Fixed points of a finite subgroup of the plane Cremona group

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Abstract

We classify all finite subgroups of the plane Cremona group which have a fixed point. In other words, we determine all rational surfaces X with an action of a finite group G such that X is equivariantly birational to a surface which has a G-fixed point.

1. Introduction

Let G be a finite subgroup of the plane Cremona group, $\mathrm{Cr}(2)$, the group of birational transformations of the complex projective plane. We say that G *has a fixed point* if there exist a smooth rational projective surface X with a faithful G-action $\rho\colon G \hookrightarrow \mathrm{Aut}(X)$, and a birational map $\phi\colon X \dashrightarrow \mathbb{P}^2$ such that X has a G-fixed point and $\phi \circ \rho(G) \circ \phi^{-1} = G$. This definition depends only on the conjugacy class of G in $\mathrm{Cr}(2)$. In this paper we present a classification of the conjugacy classes of the subgroups of $\mathrm{Cr}(2)$ with a fixed point and, for each class, we find a representative G-surface.

For abelian finite groups acting on smooth proper varieties, the presence of a fixed point is a birational invariant (see [RY00, Proposition A.2]). In general, however, this is not true; for example, the exceptional divisor of a blow-up of a fixed point may not have any fixed points. However, if $f\colon X \to X'$ is a morphism of G-surfaces, then a fixed point on X maps to a fixed point on X'. Thus, the theory of minimal models of G-surfaces tells us that it suffices to find minimal G-surfaces of one of the following two types:

- *Conic bundles*: There exists a regular map $f\colon X \to \mathbb{P}^1$ such that the general fiber is isomorphic to $\mathbb{P}^1$ and the subgroup $\mathrm{Pic}(X)^G$ of G-invariant invertible sheaves on X is generated over $\mathbb{Q}$ by the canonical class K_X and the class of a fiber of f.
- *del Pezzo G-surfaces*: The anti-canonical class $-K_X$ is ample and $\mathrm{Pic}(X)^G$ is generated over $\mathbb{Q}$ by K_X.

An important tool for solving our problem is the classification of the conjugacy classes of the finite subgroups of $\mathrm{Cr}(2)$ from [DI09]. Although we use some results from [DI09], many of our proofs do not directly rely on this work. In fact, our work led to the discovery of some gaps in the classification in that paper and we use this opportunity to fill those gaps. Note, however,

that this classification is incomplete in the case of conic bundles (see also [Tsy11]). One may also find an independent classification of the abelian subgroups of Cr(2) in [Bla07].

By considering the action on the tangent space of a fixed point, we see that any finite group G acting on a smooth surface with a fixed point must be isomorphic to a subgroup of GL(2). Also, it is well known that a cyclic group always has a fixed point on a rational variety (for example, as was noticed by J.-P. Serre, this follows from the Lefschetz fixed-point formula applied to the structure sheaf). Consequently, we restrict our attention to G not cyclic.

Recall that a del Pezzo surface has degree $d = K_X^2$. A del Pezzo surface X of degree 4 can be written by two equations in $\mathbb{P}^4$ defined by diagonal quadrics. The coordinate hyperplanes cut out five genus 1 curves $E_1, \ldots, E_5$ on X. A del Pezzo surface X of degree 3 is a cubic surface in $\mathbb{P}^3$. An *Eckardt point* on X is a point where three lines on the surface meet. A del Pezzo surface X of degree 2 is a double cover of $\mathbb{P}^2$ branched over a smooth plane quartic curve B.

THEOREM 1.1. *Suppose that G is a finite non-cyclic subgroup of the Cremona group admitting a fixed point. Then there exists a G-surface X realizing a fixed point p of G of one of the following forms:*

$\boxed{\text{L}}$ *X is $\mathbb{P}^2$.*

$\boxed{6}$ *X is a del Pezzo surface of degree 6.*

$\boxed{4}$ *X is a del Pezzo surface of degree 4 and p lies on exactly two curves E_i, E_j, both of which are equianharmonic.*

$\boxed{3}$ *X is a cubic surface and the tangent space to p contains three Eckardt points.*

$\boxed{\text{2A}}$ *X is a del Pezzo surface of degree 2 and p lies on the ramification divisor R.*

$\boxed{\text{2B}}$ *X is a del Pezzo surface of degree 2 and p is the intersection point of four exceptional curves.*

$\boxed{1}$ *X is a del Pezzo surface of degree 1 and p is the base point of the anti-canonical linear system.*

$\boxed{\text{C}}$ *X is a minimal conic bundle.*

Note that there may be some overlap between these cases as there may be more than one G-surface in an equivalence class. Occurrences of this phenomenon, along with the specific groups that occur in each case, will be discussed in the sections below. For the readers' convenience, we consolidate those groups acting on del Pezzo surfaces of degree 2–6 in Table 1. We use the notation for finite groups employed in [DI09], borrowed from [CCNPW85].

2. Preliminaries

Let G be a finite group. A *G-surface* is a pair (X, ρ) where X is a smooth projective surface and $\rho \colon G \hookrightarrow \mathrm{Aut}(X)$ is a faithful G-action. We will often refer to the pair as (X, G) or simply X when the context is clear. A *morphism* of G-surfaces $(X, \rho) \to (X', \rho')$ is a morphism of the underlying surfaces $f \colon X \to X'$ such that $\rho'(G) \circ f = f \circ \rho(G)$. Similarly, one defines rational maps, birational maps and birational morphisms of G-surfaces.

A *G-surface X* is *minimal* if any birational morphism $X \to X'$ of G-surfaces is an isomorphism. We say that an action of G on a surface X is a *minimal group of automorphisms* if the corresponding G-surface is minimal. As in the introduction, minimal G-surface are either minimal conic bundles or minimal del Pezzo G-surfaces.

Group	Order	Cases
2^2	4	2A.1
$\mathfrak{S}_3$	6	6
$\mathfrak{S}_3$	6	3.1
4×2	8	2B.2 , 2A.3
D_8	8	2B.1
Q_8	8	2B.3
3^2	9	3.3
6×2	12	2A.2
$\mathfrak{S}_3 \times 2$	12	6
$\mathfrak{S}_3 \times 2$	12	3.2
$3 : 4$	12	4
4^2	16	2B.5
8×2	16	2A.5
4.2^2	16	2B.3
6×3	18	3.3
$\mathfrak{S}_3 \times 3$	18	3.3 (twice)
12×2	24	2A.4
$2^2 : \mathfrak{S}_3$	24	4
$2 \cdot \mathfrak{A}_4$	24	2B.4
$4 \cdot D_8$	32	2B.5
$\mathfrak{S}_3 \times 6$	36	3.3
$4 \cdot \mathfrak{A}_4$	48	2B.4

TABLE 1. Non-cyclic subgroups G of $\mathrm{Cr}(2)$ with a fixed point realized by a minimal del Pezzo G-surface of degree 2–6, but not by a minimal conic bundle

A minimal conic bundle $f\colon X \to \mathbb{P}^1$ either is a minimal ruled surface with f being one of its rulings or has $k \geqslant 3$ degenerate fibers isomorphic to the union of two projective lines $\mathbb{P}^1$ intersecting transversally at one point. Recall that a del Pezzo surface X is a smooth projective surface such that the anti-canonical divisor class $-K_X$ is ample. The *degree* of a del Pezzo surface is $d = K_X^2$, which takes values $1 \leqslant d \leqslant 9$.

We caution the reader that a minimal G-surface may be a del Pezzo surface but not be a minimal del Pezzo G-surface! We shall see an example of such a surface in Section 7.

With the notable exceptions of $\mathbb{P}^2$ and $\mathbb{P}^1 \times \mathbb{P}^1$, every del Pezzo surface is a blow-up of $\mathbb{P}^2$ at $9-d$ points $x_1, \ldots, x_{9-d}$ in general position, with corresponding exceptional divisors $R_1, \ldots, R_{9-d}$. Conversely, any set of $9-d$ disjoint (-1)-curves can be blown down to $\mathbb{P}^2$, giving rise to a *plane model* of X. Each such choice is called a *geometric marking* and gives rise to a choice of basis for the orthogonal complement $\mathcal{R}_X$ of K_X in $\mathrm{Pic}(X)$.

For $d \leqslant 6$, the action of $\mathrm{Aut}(X)$ on $\mathcal{R}_X$ defines a homomorphism

$$\rho\colon \mathrm{Aut}(X) \to W_{9-d}, \tag{2.1}$$

where W_n denotes the Weyl group of a simple root system of type E_n (by definition, $E_5 = D_5$, $E_4 = A_4$, $E_3 = A_2 + A_1$). If $d \leqslant 5$, then ρ is injective. It follows that in this case any subgroup G of $\mathrm{Aut}(X)$ defines a conjugacy class of W_n which is independent of the choice of a basis in $\mathrm{Pic}(X)$.

A G-surface X is a minimal del Pezzo G-surface if $\mathrm{Pic}(X)^G$ is generated over $\mathbb{Q}$ by K_X. A minimal del Pezzo G-surface of degree 8 is isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$; the other surface of degree 8 is never minimal. Similarly, the surface of degree 7 is never minimal.

In order to determine whether a del Pezzo surface is minimal, we will use the following consequence of the Lefschetz fixed-point formula (as it was used in Section 6 of [DI09]).

PROPOSITION 2.1. *Let X be a del Pezzo surface. If σ is an automorphism of X, then the trace of σ^* on $\mathcal{R}_X$ is given by*

$$\mathrm{Tr}(\sigma^*|\mathcal{R}_X) = s - 3 + \sum_{i=1}^{n}(2 - 2g_i),$$

where s is the number of isolated fixed points and $g_1, \ldots, g_n$ are the genera of the fixed curves. Moreover, for a finite group G, the surface X is G-minimal if and only if

$$\sum_{\sigma \in G} \mathrm{Tr}(\sigma^*|\mathcal{R}_X) = 0.$$

We are classifying G-surfaces up to birational equivalence. It may happen that two minimal G-surfaces are birationally equivalent. Indeed, we will see in Sections 3–6 that all del Pezzo G-surfaces of degree at least 5 with a fixed point are birationally G-isomorphic to $\mathbb{P}^2$ with a fixed point. On the other hand, from Section 8 of [DI09] we have that any minimal del Pezzo G-surface of degree at most 3 is *rigid*; thus we have the following.

LEMMA 2.2. *Every minimal del Pezzo G-surface X of degree at most 3 is the unique minimal G-surface in its birational G-equivalence class.*

The remaining case of degree 4 is more subtle and will be discussed in Section 7.

In Theorems 8.1, 9.1 and 9.5, we will describe families of del Pezzo surfaces via normal forms involving parameters. For a given family A, there is some collection of groups G which fix a point and for which the G-surface is minimal. For certain special values of these parameters, the set of possible groups G may be larger and we have a new family B. We say that A *specializes* to B, say that B is a *specialization* of A, or write A → B. Conversely, we say that A is a *generization* of B.

This language is justified in view of the following.

PROPOSITION 2.3. *Let $X \to T$ be a flat family of del Pezzo surfaces of degree at most 5 over a base scheme T. For each conjugacy class C of subgroups in W_{9-d}, the set*

$$\{t \in T : \mathrm{Aut}(X_t) \text{ contains a subgroup } G \text{ representing } C\}$$

is closed in T.

Proof. Since the monodromy group of a smooth flat family of del Pezzo surfaces is a finite subgroup of the Weyl group W_{9-d}, after passing to a certain finite cover of T, we may trivialize the local coefficient system on T defined by the second cohomology group of fibers. Choosing simultaneously a geometric marking in each fiber, we may define a map from T to the GIT-quotient P_2^{9-d} of $(\mathbb{P}^2)^{9-d}$ by the group $\mathrm{PGL}(3)$. Since the pre-image of a closed set is closed, it suffices to assume that T is an open subset U of P_2^{9-d} parameterizing point sets whose blow-up is a del Pezzo

surface. From [DO88], the group W_{9-d} acts biregularly on U via Cremona transformations and the stabilizer of a point $t \in T$ is equal to the image of $\mathrm{Aut}(X_t)$ under the homomorphism (2.1).

Let $a \colon \Gamma \times V \to V$ be an action of a finite group Γ on an algebraic variety V. The pre-image Z of the diagonal Δ of V under the map $(a, \mathrm{id}) \colon \Gamma \times V \to V \times V$ consists of points (g, v) such that $g \in \Gamma_v$. For any subgroup H of Γ, the pre-image of H under the first projection $Z \to \Gamma$ is a closed subset W of Z. Since Γ is finite, the image of W via the second projection $Z \to V$ is a closed subset of V consisting of points whose stabilizer contains H. Applying this to our situation, where $\Gamma = W_{9-d}$ and $V = U$, we obtain the assertion of the proposition. $\qquad\square$

3. Del Pezzo surfaces of degree 9

In the degree 9 case, $X \cong \mathbb{P}^2$ and we classify finite subgroups of $\mathrm{Aut}(\mathbb{P}^2) \cong \mathrm{PGL}(3)$ that have a fixed point.

Let G be a subgroup of $\mathrm{PGL}(3)$, and let $\widetilde{G}$ be a pre-image in $\mathrm{GL}(3)$. We have a 3-dimensional representation ρ of $\widetilde{G}$. The existence of a G-fixed point on X is equivalent to the existence of a 1-dimensional subrepresentation χ of ρ. It follows that any finite group of projective transformations has either no fixed points, or one fixed point, or three isolated fixed points, or a line of fixed points plus an isolated fixed point.

THEOREM 3.1 (Case $\boxed{\mathrm{L}}$). *Conjugacy classes of finite subgroups of* $\mathrm{Aut}(\mathbb{P}^2)$ *with one isolated fixed point are in a natural bijection with conjugacy classes of finite subgroups of* $\mathrm{GL}(2)$ *with a fixed point.*

Proof. We choose coordinates x, y, z such that the fixed point is $p_0 = (0, 0, 1)$. A projective transformation g fixing this point can be uniquely represented by a transformation $(x : y : z) \mapsto (ax + by : cx + dy : z)$, where $\tilde{g} = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix} \right) \in \mathrm{GL}(2)$. Any conjugate $g' = h^{-1}gh$ must fix the point p_0 (here we use the assumption on the set of fixed points). Hence $\tilde{h}$ conjugates $\tilde{g}'$ and $\tilde{g}$. The converse is also true. $\qquad\square$

If G has three isolated fixed points, then G is an abelian group conjugate to a subgroup of transformations $(x : y : z) \mapsto (ax : by : cz)$. Finally, if G has a line of fixed points, then G is a cyclic group.

4. Del Pezzo surfaces of degree 8

There are two isomorphism classes of del Pezzo surfaces of degree 8. One is isomorphic to the blow-up of one point; hence it is not minimal. The other one is isomorphic to $X = \mathbb{P}^1 \times \mathbb{P}^1$. So we will study subgroups of $\mathbb{P}^1 \times \mathbb{P}^1$.

Assume that G has a fixed point p. Let ℓ_1, ℓ_2 be the two fibers passing through p. Their union is G-invariant. The group G contains a subgroup G' of index 1 or 2 such that each ruling $\pi_i \colon X \to \mathbb{P}^1$ is invariant.

As each ruling π_i is G'-equivariant, there must be a G'-fixed point on each image $\pi_i(X) \cong \mathbb{P}^1$. Note that any finite group of automorphisms which fixes one point on $\mathbb{P}^1$ must fix another. Thus there exists another pair of lines ℓ_1', ℓ_2' whose intersection is another G-fixed point p' on X.

If we blow up p, the strict transforms of ℓ_1 and ℓ_2 both become exceptional curves. Since they do not intersect and their union is G-invariant, we may blow them down G-equivariantly to X'.

The variety X' is isomorphic to $\mathbb{P}^2$ and has a G-fixed point since the birational map $X \to X'$ is defined at p'. Thus, we have the following.

THEOREM 4.1. *If G has a fixed point on $X \cong \mathbb{P}^1 \times \mathbb{P}^1$, then X is G-birationally equivalent to $\mathbb{P}^2$.*

5. Del Pezzo surfaces of degree 6

The surface X is isomorphic the blow-up of three non-collinear points x_1, x_2, x_3 in the plane. The strict transforms of the lines ℓ_{12}, ℓ_{13}, ℓ_{23} through each pair of points are (-1)-curves on X. Along with the exceptional divisors sitting above each point x_i, these form a hexagon of (-1)-curves.

Let p be a fixed point of G. If p is on the hexagon, then it must be one of its vertices since otherwise the side of the hexagon containing p is G-invariant and hence can be equivariantly blown down. But, if p is a vertex, then the opposite vertex is also fixed, and there will be two skew lines that are left invariant and can be equivariantly blown down. This contradicts the minimality assumption. Thus p is not on the hexagon.

Since p is not on the hexagon, it must be the pre-image of a point x_0 in the plane. The blow-up of X at p is a del Pezzo surface of degree 5. It contains ten lines; six of them are the pre-images of the sides of the hexagon, and three of them are the pre-images ℓ_1, ℓ_2, ℓ_3 of the lines in the plane joining x_0 with the vertices of the coordinate triangle. The last line is the exceptional curve $E(p)$ of the blow-up. Since the sides of the hexagon and the line $E(p)$ form a G-invariant set of lines, the set of lines ℓ_1, ℓ_2, ℓ_3 is also G-invariant. The action on this set gives a homomorphism $\rho\colon G \to \mathfrak{S}_3$. If G fixes one line, then we can blow down the pair of opposite sides of the hexagon intersecting this line. This shows that (X, G) is not minimal. So, the image of G in $\mathfrak{S}_3$ is either a cyclic group of order 3 or the whole $\mathfrak{S}_3$. An element in the kernel of ρ fixes all three lines ℓ_i, and hence fixes all pairs of opposite sides of the hexagon. Composing it with the action of the standard Cremona transformation s_1 (see [DI09, Section 6.2]) on X that permutes the opposite sides, we get the identity. This shows that $\ker(\rho)$ is either trivial or generated by s_1.

In summary, we have the following result.

THEOREM 5.1 (Case $\boxed{6}$). *Let G be a minimal finite non-cyclic group of automorphisms of a del Pezzo surface of degree 6 that fixes a point. Then G is either $\mathfrak{S}_3$, of order 6, or the group $2 \times \mathfrak{S}_3$, of order 12.*

If we blow up the fixed point p then the lines ℓ_1, ℓ_2, ℓ_3 form a G-invariant set of skew lines. Blowing these down, we obtain a G-equivariant birational equivalence from X to $\mathbb{P}^1 \times \mathbb{P}^1$. However, the fixed point is lost. This equivalence is a link of type II (see [DI09, Section 7]). From the discussion in [DI09, Section 8], we see that the only other possible minimal del Pezzo or minimal conic bundles equivariantly birational to X are del Pezzo surfaces of degree 5. But such surfaces are only minimal if G contains an element of order 5 (see below). Thus a del Pezzo surface of order 6 is the only model for G which has a fixed point.

6. Del Pezzo surfaces of degree 5

The surface is isomorphic to the blow-up of four points $x_1, \ldots, x_4$ in $\mathbb{P}^2$, no three of which are collinear. In this case we know from [DI09, Theorem 6.4] that $\mathrm{Aut}(X) \cong \mathfrak{S}_5$. The ten exceptional curves along with their intersections are in bijective correspondence with vertices and lines of the Petersen graph. Alternatively, the ten exceptional curves are in bijection with pairs of elements of $\{1, 2, 3, 4, 5\}$; two curves intersect if and only if the pairs have no common elements.

The maximal subgroups of $\mathfrak{S}_5$ are $\mathfrak{S}_3 \times 2$, $\mathfrak{S}_4$, $\mathfrak{A}_5$ and $5 : 4$. Note that $\mathfrak{S}_3 \times 2 \cong \langle (123), (45) \rangle$ is not minimal since it fixes the exceptional curve corresponding to $\{4, 5\}$. The group $\mathfrak{S}_4$ is not minimal since it leaves invariant the four skew lines $\{1, 5\}$, $\{2, 5\}$, $\{3, 5\}$, $\{4, 5\}$. The subgraph of the Petersen graph based on the orbit of any cyclic group of order 5 is a pentagon. For example, if $\sigma = (12345)$ and the vertex is $\{1, 2\}$, the orbit consists of vertices $\{1, 2\}$, $\{2, 3\}$, $\{3, 4\}$, $\{4, 5\}$, $\{1, 5\}$. This shows that any group containing an element of order 5 must be minimal. Thus, the groups $\mathfrak{S}_5$, $\mathfrak{A}_5$ and $5 : 4$ are minimal; however, they do not have 2-dimensional representations and thus cannot have fixed points. Among their subgroups, the only non-cyclic group not yet considered is $G \cong D_{10}$.

The group D_{10} is minimal and has two fixed points. To see this, we use the well-known $\mathfrak{S}_5$-equivariant isomorphism between a del Pezzo surface X of degree 5 and the GIT-quotient P_1^5 of $(\mathbb{P}^1)^5$ by $\mathrm{PGL}(2)$. Represented as point sets, the points

$$p_0 = \left(1, \epsilon_5, \epsilon_5^2, \epsilon_5^3, \epsilon_5^4 \right) \qquad \text{and} \qquad p_1 = \left(1, \epsilon_5^3, \epsilon_5, \epsilon_5^4, \epsilon_5^2 \right)$$

on X are fixed by the group $G = \langle \sigma, \tau \rangle \cong D_{10}$, where

$$\sigma = (12345) \qquad \text{and} \qquad \tau = (25)(34) \,.$$

Indeed, $\sigma(p_i) \equiv p_i$ since it amounts to multiplication by a constant, while τ corresponds to $z \mapsto z^{-1}$ on each $\mathbb{P}^1$.

While this G-surface is minimal, it is birationally equivalent to $\mathbb{P}^2$. Note that neither fixed point lies on an exceptional divisor since every G-orbit of exceptional divisors contains skew divisors. If we consider X as the blow-up of four points in $\mathbb{P}^2$, the linear system of cubic curves through the four points and a double point at the image of p_0 in the plane is of dimension 2. Thus we have an equivariant birational map from X to $\mathbb{P}^2$ which maps p_1 to a fixed point. We conclude as follows.

THEOREM 6.1. *Suppose that (X, G) is a minimal del Pezzo surface of degree 5 with a fixed point and G non-cyclic. Then $G \cong D_{10}$ and X is G-birational to $\mathbb{P}^2$ with a fixed point.*

7. Del Pezzo surfaces of degree 4

We recall several facts from Section 6.4 of [DI09]. Any quartic del Pezzo surface X is isomorphic to a smooth surface in $\mathbb{P}^4$ given by the equations

$$\sum_{i=1}^{5} t_i^2 = \sum_{i=1}^{5} a_i t_i^2 = 0 \,,$$

where $a_i \neq a_j$ whenever $i \neq j$.

The natural representation of $\mathrm{Aut}(X)$ on the Picard group of X defines an isomorphism ρ of $\mathrm{Aut}(X)$ onto a subgroup of the Weyl group $W(D_5) \cong 2^4 : \mathfrak{S}_5$. The normal subgroup 2^4 is always in the image of ρ and acts on X by multiplying an even number of coordinates by -1. The image of $\mathrm{Aut}(X)$ in $\mathfrak{S}_5$ could be one of the following groups: 1, 2, $\mathfrak{S}_3$, 4 and D_{10}.

Each element of 2^4 is represented by a subset A of $\{1, 2, 3, 4, 5\}$ corresponding to the indices of the coordinates t_i that are multiplied by -1. Since $\mathrm{Aut}(X)$ acts on the projective space $\mathbb{P}^4$, we may identify each subset of A with its complement. Thus, it suffices to assume that the cardinality of A of a non-trivial element is equal to 1 or 2. The corresponding involution ι_A is called *of the first kind* or *of the second kind*, accordingly.

We denote by E_k the genus 1 curve cut out by the hyperplane section $t_k = 0$. The group $\mathrm{Aut}(X)$ acts on the set of such curves with kernel of the action equal to 2^4. The fixed point set on X of each ι_k is precisely the corresponding genus 1 curve E_k.

We now discuss how to see the action of $W(D_5)$ on the exceptional divisors of X and its connection to the plane model. Recall that X is isomorphic to the blow-up of five points $x_1, \ldots, x_5$ in the projective plane. We label the 16 exceptional divisors of X: let $R_1, \ldots, R_5$ be the exceptional curves corresponding to the points x_i, let R_{ij} be the strict transforms of the lines $\overline{x_i, x_j}$, and let R_0 be the strict transform of the conic through the points $x_1, \ldots, x_5$. Each geometric marking corresponds to a choice of the five disjoint lines R_i. There are 2^4 such subsets and the Weyl group $W(D_5)$ has 2^4 conjugate subgroups isomorphic to $\mathfrak{S}_5$; each of them leaves invariant the set of the divisor classes of five disjoint lines.

Each of the involutions ι_k is given by a de Jonquières involution of the plane model with center at the point x_k (see [DI09, Section 2.3]). The involution is given by the linear system of cubics through the points x_i for $i \neq k$ and a singular point at x_k. The image of E_k is the unique plane cubic curve that passes through the points $x_1, \ldots, x_5$ with tangent direction at each point x_j for $j \neq k$ equal to the line $\overline{x_j, x_k}$. The de Jonquières involution preserves the pencil of lines through the point x_k.

It follows from the construction of de Jonquières involutions that ι_k interchanges R_i with R_{ik}, and R_k with R_0. The remaining set of six lines R_{ij}, where $i, j \neq k$, consist of three orbits of pairs of intersecting lines. Note that, even though no orbits of (-1)-curves can be blown down, the subgroup generated by ι_k does *not* give X the structure of a G-minimal del Pezzo surface. However, X is G-minimal when considered as a conic bundle defined by the pencil of conics given by the proper inverse transforms of the lines through x_k.

It follows that the involution $\iota_{kl} = \iota_k \circ \iota_l$ interchanges the disjoint lines R_0 and R_{kl}; thus, it does not act minimally. The fixed points of ι_{kl} are precisely the four intersection points of the two genus 1 curves E_k and E_l. The only minimal subgroups of 2^4 with fixed points are those that contain exactly two involutions of the first kind.

In [DI09, Section 8], it is shown that any minimal del Pezzo G-surface of degree 4 with a fixed point is equivariantly birationally equivalent to a G-minimal conic bundle. However, the conic bundle may not have a fixed point. We clarify the situation as follows.

LEMMA 7.1. *Suppose that X is a minimal del Pezzo G-surface.*

  (i) *If G has more than one fixed point or G is abelian, then X is birationally equivalent to a minimal conic bundle with a fixed point.*

 (ii) *If G has exactly one fixed point and G is non-abelian, then X is not birationally equivalent to a minimal conic bundle or non-isomorphic minimal del Pezzo surface with a fixed point.*

Proof. Let p be a G-fixed point on X. Blowing up the point p, we obtain a weak del Pezzo surface X' of degree 3 with $\mathrm{Pic}(X')^G \cong \mathbb{Z}^2$. The linear system $|-K_{X'} - R|$, where R is the exceptional curve of the blow-up, defines on X' a structure of a G-minimal conic bundle.

If X has more than one fixed point, then X' also has a fixed point. Also, if G is abelian, then the induced action on $R \cong \mathbb{P}^1$ is cyclic and thus X' again has a fixed point.

However, if X has a unique fixed point and G is non-abelian, then the new surface X' does not have a fixed point. Indeed, the exceptional curve R has an action of G; since G is not abelian, the image of its action is not cyclic and $R \cong \mathbb{P}^1$ cannot have a fixed point.

Conceivably, there might be a third minimal G-surface X'' birational to X which *does* have a fixed point. We consult the classification of elementary links in [DI09, Section 7.4]. From X, the link of Type I to X' as described above is the only link which changes the isomorphism class of X. The conic bundle X' satisfies $K_{X'}^2 = 3$ and the only links which change the isomorphism class are links of Type II. These are simply compositions of elementary transformations and cannot introduce new fixed points, nor change the value of $K_{X'}^2$. $\qquad\square$

We now prove the main result of this section.

THEOREM 7.2 (Case $\boxed{4}$). *Let X be a minimal del Pezzo G-surface of degree 4. Suppose that G has a fixed point and is not birationally equivalent to a minimal conic bundle with a fixed point. Then X is isomorphic to the G-surface*

$$t_1^2 + \epsilon_3 t_2^2 + \epsilon_3^2 t_3^2 + t_4^2 = t_1^2 + \epsilon_3^2 t_2^2 + \epsilon_3 t_3^2 + t_5^2 = 0\,, \qquad \epsilon_3 = e^{2\pi i/3}\,,$$

whose automorphism group (as an ordinary surface) is generated by 2^4 along with the transformations

$$g_1\colon\ (t_1 : t_2 : t_3 : t_4 : t_5) \mapsto \left(t_2 : t_3 : t_1 : \epsilon_3 t_4 : \epsilon_3^2 t_5\right),$$
$$g_2\colon\ (t_1 : t_2 : t_3 : t_4 : t_5) \mapsto (t_1 : t_3 : t_2 : t_5 : t_4)\,.$$

The group G is isomorphic to one of the following groups:

$$2^2 : \mathfrak{S}_3\,, \quad 3 : 4\,,$$

with unique fixed point $p = (1 : 1 : 1 : 0 : 0)$.

Proof. From the lemma, it suffices to find G-minimal del Pezzo surfaces with a unique fixed point and G non-abelian.

It is known that any minimal subgroup of $\mathrm{Aut}(X)$ contains a non-trivial subgroup of 2^4. Hence, a fixed point p of G must lie on one of the curves E_i. Since no three genus 1 curves among $E_1, \ldots, E_5$ have a common point, the group G contains a subgroup G' of index at most 2 that leaves E_i invariant. We may consider E_i as an elliptic curve with the zero element p. Let A be the image of G' in the automorphism group of the elliptic curve E_i. It is known that A is of order 2, 3, 4 or 6. It has 4, 3, 2 or 1 fixed point, respectively. Thus A must be of order 6, hence the order of G is divisible by 3.

Let G be a group of automorphisms of X of order divisible by 3. It is known that X is isomorphic to the surface from the assertion of the theorem. Also, the automorphism group of X is generated by involutions ι_A and the subgroup $H = \langle g_1, g_2 \rangle \cong \mathfrak{S}_3$. We fix a plane model of X as above to assume that g_1 acts on $\mathrm{Pic}(X)$ by permuting cyclically the classes of the exceptional curves R_1, R_2, R_3 and fixing the curves R_4, R_5. The element g_2 acts by switching R_2 and R_3 and switching R_4 and R_5.

There are four subgroups of order 3 in $\mathrm{Aut}(X)$:

$$\langle g_1 \iota_{12} \rangle\,, \qquad \langle g_1 \iota_{13} \rangle\,, \qquad \langle g_1 \iota_{23} \rangle\,, \qquad \langle g_1 \rangle\,,$$

but they are all conjugate. We may assume without loss of generality that g_1 is in G.

Let K be the kernel of the homomorphism $G \to \mathfrak{S}_3$, and let $\bar{G}$ be the image of this homomorphism. We enumerate all the possible subgroups K of rank at most 2 which are invariant under g_1:

$$\langle \iota_4 \rangle\,, \qquad \langle \iota_5 \rangle\,, \qquad \langle \iota_{45} \rangle\,, \qquad \langle \iota_4, \iota_5 \rangle\,, \qquad \langle \iota_{12}, \iota_{23} \rangle\,.$$

Note that $\langle \iota_{12}, \iota_{23} \rangle$ does not fix a point and can be eliminated. The remaining groups are fixed pointwise by g_1. Thus, if $\bar{G}$ is cyclic of order 3, then G is abelian and can be eliminated. It remains to consider $\bar{G} \simeq S_3$. In this case, only the subgroups $\langle \iota_{45} \rangle$ and $\langle \iota_4, \iota_5 \rangle$ are invariant under g_2; so these are the only possibilities for K.

Consider the set $\Gamma \subset 2^4$ of all elements ι_A such that $g_2 \iota_A$ is in G. Note that $g_3 g_2 \iota_A g_3 = g_2 \iota_{(132)A}$ and $(g_2 \iota_A)^{-1} = g_2 \iota_{(12)(45)A}$ are also in G. Also note that $(g_2 \iota_A)^{-1}(g_2 \iota_B) = \iota_A \iota_B$. Thus Γ is an $\mathfrak{S}_3$-invariant set such that the product of any two elements in Γ is in K. We conclude that Γ contains only id, ι_4, ι_5 and ι_{45}.

Thus the only possibilities for G are

$$\langle g_2, g_3, \iota_{45} \rangle \cong 2 \times \mathfrak{S}_3,$$
$$\langle g_2, g_3, \iota_4 \rangle \cong 2^2 : \mathfrak{S}_3,$$
$$\langle g_2 \iota_4, g_3 \rangle \cong 3 : 4.$$

All of these leave fixed the point $(1 : 1 : 1 : 0 : 0)$. Appealing to Proposition 2.1, we see that $2 \times \mathfrak{S}_3$ is not minimal while the other two groups are minimal. $\qquad\square$

REMARK 7.3. As was first noticed by Yuri Prokhorov (see [Pro15]), the groups $3 : 4$ and $2^2 : \mathfrak{S}_3$ above were missing from the classification in [DI09]. We found additional missing groups isomorphic to $2 \times D_8$, M_{16}, $2^3 : \mathfrak{S}_3$ and $L_{16} : 3$, as well as a second copy of L_{16} which is not conjugate to an existing group in the list. In addition, the group of order 32 identified as $2^2 : 8$ should instead be $2^3 : 4$. Here L_{16} and M_{16} are certain groups of order 16 whose structure is described in [DI09, Table 3]. One finds the corrected statements and the corrected proofs in an updated version of the paper at http://www.math.lsa.umich.edu/~idolga/papers.html.

8. Del Pezzo surfaces of degree 3

Recall that a del Pezzo surface of degree 3 is a smooth cubic surface in $\mathbb{P}^3$. Here we prove the following.

THEOREM 8.1 (Case $\boxed{3}$). *Suppose that G is a non-cyclic group and X is a minimal cubic G-surface with a fixed point p. Then X is equivariantly projectively equivalent to the surface in $\mathbb{P}^3$ defined by*

$$F = t_0^3 + t_1^3 + t_2^3 + t_4^3 + t_0 t_1 (a t_2 + b t_3),$$

with fixed point $p = (0 : 0 : 1 : -1)$, where a and b are parameters. The tangent plane at p contains three Eckardt points. The different possibilities are given in the following table:

Name	Possible G	Parameters	Surface type from [DI09]
$\boxed{3.1}$	$\mathfrak{S}_3$		I–VI, VIII, V
$\boxed{3.2}$	$\mathfrak{S}_3 \times 2$	$a = b$	I, II, VI
$\boxed{3.3}$	$\mathfrak{S}_3 \times 6$, $\mathfrak{S}_3 \times 3$ (twice), 6×3, 3×3	$a = b = 0$	I

The possibilities have specializations

$$\boxed{3.1} \longrightarrow \boxed{3.2} \longrightarrow \boxed{3.3}.$$

Note that we do not list those G which already occur in generizations.

Proof. We begin by considering the case $\boxed{3.3}$. Here X is the Fermat cubic surface

$$X\colon\ t_0^3 + t_1^3 + t_2^3 + t_3^3 = 0\,.$$

The automorphism group of X is $3^3 : \mathfrak{S}_4$, of order 648 (see [DI09] or [Dol12]). The surface X has 18 Eckardt points; one of these is $p = (0 : 0 : 1 : -1)$, and all the others are obtained from p by automorphisms. The stabilizer $\mathrm{Aut}(X,p)$ is isomorphic to $\mathfrak{S}_3 \times 6$, of order 36. We will show that every case specializes to this case.

Now, let X be general as in the theorem. Since G is minimal, the cardinality of any orbit on the 27 lines must be divisible by 3 (if the sum of k lines is linearly equivalent to mK_X, then $k = 3m$). Thus, G has order divisible by 3.

Let g be an element of order 3 in G. From [Dol12, Table 9.5], up to projective equivalence, we have three different options for the action of g on $\mathbb{P}^3$:

$$(3A)\colon\ g(t_0 : t_1 : t_2 : t_3) = (\epsilon_3 t_0 : t_1 : t_2 : t_3)\,,$$
$$(3C)\colon\ g(t_0 : t_1 : t_2 : t_3) = (\epsilon_3 t_0 : \epsilon_3 t_1 : t_2 : t_3)\,,$$
$$(3D)\colon\ g(t_0 : t_1 : t_2 : t_3) = (\epsilon_3 t_0 : \epsilon_3^2 t_1 : t_2 : t_3)\,,$$

where ϵ_3 is a primitive third root of unity.

If we assume that g is of class $(3C)$, then X must be the Fermat cubic (see [Dol12, Section 9.5.1]). The fixed points are all Eckardt points and so we are in case $\boxed{3.3}$.

Now, we assume that g is of class $(3D)$. As in [Dol12, Section 9.5.1], up to a projective change of coordinates, the surface X is one of the surfaces stated in the theorem. Set $\ell_1 : t_2 = t_3 = 0$ and $\ell_2 : t_0 = t_1 = 0$. Note that ℓ_1 and ℓ_2 are canonically defined given g. The three points $\ell_2 \cap X$ are the only fixed points of g on X; they are of the form $(0 : 0 : 1 : -a)$ where $a^3 = 1$. Without loss of generality we may take $p = (0 : 0 : 1 : -1)$.

There is always an involution σ which interchanges t_0 and t_1. Thus, there is an action of $\mathfrak{S}_3$ on X that fixes p. The line ℓ_1 is stable under $\mathfrak{S}_3$ and the three points $X \cap \ell_1$ are all Eckardt points by [Dol12, Proposition 9.1.27].

Consider the tangent space $T_p X \subset \mathbb{P}^3$. One checks that $T_p X$ contains ℓ_1. The intersection $C = T_p X \cap X$ is either a nodal cubic or three concurrent lines. There is a faithful action of G on $T_p X$ which must leave C invariant.

In the case where C is a nodal cubic, we see that $G \subset \mathbb{G}_m : 2$. Since G contains an element of order 3 and the three points in $\ell_1 \cap C$ must be G-invariant, we see that G is isomorphic to $\mathfrak{S}_3$ and we are in case $\boxed{3.1}$.

When C is three concurrent lines, the point p is an Eckardt point and we have $\mathfrak{S}_3 \times 2 \subset \mathrm{Aut}(X,p)$ by [Dol12, Proposition 9.1.26]. The automorphism group of three concurrent lines in $\mathbb{P}^2$ is $\mathbb{G}_m \times \mathfrak{S}_3$. The polar P of p in X is a union of two planes, the tangent plane $t_2 + t_3 = 0$ and the plane $t_2 - t_3 = 0$. Since both these planes and the line ℓ_1 must be G-invariant, the only other automorphisms fixing p must be of the form $(t_0 : t_1 : t_2 : t_3) \mapsto (t_0 : t_1 : \lambda t_2 : \lambda t_3)$, where λ is in $\mathbb{C}^\times$. This λ can be non-trivial only in the case where X is the Fermat cubic (and we are then in case $\boxed{3.3}$). If λ is forced to be trivial, then we are in case $\boxed{3.2}$.

Finally, if we assume that g is of class $(3A)$, then X is the cyclic cubic surface

$$X\colon\ t_0^3 + F(t_1, t_2, t_3) = 0\,.$$

It is a triple cover of $\mathbb{P}^2$ ramified at the smooth genus 1 curve cut out by the plane $t_0 = 0$. The Hessian quartic surface is a union of the plane $t_0 = 0$ and a cone over the Hessian cubic curve H

associated with $\mathbb{P}^2$. If X has an additional cyclic structure, then the curve H is a union of three concurrent lines and X must be isomorphic to the Fermat cubic (see [Dol12, Lemma 3.2.4]). Since the Fermat cubic was already considered above, we may assume that the cyclic structure is unique, and thus G leaves invariant a genus 1 curve containing the fixed point p. This means that G is a central extension of a cyclic group H by 3. The group G is thus cyclic unless H has order divisible by 3. In this latter case, G contains a subgroup isomorphic to C_3^2. This means that G must contain an element of order 3 whose class is not of the form $(3A)$ and thus was already discussed above.

It remains to determine which subgroups G of $\mathrm{Aut}(X, p)$ are minimal. It suffices to consider only $\boxed{3.3}$ since the others are generizations of this case. First, we list all non-cyclic subgroups of $\mathfrak{S}_3 \times 6$ up to conjugacy:

$$2^2, \quad \mathfrak{S}_3 \text{ (twice)}, \quad 3^2, \quad 6 \times 2, \quad \mathfrak{S}_3 \times 2, \quad 6 \times 3, \quad \mathfrak{S}_3 \times 3 \text{ (twice)}, \quad \mathfrak{S}_3 \times 6.$$

We compute the traces on the space $\mathcal{R}_X$ using Proposition 2.1 as in [Dol12, Table 9.4], where ϵ is a primitive third root of unity:

Eigenvalues on $\mathbb{P}^4$				$\mathrm{Tr}(\cdot|\mathcal{R}_X)$
1	1	1	1	6
1	1	1	-1	-2
1	1	-1	-1	2
1	1	1	ϵ	-3
1	1	ϵ	ϵ	3
1	1	ϵ	ϵ^2	0
1	1	ϵ	$-\epsilon$	1
1	-1	ϵ	ϵ	1
1	-1	ϵ	$-\epsilon$	-1
1	-1	ϵ	ϵ^2	-2

Note that the subgroup generated by an element with eigenvalues $1, 1, 1, \epsilon$ give traces which sum to 0, thus any group containing this element is minimal. Thus 3×3, 6×3, both classes of $\mathfrak{S}_3 \times 3$, and $\mathfrak{S}_3 \times 6$ are minimal groups. We remark that the eigenvalues of the involutions in the two different classes of $\mathfrak{S}_3 \times 3$ are different; thus the two conjugacy classes are distinct in $\mathrm{Cr}(2)$.

Additionally, the group $\mathfrak{S}_3$ generated by elements with eigenvalues $1, 1, \epsilon, \epsilon^2$ and $1, 1, 1, -1$ gives traces which sum to 0. Thus $\mathfrak{S}_3$ and $\mathfrak{S}_3 \times 2$ are minimal. It remains to establish that 6×2, 2×2 and the other conjugacy class of $\mathfrak{S}_3$ are not minimal. The group 6×2 and the group $\mathfrak{S}_3$ both have traces which sum to 12, so neither is minimal; the group 2^2 is a subgroup of 6×2 and thus is not minimal. $\qquad\square$

9. Del Pezzo surfaces of degree 2

Throughout this section, G is a non-cyclic finite group, X is a minimal del Pezzo G-surface of degree 2, and p is a G-fixed point on X.

We recall some features of such surfaces from [DI09, Section 6.6]. Any such surface has an involution γ called the *Geiser involution*. Its set of fixed points is a smooth curve R of genus 3. The quotient by γ induces a degree 2 map

$$\pi \colon X \to \mathbb{P}^2$$

with branch locus $B \cong R$ a smooth quartic curve.

We may write X as

$$F(t_0, t_1, t_2) + t_3^2 = 0$$

in the weighted projective space $\mathbb{P}(1, 1, 1, 2)$, where F is the degree 4 form which defines B in $\mathbb{P}^2$. The Geiser involution is simply the map which takes t_3 to $-t_3$. We have a decomposition $\operatorname{Aut}(X) \cong \operatorname{Aut}(B) \times \langle \gamma \rangle$. Note that $\operatorname{Aut}(B)$ is a finite subgroup of $\operatorname{PGL}(3)$ since $F = 0$ is the canonical embedding of B. The possible $\operatorname{Aut}(B)$ can be found in [Dol12, Theorem 6.5.2].

THEOREM 9.1 (Case $\boxed{\text{2A}}$). *If p lies on the ramification curve R, then the group $\operatorname{Aut}(X, p)$ is abelian of the form $H \times \langle \gamma \rangle$, where H is a cyclic subgroup of $\operatorname{Aut}(B)$. We have the following possibilities:*

Name	Possible G	Surface type from [DI09]
$\boxed{\text{2A.1}}$	2^2	*I–V, VII–X, XII*
$\boxed{\text{2A.2}}$	6×2	*III, VIII*
$\boxed{\text{2A.3}}$	4×2	*II–III, V*
$\boxed{\text{2A.4}}$	12×2	*III*
$\boxed{\text{2A.5}}$	8×2	*II*

These possibilities satisfy the specializations
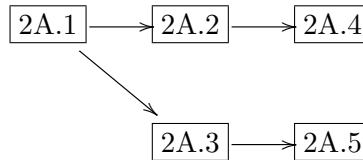
Note that we do not list those G which already occur in generizations.

Proof. Since p lies on R, the group $\operatorname{Aut}(X, p)$ contains γ. It remains only to classify the possible subgroups H. Since H acts faithfully on the tangent space to R at p, we see that H is cyclic. Since G is not cyclic, $\operatorname{Aut}(X, p)$ is not cyclic. Thus, the possible H are precisely the maximal cyclic subgroups of $\operatorname{Aut}(B)$ of even order which fix a point on B. From [Dol12, Lemma 6.5.1], we obtain the cases $\boxed{\text{2A.1}}$–$\boxed{\text{2A.5}}$ above. Minimality follows since the Geiser involution alone is minimal. $\square$

Now, suppose that G does not fix any points on the ramification curve R. Then p is not fixed by γ and we may assume that G is an isomorphic lift of a subgroup $\bar{G}$ of $\operatorname{Aut}(B)$ fixing a point $q = \pi(p)$ in $\mathbb{P}^2$ not lying on B.

A del Pezzo surface has 56 exceptional curves (lines) E_i on which G acts. Any orbit of G on the lines consists of k lines whose sum is linearly equivalent to a multiple of K_X. Since $K_X^2 = 2$, this implies that k is even. Thus the order of G is even and G contains an involution $\tilde{\tau}$, a lift of an involution τ of $\mathbb{P}^2$ that leaves B invariant. The set $(\mathbb{P}^2)^\tau$ of fixed points of τ is equal to $\{q\}$ plus a line L that intersects B at four fixed points (counted with multiplicities). The set of fixed points of $\tilde{\tau}$ is the set containing the two points p and $\gamma(p)$ along with a genus 1 curve $\pi^{-1}(L)$.

We claim that q is the intersection point of four bitangents. Choose the projective coordinates (t_0, t_1, t_2) in $\mathbb{P}^2$ such that $q = (0 : 0 : 1)$ and $L : t_2 = 0$. Then the equation of B has the form

$$t_2^4 + 2f_2(t_0, t_1)t_2^2 + f_4(t_0, t_1) = 0, \tag{9.1}$$

where the involution τ acts by $(t_0 : t_1 : t_2) \mapsto (t_0 : t_1 : -t_2)$ and f_2 and f_4 are homogeneous polynomials of degree 2 and 4, respectively. Note that we can rewrite the equation in the form

$$\left(t_2^2 + f_2(t_0, t_1)\right)^2 + \left(f_4(t_0, t_1) - f_2(t_0, t_1)^2\right) = 0.$$

This shows that each line $bt_0 - at_1 = 0$, where $(f_4(a, b) - f_2(a, b)^2) = 0$, is a bitangent of B passing through the point q. Thus q is the intersection point of four bitangents of B, as claimed.

The converse was first proven by Sonya Kowalevski [Kow84]. Although we do not use this result, we do give a proof.

PROPOSITION 9.2. *Suppose that a smooth plane quartic curve B has four bitangents meeting at a point q. Then the exists a projective involution τ of $\mathbb{P}^2$ that leaves B invariant and has the point $q \in B$ as an isolated fixed point.*

Proof. By [Dol12, Proposition 6.1.4], any three of the bitangent lines form a syzygetic triad of bitangents; that is, the corresponding six tangency points lie on a conic. This implies that all eight tangency points lie on a conic. Choose coordinates such that $q = (0 : 0 : 1)$. Let $\ell_i : l_i = a_i t_0 + b_i t_1 = 0$, and let $C_2(t_0, t_1, t_2) = 0$ be the equation of the conic K passing through the eight tangency points. Then the polynomials C_2^2 and $l_1 \cdots l_4$ define the same divisor on B, hence the equation of B can be written in the form $F = C_2^2 + l_1 l_2 l_3 l_4 = 0$. Let $C_2 = t_2^2 + 2t_2 l(t_0, t_1) + q(t_0, t_1) = (t_2 + l(t_0, t_1))^2 + q(t_0, t_1) - l(t_0, t_1)^2 = 0$. After we again change the coordinates $t_2 \mapsto t_2 + l(t_0, t_1)$, the equation of B is reduced to the form (9.1). The involution $(t_0 : t_1 : t_2) \mapsto (t_0 : t_1 : -t_2)$ is the projective involution of B. □

The involution τ has four fixed points $(a : b : 1)$ on B, where $f_4(a, b) = 0$, and the quotient $E = B/(\tau)$ is a genus 1 curve with equation

$$z^2 + 2z f_2(x, y) + f_4(x, y) = 0$$

in the weighted projective space $\mathbb{P}(1, 1, 2)$.

LEMMA 9.3. *The involution τ of B belongs to the center of the group $\bar{G}$.*

Proof. For any $\sigma \in \bar{G}$, the element $\tau' = \sigma \tau \sigma^{-1}$ fixes q and leaves invariant the set of the bitangents of B that contain q. Thus it leaves invariant the pencil of lines through q. This shows that τ' is an involution of $\mathbb{P}^2$ with the same isolated fixed point as τ. Thus τ and τ' must coincide. □

Since the polynomial f_4 has four distinct roots, we can choose projective coordinates t_0, t_1, t_2 in the plane such that

$$f_2(t_0, t_1) = at_0^2 + bt_0 t_1 + ct_1^2, \qquad f_4(t_0, t_1) = t_0^4 + dt_0^2 t_1^2 + t_1^4.$$

The only condition on the coefficients here is $d^2 \neq 4$, expressing the fact that f_4 has no multiple roots, or, equivalently, that the curve B is non-singular.

We may assume that $\bar{G}$ acts via its lift to $\mathrm{GL}(3)$ as the group of matrices of the form $\begin{pmatrix} \alpha & \beta & 0 \\ \gamma & \delta & 0 \\ 0 & 0 & 1 \end{pmatrix}$. Thus the group $\bar{G}$ is naturally identified with a subgroup of $\mathrm{GL}(2)$. The transformation τ is defined by the matrix $-I_2$.

We want to find a list of the maximal non-cyclic subgroups of $\mathrm{GL}(2)$, up to conjugacy, that leave f_2 and f_4 invariant. The automorphism τ is always present. Let H be the automorphism group of $f_4 = 0$ viewed as a set of four points in $\mathbb{P}^1$. Let K be the image of G in $\mathrm{PGL}(2)$; note that $K \subset H$. Consulting [DI09, Section 5.5], we see that for f_4 in the coordinates above, H is

either 2^2 for general d, or $\mathfrak{A}_4$ for $d^2 = -12$, or D_8 for $d = 0$. If $f_2 = 0$, then the kernel of $G \to K$ is cyclic of order 4 and the possible maximal G are, respectively, 4.2^2, $4.\mathfrak{A}_4$ and $4.D_8$.

Suppose $f_2 \neq 0$. The kernel of $G \to K$ is precisely $\langle \tau \rangle$. Since K must leave a pair of points invariant, it is isomorphic to one of 1, 2, 3, 4 or 2^2. We may discount 1 and 3 since we consider non-cyclic G. Since G must be a subgroup of 4.2^2 or $4.D_8$, all of its elements act by scaling t_0 and t_1 while possibly interchanging them. Thus, all the remaining possibilities arise when a, b or c is zero, or when $a = c$.

Note that if $a = c$, then we may instead assume $b = 0$ via the the linear change of variables

$$(t_0, t_1) \mapsto \big(\delta(t_0 - t_1), \delta(t_0 + t_1) \big)$$

for some δ satisfying $\delta^4 = (2 + d)^{-1}$. Accounting also for the symmetry between a and c, we enumerate the possibilities in Table 2:

f_4	f_2	Maximal G
any d	$a - c, a, c \neq 0, b = 0$	2^2
any d	$a = c \neq 0, b = 0$	D_8
$d = 0$	$a = b = 0, c \neq 0$	2×4
$d \neq 0, d^2 \neq -12$	0	4.2^2
$d^2 = -12$	0	$4.\mathfrak{A}_4$
$d = 0$	0	$4.D_8 \cong 4^2 : 2$

TABLE 2. Maximal non-cyclic subgroups G of $\mathrm{GL}(2)$ leaving f_2 and f_4 invariant

Our group G is a minimal isomorphic lift of a subgroup $\bar{G}$ of $\mathrm{Aut}(B)$ as above. Following [DI09, Section 6.6], we say a lift is *even* if the group G in its representation in $W(E_7)$ is contained in the normal subgroup $W(E_7)^+$ of index 2, and a lift is *odd* otherwise.

REMARK 9.4. The classification of minimal groups of automorphisms of degree 2 del Pezzo surfaces from [DI09] has the following errors:

(i) $\langle \gamma \rangle$ is missing from all types (except XII).

(ii) Type XIII is missing completely.

(iii) $2^2 \times \langle \gamma \rangle$ was omitted in Type I surfaces.

(iv) An even lift of Q_8 was omitted in Types II, III and V.

(v) $2 \cdot \mathfrak{A}_4 \cong Q_8 : 3$ in Type III (not $D_8 : 3$).

(vi) $\mathfrak{A}_4 \times \langle \gamma \rangle$ was omitted in Type IV.

(vii) $C_3 \times \langle \gamma \rangle$ was omitted in Type III.

One finds the corrected statements and the corrected proofs in an updated version of the paper at http://www.math.lsa.umich.edu/~idolga/papers.html.

THEOREM 9.5 (Case $\boxed{2\mathrm{B}}$). *Let G be a minimal group with a fixed point p that is not in the ramification curve R. Then B is isomorphic to the plane quartic curve*
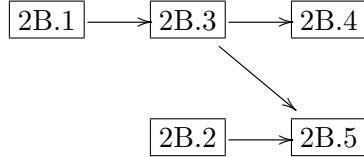
$$F = t_2^4 + t_2^2 \big(a t_0^2 + c t_1^2 \big) + t_0^4 + d t_0^2 t_1^2 + t_1^4 = 0$$

and the fixed point is a lift of $(0 : 0 : 1)$.

We have the following cases:

Name	Possible G	Parameters	Surface type from [DI09]
2B.1	D_8	$a = c \neq 0$	I–V, VII
2B.2	2×4	$a = d = 0$	II, III, V
2B.3	$4 \cdot 2^2 \cong 2.D_8$, Q_8	$a = c = 0$	II, III, V
2B.4	$4 \cdot \mathfrak{A}_4$, $2 \cdot \mathfrak{A}_4$	$a = c = 0$, $d^2 = -12$	III
2B.5	$4 \cdot D_8$, 4×4	$a = c = d = 0$	II

These cases satisfy the specializations

Note that we do not list those G which already occur in generizations.

Proof. We may assume that the fixed point is $(0 : 0 : 1)$ and that G acts as a subgroup of $\mathrm{GL}(2)$ on the coordinates (t_0, t_1). The maximal G and the appropriate parameters can be obtained from Table 2. It remains only to determine which subgroups make X minimal. Our main tools are Proposition 2.1 and the classification in [DI09, Table 7] (which was derived using the same method).

Observe that any involution in $\mathrm{PGL}(3)$ fixes an isolated point and a line. One lift to $\mathrm{Aut}(X)$ fixes points only on R and thus is excluded. The other fixes a pair of points and a genus 1 curve, thus every involution has trace -1 on $\mathcal{R}_X$. In particular, $G \cong 2^2$ has sum of traces equal to 4 and cannot be minimal.

An element of order 4 in G with eigenvalues i and $-i$ in $\mathrm{GL}(2)$ fixes two points on X and thus has trace -1 on $\mathcal{R}_X$. From this we conclude that both D_8 and Q_8 are minimal groups. In particular, the group D_8 in 2B.1 is minimal.

The element of order 4 with a generator having eigenvalues 1 and i in $\mathrm{GL}(2)$ fixes a genus 1 curve on X and thus has trace -3. The group it generates is minimal. Thus the group 2×4 from 2B.2 is minimal.

Now we refer to [DI09, Table 7]. The cases 2B.1 and 2B.2 are finished. For 2B.3 , we note that Q_8 appears and that $4 \cdot 2^2$ is minimal since it contains D_8. For 2B.4 , the groups $2 \cdot \mathfrak{A}_4$ and $4 \cdot \mathfrak{A}_4$ contain Q_8 and are therefore minimal.

In the case of 2B.5 , we only need to consider subgroups of a Sylow 2-subgroup of $\mathrm{Aut}(X)$ isomorphic to $4 \cdot D_8$. The group 4^2 contains 2×4 and is thus minimal. The group $4 \cdot D_8$ is similarly minimal. It remains to show that the even lift of M_{16} does not fix p. We will do this by showing that the cyclic subgroup of order 8 within is an odd lift.

An automorphism of order 8 of B acts by $(t_0 : t_1 : t_2) \mapsto \left(\epsilon_8^3 t_0 : \epsilon_8^{-1} t_1 : t_2 \right)$ in coordinates where B is given by the equation $t_2^4 + t_0 t_1 \left(t_0^2 + t_1^2 \right) = 0$. It has three fixed points in $\mathbb{P}^2$, two of which are on B. Thus, its lift must have four fixed points. The trace of an even lift of g is equal to -1 and has two fixed points. We conclude that an element of order 8 in G is an odd lift. Thus the subgroup isomorphic to M_{16} in 2B.5 is not minimal. $\square$

10. Del Pezzo surfaces of degree 1

THEOREM 10.1 (Case $\boxed{1}$). *Let X be a minimal del Pezzo G-surface of degree 1. Then G has a fixed point.*

This is immediate since the unique base point of $|-K_X|$ is canonical and thus must be fixed by any automorphism of X. A list of all the minimal groups in this case can be found in [DI09, Section 6.7].

11. Conic bundles

Throughout this section, $\pi\colon X \to B$ is a minimal conic G-bundle with $B \cong \mathbb{P}^1$. Let G_K be the kernel of the action of G on B, and let G_B be the image. We have an exact sequence

$$1 \to G_K \to G \to G_B \to 1\,.$$

Let $\Sigma \subset B$ be the set of points whose pre-images under π are singular.

LEMMA 11.1. *The group G_K acts faithfully on each fiber.*

Proof. Suppose that g is a non-trivial element of G_K that acts identically on a fiber F. We know that g has two fixed points on each non-singular fiber. The closure of this set of points is a 1-dimensional component C of X^g that is of relative degree 2 over the base. More precisely, this curve is the closure of a divisor of degree 2 on the general fiber X_η that defines two fixed points of G on the geometric generic fiber. Since F and C intersect and X^g is smooth, we obtain a contradiction. $\square$

Let G_0 be the kernel of the action of G on $\mathrm{Pic}(X)$. We use the trichotomy of conic bundles as in [DI09]:

(i) $X \to B$ is a minimal rational ruled surface.

(ii) $X \to B$ is *non-exceptional*: $G_0 = 1$ and X is not a minimal ruled surface.

(iii) $X \to B$ is *exceptional*: $G_0 \neq 1$ and X is not a minimal ruled surface.

We will consider each case in turn.

We begin by considering the case of a minimal ruled surface. Recall that the case of $\mathbf{F}_0 \cong \mathbb{P}^1 \times \mathbb{P}^1$ was shown to be birationally equivalent to $\mathbb{P}^2$ with a G-fixed point in Section 4. In fact, this is true for all minimal rational ruled G-surfaces.

THEOREM 11.2. *Suppose that $X \cong \mathbf{F}_n$ is a ruled G-surface with a fixed point, where $n \geqslant 2$. Then G is abelian and X is birationally equivalent to $\mathbb{P}^2$ with a G-fixed point.*

Proof. We recall some facts from the proof of [DI09, Theorem 4.10]. Let S be the exceptional section. It is invariant with respect to the group of automorphisms of $\mathbf{F}_n$. The action of G on S is isomorphic to the action of G on the base of the projection $\pi\colon \mathbf{F}_n \to \mathbb{P}^1$. Since G has a fixed point on X, its projection is fixed, hence G acts on $\mathbb{P}^1$ with two fixed points, and therefore G has two fixed points on S. Since it has two fixed points on each invariant fiber, we obtain that G has four fixed points, two on S and two outside S.

We now show that X is birationally equivalent to $\mathbb{P}^2$ with a fixed point. Since G fixes a point p not on S, we may perform an elementary transformation at that point to obtain a ruled G-surface X' isomorphic to $\mathbf{F}_{n-1}$ which must also have a fixed point since G is abelian. By applying

this procedure inductively, we eventually find a birational equivalence to a G-surface isomorphic to $\mathbf{F}_1$. Blowing down the exceptional divisor, we have the desired result. $\quad\square$

THEOREM 11.3 (Case $\boxed{\text{C.ne}}$). *Let G be a non-cyclic finite group, and let X be a non-exceptional G-minimal conic bundle. Assume that G has a fixed point. Then we have one of the following cases, where $n = mq$ is a positive integer, m is a power of 2 and q is odd:*

(i) $G_K \cong 2$, $G_B \cong 2n$, $G \cong 2 \times 2n$.

(ii) $G_K \cong 2^2$, $G_B \cong n$, $G \cong 2 \times 2n$.

(iii) $G_K \cong 2^2$, $G_B \cong n$, $G \cong (2m : 2) \times q$.

Proof. Recall that in this case G_0 is trivial and X is not a minimal ruled surface. Here $G_K \cong 2^a$ with $a = 1$ or 2 [DI09, Theorem 5.7]. Since G has a fixed point, the group G_B is cyclic.

First, consider $G_K \cong 2$. Since we assume that G is not cyclic, $G \cong 2 \times 2n$ for some positive integer n.

Now, assume $G_K \cong 2^2$. The order of G_B is $n = mq$, where m is a power of 2 and q is a positive odd integer. Thus there is a homomorphism from G to a cyclic group of order m whose kernel is a 2-group. Since the quotient and kernel have coprime orders, the extension splits. It is known that the subgroup G_K is always minimal [DI09, Lemma 5.6]. The 2-group will also be minimal since it contains G_K. Thus, it suffices to assume that the order of G_B is of the form $n = m$.

Since G must embed into $\mathrm{GL}(2)$, we see that some element z in G_K must map to the matrix $-\,\mathrm{id}$. Let x be a non-trivial element of G_K not equal to z. Let g be a lift to G of a generator of G_B.

Since g must normalize G_K and z must be central, we see that

(1) either $gxg^{-1} = x$

(2) or $gxg^{-1} = xz$.

In case (1), the group G is abelian. Since G must have rank at most 2, we see that $G \simeq 2 \times 2m$.

In case (2), rearranging, we obtain $xgx^{-1} = gz$. Note that $xg^m x^{-1} = g^m$ since m is even. Thus, the group generated by $\langle x, z, g^m \rangle$ must be abelian of rank 2. Since $G_K = \langle x, z \rangle$, and x and g do not commute, we conclude that

(a) either $g^m = 1$

(b) or $g^m = z$.

In case (a), we conclude that g has order 2 and $G \simeq D_8$; otherwise, we would have a contradiction as the abelian group $\langle x, z, g^{m/2} \rangle$ would have rank 3. In case (b), we conclude that our group G is a semidirect product $2m : 2$, where the involution x acts by $g \mapsto g^{m+1}$. $\quad\square$

EXAMPLE 11.4. Let X be a del Pezzo surface of degree 4, and let G be a subgroup of automorphisms generated by two involutions of the first kind, say ι_1 and ι_2. The group has four fixed points $E_1 \cap E_2 = \{p_1, p_2, p_3, p_4\}$. Let $\sigma \colon X' \to X$ be the blow-up of p_1. From the description of the involutions in Section 7, we see that p_1 does not lie on any exceptional divisors. Thus, the surface X' is a del Pezzo surface of degree 3. In its anti-canonical model, it is a cubic surface.

The image of the exceptional curve E of σ is a line on X' invariant with respect to G. The pencil of planes through R has R as its fixed component, and the residual pencil is a pencil of conics invariant under G. It equips X' with a structure of a minimal conic bundle G-surface with a 2-section R.

Since $G \cong 2^2$ acts faithfully on the tangent space of X at p_1 and has two invariant tangent directions corresponding to E_1 and E_2 at p_1, we see that the involution of the second kind ι_{12} acts identically on R while the other two involutions act non-trivially. We have a 2 to 1 morphism $R \cong \mathbb{P}^1 \to B \cong \mathbb{P}^1$ which is equivariant with respect to a cyclic group of order 2; this forces the action on B to be trivial. Thus the group G_B is trivial and $G = G_K \cong 2^2$.

As G acts faithfully on each fiber by Lemma 11.1, we conclude that the set of fixed points coincides with the set of singular points of the fibers of σ. A conic bundle on a cubic surface has five singular fibers, so we have five fixed points. Alternatively, we note that there are four fixed points on X, but there are two fixed points on R; thus X' has five fixed points.

Note that in the plane model of X' as the blow-up of six points $x_1, \ldots, x_5, p_1$, the pencil of conics arises from the pencil of cubics through $x_1, \ldots, x_5$ and a double point at p_1. The singular fibers are the unions of the line $\ell_i = \overline{p_1, x_i}$ and the conic C_i through the points x_k for $k \neq i$ and p_1. Three such pairs (l_i, C_i) intersect at p_1 and p_i for $i = 2, 3, 4$, and the remaining two are tangent at p_1 with the tangent directions corresponding to the cubics defined by E_1 and E_2.

THEOREM 11.5 (Case $\boxed{\text{C.ex}}$). *Let G be a non-cyclic finite group, and let X be an exceptional G-minimal conic bundle with a fixed point p. Then G_K is a dihedral group or a cyclic group of even order, G_B is cyclic or trivial, and G is a subgroup of $D_{2m} \times n$ for some integers m and n. Furthermore, p is a singular point of a singular fiber of π.*

Proof. Here G_0 is non-trivial, but X is not a minimal ruled surface. Recall from [DI09, Section 5] that X has two disjoint sections S_0 and S_∞ that can be blown down to obtain a hypersurface

$$X' : \; H_{2g+2}(t_0, t_1) + t_2 t_3 = 0$$

in the weighted projective space $\mathbb{P}(1 : 1; g+1 : g+1)$ for g a positive integer. The map $\pi \colon X \to B$ is given by the morphism $(t_0 : t_1; t_2, t_3) \to (t_0 : t_1)$.

Since G_B is cyclic or trivial, by the proof of [DI09, Proposition 5.3] we see that G is a subgroup of $G_B \times N$, where G_B acts linearly on $(t_0 : t_1)$ and N is the subgroup $\mathbb{C}^\times : 2$ of $\mathrm{SL}_2(\mathbb{C})$ which preserves $t_2 t_3$. Note that if G is minimal, then there must exist an element in G_K which swaps t_2 and t_3 and thus has even order. Since G_K is a subgroup of even order of a dihedral group, it must be of the form given in the statement of the theorem.

Finally, we establish that G fixes a singular point of a singular fiber. The subgroup G_0 leaves invariant each singular fiber and each section S_0 and S_∞. Since there are at least three singular fibers, the sections S_0 and S_∞ have at least three points fixed by G_0. Thus the action of G_0 on S_0 and S_∞ is trivial. Since G_0 is a subgroup of G_K, by Lemma 11.1 it acts faithfully on each fiber F. In particular, it can only fix the points $F \cap S_0$ and $F \cap S_\infty$ on a non-singular fiber. Since an element of G_K swaps the two sections and $X^G \subset X^{G_0}$, we see that G can only fix the singular points of singular fibers. $\square$

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