac{z^2 + w^2}{x^2 + y^2} + \frac{z^2 + w^2}{x^2 + y^2}\right) = E^2.$$

Now, we multiply Equation (10.39) by $A + B$, and, after subtracting (10.40) from the result, we obtain

$$(A + B)^2 - (C + D)^2 + E^2 = \alpha(A + B)\left(\frac{1}{x^2y^2} + \frac{1}{z^2w^2}\right).$$

Similarly, we get

$$(A - B)^2 - (C - D)^2 + E^2 = -\alpha(A - B)\left(\frac{1}{x^2y^2} + \frac{1}{z^2w^2}\right),$$

hence,

$$\frac{(A+B)^2-(C+D)^2+E^2}{(A-B)^2-(C-D)^2+E^2}+\frac{A+B}{A-B}=0.$$

From this we easily derive (10.37).

Equation (10.37) defines a cubic hypersurface in $\mathbb{P}^4$ isomorphic to the Segre cubic primal S_3 given by Equation (9.46). After substitution

$$\begin{aligned} A &= z_0+z_3, \\ B &= z_0+2z_2+2z_4+z_3, \\ C &= z_0+2z_1+2z_4+z_3, \\ D &= -z_0-z_1-2z_2-z_3, \\ E &= -2z_0+2z_3, \end{aligned} \tag{10.41}$$

we obtain the equation

$$z_0^3+z_1^3+z_2^3+z_3^3+z_4^3-(z_0+z_1+z_2+z_3+z_4)^3=0.$$

Since Kummer surfaces depend on three parameters, and the Segre cubic is irreducible, we obtain that a general point on the Segre cubic corresponds to a Kummer surface. □

Let $V=H^0(\mathbb{P}^3,\mathcal{O}_{\mathbb{P}^3}(4))^{\mathcal{H}_2}\cong\mathbb{C}^5$ with coordinates A,B,C,D,E. The linear system $|V|\subset|\mathcal{O}_{\mathbb{P}^3}(4)|$ defines a map $\Phi:\mathbb{P}^3\to|V|^\vee\cong\mathbb{P}^4$ whose image is isomorphic to the orbit space $X=\mathbb{P}^3/\mathcal{H}_2$ from (10.22). The pre-image of a hyperplane in $\mathbb{P}^4$ is singular if and only if it does not intersect X transversally. This implies that the dual of the hypersurface X is equal to the Segre primal cubic S_3, and, by Proposition 9.5.13, it is isomorphic to the Castelnuovo-Richmond quartic.

A tangent hyperplane of CR_4 at its nonsingular point is a quartic surface with 16 nodes, 15 come from the 15 singular lines of the hypersurface and one more point is the tangency point. It coincides with the surface corresponding to the point on the dual hypersurface. In this way, we see a moduli-theoretical interpretation of the set of nonsingular points of CR_4. They correspond to the Kummer surfaces of abelian surfaces equipped with some additional data. Recall that that we have chosen Klein coordinates in the Plücker space that allowed us to write the equation of a Kummer surface in $\mathcal{H}_2$-invariant form. The double plane construction model of the Jacobian Kummer surface comes with the order on the set of six lines defining the branch curve. This is the same as the order on six Weierstrass points of the corresponding curve of genus 2. As we saw in Section 5.2, the order on the Weierstrass point is equivalent to a choice of a symplectic basis in the group of 2-torsion points of the Jacobian variety. In this way, we see that a Zariski open subset of CR_4 can be identified with the moduli space of Jacobian abelian surfaces with full level 2 structure defined by a choice of a symplectic basis in the group of 2-torsion points. It turns out that the whole hypersurface CR_4 is isomorphic to a certain natural compactification $\overline{\mathcal{A}_2(2)}$ of the moduli space of abelian surface with full level 2 structure. This was proven by J. Igusa in [423], who gave an equation of the quartic CR_4 in different coordinates. The quartic hypersurface isomorphic to CR_4 is often referred to in modern literature as an *Igusa quartic*.

The 16 singular points of the Kummer surface Y given by (10.36) form an orbit of $\mathcal{H}_2$. As we know this orbit defines a (16_6)-configuration. A plane containing a set of six points cuts out on Y a plane quartic curve with 6 singular points, no three of them lying on a line. This could happen only if the plane is tangent to the surface along a conic. This conic, or the corresponding plane, is called a *trope-conic*. Again, this confirms the fact that in any general $\mathcal{H}_2$-orbit a set of coplanar six points from the (16_6)-configuration lies on a conic.

On a nonsingular model of Y isomorphic to the octavic surface S in $\mathbb{P}^5$ the exceptional curves (the singular lines of the quadratic complex) of the 16 singular points and the proper transforms of 16 tropes form the (16_6)-configuration of lines.

Consider the Gauss map from Y to its projectively dual surface $Y^\vee$ given by cubic partials. Obviously, it should blow down each trope to a singular point of $Y^\vee$. Thus, $Y^\vee$ has at least 16 singular points. It follows from the Plücker-Teissier formulas (1.2.7) that each ordinary double point decreases the degree of the dual surface by 2. Thus, the degree of the dual surface $Y^\vee$ is expected to be equal to $36 - 32 = 4$. In fact we have the following beautiful fact.

Theorem 10.3.19. *A Kummer surface is projectively isomorphic to its dual surface.*

Proof In the proof of Theorem 10.3.18 we had computed the partial cubics of Equation (10.36). The linear system of the partial cubics is invariant with respect to the action of the Heisenberg group $\mathcal{H}_2$ and defines an isomorphism of projective representations. If we choose a basis appropriately, we will be able to identify $\mathcal{H}_2$-equivariantly the dual of the linear system with the original space $\mathbb{P}^3$. We know that the image of the surface is a quartic surface with 16 singular points. Since the tropes of the original surfaces are mapped to singular points of the dual surface, we see that the two surfaces share the same configurations of nodes and tropes. Thus, they share 16 conics, and hence coincide (since the degree of intersection of two different irreducible surfaces is equal to 16). □

Remark 10.3.20. One can see the duality also from the duality of the quadratic line complexes. If we identify the space $E = \mathbb{C}^4$ with its dual space by means of the standard basis e_1, e_2, e_3, e_4 and its dual basis $e_1^*, e_2^*, e_3^*, e_4^*$, then the Plücker coordinates $p_{ij} = e_i^* \wedge e_j^*$ in $\bigwedge^2 E$ can be identified with the Plücker coordinates $p_{ij}^* = e_i \wedge e_j$ in $\bigwedge^2 E^\vee$. The Klein quadrics could be also identified. Now, the duality isomorphism $G(2, E) \to G(2, E^\vee), \ell \mapsto \ell^\perp$, becomes compatible with the Plücker embeddings. The quadratic line complex given in Klein coordinates by two diagonal quadrics (10.32) is mapped under the duality isomorphism to the quadratic line complex given by two diagonal quadrics $\sum y_i^2 = 0, \sum a_i^{-1} y_i^2 = 0$, the dual quadrics. However, the intersection of these two pairs of quadrics is projectively isomorphic under the scaling transformation $y_i \mapsto \sqrt{a_i} y_i$. This shows that, under the duality isomorphism, the singular surfaces of the quadratic line complex and its dual are projectively isomorphic. It follows from the definition of the duality that the tropes of the Kummer surface correspond to β-planes that intersect the quadratic line complex along the union of two lines.

The Kummer surface admits an infinite group of birational automorphisms. For a general one, the generators of this group have been determined in modern works of J. Keum [446] and S. Kondō [464]. We give only examples of some automorphisms.

- Projective automorphisms defined by the Heisenberg group. They correspond to translations by 2-torsion points on the abelian surface cover.
- Involutions defined by projections from one of 16 nodes.
- Switches defined by choosing a duality automorphism and composing it with elements of the Heisenberg group.
- Cubic transformations given in coordinates used in Equation (10.36) by

$$(x, y, z, w) \mapsto (yzw, xzw, xyw, xyz)$$

.

- Certain automorphisms introduced by Keum in [446] (they can be replaced by another set of automorphisms introduced by Ohashi [558]).

10.3.4 Harmonic complex of lines

Consider a pair of irreducible quadrics Q_1 and Q_2 in $\mathbb{P}^n$. A *harmonic line complex* or a *Battaglini complex* is the closure in $G_1(\mathbb{P}^n)$ of the locus of lines, which intersect Q_1 and Q_2 at two harmonically conjugate pairs. Let us see that this is a quadratic line complex and find its equation.

Let $A = (a_{ij}), B = (b_{ij})$ be two symmetric matrices defining the quadrics. Let $\ell = \overline{xy}$, where $x = [v], y = [w]$ for some $v, w \in \mathbb{C}^4$. Let $\ell = [sv + tw]$

be a parametric equation of ℓ. The restriction of Q_1 to ℓ is a binary form in s,t defined by $(vAv)s^2 + 2(vAw)st + (wAw)t^2$ and the restriction of Q_2 to ℓ is defined by the bilinear form $(vBw)s^2 + 2(vBw)st + (wBw)t^2$. By definition, the two roots of the binary forms are harmonically conjugate if and only if

$$(vAv)(wBw) + (wAw)(vBv) - 2(vAw)(vBw) = 0.$$

Let $[vw]$ be the matrix with two columns equal to the coordinate vectors of v and w. We can rewrite the previous expression in the form

$$\det\big({}^t[vw][AvBw]\big) + \det\big({}^t[v,w][BvAw]\big) = 0. \tag{10.42}$$

The expression is obviously a quadratic form on $\bigwedge^2 \mathbb{C}^{n+1}$ and also a symmetric bilinear form on the space of symmetric matrices. Take the standard basis $E_{ij} + E_{ji}, E_{ii}, 1 \le i \le j \le n+1$, of the space of symmetric matrices and compute the coefficients of the symmetric bilinear forms in terms of coordinates of v and w. We obtain

$$a_{ij;kl} = 4(x_i x_j y_k y_l + x_k x_l y_i y_j) - 2(x_k y_l + x_l y_k)(x_j y_i + x_i y_j)$$

$$= 2(p_{ik}p_{jl} + p_{il}p_{jk}),$$

where $p_{ab} = -p_{ba}$ if $a > b$. Thus, (10.42) is equal to

$$\sum (a_{ij}b_{kl} + a_{kl}b_{ij})(p_{ik}p_{jl} + p_{il}p_{jk}) = 0. \tag{10.43}$$

This is an equation of a quadratic complex. If we assume that $a_{ij} = b_{ij} = 0$ if $i \neq j$, the equation simplifies

$$\sum (a_{ii}b_{jj} + a_{jj}b_{ii})p_{ij}^2 = 0. \tag{10.44}$$

Consider the pencil $\mathcal{P}$ of quadrics $\lambda Q_1 + \mu Q_2$. Let us assume, for simplicity, that the equations of the quadrics can be simultaneously diagonalized. Then, a line ℓ is tangent to a quadric from $\mathcal{P}$ if and only if

$$\sum (\lambda a_{ii} + \mu b_{ii})(\lambda a_{jj} + \mu b_{jj})p_{ii}^2$$

$$= \sum (\lambda^2 a_{ii}a_{jj} + \lambda\mu(a_{ii}b_{jj} + a_{jj}b_{ii}) + \mu^2 b_{ii}b_{jj})p_{ij}^2 = 0.$$

The restriction of the pencil to ℓ is a linear series g_2^1 unless ℓ has a base point in which case the line intersects the base locus of the pencil. The two quadrics touching ℓ correspond to the points $[\lambda,\mu] \in \mathcal{P}$, which satisfy the equation above. Denote by $A, 2B, C$ the coefficients at $\lambda^2, \lambda\mu, \mu^2$. The map

$$G_1(\mathbb{P}^n) \to \mathbb{P}^2,\ \ell \mapsto [A,B,C]$$

is a rational map defined on the complement of codimension 3 subvariety of

$G_1(\mathbb{P}^n)$ given by the equations $A = B = C = 0$. Its general fiber is the loci of lines that touch a fixed pair of quadrics in the pencil. It is given by the intersection of two quadratic line complexes. In case $n = 2$, we recognize the well-known fact that two conics have four common tangents. The pre-image of a line $At_0 + 2Bt_1 + Ct_2 = 0$ with $AC - B^2 = 0$ is a line complex such that there is only one quadric in the pencil that touches the line. Hence, it equals the Chow form of the base locus, a hypersurface of degree 4 in $G(2, n)$.

Let us consider the case $n = 3$. In this case, a harmonic line complex is a special case of a quadratic line complex given by two quadrics

$$Q_1 := p_{12}p_{34} - p_{13}p_{24} + p_{14}p_{23} = 0,$$
$$Q_2 := a_{12}p_{12}^2 + \cdots + a_{34}p_{34}^2 = 0.$$

We assume that Q_2 is a nonsingular quadric, i.e., all $a_{ij} \neq 0$. It is easy to see that the pencil $\lambda Q_1 + \mu Q_2 = 0$ has six singular quadrics corresponding to the parameters

$$[1, \pm\sqrt{a_{12}a_{34}}],\ [1, \pm\sqrt{a_{13}a_{24}}],\ [1, \pm\sqrt{a_{14}a_{23}}].$$

Thus, we diagonalize both quadrics to reduce the equation of the quadratic line complex to the form

$$t_0^2 + \cdots + t_5^2 = 0,$$

$$k_1(t_0^2 - t_1^2) + k_2(t_2^2 - t_3^2) + k_3(t_4^2 - t_5^2) = 0.$$

The genus two curve corresponding to the intersection of the two quadrics is a special one. Its branch points are $[1, \pm k_1]$, $[1, \pm k_2]$, $[1, \pm k_3]$. The involution of $\mathbb{P}^1$ defined by $[t_0, t_1] \mapsto [t_0, -t_1]$ leaves the set of branch points invariant and lifts to an involution of the genus 2 curve. It follows from the description of binary forms invariant under a projective automorphism of finite order given in Section 8.8.2 that there is only one conjugacy class of involutions of order 2 and each binary sextic whose set of zeros is invariant with respect to an involution can be reduced to the form $(t_0^2 - t_1^2)(t_0^2 - \alpha t_1^2)(t_0^2 - \beta t_1^2)$. Thus, we see that the harmonic line complexes form a hypersurface in the moduli space of smooth complete intersections of two quadrics in $\mathbb{P}^5$. It is isomorphic to the hypersurface in $\mathcal{M}_2$ formed by isomorphism classes of genus two curves admitting two commuting involutions.

Proposition 10.3.21. *The singular surface of a harmonic line complex is projectively isomorphic to a quartic surface given by equation* (10.36) *with coefficient E equal to 0.*

Proof We use that, in Klein coordinates, our quadratic line complex has additional symmetry defined by the transformation

$$(t_0, t_1, t_2, t_3, t_4, t_5) \mapsto (-it_1, it_0, -it_3, it_2, -it_5, it_4).$$

Here, we may assume that $t_0 = i(p_{14} - p_{23}), t_1 = p_{14} + p_{23}$, etc. The transformation of $\mathbb{P}^3$ that induces this transformation is defined by $[x, y, z, w] \mapsto [-x, y, z, w]$. Equation (10.36) shows that the Kummer surface is invariant with respect to this transformation if and only if the coefficient E is zero. □

Note that under the isomorphism from the cubic (10.37) to the Segre cubic primal given by formulas (10.41), the coefficient E is equal to $-z_0 + z_3$. This agrees with a remark before Lemma 9.5.6.

Consider the Kummer surface S given by Equation (10.36) with $E = 0$. Intersecting the surface with the plane $x = 0$, we obtain the plane quartic with equation $Q(x^2, y^2, z^2) = 0$, where $Q = A(s^2 + u^2 + v^2) + 2Bsu + 2Csv + 2Duv$. Its discriminant is equal to $A(A^2 - B^2 - C^2 - D^2) + 2BCD$. Comparing it with Equation (10.37), we find that the quadratic form is degenerate. Thus, the plane section of the Kummer surface is the union of two conics with equations $(ax^2 + by^2 + cz^2)(a'x^2 + b'y^2 + c'z^2) = 0$. The four intersection points of these conics are singular points of S. This easily follows from the equations of the derivatives of the quartic polynomial defining S. Thus, we see that the 16 singular points of the Kummer surface lie by four in the coordinate planes $x, y, z, w = 0$. Following A. Cayley [103], a Kummer surface with this property is called a *Tetrahedroid*.

Note the obvious symmetry of the coordinate hyperplane sections. The coordinates of 16 nodes can be put in the following symmetric matrix:

$$\begin{pmatrix} 0 & \pm a_{12} & \pm a_{13} & \pm a_{14} \\ \pm a_{21} & 0 & \pm a_{23} & \pm a_{24} \\ \pm a_{31} & \pm a_{32} & 0 & \pm a_{34} \\ \pm a_{41} & \pm a_{42} & \pm a_{43} & 0 \end{pmatrix}.$$

The complete quadrangle formed by four nodes $p_1, \ldots, p_4$ in each coordinate plane has the property that the lines $\overline{p_i p_j}$ and $\overline{p_k p_l}$ with $\{i, j, \} \cap \{k, l\} = \emptyset$ intersect at the vertices of the coordinate tetrahedron. One can also find the 16 tropes. Take a vertex of the coordinate tetrahedron. There will be two pairs of nodes, not in the same coordinate plane, each pair lying on a line passing through the vertex. For example,

$$[0, a_{12}, a_{13}, a_{14}], [0, a_{12}, -a_{13}, a_{14}], [0, a_{21}, 0, a_{23}, a_{24}], [0, a_{21}, 0, -a_{23}, a_{24}].$$

The plane containing the two pairs contains the third pair. In our example, the third pair is $[a_{41}, -a_{42}, a_{43}, 0], [a_{41}, -a_{42}, -a_{43}, 0]$. This is one of the 16 tropes. Its equation is $a_{24}x + a_{14}y - a_{12}w = 0$. Similarly, we find the equations of all 16 tropes

$$\begin{aligned}\pm a_{34}y \pm a_{42}z \pm a_{23}w &= 0,\\ \pm a_{34}x \pm a_{41}z \pm a_{13}w &= 0,\\ \pm a_{24}x \pm a_{41}y \pm a_{12}w &= 0,\\ \pm a_{23}x \pm a_{31}y \pm a_{12}z &= 0.\end{aligned}$$

Remark 10.3.22. For experts on K3 surfaces, let us compute the Picard lattice of a general Tetrahedroid. Let $\sigma : \tilde{S} \to S$ be a minimal resolution of S. Denote by h the class of the pre-image of a plane section of S and by $e_i, i = 1, \dots, 16$, the classes of the exceptional curves. Let c_1 and c_2 be the classes of the proper transforms of the conics C_1, C_2 cut out by one of the coordinate plane, say $x = 0$. We have

$$c_1 + c_2 = h - e_1 - e_2 - e_3 - e_4.$$

Obviously, $c_1 \cdot c_2 = 0$ and $h \cdot c_i = 2$ and $c_i^2 = -2$. Consider another coordinate plane and another pair of conics. We can write

$$c_3 + c_4 = h - e_5 - e_6 - e_7 - e_8.$$

This shows that the classes of the eight conics can be expressed as linear combinations of classes h, e_i and $c = c_1$. It is known that the Picard group of a general Kummer surface is generated by the classes e_i and the classes of tropes t_i satisfying $2t_i = h - e_{i_1} - \cdots - e_{i_6}$. The Picard group of a Tetrahedroid acquires an additional class c.

The Jacobian variety of a genus 2 curve C with two commuting involutions contains an elliptic curve, the quotient of C by one of the involutions. In the symmetric product $C^{(2)}$ it represents the graph of the involution. Thus, it is isogenous to the product of two elliptic curves.

Note that the pencil of quadrics passing through the set of eight points $(C_1 \cap C_2) \cup (C_3 \cap C_4)$ defines a pencil of elliptic curves on $\tilde{S}$ with the divisor class

$$2h - e_1 - e_2 - e_3 - e_4 - e_5 - e_6 - e_7 - e_8 = c_1 + c_2 + c_3 + c_4.$$

Since $c_1 \cdot c_2 = c_3 \cdot c_4 = 0$, Kodaira's classification of fibers of elliptic fibrations shows that c_1, c_2, c_3, c_4 are the classes of irreducible components of a fiber of type I_4. This implies that the four intersection points $(C_1 \cup C_2) \cap (C_3 \cup C_4)$ lie on the edges of the coordinate tetrahedron.

The parameters A, B, C, D used to parameterize Tetrahedroid surfaces can be considered as points on the cubic surface

$$A(A^2 - B^2 - C^2 - D^2) + 2BCD = 0.$$

One can write an explicit rational parameterization of this surface using the formulas

$$A = 2abc,\ B = a(b^2 + c^2),\ C = b(a^2 + c^2),\ D = c(a^2 + b^2).$$

The formulas describe a rational map $\mathbb{P}^2 \dashrightarrow \mathbb{P}^3$ of degree 2 given by the linear system of plane cubics with three base points $p_1 = [1, 0, 0], p_2 = [0, 1, 0], p_3 = [0, 0, 1]$. It extends to a degree 2 map from a del Pezzo surface of degree 6 onto a 4-nodal cubic surface. In fact, if one considers the standard Cremona involution $[a, b, c] \mapsto [a^{-1}, b^{-1}, c^{-1}]$, then we observe that the map factors through the quotient by this involution. It has four singular points corresponding to the fixed points

$$[a, b, c] = [1, 1, 1],\ [-1, 1, 1],\ [1, -1, 1],\ [1, 1, -1].$$

of the Cremona involution. The corresponding singular points are the points $[1, 1, 1, 1],\ [1, 1, -1, -1],\ [1, -1, 1, -1],\ [1, -1, -1, 1]$.

If we change the variables $X^2 = bcx^2, Y^2 = acy^2, X^2 = abx^2, W = w$, the equation

$$A(x^4 + y^4 + z^4 + w^4) + 2B(x^2w^2 + y^2z^2) + 2C(y^2w^2 + x^2z^2)$$
$$+2D(z^2w^2 + x^2y^2) = 0$$

is transformed to the equation

$$(X^2 + Y^2 + Z^2)(a^2X^2 + b^2Y^2 + c^2Z^2)-$$
$$[a^2(b^2 + c^2)X^2W^2 + b^2(c^2 + a^2)Y^2W^2 + c^2(a^2 + b^2)Z^2W^2] + a^2b^2c^2W^4 = 0,$$

or, equivalently,

$$\frac{a^2x^2}{x^2 + y^2 + z^2 - a^2w^2} + \frac{b^2y^2}{x^2 + y^2 + z^2 - b^2w^2} + \frac{c^2z^2}{x^2 + y^2 + z^2 - c^2w^2} = 0. \quad (10.45)$$

When a, b, c are real numbers, the real points $(x, y, z, 1) \in \mathbb{P}^3(\mathbb{R})$ on this surface describe the propagation of light along the interface between two different media. The real surface with Equation (10.45) is called a *Fresnel's wave surface*. It has four real nodes

$$\Big(\pm c\sqrt{\frac{a^2 - b^2}{a^2 - c^2}}, 0, \pm a\sqrt{\frac{b^2 - c^2}{a^2 - c^2}}, 1\Big),$$

where we assume that $a^2 > b^2 > c^2$. It has four real tropes given by planes $\alpha x + \beta y + \gamma z + w = 0$, where

$$(\alpha, \beta, \gamma, 1) = \Big(\pm\frac{c}{b^2}\sqrt{\frac{a^2-b^2}{a^2-c^2}}, 0, \pm\frac{a}{b^2}\sqrt{\frac{b^2-c^2}{a^2-c^2}}, 1\Big).$$

One of the two conics cut out on the surface by coordinate planes is a circle. On the plane $w = 0$ at infinity one of the conics is the ideal conic $x^2 + y^2 + z^2 = 0$.

10.3.5 The tangential complex of lines

In the construction of the harmonic line complex defined by two quadrics Q_1 and Q_2 the quadrics are not necessarily different. In the case when $Q_1 = Q_2 = Q$, the definition of a harmonic self-conjugate pair implies that the two points in the pair coincide, i.e., the line is tangent to the quadric. This is a special case of the *harmonic line complex*, the locus of tangent lines to a quadric.

Equation (10.43) gives us the equation of the tangential line complex of a quadric Q defined by a symmetric matrix $A = (a_{ij})$:

$$\sum a_{ij}a_{kl}(p_{ik}p_{jl} + p_{il}p_{jk}) = 0. \tag{10.46}$$

Proposition 10.3.23. *The tangential quadratic line complex X_Q associated to a nonsingular quadric Q in $\mathbb{P}^n$ is singular along the variety* $\mathrm{OG}(2, Q)$ *of lines contained in Q.*

Proof It is easy to see that a line $\mathcal{P}$ in $\mathbb{G} = G_1(\mathbb{P}^n)$ is a pencil of lines in some plane Π in $\mathbb{P}^n$. The plane Π intersects Q in a conic. If the line is general, then the conic is nonsingular, and the pencil $\mathcal{P}$ contains two points represented by lines in Π that are tangent to the conic. This confirms that X_Q is a quadratic complex. Now, assume that ℓ is contained in Q. A general line $\mathcal{P}$ in $\mathbb{G}$ containing ℓ contains only one point represented by a line in $\mathbb{P}^n$ tangent to Q, namely the line ℓ. This shows that $\mathcal{P}$ is tangent to X_Q at the point ℓ. Since $\mathcal{P}$ is a general line in $\mathbb{G}$, it shows that the tangent space of X_Q at ℓ coincides with the tangent space of $\mathbb{G}$ at ℓ. This implies that X_Q is singular at ℓ. Since X_Q is a quadratic complex, ℓ is a double point of X_Q. □

Let $\mathcal{T}_Q$ be the tangent bundle of Q and let $\sigma : |\mathcal{T}_Q| \to Q$ be its projectivization. The fiber of $|\mathcal{T}_Q|$ at a point $x \in Q$ consists of lines tangent to Q at x. This defines a natural birational morphism

$$\pi : \mathbb{P}(\mathcal{T}_Q^\vee) \to X_Q$$

which is a resolution of singularities of the tangential line complex. It is easy to see that $\mathrm{OG}(2, Q)$ is of codimension 2 in X_Q. Thus, the exceptional divisor of π is isomorphic to a $\mathbb{P}^1$-bundle over $\mathrm{OG}(2, Q)$.

Remark 10.3.24. One can identify $\bigwedge^2 \mathbb{C}^{n+1}$ with the Lie algebra $\mathfrak{so}(n+1)$ of the special orthogonal group $\mathrm{SO}(n+1)$ of the space $\mathbb{C}^{n+1}$ equipped with the dot-product symmetric bilinear form and the associated quadratic form q. The orthogonal group $\mathrm{SO}(n+1)$ acts naturally on its Lie algebra $\mathfrak{so}(n+1)$ by means of the adjoint representation. One can speak about adjoint orbits of $\mathrm{SO}(n+1)$ in $|\mathfrak{so}(n+1)|$. The variety $\mathrm{OG}(2, Q)$ of lines in $Q = V(q)$ is identified with the variety of 2-dimensional isotropic linear subspaces in $\mathbb{C}^{n+1}$. It is known that this variety is the unique closed orbit. It is called the *minimal adjoint orbit*. The adjoint orbits are ordered with respect to the relation that one orbit is contained in the closure of another orbit. The tangential line complex X_Q is a *supminimal adjoint orbit* in the sense that the minimal orbit is the only orbit contained in the boundary of its closure. Considered as linear operators, points in $\mathrm{OG}(2, Q)$ are operators A of rank 2 satisfying $A^2 = 0$, and points of X_Q are operators of rank 2 satisfying $A^3 = 0$ (see, for example, [45]). In particular, we see that the variety $\mathrm{OG}(2, Q)$ can be given by quadratic equations expressing the condition that the square of the matrix (p_{ij}) is equal to 0.

Thus, both orbits are *nilpotent orbits*, i.e., they are contained in the subvariety of nilpotent linear operators. We refer for the classification of nilpotent orbits to [163]. For classical Lie algebras $\mathfrak{sl}_{n+1}, \mathfrak{so}_{n+1}, \mathfrak{so}_{n+1}$, the nilpotent orbits are classified by partition of $n + 1$ defining the Jordan form of the linear operator. Thus, the minimal orbit $\mathrm{OG}(2, Q)$ corresponds to the partition $(2, 2, 1, \ldots, 1)$ and the supminimal orbit corresponds to the partition $(3, 1, \ldots, 1)$.

Replacing the Lie algebra $\mathfrak{so}(n+1)$ by any simple complex lie algebra $\mathfrak{g}$. we obtain a generalization of the tangential line complex X_Q and its singular locus $\mathrm{OG}(2, Q)$. The latter is the unique minimal adjoint orbit in $|\mathfrak{g}|$, the former is the unique supminimal adjoint orbit. Both of these orbits are nilpotent orbits, i.e. they are contained in the subvariety of nilpotent elements of the Lie algebra. An algebraic variety isomorphic to a minimal adjoint orbit for some simple Lie algebra $\mathfrak{g}$ is called an *adjoint variety*. The adjoint varieties and, in particular, the line complexes $\mathrm{OG}(2, Q)$ of lines in a nonsingular quadric, are Fano contact varieties. Recall that a complex manifold M is called a *contact manifold* if its tangent bundle T_M contains a corank one subbundle F such that the bilinear form $F \times F \to T_M/F$ defined by the Lie bracket is nondegenerate. It is conjectured that any Fano contact variety is isomorphic to an adjoint variety (see [49]).

10.3.6 Tetrahedral complex of lines

Consider the union of four planes in $\mathbb{P}^3$ which defines a coordinate tetrahedron in the space. Let q_1, q_2, q_3, q_4 be its vertices, $\ell_{ij} = \langle q_i, q_j \rangle$ be its edges and $\pi_i = \langle q_j, q_k, q_l \rangle$ be its faces. Let $[A, B] \in \mathbb{P}^1$ and $\mathfrak{C}$ be the closure of the set of lines in $\mathbb{P}^3$ intersecting the four faces at four distinct points with the cross-ratio equal to $[A, B]$. Here, we assume that the vertices of the tetrahedron are ordered in some way. It is easy to see that $\mathfrak{C}$ is a line complex. It is called a *tetrahedral line complex* or .

Proposition 10.3.25. *A tetrahedral line complex $\mathfrak{C}$ is of degree 2. If p_{ij} are the Plücker coordinates with respect to the coordinates defined by the tetrahedron, then $\mathfrak{C}$ is equal to the intersection of the Grassmannian with the quadric*

$$Ap_{12}p_{34} - Bp_{13}p_{24} = 0. \tag{10.47}$$

Conversely, this equation defines a tetrahedral line complex.

Proof Let ℓ be a line spanned by the points $[a_1, a_2, a_3, a_4]$ and $[b_1, b_2, b_3, b_4]$. It intersects the face π_i at the point corresponding to the coordinates on the line $[s, t] = [b_i, -a_i], i = 1, \ldots, 4$. We assume that ℓ does not pass through one of the vertices. Then, ℓ intersects the faces at four points not necessarily distinct with the cross-ratio equal to $[p_{12}p_{34}, p_{13}p_{24}]$, where p_{ij} are the Plücker coordinates of the line. So, the equation of the tetrahedral line complex containing the line is $[p_{12}p_{34}, p_{13}p_{24}] = [a, b]$ for some $[a, b] \in \mathbb{P}^1$. □

Note that any tetrahedral line complex $\mathfrak{C}$ contains the set of points in $G(2, 4)$ satisfying $p_{is} = p_{it} = p_{ik} = 0$ (the lines in the coordinate plane $t_i = 0$). Also, any line containing a vertex satisfies $p_{ij} = p_{jk} = p_{ik} = 0$ and hence also is contained in $\mathfrak{C}$. Thus, we obtain that $\mathfrak{C}$ contains four planes from one ruling of the Klein quadric and four planes from another ruling. Each plane from one ruling intersects three planes from another ruling along a line and does not intersect the fourth plane.

Observe that the tetrahedral line complex is reducible if and only if the corresponding cross-ratio is equal to $0, 1, \infty$. In this case it is equal to the union of two hyperplanes representing lines intersecting one of the two opposite edges. An irreducible tetrahedral line complex has six singular points corresponding to the edges of the coordinate tetrahedron. Their Plücker coordinates are all equal to zero except one.

Proposition 10.3.26. *The singular surface of an irreducible tetrahedral line complex $\mathfrak{C}$ is equal to the union of the faces of the coordinate tetrahedron.*

Proof We know that the degree of the singular surface is equal to 4. So, it suffices to show that a general point in one of the planes of the tetrahedron belongs to the singular surface. The lines in this plane belong to the complex. So, a line in the plane passing through a fixed point p_0 is an irreducible component of the conic $\Omega(p_0) \cap \mathfrak{C}$. This shows that p_0 belongs to the singular surface of $\mathfrak{C}$. ◻

From now on, we consider only irreducible tetrahedral line complexes. There are different geometric ways to describe a tetrahedral complex.

First, we need the following fact, known as *von Staudt's Theorem* (see [720].

Theorem 10.3.27. *Let ℓ be a line belonging to a tetrahedral line complex $\mathfrak{C}$ defined by the cross-ratio R. Assume that ℓ does not pass through the vertices and consider the pencil of planes through ℓ. Then, the cross-ratio of the four planes in the pencil passing through the vertices is equal to R.*

Proof Let e_1, e_2, e_3, e_4 be a basis in $E = \mathbb{C}^4$ corresponding to the vertices of the tetrahedron. Choose the volume form $\omega = e_1 \wedge e_2 \wedge e_3 \wedge e_4$ and consider the *star-duality* in $\bigwedge^2 E$ defined by $(\alpha, \beta) = (\alpha \wedge \beta)/\omega$. Under this duality $(e_i \wedge e_j, e_k \wedge e_l) = 1(-1)$ if (i, j, k, l) is an even (odd) permutation of $(1, 2, 3, 4)$ and 0 otherwise. Let $\gamma = \sum_{1 \le i < j \le 4} p_{ij} e_i \wedge e_j$ be the 2-form defining the line ℓ and let $\gamma^* = \sum p'_{ij} e_i \wedge e_j$ define the dual line ℓ^*, where $e_i \wedge e_j$ is replaced with $(e_i \wedge e_j, e_k \wedge e_l) e_k \wedge e_i$, where i, j, k, l are all distinct. The line ℓ (resp. ℓ^*) intersects the coordinate planes at the points represented by the columns of the matrix

$$A = \begin{pmatrix} 0 & p_{12} & p_{13} & p_{14} \\ -p_{12} & 0 & p_{23} & p_{24} \\ -p_{13} & -p_{23} & 0 & p_{34} \\ -p_{14} & -p_{24} & -p_{34} & 0 \end{pmatrix}, \text{ resp. } B = \begin{pmatrix} 0 & p_{34} & -p_{24} & p_{23} \\ -p_{34} & 0 & p_{14} & -p_{13} \\ p_{24} & -p_{14} & 0 & p_{12} \\ -p_{23} & p_{13} & -p_{12} & 0 \end{pmatrix}.$$

We have $A \cdot B = B \cdot A = 0$. It follows from the proof of the previous proposition that the cross-ratio of the four points on ℓ^* is equal to $(p'_{13}p'_{24}, p'_{12}p'_{34}) = (p_{24}p_{13}, p_{24}p_{13})$. Thus, ℓ and ℓ^* belong to the same tetrahedral line complex. Now, a plane containing ℓ can be identified with a point on ℓ^* equal to the intersection point. A plane containing e_1 and ℓ is defined by the 3-form

$$e_1 \wedge \gamma = p_{23} e_1 \wedge e_2 \wedge e_3 + p_{24} e_1 \wedge e_2 \wedge e_4 + p_{34} e_1 \wedge e_3 \wedge e_4$$

and we check that $e_1 \wedge \gamma \wedge (-p_{34}e_2 + p_{24}e_3 - p_{23}e_4) = 0$ since $B \cdot A = 0$. This means that the plane containing e_1 intersects ℓ^* at the first point on ℓ^* defined by the first column. Thus, under the projective map from the pencil of planes through ℓ to the line ℓ^*, the plane containing e_1 is mapped to the intersection point of ℓ^* with the opposite face of the tetrahedron defined by $t_0 = 0$. Similarly, we check that the planes containing other vertices correspond to intersection points of ℓ^* with the opposite faces. This proves the assertion. □

Proposition 10.3.28. *A tetrahedral line complex is equal to the closure of the set of secants of rational cubic curves in $\mathbb{P}^3$ passing through the vertices of the coordinate tetrahedron.*

Proof Let R be one of those curves and $x \in R$. Projecting from x we get a conic C in the plane with four points, the projections of the vertices. Let $\ell = \langle x, y\rangle$ be a secant of R. The projection $\bar{y}$ of y is a point on the conic C and the pencil of lines through $\bar{y}$ is projectively equivalent to the pencil of planes through the secant ℓ. Under this equivalence, the planes passing through the vertices of the tetrahedron correspond to the lines connecting their projection with $\bar{y}$. Applying von Staudt's Theorem, we conclude the proof. □

Consider the action of the torus $T = (\mathbb{C}^*)^4$ on $\mathbb{P}^3$ by scaling the coordinates in $E = \mathbb{C}^4$. Its action on $\bigwedge^2 E$ is defined by

$$(t_1, t_2, t_3, t_4) : (p_{12}, \ldots, p_{34}) \mapsto (t_1t_2p_{12}, \ldots, t_3t_4p_{34}).$$

It is clear that the Klein quadric is invariant with respect to this action. This defines the action of T on the Grassmannian of lines. It is also clear that the equations of a tetrahedral line complex $\mathfrak{C}$ are also invariant with respect to this action, so T acts on a tetrahedral complex. If $\ell \in \mathfrak{C}$ has nonzero Plücker coordinates (a general line), then the stabilizer of ℓ is equal to the kernel of the action of T in $\mathbb{P}^3$, i.e. equal to the diagonal group of $(z, z, z, z), z \in \mathbb{C}^*$. Hence, the orbit of ℓ is 3-dimensional, and since $\mathfrak{C}$ is irreducible and 3-dimensional, it is a dense Zariski subset of $\mathfrak{C}$. Thus, we obtain that $\mathfrak{C}$ is equal to the closure of a general line in $G(2, 4)$ under the torus action. Since any general line belongs to a tetrahedral line complex, we get an equivalent definition of a tetrahedral line complex as the closure of a torus orbit of a line with nonzero Plücker coordinates.

Here, is another description of a tetrahedral complex. Consider a projective automorphism $\phi : \mathbb{P}^3 \to \mathbb{P}^3$ with four distinct fixed points and let $\mathfrak{C}$ be the closure of lines $\langle x, \phi(x)\rangle$, where x is not a fixed point of ϕ. Let us see that $\mathfrak{C}$ is an irreducible tetrahedral complex. Choose the coordinates in $\mathbb{C}^4$ such that the matrix of ϕ is a diagonal matrix with four distinct eigenvalues λ_i. Then, $\mathfrak{C}$ is the closure of lines defined by 2-vectors $\gamma = A \cdot v \wedge v, v \in \mathbb{C}^4$. A straightforward computation shows that the Plücker coordinates of γ are equal to $p_{ij} = t_i t_j(\lambda_i - \lambda_j)$, where $(t_1, \ldots, t_4)$ are the coordinates of the vector v. Thus, if we take v with nonzero coordinates, we obtain that $\mathfrak{C}$ contains the torus orbit of the vector with nonzero Plücker coordinates $p_{ij} = \lambda_i - \lambda_j$. As we explained in above, $\mathfrak{C}$ is an irreducible tetrahedral complex.

It is easy to see that the map which assigns to a point $x \in \mathbb{P}^3$ the line $\langle x, \phi(x)\rangle$ defines a birational transformation $\Phi : \mathbb{P}^3 \dashrightarrow \mathfrak{C}$ with fundamental points at the fixed points of ϕ. It is given by quadrics. The linear system of quadrics through four general points in $\mathbb{P}^3$ is of dimension five and defines a rational map from $\mathbb{P}^3$ to $\mathbb{P}^5$. The pre-image of a general plane is equal to the intersection of three general quadrics in the linear system. Since there are four base points, we obtain that the residual intersection consists of four points. This implies that the linear system defines a map of degree 2 onto a quadric in $\mathbb{P}^5$ or a degree 1 map onto a threefold of degree 4. Since a tetrahedral line complex is obtained in this way and any four general points in $\mathbb{P}^3$ are projectively equivalent, we see that the image must be projectively isomorphic to a tetrahedral complex. Observe that the six lines joining the pairs of fixed points of ϕ are blown down to singular points of the tetrahedral complex. Also, we see the appearance of eight planes; four of these planes are the images of the exceptional divisors of the blow-up of $\mathbb{P}^3$ at the fixed points, and the other four are the images of the planes spanned by three fixed points. The blow-up of $\mathbb{P}^3$ is a small resolution of the tetrahedral complex.

There is another version of the previous construction. Take a pencil $\mathcal{Q}$ of quadrics with a nonsingular base curve. Consider a rational map $\mathbb{P}^3 \dashrightarrow G_1(\mathbb{P}^3)$ which assigns to a point $x \in \mathbb{P}^3$ the intersection of the polar planes $P_x(Q), Q \in \mathcal{Q}$. This is a line in $\mathbb{P}^3$ unless x is a singular point of one of quadrics in $\mathcal{Q}$. Under our assumption on the pencil, there are exactly four such points which we can take as the points $[1, 0, 0, 0], [0, 1, 0, 0], [0, 0, 1, 0], [0, 0, 0, 1]$. Thus, we see that the rational map is of the same type as in the previous construction and its image is a tetrahedral complex.

10.4 Ruled Surfaces

10.4.1 Scrolls

A *scroll* or a *ruled variety* is an irreducible subvariety S of $\mathbb{P}^n$ such that there exists an irreducible family X_0 of linear subspaces of dimension r sweeping S such that a general point of S lies in unique subspace from this family. We will also assume that each point is contained only in finitely many linear subspaces. Following classical terminology, the linear subspaces are called *generators*. Note that the condition that any point lies in finitely many generators excludes cones.

We identify X_0 with its image in the Grassmann variety $\mathbb{G} = G_r(\mathbb{P}^n)$. For any $x \in X_0$ let Λ_x denote the generator defined by the point x. The universal family

$$\{(x, p) \in X_0 \times \mathbb{P}^n : p \in \Lambda_x\}$$

is isomorphic to the incidence variety Z_{X_0} over X_0. The projection $Z_{X_0} \to \mathbb{P}^n$ is a finite morphism of degree 1 which sends the fibers of the projective bundle $\mathsf{Z}_{X_0} \to X_0$ to generators. For any finite morphism $\nu : X \to X_0$ of degree 1, the pull-back $\mathcal{E} = \nu^*(\mathcal{S}_{X_0}^\vee)$ defines the projective bundle $\mathbb{P}(\mathcal{E})$ and a finite morphism $\tilde{\nu} : \mathbb{P}(\mathcal{E}) \to \mathsf{Z}_{X_0}$ such that the composition $f : \mathbb{P}(\mathcal{E}) \to \mathsf{Z}_{X_0} \to S$ is a finite morphism sending the fibers to generators. Recall that the projection $\mathsf{Z}_G \to \mathbb{P}^n = |E|$ is defined by a surjection of the locally free sheave $\alpha : E^\vee \otimes \mathcal{O}_{\mathbb{G}} \to \mathcal{S}_{\mathbb{G}}^\vee$. Thus, the morphism $f : \mathbb{P}(\mathcal{E}) \to S \subset \mathbb{P}^N$ is defined by a surjection

$$\nu^*(\alpha) : E^\vee \otimes \mathcal{O}_X \to \mathcal{E}.$$

In particular, the morphism f is given by a linear system $|E^\vee| \subset |\mathcal{O}_{\mathbb{P}(\mathcal{E})}(1)|$.

We see that any scroll is obtained as the image of a birational morphism

$$f : \mathbb{P}(\mathcal{E}) \to |E|$$

defined by a linear subsystem of $|\mathcal{O}_{\mathbb{P}(\mathcal{E})}(1)|$. The linear system can be identified with the image of $E^\vee \to H^0(X, \mathcal{E})$ under the surjective map $E^\vee \otimes \mathcal{O}_X \to \mathcal{E}$. This map also gives a finite map $\nu : X \to X_0 \subset \mathbb{G}$. The base X of the projective bundle $\pi : \mathbb{P}(\mathcal{E}) \to X$ can be always assumed to be a normal variety. Then, $\nu : X \to X_0$ is the normalization map.

A scroll defined by the complete linear system $|\mathcal{O}_{\mathbb{P}(\mathcal{E})}(1)|$ is a linearly normal subvariety of $\mathbb{P}^n$. It is called a *normal scroll*. Any scroll is a projection of a normal scroll. Note that, in many textbooks, a normal scroll is assumed to be a nonsingular variety. In Chapter 8, we classified smooth rational normal two-dimensional scrolls.

A surjective map of locally free sheaves $\mathcal{E} \to \mathcal{F}$ defines a closed embedding $\mathbb{P}(\mathcal{F}) \hookrightarrow \mathbb{P}(\mathcal{E})$. If rank $\mathcal{F} = r' + 1$, the image of $\mathbb{P}(\mathcal{F})$ under the map $f : \mathbb{P}(\mathcal{E}) \to \mathbb{P}^n$ is an *r'-directrix* of the scroll, a closed subvariety intersecting each generator along an r'-plane. If $r' = 0$, we get a section of $\mathbb{P}(\mathcal{E})$. Its image is *directrix* of the scroll, a closed subvariety of the scroll that intersects each generator at one point. Note that not every directrix comes from a section; for example, a generator could be a directrix.

Suppose $\mathcal{E} \to \mathcal{E}_1$ and $\mathcal{E} \to \mathcal{E}_2$ are two surjective maps of locally free sheaves on a smooth curve X. Let $\mathcal{E} \to \mathcal{E}_1 \oplus \mathcal{E}_2$ be the direct sum of the maps and let $\mathcal{E}'$ be the image of this map which is locally free since X is a smooth curve. Assume that the quotient sheaf $(\mathcal{E}_1 \oplus \mathcal{E}_2)/\mathcal{E}'$ is a skyscraper sheaf. The surjection $\mathcal{E} \to \mathcal{E}'$ corresponds to a closed embedding $j : \mathbb{P}(\mathcal{E}') \hookrightarrow \mathbb{P}(\mathcal{E})$. We call the projective bundle $\mathbb{P}(\mathcal{E}')$ the *join* of $\mathbb{P}(\mathcal{E}_1)$ and $\mathbb{P}(\mathcal{E}_2)$. We will denote it by $\langle \mathbb{P}(\mathcal{E}_1), \mathbb{P}(\mathcal{E}_2)\rangle$. The compositions $\mathcal{E} \to \mathcal{E}' \to \mathcal{E}_i$ are surjective maps, hence the projections $\mathcal{E}' \to \mathcal{E}_i$ are surjective and, therefore, define closed embedding $\mathbb{P}(\mathcal{E}_i) \hookrightarrow \langle \mathbb{P}(\mathcal{E}_1), \mathbb{P}(\mathcal{E}_2)\rangle$.

It follows from (2.31) that

$$\omega_{\mathbb{P}(\mathcal{E})/X} \cong \pi^*(\det \mathcal{E})(-r-1). \tag{10.48}$$

If X admits a canonical sheaf ω_X, we get

$$\omega_{\mathbb{P}(\mathcal{E})} \cong \pi^*(\omega_X) \otimes \pi^* \det \mathcal{E})(-r-1). \tag{10.49}$$

Let $\eta = c_1(\mathcal{O}_{\mathbb{P}(\mathcal{E})}(1))$. Recall from Subsection 2.4.2 that the Chern classes $c_i(\mathcal{E})$ can be defined by using the identity in $H^*(\mathbb{P}(\mathcal{E}), \mathbb{Z})$

$$(-\eta)^{r+1} + \pi^*(c_1(\mathcal{E}))(-\eta)^r + \cdots + \pi^*(c_{r+1}(\mathcal{E})) = 0. \tag{10.50}$$

Let $d = \dim X$. Multiplying the previous identity by η^{d-1}, we get

$$\eta^{d+r} = \sum_{i=1}^{r+1} (-1)^i \pi^*(c_1(\mathcal{E}))\eta^{d+r-i}. \tag{10.51}$$

Assume that $d = \dim X = 1$. Then, $c_i(\mathcal{E}) = 0$ for $i > 1$ and $c_1(\mathcal{E})$ can be identified with the degree of $\det \mathcal{E}$ (the degree of $\mathcal{E}$). Since η intersects the class of a general fiber with multiplicity 1, we obtain

$$\eta^{r+1} = \deg \mathcal{E}. \tag{10.52}$$

Since $|\mathcal{O}_{\mathbb{P}(\mathcal{E})}(1)|$ gives a finite map of degree 1, the degree of the scroll $S = f(\mathbb{P}(\mathcal{E}))$ is equal to η^{r+1}. Also, $\mathcal{E} = \nu^*(\mathcal{S}_G^\vee)$, hence

$$\deg \mathcal{E} = \nu^*(c_1(\mathcal{S}_G^\vee)) = \nu^*(\sigma_1) = \deg \nu(X) = \deg X_0,$$

where the latter degree is taken in the Plücker embedding of G. This gives

$$\deg S = \deg X_0. \tag{10.53}$$

The formula is not anymore true if $d = \dim X > 1$. For example, if $d = 2$, we get the formula

$$\deg S = \xi^{r+2} = \pi^*(c_1(\mathcal{E}))\eta^{r+1} - \pi^*(c_2(\mathcal{E}))\eta^r$$

$$= \pi^*(c_1(\mathcal{E})^2 - c_2(\mathcal{E}))\eta^r = c_1^2(\mathcal{E}) - c_2(\mathcal{E}) = \nu^*(\sigma_2),$$

where σ_2 is the special Schubert class.

Example 10.4.1. Exercise 19.13 from [375] asks us to show that the degree of S_X may not be equal to $\deg X_0$ if $\dim X_0 > 1$. An example is the scroll S of lines equal to the Segre variety $\mathsf{s}_{2,1}(\mathbb{P}^2 \times \mathbb{P}^1) \subset \mathbb{P}^5$. Its degree is equal to 3. If we identify the space $\mathbb{P}^5$ with the projective space of one-dimensional subspaces of the space of matrices of size 2×3, the Segre variety is the subvariety of matrices of rank 1. If we take homogeneous coordinates t_0, t_1, t_2 in $\mathbb{P}^2$ and homogeneous coordinates z_0, z_1 in $\mathbb{P}^1$, then S is given by

$$\text{rank}\begin{pmatrix} t_0 z_0 & t_1 z_0 & t_2 z_0 \\ t_0 z_1 & t_1 z_1 & t_2 z_1 \end{pmatrix} \leq 1.$$

When we fix (t_0, t_1, t_2), the parametric equation of the corresponding line in $\mathbb{P}^5$ is $z_0[t_0, t_1, t_2, 0, 0, 0] + z_1[0, 0, 0, t_0, t_1, t_2]$. The Plücker coordinates of the line are equal to $p_{i4+j} = t_i t_j, 0 \leq i \leq j \leq 2$, with other coordinates equal to zero. Thus, we see that the variety X parameterizing the generators of S spans a subspace of dimension 5 in $\mathbb{P}^9$ and is isomorphic to a Veronese surface embedded in this subspace by the complete linear system of quadrics. This shows that the degree of X is equal to 4.

From now on we shall assume that $X = C$ is a smooth curve C so that the map $\nu : C \to C_0 \subset G_r(\mathbb{P}^n)$ is the normalization map of the curve C_0 parameterizing generators.

Let S_1 and S_2 be two scrolls in $|E|$ corresponding to vector bundles $\mathcal{E}_1$ and $\mathcal{E}_2$ of ranks r_1 and r_2 and surjections $E^\vee \otimes \mathcal{O}_X \to \mathcal{E}_1$ and $E^\vee \otimes \mathcal{O}_X \to \mathcal{E}_2$. Let $\langle \mathbb{P}(\mathcal{E}_1), \mathbb{P}(\mathcal{E}_2) \rangle$ be the join in $\mathbb{P}(E^\vee \otimes \mathcal{O}_X) = X \times |E|$ and let S be the projection of the join to $|E| = \mathbb{P}(E^\vee)$. It is a scroll in $|E|$ whose generators are the joins of the corresponding generators of S_1 and S_2. Let $\{x_1, \ldots, x_m\}$ be the support of the sheaf $\mathcal{E}_1 \oplus \mathcal{E}_2/\mathcal{E}'$ and let h_i be the dimension of the quotient at the point x_i. Two generators corresponding to a point $x \notin \{x_1, \ldots, x_m\}$ span a linear subspace of expected dimension $r_1 + r_2 + 1$. The generators corresponding to a point x_j span a subspace of dimension $r_1 + r_2 - h_1$. The scroll S is denoted by $\langle S_1, S_2 \rangle$ and is called the *join* of scrolls S_1 and S_2. Since $\deg \mathcal{E}' = \deg \mathcal{E}_1 + \deg \mathcal{E}_2$, we obtain

$$\deg\langle S_1, S_2 \rangle = \deg S_1 + \deg S_2 - \sum_{i=1}^{m} h_i. \tag{10.54}$$

Let us consider some special examples.

Example 10.4.2. Let $E_i^\vee \otimes \mathcal{O}_C \to \mathcal{E}_i$ define scrolls S_i in $|E_i|, i = 1, 2$. Consider the surjection $E^\vee \otimes \mathcal{O}_C = (E_1^\vee \oplus E_2^\vee) \otimes \mathcal{O}_C \to \mathcal{E}_1 \oplus \mathcal{E}_2$. It defines the scroll equal to the join of the scroll $S_1 \subset |E_1| \subset |E|$ and the scroll $S_2 \subset |E_2| \subset |E|$. Its degree is equal to $\deg S_1 + \deg S_2$. For example, let $\mathcal{E}_i$ be an invertible sheaf on C defining a closed embedding $\tau_i : C \subset |E_i|$ so that $S_i = \tau_i(C)$ are curves of degree a_i spanning E_i. Then, the join of S_1 and S_2 is a surface of degree $a_1 + a_2$ with generators parameterized by C. Specializing further, we take $C = \mathbb{P}^1$ and $\mathcal{E}_i = \mathcal{O}_{\mathbb{P}^1}(a_i)$ with $a_1 \le a_2$. The scroll $\langle S_1, S_2\rangle$ is the rational normal scroll S_{a_1,a_1+a_2-1}. Iterating this construction we obtain rational normal scrolls $S_{a_1,\ldots,a_k,n} \subset \mathbb{P}^n$, where $n = a_1 + \cdots + a_k - k + 1$.

Example 10.4.3. Suppose we have two scrolls S_1 and S_2 in $\mathbb{P}^n = |E|$ defined by surjections $\alpha_i : E^\vee \otimes \mathcal{O}_{C_i} \to \mathcal{E}_i$, where rank $\mathcal{E}_i = r_i + 1$. Let $\ell_0 \subset C_1 \times C_2$ be a correspondence of bidegree (α_1, α_2) and let $\mu : \ell \to \ell_0$ be its normalization map. Let $p_i : \ell \to C_i$ be the composition of μ and the projection maps $C_1 \times C_2 \to C_i$. Consider the surjections $p_i^*(\alpha_i) : E^\vee \otimes \mathcal{O}_\ell \to p_i^*\mathcal{E}_i$. Let $\langle \mathbb{P}(p_1^*\mathcal{E}_1), \mathbb{P}(p_1^*\mathcal{E}_1)\rangle$ be the corresponding join. Let S be the image of the join in $|E|$. We assume that it is a scroll whose generators are parameterized by an irreducible curve $C_0 \subset G_{r_1+r_2-1}(|E|)$ equal to the closure of the image of the map $\phi : \ell \to G_{r_1+r_2-1}(|E|)$ defined by $\phi(z) = \langle \nu_1(p_1(z)), \nu_2(p_1(z))\rangle$. Let a be the degree of this map. Then,

$$\deg S = \frac{1}{a}(\alpha_1 \deg S_1 + \alpha_2 \deg S_2 - h),$$

where

$$h = h^0(\mathrm{Coker}(\mu^*(E^\vee \otimes \mathcal{O}_\ell \to p_1^*\mathcal{E}_1 \oplus p_2^*\mathcal{E}_2))).$$

Here, are some special examples. We can take for S_1 and S_2 two isomorphic curves in $\mathbb{P}^n$ of degrees d_1 and d_2 intersecting transversally at m points $x_1, \ldots, x_m$. Let Γ be the graph of an isomorphism $\sigma : S_1 \to S_2$. Let h be the number of points $x \in S_1$ such that $\sigma(x) = x$. Obviously, these points must be among the points x_i's. Assume that x_1 and $\sigma(t_1)$ do not lie on a common trisecant for a general point $x_1 \in S_1$. Then, $h^0 = 1$ and the scroll S is a scroll of lines of degree $d_1 + d_2 - h$. We could also take $S_1 = S_2$ and σ be an automorphism of S_1 with h fixed points. Then, the degree of the scroll S is equal to $2d - h$ if σ^2 is not equal to the identity and $\frac{1}{2}(2d - h)$ otherwise.

10.4.2 Cayley-Zeuthen formulas

From now on, until the end of this chapter, we will be dealing only with scrolls with one-parameter family C_0 of generators. A two-dimensional scroll is called a *ruled surface*. This classical terminology disagrees with the modern one, where a ruled surface means a $\mathbb{P}^1$-bundle $\mathbb{P}(\mathcal{E})$ over a smooth projective curve (see [379]). Our ruled surfaces are their images under a degree 1 morphism given by a linear system in $|\mathcal{O}_{\mathbb{P}(\mathcal{E})}(1)|$.

Let $\nu : C \to C_0$ be the normalization map and $\mathcal{E} = \nu^*(\mathcal{S}_{C_0}^\vee)$. The projective bundle $\mathbb{P}(\mathcal{E})$ is isomorphic to the normalization of the ruled surface S defined by the curve $C_0 \subset G_1(\mathbb{P}^n)$.

Let us recall some known facts about projective 1-bundles $X = \mathbb{P}(\mathcal{E})$ over smooth curves that can be found in [379, Chapter V, §2].

After tensoring $\mathcal{E}$ with an appropriate invertible sheaf we may assume that $\mathcal{E}$ is *normalized* in the sense that $H^0(C, \mathcal{E}_0) \neq \{0\}$ but $H^0(C, \mathcal{E}_0 \otimes \mathcal{L}) = \{0\}$ for any invertible sheaf $\mathcal{L}$ of negative degree. In this case, the integer $e = -\deg \mathcal{E} \geq 0$ is an invariant of the surface and the tautological invertible sheaf $\mathcal{O}_{\mathbb{P}(\mathcal{E})}(1)$ is isomorphic to $\mathcal{O}_X(E_0)$, where $E_0^2 = -e$. If $e < 0$ the curve E_0 is the unique curve on X with negative self-intersection. It is called the *exceptional section*.

Let $\sigma_0 : C \to X$ be the section of $\pi : X \to C$ with the image equal to E_0. Then, $\sigma_0^* \mathcal{O}_X(E_0) \cong \mathcal{O}_C(\mathfrak{e})$. If we identify E_0 and C by means of σ_0, then $\mathcal{O}_C(\mathfrak{e}) \cong \mathcal{O}_X(E_0) \otimes \mathcal{O}_{E_0}$ so that $\deg \mathfrak{e} = e$. A section $\sigma : C \to X$ is equivalent to a surjection of locally free sheaves $\mathcal{E} \to \mathcal{L} \cong \sigma^* \mathcal{O}_X(\sigma(C))$. In particular, $\deg \mathcal{L} = \sigma(C)^2$. The canonical class of X is given by the formula

$$K_X \sim -2E_0 + \pi^*(K_C + \mathfrak{e}), \tag{10.55}$$

which is a special case of (10.48).

Let $|H|$ be a complete linear system of dimension $N > 2$ on $\mathbb{P}(\mathcal{E})$ defined by an ample section H. Since $\pi_*(\mathcal{O}_X(H)) = \mathcal{E} \otimes \mathcal{L}$ for some invertible sheaf $\mathcal{L}$, we can write

$$H \sim E_0 + \pi^*(\mathfrak{a})$$

for some effective divisor class $\mathfrak{a}$ on C of degree a. Since H is irreducible, intersecting both sides with E_0 we find that $a \geq e$. Using the Moishezon-Nakai criterion of ampleness it is easy to see that H is ample if and only if $a > e$. We shall assume that H is ample. Assume also that $\mathfrak{a}$ is not special in the sense that $H^1(C, \mathcal{O}_C(\mathfrak{a})) = 0$ and $|\mathfrak{e} + \mathfrak{a}|$ has no base points on C. Then, the exact sequence

$$0 \to \mathcal{O}_X(\pi^*(\mathfrak{a})) \to \mathcal{O}_X(H) \to \mathcal{O}_{E_0}(H) \to 0$$

shows that the restriction of $|H|$ to E_0 is a complete linear system without base points. It is clear that any possible base point of $|H|$ must lie on E_0; hence under the assumptions from above, $|H|$ has no base points. It defines a finite map $f : X \to S \subset \mathbb{P}^N$. The surface S is a linearly normal surface in $\mathbb{P}^N$ swept by lines, the images of fibers. The family of lines is defined by the image of C in $G_1(\mathbb{P}^N)$. The next proposition shows that the map is of degree 1, hence, S is a ruled surface.

Proposition 10.4.4. *Let H be an ample section on $X = \mathbb{P}(\mathcal{E})$ as above and $|V|$*

be a linear system in $|H|$ that defines a finite map $f : \mathbb{P}(\mathcal{E}) \to S \subset \mathbb{P}^N$. Then, the degree of the map is equal to one.

Proof Suppose $f(x) = f(y)$ for some general points $x, y \in X$. Let F_x and F_y be the fibers containing x and y. Since $|H|$ has no base points, its restriction to any fiber is a linear system of degree one without base points. Suppose the degree of the map is greater than 1. Take a general fiber F; then, for any general point $x \in F$, there is another fiber F_x such that $f(F_x)$ and $f(F)$ are coplanar. This implies that there exists a divisor $H(x) \in |H - F_x - F|$. We can write $H(x) = F_x + F + R(x)$ for some curve $R(x)$ such that $R(x) \cdot F_x = R(x) \cdot F = 1$. When we move x along F we get a pencil of divisors $H(x)$ contained in $|H - F|$. The divisors of this pencil look like $F_x + R(x)$ and hence all have a singular point at $R(x) \cap F_x$. Since the fiber F_x moves with x, we obtain that a general member of the pencil has a singular point that is not a base point of the pencil. This contradicts Bertini's Theorem on singular points [379, Chapter III, Corollary 10.9]. □

Corollary 10.4.5. *Let S be an irreducible surface in $\mathbb{P}^N$ containing a one-dimensional irreducible family of lines. Suppose S is not a cone. Then, S is a ruled surface equal to the image of projective bundle $\mathbb{P}(\mathcal{E})$ over a smooth curve C under a birational morphism given by a linear subsystem in $|\mathcal{O}_{\mathbb{P}(\mathcal{E})}(1)|$.*

Proof Let $C_0 \subset G_1(\mathbb{P}^n)$ be the irreducible curve parameterizing the family of lines and let $\nu : C \to C_0$ be its normalization. The pre-image of the universal family $Z_{C_0} \to C_0$ is a projective bundle $\mathbb{P}(\mathcal{S}_{C_0}^\vee)$ over C. Since S is not a cone, the map $f : \mathbb{P}(\mathcal{E}) \to S$ is a finite morphism. The map is given by a linear subsystem of $|\mathcal{O}_{\mathbb{P}(\mathcal{E})}(1)|$. Since f is a finite morphism, the line bundle $\mathcal{O}_{\mathbb{P}(\mathcal{E})}(1) = f^*(\mathcal{O}_{\mathbb{P}(\mathcal{S}_{C_0})^\vee}(1))$ is ample. It remains to apply the previous Proposition. □

An example of a nonsingular quadric surface seems contradicts the previous statement. However, the variety of lines on a nonsingular quadric surface is not irreducible and consists of two projective lines embedded in $G_1(\mathbb{P}^3)$ as the union of two disjoint conics. So, the surface has two systems of rulings, and it is a 2-way scroll.

It follows from (10.53) that the degree of the ruled surface $S = f(\mathbb{P}(\mathcal{E}))$ is equal to the degree of C in the Plücker space. It is also equal to the self-intersection H^2 of the tautological line bundle on $\mathbb{P}(\mathcal{E})$. The latter is equal to $H^2 = (E_0 + aF)^2 = 2a - e$. The genus of C is called the *genus* of the ruled surface.

Proposition 10.4.6. *Let $S = f(\mathbb{P}(\mathcal{E})) \subset \mathbb{P}^n$ be a projection of a minimal* *ruled surface* $\mathbb{P}(\mathcal{E})$ *embedded in projective space by a linear system* $|H|$*, where* $H \sim E_0 + \pi^*(\mathfrak{a})$*. Let* D *be a directrix on* S *that is not contained in the singular locus of* S*. Then,*

$$\deg \mathsf{D} \geq \deg \mathfrak{a} - e.$$

The equality takes place if and only if the pre-image of D *on* $\mathbb{P}(\mathcal{E})$ *is in the same cohomology class as* E_0*.*

Proof The assumption on D implies that $\deg \mathsf{D} = H \cdot E$, where E is the pre-image of D on $\mathbb{P}(\mathcal{E})$. Intersecting with H we get $H \cdot E = E \cdot E_0 + a$. If $E \neq E_0$, then $H \cdot E \geq a$, if $[E] = [E_0]$, then $E \cdot E_0 = a - e$. The equality takes place if and only if $E \cdot E_0 = 0$ and $e = 0$. Since E is a section, we can write $[E] = [E_0] + m[F]$, and intersecting with E_0, we get $m = 0$. □

Since $f : \mathbb{P}(\mathcal{E}) \to S$ is of degree 1, the ruled surface is non-normal at every point over which the map is not an isomorphism.

Recall the *double-point formula* from [315, 9.3]. Let $f : X \to Y$ be a morphism of nonsingular varieties of dimensions m and n, respectively. Let Z be the blow-up of the diagonal of $X \times X$ and let R be the exceptional divisor. We think about points in R as points in X together with a tangent direction t_x at x. Let $\tilde{D}(f)$ be the proper transform in Z of the fibered product $X \times_Y X \subset X \times X$. One can view points in $\tilde{D}(f)$ either as points $x \in X$ such that there exists $x' \neq x$ with $f(x) = f(x')$, or as points (x, t_x) such that $df_x(t_x) = 0$. Let $D(f)$ be the image of $\tilde{D}(f)$ under one of the projections $X \times_Y X \to X$. This is called the *double-point set* of the morphism f. Define the *double point class*

$$\mathbb{D}(f) = f^* f_*[X] - (c(f^*\mathcal{T}_Y)c(\mathcal{T}_X)^{-1})_{n-m} \cap [X] \in H^{n-m}(X, \mathbb{Q}), \quad (10.56)$$

where c denotes the total Chern class $[X] + c_1 + \cdots + c_m$ of a vector bundle. In case $D(f)$ has the expected dimension equal to $2m - n$, we have

$$\mathbb{D}(f) = [D(f)] \in H^{n-m}(X, \mathbb{Z}).$$

Assume now that $f : X \to S$ is the normalization map and S is a surface in $\mathbb{P}^3$. Since S is a hypersurface, it does not have isolated non-normal points. This implies that $D(f)$ is either empty or it is of expected dimension $2m - n = 1$. The double-point class formula applies, and we obtain

$$[D(f)] = f^*(S) + f^*(K_{\mathbb{P}^3}) - K_X. \quad (10.57)$$

In fact, it follows from the theory of adjoints (see [479]) that the linear equivalence class of $D(f)$ is expressed by the same formula.

We say that a non-normal surface S in $\mathbb{P}^n$ has *ordinary singularities* if its singular locus is a double curve ℓ on S. This means that the completion of the local ring of S at a general point of ℓ is isomorphic to $\mathbb{C}[[z_1, z_2, z_3]]/(z_1 z_2)$. The curve ℓ may have also *pinch point* locally isomorphic to $\mathbb{C}[[z_1, z_2, z_3]]/(z_1^2 - z_2^2 z_3)$ and also triple points locally isomorphic to $\mathbb{C}[[z_1, z_2, z_3]]/(z_1 z_2 z_3)$. The curve ℓ is nonsingular outside triple points, the curve $D(f)$ is nonsingular outside the pre-images of the triple points. It has three double points over each triple point.

Under these assumptions, the map $\tilde{D}(f) \to D(f) \to \ell$ is of degree 2. It is ramified at pinch points only, and the pre-image of a triple point consists of six points.

Assume that S is a surface in $\mathbb{P}^3$ with ordinary singularities. Let $f : X \to S$ be the normalization map, and ℓ be the double curve of S. The degree of any curve on X, is the degree with respect to $f^*(\mathcal{O}_{\mathbb{P}^3}(1))$. Let us introduce the following numerical invariants in their classical notation:

- μ_0 = the degree of S;
- μ_1 = the rank of S, the class of a general plane section of S;
- μ_2 = the class of S;
- ν_2 = the number of pinch-points on S;
- t = the number of triple points on S;
- $\epsilon_0 = \deg \ell$;
- ϵ_1 = the rank of ℓ, the number of tangents to ℓ intersecting a general line in $\mathbb{P}^3$;
- ρ = the *class of immersion* of ℓ equal to the degree of the image of $D(f)$ under the Gauss map $G : X \to S \xrightarrow{\gamma} (\mathbb{P}^3)^\vee$, where γ is the Gauss map.;
- $g(\ell)$ = the genus of ℓ;
- c = the number of connected components of ℓ;
- κ = the degree of the ramification divisor $p : X \to \mathbb{P}^2$, where p is the composition of f and the general projection of S.

The following theorem summarizes different relations between the listed invariants of S. These relations are called the *Cayley-Zeuthen formulas*.

Theorem 10.4.7. *The following relations hold:*

(i) $\mu_1 = \mu_0(\mu_0 - 1) - 2\epsilon_0$;
(ii) $\epsilon_0(\mu_0 - 2) = \rho + 3t$;
(iii) $\mu_1(\mu_0 - 2) = \kappa + \rho$;
(iv) $2g(\ell) - 2c = \epsilon_1 - 2\epsilon_0$;
(v) $\nu_2 = 2\epsilon_0(\mu_0 - 2) - 6t - 2\epsilon_1$;
(vi) $2\rho - 2\epsilon_1 = \nu_2$;
(vii) $\mu_2 = \mu_0(\mu_0 - 1)^2 + (4 - 3\mu_0)\epsilon_1 + 3t - 2\nu_2$;

(viii) $2\nu_2 + \mu_2 = \mu_1 + \kappa$.

Proof (i) A general plane section of S is a plane curve of degree μ_0 with ϵ_0 ordinary double points as singularities. Thus, (i) is just the Plücker formula. Also, note that μ_1 is equal to the degree of the *contact curve*, the closure of smooth points $p \in S$ such that a general point $q \in \mathbb{P}^3$ is contained in $\mathbb{T}_p(S)$, or, equivalently, the residual curve to ℓ of the intersection of S and the first polar $P_q(S)$. Taking a general plane H and a general point $q \in H$, we obtain that $\deg \Delta$ is equal to the class of $H \cap S$.

(ii) The number ρ is equal to the number of tangent planes to S at points in ℓ that pass through a general point q in $\mathbb{P}^3$. Here, a tangent plane to a singular point $p \in \ell$ means the tangent plane to one of the two branches of S at q, or, equivalently, the image of a pre-image of p on X under the Gauss map. Consider the intersection of the second polar P_{q^2} with the contact curve ℓ. It follows from subsection 1.1.3 that $P_{q^2}(S) \cap S$ is equal to the locus of points p such that the line $\overline{pq}$ intersects S at p with multiplicity ≥ 3. This means that $P_{q^2}(S) \cap \ell$ consists of t triple points and points such that q belongs to a tangent plane of S at p. The latter number is equal to ρ. As we observed in subsection 1.1.3, P_{q^2} has a point of multiplicity 3 at p; hence each triple point enters with multiplicity 3 in the intersection of $\overline{pq}$ with ℓ. It remains to use that the degree of the second polar is equal to $\mu_0 - 2$.

(iii) Now, let us consider the intersection of the second polar $P_{q^2}(S)$ with the contact curve Δ. This intersection consists of the lines $\overline{qp}$ such that p is either one of κ ramification points of the projection of the surface from q or p is one of ρ points on $\ell \cap \Delta$, where the tangent plane contains p. In fact, these points lie on the intersection of Δ and ℓ.

(iv) - (vi) Let $\pi = h^1(\mathcal{O}_{D(f)})$ be the arithmetic genus of the curve $D(f)$ and let s be the number of connected components of $D(f)$. Applying (10.57), we get

$$-2\chi(D(f), \mathcal{O}_{D(f)}) = 2\pi - 2c = (D(f) + K_X) \cdot D(f)$$

$$= (\mu_0 - 4)\deg D(f) = 2\epsilon_0(\mu_0 - 4).$$

The curve $D(f)$ has $3t$ ordinary double points and the projection from the normalization of $D(f)$ to ℓ is a degree 2 cover ramified at ν_2 points. Applying the Riemann-Hurwitz formula, we obtain $2\pi - 2c - 6t = 2(2g(\ell) - 2c) + \nu_2$. Projecting ℓ from a general line defines a degree ϵ_0 map from the normalization of ℓ to $\mathbb{P}^1$. The number of ramification points is equal to ϵ_1. Applying the Riemann-Hurwitz formula again, we get $2g(\ell) - 2c = -2\epsilon_0 + \epsilon_1$. This gives

(iv) and also gives

$$\nu_2 = 2\epsilon_0(\mu_0 - 4) - 6t - 2(2g(\ell) - 2c) = 2\epsilon_0(\mu_0 - 4) - 6t - 2\epsilon_1 + 4\epsilon_0$$

$$= 2\epsilon_0(\mu_0 - 2) - 6t - 2\epsilon_1.$$

This is equality (v). It remains to use (ii) to get (vi).

(vii) The formula for the class of a non-normal surface with ordinary singularities has a modern proof in [315, Example 9.3.9]. In our notation, it gives (vii).

(viii) We have

$$\mu_2 = \mu_0(\mu_0 - 1)^2 + (4 - 3\mu_0)\epsilon_1 + 3t.$$

Using this and (i), we get

$$\mu_2 + 2\nu_2 = (\mu_0 - 1)(\mu_1 + 2\epsilon_0) + 4\epsilon_0 - 3\mu_0\epsilon_0$$

$$= \mu_0\mu_1 - \mu_1 + 2\epsilon_0 + \rho - \epsilon_0\mu_0 + 3t.$$

It remains to use (ii) and (iii). □

Corollary 10.4.8. *Let S be a surface in $\mathbb{P}^3$ with ordinary singularities and let X be its normalization. Then*

(i) $K_X^2 = \mu_0(\mu_0 - 4)^2 - (3\mu_0 - 16)\epsilon_0 + 3t - \nu_2$;
(ii) $c_2(X) = \mu_0(\mu_0^2 - 4\mu_0 + 6) - (3\mu_0 - 8)\epsilon_0 + 3t - 2\nu_2$;
(iii) $\chi(X, \mathcal{O}_X) = 1 + \binom{\mu_0 - 1}{3} - \frac{1}{2}(\mu_0 - 4)\epsilon_0 + \frac{1}{2}t - \frac{1}{4}\nu_2$.

Proof (i) Applying (10.57), we get

$$K_X = (\mu_0 - 4)H - D(f), \tag{10.58}$$

where $H \in |f^*(\mathcal{O}_{\mathbb{P}^3}(1))|$. The first polar of S with respect to a general point cuts out on S the union of ℓ and Δ. Taking the pre-images on X, we get

$$(\mu_0 - 1)H = D(f) + f^*(\Delta).$$

It follows from the local computation that ℓ and Δ intersect simply at ν_2 pinch points and ρ additional points (see the proof of (iii) in Theorem 10.4.7). This gives

$$D(f)^2 = (\mu_0 - 1)H \cdot D(f) - \rho - \nu_2 = 2\epsilon_0(\mu_0 - 1) - \rho - \nu_2$$

$$= 2\epsilon_0(\mu_0 - 1) - \epsilon_0(\mu_0 - 2) + 3t - \nu_2 = \epsilon_0(\mu_0 - 2) + 3t - \nu_2.$$

Hence

$$K_X^2 = (\mu_0 - 4)^2\mu_0 - 4(\mu_0 - 4)\epsilon_0 + D(f)^2$$

$$= (\mu_0 - 4)^2\mu_0 - (3\mu_0 - 16)\epsilon_0 + 3t - \nu_2.$$

(ii) The pre-image of a pinch point on X is a point in X such that the rank of the tangent map $\mathcal{T}_X \to f^*(\mathcal{T}_{\mathbb{P}^3})$ drops by 2. According to the modern theory of degeneracy loci (see [315]), this set is given by the relative second Chern class $c_2(f^*(\mathcal{T}_{\mathbb{P}^3})/\mathcal{T}_X)$. Computing this Chern class, we find

$$\nu_2 = c_1(X)^2 - c_2(X) + 4K_X \cdot H + 6\mu_0.$$

Applying (10.58), we get

$$\nu_2 = K_X^2 - c_2(X) + 4(\mu_0 - 4)\mu_0 - 8\epsilon_0 + 6\mu_0. \tag{10.59}$$

Together with (i) we get (ii). Formula (iii) follows from the *Noether formula*

$$12\chi(X, \mathcal{O}_X) = K_X^2 + c_2(X). \tag{10.60}$$

□

We know that μ_0 is equal to the degree d of C_0 in its Plücker embedding. The next theorem shows that all the numerical invariants can be expressed in terms of μ_0 and g.

Theorem 10.4.9. *Let S be a ruled surface in $\mathbb{P}^3$ of degree μ_0 and genus g. Assume that S has only ordinary singularities. Then*

(i) $\epsilon_0 = \frac{1}{2}(\mu_0 - 1)(\mu_0 - 2) - g$;
(ii) $\nu_2 = 2(\mu_0 + 2g - 2)$;
(iii) $\mu_1 = 2\mu_0 - 2 + 2g$;
(iv) $\mu_2 = \mu_0 = \mu_0$;
(v) $\kappa = 3(\mu_0 + 2g - 2)$;
(vi) $\rho = (\mu_0 - 2)(2\mu_0 - 5) + 2g(\mu_0 - 5)$;
(vii) $t = \frac{1}{6}(\mu_0 - 4)[(\mu_0 - 2)(\mu_0 - 3) - 6g]$;
(viii) $\epsilon_1 = 2(\mu_0 - 2)(\mu_0 - 3) + 2g(\mu_0 - 6)$;
(ix) $2g(\ell) - 2s = (\mu_0 - 5)(\mu_0 + 2g - 2)$.

Proof A general plane section of S is a plane curve of degree d with $k = \deg \ell$ ordinary singularities. This gives (i).

The canonical class formula gives

$$K_{\mathbb{P}(\mathcal{E})} = -2H + \pi^*(K_{\bar{C}} + \mathfrak{d}), \tag{10.61}$$

where $\mathcal{O}_{\bar{C}}(\mathfrak{d}) \cong \nu^*(\mathcal{O}_{C_0}(1))$ is of degree $d = \mu_0$.

Comparing it with formula (10.72), we find that

$$H \sim E_0 + \pi^*(\mathfrak{f}), \tag{10.62}$$

where $2\mathfrak{f} = \mathfrak{d} - \mathfrak{e}$. In particular, $e + d$ is always an even number.

Applying (10.61), we get $K^2_{\mathbb{P}(\mathcal{E})} = 4\mu_0 - 4(2g - 2 + \mu_0)$. Topologically, $\mathbb{P}(\mathcal{E})$ is the product of $\mathbb{P}^1$ and C. This gives $c_2(X) = 2(2 - 2g)$. Applying (10.59), we find

$$\nu_2 = 4\mu_0 - 4(2g-2+\mu_0) - 2(2-2g) + 4(\mu_0-4)\mu_0 - 4(\mu_0-1)(\mu_0-2) - 8g + 6\mu_0$$

$$= 2(\mu_0 + 2g - 2).$$

From (i) of Theorem 10.4.7, we get (iii).

To prove (iv) we have to show that the degree of S is equal to the degree of its dual surface. The image of a generator of S under the Gauss map is equal to the dual line in the dual $\mathbb{P}^3$, i.e. the set of hyperplanes containing the line. Since S has only finitely many torsor generators, the Gauss map is a birational map, this shows that S^* is a ruled surface. If S is defined by the vector bundle $\mathcal{E} = \mathcal{S}_G^\vee \otimes \mathcal{O}_{C_0}$, then the dual ruled surface is defined by the vector bundle $\mathcal{Q}_G \otimes \mathcal{O}_{C_0}$, where $\mathcal{O}_G$ is the universal quotient bundle. The exact sequence (11.1) shows that $\det \mathcal{Q}_G \otimes \mathcal{O}_{C_0} \cong \det \mathcal{S}_G^\vee \otimes \mathcal{O}_{C_0}$. In particular, the degrees of their inverse images under $\nu : C \to C_0$ are equal. Thus, the degrees of S and S^* are equal.

Now, (i) and (viii) of Theorem 10.4.7 and our formula (i) give (v). Using (iii) and (ii) of the same theorem, we get (vi) and (vii). Finally, (vi) gives (viii) and (ix). □

The double-point formula gives

$$\mathcal{O}_{\mathbb{P}(\mathcal{E})}(D(f)) \sim \mathcal{O}_{\mathbb{P}(\mathcal{E})}(\mu_0 - 2) \otimes \pi^*(\omega_C(1)).$$

A general point of ℓ is contained in two rulings and formula (10.57) implies that a general ruling intersects $\mu_0 - 2$ other rulings. Consider a symmetric correspondence on C defined by

$$T = \{(x, y) \in C \times C : |H - \ell_x - \ell_y| \neq \emptyset\}.$$

A point $(x, x) \in T$ corresponds to a generator that is called a *torsal generator*. The plane in $\mathbb{P}^3$ cutting out this generator with multiplicity ≥ 2 is tangent to the ruled surface at any smooth point of the generator. For a general point x, we have $\#T(x) = d - 2$. Since all generators $\ell_y, y \in T(x)$, intersect the same line ℓ_x the points $y \in T(x)$ lie in the tangent hyperplane of $G_1(\mathbb{P}^3)$ at the point x. This implies that the divisor $2x + T(x)$ belongs to the linear system $|\mathcal{O}_C(1)|$ and, in particular, T has valence equal to 2. Applying the Cayley-Brill formula from Corollary 5.5.2, we obtain the following.

Proposition 10.4.10. *The number of torsal generators of a ruled surface in* $\mathbb{P}^3$ *with ordinary singularities is equal to* $2(\mu_0 + 2g - 2)$.

Comparing with Theorem 10.4.9, we find that the number of torsal generators is equal to the number ν_2 of pinch points.

When $n = 4$, we expect that a ruled surface has only finitely many singular non-normal points, and for $n = 5$, we expect that it is nonsingular.

In Example 7.6.5, we already encountered a ruled surface S of degree 8 with a triple curve C as its singular curve. A general plane section of this surface is a plane curve of degree 8 of genus 3 with six triple points. Applying formula (10.57) we see that the linear equivalence class of the curve $D(f)$ is equal to $2H - \pi^*(K_C + \mathfrak{d})$ for some divisor $\mathfrak{d}$ of degree d. However, the curve $D(f)$ comes with multiplicity 2, so the curve C in S is the image of a curve $\tilde{C}$ on Z_C from the linear system $|H - \pi^*(\mathfrak{f})|$, where $2\mathfrak{f} \sim K_C + \mathfrak{d}$. So, each generator intersects it at three points, as expected. One can show that $\mathfrak{d} \sim K_C + 2\mathfrak{a}$, therefore, $\mathfrak{f} \sim K_C + \mathfrak{a}$. Note that the curve $\tilde{C}$ defines a $(3,3)$-correspondence on the curve C with the projections p_C and q_C to C. Its genus is equal to 19 and each projection is a degree three cover ramified at 24 points. In the case when the divisor $\mathfrak{a}$ is an even theta characteristic, the curve $\tilde{C}$ is the Scorza correspondence which we studied in Subsection 5.5.2.

The next example shows that the double curve of a ruled surface may be disconnected.

Example 10.4.11. Consider three nonsingular nondegenerate curves $X_i, i = 1,2,3$, in $\mathbb{P}^3$ with no two having common points. Let S be the set of lines intersecting each curve. Let us show that these lines sweep a ruled surface of degree $2d_1d_2d_3$, where $d_i = \deg C_i$. Recall that the set of lines intersecting a curve X of degree X is a divisor in $G_1(\mathbb{P}^3)$ of degree d. This is the *Chow form* of C (see [325]). Thus, the set of lines intersecting three curves is a complete intersection of three hypersurfaces in $G_1(\mathbb{P}^3)$, hence a curve of degree $2d_1d_2d_3$. Assuming that the curves are general enough so that the intersection is transversal, we obtain that the ruled surface must be of degree $2d_1d_2d_3$. The set of lines intersecting two curves X_1 and X_2 is a surface W in $G_1(\mathbb{P}^3)$ of degree $2d_1d_2$. Its intersection with the Schubert variety $\Omega(\Pi)$, where Π is a general plane, consists of d_1d_2 lines. It follows from the intersection theory on $G_1(\mathbb{P}^3)$ that the intersection of W with the α-plane $\Omega(p)$ is of degree d_1d_2. Therefore, we expect that, in a general situation, the number of generators of S passing through a general point on X_3 is equal to d_1d_2. This shows that a general point of X_3 is a singular point of multiplicity d_1d_2. Similarly, we show that X_1 is a singular curve of multiplicity d_2d_3 and X_2 is a singular curve of multiplicity d_1d_3.

Remark 10.4.12. According to [196], the double curve ℓ is always connected if $\mu_0 \geq g+4$. If it is disconnected, then it must be the union of two lines.

10.4.3 Developable ruled surfaces

A ruled surface is called *developable* if the tangent planes at nonsingular points of any ruling coincide. In other words, any generator is a torsal generator. One expects that the curve of singularities is a cuspidal curve. In this subsection, we will give other characterizations of developable surfaces.

Recall the definition of the vector bundle of *principal parts* on a smooth variety X. Let Δ be the diagonal in $X \times X$ and let $\mathcal{J}_\Delta$ be its sheaf of ideals. Let p and q be the first and the second projections to X from the closed subscheme Δ^m defined by the ideal sheaf $\mathcal{J}_\Delta^{m+1}$. For any invertible sheaf $\mathcal{L}$ on X, one defines the sheaf of m-th principal parts $\mathcal{P}^m(\mathcal{L})$ of $\mathcal{L}$ as the sheaf $\mathcal{P}_X^m(\mathcal{L}) = p_*q^*(\mathcal{L})$ on X. Recall that the m-th tensor power of the sheaf of 1-differentials Ω_X^1 can be defined as $p_*(\mathcal{J}_\Delta^m/\mathcal{J}_\Delta^{m+1})$ (see [379]). The exact sequence

$$0 \to \mathcal{J}_\Delta^m/\mathcal{J}_\Delta^{m+1} \to \mathcal{O}_{X\times X}/\mathcal{J}_\Delta^{m+1} \to \mathcal{O}_{X\times X}/\mathcal{J}_\Delta^m \to 0$$

gives an exact sequence

$$0 \to (\Omega_X^1)^{\otimes m} \otimes \mathcal{L} \to \mathcal{P}_X^m(\mathcal{L}) \to \mathcal{P}_X^{m-1}(\mathcal{L}) \to 0. \tag{10.63}$$

We will be interested in the case when $X_0 = C_0$ is an irreducible curve of genus g and $X = C$ is its normalization. By induction, the sheaf $\mathcal{P}_C^m(\mathcal{L})$ is a locally free sheaf of rank $m+1$, and

$$\deg \mathcal{P}_C^m(\mathcal{L}) = (m+1)\deg \mathcal{L} + m(m+1)(g-1). \tag{10.64}$$

For any subspace $V \subset H^0(C,\mathcal{L})$, there is a canonical homomorphism

$$V \to H^0(\Delta^m, q^*\mathcal{L}) = H^0(C, p_*q^*\mathcal{L}) = H^0(C, \mathcal{P}_C^m(\mathcal{L}))$$

which defines a morphism of locally free sheaves

$$\alpha_m : V_C := \mathcal{O}_C \otimes V \to \mathcal{P}_C^m(\mathcal{L}). \tag{10.65}$$

Note that the fiber of $\mathcal{P}_C^m(\mathcal{L})$ at a point x can be canonically identified with $\mathcal{L}/\mathfrak{m}_{C,x}^{m+1}\mathcal{L}$ and the map α_m at a point x is given by assigning to a section $s \in V$ the element $s \mod \mathfrak{m}_{C,x}^{m+1}\mathcal{L}$. If $m = 0$, we get $\mathcal{P}_C^m(\mathcal{L}) = \mathcal{L}$ and the map is the usual map given by evaluating a section at a point x.

Suppose that $(V,\mathcal{L})$ defines a morphism $f : C \to \mathbb{P}(V)$ such that the induced morphism $f : C \to f(C) = C_0$ is the normalization map. We have $\mathcal{L} = f^*(\mathcal{O}_\mathbb{P}(1))$. Let $\mathcal{P}^m \subset \mathcal{P}_C^m(\mathcal{L})$ be the image of α_m. Since the composition of α_1 with the projection $\mathcal{P}_C^1 \to \mathcal{L}$ is generically surjective (because C_0 spans

$\mathbb{P}(V)$), the map α_1 is generically surjective. Similarly, by induction, we show that α_m is generically surjective for all m. Since C is a smooth curve, this implies that the sheaves $\mathcal{P}^m$ are locally free of rank $m+1$. They are called the *osculating sheaves*. Let

$$\sigma_m : C \to G(m+1, V^\vee)$$

be the morphisms defined by the surjection $\alpha_m : V_C \to \mathcal{P}^m$. The morphism σ_m can be interpreted as assigning to each point $x \in C$ the m-th *osculating plane* of $f(C)$ at the point $f(x)$. Recall that it is a m-dimensional subspace of $\mathbb{P}(V)$ such that it has the highest-order contact with the branch of C_0 defined by the point $x \in C$. One can always choose a system of projective coordinates in $\mathbb{P}(V) \cong \mathbb{P}^n$ such that the branch of C_0 corresponding to x can be parameterized in the ring of formal power series by

$$t_0 = 1,\ t_i = t^{i+s_1+\cdots+s_i} + \text{ highest-order terms},\ i = 1, \ldots, n, \tag{10.66}$$

where $s_i \geq 0$. Then, the osculating hyperplane is given by the equation $t_n = 0$. The codimension 2 osculating subspace is given by $t_{n-1} = t_n = 0$ and so on. A point $x \in C$ (or the corresponding branch of $f(C)$) with $s_1 = \ldots = s_n = 0$ is called an *ordinary point*, other points are called *hyperosculating* or *stationary* points . It is clear that a point x is ordinary if and only if the highest order of tangency of a hyperplane at x is equal to n. For example, for a plane curve, a point is ordinary if the corresponding branch is nonsingular and not an inflection point.

The image $\sigma_m(C)$ in $G_m(\mathbb{P}^n)$ is called the m-th *associated curve*. Locally, the map σ_m is given by assigning to a point $x \in C$ the linear subspace of $\mathbb{C}^{n+1}$ generated by $\tilde{f}(x), \tilde{f}'(x), \ldots, \tilde{f}^{(m)}(x)$, where $\tilde{f} : C \to \mathbb{C}^{n+1}$ is a local lift of the map f to a map to the affine space, and $\tilde{f}^{(k)}$ are its derivatives (see [360], Chapter II, §4).

Let $\mathbb{P}(\mathcal{P}^m) \to C \times \mathbb{P}(V)$ be the morphism corresponding to the surjection α_m. The projection of the image to $\mathbb{P}(V)$ is called the *m-th osculating developable* of $(C, \mathcal{L}, V)$ (or of C_0). For $m = 1$ it is a ruled surface, called the *developable surface* or *tangential surface* of C_0.

Let r_m be the degree of $\mathcal{P}^m$. We have already observed that the composition of the map σ_m with the Plücker embedding is given by the sheaf $\det \mathcal{P}^m$. Thus, r_m is equal to the degree of the m-th associated curve of C_0. Also, we know that the degree of a curve in the Grassmannian $G(m+1, V^\vee)$ is equal to the intersection of this curve with the Schubert variety $\Omega(A)$, where $\dim A = n-m-1$. Thus, r_m is equal to the *m-rank* of C_0, the number of hyperosculating m-planes intersecting a general $(n-m-1)$-dimensional subspace of $\mathbb{P}(V)$. Finally, we know that the 1-rank r_1 (called the *rank* of C_0), divided by the number of tangents through a general point on the surface, is equal to the degree of the tangential surface. More generally, r_m is equal to the degree of the m-th osculating developable (see [582]). The $(n-1)$-rank r_{n-1} is called the *class* of C_0. If we consider the $(n-1)$-th associated curve in $G(n, n+1)$ as a curve in the dual projective space $|V|$, then the class of C_0 is its degree. The $(n-1)$-th associated curve $C^\vee$ is called the *dual curve* of C_0. Note that the dual curve should not be confused with the dual variety of C_0. The latter coincides with the $(n-2)$-th osculating developable of the dual curve.

Proposition 10.4.13. *Let $r_0 = \deg \mathcal{L} = \deg f(C)$. For any point $x \in C$ let $s_i(x) = s_i$, where the s_i's are defined in* (10.66)*, and $k_i = \sum_{x\in C} s_i(x)$. Then,*

$$r_m = (m+1)(r_0 + m(g-1)) - \sum_{i=1}^{m}(m-i+1)k_i$$

and

$$\sum_{i=1}^{n}(n-i+1)k_i = (n+1)(r_0 + n(g-1)).$$

In particular,

$$r_1 = 2(r_0 + g - 1) - k_1.$$

Proof Formula (10.64) gives the degree of the sheaf of principal parts $\mathcal{P}^m_C(\mathcal{L})$. We have an exact sequence

$$0 \to \mathcal{P}^m \to \mathcal{P}^m_C(\mathcal{L}) \to \mathcal{F} \to 0,$$

where $\mathcal{F}$ is a skyscraper sheaf whose fiber at $x \in C$ is equal to the cokernel of the map $\alpha^m(x) : V \to \mathcal{L}/\mathfrak{m}_{C,x}\mathcal{L}$. It follows from formula (10.66) that $\dim \mathcal{F}(x)$ is equal to $s_1 + (s_1+s_2) + \cdots + (s_1 + \cdots + s_m) = \sum_{i=1}^{m}(m-i+1)s_i$. The standard properties of Chern classes give

$$\deg \mathcal{P}^m = \deg \mathcal{P}^m_C(\mathcal{L}) - h^0(\mathcal{F}) = (m+1)(r_0 + m(g-1)) - \sum_{i=1}^{m}(m-i+1)k_i.$$

The second formula follows from the first one by taking $m = n$ in which case $r_n = 0$ (the surjection of bundles of the same rank $V_C \to \mathcal{P}^n$ must be an isomorphism). □

Adding up r_{m-1} and r_{m+1} and subtracting $2r_m$, we get the following.

Corollary 10.4.14.

$$r_{m-1} - 2r_m + r_{m+1} = 2g - 2 - k_{m+1}, \quad m = 0, \ldots, n-1, \tag{10.67}$$

where $r_{-1} = r_n = 0$.

The previous formulas can be viewed as the *Plücker formulas* for space curves. Indeed, let $n = 2$ and C is a plane curve of degree d and class $d^\vee$. Assume that the dual curve $C^\vee$ has $\delta^\vee$ ordinary nodes and $\kappa^\vee$ ordinary cusps. Applying Plücker's formula, we have

$$d = d^\vee(d^\vee - 1) - 2\delta^\vee - 3\kappa^\vee = 2d^\vee + (d^\vee(d^\vee - 3) - 2\delta^\vee - 2\kappa^\vee) - \kappa^\vee = 2d^\vee + 2g - 2 - \kappa^\vee.$$

In this case $d^\vee = r_1, d = r_0$ and $\kappa^\vee = k_1$, so the formulas agree.

Example 10.4.15. A rational normal curve R_n in $\mathbb{P}^n$ has no hyperosculating hyperplanes (since no hyperplane intersects it with multiplicity $> n$). So $r_m = (m+1)(n-m) = r_{n-m-1}$. Its dual curve is a rational normal curve in the dual space. Its tangential surface is of degree $r_1 = 2(n-1)$ and the $(n-1)$-th osculating developable is the discriminant hypersurface for binary forms of degree n. For example, for $n = 3$, the tangential surface of R_3 is a quartic surface with equation $Q_0Q_1 + Q_2^2 = 0$, where Q_0, Q_1, Q_2 are some quadrics containing R_3. To see this, one considers a rational map $\mathbb{P}^3 \dashrightarrow \mathcal{N}^\vee \cong \mathbb{P}^2$ defined by the net $\mathcal{N}$ of quadrics containing R_3. After we blow-up $\mathbb{P}^3$ along R_3, we obtain a regular map $Y \to \mathbb{P}^2$ which blows down the proper transform of the tangential surface to a conic in $\mathbb{P}^2$. Its equation can be chosen in the form $t_0t_1 + t_2^2 = 0$. The pre-image of this conic is the quartic surface $Q_0Q_1 + Q_2^2 = 0$. It contains R_3 as its double curve. Also, it is isomorphic to the discriminant hypersurface for binary forms of degree 3.

Conversely, assume that C has no hyperosculating hyperplanes. Then, all $k_i = 0$, and we get

$$\sum_{m=0}^{n-1}(n-m)(r_{m-1} - 2r_m + r_{m+1}) = -(n+1)r_0 \qquad (10.68)$$
$$= \sum_{m=0}^{n-1}(n-m)(2g-2) = n(n+1)(g-1).$$

This implies $g = 0$ and $r_0 = n$.

The computation from the previous example (10.68) can be used to obtain the formula for the number W of hyperosculating points of a curve C embedded in $\mathbb{P}^n$ by a linear series of degree d (see also [377, Lemma 5.21].

Proposition 10.4.16. *The number of hyperosculating points, counting with multiplicities, is equal to*

$$W = (n+1)(d + n(g-1)). \qquad (10.69)$$

Proof Applying (10.67), we obtain

$$
\begin{aligned}
W &= \sum_{x\in C}\sum_{i=1}^{n}(s_1(x)+\cdots+s_i(x)) = \sum_{i=0}^{n}(n-i+1)k_i = \sum_{i=0}^{n-1}(n-i)k_{i+1}\\
&= \sum_{i=0}^{n-1}(n-i)(2g-2) - \sum_{i=0}^{n-1}(n-i)(r_{i-1}-2r_i+r_{i+1})\\
&= (n+1)n(g-1)+(n+1)d = (n+1)(d+n(g-1)).
\end{aligned}
$$

The number $\sum_{i=1}^{n}(s_1(x)+\cdots+s_i(x))$ should be taken as the definition of the multiplicity of a hyperosculating point x. A simple hyperosculating point satisfies $s_i(x)=\ldots=s_{n-1}(x)=0, s_n(x)=1$. □

Example 10.4.17. Let C be an elliptic curve embedded in $\mathbb{P}^n$ by a complete linear system $|(n+1)x_0|$, where x_0 is a point on C. Then, the degree of E is equal to $n+1$ and formula (10.69) gives $W=(n+1)^2$. This is equal to the number of $(n+1)$-torsion points of C in the group law defined by the choice x_0 as the zero point. Of course, each such point x satisfies $(n+1)x\in|(n+1)x_0|$, and hence is a hyperosculating point. The formula shows that there are no other hyperosculating points.

In particular, we see that $k_i=0$ for $i<n$; hence the degree r_1 of the tangential surface is equal to $2(n+1)$. Also, if $n>2$, the dual of C is a curve of degree $r_2=3(n+1)$. It has $(n+1)^2$ singular points corresponding to $(n+1)^2$ hyperosculating planes.

Example 10.4.18. Assume C is a canonical curve in $\mathbb{P}^{g-1}$. Recall that a *Weierstrass point* of a smooth curve of genus $g>1$ is a point x such that

$$\sum_{i=1}^{g}(h^0(x)+\cdots+h^0(ix)-i)>0.$$

Let $a_i=h^0(x)+\cdots+h^0(ix)$. We have $a_1=1$ and $a_i=i$ if and only if $h^0(x)=\ldots=h^0(ix)=1$. By Riemann-Roch, this is equivalent to that $h^0(K_C-ix)=g-i$, i.e. the point x imposes the expected number of conditions for a hyperplane to have a contact with C of order i at x. A point x is a Weierstrass point if and only if there exists $i\le g$ such that the number of such conditions is less than expected by the amount equal to a_i-i. With notation (10.66), this shows that

$$s_1+\cdots+s_{i-1}=a_i-i,\ i=2,\ldots,g.$$

In particular, the point x is hyperosculating if and only if it is a Weierstrass point. Applying formula (10.69), we obtain the number of Weierstrass points

$$W=g(g^2-1). \tag{10.70}$$

Since $h^0(2x) = 1$ for all points on C (because C is not hyperelliptic), we get $k_1 = 0$. Applying Proposition 10.4.13, we obtain that the rank r_1 of C is equal to $6(g-1)$.

Assume that C is general in the sense that all Weierstrass points x are simple, i.e. $W(x) = 1$. It follows from the proof of Proposition 10.4.16 that $s_i(x) = 0, i < g-1$, and $s_{g-1}(x) = 1$. Thus, $k_m = 0, m < g-1$, and $k_{g-1} = W = g(g^2-1)$. It follows from Proposition 10.4.13 that $r_m = (m+2)(m+1)(g-1)$ for $1 \leq m < g-2$ and $r_{g-2} = g(g-1)^2$. The latter number coincides with the class of C. For example, if $g = 3$, we get $r_1 = 12$ and the 24 Weierstrass points are flex points of C. If $g = 4$, we get $r_1 = 18$ and $r_2 = 36$. We have 60 hyperosculating planes at Weierstrass points. The linear system of cubics through C defines a birational map from $\mathbb{P}^3$ to a cubic hypersurface in $\mathbb{P}^4$ with an ordinary double point. The image of the tangential surface is the enveloping cone at the node, the intersection of the cubic with its first polar with respect to the node. Its degree is equal to 6, therefore, the tangential surface is the proper inverse image of the cone under the rational map.

We refer for the proof of the following Proposition to [582].

Proposition 10.4.19. *Let $f^\vee : C \to (\mathbb{P}^n)^\vee$ be the normalization of the $n-1$-th associated curve of $f : C \to \mathbb{P}^n$, the dual curve of $f(C)$. Then,*

(i) $r_m(f^\vee(C)) = r_{n-m-1}(f(C))$*;*
(ii) $(f^\vee)^\vee = f$*;*
(iii) $k_i(f^\vee) = k_i(f)$.

Recall from Chapter 1 that the dual variety of C_0 is the closure in $(\mathbb{P}^n)^\vee$ of the set of tangent hyperplanes to smooth points of C_0. If $t_0 = f(x)$ is a smooth point, the set of tangent hyperplanes at x is a codimension 2 subspace of the dual space equal to $(n-2)$-th developable scroll of the dual curve. By the duality, we obtain that the dual of the $(n-2)$-th developable scroll of a curve C_0 is the dual curve of C_0. In particular, if $n = 3$, we obtain that the dual of the tangential surface to a nondegenerate curve C_0 in $\mathbb{P}^3$ is the dual curve of C_0, and the dual of a nondegenerate curve C_0 in $\mathbb{P}^3$ is the tangential surface of its dual curve.

Proposition 10.4.20. *Let S be a ruled surface in $\mathbb{P}^3$. The following properties are equivalent:*

(i) S is a developable surface;
(ii) S is a tangential surface corresponding to some curve C_0 lying on S;
(iii) the tangent lines of the curve $C_0 \subset G_1(\mathbb{P}^3)$ parameterizing the rulings are contained in $G_1(\mathbb{P}^3)$.

Proof (i) ⇒ (ii). Consider the Gauss map $\gamma : S \to (\mathbb{P}^3)^\vee$ which assigns to a smooth point $x \in S$ the embedded tangent plane $\mathbb{T}_x(C)$. Obviously, γ blows down generators of S; hence the image of S is a curve $\check{C}_0$ in the dual space. This curve is the dual variety of S. Its dual variety is our surface S, and hence coincides with the tangential surface of the dual curve C_0 of $\check{C}_0$.

(ii) ⇒ (iii) Let $q_C : Z_C \to C$ be the projection from the incidence variety and $D \in |\mathcal{O}_{Z_C}(1)|$. The tangent plane at points of a ruling ℓ_x cuts out the ruling with multiplicity 2. Thus, the linear system $|D - 2\ell_x)|$ is non-empty (as always, we identify a ruling with a fiber of q_C). The exact sequence

$$0 \to \mathcal{O}_{Z_C}(D - 2\ell_x) \to \mathcal{O}_{Z_C}(D - \ell_x) \to \mathcal{O}_{\ell_x}(D - \ell_x) \to 0$$

shows that $h^0(\mathcal{O}_{l_x}(D - \ell_x)) = 1$, i.e. $|D - \ell_x|$ has a base point $y(x)$ on ℓ_x. This means that all plane sections of S containing ℓ_x have residual curves passing through the same point $y(x)$ on ℓ_x. Obviously, this implies that the point $y(x)$ is a singular point of S and the differential of the projection $p_C : Z_C \to S$ at $y(x)$ is not surjective. Applying Proposition 10.1.18, we obtain that the tangent line $\mathbb{T}_x(C)$ is contained in the α-plane $\Omega(y(x)) \subset G_1(\mathbb{P}^3)$.

(iii) ⇒ (i) Applying Proposition 10.1.18, we obtain that each ℓ_x has a point $y(x)$ such that its image in S is a singular point and the differential of p_C at $y(x)$ is not surjective. This implies that $y(x)$ is a base point of the linear system $|D - \ell_x|$ on ℓ_x. As above, this shows that $|D - 2\ell_x|$ is not empty and hence there exists a plane tangent to S at all points of the ruling ℓ_x. □

The set of points $y(x) \in \ell_x, x \in C$ is a curve C_0 on S such that each ruling ℓ_x is tangent to a smooth point on C_0. So S is the tangential surface of C_0. The curve C_0 is called the *cuspidal edge* of the tangent surface. It is a curve on S such that at its general point s the formal completion of $\mathcal{O}_{S,s}$ is isomorphic to $\mathbb{C}[[z_1, z_2, z_3]]/(z_1^2 + z_2^3)$.

10.4.4 Quartic ruled surfaces in $\mathbb{P}^3$

Here, we will discuss the classification of quartic ruled surfaces in $\mathbb{P}^3$ due to A. Cayley and L. Cremona. We have already classified ruled surfaces of degree 3 in Section 9.2. They are non-normal cubic surfaces and there are two kinds of them. The double curve ℓ is a line and the map $D(f) \to \ell$ is an irreducible (reducible) degree 2 cover. The surface Z_C is isomorphic to $\mathbf{F}_1 = \mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-1))$. The linear system $|h|$ that gives the map $f : \mathbf{F}_1 \to \mathbb{P}^3$ is equal to $|2\mathfrak{f} + \mathfrak{e}|$, where, as usual, $\mathfrak{e}$ is the divisor class of the exceptional section E_0 and $\mathfrak{f}$ is the class of a fiber. The curve $D(f) \in |h - \mathfrak{f}| = |\mathfrak{f} + \mathfrak{e}|$. In the first case, the surface S has ordinary singularities, and $D(f)$ is an irreducible curve. In the second case, $D(f) \in |h|$ and consists of the exceptional section and a fiber. Now, let us deal with quartic surfaces. We do not assume that the surface has only ordinary singularities. We begin with the following.

Proposition 10.4.21. *The genus of a ruled quartic surface is equal to* 0 *or* 1.

Proof A general plane section H of S is a plane quartic. Its geometric genus g is the genus of S. If $g = 3$, the curve H is nonsingular; hence S is normal and therefore nonsingular. Since $K_S = 0$, it is not ruled. If $g = 2$, the singular curve of S is a line. The plane sections through the line form a linear pencil of cubic curves on S. Its pre-image on the normalization X of S is a pencil of elliptic curves. Since X is a $\mathbb{P}^1$-bundle over a curve of genus 2, a general member of the pencil cannot map surjectively to the base. This contradiction proves the assertion. □

So, we have two classes of quartic ruled surfaces: rational ruled surfaces ($g = 0$) and *elliptic ruled surfaces* ($g = 1$). Each surface S is defined by some curve C_0 of degree 4 in $G_1(\mathbb{P}^3)$. We denote by X the minimal ruled surface $\mathbb{P}(\mathcal{E})$ obtained from the universal family Z_{C_0} by the base change $\nu : C \to C_0$, where ν is the normalization map. We will denote by E_0 an exceptional section of X defined by choosing a normalized vector bundle $\mathcal{E}_0$ with $\mathbb{P}(\mathcal{E}_0)$ isomorphic to X.

We begin with the classification of rational quartic ruled surfaces.

Proposition 10.4.22. *A rational quartic ruled surface is a projection of a rational normal scroll* $S_{2,5}$ *or* $S_{1,5}$ *of degree* 4 *in* $\mathbb{P}^5$.

Proof Let $|h|$ be the linear system of hyperplane sections on the quartic rational normal scroll $S_{a,n} \cong \mathbf{F}_e$. We have $|h| = |k\mathfrak{f} + \mathfrak{e}|$, where $k > e$. Since $h^2 = 4$, we get $2k + e = 4$. This gives two solutions $(e, k) = (0, 2), (2, 1)$. In the first case, we get the scroll $S_{2,5} \cong \mathbf{F}_0$, in the second case, we get the scroll $S_{1,5} \cong \mathbf{F}_2$. □

Let $\mathbb{D}(f)$ be the double-point divisor class. We know that the singular curve ℓ on S is the image of a curve $D(f)$ from $\mathbb{D}(f)$ on X, where $X = S_{2,5}$ or $S_{1,5}$. Applying (10.57), this gives

$$D(f) \sim 2h - 2\mathfrak{f} = \begin{cases} 2\mathfrak{f} + 2\mathfrak{e} & \text{if } X = S_{2,5}, \\ 4\mathfrak{f} + 2\mathfrak{e} & \text{if } X \cong S_{1,5}. \end{cases}$$

Since a general plane section of S is a rational curve, $D(f)$ and ℓ consist of, at most, three irreducible components. The linear system

$$|h| = \begin{cases} |2\mathfrak{f} + \mathfrak{e}| & \text{if } X = S_{2,5}, \\ |3\mathfrak{f} + \mathfrak{e}| & \text{if } X = S_{1,5}, \end{cases}$$

maps a component D_i of $D(f)$ to an irreducible component ℓ_i of ℓ of degree $d_i = \frac{1}{m_i} H \cdot D_i$, where m_i is the degree of the map $D_i \to \ell_i$. The number m_i is equal to the multiplicity of a general point on ℓ_i as a singular point of the surface unless ℓ_i is a curve of cusps. In the latter case $m_i = 1$, but D_i enters D with multiplicity 2. A fiber $F_x = \pi^{-1}(x)$ could be also a part of D. In this case, ℓ has a singular point at $\nu(x)$. If it is an ordinary double point, the fiber component enters with multiplicity 1, if it is a cusp, it enters with multiplicity 2. Other cases will not occur. Finally, we use that $\dim |h - D_i| > 0$ if ℓ_i is contained in a plane, i.e., a line or a conic.

This gives us the following cases, which make a “rough classification” according to the possible singular locus of the surface.

1. $X = S_{2,5}$:
   - (i) $D(f) = D_1, d_1 = 3$;
   - (ii) $D(f) = D_1 + D_2, D_1 \in |\mathfrak{e}|, D_2 \in |2\mathfrak{f} + \mathfrak{e}|, d_1 = 1, d_2 = 2$;
   - (iii) $D(f) = D_1 + D_2 + F_1 + F_2, D_1, D_2 \in |\mathfrak{e}|, d_1 = d_2 = 1$;
   - (iv) $D(f) = 2D_1, D_1 \in |\mathfrak{f} + \mathfrak{e}|, d_1 = 1$;
   - (iv)' $D(f) = 2D_1, D_1 \in |\mathfrak{f} + \mathfrak{e}|, d_1 = 3$;
   - (v) $D(f) = 2D_1 + 2F_1, D_1 \in |\mathfrak{e}|, d_1 = 1$;
   - (vi) $D(f) = 2D_1 + F_1 + F_2, D_1 \in |\mathfrak{e}|, d_1 = 2$;
   - (vi)' $D(f) = 2D_1 + 2F_1, D_1 \in |\mathfrak{e}|, d_1 = 2$.
2. $X = S_{1,5}$:
   - (i) $D(f) = D_1, d_1 = 3$;
   - (ii) $D(f) = E_0 + D_1 + F, D_1 \in |3\mathfrak{f} + \mathfrak{e}|, d_1 = 1, d_2 = 2$;
   - (iii) $D(f) = 2E_0 + 2F_1 + 2F_2, d_1 = 1$;
   - (iv) $D(f) = 2D_1, D_1 \in |2\mathfrak{f} + \mathfrak{e}|, d_1 = 1$.

Theorem 10.4.23. *There are 12 different types of rational quartic ruled surfaces corresponding to 12 possible cases from above.*

Proof It suffices to realize all possible cases from above. By Proposition 10.4.22, the different types must correspond to different choices of the center of the projection in $\mathbb{P}^5$.

Let us introduce some special loci in $\mathbb{P}^5$ which will play a role in the choice of the center of the projection.

We will identify curves on $\mathbf{F}_0$ with their images in $S_{2,5}$. A conic directrix is a curve $E \in |\mathfrak{e}|$. Consider the union of planes spanning the E's. It is a scroll Σ_1 of dimension 3 parameterized by $|\mathfrak{e}| \cong \mathbb{P}^1$. Let us compute its degree. Fix two generators F_1 and F_2 of $\mathbf{F}_0$. Then, $|h - F_1 - F_2| = |\mathfrak{e}_0|$. If we fix three pairs of generators $F_1^{(i)}, F_2^{(i)}, i = 1, 2, 3$, each spanning a $\mathbb{P}^3$, then we can establish a correspondence ℓ of tri-degree $(1, 1, 1)$ on $|\mathfrak{e}| \times |\mathfrak{e}| \times |\mathfrak{e}|$ such that the point $(x, y, z) \in \ell$ corresponds to three hyperplanes from each linear system $|h - F_1^{(i)} - F_2^{(i)}|$ which cut out the same curve $E \in |\mathfrak{e}|$. The three hyperplanes intersect along the plane spanning E. This shows that our scroll is the join of three disjoint lines in the dual $\mathbb{P}^5$ which can be identified with the same $\mathbb{P}^1$. Applying formula (10.54), we obtain that the degree of Σ_1 is equal to 3.

The next scroll we consider is the union Σ_2 of three-dimensional spaces spanned by tangent planes of $S_{2,5}$ at points on a fixed generator. This 3-dimensional space is spanned by the tangent lines of two fixed conic directrices at the points where they intersect the generator. Thus, our scroll is the join of the tangential scroll of the two directrices with respect to the correspondence between the directrices defined by the generators. The degree of this scroll is given by the formula in Example 10.4.3. Since the tangent lines of a conic are parameterized by the conic, and the two conics are disjoint, the degree of Σ_2 is equal to 4. Obviously, Σ_1 is a 2-directrix of Σ_2. Since the tangent plane to $S_{2,5}$ at a point x is spanned by the generator passing through this point and the tangent line of the conic directrix passing through this point, we obtain that Σ_2 coincides with the tangential scroll of $S_{2,5}$.

One more scroll is constructed as follows. Consider directrices of $S_{2,5}$ defined by the images of curves $\ell_3 \in |\mathfrak{f} + \mathfrak{e}|$. We identify them with the images. These are directrices of degree 3. Let Σ_3 be the union of tangent planes to $S_{2,5}$ at the points of ℓ_3. These tangent planes can be obtained as joins of tangent lines of ℓ_3 at points $x \in \ell_3$ and the points x' on a conic directrix E such that the points x, x' lie on the same generator. Thus, Σ_3 is obtained by construction from Example 10.4.3 as the join of the tangential scroll of ℓ_3 and the conic. The degree of the tangential scroll has been computed there; it is equal to 4. Thus, the degree of Σ_3 is equal to $4 + 2 - 1 = 5$, where we subtracted 1 because the conic and ℓ_3 meet at one point dropping the dimension of the join by 1.

Let $p_\ell : S_{2,5} \to S$ be the projection map from a line ℓ. We will use the fact that any two points x_1, x_2 in the double locus $D(f)$ which are projected to the same point must lie on a secant of $D(f)$ that passes through these points and intersects ℓ. The secant line becomes a tangent line if $x_1 = x_2$ is a critical point of p_ℓ.

- Type 1 (i).

Take a line ℓ in $\mathbb{P}^5$ which intersects the quartic scroll Σ_2 at four distinct points and is not contained in any three-dimensional space spanned by a cubic directrix $\ell_3 \in |\mathfrak{f}+\mathfrak{e}|$. Let D be an irreducible component of $D(f)$ and let x be a general point of D. We know from the classification of all possible components of $D(f)$ that the degree of the projection map must be equal to two or three. If the degree is equal to 3, then $D \in |\mathfrak{f}+\mathfrak{e}|$ is a cubic directrix and its projection is a line. This implies that ℓ belongs to the linear span of D. By assumption on ℓ, this does not happen. So, the degree is equal to two. The map which assigns to a point $x \in D$ the intersection point of ℓ and the secant passing through x is a degree two map $D \to \ell$. The intersection points of ℓ with Σ_2 are the branch points of this map. By Hurwitz's formula, the normalization of D is a genus one curve, hence the arithmetic genus is ≥ 1. The classification of possible D's shows that this could happen only if D is a nonsingular curve from $|2\mathfrak{f}+2\mathfrak{e}|$. This realizes type 1(i).

The quartic scroll S can be described as follows. Consider a rational normal cubic curve R_3 in $\mathbb{P}^3$ and let S be the set of its secants contained in a nonspecial linear line complex. The set of secants of R_3 is a surface in $\mathbb{G} = G_1(\mathbb{P}^3)$ of degree 4 in its Plücker embedding. This can be seen by computing its cohomology class in $\mathbb{G}$. A general α-plane $\Omega(p)$ contains only one secant. A general β-plane $\Omega(\Pi)$ contains three secants. This shows that the degree of the surface of secants is equal to four. The surface must be a Veronese surface in $\mathbb{P}^5$ because it does not contain lines. We will encounter this surface again in Chapter 11 while classifying congruences of lines of order one.

The intersection of the surface with a general linear line complex is a curve C of degree four. It defines a quartic ruled surface S_C. Take a point $p \in R_3$ and consider the set of secants $\ell_x, x \in C$, such that $p \in \ell_x$. The intersection of the Schubert plane $\Omega(p, \mathbb{P}^3)$ with the Veronese surface is a conic. Its intersection with the linear line complex must consist of two lines. Thus, each point of R_3 lies on two generators of the surface S_C. The curve R_3 is the double curve of S.

- Type 1 (ii).

In this case, we take ℓ intersecting Σ_1 at some point x_0 in the plane spanned by some conic directrix $D \in |\mathfrak{e}|$. The projection of D is a line and the map is 2:1. Note that in this case the point x_0 is contained in two tangents to D so two of the four intersection points of ℓ and Σ_2 coincide. It also shows that Σ_1 is contained in the singular locus of Σ_2. The remaining two points in $\ell \cap \Sigma_2$ are the branch points of the double cover $E' \to \ell$, where $D' \in 2\mathfrak{f}+|\mathfrak{e}|$ is the residual component of $D(f)$. Arguing as above, we see that D' is a smooth rational curve of degree 4. Its projection is a conic.

- Type 1 (iii).

This time we take ℓ intersecting Σ_1 at two points p_1, p_2. These points lie in planes Π_1 and Π_2 spanned by directrix conics E_1 and E_2. The projection from ℓ maps these conics to disjoint double lines of S. Let us now find two generators F_1 and F_2, which are projected to the third double line. Consider the pencil $\mathcal{P}_i$ of lines in the plane Π_i with base point p_i. By intersecting the lines of the pencil with the conic E_i, we define an involution on E_i and hence an involution γ_i on the pencil $|\mathfrak{f}| \cong \mathbb{P}^1$ (interchanging the generators intersecting E_i at two points in the involution). Now, we have two involutions on $|\mathfrak{f}|$ whose graphs are curves of type $(1,1)$. They have two common pairs in the involution which give us two generators on $S_{2,5}$ intersecting E_i at two points on a line ℓ_i through p_i. The three-dimensional subspace spanned by ℓ, ℓ_1 and ℓ_2 contains the two generators. They are projected to a double line of S.

- Type 1 (iv), (iv)'.

The image of D_1 on $S_{2,5}$ is a rational normal cubic R_3 spanning a 3-plane M of $\mathbb{P}^5$. We project from a general line contained in M. The restriction of the projection to D_1 is a degree 3 map. So the projection of D_1 is a triple line of S.

Another possibility here is to project from a line directrix ℓ of the tangential scroll Σ_3. Each point on ℓ lies in a tangent plane to a cubic directrix $\ell_3 \in |E_0+F|$. So the projection from ℓ maps ℓ_3 to a rational curve R_3 of degree 3 and maps the tangent lines to ℓ_3 to tangent lines to R_3. Thus, the scroll S is a developable quartic surface considered in Example 10.4.15. Let us find a line directrix on Σ_3. We know that Σ_3 is equal to the image of a projective bundle $\mathbb{P}(\mathcal{E})$, where $\mathcal{E}$ is a vector bundle over $\mathbb{P}^1$ of rank 3 and degree 5. Thus, $\deg \mathcal{E}^\vee(1) = -5+3 = -2$, and applying Riemann-Roch, we obtain $h^0(\mathcal{E}^\vee(1)) \geq \deg \mathcal{E}^\vee(1) + 3 > 0$. This implies that there is a nontrivial map of sheaves $\mathcal{E} \to \mathcal{O}_{\mathbb{P}^1}(1)$. Let $\mathcal{L}$ be the image of this map. It defines a section $\sigma : \mathbb{P}^1 \to \mathbb{P}(\mathcal{E})$ such that $\sigma^*(\mathcal{O}_{\mathbb{P}(\mathcal{E})}(1)) \cong \mathcal{L}$. Thus, the restriction of $\mathcal{O}_{\mathbb{P}(\mathcal{E})}(1)$ to $\mathsf{D} = \sigma(\mathbb{P}^1)$ is of degree ≤ 1. Since Σ_3 is a scroll in our definition, the sheaf $\mathcal{O}_{\mathbb{P}(\mathcal{E})}(1)$ is ample, therefore, the degree must be equal to 1. So, the image of D in Σ_3 is a line directrix.

- Type 1 (v).

This is a degeneration of the previous case. The rational normal cubic degenerates into the union of a directrix conic and a generator. The projection is a degree 2 map on the conic and degree 1 on the line. The double curve ℓ is a triple line. It is a generator and a directrix at the same time. Through each point on ℓ passes two generators other than itself. As in the previous case, a plane containing ℓ contains only one of the other generators.

- Type 1 (vi).

Consider a hyperplane section $L \cap \Sigma_1$, where L contains two generators F_1 and F_2 of $S_{2,5}$. The quartic curve $L \cap S_{2,5}$ consists of the two generators and a directrix conic D from $|\mathfrak{e}|$. Thus, the cubic surface $L \cap \Sigma_1$ contains a plane, and the residual surface is a quadric Q containing D. Take a line ℓ in the 3-dimensional subspace M spanned by F_1 and F_2 that is tangent to the quadric $M \cap Q$. The projection from ℓ maps $S_{2,5}$ to a quartic ruled surface with double line equal to the image of the two generators F_1 and F_2 and the cuspidal conic equal to the image of the directrix conic D.

- Type 1 (vi)'.

The same as the previous case, but we choose L to be tangent along a generator F_1. The double locus is a reducible cuspidal cubic.

- Type 2 (i).

Type 2 corresponds to a projection of the rational normal quartic scroll $S_{1,5} \cong \mathbf{F}_2$ embedded in $\mathbb{P}^5$ by the linear system $|3\mathfrak{f} + \mathfrak{e}|$. The exceptional section E_0 is a line directrix on $S_{1,5}$. The curves from the linear system $| + 2\mathfrak{f} + \mathfrak{e}|$ are cubic directrices. The analog of the tangential scroll Σ_2 here is the join Σ_2' of the tangential surface of a cubic directrix D with the line E_0. It is the union of 3-dimensional spaces spanned by a tangent line to D and E_0. We know that the tangential scroll of R_3 is of degree 4. Thus, the degree of Σ_2' is equal to 4. The rest of the argument is the same as in case 1 (i). We take ℓ intersecting Σ_2' at four distinct points and not contained in a 3-space spanned by a cubic directrix. The double curve is a smooth elliptic curve of degree 6 from $|4\mathfrak{f}+2\mathfrak{e}|$. It is projected to the double curve R_3 of S. All generators of S intersect the unique line directrix ℓ, which is the projection of the exceptional section of $S_{1,5}$. So, the surface is contained in a special linear line complex $\Omega(\ell)$. This is the difference between surfaces of Type 1 (ii) and Type II (i).

- Type 2 (ii).

This time we take ℓ intersecting the plane Π spanned by E_0 and a generator F. We also do not take it in any 3-plane spanned by a cubic directrix. Then, E_0 and F will project to the same line on S, the double line. The residual part of the double locus must be a curve E from $|3\mathfrak{f} + \mathfrak{e}|$. Since no cubic directrix is a part of the double locus, we see that E is an irreducible quartic curve. Its image is a double conic on S.

- Type 2 (iii).

We choose a line ℓ intersecting two planes as in the previous case. Since the two planes have a common line E_0, they span a 3-dimensional subspace. It contains three lines which are projected to the same line on S, a triple line of S.

- Type 2 (iv).

Take a cubic curve from $|2\mathfrak{f} + \mathfrak{e}|$ and a line in the three-dimensional space spanned by the cubic. The cubic is projected to a triple line.

□

Remark 10.4.24. We have seen that a developable quartic surface occurs in case 1 (iv). Let us see that this is the only case when it may occur.

The vector bundle of principal parts $\mathcal{P}^1_C(\mathcal{L})$ must be given by an extension

$$0 \to \Omega^1_C \otimes \mathcal{L} \to \mathcal{P}^1_C(\mathcal{L}) \to \mathcal{L} \to 0, \tag{10.71}$$

where C is a rational cubic in $\mathbb{P}^3$ and $\mathcal{L} = \mathcal{O}_C(1) \cong \mathcal{O}_{\mathbb{P}^1}(3)$. It is known that the extension

$$0 \to \Omega^1_C \to \mathcal{P}^1_C \to \mathcal{O}_C \to 0,$$

from which the previous extension is obtained by twisting with $\mathcal{L}$, does not split. Its extension class is defined by a nonzero element in $\mathrm{Ext}^1(\mathcal{O}_C, \Omega^1_C) \cong H^1(C, \Omega^1_C) \cong \mathbb{C}$ (this is the first Chern class of the sheaf $\mathcal{O}_{\mathbb{P}^1}(1)$). After tensoring (10.71) with $\mathcal{O}_{\mathbb{P}^1}(-2)$ we get an extension

$$0 \to \mathcal{O}_{\mathbb{P}^1}(-1) \to \mathcal{P}^1_{\mathbb{P}^1}(\mathcal{L})(-2) \to \mathcal{O}_{\mathbb{P}^1}(1) \to 0.$$

The locally free sheaf $\mathcal{E} = \mathcal{P}^1_C(\mathcal{L})(-2)$ has 2-dimensional space of global sections. Tensoring with $\mathcal{O}_{\mathbb{P}^1}(-1)$ and using that the coboundary homomorphism

$$H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}) \to H^1(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(-2))$$

is nontrivial, we obtain that $\mathcal{E}(-1)$ has no nonzero sections, hence $\mathcal{E}$ is a normalized vector bundle of degree 0 defining the ruled surface $\mathbb{P}(\mathcal{E})$. There is only one such bundle over $\mathbb{P}^1$, the trivial bundle $\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}$. Untwisting $\mathcal{E}$, we obtain that the sheaf $\mathcal{P}^1_R(\mathcal{L})$ is isomorphic to $\mathcal{O}_{\mathbb{P}^1}(2) \oplus \mathcal{O}_{\mathbb{P}^1}(2)$, so $\mathbb{P}(\mathcal{P}^1_R(\mathcal{L})) \cong \mathbf{F}_0$ and the complete linear system defined by the tautological invertible sheaf corresponding to $\mathcal{P}^1_R(\mathcal{L})$ embeds $\mathbf{F}_0$ in $\mathbb{P}^5$ as the rational normal scroll $S_{2,5}$. The double locus class $\mathbb{D}(f)$ must be divisible by 2, and the only case when it happens is type 1 (iv)'.

We can also distinguish the previous cases by a possible embedding of the quartic curve C_0 parameterizing generators of S in $\mathbb{G} = G_1(\mathbb{P}^3)$. Since $\deg C_0 = 4$ in the Plücker embedding, the curve is always contained in a hyperplane L on $\mathbb{P}^5$. If, furthermore, C_0 lies in a codimension 2 subspace, then this subspace is either contained in one tangent hyperplane of $\mathbb{G}$ or is equal to the intersection of two tangent hyperplanes (because the dual variety of $\mathbb{G}$ is a quadric). So we have the following possibilities:

I C_0 is a rational normal quartic contained in a hyperplane L that is not tangent to $\mathbb{G}$;
II C_0 is a rational normal quartic contained in a hyperplane L which is tangent to $\mathbb{G}$ at a point O not contained in C_0;
III C_0 is a rational normal quartic contained in a hyperplane L which is tangent to $\mathbb{G}$ at a point O contained in C_0;
IV C_0 is a rational quartic curve contained in the intersection of two different tangent hyperplanes of $\mathbb{G}$;
V C_0 is a rational quartic curve contained in a three-dimensional subspace through which passes only one tangent hyperplane of $\mathbb{G}$. The tangency point is an ordinary node of C_0.

A quartic surface of type 1 (i) or 1 (iv)' from Theorem 10.4.23 belongs to type I. Following W. Edge [271], we denote types 1 (i) and 1 (iv)' with I.

In type 1 (ii), the line component of the double curve is a directrix, so all generators belong to a special linear line complex tangent to $G_1(\mathbb{P}^3)$ at the point O representing this directrix. This is Edge's type II. Through any point p on the directrix passes two generators, the point O belongs to a secant of C_0 formed by the line $\Omega(p,\Pi)$, where Π is the plane spanned by the two generators. It is a nonsingular point of C_0. We have Edge's type II (C).

In type 1 (iii), we have two directrices which are not generators. This means that C is contained in the intersection of two special linear line complexes tangent to $G_1(\mathbb{P}^3)$ at two points. This is type IV (B). The tangency points correspond to the line directrices on S. The curve C_0 is contained in the intersection of two special linear line complexes that is a nonsingular quadric. The curve C_0 has an ordinary node at the point corresponding to two generators mapped to a double line on F.

In type 1 (iv), the triple line is a directrix of S, so we are again in case II but in this case, the point O intersects the α-plane $\Omega(p)$ at three non-collinear points and intersects the β-plane $\Omega(\Pi)$ at one point. This is Edge's type II (A).

In case 1 (v), the double curve is a triple line. One of the generators F is contained in $D(f)$ with multiplicity two and is mapped to the triple line. Thus, S is contained in a unique special line complex which is tangent to G at a cusp of C_0. Since C_0 is singular, it is contained in a three-dimensional space. So, C is contained in a quadric cone equal to the intersection of $G_1(\mathbb{P}^3)$ with two linear line complexes. The singular point of this cone is the singular point of C_0. This is Edge's Type III (A).

In type 1 (vi), two generators on $S_{2,5}$ are projected to a double generator of S. The curve C_0 has an ordinary double point, hence it lies in two linear line complexes. The double generator is the only line directrix on S. Thus, there is only one special linear line complex containing S and its tangency point is an ordinary double point of C_0. This is Edge's type V (A). In case 1 (vi)', we also have type V (A), only this time the singular point of C_0 is a cusp.

In type 2 (i), the line directrix ℓ corresponding to E_0 defines a special line complex containing C. Thus, we are in type II. The α-plane $\Omega(p), p \in \ell$, contains only one point, the β-plane $\Omega(\pi), \ell \subset \pi$, contains three points. This is Edge's type II (B). This case is a degeneration of case 1(i); instead of a general line complex we take a special one.

In type 2 (ii), we have a line directrix which is at the same time a generator g. This shows that we are in type III. The curve C has a cuspidal singularity at the point O corresponding to the generator g. The curve C intersects any plane $\Omega(p, \mathbb{P}^3), p \in g$, in one point and every plane $\Omega(\pi, \mathbb{P}^3), g \subset \pi$, at two points. This is Edge's type III (B).

In type 2 (iii), we have a triple line on S formed by the projection of the line directrix E_0 of $S_{2,5}$ and its two generators. We are in case V, where the singular point of C is the singular point of the quadric cone. This is Edge's type V (B).

In type 2 (iv), we have a triple line projected from a rational cubic curve. We have two line directrices of S, one is a triple line. The curve C is nonsingular. This is Edge's type IV (A).

Next, we have to classify elliptic ruled quartic surfaces in $\mathbb{P}^3$. Let $\pi : X \to C$ be a minimal ruled surface with a base C. We write X in the form $X = \mathbb{P}(\mathcal{E}_0)$, where $\mathcal{E}_0$ is a normalized rank 2 locally free sheaf. Since $K_C = 0$ in our case, the canonical class formula (10.48) gives

$$K_X = -2\mathfrak{e} + \pi^*(\mathfrak{a}). \tag{10.72}$$

By the adjunction formula, $0 = E_0^2 + K_X \cdot E_0 = -E_0^2 + \deg \mathfrak{a}$. Thus, $a = \deg \mathfrak{a} = \mathfrak{e}_0^2 \leq 0$.

Let $|h|$ be the linear system on X which defines the normalization map $f : X \to S$. We can write $h \equiv m\mathfrak{f} + \mathfrak{e}$, where $\mathfrak{f}$ is the class of a fiber. Since h is ample, intersecting both sides with $\mathfrak{e}$, we get $m + a > 0$. We also have $h^2 = 2m + a = 4$. This gives two possibilities $a = 0, m = 2$ and $a = -2$ and $m = 3$. In the second case $h \cdot \mathfrak{e} = 1$, hence $|h|$ has a fixed point on E_0. This case is not realized (it leads to the case when S is a cubic cone). The formula for the double-point locus gives $\mathbb{D}(f) \equiv 2h - \pi^*(\mathfrak{d})$, where $d = \deg \mathfrak{d} = 4$. Thus, we obtain

$$H \equiv 2\mathfrak{f} + \mathfrak{e},\ \mathfrak{e}^2 = 0,\ D(f) \equiv 2\mathfrak{e}.$$

By Riemann-Roch, $\dim|h| = 3$. Since $\dim|h - \mathfrak{e}| = \dim|2\mathfrak{f}| = 1$, we obtain that the image of E_0 is a line. Since the restriction of $|h|$ to E_0 is a linear series of degree 2, the image of E_0 is a double line. We have two possibilities: $D(f)$ consists of two curves E_0+E_0', or $D(f)$ is an irreducible curve D with $h\cdot D = 4$. Since $|h-D| = \emptyset$, we obtain that the image of D is a space quartic, so it cannot be the double locus. This leaves us with two possible cases: $D(f)$ is the union of two disjoint curves $E_0 + E_0'$, or $D(f) = 2E_0$.

In the first case $H \cdot E_0 = H \cdot E_0' = 2$ and $\dim|h - E_0| = \dim|h - E_0'| = \dim|2\mathfrak{f}| = 1$. This shows that the images of E_0 and E_0' are two skew double lines on S. The curve C is a nonsingular elliptic curve in $G_1(\mathbb{P}^3)$. It spans a three-dimensional subspace equal to the intersection of two special linear line complexes.

Since $X = \mathbb{P}(\mathcal{E})$ has two disjoint sections with self-intersection 0, the sheaf $\mathcal{E}$ splits into the direct sum $\mathcal{L}_1 \oplus \mathcal{L}_2$ of invertible sheaves of degree 0. This easily follows from [379, Chapter V, Proposition 2.9]. One of them must have a nonzero section, i.e., must be isomorphic to $\mathcal{O}_C$. So we obtain

$$X \cong \mathbb{P}(\mathcal{O}_C \oplus \mathcal{O}_C(\mathfrak{a})),$$

where $\deg \mathfrak{a} = 0$. Note that X cannot be the direct product $C \times \mathbb{P}^1$ because, in this case, the image of any $C \times \{x\}$ must be a double line, in other words, in this case, $|H|$ defines a degree 2 map. So, we have $\mathfrak{a} \sim 0$.

In the second case, two double lines come together forming the curve of tacnodes. In this case, the curve C lies only in one special linear line complex. The pencil of hyperplanes containing C intersects the dual Klein quadric at one point.

Let $\sigma : \mathcal{E} \to \mathcal{O}_C(\mathfrak{e})$ be the surjective map of sheaves corresponding to the section E_0. Since $\deg\mathcal{E} = \deg\mathfrak{a} = 0$, we have $\deg \mathrm{Ker}(\sigma) = 0$. Thus, $\mathcal{E}$ can be given as an extension of invertible sheaves

$$0 \to \mathcal{O}_C(\mathfrak{b}) \to \mathcal{E} \to \mathcal{O}_C(\mathfrak{a}) \to 0,$$

where $\deg \mathfrak{b} = 0$. Suppose this extension splits, then X has two disjoint sections with self-intersection zero. By above, we see that the map defined by the linear system $|h|$ maps each section to a double line of S. This leads to the first case. So, in our case, there are no disjoint sections, and hence the extension does not split. This implies that $\mathrm{Ext}^1(\mathcal{O}_C(\mathfrak{a}), \mathcal{O}_C(\mathfrak{b})) = H^1(C, \mathcal{O}_C(\mathfrak{e} - \mathfrak{b})) \neq \{0\}$. This is possible only if $\mathfrak{b} \sim \mathfrak{a}$. Since $\mathcal{E}$ has a nonzero section, we also have $H^0(C, \mathcal{O}_C(\mathfrak{a})) \neq \{0\}$, i.e. $\mathfrak{e} \sim 0$. Thus, we obtain that $\mathcal{E}$ is given by a non-split extension

$$0 \to \mathcal{O}_C \to \mathcal{E} \to \mathcal{O}_C \to 0.$$

In fact, it is known that any elliptic ruled surface with $\mathfrak{e}^2 = 0$ which corresponds to a non-split vector bundle, must be isomorphic to the ruled surface $\mathbb{P}(\mathcal{E})$, where $\mathcal{E}$ is defined by the extension from above (see [379, Chapter V, Theorem 2.15]).

Let us summarize our classification in the following Table 10.3.

Type	Double curve	Edge	Cremona	Cayley	Sturm
1 (i),(iv)'	R_3	I	1	10	III
1 (ii)	L+K	II (C)	2	7	V
1 (iii)	L+L'+G	IV (B)	5	2	VII
1 (v)	3L	II (A)	8	9	IX
1 (iv)	3L	III (A)	3	-	XI
1 (vi),(vi)'	2L+G	V (A)	6	5	VIII
2 (i)	R_3	II (B)	7	8	IV
2 (ii)	L+K	III (B)	4	-	VI
2 (iii)	3L	V (B)	10	6	XII
2 (iv)	3L	IV (A)	9	3	X
$g = 1$	L+L'	VI(A)	11	1	I
$g = 1$	2L	VI(B)	12	4	II

Table 10.3 *Quartic ruled surfaces*

Here, R_3 denotes a curve of degree 3, L denotes a line, K is a conic and G is a generator.

A finer classification of quartic ruled surfaces requires the description of the projective equivalence classes. We refer to [600] for a modern work on this. Here, we explain, following [68], only the fine classification assuming that the double curve is a Veronese cubic R_3. First, by projective transformation, we can fix R_3 which will leave us only with the three-dimensional subgroup G of $\mathrm{PGL}(4)$ leaving R_3 invariant. It is isomorphic to $\mathrm{PSL}(2)$.

Let $\mathcal{N}$ be the net of quadrics in $\mathbb{P}^3$ that contains R_3. It defines a rational map $\alpha : \mathbb{P}^3 \dashrightarrow \mathcal{N}^\vee$. The pre-image of a point s in $\mathcal{N}^\vee$, i.e. a pencil in $\mathcal{N}$, is the base locus of the pencil. It consists of the curve R_3 plus a line intersecting R_3 at two points. This makes the identification between points in $\mathcal{N}^\vee \cong \mathbb{P}^2$ and secants of R_3. The pre-image of a conic K in $\mathcal{N}^\vee$ is a quartic surface which is the union of secants of R_3. It is a quartic ruled surface. Conversely, every quartic ruled surface S containing R_3 as its double curve is obtained in this way. In fact, we know that S is the union of secants of R_3 and hence the linear system of quadrics containing R_3 should blow down each secant to a point in $\mathcal{N}^\vee$. There are many direct geometric constructions of quartic ruled surfaces. The first historical one uses Cayley's construction of a ruled surface as the union of lines intersecting three space curves (see Example 10.4.11). For example, taking $(d_1, d_2, d_3) = (2, 2, 1)$ and $(a_{12}, a_{13}, a_{23}) = (2, 0, 1)$ gives a quartic ruled surface with a double conic and a double line that intersect at one point. Another construction is due to Cremona. It is a special case of the construction from Example 10.4.3, where we take the curves C_1 and C_2 of degree 2. If the two conics are disjoint, a correspondence of bidegree $(1, 1)$ gives a quartic ruled surface. In the next Subsection we will discuss a more general construction due to B. Wong [810].

Finally, we reproduce equations of quartic ruled surfaces (see [271, p. 69], p. 69).

$$
\begin{aligned}
I &: Q(xz - y^2, xw - yz, yw - z^2) = 0,\\
&\quad \text{where } Q = \sum_{1 \le i, j \le 3} a_{ij} t_i t_j \text{ is a nondegenerate quadratic form;}\\
II(A) &: zy^2(ay + bx) + wx^2(cy + dx) - ex^2y^2 = 0;\\
II(B) &: \text{same as in (I) with } a_{22}^2 + 2a_{22}a_{13} - 4a_{12}a_{23} + a_{11}a_{33} = 0;\\
II(C) &: (cyz + bxz + axy + zw - wx)^2 - xz(ax - by + cz)^2 = 0;\\
III(A) &: ax^2y^2 - (x + y)(x^2w + y^2z) = 0;\\
III(B) &: (xw + yz + azw)^2 - zw(x + y)^2 = 0;\\
IV(A) &: x(az + bw)w^2 - y(cz + dw)z^2 = 0;\\
IV(B) &: y^2z^2 + axyzw + w^2(bz + cx)x = 0,\\
&\quad (yz - xy + awx)^2 - xz(x - z + bw)^2 = 0;\\
V(A) &: : (yz - xy + axw)^2 - xz(x - z)^2 = 0;\\
V(B) &: (az^2 + bzw + cw^2)(yz - xw) - z^2w^2 = 0;\\
VI(A) &: ax^2w^2 + xy(bz^2 + czw + dw^2) + ey^2z^2 = 0;\\
VI(B) &: (xw - yz)^2 + (ax^2 + bxy + cy^2)(xw - yz) +\\
&\quad (dx^3 + ex^2y + fxy^2 + gy^3)x = 0.
\end{aligned}
$$

10.4.5 Ruled surfaces in $\mathbb{P}^3$ and the tetrahedral line complex

Fix a pencil $\mathcal{Q}$ of quadrics in $\mathbb{P}^3$ with a nonsingular base curve. The pencil contains exactly four singular quadrics of corank one. We can fix coordinate systems to transform the equations of the quadrics to diagonal forms

$$\sum_{i=0}^{3} a_i t_i^2 = 0, \quad \sum_{i=0}^{3} b_i t_i^2 = 0.$$

The singular points of four singular quadrics in the pencil are the reference points $p_1 = [1,0,0,0]$, $p_2 = [0,1,0,0]$, $p_3 = [0,0,1,0]$, $p_4 = [0,0,0,1]$. For any point not equal to one of these points, the intersection of the polar planes $P_x(Q), Q \in \mathcal{Q}$, is a line in $\mathbb{P}^3$. This defines a rational map $f : \mathbb{P}^3 \dashrightarrow G_1(\mathbb{P}^3) \subset \mathbb{P}^5$ whose image is a tetrahedral line complex K (see the end of Subsection 10.3.6). The Plücker coordinates p_{ij} of the line $f([t_0,t_1,t_2,t_3])$ are

$$p_{ij} = (a_i b_j - a_j b_i) t_i t_j.$$

For any space curve C of degree m not passing through the reference points, its image under the map f is a curve $C' = f(C)$ of degree $d = 2m$ in the tetrahedral complex. It defines a ruled surface S_C in $\mathbb{P}^3$ of degree $2d$, the union of lines $f(x), x \in C$. If we consider the graph $G_f \subset \mathbb{P}^3 \times G_1(\mathbb{P}^3)$ of f, its projection to $G_1(\mathbb{P}^3)$ over C' is the universal family $Z_{C'}$. Its projection to $\mathbb{P}^3$ is our ruled surface.

Let Π be a plane in $\mathbb{P}^3$ not containing any of the points p_i. The restriction of f to Π is given by the complete linear system of conics. Thus, its image $f(\Pi)$ is a Veronese surface embedded in $G_1(\mathbb{P}^3)$ as a congruence of secant lines of a rational normal cubic curve R_Π in $\mathbb{P}^3$. The curve R_Π is the image of the map

$$\phi_\Pi : \mathcal{Q} \cong \mathbb{P}^1 \to \mathbb{P}^3 \qquad (10.73)$$

which assigns to a quadric $Q \in \mathcal{Q}$ the intersection of polars $P_x(Q), x \in \Pi$. For any line ℓ in Π, the ruled surface S_ℓ is a quadric containing R_Π. So, one can identify the net of quadrics containing R_Π with the dual plane $\Pi^\vee$. More generally, for any curve C in Π of degree m, the ruled surface S_C is a surface of degree $2m$ containing R_Π. Consider a point $x \in \Pi$ as the intersection point of two lines ℓ_1 and ℓ_2 in Π. Then, the line $f(x)$ is contained in the intersection of the two quadrics S_{ℓ_1} and S_{ℓ_2}. Hence, it coincides with a secant of the curve R_Π. Thus, we obtain that generators of S_C are secants of R_Π. If $m = 2$, this gives that $f(C)$ is the intersection of a Veronese surface with a linear line complex, a general choice of Π gives us quartic surfaces of type I (i).

Take a line ℓ in Π. The quadric S_ℓ comes with a ruling on the quadrics whose generators are secants of R_Π. The set of lines in Π that parameterizes singular quadrics containing R_Π is a conic in $\Pi^\vee$. The dual conic $\mathfrak{C}$ in Π parameterizes pencils of quadrics containing R_Π and its line tangent. The corresponding ruled quartic surface is the developable quartic surface, a special case of type I (iii). The points on the line are pencils of quadrics containing Q_ℓ. If ℓ is tangent to $\mathfrak{C}$, then the tangency point is a pencil of quadrics which all tangent to $R_3(\Pi)$ at one point. The point is the singular point of a unique singular quadric in the pencil.

The lines $f(x), x \in \ell$, are generators of the quadric S_ℓ which intersect R_Π at two points. If ℓ is tangent to $\mathfrak{C}$, then S_ℓ is a singular quadric and all the lines $f(x), x \in \ell$, pass through its singular point. The curve R_Π also passes through this point. In this case, the line ℓ intersects a curve C of degree m in Π at m points different from $\mathfrak{C}$, all the generators of S_C corresponding to these points must pass through one point on R_Π. The converse is also true, if the generators $f(x), x \in C$, all pass through the same point on R_Π, then these points lie on a line tangent to $\mathfrak{C}$. Thus, we obtain that R_Π is m-multiple curve on S_C. This agrees with type $1(i)$ of quartic ruled surfaces. Also, C intersects $\mathfrak{C}$ at $2m$ points corresponding to generators tangent to R_Π. If $m = 2$, we get four torsal generators.

Now, let us see what happens if we choose a special plane Π. For example, let us take Π passing through one of the points $p_1, \ldots, p_4$, say p_1. Then, the map ϕ_Π defined in (10.73) is not anymore of degree 3. In fact, it is not defined at the quadric Q which has p_1 as its singular point. The map extends to a map of degree 2. Thus, the cubic R_Π degenerates to a conic. The lines in Π correspond to quadrics containing the conic R_Π and some lines intersecting the conic. This is a degeneration of the singular curve to the union of a conic and a line.

Finally, let us see how elliptic quartic surfaces arise. Take Π passing through the points p_1 and p_2. Take a nonsingular cubic C in the plane which passes through p_1 and p_2. The linear system of quadrics defining the rational map f has two of its base points on C. Thus, its image in $G_1(\mathbb{P}^3)$ is a quartic elliptic curve. We see that a ruled surface of degree 6, which corresponds to a general cubic, degenerates in this case to the union of a quartic surface and two planes (the images of the blow-ups of p_1 and p_2). The cubic R_Π degenerates to a line, one of the two double lines of S. A quadric corresponding to a line through p_1 or p_2 degenerates to a plane with a choice of a pencil of lines in it. This plane does not depend on the line, but the pencil of lines in the plane does. The line passing through p_1 and p_2 is blown down under f to a point in $G_1(\mathbb{P}^3)$ defining the second double line of S_C. This is the intersection line of the planes corresponding to p_1 and p_2.

Exercises

10.1 Let $P_n \subset \mathbb{C}[t]$ be the space of polynomials of degree $\leq n$. Let $f_0, \ldots, f_m$ be a basis of a subspace L of P_n of dimension $m+1$. Consider the *Wronskian* of the set $(f_0, \ldots, f_m)$

$$W(f_0, \ldots, f_m) = \det \begin{pmatrix} f_0 & f_1 & \ldots & f_m \\ f_0'(t) & f_1'(t) & \ldots & f_m'(t) \\ \vdots & \vdots & \vdots & \vdots \\ f_0^{(m)}(t) & f_1^{(m)}(t) & \ldots & f_m^{(m)}(t) \end{pmatrix}.$$

Show that the map

$$G_m(\mathbb{P}^n) \to \mathbb{P}^{(m+1)(n-m)}, \quad L \mapsto [W(f_0, \ldots, f_m)],$$

is well defined and is a finite map of degree equal to the degree of the Grassmannian in its Plücker embedding.

10.2 Show that any $\binom{n+1}{2} - 1$ lines in $G_1(\mathbb{P}^n)$, $n \geq 3$, lie in a linear line complex. Using this, prove that one can choose coordinates in $\mathbb{P}^n$ so that any linear line complex can be given by Plücker equations $p_{12} + \lambda p_{34} = 0$, where $\lambda = 0$ if and only if the line complex is special.

10.3 Show that the tangent lines of any smooth curve of genus g and degree d in $\mathbb{P}^n$ is contained in a linear line complex if $2(d+g-1) < \binom{n+1}{2}$.

10.4 Show that any k-plane Λ of $G_m(\mathbb{P}^n)$ coincides with the locus of m-planes in $\mathbb{P}^n$ containing a fixed $(m-1)$-plane and contained in a fixed $(m+k)$-plane or with the locus of m-planes contained in a fixed $(k+1)$-plane and containing a fixed $(k-m)$-plane. Identify these loci with appropriate Schubert varieties.

10.5 Using the previous exercise, show that any automorphism of $G_r(\mathbb{P}^n)$ arises from a unique projective automorphism of $\mathbb{P}^n$ unless $n = 2r+1$, in which case $\mathrm{PGL}(n+1)$ is isomorphic to a subgroup of index 2 of $\mathrm{Aut}(G_r(\mathbb{P}^n))$.

10.6 How many lines intersect a set of m general k-planes in $\mathbb{P}^n$?

10.7 Show that $\mathrm{Sec}_k(G_1(\mathbb{P}^n))$ is equal to the set of singular points of $\mathrm{Sec}_{k+1}(G_1(\mathbb{P}^n))$ for all $k = 0, \ldots, [\frac{n-3}{2}]$.

10.8 Using Schwarzenberger vector bundles, prove that the projective plane embedded in $G_1(\mathbb{P}^n)$ as the surface of secants of a normal rational curve of degree d in $\mathbb{P}^d$ is isomorphic to the Veronese surface V_{n-1}^2.

10.9 Let $Q_1$and Q_2 be two nonsingular quadrics in $\mathbb{P}^3$ with a choice of a ruling of lines on each of them. Any general line ℓ intersects $Q_1 \cup Q_2$ at four lines, two from each ruling. Together with ℓ, these lines span four planes in the pencil of planes through ℓ. Show that the closure of the locus of lines ℓ such that the four planes is projectively equivalent to the four intersection points of ℓ with Q_1 and Q_2 form a Battaglini line complex. Also, show that any general Battaglini line complex can be obtained in this way [697].

10.10 Show that the linear system of quadrics in $\mathbb{P}^4$ passing through a rational normal quartic curve R_4 defines a rational map $\Phi : \mathbb{P}^4 \dashrightarrow \mathbb{P}^5$ whose image is a nonsingular quadric in $\mathbb{P}^5$ identified with the Klein quadric $G(2,4)$. Show that:

(i) the secant variety $S_1(R_4)$ is mapped to a Veronese surface;

(ii) the map Φ extends to a regular map of the blow-up of $\mathbb{P}^4$ along R_4 that maps the exceptional divisor to a ruled hypersurface of degree 6 which is singular along the Veronese surface;

(iii) the image of a hyperplane in $\mathbb{P}^4$ is a tetrahedral line complex;
(iv) the image of a plane in $\mathbb{P}^4$ not intersecting R_4 is a Veronese surface;
(v) the image of a trisecant plane of R_4 is a plane in $G(2,4)$. Show that planes from another family of planes are the images of cubic ruled surfaces singular along R_4.

10.11 Show that four general lines in $\mathbb{P}^4$ determine the unique fifth one such that the corresponding points in $G_1(\mathbb{P}^4) \subset \mathbb{P}^9$ lie in the same three-dimensional subspace. Any plane which meets four lines meets the fifth line (called the *associated line*).

10.12 Show that two linear line complexes $\mathfrak{C}_\omega$ and $X_{\omega'}$ in $G_1(\mathbb{P}^3)$ are apolar to each other if and only if $i_\omega(X_{\omega'}) = \mathfrak{C}_\omega$.

10.13 Show that a general web of linear line complexes in $G_1(\mathbb{P}^4)$ contains five special line complexes.

10.14 Show that the projection of the Segre cubic primal from its nonsingular point defines a double cover with the branch locus isomorphic to a Kummer surface.

10.15 Using the Schubert calculus, show that the variety of lines contained in a cubic hypersurface in $\mathbb{P}^4$ with isolated singularities is a surface of degree 45 in the Plücker embedding of $G_1(\mathbb{P}^4)$. Show that the variety of lines contained in the Segre cubic primal S_3 is a surface of degree 45 that consists of 15 planes and six del Pezzo surfaces of degree 5.

10.16 Let $\mathcal{N}$ be a general two-dimensional linear system of quadrics in $\mathbb{P}^3$. Show that the union of lines contained in quadrics $Q \in \mathcal{N}$ is parameterized by a cubic line complex (called a *Montesano line complex*) [524].

10.17 Let $p_1, \ldots, p_{nd+1}$ be points in $\mathbb{P}^n$ in general linear position. A *monoidal line complex* consists of all codimension 2 linear subspaces Π of $\mathbb{P}^n$ for which there exists a monoidal hypersurface with singular locus containing to Π. Using the isomorphism $G_{n-2}(\mathbb{P}^n) \cong G_1(\mathbb{P}^n)$, we consider it as a line complex. Show that the degree of a monoidal line complex is equal to $\frac{1}{2}3d(d-1)$ and it coincides with a Montesano line complex when $n = d = 3$ [237].

10.18 Consider a smooth curve C of degree d and genus g in $\mathbb{P}^3$ and choose two general lines ℓ and ℓ'. Find the degree of the scroll of lines that intersect C, ℓ and ℓ'.

10.19 Let F be a surface of degree 6 in $\mathbb{P}^3$ which has the edges of the coordinate tetrahedron as its double lines. Find an equation of F and show that its normalization is an Enriques surface.

10.20 Show that the Hessian surface of a developable quartic ruled surface is equal to the surface itself taken with multiplicity two. The Steinerian in this case is the whole space [789].

10.21 Consider the embedding of the Klein quartic curve of genus three in $\mathbb{P}^3$ given by the linear system $|3\theta|$, where θ is the unique even theta characteristic invariant with respect to the group of automorphisms of the curve. Show that each hyperosculating point is of multiplicity two, and it is equal to the image of an inflection point.

10.22 Show that a generator intersecting the double curve of a ruled surface at a pinch point is a torsal generator.

10.23 Classify all ruled surfaces in $\mathbb{P}^3$ which have two line directrices.

10.24 For each type of a quartic ruled surface, find the type of its dual quartic ruled surface.

10.25 Find projective equivalence classes of quartic ruled surfaces with a triple line.

10.26 Let S be a quartic ruled surface with a twisted cubic as its double curve. Show that it contains a line directrix if and only if it contains three coplanar generators.

10.27 Let S be a ruled surface with two line directrices of multiplicities n and m. Show that the order of S is equal to $n + m$.

Historical Notes

The primary sources for these notes are [487], [495], [696], [577], and [820]. Line Geometry originates from J. Plücker, the first to consider lines in $\mathbb{P}^3$ as elements of a new four-dimensional space. These ideas appeared first in [594]. The details were published much later in [595]. The study of lines in $\mathbb{P}^3$ was very much motivated by mechanics and optics. E. Kummer gave the first differential geometrical treatment of line geometry in [470]. In 1859 A. Cayley introduced the coordinates in their modern form as six determinants of a 2×4-matrix and exhibited the quadric equation satisfied by the coordinates [110]. In a subsequent paper, under the same title, he introduced what is now called the Chow form of a space curve. The notions of linear line complex of lines and congruence of lines (the intersection of two linear line complexes) are due to Plücker and the first proofs of some of his results were given by G. Battaglini [42]. Among other earlier contributors to the theory of general line complexes, we cite M. Pash [578].

Plücker began the study of quadratic line complexes by introducing a singular quartic surface with 16 nodes. Although, in a special case, many Plïcker's results about quadratic line complexes were independently obtained by Battaglini. In his dissertation, and later published paper [453], Klein introduced the coordinate system determined by six mutually apolar linear line complexes and showed that the singular surface can be identified with a Kummer surface. The notion of the singular surface of a quadratic complex is due to Klein. We refer to [413], [429], [256] for the history of Kummer surfaces and their relationship with Line Geometry. We followed [429] in deriving the equation of a Kummer surface in the Klein coordinates.

Plücker defined a linear complex as we understand it now, i.e., as a set of lines whose coordinates satisfy a linear equation. The set of lines in a linear complex passing through a point x lies in a plane $\Pi(x)$; this defines a linear correlation from the space to the dual space. The correlations arising in this way satisfy the property $x \in \Pi(x)$. They were first considered by G. Giorgini [333] and A. Möbius [521] and were called Nullsystems by von Staudt ([720], p. 191). The notions of a null-line and a null-plane belong to Möbius. Chasles' Theorem 10.2.10 gives a purely geometric definition of a Nullsystem [126]. Linear systems of linear line complexes were extensively studied in Sturm's book [736].

In 1868, in his Inaugural dissertation at Bonn published later in [453], [456], F. Klein pointed out that Weierstrass's theory of canonical forms for a pair of quadratic forms can be successfully used for the classification of quadratic line complexes. This was accomplished later by A. Weiler (see also [792], [687]). The classification consists of 49 different types of line complexes corresponding to different Segre symbols of the pencil of quadrics. As we have already noticed earlier, A. Weiler was the first to introduce the Segre symbol [802], and Segre acknowledges this himself in [687]. In each case, the singular surface is described. For example, some of the ruled quartic surfaces can be obtained as singular surfaces of a degenerate quadratic complex. A full account of the classification, and the table can be found in Jessop's book [429]. Many special quadratic line complexes were introduced earlier by purely geometric means. Among them are the tetrahedral line complexes and Battaglini's harmonic line complexes [43] considered in the present chapter. A complete historical account of tetrahedral line complexes can be found in Lie's book [487]. Its general theory is attributed to T. Reye [614], and they are often called Reye line complexes. However, in different disguises, tetrahedral line complexes appear in much earlier works, for example, as the locus of normals to two confocal surfaces of degree 2 [58] (see a modern exposition in [702], p. 376), or as the locus of lines spanned by an argument and the value of a projective transformation [127], or as the locus of secants of twisted cubics passing through the vertices of a tetrahedron [537]. We refer to [639] and [382] for the role of tetrahedral line complexes in Lie's theory of differential equations and groups of transformations. A paper of Rowe [640] gives a nice exposition of the history of the quadratic line complex.

Modern multilinear algebra originates in Grassmann's work [352], [353]. We refer to [69] for the history of multilinear algebra. The editorial notes for the English translation of [353] are very helpful for understanding Grassmann's work. As a part of Grassmann's theory, a linear k-dimensional subspace of a linear space of dimension n corresponds to a decomposable k-vector. Its coordinates can be taken as the coordinates of the linear subspace and the associated projective subspace of $\mathbb{P}^{n-1}$. In this way, Grassmann was the first to give a higher-dimensional generalization of the Cayley-Plücker coordinates of lines in $\mathbb{P}^3$. Equations (10.4) of Grassmann varieties could not be found in his book. The fact that any relation between the Plücker coordinates follows from these relations was first proven by G. Antonelli [12] and much later by W. Young [812]. In [666] and [668] H. Schubert defines what we now call Schubert varieties, and computes their dimensions and degrees in the Plücker embedding. In particular, he finds the formula for the degree of a Grassmann variety. A modern account of Schubert's theory can be found in Hodge-Pedoe's book [404], v. II and Fulton's book [315].

The study of linear line complexes in arbitrary $[n]$ (the classical notation $[n]$ for $\mathbb{P}^n$ was introduced by Schubert in [666]) was initiated in the work of S. Kantor [439], F. Palatini [567] and G. Castelnuovo [96] (in case $n = 4$). The Palatini scroll was first studied in [568] and appeared often in modern literature on vector bundles (see, for example, [562]). Quadratic line complexes in $\mathbb{P}^4$ were extensively studied by B. Segre [691].

Although ruled surfaces were studied earlier (more from the differential geometry point of view), A. Cayley was the first to lay the foundations of the algebraic theory of ruled surfaces [105], [112], [113]. The term scroll belongs to Cayley. The study of non-normal surfaces in $\mathbb{P}^3$ and, in particular, ruled surfaces, began with G. Salmon [648], [649]. Salmon's work was extended by A. Cayley [117]. The notion of a torsal generator is due to Cayley. The formulas of Cayley and Salmon were revised in a long memoir of H. Zeuthen [817] and later in his book [818]. A modern treatment was given by R. Piene [583]. The fact that the class of a ruled surface is equal to its degree is due to Cayley. The degree of a ruled surface defined by three directrices from Example 10.4.11 was first determined by G. Salmon [647].

The classification of ruled cubic surfaces was done by A. Cayley in [113], Part II, and, independently, by L. Cremona [180]. Different projective generations of ruled cubic surfaces are given in Weyr's monograph [804]. The classification of quartic ruled surfaces was started by A. Cayley [113], Parts II and III. However, he had missed two types. Salmon's book [653, vol. II, Chapter XVI] contains a nice exposition of Cayley's classification. A complete classification was given later by L. Cremona [187]. An earlier attempt for this classification was made by M. Chasles [127]. The classification based on the theory of tetrahedral line complexes was given by B. Wong [810]. Ruled surfaces of degree 5 were classified by H. Schwarz [674]. Much later, W, Edge extended this classification to surfaces of degree 6 [271]. Edge's book and Sturm's book [735, Theil 1] give a detailed exposition of the theory of ruled surfaces. The third volume of Sturm's book contains an extensive account of the theory of quadratic line complexes.

11
Congruences of Lines in $\mathbb{P}^3$

This chapter is a continuation of the previous chapter. We will study irreducible congruences of lines in $\mathbb{P}^3$. Following Sturm and Fano, we will discuss different attributes of congruences and give the complete classification of congruences of order one and two with or without fundamental curves. We give a brief discussion of congruences of higher order. The classification of smooth surfaces in a nonsingular quadric in $\mathbb{P}^5$ can be viewed as an analog of the classification of smooth surfaces in $\mathbb{P}^3$ or $\mathbb{P}^4$; Hence, it admits a cohomological treatment, but such tools are not available when the assumption of smoothness is dropped. We are left with a purely geometrical approach that also gives a beautiful visualization of surfaces as two-dimensional familes of lines in $\mathbb{P}^3$s

11.1 Generalities on Congruence of Lines

11.1.1 The numerical invariants

Recall from the previous chapter that a *congruence of lines* in $\mathbb{P}^3$ is a reduced surface S in the Grassmann variety $\mathbb{G} = G_1(\mathbb{P}^3)$ of lines in $\mathbb{P}^3 = |E|$. The line ℓ_s in $\mathbb{P}^3$ corresponding to a point $s \in S$ is a *ray* of the congruence. The cohomology class of S in $H^2(\mathbb{G}, \mathbb{Z})$ is determined by two special Schubert classes $\sigma_2 = [\Omega(x)], \sigma_{1,1} = [\Omega(\Pi)]$, where $\Omega(x)$ is an α-plane of lines through a given point $x \in \mathbb{P}^3$, and $\Omega(\Pi)$ is a β-plane of lines contained in a given plane $\Pi \subset \mathbb{P}^3$. We can write for the cohomology class $[S]$ of S

$$[S] = m\sigma_2 + n\sigma_{1,1}, \tag{11.1}$$

for some integers m and n. Intersecting with $\sigma_{1,1}$, and using the Schubert calculus from Subsection 10.1.2, we obtain that m is equal to the number of rays in S through a general point in $\mathbb{P}^3$, it is called the *order* of the congruence.

Intersecting with σ_2, we obtain that n is the number of rays in S contained in a general plane in $\mathbb{P}^3$. It is called the *class* of S. The pair (m, n) is the *bidegree* of S.

It follows from the Schubert Calculus that $\sigma_2 + \sigma_{1,1} = \sigma_1^2$, where σ_1 is the class of a hyperplane section of $\mathbb{G}$ in its Plücker embedding in $\mathbb{P}^5$. Restricting to S, we obtain that

$$\deg(S) = m + n, \tag{11.2}$$

where $\deg(S)$ is the degree of the surface S in $\mathbb{P}^5$.

- Unless stated otherwise, we will assume that S is an irreducible surface.

The universal family $Z_S = \mathbb{P}(\mathbb{Q}_{\mathbb{G}})$ of lines parameterized by S is a closed subvariety of the universal family $Z_{\mathbb{G}}$ of lines, which was discussed in Section 10.1. It is equal to $\mathbb{P}(Q_S)$, where $\mathbb{Q}_S = i_S^*\mathbb{Q}_{\mathbb{G}}$ for the closed embedding $i_S : S \hookrightarrow \mathbb{G}$. It comes with two projections

$$\begin{array}{ccccc} & & Z_S & & \\ & \swarrow^{p_S} & & \searrow^{q_S} & \\ \mathbb{P}^3 & & & & S. \end{array} \tag{11.3}$$

The projection q_S is a projective bundle over S isomorphic to $\mathbb{P}(Q_S)$, where Q_S is the restriction of the universal quotient bundle $Q_{\mathbb{G}}$ over $\mathbb{G}$. We have

$$h := c_1(Q_S) = c_1(O_S(1)), \quad c_2(Q_S) = \sigma_{1,1} \cap [S] = n[\text{point}]. \tag{11.4}$$

It is clear that the degree of the map p_S is equal to the order m of S. The class of S is the degree of the map p_{S^*}, where S^* is the *dual congruence*, the image of S under the map $G_1(|E|) \to G_1(|E^\vee|)$. Obviously, the bidegree of the dual congruence is equal to (n, m).

Proposition 11.1.1. *A congruence of order $m = 0$ (resp. $n = 0$) coincides with an β-plane (resp. α-plane).*

Proof Assume $m = 0$. Since no ray passes through a general point in $\mathbb{P}^3$, the image of Z_S in $\mathbb{P}^3$ is a surface F. Thus, the projection $p_S : Z_S \to F$ is of relative dimension 1. This means that a general point x of F is the vertex of a cone whose generators are rays in $\Omega(x)$. This implies that F is not reduced unless it is a plane Π and S coincides with the β-plane $\Omega(\Pi)$. Using the dual arguments we prove the assertion about congruences with $n = 0$. □

- From now on, unless stated otherwise, we assume that $m, n > 0$.

For any line ℓ in $\mathbb{P}^3$, we set

$$C(\ell) := \mathbb{T}_\ell(\mathbb{G}) \cap S = \Omega(\ell) \cap S = \{s \in S : \ell_s \cap \ell \neq \emptyset\}. \tag{11.5}$$

If $\ell = \ell_s$ is a ray of S, the hyperplane $\mathbb{T}_\ell(\mathbb{G})$ is tangent to S at s, and hence $s \in \mathrm{Sing}(C(\ell))$.

The arithmetic genus $p_a(C(\ell))$ does not depend on ℓ, it is the arithmetic genus of a hyperplane section of S. It is denoted by p_a and is called the *arithmetic sectional genus* of S (we deliberately avoid to denote it by $p_a(S)$ to avoid the confusion with the arithmetic genus of S).

Assume $C(\ell)$ is reduced and let $C(\ell)^{\mathrm{nrm}}$ be its normalization. The rational map

$$f_\ell : C(\ell) \to \ell, \quad s \mapsto \ell_s \cap \ell, \tag{11.6}$$

extends to a regular map

$$\tilde{f}_\ell : C(\ell)^{\mathrm{nrm}} \to \ell. \tag{11.7}$$

We denote by m_ℓ its degree. Obviously, if ℓ is a general line, $m_\ell = m$. However, if $\ell = \ell_s$ is a general ray of S, then $m_\ell = m - 1$. The map f_{ℓ_s} is not defined at the singular point s of $C(\ell_s)$ but the map $\tilde{f}_{\ell_s}$ is defined at the pre-images of this point in $C(\ell)^{\mathrm{nrm}}$. For example, if $m = 2$, we obtain that $C(\ell_s)^{\mathrm{nrm}} \cong \ell_s \cong \mathbb{P}^1$.

For any curve $C \subset S$, we denote by $\mathcal{R}(C)$ the image of $q_S^{-1}(C)$ in $\mathbb{P}^3$. It is a ruled surface with generators $\ell_s, s \in C$, and the generatrix C. Its order is equal to the degree of $C \subset \mathbb{P}^5$.

For any line ℓ in $\mathbb{P}^3$, we set

$$\mathcal{R}(\ell) := \mathcal{R}(C(\ell)).$$

This is a ruled surface of degree $m + n$ whose generators are rays intersecting ℓ. The line ℓ intersects each of its generator, hence it is contained in it. It is a m_ℓ-multiple directrix of $\mathcal{R}(\ell)$. If $\ell = \ell_s$ is a ray, then ℓ is a generator of the ruled surface, it is a $(m_\ell - 1)$-multiple generator of the cone. A ray is called a *multiple ray* if $m_\ell < m - 1$.

If ℓ is not a ray of S, its pre-image $p_S^{-1}(\ell)$ is a directrix of the ruled surface $q_S^{-1}(C(\ell))$. The projection q_S defines a birational map of ℓ onto $C(\ell)$. If $\ell = \ell_s$ is a ray,

$$q_S^{-1}(\ell_s) = (m - m_{\ell_s})q_S^{-1}(s) + L(s), \tag{11.8}$$

where $q_S : L(s) \to C(\ell_s)$ is a birational map (on each irreducible component).

For any point $x \in \mathbb{P}^3$, the fiber $p_S^{-1}(x)$ consists of rays ℓ_s passing through x. The set of such rays is equal to the intersection $\Omega(x) \cap S$ and the projection

$$q_S : p_S^{-1}(x) \to \Omega(x) \cap S \tag{11.9}$$

is an isomorphism with the inverse map $s \to (x, s)$.

Definition 11.1.2. *A point $x \in \mathbb{P}^3$ is called a* fundamental point *of S if $p_S^{-1}(x)) = \Omega(x) \cap S$ contains infinitely many rays. The* degree $h(x)$ *of x is the degree of the one-dimensional component of $\Omega(x) \cap S$ is the* degree *of x. For any fundamental point x, we denote by $K(x)$ the one-dimensional component of $p_s(q_S^{-1}(\Omega(x)))$. It is a cone of degree $h(x)$ with vertex at x.*

We denote by $\mathrm{Fund}(S)$ the set of fundamental points of S and let $\mathrm{Fund}(S)_1$ (resp. $\mathrm{Fund}(S)_0$) denote its one-dimensional part (resp. the set of isolated fundamental points).

The dual notion is the notion of a *fundamental plane*. It is a plane in $\mathbb{P}^3$ that contains infinitely many rays. Its degree is the class of the one-dimensional part of the curve of rays.

Let r be the number of secant lines of S in $\mathbb{G}$ passing through a general point $x \in \mathbb{G}$. Here, we assume that a secant line joins two nonsingular points of S or tangent to S at its nonsingular point. The number r is called the *rank* of S. If S is a degenerate surface in $\mathbb{P}^5$, then obviously, $r = 0$. Since $\mathbb{G}$ is a quadric hypersurface in $\mathbb{P}^5$, a secant line of S through a general point $\ell \in \mathbb{G}$ is contained in $\mathbb{G}$. This gives the following geometric interpretation of the rank. A line in $\mathbb{G}$ is a pencil of rays equal to the Schubert variety $\Omega(x, \Pi) = \Omega(x) \cap \Omega(\Pi)$. We fix a general line ℓ in $\mathbb{P}^3$ and consider the pencil $\ell^\perp$ of planes through ℓ. A secant line corresponds to one of these planes that contain two of the rays from S intersecting at a point in ℓ. The rank is equal to the number of such planes.

Lemma 11.1.3. *The rank r is equal to zero if and only if S is degenerate in the Plücker embedding or non-degenerate and coincides with a Veronese surface in $\mathbb{P}^5$, hence $n + m = 4$. In particular, $r = 0$ if all rays intersect a line in $\mathbb{P}^3$.*

Proof If S is degenerate, its secant variety is contained in a hyperplane, hence a general point of $\mathbb{G}$ is not contained in a secant, hence $r = 0$. By a theorem of Zak (see Subsection 7.4.2), the secant variety of a non-degenerate surface in $\mathbb{P}^5$ is a proper subvariety only if the surface is a Veronese surface. So, if S is not-degenerate and not a Veronese surface, its secant variety is the whole $\mathbb{P}^5$. This implies that a general point in the quadric $\mathbb{G}$ is contained in a secant variety. Since the secant passing through this point meets $\mathbb{G}$ at ≥ 3 points, it is contained in $\mathbb{G}$, hence $r \neq 0$. □

A congruence S may be a singular surface. We will call ray ℓ_s, where s is a singular point of S, a *singular ray* (a *double ray* in the classical terminology).

Let $\nu : S^{\mathrm{nrm}} \to S$ be the normalization map of S, and $\sigma : \tilde{S} \to S$ be the minimal resolution of S^{nrm}. We denote by

$$\pi : \tilde{S} \to S$$

the composition $\nu \circ \sigma$.

Abusing the notation, we set

$$\mathcal{O}_{\tilde{S}}(1) := \pi^* \mathcal{O}_S(1).$$

It follows from the Bertini theorem (see [581, §4]) that a general member of $|\mathcal{O}_{\tilde{S}}(1)|$ is a smooth curve. We denote its genus by g and call it the *sectional genus* of S.

Let

$$\tilde{q}_S : \tilde{Z}_S = Z_S \times_S \tilde{S} \to \tilde{S}$$

be the pre-image of the $\mathbb{P}^1$-bundle $q_S : Z_S \to S$ to $\tilde{S}$, and let $\tilde{p}_S$ be the composition of the projections $\tilde{Z}_S \to Z_S$ and $p_S : Z_S \to \mathbb{P}^3$. We have the following commutative diagram expressing these maps:

$$\begin{array}{ccccccc} & & Z_{\mathbb{G}} & \hookleftarrow & Z_S & \longleftarrow & \tilde{Z}_S \\ & \swarrow^{p_{\mathbb{G}}} & \downarrow^{q_{\mathbb{G}}} & & \downarrow^{q_S} & & \downarrow^{\tilde{q}_S} \\ \mathbb{P}^3 & \xleftarrow{\tilde{p}_S} & \mathbb{G} & \hookleftarrow & S & \xleftarrow{\pi} & \tilde{S}. \end{array} \tag{11.10}$$

Let

$$Q_{\tilde{S}} := \pi^*(Q_{\mathbb{G}}|S).$$

The projection $\tilde{q}_S : \tilde{Z}_S \to \tilde{S}$ is the projective bundle $\mathbb{P}(Q_{\tilde{S}}) \to \tilde{S}$. Its tautological line bundle $\mathcal{O}_{\mathbb{P}(Q_{\tilde{S}})}(1)$ is isomorphic to $\tilde{p}_S^* \mathcal{O}_{\mathbb{P}^3}(1)$.

Let

$$h = c_1(Q_{\tilde{S}}) = \pi^*(c_1(\mathcal{O}_S(1))).$$

The following proposition follows from the discussion in Subsection 2.4.2:

Proposition 11.1.4. *Let* $H = c_1(\mathcal{O}_{\mathbb{P}(Q_{\tilde{S}})}(1))) = \tilde{p}_S^*(c_1(\mathcal{O}_{\mathbb{P}^3}(1)))$. *Then,*

$$H^*(\tilde{Z}_S, \mathbb{Z}) \cong H^*(\tilde{S}, \mathbb{Z})[H]/(H^2 - H \cdot \tilde{q}_S^*(h) + n\mathfrak{f}), \tag{11.11}$$

where $\mathfrak{f}$ *is the cohomology class of a fiber of the projection* $\tilde{q} : \tilde{Z}_S \to \tilde{S}$.

There is a simple geometrical interpretation of the basic relation

$$H^2 = H \cdot \tilde{q}_S^*(h) - n\mathfrak{f}.$$

To simplify the notation, we assume that $\tilde{S} = S$ and identify the fibers $q_S^{-1}(t)$ with the rays ℓ_t. A general plane Π contains n rays ℓ_t and intersects all other rays with multiplicity 1. This means that its pre-image in Z_S is a birational section of $q_S : Z_S \to S$ that contains n fibers ℓ_t. Fix a ray $\ell_s \subset \Pi$. Since all rays in Π intersect ℓ_s, all n fibers are contained in $q_S^{-1}(C(s))$ and

$$H \cdot q_S^*(h) = [p_S^{-1}(\Pi) \cap q_S^{-1}(C(s))] = [L(s) + 2\ell_s + \sum_{t \neq s, \ell_t \subset \Pi} \ell_t],$$

where we use the notation of (11.8). On the other hand, $H^2 = [q_S^{-1}(\ell_s)] = [L(s) + \ell_s]$. Comparing these two equalities, we prove the assertion.

For any line ℓ in $\mathbb{P}^3$, let $\mathsf{Bl}_\ell(\mathbb{P}^3)$ be the blow-up of $\mathbb{P}^3$ along ℓ. The linear system $|\mathcal{O}_{\mathbb{P}^3}(1) - \ell|$ defines a rational map to $\ell^\perp \cong \mathbb{P}^1$ that is resolved by the following commutative diagram:

$$\begin{array}{ccc} & \mathsf{Bl}_\ell(\mathbb{P}^3) \cong \mathbb{P} & \\ {}^{\sigma}\swarrow & & \searrow^{p} \\ \mathbb{P}^3 & \dashrightarrow & \ell^\perp \cong \mathbb{P}^1. \end{array} \tag{11.12}$$

The projection $p : \mathsf{Bl}_\ell(\mathbb{P}^3) \to \mathbb{P}^1$ is a projective $\mathbb{P}^2$-bundle $\mathbb{P} \to \ell^\perp$. Its fiber over $\Pi \in \ell^\perp$ is mapped by σ to the plane Π. We have an isomorphism

$$\mathbb{P} \cong \mathbb{P}(\mathcal{O}_{\mathbb{P}^1}^{\oplus 2} \oplus \mathcal{O}_{\mathbb{P}^1}(1)). \tag{11.13}$$

The surjection $\mathcal{E} = \mathcal{O}_{\mathbb{P}^1}^{\oplus 2} \oplus \mathcal{O}_{\mathbb{P}^1}(1) \to \mathcal{O}_{\mathbb{P}^1}^{\oplus 2}$ defines the embedding of the exceptional divisor $E \cong \mathbb{P}^1 \times \mathbb{P}^1$ as a projective $\mathbb{P}^1$-subbundle of $\mathbb{P}$.

Let $\check{p} : \mathbb{P}^* := \mathbb{P}(\mathcal{E}^\vee) \to \ell^\perp$ be the dual projective bundle. Its fibers over $\Pi \in \ell^\perp$ is the dual plane Π^* of lines in Π. The fiber of the projection $p_E : E \to \ell^\perp$ over $\Pi \in \ell^\perp$ has a distinguished line in Π, the line ℓ. This defines a section

$$\mathsf{s} : \ell^\perp \to \mathbb{P}^*$$

It corresponds to a surjection $\mathcal{E}^\vee = \mathcal{O}_{\mathbb{P}^1}^{\oplus 2} \oplus \mathcal{O}_{\mathbb{P}^1}(-1) \to \mathcal{O}_{\mathbb{P}^1}(-1)$.

Since points of $\mathbb{P}^*$ are lines in $\mathbb{P}^3$ intersecting ℓ, there is a natural map

$$\phi_1 : \mathbb{P}^* \to \Omega(\ell) \subset \mathbb{P}^4.$$

It is given by the linear system $|\mathcal{O}_{\mathbb{P}^*}(1) \otimes \check{p}^*\mathcal{O}_{\ell^\perp}(1)|$ The map ϕ_1 is a small resolution of the quadric $\Omega(\ell)$ in $\mathbb{P}^4$ with singular point ℓ. It is isomorphic to the blow-up of any β-plane in $\Omega(\ell)$ that contains the point ℓ. Its exceptional curve is the image $\mathsf{s}(\ell^\perp)$ of the section The pre-image of any β-plane $\Omega(\Pi)$ is isomorphic to the blow-up of the point $\ell \in \Omega(\Pi)$.

As is well-known a quadric of corank 1 in $\mathbb{P}^4$ has two non-isomorphic small resolutions. The second small resolution is:

$$\phi_2 : \mathsf{Bl}_{\ell^\perp}(\check{\mathbb{P}}^3) \to \Omega(\ell). \tag{11.14}$$

The composition of ϕ_1 (or ϕ_2) with the blow-up of the exceptional curve is the big resolution of the quadric, the blow-up of its vertex.

Let l is a general line in $\mathbb{P}^3$. Then, $\phi_i : \phi_i^{-1}(C(l)) \to C(l)$ is an isomorphism. This gives us two projections $\check{p} : C(l) \to l$ and $\check{p}^* : C(l) \to l^\perp$. The pre-image of a point $x \in l$ (resp. a plane $\Pi \in l^\perp$) consists of rays of S that pass through x (resp. contained in Π). This shows that the image X of the map

$$(\check{p}, \check{p}^*) : C(l) \to l \times l^\perp \cong \mathbb{P}^1 \times \mathbb{P}^1$$

is a curve X of bidegree (n, m).

The pre-image of a point $(x, \Pi) \in X$ under the map $(\check{p}, \check{p}^*)$ consists of rays ℓ such that $x \in \ell \subset \Pi$. It consists of the number of points in S that lie on the line $\Omega(x, \Pi) \subset \mathbb{G}$. Since a general secant of S intersects S at two points, we see that X has only ordinary double points and their number is equal to the rank $r > 0$ of S. The arithmetical genus $p_a(X)$ is equal to $(m - 1)(n - 1)$, this gives the following fundamental relation between p_a, m, n, and r:

$$p_a + r = (m - 1)(n - 1). \tag{11.15}$$

Remark 11.1.5. In the proof we assumed that a general point in $\mathbb{G}$ is not contained in trisecant of S. I do not know examples where it happens. But, if it happens, one has to modify the statement by replacing r with a larger number r'. In any case, we have always inequality

$$p_a \leq (m - 1)(n - 1). \tag{11.16}$$

11.1.2 Null-points and null-planes

Let $\Omega(x, \Pi)$ be a secant line of S contained in $\mathbb{G}$. The point x (resp. plane Π) is called a *null-point* (resp. *null-plane*) of S. A general secant line intersects S at two nonsingular points s, s', the rays ℓ_s and $\ell_{s'}$ intersect at a null-point of the null plane $\langle \ell_s, \ell_{s'} \rangle$.

Recall that a null-system $\omega \in |\bigwedge^2 E^\vee|$ in $\mathbb{P}^3$ defines the graph

$$\Gamma_\omega = \{(x, \omega(x)) \in |E| \times \mathbb{P}(E)$$

of $\omega : E \to E^\vee$. It satifies $x \subset \omega(x)$.

Following Sturm we give the following definition:

Definition 11.1.6. *A* null system *is a correspondence of finite degrees*

$$\Gamma \subset |E| \times \mathbb{P}(E)$$

such that $x \in \Pi$ *if* $(x, \Pi) \in \Gamma$.

If $(x, \Pi) \in \Gamma$, the point x is called the *null-point* of the *null-plane* Π.

We denote the degree of the projections $p : \Gamma \to |E|$ and $\check{p} : \Gamma \to \mathbb{P}(E)$ by α and β. The third attribute of a higher null system is its *rank* γ. It is the number of points x on a general line l such that $(x, \Pi) \in Z$ and $l \subset \Pi$.

For any point $P \in |E|$, let $(P) = p(\check{p}^{-1})(P^{\perp})$ denote the surface of points $x \in |E|$ such that one of its null planes contains P. Equivalently, it is the locus of points x such that the line $\langle x, P\rangle$ is contained in a null-plane.

A general line passing through P contains α null-lines passing through P. It also contains γ other null-points. This shows that $\deg(P) = \alpha + \gamma$. The dual notion is the curve $|l| = p(\check{p}^{-1})(l^{\perp})$. It is the locus of points x whose null-plane contains P.

A general plane Π containing a general line l contains β null-points on l and γ null-points outside l. This shows that $\deg |l| = \beta + \gamma$. Both equalities give the following expression for the cohomology class of Γ in $H^*(\mathbb{P}^3 \times \mathbb{P}^3$: t

$$[\Gamma] = \alpha h^3 + (\alpha + \gamma)h^2\check{h} + (\beta + \gamma)h\check{h}2 + \beta\check{h}^3.$$

Many examples of higher null systems is discussed in [732]. We will be concerned only with the null systems associated with a congruence S of lines in $\mathbb{P}^3$ defined by

$$\Gamma = \{(x, \Pi) \in |E| \times \mathbb{P}(E) : x = \ell_s \cap \ell_t, \Pi = \langle \ell_s, \ell_{s'}\rangle, \text{where } s, t \in S\}.$$

Here, the definition of a null-point and a null-plane coincides with the one we introduced earlier. The definition of the rank coincides with the definition of the rank of S. We also have $\alpha = \frac{1}{2}m(m-1), \beta = \frac{1}{2}n(n-1)$.

We define the surface (P) and a curve $|l|$ as above. The surface (P) is the locus of null-points whose null-plane contains P. We have

$$\deg(P) = \tfrac{1}{2}m(m-1) + r. \tag{11.17}$$

The curve $|l|$ is the locus of null-points in planes containing l. We have

$$\deg(|l|) = \tfrac{1}{2}n(n-1) + r. \tag{11.18}$$

Remark 11.1.7. If S has a fundamental curve, the curve $|l|$ and the surface (P) are reducible. Let $F_1, \ldots, F_k$ be the irreducible components of the fundamental curve and h_i be the degree $h(x)$ of a general point x on F_i. Let $d_i = \deg F_i$. Then a general plane containing l intersects F_i at d_i points of degree h_i. It contains $\delta_i = \frac{1}{2}d_i h_i(h_i - 1)$ null-points. So, each F_i is an irreducible component of $|l|$ of multiplicity δ_i. After we delete these components, we obtain the curve $|l|^*$ of degree of degree $\frac{1}{2}(n-\delta)(n-\delta-1) + \frac{1}{2}\delta + r$, where $\delta = \sum \frac{1}{2}d_i h_i(h_i - 1)$. The curve $|l|^*$ is introduced in [735, Theirl 2, p. 325], where it is called the *reduced* $|l|$.

Similarly, we define the surface $(P)^*$ by deleting from (P) the fundamantal planes of S. We have $\deg(P)^* = \frac{1}{2}m(m-1) - \frac{1}{2}\check{\delta}_i + r$, where $de\check{l}ta$ is defined as above replacing S by the dual congruence.

One define the curve $|l|^*$ and the surface $(P)^*$ by considering the correspondence Γ_S^* defined as the closure of $\{(x,\Pi) \in U \times V : x = \ell_s \cap \ell_{s'}, s, s' \in \Omega(\Pi) \cap S\}$. where $U = |E| \setminus \text{Fund}(S)$ and $V = \mathbb{P}(E) \setminus \text{Fund})S^*)$.

Proposition 11.1.8. *Assume that S has only isolated fundamental points, let x be one of them, of degree* $h(x)$. *Then, for any general line l and a general point P and a general point x,*

(i) $\text{mult}_x(|l|) = \frac{1}{2}h(x)(h(x)-1)$,
(ii) $\text{mult}_P((P)) = \frac{1}{2}m(m-1)$.
(iii) Any fundamental point x with $h(x) > 1$ *lies on* (P)
(iv) every ray passing through P is contained in (P).
(v) $\text{mult}_x(\mathcal{R}(l)) = h(x)$ *for a general line l.*
(vi) If ℓ *is a ray of S, then it enters into* $|\ell|$.

Proof (i) We choose l general enough such that $\Pi = \langle x, l\rangle$ is not a fundamental plane. It contains only finitely many fundamental points of the dual congruence S^*. The pencil $l^\perp$ of planes containing l defines a linear series of degree $\frac{1}{2}n(n-1) + r$ on $|l|$. Its divisors are the sets of null-points lying on ℓ (taken with appropriate multiplicity). The plane Π defines a divisor from this linear series that contains x with multiplicity $\frac{1}{2}h(x)(h(x)-1)$. Thus, Π intersects the curve $|l|$ at x with multiplicity $\frac{1}{2}h(x)(h(x)-1)$. By generality assumption on l, this point must be a multiple point of multiplicity $\frac{1}{2}h(x)(h(x)-1)$.

(ii) A general line passing through P contains r null-points outside P. A plane Π containing l contains $\frac{1}{2}m(m-1)$ and intersects $\Omega(P)$ with multiplicity $\frac{1}{2}m(m-1) + r$. This implies that it intersects (P) at P with multipolicity $\frac{1}{2}m(m-1)$.

(iii) If $h(x) = 1$, then the plane $K(x)$ does not contain a general point P. On the other hand, if $h(x) > 1$, the intersection of the plane spanned by P and a ray ℓ through x contains another ray ℓ' through x. Hence the null-plane $\langle \ell, \ell'\rangle$ contains x, and hence $x \in (P)$.

(iv) We assume $m > 1$. Any ray ℓ containing P intersects $m-1$ rays ℓ_i passing through P. Each plane $\langle \ell, \ell_i\rangle$ is a null-plane containing P. Thus, $\ell \subset (P)$.

(v) A fundamental point x that lies on the ruled surface $\mathcal{R}(\ell)$ is contained in $h(x)$ generators of $\mathcal{R}(\ell)$ lying in the plane $\langle x, \ell\rangle$. This shows that

$$\text{mult}_x(\mathcal{R}(\ell)) = h(x). \tag{11.19}$$

(vi) If ℓ is a ray of S, then a general point x in ℓ intersects $m-1$ other rays, hence there are $m-1$ null-planes containing ℓ with x as its null-point. This shows that $\ell \subset |l|$.

□

The proposition omits the assertion similar to (i) concerning the surface (P). We will deal with it in the next subsection.

Suppose the order $m \geq 3$. A null-plane may contain more than two rays with the same null-point. Following Schumacher [669, §8] one defines the *Tripel Fläche* $T(S)$ (a *triadic surface* in English translation from [364, §2]) to be the closure of the set of such points. Equivalently, it is the closure of points x such that the α-plane $\Omega(x)$ contains a trisecant line of S. The dual notion is triadic surface $T(S^*)$ of the dual congruence, the locus of planes in $\mathbb{P}^3$ that contain a point lying on three rays contained in the plane. In particular, we assume that S is not contained in a hyperplane section of $\mathbb{G}$ because in this case, the rank of S is equal to 0.

For example, for the congruence S of secant lines of a quartic elliptic curve C in $\mathbb{P}^3$, $T(S^*) = \check{\mathbb{P}}^3$ andt $T(S) = C$. It is conjectured in [364] that this is the only case when it can happen for a smooth congruence of lines.

The degree of $T(S)$ was given without any detail in [669, p. 119] and the details were provided by Fano [295, §22]: A modern proof for the case S is smooth can be found in [364].

Theorem 11.1.9. *Suppose S is normal, has no fundamental curves, and $T(S)$ is an irreducible surface of degree t_S. Then*

$$t_S \leq (m-2)[r+\frac{1}{6}(m-1)(m-3n)] = (m-2)((m-1)(n-1)-g+\frac{1}{6}(m-1)(m-3n) \tag{11.20}$$

Proof Let l be a general line in $\mathbb{P}^3$ and $C(l)$ be the corresponding hyperplane section $\mathfrak{G}(l)$ of S. The pre-image C of l under p_S is isomorphic to $C(l)$ under the projection q_S. The cover $C \to l = \mathbb{P}^1$ defines a g^1_m on C. The degree t_S of $T(S)$ is equal to the number of members of the g^1_m that contain a triple of points spanning a line. The number was computed by C. Segre [689] and [695, §17], Note that our curve lies on a quadric $\mathbb{G} \cap \langle C(l) \rangle$. A modern proof of the formula can be found in [330]. Note that the formula is given in terms of the degree of a certain algebraic cycle on the symmetric product $C^{(m)}$, so the number t_S could be less if the cycle is not reduced. □

Corollary 11.1.10. *Under the assumptions of the theorem,*

$$g \leq (m-2)[(m-1)(n-1) + \frac{1}{6}(m-1)(m-3n)].$$

Example 11.1.11. Assume $m = 3$. We can confirm the inequality for the sectional genus by the following argument taken from [295, §21]. Let $|h|$ be the linear system of hyperplane sections of $C(l)$. Let D be a general divisor from the g_3^1. The assumption that a general member of the curve $C(l)$ does not span a line in $\mathbb{P}^4$ means that $\dim|h-D| = 4-3 = 1$. By Riemann-Roch, $1 \le \dim|h-D| = m+n-3-g = n-g$. This gives $g \le n-1$.

11.1.3 The focal locus of a congruence of lines

Since $\tilde{S}$ is smooth, the threefold $\tilde{Z}_S$ is also smooth, and the projection $\tilde{p}_S$ is a map of smooth 3-folds of finite degree equal to the order m of S.

Let

$$0 \to \tilde{p}_S^*\Omega^1_{\mathbb{P}^3} \to \Omega^1_{\tilde{Z}_S} \to \Omega^1_{\tilde{Z}_S/\mathbb{P}^3} \to 0 \tag{11.21}$$

be the standard exact sequence of the sheaves of Kähler differentials. The map $\tilde{p}_S$ is not smooth at a point $z \in \tilde{Z}_S$ if and only if z is contained in the support of the sheaf $\Omega^1_{\tilde{Z}_S/\mathbb{P}^3}$. The set of points where $\tilde{p}_S$ is not smooth has a structure of a closed subscheme, not necessarily reduced. Its ideal sheaf is the Fitting ideal $F_0(\Omega^1_{\tilde{Z}_S/\mathbb{P}^3})$. It is locally given by the determinant of the matrix M_z defining the map $(\tilde{p}_S^*\Omega^1_{\mathbb{P}^3})_z \to (\Omega^1_{\tilde{Z}_S})_z$ of free modules of rank 3. Globally, the Fitting Ideal is equal to $\tilde{p}_S^*\Omega^3_{\mathbb{P}^3} \otimes (\Omega^3_{\tilde{Z}_S})^{\otimes -1} = \tilde{p}_S^*\omega_{\mathbb{P}^3} \otimes \omega_{\tilde{Z}_S}^{-1}$. It defines a positive divisor $R(S)$ such that

$$\mathcal{O}_{\tilde{Z}_S}(R_S) \cong \omega_{\tilde{Z}_S} \otimes \tilde{p}_S^*\omega_{\mathbb{P}^3}^{-1}. \tag{11.22}$$

We get the familiar formula

$$K_{\tilde{Z}_S} \sim \tilde{p}_S^*(K_{\mathbb{P}^3}) + R(S), \tag{11.23}$$

where we identify $R(S)$ with its linear equivalence class. Taking the dual of exact sequence (11.21), we obtain an exact sequence of tangent sheaves

$$0 \to \Theta_{\tilde{Z}_S} \xrightarrow{d\tilde{p}_S} \tilde{q}_S^*\Theta_{\mathbb{P}^3} \to \mathcal{E}xt^1(\Omega^1_{\tilde{Z}_S/\tilde{S}}, \mathcal{O}_{\tilde{Z}_S}) \to 0. \tag{11.24}$$

For any point $z \in \tilde{Z}_S$, the map of fibers $(\Theta_{\tilde{Z}_S})(z) \to (\tilde{p}_S^*\Theta_{\mathbb{P}^3})(z)$ at z coincides with the map

$$(d\tilde{p}_S)_z : T_z\tilde{Z}_S \to T_s\mathbb{P}^3. \tag{11.25}$$

It is locally given by the transpose of the matrix M_z and hence

$$F_0(\mathcal{E}xt^1(\Omega^1_{\tilde{Z}_S/\tilde{S}}, \mathcal{O}_{\tilde{Z}_S})) = \mathcal{O}_{\tilde{Z}_S}(-R(S)).$$

In particular, the support of $R(S)$ is equal to the set of points $z \in \tilde{Z}_S$ where the differential $(d\tilde{p}_S)_z$ is not surjective.

Proposition 11.1.12. *Let s be a nonsingular point of $S \subset \mathbb{P}^5$ and $\mathbb{T}_s(S)$ be its embedded tangent plane at s. For any $z = (x, s) \in Z_S$,*

$$\mathrm{rank}(dp_S)_z = 3 - \dim(\mathbb{T}_s(S) \cap \Omega(x)). \tag{11.26}$$

Proof Let $p_{\mathbb{G}} : Z_{\mathbb{G}} \to \mathbb{P}^3$ and $q : Z_{\mathbb{G}} \to \mathbb{G}$ be the projection maps for the universal family of lines in $\mathbb{P}^3$. The fiber $p_S^{-1}(x)$ is contained in the fiber $p^{-1}(x)$ that is mapped isomorphically under the projection q to $\Omega(x) \subset \mathbb{G}$. The kernel $\mathrm{Ker}((dp_S)_z)$ of the map (11.25) embeds in the kernel of the differential of the map $p_{\mathbb{G}}\, dp_z : T_z(Z_{\mathbb{G}}) \to T_x(\mathbb{P}^3)$. Under the differential $dq_z : T_z(Z_{\mathbb{G}}) \to T_s(\mathbb{G})$ of the map $q_{\mathbb{G}}$, it is mapped isomorphically onto a two-dimensional subspace of $T_s(\mathbb{G})$ that coincides with $T_s(\Omega(x))$. Under the differential $(dq_S)_z : T_z(Z_S) \to T_s(S)$, the linear subspace $\mathrm{Ker}((dp_S)_z)$ of $T_z\mathbb{Z}_G$ is mapped isomorphically onto a linear subspace of $T_s(S)$. Passing to the embedded tangent spaces, we obtain $\mathbb{T}_s(S) \subset \mathbb{T}_s(\Omega(x)) = \Omega(x)$. Thus, $\dim \mathrm{Ker}((dp_S)_z) = \dim \mathbb{T}_s(S) \cap \Omega(x)$. This proves (11.26) □

Let

$$\tilde{p}_S : \tilde{Z}_S \xrightarrow{\nu} \tilde{Z}'_S \xrightarrow{\tilde{p}'_S} \mathbb{P}^3 \tag{11.27}$$

be the *Stein factorization* of $\tilde{p}_S$ [379, Chapter III, Corollary 11.5]. Here, the morphism ν has connected fibers, and $\tilde{p}'_S$ is a finite morphism of degree m. Since $\dim Z_S = \dim \mathbb{P}^3$, ν is of degree one, hence it is a birational morphism. Since $\tilde{Z}_S$ is smooth and, in particular, normal, we may assume that $\tilde{Z}'_S$ is normal.

The morphism ν is the universal for the property $\nu_*(\mathcal{O}_{\tilde{Z}_S}) \cong \mathcal{O}_{\tilde{Z}'_S}$. This implies that σ factors through $\tilde{Z}_S \to Z_S^{\mathrm{nrm}}$.

The image of $R(S)$ in $\mathbb{P}^3$ is denoted by $\mathrm{Foc}(S)$, its points are *focal points* of S. Obviously,

$$\mathrm{Fund}(S) \subset \mathrm{Foc}(S).$$

It follows from the Stein factorization (11.27) of $\tilde{q}_S$ that

$$\mathrm{Fund}(S) = \mathrm{Foc}(S) \text{ if } m = 1$$

We will classify congruences of order one in the next section.

Assume $m > 1$. The (reduced) branch divisor

$$\Phi(S) \subset \mathbb{P}^3$$

of the finite morphism $\tilde{p}'_S : \tilde{Z}'_S \to \mathbb{P}^3$ is called the *focal surface* of S.

Proposition 11.1.13. *Each isolated fundamental point is contained in the focal surface.*

Proof Suppose $x \in \mathrm{Fund}(S)$ but $x \notin \Phi(S)$, then $\tilde{Z}'_S \to \mathbb{P}^3$ is unramified over x, and each point in the pre-image of x is a non-singular point of $\tilde{Z}'_S$. Since $x \in \mathrm{Foc}(S)$, the pre-image of one of its points under the birational morphism $\tilde{Z}_S \to Z_S$ is one-dimensional or coincides with $p_S^{-1}(x)$. In the latter case, the whole plane $\Omega(x)$ is contained in S, and hence $m = 0$. Since the exceptional locus of a birational morphism to a smooth variety is a divisor, the point x belongs to the fundamental curve of S.

□

We will see later that $\mathrm{Fund}(S)$ is not necessary contained in $\Phi(S)$.

Assume that S^{nrm} is nonsingular at any point in the pre-image of a singular point $s \in S$. A singular point $s \in S$ is called *ordinary non-normal singular point* if the morphism $\pi : \tilde{S} \to S$ is unramified at each point in $\pi^{-1}(s)$, i.e. the differential of the map $\tilde{S} \to S \to \mathbb{P}^5$ at $\tilde{s}$ is injective. In this case, the tangent cone of S at s is equal to the union of the images of the tangent spaces of $\tilde{S}$ at all points in the pre-mage of s [407]. For any ordinary non-normal singular point, the ray ℓ_s is not focal. However, if s is not ordinary, the projection is ramified at some point $\tilde{s} \in \tilde{S}$, the fiber $\tilde{q}_S^{-1}(\tilde{s})$ is contained in $R(S)^{\mathrm{dm,ndm}}$, and its image in $\mathbb{P}^3$ is a focal ray.

In particular, assume that $S^{\mathrm{nrm}} = \tilde{S}$, and S is not normal along an irreducible curve C. If its general point is an ordinary singularity, then there will be only finitely many singular focal rays. Otherwise, there will be an irreducible component of $\Phi(S)$ that consists of singular focal rays.

Assume now that a point $s' \in S^{\mathrm{nrm}}$ is singular. Let E be an irreducible component of the exceptional curve of $\tilde{S} \to S^{\mathrm{nrm}}$ over s'. Its pre-image $\tilde{q}_S^{-1}(E)$ in $\tilde{Z}_S$ is a $\mathbb{P}^1$-bundle over E. Its image under the projection $\tilde{p}_S$ is the singular ray ℓ_s. It follows that $\tilde{q}_S^{-1}(E)$ is an irreducible component of $R(S)^{\mathrm{nd,nd}}$. It is easy to see that irreducible components of $R(S)^{\mathrm{nd,nd}}$ are obtained in this way.

We will see later that, in general, $\mathrm{Fund}(S)$ is not a subset of $\Phi(S)$.

Proposition 11.1.14. *Suppose s is a smooth point of S. Then, $\mathbb{T}_s(S) \cap \mathbb{G}$ is either an α-plane $\Omega(x)$, or a β-plane $\Omega(\Pi)$, or a singular conic. If the first case, $x \in$ Fund(S) and the fiber $p_S^{-1}(x)$ is singular at some point on $q_S^{-1}(s)$, and x is the only focal point on ℓ_s. In the second case, ℓ_s is a focal ray. In the third case, ℓ_s intersects $\Phi(S)$ at the number of points equal to the number one or two of irreducible components of the singular conic.*

Proof Since $\mathbb{T}_s(S)$ is contained in $\mathbb{T}_s(\mathbb{G})$, its intersection with $\mathbb{G}$ is either the whole plane, or a conic with a singular point at s. Any plane contained in $\mathbb{G}$ is either an α-plane or a β-plane. This gives the three possibilities for $\mathbb{T}_s(S)$.

Assume that $\mathbb{T}_s(S) \cap \mathbb{G} = \Omega(x)$, hence $\mathbb{T}_s(S) = \Omega(x)$, for some point $x \in \ell_s$.

Since $q_S : p_S^{-1}(x) \to F(x) = \Omega(x) \cap S$ is an isomorphism and s is a singular point of $\Omega(x) \cap S$, we obtain that $z = (x, s)$ is a singular point of $p_S^{-1}(x)$. Since, for any $x, x' \in \ell_s$, $\Omega(x) \cap \Omega(x') = \{s\}$, we see from (11.26) that x is the only focal point on ℓ_s.

Assume that $\mathbb{T}_s(S) = \Omega(\Pi)$ for some $\Pi \in \ell_s^{\perp}$. Then, for any $x \in \ell_s$, the pencil $\Omega(\Pi) \cap \Omega(x)$ is contained in $\Omega(x)$. Thus, Proposition 11.1.14 gives $\mathrm{rank}(dp_S)_{(x,s)} = 2$. This shows that any point on ℓ_s is focal.

Suppose $\mathbb{T}_s(S) \cap \mathbb{G}$ is the union of two lines (maybe equal) intersecting at s. Each line in $\mathbb{G}$ is a pencil of rays $\Omega(x, \Pi)$, i.e. $x \in \ell_s \subset \Pi$. Thus, $\dim \mathbb{T}_s(S) \cap \Omega(x) = 1$. So, each component of the singular conic determines a focal point on the ray ℓ_s. □

Corollary 11.1.15. *Let s be a smooth point of S. The following properties are equivalent.*

(i) $\mathbb{T}_s(S)$ is a β-plane $\Omega(\Pi)$;
(ii) The ray ℓ_s is a focal ray;

Any of these properties implies
(iii) each point $x \in \ell_s$ is a null-point with the same null plane Π containing ℓ_s.

Proof We have already proved that (i) → (ii). Suppose ℓ_s is a focal ray. By (11.26), for any $x \in \ell_s$, the intersection $\mathbb{T}_s(S) \cap \Omega(x)$ contains a line in $\mathbb{G}$. When we let x run ℓ_s, these lines will span the whole plane $\mathbb{T}_s(S)$ showing that $\mathbb{T}_s(S) \subset \mathbb{G}$. It follows from the previous proposition that $\mathbb{T}_s(S) = \Omega(\Pi)$. Also, it implies that each line $\Omega(x) \cap \Pi$ is tangent to S at s, hence it is a secant line of S. So, (ii) implies (i) and implies (iii). □

Let us compute the divisor class $[R(S)]$ of $R(S)$ in $\tilde{Z}_S$.

Proposition 11.1.16.

$$[R(S)] = 2H + \tilde{q}_S^*(h + K_{\tilde{S}}),$$
$$\omega_{R(S)} \cong \tilde{q}_S^* \omega_{\tilde{S}}(1)^{\otimes 2} \otimes \mathcal{O}_{R(S)}.$$

Proof Let $i : \tilde{Z}_S \to Z_{\mathbb{G}}$ be the composition of the maps $\tilde{Z}_S \to Z_S$ and the closed embedding $Z_S \hookrightarrow Z_{\mathbb{G}}$. Using the formula for the sheaf $\omega_{Z_{\mathbb{G}}/\mathbb{G}}$ from Subsection 10.1.1, we get

$$\omega_{\tilde{Z}_S/\tilde{S}} \cong i^*\omega_{Z_{\mathbb{G}}/\mathbb{G}} = i^*(p^*\mathcal{O}_{\mathbb{P}^3}(-2) \otimes q^*\mathcal{O}_{\mathbb{G}}(1))$$
$$= \tilde{p}_S^*\mathcal{O}_{\mathbb{P}^3}(-2) \otimes \tilde{q}_S^*\mathcal{O}_{\tilde{S}}(1),$$

where $\mathcal{O}_{\tilde{S}}(1) := \pi^*\mathcal{O}_S(1)$. This gives

$$\omega_{\tilde{Z}_S} \cong \omega_{\tilde{Z}_S/\tilde{S}} \otimes \omega_{\tilde{S}} \cong \tilde{p}_S^*\mathcal{O}_{\mathbb{P}^3}(-2) \otimes \tilde{q}_S^*\omega_{\tilde{S}}(1) \qquad (11.28)$$

and, applying (11.22), we obtain

$$\mathcal{O}_{\tilde{Z}_S}(R(S)) \cong \tilde{p}_S^*\mathcal{O}_{\mathbb{P}^3}(-2)) \otimes \tilde{q}_S^*\omega_{\tilde{S}}(1) \otimes \tilde{p}_S^*\omega_{\mathbb{P}^3}^{-1}$$

$$\cong \tilde{p}_S^*\mathcal{O}_{\mathbb{P}^3}(2) \otimes \tilde{q}_S^*\omega_{\tilde{S}}(1) \cong \mathcal{O}_{\tilde{Z}_S}(2) \otimes \tilde{q}_S^*\omega_{\tilde{S}}(1).$$

Taking the first Chern classes, we get the first assertion.

To prove the second assertion, we apply the adjunction formula to get

$$\omega_{R(S)} \cong \omega_{\tilde{Z}_S}(R(S)) \otimes \mathcal{O}_{R(S)} \cong \mathcal{O}_{\tilde{Z}}(-2) \otimes \tilde{q}_S^*\omega_{\tilde{S}}(1) \otimes \mathcal{O}_{\tilde{Z}}(R(S)) \otimes \mathcal{O}_{R(S)}.$$

It remains to apply the formula for $\mathcal{O}_{\tilde{Z}_S}(R(S))$. □

Let $R(S)_i$ be an irreducible component of $R(S)$ and η_i be its generic point. then

$$\mathrm{codim}(p_S(\eta_i), \mathrm{codim}(q_S(\eta_i)) \in \{(1,0),(1,1),(2,0),(2,1)\}.$$

We denote by $R(S)^{a,b}$ the union of irreducible components of $R(S)_i$ with $(a,b) = \mathrm{codim}(p_S(\eta_i), \mathrm{codim}(q_S(\eta_i))$.

Corollary 11.1.17. *One of the following cases occurs:*

1. *Fund(S) is a finite set of points (maybe empty), and $R(S) = R(S)^{1,0}$ is reduced and irreducible. There are only finitely many focal rays, and any non-focal ray is tangent to $\Phi(S)$ at two points (which may coincide).*
2. *Fund(S) is a finite set of points, and $R(S) = 2R(S)^{1,0}$. There are only finitely many focal rays, and any non-focal ray is tangent to $\Phi(S)$ at one point*
3. *$R(S)^{1,0} \neq \emptyset$ and $R(S)^{2,0} \neq \emptyset$. In this case, the fundamental curve and the focal surface are irreducible. Each ray intersects Fund(S) and tangent to $\Phi(S)$ at one point.*
4. *$R(S)^{1,0} = \emptyset, R(S)^{2,0} \neq \emptyset$. This happens only if $m = 1$. The fundamental curve consists of one or two irreducible components and a general ray intersects each irreducible component and has two focal points on it.*

   Moreover,
5. *If $R^{1,1} \neq \emptyset$, the focal surface has an irreducible component which consists of focal rays.*
6. *If $R^{2,1} \neq \emptyset$, S^{nrm} is singular, and $\tilde{q}_S(R^{2,1})$ is the exceptional curve of $\tilde{S} \to S^{nrm}$.*

In the first case, a general ray is tangent to the focal surface at two points which may coincide. The closure of lines tangent to an irreducible surface X of degree d in $\mathbb{P}^3$ is a surface $\mathrm{Bit}(F)$ in $G_1(\mathbb{P}^3)$, called the *bitangent surface* of X. It follows that our congruence S is an irreducible component of $\mathrm{Bit}(\Phi(S))$.

The following fact from [653, Volume 2, p. 281]) is useful. We follow Salmon's proof (see also [18, Proposition 3.3]).

Proposition 11.1.18. *Let Φ be an irreducible normal surface of degree d in $\mathbb{P}^3$. The closure $Bit(\Phi)$ in $\mathbb{G}$ of the set of lines that are tangent to Φ has the cohomology class in $H^4(\mathbb{G},\mathbb{Z})$ equal*

$$(d+2)(d-3)\sigma_{1,1} + \tfrac{1}{2}d(d-1)(d-2)(d-3)\sigma_2.$$

Proof Let q be a general point in $\mathbb{P}^3$ and let $\ell = \langle q, q'\rangle$ be the line containing q and tangent to X at some point $q' = [x_0, y_0, z_0, w_0]$. Without loss of generality, we may assume that $q = [0, 0, 0, 1]$ and

$$F = w^d + A_1 w^{d-1} + \cdots + wA_{d-1} + A_d,$$

where A_k are homogeneous forms of degree k in x, y, z.

Plugging in the parametric equation $[s,t] \mapsto [sv+tv']$, where $[v] = q$, $[v'] = q'$, we get

$$f := F(sv+tv') = (s+tw_0)^d + \sum_{i=1}^{d} t^i A_i(x_0, y_0, z_0)(s+tw_0)^{d-i}.$$

By polarizing, we can rewrite it in the form

$$f = \sum_{k+m=d} s^k t^m P_{q^k}(q'),$$

where $P_{v^k}(v') = P_{v'^m}(v)$ is the value of k-th polar of F with pole at q at the vector v' (or, equivalently, the value of the totally polarized symmetric multilinear form defined by f at $(v, \ldots, v, v', \ldots, v')$. Since $q' \in X$, we get $P_{v^0}(v') = F(v') = 0$. Moreover, because ℓ is tangent to X at q', we obtain $P_v(v') = 0$. Thus, we can rewrite

$$f = s^2 g_{d-2}(s,t).$$

The line ℓ is tangent to X at some other point if and only if the binary form g_{d-2} of degree $d-2$

$$g_{d-2}(s,t) = \sum_{k=0}^{d-2} s^k t^{d-2-k} P_{q^{k+2}}(q')$$

has a multiple root.

Recall that the discriminant polynomial $D(a_0, \ldots, a_d)$ of a binary form of degree d $\sum_{i=0}^{d} a_i u^{d-i} v^i$ is a homogenous polynomial of degree $2(d-1)$. It is also a bi-homogeneous polynomial of bidegree $(d(d-1), d(d-1))$ with respect to the action of $\mathbb{G}_m^2$ via

$$(a_0, \dots, a_d) \mapsto (\lambda^d a_0, \lambda^{d-1}\mu a_1, \dots, \lambda\mu^{d-1}a_{d-1}, \mu^d a_d).$$

Thus, we obtain that the locus of points q' such that the line $\langle q, q'\rangle$ is tangent to X at two points, including q' is contained in the intersection of hypersurfaces of degrees $d, d-1$, and $(d-2)(d-3)$. This implies that the expected number of bitangent lines passing through q is equal to $\frac{1}{2}d(d-1)(d-2)(d-3)$.

The class n of $\mathrm{Bit}(X)$ is equal to the number of bitangent lines to a general plane section H of X. Since X is normal, it is a smooth plane curve of degree d. We know from Subsection 5.5.1 that their number is is is equal to $d(d-2)(d^2-9)$.

□

Let $f : X \to Y$ be a finite morphism of a normal variety X to a smooth variety Y. For any point $x \in X$ (not necessary closed), let $f^* : \mathcal{O}_{Y,f(x)} \to \mathcal{O}_{X,x}$ be the corresponding homomorphism of local rings. Passing to the formal completions, we obtain a homomorphism $\hat{f}^* : \hat{\mathcal{O}}_{Y,f(x)} \to \hat{\mathcal{O}}_{X,x}$. By definition, the *index of ramification* $e_f(x)$ is the degree of the corresponding finite morphism of local schemes $\mathrm{Spec}(\hat{\mathcal{O}}_{X,x}) \to \mathrm{Spec}(\hat{\mathcal{O}}_{Y,f(x)})$. If x is a point of codimension one, then $\mathcal{O}_{X,x}$ and $\mathcal{O}_{Y,f(x)}$ are discrete valuation rings, and the index of ramification $e_d(x)$ coincides with the usual one, defined by $f^*(\mathfrak{m}_{Y,f(x)}) = \mathfrak{m}_{X,x}^{e_f(x)}$. Intuitively, $e_f(x)$ is the number of sheets of the cover that come together at x.

For any point $y \in Y$, we have

$$\deg(f) = \sum_{f(x)=y} [\kappa(x) : \kappa(y)] e_f(x), \tag{11.29}$$

where $[\kappa(x) : \kappa(y)]$ is the degree of the extension of the residue fields at the points $x \in X$ and $y \in Y$.

The function $x \to e_f(x)$ is upper-semicontinuous, i.e., $e_f(x) \le e_f(x')$ if x' is a specialization of x. Also, if x_0 is a specialization of two different points x, x', then $e_f(x_0) \ge e_f(x) + e_f(x')$ [318, Lemma 1 and Lemma 2].

We apply this to our finite morphism $\tilde{p}'_S : \tilde{Z}'_S \to \mathbb{P}^3$. To shorten the notation, we set $e_{\tilde{p}'_S}(z) := e(z)$.

Localizing the exact sequence (11.21), for any codimension one point $z \in \tilde{Z}'_S$, we get an exact sequence

$$0 \to (\tilde{p}_S^*\Omega^1_{\mathbb{P}^3})_z \to (\Omega^1_{\tilde{Z}_S})_z \to (\Omega^1_{\tilde{Z}_S/\mathbb{P}^3})_z \to 0. \tag{11.30}$$

Let t be a generator of the maximal ideal of a codimension one point $x = f(z) \in \mathbb{P}^3$ and $p_S^*(t) = u^{e(x)}$, where u is a generator of the maximal ideal of a codimension one point $z \in \tilde{Z}'_S$. We have $p_S^*(dt) = e(z)u^{e(z)-1}du$ that shows that $(\Omega^1_{\tilde{Z}_S/\mathbb{P}^3})_z \cong \mathcal{O}_{\tilde{Z}'_S,z}/(u^{e(z)-1})$.

Since any irreducible component of $R^{\mathrm{dm,dm}}$ enters with multiplicity one or two and $q_S : R^{1,0} \to S$ is of degree one or two, we obtain the following:

Proposition 11.1.19. *The index of the ramification of the finite map $\tilde{Z}'_S \to \mathbb{P}^3$ at the generic point of an irreducible component $R(S)_i$ of $R(S)^{1,0}$ is equal to* 2 *or* 3. *In the former case, $R(S)_i$ enters with multiplicity* 1*, and, in the latter case, it enters with multiplicity* 2.

Let $\Phi(S)^{(1)}$ denote the set of general points of irreducible components of $\Phi(S)$. It follows from (11.29) that, for any $\zeta \in \Phi(S)^{(1)}$,

$$m = \sum_{\tilde{q}'_S(\eta_i)=\zeta} [\kappa(\eta_i) : \kappa(\zeta)]e(\eta_i).$$

We know that an irreducible component $R(S)_i$ with generic point η_i enters with multiplicity $e(\eta_i) - 1$. Let us assign to ζ_i the multiplicity

$$\mu_\zeta(\Phi(S)) := [\kappa(\eta_i) : \kappa(\zeta)](e(\eta_i) - 1)$$

and set

$$\Phi(S)_{\mathrm{sch}} = \sum_{\zeta \in \Phi(S)^{(1)}} \mu_\zeta(\Phi(S))\overline{\zeta}$$

This puts a scheme-theoretical structure on the focal surface.

Let us compute the degree of $\Phi(S)_{\mathrm{sch}}$.

Proposition 11.1.20. *Let S be a congruence of order $m > 1$. Then*

$$\deg(\Phi(S)_{sch}) = 2m + 2g - 2, \tag{11.31}$$

where g is the sectional genus of S.

Proof Let ℓ be a general line in $\mathbb{P}^3$. Its pre-image $\tilde{p}_S^{-1}(\ell)$ in $\tilde{Z}_S$ intersects only the parts $R(S)^{\mathrm{dm,dm}}$ and $R(S)^{\mathrm{dm,ndm}}$ of $R(S)$. If there are only finitely many focal rays, the second part is empty. Since $[\tilde{p}_S^{-1}(\ell)] = H^2$, applying the projection formula, we obtain:

$$\begin{aligned}
&\deg(\Phi(S)_{\mathrm{sch}}) = H^2 \cdot (R(S)^{\mathrm{dm,dm}} + R(S)^{\mathrm{dm,ndm}}) = H^2 \cdot R(S) \\
&= H^2 \cdot (2H + \tilde{q}_S^*(h + K_{\tilde{S}})) = 2H^3 + H^2 \cdot \tilde{q}_S^*(h + K_{\tilde{S}}) \\
&= 2m + (\tilde{q}_S)_*(H^2) \cdot (h + K_{\tilde{S}}) = 2m + [\pi^*(C(\ell))] \cdot (h + K_{\tilde{S}}) \\
&= 2m + 2 + h \cdot (h + K_{\tilde{S}}) = 2m + 2g - 2.
\end{aligned} \tag{11.32}$$

□

In the case when the sectional genus is equal to the sectional arithmetic genus, i.e. S is smooth or has only isolated singular points, formula (11.15) gives

$$\deg(\Phi(S)_{\mathrm{sch}}) = 2n(m-1) - 2r. \tag{11.33}$$

Corollary 11.1.21. *The focal surface $\phi(S^*)$ of the dual congruence is the dual surface of $\Phi(S)$. The class of $\Phi(S)$ is equal to to $2n+2g-2 = \deg(\Phi(S))+n-m$.*

Let

$$R(S) \xrightarrow{\mu} R(S)' \xrightarrow{\tilde{q}'_S} \tilde{S} \tag{11.34}$$

be the Stein factorization of the morphism $\tilde{q}_S$. Assume that $R(S)$ is reduced. The morphism $\tilde{q}'_S$ is a finite cover of degree 2. We have

$$(\tilde{q}'_S)_* \mathcal{O}_{R(S)'} \cong \mathcal{O}_{\tilde{S}} \oplus \mathcal{L},$$

where $(\tilde{q}_S)_*\mathcal{O}_{R(S)'} = (\tilde{q}_S)_*\mathcal{O}_{R(S)}$ and $\mathcal{L}$ is an invertible sheaf. The cover is ramified over the scheme of zeros $B(S)$ of a section of $\mathcal{L}^{\otimes -2}$.

The well-known formula for the canonical sheaf of a double cover gives

$$\omega_{R(S)'} \cong \tilde{q}'^*_S(\omega_{\tilde{S}} \otimes \mathcal{L}^{-1}).$$

The fibers of the map $\mu : R(S) \to R(S)'$ are closed subsets of $\mathbb{P}^1$. This implies that μ is an isomorphism over a point $z' \in R(S)'$ over $\tilde{s} \in \tilde{S}$, or the fiber coincides with the fiber $\tilde{q}_S^{-1}(\tilde{s})$. In the latter case, the ray ℓ_s is a focal ray. It follows from Proposition 11.1.14 that this could happen only if $\mathbb{T}_s(S)$ is a β-plane.

Proposition 11.1.22. *Suppose S has no focal rays. Then, $\tilde{p}_S : R(S) \to \tilde{S}$ is a finite cover of degree two with the branch divisor*

$$B(S) \in |2K_{\tilde{S}} + 4h|.$$

Its degree $B(S) \cdot h$ with respect to h is equal to $2(2g-2+m+n)$.

Proof Applying Proposition 11.1.16, we obtain

$$\omega_{R(S)} = \tilde{q}_S^*(\omega_{\tilde{S}}(1)^{\otimes 2}) = \tilde{q}_S^*(\omega_{\tilde{S}} \otimes \mathcal{L}^{-1}).$$

This gives

$$\mathcal{L}^{-1} \cong \omega_{\tilde{S}} \otimes \mathcal{O}_{\tilde{S}}(2),$$

hence

$$\mathcal{O}_{\tilde{S}}(B(S)) \cong \mathcal{L}^{-2} \cong \omega_{\tilde{S}}^{\otimes 2} \otimes \mathcal{O}_{\tilde{S}}(4).$$

Taking the first Chern classes, we obtain the first assertion. Intersecting it with h, we get its degree. □

Corollary 11.1.23. *Let* $\mathrm{Sing}(B(S))$ *be the locus of singular points of* $B(S)$*. Then,* $\mathrm{Sing}(R(S)) = \tilde{q}_S^{-1}(\mathrm{Sing}(B(S))$*. In particular,* $R(S)$ *is smooth if* $B(S)$ *is smooth.*

Proof It follows from local equations of a double cover of a nonsingular surface, that $\mathrm{Sing}(R(S)')$ is equal to the pre-image of $\mathrm{Sing}(B(S))$. The birational morphism $\tau : R(S) \to R(S)'$ is either a local isomorphism over a nonsingular point or is a homeomorphism over the blow-up of this point. □

Let us consider again the Stein factorization (11.27). Let x be a fundamental point of S. Then the fiber $\tilde{p}_S'^{-1}(x)$ contains some points over which the fiber of $\nu : \tilde{Z}_S \to \tilde{Z}_S'$ are one-dimensional. The pre-images in $\tilde{Z}_S$ of the rest of the points are projected to a finite set of points in $\tilde{S}$. Their images in S are called *isolated rays*. They come with multiplicities equal to ramification indices of $\tilde{p}_S'$ at these points.

For any point $\tilde{s} \in B(S)$, the fiber $\tilde{q}_S^{-1}(\tilde{s})$ intersects $R(S)$ at one point $(x, \tilde{s})$. Thus, the two focal points on the ray ℓ_s collide and ℓ_s intersects the focal surface at the point x with multiplicity 4. The rays ℓ_s sweep a ruled surface of degree $2(2g-2+m+n)$ equal to the degree of $B(S)$.

Suppose that $\deg(\tilde{p}_S : R(S) \to \mathrm{Foc}(S)) = 1$ and $\mathrm{Foc}(S) = \Phi(S)$. The ruled surface $\mathcal{R}(B(S))$ touches $\Phi(S)$ along a curve $\mathrm{Foc}(S)_0$ with multiplicity 4. We have

$$\deg(\mathrm{Foc}(S)_0) = \frac{1}{4}(2m+2g-2)(2(2g-2+m+n)) = (m+g-1)(m+n+2g-2). \tag{11.35}$$

Proposition 11.1.24. *Let* x *be a fundamental point of* S*. Then,* x *is a singular point of the focal surface* $\Phi(S)$*.*

Proof Suppose x is a smooth point of $\Phi(S)$. For a sufficiently small connected open neighborhood of $U(x)$ of x in $\mathbb{P}^3$ and any point z in its pre-image in $\tilde{Z}_S'$, there exists an open connected neighborhood $V(y)$ of z such that the map $\tilde{p}_S'$ is a finite map $V(z) \to U(x)$ ramified over a smooth codimension manifold D containing x with local equation $u_1 = 0$, where u_1, u_2, u_3 are analytic coordinates in $U(x)$. Shrinking $U(x)$ further, we may assume that the local fundamental group of $U(x) \setminus D$ is isomorphic to $\mathbb{Z}$. Thus, $V(z) \to U(x)$ is a cyclic cover defined by $u_1 = v_1^e, u_2 = v_2, u_3 = v_3$, and hence $V(y)$ is a complex manifold.

It follows from the Stein factorization (11.27) that the fiber of $\nu : \tilde{Z}_S \to \tilde{Z}_S'$ over some point in the pre-image of x in $\tilde{Z}_S'$ is a (-1)-curve E contained in $R(S)$. Using the formula for the canonical class of $R(S)$ from Proposition 11.1.16, we find that $K_{R_S} \cdot E$ is even, contradicting the fact that E is a (-1)-curve.

□

The next proposition gives the promised addition to Proposition 11.1.8.

Proposition 11.1.25. *Let x be an isolated fundamental point of S of degree h. For a general point $P \in \mathbb{P}^3$,*

$$\mathrm{mult}_x(P) = h(m-1) - \tfrac{1}{2}\mathrm{mult}_x\Phi(S).$$

In particular, $\mathrm{mult}_x\Phi(S)$ *is even.*

Proof Let us first, following [295, §26], give another proof of the formula for the degree of the surface (P). Take a general line l in $\mathbb{P}^3$ and a general plane Π containing P. The intersection of Π with the ruled surface $\mathcal{R}(l)$ is a directrix C that intersects l at a m-multiple point x_0. We take P and l general enough such that x_0 does not belong to the focal surface $\Phi(S)$ of S. In this case, x_0 is an ordinary m-multiple point of C. We have a rational map $C \to l$ that assigns to a point $x \in C$ the intersection point of the generator of $\mathcal{R}(C)$ passing through x with l. It defines a g^1_m on the normalization C' of C. Its double points are the intersection points of l with $\Phi(S)$.

Let $\mathcal{P}$ be the pencil of lines $\Omega(P,\Pi)$. We can identify it with l. For any $x \in l$, the line $\langle P, x\rangle$ cuts out in C a divisor $A(x) = x_1 + \cdots + x_{m+n}$ on C from g^1_{m+n} contained in the complete linear system of hyperplne sections of the generatrix of $\mathcal{R}(l)$. For any x_i, let $B(x_i) = D(x) - x_i$, where $D(x)$ is a member of the g^1_m that contains x. Consider the correspondence $T \subset l \times l$ such that $T(x)$ is the image in l of the union of the divisors $A(x)$ and $B(x_i), x_i \in A(x)$. This is a symmetric correspondence of bidegree $(m-1)(m+n)$. It has $c = 2(m-1)(m+n)$ coincidence points. Each coincidence point defines a point x in l such that there are two rays passing through this point that span a plane containing P. We have

$$c = 2\deg((P)) + \#l \cap \Phi(S) + m(m-1).$$

Here, the second summand takes into account double points of the g^1_m and the third summand gives the contribution of the point $x+0$. This gives

$$\deg((P)) = (m-1)(m+n) - \tfrac{1}{2}(2n(m-1)+2r) + \tfrac{1}{2}m(m-1) = \tfrac{1}{2}m(m-1)+r$$

in agreement with (11.17).

Now, we take l to be a general line passing through a fundamental point x of degree h. We repeat the argument to compute the number of focal points $x' \neq x$ on l with the focal plane containing P. To do this we have to replace $m+n$ with $m+n-h$ and the degree $n(m-1)+2r$ of $\Phi(S)$ with $(m-1)+2r-s$, where $s = \mathrm{mult}_x\Phi(S)$, and obtain the number is equal to

$$(m-1)(m+n-h) - \tfrac{1}{2}(2n(m-1)+2r-s) + \tfrac{1}{2}m(m-1).$$

The difference between $\deg((P))$ and this number is equal to $\mathrm{mult}_x(P)$. This gives

$$\mathrm{mult}_x(P) = h(m-1) - \tfrac{1}{2}s. \tag{11.36}$$

□

Proposition 11.1.26. *Let $x \in Fund(S)$, and let $\gamma(x)$ be the number of isolated rays through x. Then*

$$\gamma(x) \le \#\tilde{p}_S'^{-1}(x) - 1.$$

The equality holds if $K(x)$ is irreducible and S^{nrm} is smooth.

Proof Here we use notation from the Stein factorization (11.27) of $\tilde{p}_S$. We see that $\gamma(x)$ is equal to the number of points $z \in \tilde{p}_S^{-1}(x)$ such that the differential $(d\tilde{p}_S)_y$ is of rank > 1. For example, if $x \notin \Phi(S)$, the morphism $\tilde{Z}_S' \to \mathbb{P}^3$ is unramified, hence consists of m points. The morphism $\nu : \tilde{Z}_S \to \tilde{Z}_S'$ has connected fibers, some of which are of positive dimension. Each irreducible component of some fiber is mapped, under the projection $\tilde{q}_S$, to an irreducible component of the proper transform of the curve $C(x) = \Omega(x) \cap S$ to $\tilde{S}$. If the normalization of S is smooth, and $K(x)$ is irreducible, then $C(x)$ is irreducible, and there is only one fiber of positive dimension, and it is irreducible. This proves the assertion.

□

The following proposition is due to Sturm [735, Thiel 2, n. 295] (see also [295, §6]).

Proposition 11.1.27. *Let x be an isolated fundamental point of S of degree $h(x)$. Then the tangent cone of $\Phi(S)$ at x is of class $m - \gamma(x)$.*

Proof Let l be a general line throigh the point x. The cone $K(x)$ is a component of the ruled surafce $\mathcal{R}(l)$. The degree of the residual part $\mathcal{R}(l)'$ is equal to $m + n - h(x)$. Since a general point of l is contained in m rays, the ruled surface $\mathcal{R}(l)'$ has l as its m-multiple directrix. There are $\gamma(x)$ generators of $\mathcal{R}(l)'$ corresponding to the isolated rays and $m - \gamma(x)$ generators g_i of $K(x)$. The pencil of rays in the plane $\langle l, g_i \rangle$ passing through x is a line in $\Omega(x)$ that is tangent to the generatrix $C(x)$ of $K(x)$ at the point g_i which is the common generator of the parts $K(x)$ and $\mathcal{R}(l)'$ of $\mathcal{R}(l)$. This shows that the class of $C(l)$ is equal to $m - \gamma(x)$. This is the assertion of the proposition. □

The following important theorem is also due to Sturm; we will follow his beautiful proof.

Theorem 11.1.28. *Assume that the one-dimensional part F of Fund(S) is irreducible curve of degree d. Let $\gamma = \gamma(x)$ and $h = h(x)$ for a general point $x \in F$. Then*

$$m = h + \gamma.$$

In particular, $h \leq m$.

Proof Let l be a general line in $\mathbb{P}^3$. By (11.18), $\deg(|l|) = r + \frac{1}{2}(n(n-1) - dh(h-1)$.

Now, assume that l intersects F at its general point x. Each plane Π containing ℓ contains $n-h$ rays not passing through x. They contribute ' $\frac{1}{2}(n-h)(n-h-1)$ intersection points to the curve $|l|$. Each of $d-1$ other intersection points of Π with F contribute further further $\frac{1}{2}(d-1)h(h-1)$ intersection points. Let r be the rank of S and $r' \leq r$ be the number of intersection points of rays not passing through x lying on ℓ. When we move Π in the pencil $\ell^{\perp}$ we obtain a component $|\ell|_1$ of $|\ell|$ of degree

$$\tfrac{1}{2}(n-h)(n-h-1) - \tfrac{1}{2}(d-1)h(h-1) + r'.$$

The ruled surface $\mathcal{R}(\ell)$ contains $K(x)$ with multiplicity h and also contains γ isolated rays passing through x. A general point of ℓ contains m rays of S. The residual part of $\mathcal{R}(\ell)$ intersects $K(x)$ at $h(n+m-h)-(m-\gamma)$ rays. They define the residual part $|\ell|_2$ of $|\ell|$ of this degree.

Adding up the degrees of $|\ell|_1$ and $|\ell|_2$,

$$\tfrac{1}{2}(d-1)h(h-1) = (\tfrac{1}{2}(n-h)(n-h-1) - \tfrac{1}{2}(d-1)h(h-1) + r') + h(n+m-h) - (m-\gamma).$$

Expanding the expressions, we find

$$r - r' = m(h-1) + \gamma. \qquad (11.37)$$

Next, we choose a general line l and consider a symmetric involution T on F. It is the closure in $F^{\mathrm{nrm}} \times F^{\mathrm{nrm}}$ of pairs of nonsingular points (x, x') on F such that a generator of $K(x)$ and a generator of $K(x')$ intersect at a point on l. It is easy to see that the bidegree of T is equal to $(h(m-1), h(m-1)$. Let X be the closure of the union of lines $\langle x, x'\rangle$. The line $\ell = \langle x, x'\}$ connecting two points (x, x') intersects l at a point t if and only if t is the null-point of the rays $\langle x, t\rangle$ and $\langle x', t\rangle$ spanning their null-plane. This shows that the degree of X is equal to the rank r of S.

Now, we specialize and take l passing through a general point $x \in F$. Then, we can still define the correspondence T' and the surface X'. However, it contains the curve $\{(x, x'), x' \in F\}$ with multiplicity $h(m-1)$. The degree of the residual surface is equal to r', where r' was defined above. This gives

$$r = r' + h(m-1). \tag{11.38}$$

The equalities (11.37) and (11.38) imply the assertion.

□

Even when S and $R(S)$ are smooth, the focal surface $\Phi(S)$ may be singular outside fundamental points. In fact, if $m > 2$, the surface $\mathrm{Foc}(S)$ is a non-normal surface and hence singular along a curve. The curve is the locus of points $x \in \mathrm{Foc}(S)$ such that the map $R(S) \to \mathrm{Foc}(S)$ is not a local isomorphism. These could be the images of points $z \in R(S)'$ with the ramification index ≥ 3. According to [318, Theorem 1], this locus is never empty and contains one-dimensional components. These one-dimensional components form the cuspidal curve of $\mathrm{Foc}(S)$. Also, $\mathrm{Foc}(S)$ may contain nodal curve, the locus of points $x \in \mathrm{Foc}(S)$ such that their pre-image in $R(S)'$ consists of more than one point.

If $n \geq 3$, the focal surface is non-normal. It contains a cuspidal curve, a one-dimensional part of the closure of non-fundamental points $y \in \Phi(S)$ such that $\tilde{p}_S^{-1}(y)$ contains a point with the ramification index $e_{\tilde{p}_S}(x) \geq 3$. It is known that the set of such points is non-empty and contains an irreducible component of codimension 1 in $\Phi(S)$ [318, Theorem 1]. If $m \geq 4$, the locus also contains isolated points and may contain the double curve, the locus of points $y \in \Phi(S)$ such that $\tilde{p}_S^{-1}(y)$ contains two points with the ramification degree 2.

The following formula for the degree of the cuspidal curve R, under the assumption that S is smooth, was given by Schumacher [669, p. 124]:

$$\deg(R) = 3(m+n-2)(\frac{mn}{2} - r) + t_S - 2n), \tag{11.39}$$

where t_S is the degree of the triadic surface $T(S)$.

11.2 Linear Congruences of Lines

11.2.1 Examples

A congruence of lines in $\mathbb{P}^3$ of order $m = 1$ is classically known as a *linear congruence*. It follows from (11.15) that

$$p_a = g = r = 0. \tag{11.40}$$

Obviously, $\mathrm{Foc}(S) = \mathrm{Fund}(S)$.

Let us start with examples.

Example 11.2.1. Let ℓ_1 and ℓ_2 be two skew lines in $\mathbb{P}^3$. For any point $x \notin \ell_1 \cup \ell_2$, there is a unique line ℓ that intersects ℓ_1 and ℓ_2. It is equal to the intersection of the planes $\langle \ell_1, x\rangle$ and $\langle \ell_2, x\rangle$.

If $x \in \ell_1 \cup \ell_2$, there is a pencil of such lines. We see that the lines intersecting ℓ_1 and ℓ_2 are parametrized by a congruence S of lines of order one and class one. Of course, it also follows from the relation:

$$[\Omega(\ell_1) \cap \Omega(\ell_2)] = \sigma_1^2 = \sigma_2 + \sigma_{1,1}.$$

The congruence S is isomorphic to a smooth quadric embedded into $\mathbb{G}$ as the complete intersection $\Omega(\ell_1) \cap \Omega(\ell_2)$. The fundamental curve of S consists of the union of the lines ℓ_1 and ℓ_2.

The universal family Z_S is isomorphic to the blow-up $\mathsf{Bl}_{\ell_1 \cup \ell_2}(\mathbb{P}^3)$. The linear system $|\mathcal{O}_{\mathbb{P}^3}(2) - \ell_1 - \ell_2|$ defines a rational map $\mathbb{P}^3 \dashrightarrow S = \mathbb{P}^1 \times \mathbb{P}^1 \subset \mathbb{P}^3$ that lifts to the projection $q_S : Z_S = \mathbb{P}(\mathcal{Q}_S) \to S$. The universal quotient vector bundle $\mathcal{Q}_S$ is isomorphic to $\mathcal{O}_S(\mathfrak{f}_1) \oplus \mathcal{O}_S(\mathfrak{f}_2)$, where $\mathfrak{f}_i$ are the divisor classes of the rulings on S. The exceptional divisors $E_i, i = 1, 2$, correspond to the projections maps $\mathcal{O}_S(\mathfrak{f}_1) \oplus \mathcal{O}_S(\mathfrak{f}_2) \to \mathcal{O}_S(\mathfrak{f}_i)$. They are sections of the projective bundle.

We have $\mathcal{O}_{\mathbb{P}(\mathcal{Q}_S)}(1) = \mathcal{O}_{Z_S}(H)$, where H is a section of Z_S equal to the pre-image of a plane in $\mathbb{P}^3$. It corresponds to the surjection $\mathcal{Q}_S \to \mathcal{O}_S(1) = h$ with the normal bundle $\mathcal{O}_H(H) \cong \mathcal{O}_S(1)$. We have $h = \mathfrak{f}_1 + \mathfrak{f}_2$ and $c_2(\mathcal{Q}_S) = [\text{point}]$. This checks formula (11.11) $H \cdot (H^2 - H \cdot q_S^*(h) + q_S^*(c_2)) = H^3 - H \cdot (q_S^*(c_2)) = n - 1 = 0$. We also check the formula for $[R(S)]$ from Proposition 11.1.16:

$$[R(S)] = [E_1] + [E_2] = (H - q_S^*(\mathcal{O}_S(\mathfrak{f}_1)) + (H - q_S^*(\mathcal{O}_S(\mathfrak{f}_2))$$

$$= 2H - q_S^*(h) = 2H + q_S^*(h + K_S)$$

Example 11.2.2. Let R_3 be a twisted cubic in $\mathbb{P}^3$. Then, any point $x \notin R_3$ is contained in a unique secant line of R_3. This shows that the closure in $\mathbb{G}$ of the set of secant lines is a congruence of lines S of order one. A general plane intersects R_3 at three points and hence contains three secant lines. This shows that the class of S is equal to 3. Thus, S is a quartic surface in the Plücker embedding. It is isomorphic to the symmetric square of $R_3 \cong \mathbb{P}^1$, and its image in $\mathbb{P}^5$ is a Veronese surface. The fundamental curve of S is equal to R_3. The universal family Z_S is isomorphic to the blow-up $\mathsf{Bl}_{R_3}(\mathbb{P}^3)$. The surface $R(S)$ is its exceptional divisor. It is known that the normal bundle $\mathcal{N}_{R_3}$ is isomorphic to $\mathcal{O}_{\mathbb{P}^1}(5)^{\oplus 2}$ (see, for example, [280, Theorem 4]). This implies that $R(S) \cong \mathbb{P}^1 \times \mathbb{P}^1$. The projection $q_S : R(S) \to S$ is a finite map of degree 2. Its branch divisor is the curve parameterizing tangent lines of R_3. Its pre-image under the Veronese map $\mathbb{P}^2 \to S$ is a conic.

Example 11.2.3. Fix a line ℓ in $\mathbb{P}^3$ and consider diagram 11.12 from Subsection 11.1.1.

Let $\phi : \mathbb{P}^1 \to C$ be the normalization map of a reduced rational curve C in $\mathbb{P}^3$ of degree n, and

$$\mathcal{L} = \phi^* \mathcal{O}_C(1) \cong \mathcal{O}_{\mathbb{P}^1}(n).$$

Fix an invetible sheaf $\mathcal{L}$ on $\ell^\perp \cong \mathbb{P}^1$ of degree $n > 0$. Let

$$\mathfrak{s} : \mathbb{P}^1 \to \mathbb{P}(\mathcal{E})$$

be the section of $\mathbb{P} = \mathbb{P}(\mathcal{E}) \cong \mathsf{Bl}_\ell(\mathbb{P}^3)$ corresponding to a surjection $\mathcal{E} = \mathcal{O}_{\ell^\perp}^{\oplus 2} \oplus \mathcal{O}_{\ell^\perp}(1) \to \mathcal{L}$. The image of $\mathfrak{s}$ is a rational curve $C \subset \mathbb{P}$. We have $\mathfrak{s}^* \mathcal{O}_{\mathbb{P}}(1) \cong \mathcal{L}$.

If the surjection factors through $\mathcal{O}_{\ell^\perp}^{\oplus 2} \to \mathcal{L}$, the curve C is a smooth rational curve contained in the exceptional divisor $E \cong \mathbb{P}^1 \times \mathbb{P}^1$ of $\mathbb{P} \cong \mathsf{Bl}_\ell(\mathbb{P}^3) \xrightarrow{\sigma} \mathbb{P}^3$ and the projection map $\sigma : C \to \ell^\perp$ is of degree n. The bidegree of C in E is equal to $(1, n)$.

Otherwise, $C' = \sigma(C)$ is a curve in $\mathbb{P}^3$ of degree n and the map

$$\sigma \circ \mathfrak{s} : \ell^\perp \to C' \subset \mathbb{P}^3$$

coincides with the normalization map of C'. It is easy to see that the normalization map is just the map $\Pi \mapsto \Pi \cap C'$. The fact that it is of degree 1 implies that

$$\# C' \cap \ell = n - 1.$$

More precisely, the intersection number $E \cdot C'$ in $\mathbb{P}$ is equal to $n - 1$. In the following, we assume, for the simplicity of exposition, that C' intersects E at $n - 1$ distinct points $p_1', \ldots, p_{n-1}'$, hence its projection C is a smooth rational curve in $\mathbb{P}^3$ intersecting ℓ at $n - 1$-distinct points $p_1, \ldots, p_{n-1}$ and leave it to the reader to consider the general case.

Let

$$\mathrm{Pic}(\mathbb{P}) = \mathbb{Z}H \oplus \mathbb{Z}F,$$

where $H = c_1(\mathcal{O}_{\mathbb{P}}(1))$ and F is the divisor class of a fiber of the protection $\pi : \mathbb{P} \to \ell^\perp$.

Let $S(\ell, C)$ be a congruence of lines equal to the closure in $\Omega(\ell) \subset \mathbb{G}$ of the locus of lines $\ell' \neq \ell$ whose proper transform in $\mathbb{P}$ intersects C.

Let x be a general point in $\mathbb{P}^3$, the proper transform of the plane in $\mathbb{P}$ intersects C transversally at one point. This shows that there is a unique ray ℓ_s through x, it intersects ℓ at one point p and its proper transform in $\mathbb{P}$ intersects C' at one point over p. Thus, the order of the congruence $S(\ell, C)$ is equal to 1.

Take a general plane Π in $\mathbb{P}^3$. It intersects ℓ at one point p and its proper transform in $\mathbb{P}$ intersects C' at n points. Thus, there are n rays in Π, hence the class of $S(\ell, C)$ is equal to n.

If $n = 1$, and $S(\ell, C)$ is of type (I), then ℓ and C are skew lines, and the congruence coincides with the congruence from Example 11.2.1. If $S(\ell, C)$ is of type (II), then each ray in $S(\ell, C)$ is contained in the plane Π whose proper transform in $\mathsf{Bl}_\ell(\mathbb{P}^3)$ intersects E along C. Thus, the congruence is a β-plane. So, we will assume that $n \geq 2$.

For any point $x \in C$, let $\langle x, \ell \rangle$ be the unique plane in $\ell^\perp$ whose proper transform in $\mathbb{P}$ contains the pre-image $x' \in C'$ of x. Each line in the plane $\langle x, \ell \rangle$ passing through x is a ray of $S(\ell, C)$. Thus, $S(\ell, C)$ is a ruled surface of degree $n + 1$ in $\Omega(\ell) \subset \mathbb{G}$.

The definition of $\langle x, \ell \rangle$ makes sense even if $C \not\subset E$ and $x \in \{p_1, \ldots, p_{n-1}\}$. In this case, $\langle x, \ell \rangle$ is spanned by ℓ and the tangent line of C at x. If $n \geq 3$, the line ℓ belongs to $S(\ell, C)$, hence $S(\ell, C) \subset \mathbb{T}_\ell(\mathbb{G}) \cap \mathbb{G}$ is singular at the point s_0 such that $\ell = \ell_{s_0}$.

For any point $x \in \ell$, the proper transform of a plane $\Pi \in \ell^\perp$ in $\mathbb{P}$ intersects C at one point p. The line $\langle x, p \rangle$ is a ray of the congruence. These rays define a directrix of the ruled surface $S(\ell, C)$ of degree n.

If S is of type (II), then $R(S)$ consists of two irreducible components $R(S)^{2,0}$ and $R(S)^{2,1}$. Both are projected to ℓ under the map $\tilde{p}_S$. The image of $\tilde{q}_S(R(S)^{2,1})$ is the exceptional curve of $\tilde{S} \to S$ isomorphic to C.

11.2.2 Classification

We will show now that all examples from the previous subsection cover all possible congruences of order one.

We will need the following classical fact:

Lemma 11.2.4. *Let C be an irreducible nondegenerate curve in $\mathbb{P}^3$ of degree d and s be the number of secants of C passing through a general point in $\mathbb{P}^3$. Then*

$$s \geq d - 2.$$

Proof We project C from a general point of $\mathbb{P}^3$. The image of the projection is a plane curve X of degree d. Let $\mu(X, x)$ and $m(x)$ be the Milnor number and the multiplicity of a point $x \in X$. Applying the Plücker formula (1.52), we get

$$d^\vee = d(d-1) - 2s - \sum_{x \in X} \mu(X, x) - \sum_{x \in C} (m(x) - 1).$$

Next, we project C from a general point $c \in C$, and apply the same formula to the image X' of the projection. We get

$$d^\vee - 2 = (d-1)(d-2) - 2t - \sum_{x\in X} \mu(X,x) - \sum_{x\in C} (m(x)-1), \qquad (11.41)$$

where t is the number of the 3-secants passing through c. After subtracting the two equalities, we get

$$s \geq d-2.$$

□

Theorem 11.2.5. *A congruence S of order 1 and class $n \geq 1$ coincides with a congruence from one of Examples 11.2.1, 11.2.2, and 11.2.3. We have*

1. *S is a smooth surface.*
   - (a) *S is of class 1 from Example 11.2.1;*
   - (b) *S is of class 3 from Example 11.2.2;*
   - (c) *S is of class 2 from Example 11.2.3 with $\ell \not\subset E$;*
2. *S is a singular surface.*
   - (a) *S is of class $n \geq 3$ from Example 11.2.3 with $\ell \not\subset E$. It has an isolated non-normal singular point;*
   - (b) *S is of class $n \geq 1$ from Example 11.2.3 with $\ell \subset E$. It is is isomorphic to a cone over a rational normal curve of degree $n+1$.*

Proof We already know that $p_a = r = 0$ and $\mathrm{Foc}(S) = \mathrm{Fund}(S)$. Since $p_a = 0$, we have $p_a = g = 0$ and S is a rational surface with only isolated singularities.

We apply Corollary 11.1.17. Assume $R(S)$ is reduced. Since $m = 1$, a general ray intersects $\mathrm{Fund}(S)$ at two points. Assume $C := \mathrm{Fund}(S)$ is irreducible, This means that the number s of secants of C passing through a general point $x \in \mathbb{P}^3$ is equal to 1. Obviously, C is not a plane curve, thus Lemma 11.2.4 implies that $d = 3$. If C is singular, then the projection of C from x is a cubic curve with two singular points, a contradiction. Therefore, C is a twisted cubic and S is the congruence from Example 11.2.2.

Assume that $R(S)$ is reduced and consists of two irreducible components. In this case, $\mathrm{Fund}(S)$ is the union of two irreducible curves F_1 and F_2 of degrees $d_1 \leq d_2$. A general plane Π intersects F_i at d_i distinct points. A general point in Π is contained in d_1 rays in Π intersecting F_1. Since $m = 1$, this gives $d_1 = 1$. Thus, we may assume that $F_1 = \ell$ is a line. If $d_2 = 1$, F_2 is also a line, obviously skew to F_1. Thus, S is congruence from Example 11.2.1. In this case S is smooth.

So, we may assume that $d_2 > 1$. A general plane Π intersects F_2 and contains

n rays, no two of which intersect outside $F_1 \cup F_2$. Since Π intersects F_2 at $d_2 > 1$ points, $n = d_2$ and all rays in Π pass through a unique point x_Π. We see that our congruence is contained in the intersection of the Chow complexes of F_1 and F_2. The intersection of the two complexes is a surface with the cohomology class $n\sigma_2 + n\sigma_{1,1}$. Since $[S] = \sigma_2 + n\sigma_{1,1}$, the intersection consists of S and the union of $n-1$ α-planes. They consist of lines passing through the intersection points of ℓ with $C := F_2$. Thus, S is as in Example 11.2.3. As we saw in this example, the line ℓ is a singular ray of S.

Assume Fund(S) is irreducible but not reduced, i.e., $R(S) = 2(R(S)^{2,0})$. In this case, the fundamental curve F is an irreducible curve of degree d and a general ray intersects F at one points. Obviously, F is not a plane curve. Take a general plane that intersects F at d points. It contains n rays, a pair of them passing through different intersection points intersect outside F. This contradicts the assumption that $m = 1$. Thus, $F = \ell$ is a line.

Let us consider the pencil $\ell^\perp$. The same argument shows that all rays contained in $\Pi \in \ell^\perp$ form a pencil with the base point $x_\Pi \in \ell$. This defines a map $f : \ell^\perp \to \ell, \Pi \mapsto x_\Pi$, of some degree k. For any point $x \in \ell$, there are k pencils of rays passing through x. A general plane through a point x has k lines, one ray from each of k pencils. Since S is of class n, we obtain that $k = n$. Consider the graph of the map f. It is a smooth curve C of bidegree $(1, n)$ whose projection to ℓ is of degree n. Considered as a section of $\mathbb{P}(\mathcal{O}_{\mathbb{P}^1}^{\oplus 2})$, it defines a section of $\mathbb{P} = \mathbb{P}(\mathcal{E})$ corresponding to a surjection $\mathcal{E} \to \mathcal{O}_{\mathbb{P}^1}(n)$ that factors through a surjection $\mathcal{O}_{\mathbb{P}^1}^{\oplus 2} \to \mathcal{O}_{\mathbb{P}^1}(n)$. It follows that $S = S(\ell, C)$ is a congruence from example 11.2.3

One of the rays is the line ℓ_{s_0}, which belongs to all pencils of rays. Let ℓ be a general line that we could identify with $\ell_0^\perp$ by taking the intersection point of $\Pi \in \ell_0^\perp$ with ℓ. Each point on ℓ intersects the unique ray that passes through this point and lies in the corresponding plane $\Pi \in l_0^\perp$. This shows that ℓ is the directrix of the ruled surface $\mathcal{R}(\ell)$. Its degree is equal to $n+1$. Since all rulings in this cone contain ℓ_0, we see that $\mathcal{R}(\ell)$ is a cone of degree $n+1$ with the vertex s_0. This shows that S is a congruence from Example 11.2.3.

It remains to check the assertions about singularities of $S(\ell, C)$. We already know that congruences from Examples 11.2.1 and 11.2.2 are smooth congruences.

Let

$$\phi_1 : \mathbb{P}^* \to \Omega(\ell)$$

be one of the small resolutions of $\Omega(\ell)$ from (11.14). Since all rays intersect ℓ, $S(\ell, C) \subset \Omega(\ell)$. The pre-image of the singular point s_0 of $\Omega(\ell)$ is the exceptional curve E. If $C \not\subset E$, the proper transform of $S(\ell, C)$ contains $n-1$ intersection points with E. Thus, S is singular at s_0 if $n \geq 3$.

Assume $C \subset E$. A general point $x \in \ell$ has n pre-images $c_1, \ldots, c_n \in C$, each of them defines a plane $\Pi_i \in \ell^\perp$ whose proper transform in $\mathsf{Bl}_\ell(\mathbb{P}^3)$ passes through c_i. Each plane Π_i contains a pencil of rays passing through x. These lines intersect at the point s_0 corresponding to ℓ. This shows that $S(\ell, C)$ is always singular if $n > 1$. The surface S is a cone of degree $n+1$ with vertex at s_0. □

We know from Theorem 10.1.2 that $\mathrm{Aut}(\mathbb{P}^3)$ is isomorphic to the connected component of the identity of $\mathrm{Aut}(\mathbb{G})$. It follows that congruences of order one of type 1 (a) and 1 (b) are projectively isomorphic in $\mathbb{G}$. The same is true for congruences of type 1 (c), because they are cubic ruled surfaces in $\mathbb{P}^4$ and all projectively equivalent in $\mathbb{G}$.

Proposition 11.2.6. *The number of moduli M of congruences of order one from Case 2 is equal to*

$$M = \begin{cases} \max\{0, 3n-10\} & \textit{in Case 2 (a)}, \\ \max\{0, 2n-10\} & \textit{in Case 2 (b)}. \end{cases}$$

Proof Fixing the line ℓ, we are left with the subgroup of $\mathrm{Aut}(\mathbb{P}^3)$ of dimension 11 that acts on the projectivization of the linear space

$$\mathrm{Hom}(\mathcal{E}, O_{\mathbb{P}^1}(n)| \cong H^0(\mathbb{P}^1, O_{\mathbb{P}^1}(n-1)) \oplus O_{\mathbb{P}^1}(n)^{\oplus 2}).$$

of dimension $3n+1$. If $C \not\subset E$, then $S(\ell, C)$ is determined uniquely by an element of this space, and we obtain $M = 3n - 10$, unless $n = 3$, and $M = 0$. This can be checked directly; we fix a twisted cubic with a 3-dimensional stabilizer subgroup of $\mathrm{Aut}(\mathbb{P}^3)$. Then, fixing two points on R_3 that determine the secant line ℓ, we are left with a one-dimensional stabilizer subgroup isomorphic to $\mathrm{PO}(2) \cong \mathbb{G} \rtimes (\mathbb{Z}/2\mathbb{Z})$.

If $C \subset E$, then $S(\ell, C)$ belongs to the subspace $|\mathrm{Hom}(O_{\mathbb{P}^1}^{\oplus 2}, O_{\mathbb{P}^1}(n))|$ of dimension $2n+1$, and we obtain $M = \max\{0, 2n-10\}$. We can also argue more geometrically. After we fix ℓ, we are left with a choice of a curve C in E of bidegree $(1, n)$. The curves depend on $2n+1$ parameters.

□

11.2.3 Monoidal surfaces in $\mathbb{P}^3$

Let Φ_d be a monoidal surface of degree d in $\mathbb{P}^3$ with singular line ℓ of multiplicity $d-1$ (see Subsection 7.2.3). If we choose projective coordinates such that ℓ is given by equations $x_0 = x_1 = 0$, the equation of Φ_d must be of the form

$$A_d(x_0, x_1) + B_{d-1}(x_0, x_1)x_2 + C_{d-1}(x_0, x_1)x_3 = 0, \tag{11.42}$$

where A_d, B_{d-1}, C_{d-1} are binary forms of degrees indicated by the subscripts. Counting parameters, we easily obtain that monoidal surfaces depend on

$$M = \max\{3d - 11, 0\}$$

moduli.

Any plane $\Pi \in \ell^\perp$ cuts out Φ_d along ℓ taken with multiplicity $d-1$ and the residual line (which may coincide with ℓ). This shows that Φ_d has a structure of a ruled surface with the ruling defined by the residual lines and the generatrix equal to $\ell^\perp$.

Let $S = S(\ell, C_n)$ be a congruence of lines of bidegree $(1, n)$ from Example 11.2.3. We assume that C_n is a smooth rational curve in $\mathbb{P}^3$ that intersects ℓ transversally at a set Σ of $n-1$ points $p_1, \ldots, p_{n-1}$. For any point $x \in \ell \setminus \Sigma$, the rays in a plane $\Pi \in \ell^\perp$ passing through x form a pencil. One of the rays in this pencil is a residual line of Φ_d. This shows that the ruled surface Φ_d defines a directrix of degree d of the ruled surface $S \subset \mathbb{P}^4 = \mathbb{T}_\ell(\mathbb{G})$. Conversely, any directrix of S of degree d defines a ruled surface in $\mathbb{P}^3$. It contains a unique generator ℓ in a general plane $\Pi \in \ell^\perp$ which is a ray of S contained in ℓ. Since ℓ intersects C_n at one point, the curve C_n is a directrix of the ruled surface. This shows that the ruled surface contains ℓ with multiplicity $d-1$, and hence it is a monoidal surface Φ_d of degree d. Since a general ray of S intersects C_n with multiplicity 1, the curve C_n is a directrix of Φ_d of multiplicity one.

The following proposition follows from our discussion.

Proposition 11.2.7. *There is a bijection between the set of directrices of degree d of the ruled surface $S(\ell, C_n) \subset \mathbb{P}^4$ and monoidal surfaces of degree d containing ℓ with multiplicity $d-1$ and containing the curve C_n with multiplicity one.*

Consider a monomial surface Φ_d of degree d containing ℓ as a line of multiplicity $d-1$. It intersects C_n at $n-1$ points with multiplicity $d-1$. Thus, if we let Φ_d contain $dn - (n-1))(d-1) + 1 = d + n$ generic points on C_n, we obtain that $C_n \subset \Phi_d$. Since monoidal surfaces Φ_d with fixed line ℓ form a linear system $|\mathcal{O}_{\mathbb{P}^3}(d) - (d-1)\ell|$ of dimension $3d$, we see that we can always find a monoidal surface of degree d containing C_n provided that $2d \geq n$.

Let $|\mathcal{O}_{\mathbb{P}^3}(d) - (d-1)\ell - C_n|$ be the linear system of monoidal surfaces Φ_d containing the curve C_n as its directrix. We obtain

$$\dim |\mathcal{O}_{\mathbb{P}^3}(d) - (d-1)\ell - C_n| \geq 2d - n. \tag{11.43}$$

We expect that a general congruence $S(\ell, C_n)$ contains a unique directrix of degree k if $n = 2k$ and a pencil of such directrices if $n = 2k - 1$. Moreover, there are no directrices of smaller degree.

Let s be the smallest degree of a directrix of $S = S(\ell, C_n)$. Applying Corollary 10.4.5, we obtain that the normalization of S^{nrm} is isomorphic to some minimal ruled surface $\mathbf{F}_k$, and S is obtained by the projection of S^{nrm} embedded in $\mathbb{P}^{k+2s+1}$ by the complete linear system $|(k+s)\mathfrak{f} + \mathfrak{e}|$. Since $\deg(S) = n+1$, $n+1 = k+2s$, we see that $s \le \frac{1}{2}(n+1)$. In particular, if $n = 2k-1$ is odd (resp. $n = 2k$ is even), and $s = k$, we obtain that $S^{\mathrm{nrm}} \cong \mathbf{F}_0$ if n is odd (resp. $\mathbf{F}_1$ if n is even).

Applying this to our case when $X = S(\ell, C_n)$, we obtain the following:

Proposition 11.2.8. *Let $S = S(\ell, C_n)$ be a general linear congruence of class n not of type 1 (a). Then,*

$$\tilde{S} \cong \begin{cases} \mathbf{F}_0 & \textit{if } C \not\subset E, n \textit{ is odd}, \\ \mathbf{F}_1 & \textit{if } C \not\subset E, n \textit{ is even}, \\ \mathbf{F}_{n+1} & \textit{if } C \subset E. \end{cases}$$

11.3 Quadratic Congruences Without Fundamental Curves

In classical terminology, a congruence of lines o forder 2 is called a *quadratic congruence*. In this section, we present Kummer's classification of quadratic congruences of lines with isolated fundamental points.

11.3.1 Fundamental points

It follows from Proposition 11.1.16 that

$$\omega_{\tilde{S}} \cong \mathcal{O}_{\tilde{S}}(-1). \tag{11.44}$$

The surface $\tilde{S}$ is a del Pezzo surface of degree $n+2$ which is projected to a surface $S \subset \mathbb{P}^5$ of degree $n+2$. In particular,

$$n \le 7.$$

By Proposition 11.1.13

$$\mathrm{Foc}(S) = \Phi(S).$$

Since the index of ramification of every point of codimension 1 in $\tilde{Z}_S$ is at most two, we have $\Phi(S) = \Phi(S)_{\mathrm{sch}}$. Since the sectional genus of a del Pezzo surface is equal to one, we obtain

$$g = 1,\ r = n - 2,\ \deg(\Phi(S)) = 4.$$

It follows from Proposition 11.1.16 that

$$[R(S)] = 2H, \quad \omega_{R(S)} \cong \mathcal{O}_{R(S)}, \quad B(S) \in |-2K_{\tilde{S}}|. \tag{11.45}$$

We leave it to the reader to deduce the following properties of S from the formulas obtained in Subsection 11.1.1.

1. $\deg(|l|) = \frac{1}{2}n(n+1) - 2$.
2. $\mathrm{mult}_x|l| = \frac{1}{2}h(x)(h(x) - 1;$
3. $\deg((P)) = n - 1$.
4. $\mathrm{mult}_x(P) = h(x) - 1;$
5. $\mathrm{mult}_P(P) = 1;$
6. Each ray is tangent to $\mathrm{Foc}(S)$ at two points which coincide at points on a curve $\mathrm{Foc}(S)_0$ of degree $2(n+2)$.

Proposition 11.3.1. *S has no focal rays.*

Proof The asserion is true if $n = 2$ since a Jacobian Kummer surface does not contain lines. Assume $n > 2$. By Lemma 12.2.2 from the next chapter, a line on the quartic surface $\Phi(S)$ contains at most three nodes. Therefore, we can always find a fundamental point x not lying on ℓ_s. Then, ℓ_s intersects $K(x)$ at a ray different from ℓ_s. Since ℓ is contained in the focal surface, we get a contradiction. □

Let

$$\mathrm{Fund}(S)_k = \{x \in \mathrm{Fund}(S) : h(x) = k\}.$$

I hope that this notation will not confuse the reader with the notation $\mathrm{Fund}(S)_0$ and $\mathrm{Fund}(S)_1$ of the one-dimensiuonal and zero-dimensional part of the locus of fundamental points of S. In our case, $\mathrm{Fund}(S)$ consists only of isolated points.

Recall that, for any $x \in \mathrm{Fund}(S)_k$, the curve $C(x) = \Omega(x) \cap S$ is a plane curve of degree k such that $K(x) = \mathcal{R}(C(x))$. Since each generator of $K(x)$ is a ray tangent to $\Phi(S)$ at some point, we have

$$K(x) \cap \Phi(S) = 2T_x, \tag{11.46}$$

where T_x is a curve of degree $2k$. Following the classical terminology, we call this curve a *trope* of degree $2k$ (a trope-conic, a trope-quartic, etc.).

Let E_x be the exceptional curve of the birational map $\tilde{p}_S : R(S) \to \Phi(S)$. It consists of points $z = (x, \tilde{s})$ where ℓ_s is a generator of the cone $K(x)$. We have

$$\tilde{p}_S^{-1}(K(x)) = E_x + \bar{T}_x, \tag{11.47}$$

where $\bar{T}_x$ is the proper transform of T_x in $R(S)$. The curve $\bar{T}_x$ consists of points $z = (x', \tilde{s}) \in R(S)$ such that $x' \in T_x$ and ℓ_s is a generator of $K(x)$ that intersects $\Phi(S)$ at x' with multiplicity two. In particular,

$$E_x \cap \bar{T}_x = \{(x, \tilde{s}) : \tilde{s} \in B(S), x \in \ell_s\}.$$

The ray ℓ_s here is a tangent line to one of the branches of the trope $T(x)$ at x. In particular, we expect that there are $h(x)$ branches and hence $\#E_x \cap \bar{T}_x = h(x)$.

The projection $\tilde{q}_S$ maps E_x and $\bar{T}_x$ isomorphically to the curve $\tilde{C}(x)$ equal to the pre-image of the generatrix of $C(x) = \Omega(x) \cap S$ of the cone $K(x)$ in $\tilde{S}$. The curve $C(x)$ is a plane curve of degree $h(x)$. Since it is birationally isomorphic to E_x, and x is a double rational point, it is a rational curve. In particular, it is singular if $h(x) \geq 3$. The curve $\tilde{C}(x)$ splits under the cover $R(S) \to \tilde{S}$ into the union $E_x + \bar{T}_x$. We expect that $\tilde{C}(x)$ is a smooth rational curve everywhere tangent to $B(S)$.

Recall that the rays $\ell_s, \tilde{s} \in B(S)$, intersect $\Phi(S)$ at one point with multiplicity 4, and the intersection points form the curve $\Phi(S)_0$ of degree equal to $2(n+2)$. Since $B(S) \in |\mathcal{O}_{\tilde{S}}(2)|$, the curves $\tilde{C}(x)$ intersects $B(x)$, hence E_x contains a point $(x, \tilde{s}), \tilde{s} \in B(S)$. This implies that

$$\mathrm{Fund}(S) \subset \Phi(S)_0. \tag{11.48}$$

We know that $n - 1 = \deg((P)) \geq \mathrm{mult}_x(P) = h(x) - 1$, hence $h(x) \leq n$. If $h(x) = n$, the linear system of surfaces (P) consists of cones with a singular point at x. Thus, all focal planes pass through x, obviously absurd. This gives

$$h(x) \leq n - 1$$

Theorem 11.3.2. *The number α_k of fundamental points of degree k on a congruence of bidegree $(2, n)$ without fundamental curve is given in Table 11.1 below.*

Proof Let ℓ and ℓ' be two general rays of S. The ruled surfaces $\mathcal{R}(\ell)$ and $\mathcal{R}(\ell')$ intersect at $\deg(S) = n+2$ rays. Thus, the two surfaces residually intersect along a curve C of degree $(n+2)^2 - (n+2) = n^2 + 3n + 2$. If $x \in C$ and it does lie on any common ray, then x is the intersection point of two different rays, and hence no other ray passes through it. This shows that three general ruled surfaces $\mathcal{R}(\ell), \mathcal{R}(\ell')$ and $\mathcal{R}(\ell'')$ can intersect only at fundamental points of $\mathrm{Foc}(S)$ and at $3(n+2)^2$ points on the common rays (a ray common to two ruled

	(2,2)	(2,3)	(2,4)	(2,5)	$(2,6)_I$	$(2,6)_{II}$	(2,7)
α_1	16	10	6	3	1	0	0
α_2		5	6	6	4	8	0
α_3			2	3	6	0	10
α_4				1	0	4	0
α_5					1		
α_6							1
$\sum \alpha_i$	16	15	14	13	12	12	11

Table 11.1 *Fundamental points of congruences of bidegree* $(2, n)$

surfaces must meet the third). It follows from (11.19) that x is a point of $\mathcal{R}(\ell)$ of multiplicity $h(x)$. Thus, the intersection curve has multiplicity $h(x)^2$ at x. Intersecting with the third ruled surface $\mathcal{R}(\ell'')$, we get

$$(n+2)^3 = 3(n+2)^2 + \sum k^3 \alpha_k, \tag{11.49}$$

where α_k is the number of fundamental points x with $h(x) = k$.

Let $x \in \mathrm{Fund}(S)_k$ and let ℓ and ℓ' be two general lines in $\mathbb{P}^3$. Then, the hyperplane section $\Omega(\ell')$ intersects the plane curve $C(x)$ at k points, hence x is a point of multiplicity k on $\mathcal{R}(\ell')$. We also know from Proposition 11.1.8 that x is a point of multiplicity $\frac{1}{2}k(k-1)$ on the curve $|\ell|$. Thus, $\mathcal{R}(\ell')$ and $|\ell|$ intersect at x with multiplicity $\frac{1}{2}k^2(k-1)$. If $y \notin \mathrm{Fund}(S)$ is an intersection of point of $\mathcal{R}(\ell')$ and $|\ell|$, then one of two rays passing through y must be a common generator of $\mathcal{R}(\ell')$ and $\mathcal{R}(\ell)$. Thus, all non-fundamental intersection points of $\mathcal{R}(\ell')$ and $|\ell|$ lie on $n+2$ rays intersecting ℓ and ℓ'. On each such ray, the intersection point lies on the remaining $n-1$ rays in the plane spanned this ray and the line ℓ.

This gives us the second equality

$$\#\mathcal{R}(\ell') \cap |\ell| = (n+2)(\tfrac{1}{2}n(n-1)+n-2) = (n+2)(n-1) + \tfrac{1}{2}\sum_k k^2(k-1)\alpha_k. \tag{11.50}$$

Now, we can find all possible solutions of (11.49).

We know that $h(x) \leq n-1$. If $n = 2$, this gives $\alpha_1 = 16$.

If $n = 3$, then (11.50) gives $\alpha_2 = 5$ and (11.49) gives $\alpha_1 = 10$.

If $n = 4$, then (11.50) gives $30 = 2\alpha_2 + 9\alpha_3$ and (11.49) gives $108 = \alpha_1 + 8\alpha_2 + 27\alpha_3$. The first equality implies $\alpha_3 = 2$ and $\alpha_1 = 6$, hence $\alpha_1 = 6$.

If $n = 5$, then (11.50) gives $63 = 2\alpha_2 + 9\alpha_3 + 24\alpha_4$, hence $\alpha_3 \in \{1, 3\}$. If $\alpha_3 = 1$, then $(\alpha_2, \alpha_4) = (15, 1)$ or $(3, 2)$. The second equality gives $196 = \alpha_1 + 8\alpha_2 + 27\alpha_3 + 64\alpha_4$ and shows that the solution $(15, 1)$ is impossible and

$\alpha_1 = 15$. However, by Proposition 12.2.14, a normal quartic surface has at most 16 singular points. Hence, we may assume that $\alpha_3 = 3$. The first equality gives $18 = \alpha_2 + 12\alpha_4$, hence $\alpha_2 = 6, \alpha_4 = 1$, and the second one gives $\alpha_1 = 3$.

If $n = 6$, (11.50) gives

$$112 = 2\alpha_2 + 9\alpha_3 + 24\alpha_4 + 50\alpha_5$$

and the first equality gives

$$320 = \alpha_1 + 8\alpha_2 + 27\alpha_3 + 64\alpha_4 + 125\alpha_5.$$

If $\alpha_5 = 2$, then

$$12 = 2\alpha_2 + 9\alpha_3 + 24\alpha_4$$

gives $\alpha_4 = \alpha_3 = 0, \alpha_2 = 6$. Equality (11.49) gives a contradiction. Thus, we may assume that $\alpha_5 \leq 1$.

If $\alpha_5 = 1$, we get $62 = 2\alpha_2 + 9\alpha_3 + 24\alpha_4$ and 195 =. We check all solutions for $\alpha_4 \neq 0$ and find that the number of singular points is larger than 16. So, $\alpha_4 = 0$, and, for the same reason, we get that $2 = 2\alpha_2 + 9\alpha_3$ implies $\alpha_3 = 6$, hence

$$\alpha_2\alpha_1 + 8\alpha_2 + 27\alpha_3 + 64\alpha_4 = 4, \alpha_1 = 1.$$

This corresponds to the column $(2, 6)_I$ from the Table.

If $n = 6$ and $\alpha_5 = 0$, we get

$$112 = 2\alpha_2 + 9\alpha_3 + 24\alpha_4,$$

and

$$320 = \alpha_1 + 8\alpha_2 + 27\alpha_3 + 64\alpha_4.$$

This gives $\alpha_4 \leq 4$, and we find that the only possible solution is $(\alpha_1, \alpha_2, \alpha_3, \alpha_4) = (0, 8, 0, 4)$. This gives column $(2, 6)_{II}$.

Finally, if $n = 7$, we get

$$180 = 2\alpha_2 + 9\alpha_3 + 24\alpha_4 + 50\alpha_5 + 90\alpha_6,$$

and

$$486 = \alpha_1 + 8\alpha_2 + 27\alpha_3 + 64\alpha_4 + 125\alpha_5 + 216\alpha_6.$$

The first equality implies that $\alpha_6 \leq 2$, and if $\alpha_6 = 2$, we get $\alpha_1 > 16$. If $\alpha_6 = 0$, we gain get too many singular points. So, $\alpha_6 = 1$, and we get

$$270 = \alpha_1 + 8\alpha_2 + 27\alpha_3 + 64\alpha_4 + 125\alpha_5,$$

and

$$90 = 2\alpha_2 + 9\alpha_3 + 24\alpha_4 + 50\alpha_5.$$

If $\alpha_5 = 1$, then $\alpha_4 = \alpha_3 = 0, \alpha_2 = 8$, hence the first equality gives $\alpha_1 > 16$. So, $\alpha_5 = 0$. By enumerating all possible cases, we find the only solution is $\alpha_1 = \alpha_2 = \alpha_4 = 0, \alpha_3 = 10$. □

Proposition 11.3.3. *Let x, x' be two fundamental points of S with $h(x)+h(x') = n+a$, where $a > 0$. Then, $a \le 2$, and there exists a ray of S containing x and x'. If $a = 2$, then the ray is a singular ray corresponding to an ordinary singular point of S. Conversely, if $s \in \mathrm{Sing}(S)$, then it is an ordinary non-normal singular point of multiplicity 2 and it contains two fundamental points x, x' with $h(x) + h(x') = n + 2$.*

Proof A general plane through x and x' contains n rays, among them are $h(x)+h(x')$ rays coming from generators of $K(x)$ and $K(x')$. So, if $h(x) + h(x') > n$, the cones must share a ray ℓ_s. Since the degree of the curve $C(\ell_s) = \Omega(\ell_s) \cap S$ is equal to $n+2$ and there is only one ray connects x with x', we obtain that $a \le 2$. If $a = 2$, then a general plane containing ℓ_s cuts out ℓ_s with multiplicity 2. By Proposition 11.3.1. ℓ_s cannot be a focal ray. Thus, ℓ_s is a singular ray whose pre-image in $\tilde{Z}_S$ is not contained in $R(S)$. This shows that s is an ordinary non-normal singularity of multiplicity two.

□

By inspection of Table 11.1, we get the following.

Corollary 11.3.4. $\mathrm{Sing}(S)$ *consists of* $\frac{1}{2}(n-1)(n-2)$ *ordinary non-normal singular points of multiplicity* 2.

Corollary 11.3.5. *For any $x \in Fund(S)_k$ with $k \le 3$, the cone $K(x)$ is irreducible, hence the curves E_x, T_x and $C(x)$ are irreducible.*

Proof If $h(x) = 1$, $K(x)$ is a plane, hence it is irreducible. If $h(x) \le 3$, and $K(x)$ is reducible, then one of the irreducible components of $K(x)$ is a plane, hence $C(x)$ contains a line as it irreducible components. But Table 11.1 shows that $\#\mathrm{Fund}(S)_1$ is equal to the number of lines on $\tilde{S}$. □

Proposition 11.3.6. $\Phi(S)$ *is a normal quartic surface birationally isomorphic to a K3 surface. A fundamental point of S is a rational double point, and its enveloping cone contains the cone $K(x)$ of degree $h(x)$.*

Proof Suppose $\Phi(S)$ is irreducible and non-normal. Let $\Phi(S)^{\mathrm{nrm}}$ be its normalization. The pre-image of any fundamental point on $\Phi(S)^{\mathrm{nrm}}$ is a singular point. We know that the number of them is equal to $18 - n$. As we will learn in Section 12.1, a non-normal quartic surface does not contain so many isolated singular points.

Let us assume that $\Phi(S)$ is normal. Then, the projection $\tilde{p}_S : R(S) \to \Phi(S)$

is a birational isomorphism. Since $\omega_{R(S)} \cong \mathcal{O}_{R(S)}$, and $\tilde{q}_S : R(S) \to \tilde{S}$ is a double cover of a del Pezzo surface branched over $B(S) \in |-2K_{\tilde{S}}|$, we obtain that $R(S)$ is birationally isomorphic to a K3 surface. Thus, $\Phi(S)$ is birationally isomorphic to a K3 surface. □

The surface $R(S)$ is smooth if and only if the branch cover $B(S)$ is smooth. Since all its singular points are rational double points, the curve $B(S)$ has only simple singularities (see Definition 4.2.16). A local equation of the singularity is $z^2 + f(x, y) = 0$, where $f(x, y) = 0$ is a local equation of a simple singularity of $B(S)$. If the latter is of type a_n, d_n, e_n, then the singularity of the surface is of type A_n, D_n, E_n.

It follows that both $\Phi(S)$ and $R(S)$ ate birationally isomorphic to a K3 surface. Let

$$\pi : \tilde{R}(X) \to R(X) \to \Phi(S)$$

be a minimal resolution of $R(S)$ and hence of $\Phi(S)$. Of course, we expect that for a general S, $R(X) \cong \tilde{R}(X)$.

Recall that we also have the Stein factorization of the map $\tilde{q}_S$:

$$R(S) \to R(S)' \to \tilde{S}.$$

Proposition 11.3.7. *The following assertions are equivalent:*

(i) *$B(S)$ is smooth;*
(ii) *$R(S)'$ is smooth;*
(iii) *$R(S)$ is smooth.*

Proof (i) ⇔ (ii) The double cover $R(S)' \to \tilde{S}$ is given locally by equation $y^2 - \phi(x) = 0$, where $\phi(x) = 0$ is a local equation of $B(S)$. It is smooth if and only if $B(S)$ is smooth.

(ii) ⇔ (iii) If $R(S)'$ is smooth, then it is isomorphic to $\tilde{R}(S)$, and hence $R(S) \cong \tilde{R}(S)$. Conversely, if $R(S)$ is smooth, and $R(S)'$ is not, then one of the fibers of $R(S) \to R(S)'$ is equal equal to $\tilde{q}_S^{-1}(s)$. Its image in $\Phi(S)$ is a focal ray, contradicting Proposition 11.3.1.

□

Definition 11.3.8. *We say that S is a* general congruence *if the following equivalent conditions are satisfied:*

(i) *$B(S)$ is nonsingular and the cones $K(x), x \in Fund(S)$, are irreducible;*
(ii) *All fundamental points are ordinary double points of $\Phi(S)$ and $\mathrm{Sing}(\Phi(S)) = Fund(S)$.*

11.3.2 The conjugacy graphs

Following the classical terminology, we say that two fundamental points are *conjugate* if they are contained in a ray of the congruence. For example, by Proposition 11.3.3, two fundamental points x, x' with $h(x) + h(x') \geq n + 1$ are conjugate.

Let $\Gamma(S)$ be the *conjugacy graph* of S. Its vertices are fundamental points and its edges are rays connecting two fundamental points. The singular ray is a double edge. It comes with a natural labelling of vertices: we mark a vertex with its degree. The full subgraph with vertices $\mathrm{Fund}(S)_k$ will be denoted by $\Gamma(S)_k$.

Let us describe the conjugacy graphs of general congruences of bidegree $(2, n)$ without fundamental curves.

$(2, 2)$:

For each fundamental point $x \in \Phi(S)$, the cone $K(x)$ is a plane that intersects $\Phi(S)$ along the trope-conic T_x. A point x' conjugate to x lies on T_x. This happens if and only if the image of $p_S^{-1}(T_x)$ (that coincides with the image of $p_S^{-1}(E_x)$ in S intersects the image of $p_S^{-1}(T_{x'})$. The surface S is a del Pezzo surface of degree 4. It contains 16 lines with the incidence graph pictured in Figure 8.6.3. This graph is isomorphic to $\Gamma(S)_1$.

$(2, 3)$:

The surface S is a del Pezzo surface of degree 5. It contains 10 lines and 5 pencils of conics. The subgraph $\Gamma(S)_1$ is the incidence graph of the set of 10 lines. We know from Section 8.5 that this graph is isomorphic to the Petersen graph from Figure 8.5.1.

The focal surface $\Phi(S)$ has 5 fundamental points $x \in \Phi(S)$ with $h(x) = 2$. The concern $K(x)$ is of degree 2 and it intersects $\Phi(S)$ along a trope-quartic T_x with a double point at x. All fundamental points x' conjugate to x lie on these tropes. The image of $p_S^{-1}(T_x)$ on S is a conic. We have 5 conics from different pencils that split under the cover $R(S) \to S$ and define the trope-quartics. This shows that the graph $\Gamma(S)_2$ is isomorphic to the complete graph $K(5)$.

It is easy to see that each line on a quintic del Pezzo surface S is realized as a section of exactly two conic bundles. This shows that each vertex of $\Gamma(S)_1$ is connected to two vertices of $\Gamma(S)_2$. Also, each conic bundle has four sections, hence each vertex of $\Gamma(S)_2$ is connected to four vertices of $\Gamma(S)_1$. In other words, the incidence graph of the sets of vertices of $\Gamma(S)_1$ and $\Gamma(S)_2$ is of type $(10_2, 5_4)$.

$(2, 4)$:

The surface S is a projection of del Pezzo surface of degree 6 from a point in $\mathbb{P}^6$. It contains one singular point s that corresponds to the singular ray ℓ_s connecting two fundamental points y_1, y_2 of degree 3. Six fundamental points $x_1, \ldots, x_6$ of degree 2 are conjugate to each of these points. The images of the exceptional curve E_{y_i} is a plane cubics in $\Omega(y_i)$ with double point at $s = \Omega(y_1) \cap \Omega(y_2)$. The proper transforms of these cubics in $\tilde{S}$ are two rational cubic curves $R_1 \in |e_0|, R_2 \in |2e_0 - e_1 - e_2 - e_3|$. They intersect at two points with the center of the projection contained in a line joining these two points.

The images of E_{x_i} in S are the projections of six conics on $\tilde{S}$, two from each conic pencil on $\tilde{S}$. They form the graph $\Gamma(S)_2$ isomorphic to a tripartite graph. The images of $E_x, h(x) = 1$ are the projections of six lines on $\tilde{S}$. They form the graph $\Gamma(S)_1$ isomorphic to a hexagon.

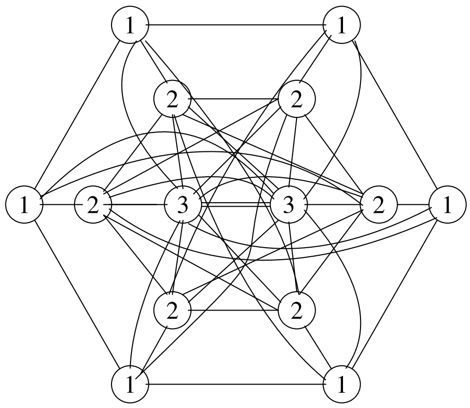


Figure 11.1 Conjugacy graph of a congruence of order $n = 4$

Projecting from one of the fundamental points of degree 3, we obtain a birational model of $\tilde{S}$ as the double cover of the plane with the branch curve equal to the union of three lines and a nodal cubic.

(2, 5)

The congruence S is a projection of a del Pezzo surface $\tilde{S} \subset \mathbb{P}^7$ of degree 7 from a line. The surface $\tilde{S}$ has 3 lines representing the divisor classes $e_0 - e_1 - e_2, e_1, e_2$, two pencils of conics representing $e_0 - e_1$ and $e_0 - e_2$, the net of cubics representing e_0, and a web of quartics representing $2e_0 - e_1 - e_2$. They are the exceptional curves E_x over three 3 fundamental points of degree 1, six fundamental points of degree 2, three fundamental points of degree 3, and one fundamental point of degree 4

There are three singular rays $\ell_{s_1}, \ell_{s_2}, \ell_{s_3}$ connecting the unique fundamental point x_0 of degree 4 with three fundamental points x_1, x_2, x_3 of degree 3. The image of the curves $E_{x_i}, i = 1, 2, 3$ is a plane cubic in the plane $\Omega(x_i)$ with singular point at s_i. The image of E_{x_0} is a plane quartic in the plane $\Omega(x_0)$ that passes through the singular points s_1, s_2, s_3.

The image of the quartic curve span a 4-dimensional subspace M in $\mathbb{P}^7$. The images of cubics span 3-dimensional subspaces L_1, L_2, L_3. Since the union of the quartic and a cubic is a hyperplane section of $\tilde{S}$, the subspaces M and L_i intersect along a line. This the line that is projected to a singular point of S. The center of the projection is a line in M that intersects the lines $L_i \cap M$.

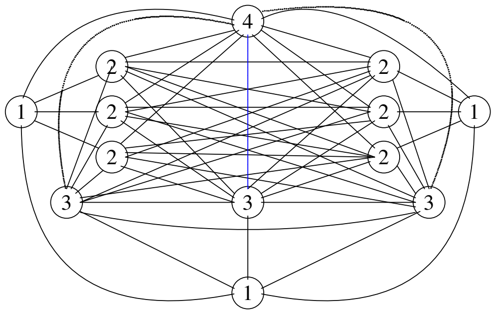


Figure 11.2 Conjugacy graph of a congruence of class $n = 5$

The projection from the fundamental point of degree 4 gives a birational model of $\tilde{S}$ as a double cover of the plane branched along the union of a 3-nodal quartic and two lines.

$(2, 6)_I$

The congruence S is the projection of a del Pezzo surface $\tilde{S} \cong \mathsf{Bl}_{p_1}(\mathbb{P}^2)$ of degree 8 in $\mathbb{P}^8$ from a plane.

It has a unique fundamental point x_0 of degree 5, the image of E_{x_0} in S is a plane quintic in $\Omega(x_0)$. It has six singular points $s_1, \ldots, s_6$ corresponding to the singular rays connecting x_0 with six fundamental points $y_1, \ldots, y_6$ of degree 3. The image of E_{y_i} is a cubic in $\Omega(y_i)$ with singular point at s_i. The image of E_{x_0} in $\mathbb{P}^8$ is a smooth rational quintic curve that spans a linear subspace M of dimension 5. The images of E_{y_i} is a smooth rational quintic curve that spans a linear space L_i of dimension 3. The quintic curve is the image in the blow-up of a conic passing through the point p_1. Together with a cubic represented by a general line in the plane, they form an anti-canonical divisor. This shows that $M \cap L_i$ is a line, the secant line of the cubic spanning L_i The center of the projection is a plane in M that intersects the six secant lines.

The projection of $\Phi(S)$ from a fundamental point of degree 5 defines a birational model of $\Phi(S)$ as a double cover of $\mathbb{P}^2$ branched over the union of a rational plane quintic and a line. They are the images of trope T_{x_0} and the unique trope-conic.

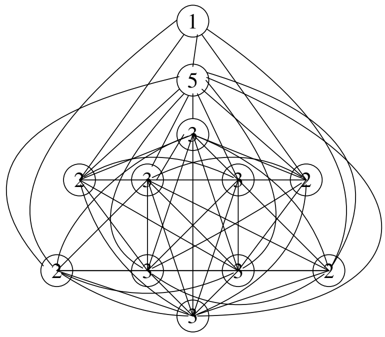


Figure 11.3 Conjugacy graph of a congruence $(2,6)_I$

$(2,6)_{II}$

The congruence S is the projection of a del Pezzo surface $\tilde{S} \cong \mathbf{F}_0$ of degree 8 in $\mathbb{P}^8$ from a plane.

It has four fundamental points x_1, x_2, x_3, x_4 of degree 4. The rays $\ell_{ij} = \langle x_i, x_j \rangle$ are the six singular rays of S. The four trope-octics T_{x_i} have double points at $x_j, j \neq i$, and pass through eight fundamental points of degree two. The images of E_{x_i} in S are plane quartics with nodes at three singular rays ℓ_{jk}. Their proper transforms in $\tilde{S}$ are quartics curves $Q(x_i)$, the images of four conics under the anti-bicanonical map from $\mathbf{F}_0 \to \mathbb{P}^8$.

The eight trope-quartics T_{y_i} pass through all fundamental points of degree 4. The images of E_{y_i} in S are conics. Their proper transforms in $\tilde{S}$ are conics $R(y_i)$, the image of eight lines in $\mathbf{F}_0$ under the anti-bicanonical map, four from each family of lines. This divides the set of fundamental points into two subsets $\{y_1, \ldots, y_4\}$ and $\{y_5, \ldots, y_8\}$ such that each trope $T_i, i \leq 4$, passes through the points $y_j, j > 4$, and each trope $T_i, i > 4$, passes through the points $y_j, j \leq 4$.

The conjugacy graph $\Gamma(S)$ is the join of $\Gamma(S)_2$ isomorphic to a complete bipartite graph $K_{4,4}$ and $\Gamma(\Phi)_4$ isomorphic to the complete graph $K(4)$.

The quartic $Q(x_i)$ (resp. conic $R(y_i)$) spans a 4-dimensional linear subspace M_i in $\mathbb{P}^8$. The six intersection lines $M_i \cap M_j$ are in a special position such that they intersects a plane in $\mathbb{P}^8$, the center of the projection $\mathbb{P}^8 \dashrightarrow \mathbb{P}^5$.

$(2,7)$

The congruence S is the projection of the Veronese surface $\tilde{S}$ in $\mathbb{P}^9$. It has one fundamental point x_0 of degree 6 and 10 singular rays that connect this point to ten fundamental points y_i of degree 3. The image of E_{x_0} in S is a 10-nodal plane sextic in $\Gamma(x_0)$. Its proper transform in $\tilde{S}$ is a smooth rational sextic, the image of a conic in $\mathbb{P}^2$ under the Veronese map. The image of E_{y_i} in S is a plane cubic with one node at one of the singular points of S. Its proper transform in $\tilde{S}$ is a smooth rational cubic, the image of a line in $\mathbb{P}^2$ under the Veronese map.

The conjugacy graph $\Gamma(S)$ is the join of the complete graph $\Gamma(S)_2$ with $\Gamma(S)_6$.

The surface S is a projection of $\tilde{S}$ from a 3-dimensional subspace contained in the linear span of the rational sextic.

The projection of $\Phi(S)$ from the fundamental point of degree 6 defines a birational model of $\Phi(S)$ as the double cover of the plane branched over a rational plane sextic W. The images of the trope-sextics T_{y_i} are rational quartics passing through nine nodes of W with a triple point at the remaining node of W. They are split under the double cover. The sextic branch curve admits a contact conic (see Section 12.2), this imposes one condition on the sextic.

11.3.3 Confocal congruences

We know that a del Pezzo surface $\tilde{S}$ of degree $2+n$ has $N(n)$ pencils of conics $\{C_t\}_{t\in\mathbb{P}^1}$, where $N(n) = 10, 5, 3, 2, 1, 2, 0$ if $n = 2, 3, 4, 5, 6_I, 6_{II}, 7$, respectively. For any conic C_t, the image $\mathcal{R}(C_t)$ of $\tilde{q}_S^{-1}(C_t)$ in $\mathbb{P}^3$ is a ruled surface of degree 2, a quadric surface. If it is smooth, it comes with a choice of one of the ruling. The rays from this rules is an infinite set of disjoint lines in $\mathbb{P}^3$, in classical terminology, it is a *regulus*.

Note that the one-dimensional algebraic system of quadrics $\{\mathcal{R}(C_t)\}_{t\in\mathbb{P}^1}$ is a quadratic pencil (see 4.1.4). For any general point $x \in \mathbb{P}^3$, its pre-image $\tilde{p}_S^{-1}(x)$ consists of two points $(x_1, s_1), (x_2, s_2)$, and there will be two conics C_{t_1}, C_{t_2} passing through s_1, s_2 in the pencil, hence two quadrics $\mathcal{R}(C_{t_1}), \mathcal{R}(C_{t_1})$ passing though x.

A general ray from a regulus $\mathcal{R}(C)$ is touching the focal service $\Phi(S)$ at two points. This implies that $\mathcal{R}(C) \cap \Phi(S)$ is a curve of degree 4 taken with multiplicity 2. It is a curve of bidegree $(2,2)$ on $\mathcal{R}(C)$, i.e. a curve of arithmetic genus one. When C varies in a pencil, we obtain a pencil of elliptic curves on $\Phi(S)$, its pre-image on the K3 surface $\tilde{R}(S)$ defines an elliptic fibration.

In fact, the algebraic family $\mathcal{R}(C_t)_{t\in\mathbb{P}^1}$ is a family of contact quadrics to the focal surface. We can write its member C_t in the form

$$u^2Q_1 + 2uvQ_2 + v^2Q_3 = 0,$$

where $[u, v]$ are homogenous coordinates of $t \in \mathbb{P}^1$. The equation

$$Q_2^2 - Q_1Q_3 = 0 \tag{11.51}$$

describes the closure of the locus of points $x \in \mathbb{P}^3$ such that $\tilde{q}_S^{-1}(x)$ consists of one point. It coincides with the focal surface $\Phi(S)$. This gives the following.

Proposition 11.3.9. *A choice of a pencil of conics in $\tilde{S}$ puts the equation of $\Phi(S)$ in the form* (11.51)

We expect that $Q_1 \cap Q_2 \cap Q_3$ consists of 8 points, they are the base points of the quadratic pencil of quadrics $\mathcal{R}(C_t)$. They all lie on the quartic curve $Q_1 \cap Q_3$ along which any $\mathcal{R}(C_t)$ is tangent to $\Phi(S)$.

Let $x \in \mathrm{Fund}(S)$, the image of the exceptional curve $\tilde{p}_S^{-1}(x)$ on $\tilde{S}$ is either a section or a line component of a pencil of conics $\{C_t\}_{t\in\mathbb{P}^1}$, viewed as a conic bundle $\tilde{S} \to \mathbb{P}^1$. If it is a section, it defines a base point of the pencil of quadric surfaces $\mathcal{R}(C_t)$.

Proposition 11.3.10. *Let $\{C_t\}_{t\in\mathbb{P}^1}$ be a pencil of conics on $\tilde{S}$. Then, the pencil of quadric surfaces $\mathcal{R}(C_t)$ has exactly eight fundamental points as its base points. If S is not of type $(2,6)_{II}$, there are two conjugate points $x \in \mathrm{Fund}(S)_{n-1}$ and $x' \in \mathrm{Fund}(S)_1$ among these eight points. The remaining six fundamental points are all points from $\mathrm{Fund}(S)_3$ and points of degree ≤ 2 not conjugate to x'.*

Proof We will check this case-by-case.

Suppose $n = 2$. Then, $S \cong \mathsf{Bl}_{p_1,\ldots,p_5}(\mathbb{P}^2)$. In the corresponding geometric basis of $\mathrm{Pic}(S)$, the surface S has 10 pencils of conics $|e_0 - e_i|$ and $|2e_0 - \sum_{i=1}^5 e_i + e_j|$. Since they are all Cremona-equivalent, we may choose one of them, say $|e_0 - e_1|$. The set $\mathrm{Fund}(S)_1$ corresponds to 16 lines on S. There are eight lines contained in reducible members of the pencil. They represent $e_i, i \neq 1, e_0 - e_1 - e_j$. The remaining 8 lines representing $e_1, e_0 - e_i - e_j, 2e_0 - \sum_{i=1}^5 e_i$ are sections of the conic bundle. We may choose x corresponding to $l(x) \in |e_1|$ and x' corresponding to $l(x') \in |2e_0 - \sum_{i=1}^5 e_i|$. Since $l(x')$ does not intersect the six lines representing $e_0 - e_i - e_j$, we see that the corresponding points from $\mathrm{Fund}(S)_1$ are not conjugate to x'.

Suppose $n = 3$. Then, $S \cong \mathsf{Bl}_{p_1,\ldots,p_4}(\mathbb{P}^2)$. In the corresponding geometric basis of $\mathrm{Pic}(S)$, the surface S has 5 pencils of conics $|e_0 - e_i|$ and $|2e_0 - \sum_{i=1}^4 e_i|$. Without loss of generality, we may choose one of them, say $|e_0 - e_1|$. The set $\mathrm{Fund}(S)_2$ corresponds to 5 conics $C_1, \ldots, C_5$ from different pencils of conics. Four of them $C_1, C_2, C_3, C_4 \notin |e_0 - e_1|$ are sections of the conic bundle. The set $\mathrm{Fund}(S)_1$ corresponds to 10 lines $l_1, \ldots, l_{10}$. Four lines l_1, l_2, l_3, l_4 representing $e_1, e_0 - e_i - e_j, i, j > 1$, are line sections of the conic bundle. All together we found 8 sections represented by fundamental points of S. We choose x corresponding to $C_4 \in |2e_0 - \sum_{i=1}^4 e_i|$ and x' corresponding to the line $l_4 \in |e_1|$. The three lines l_1, l_2, l_3 and the four conics $C_1, \ldots, C_3$ do not intersect l. So, we have found 8 fundamental points satisfying the assertion of the proposition.

Suppose $n = 4$. Then, $\tilde{S} \cong \mathsf{Bl}_{p_1,\ldots,p_3}(\mathbb{P}^2)$. In the corresponding geometric basis of $\mathrm{Pic}(\tilde{S})$, the surface $\tilde{S}$ has three pencils of conics $|e_0 - e_i|$. Without loss of generality, we may choose one of them, say $|e_0 - e_1|$. The set $\mathrm{Fund}(S)_3$ corresponds to two rational cubic curves R_1, R_2 from $|e_0|$. Both of them are sections of the conic bundle. The set $\mathrm{Fund}(S)_2$ corresponds to the set of six conics $C_i, C_i' \in |e_0 - e_i|, i = 1, 2, 3$. Four of them C_2, C_2', C_3, C_3' are sections of $|e_0 - e_1|$. Other two sections of $|e_0 - e_1|$ are defined by two lines l_1, l_2 representing $e_0 - e_2 - e_3, e_1$. Thus, we have altogether 8 sections defined by $R_1, R_2, C_2, C_2', C_3, C_3', l_1, l_2$. We may choose $x \in \mathrm{Fund}(S)_3$ corresponding to R_1 and $x' \in \mathrm{Fund}(S)_1$ corresponding to l_1. The line l_2 and the conics C_2, C_2', C_3, C_3' do not intersect, and hence define fundamental points of degree one or two that are not conjugate to x'. So, the assertion is checked.

Suppose $n = 5$. Then, $\tilde{S} \cong \mathsf{Bl}_{p_1,p_2}(\mathbb{P}^2)$. There are two pencils of conics $|e_0 - e_1|$ and $|e_0 - e_2|$. We may assume that our pencil is $|e_0 - e_1|$. It has 8 sections defined by fundamental points represented by the line l from $|e_1|$, conics $C_1, C_2, C_3 \in |e_0 - e_2|$, cubics R_1, R_2, R_3 from $|e_0|$, and the rational quartic Q from $|2e_0 - e_1 - e_2|$. We choose x to be unique point from $\mathrm{Fund}(S)_4$ corresponding to Q and $x' \in \mathrm{Fund}(S)_1$ to be the point corresponding to l.

Using the conjugacy graph from Figure 11.3 we check the rest of the assertion.

Suppose $n = 6$ and the congruence is of type $(2, 6)_I$. Then, $\tilde{S} \cong \mathsf{Bl}_{p_1}(\mathbb{P}^2)$. There is only one pencil of conics $|e_0 - e_1|$. It has 8 sections defined by fundamental points. They are defined by one line in $\tilde{S}$ representing e_1, one rational quintic from $|2e_0 - e_1|$ and all six rational cubics from $|e_0|$. The choice of x and x' is obvious, and we leave it to the reader to verify the remaining assertion.

If $n = 6$ and $\tilde{S} \cong \mathbf{F}_0$, we have two pencils of conics corresponding to two rulings of $\mathbf{F}_0$. We choose one of them as a pencil of conics. The four fundamental points of degree 2 corresponding to the other pencil come from four sections of the pencil. Other four sections are defined by the four quartics equal to the images of four conics under the Veronese map $\mathbf{F}_0 \to \tilde{S} \subset \mathbb{P}^8$. They define four fundamental base points of degree 4. Altogether we found eight fundamental base points.

□

Remark 11.3.11. One can describe the quadratic pencil of contact quadrics as follows. We know that among the eight base fundamental points, there are two distinguished points $x \in \mathrm{Fund}(S)_{n-1}$ and $x' \in \mathrm{Fund}(S)_1$ uniquely characterized by the property that other six base fundamental points are not conjugate to x'. Let $\Omega(x, K(x'))$ be the pencil of lines in the plane $K(x')$ with base point x and $\ell_s = \langle x, x' \rangle$. For any line $\Omega(x, K(x'))$, the special hyperplane section $S \cap \mathbb{T}_\Gamma(\mathbb{G})$ is of degree $n + 2$, and it contains an irreducible component of degree $n - 1$ of rays passing through x and an irreducible component of degree 1 of rays passing through x. The residual component is a conic. So, varying ℓ in the pencil, we obtain a pencil of conics on S.

Corollary 11.3.12. *A pencil of conics $\{C_t\}_{t\in\mathbb{P}^1}$ on S divides the set of $18 - n$ fundamental points in three disjoint subsets: eight base points of the quadratic pencil $\{Q_t\}_{t\in\mathbb{P}^1}$, $2(6-n)$ points corresponding to line-components of reducible conics in the pencil, and $n - 2$ vertices of singular irreducible quadrics $\mathcal{R}(C_t)$.*

Proof A pencil of conics on a del Pezzo surface X of degree d has $8 - d$ reducible members. This can be verified in each case, or one can use that the Euler-Poincaré characteristic $e(X)$ is equal to $e(\mathbb{P}^2) + 9 - d = 12 - d$, and formula (8.26) for the Euler-Poincaré characteristic of a fibered surface gives $12 - d = 4 + N$, where N is number of reducible conics. This implies that $N = 8 - d$. Applying this to our case $d = 2 + n$, we obtain that there are $2(6 - n)$ line components of reducible conics. They define $2(6 - n)$ fundamental points of degree one different from the base fundamental points. The pencil of conics has $n - 2$ smooth members which define fundamental points $x_1, \ldots, x_{n-2}$ of degree two corresponding to conics $C_1, \ldots, C_{n-2}$ from the pencil that split under the double cover $R(S) \to S$. The exceptional curves E_{x_i} are contained in $\tilde{q}_S^{-1}(C_i)$, and the image of $\tilde{q}_S^{-1}(C_i)$ is a quadric cone with vertex at q_i. □

Definition 11.3.13. *Two congruences of lines with the same focal surface are called* confocal *congruences.*

Let $\{C_t\}_{t\in\mathbb{P}^1}$ be a pencil of conics. We know that it defines a pencil of quadric surface $\mathcal{R}(C_t) = \{Q_t\}_{t\in\mathbb{P}^1}$ in $\mathbb{P}^3$ which comes with a choice of a regulus on each smooth quadric. The other ruling of $\mathcal{R}(C_t)$ defines a *complementary regulus*.

Lemma 11.3.14. *The closure of the set of complementary reguli of smooth quadrics Q_t form a confocal congruence of the same bidegree.*

Proof We know that a general generator g_1 of the regulus of Q_t is a ray of S that is tangent to $\Phi(S)$ at two points. Through each point passes one generator g_1' of the complementary regulus. The plane spanned by g_1 and g_1' is tangent to $\Phi(S)$ at the intersection point. This implies that g_1' is also tangent to $\Phi(S)$ at this point. So, g_1' is tangent to $\Phi(S)$ at two points, and hence these rays form a confocal congruence isomorphic to S.

Since through each point on a ray from the first regulus passes a unique ray from the complementary regulus, the orders of the two congruences are equal. A similar argument shows that their classes are equal. □

Let $x \in \mathrm{Fund}(S)_{n-1}$ and $C(x)$ be the corresponding plane curve of degree $n-1$ equal to the image of the trope curve T_x on S. We know that, if $n \geq 3$, it is a rational plane curve of degree $n-1$ in $\Omega(x)$ with singular poinst at the singular points of S. A hyperplane section of S containing the plane $\Omega(x)$ has a rational cubic curve as the residual curve. Thus, x defines a net $\mathcal{N}(x)$ of cubic curves on $\tilde{S}$. It belongs to the net $|-K_{\tilde{S}} - \tilde{C}(x)|$, where $\tilde{C(x)}$ is the proper transform of $C(x)$ on $\tilde{S}$. We can also see these cubics as the residual curves of the intersection $\Omega(\ell) \cap S$, where ℓ is a ray passing through x.

If $\tilde{S} \cong \mathsf{Bl}_{p_1,\ldots,p_{7-n}}(\mathbb{P}^2)$, and $\tilde{C}(x) \in |ke_0 - \sum_{i=1}^{7-n} k_i e_i|$, where $0 \leq k_i \leq 1$ and $3k - \sum_{i=1}^{7-n} k_i = n-1$, then the net of rational cubic curves is $|(3-k)e_0 - \sum_{i=1}^{7-n}(1-k_i)e_i|$. For example, if $n = 4$, the net is $|e_0|$ and S contains two fundamental points of degree 3, one of them is x and another belongs to the residual net of cubics.

Lemma 11.3.15. *Let $x \in Fund(S)_{n-1}$ and suppose that the cubic $C(x)$ is contained in $\mathcal{N}(x)$. Let $C \in |-K_{\tilde{S}} - \tilde{C}(x)|$ be the residual rational cubic. Then, the union of rays $\ell_s, s \in C$, is a cubic ruled surface $\mathcal{R}(C)$ in $\mathbb{P}^3$ which is tangent to the focal surface. The unique line directrix of $\mathcal{R}(C))$ is tangent to $\Phi(S)$ and the closure of the union of these directrices for all residual cubics C fis a congruence of lines of bidegree $(2(n-2), \frac{1}{2}n(n-1))$.*

Proof A generator of $\mathcal{R}(C)$ is a ray of S, hence it is tangent to S at two points. It also passes through the double point x of $\Phi(S)$. Thus, it intersects S at three points with multiplicity 2. This implies that the ruled surface $\mathcal{R}(C)$ is tangent to $\Phi(S)$ along a curve of degree 6. We know that a cubic ruled surface has a unique line directrix, the image of the exceptional section of $\mathbf{F}_1$.

Each directrix passing through the intersection point of $\mathcal{R}(C)$ with $\Phi(S)$ is tangent to $\Phi(S)$ at this point. Hence, it is tangent to $\Phi(S)$ at two points, and hence, the set of the directrices is a congruence of lines.

□

Proposition 11.3.16. *Assume that S is not of type $(2,6)_I$. Then, S is contained in a tetrahedral quadratic line complex $\mathfrak{T}$.*

Proof Let x_1, x_2, x_3, x_4 be four points in $\mathbb{P}^3$ in a general position meaning that they can be realized as the vertices of a tetrahedral T with vertices $x_1, \ldots, x_4$. We know from Subsection 10.4.5 that there exists a map $f : \mathbb{P}^3 \to \mathbb{G} \subset \mathbb{P}^5$ whose image is the tetrahedral line complex $\mathfrak{T}$. The map is defined by a linear system of quadrics $|\mathcal{O}_{\mathbb{P}^3}(2) - x_1 - \cdots - x_4|$ and depends on a choice of a pencil of quadrics Q whose base locus is a smooth quartic curve. Let Q be a smooth quadric in $\mathbb{P}^3$ passing through a subset Σ of cardinality $1 \le k \le 4$ of the set of the vertices of T. The restriction of f to Q defines a map

$$f : \mathsf{Bl}_\Sigma(Q) \to K \subset \mathbb{G} \subset \mathbb{P}^5.$$

Now, we use that a del Pezzo surface $\tilde{S}$ of degree $n+2 \le 8$ different from the blow-up of one point in $\mathbb{P}^2$ is isomorphic to the blow-up of $6-n$ points on Q in a general position. It is obvious for $n = 6$ since $\tilde{S} \cong Q$ in this case. For $n < 6$, we have $\tilde{S} \cong \mathsf{Bl}_{p_1,\ldots,p_{7-n}}(\mathbb{P}^2)$, so we can isolate two of the points, say p_1, p_2, and then the linear system $|\mathcal{O}_{\mathbb{P}^2}(2) - p_3 - \cdots - p_{7-n}|$ will define a map $\mathbb{P}^2 \dashrightarrow Q$ that can be extended to an isomorphism $\tilde{S} \cong \mathsf{Bl}_\Sigma(Q)$, where Σ is the image of the points $p_3, \ldots, p_{7-n}$ and the image of the line $\langle p_1, p_2\rangle$.

In fact. we see that a choice of Σ shows that the congruence S is contained in $\binom{4}{6-n}$ tetrahedral complexes. Also, observe, that in the case when $n = 2$, the quadric Q belongs to the linear system $|\mathcal{O}_{\mathbb{P}^3}(2) - x_1 - \cdots - x_4|$, and hence the image of Q lies in a hyperplane of $\mathbb{P}^5$. □

Corollary 11.3.17. *The number of moduli of congruences S of class $n \le 6$ and not of type $(2,6)_I$ is equal to $n+1$.*

Proof This is well-known for $n = 2$ and can be seen in many ways. Congruences of class 2 are quartic del Pezzo surfaces. They depend on two moduli and one has also make a choice of a smooth quadric containing it in its anti-canonical embedding. This gives 3 moduli. Another way to see it is to use that Kummer quartic surfaces depend on 3 parameters, and each surface defines six congruences of bidegree $(2,2)$.

Suppose S is not of type $(2,6)_I$. Then, we can apply the previous Proposition and see that the dimension of the space of quadrics passing through $6-n$ vertices of T is equal to $9-(6-n) = 3+n$. We also have a choice for a pencil Q which we used to define the map f. It depends on one parameter, the cross-ratio of the set of four singular quadrics in $\mathcal{W}$. Finally, we use that the automorphism group of $\mathbb{P}^3$ leaving invariant the tetrahedron T is a 3-dimensional torus. We have to subtract 3 from $4+n$ to get $n+1$. Thus, we have found a $n+1$-dimensional irreducible family subvariety of the moduli space. We will show in the next section that the dimension of any irreducible component of the moduli space of μ-nodal quartic surfaces is equal to $19-\mu$. Since the focal surface of S has $18-n$ nodes. and each S is an irreducible component of its bitangent surface, we find that the number of moduli cannot exceed $19-(18-n) = n+1$. □

Remark 11.3.18. Assume now that $n = 6$ and $\tilde{S} \cong \mathbf{F}_1$. We have to argue differently. We know from Corollary 8.3.7 that $\tilde{S}$ is contained in a 19-dimensional linear system of quadrics. By Theorem 2.4.19, the subvariety of quadrics whose singular locus contains a plane is of codimension 6. Fixing a plane in $\mathbb{P}^8$, we get 6 conditions for a quartc to have this plane contained in its singular locus. This shows that there is a variety of planes in $\mathbb{P}^8$ of dimension $19 - 12 = 7$ such that there exists a quadric with singular locus containing this plane. Projecting $\tilde{S}$ from a general such plane, we find that the image of the projection is contained in a smooth quadric in $\mathbb{P}^5$. After identifying this quadric with the Grassmannian $\mathbb{G}$, we obtain that this surface is a congruence of type $(2,6)_I$. So, we have $\le 7 = n + 1$ moduli.

If $n = 7$, a similar computation shows that the number of moduli $\le 26 - 10 - 10 = 6$. We will return to this in the next section.

Theorem 11.3.19. *The bitangent surface Bit*$(\Phi(S))$ *is always reducible.The number of irreducible components and their bidegrees is given in Table 11.2 below.*

n	bidegree	Number
2	$(2,2)$	6
	$(0,1)$	16
3	$(2,3)$	6
	$(0,1)$	10
4	$(2,4)$	4
	$(0,1)$	6
	$(4,6)$	1
5	$(2,5)$	4
	$(0,1)$	3
	$(6,10)$	1
6_I	$(2,6)$	2
	$(0,1)$	1
	$(8,15)$	1
6_{II}	$(2,6)$	3
	$(6,10)$	1
7	$(2,7)$	1
	$(10,21)$	1

Table 11.2 *Irreducible components of* $\mathrm{Bit}(\Phi(S))$

Proof We do each case separately.

$(2,2)$

The Kummer surface $\Phi(S)$ has 10 pencils of conics. They are divided into two complementary parts in a hyperplane section of S. If we choose an isomorphism $S \cong \mathsf{Bl}_{p_1,\ldots,p_5}(\mathbb{P}^2)$, the pairs of pencils are $|e_0 - e_i|$ and $|2e_0 - e_1 - \cdots - e_5 + e_i|$. Two general members C and C' of the complementary pencils intersect at two points. It is immediate to see that base points x, x' of base points from Proposition 11.3.10 are also the base points of the complementary pencils. This implies that the set of fundamental base points of the quadratic pencils corresponding to the complementary pencils of conics coincide. From this, we can easily deduce that the quadratic pencils coincide. Hence, ten pencils of conics on S define only five confocal correspondences. Since $6(2,2)+16(0,1) = (12,28)$, we see that there are six irreducible components of $\mathrm{Bit}(\Phi(S))$.

We can see the six congruences of bidegree $(2,2)$ associated to a quartic Kummer surface in a different way. Let S_3 be the Segre cubic primal and $p : \mathsf{S}_3 \dashrightarrow \mathbb{P}^3$ be the projection from one of its nonsingular point x. The branch surface Φ of the projection map of degree 2 is a quartic surface with 16 nodes. Ten of them are the projections of the nodes of S_3, the remaining 6 nodes are the projections of the lines passing through x. This identifies Φ with a Kummer surface (see Exercise 10.14). In fact, one can show that this is the Kummer surface $\mathrm{Kum}(J(C))$, where C is a genus two curve associated with 6 points corresponding to x under an isomorphism from $\mathrm{P}_1^6 \to \mathsf{S}_3$ (see Section 9.5

It is known that the Fano surface $F(\mathsf{S}_3)$ of lines of S_3 consists of 15 planes and 6 surfaces $D_i \subset G_1(\mathbb{P}^4)$ isomorphic to quintic del Pezzo surfaces [250] (see Exercise 10.15). Under the map $\mathbb{P}^3 \dashrightarrow \mathsf{S}_3$ given by quadrics through five points $p_1, \ldots, p_5$, they are the images of lines through p_i and the twisted cubics through $p_1, \ldots, p_5$. The center of the projection x is contained in six lines $\ell_i \in D_i, i = 1, \ldots, 6$. The image of each D_i under the projection $G_1(\mathbb{P}^4) \dashrightarrow G_1(\mathbb{P}^3)$ from the 3-dimensional linear Schubert subvariety $\Omega(\ell_i)$ of $\mathbb{P}^9$ is a congruence of lines S_i of bidegree $(2,2)$. Each ray of S_i is equal to the projection in $\mathbb{P}^3$ of a line in S_3. This line is a bitangent line of $\Phi(S)$. This gives another explanation of the fact that $\mathrm{Bit}(\Phi)$ has six irreducible components isomorphic to del Pezzo surfaces of degree 4.

$(2,3)$

In this case, S has five pencils of conics, and the points $x \in \mathrm{Fund}(S)_2$ and $x' \in \mathrm{Fund})S)_1$ associated with each pencil cannot be interchanged. So, we have five confocal congruencies of bidegree $(2,3)$, and since $6(2,3) + 10(0,1) = (12,28)$, all irreducible components of $\mathrm{Bit}(\Phi(S))$ are accounted for. We refer for mode detailed geometry of these components to [255].

$(2,4)$

In this case, $\tilde{S} \cong \mathsf{Bl}_{p_1,p_2,p_3}(\mathbb{P}^2)$ has three pencils of conics. Together with S, we obtain four confocal congruences of bidegree $(2,4)$. We also have two fundamental points of degree 3, and each defines a confocal congruence of line directrices of a cubic ruled surface associated to the net of rational cubics on

S. By Lemma 11.3.15, the bidegree of this congruence is equal to $(4,6)$. Since $4(2,4)+(4,6)+6(0,1) = (12,28)$, all irreducible components of $\mathrm{Bit}(\Phi(S))$ are accounted for.

$(2,5)$

In this case, $\tilde{S} \cong \mathsf{Bl}_{p_1,p_2}(\mathbb{P}^2)$ has two pencils of conics and $\mathrm{Fund}(S)_3$ consists of three points. By Lemma 11.3.15, the congruence defined by the fundamental point of degree 3 is of bidegree $(6,10)$, so we have $3(2,5)+(0,3)+(6,10) = (12,28)$, so all irreducible components of $\mathrm{Bit}(\Phi(S))$ are accounted for.

$(2,6)_I$

The surface $\tilde{S}$ is isomorphic to $\mathbf{F}_1 \cong \mathsf{Bl}_{p_1}(\mathbb{P}^2)$. It has one pencil of conics and six fundamental points of degree 3. The bidegree of the congruence defined by any of these points is equal to $(8,15)$. So $2(2,6)+(0,1)+(8,15) = (12,28)$, hence all irreducible components of $\mathrm{Bit}(\Phi(S())$ have been found.

$(2,6)_{II}$

In this case, $\tilde{S} \cong \mathbf{F}_0$, hence it has two pencils of conics (in the Plücker space) but no rational cubics. The bidegree of the residual part of $\mathrm{Bit}(\Phi(S))$ is equal to $(12,28)-3(2,6) = (6,10)$. Since there are no lines on $\tilde{S}$, we have to prove that the part of bidegree $(6,10)$ is irreducible.

We know from Proposition 11.3.16 that S is contained in a tetrahedral line complex $\mathfrak{T}$ equal to the image of $\mathbb{P}^3$ by a linear system of quadrics passing through its vertices defined by a choice of a pencil of quadrics. Let C_i be the intersections of the faces of $\mathfrak{T}$ with Q. Their images are quartic curves with three singular points equal to the images of the edges of $\mathfrak{T}$ contained in the face. The images of the faces are four α-planes of the Grassmannian $\mathbb{G}$ which contain the quartics. So, we realize the vertices of T as the four fundamental points of S of degree 4.

Let $y_1,\ldots,y_4$ and $y_4',\ldots,y_4'$ be the set $\mathrm{Fund}(S)_2$, where y_i, y_j' are conjugate fundamental points. We know that the four quadric cones $K(y_i)$ pass through eight fundamental points $x_1,\ldots,x_4$ and $y_1',\ldots,y_4'$. This implies that the linear system of quadrics spanned by these cones is a net $\mathcal{N}$ of quadrics with eight base points. We know that the set of lines contained in some quadric from the net form a cubic line complex $\mathfrak{C}$ (the Montesano complex from Exercise 10.16). Any line containing a base point of the net is contained in a quadric from the net, hence it is a ray of the Montesano line complex. The intersection $S \cap \mathfrak{C}$ contains the four quartic cones $K(y_i)$, and four quadric cones $K(y_i')$ of total degree equal $\deg(S \cap \mathfrak{C} = (2\sigma_{1,1}+6\sigma_2)\cdot 3\sigma_1 = 24$, we get $S \subset \mathfrak{C}$, as soon as will find another ray contained in the intersection. Take a ray $\ell = \langle x_i, y_j' \rangle$. Since $\Omega(\ell)$ contains $K(x_i)$ and $K(y_i')$, the residual curve is a conic C The ray

ℓ is also contained in a pencil of quadrics from $\mathcal{N}$, its base locus is the union of ℓ and a rational cubic curve. This curve intersects $\mathcal{R}(C)$ and the generator of $\mathcal{R}(C)$ coincides with the ray in $\mathfrak{C}$ passing through the intersection point.

Replacing $y_1', \dots, y_4'$ with $y_1, \dots, y_4$, we obtain another net of quadrics and another cubic line complex $\mathfrak{C}'$ containing S. The intersection $K \cap \mathfrak{C} = K \cap \mathfrak{C}'$ is a congruence of bidegree $(6, 6)$, it contains four α-planes $\Omega(x_i)$ and the residual part is our congruence S.

Now, after revealing such a beautiful geometry of a congruence of type $(2, 6)_{II}$, we are ready to prove that the residual part of $\mathrm{Bit}(S)$ is irreducible.

Let Π be a general plane in $\mathbb{P}^3$. It intersects the quadric Q along a conic, and its image $f(\Pi \cap Q)$ is a quartic $\gamma \subset S$. We know from Subsection 10.4.5 that the image $f(\Pi)$ is the congruence of bidegree $(1, 3)$ of secant lines of a rational cubic curve R_Π. The ruled quartic surface $\mathcal{R}(\gamma)$ has the curve R_Π as its double curve. We know from the subsection 10.4.5 that there are two types of such ruled surfaces, Types I and Types II(B) in Edge's notations. Type II(B) surfaces form a subfamily of codimension one of the family of surfaces of type I, they are distinguished by the property that there exists a line directrix.

More explicitly, the equations at the very end of the subsection show that quartics of Type I are the pre-images of the conics under the map given the net of quadrics containing R_Π. One can find an explicit quadratic relation between the coefficients of the conic such that its pre-image is a quartic ruled surface of type II(B).

Thus, we have an irreducible variety of plane sections C of Q such that the ruled surface $\mathcal{R}(\gamma)$ admits a line directrix. Since $\mathcal{R}(C)$ is touching $\Phi(S)$ everywhere, the line directrix is tangent to $\Phi(S)$ at two points, and hence belongs to $\mathrm{Bit}(\Phi(S))$. The closure of such line directrices defines an irreducible component of the bitangent surface. congruence of S.

It remains to compute its bidegree. In fact, it is enough to prove that its class is equal to 10 since $\mathrm{Bit}(\Phi(S))$ has no α-planes $\Omega(x)$ (otherwise, the projection from x defines a finite map $\Phi(S) \to \mathbb{P}^2$ everywhere ramified).

Let ℓ be a line directrix of some quartic ruled surface $\mathcal{R}(\gamma)$. The ruled surface $\mathcal{R}(\ell)$ of degree 8 contains $\mathcal{R}(\gamma)$ and another quartic ruled surface of the same type II(B). A plane Π containing ℓ contains $6 = 3 + 3$ generators of both ruled surfaces. Each subset of three defines a plane Π such that $\Pi \cap Q$ defines a quartic ruled surface with a line directrix. Thus, we have $\frac{1}{2}\binom{6}{3} = 10$ paired quartics with the same directrix contained in Π. So, the class of the congruence of the line directrices is equal to 6.

(2, 7)

In this case, $\tilde{S}$ is a Veronese surface in $\mathbb{P}^9$. It has one net of rational cubics, the images of lines in the plane under the Veronese map. The bidegree of the corresponding confocal congruence is $(10, 21)$. So $(2, 7) + (10, 21) = (12, 28)$, and we have only one irreducible component besides the confocal congruences.

□

Remark 11.3.20. In this remark we will explain that the dual of the congruences of bidegrees $(2, 5)$, $(2, 6)_I$ and $(2, 7)$ are special cases of a certain construction of congruences of lines due to Caporali [83]. Let W be a web of plane curves of degree d with with ordinary base points $p_1, \ldots, p_r$ of multiplicities $m_1, \ldots, m_r$. Let $f_W : \mathbb{P}^2 \dashrightarrow W^*$ be the rational map defined by W. We assume that it is of degree 1, so its image is a surface Σ of degree $N = n^2 - \sum_{i=1}^r m_i^2$. Fix an isomorphism $W^* \cong \mathbb{P}^3$ and also fix an isomorphism $i : \mathbb{P}^2 \to \Pi_0$ to a plane in W^*. We will identify $\mathbb{P}^2$ with Π_0 and W^* with $\mathbb{P}^3$. Thus, W defines a rational map $f_W : \Pi_0 \to \Sigma$ and we define a congruence of lines S_W equal to the closure of the lines $\langle x, f_W(x)\rangle$, where $x \notin \{x_1, \ldots, x_r\}$.

Let us compute the bidegree of S_W. Let Γ be the graph of the rational map f_W. We can compute its mulidegree defined in Subsection 7.1.3. Since the image of a general line in Π_0 is a curve of degree n, and the pre-image of a hyperplane Its cohomology class in $\mathbb{P}^2 \times \mathbb{P}^3$. In a basis $(h_1^0 h_2^3, h_1 h_2^2, h_1^2 h_2)$ of $H^3(\mathbb{P}^2 \times \mathbb{P}^3, \mathbb{Z})$, the cohomology class of Γ is equal to $(1, d, N)$. The 3-dimensional subvariety $X(p)$ of points $(x_1, x_2) \in \mathbb{P}^2 \times \mathbb{P}^3$ such that the line $\langle x_1, x_2\rangle$ passes through a fixed point x has the cohomology class $(h_1^0 h_2^2 + h_1 h_2 + h_1^2 h_2^0$. Intersecting $[\Gamma]$ with $[X(p)]$ we get $m = d + N + 1$.

Let Π be a general plane in $\mathbb{P}^3$. It intersects Π_0 along a line ℓ. Consider the subvariety of $\mathbb{P}^2 \times \mathbb{P}^3$ of points $(x_1, x_2) \in \ell \times \Pi$. Its cohomology class is equal to $h^1 h_2$. Intersecting it with Γ we obtain that $n = d$.

Let $C = \Pi_0 \cap \Sigma$ and C_0 be a curve in W that corresponds to Π_0 under our identification W with $\check{\mathbb{P}}^3$. Then, the rays $\langle x, f_W(x)\rangle, x \in C_0$ lie in Π_0. The form a curve in $\check{\Pi}_0$ of degree $N + d$. To see this, we take a general pencil of lines in Π_0 with a base point x_0. It cuts out in C_0 a linear pencil g_d^1 and cuts in C a linear pencil g_N^1. They can be identified with linear pencils on their isomorphic normalizations C^{nrm}. The image of the map $C^{\mathrm{nrm}} \to \mathbb{P}^1 \times \mathbb{P}^1$ given by the two pencils is a curve of bidegree (d, N), and its intersection number with the diagonal is equal to $N + d$. This is the number of lines ℓ through x_0 such that $\ell \cap C$ and $\ell \cap C_0$ contain a pair of points x and $x' = f_W(x)$.

We will be interested in the dual congruences S_W^*. Its bidegree is equal to $(d, N_d + 1)$ and, by above, they contain a fundamental point of degree $N + d$. Obviously, S_W and S_W^* are rational surfaces. The plane Π_0 defines a fundamental point on S_W^* of degree $N + d$.

Now, let us specialize.

Assume $d = 2$ and consider the dual congruence S_W^* of bidegree $(2, N+3)$. Since $N = 4 - \sum_{i=1}^3 m_i^2$, we get $N = 1, 2, 3, 4$, hence the class n of S_W^* is equal to $4, 5, 6, 7$.

Assume $N = 4$, i.e. W has no base points. The surfaceΣ is a Steiner quartic surface, a projection of the Veronese surface to $\mathbb{P}^3$. We discussed these surfaces in Subsection 2.1.1. The map $f_W : \mathbb{P}^2 \to \Sigma$ can be identified with the projection map. The double curve on $\mathbb{P}^2$ is the union of three concurrent lines. The congruence S_W^* has one fundamental point of degree 6. It corresponds to the triple point of Σ. It has no fundamental curves. We know that a congruence of bidegree $(2, 7)$ without fundamental curves is isomorphic to a projection of the third Veronese surface of $\mathbb{P}^2$ to $\mathbb{G} \subset \mathbb{P}^5$. It has ten isolated singular points and 10 more fundamental points of degree 3. The ten fundamental points of degree 3 correspond to 10 fundamental planes in S_W.

Let us see these fundamental planes in S_W. They correspond to ten nets in W with two base points. Such a net in W defines a net of planes in $\mathbb{P}^3$ whose base point x lies on one of the three double lines Γ_i of the Steiner surface Σ. The pre-image of this point is the pair of base points y, y' of the net. The pencil of planes containing Γ_i correspond to a pencil in W with the base line $\langle y, y' \rangle$ and the residual lines are the pre-images of conics on Σ. A special net corresponds to the case when the line $\langle y, y' \rangle$ is contained in the plane spanning the conic. Such a plane is one of the fundamental planes of S_W. We refer to [735, Tiele 2, Nr. 454] for showing that there are indeed 10 such planes.

If $N = 3$, i.e. W has one base point p_1, the congruence S_W is of bidegree $(6, 2)$. Its normalization is isomorphic to $\mathsf{Bl}_p(\mathbb{P}^2) \cong \mathbf{F}_1$, and it contains one fundamental plane of degree 5. Its dual congruence S_W^* is isomorphic to the congruence $(2, 5)_I$ without base curves.

If $N = 2$, i.e. W has two base points p_1, p_2, the congruence S_W is of bidegree $(5, 2)$. Its normalization is isomorphic to $\mathsf{Bl}_{p_1,p_2}(\mathbb{P}^2)$, and it contains one fundamental plane of degree 4. The dual congruence S_W^* is a congruence of bidegree $(2, 5)$ without fundamental curves.

11.4 Quadratic Congruences With a Fundamental Curve

In this section, following Kummer [473] and [735, Theil 2], we will classify quadratic congruences with a fundamental curve.

11.4.1 Kummer's examples

Let S be a congruence of bidegree (m, n). If $n = 0$, this is a β-plane. If $n = 1$, it is dual to a congruence of bidegree $(1, 2)$. So, we may assume that $n \geq 2$.

Let $F = \mathrm{Fund}(S)_1$ be the fundamental curve of S. Applying Corollary 11.1.17, we have the following possible types of congruences:

(I) F is irreducible and a general ray intersects F at two points, the focal surface $\Phi(S)$ is ruled by focal rays of S.

(II) F consists of two irreducible components and a general ray intersects each at one point, the focal surface $\Phi(S)$ is ruled by focal rays of S.

(III) F is irreducible and a general ray intersects F at one point and is tangent to the focal surface $\Phi(S)$.

Let us give an example of a congruence of lines of each type (see [473]).

Example 11.4.1. [1] This is an example of a congruence of Type (I), where the fundamental curve F is an elliptic quartic curve embedded in $\mathbb{P}^3$ by a complete linear system of degree 4. Consider the congruence of secant lines of F. Since the projection of F from a general point x is a plane quartic curve C_4 of genus one, x is contained in two secant lines that are projected to two double points of C_4. This shows that $m = 2$. Since a general plane intersects F at four points, the class of S is equal to $\binom{4}{2} = 6$. The surface S is a smooth surface of degree 8 in $\mathbb{P}^5$ isomorphic to the symmetric product $F^{(2)}$. The map

$$\pi : S \to \mathrm{Pic}^2(F) \cong F, \ \{x, y\} \to [x + y],$$

defines a structure of an elliptic ruled surface on S. Let $|I_S(2)|$ be the pencil of quadrics containing F, and $C \to |I_S(2)|$ be its double cover obtained by fixing a ruling on a quadric from $|I_F(2)|$. Its branch divisor consists of four singular quadrics $Q_1, \ldots, Q_4$. Each secant line chooses a unique quadric from $|I_S(2)|$ containing it and its ruling containing it as a generator. All secant lines from the same ruling of the quadric are linearly equivalent divisors. This identifies $\mathrm{Pic}^2(F)$ with the curve C.

Let E_p be the curve on S parametrizing secant lines passing through a point $p \in E$. It is a section of the projective bundle $\pi : S \to C$. Since $E_p \cap E_q = \langle p, q \rangle$, we obtain $[E(p)] \cdot [E(q)] = 1$. This is also true for $p = q$.

It follows from [13, Chapter VII, §2] that

$$S = F^{(2)} \cong \mathbb{P}(\mathcal{E}_p),$$

with $\mathcal{O}_{\mathbb{P}(\mathcal{E}_p)}(1) \cong \mathcal{O}_S(E_p)$. The formula (2.29) for the canonical class of a

[1] Lehrsatz VII from [473].

projective bundle gives

$$\omega_S \cong \pi^* \det(\mathcal{E}_p) \otimes \mathcal{O}_S(-2E_p).$$

By the adjunction formula, $\omega_S \otimes \mathcal{O}_{E_p} \cong \mathcal{O}_{E_p}(-E_p)$. Thus, $\det(\mathcal{E}_p) \cong i_p^* \mathcal{O}_{E_p}(E_p)$, where $i_p : C \to S$ is the section defined by E_p. In particular, we obtain that $\deg(\det(\mathcal{E}_p)) = 1$.

The section i_p describes the rank two bundle $\mathcal{E}_p$ as an non-split extension

$$0 \to \mathcal{O}_C \to \mathcal{E}_p \to i_p^* \mathcal{O}_{E_p}(E_p) \to 0.$$

The locally free sheaf $\mathcal{E}_p$ depends on p, replacing p with some other point q, we see that the linear systems $|\pi^* \det(\mathcal{E}) \otimes \mathcal{O}_S(2E_p)|$ and $|\pi^* \det(\mathcal{E}_q) \otimes \mathcal{O}_S(2E_q)|$ coincide. Each defines an embedding

$$S \hookrightarrow \mathbb{P}^5$$

whose image is a surface of degree $m + n = 8$. The fibers of the ruled surface go to conics parametrizing the rulings of quadrics in the pencil $|I_F(2)|$.

A hyperplane section H of S satisfies $H^2 = 8, H \cdot K_S = -4$. It follows that

$$p_a = g = 3, \ r = 2.$$

The focal surface $\Phi(S)$ is the union of four cones $Q_1, \ldots, Q_4$ in the pencil $|I_F(2)|$. In this case, $R(S) = R(S)^{1,1}$, that is the focal surface is swept by focal rays. The vertices of the cones are isolated fundamental points of S of degree 2. We have $\deg(\Phi(S)) = 8 = 2m + 2p_a - 2$ that agrees with Proposition 11.1.20.

The correspondence

$$I = \{(s, p) \in S \times F : \ell_s \in E_p\}$$

is a double cover of S branched over the curve $T(F)$ of rays tangent to F. It is a 4-section of the projective bundle. In the familiar picture of the surface of secant lines of F considered as the symmetric product $F^{(2)}$, the projection $I \to S$ corresponds to the projection $F^2 \to F^{(2)}$.

Example 11.4.2. This is a degeneration of the previous example; we take F to be an irreducible quartic of arithmetic genus one with one double point. Kummer does not discuss this degenerate case. Instead of the four cones in the pencil $|I_F(2)|$, we have only three cones Q_1, Q_2, Q_3, one of them, say Q_1, has its singular point at the singular point p_0 of F. The congruence S is singular along the conic K of generators of this cone. So, this is an example of a non-normal congruence of lines.

The sections $E(p)$, $p \neq p_0$, are now rational cubics with a double point on the conic D. The normalization $\tilde{S} \cong S^{\mathrm{nrm}}$ is a ruled surface over the double cover

C of $|I_F(2)| \cong \mathbb{P}^1$ ramified over the cones Q_2, Q_3. The curve C is isomorphic to the normalization of the curve $E(p)$.

The rank of S is still equal to 2, the proof is the same as in the nonsingular case. Hence, $p_a = 3$ but the geometric sectional genus now is equal to one. The sections $E(p)$ now are plane cubics with singular point at the double conic. They intersect at one nonsingular point. Their pre-images in $S^{\mathrm{nrm}} = \tilde{S}$ are sections E of a rational minimal ruled surface intersecting at one point. Let us identify h with its pre-image in $\tilde{S}$. As in the nonsingular case, we get $h = \mathfrak{f}+2[E]$, where $\mathfrak{f}$ is the divisor class of a generator. Since $8 = h^2 = 4+4E^2$, we get $E^2 = 1$. Writing $[E] = a\mathfrak{f} + \mathfrak{e}$, where $\mathfrak{e}$ is the class of the exceptional section with $\mathfrak{e}^2 = -n$, we get $1 = E^2 = a + E\cdot\mathfrak{e}$. This gives $a = 1, E\cdot\mathfrak{e} = 0$. This could happen only if $n = 1$ and $E \in |\mathfrak{f} + \mathfrak{e}|$.

Note that the focal surface of S is the union of the cones Q_1, Q_2, Q_3. However, the cone Q_1 enters into Φ_{sch} with multiplicity 2. In particular, $R(S)^{0,1}$ is not reduced. The singular points of Q_2, Q_2 are isolated fundamental points of degree two.

Example 11.4.3. [2] This is again a degeneration of Example 11.4.1: we take F to be the union of two smooth conics F_1 and F_2 intersecting at two points p_1, p_2. We assume that $p_1 \neq p_2$ and let the reader modify the discussion in the case the conics have a common tangent at the intersection point. Note that another special case, where F is the union of a line and a twisted cubic, leads to a congruence of order one.

Let Π_1 and Π_2 be the planes spanned by the conics. They intersect along the line $\ell_0 = \langle p_1, p_2\rangle$. It is clear that the join congruence $J(F_1, F_2)$ is of order 4 and class 4. It is reducible and contains α-planes $\Omega(p_1)$ and $\Omega(p_2)$. The residual part is our congruence of bidegree $(2, 4)$.

The pencil $|I_{F_1+F_2}(2)|$ of quadrics containing the union $F_1 + F_2$ contains two singular quadrics Q_1 and Q_2 of corank 1 and one reducible quadric $\Pi_1 + \Pi_2$. The fundamental curve is the union $F_1 + F_2$. A general point on the fundamental curve is of degree 2. However, the points p_1 and p_2 are of degree 3. The cone of rays $K(p_i)$ consists of three planes Π_1, Π_2, and $\langle \mathbb{T}_{p_i}(F_1), \mathbb{T}_{p_i}(F_2)\rangle$. There are two isolated fundamental points of degree 2, the vertices of the cones Q_1 and Q_2.

For any general point $x \in \mathbb{P}^3$, there is a unique quadric $Q \in |I_{F_1+F_2}|$ containing x. The two lines in Q passing through x are rays of S. A general plane intersects F_1 at two points and intersects F_2 at two points. It contains four rays in the plane. This confirms that S is a congruence of bidegree $(2, 4)$.

A general line ℓ is tangent to two quadrics in the pencil. Each tangency point

[2] Lehrsatz IX from [473].

is equal to the intersection of two rays that lie in the tangent plane. The pencil of lines in the tangent plane passing through the tangency points is a secant of S contained in $\mathbb{G}$. Conversely, every secant is obtained in this way. This shows that $r = 2$. It follows from (11.15) that $p_a = 1$. This agrees with the formula for the degree of the focal surface, and we get

$$\Phi(S) = \Phi(S)_{\mathrm{sch}} = Q_1 + Q_2.$$

As in the previous case, $\mathrm{Fund}(S) \subset \Phi(S)$. Note that $R(S) = R(S)^{1,1}$ and $\Phi(S)$ is swept by focal rays.

The line $\ell = \langle p_1, p_2\rangle$ is a singular ray ℓ_{s_0} of S. The hyperplane spanned by $\Omega(p_1)$ and $\Omega(p_2)$ cuts out S along the union of 6 lines, three in each plane. This shows that the tangent cone of S at s_0 is the union of two planes. The point s_0 is an ordinary non-normal singular point of multiplicity 2.

The degree of the union $S + \Omega(p_1) + \Omega(p_2)$ is equal to 8. It is a degeneration of the previous example. The general hyperplane section is the union of an elliptic sextic curve and two of its secant lines. Its arithmetic genus is equal to 3.

Example 11.4.4. [3] This is an example of congruences of Type (II) where one of the irreducible components of $\mathrm{Fund}(S)$ is a line. Let C_d be an irreducible curve of degree $d \geq 2$ that intersects transversally a line ℓ at $d-2$ points $p_1, \ldots, p_{d-2}$. The restriction of the pencil $\ell^\perp$ to C_d is the linear series $g_2^1 + p_1 + \ldots + p_{d-2}$ on the normalization C_d^{nrm} of C_d with base points $p_1, \ldots, p_{d-2}$. In particular, C_d^{nrm} is either rational, or elliptic or a hyperelliptic curve of genus $g \geq 2$ equipped with an involution $\iota : C_d^{\mathrm{nrm}} \to C_d^{\mathrm{nrm}}$ defined by the g_2^1. We assume, for simplicity of the exposition, that C_d is a nonsingular curve of genus g.

The congruence $S = S(\ell, C_d)$ is defined to be the closure of the set of lines that intersect ℓ and C_d at points different from the points in $\ell \cap C_d$. Since every general ray spans a plane $\Pi \in \ell^\perp$ that intersects C_d at two points, we see that the order m is equal to 2. A general plane intersects C_d at d points and intersects ℓ at one point. It contains d rays. So, the class n is equal to d.

The surface S is a ruled surface of degree $d + 2$. Its generators are pencils $\Omega(x, \Pi)$, where $x \in C_d$ and $\Pi = \langle x, \ell\rangle$. So, the curve C_d is a generatrix of S. The ray $\langle x, \iota(x)\rangle$ is a point on S contained in two generators. This shows that the ray is a singular point of S. The set of such rays is a singular directrix D of S isomorphic to $g_2^1 \cong \mathbb{P}^1$.

Let us show that D is a conic if $d \geq 3$ and a line if $d = 2$. Assume $d \geq 3$. Take a general line L in $\mathbb{P}^3$. For any point $x \in L$, the plane $\langle x, \ell\rangle$ intersects C_d at 2 points c_1, c_2. The lines $\langle x, c_1\rangle$ and $\langle x, c_2\rangle$ intersect ℓ at two points. In

[3] Lehrsatz X from [473]

particular This defines a map $L \to \ell^{(2)} \cong \mathbb{P}^2$. Its image is a line in $\mathbb{P}^2$ identified with the g^1_2 on C_d. The image of the diagonal in ℓ^2 in $\ell^{(2)}$ is a conic. This shows that there are two points x, x' on L such that the points x, c_1, c_2 are collinear. It follows that the hyperplane $\Omega(L)$ intersects the generatrix of the ruled surface at two points, hence it is a conic.

If $d = 2$, then C_2 is disjoint from ℓ. The secant lines $\langle x, \iota(x)\rangle$ lie in the plane $\Pi_0 := \langle C_2\rangle$. A general line l intersects Π_0 at one point x_0 and there is only one ray passing through this point; the line $\langle x_0, \ell \cap \Pi_0\rangle$. So, D is a line.

Since $r = 0$, we get $p_a = d - 1$. Since a general hyperplane section has two ordinary nodes, the sectional genus is equal to $d - 3$, or 0 if $d = 2$. It follows from Proposition 11.1.20 that $\deg(\Phi(S)) = 2d - 2$ if $d \geq 3$ and $\deg(\Phi(S)) = 4$ if $d = 2$.

Assume $d \geq 3$. Define a map $j : C_d \to \ell^\perp \times \ell$ that assigns to a point $x \in C_d$ the pair (Π, y), where $\Pi = \langle x, \ell\rangle$ and $y = \langle x, \iota(x)\rangle \cap \ell$. If $x = \iota(x)$, the line $\langle x, \iota(x)\rangle$ is the tangent line of C_d at x. If $x \in C_d \cap \ell$, we take Π to be the plane spanned by ℓ and $\mathbb{T}_x(C_d)$. It is easy to see that the map j defines an isomorphism from C_d to a curve of bidegree $(2, d-2)$. The well-known formula for the genus of a curve on a quadric confirms that the genus of C_d is equal to $d - 3$. Thus, C_d contains $2(d - 3) + 2 = 2d - 4$ fixed points of the involution ι. This coincides with the degree of the focal surface. The explanation of this coincidence is easy: $\Phi(S)$ is equal to the union of $2g + 2$ planes from $\ell^\perp$ that are tangent to C_d. Each plane is swept by focal rays passing through the tangency point. The focal surface is the image under $\tilde{p}_S$ of the divisor $R(S)^{1,1} \subset R(S)$. Its projection to S is the union of $2g + 2$ generators.

The planes from $\ell^\perp$ are fundamental planes of S of degree 2. In the dual congruence they form the fundamental line whose cones $K(x)$ are the unions of two planes.

Finally, note that the line ℓ is an isolated singular ray of S if $d \geq 4$.

Example 11.4.5. [4] The fundamental curve in this example is irreducible, however, we treat it as a congruence of Type (II) because it has a hidden infinitely near irreducible component.

In the previous example, Fund(S) was the union of a line ℓ and a curve C_d in $\mathbb{P}^3$ isomorphic to a curve of bidegree $(2, d - 2)$ in $\ell^\perp \times \ell$. Similarly to example of a congruence of order 1 of type (II), we consider the case when the curve C is infinitely near to ℓ; we let C to lie in the exceptional locus $E \cong \ell^\perp \times \ell$ of the blow-up $\mathsf{Bl}_\ell(\mathbb{P}^3) \to \mathbb{P}^3$ as a smooth curve of bidegree $(2, d - 2)$. We assume that $d \geq 3$.

[4] Lehrsatz X, Lehhsatz XII and Lehrsatz XV from [473]

We define S to be the closure in $\Omega(\ell)$ of the set of lines whose proper transform in $\mathsf{Bl}_\ell(\mathbb{P}^3)$ intersects C.

The curve $C \subset \ell^\perp \times \ell$ is a correspondence between $\ell^\perp$ and ℓ. The proper transform of a plane $\Pi \in \ell^\perp$ in $\mathsf{Bl}_\ell(\mathbb{P}^3)$ intersects C at two points which are projected to two points on ℓ. The lines in Π passing through these two points are rays of S. For each point $p \in \ell$, there are $d-2$ planes $\Pi_1(p), \dots, \Pi_{d-2}(p) \in \ell^\perp$, and the lines in $\Pi_i(p)$ passing through p form a pencil of rays of S. Note that ℓ is a member of all pencils, and here $\ell \in S$. The congruence S is a cone of degree $m + n = d$ in $\mathbb{P}^4$.

For a general point $x \in \mathbb{P}^3$, we find the plane $\langle \ell, x\rangle \in \ell^\perp$. There will be two points p_1, p_2 on ℓ, and the lines $\langle x, p_1\rangle$ and $\langle x, p_2\rangle$ are the rays in S passing through x. This shows that the order of S is equal to 2.

A general plane Π in $\mathbb{P}^3$ intersects ℓ at one point p; there are $d-2$ corresponding planes $\Pi_1(p), \dots, \Pi_{d-2}(p)$ in $\ell^\perp$, and $d-2$ lines $\Pi \cap \Pi_i(p)$ are the $d-2$ rays contained in Π. So, the class of S is equal to $d-2$.

Since all rays intersect ℓ, we see that $r = 0$, and hence $p_a = d - 3$. Since S has an isolated singular point, $g = p_a$.

The cone S has $2g + 2$ generators corresponding to branch points of the projection $C \to \ell^\perp$. The pre-image of these generators under q_S are irreducible components of $R(S)^{1,1}$. They are projected to planes in $\mathbb{P}^3$. The union of these planes is the focal surface $\Phi(S)$.

The general fiber of the projection $\tilde{p}_S : R(S)^{2,1} \to \mathrm{Fund}(S) = \ell$ is equal to the union of $d-2$ lines. In the Stein factorization $\tilde{Z}_S \to \tilde{Z}'_S \to \mathbb{P}^3$ of $\tilde{p}_S$, the pre-image of ℓ in $\tilde{Z}'_S$ is the curve of singularities of $\tilde{Z}_{S'}$ isomorphic to C. The projection $C \to \ell$ is of degree $d-2$. The birational morphism $\tilde{Z}_S \to \tilde{Z}'_S$ is the blow-up the singular line followed by the blow-up a curve on the exceptional divisor isomorphic to C. The exceptional divisor of the second blow-up is equal to $R(S)^{2,0}$. The proper transform of the exceptional divisor of the first blow-up is the divisor $R(S)^{2,1}$. Both divisors are isomorphic to $\mathbb{P}^1$-bundle over C. The divisor $R(S)^{1,1}$ is the proper transform of the focal surface. It consists of $2g + 2$ irreducible components. Each component is isomorphic to the blow-up one point in $\mathbb{P}^2$. It intersects $R(S)^{2,1}$ along the exceptional curve and intersects $R(S)^{2,0}$ along a fiber of its ruling.

Example 11.4.6. [5] This is an example of a congruence of Type (III) with a line as its fundamental curve.

Let Φ_d be a submonoidal surface in $\mathbb{P}^3$ (see Subsection 7.7.2). We assume that Φ_d contains a line ℓ with multiplicity $d-2$. Any plane $\Pi \in \ell^\perp$ intersects Φ_d along the line ℓ with multiplicity $d-2$ and a conic. We call it a *residual*

[5] Lehrsatz XV of Kummer.

conic. It may be reducible and may contain ℓ as its irreducible component. Let us assume for simplicity that Φ_d is nonsingular outside ℓ. In this case, all singular residual conics are line-pairs.

We define a *submonoidal congruence* S to be the closure of lines intersecting ℓ and tangent to Φ_d at some nonsingular point.

A general point $x \in \mathbb{P}^3$ is contained in two rays, the tangents to the section of S by the plane $\langle \ell, x \rangle$. So, $m = 2$, and, since all rays intersect the line ℓ, the rank r is equal to zero.

A general plane Π intersects Φ_d along an irreducible curve H of degree d, the point $x_0 = \Pi \cap \ell$ is of multiplicity $d - 2$ on H. I

Projecting from this point, we find that there are $2d - 2$ ramification points, hence $2d - 2$ rays in Π. So, the class of S is equal to $2d - 2$. It follows from (11.15) that $p_a = n - 1 = 2d - 3$.

Obviously, $\Phi_d \subset \Phi(S)$. A general point in the plane spanned by a singular residual conic is contained in a unique ray that it connects to the singular point of the conic. One can show that there are $3d - 4$ singular residual conics [258, 4.1]. This gives additional $3d - 4$ irreducible components of $\Phi(S)$ which are planes. This agrees with formula (11.31) for the degree of the focal surface $\deg(\Phi(S)) = d + (3d - 4) = 2n = 2(2d - 2)$. Note that the singular points of residual conics are isolated fundamental points.

There are $2(d - 2)$ residual conics that intersects ℓ at one point. Thus, if $d \geq 3$, the line ℓ belongs to S. It is a non-normal ordinary singular point of S of multiplicity $2(d - 2)$.

Assume $d = 2$, i.e. ℓ does not lie in a quadric Φ_2. [6] An irreducible quadric Φ_2 is a submonoidal surface with respect to any line $\ell \not\subset \Phi_2$.

Suppose Φ_2 is smooth. Then, the submonoidal congruence of lines defined by (Φ_2, ℓ) is of bidegree $(2, 2)$. The surface S is an anti-canonical quartic del Pezzo surface. The tangential line complex of Φ_2 has two disjoint conics as its singular locus. They are the generatrices of the two rulings of Φ_2. If ℓ intersects Φ_2 transversally at two points x_1, x_2, then two pairs of generators of Φ_2 passing through x_1, x_2 are four ordinary double points of S. If Γ is tangent to Φ_2 at one point, then S has two rational double points of type A_2.

Assume Φ_2 is a quadric cone. In this case, the tangential line complex has only one conic as its singular locus. If ℓ intersects it transversally, S is an anti-canonical quartic del Pezzo surface with two nodes. Otherwise, it has one rational double point of type A_2.

Remark 11.4.7. In the previous example, we assumed that Φ_d is nonsingular outside ℓ. In general, Φ_d may acquire torsal lines corresponding to non-reduced

[6] Kummer considers this example separately, Lehrsatz XII.

residual conics. In this case, the plane spanned by ℓ and a torsal line is a tangent plane to Φ_d and all lines in this plane belong to the congruence. In fact, these planes enter with multiplicity 2. Since we assume that a congruence is an irreducible surface in $G_1(\mathbb{P}^3)$, we have to delete these planes (leaving the torsal lines which are focal lines). The class n decreases by the number τ of torsal lines and becomes equal to $2d - 2 - 2\tau$. This agrees with the formula $\deg(\Phi(S)) = 4d - 4 - 2\tau = 2(2d - 2 - \tau)$ for the degree of the focal surface.

For example, there exists a quartic submonoidal surface with four torsal generators, each containing two ordinary nodes. These quartic surfaces are classically known as *Plücker Complex surfaces* (see [653, Article 455]). Let $\mathfrak{C}$ be a quadratic line complex and ℓ be a general line in $\mathbb{P}^3$. A Plücker Complex surface Φ_4 of $\mathfrak{C}$ is defined to be the locus of points $x \in \mathbb{P}^3$ such that the pencil of lines $\Omega(x, \langle x, \ell\rangle)$ is tangent to the conic $\Omega(\Pi) \cap \mathfrak{C}$. The line ℓ is a double line of the surface. The residual conic in a plane $\Pi \in \ell^\perp$ is the dual conic $\Omega(\Pi) \cap \mathfrak{C}$. There will be four planes such that $\Omega(\Pi) \cap \mathfrak{C}$ is a reducible conic (since the degree of the singular surface of $\mathfrak{C}$ is equal to 4). The corresponding residual conics are the double torsal lines.

Next, we will give examples of congruences S of Type (III) with an irreducible fundamental curve C_d of degree $d > 1$.

Example 11.4.8. [7] We take for C_d an irreducible plane monoidal curve of degree d with a fixed point c_0 of multiplicity $d - 1$. The congruence S consists of the closure in $\mathbb{G}$ of the set of lines that intersect C_d and tangent to a fixed irreducible quadric cone K with its vertex at c_0. If $d = 2$, we assume that c_0 lies in the same plane as C_2.

Assume first that the plane containing C_d Π_0 intersects K along two different generators.

For any point $x \notin K$, the polar plane $P_x(K)$ contains the vertex of K and hence intersects K along two generators $g_1(x)$ and $g_2(x)$. It also intersects the plane Π_0 along a line $\ell_0(x)$. The latter line intersects C_d at a unique point $c \neq c_0$ (or a branch of C_d at c_0). A ray of S passing through x also passes through the point c. It is contained in one of the plane $\langle x, g_1(x)\rangle$ or $\langle x, g_2(x)\rangle$ tangent to K. This shows that the order of S is equal to 2. A general plane intersects K along a conic and intersects C_d at d points. From each point we can draw two tangents to the conic, hence the class of S is equal to $2d$.

The surface S is contained in the intersection of the tangential quadratic line complex of K and the Chow line complex of C_d of degree d. It is a surface

[7] Lehrsatz XIII from [473]

of bidegree $(2d, 2d)$. The residual part of S is the plane $\Omega(c_0)$ taken with multiplicity $2d-2$. It intersects S along the line $\Omega(c_0, \Pi_0)$.

The conic generatrix of K intersects S at two points corresponding to the generators in $K \cap \Pi_0$.

Let $c \neq c_0$ be a point in C_d, the two tangent planesΠ_c, Π_c' of K spanned by c and the generators $P_c(K) \cap K$ intersect Π_0 along the line $\langle c, c_0 \rangle$. It is a ray of S equal to the intersection of two pencils of lines $\Omega(c, \Pi_c)$ and $\Omega(c, \Pi_c')$. So, we see that S is a ruled surface and the pencil of rays $\Omega(c_0, \Pi_0)$ is its double directrix. A general hyperplane section of S This implies that sectional genus is equal to 0.

The proper transform of C_d under the minimal resolution $\mathbf{F}_2 \to K$ intersects the exceptional curve at $d-1$ points. These points correspond to the branches of C_d at c_0. The generators of K whose proper transform on $\mathbf{F}_2$ passes through these points are rays of S. These are points on the double line $\Omega(c_0, \Pi_0)$ of S. This shows that we can identify the double line with the normalization $C_d^{\mathrm{nrm}} \cong \mathbb{P}^1$.

The generatrix of S is a double cover of C_d^{nrm} ramified over the two points $c \in C_d \cap K, c \neq c_0$. Thus, S is a rational minimal ruled surface of degree $2+2d$ with a double line, and $\tilde{S} = S^{\mathrm{nrm}}$ is a minimal rational ruled surface. This implies that the sectional genus g of S is equal to 0 and the arithmetic sectional genus p_a is equal to 1. Applying (11.15), we obtain that the rank is equal to $n-2 = 2d-2$.

Since each ray is tangent to K, the focal surface $\Phi(S)$ contains K as its irreducible component. By Proposition 11.1.20, $\deg(\Phi(S)) = 2m-2 = 2$. Thus, $\Phi(S) = K$. The generators of K corresponding to the branches of C_d at c_0 are the focal rays of S. The point c_0 and two points from $C_d \cap K$ are isolated fundamental points of S.

Let ℓ be a general line in $\mathbb{P}^3$. It intersects K at two points x_1, x_2. Suppose $x \neq x_1, x_2$ is a point in ℓ , which is the null-point with the null-plane $\Pi \in \ell^\perp$. Then, each of the rays in Π passing through x intersects one of the generators g_1, g_2 of K contained in Π. Thus, the plane Π is tangent to K along two generators, a contradiction. This shows that the null-point of a null-plane Π in $\ell^\perp$ must be one of the two points x_1, x_2. One of the rays passing through x_i must be the generator g_i and Π must be tangent to K along this generator. Again we find a contradiction because ℓ is not tangent to K. We conclude that $r = 0,\ p_a = 2d-1$.

Assume now that the plane Π_0 containing C_d is tangent to K along a generator g. Then, any line in Π_0 intersects C_d and tangent to K. Since we are interested in irreducible congruences, we consider S to be the closure of lines intersecting C_d and tangent to K but not lying in Π_0. The order of S is still equal to 2 but the class now is equal to d instead of $2d$. Indeed, a general plane Π intersects Π_0 along a line ℓ tangent to the conic $K \cap \Pi$. So, we can drop only one tangent from a point from $\ell \cap C_d$. The surface S is still a rational ruled surface with a double line. So, $p_a = 1, g = 0$. This gives $r = d - 2$ and $\deg(\Phi(S)) = 2$. Since C_d intersects K at one point different from c_0, the number of isolated fundamental points is equal to 2.

Note that the two rays along which $K(x)$ intersect Π_0 are singular rays.

Example 11.4.9. [8] Let K be as in the previous example with vertex at c_0 and let C_d be a submonoidal curve of degree $d \geq 2$ with c_0 as its point of multiplicity $d - 2$. If $d = 2$, we assume that $c_0 \in \langle C_s \rangle$.

We assume that C_d is tangent to K at two points x_1, x_2 outside c_0. Let S' be the closure of the set of lines that are tangent to K and intersect C_d at its nonsingular point.

Take a general point $x \in \mathbb{P}^3$. The polar plane $P_x(K)$ intersects K along two generators g_1, g_2. Each plane $\langle x, g_1 \rangle$ and $\langle x, g_2 \rangle$ intersects C_d at two points. The four lines joining x with these points are tangent to K at points on g_1, g_2. This shows that $m = 4$. Taking a general plane Π that intersects C_d at d points, we see that the two tangents to the conic $\Pi \cap K$ dropped from these points belong to S. Thus, $n = 2d$. Let us see that the congruence splits into the union of two congruences of bidegree $(2, d)$.

Let K_1 be the quadric cone K' over the projection of C_d from the point c_0 to a plane Π that does not contain c_0. The intersection $\Pi \cap K$ is a conic C. Its tangent lines can be identified with the tangent planes to K. The intersection $\Pi \cap K_1$ is a conic C_1, its points can be identified with points on $C_d \setminus \{c_0\}$ and its branches at c_0.

We define a $(2, 2)$-correspondence:

$$\mathfrak{C} := \{(x, y) \in K \times K_1 : x_1 \in T_y(C)\}.$$

Since K is tangent to K_1 at two generators, the projection $\mathfrak{C} \to K$ is ramified at two points. An irreducible curve $\mathfrak{C}$ ramifies over four points if it is smooth or it ramifies over three points if it is singular. So, we conclude that $\mathfrak{C}$ is the union of two curves of bidegree $(1, 1)$. This defines a bijective correspondence between points of C_d and tangent planes of K and shows that the congruence of bidegree $(4, 2d)$ splits into the union of two congruences of bidegree $(2, d)$. If $d = 2$, one can show that each congruence is isomorphic to the minimal ruled surface $\mathbf{F}_2$ embedded in $\mathbb{P}^5$ by the linear system $|2\mathfrak{f}_1 + \mathfrak{f}_2|$. The conic C_2 is its directric of of degree 1.

[8] Lehrsatz XIV from [473].

Example 11.4.10. [9] Let K and C_d be as in the previous example but we assume that $C_d \subset K$. If $d = 2$, we assume that c_0 does not lie in the plane $\langle C_2 \rangle$. The proper transform of C_d on the minimal resolution $\mathbf{F}_2$ of K belongs to the linear system $|d\mathfrak{f} + \mathfrak{e}|$.

We define a congruence S to be the closure of the set of lines that are tangent to K at a nonsingular point of C_d. The congruence is contained in the intersection of the tangential quadratic line complex of K and the Chow line complex $\mathrm{Chow}(C_d)$. The intersection of these line complexes is a congruence of bidegree $(2d, 2d)$.

Let x be a general point in $\mathbb{P}^3$, there are two generators g_1, g_2 of K such that the planes $\langle x, g_1 \rangle$ and $\langle x, g_2 \rangle$ are tangent to K. There are two rays passing through x that join it with the points $g_1 \cap C_d$ and $g_2 \cap C_d$. So, S is of order $m = 2$.

A general plane intersects K along a smooth conic and intersects C_d at d points lying on the conic. The tangent lines of the conic at the intersection points are the rays of S contained in the plane. This shows that the class of S is equal to d.

The surface S enters with some multiplicity k_1 in the intersection of two line complexes and the plane $\Omega(x_0)$ enters with some multiplicity k_2

Since $(2d, 2d) = k_1(2, d) + k_2(1, 0)$, the only possible solution is $k_1 = 2, k_2 = 2d - 4$. If $d = 2$, the congruence is isomorphic to $\mathbf{F}_2$ embedded in $\mathbb{P}^5$ by the complete linear system $|3\mathfrak{f} + \mathfrak{e}|$. The conic C_2 is its directrix of degree 2.

11.4.2 New examples

The next three examples were absent in Kummer's paper. They were discovered, independently, by D. Montesano [522], [523] and R. Schumacher [669], and R. Sturm [734], [735, Theil 2].

Example 11.4.11. [10] Let Q be a Dupin cyclide quartic surface from Example 8.6.8. It is the projection of a 4-nodal quartic del Pezzo surface in $\mathbb{P}^4$. Let p_1, p_2, p_3, p_4 be the projections of the nodes on Q and $C \subset Q$ be the double conic.

The projection of Q from a point p on C exhibits Q as a double cover of the plane branched along the union of a line ℓ taken with multiplicity two (the image of C) and the union of two conics intersecting at four points (the projections of the nodes). The line ℓ intersects each conic at two points. They are the projections of the pinch points of C. Thus, the tangent cone of Q at

[9] Lehrsatz XVI from [473].
[10] [735, Theil II:Art. 503]

p splits into the union of two quadric cones passing through $p_1, \ldots, p_4$. Each cone also contains two of the pinch points of C. The congruence of generators of these cones is of bidegree $(4, 8)$, and it splits into two congruences of bidegree $(2, 4)$. The splitting is defined by the choice of two base points among the four pinch points.

Each cone in the congruence is uniquely determined by the conditions that it passes through $p_1, \ldots, p_4$, passes through a fixed pinch point p_5, and has a double point at a fixed point on the conic. The cones $K(x)$ form a quadratic pencil of contact quadrics to the surface Q. However, it is different from the contact family of quadrics (generally nonsingular) we introduced in Example 8.6.8.

Let T be the tetrahedron with vertices $p_1, \ldots, p_4$. We may assume that the lines $\langle p_1, p_2\rangle, \langle p_2, p_3\rangle, \langle p_3, p_4\rangle, \langle p_1, p_4\rangle$ are the projections of four lines on the 4-nodal quartic del Pezzo surface X. The diagonals $\langle p_1, p_3\rangle$ and $\langle p_2, p_4\rangle$ are the projections of the double lines of two quadrics in the pencil $|I_X(2)|$ of rank 3.

The surface Q is the focal surface of the congruence. The nodes $p_1, \ldots, p_4$ are fundamental points of degree 2 of the congruence. The four planes $\langle p_i, p_j, p_k\rangle$ are the fundamental planes of degree two.

The unions of two planes intersecting at the diagonal are reducible quadrics in the quadratic pencil of cones $K(x)$ of the congruence. lines are the irreducible components of the two reducible cones on the quadratic pencil.

Example 11.4.12. [11] Let Q be a quartic surface with a double rational normal cubic R_3. Since a general point on Q lies on a unique secant line of R_3 that must be contained in Q, we see that Q is a ruled surface. There are two types of such quartic ruled surfaces; type 1(i) and 2(i) from the classification of quartic ruled surfaces in Subsection 10.4.4. Recall that the ruled surface is swept by secant lines of R_3 contained in a linear line complex (non-special or special). A general point of R_3 is contained in two generators. This defines a double cover of $D \to R_3$ isomorphic to an elliptic curve. Its branch points $p_1, \ldots, p_4$ are the pinch points. The unique generator containing one of these points is a cuspidal generator.

For a general point $x \in R_3$, the polar $P_x(Q) \cap Q$ is a curve of degree 6 that contains the quadratic cone over the projection of R_3 from R_3 with multiplicity 2. The residual part is a quadratic cone $K(x)$ with its vertex at x. In other terms, the projection from x presents Q as a double cover of $\mathbb{P}^2$ with the branch curve equal to the union of a conic K_0 taken with multiplicity 2 and a conic K, the projection of $K(x)$. The intersection $K_0 \cap K$ consists of four points. They are

[11] [735, Theil II:Art. 509]

the projections of the pinch points p_i. The corresponding lines are generators of $K(x)$ that touch the cuspidal generators ℓ_i of Q at p_i.

We define a congruence S as the closure of the set of generators of quadric cones $K(x), x \in R_3$. Since a general plane intersects R_3 at three points, it contains six rays. Thus, $n = 6$.

For any point $x \in R_3$ which is not one of the pinch points, Q contains two generators passing through x. The cone $K(x)$ must contain these generators. Any other generator of $K(x)$ is tangent to Q at its intersection point with Q. Thus the algebraic family of cones $K(x)$ is a family of contact cones of Q. For a general point $x \in Q$, the embedded tangent plane $\mathbb{T}_x(Q)$ intersects R_3 at three points, two of which are the intersection points of a generator passing through x. Thus, there is only one cone $K(x)$ whose generator is tangent to Q at x. This shows that Q is the discriminant surface of the family of cones $K(x)$. Since Q is a quartic surface, the cones form quadric pencil, and hence, the order m of the congruence is equal to two.

Note that the quadratic pencil $\mathcal{K} = \{K(x)\}_{x\in R_3}$ can also be defined as the set of quadric cones with vertices on R_3 that pass through $p_1, \ldots, p_4$ and tangent to the cuspidal generators at these points. This is Sturm's definition of S.

Each line $\langle p_i, p_j\rangle$ is a generator of two cones from $\mathcal{K}$ that contain a general point on this line. The corresponding rays are singular points of S.

Example 11.4.13. [12] Let Q be a 4-nodal cubic surface and C_3 be its intersection with a plane tangent to Q at a nonsingular point q. So, C_3 is a nodal plane cubic curve. We consider the family of quadric cones $K(x)$ with vertices on Q that pass through the four nodes $p_1, \ldots, p_4$ of Q and the point q. Counting constants, we see that, for a general point $x \in C_3$ there exists a unique such cone. So, the cones form a pencil of some order r..

Let us see that $r = 2$. Let x be a general point in $\mathbb{P}^3$. Consider the web of quadrics with base points $p_1, \ldots, p_4, q, x$. The determinant surface of this web is a Weddle surface X whose points are singular points of quadrics from the web. The cubic surface and the quartic surface X contain the lines $\ell_{ij} =\leq p_i, p_j\rangle$. The curve C_3 intersects X at the point q with multiplicity 4 and also intersects it at six points where the lines ℓ_{ij} intersect the plane $\langle C_3\rangle$. Thus, C_3 intersects X at $12 - 10 = 2$ addional points. This shows that there will be two points p, p' on C_3 such that the cones $K(p)$ and $K(p')$ pass through x.

As before, we define a congruence S to be the closure of generators of cones $K(x)$. Its order $m = 2$, and its class $n = 6$.

Projecting Q from a general point $p \in C_3$, we find that the tangent cone at p is equal to the union of two quadric cones intersecting along four common

[12] [735, Theil II:Art. 507]

generators, each containing one of the points $p_1, \dots, p_4$. They also contain the line $\langle p, q \rangle$ because it is contained in the tangent plane at q. Our cones are among these cones; they are the ones that contain the generator $\langle p, q \rangle$.

In particular, we see that each ray of the congruence S is touching the cubic surface. Again, we constructed a quadratic pencil of cones whose discriminant surface contains the cubic surface Q.

Finally, observe that the points $p_1, \dots, p_4$ are isolated fundamental points of degree 3.

11.4.3 Classification

Recall that we divided all congruences of order 2 with fundamental curve $\mathrm{Fund}(S)$ into three different types (I), (II), and (III) according to the properties of the fundamental curve.

Proposition 11.4.14. *Let S be a congruence of Type (I). Then, it coincides with the congruence of secants of an irreducible quartic curve of arithmetic genus one.*

Proof Let F be the irreducible fundamental curve of S. Since $m = 2$, the number of secants of F passing through a general point of $\mathbb{P}^3$ is equal to two. By Proposition 11.2.4, F is a curve of degree 4 whose projection from x is a plane quartic with two (or three) nodes. In the first case, F is smooth, and in the second case, F is a nodal space quartic. Thus, S is a congruence from Examples 11.4.1 or 11.4.2 □

Proposition 11.4.15. *Let S be a congruence of Type (II). Then, it coincides with a congruence from Example 11.4.2 or 11.4.3.*

Proof Let F_1 and F_2 be two irreducible components of $\mathrm{Fund}(S)$, and let $d_1 \leq d_2$ be their degrees. The congruence is contains in the intersection of the Chow congruences $\mathrm{Chow})(F_1) \cap \mathrm{Chow})(F_1)$. Its bidegree is equal to (d_1d_2, d_1d_2). A general plane intersects F_i at d_i points, hence it contains d_1d_2 rays of S.This shows that the residual part should consists of $d_1d_2 - 2$ of α-planes $\Omega(x_i)$, where $x_i \in F_1 \cap F_2$. If $d_1d_2 \leq 2$, then $d_1 = 1, d_2 = 2$, and the assertion is true. Suppose that $d_1 > 1$ and $d_1d_2 > 2$. Projecting from a common point, we obtain the plane curves of degree $d_1 - 1$ and $d_2 - 1$ with common $d_1d_2 - 3$ points. The gives $(d_1 - 1)(d_2 - 1) - d_1d_2 + 3 = 4 - d_1 - d_2 \geq 0$, hence $d_1 = d_2 = 2$. Thus, the only possibilities are either F_1 is a line, or F_1 and F_2 are two conics intersecting at two points. These are our examples Example 11.4.2 and Example 11.4.3. □

Proposition 11.4.16. *Let S be a congruence of Type (III). Suppose that Fund*(S) *is a line* ℓ. *Then, S is a congruence from Example 11.4.5, or Example 11.4.6, or Example 11.4.8 (*$d = 1$*).*

Proof Let d be the degree of a general point on ℓ. Let ℓ be a general ray of S, it intersects ℓ at one point p_ℓ and spans with ℓ a plane $\Pi_\ell = \langle \ell, \ell \rangle$. This defines a rational map $f : S \dashrightarrow \ell \times \ell^\perp$. Let $C = f(S)$. For any $c = (p, \Pi) \in C$, the fiber $f^{-1}(c)$ is the pencil of lines $\Omega(p, \Pi)$. This shows that C is an irreducible curve and S is a ruled surface. Let $\mathrm{pr}_1 : C \to \ell$ and $\mathrm{pr}_2 : C \to \ell^\perp$ be the two projections.

The pre-image of a general point $p \in C$ under the map $\mathrm{pr}_1 \circ f : S \dashrightarrow \ell$ consist of $d = \deg(\mathrm{pr}_1)$ pencils of rays. This shows that d is the degree of a general point of the fundamental curve ℓ. It is also equal to the number of rays in a general plane passing through p. Thus, $d = n$ is the class of S.

The pre-image of a general plane $\Pi \in \ell^\perp$ under $\mathrm{pr}_2 \circ f$ is the curve $\Omega(\Pi) \cap S$ of rays containing in Π. Since $m = 2$, a line in the dual plane $\Omega(\Pi)$ corresponding to a general ray in Π intersects this curve at two points. Hence, $\Omega(\Pi) \cap S$ is a conic, maybe reducible or a double line. Taking Π general enough, we obtain that $\deg(\mathrm{pr}_2) = 2$.

Now, we have three possibilities for a general $\Pi \in \ell^\perp$:

- $\Omega(\Pi) \cap S$ is a smooth conic;
- $\Omega(\Pi) \cap S$ is the union of two pencils of rays with base points on ℓ;
- $\Omega(\Pi) \cap S$ is the union of two pencils of rays with base points outside ℓ;

We leave it to the reader to check that the first case leads to a submonoidal congruence from Example 11.4.6, the second case leads to a congruence from Example 11.4.5, and the last case leads to a congruence from Example 11.4.8, where $d = 1$.

If $\Omega(\Pi) \cap S$ is a double line, each ray in Π is a focal ray. So, there is only finitely many such planes.

□

The idea of the proof of the next proposition is taken from [17].

Lemma 11.4.17. *Let S be a congruence of Type (III) with irreducible fundamental curve F. Suppose that a general point of F is of degree one. Then, the planes* $K(x)$ *of rays passing through a point* $x \in F$ *form either a pencil of planes or a smooth conic of planes. In the first case, F coincides with the base line of the pencil of planes. In the second case, the planes are tangent planes of a quadric cone K, and S consists of lines tangent to K and intersecting F.*

Proof We have a rational map $f : F \dashrightarrow \hat{\mathbb{P}}^3$ which assigns to $p \in F$ the plane $K(p)$. Let x be a general point of $\mathbb{P}^3$, there are two rays passing through x and intersecting F at two points p_1, p_2. This shows that the pre-image of a hyperplane $x^\perp$ in $\check{\mathbb{P}}^3$ consists of two points. In other words, either the degree of the map f is equal to two and the image is a line in $\check{\mathbb{P}}^3$, i.e. a pencil of planes, or the degree is equal to one, and the image is a conic $\check{\mathbb{P}}^3$.

Suppose the planes $K(x)$ form a pencil of planes and let ℓ be its base line. Since, by definition of a congruence of type (III), all rays intersect F and each ray is contained in one of the planes $\Pi \in \ell^\perp$. This is possible only if $F = \ell$.

Suppose the planes $K(x)$ form a conic. A conic in $\check{\mathbb{P}}^3$ is contained in a plane $c_0^\perp$. The point c_0 is contained in all planes $K(x)$. Fixing a plane $\Pi \subset \mathbb{P}^3$ not containing c_0, we see that the intersection of the planes $K(x)$ with Π are lines passing through c_0 which sweep a quadric cone K. The planes $K(x)$ are tangent to K along a generator. We see that all rays are tangent to K and also intersect F.

□

Proposition 11.4.18. *Under the assumptions of Lemma 11.4.17, S is a congruence from Example 11.4.5, or Example 11.4.8, or Example 11.4.9.*

Proof Let us see that, under the assumption of Lemma 11.4.17, the congruence S with pencil of planes $K(x)$ is a congruence from Example 11.4.5. We define a correspondence on $C \subset \ell \times \ell^\perp$ by

$$C = \{(p, \Pi) \in \ell \times \ell^\perp : K(p) = \Pi\}.$$

The projection $\mathrm{pr}_2 : C \to \ell^\perp$ is of degree 2, and the projection $\mathrm{pr}_1 : C \to \ell$ is of degree equal to some $d > 0$. The congruence S is a ruled surface of degree $d + 2$. Its rulings are the pencil of lines in $\mathbb{P}^3$ passing through a point $p \in \ell$ and contained in one of d planes $\Pi \in \mathrm{pr}_2(\mathrm{pr}_1^{-1}(p))$. This is a congruence from Example 11.4.5.

Next, assume that, under the assumption of Lemma 11.4.17, the congruence S with $\{K(x), x \in F\}$ forms a conic. It follows from the proof of the lemma that F is birationally isomorphic to the conic, hence it is a rational curve of planes tangent to a quadric cone K. The fundamental curve F is a curve C_d of some degree d such that the closure of lines intersecting C_d and tangent to K is a congruence of order 2.

There are two possibilities: C_d lies on K or does not lie on K.

In both cases, S is contained in the intersection of the quadratic line complex $\mathfrak{T}$ of tangents to K and the Chow complex $\mathrm{Chow}(C_d)$. Its bidegree is $(2d, 2d)$. A general plane intersects C_d at d points $x_1, \ldots, x_d$ and intersects K along a conic.

Assume $C_d \subset K$, then the points $x_1, \ldots, x_d$ are contained in the conic. The rays of S contained in the plane are tangent lines of the conic at the points x_i. This shows that the class of S must be equal to d. Let x be a general point of $\mathbb{P}^3$. The polar plane $P_x(K)$ intersects K along two rays. A ray passing through x intersects one of the two generators g at one point lying on C_d. Since $m = 2$, there must be only one point of C_d on g besides the vertex. Its proper transform $\bar{\ell}_d$ on the minimal resolution $\mathbf{F}_2$ of K belongs to $|a\mathfrak{f}+\mathfrak{e}|$. Since $\deg(C_d) = (a\mathfrak{f} + \mathfrak{e}) \cdot (2\mathfrak{f} + \mathfrak{e}) = d$, we obtain that $a = d$. Thus, $\bar{\ell}_d$ intersects the exceptional section with multiplicity $d-2$. This means that c_0 is a point on C_d of multiplicity $d-2$. Conversely, this property guarantees that $m = 2$. So, the bidegree of S is equal to $(2, d)$. We have $(2d), 2d) = 2(2, d) + (2d - 4)(1, 0)$, i.e $\mathfrak{T} \cap \mathrm{Chow}(C_d)$ contains S with multiplicity 2 and contains $\Omega(c_0)$ with multiplicity $2d - 4$. The surface S is one from Example 11.4.10.

Next, assume that $C_d \not\subset K$. Suppose first that C_d is contained in a plane Π_0. Using the previous argument, we see that a general plane tangent to K intersects Π_0 along a line that interests C_d at one point outside c_0. Thus, C_d is a monoidal curve of degree d. We get a congruence from Example 11.4.9.

Finally, assume that C_d does not lie on K and C_d is not a plane curve. Since we expect that $C_d \cap K$ intersects at $2d$ points, we have to require that c_0 is a multiple point of C_d of multiplicity $d-1$, however in this case C_d is contained in a plan (the pre-image of the projection of C_d from c_0 which is a line). Since we assume that C_d is not a monoidal curve. Assume that c_0 is of multiplicity $d - 1 - k$, so $m = 2 + k > 2$. The projection form c_0 is a plane curve of degree $k + 1$. Since C_d is rational this implies that $k = 1$. A general plane Π intersects K along a conic, and there are two rays in S passing through one of d points in $\Pi \cap C_d$. They are tangents to the conic. This shows that the class $n = 2d$. □

Suppose the fundamental curve F is irreducible of degree $d > 1$. It follows from Theorem 11.1.28 that its general point is of degree $h = 2$. Thus it defines an algebraic family $\mathcal{K} = \{K(x)\}_{x \in F}$ of quadric cones.

The proof of the following lemma uses the theory of complete quadrics from Section 2.4.

A general one-dimensional family of quadrics in $\mathbb{P}^3$ (and in any $\mathbb{P}^n$) contains only quadrics of corank ≤ 1. On the other hand, a general one-dimensional family of quadric cones in $\mathbb{P}^3$ lifted to CQ_3 is contained in E_3 and intersects E_2° (resp. E_1°) at δ points η (resp. δ points) corresponding to reducible cones (resp. point-pairs).

Lemma 11.4.19. *Let $\mathcal{K}$ be an irreducible algebraic family of quadric cones whose proper transform in CQ_3 has the cohomology class $\mu^a \nu^b \rho^c$, $a + b = c =$*

8. *Let* $\alpha = \mu^{a+1}\nu^b\rho^c, \beta = \mu^a\nu^{b+1}\rho^c, \gamma = \mu^a\nu^b\rho^{c+1}$ *be the number of quadric cones in the family passing through a general point, tangent to a general line, and tangent to a general plane, respectively. Then*

$$2\beta = \gamma + 2\alpha + \eta, \quad 2\rho = \beta + \delta.$$

Proof We have

$$\eta = e_2\mu^a\nu^b\rho^c, \quad \delta = e_3\mu^a\nu^b\rho^c.$$

Applying Proposition 2.4.28 and using the computation of characteristic numbers for quadric cones, we find

$$\begin{aligned}\eta &= (2\nu - \mu - 2\rho)\mu^a\nu^b\rho^c = 2\mu^a\nu^{b+1}\rho^{c+1} - \mu^{a+1}\nu^b\nu^c - 2\mu^a\nu^b\rho^{c+1} \\ &= 2\nu - \alpha - 2\gamma, \\ \delta &= (2\rho - \nu)\mu^a\nu^b\rho^c = 2\mu^a\nu^b\rho^{c+1} - \mu^a\nu^{b+1}\rho^c = 2\gamma - \beta.\end{aligned} \tag{11.52}$$

This proves the lemma. □

Note that the number γ is equal to the degree of C, the number of the cones from the family passing through a general point in $\mathbb{P}^3$. Since the order m of S is equal to 2, we obtain that the two equalities imply

$$6 - 2\delta = 2\gamma + \eta. \tag{11.53}$$

The number γ is equal to the number of cones in the family whose vertex lies in a fixed general plane, so this number equals $d = \deg(F)$.

This gives us the following possibilities:

$$\begin{aligned}&\delta = 1,\ \beta = 3,\ d = 2,\ \eta = 0, \\ &\delta = 0,\ \beta = 4,\ d = 2,\ \eta = 2. \\ &\delta = 0,\ \beta = 4,\ d = 3,\ \eta = 0,\end{aligned} \tag{11.54}$$

Here, we used that $\gamma = d > 1$ since F is not a line.

In particular, $\deg(F)$ is equal to 2 or 3. If F is a plane cubic, then it must be a singular curve. Indeed, the universal family of the curves $C(x) = \Omega(x) \cap S, x \in F$, is projected to $\mathbb{P}^3$ with a general fiber isomorphic to a conic, hence it is rational. It is projected to F, hence F is rational.

Now, we are in business.

The family $\mathcal{K}$ of cones $K(x)$ is an example of a quadratic pencil of quadrics. We discussed such pencils in Subsection 4.1.4. The fundamental curve F is a rational curve, we fix its rational parameterization $\nu : \mathbb{P}^1 \to F$, and consider the quadratic pencil with parameters $[u, v] \in \mathbb{P}^1$.

We can write our pencil in the form

$$K(u,v) = \sum_{0\le i,j\le 3} a_{ij}(u,v)t_it_j = 0,$$

where $a_{ij}(u,v)$ are quadratic forms in t_0,t_1,t_2,t_3. Since all quadrics $K(u,v)$ are singular, we get $\det(a_{ij}) = 0$. The set of singular points of the cones $V(K(u,v))$ coincides with the fundamental curve F of S. The singular point x of $K(u,v)$ is defined by the null-space of the matrix $M = (a_{ij})$. It is given by any column (or a row) of the adjugate matrix $\mathrm{adj}(M)$ of cofactors of M. Thus, the map

$$\nu : \mathbb{P}^1 \to F \subset \mathbb{P}^3, \quad t \mapsto \mathrm{Sing}(K(x))$$

of degree $k \le 3$. Since any point on F is contained in only one cone $K(x)$, it is contained in the surface $V(D)$. Taking the partials, we see that

$$F \subset \mathrm{Sing}(V(D)).$$

We also explained that all cones $K(x)$ are tangent to $V(D)$, hence the focal surface $\Phi(S)$ is contained in the quartic surface $V(D)$.

The possible degrees of F are confirmed by (11.54).

Case 1: F is a conic.

Suppose F is a conic. Recall from the proof of Theorem 11.1.28 that a general line ℓ defines a correspondence of bidegree $(2,2)$ on $F \cong \mathbb{P}^1$. The united points of this correspondence are the intersection points of ℓ with the focal surface. Since a correspondence of bidegree $(2,2)$ has four united points, we see that the focal surface must be a quartic surface and hence coincides with $D(\mathcal{K})$. The curve F is the singular locus of $\Phi(S)$. So, $\Phi(S)$ is a non-normal quartic surface with a double conic, a cyclide quartic surface from Subsection 8.6.2. We know that $\beta = 3$ or 4. The surface $Q = D(\mathcal{K})$ must be a cyclide quartic surface. The number β coincides with the class of the surface. In Subsection 8.6.34 we computed the possible class of a cyclide quartic surface. It shows that the case $\beta = 3$ is impossible, and the case $\beta = 4$ leads to the case of a Dupin cyclide quartic surface from Example 8.6.8. It also agrees with the number $\eta = 2$ that shows the quadratic pencil contains two reducible cones. The quartic surface Q is the focal surface of S. A general plane intersects F at two points and intersects the corresponding cones $K(x)$ at two lines. Hence, the bidegree of the congruence is equal to $(2,4)$. We see that the congruence coincides with a Surm congruence from Example 11.4.11.

Case 2: $\deg F = 3$c.

In this case, $\delta = \eta = 0$, so the quadratic pencil consists of cones of corank one. This means that the linear series defining the map ν is of degree 3 without base points. If it is a complete linear system, the map ν is a Veronese map and F is a twisted cubic. The discriminant surface $D(\mathcal{K})$ is a quartic surface with the double curve R_3. It coincides with the focal surface of S. The congruence is a Sturm's congruence from Example 11.4.12.

If the series is not complete, and its dimension is one, then F is a line, contradicting our assumption on F. Thus, the dimension is equal to two, and F is a plane nodal cubic. The discriminant surface $D(\mathcal{K})$ is the union of a cubic surface Q_3 and a plane intersecting along F. It coincides with the focal surface of S. Since $\beta = 4$, the class of the cubic surface is equal to 4. A cone $K(x) \in \mathcal{K}$ is a contact cone of Q_3. Hence, it is contained in the tangent cone of Q_3 at x. This implies that the quartic branch curve of the projection of Q_3 from x contains a conic as its irreducible component. The residual component is another conic. The intersection point of the two conics must be singular points of the cubic surface. Since all cones are of corank one, it is easy to see that the conics intersect transversally at four points, so Q_3 is a 4-nodal cubic surface. Thus S is a Sturm's congruence from Example 11.4.13.

This concludes the classification. If S is of Type (I) or Type (II), then Propositions 11.4.14 and 11.4.15 show that all appear in Examples Examples 11.4.1, 11.4.2 and 11.4.3.

If S is of type (III), and $\mathrm{Fund}(S)$ is a line, we use Proposition 11.4.16 to see all possible cases are covered in Examples 11.4.5–11.4.8. If $\mathrm{Fund}(S)$ is not a line, by Proposition 11.1.28, $h(x) \le 2$ for a general point on the fundamental curve. If $h(x) = 1$, we use Proposition 11.4.18 to get Examples 11.4.5, 11.4.8, and 11.4.9, if $h(x) = 2$, we get new Sturm's example.

The classification of quadratic congruences with fundamental curves is summarized in Table 11.3.

11.5 Congruences of Lines of Higher Order

The classification of congruences of lines of any order $m \ge 3$ is unknown. In this section, we will give some classical examples and discuss the known classification of smooth congruences of degree ≤ 10.

11.5.1 Reye congruences

We have already encountered Reye varieties of lines S in Subsection 1.1.7. In the special case of lines in $\mathbb{P}^3 = |E|$, it is defined as the variety of lines

Type	Fund.curve	Fund.points	n	r	p_a	g	Example
I_1	elliptic C_4	4	6	2	3	3	11.4.1
I_2	nodal C_4	2	6	2	3	1	11.4.2
II_1	$C_2 + C_2$	0	4	2	1	1	11.4.3
II_2	$\ell + C_d, d \geq 2$	0	d	0	$d-1$	$d-3$	11.4.4
II_3	$C > \ell$	0	d	0	$d-1$	$d-3$	11.4.5
III_1	ℓ	3d-4	2d-2	0	2d-3	2d-3	11.4.6
III_2	$\mathrm{mon}(C_d, c_0) \subset \mathbb{P}^2$	3	2d	2d-2	1	0	11.4.8
III_2'	mon $(C_d, c_0) \subset \mathbb{P}^2$	2	d	d-2	1	0	11.4.8
III_3	smon (C_d, c_0)	2	d	$d-1$	0	0	11.4.9
III_4	smon $(C_d, c_0) \subset K$	1	d	d-1	0	0	11.4.10
III_5	C_2	4	4	6	4	1	11.4.11
III_6	R_3	4	6	4	1	1	11.4.12
III_7	C_3	4	6	4	1	1	11.4.13

Table 11.3 *Quadratic Congruences with fundamental curve*

contained in a subpencil of a web $W = |V|$ of quadrics in $\mathbb{P}^3$. For general W it is a smooth irreducible surface S in $G_1(\mathbb{P}^3)$. In fact, a weaker condition for this to be true is that W is a regular web. In this case, S is a smooth Enriques surface of bidegree $(7, 3)$ in $G_1)(\mathbb{P}^3)$. Its canonical cover is a K3 surface isomorphic to the subvariety

$$\mathrm{PB}(W) = \{([v], [v']) \in \mathbb{P}^3 \times \mathbb{P}^3 : b_q(v, v') = 0, \ \forall v, v' \in V\}.$$

where b_q is the polar symmetric bilinear forms associated with $q \in V$. The rays of S are called Reye lines. The regularity assumption implies that $|W|$ has no base points. Any line through a base point is a Reye line. Since we assume that the congruence is irreducible, we must delete the plane components containing these lines.

We refer to [259, Chapter 7] for a detailed exposition on Reye congruences of regular webs of quadrics. The Reye congruence has no fundamental points if and only if W is an excellent web. The focal surface of a smooth Reye congruence is a surface of degree 24. Since $\deg S = 10$, and S is an Enriques surface embedded in $\mathbb{P}^5$ by a complete linear system. A general hyperplane section H satisfies $H^2 = 10$, hence the sectional genus $p_a = g = 6$. The focal surface $\Phi(\mathrm{Rey}(W))$ is of degree $14 + 10 = 24$. The rank $r = (m-1)(n-1) - g = 6$.

Now, let us drop the condition that W has no base points. We will make the following assumptions

1. The base scheme of W is reduced and consists of k isolated points $b_1, \dots, b_k$.
2. W contains ten reducible quadrics and no double planes.
3. The discriminant surface $\mathcal{D}(W)$ has no lines, or, equivalently, W does not contain pencils of singular quadrics.

Note that the first assumption implies that a general member of W is a smooth quadric, the second one implies that no three points are on a line, and (iii) implies that no four base points are coplanar. Since $\dim W = 3$, these properties imply that $k \le 6$. By definition, a regular web is excellent if it satisfies the last condition.

We say that a Reye line is *general* if $|\mathcal{O}_{\mathbb{P}^3}(2) - \ell|$ is a pencil with the base locus equal to the union of ℓ and a twisted cubic intersecting it at two points. We define the Reye congruence $\mathrm{Rey}(W)$ to be the closure of the set of general Reye lines in $G_1(\mathbb{P}^3)$. All lines from $\mathrm{Rey}(W)$ are called Reye lines. For example, a line passing through a point b_i is a Reye line if and only if it is a generator of the tangent cone of $\mathrm{St}(W)$ at b_i.

As is in the case of regular webs we can define the *Reye involution*:

$$\mathfrak{r}_W : \mathrm{St}(W) \dashrightarrow \mathrm{St}(W), \quad x \mapsto x \mapsto \cap_{Q \in W} P_x(Q).$$

By property (3), the involution is a regular involution and its set of fixed points is equal to $\mathrm{Bs}(W)$.

Each line $\ell_{ij} = \langle b_i, b_j \rangle$ is invariant with respect to $\mathfrak{r}_W$. Indeed, ℓ_{ij} is contained in a net $\mathcal{N}_{ij}$ of quadrics from W. For a general point $x \in \ell_{ij}$, the intersection $\cap_{Q \in \mathcal{N}_{ij}} P_x(Q)$ contains ℓ_{ij}, hence $\mathfrak{r}_W(x) \in \ell_{ij}$.

Also, in this case, the projection

$$p : \mathrm{PB}(W) \to \mathrm{St}(W)$$

is an isomorphism. Under this isomorphism, the Reye involution is defined by the switch of factors in the ambient $\mathbb{P}^3 \times \mathbb{P}^3$. The linear system $|\mathcal{O}_{|E|}(1) \otimes \mathfrak{r}_W^* \mathcal{O}_{|E|}(1)|$ on $\mathrm{PB}(W)$ is the restriction of the linear subsystem $|\mathcal{O}_{\mathbb{P}^3 \times \mathbb{P}^3}(1,1)| = |E^\vee \otimes E^\vee|$ to $\mathrm{PB}(W)$. It can be identified with the linear system $|\bigwedge^2 E^\vee \oplus V^\perp|$, where $V^\perp \subset S^2(E)$ is of dimension 6. It defines two rational projections

$$r : \mathrm{PB}(W) \dashrightarrow |\bigwedge^2 E|, \quad c : \mathrm{PB}(W) \dashrightarrow |V^\perp|.$$

They are the restrictions of the projection maps of the Segre subvariety $\mathsf{s}_3(|E| \times |E|) \subset |E \otimes E|$. The first projection is not defined at the intersection of the diagonal with $\mathrm{PB}(W)$ that coincides with the set of its singular points. The image of the first projection is the Reye congruence $\mathrm{Rey}(W)$. The second projection is regular and its image is the *Cayley model* $\mathrm{Cay}(W)$ of $\mathrm{Rey}(W)$.

Since the Segre variety is of degree $\binom{6}{3} = 20$, and the both projections are of degree 2, we obtain that

$$\deg \mathrm{Rey}(W) = 10 - k, \quad \deg \mathrm{Cay}(W) = 10.$$

Let

$$\pi_W : X \to \mathcal{D}(W), \quad \pi_{\mathrm{st}} : X \to \mathrm{St}(W)$$

be the common minimal resolutions of singularities of the surfaces $\mathcal{D}(W)$ and $\mathrm{St}(W)$. The involution $\mathfrak{r}_W$ lifts to a biregular involution $\tilde{\mathfrak{r}}_W$ of X. Its set of fixed points is equal to the union of the exceptional curves over $\mathrm{Bs}(W)$. Both projections lift to regular maps

$$\tilde{r} : X \to |\bigwedge^2 E|, \quad \tilde{c} : X \to |V^{\perp}|.$$

Suppose $k \geq 2$. Then each line $\ell_{ij} = \langle p_i, p_j \rangle$ is invariant with respect to the Reye involutions. Indeed, let $x \in \ell_{ij} = \langle p_i, p_j \rangle$ different from p_i, p_j. There is a net of quadrics containing ℓ_{ij}. A general member of this net $N(x)$contains ℓ_{ij} in the tangent space at x. This shows that $\ell_{ij} \subset \cap_{Q \in N(x)} P_x(Q)$. Thus, $\mathfrak{r}_W(x) = \cap_{Q \in W} P_x(Q) \in \ell$. The map $\tilde{r} : X \to \mathrm{Rey}(W)$ blows down each line ℓ_{ij} to a nonsingular point $s_{ij} \in \mathrm{Rey}(S)$. The pre-image of ℓ_{ij} in X is a (-2)-curve R_{ij} invariant with respect to the Reye involution. Its image in $Y = X/(\tilde{\mathfrak{r}}_W)$ is a (-1)-curve. The Reye congruence is obtained from Y by blowing down the curves R_{ij}.

It follows from the formula for the canonical class of a double cover of smooth surfaces that $|-K_Y| = \emptyset$ but $|-2K_Y|$ consists of the union of k disjoint smooth rational curves B_i with $B_i^2 = -4$ (a (-4)-curve, for short). They are the images of the exceptional curves of π_{st}. The surface is an example of a *Coble surface* with k boundaries (see [259, Chapter 9]). It satisfies

$$K_Y^2 = -k.$$

So, if $k \geq 1$, we obtain a description of the Reye congruence $\mathrm{Rey}(W)$ as the blow-down of $\binom{k}{2}$ (-1)-curves on a Coble surface, each intersecting two boundary components.

Let $\eta_{\mathrm{st}} = \pi_{\mathrm{st}}^* c_1(\mathcal{O}_{|E|}(1))$ and $\eta_W = \pi_W^* c_1(\mathcal{O}_{|W}(1))$. It is known that the

$$\eta_{\mathrm{st}} = \tfrac{1}{2}(3\eta_W - E_1 - \cdots - E_{10}), \tag{11.55}$$

where E_i are the exceptional curves of π_W over the reducible quadrics in W. Let $R_1, \ldots, R_k$ be the exceptional curves over the base points of W. The pre-image of $c_1(\mathcal{O}_{\mathrm{Rey}(W)}(1))$ (where $\mathrm{Rey}(W)$ is embedded in the Plücker space) under map $r : X \to \mathrm{Rey}(W)$ is equal to $\eta + \tilde{\mathfrak{r}}^*(\eta) - R_1 - \cdots - R_k$. It follows that the degree of the image of the exceptional curve E over a singular point of $\mathrm{St}(W)$ is equal to $\frac{1}{2}(3+1) = 2$.

In the realization of $\mathrm{Rey}(W)$ as a congruence of lines in $\mathbb{P}^3$, the cone of rays corresponding to points of C_i is the tangent cone of $\mathrm{St}(W)$ at a base point.

It follows from above that we can identify the Reye congruence with the orbits of $\tilde{\mathfrak{r}}_W$, two orbits on the same line define the same point in $R(W)$. Consider the map

$$\nu_W : \mathrm{Rey}(W) \dashrightarrow G_1(W), \quad \ell \mapsto |W - \ell|. \tag{11.56}$$

This map is not defined at the point corresponding to the lines ℓ_{ab}. We know that we can lift ν_W to a regular map

$$\tilde{\nu}_W : Y \to G_1(W). \tag{11.57}$$

from the Coble surface Y to $G_1(\mathbb{P}^3)$. Each Reye line except the one corresponding to ℓ_{ab} has two quadrics with singular points on it. This shows that its image is a bisecant of $\mathcal{D}(W) \subset W$. Since Y is irreducible, the image of the map is one of the irreducible components $\mathrm{Bit}(\mathcal{D}(W))_i$ of the bitangent surface $\mathrm{Bit}(\mathcal{D}(W))$. A general net of quadrics in W contains 8 base points. It contains $\binom{8-k}{2}$ Reye lines joining the pairs base points of the net different from the base points of W. This shows that the image of the map $\tilde{\mathfrak{l}}_W$ is an irreducible component of the bitangent surface of the class $\binom{8-k}{2} \in \{21, 15, 10, 6, 3, 1\}$. The map $\tilde{\nu}_W$ is the normalization map of $\mathrm{Bit}(\mathcal{D}(W))_i$.

Before we give more details in each case $k = 1, \ldots, 6$, we prove the following lemma.

Lemma 11.5.1. *Let $x \in \mathbb{P}^3 \setminus \mathrm{Bs}(W)$ and $N(x)$ be the net of quadrics in W containing x. Then $\mathrm{Bs}(N(x))$ is one of the followings sets:*

- *(i) $\{x, p_1, \ldots, p_k, y_1, \ldots, y_{7-k}\}$, where some of the points may coincide.*
- *(ii) $\ell \cup \{y_1, y_2, y_3, y_4\}$, where ℓ is a line.*
- *(iii) a conic K (maybe reducible) and $\{y_1, y_2\}$, where y_i are lying on a conic intersecting ℓ and ℓ'.*
- *(iv) a line ℓ and an irreducible conic intersecting at one point.*
- *(v) two skew lines.*

Proof By assumption on W, it cannot contain a net of singular quadrics. If $\mathrm{Bs}(N(x))$ consists of isolated points, then we get the case (i). Suppose $\mathrm{Bs}(N(x))$ contains an irreducible curve C. Then, the residual curve is of degree $< 4 - \deg(C)$. It follows that $\mathrm{Bs}(N(x)$ contains a line.and the residual curve R of $\mathcal{P}$ is of degree ≤ 2.

Suppose R does not contain irreducible components from $\mathrm{Bs}(N(x)$. Then R

intersect ℓ at two points, hence, $N(x)$ has addionally four base points on R. The number of base points different from base points of W is equal to $4 - s$, where $s = \#\ell \cap \mathrm{Bs}(W)$. This gives case (ii).

Suppose R is the union of a line ℓ' and a conic K. If $\ell' \subset \mathrm{Bs}(N(x)$, we have 2 additional base points on K. This also gives case (iii).

If K is contained in $\mathrm{Bs}(N(x))$, then we do not have any more base points. This gives case (iv).

Suppose R is the union of three lines ℓ_1, ℓ_2, ℓ_3 which, together with ℓ form a quadrangle of lines with opposite skew lines ℓ, ℓ_1 and ℓ_2, ℓ_3. If none of ℓ_i is contained in $\mathrm{Bs}(N(x))$, then we have four base points, two on ℓ_1, and one on each ℓ_2, ℓ_3. If ℓ_2 (or ℓ_3) is contained in $\mathrm{Bs}(N(x))$, then we have two new base points, one on each ℓ_1 and ℓ_3 (or ℓ_2). If $\ell_1 \subset \mathrm{Bs}(N(x))$, we have no more base points. This gives case (v).

□

Proposition 11.5.2. *The bideger* (m, n) *of the Reye surface* $\mathrm{Rey}(W)$ *is equal to* $(7 - k, 3)$.

Proof Let x be a general point in $\mathbb{P}^3$. If $N(x)$ has a curve of base points, a general quadric in W intersects at a point different from x and different from its base points. This contradicts our assumptions on W. It follows from our assumptions that x is not a singular point of a quadric from the web. Thus, $N(x)$ has 8 base points, the points $p_1, \ldots, p_k, x$ are among them. Any line spanned by x and one of the base points is a Reye line. A line imposes three conditions on quadrics to contain it. So, any Reye line must be one of them. Since we excluded lines passing through $p_1, \ldots, p_k$, we obtain that the order of S equals $7 - k$.

Since we already know that $\deg \mathrm{Rey}(W) = 10 - k$, we get $n = 3$. However, we can see it also as follows. Take a general plane Π. The restriction of W to Π is a web of conics without base points. Its Reye variety consists of 3 points. In fact, the variety of

$$\mathrm{PB}(W) = \{([v], [v']) \in \Pi \times \Pi : b_q(v, v') = 0, \quad \text{for any } V(q) \in |W|\}.$$

is equal to the intersection of four divisors of bidegree $(1, 1)$. They intersect at 6 points, the Reye variety is the set of 3 pairs invariant under the involution of $\Pi \times \Pi$ that switches the factors.

□

Proposition 11.5.3. *A point* x *is a fundamental point of* $\mathrm{Rey}(W)$ *if and only if* $\mathrm{Bs}(N(x))$ *contains a line or a twisted cubic. In the former case,* $h(x) = 1$ *and in the latter case* $h(x) = 2$.

Proof Suppose x is a fundamental point and let $N(x)$ be the net of quadrics in W containing x. Let ℓ be a Reye line containing x. Then, it is a Reye line of the net $N(x)$, and hence, $N(x)$ has a curve of Reye lines containing x. The restriction of $N(x)$ to a Reye line ℓ of $N(x)$ is a divisor $x + x_\ell$ on ℓ. If $x = x_\ell$ for all Reye lines ℓ, then $N(x)$ conntains a pencil of quadrics with singular point x. This contradicts our assumption on W.

Otherwise, the residual points x_ℓ form an irreducible curve F in $\mathrm{Bs}(N(x)$. By Lemma 11.5.1 F is a line, smooth conic, or a twisted cubic. In the first case, the cone $K(x)$ of rays through x is a pencil of rays and $h(x) = 1$. In the second case, if x either lies on F, we get again a pencil of rays; otherwise, $h(x) = 2$, and we have a quadric cone $K(x)$ of rays with vertex at x. In the last case, x must lie on F, and we have $h(x) = 2$. □

In the following examples, we use that a nodal quartic surface Q with $\mu \geq 10$ nodes admits a symmetric determinantal representation if and only if it contains a subset of 10 nodes such that (11.55) holds. In other words (see Section 12.2) the set of 10 nodes is a weakly even set. We will find in Table 12.1 all quartic surfaces admitting weak even sets of ten nodes. We will see that all surfaces are realized as the focal surfaces of quadratic congruences without fundamental curves. The conjugacy graph of fundamental points allows one to determine all subsets of 10 weakly even nodes.

Example 11.5.4. Assume $k = 1$. The Reye congruence is of bidegree $(6, 3)$ and sectional genus $g = 5$. The Steinerian surface is a quartic with one node. The discriminant surface $\mathcal{D}(W)$ is a 11-nodal quartic. It is the focal surface of a quadratic congruence of class 7. It has only one weakly even set of nodes formed of fundamental points of degree 3.

The image of the map ν_W is an irreducible component of $\mathrm{Bit}(\mathcal{D}(W))$ of bidegree $(10, 21)$ It is isomorphic to its normalization. The Reye congruence $\mathrm{Rey}(W)$ is isomorphic to the Coble surface Y with one boundary component B_1,

$$\mathrm{Rey}(W) \cong Y \cong \mathsf{Bl}_{p_1,\ldots,p_{10}}(\mathbb{P}^2),$$

where the points $p_1, \ldots, p_{10}$ are double points of an irreducible plane sextic, the image of B_1 in $\mathbb{P}^2$. The Plücker map $\mathrm{Rey}(W) \to G_1(\mathbb{P}^3)$ is defined by the linear system

$$|7e_0 - 2(e_1 + \cdots + e_{10})|,$$

where $(e_0, e_1, \ldots, e_{10})$ is the geometric basis of $\mathrm{Pic}(\mathrm{Rey}(W))$ defined by the blow-up. The linear system $|10e_0 - 3(e_1 + \cdots + e_{10})|$ defines the Cayley model of $\mathrm{Rey}(W)$. The image of B is a singular point of the Cayley model. The degree of the Cayley model is equal to 10.

The congruence contains ten conics corresponding to the exceptional curves of the blow-up. The span of each conic is a fundamental plane of the congruence of degree 2. The plane spanned by the image of B in $S = \mathrm{Rey}(W) \subset G_1(\mathbb{P}^3)$ is the unique fundamental point of S. Its degree is equal to 2. The dual congruence of bidegree $(3, 6)$ has 10 fundamental points and one fundamental plane.

Example 11.5.5. Assume $k = 2$. The Reye congruence is of bidegree $(5, 3)$ and sectional genus $g = 4$. The Steinerian surface is a quartic with two nodes P_1, P_2. The discriminant surface $\mathcal{D}(W)$ is a 12-nodal quartic. It is the focal surface of a congruence S of bidegree $(2, 6)$. It has only one set of 10 weakly even nodes. It consists of four fundamental points of degree 4 and six fundamental points of degree three on S. The points P_1 and P_2 are the images of the remaining fundamental points of degrees one and five.

Since Reye congruence $\mathrm{Rey}(W)$ is the normalization of an irreducible component of $\mathrm{Bit}(\mathcal{D}(W))$ of bidegree $(10, 15)$, the quadratic congruence S must be of type $(2, 6)_I$.

The Reye congruence $\mathrm{Rey}(W)$ is isomorphic to the blow down of one (-1)-curve on a Coble surface Y with two boundary components. The surface Y is isomorphic to the blow-up of ten intersection points $q_1, \ldots, q_{10}$ of two curves F_1, F_2 of bidegree $(1, 3)$ and $(3, 1)$ on a quadric $Q \cong \mathbb{P}^1 \times \mathbb{P}^1$ [259, Chapter 9]. The curves F_1 and F_2 are the images of the boundary curves B_1 and B_2. In our situation,

$$\mathrm{Rey}(W) \cong \mathsf{Bl}_{q_1,\ldots,q_9}(Q).$$

The linear system $|\mathcal{O}_Q(3) - q_1 - \ldots - q_{10}|$ maps Y to $\mathbb{P}^5$. We can put the image on a smooth quadric and realize Y as a congruence of bidegree $(4, 4)$ and sectional genus 4.

The Cayley model of Y is given by the linear system $|\mathcal{O}_Q(5) - 2(q_1 + \ldots + q_{10})|$. It maps Y to a surface of degree 10 in $\mathbb{P}^5$ with two singular points equal to the images of the boundary components.

We may assume that $\mathrm{Rey}(W)$ is obtained from Y by blowing down the exceptional curve over q_{10}. Thus

$$\mathrm{Rey}(W) \cong \mathsf{Bl}_{q_1,\ldots,q_9}(Q).$$

It is embedded into $\mathbb{P}^5$ by the linear system

$$|\mathcal{O}_Q(4) - (q_1 + \cdots + q_4) - 2(q_5 + \ldots q_9)|$$

of curves of degree eight and genus four. The image lies on a unique smooth quadric which we can realize as the Grassmanian. The images of B_1 and B_2 in $\mathrm{Rey}(W)$ are (-3) curves representing $|\mathfrak{f}_1 + 3\mathfrak{f}_2 - q_1 \ldots, -q_9|$ and $|3\mathfrak{f}_1 + \mathfrak{f}_2 - q_1 \ldots, -q_9|$. Their images in $\mathbb{P}^5$ are conics.

Projecting from the point q_9, we get another birational model of the Rey congruence:

$$\mathrm{Rey}(W) \cong \mathsf{Bl}_{p_1,\ldots,p_{10}}(\mathbb{P}^2).$$

The linear system that embeds $\mathrm{Rey}(W)$ into $\mathbb{P}^5$ is now

$$|6e_0 - (e_1 + \cdots + e_4) - 2(e_5 + \cdots + e_{10})|.$$

The images of F_1 and F_2 are cubic curves with double points at p_9 and p_{10}. The images of the exceptional curves $E_i, i = 1, \ldots, E_4$, in the congruence $\mathrm{Rey}(W)$ are pencils of lines. Their base points are fundamental points of $\mathrm{Rey}(W)$. The images of the curves $C_i \in |3e_0 - (e_1 + \cdots + e_{10}) + e_i|, i = 1, \ldots, 4$, are plain cubics. They define six fundamental planes of degree 3. The images of the exceptional curves E_i over $p_5, \ldots, p_{10}$, are conics of rays. There are also 15 conics corresponding to the lines passing through two points $p_i, p_j, i, j \geq 5$, and 6 conics corresponding to conics through the points p_5, p_6. They define 27 fundamental planes of degree two. The dual congruence $\mathrm{Rey}(W)^*$ is of bidegree $(3, 5)$. It has 6 fundamental points of degree 2 and 4 fundamental points of degree 3. It has 4 fundamental planes of degree one.

Let θ_i be ten lines in $\mathrm{St}(W)$, the images of the singular lines of 10 reducible quadrics in W corresponding to a weakly even set of nodes of $\mathcal{D}(W)$. Let $\theta_1, \ldots, \theta_4$ correspond to fundamental points of degree two in $\mathcal{D}(W)$ and the rest correspond to fundamental points of degree 3. The line ℓ_{12} is the image of the unique trope-conic corresponding to the fundamental point of degree one. The conjugacy graph $\Gamma(S)$ shows that ℓ_{12} intersects $\theta_1, \ldots, \theta_4$.

The pencils of cubic curves cut out by planes through the lines θ_i are represented in the plane blow- up model of $\mathrm{Rey}(W)$ by $|f_i| = |6e_0 - 2(e_1 + \cdots + e_{10} - e_i)|$. The pencils $|f_i|, i = 1, \ldots, 4$ contain the reducible members represented by $F_1 + F_2 + E_i$. Other six pencils $|f_i|$ contain two members represented by $|3e_0 - (e_1 + \cdots + e_{10}) + e_i|, i = 5, \ldots, 10$.

The net of quadrics containing ℓ_{12} has four addional base points $y_1, \ldots, y_4$. The planes $\Pi_i = \langle y_i, \ell_{12}\rangle$ intersect $\mathrm{St}(W)$ along $\ell_{12} + \theta_i + K_i$, where K_i is a conic containing the nodes P_1, P_2. The conics are invariant with respect to the involution $\mathfrak{r}_W$. Their images in $\mathrm{Rey}(W)$ are the curves E_i. The pencils of rays in $R(W)$ is $\Omega(y_i, \Pi_i)$. The points y_i are the fundamental points of $R(W)$ of degree 1. The plane Π_i is one of the two irreducible components of a reducible quadric from W. The other plane component cuts out in $\mathrm{St}(W)$ the union of θ_i and a plane cubic. Its image in $\mathrm{Rey}(W)$ is a cubic C_i.

Each of the plane components of the reducible quadric with the double line $\theta_i, i = 5, \ldots, 10$, contains one of the nodes of $\mathrm{St}(W)$. It cuts $\mathrm{St}(W)$ along the union of θ_i and a cubic with a node at P_1 or P_2. Its

The elliptic fibrations defined by planes through $\theta_i, i = 1, \ldots, 10$, are represented on $\mathrm{Rey}(W)$ by for base-point-free pencils $|\mathcal{O}_Q(4) - 2\sum_{i=1}^{9} +2q_i|, i = 1, \ldots, 4$ and six pencils $|\mathcal{O}_Q(4) - 2\sum_{i=1}^{9} q_i|, i = 5$. They have the base point at q_{10}. The other six elliptic fibrations are base-point-free pencils $|\mathcal{O}_Q(2) - e_i - e_j - (e_5 + \cdots + e_9)|$.

Example 11.5.6. Assume $k = 3$. The Reye congruence is of bidegree $(4, 3)$ and sectional genus $g = 3$. The Steinerian surface is a 3-nodal quartic. The discriminant surface $\mathcal{D}(W)$ is a 13-nodal quartic Q. It is the focal surface of a congruence S of bidegree $(2, 5)$. There is only one structure of a discriminant surface on $\mathcal{D}(W)$ defined by choosing the set of six fundamental points of degree 2, three fundamental points of degree 3 on S, and one fundamental point of order 1 that is conjugate the other fundamental points of order one.

The images of the remaining two fundamental points of degree 1 are singular points P_1, P_2, and the image of the fundamental point of degree 4 is the point P_3.

The surface $\mathrm{Rey}(W)$ is the normalization of an irreducible component of $\mathrm{Bit}(Q)$ of bidegree $(6, 10)$.

The line ℓ_{23} is the image of the trope-conic corresponding to the fundamental point of degree one conjugate to other fundamental points of degree one. The lines $\ell_{12} = \langle P_1, P_2\rangle$ and $\ell_{13} = \langle P_1, p_3\rangle$ are the images of the other trope-conics.

There are ten reducible quadrics in W with the double lines equal to the images of the exceptional curves of the minimal resolution of Q. Let θ_1 be the exceptional curve over the fundamental point of degree one, $\theta_2, \theta_3, \theta_4$ are the exceptional curve over the fundamental point of degree three, and $\theta_5, \theta_6, \theta_7$ and $\theta_8, \theta_9, \theta_{10}$, are the exceptional curves over the fundamental points of degree two. The points in each set of three are pairwise non-conjugate.

The line ℓ_{12} intersects the lines $\theta_1, \theta_2, \theta_3, \theta_4$. The line ℓ_{13} intersects $\theta_1, \theta_5, \theta_6, \theta_7$. The line ℓ_{23} intersects $\theta_1, \theta_8, \theta_9, \theta_{10}$. It follows that the plane $\langle P_1, P_2, P_3\rangle$ intersects $\mathrm{St}(W)$ along the union of the lines ℓ_{ij} and the line θ_1.

The Reye congruence $\mathrm{Rey}(W)$ is isomorphic to the blow-down of three (-1)-curves on the Coble surface Y with three boundary components C_1, C_2, C_3. Each (-1)-curve intersects two components. The surface Y is the blow-up of $\mathbb{P}^2$ at 12 nodes $p_1, \ldots, p_{12}$ of the union of three conics K_1, K_2, and K_3 [259, Chapter 9]. The Reye congruence is obtained by blowing down E_1, E_2, and E_3. We have an isomorphism

$$\mathrm{Rey}(W) \cong \mathsf{Bl}_{p_1,\ldots,p_9}(\mathbb{P}^2).$$

The images of the boundary components are three conics F_1, F_2, F_3. Each conic passes through 2of the points p_{10}, p_{11}, p_{12}. Together with the points $p_1, \ldots, p_{12}$ the conics form an abstract configuration $(3_8, 12_2)$. The linear system that defines the Plücker embedding is equal to

$$|4e_0 - e_1 - \cdots - e_9|.$$

The conics B_1, B_2, B_3 are the images of F_1, F_2, F_3 under the map given by this linear system. There are 10 elliptic pencils $|f_i|$ on $R(W)$ corresponding to the pencils of planes through θ_i. The pencil $|f_1|$ is equal to the image of the pencil $|6e_0 - 2(e_1 + \cdots + e_9)|$. One of its members is the union of three conics. It is a Halphen pencil of index 2 (see Subsection 3.2.2). The image of the cubic from $|3e_0 - e_1 - \cdots - e_9|$ spans a fundamental plane of degree 3.

The other fiber is the image of the cubic curve from $|3e_0 - (e_1 + \cdots + e_9)|$ taken with multiplicity two. The other nine pencils are the images of the linear systems $|3e_0 - (e_1 + \cdots + e_9) + e_i|$. Their ninth base point is among the points p_{10}, p_{11}, p_{12}.

Each of the three nets of quadrics containing one of the lines ℓ_{ij} has three isolated base points $q_{ij}^{(s)}, s = 1, 2, 3$. The pencil of rays $\Omega(q_{ij}^{(s)}, \ell_{ij})$ through $q_{ij}^{(s)}$ correspond to the exceptional curves over the points $p_1, \ldots, p_9$.

Thus, $R(W)$ has nine fundamental points of degree one and nine fundamental planes of degree one. It also has $\binom{9}{2}$ conics equal to the proper transforms of the $\langle p_i, p_j\rangle$, $1 \le i < j \le 9$, and three conics C_1, C_2, C_3. This gives $36 + 3 = 30$ fundamental planes of degree 2.

Example 11.5.7. Assume $k = 4$. The congruence is of bidegree $(3, 3)$. We know that the sectional genus $g = 2$. This agrees with Example 11.1.11. The Steinerian surface $\mathrm{St}(W)$ is a 4-nodal quartic surface, and the discriminant surface $\mathcal{D}(W)$ is a 14-nodal quartic surface Q. It is the focal surface of a congruence S of bidegree $(2, 4)$ without fundamental curves. There are two structures of a discriminant quartic surface on Q. They correspond to a choice of ten nodes, the first corresponds to the fundamental point w_1 of degree 3, the next three correspond to pairwise non-conjugate fundamental points w_2, w_3, w_4 which are conjugate to w_1. and the remaining six correspond to the fundamental points of degree two.

The singular points P_1, P_2, P_3, P_4 of Q are the images of the remaining four fundamental points of S, three of order 1 and one of order three, respectively.

The images of six trope-conics are the lines $\ell_{ij} = \langle P_i, P_j\rangle$. The conjugacy graph $\mathrm{Fund}(S)_1$ is a hexagon, so two incident vertices of this graph correspond to two incident lines ℓ_{ij}.

Let $\theta_1, \ldots, \theta_{10}$ be the double lines of reducible quadrics corresponding in this order to the nodes $q_1, \ldots, q_{10}$ of $\mathcal{D}(W)$. It follows from the known conjugacy graph of S that the lines $\theta_1, \theta_2, \theta_3, \theta_4$ lie in the faces of the tetrahedron T with vertices at $P_1, \ldots, P_4$. The remaining lines θ_i intersect by pairs two opposite edges of the tetrahedron. They are double lines of reducible quadrics with the plane components $\langle \theta_i, \ell_{ab} \rangle$ and $\langle \theta_i, \ell_{cd} \rangle$, where θ_i intersects the opposite edges ℓ_{ab} and ℓ_{cd}.

The lines $\theta_1, \ldots, \theta_4$ are the double lines of reducible quadrics equal to the union of the face of T containing the line and the plane Π_i containing the opposite vertex of T. The net N_{ij} of quadrics from W that contain the line ℓ_{ab} contains two base points x_{ab}, x'_{ab} besides the base points y_c, y_d. The 12 points p_{ab}, p'_{ab} are fundamental points of degree one of $\mathrm{Rey}(W)$. The corresponding pencils of Reye lines are lines in the plane $\langle \ell_{ab}, p_{ab} \rangle$ (or $\langle \ell_{ab}, p'_{ab} \rangle$) passing through p_{ab} (or p'_{ab}). The planes are fundamental planes of degree one.

The Coble surface Y has four boundary components B_1, B_2, B_3, B_4. It is isomorphic to the blow-up of 13 double points of a reducible sextic curve equal to the union of two smooth conics and two lines. We have

$$\mathrm{Rey}(W) \cong \mathsf{Bl}_{p_1,\ldots,p_7}(\mathbb{P}^2),$$

i.e., the Reye congruence is isomorphic to a del Pezzo surface of degree two. The images of the boundary components B_i in $\mathrm{Rey}(W)$ are the proper transforms of two conics K_1, K_2 containing p_1, p_2, p_3, p_4, p_5 and p_1, p_2, p_3, p_6, p_7, respectively, and two lines $\langle p_4, p_6 \rangle$ and $p_5, p_7 \rangle$. The four curves intersect pairwise at one point outside the set $\{p_1, \ldots, p_7\}$. The additional blow-up of these six points is the Coble surface Y.

The linear system that maps $\mathrm{Rey}(W)$ to the Grassmannian is

$$|4e_0 - 2e_1 - e_2 - \cdots - e_7|. \tag{11.58}$$

The 12 pencils of Reye lines from above are the pre-images of the six lines $\langle p_1, p_i \rangle$. They intersect at the points corresponding to the Reye lines $\langle p_{ab}, p'_{ab} \rangle$.

The Reye congruence $\mathrm{Rey}(W)$ has 10 pencils of elliptic curves of degree 6 in $\mathbb{P}^5$. They correspond to the pencils of planes through the lines θ_i. Four of them are the proper transforms of 4 pencils of 5-nodal quintics passing through the points p_i, with nodes at p_1, p_2, p_3, and two other nodes, one is from the set $\{p_4, p_5\}$ and the other one from the set $\{p_6, p_7\}$.with first four pencils $|f_i|$ are the pencils The remaining pencils are the proper transforms of 2-nodal quartics passing through the points p_i, with two nodes from the set $\{p_4, p_5, p_6, p_7\}$.

Remark 11.5.8. The image of the embedding of a del Pezzo surface of degree 2 defined by the linear system (11.58) is known in the modern literature as a *Castelnuovo surface* (probably due to his extensive work on surfaces with sectional genus ≤ 3). The author has never encountered this terminology in the classical literature.

Example 11.5.9. Assume $k = 5$. The congruence coincides with the quadratic congruence S of class $n = 3$ with isolated fundamental points.

The isomorphism $\mathrm{Rey}(W) \cong S$ depends on a choice of the focal surface $\Phi(S)$ as the discriminant surface $\mathcal{D}(W)$. There are six such realizations corresponding to the set of six weakly even sets of 10 nodes. The explicit isomorphism from $\mathrm{Rey}(W)$ to the corresponding irreducible component of $\mathrm{Bit}(\Phi(S))$ is given by (11.56). We refer to [255] for more details about the geometry of a 15-nodal quartic surface $\mathcal{D}(W)$.

The Coble surface Y has five boundary components $B_1, \ldots, B_5$. It is obtained as the blow-up of 14 nodes of a reducible plane sextic equal to the union of four lines and a conic. The Reye surface is obtained from Y by blowing down ten (-1)-curves each intersecting two boundary components. We have

$$\mathrm{Rey}(W) \cong \mathsf{Bl}_{p_1,p_2,p_3,p_4}(\mathbb{P}^2).$$

This agrees with our description of S as a quintic del Pezzo surface.

Example 11.5.10. Assume $k = 6$. The Reye congruence is of bidegree $(1, 3)$. The discriminant surface $\mathcal{D}(W)$ is a Kummer 16-nodal quartic surface. The map ν_W maps $\mathrm{Rey}(W) \cong \mathbb{P}^2$ to one of the fifteen irreducible components of $\mathrm{Bit}(\mathcal{D}(W))$ isomorphic to a plane. The Reye involution coincides with the deck involution of the familiar representation of a Jacobian Kummer surface as the double cover of the plane branched along six lines $l_1, \ldots, l_6$

The Coble surface Y has 6 boundary components, it is isomorphic to the blow-up of 15 double points of the union $l_1 + \ldots + l_6$.

11.5.2 Projections of congruences of lines in $\mathbb{P}^4$

In this subsection, we discuss Castlenuovo's constructions of congruences of order 3 [94], [95].

Let $\Lambda_1, \Lambda_2, \Lambda_3$ be three disjoint planes in $\mathbb{P}^4$ and $\mathcal{P}_i$ be three pencils of hyperplanes with base locus Λ_i. We fix an isomorphisms $\alpha_i : \mathbb{P}^2 \to \mathcal{P}_i$ and consider a family of lines in $G_1(\mathbb{P}^4)$, the closure of the set of lines of the form

$$\gamma(t) = \alpha_1(t) \cap \alpha_2(t) \cap \alpha_3(t), \quad t \in \mathbb{P}^2.$$

The image of the rational $\gamma : \mathbb{P}^2 \dashrightarrow G_1(\mathbb{P}^4)$ is a congruence (a surface) Γ of lines in $\mathbb{P}^4$ (see Example 10.1.6). Its cohomology class in $H^2(G_1(\mathbb{P}^4), \mathbb{Z})$ is equal to $m\sigma_{3,1} + n\sigma_{2,2}$, where $\sigma_{3,1}$ (resp. $\sigma_{2,2}$) is the cohomology class of the Schubert variety of lines contained in 3-dimensional subspace and containing a fixed point in it (resp. contained in a plane). The number m (resp. n) is called the *order* (resp. *class*) of the congruence. It is equal to the number of rays of the congruence that intersect a fixed general line (resp. contained in a fixed general hyperplane). As in the case of congruences of lines in $\mathbb{P}^3$, the sum $m+n$ is equal to the degree of the congruence in the Plücker embedding $G_1(\mathbb{P}^4) \hookrightarrow \mathbb{P}^9$.

In coordinates, the line $\gamma(t), t = [0, t_1, t_2]$ corresponds to the null space of a system of linear equations

$$\sum_{j=0}^{3} a_{ij}(t_0, t_1, t_2)x_j + \cdots + a_{04}(t_0, t_1, t_2)x_4 = 0, \quad i = 0, 1, 2,$$

where a_{ij} are linear forms. We can rewrite these equations in the form

$$L_{i0}(x_0, \dots, x_4)t_0 + L_{i1}(x_0, \dots, x_4)t_1 + L_{i2}(x_0, \dots, x_4)t_2 = 0, \quad i = 0, 1, 2,$$

where L_{ij} are linear forms in $x_0, \dots, x_4$. It shows that the image of the map γ is a determinantal cubic hypersurface V_3 in $\mathbb{P}^4$ expressed by the vanishing of the determinant of the matrix (L_{ij}). It follows from Theorem 4.1.1 that V_3 has six singular points $p_1, \dots, p_6$ corresponding to the condition that the corank of the matrix larger than 1. The points are in a special linear position: each subset of five nodes spans a hyperplane. Nodal cubic hypersurfaces in $\mathbb{P}^4$ were extensively discussed in Segre's memoir [688]. We refer to a modern treatment in [250] that is especially concerned with 6-nodal and 10-nodal cubic threefolds.

The blow-up of V_3 at one of the nodes is isomorphic to the blow-up of $\mathbb{P}^3$ along the union of two curves C_1 and C_2 of bidegrees $(2, 1)$ and $(1, 2)$ on a smooth quadric Q. It is equal to the image of the rational map $\mathbb{P}^3 \dashrightarrow \mathbb{P}^4$ given by the linear system of cubic surfaces containing the curve $C_1 + C_2$. The linear subsystem that consists of the quadric Q corresponds to the node of V_3. It is clear that the congruence S defined above coincides with the Fano surface of lines in V_3. It is explained in loc. cit. that it consists of three irreducible components F_1, F_2, and F_3. The first two components are of bidegree $(3, 6)$. They arise from secants of C_1 and C_2 and isomorphic to the third Veronese surface $\mathsf{v}_3(\mathbb{P}^2)$ embedded in $\mathbb{P}^9$. The third component F_3 arises from lines intersecting both C_1 and C_2. It is of bidegree $(12, 15)$. The total bidegree is $2(3, 6) + (12, 15) = (18, 27)$. It is a reducible surface in $\mathbb{P}^9$ of degree 45. The same degree as the Fano surface of lines of a general cubic threefold.

Fix a general hyperplane H in $\mathbb{P}^4$ and consider the map $\mathbb{P}^4 \dashrightarrow \mathbb{P}^4$ given by the linear system $|\mathcal{O}_{\mathbb{P}^4}(3) - 2p_1 - \cdots - 2p_6|$ of cubic hypersurfaces with nodes at the points $p_1, \dots, p_6$. Our cubic $V_3(4)$ is a member of this linear system, and hence, it is equal to the pre-image of a hyperplane. Each rational cubic through $p_1, \dots, p_6$ intersects a general member only at the points p_i, hence the image of the map coincides with the Segre cubic primal $\mathsf{S}_3 \subset \mathbb{P}^4$. The lift of the map to the blow-up of the six points, the lines $\langle p_i, p_j\rangle$, and the planes $\pi_i, p_j, p_k\rangle$ is equal to the composition

$$\overline{M}_{0,7} \to \overline{M}_{0,6} \to \mathsf{S}_3,$$

where $\overline{M}_{0,n}$ denote the moduli space of stable rational n-marked points on their Kapranov's realizations as the blow-up of $\mathbb{P}^{n-3}$, the first map is the projection map that forgets the last point, and the second map is the lift of the familiar map $\mathbb{P}^3 \to \mathsf{S}_3$ given by the linear system $|\mathcal{O}_{\mathbb{P}^3}(2) - p_1 - \ldots - p_5|$. It follows that our cubic hypersurface is the pre-image of a hyperplane section F of S_3. It is proven in [250] that the cubic surface F is isomorphic to the blow-up of the quadric Q at five intersection points $C_1 \cap C_2$ on the quadric. The hyperplane H cuts out each cubic cone over C_i at a rational normal curve γ_i. The lines on the cubic surface $H \cap V_3$ are divided in two sets 6 lines intersecting γ_1 at two points, 6 lines intersecting γ_2 at two points, and 15 lines intersecting both γ_1 and γ_2 [250, Lemma 3.3].

Let x_0 be a point in $\mathbb{P}^4 \setminus V_3(6)$. Consider the projection $p_{x_0} : \mathbb{P}^4 \dashrightarrow \mathbb{P}^3$. Each congruence of lines F_1, F_2, F_3 is projected to a congruence S_i of lines in $\mathbb{P}^3$. A general line ℓ containing x_0 in $\mathbb{P}^4$ intersects the cubic V_3 at three points. Each of the three points lies in one of the rays from F_1, F_2. This shows that the projection of ℓ is a point on three rays from S_1, S_2. Thus, the order of S_i is equal to 3. A general hyperplane H containing x_0 intersects V_3 along a cubic surface. By above, it contains six lines from S_1, S_2. hence it contains six rays of F_1, F_2. They are projected to six rays of S_1, S_2. Thus, the class of S_1, S_2 is equal to six. This gives us a construction of a congruence of lines in S_3 of bidegree $(3, 6)$.

Let us look at the projection of F_3. The line ℓ intersects 12 rays from F_3, four passing through each intersection point of ℓ with V_3. The hyperplane H contains 15 lines on the surface $H \cap F_3$. Thus the projection of F_3 is a congruence of bidegree $(12, 15)$.

We denote by S one of the congruences S_1, S_2. Since a general hyperplane section of the Veronese surface $\mathsf{v}_3(\mathbb{P}^2)$ is equal to 1, we obtain that the sectional gebys of S_1 is equal to 1. Thus $\deg(\Phi(S)) = 6$ and the rank $r = 9$. Note that the congruence S is different from the dual of a Reye congruence of bidegree $(6, 3)$. The focal surface of the latter congruence is of degree $2 \cdot 6 + 8 = 20$.

One can degenerate the construction of the congruence S by allowing the cubic V_3 to acquire an additional number $k \leq 4$ of nodes [95]. In terms of the projective generation, this means that there will be k points t_i in $\mathbb{P}^2$ such that the planes $\alpha_i(t_i)$ intersect along a plane Π_i. This leads to congruences of bidegree $(3, 6 - k)$. The Fano surface of V_3 acquires k planes that consist of lines in the planes Π_i.

11.5.3 Cremonian congruences

In this subsection, we will discuss a construction of congruences using Cremona transformation between two fixed planes in $\mathbb{P}^3$ [395], [397].

Fix two planes Π_1 and Π_2 and isomorphisms $\alpha_i : \Pi_i \to \mathbb{P}^2$. Let T be a Cremona transformation of $\mathbb{P}^2$ of algebraic degree d with finitely many fixed points. By Corollary 7.2.14, the number of fixed points is equal to $d+2$. It defines a birational map $\phi = \alpha_2^{-1} \circ T \circ \alpha_1 : \Pi_1 \dashrightarrow \Pi_2$. Define a congruence S_T of lines in $\mathbb{P}^3$ to be the closure of the set of lines $\langle x, T(x)\rangle, x \in \Pi_1$. Following Hirst, we call S_T a *Cremoniam congruence*.

Proposition 11.5.11. *The bidegree of a Cremonian congruence S_T of lines is equal to $(d+2, d)$. The line $\ell_0 = \Pi_1 \cap \Pi_2$ is a singular point of S_T of multiplicity d. The planes Π_1, Π_2 are fundamental planes of S_T of degree d.*

Proof Take a general line ℓ in Π_1. The image of $\alpha_1(\ell)$ under T is a curve of degree d. Thus the image of ℓ under ϕ is a curve C of degree d in Π_2. A general plane Π intersects C at d points, so there will be d rays of S contained in Π. They connect the intersection points with their pre-images under ϕ. Thus, the order of S is equal to d.

Take a general point $p \in \mathbb{P}^3$. Assume $p \in \langle x, \phi(x)\rangle$ for some $x \in \Pi_1$. Projection from p to Π_1 and Π_2, we get a correspondence $\Gamma \subset \mathbb{P}^2 \times \mathbb{P}^2$ isomorphic to the graph Γ_T. The order of S_T is equal to the intersection number of Γ_T with the diagonal. It is equal to the number $d+2$ of fixed points.

Let $x \in \ell_0 \cap \phi(\ell_0)$, then $y = \phi^{-1}(x) \in \ell_0$ and hence $\langle x, y\rangle = \ell_0$. So, ℓ_0 is a ray of the congruence. Since the curve $C(\ell_0) \subset G_1(\mathbb{P}^3)$ of rays intersecting ℓ_0 intersects d other rays, the ray is singular of multiplicity d (see Section 1.1). The pre-image $\alpha^{-1}(\ell_0)$ is a curve in Π_1 of degree d. The fundamental curve consists of rays $\langle x, \phi(x)\rangle, x \in \alpha^{-1}(\ell_0)$. Its degree is equal to d.

□

Let $p_1, \ldots, p_k$ (resp. $q_1, \ldots, q_k$) be the fundamental points of T (resp. T^{-1}), and $x_1, \ldots, x_k$ (resp. $y_1, \ldots, y_k$) be the corresponding points in Π_1 (resp. Π_2). The points x_i (resp. y_i) are fundamental points of S_T. The degree $h(x_i)$ (resp. $h(y_i)$ is equal to the degree of the principal curve of T^{-1} (resp. T) over x_i (resp. y_i).

We omit the proof of the next proposition (see [395]).

Proposition 11.5.12.

$$\deg(\Phi(S_T)) = 4d, \quad \deg(\Phi(S_T^*) = 4(n-1).$$

Example 11.5.13. Assume T is a projective transformation. Then, $\phi : \Pi_1 \to \Pi_2$

is a projective isomorphism. Choose projective coordinates such that $\Pi_1 = V(t_0), \Pi(t_1)$. The we can extend the map ϕ to a projective transformation $\tilde{\phi}$ given by the matrix

$$\begin{pmatrix} 0 & a_1 & b_1 & c_1 \\ 1 & 0 & 0 & 0 \\ 0 & a_2 & b_2 & c_2 \\ 0 & a_3 & b_3 & c_3 \end{pmatrix}$$

The matrix has 4 eigenvectors corresponding to 4 fixed points $p_1, \ldots, p_4$ of $\tilde{\phi}$. Two of them lie on the line $\ell_0 = \Pi_1 \cap \Pi_2$. Each fixed point on ℓ_0 is a fundamental point of S_T. The cone $K(x)$ is the union of two planes $\Pi_1 + \Pi_2$.

Recall from Subsection 10.3.6 that the closure of the set of lines $\langle x, \tilde{\phi}(x)\rangle$ in $G_1(\mathbb{P}^3)$ is a tetrahedral line complex $\mathfrak{G}$. The blow-up $\mathsf{Bl}_{p_1,\ldots,p_4}(\mathbb{P}^3)$ is a small resolution of $\mathfrak{G}$, the map is given by the linear system of quadrics with base points $p_1, \ldots, p_4$. The image under $\tilde{\phi}$ of any curve C in $\mathbb{P}^3$ not containing any points p_i is a curve of degree s, its image is a curve of degree $2s$ in $\mathfrak{G}$.

11.5.4 Smooth congruences of low degree

The assumption of smoothness allows us to use the classification of algebraic surfaces, and, in particular surfaces embeddable as smooth surfaces of small degree in $\mathbb{P}^5$. One of the main tools is the theory of liasons and the theory of ample rank 2 vector bundlea that embedd surfaces into $G_1(\mathbb{P}^3)$. This approach was undertaken at the end of the last century by many geometers that led to the classification of smooth congruences of degree $m + n \leq 10$ [15], [14], [362], [363], [364], [787].

In the case where S is a smooth surface, the following proposition from [385, p.129] relates the bidegree of S, sectional genus g and the standard invariants of smooth surface as K_S^2 and $\chi(S, \mathcal{O}_S)$.

Proposition 11.5.14. *Let S be a smooth congruence of order (m, n) and let g be its sectional genus. Then,*

$$m^2 + n^2 = 3(m + n) + 8(g - 1) + 2K_S^2 - 12\chi(\mathcal{O}_S). \tag{11.59}$$

Proof Using the intersection theory on $\mathbb{G}$, we obtain $[S]^2 = m^2 + n^2$. On the other hand, this number is equal to the second Chern class of the normal sheaf $\mathcal{N}_{S/\mathbb{G}} = (\mathcal{I}_S/\mathcal{I}_S^2)^\vee$ of S in $\mathbb{G}$. The standard exact sequence

$$0 \to \mathcal{I}_S/\mathcal{I}_S^2 \to \Omega^1_{\mathbb{G}} \otimes \mathcal{O}_S \to \Omega^1_S \to 0, \tag{11.60}$$

after passing to the dual exact sequence and taking the Chern classes, gives

$$c_1(\mathcal{N}_{S/\mathbb{G}}) = -K_{\mathbb{G}} \cdot S + K_S = -4 \deg c_1(\mathcal{O}_S(1)) + K_S = 4c_1(\mathcal{O}_S(1)) + K_S,$$
$$c_2(\mathcal{N}_{S/\mathbb{G}}) = c_2(\mathbb{G}) \cdot S - c_2(S) + K_S \cdot c_1(\mathcal{N}_{S/\mathbb{G}}).$$

The second Chern class of the quadric $\mathbb{G}$ in $\mathbb{P}^5$ is computed using th exact sequence

$$0 \to \Theta_{\mathbb{G}} \to \Theta_{\mathbb{P}^5} \otimes \mathcal{O}_{\mathbb{G}} \to \mathcal{O}_{\mathbb{G}}(2) \to 0.$$

It yields $c_2(\mathbb{G}) = 7c_1(\mathcal{O}_{\mathbb{G}}(1))^2$. Next, we apply the Noether formula $c_2(S) + K_S^2 = 12\chi(\mathcal{O}_S)$ and obtain

$$\begin{aligned} c_2(\mathcal{N}_{S/\mathbb{G}}) &= 7(m+n) - (12\chi(\mathcal{O}_S) - K_S^2) + 4c_1(\mathcal{O}_S(1)) \cdot K_S + K_S^2 \\ &= 7(m+n) - 12\chi(\mathcal{O}_S) + 2K_S^2 + 4(2g-2) - 4(m+n) \\ &= 3(m+n) - 12\chi(\mathcal{O}_S) + 2K_S^2 + 8(g-1). \end{aligned}$$

□

Remark 11.5.15. It follows from the proof and the Riemann-Roch theorem that

$$\begin{aligned} &\chi(S, \mathcal{N}_{S/\mathbb{G}}) = \tfrac{1}{2}(c_1(\mathcal{N}_{S/\mathbb{G}})^2 - K_S \cdot c_1(\mathcal{N}_{S/\mathbb{G}})^2) - c_2(\mathcal{N}_{S/\mathbb{G}}) + 2\chi(\mathcal{O}_S) \\ &= \tfrac{1}{2}((4h + K_S)^2 - (4h \cdot K_S) \cdot K_S) - (m^2 + n^2) + 2\chi(\mathcal{O}_S) \\ &= 8(m+n) + 2K_S \cdot h - (m^2 + n^2)) = 6(m+n) + 4(g-1) - (m^2 + n^2) + 2\chi(\mathcal{O}_S). \end{aligned} \tag{11.61}$$

Here, $h = c_1(\mathcal{O}_S(1))$ and we used the adjunction formula $h \cdot K_S + h^2 = 2g - 2$. If we assume that $h^i(\mathcal{N}_{S/\mathbb{G}}) = 0, i = 1, 2$, i.e., the deformations of S inside $\mathbb{G}$ are unonstruced, the formula gives the dimension of the irreducible component of the Hilbert scheme of $\mathbb{G}$ that contains S. Subtracting $15 = \dim \mathrm{Aut}(\mathbb{G})$ this gives the expected number of moduli of the congruence S.

For example, if S is a smooth quadratic congruence with isolated fundamental points of class $n = 2, 3$, we obtain $\chi(S, \mathcal{N}_{S/\mathbb{G}}) = 18, 19$, and subtracting 15, we get $3, 4$ moduli that agrees with the number of moduli of such congruences. We come to the same number of moduli if we use the realization of these congruences as the Reye congruences of a web of quadrics with $k = 5, 6$ base points.

Smooth congruences of lines may have a fundamental curve. If Fund(S) contain a curve $F$of degree $d > 1$, then the corresponding component of $R(S)$ does not contain fibers of $q_S : Z_S \to S$, hence $q_s : R(S) \to S$ is an isomorphism. In particular, F is smooth and S admits a fibration $S \to F$ with the degree of a fiber equal to $h(x)$, where x is a point on F. Examples are linear congruences from Theorem 11.2.5 or quadratic correspondences from Example 11.4.1.

The following theorem from [15] gives the classification of smooth congruences of lines with a fundamental curve.

Theorem 11.5.16. *Let S be a smooth congruence of lines in $\mathbb{P}^3$ that admit a fundamental curve F. Then, one of the following three cases may accir:*

1. *F is a line;*
2. *S is the congruence of secants lines of a rational normal cubic or an elliptic quartic curvee.*
3. *F is a smooth plane curve of degree $d \geq 2$ with $h(x) = e$, for any point $x \in F$, such that there exists an effective divisor D on F satisfying*

$$d(d-1)(e-1) + (1 + 2e - 2ed \deg(D) + \deg(D)^2 = 0. \tag{11.62}$$

The bidegree of S is equal to $(ed - \deg(D), ed)$.

Note that we considered many examples of quadratic congruences with plane fundamental curve C_d. However, in most examples C_d was singular, and the only example with a smooth C_d was considered in Example 11.4.9 with $d = 2$. In fact, the formula (11.62) easily shows that the only solution if $m = 2$ is $D = 0, e = 1, d = 2$, and $n = 2$.

The focal surface of a congruence of lines of order ≥ 3 is a non-normal surface.

We refer to [18] for the proof of the following Proposition.

Proposition 11.5.17. *Let S be a smooth congruence. Assume that the one-dimensional part of $Foc(S)$ is a nodal curve D and a cuspidal curve C. Then*

$$\deg(D) = 2m^2 - 10m + 4n + 4mg + 2g^2 - 34g + 32 - 4K_S^2 + 12\chi(O_S),$$
$$\deg(C) - 3m - 3n + 18g - 18 + 3K^2 - 12\chi(O_S).$$

One can compare the formula for the degree of C with Schumacher's formula (11.39) and use formula (11.59) to express the numerical invariants $\chi(O_S)$ and K_S^2 in terms of m, m, g.

For example, the focal surface of a Reye congruence of bidegree $(3, 3)$ is an octic surface with a cuspidal curve of degree 12 and no double curve. Recall from Subsection 11.1.3 that the normalization of the focal surface is the surface $R(S)$ isomorphic to the double cover of S branched along a curve B from the linear system $|2K_S + 4h|$. For example, in the previous example, S is a del Pezzo surface of degree 2 and $h = 4e_0 - 2e_1 - e_2 \cdots - e_7$. This gives $K_{R(S)} = q_S^*(2K_S + 2h) = q_S^*(2e_0 - 2e_1)$. Assuming B is smooth, we obtain that $R(S)$ is a nonsingular model of $\Phi(S)$ with $K_{R(S)}^2 = 0$ and $p_g = 3$.

In Table 11.4 below we summarize the known classification of smooth congruences of degree ≤ 10. We list only congruence with $m \leq n$.

(m,n)	g	r	$\deg(\Phi(S))$	smooth model	embedding	fund.curve	# fund.points	credit
$(1,1)$	0	0	$\emptyset$	$\mathbf{F}_0$	$(1,1)$	$\ell_1+\ell_2$	0	[473]
$(1,2)$	0	0	$\emptyset$	$\mathbf{F}_1$	$(2,1)$	$\ell+K$	0	[473]
$(1,3)$	1	1	$\emptyset$	Veronese	$[2]$	R_3	0	[473]
$(2,2)$	1	0	4	DP_4	$\lvert -K_S\rvert$	$\emptyset$	(1^{16})	[473]
$(2,2)$	0	1	2	$\mathbf{F}_0$	$\lvert 2\mathfrak{f}_1+\mathfrak{f}_2\rvert$	$\emptyset$	(2^1)	[473]
$(2,2)$	0	1	2	$\mathbf{F}_1$	$\lvert 3\mathfrak{f}+\mathfrak{e}\rvert$	$\emptyset$	(2^1)	[473]
$(2,3)$	1	1	4	DP_5	$\lvert -K_S\rvert$	$\emptyset$	$(2^5,1^{10})$	[473]
$(2,3)$	2	0	2	rational	$[4;2,1^7]$	$\ell+R_3$	0	[473]
$(2,6)$	3	2	8	$E^{(2)}=\mathbb{P}(\mathcal{E})$	$\lvert\pi^*\det(\mathcal{E})\rvert$	E	$(2^4)4$	[473]
$(3,3)$	1	3	6	DP_6	ω_S^{-1}	$\emptyset$	$(2^3,1^6)$	[294],[295]
$(3,3)$	1	3	6	ell.ruled	$\lvert 3\mathfrak{f}+\mathfrak{e}\rvert$	$\emptyset$	$(3^1,1^9)$	[294],[295]
$(3,3)$	2	2	8	DP_2	$[4;2,1^6]$	$\emptyset$	(1^{12})	[294],[295]
$(3,3)$	4	0	12	$V_{1,2,3}$	$\lvert\mathcal{L}_6\rvert$	$\emptyset$	?	
$(3,4)$	3	3	10	rational	$[4;1^9]$	$\emptyset$	$(3^1,1^9)$	[294],[295]
$(3,4)$	6	0	16	elliptic(q=0,$p_g=2$)	$\emptyset$		?	[294],[295]
$(3,5)$	4	4	12	rational	$[6;2^6,1^4]$	$\emptyset$	(3^4)	[294],[295]
$(3,6)$	5	5	14	rational	$[6;2^{10}]$	$\emptyset$	(3^{10})	[295]
$(3,6)$	4	6	12	rational	$[6;2^6,1^4]$	C_3	?	[15]
$(3,7)$	6	6	24	Enriques	$\lvert\mathcal{L}_{10}\rvert$	$\emptyset$	(3^{20})	[614],[295]
$(4,4)$	3	6	12	DP_2	$\lvert -2K_S\rvert$	$\emptyset$	?	[294]
$(4,4)$	4	5	14	rational	$[5;2,2,1^9]$	$\emptyset$	?	[294]
$(4,4)$	5	4	16	$V_{2,2,2}$	$\lvert\mathcal{L}_8\rvert$	$\emptyset$	?	[294]
$(4,4)$	9	0	24	$V_{1,2,4}$	$\lvert K_S\rvert$	$\emptyset$	?	[294]
$(4,5)$	5	7	16	rational	$[6,2^5,1^7]$	$\emptyset$	?	[787]
$(4,5)$	6	6	18	$\mathsf{Bl}_p(K3)$	$\lvert\mathcal{L}_{10}-p\rvert$	$\emptyset$	?	[787]
$(4,6)$	6	9	18	rational	$[7,2^9,1^3]$	$\emptyset$	?	[363]
$(4,6)$	7	8	20	$\mathsf{Bl}_p(K3)$	$\mathcal{L}_{10}$	$\emptyset$	?	[363]
$(5,5)$	4	12	16	ell.ruled		$\emptyset$	?	[363]
$(5,5)$	5	11	18	ell.ruled		$\emptyset$	?	[363]
$(5,5)$	6	10	20	rational	$[7;3,2^6,1^6]$	$\emptyset$	?	[363]
$(5,5)$	6	10	20	ell.ruled		$\emptyset$	?	[363]
$(5,5)$	7	9	22	$\mathsf{Bl}_{p,q}(K3)$	$\lvert\mathcal{L}_{12}-p-q\rvert$	$\emptyset$	?	[363]
$(5,5)$	7	9	22	rational	$[6;2^3,1^{14}]$	$\emptyset$	?	[363]
$(5,5)$	8	9	24	elliptic($q=0,p_g=2$)		$\emptyset$	?	[363]
$(5,5)$	16	0	20	$V_{1,2,5}$	$\lvert\frac{1}{2}K_S\rvert$	$\emptyset$	?	

Table 11.4 *Classification of smooth line congruences of degree* ≤ 10

Here, ℓ, K, R_3, E, C_d stands for a line, a conic, a twisted cubic, an elliptic curve, and a plane curve of degree d, respectively. Also $[d; m_1, \ldots, m_k]$ denotes the linear system of curves of degree d passing through points $p_1, \ldots, p_k$ with multiplicities $\geq m_i$. We abbreviate m^s if the same multiplicity repeats s times. Similar notation is used for the configuration of fundamental points. We denote by $\mathcal{L}_n$ an ample invertible sheaf on a surface with $(\mathcal{L}_n, \mathcal{L}_n) = n$. Also (a, b) denotes the linear system $a\mathfrak{f} + b\mathfrak{e}$ on $\mathbf{F}_n$.

Remark 11.5.18. It was conjectured by the author, and Igor Reider in [236], based on Bogomolov's instability criterion of vector bundles on algebraic surfaces, that for a smooth congruence S not contained in a linear complex of lines,

$m \leq 3n$, and hence $n \leq 3m$. It was checked for congruences of lines, which are surfaces with Kodaira dimension ≤ 1 [362].

Exercises

11.1 An irreducible subvariety X of the Grassmannian $G_r(\mathbb{P}^n)$ is called a *congruence* if $\dim X = n - r$ [691].

(i) Show that the number of r-planes from X passing through a general point (resp. contained in a general hyperplane) is finite. It is called the *order* of X.

(ii) Show that the subvariety of r-planes from X contained in a general hyperplane is an irreducible variety of degree $n - 2r - 1$. Its degree is called the *class* of X.

(iii) Extend to X the notion of a fundamental point and the focal hypersurface, and prove that, if $r = 1$ and X has only finitely many fundamental points, a general ray from X is tangent to the focal hypersurface at $n - 1$ points.

(iv) Using (iii), show that, if $n \geq 4$, the order of a congruence of lines in $\mathbb{P}^n$ is not equal to 2 [311].

11.2 Consider a subvariety $X(N_1, \ldots, N_{n-1})$ of $\mathbb{P}^n$ projectively generated by a general set of nets $N_1, \ldots, N_{n-1}$ of hyperplanes with a fixed isomorphism to $N_i \to \mathbb{P}^{2}$' (see Subsection 3.3.1).

(i) Show that the lines $H_1(\lambda) \cap \cdots \cap H_{n-1}(\lambda), \lambda \in \mathbb{P}^2$, form of a surface $\mathcal{S}(N_1, \ldots, N_{n-1}; n)$ of order $\frac{1}{2}(n-1)(n-2)$ and class $\frac{1}{2}n(n-1)$ of lines in $\mathbb{P}^n$.

(ii) Show that the surface $\mathrm{Sec}^1(R_n)$ formed by secant lines of a rational normal curve R_n in $\mathbb{P}^n$ is an example of a surface $\mathcal{S}(N_1, \ldots, N_{n-1}; n)$. Show that its Plücker embedding is projectively isomorphic to the Veronese surface $\nu_{n-1}(\mathbb{P}^2)$.

(iii) Show that rays of $\mathcal{S}(N_1, \ldots, N_{n-1}; n)$ sweep a 3-dimensional variety F_d of degree $d = \frac{1}{2}(n-1))(n-2)$.

(iv) Assume $n = 4$, show that, for a general $\mathcal{S}(N_1, N_2, N_3; 3)$, the cubic F_3 has six singular points (in particular, a general $\mathcal{S}(N_1, N_2, N_3; 3)$ does not coincide with the congruence $\mathrm{Sec}^1(R_4)$).

(v) Show that the cubic scroll F_3 has two rulings whose generators define two congruence of lines in $\mathbb{P}^4$ of order 3 and class 6

11.3 (i) Show that the projection $\mathbb{P}^4 \dashrightarrow \mathbb{P}^3$ from a point not in F_3 maps the rays of $\mathcal{S}(N_1, N_2, N_3; 3)$ to rays a congruence of lines Γ_6^3 in $\mathbb{P}^3$ of order $n = 3$, class $m = 6$, and sectional genus 4.

(ii) Show that the projection $\mathbb{P}^4 \dashrightarrow \mathbb{P}^3$ from a general point in F_3 maps rays of $\mathcal{S}(N_1, N_2, N_3; 3)$ to rays of a congruence of lines without fundamental curve of bidegree $(2,6)_{II}$ in $\mathbb{P}^3$.

(iii) Find the bidegree of the congruence of lines in $\mathbb{P}^3$ obtained by the projection of $\mathcal{S}(N_1, N_2, N_3; 3)$ from a point on a line ℓ_i.

(iv) Assume that the nets N_1, N_2, N_3 contain k common planes $\Pi_i : H_1(\lambda_i) \cap H_2(\lambda_i) \cap H_3(\lambda_i), i = 1, \ldots, k$. Show that the class of the degenerate congruence $\mathcal{S}(N_1, N_2, N_3; 3)$ is equal to $6 - k$. Show that by projecting these congruences one obtain congruences of lines in $\mathbb{P}^3$ of bidegree $(2, 6-k)$ [95].

11.4 Let C_1, C_2, C_3, C_4 be irreducible curves in $\mathbb{P}^3$ of degrees d_1, d_2, d_3, d_4 intersecting each other transversally at n_{ij} points. Show that the number of lines intersecting the four curves is finite and equal to

$$2d_1d_2d_3d_4 + d_{14}d_{23} + d_{13}d_{24} + d_{14}d_{23} - \sum_{\{i,j\}\cap\{k,l\}=\emptyset} n_{ij}d_{kl}.$$

[735, Theil 1, Art. 9].

11.5 Show that the closure of the set of trisecant lines of a Bordiga sextic surface in $\mathbb{P}^4$ is a congruence of order one and class three.

11.6 Let X be a del Pezzo surface of degree 5 and $\{L_1, \ldots, L_{10}\}$ be the set of lines on X. Let $\mathcal{E} = \Omega^1_X(\log D)$ be the sheaf of logarithmic differential 1-forms on X, where D is the sum of the ten line on X.

(i) Let $\mathcal{E} = \Omega^1_X(\log D)$ be the sheaf of logarithmic differentials that fits in the short exact sequence

$$0 \to \Omega^1_X \to \Omega^1_X(\log D) \xrightarrow{\text{res}} \bigoplus_{i=1}^{10} \mathcal{O}_{L_i} \to 0,$$

where res is the residue map. Show that $\dim H^0(X, \mathcal{E}) = 5$ and $\mathcal{E}$ is generated by its global sections.

(ii) Show that the map $x \mapsto \mathrm{Ker}(H^0(X, \mathcal{E}) \xrightarrow{\text{ev}} \mathcal{E}(x))^\perp$, where ev is the evaluation map, defines a closed embedding $j : X \to G(2, H^0(X, \mathcal{E})^\vee) \cong \mathbb{G} = G_1(\mathbb{P}^4)$ such that $\mathcal{E} = j^*Q_{G_1(\mathbb{P}^4)}$.

(iii) Show that the image S of X is a congruence of lines in $\mathbb{P}^4$ of order 2 and class 3.

(iv) Show that the rays of S sweep a three-dimensional hypersurface $p_{\mathbb{G}}(q_{\mathbb{G}}^{-1}(S))$ in $\mathbb{P}^4$ isomorphic to the Segre cubic primal.

(v) Let $S^* \subset G(3, H^0(S, \mathcal{E})) \cong \mathbb{G}^* = G_2(\check{\mathbb{P}}^4)$ be the image of S under the duality map $\mathbb{G} \to \mathbb{G}^*$. Show that $p_{\mathbb{G}^*} : q_{\mathbb{G}^*}^{-1}(S^*)) \to \check{\mathbb{P}}^4$ is a degree 2 map whose branch divisor is the Castelnuovo-Richmond quartic hypersurface CR_4.

11.7 Let S_1 and S_2 be two congruences of lines in $\mathbb{P}^3$ of order 1 and class 2. Define a Cremona transformation $T : \mathbb{P}^3 \dashrightarrow \mathbb{P}^3$ as follows. A general point $x \in \mathbb{P}^3$ contains in a unique ray ℓ_1 of S_1 and a unique ray ℓ_2 of S_2. Since the class is equal to 2, the plane spanned by ℓ_1 and ℓ_2 contains a unique ray ℓ_1' of S_1 and a unique ray ℓ_2' of S_2 that intersect at a unique point $T(x)$ [603].

(i) Show that T is a Cremona involution.

(ii) Find the F-locus and F-locus of T.

(iii) Find the multidegree of T.

11.8 Let S be the congruence of order two of secant lines of an elliptic quartic curves F in $\mathbb{P}^3$.

(i) Show that, for a general point $P \in \mathbb{P}^3$, the surface (P) of centers of the null-planes containing P coincides with the cubic surface from Exercise 9.25. It associated to the point P and the pencil of quadrics $|I_F(2)|$ containing the curve F.

(ii) Show that the curve $|\ell|$ of centers of null-planes containing ℓ is the union of the curve F taken with multiplicity 3 and a curve R of degree 5 taken with multiplicity one.

(iii) Show that the curve R passes through the tangency points of planes from $\ell^\perp$ and quadrics from the pencil $|I_F(2)|$.

(iv) Find the genus of the curve R.

11.9 Show that congruences S^* dual to singular congruences S of bidegree $(1,3)$ and $(2,4)$ are Cremonian congruences.

11.10 Let S be a congruence of lines in $\mathbb{P}^3$ of order one and let Π, Π' be two general planes. Show that the rational map $T : \Pi \dashrightarrow \Pi'$ that assigns to a general point $x \in \Pi$ the intersection of the unique ray of S passing through x with the plane Π' is a birational transformation. Find its degree and fundamental points.

11.11 Let S be a congruence of lines in $\mathbb{P}^3$ of bidegree $(3,3)$ without fundamental curve.

(i) Show that its arithmetical sectional genus p_a is less than or equal to 4 and the equality holds only if S is a complete intersection of a cubic and linear complexes of lines.

(ii) Show that a smooth congruence S with $p_a = 4$ is a K3 surface.

(iii) Show that the focal surface of a general congruence S with $p_a = 4$ is a surface of degree 12 isomorphic to its dual surface.

(iv) Show that a smooth S with sectional genus one is equal to the projection of an anti-canonical del Pezzo surface of degree six [294].

11.12 Let S be a general congruence of lines in $\mathbb{P}^3$ of bidegree $(3,4)$ and arithmetic sectional genus equal to 3.

(i) Show that S is equal to the residual surface of the intersection of two quadratic line complexes containing a common plane Λ.

(ii) Show that the intersection $S \cap \Lambda$ is a curve C of degree 3.

(iii) Show that S is a rational surface, the image of $\mathbb{P}^2$ under a map given by the linear system of curves of degree four passing trough a general set of nine points and the curve C is the image of the unique cubic curve through the nine points [294].

11.13 Let S be a general congruence of lines in $\mathbb{P}^3$ of bidegree $(3,5)$ and arithmetic sectional genus equal to 4.

(i) Show that S is contained in the intersection of a quadratic line complex and a cubic quadratic complex with the residual surface of order 4.

(ii) Show that the residual surface of a smooth S is a congruence of bidegree $(2,2)$ isomorphic to a Veronese surface, and S is a rational surface equal to the image of $\mathbb{P}^2$ under a rational map given by a linear system of curves of degree 6 with 6 double base points and four simple base points [294].

11.14 Show that the intersection of two line complexes of degree d_1 and d_2 is a congruence of bidegree (d_1d_2, d_1, d_2) and of rank $d_1d_2(d_1-1)(d_2-1)$.

11.15 Let S be a congruence of order two without fundamental curve and S' be a confocal congruence. Find the intersection of the corresponding irreducible components of the bitangent surface $\mathrm{Bit}(S)$.

11.16 Let $\mathcal{R}_3$ be the set of stable Veronese curves of degree 3 passing through the five reference points p_i.

(i) Show that there are ten curves from $\mathcal{R}_3$ that intersect two general lines in $\mathbb{P}^3$.

(ii) Show that the union of curves in $\mathcal{R}_3$ which are tangent to a general plane Π is a surface F_Π of degree 10.

(iii) Show that the reference points are 6-fold singular points of F_Π and the lines $\langle p_i, p_j\rangle$ are double lines of F_Π.

11.17 Let Π be a plane in $\mathbb{P}^3$ and W a web of plane curves of degree d with with simple base points $p_1, \ldots, p_r$ of multiplicities $m_1, \ldots, m_r$. Fix a basis in W to assume that W defines a rational map $f : \Pi \to \mathbb{P}^3$.

(i) Show that the closure of the set of lines $\langle x, f(x)\rangle$, where $x \notin \{x_1, \ldots, x_r\}$, is a congruence S_W of bidegree $(m, n) = (N + d + 1, d)$.

(ii) Show that, for $N = 2, 3, 4$ and $d = 2$, the dual congruence S_W^* coincides with a quadratic congruence $(2, 5), (2, 6)_{II}, (2, 7)$ without fundamental curves.

[83]

11.18 Let $\mathcal{R}_3$ be the family of stable rational normal curves of degree 3 with 5 marked points with the universal family isomorphic to $\overline{M}_{0,6} \to \overline{M}_{0,5}$. We identify curves from $\mathcal{R}_3$ with stable rational curves of degree 3 in $\mathbb{P}^3$ passing through the reference points $p_1, \ldots, p_5$. The space $\overline{M}_{0,5}$ isomorphic to a smooth del Pezzo surface of degree 5 is identified with one of the irreducible components of the Fano surface of lines of the Segre cubic primal.

(i) Let Π be a general plane. Show that the set of tangency points of curves from $\mathcal{R}_3$ with Π is a conic $K(\Pi)$ in Π (the *Reye conic* [615]).

(ii) Let ℓ be a line in $\mathbb{P}^3$ which is not contained in any plane $\langle p_i, p_j, p_k\rangle$. Show that all curves $\mathcal{R}_3(\ell)$ from $\mathcal{R}_3$ intersecting ℓ form a hyperplane section of $\overline{M}_{0,5}$ in its Plücker embedding that coincides with its anticanonical embedding. Show that there is a unique curve from $\mathcal{R}_3$ that intersects ℓ at two points and this curve is the unique singular point of the curve $\mathcal{R}_3(\ell)$.

(iii) Show that the closure of the set of tangent lines to smooth curves from $\mathcal{R}_3$ is a complex of lines of degree 6.

(iv) Show that the set of tangent lines of curves from $\mathcal{R}_3$ at their intersection points with a general plane Π is a congruence of lines of bidegree $(7, 2)$ isomorphic to the dual of the quadratic congruence $(2, 7)$ without fixed curves.

(v) Show that, taking planes passing through $k = 1, 2, 3$ of the points $p_1, \ldots, p_5$ we btain a congruence of bidegree $(2, 7 - k)$ dual to congruences $(2, 7 - k)$ without fundamental points.

[735, Theil 2, Art. 459].

11.19 Show that the bidegree of the congruence of secant lines of a non-plane irreducible smooth curve C of degree d and genus π is equal to $(\frac{1}{2}(d-1)(d-2) - \pi, \frac{1}{2}d(d-1))$ and its sectional genus g is equal to $\frac{1}{2}(d-2))(d-) + 2\pi)$.

Historical Notes

The main sources for the theory of congruences of lines are [429], [735, Theil II], [820, §§42–54]. From an analytical point of view, a ray in $\mathbb{R}^3$ is defined by six numbers $(x, y, z, \xi, \eta, \zeta$, where (x, y, z) are the coordinates of a point on it, and (ξ, η, ζ) the cosines of the angles between the ray and the coordinate axes. A congruence is a two-parametric family of rays in $\mathbb{R}$. The analytical theory of rays played an important role in optics and differential geometry [13] of surfaces (e.g by consideration the congruence of normals of a surface), we refer to Kummer's work in 1860 [470] where he gives a historical account and also gives an exposition and further development of this theory.

In a large memoir [473] , Kummer gave the first foundation of the algebraic theory of congruences of lines in $\mathbb{P}^3$. Kummer used *Strahlensysteme* for a congruences of line, they became later *Strahlenkongruenzen*. For exsmple, Kummer was the first to introduce the notions of the order (Ordnung), class (Klass), focal surface (Brennfläche), singular point (singuläre Punkt)(=fundamental point) and its degree, singular curve (singuläre Linien, or Leitkurven, or Brenncurve) (=fundamental curve), and singular plane (singulr̈e Ebene) (=fundamental plane). Further development of the general theory of line congruences was given in a Münich dissertation of R. Schumacher in 1885 that appeared in his paper [669] in 1890. He was the first to introduce the rank r, he called it Art (changed to Rang by Sturm) and some other characteristics of a line congruence. In the same memoir Schumacher introduced the Triple Fläche and computed its degree (assuming, as always, not expicitly stated generality condition). Schumacher also computes the degree and the class of the focal surface, the degree of the ruled surface whose generators are rays tangent to the focal surface with multiplicity four. The sectional genus g of a line congruence was introduced by Fano [294] in 1893. The three Parts of Sturm's treatise [736] is the most comprehensive treatise on the line geometry that uses a synthetic approach (in words of Jessop: it is a storehouse of information). In particular, volume 2 is entirely devoted to congruences of lines. Our exposition heavily relies on Sturm's treatise. Based on his earlier work on higher Nullsystem, Sturm introduces the surface (P) and a curve (l) which plays an important role in the theory of line congruences.

The classification of congruences of lines was started by Kummer [473]. He claims that any congruence o order one is either the congruence of secant lines of a twisted cubic or secant lines of the union of a line and a rational curve C_d of order n intersecting it at $n-1$-points. As we see from our classification Theorem 11.2.5, the only missing is the cae where C_d is infinitely near to the line. This gap was corrected by Sturm [736, II. Theil, p. 31] (although in modern literature, it is attributed to Z. Ran [606]).

The extension of the notion of a congruence of lines in $\mathbb{P}^3$ to higher-dimensional projective space was first suggested by C. Segre [691]. Castelnuovo in his papers on surfaces of lines in $\mathbb{P}^4$ [94], [95] continued to call them congruences. The first attempt to classify congruences of lines in $\mathbb{P}^4$ is due to

[13] In the spirit of differential geometry it was pursued by a Russian school of S. P. Finikov with almost no connection to the algebraic geometry aspect of the theory (see [[301]).

G. Marletta [505], [506]. His work was continued in a series of papers by P. de Poi (see [216] and the references there) and a paper by C. Peskine [581]. Some examples of congruences of lines in $\mathbb{P}^4$ of low degree can be found in [17]. One cal also consider surfaces or codimension two subvarieties in $G_1(\mathbb{P}^n)$ as a generalization of congruences of lines in $\mathbb{P}^3$. Their cohomology class is determined by two numbers: the order and the class. Surfaces of order 1 were classified by Z. Ran [606].

The classification of congruences of lines in $\mathbb{P}^3$ of order two without a fundamental curve is essentially due to Kummer [473]. Kummer's arguments are based on the analysis of possible equations for the coordinates $(x, y, z, \xi, \eta, \zeta)$ in the form $L(\xi, \eta, \zeta) = Q(\xi, \eta, \zeta) = 0$, where L is a linear form, and Q is a quadratic form whose coefficients are polynomials in x, y, z. Kummer gives a rather detailed discussion of congruences of different classeses, in particular, find a relation between congruences of class 2 and 16-nodal quartic surfaces, the Kummer surfaces from his original paper [471] two years yearly. Table 11.1 that contains the classification in terms of possible fundamental points can be found in [735, Teil II, p.51], and it is also reproduced in Jessop's treatise [429, p. 280]. However, it is essentially contained in Kummer's paper. Sturm's treatise contains the first synthetic treatment of Kummer's results and contains a lot of beautiful geometry that one cannot find in Kummer's paper.

Almost all known congruencies of order two with fundamental curve can be found in Kummer's paper. However, his missed three new type discovered by Sturm.

The construction of congruences of lines defined by a birational transformation between two planes is due to Hirst [395], [397]. There are numerous other ingenious geometric constructions can be found in references from [820].

Classification of congruences of lines of degree ≤ 8 and order ≥ 3 without a fundamental curve is due to Fano [294]. In a later memoir [295] Fano extends it to the classification of all congruences of order 3. In particular, he classified smooth congruences among them. As always, some of his geometrical methods do not satisfy the modern rigor. A modern proof of Fano classification based on the known results about surfaces of small degree in $\mathbb{P}^5$ and using the cohomological methods was given E. Arrondo and I. Sols ($m + n \leq 8$), by A. Verra [**?**] for $m + n = 9$ and by M. Gross [363] ($m + n = 10$). In [364], gives a modern treatment of Fano's classification (adding the possibility of the existence of a fundamental curve) of smooth congruences of order 3. It turns out that the class of such congruences is less than or equal to 7.

12
Quartic Surfaces

In the previous chapter, we found that the focal surface $\Phi(S)$ of quadratic congruences S without a fundamental curve is always a quartic surface in $\mathbb{P}^3$ with $\mu = 18 - n$ rational double points, which, generically, are expected to be ordinary nodes. We also saw that, in many cases, the focal surface of a quadratic congruence with a fundamantal curve is also a quartic surface but, this time, with non-isolated singularities. We also encountered quartic surfaces on several occasions in this book, for example, ruled quartic surfaces, Cayley quartic symmetroids, quartic cyclide surfaces, Steiner quartic surfaces, Kummer and Weddle quartic surfaces. In this chapter, we will find their place in the classification of irreducible quartic surfaces Q in $\mathbb{P}^3$. We will also assume that Q is not a cone.

12.1 Rational and Ruled Quartic Surfaces

A quartic surface that has singular points besides rational double points is a ruled surface, rational or irrational. In this section, we will discuss their classification.

12.1.1 Quartic surfaces with isolated non-rational singular points

Let Q be a quartic surface, assumed here and in the sequel, to be irreducible, reduced, and not a cone. Let q be its singular point. Choose the projective coordinates to assume that $q = [0, 0, 0, 1]$, then the equation of Q can be written in the form

$$t_3^2 F_2(t_0, t_1, t_2) + t_3 F_3(t_0, t_1, t_2) + F_4(t_0, t_1, t_2) = 0, \qquad (12.1)$$

where $F_k(t_0,t_1,t_2)$ is a homogeneous form of degree k. If q is a triple point, then Q is a monoidal surface. In this case, $F_2 = 0$. Projecting from q, we obtain a rational map $\mathrm{pr}_q : Q \dashrightarrow \mathbb{P}^2$ that blows down the curve $V(F_3,F_4)$ in $\mathbb{P}^3$ to the 0-dimensional subscheme $Z = V(F_3,F_4)$ in $\mathbb{P}^2$. The map pr_q regularizes on the blow-up $X = \mathsf{Bl}_q(Q)$. It maps the exceptional curve E_q to the cubic curve $C_3 = V(F_3)$. The singular point q is an elliptic Gorenstein triple singular point of degree 3. All such singularities have been classified [556, 7.2], [795].

The inverse rational map $\mathrm{pr}_q^{-1} : \mathbb{P}^2 \to Q \subset \mathbb{P}^3$ is given by the linear system $|I_Z(4)|$. Any Q as above can be obtained as the image of the rational map given by such linear system. If Z is reduced and consists of 12 distinct points, the surface $\mathsf{Bl}_q(Q)$ is nonsingular, and the exceptional curve is isomorphic to $V(F_3)$. For example, take $V(F_3)$ to be a nonsingular plane cubic curve, and $V(F_4)$ be a quartic curve intersecting it transversally at 12 points. We obtain a quartic surface with a *simple elliptic singularity* of degree 3 [556, Example 7.2.18].

Now, assume that q is an isolated double point different from a rational double point. Let $\pi : X \to Q$ be its minimal resolution. Recall that the *genus* $p_g(Q,q)$ is the dimension of the linear space $H^0(Q, R^1\pi_*\mathcal{O}_X)$. A rational double point of a surface is characterized by the property that its genus is equal to zero.

Proposition 12.1.1. *There are the following three possibilities:*

1. *If* $\sum_{q\in \mathrm{Sing}(Q)} p_g(Q,q) = 0$, *then* $\mathrm{Sing}(Q)$ *consists of rational double points, and X is a K3 surface.*
2. *If* $\sum_{q\in \mathrm{Sing}(Q)} p_g(Q,q) = 1$, *then X is a rational surface with an effective anti-canonical divisor* $D \in |-K_X|$.
3. *If* $\sum_{q\in \mathrm{Sing}(Q)} p_g(Q,q) = g$, *then X is birationally equivalent to a minimal ruled surface over a curve of genus* $g-1$.

Proof We apply the Leray spectral sequence $E_2^{i,j} = H^i(Q, R\pi_*^j) \Rightarrow H^{i+j}X, \mathcal{O}_X)$ to obtain an exact sequence

$$0 \to H^1(Q,\mathcal{O}_Q) \to H^1(X,\mathcal{O}_X) \to H^0(X,R^1\pi_*\mathcal{O}_X) \to H^2(Q,\mathcal{O}_X) \to H^2(X,\mathcal{O}_X) \to 0.$$

In case 1), we get $H^0(X,R^1\pi_*\mathcal{O}_X) = \{0\}$, hence $H^2(X,\mathcal{O}_X) \cong H^2(Q,\mathcal{O}_Q)$. By Serre's duality, $H^2(Q,\mathcal{O}_Q) = H^0(Q,\omega_Q) = H^0(Q,_Q) \cong \mathbb{C}$. Also, we have $H^1(X,\mathcal{O}_X) \cong H^1(Q,\mathcal{O}_Q) = 0$. Since π is a minimal resolution $\omega_X \cong \mathcal{O}_X(-D)$, where D is supported on the exceptional curves [556, 6.3]. This implies that $D = 0$, hence $\omega_X = \mathcal{O}_X$, $H^1(X,\mathcal{O}_X) = 0$, and X is a K3 surface.

In case 2), we get $H^0(X,R^1\pi_*\mathcal{O}_X) \cong \mathbb{C}$. Since $H^1(Q,\mathcal{O}_Q) = 0$, we obtain either $H^1(X,\mathcal{O}_X) \cong \mathbb{C}$ and $H^2(X,\mathcal{O}_X) \cong H^2(Q,\mathcal{O}_Q) \cong \mathbb{C}$, or $H^1(X,\mathcal{O}_X) =$

$\{0\}$, $H^2(X, \mathcal{O}_X) = \{0\}$. In the first case, $H^2(X, \mathcal{O}_X) = H^0(X, \omega_X) \neq 0$, hence $D = 0$. This implies that all irreducible exceptional curves E satisfy $E \cdot K_X = 0, E^2 < 0$, hence they are (-2)-curves and q is a rational double point. In the second case, we get $p_g(X) = 0, h^1(X, \mathcal{O}_X) = 0$. Since $K_X < 0$, X must be a rational surface.

Finally, in case 3), by a similar argument, we get $p_g(X) = 0$ and $h^1(X, \mathcal{O}_X) = g - 1$. By classification of algebraic surfaces, X is birationally isomorphic to a ruled surface over a curve of genus $g - 1$. □

Suppose Q is a rational quartic surface with isolated singularities. Then, it has exactly one singular point with $p_g = 1$ and all other singular points are rational double points. A Gorenstein singular point with $p_g = 1$ is called *minimal elliptic singularity*. Since a quartic surface Q is a deformation of a smooth quartic surface with the second Betti number equal to 22, the Milnor number of an isolated surface singularity on Q is less than or equal to 21. All possible singularities can be found among uni-modal and bi-modal surface singularities in Arnol'd classifiation of critical points of analytic functions (see [209]).

Consider again the projection from a non-rational double singular point q. Thus, the map pr_q is of degree 2. We will study the projection with more detail in the next subsection and will draw from this some conclusions about possible non-rational double points. However, before we do it, let us give an example.

Example 12.1.2. Let $C_6 = V(F_6(x_0, x_1, x_2))$ be a plane curve of degree six with a point P of multiplicity 4. For, example, we may take

$$F_6 = (x_0^2 + x_1x_2)x_1x_2(x_1^2 + x_2^2) + f_3(x_1, x_2)^2.$$

Consider the double cover of $\mathbb{P}^2$ branched along C. It can be defined as a hypersurface of degree 6

$$x_3^2 + F_6(x_0, x_1, x_2) = 0$$

in the weighted projective space $\mathbb{P} = \mathbb{P}(1, 1, 1, 3)$. The linear system $\mathcal{O}_{\mathbb{P}}(3)$ maps P onto a cone in $\mathbb{P}^{10}$ over the Veronese surface $\mathsf{v}_3(\mathbb{P}^3) \subset \mathbb{P}^9$. The projection to the Veronese surface is our double cover. Taking the affine equation with affine coordinates $z = x_3/x_0^2, x = x_1/x_0, y = x_2/x_0$, we obtain an affine equation

$$z^2 + f_6(x, y) = 0.$$

This is the usual equation of a *double plane*. We can resolve the singular point by first blowing up the point $(0, 0) \in \mathbb{A}^2$ and then take the double cover X of the blow-up branched over the proper transform of the branch curve. We leave it to the reader to do the computation to obtain the exceptional curve of the minimal resolution of the singular point is a smooth elliptic curve with self-intersection -2. This is a simple elliptic singularity of degree 2. It remains to show that the surface X admits a birational model isomorphic to a quartic surface with a simple elliptic singularity of degree 2. To do this, we consider the pre-image of the conic $V(x_0^2+x_1x_2)$ under the double cover. Obviously, it is tangent to C_6 at each intersection point. It follows that it splits in the cover into the union of two curves C_1+C_2 intersecting at 6 points. We find $8=2C^2=(C_1+C_2)^2=2C_1+12$, hence $C_1^2=C_2^2=-2$. Let h be a general line in the plane, the linear system $|H|=|\pi^*(h)+C_1|$ satisfies $H^2=4$ and $H\cdot C_1=0$. It defines a birational map from X to a quartic surface Q which is an isomorphisms outsider C_1 and blows down C_1 to an ordinary double point. The surface has two singular double points, one is elliptic and one is a rational double point.

Example 12.1.3. This example is due to Cremona [191]. The surface Q is given by the equation

$$x^2w^2+2wxf_2(x,y,z)+f_4(x,y,z)=0.$$

The plane $V(x)$ intersects Q along the union of four lines intersecting at the singular point $P=[0,0,0,1]$. It is a simple elliptic singularity of degree 2 locally analytically isomorphic to the singular point $t^2+\phi_4(u,v)=$. Cremona shows that the surface is isomorphic to the image of the plane under the birational map given by the linear system of curves $|6h-2(p_1+\ldots+p_7)-(q_1+q_2+q_3+q_4)|$. Here, the eleven points lie on a cubic curve C. The surface is the projection of the del Pezzo surface $V=\mathsf{Bl}_{p_1,\ldots,p_7}(\mathbb{P}^2)$ embedded in $\mathbb{P}^6$ by $|-2K_V|$ from the plane intersecting V at four points. It is spanned by the image C' of the cubic curve C. The images of the exceptional curves over these points are the four lines through P. The projection of C' is the singular point.

Similar examples were given by Noether [555]. He considered the linear system $|7h-2(p_1+\cdots+p_9)+3q|$ or $|9h-3(p_1+\cdots+p_8)-3q_1-2q_2-q_3|$. Here, again the points lie on a cubic curve C. The singularities are simple elliptic singularities of degree 1 and 2, respectively.

In Subsection 12.2.1 we will discuss the construction of a quartic surface as a double cover of the plane branched along a curve of degree six that admits a contact conic. Putting non-simple singularities on the branch curve, gives a quartic surfaces with non-rational double points. We will not pursue this further, and refer to [772], [774] and [426] for modern work on the classification of non-rational singularities of quartic surfaces.

Assume now that Q is a non-normal quartic surface. We have already encountered many examples of such surfaces. Let C be the curve of singularities of Q. Obviously, there are the following possible cases:

1. C is a double line;
2. C is a smooth conic;
3. C is the union of three concurrent lines .
4. C is a rational normal curve;
5. C is a triple line;
6. C is the union of two lines;
7. C is the union of a line and a conic;

By taking the pencil of planes through a double line, we see that all surfaces of types 5)-7) are ruled surfaces. A surface of type 4) is also ruled since the secant line of the double curve through a general point on the surface is contained in the surface. We classified and discussed quartic ruled surfaces in Section 10.4.

Surfaces of type 1) are submonoidal surfaces. We refer to [258] for a modern treatment of such surfaces. Note that the maximal number of isolated ordinary nodes on such surfaces is equal to 8. It achieved for Kummer Complex quartic surfaces which we encountered before.

Surfaces of type 2) are cyclide quartic surfaces. We proved in Theorem 8.6.4 that a cyclide quartic surface is a projection of a quartic del Pezzo surface in $\mathbb{P}^4$ to $\mathbb{P}^3$.

A Steiner surface is an example of a surface of type 3). In fact, any surface of type 1) must be a Steiner surface. Recall that in Example 7.4.4 we constructed a quadratic Cremona transformation of bidegree $(2,4)$ defined by the linear system of quadrics through 4 points and tangent to a fixed plane at one of the points. The restriction of this homaloidal linear system to the plane is a three-dimensional linear system of conics. It maps the plane to the projection of the Veronese surface from a line. So, the image of a general plane is a Steiner surface. The inverse map is given by Steiner quartic surfaces containing a fixed cross of double lines. They are mapped to quartics from the linear system $|\mathcal{O}_{\mathbb{P}^3}(4) - \mathfrak{b}|$, where $\mathfrak{b}$ is the normalization of the ideal of the (xy, yz, xz). The proper transform of a quartic surface Q containing $V(xy, yz, xz)$ in the blow-up $\mathsf{Bl}_{V(\mathfrak{b})}(\mathbb{P}^3)$ is the normalization of Q. It maps it isomorphically to a plane in $\mathbb{P}^3$. Thus, the normalization of Q is isomorphic to $\mathbb{P}^2$ and this proves that Q is isomorphic to a Steiner surface.

Note that there is only one Steiner surface up to projective isomorphism, and it is given by equation (2.3). Its rational birational parametrization is given explicitly in (2.4).

A non-normal quartic surface may have isolated singular points. We refer to their classification to [774].

12.2 Nodal Quartic Surfaces

12.2.1 Plane sextics and quartic surfaces

Suppose Q has a double point q. Projecting from q to $\mathbb{P}^2$, we get a degree two rational map

$$\mathrm{pr}_q : Q' \to \mathbb{P}^2, \tag{12.2}$$

where $\pi' : Q' \to Q$ is the proper transform of Q in the blow-up $\mathsf{Bl}_q(\mathbb{P}^3) \to \mathbb{P}^3$. The map pr_q is a finite morphism if and only if Q does not contain lines containing q. Let $\mathrm{I} = \{q_1, \ldots, q_k\}$ be the set of images of these lines, or the empty set if Q does not contain such lines. Let

$$f' : Q' \to \bar{Q}' \overset{\mathrm{pr}'_q}{\to} \mathbb{P}^2$$

be the Stein factorization of the map pr_q. The map pr'_q is a finite morphism of degree 2. Let B be its branch curve.

Let $\pi' : \tilde{Q} \to Q'$ be a minimal resolution of Q'. The composition

$$\pi = \mathrm{pr}_q \circ \pi' : \tilde{Q} \to Q$$

is a minimal resolution of Q

Let

$$f = f' \circ \pi' : \tilde{Q} \to \mathbb{P}^2. \tag{12.3}$$

It is a map of degree 2.

The proper transform of a line through q in $\tilde{Q}$ is a smooth rational curve with self-intersection -2. Since it is blown down under σ', its image in $\bar{Q}'$ is a singular point. Since all singularities of Q' are rational double points, $\bar{Q}'$ has only rational double points. Hence, the points q_i are simple singularities, as well as other possible singular points of B. The formula for the canonical bundle of a double cover shows that B must be a curve of degree 6.

This can be, of course, deduced from the equation of Q

$$t_0^2 F_2(t_0, t_1, t_2) + t_0 F_3(t_0, t_1, t_2) + F_4(t_0, t_1, t_2) = 0,$$

where $q = [1, 0, 0, 0]$ and $F_2 \neq 0$.

The branch curve has the equation

$$F_6(t_0, t_1, t_2) := F_3(t_0, t_1, t_2)^2 - F_2(t_0, t_1, t_2) F_4(t_0, t_1, t_2) = 0. \tag{12.4}$$

We immediately see that $\mathrm{I} = V(F_2, F_3, F_4)$.

The conic $K = V(F_2)$ is equal to the image of the exceptional curve of the birational morphism $Q' \to Q$. It is nonsingular if and only if q is an ordinary node. It follows from the local equations of rational double points that K is the union of two distinct lines, and then q is a singular point of type $A_k, k > 1$. Otherwise, it is a singular point of some type D_n or E_k. Since $\tilde{Q}$ is a K3 surface, the surface Q has a basket of rational double points with the sum μ of Milnor numbers at most 19. Each singular point different from q is a double rational point of type A_n, D_n, E_k and its projection to the plane is a simple singular point of B of type a_n, d_n, e_k. We refer to [773] and [811] for the classification of possible rational double points on a quartic surface.

- *From now on, we assume that Q contains only ordinary nodes $x_1, \ldots, x_\mu = q$ as singularities.* Let $E_1, \ldots, E_\mu$ be the exceptional curves over the nodes.

Lemma 12.2.1. *The pre-image of the contact conic in $\tilde{Q}$ splits into the sum of two (-2)-curves $E_\mu + E'_\mu$ intersecting with multiplicity $6-k$, where $k = \#\mathrm{I}$. The linear system $|f^*\mathcal{O}_{\mathbb{P}^2}(1) \otimes \mathcal{O}_{\tilde{Q}}(E_\mu)|$ coincides with $|\pi^*\mathcal{O}_Q(1)|$ and the linear system $|f^*\mathcal{O}_{\mathbb{P}^2}(1) \otimes \mathcal{O}_{\tilde{Q}}(E'_\mu)|$ defines an involution τ_q of $\tilde{Q}$ induced by the deck transformation of the double cover* $\mathrm{pr}_q : Q' \to \mathbb{P}^2$. *It switches the curves E_μ and E'_μ.*

Proof The double cover $\mathrm{pr}_q : Q' \to \mathbb{P}^2$ induces a double cover $K' \to K$ defined by an invertible sheaf $\mathcal{L}$ such that $\mathcal{L}^{\otimes 2} \cong \mathcal{O}_{\mathbb{P}^2}(B) \otimes \mathcal{O}_K$. This shows that K splits in the double cover into two smooth rational curves $K_1 + K_2$. We may assume that the proper transform of K_1 in Q' is the curve E. We have $\mathrm{pr}_q^*\mathcal{O}_{\mathbb{P}^2}(1) \cong \mathcal{O}_{Q'}(H' - E_\mu)$, where H' is the divisor class of the pre-image of a plane section of Q in Q'. Thus, $H' = \mathrm{pr}_q^*(l) + E_\mu$, where l is a line in $\mathbb{P}^2$, and $\tau'^*(H') = \mathrm{pr}_q^*(l) + E'_\mu$. The linear system $|H'|$ defines the identity map of Q' and the linear system $\tau'^*(H')$ defines the involution τ'. The pre-images of these linear systems on $\tilde{Q}$ are the linear systems $|H + E_\mu|$ and $|H + E'_\mu|$, where H is the preimage of H' and we identify the pre-images of E and E'_μ with E_μ and E'_μ. The curves now intersect only at the pre-images of points in $K \cap B$ outside I.

We have $(H + E_\mu) \cdot E'_\mu = 2 = 2 = 0$, as it should be, and $(H + E_\mu) \cdot E'_\mu = 2 + 6 - kL = 8 - k$.

□

Let

$$\tau_q : \tilde{Q} \to \tilde{Q}$$

be the biregual involution of $\tilde{Q}$ induced by the deck transformation of the double cover $\mathrm{pr}_q : Q' \to \mathbb{P}^2$. Its locus of fixed points contains a curve. It is known that the set X^τ of fixed points of a biregular involution of a K3 surface X is either empty or consists of isolated fixed points, or is a smooth curve. In our case, $\tilde{Q}^{\tau_q}$ certainly contains a curve, hence $\tilde{Q}^{\tau_q}$ is a curve. Its image in Q is contained in the intersection of Q with the first polar cubic surface $P_q(Q)$.

Let $Z = \tilde{Q}/(\tau_q)$ be the quotient by the involutions. It is a smooth rational surface. Let $\sigma : Z \to \mathbb{P}^2$ be the blowing down to $\mathbb{P}^2$, so that we have a commutative diagram

$$\begin{array}{ccc} \widetilde{Q} & \xrightarrow{\widetilde{\mathrm{pr}}_q} & Z \\ \downarrow{\scriptstyle \pi} & \overset{f}{\searrow} & \downarrow{\scriptstyle \sigma} \\ Q' & \overset{\mathrm{pr}_q}{\dashrightarrow} & \mathbb{P}^2 \end{array}$$

The birational morphism $\sigma : Z \to \mathbb{P}^2$ is a log-resolution of the branch curve B. Its proper transform $\bar{B}$ in Z is a smooth branch curve of the map $\widetilde{\mathrm{pr}}_q$. The map $\sigma : \bar{B} \to B$ is the normalization of the plane sextic curve B. Since we assume that Q has only ordinary nodes as singularities, all singular points of B outside the set I are ordinary nodes.

Lemma 12.2.2. *Let k_i be the number of nodes of Q lying on a line ℓ_i containing q. Then $k_i \le 3$.*

Proof Taking the pencil of planes through the line ℓ_i we obtain the pencil of residual cubics. A general cubic from the pencil intersects ℓ at three points. These are the singular points of a plane quartic plane section of Q. It has, at most, two singular points outside the point q. It is clear that singular points on Q on ℓ_i are among them. □

Assume that $\mathrm{I} \neq \emptyset$. We denote by $\mathcal{E}_i$ the sum of the exceptional curves over the nodes lying on the line ℓ_i. By the previous lemma, $\mathcal{E}_i$ is the sum of $2 \le k_i \ge 0$ exceptional curves.

Proposition 12.2.3. *Let $R_1, \ldots, R_k$ be the proper transforms of the lines $\ell_1, \ldots, \ell_k$ in Q containing the point $x_\mu = q$. Let $\mathcal{E}_1, \ldots, \mathcal{E}_k$ be the sum of $0 \le k_i \le 2$ exceptional curves $E_j, j \neq \mu$ over nodes of Q lying on ℓ_i. Then*

$$\begin{aligned} H' &:= \tau_q(H) = 3H - 4E - \sum_{i=1}^{k} \big((k_i+1)R_i + \mathcal{E}_i\big), \\ E' &= \tau_q(E) = 2H - 3E - \sum_{i=1}^{k} \big((k_i+1)R_i + \mathcal{E}_i\big), \end{aligned} \tag{12.5}$$

where H is the inverse image in $\mathrm{Pic}(\tilde{Q})$ *of $c_1(\mathcal{O}_Q(1))$. Moreover, $\tau_q(R_i) = R_i$ if $k_i = 0, 2$, and $\tau_q(R_i) = \mathcal{E}_i$ if $k_i = 1$.*

Proof Let K be the contact conic.

We have $H = f^*(l) + E$ and $2f^*(l) = f^*(K) = E + E' + R$, where

$$R = \sum_{i=1}^{k} m_i R_i + n_i \mathcal{E}_i$$

and $f(R) = \mathfrak{I}$. Since $\tau_q(E) = E'$ and $\tau_q^*(f^*(K)) = f^*(K)$, we obtain that $\tau_q(R) = R$. Also,

$$\begin{aligned} H' &:= \tau_q^*(H) = \tau_q^*(f^*(l) + E) = f^*(l) + E' = f^*(l) + (2f^*(l) - E - R) \\ &= 3f^*(l) - E - R = 3(H - E) - E - R = 3H - 4E - R. \end{aligned} \tag{12.6}$$

Similarly, we get

$$E' = 2f^*(l) - E - R) = 2(H - E) - E - R = 2H - 3E - R.$$

Obviously, each R_i is contained in R, and we get $1 = H \cdot R_i = H' \cdot R_i = -1 - R_i \cdot R$, hence $R_i \cdot R = -2$. Suppose that $k_i = 0$. Then $R_i \cdot (R - R_i) = -2 + 2 = 0$. Hence R_i enters in R with the coefficient 1.

If $k_i = 1$, then $\mathcal{E}_i = E_i$, the exceptional curve over one node of Q lying on ℓ_i. We have $R_i \cdot E_i = 1$, and $R_i \cdot (R - R_i - E_i) = -1$. Thus R_i enters in $R - R_i$, and we obtain $R_i \cdot (R - 2R_i - E_i) = -2 + 4 - 1 = 1$. So, R_i enters with the multiplicity 2 and E_i enters with multiplicity 1.

If $k_i = 2$, we, similarly, get that $R_i \cdot (R - R_i - \mathcal{E}_i = -2 + 2 - 2 = -1$ and $R_i \cdot (R - 2R_i - \mathcal{E}_i) = 0$. Both contradict the fact that R must intersect positively with $R - m_i R_i$. Thus, $R \cdot (R - 3R_i - \mathcal{E}_i) = -2 + 6 - 2 = 2$, and $R = 3R_+ \mathcal{E}_i$. We have R_i intersects each of the two components of $\mathcal{E}_i$ with multiplicity one.

This proves the first equality. Since $H' = 3H - 4E - R$ and $E' = 2H - E - R'$. After subtracting, we get $R' = R$.

Similarly, we have $\sum_{i=1}^k \big((k_i + 1)R_i + \mathcal{E}_i\big) = 3H - 4E - H'$ and $\tau_q^*(\sum_{i=1}^k \big((k_i + 1)R_i + \mathcal{E}_i\big) = 3H' - H - 4E'$. Since $H' = E' = H - E$, the sums coincide. The sum could be invariant only if $\tau_q^*(R_i)$ is as in the assertion of the proposition.

□

Corollary 12.2.4. *Let ℓ_i be a line on Q passing through q and k_i be the number of nodes on $\ell_i, i \neq \mu$ lying on ℓ_i. The point q_i is an ordinary double point if $k_i = 0$, a cusp if $k_i = 1$ (a simple singularity of type a_2), or a tacnode corresponding to an infinitely near ordinary double point $q_i' > q_i$ (a simple singularity of type a_3) if $k_i = 2$. The contact conic intersects B at q_i with multiplicity 2 if $k_i = 1$ and multiplicity 4 if $k_i = 2$ passing through q_i and q_i'.*

Corollary 12.2.5. *Any intersection point $x \notin \mathfrak{I}$ of K with B is a nonsingular* *point of B. Moreover, K and B are tangent at this point with some even multiplicity $2n_x$. The sum $\sum_{x \notin \mathfrak{I}} n_x = 6 - n_x$ and the total multiplicity of nonsingular intersection points is equal to $6 - \#\mathfrak{I}$ if each line through q contains at most two nodes, and equal to $6 - \#\mathfrak{I} - s$, where s is the number of lines that contain three nodes.*

Proof Suppose x is such an intersection point. Taking partials of F_6 at this point, we get that they all vanish if and only if F_4 vanishes at this point. This contradicts to the assumption that $x \notin \mathfrak{I} = V(F_2, F_3, F_4)$. We know how K intersects B at points from $\mathfrak{I}$. The multiplicity is always even since K splits in the cover. □

The condition that a nodal plane sextic admits a contact conic is rather implicit. The linear system of conics is of dimension 5, and the condition that a conic is tangent to B at some point imposes one condition on the conic. Since we need six tangency points, it imposes one condition on the plane sextic. In particular, it imposes one condition on the position of its nodes.

We can reverse the construction. We leave the proof of the next proposition to the reader.

Proposition 12.2.6. *Let B be a plane curve of degree 6 with n ordinary nodes as its singularities. Assume that there exists a smooth conic K passing through k nodes $p_1, \ldots, p_k$, and tangent to B at $6 - k$ nonsingular points. Let $X \to \mathbb{P}^2$ be the double cover of $\mathbb{P}^2$ branched along B and $\tilde{X}$ be its minimal nonsingular model. Then, there exists a birational morphism $\phi : \tilde{X} \to Q$, where Q is a quartic surface that contains $\mu = n + 1 - k$ nodes. The curve B splits in the cover $\tilde{X} \to \mathbb{P}^2$ into the union of two smooth rational curves intersecting at $6 - n$ points. The birational morphism ϕ maps one of the curves to a node q of Q. The images of the exceptional curves over the points p_i are lines in Q passing through q. The images of other exceptional curves over $n - k$ nodes of B are nodes on Q.*

Example 12.2.7. Assume $B = V(F_6)$ has $n \geq 6$ nodes and a conic $C = V(F_2)$ passes through six nodes $p_1, \ldots, p_6$ of B. If we take the double cover X of $\mathbb{P}^2$ branched along B, and take C as a contact conic, we obtain a quartic surface with $n + 1 - 6$ lines passing through one of the nodes. It also contains a conic, the image of one of the components of the pre-images of C in the cover.

Now, taking the partials of F_6 and using equation (12.1), we find that $V(F_4)$ passes through the six points. Let $V(F_3)$ be a cubic curve passing through $p_1, \ldots, p_6$. Since the restriction of F_6 to C is a divisor of the form $2(p_1 + \cdots + p_6)$

F_3 by some nonzero constant)

$$F_6 = F_3^2 + 2F_2F_4 = F_3^2 + 2F_2F_4(L_1F_3 + F_2G_2) = (F_3 + F_2L_1)^2 + F_2^2(2G_2 - L_1^2).$$

This shows that $K = V(2G_2 - L_1^2)$ is a contact conic of B not passing through its singular points. Now, we repeat the construction and obtain another birational model of X that has $n+1$ nodes.

The assumption is made only to simplify the exposition and the notation. If needed, the reader can easily drop the assumption and modify the following discussion accordingly.

The map $\sigma : Z \to \mathbb{P}^2$ is the blowing up of $\mu - 1$ ordinary nodes of the branch curve B. Let $(e_0, e_1, \ldots, e_{\mu-1})$ be the corresponding geometric basis of Z. Then, Let

$$[E_i] = \widetilde{\mathrm{pr}}_q^*(e_i), \quad i = 1, \ldots, \mu - 1.$$

Let

$$h := \widetilde{\mathrm{pr}}_q^*(e_0).$$

Then, the branch curve of $\widetilde{\mathrm{pr}}_q$ is equal to the proper trasnform of B, and

$$[\bar{B}] = 6e_0 - 2(e_1 + \cdots + e_{\mu-1}).$$

The ramification curve R of $\widetilde{\mathrm{pr}}_q$ is its preimage in $\tilde{Q}$, and

$$[R] = 3h - E_1 - \cdots - E_{\mu-1}.$$

Let τ be the biregular involution of $\tilde{Q}$ defined by the deck transformation of the double cover $Q' \to \mathbb{P}^2$. We know from Lemma 12.2.1 that it is given by the linear system $|h + E'_\mu|$, where $E'_\mu = \tau_q(E)$. It follows from Proposition 12.2.3 that

$$\begin{aligned} H' = \tau_q^*(H) = 3H - 4E, \quad E'_\mu = 2H - 3E, \\ \tau_q^*(E_i) = E_i,\ i \neq \mu, \quad \tau^*(h) = h. \end{aligned} \tag{12.7}$$

We know that $R \cdot E_\mu = 6$ and the images of the six points (some of them may coincide) to $\mathbb{P}^2$ is the intersection points of the contact conic K with B.

Lemma 12.2.8. *There exists an isomorphism $f : W \to Q'$, where W is a surface of degree 6 in the weighted projective space $\mathbb{P}(1,1,1,3)$ given by equation*

$$z^2 - F_6(t_0, t_1, t_2) = z^2 - F_3^2 + F_2F_4 = 0.$$

Under this isomorphism, the image of the hyperplane section $V(z)$ is equal to the ramification curve R' of the double cover $Q' \to \mathbb{P}^2$. The images of the curves $K_1 = W \cap V(z + F_3)$ and $K_2 = W \cap V(z + F_3)$ are the curves E_μ and E'_μ.

Proof Consider a rational map $T : \mathbb{P}(1,1,1,3) \dashrightarrow \mathbb{P}^3$ given by

$$[t_0, t_1, t_2, z] \mapsto [t_0F_2(t_0,t_1,t_2), t_1F_2(t_0,t_1,t_2), t_2F_2(t_0,t_1,t_2), z - F_3(t_0,t_1,t_2)]$$

It is not defined along the curve K_1 and blows down the curve K_2 to a point $[0,0,0,1]$. Since

$$(z - F_3)^2 + 2(z - F_3)F_3 + F_4F_2 = z^2 - F_3^2 + F_4F_2.$$

the image of W is equal to Q. The rational map T restricts to a birational morphism $f : W \to Q$ which blows down K_1 to the singular point x_μ of Q. It may be undefined only at the intersection points of K_1 with K_2. However, it follows from the non-degeneracy assumption that we may take F_2 and F_3 as local parameters of the surface at any indeterminacy point of T. It also implies that the map T restricts to a regular birational map on X. This shows that both Q' and X are isomorphic to a minimal resolution of the singular point q, and hence $W \cong Q'$. □

We will identify Q' with the surface W in $\mathbb{P}(1,1,1,2)$. Local computation show that all singular points $q_i, i \neq \mu$, of $Q' \cong X$ are the pre-images of singular points of $B = V(F_6)$. Moreover, the singular points of Q' different from q_μ lie over singular points of B and formally isomorphic to singularities $w^2 + \phi(u,v) = 0$, where the singular point of its image in B is formally isomorphic to the singularity $\phi(u,v) = 0$. In particular, if a singular points p of B is a simple point of type a_n, d_n, e_n, then the singular point of Q' over p is a rational double point of type A_n, D_n, E_n, respectively.

Another birational model of Q is a closed subvariety X' of the geometric line bundle $\mathbb{V}(\mathcal{L}^{-1})$, where $\mathcal{L} = \mathcal{O}_{\mathbb{P}^2}(3)$. The subvariety X' is given by local equations $z^2 - \phi(x,y) = 0$, where $\phi(x,y) = 0$ are local equations of B. It is isomorphic to the affine spectrum of the sheaf of quadratic algebras $\mathcal{O}_{\mathbb{P}^2} \oplus \mathcal{L}^\vee$. The projection of $\mathbb{V}(\mathcal{L}^{-1})$ to the base restricts to a finite map $f : X' \to Q$ of degree 2. The formula for the canonical sheaf

$$\omega_X \cong f^*(\omega_{\mathbb{P}^2} \otimes \mathcal{L})$$

gives in our case $\omega_X \cong \mathcal{O}_X$ confirming that a minimal resolution of X is a K3 surface provided that all singular points of B are simple curve singularities.

12.2.2 Apparent nodes

Let C be an irreducible plane nodal curve that admits a contact conic K that intersects it at nonsingular points of C. Let $\phi : X \cong \mathbf{F}_0 \to \mathbb{P}^2$ be the double cover of $\mathbb{P}^2$ branched over K. We say that a node of C is an *apparent node* if its pre-image in the cover consists of two smooth points on $\pi^{-1}(C)$.

Lemma 12.2.9. *Let C be a plane nodal curve of degree d that admits a contact conic K intersecting it at nonsingular points. If C is rational, then it is equal to a projection of a curve of degree d lying on a smooth quadric. If C is irrational, then the same is true if one of the nodes of C is an apparent node.*

Proof We identify X with a smooth quadric in $\mathbb{P}^3$. The double cover $\phi : X \to \mathbb{P}^2$ is the projection from a point x_0 outside X, its ramification curve is the intersection $L = X \cap P_{x_0}(X)$ of X with the polar plane. The covering involution ι switches the two rulings of X, and a generator of each ruling projects to a tangent line of K.

Suppose $C' = \pi^{-1}(C)$ does not split in the cover, i.e. $\iota(C) = C$. Then, $C' \in |\mathcal{O}_X(d)|$ is an irreducible curve of arithmetic genus equal to $p_a(C') = (d-1)^2$. Let $\delta = \delta_1 + \delta_2$ be the number of nodes of C, among which δ_1 is the number of apparent nodes. The curve C' has d nodes at the intersection points with L and $2\delta_2$ nodes outside. Let $Q' \to Q$ be the blow-up of the set of nodes of C' and $Z \to \mathbb{P}^2$ be the blow-up of δ_2 nodes of C, and d intersection points with B. Then, the proper transform of C in Z is a smooth curve $\tilde{C}$ of genus $g = p_a(C) - \delta_1 - \delta_2 = \frac{(d-1)(d-2)}{2} - \delta_1 - \delta_2$ and the proper transform $\tilde{C}'$ of C' on Q' is a smooth curve of genus $g' = p_a(C') - d - 2\delta_2 = (2d-1)^2 - d - 2\delta_2$. The double cover $Q' \to Z$ restricts to an unramified cover $\tilde{C}' \to \tilde{C}$.

If C is a rational curve, then $\tilde{C}$ is a smooth rational curve isomorphic to the normalization of C. Since it does not admit an irreducible unramified cover, we get a contradiction. If C is not rational, then the Hurwitz formula $g' = 2g - 1$ gives the equality

$$(d-1)^2 - d - 2\delta_2 = 2\left(\frac{(d-1)(d-2)}{2} - \delta_1 - \delta_2\right) - 1$$

that yields $\delta_1 = 0$. This contradicts our assumption.

□

Suppose now that C splits in the cover $\phi : X \to \mathbb{P}^2$. Then, its pre-image in the cover is equal to the union $C_1 + C_2$, where $C_2 = \iota(C_1)$. Let $(a, d-a)$ be the bodegree of C_1, hence $(d-a, a)$ is the bidegree of C_2. The projection $C_i \to C$ is a partial normalization, it is an isomorphism outside apparent nodes. There are two nonsingular points in $C_1 \cap C_2$ over each apparent node of C. We have

$$C_1 \cdot C_2 = a^2 + (d-a)^2 = d + 2\delta_1.$$

Using that the arithmetic genus of a curve of bidegree $(a, d-a)$ in $X \cong \mathbb{P}^1 \times \mathbb{P}^1$ is equal to $a(d-a) - d + 1$, we get the following.

Corollary 12.2.10. *Let C be an irreducible nodal plane curve that a admits a contact conic intersecting it at nonsingular points. Assume C splits in the cover*

$\phi : X \to \mathbb{P}^2$ ramified over K. Let δ_1 be the number of apparent nodes of C, and δ_2 be the number of remaining nodes of C. Then,

$$\delta_1 = \tfrac{1}{2}((d-a)^2 + a^2 - d), \quad 0 \le \delta_2 \le (a-1)(d-a-1), \qquad (12.8)$$

where $(a, d-a)$ is the bidegree of one of its splitting component in Q.

Proposition 12.2.11. *Let C be an irreducible nodal curve of degree $d = 2n$. Suppose it is a projection of a curve C' of bidegree (n,n) on a smooth quadric X. Then, it has $n(n-1)$ apparent nodes, and they are intersection points of curves of degree n and $n-1$.*

Proof The projection map defines a birational isomorphism $C' \to C$. This implies that C splits in the cover. Applying (12.8), we obtain that $\delta_1 = n(n-1)$.

The curve C' is a complete intersection of X and a surface F of degree n. Let us choose projective coordinates (x,y,z,w) such that the center of the projection $x_0 = [1,0,0,0]$ and $\Pi = P_{x_0}(X) = V(x)$. We may assume that $x_0 \in F$, so we can write

$$F = V(xA_{n-1}(x,y,z,w) + B_n(y,z,w)).$$

The cone over C with vertex at x_0 intersects the cone $V(B_n(y,z,w))$ at $2n^2$ rulings. There are $2n$ generators that pass through the intersection points $\Pi \cap C'$, the remaining $2\delta_1 = 2n(n-1)$ generators (counted with multiplicity 2 since they are the pre-images of nodes of C). So the intersection of $V(A_{n-1}) \cap V(B_n)$ with X consists of two points on $2\delta_1$ generators. They are the pre-images of apparent points of C. This shows that the set of $n(n-1)$ apparent nodes is a complete intersection of curves of degrees n and $n-1$, the curve $V(B_n(y,z,w))$ and the projection of the curve $V(A_{n-1}) \cap F$ to the plane. □

Example 12.2.12. Let C be an irreducible plane curve of degree d equal to the projection of a curve C' of degree d on a smooth quadric X. We know from Lemma 12.2.9 that C is either rational curve or has at least one apparent node.

Suppose $d = 3$. Then, C must have an apparent node, and C' is a curve of bidegree $(1,2)$ or $(2,1)$. smooth rational curves of bidegree $(1,2)$ and $(2,1)$.

If $d = 4$, then we may choose C' of bidegree $(1,3)$ or $(2,2)$. In the former case C is rational and hence has $\delta = \delta_1 = 3$ apparent nodes. In the latter case $\delta_1 = 2$, and $\delta_2 = 0$ if C' is smooth, and $\delta_2 = 1$ if C' has a node. In particular, a smooth or one-nodal quartic curve is not a projection of a quartic curve on X.

If $d = 5$, then C is of bidegree $(1,4)$ or $(2,3)$. In the first case, C' is smooth and rational, hence $\delta = \delta_1 = 6$. In the second case, $\delta_1 = 4$. Since $p_a(C_1) = 2$, we have $\delta_2 \in \{0,1,2\}$. In particular, a nodal quintic can be a projection of a curve of degree 5 on a quadric only if $\delta \ge 4$.

If $d = 6$, then C_1 is of bidegree $(3,3),(2,4)$, or $(1,5)$. In the classical terminology, C is of the first, the second, or the third *species*.

Note that the number of parameters of such sextics is the same and equals 8. We get $\delta_1 = 6, 7, 10$, respectively. The arithmetic genus of C_1 is equal to $4, 3, 0$, so the number δ of possible nodes of C is greater than or equal to 6. In particular, there are three kinds of 10-nodal sextics that admit a contact conic.

If C is of the first species, Proposition 12.2.11 implies that six apparent nodes are on a conic.

Assume that C is of the second species. Adding to C_1 two generators of X intersecting C_1 at four points, we obtain a reducible curve $\tilde{C}$ of degree 8 with two line components ℓ_1 and ℓ_2, each intersecting C at six points. The curve $\tilde{C}$ has 7 apparent nodes $p_1, \ldots, p_7$ on C, two apparent nodes q_i, q_i' on each line, and one apparent node equal to the intersection of the two lines. The plane cubic curve R passing through the twelve apparent nodes passes through the points $p_1, \ldots, p_7$ and two points q_i, q_i' in $\ell_i \cap C$.

Let us take $\ell_1 = \ell_2$ passing through a node p_i of C and tangent to the contact conic K. Then, we obtain a cubic curve that passes through apparent nodes of C and intersects C at one of the nodes p_i with multiplicity 4. It intersect each line at this point with multiplicity 2. This implies that p_i is a double point of the cubic. We saw the existence of such cubics when C is the projection of a trope curve of degree 12 from the fundamental points of degree 6 on the focal surface of a congruence lines of bidegree $(2,7)$.

Assume that C is of the third species. In this case, C' is a smooth rational curve of bidegree $(1,5)$ on X. The curve C is a 10-nodal sextic, all its nodes are apparent. We add to C' four generators intersecting it at 5 points to obtain a nodal curve of bidegree $(5,5)$. Its projection $\bar{C}$ is a reducible curve of degree 10 with four line components $\ell_1, \ldots, \ell_4$ which are tangent to the contact conic. A choice of four generators is equivalent to a choice of four points on the contact cubic. The curve $\bar{C}$ has 10 apparent nodes besides apparent nodes of C. They are the intersection points of four lines and one point among the intersection of the line with C.

Let us specialize the four lines to two double lines passing through a node p_i of C. Similar to the previous case, we obtain a quartic curve that passes through the nodes of C and intersects C at p_i with multiplicity 6. The quartic curve intersects each line with multiplicity 3 at p_i. This implies that p_i is a triple point of the quartic.

Remark 12.2.13. We can extend the previous discussion to reducible nodal curves admitting a contact conic. Let $C = C_1 + \cdots + C_k$ be a nodal curve of degree d, the union of irreducible curves C_i' of degrees d_i. Assume that each

C_i splits in $Q \to \mathbb{P}^2$ into the union of curves of bidegree $(a_i, d_i - a_i)$ such that $\sum_{i=1}^k a_i = \sum_{i=1}^k (d_i - a_i)$, i.e. $2\sum_{i=1}^k a_i = d$. The set of apparent nodes of C is the union of the set of apparent nodes of each C_i and $C_i \cdot C_j - C_i' \cdot C_i = d_i d_j - a_i d_j - a_j d_i - 2a_i a_j$ points among the intersection points of the components C_i. They are the intersection points of a curve of degree d and $d-1$.

For example, assume $d = 6$ and C is the union of a line C_1 and a 5-nodal quintic C_2 with four apparent nodes. Additionally, two apparent points among the five intersection points. There must be a conic passing through the six apparent points.

Another example, is when C is the union of two lines C_1, C_2 and a 3-nodal quartic C_3 with three apparent nodes p_1, p_2, p_3. We have additional three apparent nodes $p_4 = C_1 \cap C_2, p_5 \in C_1 \cap C_3, p_6 \in C_2 \cap C_3$. There is a conic containing $p_1, \ldots, p_6$.

12.2.3 The nodal lattice

The moduli space of μ-nodal quartic surfaces may consist of different irreducible components. Here, we introduce the important invariant of general nodal quartic surfaces which distinguishes these components.

Proposition 12.2.14. *Let Q be a quartic surfaces with μ ordinary double points. Then,*

$$\mu \leq 16.$$

Proof We apply the Plücker-Teissier formula 1.2.7. By taking a general hyperplane section passing through q_i, we find that $H \cap Q$ is a an ordinary node at q_i. This implies that the degree $d^\vee$ of the dual surface $Q^\vee$ satisfies

$$d^\vee = 36 - \sum_{i=1}^{k} (\mu(Q, q_i) + 1) = 36 - 2k.$$

If $k > 16$, $d^\vee \leq 3$, hence $Q^\vee$ is a rational surface. However, Q is birationally isomorphic to $Q^\vee$ but Q is birationally isomorphic to a K3 surface. □

It follows from the previous discussion, that the projection from a node q of Q defines a pair (B, K), where B is a plane sextic with $\mu - 1$ ordinary nodes and K is a smooth contact conic of B not passing through singular points of B. Conversely, given such a pair (B, K), we define Q' by equation from Lemma 12.2.8. Blowing down one of the splitting components of the pre-image of the contact conic, we obtain Q.

Let

$$\mathcal{N} = \mathbb{Z}H \oplus \mathbb{Z}[E_1] \oplus \cdots \oplus \mathbb{Z}[E_\mu] \subset \mathrm{Pic}(\widetilde{Q}) \qquad (12.9)$$

be the sublattice of $\mathrm{Pic}(\widetilde{Q})$ spanned by the divisor classes $[E_1],\ldots,[E_\mu]$ and H. Since $|E_i| = \{E_i\}$, we will identify the divisor classes of E_i with the curves E_i. The lattice $\mathcal{N}$ has $(H, E_1, \ldots, E_\mu)$ as an orthogonal basis with $H^2 = 4, E_i^2 = -2$. As an abstract quadratic lattice, $\mathcal{N}$ is isomorphic to the lattice

$$\mathsf{N}_\mu = \langle 4\rangle \oplus \langle -2\rangle^{\oplus\mu}.$$

Here, we use the standard notation $\langle a\rangle$ for a lattice of rank 1 generated by a vector with norm-square equal to a.

Note that the lattice embedding $j : \mathsf{N}_m \hookrightarrow \mathrm{Pic}(\widetilde{Q})$ may not be primitive because its image $\mathcal{N}$ may not be a primitive sublattice of $\mathrm{Pic}(\widetilde{Q})$. Let $\mathcal{N}^{\mathrm{s}}$ be the primitive saturation of $\mathcal{N}$ in $\mathrm{Pic}(\widetilde{Q})$, the largest primitive sublattice of $\mathrm{Pic}(\widetilde{Q})$ containing $\mathcal{N}$. It is known that an even lattice M' containing an even lattice M of finite index $[M' : M]$ corresponds to an isotropic subgroup A of the discriminant group $\mathrm{Disc}(M)$ equipped with a quadratic map

$$q : \mathrm{Disc}(M) \to \mathbb{Q}/2\mathbb{Z}, \quad v \mapsto v^2 \mod 2\mathbb{Z}$$

[550]. We already saw in Lemma 8.2.1 that $\mathrm{disc}(M) = [M' : M]^2\mathrm{disc}(M')$. The index $[M' : M]$ here is the order of the group $H^\perp/H$.

The discriminant group $\mathrm{Disc}(\mathsf{N}_\mu)$ is equal to the direct sum of the discriminant groups of its direct summands isomorphic to $\langle 4\rangle$ or $\langle -2\rangle$. It is generated by $r = \frac{1}{4}H \mod \mathsf{N}_\mu$ and $r_i = \frac{1}{2}[E_i] \mod \mathsf{N}_\mu$, and it is isomorphic to the direct sum

$$\langle \frac{1}{4}\rangle \oplus \langle -\tfrac{1}{2}\rangle^{\oplus\mu}, \tag{12.10}$$

We have a chain of inclusions

$$0 \subset \mathsf{N}_\mu \subset \mathsf{N}_\mu^s \subset (\mathsf{N}_\mu^s)^\vee \subset \mathsf{N}_\mu^\vee \tag{12.11}$$

The isotropic subgroup H corresponding to the overlattice $\mathsf{N}_\mu^{\mathrm{s}}$ is the the 2-elementary abelian group

$$\mathsf{N}_\mu^s/\mathsf{N}_\mu \subset \mathsf{N}_\mu^\vee/\mathsf{N}_\mu = \mathrm{Disc}(\mathsf{N}_\mu).$$

Let $\mathcal{P}$ be the set of nodes of Q. Recall from Remark 4.3.7 that a subset $\{x_i, i \in I\}$ of nodes is called *even* (resp. *weakly even*) if $\frac{1}{2}\sum_{i\in I} E_i \in 2\mathrm{Pic}(\widetilde{Q})$ (resp. $H + \sum_{i\in I} E_i \in 2\mathrm{Pic}(\widetilde{Q})$). We define a homomorphism

$$\mathsf{N}_\mu^s/\mathsf{N}_\mu \to 2^{\mathcal{P}} \cong \mathbb{F}_2^\mu$$

that assigns to $\sum_{i\in I} r_i, \#I = 4s$, the even set of nodes $w_I = \{x_i, i \in I\}$ and assigns to $2r + \sum_{j\in J} r_j, \#J = 4s+2$, the weakly even set of nodes $w_J = \{x_i, i \in J\}$. It is immediate to see that the homomorphism is injective that allows us to identify $\mathsf{N}_\mu^s/\mathsf{N}_\mu$ with a binary linear code in $\mathbb{F}_2^\mu$. The addition law here is of course the symmetric sum $A + B = (A \cup B) \setminus (A \cap B)$. We will denote this binary linear code by C_μ.

Recall from the theory of linear codes that elements of a binary linear code $V \subset \mathbb{F}_2^n$ are called words, and the *weight* of a word w is the number $|w|$ of its non-zero coordinates in the standard basis of $\mathbb{F}_2^n$. The minimal weight d of a nonzero word coincides with the minimal distance of the code, the smallest weight $|w + w'|$, where $w, w' \in V$. We say that V is of of type $[n, k, d]$, where $k = \dim V$ and d is the minimal distance of V. In our case, $n = \mu$.

The linear space $2^{\mathcal{P}}$ is equipped with the standard symplectic bilinear form $b(A, B) = \#(A \cap B) \mod 2$ and the code C_μ is its isotropic subspace. In terminology of the theory of linear codes, C_μ is a self-orthogonal code. The dual linear code $C_\mu^\perp$ contains C_μ, and we get

$$\mathrm{Disc}(\mathsf{N}_\mu^{\mathrm{s}}) \cong H^\perp/H \cong \mathbb{F}_2^{1+\mu-2k} \cong \langle \frac{1}{4} \rangle \perp C_\mu^\perp/C_\mu \qquad (12.12)$$

with the quadratic form on $C_\mu^\perp/C_\mu$ inherited from the quadratic form on $2^{\mathcal{P}}$.

Lemma 12.2.15. *The cardinality of an even (resp. weakly even) set of nodes belongs to the set* $\{8, 16\}$ *(resp.* $\{6, 10\}$*).*

Proof An isotropic element v of $\mathrm{Disc}(\mathsf{N}_\mu)$ is equal to $2r_0 + \sum_{i \in I} r_i$ with $\#I \equiv 2 \mod 4$ or $\sum_{i \in I} r_i, \#I \equiv 0 \mod 4$. It shows that $2\mathcal{N}^{\mathrm{s}} \subset \mathcal{N}$ always. So, $v = \frac{1}{2}(H - \sum_{i \in I} E_i)$ with $\#I \equiv 2 \mod 4$ or $v = \sum_{i \in I} E_i$, with $\#I \equiv \mod 4$. Since $\mathrm{Pic}(\widetilde{Q})$ is an even lattice, in the first case, $v^2 = \frac{1}{4}(4 - 2\#I) = 1 - \frac{1}{2}\#I$ must be equal to $-2s$, hence $\#I \in \{6, 10, 14\}$. In the second case, $v^2 = -\frac{1}{2}(\#I) = -2s$, hence $\#I \in \{4, 8, 12, 16\}$.

Let us exclude the case $\#I = 4$. Suppose such a set exists, then, after reordering the set of nodes, we find a divisor class D with $D^2 = -2$ and $2D \sim E_1 + E_2 + E_3 + E_4$. Since $\mathcal{O}_{\widetilde{Q}}(E_1 + E_2 + E_3 + E_4) \cong \mathcal{O}_{\widetilde{Q}}(2D)$, there exists a double cover $\phi : X' \to \widetilde{Q}$ branched over $E_1 + E_1 + E_3 + E_4$. We have $K_{X'} = \phi^*(K_{\widetilde{Q}} + D) = \pi^*(D)$. Since $\phi^*(E_i) = 2\tilde{E}_i$, where $\tilde{E}_i^2 = -1$, we get $K_{X'}^2 = -4$. After blowing down the four (-1)-curves $\bar{E}_i$, we get a surface X with $K_X = 0$. On the other hand, the standard Hurwitz formula for the double cover $X' \to \widetilde{Q}$ allows us to compute the topological Euler-Poincaré characteristic $e(X')$. We get $e(X') = 2e(v) - \sum_{i=1}^4 e(R_i) = 40$. This gives $e(X) = e(X') - 4 = 36$. It follows from the classification of algebraic surfaces X must be either a K3 surface or an abelian surface. In the former case, $e(X) = 24$, and, in the latter case, $e(X) = 0$. This contradiction proves the assertion.

Let us exclude $\#I = 12$. Arguing as in the previous case, we find a surface Y with $e(X') = 24$ and $K_{X'}^2 = -12$ that contains 12 disjoint (-1)-curves. Blowing them down we get a surface X with $K_X^2 = 12$ and $K_X = 0$. The classification of algebraic surfaces shows that such a surface does not exist.

We also can exclude the case $\#I = 14$. We can refer to a general results about the cardinality of the sets of even or weakly even nodes to [317], however, in our special case, we can use the fact that each quartic surface with $\mu \geq 14$ nodes can be realized as the focal surface of a quadratic congruence of lines of class $18 - \mu$. Our description of the set of trope conics easily shows that there are no weakly even sets of 14 nodes.

□

Lemma 12.2.16. *Let* $[\mu, k, d]$ *be the type of the linear code* C_μ. *Then,*

$$k \geq \mu - 10,$$

and, if $k \geq 2$,

$$d \in \{6, 8\}.$$

Proof We use that the composition of a primitive embedding

$$\mathsf{N}_\mu^{\mathrm{s}} \hookrightarrow \mathrm{Pic}(\widetilde{Q}) \hookrightarrow H^2(\widetilde{Q}, \mathbb{Z})$$

is a primitive embedding $\mathsf{N}_\mu^{\mathrm{s}} \hookrightarrow H^2(\widetilde{Q}, \mathbb{Z})$. Here, $H^2(\widetilde{Q}, \mathbb{Z}) \cong \mathbb{Z}^{22}$ is a unimodular lattice with respect to the cup-product.

We know from Subsection 8.2.1 that the discriminant groups of a primitive sublattice M and its orthogonal complement $N^\perp$ in a unimodular lattice are isomorphic. Also, it is easy to see that $q_M = -q_{M^\perp}$.

It follows from (12.12) that $l(\mathrm{Disc}(\mathsf{N}_\mu^{\mathrm{s}}) = \mu - 2k$. On the other hand, $l(\mathrm{Disc}((\mathsf{N}_\mu^{\mathrm{s}})^\perp) \leq 22 - (1 + \mu) = 21 - \mu$. This gives $21 - \mu \leq \mu - 2k$, hence $k \geq \mu - 10$.

If $k \geq 2$, the subspace C_μ^{ev} contains a non-trivial word, and Lemma 12.2.15 implies that its weight is equal 10. □

The following Lemma follows immediately from the projection formula for the divisor classes [315, Chapter 2].

Lemma 12.2.17. *Let* $\phi : X \to S$ *be a finite map of degree* 2 *of smooth projective surfaces. Let* σ *be the involution of* X *defined by the deck transformation of the cover. Then, the homomorphism* $\phi^* : \mathrm{Pic}(S) \to \mathrm{Pic}(X)$ *is injective, its image is the subgroup* $\mathrm{Pic}(X)^\sigma$ *of* σ*- invariant divisor classes, and cokernel of* $\phi^* : \mathrm{Pic}(S) \to \mathrm{Pic}(X)^\sigma$ *is a* 2*-elementary abelian group generated by the cosets of divisor classes of irreducible components of the ramification divisor of* ϕ.

Let $\{x_i, i \in I\}$ be an even set of eight nodes. The divisor

$$D_I = \tfrac{1}{2}(2H - \sum_{i \in I} E_i)$$

has $D^2 = 0$ and $D \cdot H = 4$. By Riemann-Roch Theorem, $\dim |D| = 1$, the pencil of quadrics passing through the eight points is a pencil of contact quadrics. Its members cut out Q along a quartic curve of arithmetic genus 1 taken with multiplicity 2.

Fix a node x_i and let τ_i be the biregular involution of $\widetilde{Q}$ defined by the deck transformation of the projection from x_j. Formulas (12.7) show how it acts on $\mathrm{Pic}(v)$.

Suppose $j \notin I$ and $\#I = 8$, then a contact quadric passing through x_j defines a τ_j-invariant (-2)-curve $C_j \sim D - E_j \sim \frac{1}{2}(2H - 2E_j - \sum_{i \in I} E_i)$. It is the proper transform of a quartic curve on Q passing through $x_i \in I$ and having x_j as its double point.

Since C_j intersects 8 curves $E_i, i \in I$, with multiplicity one, and these curves are τ_{x_j}-invariant, we obtain 8 fixed points on it. Hence, C is fixed pointwise by τ_j. It is an irreducible component of the ramification curve R of $\pi : \widetilde{Q} \to Z$. The projection of C_j to the plane is a conic passing through 8 points of the branch curve B. We encountered such curves on the focal surface $\Phi(S)$ and called them trope-quartics. We continue to call them trope-quartics.

Let $\{x_i, i \in I\}$ be a weakly even set of 6 nodes. The divisor

$$D_I = \tfrac{1}{2}(H - \sum_{i \in I} E_i)$$

satisfies $D_I^2 = -2$ and $D \cdot H = 2$. It defines a trope conic on Q passing through the six nodes. If $j \in I$, then D_I is τ_j-invariant, and, as above, we see that it is fixed by τ_j pointwise. The projection of D_I to the plane from x_j is a line passing through 5 singular points of B.

Theorem 12.2.18. *The sublattice $\mathcal{N}$ is a primitive sublattice of* $\mathrm{Pic}(\widetilde{Q})$ *if and only if the branch curve of the projection from each node is an irreducible plane sextic.*

Proof We apply the lemma to our case of the map $\pi : \widetilde{Q} \to Z$ whose branch curve $\bar{B}$ belongs to $|6e_0 - 2\sum_{i \neq \mu} e_i|$. We have $e_0 = h'$ and $\pi^*(e_0) = h = H - E_\mu$. Thus,

$$R \sim 3h - \sum_{i \neq \mu} E_i \sim 3H - 3E_\mu - \sum_{i \neq \mu} E_i.$$

Suppose B is reducible, then one of the nodes p_k of B is the intersection point of two irreducible components B_1 and B_2 of B. Thus, $R = R_1 + R_2$, where

$\pi^*(B_i) = 2R_i$. Since $R \cdot E_k = 2$, we get $R_1 \cdot E_k = 1$. However, $E_k \cdot D$ is even for any $D \in \mathcal{N}$. This contradiction. shows that R_1 defines a non-trivial element of $\mathcal{N}^s$.

Conversely, suppose the linear code $C_\mu \subset \mathbb{F}_2^\mu$ is non-trivial but the branch curve B_s of the projection map pr_{x_s} is irreducible. Since an irreducible plane sextic has at most 10 nodes, $\mu \le 11$. By Lemma 12.2.16, any non-trivial word in C_μ has weight equal to $6, 8$, or 10.

The Griesmer bound (12.13) from below gives $\dim C_\mu = 1$. Suppose $|w_I| = 6$, then the trope-conic $D_I = \frac{1}{2}(H - \sum_{i \in I} E_i)$ must be τ_j-invariant for any $j \in I$. As we explained in above, its projection from p_j to the plane is a line component of the branch curve B_j.

Suppose $|w_I| = 8$. Then, the argument is the same applied to the trope-quartic $C \sim \frac{1}{2}(2H - \sum_{i \in I} E_i) - 2E_j \in |D_i - 2E_j|$, where $j \notin I$.

Finally, if $|w| = 10$, then $\mu = 11$, and we may assume that $I = \{1, \ldots, 10\}$. Let $D = \frac{1}{2}(3H - 3E_{10} - \sum_{i=1}^{9} E_i)$. Then, $D^2 = 0$, and $|D| = 1$ is a pencil. The projection of D from p_1 is a pencil of cubic curves passing through 9 nodes of the irreducible 10-nodal sextic B. We find a cubic passing through an additional node and get a contradiction with the Bezout Theorem.

□

The curve B_1 intersects $B - B_1$ at $a(6 - a)$ simple points. If a is even, this set defines an even set of nodes of Q. If a is odd, it defines a weakly even set of nodes of cardinality $1 + a(a - 6) = 6$ or 10.

Theorem 12.2.19. *Let Q be a quartic surface with μ nodes with non-primitive sublattice $\mathcal{N}$. The following Table 12.1 describes possible binary linear codes C defined by even and weakly even sets of nodes.*

Here, $[w_1^{n_1}, \ldots, w_k^{n_k}]$ means that the code consists of n_i nonzero words of weight n_i.

Proof First, it is easy to list all possible reducible sextics with a given number of nodes. All the possibilities can be found in the last column of the Table.

Second, we use the proof of Lemma 12.2.15 to identify words $w_I = \{x_i, i \in I\}$ in C_μ with effective divisor classes D_I. It follows from (12.7) that D_I is τ_k-invariant if and only if the multiplicity at H is equal to the multiplicity at E_k.

Third, we will also use the *Griesmer bound* for a code $[n, k, d]$ of dimension k in $\mathbb{F}_2^n$ with minimal distance k [488, (5.2.6)]:

$$n \ge \sum_{i=0}^{k-1} [\frac{d}{2^i}]. \tag{12.13}$$

μ	Type	$\dim C$	C_μ	B	$(2,n)$
$\mu \le 5$	$\{0\}$	0	$\{0\}$	$6_{\mu-1}$	-
6	VI_a	0	$\{0\}$	6_5	-
	VI_b	1	$[6^1]$	$1_0 + 5_0$	-
7	VII_a	0	$\{0\}$	6_6	-
	VII_a	1	$[6^1]$	$1_0 + 5_1$	-
8	VIII_a	0	$\{0\}$	6_7	-
	VIII_b	1	$[6^1]$	$1_0 + 5_2$	-
9	IX	0	$\{0\}$	6_8	-
	IX_a	1	$[6^1]$	$1_0 + 5_3$	-
	IX_b	1	$[8^1]$	$2_0 + 4_0$	-
10	X	$\{0\}$	$\{0\}$	6_9	-
	X_a	1	$[10^1]$	$3_0 + 3_0$	-
	X_b	1	$[8^1]$	$2_0 + 4_1$	-
	X_c	1	$[6^1]$	$1_0 + 5_4$	-
	X_d	2	$[6^2, 8^1]$	$1_0 + 1_0 + 4_0$	-
11	XI_a	1	$[10^1]$	$3_0 + 3_1$	$(2,7)$
	XI_b	1	$[8^1]$	$2_0 + 4_2$	-
	XI_c	1	$[6^1]$	$1_0 + 5_5$	-
	XI_d	2	$[6^2, 8^1]$	$1_0 + 1_0 + 4_1$	-
12	XII_a	2	$[6^1, 8^1, 10^1]$	$1_0 + 2_0 + 3_0, 3_1 + 3_1, 1_0 + 5_6$	$(2.6)_I$
	XII_b	2	$[8^3]$	$2_0 + 4_3$	$(2,6)_{II}$
	XII_c	3	$[6^4, 8^3]$	$1_0 + 1_0 + 4_2$	-
	XII_d	2	$[6^2, 8]$	$1_0 + 1_0 + 4_2$	-
13	XIII_a	3	$[6^4, 8^3]$	$1_0 + 1_0 + 4_3, 2_0 + 2_0 + 2_0$	-
	XIII_b	3	$[6^3, 8^3, 10^1]$	$1_0 + 2_0 + 3_1, 3\ 1_0 + 3_0, 2\ 1_0 + 4_3$	$(2,5)$
14	XIV	4	$[6^7, 8^7, 10^2]$	$1_0 + 1_0 + 2_0 + 2_0, 3\ 1_0 + 3_1$	$(2,4)$
15	XV	5	$[6^{10}, 8^{15}, 10^6]$	$5\ 1_0 + (2,0)$	$(2,3)$
16	XVI	6	$[6^{16}, 8^{30}, 10^{15}, 16^1]$	$6\ 1_0$	$(2,2)$

Table 12.1 *Linear codes of general quartic μ-nodal surfaces*

$1 \le \mu \le 5$:

A plane sextic with less than five nodes is irreducible. So, the linear code is trivial if $\mu \le 5$.

$6 \le \mu \le 8$:

Since $d \ge 6$, the Griesner bound (12.13) gives $k = 1$. The linear code C_μ is generated by a word of weight 6 or 8. Since $\mu \le 8$, the reducible B could be only of type $1_0 + 5_{\mu-6}$, hence there is a word of weight 6.

Let $D_I \sim \frac{1}{2}(H - \sum_{i\in I} E_i)$ be the proper transform of a trope-conic. It is projected from any point $x_i, i \in I$, to a line component of the branch curve. So, its type is $1_0 + 5_{\mu-6}$. Note that projecting from a point $p_j, j \notin I$, we get a conic passing through 6 nodes of the irreducible branch curve.

$\mu = 9$:

If $k = 2$, then $d = 6$ and $\dim C_\mu^{\mathrm{ev}} = 1$, hence there is a word of weight 8 with distance 6 from the word of weight 6. It is easy to see that this is impossible because μ is too small. So, $k = 1$ and we have a new possibility that $d = 8$. We represent $D_I = \frac{1}{2}(2H - \sum_{i\in I} E_i - 2E_j)$ by a (-2)-curve. Its projection from x_j is a conic component of B. This leads to B of type $2_0 + 4_0$.

***μ* = 10, 11**:

If $k = 1$, then $d = 6$, or $d = 8$. We use the same argument as before. This gives the corresponding rows of the table with $k = 1$.

Suppose $k > 1$. The Griesmer bound gives $k = 2$, and $d = 6$. Since $\dim C_\mu^{\mathrm{ev}} = k - 1 \geq 1$, there is an even set $\{x_i : i \in I\}$ of 8 nodes. It must be the sum of two words of weight 6. This gives $D_I \sim \frac{1}{2}(2H - \sum_{i\in I} E_i - 2E_j), j \notin I$, is the sum of two trope-conics $\frac{1}{2}(H - \sum_{i\in I_1} E_i - E_j) + \frac{1}{2}(H - \sum_{i\in I_2} E_i - E_j)$, where $I = I_1 + I_2$. The projection of D_I from x_j is the union of two line components of B, the residual part is an irreducible quartic with $\mu - 10$ nodes.

***μ* = 12**:

By (12.2.16), $k \geq 2$. By Griesmer bound $k \leq 3$.

Suppose $k = 2$ and $d = 8$. Then, all non-trivial words are of weight 8. Let $\{x_i, i \in I\}$ be a weakly even set of 8 nodes. The effective divisor $D_I \sim \frac{1}{2}(2H - \sum_{i\in I} E_i - 2E_j)$ is defined by a trope-quartic. Its projection from x_j is a degree two irreducible component B_1 of the branch curve B. The residual component must also be defined by a weakly even set, and it is easy to see that it must be a 3-nodal quartic B_2.

Assume that $k = 2$ and $d = 6$. Then, $\dim C_\mu^{\mathrm{ev}} = 1$, and there exists a weakly even set of 8 nodes. We argue as above, and obtain that it defines a reducible effective divisor D_I, and its projection is the union of two lines. This corresponds to the code of type $[6^2, 8]$.

Assume that $k = 3$. The Griesmer bound implies that the minimal distance $d = 6$ and $\dim C_\mu^{\mathrm{ev}} = 2$. We may assume that the three even sets of nodes correspond to subsets I, J, K as above. There will be a weakly even set of six nodes. It has four common nodes with each even set of nodes. Without loss of generality, we may assume that it corresponds to a subset $L = \{1, 2, 5, 6, 9, 10\}$. Projecting from x_{12}, we find that the image of the trope-conic $\frac{1}{2}(H - \sum_{i\in L} E_i)$ is a conic passing through four nodes of the projection B_1 of D_I. It also passes through two nodes of the residual 3-nodal quartic B_2.

***μ* = 13**:

Applying (12.2.16), we get $k \geq 3$. By Griesmer bound, the minimal distance $d = 6$. If $k \geq 4$, then $\dim C_\mu^{\mathrm{ev}} = 3$ and the minimal distance is 8, contradicting the Griesmer bound. So, we obtain that $k = 3$ and $\dim C_\mu^{\mathrm{ev}} = 2$. As in the case $\mu = 12, k = 3$, we may assume that B has a smooth conic as its irreducible component. In this case, the residual part becomes the union of two smooth conics or a line and a nodal cubic. The second possibility implies that C_μ has a word of weight 10. Since $\mu = 13$, it is a unique word of weight 10 (since otherwise they have to share 6 nonzero coordinates that implies that $\mu \geq 15$). We have two more possible types of $B : 1_0 + 1_0 + 1_0 + 3_0$ and $1_0 + 1_0 + 4_3$. They correspond to a different choice of the center of the projection.

μ = 14:

By (12.2.16) $k \geq 4$. By Griesmer bound $\dim C_\mu^{\mathrm{ev}} \leq 3$, hence $k = 4$ and $\dim C_\mu^{\mathrm{ev}} = 3$. A plane sextic with 13 nodes must be of type $1_0 + 1_0 + 1_0 + 3_1$ or $1_0 + 1_0 + 2_0 + 2_0$. Assume that B is of the first type. The irreducible component of the ramification curve R defined by the nodal cubic component of B defines a weakly even set of 10 nodes. It follows from the discussion of the previous case that it is a unique weakly even set of 10 nodes. We must also have $7 = 2^3 - 1$ even sets of 8 nodes. The rest is the set of 7 weakly even sets of 6 nodes. This gives us the binary linear code of type $[6^7, 8^7, 10]$.

Assume that B is of the second type. Then, deleting one conic component of B, and taking the proper transform of the other three components, we will find a weakly even set of 10 nodes. The rest of the argument is the same as before and leads to the same before and leads to the same linear code C_μ.

μ = 15:

We argue as in the previous case. We get $k = 5$, and $C_\mu^{\mathrm{ev}} = 4$. In this case, B consists of four lines and a conic. We can identify Q with the focal surface of a congruence of lines of bidegree $(2, 3)$. There are 10 trope-conics that give 10 words of weight 6. This gives that C_μ is of type $[6^{10}, 8^{15}, 10^6]$.

μ = 16:

This is, of course, a frequently discussed case of Kummer quartic surfaces. We have $k \geq 6$. In fact, since the orthogonal complement of $\mathcal{N}$ in $H^2(\widetilde{Q}, \mathbb{Z})$ is of rank 6, the proof of Proposition 12.2.16 show that $k = 6$.

We have 16 trope-conics T_i whose sum $\sum_{i=1}^{16} T_i$ is equal to $8H - 3\sum_{i=1}^{16}$ (since each node is contained in six trope-conics). The sum $\sum_{i=1}^{16} E_i$ represents the unique word w_{16} of weight 16. This implies that C_μ contains 16 words of weight 6. The map $w_6 \mapsto w_6 + w_{16}$ is a bijection from the set of words of weight 6 to the set of words of weight 10, and the map $w_8 \mapsto w_8 + w_{16}$ is an involution on the se if words of weight 8. Since $\dim C_\mu^{\mathrm{ev}} = 5$, this implies that the linear code C_μ is of type $[6^{16}, 8^{30}, 10^{16}, 16^1]$. □

12.2.4 Moduli space of nodal quartic surfaces

Let Q be a μ-nodal quartic surface in $\mathbb{P}^3$ and $\widetilde{Q}$ be its minimal nonsingular model, a K3 surface. The birational morphism $\widetilde{Q} \to Q \subset \mathbb{P}^3$ defines a quasi-polarization h of $\widetilde{Q}$ of degree $h^2 = 4$. Recall that there are three type of quasi-polarization of degree 4 on a K3 surface (i.e. a pseudo-ample divisor class h with $h^2 = 4$). They are distinguished by the equivalence classes of embeddings of the lattice

$$\iota : \langle 4 \rangle \hookrightarrow \mathrm{Pic}(\widetilde{Q}) \subset H^2(\widetilde{Q}, \mathbb{Z}).$$

A *general quasi-polarization* of degree 4 has the property that $|h \cdot f| \geq 3$ for any isotropic vector $v \in \mathrm{Pic}(\widetilde{Q})$. Other possibilities are that there exists an isotropic vector f in $\mathrm{Pic}(\widetilde{Q})$ such that $h \cdot f = 2$ (a *hyperelliptic* quasi-polarization) or 1 (a *unigonal* quasi-polarization). The polarization coming from a nodal quartic surface is a general quasi-polarization of degree 4.

Two polarized K3 surfaces $(\widetilde{Q}, h)$ and $(\widetilde{Q}', h')$ are isomorphic if there exists an isomorphism $f : X \to X'$ such that $f^*(h') = h$. This corresponds to projective equivalence of nodal quartic surfaces. So, we are interested in the moduli space $\mathcal{K}3_{4,\mu}$ of degree 4 polarized K3 surfaces $(\widetilde{Q}, h)$ such that $|h|$ defined a birational morphism onto a quartic surface with μ nodes. It is known that a nodal quartic surface represents a stable point in the action of $\mathrm{PGL}(4)$ in the space of quartic surfaces [709]. The moduli space $\mathcal{K}3_{4,\mu}$ is constructed as the closure of the $\mathrm{PGL}(4)$-orbits of μ-nodal quartic surfaces in the GIT-quotient $|\mathcal{O}_{\mathbb{P}^3}(4)|//\mathrm{PGL}(4)$. As we already know, $\mathcal{K}3_{4,\mu}$ may consist of several irreducible components.

Theorem 12.2.20. *Let Q be a general μ-nodal quartic surface and $\widetilde{Q}$ be the associated K3 surface with the quasi-polarization H of degree 4. The isomorphism class $\mathsf{N}_\mu^{\mathrm{s}}$ of the saturation $\mathcal{N}^{\mathrm{s}}$ of the sublattice $\mathcal{N} \subset \mathrm{Pic}(\widetilde{Q})$ of rank $\mu+1$ generated by H and the exceptional curves E_i is uniquely determined by the binary linear code C_μ of $\widetilde{Q}$. The primitive embedding $\mathsf{N}_\mu^{\mathrm{s}} \hookrightarrow H^2(\widetilde{Q}, \mathbb{Z})$ is uniquely determined by the code too.*

Proof We only sketch the proof since we do not intend to go into the technicalities of the theory of K3 surfaces.

We know from the discussion in Subsection 11.5.3 that the binary linear code C_μ determine an isotropic subgroup A of the discriminant group $\mathrm{Disc}(\mathsf{N})$ which defines the lattice $\mathsf{N}_\mu^{\mathrm{s}}$ with discriminant group $A^\perp/A$. It follows from the theory of quadratic lattices that two isotropic subgroup A_1, A_2 of define isomorphic overlattices of N if and only if they are conjugate under an automorphism of

$\mathrm{Disc}(\mathsf{N})$ from the image of the natural homomorphism

$$\rho_{\mathsf{N}} : \mathsf{O}(\mathsf{N}) \to \mathsf{O}(\mathrm{Disc}(\mathsf{N})).$$

[550, Proposition 1.4.2]. It follows from [550, Theorem 1.14.2] that the discriminant group $(\mathrm{Disc}(M), q)$ together with its quadratic form determine the isomorphism class of M provided that $t_+ + t_- \geq l(\mathrm{Disc}(M)) + 2$. Also, the equivalence class of a primitive embedding of M into a unimodular lattice L of signature $(l_+ \geq t_+, l_- \geq t_-)$ is determined uniquely provided $(l_+ + l_-) - (t_+ + t_-) \geq l(\mathrm{Disc}(M)) + 2$.

In our case, $t_+ = 1, t_- = \mu$ and $l(\mathrm{Disc}(\mathsf{N}_\mu^{\mathrm{s}})) = 1 + \mu - 2k$, so the isomorphism class of $\mathsf{N}_\mu^{\mathrm{s}}$ is determined uniquely if $k > 0$. Also, we have $N = H^2(\widetilde{Q}, \mathbb{Z})$ with $l_+ = 3, l_- = 19$, so the equivalence class of a primitive embedding $\mathsf{N}_\mu^{\mathrm{s}} \hookrightarrow H^2(\widetilde{Q}, \mathbb{Z})$ is determined uniquely if $22 - (1 + \mu) \geq (1 + \mu - 2k) + 2$, i.e. $\mu \leq k + 9$. In Proposition 12.2.16, we have proved a weaker inequality $\mu \leq k + 10$. Inspecting the Table, we see that, $\mu > k + 9$ implies $\mu = k + 10$ and $(22 - (1 + \mu) = l(\mathrm{Disc}(\mathsf{N}_s^{\mathrm{s}})) = 1 + \mu - k$. A finer result of Nikulin tells that the uniqueness of a primitive embedding still holds provided that $\mathcal{N}_\mu^{\mathrm{s}}$ contains two vectors f_1, f_2 with the Gram matrix $\left(\begin{smallmatrix} 0 & 2 \\ 2 & 0 \end{smallmatrix}\right)$ or $\left(\begin{smallmatrix} 0 & 2 \\ 2 & -2 \end{smallmatrix}\right)$. If $\mu \geq 2$, this condition is satisfied in our case if we take $f_1 = H - E_1 - E_2, f_2 = E_1$.

The assertion is obvious if $\mu = 0$, since $\mathcal{K}3_{4,0}$ is the GIT-quotient of $|\mathcal{O}_{\mathbb{P}^3}(4)|$. If $\mu = 1$, we use that the discriminant hypersurface of quartic surfaces is irreducible, its Zariski open set parameterizes one-nodal quartic surfaces.

□

A *lattice M polarization* of a K3 surface $\widetilde{Q}$ is a primitive embedding of lattices $j : M \hookrightarrow \mathrm{Pic}(\widetilde{Q})$ such that the image of $C(M)$ contains a nef and big divisor class. A lattice M polarization is called *ample* if $j(C(M))$ contains an ample divisor. In this case, one may omit a choice of $C(M)$ by fixing it uniquely in such a way that its image contain an ample divisor.

Two lattice M polarizations $(\widetilde{Q}, j)$ and $(\widetilde{Q}', j')$ are said to be isomorphic if there exists an isomorphism of surfaces $f : \widetilde{Q} \to \widetilde{Q}'$ such that $f^* \circ j' :\hookrightarrow \mathrm{Pic}(\widetilde{Q})$ coincides with j. One constructs a course moduli space $\mathcal{M}_{K3,M}$ of *lattice M polarized* K3 surfaces $(\widetilde{Q}, j)$ [239]. The moduli space $\mathcal{M}_{K3,M}$ is a quasi-projective variety, each of its irreducible components is of dimension $19 - m$. It is known that $\mathcal{M}_{K3,M}$ is irreducible if M contains as a primitive sublattice of rank 2 generated by isotropic vectors v_1, v_2 with $v_1 \cdot v_2 \leq 2$ [239, Proposition 5.6].

Applying this in our case when $M = \mathsf{N}_\mu^{\mathrm{s}}$, we see that the condition is satisfied when $\mu \leq 3$ since we can take $v_1 = H - E_1 - E_2, v_2 = H - E_2 - E_3$. Using

Proposition 12.2.20, we obtain that $\mathcal{M}_{K3,\mathsf{N}^s_\mu}$ is an irreducible component of $\mathcal{K}3_{4,\mu}$. Thus, Table 12.1 describes irreducible components of $\mathcal{K}3_{4,\mu}$.

Corollary 12.2.21. *The number N of irreducible components of the moduli space $\mathcal{K}3_{4,\mu}$ of μ-nodal quartic surfaces is given in the following Table.*

μ	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
N	1	1	1	1	1	1	2	2	2	3	5	5	4	2	1	1	1

12.2.5 Equations of nodal quartics

Let us give some additional geometric information about general nodal quartic surfaces from different irreducible components of the moduli space $\mathcal{K}3_{4,\mu}$ of μ-nodal quartic surfaces. Recall that each irreducible component of $\mathcal{K}3_{4,\mu}$ coincides with the moduli space $\mathcal{M}_{K3,\mathsf{N}^s_\mu}$ of lattice N^s_μ polarized K3 surfaces.

1. If $\mu \le 5$, $\mathcal{K}3_{4,\mu}$ is irreducible and the branch curve of any projection map is a plane sextic with $\mu - 1$ nodes that admit a contact conic. The blow-up of its nodes identifies its proper transform with an anti-canonical divisor of a del Pezzo surface of degree $d = 10 - \mu$.
2. If $\mu = 6$, there are two families of quartic surfaces with 6 nodes dependent on whether the nodes are coplanar or not. If they are non-coplanar, the linear code is trivial.

   Assume that the nodes are not coplanar. The linear system $|L| = |\mathcal{O}_{\mathbb{P}^3}(2) - x_1 - \cdots - x_6|$ of quadrics containing the nodes is of dimension 3. It defines a degree 2 rational map $f : \mathbb{P}^3 \dashrightarrow \mathbb{P}^3$. The equation of Q is of the form

$$F_2(q_0, q_1, q_2, q_3) + J(q_0, q_1, q_2, q_3) = 0, \tag{12.14}$$

   where (q_0, q_1, q_2, q_3) defines a basis of the linear system $|L|$, and $J(q_0, q_1, q_2, q_3)$ is their jacobian. In this way, we see the twelve moduli of such surfaces. The surface $W = V(J(f_0, \ldots, f_3))$ is a *Weddle quartic surface*. We will discuss it in Section 12.3

   On the other hand, we know that quartics that contain a weakly even set of six nodes, contain a trope-conic, in particular, the nodes are coplanar. The linear system $|L|$ is of dimension 4, it contains a codimension one subspace of reducible quadrics. The linear system defines a rational map $f : \mathbb{P}^3 \dashrightarrow \mathbb{P}^4$. It blows down the plane Π containing the nodes to a point $P \in \mathbb{P}^4$. The image of $\mathbb{P}^3$ is a cone over a quadric in $\mathbb{P}^3$ isomorphic to the weighted projective space $\mathbb{P}(1, 1, 1, 2)$. The equation of Q can be put in the form

$$F_2(x, y, z)^2 + xF_3(x, y, z, w) = 0, \tag{12.15}$$

where the nodes lie in the plane $V(x)$ and the cubic $V(F_3)$ contains the nodes.

3. If $\mu = 7$, we have already discussed the family of asyzygetic 7-nodal quartics given by equation (12.3.2). They describe quartics from the irreducible component $\mathcal{M}_{K3,\mathsf{N}_7}$.

   Another component $\mathcal{M}_{K3,\mathsf{N}_7^s}$ consists of isomorphism classes of 7-nodal quartics with six of the nodes lying on a conic. The linear system of quadrics through the nodes is of dimension 4. It defines a rational map $f : \mathbb{P}^3 \dashrightarrow \mathbb{P}^3$ that blows down the plane spanned by the conic to a point P. The image of the map is again a quadratic cone with vertex at at P. The equation of Q can be put in the form (12.15), where F_3 contains six of the nodes lying on a conic in $V(x_0)$ and tangent to the quadric $V(F_2)$ at the remaining node.

   The branch curve B of a projection from a node is either of type $1_0 + 5_1$ or irreducible but admits a conic passing through six nodes.

4. Assume $\mu = 8$. If we fix a subset of 7 nodes, then we know from the previous subsection that the linear system of quartics $\mathcal{Q}_7$ with the seven nodes contains a subfamily of quartics with an additional node at the Cayley dianode surface $\mathfrak{D}$. They are the pre-images of hyperplane sections of the cone $\mathbb{P}(1,1,1,2) \subset \mathbb{P}^6$ which are tangent to the image of $\mathfrak{D}$ under the rational map $\mathbb{P}^3 \to \mathbb{P}^6$ given by $\mathcal{Q}_7$.

   So, we see that the locus of sets of 8 points realized as the set of nodes of an irreducible quartic surface is of codimension one in the variety of sets of eight points in $\mathbb{P}^3$. Its dimension (modulo projective equivalence) is equal to 8, and the dimension of the linear space of quartics that contains a fixed set of 8 points is equal to 3. This adds up to the correct number of moduli of an irreducible component of 8-nodal quartic surfaces.

   We can write a general quartic with a given set of 8 nodes by an equation

$$F_2(q_0(x,y,z,w), q_1(x,y,z,w)) + F_4(x,y,z,w) = 0, \tag{12.16}$$

   where $V(q_0), V(q_1)$ are quadrics generating the pencil of quadrics passing through 8 nodes, and F_4 is any 8-nodal quartic surface not belonging to this pencil.

   We have another family of 8-nodal quartic surfaces of Type VIII_a that contain one weakly even set of nodes. In this case six of the eight nodes are lie on a conic. A general quartic surface from this family is given by equation (12.15), where where $V(F_3(x,y,z,w))$ is tangent to $V(F_2)$ at two points.

   The branch curve B in this case is with irreducible with six nodes on a conic, or reducible of type $1_0 + 5_2$.

5. Assume $\mu = 9$. Surfaces of type IX with trivial linear code C_μ are quartics in the family (12.16) of quartics with nodes at $x_1, \ldots, x_8$ that acquire an additional node x_9. The locus of these nodes lie on the determinantal curve $D(x_1, \ldots, x_8)$, the locus of points such that

$$\text{rank } \mathcal{J}(q_0, q_1, F_4) \leq 3,$$

where $\mathcal{J}(q_0, q_1, F_4)$ is the jacobian matrix of polynomials q_0, q_1, F_4. The curve $D(x_1, \ldots, x_8)$ is called the *dianodal curve*[430, p. 12]. If we fix seven of the eight nodes $x_j, j \leq i, 8$, we obtain that a 9-nodal quartic lies on the Cayley dianodal surface $\mathfrak{D}_i$ defined by the seven points. Thus,

$$D(x_1, \ldots, x_8) = \bigcap_{i=1}^{8} \mathfrak{D}_i.$$

In fact, we know that $\mathfrak{D}_7 \cap \mathfrak{D}_8$ contains the lines $\ell_{ij} = \langle x_i, x_j \rangle, 1 \leq i < j \leq 6$ and the twisted cubic γ_{ij} through $x_1, \ldots, x_6$. The residual curve is of degree 18. Changing a pair of points, we obtain that the residual curve of degree 18 coincides with the dianodal curve. Since $\mathfrak{D}_7 \cap \mathfrak{D}_8$ have 6 common triple points through which passes 6 lines ℓ_{ij} and the twisted cubic γ_{ij}, we find that $D(x_1, \ldots, x_8)$ has triple points at $x_1, \ldots, x_8$. We refer for details to [430, p.12] and [630, §8].

If $V(F_4)$ is one of a 9-nodal surface with the ninth node on the dianodal curve, other surfaces can be written in the form

$$F_2(x, y, z, w)^2 + F_4(x, y, z, w) = 0, \tag{12.17}$$

where $V(F_2)$ is the unique quadric through the nine nodes.

Surfaces of Type IX$_a$ admit one weakly even set of 6 nodes lying on a conic. A general quartic surface from this family is given by equation (12.15), where $V(F_3(x, y, z, w))$ is tangent to $V(F_2)$ at three points.

Surfaces of Type IX$_b$ admit one even set of nodes. We know that they contain a trope-quartic passing through $p_1, \ldots, p_8$ and having p_9 as its double point. The projection of this curve from x_9 is a conic. A general member of the family can be given by equation

$$F_2(x, y, z, w)^2 + C_2(x, y, z)G_2(x, y, z, w) = 0, \tag{12.18}$$

where $x_9 = [0, 0, 0, 1]$, $V(F_2)$ is the unique quadric containing $x_1, \ldots, x_9$, $V(C_2)$ is the cone with vertex at p_9 passing through $p_1, \ldots, p_8$, and $V(G_2)$ is any quadric from the net of quadrics with baee points $x_1, \ldots, x_8$.

6. Assume $\mu = 10$. One of the families here is very important and occurs in many constructions in algebraic geometry. This is the family of surfaces of

Type X_c that admit a weakly even set of 10 nodes. The branch curve B of the projection of this set from any node is the union of two cubic curves intersecting at 9 points. According to a theorem of Cayley this property characterizes 10-nodal quartic symmetroid surfaces. The surfaces are given by the determinant of a symmetric matrix with linear forms as its enties. A general matrix defines a quartic symmetroid.

Surfaces of Type X with trivial linear code C_{10} are contained in the family of 9-nodal surfaces given by equation (12.17). We may fix first 8 nodes such that the ninth node lies on the dianodal curve $D(x_1, \dots, x_8)$. We find the tenth node at the intersection of this curve with the dianodal surface $\mathfrak{D}$ corresponding to seven points $x_1, \dots, x_6, x_9$. The intersection of the curve with the surface is of multiplicity 108, and it contains among them the triple points $x_1, \dots, x_6$ of the curve and the surface. So, we have to subtract $9 \times 6 = 54$ from 108. However, we have to subtract more. The surface $\mathfrak{D}$ contains 15 lines $\langle x_i, x_j \rangle$ joining two of the points $x_1, \dots, x_6, x_9$. The dianodal curve has two points on it. Also it has two points on the twisted cubic through these seven points. If a quartic contains one of these 32 points, then it will contain a line or the twisted cubic and this put an extra condition on the surface. A general 10-nodal quartic does not pass through these points. This leaves us with $13 = 108 - 54 - 32$ choices of the tenth point. Note that Jessop shows that one of these choices leads to the Cayley quartic symmetroid [430, p. 15].

Surfaces of Type X_a admit only one weakly even set of 6 nodes. A general member of this family admits equation of the form (12.15) where $V(F_2)$ is tangent to $V(F_3)$ at 4 points.

Surfaces of Type X_b admit an even set of 8 nodes $x_1, \dots, x_8$. Their ten nodes lie on a quadric $V(F_2)$. A general member of this family has equation of the form

$$F_2(x, y, z, w)^2 + K_1(x, y, z, w)K_2(x, y, z, w) = 0, \tag{12.19}$$

where K_1 and K_2 are cones with vertices at x_9, x_{10} over the conic component of the branch curves of the projections $\mathrm{pr}_{x_9}, \mathrm{pr}_{x_{10}}$.

Surfaces of Type X_d have two-dimensional linear code C_{10}. They admit two weakly sets of 6 nodes with common two nodes. We may assume that they lie in planes $V(x)$ and $V(y)$. The equation of a surface is of the form

$$F_2(x, y, z, w)^2 + xyG_2(x, y, z, w) = 0. \tag{12.20}$$

7. Assume $\mu = 11$. Starting from $\mu = 11$, the binary linear code C_μ is non-trivial. We have four families corresponding to different reducible nodal plane sextics. We also know that irreducible plane sextics that admit a

contact conic intersecting it at nonsingular points are of three different species dependent on 6, 7, or 10 apparent nodes. They appear as projections from other nodes.

The surfaces of Type XI_a are familiar to us since they are realized as the focal surfaces of congruences of lines of bidegree $(2,7)$. The projection from its unique fundamental point of degree 6 has an irreducible 10-nodal plane sextic. Also, for each its node there exists a plane quartic with triple point at it and simple points at other 9 nodes. The latter conditions guarantees that the sextic admits contact conic. Projecting from a fundamental point of degree 3, we obtain the branch curve of type $3_0 + 3_1$. We can consider surfaces from this family as Cayley quartic symmetroids acquiring an additional node.

Surfaces of type XI_b has a unique even set of 8 nodes $\{x_i, i \in I\}$. The projection from a node outside of this set has an irreducible branch sextic that admits a curve of degree 4 which passes through eight nodes and has a triple point at some other node. This is a plane sextic of the third species with 10 apparent nodes.

Surfaces of type XI_c form a hypersurface in the family X_c of 10-nodal surfaces which admit a unique trope-conic. We know that the projection from a node not lying on the trope-conic is branched over an irreducible plane sextic of the first species with 6 apparent nodes. The projection of the trope conic is a conic containing 6 of the nodes.

The equation of the irreducible sextic is

$$f_3(x,y,z)^2 + f_2(x,y,z)g_2(x,y,z) = 0,$$

where $V(f_2)$ is the contact conic and $V(g_2)$ is the conic through six nodes. This leads to an equation of the quartic of Type XI_a general member of the form

$$w^2 f_2(x,y,z) + 2f_3(x,y,z)w + g_2(x,y,z)^2 = 0. \tag{12.21}$$

A surface of Type XII_d has two subsets A, B of six nodes lying on a conic. The subsets have two common nodes. The projection from the unique node x_i not belonging to these sets has branch curve of type $1_0 + 1_0 + 4_1$. The point x_i is a double points of a quartic curve that passes through the even set of nodes $A + B$. The pencil of quadrics through the even set of nodes contains the union of two conics as its base points. Let $V(F_2)$ be the quadric in the pencil that passes through x_i, then the equation of a 11-nodal quartics can be put in the form

$$F_2(x,y,z,w)^2 + \lambda xyK_2(x,y,z,w) = 0, \tag{12.22}$$

where $V(K_2)$ is a quadratic cone with vertex at x_i.

8. Assume $\mu = 12$. Surfaces of type XII_a are specializations of the Cayley quartic symmetroid. They acquire two additional nodes when each of the cubic components of the branch curve acquires a node. The surfaces are also realized as the focal surfaces of congruences of bidegree $(2, 6)_I$. The projection from the fundamental point of degree 5 has the branch divisor equal to the union of a 5-nodal quintic, and the line equal to the projection of a trope-conic.

   A trope-octic contains an even set of 8 nodes: six fundamental points of degree 3, one of degree 1 and one of degree 5. The projection from a fundamental point of degree 2 is a conic. It is an irreducible component of a branch curve of type $1_0 + 2_0 + 3_0$. The projection from a fundamental point of degree 3 is a nodal cubic. The projection of a trope-quintic from the same point is another cubic component of the branch curve of type $3_1 + 3_1$.

   Surfaces of Type XII_b are realized as the focal surfaces of congruencies of bidegree $(2, 6)_{II}$. The projection from a fundamental points of degree 4 is a quartic curve with three nodes, the projections of other three fundamental points of degree 4. It is an irreducible component of the branch curve of type $2_0 + 4_3$. The conic component is the projection of a trope-quartic.

   Surfaces of Type XII_c have three sets of weakly even nodes intersecting each other at two nodes. The union of these sets is the set of twelve nodes. We may assume that the equations of the corresponding trope planes are $x = 0, y = 0, z = 0$. Then, the equation of the surface can be put in the form

$$F_2(x, y, z, w)^2 + xyzL(x, y, z, w) = 0, \tag{12.23}$$

   where $V(F_2)$ is the quadric containing all nodes, and $V(L, F_2)$ is a trope-tonic with the set of nodes equal to the symmetric sum of the sets of 6 nodes on the planes $x = 0, y = 0, z = 0$. The surfaces are specializations of surfaces of Type XI_c when the cone $V(g_2)$ splits into two planes.

   Surfaces of Type XII_d are obtained when a surface of Type XI_d acquires an additional node.

9. Assume $\mu = 13$. We have two families of 13-nodal quartic surfaces. Suppose the branch curve of the projection from a node x is equal to the union B of three conics with a common contact conic. One can consider B as a degeneration of a 10-nodal sextic of the first species. It is the projection of the union of three conics on a smooth quadric. A choice of a splitting component of each conic defines six apparent nodes of B and hence a conic through these nodes. In this way we find four conics passing through six nodes, two on each irreducible component. each of these conic splits under the projection pr_x, and one of the splitting component is a trope-conic This leads to the binary linear code C_μ of type $[6^4, 8^3]$ and implies that the

surface is of Type XIII_a. One can also project from the point common to two trope-conics. The branch curve of this projection is of type $1_0 + 1_0 + 4_3$.

The surfaces of Type XIII_a are specializations of surfaces of Type XII_c. They are given by equation (12.23), where L is is chosen in a such a way that the surface acquires an additional node.

A general surface of Type XIII_b is the focal surface of a congruence of lines of bidegree $(2,5)$. In fact, the trope-octic T_x corresponding to a fundamental points of degree 3, contains a weakly even set of 10 points formed by the fundamental point of degree 4, two fundamental points of degree 3, six fundamental points of degree 2, and one fundamental point of degree 1. The existence of a weakly set of 10 nodes distinguishes the two irreducible components of the moduli space of 13-nodal quartic surfaces.

The projection from x maps T_x to a component of the branch curve of type 3_1. The singular point is the projection of the fundamental point of degree 4. The branch curve has two other irreducible components, the images of the unique trope-conic corresponding to the fundamental point of degree 1 conjugate to x, and the image of a quartic conic that contains x and six fundamental points of degree two (they are trope=conics of a confocal congruence), This realizes the branch curve of type $1_0 + 2_0 + 3_1$.

Projecting from the unique fundamental point of degree four has the branch curve of type $1_0 + 1_0 + 4_3$, same as for surfaces of Type XIII_a. The line components are the projections of trope-conics corresponding to non-conjugate fundamental points of degree one. Projecting from a fundamental point of degree two also gives the branch curve of type $1_0 + 2_0 + 3_1$.

We know that one of the fundamental points of degree one is conjugate to two non-conjugate fundamental points of degree one. This implies that three weakly even sets w_I of 6 nodes have a common node. We may assume that the sets I are $\{1,2,3,4,5,6\}$, $\{1,2,7,8,9,10\}$, $\{1,3,7,11,12,13\}$. The symmetric sum of these sets is the weakly even set w_J of 10 nodes, where $J = \{1,4,5,6,8,9,10,11,12,13\}$. The linear system $|\frac{1}{2}(3H - 3E_1 - \sum_{j\in j, j\neq 1} E_i|$ is a pencil of elliptic curves on $\widetilde{Q}$. It contains one nonsingular member which is invariant under the involution τ_{x_1}. Projecting from x_1, we obtain the branch curve of type $1_0 + 1_0 + 1_0 + 3_0$.

Assume that the three trope-conics lie in planes $x = 0, y = 0, z = 0$. Then, the point $[0,0,0,1]$ is the common node of three trope-conics. The quadric containing the first two trope-conics gas an equation $wz + xl + ym = 0$, where l, m are some linear forms in x, y, z, w. We can put the equation of Q in the form

$$(wz + xl + ym)^2 + xyF_2(x, y, z, w) = 0.$$

Substituting $z = 0$, we must get the square of a quadratic form. This allows us to write F_2 in the form $-4lm + zn$ and get the equation of Q in the form:

$$w^2z^2 + x^2l^2 + y^2m^2 - 2xylm - 2zwxl - 2wzym + 4xyzn$$
$$= \det\begin{pmatrix} 0 & z & y & l \\ z & 0 & x & m \\ y & x & 0 & w \\ l & m & w & n \end{pmatrix} = 0. \tag{12.24}$$

So, we see that surfaces of Type XIII_b are specialization of the Cayley quartic symmetroid.

10. Assume $\mu = 14$. The moduli space of 14-nodal quartics is irreducible. A reducible branch curve is of type $1_0+1_0+2_0+2_0$ or $1_0+1_0+1_0+3_1$. A general surface of Type XIV is isomorphic to the focal surface of a congruence of lines of bidegree $(2,4)$. We know that it has two conjugate fundamental points of degree 3, each conjugate to three fundamental points of degree one. Projecting from one of these points, we get the branch curve of type $1_0 + 1_0 + 1_0 + 3_1$.

   We may choose a basis of the linear code C_{14} formed by four weakly even sets w_I of six nodes, where I is one of the sets

$$I_1 = \{1,2,3,4,5,6\},\ I_2 = \{1,2,7,8,9,10\},$$

$$I_3 = \{3,4,7,8,11,x12\},\ I_4 = \{1,3,9,11,13,14\}.$$

   The sum $w_{I_1} + w_{I_2}$ is an even set of 8 nodes that has two common nodes with w_{I_4}. Similarly, $w_{I_3} + w_{I_4}$ has two common nodes with w_1. Projecting from the common node x_1 of w_{I_1} and w_{I_4}, we get the branch curve of type $1_0 + 1_0 + 2_0 + 2_0$.

   A surface of type XIV lies in the intersection of the families of surfaces of Types XIII_a and XIII_b when they acquire an additional node. The branch curve of type $2_0+2_0+2_0$ (resp. $1_0+2_0+3_1$) becomes of type $1_0+1_0+2_0+2_0$ (resp. $1_0 + 1_0 + 1_0 + 3_1$) when one of the conics becomes reducible.

   Similar to the previous case, one can show that the equation of a surface of Type XIV can be put in the form

$$\det\begin{pmatrix} 0 & z & x' & l \\ z & 0 & x & y' \\ y & x & 0 & z' \\ x' & y' & z' & 0 \end{pmatrix} = 0. \tag{12.25}$$

   where $x = 0, y = 0, z = 0, x' = 0, y' = 0, z' = 0$ are equation of the six tropes cutting out the six trope-conics. (see [430, p. 21].

11. Assume that $\mu = 15$. A general surface of Type XV is isomorphic to the focal surface of a congruence S of bidegree $(2,3)$. The branch curve of any projection is of type $1_0 + 1_0 + 1_0 + 1_0 + 2_0$. Recall that the conjugacy graph $\ell(S)_1$ is the Petersen graph. Each fundamental point is congruent to three fundamental points of degree one. Projecting from one of them, we get the four lines components of the branch curve. They are the images of the four trope-conics. The conic component is the projection of a quartic with double point at the center of the projection. It is one of five trope-quartics of a confocal congruence.

    The equation of a general 15-nodal quartic surface is obtained by adding a linear equation to the equation of the Castelnuovo-Richmond quartic hypersurface CR_4 [255, §6].
12. A quartic with $\mu = 16$ is a jacobian Kummer surface. We cannot add here anything that has not be already discussed about these surfaces.

Remark 12.2.22. 1. Let $X \subset \mathbb{P}^3$ be surface of degree d and Σ be an even of weakly even set of nodes on X. Then, there exists a vector bundle $\mathcal{E}$ over $\mathbb{P}^3$ and a section $s \in H^0(\mathbb{P}^3, S^2(\mathcal{E}(d+\epsilon))) \subset \mathrm{Hom}(\mathcal{E}^\vee(-d-\epsilon), \mathcal{E})$ such that the scheme $Z(s)$ of its zeros is equal to Σ [93]. Here, $\epsilon = 0$ if Σ is even, and equal to 1 otherwise. It was shown earlier by Barth [33] that s always defines a pair (X, Σ) as above. The surface X is equal to the support of the cokernel of the map $s : \mathcal{E}^\vee(-d-\epsilon) \to \mathcal{E}$ and Σ is the locus where the rank of the map drops by 2.

For example, if X is a quartic symmetroid with 10 nodes (Type X_c surface from Table 12.1), we know from Example 4.3.8 that X is the determinant of a symmetric matrix with linear forms as its entries. This corresponds to a section of $S^2(\mathcal{O}_{\mathbb{P}^3}(1)^{\oplus 4})$. Since in this case $d = 4, \epsilon = 1$, we get $\mathcal{E} = \mathcal{O}_{\mathbb{P}^3}(-2)^{\oplus 4}$.

Another example is a jacobian Kummer quartic surface. We know from (10.6) that the projection $p : Z_{\mathbb{G}} \to \mathbb{P}^3$ is the projective bundle $\mathbb{P} = \mathbb{P}(\Omega^1_{\mathbb{P}^3}(1))$. The projection $q : Z_{\mathbb{G}} \to \mathbb{G}$ is given by the complete linear system $|\mathcal{L}| = |\mathcal{O}_{\mathbb{P}}(1) \otimes p^*\mathcal{O}_{\mathbb{P}^3}(1)|$. The pre-image of a quadratic line complex $\mathbb{G} \cap X$ is the zero scheme of a section of $\mathcal{L}^{\otimes 2}$. It is a conic bundle over $\mathbb{P}^3$ defined by a section of a vector bundle

$$p_*(\mathcal{L}^{\otimes 2}) = p_*(\mathcal{O}_{\mathbb{P}}(2)) \otimes p^*\mathcal{O}_{\mathbb{P}^3}(2) = S^2\Omega^1_{\mathbb{P}^3}(1)(2) = (S^2\Omega^1_{\mathbb{P}^3})(4).$$

Since in this case $d = 4, \epsilon = 1$, we get $\mathcal{E} = \Omega^1_{\mathbb{P}^3}(1)$.

2. Let $\mathrm{Cl}(X)$ be the class group of Weil divisors on Q modulo linear equivalence. Since all singularities of Q are ordinary double points, the local class group is isomorphic to the local class group of the vertex of an irreducible

quadric cone. It is generate day the germ of a line and isomorphic to $\mathbb{Z}/2\mathbb{Z}$ [379, Chapter 2, §6]. There is a natural exact sequence

$$0 \to \mathrm{Pic}(Q) \to \mathrm{Cl}(Q) \xrightarrow{r} \bigoplus_{x \in \mathrm{Sing}(Q)} \mathbb{Z}/2\mathbb{Z}. \tag{12.26}$$

Since $\mathrm{Cl}(Q) \cong \mathrm{Cl}(U)$, where $U = Q \setminus \mathrm{Sing}(Q)$, and U is isomorphic to the complement $\widetilde{Q} \setminus \bigcup_{i=1}^{\mu} E_{x_i}$, we obtain

$$\mathrm{Cl}(Q) \cong \mathrm{Pic}(\widetilde{Q})/\langle E_{x_1}, \ldots, E_{x_\mu}\rangle.$$

Assume that $\mathrm{Pic}(Q)$ is generated by the class of a plane section. In this casee $\mathrm{Cl}(Q) \cong \mathcal{N}$, where $\mathcal{N}$ is defined in (12.9). The natural homomorphism $\pi^* : \mathrm{Cl}(Q) \to \mathrm{Pic}(\widetilde{Q})$ has the covered equal to $\mathcal{N}^{\mathrm{s}}/\mathcal{N}$ isomorphic to the binary linear cone $\mathsf{N}_\mu^{\mathrm{s}}/\mathsf{N}$. It is isomorphic to the image of the homomorphism r in exact sequence (12.26).

3. Let $f : X \to \mathbb{P}^3$ be the double cover ramified over a μ-nodal quartic Q (*a quartic double solid*). It has singularities locally isomorphic to the singular point of the affine cone over a smooth 2-dimension quadric. We can view it as a quartic hypersurface in $\mathbb{P}(1,1,1,1,2)$. We have the following commutative diagram

$$\begin{array}{ccccc} \widetilde{X} & \xrightarrow{\tilde{\pi}} & \widetilde{\mathbb{P}}^3 & \xleftarrow{\tilde{j}} & \widetilde{Q} \\ \downarrow{\scriptstyle \tilde{\sigma}} & & \downarrow{\scriptstyle \sigma} & & \downarrow{\scriptstyle \alpha} \\ X & \xrightarrow{\pi} & \mathbb{P}^3 & \xleftarrow{j} & Q \end{array}$$

where $\widetilde{\mathbb{P}}^3 := \mathrm{Bir}_{x_1,\ldots,x_\mu}(\mathbb{P}^3)$. The group $\mathrm{Pic}(\widetilde{\mathbb{P}}^3)$ is freely generated by $\sigma^*\mathcal{O}_{\mathbb{P}^3}(1)$ and the classes of the exceptional divisors $\tilde{E}_{x_i}$. Each $\tilde{E}_{x_i}$ is isomorphic to a smooth quadric containing the exceptional divisors E_{x_i} as its hyperplane sections. The group $\mathrm{Pic}(\tilde{X})$ contains $\tilde{\pi}^*(\mathrm{Pic}(\widetilde{\mathbb{P}}^3))$. It is a free abelian group of rank $1 + \mu + d$, where d is called the *defect* of the double solid X. It is known that

$$d = \mu - 9 + \dim |\mathcal{O}_{\mathbb{P}^3}(2) - \mathrm{Sing}(Q)|. \tag{12.27}$$

[151]. When $\mu \le 9$, d is the number of extra linearly independent quadrics vanishing on singular points of Q. The defect is equal to zero if the nodes of Q impose independent conditions on quadrics passing through them. We already saw that $d > 0$ if six of the nodes of Q are coplanar. In fact the converse is true, if $d > 0$ then six of the nodes are coplanar. The restriction homomorphism $\tilde{j}^* : \mathrm{Pic}(\widetilde{\mathbb{P}}^3) \to \mathrm{Pic}(\widetilde{Q})$ defines a homomorphism

$$\mathrm{Pic}(\widetilde{\mathbb{P}}^3)^{(2)} \to \mathrm{Pic}(\widetilde{Q})^{(2)},$$

where, for any abelian group A, we denote by $A^{(2)}$ the cokernel of the homomorphism $[2] : a \mapsto 2a$. The cokernel of this homomorphism is our binary linear code C_μ of the quartic Q. One can also show that

$$C_\mu \cong \mathrm{Coker}(H^3(\widetilde{\mathbb{P}}^3), \mathbb{Z})^{(2)} \to H^2(\widetilde{X}, \mathbb{Z})^{(2)}) \cong \mathbb{F}_2^d \oplus T_2,$$

where

$$T_2 = \mathrm{Tors}(H^3(\widetilde{X}, \mathbb{Z}))$$

[289]. The latter group is the 2-torsion group of the *Brauer group* of the threefold $\tilde{X}$ which is an important birational invariant of a Fano variety $\tilde{X}$. Also it is known that

$$d = \mathrm{rank}(\mathrm{Cl}(X)) - \mathrm{rank}(\mathrm{Pic}(X)) = \beta_4(X) - \beta_2(X)$$

[151], [436], so $d = 0$ if and only if the double solid X is factorial. In particular, we see that X could be factorial only if $\mu \le 9$. It is factorial if C_μ is trivial, for example. if $\mu \le 5$. It is proven in [289, Theorem 3.6] that $T_2 = 0$ in all cases except the case of the Cayley quartic symmetroid with 10 nodes. Its binary linear code is $[10^1]$. In tis case $T_2 \cong \mathbb{F}_2$ as was computed by Artin and Mumford who used X to construct a counter-example to the Lüroth Problem on rationality of unirational threefolds [25].

Finally, let me state some questions to which I would like to know an answer.

Questions 12.2.23. 1. What is the bitangent surface $\mathrm{Bit}(Q)$ of a general μ-nodal quartic surface. We know the answwr when Q coincides with the focal surface of a congruence of lines of order 2 without fundamantal curves. It is known that it is irreducible and smooth if $\mu = 0$ [751]. Its Albanese variety is isomorphic to the Intermediate Jacobian variety $J(\widetilde{X})$, where X is a quartic double solid. For a general quartic surface with defect zero and $\mu \le 6$, the Intermediate Jacobian iis a prinicipally polarized abelian variety of dimensions equal to $10 - \mu$ [151]. Clemens proves that $\mathrm{Bit}(Q)$ is irreducible for general quartic surfaces with $\mu \le 6$ in the irreducible family of quartics with trivial binary linear code C_μ.

2. Which irreducible components of the moduli space of nodal quartic surfaces are rational varieties? As far as I know, the rationality is known only for $\mu = 15, 16$.
3. What is the group of birational $\mathrm{Bir}(Q)$ automorphisms of a general nodal quartic surface. It contains a subgroup a subgroup generated by the deck transformations of the projections from a node. Since no two nodes lie on a line contained in a surface, this group is free with μ generators. The group $\mathrm{Bir}(Q)$ is known if $\mu = 15$ [257] or $\mu = 16$ [464]

12.3 Special quartic surfaces

In this section, we will discuss some interesting special nodal quartic surfaces. We have already discussed on several occasions the Kummer 16-nodal quartic surfaces. This time we give more attention to other frequently encountered nodal quartic surfaces, for example, Weddle 6-nodal surface and Cayley 10-nodal quartic symmetroid surface.

12.3.1 Weddle surface

We already encountered a Weddle quartic surface in the discussion of quadratic congruences of lines and in Example 7.7.17 of a Cremona involution. Here we will give a more detailed discussion.

Let $\mathcal{P} = \{p_1, \dots, p_6\}$ be a set of six points in $\mathbb{P}^3$, no four of which are coplanar. Let $L = |V|$ be the web of quadrics with the base points $p_1, \dots, p_6$. We define the Weddle surface $\mathsf{W}_\mathcal{P}$ to be the Jacobian surface of $L|$, the set of singular points of quadrics from L.

Let $\sigma : X = \mathsf{Bl}_\mathcal{P}(\mathbb{P}^3) \to \mathbb{P}^3$ and $f : Y \to L^*$ be the morphism given by the linear system $|D| = |2H-E|$, where $H = \sigma^*(c_1(\mathcal{O}_{\mathbb{P}^3}(1)$ and $E = E_1+\cdots+E_6$ is the exceptional divisor. Since $D^2 = (2H-E)^3 = 8-6 = 2$, the map f is of degree 2. Obviously, it blows down the proper transforms ℓ_i' of 15 lines $\ell_i = \langle p_i, p_j\rangle$ to points $q_i \in L^*$. It also blows down to a point q the proper transform R' of the unique rational normal curve R_3 containing the set $\mathcal{P}$. Moreover, f maps the exceptional divisors $E_i \cong \mathbb{P}^2$ to planes Π_i. Among quadrics from L, there are 10 reducible quadrics $Q_{ijk} = \langle p_i, p_j, p_k\rangle\rangle \cup \langle p_k, p_l, p_m\rangle\rangle$. They intersect along a line $\ell_{ijk} = \ell_{k,l.m}$, and the restriction of L to such a line is defined by a complete series of degree two. The images of these lines are 10 conics in the target $\mathbb{P}^3$ which span 10 planes Π_{ijl}. Thus, we o btain a configuration of 16 points and 16 planes in the target $\mathbb{P}^3$. It is immediate to see that this is a Kummer configuration (16_6). Since $K_X = -2D$, the ramification divisor is a quartic surface in $\mathbb{P}^3$ and the branch divisor of f is a quartic surface K. It contains 16 singular points q_i, q, and hence, it realizes the Kummer configuration. Let $\mathsf{K}_\mathcal{P}^*$ be the dual Kummer surface in L. A quadric $Q \in L$ is singular if and only if Q is equal to the pre-image of a hyperplane in L^* which is tangent to $\mathsf{K}_\mathcal{P}$. Its singular point is the tangency point of the hyperplane with K. This shows that the Weddle surface $\mathsf{W}_\mathcal{P}$ coincides with the proper tranform of $\mathsf{K}_\mathcal{P}$ which is the ramification divisor of f. The restriction of f to $\mathsf{W}_\mathcal{P}$ is a birational map, and its lift to the proper transform $\widetilde{\mathsf{W}}_\mathcal{P}$ is a minimal resolution of the Kummer surface $\mathsf{K}_\mathcal{P}$. We also see that $\mathsf{K}_\mathcal{P}^*$ can be identified with the discriminant surface of L.

The exceptional curves of $\sigma : \widetilde{\mathsf{W}}_{\mathcal{P}} \to \mathsf{W}_{\mathcal{P}}$ are the trope-conics $\Pi_i \cap \mathsf{K}_{\mathcal{P}}$. This implies that $\mathcal{P}$ is the set of ordinary nodes of $\mathsf{W}_{\mathcal{P}}$.

Let us summarize our discussion.

Theorem 12.3.1. *Let $\mathsf{W}_{\mathcal{P}}$ be the Jacobian surface of a web of quadrics with the set of base points $\mathcal{P}$.*

1. *$\mathsf{W}_{\mathcal{P}}$ is a quartic surface with the set of nodes $\mathcal{P}$ as it set of singular points.*
2. *$\mathsf{W}_{\mathcal{P}}$ contains 15 lines $\ell_{ij} = \langle p_i, p_j \rangle$.*
3. *$\mathsf{W}_{\mathcal{P}}$ contains the unique rational normal curve R_3 through the points p_i.*
4. *$\mathsf{W}_{\mathcal{P}}$ contains 10 lines $\ell_{ijk} = \langle p_i, p_j, p_k \rangle \cap \langle p_l, p_m, p_n \rangle$.*
5. *$\mathsf{W}_{\mathcal{P}}$ is the ramification divisor of the degree two map $f : \mathbb{P}^3 \dashrightarrow L^*$ given by the web of quadrics L.*
6. *The branch divisor of f is a Kummer surface $\mathsf{K}_{\mathcal{P}}$ and f lifts to a minimal resolution of singularities $f' : \widetilde{\mathsf{W}}_{\mathcal{P}} = \mathsf{Bl}_{\mathcal{P}}(\mathsf{W}_{\mathcal{P}}) \to \mathsf{K}_{\mathcal{P}}$.*
7. *The discriminant surface of L is the dual Kummer surface $\mathsf{K}_{\mathcal{P}}^*$.*

Corollary 12.3.2. *The restriction of the map $f : \mathbb{P}^3 \to L^* \cong \mathbb{P}^3$ to the Weddle surface defines a birational map $\mathsf{W}_{\mathcal{P}}$ to a Kummer surface $K_{\mathcal{P}} \subset L^*$. It blows down 15 lines ℓ_{ij} and the rational cubic curve R_3 to 16 double points. It maps the 10 lines ℓ_{ijk} and 6 exceptional curves E_i of $\mathsf{Bl}_{\mathcal{P}}(\mathbb{P}^3) \to \mathbb{P}^3$ to 16 trope-conics in K.*

Recall that the Kummer surface $K_{\mathcal{P}}$ comes with a choice of six disjoint conics $f(E_i)$ containing the node $f(R_3)$. The dual Kummer surface $K_{\mathcal{P}}^*$ comes with a choice of six nodes on a fixed trope trope-conic. It follows that $K_{\mathcal{P}} \cong K_{\mathcal{P}}^* \cong \mathrm{Kum}(\mathrm{Jac}(C))$, where C is the hyperelliptic curve of genus 2 with the set $\mathcal{P}$ of Weierstrass points.

Let us find a basis in L. Clearly, L contains the net N of quadrics with R_3 as its base curve. We consider R_3 as the image of the dual Veronese map

$$\mathsf{v}_3^* : |U| \cong \mathbb{P}^1 \to |S^3(U^\vee)|, [\alpha, \beta] \mapsto [-\beta^3, \alpha\beta^2, -\alpha^2\beta, \alpha^3]$$

from Example 1.3.15. The space $S^3(U\vee$ is the spance of binary cubics, and the image of a point $[\alpha, \beta] \in \mathbb{P}^1$ is the point $[(\beta u_0 - \alpha u_1)^3]$.

Lemma 12.3.3. *Any $b_3 \in S^3(U^\vee)$ can be written in the form*

$$b_3 = (\alpha u_0 + \beta u_1)^3 + (\delta u_0 + \gamma u_1)^3,$$

where

$$Hess(b_3) = c(\alpha u_0 + \beta_1)(\delta u_0 + \gamma u_1), c \in \mathbb{C}^*.$$

Proof The assertion is obvious if $b_3 = (\alpha u_0 + \beta)^3$. In this case $\mathrm{Hess}(b_3) = 0$. Also, we check the assertion in the case where $b_3 = u_0^3 + u_1^3$. We have $\mathrm{Hess}(b_3) = 6u_0u_1$. So, we may assume that $[b_3] \notin R_3$. There is a unique secant line containing $[b_3]$ with the end-points $[(\alpha u_0 + \beta u_1)^3]$ and $[(\alpha u_0 + \beta u_1)^3]$. This shows that b_3 can be written as in the assertion of the lemma. Since the end-points are different, we can make a linear transformation $u_0 \mapsto \alpha u_0 + \beta u_1, u_1 \mapsto \delta u_0 + \gamma u_1$, to transform b_3 to the form u_0u_1. It remains to use that the Hessian is a covariant of binary forms, we are done. □

Remark 12.3.4. The lemma implies that the map $\mathrm{Hess} : S^3(U^\vee) \to S^2(U^\vee)$ defined by the Hessian covariant defines a rational map

$$\mathbb{P}^3 \cong |S^3(U^\vee)| \dashrightarrow \mathbb{P}^1 \cong |S^2(U^\vee|.$$

Its locus of points of indeterminacy coincides with R_3. Its fibers are secant lines if R_3. So, we can identify this map with the rational map defined by the linear system $|I_{R_3}(2)|$.

We know from Subsection 1.5.1 that the Hessian covariant of a binary cubic form is equal to

$$(a_0a_2 - a_1^2)u_0^2 + (a_0a_3 - a_1a_2)u_0u_1 + (a_1a_3 - a_2^2)u_1^2. \tag{12.28}$$

It can be considered as a quadratic pencil of quadrics in $\mathbb{P}^3$. The coefficients form a basis of the net of quadrics containing R_3. A quadric corresponding to a root $[u_0, u_1]$ of the Hessian contain a secant line whose end-points are the vertices of singular quadrics in the net. Thus, (12.28) is a quadratic pencil of quadric cones.

The net N of quadrics containing R_3 is obviously contained in the web L. Since it contains cones with vertices on R_3, we obtain another proof that the Weddle surface contains R_3.

We encountered a quadratic pencil of cones with vertices on R_3 in Example 11.4.12. The equation

$$(a_0a_3 - a_1a_2)^2 - 4(a_0a_2 - a_1^2)(a_1a_3 - a_2^2) = 0$$

is the equation of a quartic surface singular along R_3. The points $p_1, \ldots, p_6$ are the images of 6 points $[\alpha_i, \beta_i]$ equal to the points $[(u_0\beta_i - u_1\alpha_i)^3]$ (see Example 1.3.15). Let

$$P(u_0, u_1) = \prod_{i=1}^{6}(\beta_i u_0 - \alpha_i u_1) = \sum_{k=0}^{6} a_i(-1)^k u_0^k u_1^{6-k}.$$

Consider the quadric

$$B(x) = a_6t_0^2 - a_5t_0x_1 + a_4t_1^2 - a_3t_0x_2 + t_2^2 - a_1t_2t_3 + a_0t_3^2 = 0.$$

Substituting $(t_0, t_1, t_2, t_3) = (-\alpha^3, \alpha^2\beta, -\alpha\beta^2, \beta^3) \in R_3$, we obtain $B(\alpha, \beta)$. This shows that the quadric vanishes at a point of R_3 if and only if this point is one of the points p_i.

So, we have found a basis in V

$$(t_0t_3 - t_1t_2, t_1^2 - t_0t_2, t_2^2 - t_1t_3, B(t_0, t_1, t_2, t_3)).$$

The discriminant surface of the web is isomorphic to a Kummer surface with equation

$$\det\begin{pmatrix} 2a_6w & -a_5w & -a_3w - y & x \\ -a_5w & 2a_4w + 2y & -x & -z \\ -a_3w - y & -x & 2a_2w + 2z & -a_1w \\ x & -z & -a_1w & 2a_0w \end{pmatrix} = 0 \tag{12.29}$$

Here (x, y, z, w) are coordinates in W corresponding to our choice of a basis.

We also can find the equation of the Weddle surface as the Jacobian surface of the web that coincides with its Steinerian surface. If we write the matrix from above in the form $xA_1 + yA_2 + zA_3 + wA_4$, then, the equation of the Jacobian hypersurface is $\det(xA_1 + yA_2 + zA_3 + wA_4) = 0$. This gives the following equation of $\mathsf{W}_{\mathcal{P}}$:

$$\det\left[A_0\cdot\begin{pmatrix} t_0 \\ t_1 \\ t_2 \\ t_3 \end{pmatrix}, A_1\cdot\begin{pmatrix} t_0 \\ t_1 \\ t_2 \\ t_3 \end{pmatrix}, A_2\cdot\begin{pmatrix} t_0 \\ t_1 \\ t_2 \\ t_3 \end{pmatrix}, A_3\cdot\begin{pmatrix} t_0 \\ t_1 \\ t_2 \\ t_3 \end{pmatrix}\right] = \det\begin{pmatrix} t_3 & -t_2 & 0 & 2a_6t_0 - a_5t_1 - a_3t_2 \\ -t_2 & 2t_1 & -t_3 & -a_5t_0 + 2a_4t_1 \\ -t_1 & -t_0 & 2t_2 & -a_3t_0 + 2a_2t_2 - a_1t_3 \\ t_0 & 0 & -t_2 & -a_1t_2 + 2a_0t_3 \end{pmatrix} = 0 \tag{12.30}$$

There is another determinantal equation of $\mathsf{W}_{\mathcal{P}}$ discovered by Caspary [86]. (see also [418], [413, §97]. We may choose the projective coordinates to assume that

$$\begin{aligned} p_1 &= [1, 0, 0t, 0],\ p_2 = [0, 1, 0, 0],\ p_3 = [0, 0, 1, 0], \\ p_4 &= [0, 0, 0, 1],\ p_5 = [1, 1, 1, 1],\ p_6 = [a, b, c, d], \end{aligned} \tag{12.31}$$

where the point p_6 does not lie in any plane spanned by three of the points $p_i, i < 6$. The parametrical equation of a rational normal curve R_3 containing the points p_i is

$$[t_0, t_1] \mapsto [\frac{a_1}{t_0 - at_1}, \frac{b}{t_0 - b_1t_1}, \frac{c}{t_0 - ct_1}, \frac{d}{t_0 - dt_1}] \tag{12.32}$$

(see [375, Example 1.17]). Consider a surface given by the following equation:

$$\det\begin{pmatrix} ayzw & x & 1 & a \\ bxzw & y & 1 & b \\ cxyw & z & 1 & c \\ dxyz & w & 1 & d \end{pmatrix} = 0 \tag{12.33}$$

[413, §97]. One immediately checks the surface contains the lines $\langle p_i, p_j\rangle$, and using our parametrization of the curve R_3, we also check that it contains R_3. Thus it intersects the Weddle surface along a curve of degree $18 > 16$, and hence, must coincide with it.

Finally, we give another equation of a Weddle surface in terms of bracket determinant $ijkx|$, where $i, j, k \in \{1, \dots, 6\}$ correspond to the points p_i and x corresponds to a general point in $\mathbb{P}^3$;

$$\det\begin{pmatrix} |135x||425x| & |145x||235x| \\ |426x||136x| & |145x||236x| \end{pmatrix} = 0.$$

(see [159, p. 118]). It is clear that the equation describes a quartic surface containing the points $x \in \mathcal{P}$. If we choose the points p_i as above, and use the parameterization (12.32), we find that the curve R_3 is also contained in this quartic surface. Thus, it must coincide with the Weddle surface $\mathsf{W}_{\mathcal{P}}$.

The following theorem relates a Weddle surface with the Segre cubic primal S_3. Recall that S_3 is isomorphic to the image of $\mathbb{P}^3$ under the map $\phi : \mathbb{P}^3 \dashrightarrow \mathbb{P}^4$ given by quadrics through teh first five points $p_1, \dots, p_5$.

Theorem 12.3.5. *Let $x \in \mathsf{S}_3 \subset \mathbb{P}^4$ be a nonsingular point, and Q_x be the polar quadric of S_3 with pole at x. The pre-image X of Q_x under the map ϕ is isomorphic to the Weddle surface associated with* 6 *points* $(p_1, \dots, p_5, p_6 = \phi^{-1}(x))$.

Proof Recall that the ten nodes of S_3 are the images of the lines ℓ_{ij}, $1 \le i, j \le 5$. Since Q_x contains singular points of S_3, and the map ϕ is defined by quadrics, the pre-image of Q_x is quartic surface X that contains the lines ℓ_{ij}. Since each of these lines meets four other lines, the points $p_1, \dots, p_5$ are singular points of X. Since Q_x is tangent to Σ_3 at the point x, the quartic acquires an additional double point at $p_6 = \phi^{-1}(x)$. It is known that the polar quadric Q_x intersects Σ_3 at points y such that the tangent hyperplane of Σ_3 at y contains x (see Theorem 1.1.5). This implies that the five lines $\phi(\ell_i)$ are contained in Q_x, and hence the lines ℓ_i are contained in W.

Let R_3 be the unique twisted cubic through the six points $p_1, \dots, p_6$. Its image in Σ_3 is the sixth line in Σ_3 passing through x. By above, it is also contained in Q_x, hence W contains R_3. Now, let W be the Weddle surface associated with the points $p_1, \ldots, p_6$. Then $X \cap W$ contains 15 lines and R_3, a curve of degree > 16. This shows that $X = W$.

□

Recall that the polar quadrics of S_3 define a birational map onto the Castelnuovo-Richmond quartic threefold. CR_4. It follows from the theorem that W is the pre-image of a hyperplane section of CR_4 tangent at the image of the point x. This is, of course, isomorphic to the Kummer surface associated with W.

The following proposition gives another explicit relationship between the Weddle and Kummer surfaces. Recall from Section 10.3.3 that a Kummer surface admits a nonsingular model X as a complete intersection of three quadrics in $\mathbb{P}^3$. It contains 32 lines forming the Kummer configuration (16_6).

Proposition 12.3.6. *Let $X \subset \mathbb{P}^5$ ba an octic nonsingular model of a Kummer surface. Then the projection of X to $\mathbb{P}^3$ from a line on X is a Weddle quartic surface.*

Proof Let h be the divisor class of a hyperplane section of X and ℓ be a line on X. The projection map is given by the linear system $|h - \ell|$. We have $(h-\ell)^2 = 8-2-2 = 4$, so the image of the projection map is a quartic surface Y. Since $(h-\ell)\cdot\ell = 3$, the image of ℓ is a rational normal cubic curve. The line ℓ intersects six other lines $\ell_1, \ldots, \ell_6$. Since $(h-\ell)\cdot\ell_i = 0$, they are blown down to six nodes $p_1, \ldots, p_6$ of Y. There are $16-6$ other skew lines that do not intersect ℓ and any ℓ_i. They intersect $h-\ell$ with multiplicity one, hence are mapped to lines. Any other of the remaining 15 lines is projected to a line containing two of the points p_i. Let W be the Weddle surface associated with the set of points $p_1, \ldots, p_6$. We finish as in the proof of the previous theore, □

The Weddle surface is a peculiar quartic surface in the following sense. It is known that, for any nonsingular quartic surface X, the image of the rational map

$$\phi_X : |\mathcal{O}_X(1)| \dashrightarrow \mathcal{M}_3 \tag{12.34}$$

is three-dimensional.

Recall that an *asymptotic curve* on a surface is an irreducible curve such that the tangent plane to the surface at its general point is an osculating tangent plane to the curve.

Lemma 12.3.7. *The curve R_3 is an asymptic curve of* $\mathsf{W}_{\mathcal{P}}$.

Proof Let Π be a tangent plane of $\mathsf{W}_{\mathcal{P}}$ at some point $x \in R_3$. It intersects the surface along a quartic curve C with a singular point at x. The tangent line ℓ

of R_3 at x intersects the surface at two additional points a, b not lying on R_3. They lie on $\Pi \cap \mathsf{W}_{\mathcal{P}}$, and hence, on C. If Π intersects R_3 at a point $x' \neq x$, then $C \cdot R_3 > 4$, a contradiction. □

Theorem 12.3.8. *If X is a general Kummer quartic surface, the map ϕ_X is of finite degree on its image. However, if X is a general Weddle surface, the map ϕ_X has one-dimensional fibers.*

Proof A general plane section P of a Kummer quartic K is a plane curve of genus 3. It also comes with an étale cover of degree 2 induced by the double cover of $\mathrm{Jac}(C) \to K$ ramified at the nodes. The isomorphism class of (P, θ), where $\theta \in \mathrm{Jac})C)[2] \setminus \{0\}$. Let $\mathcal{R}_3 \to \mathcal{A}_2$ be the Prym map in genus 3 (see Remark 6.2.2. Let s be the point in $\mathcal{A}_2$ represented the isomorphism class of the curve C. It is proved in [786] that the fibers of the Prym map are three-dimensional and the image of the map ϕ_K is mapped dominantly. In particular, for a general quartic surface the map is also three-dimensional.

Assume now that X is a Weddle surface $\mathsf{W}_{\mathcal{P}}$. We follow the proof from [527]. Let $C_4 = P \cap \mathsf{W}_{\mathcal{P}}$, and $P \cap R_3 = \{p, q, r\}$. We choose coordinates in P to assume that $q_1 = [1, 0, 0], q_2 = [0, 1, -], q_3 = [0, 0, 1]$. The net N cuts out in P a net of There will be a conic containing each coordinate line, hence the coordinate triangle is self-polar.

By Lemma 12.3.7, the tangent planes Π_i of the surface at points in R_3 are osculating planes of R_3. They form the dual Veronese curve R_3^* in the dual projective space. The planes Π considered as points on R_3^* span a plane dual a point $x \in \mathbb{P}^3$. Since three osculating planes Π_1, Π_2, Π_3 at points q_i are cut out by the plane P, hence the corresponding tangent planes intersect P at a common points. This shows that the tangent lines to C_4 at the points q_i are concurrent. Thus, Salmon's equation (6.39) of C_4 is in the form

$$a_2x^3y + a_3x^2z + b_1xy^3 + b_3y^3z + c_1xz^3 + c_2yz^3 + 3xyz(lx + my + nz) = 0,$$

where $a_2b_2c_1 + a_3b_1c_2 = 0$. Using the explicit formula (6.40) for the invariant I_3 and the formula for the catalecticant invariant I_6 from 1.4.1, we obtain

$$I_3 = 12(-lmn + lb_1c_1 + mc_2a_2 + na_3b_3,$$

$$I_6 = -\det\begin{pmatrix} l & a_3 & a_2 \\ b_3 & m & b_1 \\ c_2 & c_1 & n \end{pmatrix}.$$

This shows that the invariant $I_4^2 + 144I_6$ vanishes on C_4. Varying $\mathcal{P}$ in a 3-dimensional family and taking plane section we obtain that their images in $\mathcal{M}_3$

belong to a hypersurface defined by the invariant I_4+144I_6. Thus, we expect that the images of a plane section of a general Weddle surface is two-dimensional.

Morley and Conner go further and find another distinguished property of a general plane section of a Weddle surface. The quartic plane curve contains a configuration of 15 points (the intersection points with the lines ℓ_{ij} 10 pairs of lines (the intersection the planes $\langle p_i, p_j, p_k\rangle$ with the intersection points $x_{ijk} = x_{lmn}$ and each line meeting three of the 15 points. In fact, they show that there are ∞^1 of such configurations on the curve. □

Finally, let us discuss birational automorphisms of a Weddle surface $\mathsf{W}_{\mathcal{P}}$. In Remark 7.7.17 we discussed a 2-elementary group 2^5 of Cremona transformations that leave the surface invariant. One of them, the transformation T_7 from Example 7.7.17 acts identically on the surface. Thus, $\mathrm{Bir}(\mathsf{W}_{\mathcal{P}})$ conatins a subgroup isomorphic to 2^4. Under the birational isomorphism $\mathsf{W}_{\mathcal{P}} \cong K_{\mathcal{P}}$, the group corresponds to the group of birational automorphisms of the Kummer surfaces induced by translations by 2-torsion points of teh corresponding abelian surface. Another element of this group is a cubic Cremona transformation with four of the points in $\mathcal{P}$ as its fundamental points.

If we use (12.33) as an equation of $\mathsf{W}_{\mathcal{P}}$, then this transformation is given by

$$H : [x, y, z, w] \mapsto \Big[\frac{a}{x}, \frac{b}{y}, \frac{c}{z}, \frac{d}{w}\Big].$$

It switches the points p_5, p_6.[1] This birational automorphism of the Weddle surface lifts to a biregular automorphism of its minimal nonsingular model X. The quotient of X by this biregular involution is an Enriques surface. The complex of lines $\mathfrak{G}$ is the Reye congruence of lines.

Note that the group $\mathrm{Bir}(K_{\mathcal{P}})$, for a general set $\mathcal{P}$ of six points is known in terms of its a finite set of geometrically defined generators. [464], [259, Chapter 8]. so, it gives a finite set of generators of $\mathrm{Bir}(\mathsf{W}_{\mathcal{P}})$.

12.3.2 Cayley octads and 7-nodal quartic surfaces

Here, we give some additional comments to Remark 6.3.15 which we will use in the next Subsection. In this Remark we discussed the elliptic fibration

$$f : X = \mathsf{Bl}_{x_1,\ldots,x_7,x_8}(\mathbb{P}^3) \to L^* \cong \mathbb{P}^2, \tag{12.34}$$

where L is the net of quadrics in $\mathbb{P}^3$ with base points $x_1, \ldots, x_8$ forming a Cayley octad. As we know, such a net defines the discriminant quartic curve

[1] It is puzzling that Coble discussing this transformation does not mention the Hutchinson's paper [420] where this transformation was first introduced.

$\Delta \subset L$ together with an even theta characteristic ϑ on it. We assume that the Cayley octad $\{x_1, \ldots, x_8\}$ is general so that the curve Δ is nonsingular. A singular quadric in the net L is a point $q \in \Delta$, and the fiber $f^{-1}(q)$ is singular if and only if, considered as a line in L, it is tangent to Δ. This shows that the discriminant curve of f parameterizing singular fibers is equal to the dual curve Δ^* of the plane quartic Δ. It is a curve of degree 12 with 28 ordinary nodes and 24 ordinary cusps. Recall that the closure of the set of nodes of singular quadrics from L is the Steinerian curve of L. It is a smooth curve R of degree 6, the image of Δ by a map given by the linear system $|\vartheta(1)|$. The pencil corresponding to a node of Δ^* is a bitangent of Δ. It has two singular quadrics with nodes at R.

A general quadric from L does not intersect the lines $\langle x_i, x_j\rangle$ outside the base points. This shows that the proper transforms l_{ij} of these 28 lines are blown down to points. These are the nodes of $q_{ij} \in \Delta^*$. The fiber $f^{-1}(q_{ij})$ is equal to the union of l_{ij} and the proper transform r_{ij} of the unique twisted cubic through the remaining six base points of L. The two irreducible components intersect at the pre-images of the two nodes of singular quadrics in the pencil in L defined by the corresponding bitangent of Δ.

The pre-image of a conic in L^* is a quartic surface with nodes at $x_1, \ldots, x_8$. It is called a *syzygetic* 8*-nodal quartic*. It is given by equation

$$F_2(q_0(x, y, z, w), q_1(x, y, z, w), q_2(x, y, z, w)) = 0 \qquad (12.34)$$

where q_0, q_2, q_2 define a basis of the net L, and F_2 is a quadratic form. The dimension of the linear system of such quartics (with fixed seven nodes $x_1, \ldots, x_7$) is equal to 5. Since a node x_i imposes four conditions on quartic surfaces, the dimension of the family of quartics with 7 nodes is equal to 6. This shows that a quartic with nodes at $x_1, \ldots, x_7$ can be given by equation

$$F_2(q_0(x, y, z, w), q_1(x, y, z, w), q_2(x, y, z, w)) + \lambda F_4(x, y, z, w) = 0. \quad (12.34)$$

A quartic with $\lambda \neq 0$ is called an *asyzygetic* 7*-nodal quartic*.

Note that there are no irreducible quartic surfaces with a fixed set of 8 nodes in general position. In fact, counting constants, we find that such quartics depend on two parameters and hence coincide with the reducible quartic $V(q_0, q_1)$, where $V(q_0)$ and $V(q_1)$ generate the pencil of quadrics through the eight nodes.

The surface $\mathfrak{D} := V(J(q_0, q_1, q_2, F_4))$ is called the *Cayley dianode surface*[2]. It is a surface of degree 6. By writing local equations of the surface at points $x_1, \ldots, x_7$, we find that these points are triple points.

[2] It is Rohn's *Knotenflächen* [630]

The 6-dimensional linear system Q_7 of quartics with nodes at $x_1, \dots, x_7$ defines a map of degree 2

$$\phi : \mathsf{Bl}_{x_1,\dots,x_7} \to Q_7^* \cong \mathbb{P}^6. \tag{12.34}$$

The linear subsystem of syzygetic 8-nodal quartics defines a map $f' : X = \mathsf{Bl}_{x_1,\dots,x_7}(\mathbb{P}^3) \to \mathbb{P}^5$ whose image is the quartic Veronese surface V_4. It shows that the image of ϕ is equal to the cone $\mathbb{P}(1,1,1,2)$ over the Veronese surface with vertex at the point $\phi(x_8)$. This is analogous to the anti-bicanonical map of a del Pezzo surface $\mathsf{Bl}_{p_1,\dots,p_8}(\mathbb{P}^2)$ of degree one to the quadratic cone $\mathbb{P}(1,1,2)$.

We have a commutative diagram

$$\begin{array}{ccccc}
\mathsf{Bl}_{x_1,\dots,x_7}(\mathbb{P}^3) & \xrightarrow{\sigma} & \mathsf{V}_6(1,1,1,2,3) & \xrightarrow{\phi'} & \mathbb{P}(1,1,1,2) \\
\downarrow{\scriptstyle \pi} & \overset{f'}{\searrow} & \downarrow{\scriptstyle \mathrm{pr}_{\sigma(x_8)}} & & \downarrow{\scriptstyle \mathrm{pr}_{\phi(x_8)}} \\
\mathbb{P}^3 & \overset{\phi_L}{\dashrightarrow} & \mathbb{P}^2 & \xrightarrow{\mathsf{v}_2} & \mathsf{V}_4
\end{array}$$

The top morphisms are define by the Stein factorization of ϕ, where σ is a birational morphism and ϕ' is a degree two finite morphism. The threefold $\mathsf{V}_6(1,1,1,2,3)$ is a hypersurface of degree 6 in $\mathbb{P}(1,1,1,2,3)$ that can be given by an equation

$$y^2 + x^3 + a_4(t_0,t_1,t_2)x + a_6(t_0,t_1,t_2) = 0, \tag{12.34}$$

where a_4 and a_6 are homogeneous forms of degree 4 and 6 [159, p. 188]. The image of x_8 under the map σ is the point $[0,0,0,1,-1]$. Its image under ϕ' is the vertex $\phi(x_8)$ of the cone $\mathbb{P}(1,1,1,2)$. The map f' is the map given by the linear system L of quadrics with base points $x_1, \dots, x_8$. The rational maps in the diagram are not defined at $x_8, \sigma(x_8)$ and $\phi(x_8)$.

The birational morphism σ blows down proper transforms of 21 lines $\ell_{ij} := \langle x_i, x_j \rangle, i, j \le 7$, and the proper transforms of 7 twisted cubics R_i through the point $x_k, k \ne i$.

Equation (12.3.2) can be considered as the Weierstrass equation of the elliptic fibration (12.3.2), where the zero section is the exceptional divisor over x_8. It is analogous to the equation of a del Pezzo surface of degree one and the Weierstrass form of the elliptic fibration $\mathsf{Bl}_{p_1,\dots,p_8,p_9}(\mathbb{P}^2) \to \mathbb{P}^1$, where p_9 is the ninth base point of the pencil of cubic curves through $p_1, \dots, p_8$.

The involution

$$\beta : \mathsf{V}_6(1,1,1,2,3) \to \mathsf{V}_6(1,1,1,2,3). \quad y \mapsto -y$$

is a 3-dimensional analog of the Bertini involution. Its restriction to the general fiber of f is the negation involution of the elliptic curve. This shows that its locus of fixed points consists of the isolated point $\sigma(x_8)$ and the closure $\mathfrak{R} = V(y)$ of singular points of irreducible singular fibers of f.

The surface $\mathfrak{R}$ is the ramification divisor of the double cover ϕ'. Its image under the map ϕ' is equal to a surface $V(x^3 + a_4x + a_6)$ in $\mathbb{P}(1,1,1,2) \subset \mathbb{P}^6$. It is cut out by a cubic hypersurface in $\mathbb{P}^6$. The pre-image of a hyperplane section of $\mathbb{P}^6$ is a quartic from the linear system Q_7. It acquires an additional node if it is tangent to the branch divisor. This shows that $\mathfrak{R}$ is the image of the Cayley dianode surface $\mathfrak{D}$ under the rational map $\sigma \circ \pi^{-1} : \mathbb{P}^3 \dashrightarrow \mathsf{V}_6(1,1,1,2,3)$. It coincides with the image of the proper transform $\widetilde{\mathfrak{D}}$ of $\mathfrak{D}$ in X under the map σ.

The surface $\mathfrak{R}$ contains the images of the exceptional curves of σ. Their images under the map $\mathrm{pr}_{\sigma(x_8)} : \mathsf{V}_6(1,1,1,2,3) \to \mathbb{P}^2$ are equal to 28 nodes of Δ^*. This shows that $\mathfrak{D}$ contains the 21 lines $\langle x_i, x_j\rangle$, $1 \le i < j \le 7$ and the 7 twisted cubics through six points among $x_1, \ldots, x_7$. The surface $\widetilde{\mathfrak{D}}$ contains the corresponding irreducible components of reducible fibers of f over the nodes of Δ^*.

The map $\sigma : \widetilde{\mathfrak{D}} \to \mathfrak{R}$ is a minimal resolution of singularities of 28 nodes of $\mathfrak{R}$. By the adjunction formula for hypersurfaces in a weighted projective space, $\omega_{\mathfrak{B}} \cong \mathcal{O}_{\mathfrak{R}}(1)$ [233]. This shows that the map σ is given by the canonical linear system. Its nonsingular minimal birational model is a surface $\mathfrak{R}'$ of general type with $K^2_{\mathfrak{B}'} = p_g(\mathfrak{B}') = 3$. (see [234, p. 189, Remark 6]). It is shown by Coble [159, §47-49] (see also [234, Chapter VII, §6]) that the surface $\mathfrak{R}$ is isomorphic to the quotient of the symmetric theta divisor Θ of $\mathrm{Jac}(\Delta)$ by the negation involutions.

Note that $\mathfrak{D}$ contains the Steinerian curve C of the net of quadrics L. The proper transform of C in X is the closure of singular points of irreducible fibers. Its image $\sigma(C)$ on $\mathfrak{B}$ is the ramification curve of the triple cover $\mathfrak{B} \to \mathbb{P}^2$, and the map $\sigma(C) \to \Delta^*$ is the normalization map. Note that a sextic surface may have up to ten isolated triple points [290].

12.3.3 Quartic surfaces containing lines

Let us first consider quartic surfaces Q containing N skew lines $\ell_1, \ldots, \ell_N$. We assume that Q has only μ ordinary nodes as possible singularities. Let $\pi : \tilde{Q} \to Q$ be a minimal resolution of singularities of Q. The proper transforms of the lines ℓ_i in $\tilde{Q}$ are disjoint (-2)-curves.

We say that a set of lines in a nodal quartic surface is *weakly skew* if their proper transform in $\tilde{Q}$ is a set of disjoint (-2)-curves. The following proposition is proved in [551, Application, Corollary 1].

Theorem 12.3.9. *The number of weakly skew lines on a nodal quartic surface is less than or equal to* 16*. A surface with* 16 *skew lines is birationally isomorphic to a Kummer surface of some abelian surface (not necessarily principally polarized).*

Example 12.3.10. A quartic Kummer surface with 16 nodes does not contain lines. However, there exists a smooth quartic surface with two sets of 16 disjoint lines forming a symmetric abstract configuration (16_{10}). These are classically known as *Traynard quartic surfaces*. These surfaces were constructed by Traynard [766] (also see [340], where the surfaces are named after Traynard). Not being aware of Traynard's work, W. Barth and I. Nieto rediscovered the Traynard surfaces in [36]. The surfaces are embedded Kummer surfaces of simple abelian surfaces A with polarization of type $(1, 3)$. The negation involution acts on the linear space $H^0(A, \mathcal{O}_A(2\Theta))$, where Θ is a symmetric polarization divisor. The eigensubspace V with eigenvalue equal to -1 is of dimension 4. The linear system $|V| \subset |2\Theta|$ has base points at all 2-torsion points of A and defines a finite map of degree 2 of the blow-up of these points to $\mathbb{P}^3$ with image a smooth quartic surface X. The images of the exceptional curves over the torsion points form a set $\mathcal{A}$ of 16 lines on X. The unique symmetric theta divisor Θ is a curve of genus 4, it passes through 10 torsion points, and the images of the translates of Θ by 2-torsion points provides another set $\mathcal{B}$ of 16 disjoint lines on X.

Since there are five conditions for a quartic surface to contain a line, a quartic surface with more than 6 lines must be special. So, we assume that $N \geq 7$.

Example 12.3.11. At the end of Subsection 2.3.1, we discussed desmic 12-nodal quartic surface. Here, following [430, Chapter II], we give some more detail. In particular, it turns out that a desmic quartic surface contains a weakly even set of 16 skew lines.

Recall that a desmic surface is a member of a pencil of quartic surfaces that contain three desmic tetrahedra. Its equation can be written in the form:

$$a(x^2-y^2)(z^2-w^2)+b(x^2-z^2)(y^2-w^2)+c(x^2-w^2)(y^2-z^2) = 0, \quad a+b+c = 0. \tag{12.34}$$

The surfaces have an obvious group of symmetry isomorphic to the 2-elementary group 2^5. It is defined by changing the signs of the coordinates and permutations $(12)(34), (24)(13), (14)(23)$ of the coordinates. It has 12 singular points forming the orbits of $[1, 0, 0, 0], [1, 1, 1, -1], [1, 1, -1, -1]$ and $[1, 1, 1, 1]$. They lie by two on the edges of the desmic tetrahedra. The 16 lines are the intersections of the faces on two different desmic tetrahedra. They are given by equations $x - y = 0, x + z = 0$ and similar equations. The nodes and lines form an abstract configuration $(12_4, 16_3)$. It is isomorphic to the *Reye configuration* of points and lines in $\mathbb{R}^3$ and also Hesse-Salmon configuration of points and lines in the plane (see [245]).

Every node of the surface is a degenerate node, i.e. the exceptional curve E over the node is contained in the ramification divisor of the projection map pr_q. For example, the equation of the branch curve B of the projection from the point $q = [0, 0, 0, 1]$ is

$$\big((a+b)x^2+(a-c)by^2+(b+c)z^2\big)\big((b+c)x^2y^2+(a-c)x^2z^2-(a+b)y^2z^2\big) = 0. \tag{12.34}$$

The cubic $V(a_1(y^2z+yz^2)+a_2(x^2z+xz^2)+a_3(x^2y+xy^2))$ is nonsingular and it is tangent to K at the cusps. We see a peculiar example, where the contact conic intersects the branch curve only at its singular points.

Theorem 12.3.12. *The desmic quartic surface is isomorphic to the Kummer surface* $\mathrm{Kum}(E \times E)$ *of the self-product of an elliptic curve* E.

Proof Choose two singular points p_1, p_2 on an edge of one of the tetrahedra. Let E_1, E_2 be the exceptional curves over these points on a minimal resolution X of the desmic quartic surface Q. The eight lines that pass through p_1 or p_2 have one common line F_0. Let $F_1, \ldots, F_6$ be the remaining lines. Consider the divisor

$$H = 2(E_1 + E_2) + F_1 + \cdot + F_6 + 2F_0.$$

Then we check that $H^2 = 4$ and the linear system $|H|$ defines a degree two map

$$\Phi : X \to \mathbf{F}_0 \subset \mathbb{P}^3.$$

Its branch divisor is the union of 8 lines, four from each ruling. On the other hand, the linear system $|2(E\times\{\mathrm{pt}\}+\{\mathrm{pt}\}\times E)|$ defines a double cover $E\times E \to \mathbb{P}^1 \times \mathbb{P}^1$ branched along the union of 8 lines, from from each ruling. Since $\mathrm{Pic}(\mathbf{F}_0)$ has no torsion, there is only one isomorphism class of such a cover. This proves the assertion.

□

Example 12.3.13. This is an example of a nodal quartic surface with 12 weakly skew lines. The lines are the edges of the cube with the faces $x = 0, y = 0, z = 0, x - w = 0, y - w = 0, z - w - 0)$. The eight vertices are ordinary nodes of teh surface. and its edges are the lines. The surfaces depend on two parameters and are given by the equation

$$\begin{aligned}&(a_1yz + a_2xy + a_3xz)w^2 + (a_1(y^2z + yz^2) + a_2(x^2z + xz^2)\\&+ a_3(x^2y + xy^2))w + a_1y^2z^2 + a_2x^2y^2 + a_3x^2z^2 = 0.\end{aligned} \tag{12.34}$$

The equation of of the branch curve of the projection from the point $[0,0,0,1]$ is

$$(a_1(y^2z+yz^2)+a_2(x^2z+xz^2)+a_3(x^2y+xy^2))^2- \\ 4(a_1y^2z^2+a_2x^2y^2+a_3x^2z^2)(a_1y^2z^2+a_2x^2y^2+a_3x^2z^2)=0. \tag{12.34}$$

We check that it has three simple singular points of type a_5 (a cusp and infinitely near cusp) at $[1,0,0],[0,1,0],[0,0,1]$, three ordinary cusps at the points $[1,1,0],[0,1,1],[1,0,1],[1,1,1]$ and an ordinary node at $[1,1,1]$. The contact conic $K=V((a_1y^2z^2+a_2x^2y^2+a_3x^2z^2)$. It intersects B at the cusps $[0,0,1],[1,0,0],[0,1,0]$ with multiplicity four.

By Theorem 12.3.9, the minimal resolution $\tilde{Q}$ of Q is isomorphic to the Kummer surface of some abelian surface. It turns out that the abelian surface is the self-product $A=F\times F$ of an elliptic curve F. Let Θ_i be the proper transforms of the 16 lines in $\tilde{Q}$. The proof of Nikulin's theorem shows that the sum $\sum\Theta_i\sim 2D$ is divisible by 2 in $\mathrm{Pic}(\tilde{Q})$. Let A' be the double cover of $\tilde{Q}$ branched over $\sum\Theta_i$. The ramification divisor consists of sixteen (-1)-curves on A'. The proper transform of exceptional curve E_i over the nodes intersects the branch divisor at four points. Its pre-image sin A' is an elliptic curve. Blowing down the (-1)-curves, we obtain an abelian surface A with a set of 12 elliptic curves on it, intersecting by three at each of the 2-torsion points of A. Since we can find two of them, say F_1,F_2, intersecting at one point, we obtain that $A\cong F_1\times F_2$. On the other hand, we can find another one F_3 that intersects F_1 and F_2 at one point. It defines a graph of an isomorphism $F_{\rightarrow}F_2$. Thus, $A\cong F\times F$.

The isomorphism class of the elliptic curve F is determined by the cross-ratio of the tangent directions of the four lines passing through the node on the exceptional curve over the node identified with $\mathbb{P}^1$. Thus, we obtain a remarkable property that all the 12 cross-ratios coincide.

Note that, if we take the parameters $(a,b,c)=(1,\eta_3,\eta_3^2)$, where $\eta_3^3=1,\eta\neq 1$, we obtain an automorphism of Q defined by a cyclic permutation of x,y,z. In this case, the elliptic curve F acquires a complex multiplication of order 3. The desmic surface of this kind is discussed in detail by Hunt [416]. Similarly, we take $(a,b,c)=(1,-1,0)$, we obtain an extra automorphism defined by a cyclic permutation of (x,y,z,w). In this case, F acquires complex multiplication by $\sqrt{-1}$.

Let R_3 and R_3' be two disjoint twisted cubics in $\mathbb{P}^3$. We know from Proposition 10.1.8 that they have 10 common secants lines. Taking R_3 and R_3' general enough we may assume that they are skew. Since containing an R_3 imposes 13 conditions on quartics, quartics containing R_3 and R_3' depend on $34-26=8$ parameters. Since each secant line intersects R_3+R_3' at four points, there is one additional condition to contain a secant. Thus, we can find a quartic surface containing 7 or 8 disjoint lines. We refer to [796] for an explicit condition for a quartic containing 6 skew lines to contain another skew line.

An example of a nonsingular quartic surface with 10 skew lines is the Steinerian surface of a general web of quadrics in $\mathbb{P}^3$. They are the singular lines of ten reducible quadrics in the web.

A nonsingular quartic surface has at most 64 lines. This is a theorem of Beniamono Segre [682] (his proof corrected in [605]). An example of a nonsingular surface with 64 lines was given much earlier by F. Schur [671]. It is given by the equation

$$x(x^3 - y^3) - z(z^3 - w^3) = 0. \tag{12.34}$$

The surface has a large linear automorphism group. It is generated by the transformation $(x, y, z, w) \mapsto (z, w, x, y)$ and the symmetry of the binary form $u(u^3 - v^3)$ isomorphic to $\mathfrak{A}_4$. The set of lines is the orbit of the obvious lines: line $x - z = y - w = 0$ and $y - \alpha_i x = w - \alpha_j = 0$, where α_i are roots of the polynomial $t(1 - t^3)$.

In the same paper, Schur gave an example of a quartic surface with 52 lines.

An easy example is, of course, the Ferma quartic surface that contains $3 \times 16 = 48$ lines. We refer to [605] for other examples and the classification of quartic surfaces with more than 52 lines.

Assume now that Q is a μ-nodal quartic surface. By Theorem 12.3.9, it contains at most 16 skew lines. For, example, a Weddle surface contains 15 weakly skew lines, or the Hessian quartic surface contains 10 weakly skew lines.

Let us construct a quartic surface with 12, 13, or 14 skew lines. Its set of nodes will be the set of vertices of a cube whose edges are lines. We require that a quartic surface contains the 8 vertices and three points on each of the 12 edges. This require $8 + 24 = 32$ conditions. So, we can find a quartic surface. Since the edge lines at the vertices are not coplanar, the vertices are singular points. For a general choice of Q they are ordinary nodes. We can also add one or two diagonals of the cube and use 2 extra parameters to require that the quartic contains them.

Exercises

12.1 An irreducible subvariety X of the Grassmannian $G_r(\mathbb{P}^n)$ is called a *congruence* if $\dim X = n - r$ [691].

(i) Show that the number of r-planes from X passing through a general point (resp. contained in a general hyperplane) is finite. It is called the *order* of X.

(ii) Show that the subvariety of r-planes from X contained in a general hyperplane is an irreducible variety of degree $n - 2r - 1$. Its degree is called the *class* of X.

(iii) Extend to X the notion of a fundamental point and the focal hypersurface, and prove that if $r = 1$ and X has only finitely many fundamental points) a general ray from X is tangent to the focal hypersurface at $n - 1$ points.

(iv) Using (iii), show that, if $n \geq 4$, the order of a congruence of lines in $\mathbb{P}^n$ is not equal to 2 [311].

12.2 Consider a subvariety $X(N_1, \ldots, N_{n-1})$ of $\mathbb{P}^n$ projectively generated by a general set of nets $N_1, \ldots, N_{n-1}$ of hyperplanes with a fixed isomorphism to $N_i \to \mathbb{P}^{2}$' (see Subsection 3.3.1).

(i) Show that the lines $H_1(\lambda) \cap \cdots \cap H_{n-1}(\lambda), \lambda \in \mathbb{P}^2$, form of a surface $\mathcal{S}(N_1, \ldots, N_{n-1}; n)$ of order $\frac{1}{2}(n-1)(n-2)$ and class $\frac{1}{2}n(n-1)$ of lines in $\mathbb{P}^n$.

(ii) Show that the surface $\mathrm{Sec}^1(R_n)$ formed by secant lines of a rational normal curve R_n in $\mathbb{P}^n$ is an example of a surface $\mathcal{S}(N_1, \ldots, N_{n-1}; n)$. Show that its Plücker embedding is projectively isomorphic to the Veronese surface $\nu_{n-1}(\mathbb{P}^2)$.

(iii) Show that rays of $\mathcal{S}(N_1, \ldots, N_{n-1}; n)$ sweep a 3-dimensional variety $F_{\frac{1}{2}(n-1))(n-2)}$ of degree $\frac{1}{2}(n-1))(n-2)$.

(iv) Assume $n = 4$, show that, for a general $\mathcal{S}(N_1, N_2, N_3; 3)$, the cubic F_3 has six singular points (in particular, a general $\mathcal{S}(N_1, N_2, N_3; 3)$ does not coincide with the congruence $\mathrm{Sec}^1(R_4)$.

(v) Show that the cubic scroll F_3 has two rulings whose generators define two congruence of lines in $\mathbb{P}^4$ of order 3 and class 6

12.3 (i) Show that the projection $\mathbb{P}^4 \dashrightarrow \mathbb{P}^3$ from a point not in F_3 maps the rays of $\mathcal{S}(N_1, N_2, N_3; 3)$ to rays a congruence of lines ℓ_6^3 in $\mathbb{P}^3$ of order $n = 3$, class $m = 6$, and sectional genus 4.

(ii) Show that the projection $\mathbb{P}^4 \dashrightarrow \mathbb{P}^3$ from a general point in F_3 maps rays of

$$\mathcal{S}(N_1, N_2, N_3; 3)$$

to rays of a congruence of lines without fundamental curve of bidegree $(2, 6)_{II}$ in $\mathbb{P}^3$.

(ii) Show that F_3 contains the three base lines ℓ_1, ℓ_2, ℓ_3 of the nets N_1, N_2, N_3 are not rays of the congruence St_4, each containing two of singular points of F_3.[94]

(iii) Find the bidegree of the congruence of lines in $\mathbb{P}^3$ obtained by the projection of $\mathcal{S}(N_1, N_2, N_3; 3)$ from a point on a line ℓ_i.

(iv) Assume that the nets N_1, N_2, N_3 contain k common planes $\Pi_i : H_1(\lambda_i) \cap H_2(\lambda_i) \cap H_3(\lambda_i), i = 1, \ldots, k$. Show that the class of the degenerate congruence $\mathcal{S}(N_1, N_2, N_3; 3)$ is equal to $6 - k$. Show that by projecting these congruences one obtains congruences of lines in $\mathbb{P}^3$ of bidegree $(2, 6-k)$ [94].

12.4 Show that the closure of the set of trisecant lines of a Bordiga sextic surface in $\mathbb{P}^4$ is a congruence of order one and class three.

12.5 Let X be a del Pezzo surface of degree 5 and $\{L_1, \ldots, L_{10}\}$ be the set of lines on X. Let $\mathcal{E} = \Omega_X^1(\log D)$ be the sheaf of logarithmic differential 1-forms on X, where D is the sum of the ten line on X.

(i) Let $\mathcal{E} = \Omega_X^1(\log D)$ be the sheaf of logarithmic differentials that fits in the short exact sequence

$$0 \to \Omega_X^1 \to \Omega_X^1(\log D) \xrightarrow{\text{res}} \bigoplus_{i=1}^{10} \mathcal{O}_{L_i} \to 0,$$

where res is the residue map. Show that $\dim H^0(X, \mathcal{E}) = 5$ and $\mathcal{E}$ is generated by its global sections.

(ii) Show that the map $x \mapsto \mathrm{Ker}(H^0(X, \mathcal{E}) \xrightarrow{\text{ev}} \mathcal{E}(x))^\perp$, where ev is the evaluation map, defines a closed embedding $j : X \to G(2, H^0(X, \mathcal{E})^\vee) \cong \mathbb{G} = G_1(\mathbb{P}^4)$ such that $\mathcal{E} = j^*\mathcal{Q}_{G_1(\mathbb{P}^4)}$.

(iii) Show that the image S of X is a congruence of lines in $\mathbb{P}^4$ of order 2 and class 3.

- Show that the rays of S sweep a three-dimensional hypersurface $p_{\mathbb{G}}(q_{\mathbb{G}}^{-1}(S))$ in $\mathbb{P}^4$ isomorphic to the Segre cubic primal.

(iv) Let $S^* \subset G(3, H^0(S, \mathcal{E})) \cong \mathbb{G}^* = G_2(\check{\mathbb{P}}^4)$ be the image of S under the duality map $\mathbb{G} \to \mathbb{G}^*$. Show that $p_{\mathbb{G}^*} : q_{\mathbb{G}^*}^{-1}(S^*)) \to \check{\mathbb{P}}^4$ is a degree 2 map whose branch divisor is the Castelnuovo-Richmond quartic hypersurface CR_4.

12.6 Let S be a congruence of lines in $\mathbb{P}^3$. A plane Π in $\mathbb{P}^3$ is called a *fundamental plane* of S if it contains infinitely many rays..

(i) Show that a fundamental plane Π is a fundamental point $p = \Pi^\perp$ of the dual congruence of lines.

(ii) Show that the degree of the fundamental p is equal to the degree of the one-dimensional part of $\Omega(\Pi) \cap \Pi$. By the definition, this is the degree of a fundamental plane.

(iii) Find fundamental planes for congruences of order one or two discussed in this chapter.

12.7 Let S_1 and S_2 be two congruences of lines in $\mathbb{P}^3$ of order 1 and class 2. Define a Cremona transformation $T : \mathbb{P}^3 \dashrightarrow \mathbb{P}^3$ as follows. A general point $x \in \mathbb{P}^3$ contains in a unique ray ℓ_1 of S_1 and a unique ray ℓ_2 of S_2. Since the class is equal to 2, the plane spanned by ℓ_1 and ℓ_2 contains a unique ray ℓ_1' of S_1 and a unique ray ℓ_2' of S_2 that intersect at a unique point $T(x)$ [603].

(i) Show that T is a Cremona involution.

(ii) Find the F-locus and F-locus of T.

(iii) Find the multidegree of T.

12.8 Let S be the congruence of order two of secant lines of an elliptic quartic curves F in $\mathbb{P}^3$.

(i) Show that, for a general point $P \in \mathbb{P}^3$, the surface (P) of centers of the null-planes containing P coincides with the cubic surface from Exercise 9.25. It is associated to the point P and the pencil of quadrics $|I_F(2)|$ containing the curve F.

(ii) Show that the curve $|\ell|$ of centers of null-planes containing ℓ is the union of the curve F taken with multiplicity 3 and a curve R of degree 5 taken with multiplicity one.

(iii) Show that the curve R passes through the tangency points of planes from $\ell^\perp$ and quadrics from the pencil $|I_F(2)|$.
(iv) Find the genus of the curve R.

12.9 Let $|L|$ be a general web of quadrics in $\mathbb{P}^3$ and S be the Reye congruence whose rays are Reye lines of $|L|$ (see Subsection 1.1.7).

(i) Using Exercise 10.16 show that S is contained in the intersection of two cubic line complexes which intersect along a congruence of bidegree $(2, 6)$ with a fundamental curve. Deduce from this that the bidegree of S is equal to $(7, 3)$.
(ii) Show that S is smooth and its sectional genus p_a equals 6.
(iii) Show that S has no fundamental points but has 20 fundamental planes.
(iv) Show that the focal surface of S is a smooth quartic surface but the focal surface of the dual con
(v) Show that the map $f : \mathbb{P}^3 \to |L^\vee|$ given by the linear system $|L|$ is of degree 8, and its branch divisor $\Delta$is the dual surface of the Cayley quartic symmetroid Q in $L|$.
(vi) Show that the focal surface of S is a surface of degree 24 which is one of the two irreducible components of $f^{-1}(\Delta)$.
(vii) Specialize $|L|$ to a web of quadrics with k base points. Show that the Reye congruence specializes to a congruence of bidegree $(7 - k, 3)$.

12.10 Let $T : \Pi \dashrightarrow \Pi'$ be a birational transformation of degree n between two planes in $\mathbb{P}^3$. The closure of lines $\langle x, T(x)\rangle$ is a congruence S in $\mathbb{P}^3$, called a *Cremonian correspondence* [395], [397].

(i) Show that the class of S is equal to n, i.e. the degree of the image of a general line in Π.
(ii) Show that the line $\Pi \cap \Pi'$ is a singular ray of S of multiplicity n and the planes Π and Π' are its fundamental planes of S.
(iii) Show that a base point of multiplicity k of the homaloidal line system that defines T is a fundamental point of S of degree k.
(iv) Show that congruences S^* dual to singular congruence S of bidegree $(1, 3)$ and $(2, 4)$ are Cremonian correspondences.

12.11 Let S be a congruence of lines in $\mathbb{P}^3$ of order one and let Π, Π' be two general planes. Show that the rational map $T : \Pi \dashrightarrow \Pi'$ that assigns to a general point $x \in \Pi$ the intersection of the unique ray of S passing through x with the plane Π' is a birational transformation. Find its degree and fundamental points.

12.12 Let S be a congruence of lines in $\mathbb{P}^3$ of bidegree $(3, 3)$.

(i) Show that its arithmetical sectional genus p_a is less than equal than 4 and the equality holds only if S is a complete intersection of a cubic and linear complexes of lines.
(ii) Show that a smooth congruence S with $p_a = 4$ is a K3 surface.
(iii) Show that the focal surface of a general congruence S with $p_a = 4$ is a surface of degree 12 isomorphic to its dual surface.
(iv) Show that a smooth S with sectional genus one is equal to the projection of an anti-canonical del Pezzo surface of degree six.

12.13 Let S be a general congruence of lines in $\mathbb{P}^3$ of bidegree $(3, 4)$ and arithmetic sectional genus equal to 3.

(i) Show that S is equal to the residual surface of the intersection of two quadratic line complexes containing a common plane Λ.
(ii) Show that the intersection $S \cap \Lambda$ is a curve C of degree 3.

(iii) Show that S is a rational surface, the image of $\mathbb{P}^2$ under a map given by the linear system of curves of degree four passing trough a general set of nine points and the curve C is the image of the unique cubic curve through the nine points [294].

12.14 Let S be a general congruence of lines in $\mathbb{P}^3$ of bidegree $(3,5)$ and arithmetic sectional genus equal to 4.

(i) Show that S is contained in the intersection of a quadratic line complex and a cubic quadratic complex with the residual surface of order 4.
(ii) Show that the residual surface of a smooth S is a congruence of bidegree $(2,2)$ isomorphic to a Veronese surface, and S is a rational surface equal to the image of $\mathbb{P}^2$ under a rational map given by a linear system of curves of degree 6 with 6 double base points and four simple base points [294].

12.15 Show that the intersection of two line complexes of degree d_1 and d_2 is a congruence of bidegree (d_1d_2, d_1, d_2) and of rank $d_1d_2(d_1-1)(d_2-1)$.

12.16 Let S be a congruence of order two without fundamental curve and S' be a confocal congruence. Find the intersection of the corresponding irreducible components of the bitangent surface $\mathrm{Bit}(S)$.

12.17 Show that the projection of a general hyperplane section of a nodal quartic from one of the nodes is a plane quartic which is tangent to the branch curve B of the projection at 12 nonsingular points. Prove that the points are lie on a cubic curve that passes through the six points whether the contact conic intersects B.

12.18 Let B be a plane sextic with two nodes p_1, p_2 and q_1, q_2 are the residual intersections of the line $\langle p_1, p_2\rangle$ with C. Show that there exists a quartic curve C that passes through p_1, p_2, q_1, q_2 and intersects B at the remaining $24-6=8$ points such that there exists a sextic that B' that is tangent to B at these points and admits a contact conic that passes through p_1, p_2.

12.19 Let B be an irreducible plane sextic with 9 (resp. 10) nodes. Show that there exists 4 (resp. 2) contact-cubics that touch B at 9 nonsingular points.

12.20 Prove that a Reye line of a general web $|L|$ of quadrics is touching the the quartic symmetroid Q defined by $|L|$. Using this show that the bitangent surface $\mathrm{Bit}(Q)$ is irreducible.

12.21 Let $\mathfrak{D}$ be a Cayley dianode surface with triple points $x_1, \ldots, x_7$. Show that, for any x_i there exists a unique quartic surface with a triple point at x_i and double points at x_j, $j \neq i$. Show that its proper transform on $\mathsf{Bl}_{x_1,\ldots,x_7}$ is a section of the elliptic vibration defined by the net of quadrics through the seven points.

12.22 Let $\mathcal{R}_3$ be the set of stable Veronese curves of degree 3 passing through the five reference points p_i.

(i) Show that there are ten curves from $\mathcal{R}_3$ that intersect two general lines in $\mathbb{P}^3$.
(ii) Show that the union of curves in $\mathcal{R}_3$ which are tangent to a general plane Π is a surface F_Π of degree 10.
(iii) Show that the reference points are 6-fold singular points of F_Π and the lines $\langle p_i, p_j\rangle$ are double lines of F_Π.

12.23 Let F be an elliptic curve. Find the linear system on $A = F \times F$ that defines a rational map $A \dashrightarrow \mathbb{P}^3$ with the image equal to a desmic quartic surface.

12.24 Show that the projection of a general hyperplane section of a nodal quartic from one of the nodes is a plane quartic which is tangent to the branch curve B of the projection at 12 nonsingular points. Prove that the points lie on a cubic curve that passes through the six points whether the contact conic intersects B.

12.25 Let B be a plane sextic with two nodes p_1, p_2 and q_1, q_2 are the residual intersections of the line $\langle p_1, p_2\rangle$ with C. Show that there exists a quartic curve C

that passes through p_1, p_2, q_1, q_2 and intersects B at the remaining $24 - 6 = 8$ points such that there exists a sextic that B' that is tangent to B at these points and admits a contact conic that passes through p_1, p_2.

12.26 Let B be an irreducible plane sextic with 9 (resp. 10) nodes. Show that there exists 4 (resp. 2) contact-cubics that touch B at 9 nonsingular points.

12.27 Show that the linear system of quadrics in $\mathbb{P}^3$ containing a smooth conic and tangent to a point on it to the plane spanned by the conic maps a cubic surface containing the conic to a quartic surface with a double conic ([188]).

12.28 Let $\mathfrak{D}$ be a Cayley dianode surface with triple points $x_1, \ldots, x_7$. Show that, for any x_i there exists a unique quartic surface with a triple point at x_i and double points at $x_j, j \neq i$. Show that its proper transform on $\mathsf{Bl}_{x_1,\ldots,x_7}$ is a section of the elliptic vibration defined by the net of quadrics through the seven points.

12.29 Prove that a normal quartic surface containing a set of seven nodes in a general linear position has at most ten nodes.

Historical Notes

An excellent source for the history of the theory of algebraic surfaces is Loria's book [495, Capiotolo 3]. We will discuss only the part concerning quartic surfaces. We refer to the Historical Notes in Chapter 10 for the history of the study of ruled quartic surfaces. Jessop's book [430] gives a brief account of nodal and non-normal quartic surfaces. on

The study of non-ruled quartic surfaces with a double line began with Kummer's paper [472]. Here, he introduced the Kummer Complex surface with eight isolated nodes. In the same memoir, Kummer introduced quartic surfaces with a double conic. In [120] Cayley constructed a quartic surface with a double line obtained as a degeneration of Cayley's quartic symmetroid.

The study of quartic surfaces with a double circle began with a paper by M. Moutard [530]. He called such surfaces *surfaces analagmatique*. Later Darboux gave them the name *cyclide* [201]. A long memoir by Humbert [410] contains the history and a detailed study of cyclide from the view of differential geometry.

Cremona [188] and Noether [553] constructed such surfaces as the images of a birational map from a cubic surface or the projective plane. The first examples of quartic surfaces with a cuspidal double conics were given by Cremona [193] and Bela Tötössy [763]. As we already noticed the classification of quartic surfaces with a double conic as the projections of a quartic del Pezzo surface was given by C. Segre [685]. Classification of real quartic cyclide surfaces was given by Loria [496]. Loria's book [495] contains a vast bibliography on cyclide quartic surfaces, including Dupin cyclides.

Rational quartic surfaces with an isolated double point were studied by Cremona [191], [190] and Noether [555]. Modern classification of possible isolated singularities of ruled quartic surfaces can be foun in [209], [770], [772], [774].

The first systematic study of nodal quartic surfaces started in Kummer's work on quadratic congruences of lines in $\mathbb{P}^3$. In his first memoir on quartic surfaces [119], Cayley acknowledges the significant influence on his work of Kummer's work. As we saw in the previous chapter, Kummer discussed quartic surfaces with 11-16 nodes realized as the focal surfaces. Cayley shows in his memoir that these quartic surfaces arise by a degeneration of the 10-nodal Cayley quartic symmetroid. He systematically investigated quartic surfaces with all possible $\mu \leq 16$ nodes. Introducing in the way, the dianodal sextiic surfaces and shows that a quartic surface with seven nodes in a general linear position has at most ten nodes. He also introduced the notion of a dianome quartic surface and the dianodal sextic surfaces, the locus of nodes of quadrics from the linear system of quadric with fixed seven base points. The systematic study of nodal quartic surfaces via the projections from a node was done by K. Rohn [630], [629]. Rohn also gave a classification of triple points on quartic surfaces [630].

The Steiner quartic surface is also known as the *Steiner Roman surface*. Steiner discovered it during his visit of Rome in 1844. Steiner himself had never documented his discovery. It was rediscovered by Kummer [472], a short note added by Weierstrass to his paper gives Steiner's synthetic construction of these surfaces (see more history in [641]).

The main topic of Cayley's memoir is what we now call the Cayley symmetroid quartic surface. In fact, Cayley discusses general determinantal varieties defined by unnecessary square matrices.

We already gave some bibliographical references to Kummer quartic surfaces in Chapter 10. We may add new recent references [256] and [641] on the history of Kummer surfaces.

Although Cayley discusses general 6-nodal quartic surfaces, the special quartic surface birationally isomorphic to the Kummer surface was introduced much earlier in 1850 by T. Weddle [800]. We already gave some references to the history of desmic quartic surfaces in desmic quartic surfaces in Subsection 2.3.1. Humbert was the first to establish a birational isomorphism between desmic quartic surfaces and the Kummer surfaces of the self-product of an elliptic curve. In his paper [411], he gives many other geometrical facts about the surfaces, and, in particular, introduces their relationship to an earlier Cremona's construction of some quartic surfaces related to a tritangent plane of a cubic surface [186]. Jessop's book contains a chapter on desmic surfaces.

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