Journal of Algebra 436 (2015) 35–60

Contents lists available at ScienceDirect

Journal of Algebra

www.elsevier.com/locate/jalgebra

Real reductive Cayley groups of rank 1 and 2 ☆

Mikhail Borovoi [1]

Raymond and Beverly Sackler School of Mathematical Sciences, Tel Aviv University, 69978 Tel Aviv, Israel

A R T I C L E   I N F O

Article history:
Received 20 September 2013
Available online xxxx
Communicated by Eva Bayer-Fluckiger

MSC:
20G20
20G15
14E05

Keywords:
Linear algebraic group
Cayley group
Cayley map
Equivariant birational isomorphism
Algebraic surface
Elementary link

A B S T R A C T

A linear algebraic group G over a field K is called a Cayley K-group if it admits a Cayley map, i.e., a G-equivariant K-birational isomorphism between the group variety G and its Lie algebra. We classify real reductive algebraic groups of absolute rank 1 and 2 that are Cayley $\mathbb{R}$-groups.

1. Introduction

Let G be a connected linear algebraic group defined over a field K, and let $\mathrm{Lie}(G)$ denote its Lie algebra. The following definitions are due to Lemire, Popov and Reichstein [12]:

☆ With an appendix by Igor Dolgachev.

E-mail address: borovoi@post.tau.ac.il.

[1] Borovoi was partially supported by the Hermann Minkowski Center for Geometry.

http://dx.doi.org/10.1016/j.jalgebra.2015.03.034

Definitions 1.1. (See [12].) A *Cayley map for G* is a K-birational isomorphism $G \overset{\simeq}{\dashrightarrow} \mathrm{Lie}(G)$ which is G-equivariant with respect to the action of G on itself by conjugation and the action of G on $\mathrm{Lie}(G)$ via the adjoint representation. A linear algebraic K-group is called a *Cayley group* if it admits a Cayley map. A linear algebraic K-group is called a *stably Cayley group* if $G \times_K (\mathbb{G}_{m,K})^r$ is Cayley for some $r \geq 0$, where $\mathbb{G}_{m,K}$ denotes the multiplicative group.

Lemire, Popov and Reichstein [12] classified Cayley and stably Cayley simple groups over an algebraically closed field k of characteristic 0. Borovoi, Kunyavskiĭ, Lemire and Reichstein [2] classified stably Cayley simple K-groups, and later Borovoi and Kunyavskiĭ [3] classified stably Cayley semisimple K-groups, over an arbitrary field K of characteristic 0. Clearly any Cayley K-group is stably Cayley. In the opposite direction, some of the stably Cayley K-groups are known to be Cayley, see [12, Examples 1.9, 1.11 and 1.16]. For other stably Cayley K-groups, it is a difficult problem to determine whether they are Cayley or not. By [2, Lemma 5.4(c)] the answer to the question whether a K-group is Cayley depends only on the equivalence class of G up to inner twisting.

By [2, Corollary 7.1] all the reductive K-groups of rank ≤ 2 over a field K of characteristic 0 are stably Cayley (by the rank we always mean the *absolute* rank). We would like to know, which of those stably Cayley K-groups of rank ≤ 2 are Cayley.

The case of a simple group of type $\mathbf{G}_2$ was settled in [12, §9.2] and Iskovskikh's papers [9,10]. Namely, a simple group of type $\mathbf{G}_2$ over an algebraically closed field k of characteristic 0 is not Cayley. Hence no simple K-group of type $\mathbf{G}_2$ over a field K of characteristic 0 is Cayley.

Popov [15] proved in 1975 that, contrary to what was expected (cf. [13, Remarque, p. 14]), the group $\mathbf{SL}_3$ over an *algebraically closed field* k of characteristic 0 is Cayley; see [12, Appendix] for Popov's original proof, and [12, §9.1] for an alternative proof.

Here we are interested in $\mathbb{R}$-groups, where $\mathbb{R}$ denotes the field of real numbers. If G is an inner form of a split reductive $\mathbb{R}$-group, and $G_{\mathbb{C}} := G \times_{\mathbb{R}} \mathbb{C}$ is *stably Cayley* over $\mathbb{C}$, then by [2, Remark 1.8] G is *stably Cayley* over $\mathbb{R}$. Similarly, since $\mathbf{SL}_{3,\mathbb{C}}$ is *Cayley* over $\mathbb{C}$ by Popov's theorem, one might expect that the split $\mathbb{R}$-group $\mathbf{SL}_{3,\mathbb{R}}$ is *Cayley* over $\mathbb{R}$. However, this turns out to be false, see Theorem 8.1 of Appendix A. On the other hand, the outer form $\mathbf{SU}_3$ of the split group $\mathbf{SL}_{3,\mathbb{R}}$ is Cayley, see Theorem 7.1 of Appendix A and Corollary 4.4.

We recall the classification of reductive K-groups of rank ≤ 2. A reductive K-group of rank 1 is either a K-torus or a simple K-group of type $\mathbf{A}_1$. A reductive K-group of rank 2 is either not semisimple, or semisimple of type $\mathbf{A}_1 \times \mathbf{A}_1$, or simple of one of the types $\mathbf{A}_2$, $\mathbf{B}_2 = \mathbf{C}_2$, or $\mathbf{G}_2$. If a reductive K-group of rank 2 is not semisimple, then either it is a K-torus or it is isogenous to the product of a one-dimensional K-torus and a simple K-group of type $\mathbf{A}_1$.

We recall the classification of *real* simple groups of type $\mathbf{A}_2$. Such an $\mathbb{R}$-group is isomorphic to one of the groups $\mathbf{SL}_{3,\mathbb{R}}$, $\mathbf{PGL}_{3,\mathbb{R}}$, $\mathbf{SU}_3$, $\mathbf{PGU}_3$, $\mathbf{SU}(2,1)$, or $\mathbf{PGU}(2,1)$. Here, following the Book of Involutions [11, §23], we write $\mathbf{PGU}_n$ rather than $\mathbf{PSU}_n$ for the corresponding adjoint group. We write $\mathbf{SU}(2,1)$ and $\mathbf{PGU}(2,1)$ for the (inner) forms of $\mathbf{SU}_3$ and $\mathbf{PGU}_3$, respectively, corresponding to the Hermitian form with diagonal matrix $\mathrm{diag}(1,1,-1)$.

In this paper we classify real reductive algebraic groups of rank ≤ 2 that are Cayley. To be more precise, for each real reductive group of rank 1 or 2 (up to an isomorphism) we determine whether it is Cayley or not:

Theorem 1.2. *Let G be a connected reductive algebraic $\mathbb{R}$-group of absolute rank ≤ 2 over the field $\mathbb{R}$ of real numbers. If G is simple of type $\mathbf{G}_2$ or is isomorphic to $\mathbf{SL}_{3,\mathbb{R}}$, or $\mathbf{PGU}_3$, or $\mathbf{PGU}(2,1)$, then G is* not *Cayley. Otherwise G is Cayley.*

Theorem 1.2 will be proved case by case. The cases when G is Cayley will be treated by the author in the main text of the paper. In the case when G is of type $\mathbf{G}_2$ it is known that G is not Cayley, see above. The other cases when G is not Cayley (and again the case of $\mathbf{SU}_3$ when G is Cayley) will be treated by Igor Dolgachev in Appendix A.

Note that by [2, Corollary 7.1] any K-group G of absolute rank ≤ 2 over a field K of characteristic 0 is stably Cayley, that is, there exists $r \ge 0$ such that the group $G \times_K \mathbb{G}_{m,K}^r$ is Cayley, where $\mathbb{G}_{m,K}$ denotes the multiplicative group over K. The following theorem, which generalizes [12, Proposition 9.1], shows that one can always take $r = 2$.

Theorem 1.3. *Let G be a connected reductive algebraic K-group of absolute rank ≤ 2 over a field K of characteristic 0. If G is of absolute rank 1, then G is Cayley. If G is of absolute rank 2, then $G \times_K \mathbb{G}_{m,K}^2$ is Cayley.*

The following question generalizes [12, Remark 9.13].

Question 1.4. Let G be a reductive K-group of absolute rank 2 that is not Cayley, for example $\mathbf{SL}_{3,\mathbb{R}}$. Is $G \times_K \mathbb{G}_{m,K}$ a Cayley group?

Question 1.5. Are the $\mathbb{R}$-groups $\mathbf{PGU}_{2n+1}$ Cayley for $n \ge 2$? (Note that these $\mathbb{R}$-groups are stably Cayley, see [2, Thm. 1.4].)

The plan of the rest of the paper is as follows. In Section 2 we reproduce some examples of Cayley groups from [12], and state some known properties of Cayley groups. In Section 3 we prove Theorem 1.2 modulo results of Section 4 and of Appendix A. In Section 4 we treat the case $\mathbf{SU}_3$ of Theorem 1.2, using explicit calculations. In Section 5 we prove Theorem 1.3 (case by case). In Appendix A, Igor Dolgachev treats the difficult cases $\mathbf{SL}_{3,\mathbb{R}}$ and $\mathbf{PGU}_3$ of Theorem 1.2 (and again the case $\mathbf{SU}_3$), using the theory of elementary links due to Iskovskikh [8–10]. In Appendix B, contributed by the anonymous referee, the case of the group $\mathbf{PGL}_1(A)$ for a central simple algebra A of degree n over a field K of positive characteristic p dividing n is considered.

The author is very grateful to Igor Dolgachev for writing Appendix A. The author thanks the referee for writing Appendix B. The author thanks Zinovy Reichstein for most helpful comments.

2. Preliminary remarks

We reproduce some examples from [12]. Note that in [12] it is always assumed that the characteristic of K is zero, while we attempt to state these results assuming that K is a field of arbitrary characteristic.

Example 2.1. (Cf. [12, Ex. 1.9].) Consider a finite-dimensional associative K-algebra A with unit element 1, over a field K of arbitrary characteristic, and the K-group $A^\times$ of invertible elements of A. Then clearly $A^\times$ is Cayley. In particular, the K-group $\mathbf{GL}_{n,K}$ is Cayley.

Example 2.2. (Cf. [12, Ex. 1.11].) Let A be a *central simple* K-algebra of degree n, and assume that $\mathrm{char}(K)$ does not divide n. For any element $a \in A$ denote by $\mathrm{tr}\, a$ the trace of the linear operator L_a of left multiplication by a in A. Then $\mathrm{tr}\, 1 = n^2 \neq 0 \in K$. The argument in [12] shows that the quotient group $\mathbf{PGL}_1(A) := A^\times/\mathbb{G}_{m,K}$ is Cayley.

Now assume that $\mathrm{char}(K)$ divides n and that $4|n$ when $\mathrm{char}(K) = 2$, then again $\mathbf{PGL}_1(A)$ is Cayley, see Theorem B.1 in Appendix B below.

We see that if $\mathrm{char}(K) \neq 2$ or if n is odd, then the group $\mathbf{PGL}_{n,K}$ is Cayley. In particular, in arbitrary characteristic the group $\mathbf{PGL}_{3,K}$ is Cayley. Moreover, if $\mathrm{char}(K) \neq 2$, then $\mathbf{PGL}_{2,K}$ is Cayley. On the contrary, if $\mathrm{char}(K) = 2$, then $\mathbf{PGL}_{2,K}$ is not Cayley, see Proposition B.3 in Appendix B below.

Example 2.3. (Cf. [12, Ex. 1.16], [18, p. 599].) Let A be a finite-dimensional associative K-algebra with unit element 1 over a field K of characteristic $\neq 2$, and let ι be an involution (over K) of A. Set

$$G = \{a \in A^\times \mid a^\iota\, a = 1\}^0,$$

where S^0 denotes the identity component of an algebraic group S. The Lie algebra of G is the subspace of odd elements of A for ι,

$$\mathrm{Lie}(G) = \{a \in A \mid a^\iota = -a\}.$$

Since $\mathrm{char}(K) \neq 2$, the formula

$$a \mapsto (1-a)(1+a)^{-1}$$

defines an equivariant rational map $\lambda\colon G \dashrightarrow \mathrm{Lie}(G)$, and the formula

$$b \mapsto (1-b)(1+b)^{-1}$$

defines its inverse $\lambda^{-1}\colon \mathrm{Lie}(G) \dashrightarrow G$. Thus λ is a Cayley map and G is a Cayley group over K.

We see that if L/K is a separable quadratic extension, then the group $\mathbf{U}_{n,L/K}$ of $n \times n$ unitary matrices in $M_n(L)$ is Cayley over K; that the group $\mathbf{Sp}_{2n,K}$ is Cayley over K, in particular, $\mathbf{SL}_{2,K} \simeq \mathbf{Sp}_{2,K}$ is Cayley; that the group $\mathbf{SO}(m,n)$ is Cayley over K, in particular, the groups $\mathbf{PGL}_{2,K} \simeq \mathbf{SO}(2,1)$ and $\mathbf{Sp}_{4,K}/\mu_{2,K} \simeq \mathbf{SO}(3,2)$ are Cayley, where $\mu_{2,K} = \{\pm 1\}$ is the group of roots of unity of order dividing 2 in K. Here we write $\mathbf{SO}(m,n)$ for the special orthogonal group over K of the diagonal quadratic form $x_1^2 + \cdots + x_m^2 - x_{m+1}^2 - \cdots - x_{m+n}^2$.

We state some known properties of Cayley groups.

Remark 2.4. If G_1 and G_2 are Cayley K-groups over an arbitrary field K, then evidently $G_1 \times_K G_2$ is a Cayley K-group.

Remark 2.5. If L/K is a finite separable field extension and H is a Cayley L-group, then evidently the Weil restriction $R_{L/K}H$ is a Cayley K-group.

Remark 2.6. If G is a Cayley K-group over an arbitrary field K, and L/K is an arbitrary field extension, then $G \times_K L$ is evidently a Cayley L-group.

Proposition 2.7. *(See [2, Lemma 5.4(c)].) If G is a Cayley K-group over an arbitrary field K, then all the inner forms of G are Cayley. In particular, if all the automorphisms of G are inner, then all the twisted forms of G are inner forms, hence they all are Cayley K-groups.*

The following lemma is a version of [2, Lemma 5.4(a)] and can be proved similarly.

Lemma 2.8. *Let G be a reductive K-group and M be a K-group, not necessarily connected, acting on G, over a field K of characteristic 0. Consider the induced action of M on $\mathrm{Lie}(G)$. Let L/K be a Galois extension, and $c\colon \mathrm{Gal}(L/K) \to M(L)$ be a cocycle. Assume that there exists an M-equivariant birational isomorphism $f\colon G \overset{\simeq}{\dashrightarrow} \mathrm{Lie}(G)$ over K. Then there exists a ${}_cM$-equivariant birational isomorphism of the twisted varieties ${}_cf\colon {}_cG \overset{\simeq}{\dashrightarrow} \mathrm{Lie}({}_cG)$, where ${}_cM$ is the twisted group.*

Proposition 2.9. *(See [2, Corollary 6.5].) Let G be a reductive K-group over a field K of characteristic 0, and let $T \subset G$ be a maximal K-torus. Then G is Cayley if and only if there exists a $W(G,T)$-equivariant birational isomorphism $T \overset{\simeq}{\dashrightarrow} \mathrm{Lie}(T)$ defined over K, where the Weyl group $W(G,T)$ is viewed as an algebraic K-group.*

Note that the proof of this (difficult) result uses [4], where it is assumed that $\mathrm{char}(K) = 0$.

3. Proof of Theorem 1.2, easy cases

We start proving Theorem 1.2 case by case.

Proposition 3.1. *Any connected reductive K-group G of (absolute) rank 1 over a field K of characteristic $\neq 2, 3$ is a Cayley group.*

Proof. If G is a torus of rank 1, then G is K-rational, see e.g. [17, §4.9, Example 6], hence it is Cayley over K. If G is not a torus, then G is simple of rank 1, hence G is a twisted form of one of the groups $\mathbf{SL}_{2,K}$, $\mathbf{PGL}_{2,K}$. Both these groups are Cayley over K, see Example 2.3 and Example 2.2. Since all the automorphisms of $\mathbf{SL}_{2,K}$ and $\mathbf{PGL}_{2,K}$ are inner, by Proposition 2.7 G is Cayley. □

Proposition 3.2. *Any connected, reductive and not semisimple K-group G of absolute rank 2 over a field K of characteristic $\neq 2, 3$ is a Cayley group.*

Proof. If G is a torus of rank 2, then G is K-rational, see [17, §4.9, Example 7], hence it is Cayley over K. If G is not a torus, denote by $R := Z(G)^0$ its radical and by $G^{\mathrm{der}} := [G, G]$ its commutator subgroup. Since G is not a torus and not semisimple, R is a one-dimensional torus and G^{der} is a simple group of absolute rank 1. Set $\mu = R \cap G^{\mathrm{der}}$. The multiplication in G gives a canonical epimorphism $\pi \colon R \times_K G^{\mathrm{der}} \twoheadrightarrow G$ with kernel isomorphic to μ.

If this epimorphism is an isomorphism, then G is isomorphic to the product of two K-groups R and G^{der} of absolute rank 1. By Proposition 3.1, R and G^{der} are Cayley over K, hence by Remark 2.4 G is Cayley.

If the epimorphism $\pi \colon R \times_K G^{\mathrm{der}} \twoheadrightarrow G$ is not an isomorphism, then $\mu \neq \{1\}$. It follows that the center $Z(G^{\mathrm{der}}) \neq 1$, hence the simple group G^{der} of absolute rank 1 is not adjoint, hence it is simply connected. We see that $G^{\mathrm{der}}_{\overline{K}} \simeq \mathbf{SL}_{2,\overline{K}}$, and $\mu_{\overline{K}} = Z(G_{\overline{K}}) = \mu_{2,\overline{K}} = \{\pm 1\}$, where $\overline{K}$ is an algebraic closure of K. Thus $G_{\overline{K}} = (\mathbb{G}_{m,\overline{K}} \times_{\overline{K}} \mathbf{SL}_{2,\overline{K}})/\mu_{2,\overline{K}} \simeq \mathbf{GL}_{2,\overline{K}}$ (here $\mu_{2,\overline{K}}$ is embedded diagonally). This means that G is a K-form of $\mathbf{GL}_2$. By Lemma 3.3 below all the K-forms of $\mathbf{GL}_2$ are Cayley and hence, G is Cayley. □

Lemma 3.3. *Any K-form of $\mathbf{GL}_2$ over a field K of characteristic $\neq 2, 3$ is a Cayley group.*

Proof. Write $\mathrm{Out}(G) := \mathrm{Aut}(G)/\,\mathrm{Inn}(G)$ for “the group of outer automorphisms” of G. Write $G^{\mathrm{tor}} := G/G^{\mathrm{der}}$. The canonical homomorphism

$$\mathrm{Aut}(G) \to \mathrm{Aut}(G^{\mathrm{der}}) \times \mathrm{Aut}(G^{\mathrm{tor}})$$

gives for $G = \mathbf{GL}_2$ a canonical isomorphism

$$\mathrm{Aut}(\mathbf{GL}_2) \xrightarrow{\sim} \mathrm{Aut}(\mathbf{SL}_2) \times \mathrm{Aut}(\mathbb{G}_m).$$

Since all the elements of $\mathrm{Aut}(\mathbf{SL}_2)$ are *inner* automorphisms, we obtain a canonical isomorphism

$$\mathrm{Out}(\mathbf{GL}_2) \xrightarrow{\sim} \mathrm{Aut}(\mathbb{G}_m)$$

taking the class of an automorphism of $\mathbf{GL}_2$ to the induced automorphism of $(\mathbf{GL}_2)^{\mathrm{tor}} = \mathbb{G}_m$. Thus we obtain a bijection of the set of K-forms of $\mathbf{GL}_{2,K}$ up to inner twisting onto the set of K-forms of $\mathbb{G}_{m,K}$ up to an isomorphism. One can easily see that this bijection takes $[\mathbf{GL}_{2,K}]$ to $[\mathbb{G}_{m,K}]$ and $[\mathbf{U}_{2,L/K}]$ to $[\mathbf{U}_{1,L/K}]$, where L runs over the separable quadratic extensions of K and we denote by $[\ ]$ the corresponding equivalence classes. Since the K-groups $\mathbb{G}_{m,K}$ and $\mathbf{U}_{1,L/K}$ are *all* the K-forms of $\mathbb{G}_m$ up to an isomorphism, we see that $\mathbf{GL}_{2,K}$ and $\mathbf{U}_{2,L/K}$ are all the K-forms of $\mathbf{GL}_2$ up to inner twisting. Since all these K-groups, $\mathbf{GL}_{2,K}$ and $\mathbf{U}_{2,L/K}$, are Cayley, see Examples 2.1 and 2.3, we conclude, using Proposition 2.7, that all the K-forms of $\mathbf{GL}_2$ are Cayley. □

Proposition 3.4. *Any connected semisimple K-group G of absolute rank 2 of type $\mathbf{A}_1 \times \mathbf{A}_1$ over a field K of characteristic $\neq 2, 3$ is a Cayley group.*

Proof. In this case the group G decomposes into an almost direct product of two groups of type $\mathbf{A}_1$ defined either over K or over a separable quadratic extension L of K. If this almost direct product is direct, then G is either a direct product of two simple K-groups of type $\mathbf{A}_1$, and hence is Cayley by Proposition 3.1 and Remark 2.4, or G is of the form $R_{L/K}G'$, where G' is a simple L-group of type $\mathbf{A}_1$, and we conclude by Proposition 3.1 that G' is Cayley over L, and conclude by Remark 2.5 that G is Cayley over K. If this almost direct product is not direct, then G is a twisted form of $\mathbf{SO}_4$, hence G is an inner form of a special orthogonal group of the form $\mathbf{SO}(K^4, q)$ for some nondegenerate quadratic form q in 4 variables, and G is Cayley by Example 2.3 and Proposition 2.7. □

Proposition 3.5. *Any connected simple K-group G of absolute rank 2 of type $\mathbf{B}_2 = \mathbf{C}_2$ over a field K of characteristic $\neq 2, 3$ is a Cayley group.*

Proof. In this case G is an (inner) twisted form of one of the K-groups $\mathbf{Sp}_{4,K}$ and $\mathbf{Sp}_{4,K}/\mu_{2,K}$. Both these groups are Cayley by Example 2.3, and using Proposition 2.7, we conclude that G is Cayley. □

3.6. *Proof of Theorem 1.2 modulo Theorem 4.3 and results of Appendix A.*

The cases when G is not a simple group of type $\mathbf{G}_2$ or $\mathbf{A}_2$ were treated in Propositions 3.1, 3.2, 3.4, and 3.5.

Any connected simple K-group of absolute rank 2 of type $\mathbf{G}_2$ over a field K of characteristic 0 is not Cayley, see [12, §9.2] and Iskovskikh's papers [9,10] (this was explained in our Introduction).

Let G be a connected simple $\mathbb{R}$-group of rank 2 of type $\mathbf{A}_2$. We consider all the possible cases.

The group $\mathbf{PGL}_{3,\mathbb{R}}$ is Cayley by Example 2.2. The group $\mathbf{SU}_3$ is Cayley by Corollary 4.4 of Theorem 4.3, see also Theorem 7.1 of Appendix A. Since the group $\mathbf{SU}(2,1)$ is an inner form of $\mathbf{SU}_3$, by Proposition 2.7 it is Cayley as well.

The group $\mathbf{SL}_{3,\mathbb{R}}$ is not Cayley by Theorem 8.1 of Appendix A. The group $\mathbf{PGU}_3$ is not Cayley by Theorem 7.2 of Appendix A. Since the group $\mathbf{PGU}(2,1)$ is an inner form of $\mathbf{PGU}_3$, by Proposition 2.7 it is not Cayley either. □

4. The group $\mathbf{SU}_3$

4.1. Let W be a finite group. Let L/K be a finite Galois extension with Galois group $\Gamma = \mathrm{Gal}(L/K)$. We shall consider W-varieties defined over K and (W, Γ)-varieties defined over L. By a W-variety defined over K we mean a K-variety X with a W-action $W \to \mathrm{Aut}(X)$. By a semilinear action of Γ on an L-variety Y we mean a homomorphism $\rho: \Gamma \to \mathrm{SAut}_{L/K}(Y)$ into the group $\mathrm{SAut}_{L/K}(Y)$ of L/K-semilinear automorphisms of Y, such that $\rho(\gamma)$ is a γ-semilinear automorphism of Y for any $\gamma \in \Gamma$ (see [1, §1.1] and [6, §1.2] for the definitions of semilinear automorphisms). By a (W, Γ)-variety defined over L we mean an L-variety Y with two commuting actions: an L-action of W and a semilinear action of Γ. One defines morphisms and rational maps of (W, Γ)-varieties. We have a base change functor $X \mapsto X \times_K L$ from the category of W-varieties over K to the category of (W, Γ)-varieties over L, and it is well known that this functor is fully faithful, i.e., the natural map

$$\mathrm{Hom}_W(X, X') \to \mathrm{Hom}_{(W,\Gamma)}(X \times_K L, X' \times_K L)$$

is bijective for any two W-varieties X, X' defined over K. Similarly, W-varieties X and X' over K are W-equivariantly birationally isomorphic over K if and only if $X \times_K L$ and $X' \times_K L$ are (W, Γ)-equivariantly birationally isomorphic over L. Note that, by Galois descent (see Serre [16, Ch. V.20, Cor. 2 of Prop. 12]), any *quasi-projective* (W, Γ)-variety over L comes from a W-variety over K; we shall not use this fact, however.

4.2. Let K be a field of characteristic 0. Assume that K does not contain non-trivial roots of unity of order 3. Set $L = K(\zeta)$, where $\zeta^3 = 1$, $\zeta \neq 1$. We can also write $L = K(\sqrt{-3})$. (For example, one can take $K = \mathbb{R}$, $L = \mathbb{R}(\sqrt{-3}) = \mathbb{C}$.) We set $\Gamma = \mathrm{Gal}(L/K)$, $\Gamma = \{\mathrm{id}, \gamma\}$, and we write the action of γ on $a \in L$ as $a \mapsto {}^{\gamma}a$.

Let $G = \mathbf{SU}(3, L/K, H) := \mathbf{SU}(L^3, H)$, the special unitary group of the L/K-Hermitian form with matrix H, where $H \in M_3(L)$ is a nondegenerate 3×3 Hermitian matrix. Then G is a simple K-group, an outer L/K-form of the split K-group $\mathbf{SL}_{3,K}$. Note that

$G = \mathbf{SU}(3, L/K, H)$ is an *inner* form of the K-group $\mathbf{SU}_{3,L/K} := \mathbf{SU}(3, L/K, I_3)$, where $I_3 = \mathrm{diag}(1, 1, 1)$.

Theorem 4.3. *Let a field K, the quadratic field extension $L = K(\zeta)$ of K, and a Hermitian matrix $H \in M_3(L)$ be as in § 4.2. Then the K-group $G = \mathbf{SU}(3, L/K, H)$ is Cayley.*

Theorem 4.3 will be proved below.

Corollary 4.4. *The $\mathbb{R}$-groups $\mathbf{SU}_3$ and $\mathbf{SU}(2, 1)$ are Cayley.* $\square$

4.5. Let K, L be as in § 4.2. Consider the torus $\mathbb{G}^3_{m,K}$ and write the standard action of the symmetric group $\mathfrak{S}_3$ on it, given by:

$$\sigma(x_1, x_2, x_3) := (x_{\sigma^{-1}(1)}, x_{\sigma^{-1}(2)}, x_{\sigma^{-1}(3)}) \quad \text{for } \sigma \in \mathfrak{S}_3. \tag{4.1}$$

We consider the K-subtorus

$$T := \{(x_1, x_2, x_3) \in \mathbb{G}^3_{m,K} \mid x_1x_2x_3 = 1\}$$

and we set $\mathfrak{t} = \mathrm{Lie}(T)$.

We set $T_L = T \times_K L$, $\mathfrak{t}_L = \mathrm{Lie}(T_L) = \mathfrak{t} \otimes_K L$, then

$$\mathfrak{t}_L = \{(x_1, x_3, x_3) \in L^3 \mid x_1 + x_2 + x_3 = 0\}.$$

The group $\mathfrak{S}_3$ acts on T_L and $\mathfrak{t}_L$ by formula (4.1), and Γ acts by

$$^\gamma(x_1, x_2, x_3) = ({}^\gamma x_1, {}^\gamma x_2, {}^\gamma x_3).$$

We consider also the Γ-twisted $(\mathfrak{S}_3, \Gamma)$-varieties T'_L and $\mathfrak{t}'_L$: the same L-varieties T_L and $\mathfrak{t}_L$ with the same $\mathfrak{S}_3$-actions, but with the twisted actions of γ:

$$(x_1, x_2, x_3) \mapsto ({}^\gamma x_1^{-1}, {}^\gamma x_2^{-1}, {}^\gamma x_3^{-1}) \quad \text{for } T'_L,$$
$$(x_1, x_2, x_3) \mapsto (-{}^\gamma x_1, -{}^\gamma x_2, -{}^\gamma x_3) \quad \text{for } \mathfrak{t}'_L.$$

These $(\mathfrak{S}_3, \Gamma)$-varieties over L come from some $\mathfrak{S}_3$-varieties T' and $\mathfrak{t}'$ defined over K which are easy to describe, see below.

4.6. Let $T_{\mathbf{SU}_3}$ denote the diagonal maximal torus of $\mathbf{SU}_{3,L/K}$, and let $\mathfrak{t}_{\mathbf{SU}_3}$ denote its Lie algebra. Let $N_{\mathbf{SU}_3}$ denote the normalizer of $T_{\mathbf{SU}_3}$ in $\mathbf{SU}_{3,L/K}$, and set $W = N_{\mathbf{SU}_3}/T_{\mathbf{SU}_3}$. The finite algebraic group W is canonically isomorphic to the symmetric group $\mathfrak{S}_3$ with trivial Galois action. We see that $T_{\mathbf{SU}_3}$, and $\mathfrak{t}_{\mathbf{SU}_3}$ are $\mathfrak{S}_3$-varieties over K. Furthermore, it is well known that $T_{\mathbf{SU}_3} \times_K L$ is canonically isomorphic to T'_L and that $\mathfrak{t}_{\mathbf{SU}_3} \otimes_K L$ is canonically isomorphic to $\mathfrak{t}'_L$ as $(\mathfrak{S}_3, \Gamma)$-varieties. Therefore we set

$$T' := T_{\mathbf{SU}_3}, \quad \mathfrak{t}' := \mathfrak{t}_{\mathbf{SU}_3}.$$

Proposition 4.7. *Let K be a field of characteristic 0. We assume that K contains no nontrivial cube root of 1, and we set $L = K(\zeta)$, where $\zeta^3 = 1$, $\zeta \neq 1$. Then the $(\mathfrak{S}_3, \Gamma)$-varieties T'_L and $\mathfrak{t}'_L$ are $(\mathfrak{S}_3, \Gamma)$-equivariantly birationally isomorphic over L.*

4.8. *Reduction of Theorem 4.3 to Proposition 4.7.* Since our group $\mathbf{SU}(3, L/K, H)$ is an inner form of $\mathbf{SU}_{3,L/K}$, by Proposition 2.7 in order to prove that the group $\mathbf{SU}(3, L/K, H)$ is Cayley, it suffices to prove that $\mathbf{SU}_{3,L/K}$ is Cayley. By Proposition 2.9, the group $\mathbf{SU}_{3,L/K}$ is Cayley if and only if the $\mathfrak{S}_3$-varieties $T' = T_{\mathbf{SU}_3}$ and $\mathfrak{t}' = \mathfrak{t}_{\mathbf{SU}_3}$ are $\mathfrak{S}_3$-equivariantly birationally isomorphic over K. The discussion in § 4.1 shows that they are $\mathfrak{S}_3$-equivariantly birationally isomorphic over K if and only if the $(\mathfrak{S}_3, \Gamma)$-varieties T'_L and $\mathfrak{t}'_L$ are $(\mathfrak{S}_3, \Gamma)$-equivariantly birationally isomorphic over L. Therefore, Theorem 4.3 follows from Proposition 4.7.

We give here a proof of Proposition 4.7 which is close to the proof of Proposition 9.1 in [12]. For an alternative proof (in the case $K = \mathbb{R}$) see Appendix A, Theorem 7.1.

4.9. We consider the variety $(\mathbb{G}^3_{m,L}/\mathbb{G}_{m,L})_{(\mathfrak{S}_3,\Gamma)\text{-twisted}}$, which is just $\mathbb{G}^3_{m,L}/\mathbb{G}_{m,L}$ (with $\mathbb{G}_{m,L}$ imbedded diagonally in $\mathbb{G}^3_{m,L}$) with the following (twisted) $\mathfrak{S}_3$-action and *twisted* Γ-*action*:

$$\sigma([x]) = [\sigma(x)^{\operatorname{sign}\sigma}], \quad {}^{\gamma}[x] = [{}^{\gamma}x^{-1}] \quad \text{for } x \in \mathbb{G}^3_{m,L},\ \sigma \in \mathfrak{S}_3.$$

Here we write $[x] \in \mathbb{G}^3_{m,L}/\mathbb{G}_{m,L}$ for the class of $x \in \mathbb{G}^3_{m,L}$. We have an $(\mathfrak{S}_3, \Gamma)$-equivariant isomorphism

$$(\mathbb{G}^3_{m,L}/\mathbb{G}_{m,L})_{(\mathfrak{S}_3,\Gamma)\text{-twisted}} \xrightarrow{\sim} T'_L, \quad [x_1, x_2, x_3] \mapsto (x_2/x_3, x_3/x_1, x_1/x_2).$$

It remains to prove that $(\mathbb{G}^3_{m,L}/\mathbb{G}_{m,L})_{(\mathfrak{S}_3,\Gamma)\text{-twisted}}$ is (S_3, Γ)-equivariantly birationally isomorphic to $\mathfrak{t}'_L$.

4.10. Consider the following (twisted) $\mathfrak{S}_3$-action and *twisted* Γ-*action* on the set $\mathfrak{t}_L \times \mathfrak{t}_L$:

$$\sigma(x, y) := \begin{cases} (\sigma(x), \sigma(y)) & \text{if } \sigma \text{ is even,} \\ (\sigma(y), \sigma(x)) & \text{if } \sigma \text{ is odd,} \end{cases} \quad \text{where } \sigma \in \mathfrak{S}_3,\ x, y \in \mathfrak{t}_L,$$
$${}^{\gamma}(x, y) := ({}^{\gamma}y, {}^{\gamma}x).$$

These actions of $\mathfrak{S}_3$ and Γ on $\mathfrak{t}_L \times \mathfrak{t}_L$ induce actions on the surface $\mathbb{P}(\mathfrak{t}_L) \times_L \mathbb{P}(\mathfrak{t}_L)$, on the tensor product $\mathfrak{t}_L \otimes_L \mathfrak{t}_L$ and on the 3-dimensional projective space $\mathbb{P}(\mathfrak{t}_L \otimes_L \mathfrak{t}_L)$, and we write

$$(\mathbb{P}(\mathfrak{t}_L) \times_L \mathbb{P}(\mathfrak{t}_L))_{(\mathfrak{S}_3,\Gamma)\text{-twisted}},\ (\mathfrak{t}_L \otimes_L \mathfrak{t}_L)_{(\mathfrak{S}_3,\Gamma)\text{-twisted}} \text{ and } \mathbb{P}(\mathfrak{t}_L \otimes_L \mathfrak{t}_L)_{(\mathfrak{S}_3,\Gamma)\text{-twisted}}$$

for the corresponding $(\mathfrak{S}_3, \Gamma)$-varieties.

4.11. We claim that the $(\mathfrak{S}_3,\Gamma)$-varieties $(\mathbb{G}^3_{m,L}/\mathbb{G}_{m,L})_{(\mathfrak{S}_3,\Gamma)\text{-twisted}}$ and $(\mathbb{P}(\mathfrak{t}_L)\times_L \mathbb{P}(\mathfrak{t}_L))_{(\mathfrak{S}_3,\Gamma)\text{-twisted}}$ are $(\mathfrak{S}_3,\Gamma)$-equivariantly birationally isomorphic. We write $[t]\in\mathbb{P}(\mathfrak{t}_L)$ for the class of $t\in\mathfrak{t}_L$. Consider the rational map

$$\begin{aligned}\varphi\colon (\mathbb{G}^3_{m,L}/\mathbb{G}_{m,L})_{(\mathfrak{S}_3,\Gamma)\text{-twisted}} &\dashrightarrow (\mathbb{P}(\mathfrak{t}_L)\times_L \mathbb{P}(\mathfrak{t}_L))_{(\mathfrak{S}_3,\Gamma)\text{-twisted}}\\ [x] &\mapsto \big([x-\tau(x)\mathbf{1}_3],\,[x^{-1}-\tau(x^{-1})\mathbf{1}_3]\big),\end{aligned}$$

where $\tau(x_1,x_2,x_3)=(x_1+x_2+x_3)/3$ and $\mathbf{1}_3=(1,1,1)\in L^3$. It is immediately seen that φ is $(\mathfrak{S}_3,\Gamma)$-equivariant. An inverse rational map to φ was constructed in [12, Proof of Prop. 9.1, Step 1]. Thus φ is an $(\mathfrak{S}_3,\Gamma)$-equivariant birational isomorphism.

4.12. Consider the Segre embedding

$$(\mathbb{P}(\mathfrak{t}_L)\times_L \mathbb{P}(\mathfrak{t}_L))_{(\mathfrak{S}_3,\Gamma)\text{-twisted}} \hookrightarrow \mathbb{P}(\mathfrak{t}_L\otimes_L \mathfrak{t}_L)_{(\mathfrak{S}_3,\Gamma)\text{-twisted}}$$

given by $([x],[y])\mapsto[x\otimes y]$, it is $(\mathfrak{S}_3,\Gamma)$-equivariant. Its image is a quadric Q in $\mathbb{P}(\mathfrak{t}_L\otimes_L\mathfrak{t}_L)$ described as follows. Choose a basis $D_1:=\operatorname{diag}(1,\zeta,\zeta^2)$, $D_2:=\operatorname{diag}(1,\zeta^2,\zeta)$ of $\mathfrak{t}_L$, where ζ is our primitive cube root of unity. Set $D_{ij}=D_i\otimes D_j$. Then

$$Q=\{(\alpha_{11}:\alpha_{12}:\alpha_{21}:\alpha_{22})\mid \alpha_{11}\alpha_{22}=\alpha_{12}\alpha_{21}\},\tag{4.2}$$

where $(\alpha_{11}:\alpha_{12}:\alpha_{21}:\alpha_{22})$ is the point of $\mathbb{P}(\mathfrak{t}_L\otimes_L\mathfrak{t}_L)$ corresponding to

$$\alpha_{11}D_{11}+\alpha_{12}D_{12}+\alpha_{21}D_{21}+\alpha_{22}D_{22}\in\mathfrak{t}_L\otimes_L\mathfrak{t}_L.$$

4.13. We denote by $V_{11,22}$ the 2-dimensional subspace in $(\mathfrak{t}_L\otimes_L\mathfrak{t}_L)_{(\mathfrak{S}_3,\Gamma)\text{-twisted}}$ with the basis D_{11},D_{22}, and we denote by V_{12} and V_{21} the one-dimensional subspaces generated by D_{12} and D_{21}, respectively. An easy calculation shows that the subspace $V_{11,22}$ is $\mathfrak{S}_3$-invariant and Γ-invariant, and that the basis vectors D_{12} and D_{21} are $\mathfrak{S}_3$-fixed and Γ-fixed.

Consider the stereographic projection $Q\dashrightarrow\mathbb{P}(V_{11,22}\oplus V_{12})$ from the $(\mathfrak{S}_3,\Gamma)$-fixed L-point $x_{21}:=[D_{21}]=(0:0:1:0)\in Q(L)$ to the $(\mathfrak{S}_3,\Gamma)$-invariant plane $\mathbb{P}(V_{11,22}\oplus V_{12})$. This stereographic projection is an $(\mathfrak{S}_3,\Gamma)$-equivariant birational isomorphism. Furthermore, the embedding

$$V_{11,22}\hookrightarrow\mathbb{P}(V_{11,22}\oplus V_{12}),\quad x\mapsto[x+D_{12}]$$

is an $(\mathfrak{S}_3,\Gamma)$-equivariant birational isomorphism. Thus the quadric Q is $(\mathfrak{S}_3,\Gamma)$-equivariantly birationally isomorphic to the vector space $V_{11,22}$. Since the 2-dimensional $(\mathfrak{S}_3,\Gamma)$-vector spaces $V_{11,22}$ and $\mathfrak{t}_L$ are isomorphic (the map of bases $D_{11}\mapsto D_2$, $D_{22}\mapsto D_1$ induces an $(\mathfrak{S}_3,\Gamma)$-isomorphism $V_{11,22}\xrightarrow{\sim}\mathfrak{t}_L$), and $\mathfrak{t}_L$ is isomorphic to $\mathfrak{t}'_L$ (an isomorphism is given by $(x_i)\mapsto(\sqrt{-3}\cdot x_i)$), we conclude that Q is $(\mathfrak{S}_3,\Gamma)$-equivariantly birationally isomorphic to $\mathfrak{t}'_L$.

Thus T'_L is $(\mathfrak{S}_3, \Gamma)$-equivariantly birationally isomorphic to $\mathfrak{t}'_L$. This completes the proofs of Proposition 4.7, Theorem 4.3, and Corollary 4.4. □

5. The groups $G \times \mathbb{G}_m^2$

In this section we prove Theorem 1.3. Let K be a field of characteristic 0, and let $\overline{K}$ be a fixed algebraic closure of K.

Let $\mathbf{G}_{2,K}$ denote the *split* K-group of type $\mathbf{G}_2$. Since by [12, Proposition 9.10], the group $\mathbf{G}_{2,\overline{K}}$ is not Cayley over $\overline{K}$, we see that $\mathbf{G}_{2,K}$ is not Cayley.

Proposition 5.1. *For any field K of characteristic 0, the split K-group $\mathbf{G}_{2,K} \times_K \mathbb{G}_{m,K}^2$ is Cayley.*

Corollary 5.2. *For any K-group G of type $\mathbf{G}_2$ over a field K of characteristic 0, the K-group $G \times_K \mathbb{G}_{m,K}^2$ is Cayley.*

Proof. Since $G \times_K \mathbb{G}_{m,K}^2$ is an inner form of $\mathbf{G}_{2,K} \times_K \mathbb{G}_{m,K}^2$, by Proposition 2.7 the corollary follows from Proposition 5.1. □

5.3. Let K be a field of characteristic 0. We define a K-torus T by

$$T := \{(x_1, x_2, x_3) \in \mathbb{G}_{m,K}^3 \mid x_1 x_2 x_3 = 1\}.$$

We define a K-action of $\mathfrak{S}_3$ on T by

$$\sigma(x_1, x_2, x_3) := (x_{\sigma^{-1}(1)}, x_{\sigma^{-1}(2)}, x_{\sigma^{-1}(3)}) \quad \text{for } \sigma \in \mathfrak{S}_3.$$

We define a K-action of $\mathfrak{S}_2$ on T by

$$\varepsilon(t) = t^{-1} \quad \text{for } t \in T,$$

where ε is the nontrivial element of $\mathfrak{S}_2$. We obtain a K-action of $\mathfrak{S}_3 \times \mathfrak{S}_2$ on T. Set $\mathfrak{t} = \mathrm{Lie}(T)$, then $\mathfrak{S}_3 \times \mathfrak{S}_2$ acts on $\mathfrak{t}$. We may regard T as a split maximal torus of $\mathbf{G}_{2,K}$, and $\mathfrak{S}_3 \times \mathfrak{S}_2$ as the corresponding Weyl group, then $T \times_K \mathbb{G}_{m,K}^2$ is a maximal torus of $\mathbf{G}_{2,K} \times_K \mathbb{G}_{m,K}^2$.

Proposition 5.4. *(See [12].) For an arbitrary field K of characteristic 0, the K-varieties $T \times_K \mathbb{G}_{m,K}^2$ and $\mathfrak{t} \times_K \mathbb{A}_K^2$ are $\mathfrak{S}_3 \times \mathfrak{S}_2$-equivariantly birationally isomorphic over K.*

Proof. This is proved in [12] in the proof of Proposition 9.11. The authors assume that K is an algebraically closed field of characteristic 0, but the proof goes through for any field K of characteristic $\neq 2, 3$. □

Proof of Proposition 5.1. By Proposition 2.9, our proposition follows from Proposition 5.4. □

Corollary 5.5. *The K-varieties $T \times_K \mathbb{G}^2_{m,K}$ and $\mathfrak{t} \times_K \mathbb{A}^2_K$ of Proposition 5.4 are $\mathfrak{S}_3$-equivariantly birationally isomorphic over K (with respect to the standard embedding $\mathfrak{S}_3 \hookrightarrow \mathfrak{S}_3 \times \mathfrak{S}_2$).*

Proof. The $\mathfrak{S}_3 \times \mathfrak{S}_2$-equivariant birational isomorphism of Proposition 5.4 is, in particular, $\mathfrak{S}_3$-equivariant. □

Proposition 5.6. *For any field K of characteristic 0, the K-group $\mathbf{SL}_{3,K} \times_K \mathbb{G}^2_{m,K}$ is Cayley.*

Proof. We regard T as a split maximal torus of $\mathbf{SL}_{3,K}$ and $\mathfrak{S}_3$ as the corresponding Weyl group, then $T \times_K \mathbb{G}^2_{m,K}$ is a maximal torus of $\mathbf{SL}_{3,K} \times_K \mathbb{G}^2_{m,K}$. Now by Proposition 2.9, our proposition follows from Corollary 5.5. □

5.7. Let T be the $\mathfrak{S}_3 \times \mathfrak{S}_2$-torus over K of Section 5.3. Let L/K be an arbitrary quadratic extension. Write $\Gamma = \mathrm{Gal}(L/K) = \{1, \gamma\}$. Define a cocycle (homomorphism)

$$c\colon \Gamma \to \mathfrak{S}_3 \times \mathfrak{S}_2$$

taking γ to the nontrivial element $\varepsilon \in \mathfrak{S}_2$. We obtain a twisted torus ${}_cT$. Let ${}_cT_L$ denote the corresponding $(\mathfrak{S}_3 \times \mathfrak{S}_2, \Gamma)$-variety over L, it is $T_L := T \times_K L$ with the following actions:

$$\sigma(x_1, x_2, x_3) := (x_{\sigma^{-1}(1)}, x_{\sigma^{-1}(2)}, x_{\sigma^{-1}(3)}) \quad \text{for } \sigma \in \mathfrak{S}_3, \tag{5.1}$$

$$\varepsilon(x_1, x_2, x_3) = (x_1^{-1}, x_2^{-1}, x_3^{-1}), \tag{5.2}$$

$$^{\gamma}(x_1, x_2, x_3) = ({}^{\gamma}x_1^{-1}, {}^{\gamma}x_2^{-1}, {}^{\gamma}x_3^{-1}). \tag{5.3}$$

Note that ${}_c(\mathfrak{S}_3 \times \mathfrak{S}_2) = \mathfrak{S}_3 \times \mathfrak{S}_2$, because $c(\gamma) = \varepsilon$ is central in $\mathfrak{S}_3 \times \mathfrak{S}_2$.

Proposition 5.8. *There exists a birational $(\mathfrak{S}_3 \times \mathfrak{S}_2, \Gamma)$-isomorphism between the $(\mathfrak{S}_3 \times \mathfrak{S}_2, \Gamma)$-varieties ${}_cT_L \times_L \mathbb{G}^2_{m,L}$ and $\mathrm{Lie}({}_cT_L) \times_L \mathbb{A}^2_L$.*

Proof. This follows from Proposition 5.4 and Lemma 2.8. □

5.9. We define two embeddings $\mathfrak{S}_3 \hookrightarrow \mathfrak{S}_3 \times \mathfrak{S}_2$, the standard one and the twisted one:

$$\mathrm{St}(\sigma) = (\sigma, 1) \quad \text{for } \sigma \in \mathfrak{S}_3,$$

$$\mathrm{Tw}(\sigma) = (\sigma, \varepsilon^{\mathrm{sign}(\sigma)}) = \begin{cases} (\sigma, 1) & \text{if } \mathrm{sign}(\sigma) = 1, \\ (\sigma, \varepsilon) & \text{if } \mathrm{sign}(\sigma) = -1. \end{cases}$$

These two embeddings define two $\mathfrak{S}_3$-actions on ${}_cT_L$. We denote the corresponding $(\mathfrak{S}_3, \Gamma)$-varieties (with the twisted Γ-action (5.3)) by ${}_{\mathrm{St}}T'_L$ and ${}_{\mathrm{Tw}}T'_L$, respectively.

Corollary 5.10. *There exist birational* $(\mathfrak{S}_3, \Gamma)$*-isomorphisms*

$$ {}_{\mathrm{St}}T'_L \times_L \mathbb{G}^2_{m,L} \overset{\simeq}{\dashrightarrow} \mathrm{Lie}({}_{\mathrm{St}}T'_L) \times_L \mathbb{A}^2_L \quad \textit{and} \quad {}_{\mathrm{Tw}}T'_L \times_L \mathbb{G}^2_{m,L} \overset{\simeq}{\dashrightarrow} \mathrm{Lie}({}_{\mathrm{Tw}}T'_L) \times_L \mathbb{A}^2_L. $$

Proof. The $(\mathfrak{S}_3 \times \mathfrak{S}_2, \Gamma)$-equivariant birational isomorphism of Proposition 5.8 is, in particular, $(\mathfrak{S}_3, \Gamma)$-equivariant with respect to each of the two embeddings $\mathrm{St}, \mathrm{Tw}\colon \mathfrak{S}_3 \hookrightarrow \mathfrak{S}_3 \times \mathfrak{S}_2$. □

5.11. Let L/K be an arbitrary quadratic extension of fields of characteristic 0. Let $G = \mathbf{SU}(3, L/K, H) := \mathbf{SU}(L^3, H)$, the special unitary group of the L/K-Hermitian form with matrix H, where $H \in M_3(L)$ is a nondegenerate Hermitian matrix. Then G is a simple K-group, an outer L/K-form of the split K-group $\mathbf{SL}_{3,K}$. Note that $G = \mathbf{SU}(3, L/K, H)$ is an *inner* form of the K-group $\mathbf{SU}_{3,L/K} := \mathbf{SU}(3, L/K, I_3)$, where $I_3 = \mathrm{diag}(1, 1, 1)$.

Proposition 5.12. *Let a quadratic extension L/K and a Hermitian matrix $H \in M_3(L)$ be as in §5.11. Let $G = \mathbf{SU}(3, L/K, H)$, then $G \times_K \mathbb{G}^2_{m,K}$ is Cayley.*

Proof. Since G is an inner form of $\mathbf{SU}_3 := \mathbf{SU}(3, L/K, I_3)$, by Proposition 2.7 it suffices to consider the case of $\mathbf{SU}_3$. Let $T_{\mathbf{SU}_3}$ denote the diagonal maximal torus of $\mathbf{SU}_3$, we can identify it with the torus ${}_{\mathrm{St}}T'_L$ of Corollary 5.10. Now our proposition follows from Corollary 5.10 and Proposition 2.9. □

Proposition 5.13. *Let a quadratic extension L/K and a Hermitian matrix $H \in M_3(L)$ be as in §5.11. Let $G = \mathbf{PGU}(3, L/K, H)$ be the adjoint K-group corresponding to the simply connected K-group $\mathbf{SU}(3, L/K, H)$. Then $G \times_K \mathbb{G}^2_{m,K}$ is Cayley.*

Proof. Since G is an inner form of $\mathbf{PGU}_3 := \mathbf{PGU}(3, L/K, I_3)$, by Proposition 2.7 it suffices to consider the case of $\mathbf{PGU}_3$. Let $T_{\mathbf{PGU}_3} \subset \mathbf{PGU}_3$ denote the image of the diagonal maximal torus of $\mathbf{SU}_3$, we can identify the corresponding L-torus $T_{\mathbf{PGU}_3} \times_K L$ with the torus $(\mathbb{G}^3_{m,L}/\mathbb{G}_{m,L})_{\Gamma\text{-twisted}}$ endowed with the following actions of $\mathfrak{S}_3$ and Γ:

$$ \sigma([x_1, x_2, x_3]) := [x_{\sigma^{-1}(1)}, x_{\sigma^{-1}(2)}, x_{\sigma^{-1}(3)}] \quad \text{for } \sigma \in \mathfrak{S}_3, $$
$$ {}^\gamma[x_1, x_2, x_3] = [{}^\gamma x_1^{-1}, {}^\gamma x_2^{-1}, {}^\gamma x_3^{-1}]. $$

We define a homomorphism $\mathbb{G}^3_{m,L}/\mathbb{G}_{m,L} \to T_L$ by

$$ [x_1, x_2, x_3] \mapsto (x_2/x_3, x_3/x_1, x_1/x_2). $$

One checks immediately that we obtain an $(\mathfrak{S}_3, \Gamma)$-equivariant isomorphism

$$(\mathbb{G}^3_{m,L}/\mathbb{G}_{m,L})_{\Gamma\text{-twisted}} \xrightarrow{\sim} {}_{\mathrm{Tw}}T'_L,$$

and its differential, which is also an $(\mathfrak{S}_3, \Gamma)$-equivariant isomorphism,

$$\mathrm{Lie}(\mathbb{G}^3_{m,L}/\mathbb{G}_{m,L})_{\Gamma\text{-twisted}} \xrightarrow{\sim} \mathrm{Lie}\, {}_{\mathrm{Tw}}T'_L.$$

By Corollary 5.10 there exists an $(\mathfrak{S}_3, \Gamma)$-equivariant birational isomorphism

$${}_{\mathrm{Tw}}T'_L \times_L \mathbb{G}^2_{m,L} \overset{\simeq}{\dashrightarrow} \mathrm{Lie}\, {}_{\mathrm{Tw}}T'_L \times_L \mathbb{A}^2_L.$$

Combining these birational isomorphisms, we obtain an $(\mathfrak{S}_3, \Gamma)$-equivariant birational isomorphism

$$(\mathbb{G}^3_{m,L}/\mathbb{G}_{m,L})_{\Gamma\text{-twisted}} \times_L \mathbb{G}^2_{m,L} \overset{\simeq}{\dashrightarrow} \mathrm{Lie}(\mathbb{G}^3_{m,L}/\mathbb{G}_{m,L})_{\Gamma\text{-twisted}} \times_L \mathbb{A}^2_L,$$

that is, an $\mathfrak{S}_3$-equivariant birational isomorphism

$$T_{\mathbf{PGU}_3} \times_K \mathbb{G}^2_{m,K} \overset{\simeq}{\dashrightarrow} \mathrm{Lie}(T_{\mathbf{PGU}_3}) \times_K \mathbb{A}^2_K.$$

Now Proposition 5.13 follows from Proposition 2.9. □

Proof of Theorem 1.3. If G is of absolute rank 1, then by Proposition 3.1 the group G is Cayley (and hence, the group $G \times_K \mathbb{G}^2_{m,K}$ is Cayley). Now assume that G is of absolute rank 2. If G is not semisimple, or is of type $\mathbf{A}_1 \times \mathbf{A}_1$, or is of type $\mathbf{B}_2 = \mathbf{C}_2$, then by Propositions 3.2, 3.4, and 3.5 the group G is Cayley, hence the group $G \times_K \mathbb{G}^2_{m,K}$ is Cayley. Otherwise G is of type $\mathbf{G}_2$ or $\mathbf{A}_2$, and by Example 2.2 and Propositions 5.1, 5.6, 5.12, and 5.13 the group $G \times_K \mathbb{G}^2_{m,K}$ is Cayley. □

Appendix A. Elementary links

Igor Dolgachev

Department of Mathematics, University of Michigan, 525 E. University Av., Ann Arbor, MI, 49109, USA

E-mail address: idolga@umich.edu.

In this appendix we will follow the ideas from Iskovskikh's papers [8–10] to study the Cayley property of the groups $\mathbf{SU}_3$, $\mathbf{PGU}_3$ and $\mathbf{SL}_3$ over $\mathbb{R}$.

6. Elementary links for *G*-surfaces

Let X be a smooth projective surface over a perfect field K and G be a finite group of K-automorphisms of X. We say that the pair (X, G) is a G-surface. Two G-surfaces

(X, G) and (X', G) are called birationally (biregularly) isomorphic if there exists a birational (biregular) G-equivariant map $\phi : X \dashrightarrow X'$ defined over K. A G-surface (X, G) is called *minimal* if any birational G-equivariant morphism $X \to X'$ is an isomorphism. Any birational G-map between two G-surfaces can be factored into a sequence of birational G-morphisms and their inverses. A birational G-morphism $f : X \to Y$ is isomorphic to the blow-up of a closed G-invariant 0-dimensional subscheme $\mathfrak{a}$ of Y. For the future use let us remind that the degree of $\mathfrak{a}$ is the number $\deg(\mathfrak{a}) = h^0(\mathcal{O}_{\mathfrak{a}})$. If $\mathfrak{a}$ is reduced and consists of closed points $y_1, \ldots, y_k$ with residue fields $\kappa(y_i)$, then $\deg(\mathfrak{a}) = \sum \deg(y_i)$, where $\deg(y_i) = [\kappa(x_i) : K]$. The G-invariance of $\mathfrak{a}$ means that $\mathfrak{a}$ is the union of G-orbits.

The birational classification of G-surfaces over K is equivalent to the classification of minimal G-surfaces up to birational isomorphisms.

From now on we assume that X is a rational surface, i.e. after a finite base change L/K, the surface is birationally isomorphic to $\mathbb{P}^2_L$. It is known (see [8]) that a minimal rational surface belongs to one of the following two classes:

($\mathcal{D}$) X is a del Pezzo surface with $\mathrm{Pic}(X)^G \cong \mathbb{Z}$;
($\mathcal{C}$) X is a conic bundle with $\mathrm{Pic}(X)^G \cong \mathbb{Z}^2$.

Recall that X is called a *del Pezzo surface* if the anti-canonical sheaf ω_X^{-1} is ample. The self-intersection number (ω_X, ω_X) takes its value between 1 and 9 and is called the *degree* of a del Pezzo surface. Also X is a called a *conic bundle* if there exists a K-morphism $f : X \to C$ such that each fiber is reduced and is isomorphic to a conic over K (maybe reducible).

In the case when K is an algebraically closed field, the problem of birational classification of minimal G-surfaces is equivalent to the problem of classification of conjugacy classes of finite subgroups of the Cremona group $\mathrm{Cr}_K(2)$ of birational automorphisms of $\mathbb{P}^2_K$. We refer to [5] for the results in this direction. When $G = \{1\}$, the problem of classification of rational K-surfaces has been addressed in fundamental works of V.A. Iskovskikh [7] and Yu.I. Manin [14]. In both cases a modern approach uses the theory of elementary links [8].

We will be dealing with minimal del Pezzo G-surfaces or minimal conic bundles G-surfaces. In the G-equivariant version of the Mori theory they are interpreted as extremal contractions $\phi : S \to C$, where $C = \mathrm{pt}$ is a point in the first case and C is a curve in the second case. They are also two-dimensional analogs of rational Mori G-fibrations.

A birational G-map between Mori fibrations is a diagram of G-equivariant rational K-maps

$$\begin{array}{ccc} S & \overset{f}{\dashrightarrow} & S' \\ {\scriptstyle\phi}\downarrow & & \downarrow{\scriptstyle\phi'} \\ C & & C' \end{array} \qquad (6.1)$$

which in general do not commute with the fibrations. Such a map is decomposed into *elementary links*. These links are divided into the four following types.

- Links of type I:

They are commutative diagrams of the form

$$\begin{array}{ccc} S & \xleftarrow{\sigma} & Z = S' \\ \phi \downarrow & & \downarrow \phi' \\ C = \mathrm{pt} & \xleftarrow{\alpha} & C' = \mathbb{P}^1 \end{array} \tag{6.2}$$

Here $\sigma : Z \to S$ is a blow-up of a closed G-invariant 0-dimensional subscheme G-orbit, S is a minimal del Pezzo surface, $\phi' : S' \to \mathbb{P}^1$ is a minimal conic bundle, α is the constant map. For example, the blow-up of a G-fixed K-rational point on $\mathbb{P}^2$ defines a minimal conic G-bundle $\phi' : \mathbf{F}_1 \to \mathbb{P}^1$ with a G-invariant exceptional section. Here and in the sequel we denote by $\mathbf{F}_n$ a K-surface which becomes isomorphic over the algebraic closure of K to a minimal ruled surface $\mathbb{P}(\mathcal{O}_{\mathbb{P}^1_K} \oplus \mathcal{O}_{\mathbb{P}^1_K}(-n))$.

- Links of type II:

They are commutative diagrams of the form

$$\begin{array}{ccccc} S & \xleftarrow{\sigma} & Z & \xrightarrow{\tau} & S' \\ \phi \downarrow & & & & \downarrow \phi' \\ C & & = & & C' \end{array} \tag{6.3}$$

Here $\sigma : Z \to S$, $\tau : Z \to S'$ are the blow-ups of G-invariant closed 0-dimensional subschemes such that $\operatorname{rank}\operatorname{Pic}(Z)^G = \operatorname{rank}\operatorname{Pic}(S)^G + 1 = \operatorname{rank}\operatorname{Pic}(S')^G + 1$, $C = C'$ is either a point or a curve. An example of a link of type II is the a link between $\mathbb{P}^2$ and $\mathbb{F}_0$ where one blows up a G-invariant closed subscheme $\mathfrak{a}$ of $\mathbb{P}^2_K$ of degree 2 and then blows down the proper transform of the line spanned by $\mathfrak{a}$. Another frequently used link of type II is an elementary transformation of minimal ruled surfaces and conic bundles.

- Links of type III:

These are the birational maps which are the inverses of links of type I.

- Links of type IV:

They exist when S has two different structures of G-equivariant conic bundles. The link is the exchange of the two conic bundle structures

$$\begin{array}{ccc} S & = & S' \\ {\scriptstyle\phi}\downarrow & & {\scriptstyle\phi'}\downarrow \\ C & & C' \end{array} \tag{6.4}$$

Theorem 6.1. *Let $f : S \dashrightarrow S'$ be a birational map of minimal G-surfaces. Then f is equal to a composition of G-equivariant elementary links.*

The proof of this theorem is the same as in the arithmetic case considered in [9, Theorem 2.5].

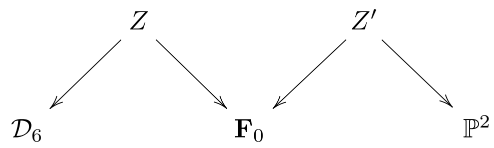
To start an elementary link, one has to blow up a G-invariant subscheme of maximal multiplicity of a linear system defining the birational map.

The classification of possible elementary links can be found in [8]. It is stated in the case $G = \{1\}$, however it can be extended to the general case in a straightforward fashion. The case when $G \neq \{1\}$ but K is algebraically closed is considered in [5, 7.2].

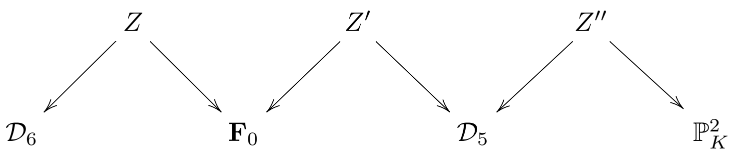
Example 6.2. Assume X is a del Pezzo surface $\mathcal{D}_6$ of degree 6 and $X' = \mathbb{P}^2_K$. We want to decompose a birational G-equivariant map $X \dashrightarrow X'$ into a composition of elementary links. From Propositions 7.12 and 7.13 in [5] we obtain that the only elementary link starting at (X, G) ends either at a del Pezzo surface Y of degree 6 or at $\mathbf{F}_0$. Since we do not want to stay on some $(\mathcal{D}_6, G)$, we may assume that $Y = \mathbf{F}_0$. Now we need an elementary link starting at Y. The same propositions tell us that the end of the next elementary link is either a conic bundle $Y' \to C$, or $\mathbf{F}_0$, or $\mathbb{P}^2_K$, or a del Pezzo surface of degree 5 or 6. Since we do not want to return back to X or $\mathbf{F}_0$ we may assume that the end of the link Y' is either a conic bundle or a del Pezzo surface of degree 5, or $\mathbb{P}^2$. If $Y' = \mathbb{P}^2$, then Proposition 7.13, case 2, tells us that $\mathbf{F}_0$ must contain a G-invariant K-rational point. If Y' is a del Pezzo surface of degree 5, then the same proposition tells us that $Z \to Y'$ is the blow-up of a G-invariant subscheme of degree 5. Finally, if Y' is a conic bundle, we may continue to do elementary links staying in the class $\mathcal{C}$ and at some point we have to link a conic bundle with a del Pezzo surface Y''. Proposition 7.12 tells us that Y'' is either a del Pezzo surface of degree 4 or $\mathbf{F}_0$. Since we do not want to return back to $\mathbf{F}_0$, we may assume that Y'' is a del Pezzo surface of degree 4. However, we find from Proposition 7.13, case 5, that we are stuck here since any elementary link relates Y'' only with itself.

Assume X is birationally G-isomorphic to $\mathbb{P}^2_K$. Then the previous analysis shows that X must have a G-invariant rational K-point allowing us to find an elementary link with $\mathbf{F}_0$. To continue, we need to find either a K-rational G-equivariant point on $\mathbf{F}_0$ to link the latter with $\mathbb{P}^2_K$, or to find a G-invariant 0-dimensional subscheme of length 5 to link $\mathbf{F}_0$ with a del Pezzo surface $\mathcal{D}_5$ of degree 5. The only elementary link which ends not at a del Pezzo surface of degree 5 or $\mathbf{F}_0$ is a link connecting to $\mathbb{P}^2_K$. It follows from Proposition 7.13, case 4, that to perform this link we need a K-rational G-invariant point on $\mathcal{D}_5$.

Here we exhibit possible elementary links relating a del Pezzo G-surface $(\mathcal{D}_6, G)$ with $(\mathbb{P}^2, G)$.



This is possible only if $\mathbf{F}_0$ has a G-invariant K-rational point.



This is possible only if $\mathbf{F}_0$ has a G-invariant closed subscheme of degree 5, and also $\mathcal{D}_5$ has a K-rational G-invariant point.

7. Maximal tori in $\mathbf{SU(3), PGU(3)}$

Let $\mathbf{SL}_3$ be the split simply connected simple group of type $\mathbf{A}_2$ over the field of real numbers. Let $\mathbf{SU}_3$ be its real form defined by the element of $H^1(\mathrm{Gal}(\mathbb{C}/\mathbb{R}), \mathbf{SL}_3(\mathbb{C}))$ represented by the map $A \mapsto \bar{A}^{-1}$. Its group of real points $\mathbf{SU}_3(\mathbb{R})$ is isomorphic to the group SU(3) of unitary 3×3 complex matrices. A maximal torus $\mathbb{T}$ in $\mathbf{SU}_3$ is a real form of the standard torus $(\mathbb{C}^*)^2 = \{(z_1, z_2, z_3) \in (\mathbb{C}^*)^3 : z_1 z_2 z_3 = 1\}$. It is defined by the map $(z_1, z_2, z_3) \mapsto (\bar{z}_1^{-1}, \bar{z}_2^{-1}, \bar{z}_3^{-1})$ and it is isomorphic to $(\mathbb{S}^1)^2$, where $\mathbb{S}^1 = \operatorname{Spec} \mathbb{R}[x, y]/(x^2 + y^2 - 1)$ with the natural structure of an algebraic group over $\mathbb{R}$. The group of real points of $\mathbb{S}^1$ is the circle $\mathrm{SU}(1) = \{z \in \mathbb{C} : |z| = 1\}$. Its complex points are $\{(z_1, z_2) \in \mathbb{C}^2 : z_1^2 + z_2^2 = 1\}$. The isomorphism $\mathbb{S}^1(\mathbb{C}) \to \mathbb{C}^*$ is given by $(z_1, z_2) \mapsto z = z_1 + iz_2$.

Let $C = \operatorname{Proj} \mathbb{R}[t_0, t_1, t_2]/(t_1^2 + t_2^2 - t_0^2)$ be the standard compactification of $\mathbb{S}^1$. It is a plane nonsingular conic defined over $\mathbb{R}$. Its real points satisfying $t_0 \neq 0$ are identified with SU(1) via the map $a + bi \mapsto [a, b, 1]$. Let

$$f : \mathbb{P}^1 \to C, \quad [u, v] \mapsto [u^2 - v^2, 2uv, u^2 + v^2]$$

be the rational parameterization of $\mathbb{S}^1$ defined over $\mathbb{R}$. We have

$$[u, v] \cdot [u', v'] := [uu' - vv', uv' + u'v]$$

is mapped to

$$[(uu' - vv')^2 - (uv' + u'v)^2, 2(uu' - vv')(uv' + u'v), (uu' - vv')^2 + (uv' + u'v)^2]$$
$$= [(u^2 - v^2)(u'^2 - v'^2) - 4uvu'v', (u^2 - v^2)2u'v' + (u'^2 - v'^2)2uv,$$
$$(u^2 + v^2)(u'^2 + v'^2)].$$

This shows that the restriction of the map f to the open subset $D^+(u^2 + v^2)$ is a homomorphism of groups.

Now let us consider the subvariety X of $(\mathbb{P}^1)^3$ given by the condition that $x \cdot y \cdot z = (1, 0)$. It is given by the equation

$$uu'v'' - vv'v'' + uv'u'' + u'vu'' = 0.$$

This is a compactification of the maximal torus $\mathbb{T}$ in $\mathbf{SU}_3$. The equation is given by a trilinear function, hence X is a hypersurface in $(\mathbb{P}^1_K)^3$ of type $(1, 1, 1)$. By the adjunction formula,

$$K_X = (K_{(\mathbb{P}^1)^3} + X) \cdot X = -(h_1 + h_2 + h_3).$$

This shows that X is a del Pezzo surface, anticanonically embedded in $\mathbb{P}^7$ by means of the Segre map $(\mathbb{P}^1)^3 \hookrightarrow \mathbb{P}^7$. Here h_i are the preimages of $\mathcal{O}_{\mathbb{P}^1}(1)$ under the projections $p_i : X \to \mathbb{P}^1$. The degree of the del Pezzo surface X is equal to $(h_1 + h_2 + h_3)^3 = 6h_1h_2h_3 = 6$. Over $\mathbb{C}$, a del Pezzo surface of degree 6 is isomorphic to the blow-up of three non-collinear points in $\mathbb{P}^2$.

The boundary $X \setminus \mathbb{T}$ of the torus $\mathbb{T}$ consists of three irreducible (over $\mathbb{R}$) components $p_i^{-1}(V(u^2+v^2))$. Over $\mathbb{C}$, each such component splits into two disjoint curves isomorphic to $\mathbb{P}^1$. The boundary becomes a hexagon of lines in the anticanonical embedding. The opposite sides are the pairs of conjugate lines. The group of automorphisms of the root system of type $\mathbf{A}_2$ of the group $\mathbf{SU}_3$ is isomorphic to the dihedral group D_6 of order 12 (also isomorphic to the direct product $\mathfrak{S}_3 \times \mathbb{Z}/2\mathbb{Z}$). Its standard action on $\mathbb{T}$ extends to a faithful action on the compactification X. It acts on the hexagon via its obvious symmetries.

Note that the Picard group $\operatorname{Pic}(X_{\mathbb{C}})$ is generated by the classes e_0, e_1, e_2, e_3, where e_0 is the class of the preimage of a line under the blow-up $X_{\mathbb{C}} = X_{\mathbb{C}} \to \mathbb{P}^2_{\mathbb{C}}$, and e_i are the classes of the exceptional curves. The hexagon of lines on X consists of the six lines with the divisor classes

$$e_1,\ e_2,\ e_3,\ f_1 = e_0 - e_2 - e_3,\ f_2 = e_0 - e_1 - e_3,\ f_3 = e_0 - e_1 - e_2.$$

The pairs of opposite sides are $\{f_i, e_i\}$. The group $\mathfrak{S}_3$ acts on $\operatorname{Pic}(X)$ by permuting e_1, e_2, e_3, and the Galois group acts on $\operatorname{Pic}(X)$ by $f_i \mapsto e_i$. Note that $-K_X = 3e_0 - e_1 - e_2 - e_3$ and, since K_X is Galois invariant, the conjugation isometry of $\operatorname{Pic}(X_{\mathbb{C}})$ sends e_0 to $2e_0 - e_1 - e_2 - e_3 = -K_X - e_0$ and $e_0 - e_i$ to $-K_X - e_0 - (e_0 - e_j - e_k) = e_0 - e_i$.

This shows that the pencil of conics $|e_0 - e_i|$ defines a map $p_i : X \to \mathbb{P}^1$ over $\mathbb{R}$. This defines our embedding

$$X \hookrightarrow (\mathbb{P}^1)^3 \hookrightarrow \mathbb{P}^7.$$

Also note that the invariant part $\mathrm{Pic}(X)^{\mathfrak{S}_3\times \mathrm{Gal}(\mathbb{C}/\mathbb{R})} = \mathbb{Z}K_X$, i.e. X is a minimal $\mathfrak{S}_3$-surface over $\mathbb{R}$.

Consider the real point $e \in \mathbb{T}(\mathbb{R})$, the unit element of the torus. The tangent plane to the Segre variety $s(\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1)$ in $\mathbb{P}^7$ at the point e is spanned by the images of $e \times \mathbb{P}^1 \times \mathbb{P}^1$, $\mathbb{P}^1 \times e \times \mathbb{P}^1$ and $\mathbb{P}^1 \times \mathbb{P}^1 \times e$. Its intersection with X is the point e. Consider the projection $\mathbb{P}^7 \dashrightarrow \mathbb{P}^3$ from the tangent plane of X at e. Its restriction to X defines a rational map $X \dashrightarrow Q$, where Q is a nonsingular quadric Q in $\mathbb{P}^3$. In fact, the rational map is the composition $\tau \circ \pi^{-1}$, where $\pi : X' \to X$ is the blow-up of the point e, and $\tau : X' \to Q$ is the blow-down of the proper transforms of three conics R_i, the images of $(\mathbb{P}^1 \times \mathbb{P}^1 \times \{e\}) \cap X$, $(\{e\} \times \mathbb{P}^1 \times \mathbb{P}^1) \cap X$, and $(\mathbb{P}^1 \times \{e\} \times \mathbb{P}^1) \cap X$. Note that, $R_i^2 = 0$ on X, and $\bar{R}_i^2 = -1$ on X'. We have $K_{X'}^2 = K_X^2 - 1 = 6 - 1 = 5$, and $K_Q^2 = 5 + 3 = 8$, so Q is a del Pezzo of degree 8, i.e. a quadric or $\mathbf{F}_1$. But the latter is not embedded in $\mathbb{P}^3$ as a normal surface.

The surface X has three $\mathfrak{S}_3$-invariant points $e, \eta, \eta^2 \in \mathbb{T}(\mathbb{R})$ corresponding to the diagonal matrices in SU(3). The image of η is a $\mathfrak{S}_3$-invariant real point in the real structure of Q defined by the map $X \to Q$. Projecting from this point, we see that Q is birationally trivial over $\mathbb{R}$ as an $\mathfrak{S}_3$-surface.

Applying Proposition 2.9, we obtain

Theorem 7.1. *The group* $\mathbf{SU}_3$ *is a Cayley group.*

Next we consider the group $\mathbf{PGU}(3)$. It is the quotient of $\mathbf{SU}_3$ by the cyclic group μ_3 of order 3. Its group $\mathbf{PGU}_3(\mathbb{R})$ of real points is isomorphic to the group PSU(3). A maximal torus of $\mathbf{PGU}_3$ is isomorphic to $\mathbb{T}/\mu_3$, where $\mathbb{T}$ is a maximal torus of $\mathbf{SU}_3$. In the real picture from the previous section, the action of μ_3 on $\mathbb{T}$ is the multiplication map $\sigma : (u, v) \mapsto (u, v) \cdot (1/2, \sqrt{3}/2)$. The action of μ_3 extends to the compactification X of the maximal torus $\mathbb{T}$ of $\mathbf{SU}_3$. Obviously, it leaves invariant the boundary, and has six isolated fixed points on the boundary; they are the vertices of the hexagon. The automorphism group of the del Pezzo surface X (over $\mathbb{C}$) is $(\mathbb{C}^*)^2 \times D_6$, and σ belongs to the connected part, and hence acts identically on $\mathrm{Pic}(X)$. In particular, it acts identically on the sides of the hexagon of lines. The quotient $Y = X/\mu_3$ is a singular compactification of a maximal torus of $\mathbf{PGU}_3$. It has six singular quotient singularities of type $\frac{1}{3}(1, 1)$, a minimal resolution $Y' \to Y$ has six exceptional curves E_i with $E_i^2 = -3$. The proper transforms of the images of the sides of the hexagon are six disjoint (-1)

curves.[2] Together with E_i's they form a 12-gon.[3] All of this is defined over $\mathbb{R}$, the Galois group switches opposite (-3)-sides and opposite (-1)-sides of the 12-gon. Now we can blow down the (-1)-sides to get a nonsingular surface Z with a hexagon of (-1)-curves formed by the images of the (-3)-sides. So, Z is a del Pezzo surface of degree six again! We have found a nonsingular $\mathfrak{S}_3 \times \mathrm{Gal}(\mathbb{C}/\mathbb{R})$-invariant minimal compactification of a maximal torus of $\mathbf{PGU}_3$ which is a del Pezzo surface of degree six.

Note that the group $\mathfrak{S}_3 \times \mathrm{Gal}(\mathbb{C}/\mathbb{R})$ acts on $\mathrm{Pic}(Z)$ in the same way as it acts in the case of $\mathbf{SU}_3$. So, as before, we have an $\mathfrak{S}_3$-invariant embedding $Z \hookrightarrow \mathbb{P}^7$ defined over $\mathbb{R}$ with a rational point equal to the orbit $\bar{e}$ of the origin $e \in X$ which consists of the diagonal matrices of $\mathrm{SU}(3)$. This time we have no any other $\mathfrak{S}_3$-invariant rational points on X (they obviously do not lie on the boundary). By projection from the point $\bar{e}$, we obtain a quadric Q.

The projection defines an $\mathfrak{S}_3$-equivariant isomorphism over $\mathbb{R}$ between the complement of the three conics on X and the complement of the image of the exceptional curve over $\bar{e}$ in Q. The latter curve is a conic section R' of Q. The three conics are permuted under $\mathfrak{S}_3$, so $\mathfrak{S}_3$ acts on R' without fixed points. Thus an $\mathfrak{S}_3$-invariant real point on Q must be the projection of a real $\mathfrak{S}_3$-invariant point on the del Pezzo surface X. There is none except the point which has been blown up. Thus the quadric Q has no $\mathfrak{S}_3$-invariant real points. It follows from Example 6.2 that there is no birational $\mathfrak{S}_3$-equivariant map from Z to $\mathbb{P}^2_{\mathbb{R}}$ (we are stuck at the first elementary link!).

Using Proposition 2.9, we obtain

Theorem 7.2. *The group* $\mathbf{PGU}_3$ *is not Cayley.*

8. Maximal tori in $\mathbf{SL}_3$

The group $\mathbf{SL}_3$ is a simple algebraic group split over $\mathbb{R}$. Its group of real points $\mathbf{SL}_3(\mathbb{R})$ is the group of unimodular real 3×3-matrices. Its maximal torus is the standard torus $\mathbb{T} = \mathrm{Spec}\,\mathbb{R}[z_1, z_2, z_3]/(z_1z_2z_3 - 1)$. The group $\mathbb{T}(\mathbb{R})$ of its real points is naturally isomorphic to $\{(a, b, c) \in (\mathbb{R}^*)^3 : abc = 1\}$ with the $\mathfrak{S}_3$-action defined by permutation of the coordinates. Obviously, a real $\mathfrak{S}_3$-invariant point on $\mathbb{T}$ must be equal to the identity point $(1, 1, 1)$.

A natural $\mathbb{T}$-equivariant compactification of $\mathbb{T}$ is the cubic surface $Y = \mathrm{Proj}\,\mathbb{R}[t_0, t_1, t_2, t_3]/(t_1t_2t_3 - t_0^3)$. It has three quotient singularities of type $\frac{1}{3}(1, 2)$, rational double points of type $\mathbf{A}_2$. They are defined over $\mathbb{R}$. The exceptional curve over each singular point consists of two (-2)-curves $E_i + E_i'$ intersecting transversally at one point. The intersection point $E_i \cap E_i'$ is a real point, hence the curves are isomorphic to $\mathbb{P}^1$ over $\mathbb{R}$. The group $\mathfrak{S}_3$ permutes the pairs (E_i, E_i'). After we minimally resolve Y over $\mathbb{R}$, we obtain a

[2] An $(-n)$-curve is a smooth rational curve on a nonsingular projective surface with self-intersection equal to $-n$.

[3] One can also arrive at this 12-gon by first blowing up the vertices of the hexagon, then extend the action of μ_3 to the blow-up, and then taking the quotient.

surface isomorphic to the blow-up of a del Pezzo surface of degree 6 at three vertices of the hexagon of lines. The boundary consists of a 9-gon with 9 consecutive sides $R_1, \dots, R_9$, where $R_1, R_2, R_4, R_5, R_7, R_8$ are (-2)-curves and the sides R_3, R_6, R_9 are (-1)-curves. The latter curves are the proper transforms of the three lines on the cubic surface Y that join the pairs of the singular points. After we blow down ($\mathfrak{S}_3$-equivariantly) the (-1) curves, we obtain a del Pezzo surface X of degree 6 with a hexagon of lines at the boundary. The linear system that defines the rational map $Y \dashrightarrow X$ consists of quadric sections of Y passing through the singular point. Note that both X and Y are $\mathfrak{S}_3$-equivariant compactifications of $\mathbb{T}$.

Theorem 8.1. $\mathbf{SL}_3$ *is not Cayley.*

Proof. By Proposition 2.9 it suffices to prove that $(X, \mathfrak{S}_3)$ is not birationally isomorphic to a $(\mathbb{P}^2_{\mathbb{R}}, \mathfrak{S}_3)$. Suppose they are birationally isomorphic. It follows from Example 6.2 that the first link must end at $\mathbf{F}_0 \cong \mathbb{P}^1_{\mathbb{R}} \times \mathbb{P}^1_{\mathbb{R}}$ which we identify with a split nonsingular quadric Q in $\mathbb{P}^3_{\mathbb{R}}$. The link consists of blowing up the unique $\mathfrak{S}_3$-invariant real point on X, namely the point e, and then blowing down three (-1)-curves. They are the images of the conics on Y that, together with the three lines, are cut out by the quadrics $t_i t_j - t_0^2 = 0$. The conics are left invariant under the conjugation but permuted by $\mathfrak{S}_3$. The action of $\mathfrak{S}_3$ on X shows easily that the induced action of $\mathfrak{S}_3$ on Q permutes the two rulings (i.e. the two projections to $\mathbb{P}^1$). It is easy to see, using the description of automorphisms of $\mathbb{P}^1_{\mathbb{R}} \times \mathbb{P}^1_{\mathbb{R}}$, that the quadric Q has no real $\mathfrak{S}_3$-invariant points, so the next elementary link relates Q with a del Pezzo surface $\mathcal{D}_5$ of degree 5. For this we need an $\mathfrak{S}_3$-invariant 0-dimensional subscheme $\mathfrak{a}$ of degree 5. It must consist of an $\mathfrak{S}_3$-invariant point of degree 2 and an $\mathfrak{S}_3$-orbit of three real points. It is easy to see that the only $\mathfrak{S}_3$-invariant point of degree 2 is the image of two conjugate scalar matrices in $\mathbf{SL}_3(\mathbb{C})$. There are plenty of $\mathfrak{S}_3$-orbits of three real points. Now we have to apply the elementary link $Q \leftarrow Z \rightarrow \mathcal{D}_5$ with the target equal to a del Pezzo surface $\mathcal{D}_5$ of degree 5. Either we are stuck here and hence prove the assertion or we find a real $\mathfrak{S}_3$-invariant point on $\mathcal{D}_5$ to make the final elementary link with $(\mathbb{P}^2, \mathfrak{S}_3)$. Since Q has no such points, a real $\mathfrak{S}_3$-invariant point q on $\mathcal{D}_5$ lies on the image of an exceptional curve of $Z \rightarrow Q$ or on the image of an exceptional curve of $Z \rightarrow \mathcal{D}_5$. The three exceptional curves on Z over real points in Q are permuted by $\mathfrak{S}_3$, so q cannot lie on them. Also the exceptional curve on Z over the complex point in Q consists of two disjoint conjugate curves. So, q is not on them either. It follows from the description of the linear system defining the link, that the exceptional curves of $Z \rightarrow \mathcal{D}_5$ are the proper transforms $\bar{R}_1$ and $\bar{R}_2$ of the two rational curves R_1 and R_2 of degree 3 (of bidegrees $(2,1)$ and $(1,2)$) on Q. Since $\mathfrak{S}_3$ permutes the two rulings on Q, it cannot leave R_1 or R_2 invariant. Thus the images of the exceptional curves $\bar{R}_1$ and $\bar{R}_2$ are not fixed under $\mathfrak{S}_3$. Thus the point q cannot be one of these points. This shows that the last elementary link $\mathcal{D}_5 \dashrightarrow \mathbb{P}^2$ is not possible. $\square$

Remark 8.2. The real split group $\mathbf{PGL}_3$ is known to be a Cayley group (see [12, Example 1.11]). Using Proposition 2.9, this fact immediately follows from the existence of an $\mathfrak{S}_3$-equivariant compactification of a maximal torus of $\mathbf{PGL}_3$ isomorphic to the projective plane. In fact, consider the cubic surface X from the proof of the previous theorem. The quotient of this surface by the cyclic group generated by the transformation $[t_0, t_1, t_2, t_3] \mapsto [\eta_3 t_0, t_1, t_2, t_3]$ is isomorphic to $\mathbb{P}^2_{\mathbb{R}}$ via the projection map from the point $[1, 0, 0, 0] \in \mathbb{P}^3 \setminus X$. Its maximal torus $\mathbb{T}$ is the standard torus in $\mathbb{P}^2_{\mathbb{R}}$.

Appendix B. Bad characteristics

This appendix was contributed by the anonymous referee. Since the referee's original exposition has been changed, the responsibility for possible inaccuracies or mistakes lies on the author of the paper.

Theorem B.1. *Let K be a field of characteristic $p > 0$. We write $\mathbb{G}_m$ for the multiplicative group $\mathbb{G}_{m,K}$, and $\mathbb{G}_a$ for the additive group $\mathbb{G}_{a,K}$. Let A be a central simple algebra of degree n over K. Assume that $p|n$ and that $4|n$ if $p = 2$. Then the group $G = \mathbf{PGL}_1(A) := A^\times/\mathbb{G}_m$ is Cayley.*

Proof. For two G-varieties X and Y over K, we write $X \sim Y$ and say that X is equivalent to Y if X is G-equivariantly birationally equivalent to Y.

We regard A also as a linear K-space, and we consider the projective space $\mathbb{P}(A)$. Clearly $G \sim \mathbb{P}(A)$.

We denote by $t\colon A \to K$ the reduced trace. Set

$$V = \{a \in A \mid t(a) = 1\},$$

then V is a G-variety, and it is easy to see that $\mathbb{P}(A) \sim V$, hence $G \sim V$. Since $p|n$, we have $t(x) = 0$ for any $x \in K \subset A$, hence the additive group $\mathbb{G}_a$ acts on V by translations:

$$x.a = x + a, \quad x \in K,\ a \in V \subset A.$$

Since $t(1) = 0$, we can define the linear function t also on $\mathrm{Lie}(G) = A/\langle 1\rangle$. Set

$$W = \{b \in A/\langle 1\rangle \mid t(b) = 1\}.$$

Clearly $W = V/\mathbb{G}_a$.

Note that the rational map

$$\mathrm{Lie}(G) = A/\langle 1\rangle \to W \times_K \mathbb{G}_m, \quad b \mapsto (b/t(b), t(b)) \quad \text{for } b \in A/\langle 1\rangle$$

gives an equivalence $\mathrm{Lie}(G) \sim W \times_K \mathbb{G}_m$. On the other hand, by Lemma B.2 below we have $V \sim W \times_K \mathbb{G}_a$. Thus

$$G \sim V \sim W \times_K \mathbb{G}_a \sim W \times_K \mathbb{G}_m \sim \mathrm{Lie}(G),$$

which proves the theorem. □

Lemma B.2. $V \sim W \times_K \mathbb{G}_a$.

Proof. Consider the projection

$$V \to V/\mathbb{G}_a = W.$$

It is enough to show that q has an equivariant section. Denote by $c_2(a)$ the second coefficient of the reduced characteristic polynomial of an element $a \in A$. In characteristic 0, c_2 is of course quadratic. But here, with our assumptions on p and n, we have

$$c_2(a+x) = \frac{n(n-1)}{2}x^2 + (n-1)t(a)x + c_2(a) = -t(a)x + c_2(a)$$

for $a \in A$ and $x \in K$ (check it in the split case for diagonal matrices, this is easy and implies the general formula). Hence the map $s\colon V/\mathbb{G}_a \to V$ sending a class $y = a + \mathbb{G}_a \in V/\mathbb{G}_a$ to the element $s(y) := a + c_2(a) \in y \subset V$ is well defined and is an equivariant section of q, as required. □

Proposition B.3. *If $p = 2$, then $G = \mathbf{PGL}_{2,K}$ is not Cayley.*

Proof. Indeed, assume for the sake of contradiction that there exists a G-equivariant birational isomorphism

$$\varphi\colon V := \{a \in M_2(K) \mid t(a) = 1\} \overset{\simeq}{\dashrightarrow} M_2(K)/\langle 1 \rangle.$$

Pick a generic invertible matrix $b \in V$. If $a \in V$ commutes with b and $\varphi(a)$ is defined, then $\varphi(a)$ commutes with b, hence φ restricts to a $\mathbb{Z}/2\mathbb{Z}$-equivariant birational isomorphism

$$\psi\colon \{a \in L \mid t(a) = 1\} \overset{\simeq}{\dashrightarrow} L/\langle 1 \rangle,$$

where L is the centralizer of b in $M_2(K)$, which is the maximal étale subalgebra of $M_2(K)$ generated by b, and $\mathbb{Z}/2\mathbb{Z}$ is the Weyl group of G with respect to its maximal torus $L^\times/\mathbb{G}_m$. We split the étale algebra L by a field extension K'/K (quadratic or trivial), then the Weyl group $\mathbb{Z}/2\mathbb{Z}$ acts on $L \otimes_K K' = K' \times K'$ by transposition of the factors, and we obtain a $\mathbb{Z}/2\mathbb{Z}$-equivariant birational isomorphism

$$\psi\colon \{(a_1, a_2) \in (K')^2 \mid a_1 + a_2 = 1\} \overset{\simeq}{\dashrightarrow} (K')^2/\langle (1,1) \rangle.$$

But this is absurd: the Weyl group $\mathbb{Z}/2\mathbb{Z}$ acts faithfully on the left, but trivially on the right. □

Remark B.4 *(of the referee).* If $p = 2$, $2|n$, $n \equiv 2 \bmod 4$, and $n \geq 6$, then $\mathbf{PGL}_1(A)$ is Cayley. (If $n \equiv 2 \bmod 8$, then one may use c_4 instead of c_2 in the proof; otherwise one may cook something 'linear' out of c_4 and powers of c_2.)

References

[1] M. Borovoi, Abelianization of the second nonabelian Galois cohomology, Duke Math. J. 72 (1993) 217–239.
[2] M. Borovoi, B. Kunyavskiĭ, N. Lemire, Z. Reichstein, Stably Cayley groups in characteristic zero, Int. Math. Res. Not. 2014 (19) (2014) 5340–5397.
[3] M. Borovoi, B. Kunyavskiĭ, Stably Cayley semisimple groups, Doc. Math. Extra Volume: Alexander S. Merkurjev's Sixtieth Birthday (2015), in press, arXiv:1401.5774 [math.AG].
[4] J.-L. Colliot-Thélène, B.È. Kunyavskiĭ, V.L. Popov, Z. Reichstein, Is the function field of a reductive Lie algebra purely transcendental over the field of invariants for the adjoint action?, Compos. Math. 147 (2011) 428–466.
[5] I. Dolgachev, V. Iskovskikh, Finite subgroups of the plane Cremona group, in: Algebra, Arithmetic, and Geometry: In Honor of Yu.I. Manin, vol. I, in: Progr. Math., vol. 269, Birkhäuser Boston, Inc., Boston, MA, 2009, pp. 443–548.
[6] Y.Z. Flicker, C. Scheiderer, R. Sujatha, Grothendieck's theorem on non-abelian H^2 and local-global principles, J. Amer. Math. Soc. 11 (1998) 731–750.
[7] V.A. Iskovskikh, Minimal models of rational surfaces over arbitrary fields, Izv. Akad. Nauk SSSR Ser. Mat. 43 (1) (1979) 19–43, 237 (in Russian); English transl.: Math. USSR Izv. 14 (1) (1980) 17–39.
[8] V.A. Iskovskikh, Factorization of birational mappings of rational surfaces from the point of view of Mori theory, Uspekhi Mat. Nauk 51 (4) (1996) 3–72 (in Russian); English transl.: Russian Math. Surveys 51 (1996) 585–652.
[9] V.A. Iskovskikh, Two nonconjugate embeddings of the group $S_3 \times \mathbb{Z}_2$ into the Cremona group, in: Teor. Chisel, Algebra i Algebr. Geom., Tr. Mat. Inst. Steklova 241 (2003) 105–109 (in Russian); English transl.: Proc. Steklov Inst. Math. 241 (2003) 93–97.
[10] V.A. Iskovskikh, Two non-conjugate embeddings of $S_3 \times \mathbb{Z}_2$ into the Cremona group II, Adv. Stud. Pure Math. 50 (2008) 251–267.
[11] M.-A. Knus, A. Merkurjev, M. Rost, J.-P. Tignol, The Book of Involutions, Amer. Math. Soc. Colloq. Publ., vol. 44, AMS, Providence, RI, 1998.
[12] N. Lemire, V.L. Popov, Z. Reichstein, Cayley groups, J. Amer. Math. Soc. 19 (2006) 921–967.
[13] D. Luna, Slices étales, Mém. Bull. Soc. Math. Fr. 33 (1973) 81–105.
[14] Yu.I. Manin, Rational surfaces over perfect fields. II, Mat. Sb. (N.S.) 72(114) (2) (1967) 161–192 (in Russian); English transl.: Math. USSR Sb. 1 (2) (1967) 141–168.
[15] V. L. Popov, Letter to D. Luna, March 12, 1975.
[16] J.-P. Serre, Groupes algébriques et corps de classes, Hermann, Paris, 1975.
[17] V.E. Voskresenskiĭ, Algebraic Groups and Their Birational Invariants, Transl. Math. Monogr., vol. 179, Amer. Math. Soc., Providence, RI, 1998.
[18] A. Weil, Algebras with involutions and the classical groups, J. Indian Math. Soc. (N.S.) 24 (1961) 589–623.