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Enriques Surfaces I

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Preface

The book gives a contemporary account of the study of the class of projective algebraic surfaces known as Enriques surfaces. These surfaces were discovered more than 125 years ago in an attempt to extend the characterization of rational algebraic curves via the absence of regular (or holomorphic) differential 1-forms to the two-dimensional case.

The theory of differential forms on complex algebraic varieties of arbitrary dimension and their birational invariance was laid out in the works of Clebsch and Noether between 1870 and 1880. Further developments of these ideas and clarification of their geometric meaning were undertaken by the school of Italian algebraic geometers, who were probably the first to define one of the main goals of algebraic geometry, namely the classification of algebraic varieties up to birational equivalence. They also understood the significance of vector spaces of regular differential forms. One of the main achievements of their work was the classification of algebraic surfaces, mainly due to Castelnuovo and Enriques. Central results of this classification are achieved via the analysis of the canonical and pluri-canonical linear systems and the Albanese map. The main numerical invariants are q, p_g, and P_n, which are, by definition, the dimensions of the vector spaces of regular 1-forms, regular 2-forms, and regular n-pluri-canonical forms, respectively. A rational variety, that is, an algebraic variety birationally equivalent to projective space, has no nonzero regular forms, and the converse is true for algebraic curves. In 1894, Castelnuovo proved that the vanishing of q, p_g, and P_2 is sufficient for the rationality of an algebraic surface. In discussions with Enriques about whether the condition $P_2 = 0$ can be eliminated, each came up with an example that shows that it cannot be done. In the example of Enriques, one has $P_{2n} = 1$ and $P_{2n+1} = 0$ for all $n \geq 0$, and in the example of Castelnuovo, one has $P_n = [1 + \frac{n}{2}]$, that is, linear growth as n tends to infinity. Enriques mentions this example in a letter to Castelnuovo on July 22, 1894 [222, Letter 11], and he also mentions it in his 1896 paper [218, §39]. Castelnuovo's example is discussed in his 1896 paper [109]. In the later development of the classification of algebraic surfaces, these two examples occupy different places: Enriques' example is of Kodaira dimension 0 and shares this class with abelian surfaces, K3 surfaces, and hyperelliptic surfaces. On the other hand, Castelnuovo's example is a surface of Kodaira dimension 1. The Enriques construction has a birational model that is a non-normal surface of degree 6 in $\mathbb{P}^3$ that passes through the edges of the coordinate tetrahedron with multiplicity 2. It was dubbed an Enriques sextic surface and the notion of an Enriques surface as a smooth projective surface with $q = 0$ and $P_2 = 1$ occurs in Artin's thesis from 1960 [18], in Shafarevich's seminar in 1961–1963 [5], as well as in Kodaira's 1963 paper [401, part 3, p. 719].

In 1906, Enriques proved that every (general) surface with invariants $p_g = q = 0$ and $P_2 = 1$ is birationally equivalent to an Enriques sextic. He also gave other birational models of his surfaces, for example, as double planes branched along a certain curve of degree 8, an Enriques octic. A special case of the double plane construction was known to Enriques already in 1896 [222, Letter 302].

Still over the complex numbers, a minimal, smooth, and projective surface satisfies $p_g = q = 0$, $P_2 = 1$ if and only if its fundamental group is of order two and its universal cover is isomorphic to a K3 surface, which is characterized by being a minimal, smooth, and projective projective surface with invariants $q = 0$ and $p_g = P_2 = 1$. Enriques already understood this and proved that the pre-image of his sextic surface under the double cover of $\mathbb{P}^3$ branched along the union of four coordinate planes is birationally equivalent to a K3 surface [220]. This result leads to the modern definition of an Enriques surface as the quotient of a K3 surface by a fixed-point-free involution. This point of view suggests that the theory of Enriques surface may be understood as a part of the theory of K3 surfaces, which is widely discussed and used in the modern literature, see, for example, [43], [321], or [417]. However, most usage of K3 surfaces in the study of Enriques surfaces consists of applying transcendental methods related to the theory of periods of K3 surfaces, which has little to do with the fascinating intrinsic geometry of Enriques surfaces.

The classification of algebraic surfaces was extended to algebraically closed fields of positive characteristic in the work of Bombieri and Mumford [539], [78] and [77]. In particular, they gave a characteristic-free definition of Enriques surfaces. It turns out that Enriques surfaces in characteristic two live in a completely different and beautiful world that has many features that have no analogs in characteristic $\neq 2$. For example, the canonical double cover still exists but is a torsor under one of the three finite group schemes $\boldsymbol{\mu}_2$, $\mathbb{Z}/2\mathbb{Z}$, $\boldsymbol{\alpha}_2$ of order 2. Accordingly, this divides Enriques surfaces in characteristic two into three different classes, which are called classical, $\boldsymbol{\mu}_2$-surfaces (or singular surfaces), and $\boldsymbol{\alpha}_2$-surfaces (or supersingular surfaces). In the case where the canonical cover is inseparable, it is never a smooth surface, and in some cases, it is a rational surface, so it is not even birationally equivalent to a K3 surface. There are many good modern expositions of the theory of algebraic surfaces, and, in particular, Enriques surfaces over the complex numbers (see, for example [43]). Our priority is to provide the first complete as possible treatment of Enriques surfaces over fields of arbitrary characteristic. The price that we have to pay for this goal is reflected in the size of our book and also in requiring many more technical tools that we use. We collect all these needed tools in Chapter 0 and, in fact, more than we need in the hope that this may serve as a helpful reference for the study of algebraic surfaces over fields of arbitrary characteristic.

The authors have to admit that the initial goal of providing a complete exposition of the theory of Enriques surfaces over fields of arbitrary characteristic turned out to be too ambitious. Among the important topics that had to be left out are vector bundles on Enriques surfaces, derived categories of coherent sheaves on Enriques surfaces, arithmetic properties such as the (non-)existence of rational points on Enriques surfaces over number fields, as well as the theory of special subvarieties of the moduli spaces of algebraic curves that represent curves lying on Enriques surfaces.

Each chapter ends with a bibliographical note, where we tried our best to give credit to the original research discussed in the chapter.

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Contents

Introduction to Volume One

The first of the two volumes is a significant revision of the book *Enriques Surfaces I* published in 1989 by the two first-named authors [138]. Some of the material of this book has been moved to Volume II, and some of it has been eliminated because it became obsolete.

Many (too many!) typographical and mathematical errors of the first edition have been corrected in the new edition. Furthermore, a new chapter about the moduli spaces of Enriques surfaces and the Appendix written by S. Kondō has been added to Volume I. The addition of new material and updating of old material is the main reason for a substantial increase of the size of Volume I compared to the first edition.

We start with Chapter 0, collecting many technical tools we constantly use in the book. Compared to the first edition, we have replaced the section on double covers with three sections on finite group schemes, cyclic covers, inseparable morphisms, and vector fields on surfaces over fields of positive characteristic. More precisely, we first give an overview of the theory of group schemes and torsors under them, which then leads to the explicit description and construction of cyclic covers – separable and inseparable – in arbitrary characteristics. Then, we treat the correspondence between purely inseparable field extensions, derivations, purely inseparable morphisms, and p-closed vector fields. From there, we proceed to the arguably most important class of normal surface singularities, namely rational double points, where the emphasis is again on a characteristic-free approach. In the following three sections, we discuss nondegenerate surfaces in projective space of minimal and next to minimal degree. We then discuss (weak) del Pezzo surfaces of degrees 3 and 4 in detail, including a discussion of symmetroid surfaces. The sections about symmetroid cubic surfaces and symmetroid quartic del Pezzo surfaces have only changed slightly compared to the first edition; however, we correct the results about their automorphism groups. It follows a section on quadratic lattices, reflection groups, and root bases that was included in Chapter 1 of the first edition. A discussion of Picard schemes and Albanese varieties in arbitrary characteristic is followed by a lengthy new section, where we collect all needed facts about different cohomology theories (de Rham, ℓ-adic, crystalline) and some of their geometric implications used in the theory of algebraic surfaces over fields of arbitrary characteristic.

In Chapter 1, we introduce Enriques surfaces and discuss some of their more elementary properties. We start with an exposition of the classification of algebraic surfaces over fields of arbitrary characteristic, where we also define the main object of this book, namely Enriques surfaces. We proceed with a discussion of the Picard scheme and the Brauer group of an Enriques surface and then turn to its canonical double cover, which is isomorphic to a K3 surface in characteristic $p \neq 2$. Here, we pay special attention to characteristic 2, where the canonical cover no longer needs to be a smooth K3 surface. Compared to the first edition, we added some material about Brauer groups and new results about their K3-covers. We then compute all sorts of cohomological invariants and differential invariants of an Enriques surface. There follows a section on the Enriques lattice, that is, the Néron–Severi lattice of an Enriques surface, which was in Chapter 2 of the first edition. We end the chapter by discussing some classical examples and constructions of Enriques surfaces.

Most of Chapter 3 'The Geometry of the Enriques Lattice' from the first edition, has been moved to Chapter 6 of Volume II. Chapters 4 'Projective Models' of the first edition, has been now spilt into two chapters, namely 'Linear Systems on Enriques Surfaces' (Chapter 2 of Volume I) and 'Projective models of Enriques surfaces' (Chapter 3 of Volume I). These are organized as follows.

In Chapter 2, we first establish and discuss a vanishing theorem for Enriques surfaces, which holds in an arbitrary characteristic and is central for studying linear systems. Then, we discuss nef divisors, various cones of divisors, fundamental chambers, Weyl groups, and isotropic vectors. The latter leads to the analysis of indecomposable divisors of canonical type, genus one fibrations, and their degenerate fibers. In the next section, we discuss Enriques' Reducibility Lemma. It states that every effective divisor on an Enriques surface is linearly equivalent to a sum of smooth rational curves and genus one curves. Then, we introduce the Φ-function, which is a sort of positivity measure for linear systems on Enriques surfaces. Moreover, it is crucial for understanding the (rational) maps associated with linear systems. We then discuss the (higher) numerical connectedness of a divisor and its relation to the function Φ. Finally, we classify big and nef divisors with $\Phi \leq 2$, similar to the discussion in the first edition, although we omit some details. Here, we systematically use the Reider theorem that is now valid for Enriques surfaces in arbitrary characteristic.

In Chapter 3, we start with some general results about projective models of Enriques surfaces and their K3-covers, which lay out the scene for projective models of Enriques surfaces. Then, we discuss some cases of low degrees in greater detail. We start with hyperelliptic maps from Enriques surfaces, which are rational maps that are generically finite of degree two onto certain rational surfaces. The central part of Chapter 3 is occupied with bielliptic maps (formerly called superelliptic maps), which are degree two morphisms onto certain weak del Pezzo surfaces of degree 3 or 4, namely the symmetroid cubic surfaces or symmetroid quartic surfaces studied in Chapter 0. These maps are one of the main tools to study Enriques surfaces and are especially important in the analysis of their automorphism groups. This analysis also leads to Enriques's double plane construction and Horikawa's models. Compared to the first edition, we add a new section on linear systems of degree 4. It is used for the construction of Enriques surfaces as the quotient of a complete intersection of three quadrics in $\mathbb{P}^5$ by a group scheme of order 2 in all characteristics. We end with a section on birational models of Enriques surface of small degree, which includes Enriques' original model of degree 6 in $\mathbb{P}^3$, and which is substantially extended compared to the first edition. For example, we give more information about the Fano model of an Enriques surface as a surface of degree 10 in $\mathbb{P}^5$. Moreover, we discuss a new model of degree 18 in $\mathbb{P}^9$, which is due to Mukai.

The contents of Chapter 4 'Genus One Fibrations' expands the contents of Chapter 5 of the first edition. The new material includes the computation of the torsion invariant of a wild multiple fiber (due to Michele Raynaud), the Mordell–Weil lattice, a new geometric approach to Lang's (for elliptic vibrations) and Ito's (for quasi-elliptic fibrations) classification of extremal jacobian elliptic fibrations on rational surfaces, and the p-torsion part of the Tate–Shafarevich group of elliptic and quasi-elliptic fibrations. Here, we treat the theory of torsors of a jacobian quasi-elliptic fibrations separately, which reveals its close connection with the theory of integral models and torsors of wound unipotent algebraic groups.

Finally, in Chapter 5 'Moduli Spaces', the reader finds a discussion of moduli spaces of Enriques surfaces, marked, polarized and unpolarized. First, we introduce general moduli problems via stacks, and their coarse and fine moduli spaces. Next, we recall the theory of lattice polarized K3 surfaces and extend it to fields of arbitrary characteristic by introducing the corresponding stack of lattice polarized K3 surfaces. Over the complex numbers, we then construct coarse moduli spaces via the theory of periods of K3 surfaces and with analytic methods. After that, we discuss various types of moduli spaces of Enriques surfaces over the complex numbers. These are constructed via moduli spaces of their K3 covers equipped with a lattice polarization defined by the Enriques lattice (with quadratic form multiplied by 2). We also give applications to the structure of the automorphism groups of complex Enriques surfaces, which we later extend to fields of arbitrary characteristic in Volume II.

In Section 6 we introduce the notion of the Nikulin root invariant that describes the set of smooth rational curves on an Enriques surface. This leads to the moduli spaces for nodal Enriques surfaces with fixed Nikulin root invariant. In Sections 8 and 9, we discuss the moduli spaces of polarized Enriques surfaces and compute the number of non-isomorphic polarizations of small degree for general Enriques surfaces. After that, we discuss the birational properties of moduli spaces of polarized Enriques surface and include some previously unpublished results on the rationality of some of these spaces.

In Sections 7 and 8, we discuss compactifications of moduli spaces and give an interpretation of the boundaries via degenerations. In particular, we discuss Kulikov's results on degenerations of Enriques surfaces. In the final section, we discuss the deformation theory of Enriques surfaces, their formal deformation spaces, and the algebraic construction of moduli spaces over arbitrary base schemes.

Volume I ends with an appendix written by S. Kondō on Borcherds' theory of automorphic forms that allows one to describe explicit projective embeddings of some of the moduli spaces.

Chapter 0
Preliminaries

In this chapter, we collect many results with a special view towards algebraic geometry over algebraically closed fields of positive characteristic. This includes construction techniques of algebraic varieties using group schemes, finite covers, and purely inseparable morphisms. We discuss rational double point singularities, and after that, we turn to del Pezzo surfaces and two special subclasses, namely symmetroid surfaces. We discuss some general aspects of lattice theory and Picard schemes. We finally discuss various cohomology theories and give interpretations of cohomology groups of small degree. These results will be needed for the study of Enriques surfaces. However, since many of these results are scattered over the literature and sometimes hard to find, we also provide references and give more background than needed later in the book.

0.1 Group Schemes

In this section, we discuss group schemes. In particular, we discuss Hopf algebras, actions of group schemes and fixed loci, torsors, and quotients, give examples, and discuss some structure results. We pay special attention to the group schemes $\boldsymbol{\mu}_p$, α_p, $\mathbb{Z}/p\mathbb{Z}$, $\mathbb{G}_m$, and $\mathbb{G}_a$. On our way, we also discuss various Frobenius morphisms, Witt vectors, and a little bit of Dieudonné theory. We end the section by briefly treating formal group laws and perfect group schemes.

For simplicity, we will assume that all schemes in this section are separated and locally noetherian. Moreover, by a *variety* over a field $\Bbbk$ we will mean a geometrically integral and separated scheme of finite type over $\Bbbk$.

First, we introduce *group schemes* and refer to [245], [508], [541], or [733] for details and proofs. By definition, an S-group scheme G is a scheme over some fixed base scheme S such that the Yoneda functor $h_G : T \to G(T)$ from the category of S-schemes to the category of sets takes values in the subcategory of groups. Moreover, G is said to be *commutative* if it takes values in the subcategory of commutative groups. Equivalently, one can define an S-group scheme G by requiring that there are morphisms of S-schemes $\mu : G \times_S G \to G$ (multiplication), an automorphism $\iota : G \to G$ (inverse), and a section $e : S \to G$ (zero or unit section) that satisfy the usual axioms of a group, which is expressed in the commutativity of some diagrams. Moreover, an S-group scheme G is said to be *affine* (resp. *finite*, *flat*, *étale*, *proper*, *separated*, *smooth*,...) if the structure morphism $G \to S$ has this property. Morphisms and isomorphisms of group schemes and group actions are defined in the obvious way.

Given an abstract group G, we define the *constant group scheme* associated to G to be the group scheme $\underline{G} \to \operatorname{Spec} \mathbb{Z}$ that is represented by the functor $T \to G^{\pi_0(T)}$ for all $T \to \operatorname{Spec} \mathbb{Z}$. This is an étale group scheme over $\operatorname{Spec} \mathbb{Z}$ and it is finite (resp. commutative) if and only if G is finite (resp. commutative) as an abstract group. Moreover, if G is finite, then the structure morphism $\underline{G} \to \operatorname{Spec} \mathbb{Z}$ is a finite and flat morphism of *length* equal to the order of G. Also, if G is finite, then $\underline{G}$ is an example of an affine group scheme over $\operatorname{Spec} \mathbb{Z}$ (finite morphisms are affine). Given an abstract group G and a scheme S, we will also write by abuse of notation $G \to S$ for the flat group scheme $\underline{G}_S := \underline{G} \times_{\operatorname{Spec} \mathbb{Z}} S \to S$. The category of finite and étale group schemes over an algebraically closed field is equivalent to the category of finite groups:

Theorem 0.1.1 *Let $\Bbbk$ be an algebraically closed field and let G be a finite and étale group scheme over $\Bbbk$. Then, G is isomorphic to a constant group scheme.*

Let us note two special cases, where G is a group scheme over $S = \operatorname{Spec} \Bbbk$, where $\Bbbk$ is a field. If G is smooth, proper, and geometrically connected over $\Bbbk$, then the group scheme is automatically commutative (see [541], Chapter II.4) and in this case, G is called an *abelian variety* over $\Bbbk$. Moreover, one-dimensional abelian varieties are called *elliptic curves* and two-dimensional abelian varieties are called *abelian surfaces*. On the other hand, if G is affine and of finite type over $\Bbbk$, then G is isomorphic to a closed subgroup scheme of the *general linear group* scheme GL_n for some n (see Example 0.1.6 below for the special case $\mathbb{G}_m = \mathrm{GL}_1$). If G is affine and smooth over $\Bbbk$, then G is said to be a *linear algebraic group*.

Now, assume that G is an affine group scheme over some affine base scheme $S = \operatorname{Spec} R$. Then, $G = \operatorname{Spec} A$ for some R-algebra A and the property of being a group scheme is equivalent to A carrying the structure of a commutative *Hopf algebra* over R. This means that there exist R-algebra homomorphisms $\mu^\dagger : A \to A \otimes_R A$ (comultiplication), $\iota^\dagger : A \to A$ (coinverse or antipode), and $e^\dagger : A \to R$ (counit or augmentation) subject to the following axioms:

- the compositions $(\mu^\dagger \otimes \mathrm{id}_A) \circ \mu^\dagger : A \to A \otimes_R A \to A \otimes_R A \otimes_R A$ and $(\mathrm{id}_A \otimes \mu^\dagger) \circ \mu^\dagger : A \to A \otimes_R A \to A \otimes_R A \otimes_R A$ coincide,
- the composition $\mu^\dagger \circ (\iota^\dagger \otimes \mathrm{id}_A) : A \to A \otimes_R A \to A$ is equal to the composition $p \circ e^\dagger : A \to R \to A$, where $p : R \to A$ denotes the structure homomorphism of the R-algebra A, and
- the composition $(e^\dagger \otimes \mathrm{id}_A) \circ \mu^\dagger : A \to A \otimes_R A \to R \otimes_R A$ is equal to the map $a \mapsto 1 \otimes a$.

Moreover, the group scheme G is commutative if and only if the Hopf algebra is cocommutative. For examples, we refer to Example 0.1.5 and the ones thereafter, as well as to [733].

Before proceeding, let us briefly digress on Frobenius morphisms in positive characteristic: let X be a scheme of characteristic $p > 0$, that is, p is a prime number and $p\mathcal{O}_X = 0$. This is equivalent to saying that the natural structure morphism $X \to \operatorname{Spec}\mathbb{Z}$ factors over $\operatorname{Spec}\mathbb{F}_p \to \operatorname{Spec}\mathbb{Z}$. Then, the *absolute Frobenius morphism* is defined to be the morphism $\mathbf{F} = \mathbf{F}_{\mathrm{abs}} : X \to X$ of schemes that is the identity on the underlying topological spaces and where $\mathbf{F}^{\#} : \mathcal{O}_X \to \mathbf{F}_*\mathcal{O}_X$ is defined to be $s \mapsto s^p$ for all open subsets $U \subseteq X$ and all sections $s \in \mathcal{O}_X(U)$. Next, let $f : X \to S$ be a morphism of schemes of characteristic $p > 0$ and let $X^{(p)} := X \times_S S$, where the fiber product is taken with respect to $f : X \to S$ and $\mathbf{F} : S \to S$. Using the universal property of fiber products, we obtain the following commutative diagram

$$
\begin{tikzcd}
X \arrow[dr, "\mathbf{F}_{X/S}"] \arrow[drr, "\mathbf{F}"] \arrow[ddr, "f"'] & & \\
& X^{(p)} \arrow[r] \arrow[d, "f^{(p)}"] & X \arrow[d, "f"] \\
& S \arrow[r, "\mathbf{F}"] & S
\end{tikzcd}
$$

The induced morphism $\mathbf{F}_{X/S} : X \to X^{(p)}$ is called the *relative Frobenius morphism* over S or *S-linear Frobenius morphism*, which is a morphism of schemes over S. If no confusion is likely to arise, we will drop the subscript X/S from the notation $\mathbf{F}_{X/S}$ in the sequel. Let us illustrate these two Frobenius morphisms in the case where $S = \operatorname{Spec}\mathbb{k}$ for some field $\mathbb{k}$ of positive characteristic p and $X = \mathbb{A}^n_{\mathbb{k}} = \operatorname{Spec} R$ with $R = \mathbb{k}[x_1, ..., x_n]$: then, on the level of rings, the absolute Frobenius map of X is the ring homomorphism $R \to R$, $r \mapsto r^p$. On the other hand, the $\mathbb{k}$-linear Frobenius morphism corresponds to the ring homomorphism $R \to R$ that is the identity on $\mathbb{k}$ and that sends $x_i \mapsto x_i^p$ for all i. If X is a d-dimensional variety over a perfect field $\mathbb{k}$, then $\mathbf{F}_{X/\mathbb{k}} : X \to X^{(p)}$ is a finite morphism of degree p^d and the induced extension of function fields is $\mathbb{k}(X^{(p)}) = \mathbb{k}(X)^p \subseteq \mathbb{k}(X)$ (if L is a field of characteristic p, then we recall that the set $L^p := \{x^p, x \in L\}$ is a subfield of L). We also note that if $\mathbb{k}$ is the finite field $\mathbb{F}_q$ with $q = p^n$ elements, then sometimes, also the map $x \mapsto x^q$ is called the Frobenius. In particular, the name Frobenius morphism may refer to different morphisms and so, a little care is needed.

Coming back to group schemes, the relative Frobenius morphism gives rise to a homomorphism of group schemes: let G be an S-group scheme where the base scheme S is of characteristic $p > 0$. Then, we have the S-linear *Frobenius morphism*

$$\mathbf{F}_{G/S} \; : \; G \;\to\; G^{(p)}$$

and we note that some authors denote $G^{(p)}$ by $G^{(1)}$. Now, also $G^{(p)}$ naturally carries the structure of an S-group scheme and $\mathbf{F}_S$ is a homomorphism of S-group schemes.

If G is a group scheme over $S = \operatorname{Spec} \mathbb{F}_p$, then the S-linear Frobenius morphism coincides with the absolute Frobenius morphism and then, $G^{(p)}$ is isomorphic to G over S (note, however, that $\mathbf{F}_S$ need not be an isomorphism). More generally, if G is a group scheme over $\mathbb{F}_p$ and S is an arbitrary scheme of characteristic $p > 0$, then $G_S^{(p)} \cong G_S$ and we obtain the S-linear Frobenius morphism $G_S \to G_S$ by base change. For example, this applies to the group schemes $\mathbb{G}_{a,S}$, $\mathbb{G}_{m,S}$, $\boldsymbol{\alpha}_{p,S}$, and $\boldsymbol{\mu}_{p,S}$, which we will discuss below in some detail. On the other hand, $G^{(p)}$ is usually not isomorphic to G as the following example shows.

Remark 0.1.2 If E is an elliptic curve over a field $\mathbb{k}$ of characteristic $p > 0$, then the $\mathbb{k}$-linear Frobenius morphism $\mathbf{F}_{\mathbb{k}} : E \to E^{(p)}$ is a finite morphism of degree p between elliptic curves over $\mathbb{k}$. Their j-invariants satisfy $j(E^{(p)}) = j(E)^p$. In particular, if $j(E) \notin \mathbb{F}_p$, then $j(E) \neq j(E^{(p)})$, and then, E and $E^{(p)}$ are not isomorphic as elliptic curves over $\mathbb{k}$.

Next, let S be an arbitrary base scheme, let $X \to S$ be a morphism of schemes, and let G be an S-group scheme. Then, an *action* of G on X over S is a morphism of schemes over S,

$$a \ : \ G \times_S X \ \to \ X,$$

such that the morphisms $a \circ (\mu \times \mathrm{id}_X)$, $a \circ (\mathrm{id}_G \times a) : G \times_S G \times_S X \to G \times_S X \to X$ coincide, plus some axioms related to the inverse $G \to G$ and the identity section $S \to G$. Given a scheme T over S, $x \in X(T)$, and $g \in G(T)$, we denote by $g \cdot x$ the image of (g, x) under the map of sets $G(T) \times X(T) \to (G \times_S X)(T) \stackrel{a(T)}{\to} X(T)$. The axioms of the action for a imply that this defines an action of the group $G(T)$ on the set $X(T)$ in the classical sense. If all schemes are affine, say $S = \operatorname{Spec} R$, $G = \operatorname{Spec} A$, and $X = \operatorname{Spec} B$, then A is a Hopf algebra over R and the action a induces a coaction $a^\dagger : B \to B \otimes_R A$ such that the R-algebra homomorphisms $(\mathrm{id}_B \otimes \mu^\dagger) \circ a^\dagger, (a^\dagger \times \mathrm{id}_A) \circ a^\dagger : B \to B \otimes_R A \to B \otimes_R A \otimes_R A$ coincide.

Given an action $a : G \times_S X \to X$ of an S-group scheme G on a scheme X over S, we say that a closed subscheme $j : Y \hookrightarrow X$ of X is *G-invariant* with respect to the action a if the image of the morphism $a \circ (\mathrm{id}_G \times j) : G \times_S Y \hookrightarrow G \times_S X \to X$ lies in Y. In the case where all schemes are affine, say $G = \operatorname{Spec} A$, $X = \operatorname{Spec} B$, and $Y = V(I)$ for some ideal $I \subseteq B$, then, being a G-invariant subscheme translates into $a^\dagger(I) = I \otimes A$. Coming back to the general case, we say that $Y \subseteq X$ is scheme-theoretically *pointwise fixed* if the induced action $G \times_S Y \to Y$ is trivial. The largest closed subscheme X^G with this property is called the *fixed locus* of the G-action. We note that this subscheme is not necessarily reduced. It represents the functor that assigns to a S-scheme T the set of G-equivariant morphisms $T \to X$, where G acts as identity on T. In the affine situation, it is defined to be the smallest ideal $I \subseteq B$, such that $a^\dagger(i) = i \otimes 1$ for all $i \in I$. The action is called *fixed-point-free* if X^G is empty. The action is fixed-point-free if and only if the morphism

$$\Phi \ = \ (a, \mathrm{pr}_X) \ : \ G \times_S X \ \to \ X \times_S X \tag{0.1.1}$$

is a closed embedding of S-schemes. For example, the multiplication $\mu : G \times_S G \to G$, considered as an action of G on itself, is fixed-point-free. We refer to [235] for more about fixed–point schemes.

Turning back to the affine situation, let $S = \operatorname{Spec} R$ be an affine base scheme, let $G = \operatorname{Spec} A$ be an affine S-group scheme that acts on an affine S-scheme $X = \operatorname{Spec} B$ and assume that all morphisms are over S. Then, the subring

$$B^G \;:=\; \{b \in B \;:\; a^\dagger(b) = b \otimes 1\}$$

is called the *ring of invariants*, which is an R-subalgebra of B. If G is finite and flat over S, then we define $X/G := \operatorname{Spec} B^G$. This is a *geometric quotient* of X by G in the sense of Mumford [542]. If X is not necessarily affine, but G is still assumed to be finite and flat over S, and if we assume moreover that for every point $x \in X$, the set $a(\Phi^{-1}(x))$ is contained in an affine subset of X, then the geometric quotient of X by G also exists (see [159], Chapitre III, Théorème 3.2), and it is obtained by gluing together the affine quotients. For example, the quotient X/G exists if R is a field, G is finite and flat over $S = \operatorname{Spec} R$, and X is a quasi-projective over S. In this case, many properties of X are inherited by the quotient $Y = X/G$. For example, if X is normal, then so is Y.

From now on, we will assume that every S-group scheme G is flat and locally of finite type over S. The existence of the unit section $e : S \to G$ implies that G is faithfully flat, that is, the structure morphism $G \to S$ is surjective. Let us recall that the *flat topology* (resp. *étale topology*) S_{fl} (resp. $S_{\text{ét}}$) on the category of S-schemes is defined by covering families $\{U_i\}_{i\in I}$ of flat (resp. étale) morphisms that are locally finite type $\phi_i : U_i \to X$ such that $X = \cup_{i\in I}\phi_i(U_i)$, see also Section 0.10. For a group scheme G, we denote by $\tilde{G}$ the associated sheaf in the flat topology (or another Grothendieck topology if this is clear from the context) associated to the Yoneda functor h_G. If G is commutative, then sheaf $\tilde{G}$ on S_{fl} is abelian and then, the cohomology groups $H^i(S_{\mathrm{fl}}, \tilde{G})$ are defined for all $i \geq 0$ and we refer to Section 0.10 or [508]. We denote these cohomology groups by $H^i_{\mathrm{fl}}(S, G)$, $H^i_{\text{ét}}(S, G)$, or $H^i_{\mathrm{Zar}}(S, G)$ depending on the Grothendieck topology chosen. We remark that if G is not commutative, then $\check{H}^i(S_{\mathrm{fl}}, G)$ is at least defined for $i \in \{0, 1\}$ using limits over all covers in S_{fl} and Čech-cohomology.

If an S-group scheme G acts on a scheme $X \to S$, then X is a *torsor*, or, a *principal homogeneous space*, of G over S, or simply a G-*torsor* if one of the following two equivalent conditions is satisfied (we refer to [508], Chapter IV, Proposition 4.1 for details):

- the structure morphism $X \to S$ is faithfully flat and the morphism Φ from (0.1.1) is an isomorphism, or
- there exists a covering $\{U_i \to S\}_{i\in I}$ of S in the flat topology such that the base changes $G_i = G \times_S U_i \to U_i$ and $X_i = X \times_S U_i \to U_i$ are isomorphic as U_i-schemes together with the induced G_i-actions on G_i and X_i for all $i \in I$.

The second condition says that X is *locally trivial* in the flat topology. If we can find such a trivializing cover $\{U_i \to S\}_{i\in I}$ even in the étale or Zariski topoology of S, we say that the G-torsor X is a torsor in the étale or Zariski topology, respectively. Finally, a G-torsor X is called *(globally) trivial* if there exists a *trivialization*, that is, an isomorphism $X \to G$ of S-schemes that is compatible with the G-actions on both sides. Such a trivialization exists if and only if X admits a section $e' : S \to X$, in which case the required isomorphism is the composition $\Phi \circ (\mathrm{id}_G \times e') = G \times_S S \to G \times_S X \to X \times_S S = X$. The following result classifies torsors under commutative group schemes in terms of flat cohomology.

Theorem 0.1.3 *Let G be a commutative S-group scheme that is flat and locally of finite type over a noetherian and separated base scheme S.*

1. *We denote by* $\mathrm{PHS}_S(G)$ *the set of isomorphism classes of G-torsors over S. Then, there exists a natural and injective map*

$$c \;:\; \mathrm{PHS}_S(G) \;\to\; H^1_{\mathrm{fl}}(S, G).$$

In the following cases, this map is even bijective:

a. *G is affine over S, or*
b. *G is smooth and separated over S and* $\dim S \leq 1$*, or*
c. *G is smooth and proper over S with geometrically connected fibers.*

2. *If G is smooth and quasi-projective over S, then the canonical maps*

$$H^i_{\acute{e}t}(S, G) \;\to\; H^i_{\mathrm{fl}}(S, G)$$

are isomorphisms for all $i \geq 0$. In particular, every G-torsor is locally trivial in the étale topology.

Proof We start with Claim (1). Here, we only explain the map c and refer to [508], Chapter III.4 for a complete proof. Suppose that X is a G-torsor, and let $\{U_i\}_{i \in I}$ be a trivializing covering in the flat topology. Then, $X(U_i) \neq \emptyset$ and we can choose sections $e_i \in X(U_i)$. Let e_i^j be the restriction of e_i to $U_i \times_X U_j$. Then, the surjectivity of the map Φ from (0.1.1) implies that there exist $g_{ij} \in G(U_i \times_X U_j)$ such that $e_i^j = g_{ij} \cdot e_j^i$. It is not difficult to see that (g_{ij}) defines a 1-cocycle of the sheaf $\tilde{G}$ and that a different choice of trivialization leads to a cohomologuous 1-cocycle. This shows that c exists and that it is well-defined. To show the injectivity of c, we note that this 1-cocycle is trivial if and only if $g_{ij} = g_i^{-1} \cdot g_j$ in $U_i \cap U_j$ for some collection of $g_i \in G(U_i)$, $i \in I$ (possibly after refining the original cover U_i). Replacing e_i by $f_i := g_i \cdot e_i$, we obtain that $f_i = f_j$ on $U_i \cap U_j$, and thus, the sections $f_i \in X(U_i)$ glue together to a global section $f \in X(S)$, which trivializes the G-torsor X. Conversely, a 1-cocycle (g_{ij}) defines an abelian sheaf $\mathcal{F}$ on S_{fl} together with an action of the sheaf $\tilde{G}$. Under one of the extra conditions of the theorem, this sheaf $\mathcal{F}$ is representable by an S-scheme X, which then carries the structure of a G-torsor. (We remark that it is sometimes easier to represent an abelian sheaf $\mathcal{F}$ in the category of algebraic spaces.) Thus, in these latter cases, c is a bijection. For claim (2), we refer to [508], Theorem III.3.9. □

Concerning torsors over regular schemes, we have the following useful extension and purity results. For example, these apply to torsors over smooth varieties.

Theorem 0.1.4 (Purity) *Let S be a regular, noetherian, and separated scheme and let G be a finite and flat S-group scheme.*

1. *Let $f : X \to S$ be a G-torsor. If f has a section over an open and dense subset, then f has a section.*
2. *Let $U \subseteq S$ be an open and dense subset whose complement $S \backslash U$ is of codimension ≥ 2. Then, the restriction map*

$$\mathrm{PHS}_S(G) \to \mathrm{PHS}_U(G|_U)$$

from G-torsors over S to $G|_U$-torsors over U is an equivalence of categories. In particular, every $G|_U$-torsor $V \to U$ extends uniquely to a G-torsor over S.

Proof To prove Claim (1), assume that we are given an open and dense subset $U \subseteq S$ and a section $s : U \to Y$ of f over U. Let Γ be the closure of the graph of s inside $S \times X$. Then, the projection $p_1 : \Gamma \to S$ is a birational morphism that is an isomorphism over U and thus, p_1 is an isomorphism by Zariski's Main Theorem. In particular, using p_1 we can extend the section from U to S.

We only sketch the proof of Claim (2) and refer the reader to [487], Theorem 3.1 for the details, as well as to [211], Proposition 1.4 and [224], Section 2 for different approaches. First, we note that injectivity follows from Zariski's Main Theorem as in the proof of Claim (1). To prove surjectivity, let $j : U \to S$ be the inclusion, let $\pi : G \to S$ be the structure morphism and set $\mathcal{A} := \pi_* \mathcal{O}_G$. Given a $G|_U$-torsor $g : V \to U$, we set $f : X := \operatorname{Spec} j_* g_* \mathcal{O}_V \to S$. The $G|_U$-action on V corresponds to a coaction

$$g_* \mathcal{O}_V \to \mathcal{A}|_U \otimes_{\mathcal{O}_U} g_* \mathcal{O}_V .$$

It is easy to see that we have an isomorphism $j_*(\mathcal{A}|_U \otimes_{\mathcal{O}_U} g_* \mathcal{O}_V) \cong \mathcal{A} \otimes_{\mathcal{O}_S} \mathcal{O}_X$, from which we obtain a morphism of $\mathcal{O}_S$-modules

$$f_* \mathcal{O}_X \to \mathcal{A} \otimes_{\mathcal{O}_S} f_* \mathcal{O}_X .$$

One can check that this is a coaction, that is, we obtain an action of G on X. Moreover, one can also check that $f : X \to S$ is a finite and flat morphism and that the G-action on X turns f into a G-torsor. □

We now give a couple of examples of group schemes, which will be the most important ones for the purposes of this book.

Example 0.1.5 The *additive group scheme* $\mathbb{G}_a$ is defined to be the affine scheme $\operatorname{Spec} \mathbb{Z}[u]$ over $\operatorname{Spec} \mathbb{Z}$ with comultiplication defined by

$$\mu^\dagger : \mathbb{Z}[u] \to \mathbb{Z}[u] \otimes_{\mathbb{Z}} \mathbb{Z}[u], \quad u \mapsto u \otimes 1 + 1 \otimes u,$$

with counit $e^\dagger : \mathbb{Z}[u] \to \mathbb{Z}, u \mapsto 0$, and inverse $\iota^\dagger : \mathbb{Z}[u] \to \mathbb{Z}[u], u \mapsto -u$. For an arbitrary scheme S, we define $\mathbb{G}_{a,S}$ to be the S-scheme obtained from $\mathbb{G}_a$ by the base change $S \to \operatorname{Spec} \mathbb{Z}$. If no confusion arises, we will drop S from the notation of this group scheme.

The additive group scheme $\mathbb{G}_a$ has the following generalization: let $\mathcal{E}$ be a locally free $\mathcal{O}_S$-Module on S of rank r and let $\mathbb{V}(\mathcal{E}) = \operatorname{Spec}\operatorname{Sym}^\bullet(\mathcal{E}) \to S$ be the spectrum of its symmetric $\mathcal{O}_S$-algebra (see also Section 0.3 for details, notation, and our sign conventions). It defines a commutative group scheme over S that represents the functor $(p : T \to S) \mapsto \mathcal{O}_T(p^*\mathcal{E}^\vee)$. It is a smooth commutative group scheme over S. We will refer to it as a *vector group scheme*. In the special case $\mathcal{L} \cong \mathcal{O}_S$, this group scheme coincides with the additive group scheme $\mathbb{G}_{a,S}$. In general, $\mathbb{V}(\mathcal{L})$ is a *twisted form* of $\mathbb{G}_a^r$ in the Zariski topology of S, that is, locally in the Zariski topology, it is isomorphic to the group scheme $\mathbb{G}_a^r$.

It is known that the cohomology groups $H^i(S, \mathbb{V}(\mathcal{E}))$ with respect to flat, étale, and Zariski topology all coincide with the sheaf cohomology $H^i(S, \mathcal{E}^\vee)$ of the $\mathcal{O}_S$-module $\mathcal{E}^\vee$, see, for example, [508, Chapter III.3]. Thus, $\mathbb{V}(\mathcal{E})$-torsors are classified by the following cohomology groups

$$H^1_{\text{fl}}(S, \mathbb{V}(\mathcal{E})) \;\cong\; H^1_{\text{ét}}(S, \mathbb{V}(\mathcal{E})) \;\cong\; H^1_{\text{Zar}}(S, \mathbb{V}(\mathcal{E})) \;\cong\; H^1(S, \mathcal{E}^\vee), \tag{0.1.2}$$

all of which are mutually isomorphic as abelian groups.

Example 0.1.6 The *multiplicative group scheme* $\mathbb{G}_m$ is defined to be the affine scheme $\operatorname{Spec}\mathbb{Z}[u, u^{-1}]$ over $\operatorname{Spec}\mathbb{Z}$ with comultiplication defined by

$$\mu^\dagger \;:\; \mathbb{Z}[u, u^{-1}] \;\to\; \mathbb{Z}[u, u^{-1}] \otimes_{\mathbb{Z}} \mathbb{Z}[u, u^{-1}], \quad u \mapsto u \otimes u,$$

with counit $e^\dagger : \mathbb{Z}[u, u^{-1}] \to \mathbb{Z}$, $u \mapsto 1$, and inverse $\iota^\dagger : \mathbb{Z}[u, u^{-1}] \to \mathbb{Z}[u, u^{-1}]$, $u \mapsto u^{-1}$. For an arbitrary scheme S, we define $\mathbb{G}_{m,S}$ to be the S-scheme obtained from $\mathbb{G}_m$ by the base change $S \to \operatorname{Spec}\mathbb{Z}$. Again, if no confusion arises, we will drop S from the notation of this group scheme. Here, the associated sheaf $\tilde{\mathbb{G}}_{m,S}$ is $\mathcal{O}_S^\times$, which is not a coherent $\mathcal{O}_S$-module. Nevertheless, $\mathbb{G}_{m,S}$-torsors are described by the following mutually isomorphic cohomology groups

$$H^1_{\text{fl}}(S, \mathbb{G}_{m,S}) \;\cong\; H^1_{\text{ét}}(S, \mathbb{G}_{m,S}) \;\cong\; H^1_{\text{Zar}}(S, \mathcal{O}_S^\times) \;\cong\; \operatorname{Pic}(S),$$

where $\operatorname{Pic}(S)$ denotes the group of isomorphism classes of invertible sheaves on S. The first isomorphism follows from Theorem 0.1.3, and the second isomorphism is Hilbert's Theorem 90, see [508], Proposition III.4.9. The last isomorphism is well-known and has the following interpretation in terms of torsors: for an invertible sheaf $\mathcal{L} \in \operatorname{Pic}(S)$, we have the associated line bundle $\pi : \mathbb{L} := \mathbb{V}(\mathcal{L}^\vee) \to S$. Then, the $\mathbb{G}_{m,S}$-action on the complement of the tautological section of $\pi^*\mathcal{L}^\vee$ in $\mathbb{L}$ is fixed-point free, and we obtain a $\mathbb{G}_{m,S}$-torsor. This construction gives rise to a bijection $\operatorname{Pic}(S) \to \operatorname{PHS}(\mathbb{G}_{m,S})$. We will discuss Picard groups and Picard schemes in detail in Section 0.9.

Example 0.1.7 The group scheme $\mathbb{G}_{a,S}$ has no nontrivial subgroup schemes, unless S is a scheme of characteristic $p > 0$. In this latter case, given a global section $a \in H^0(S, \mathcal{O}_S)$, we define

$$\alpha_{p,a} \;:=\; \operatorname{Spec}\mathcal{O}_S[u]/(u^p - au)$$

with comultiplication, coinverse, and counit inherited from $\mathbb{G}_{a,S}$. In the special cases $a = 0$ and $a = 1$, we define

$$\boldsymbol{\alpha}_p := \boldsymbol{\alpha}_{p,0} \quad \text{and find} \quad \mathbb{Z}/p\mathbb{Z} \cong \boldsymbol{\alpha}_{p,1},$$

that is, $\boldsymbol{\alpha}_{p,1}$ is the constant group scheme associated to the finite group $\mathbb{Z}/p\mathbb{Z}$. Since S is of characteristic $p > 0$, we have the S-linear Frobenius morphism $\mathbf{F} = \mathbf{F}_S : \mathbb{G}_{a,S} \to \mathbb{G}_{a,S}$. Then, after identifying a commutative group scheme with the associated abelian sheaf in the flat topology, we obtain the *Artin–Schreier sequence*

$$0 \to \boldsymbol{\alpha}_{p,a} \to \mathbb{G}_{a,S} \xrightarrow{\mathbf{F}-a} \mathbb{G}_{a,S} \to 0, \tag{0.1.3}$$

which is an exact sequence of sheaves in the flat topology. Passing to cohomology, we obtain an exact sequence

$$0 \to \mathrm{Coker}\left(H^0(S, \mathcal{O}_S) \xrightarrow{\mathbf{F}-a} H^0(S, \mathcal{O}_S)\right) \to H^1_{\mathrm{fl}}(S, \boldsymbol{\alpha}_{p,a}) \to \mathrm{Ker}\left(H^1(S, \mathcal{O}_S) \xrightarrow{\mathbf{F}-a} H^1(S, \mathcal{O}_S)\right) \to 0. \tag{0.1.4}$$

This can be generalized as follows: let $\mathcal{L}$ be an invertible sheaf on S and let $\mathbb{L} := \mathbb{V}(\mathcal{L}^\vee) \to S$ be the associated line bundle, which we consider as a group scheme over S. Then, we have an S-linear Frobenius morphism $\mathbf{F} : \mathbb{V}(\mathcal{L}^{\otimes -1}) \to \mathbb{V}(\mathcal{L}^{\otimes -p})$, which is a morphism of group schemes over S. For every global section $a \in H^0(S, \mathcal{L}^{\otimes(p-1)})$, we obtain an exact sequence of abelian sheaves in the flat topology

$$0 \to \boldsymbol{\alpha}_{\mathcal{L},a} \to \mathcal{L} \xrightarrow{\mathbf{F}-a} \mathcal{L}^{\otimes p} \to 0,$$

where $\boldsymbol{\alpha}_{\mathcal{L},a}$ is, by definition, the kernel of $(\mathbf{F} - a)$. As before, we obtain an exact sequence

$$0 \to \mathrm{Coker}\left(H^0(S, \mathcal{L}) \xrightarrow{\mathbf{F}-a} H^0(S, \mathcal{L}^{\otimes p})\right) \to H^1_{\mathrm{fl}}(S, \boldsymbol{\alpha}_{\mathcal{L},a}) \to \mathrm{Ker}\left(H^1(S, \mathcal{L}) \xrightarrow{\mathbf{F}-a} H^1(S, \mathcal{L}^{\otimes p})\right) \to 0. \tag{0.1.5}$$

The group scheme $\boldsymbol{\alpha}_{\mathcal{L},a}$ is a finite flat group scheme of length p over S and a subgroup scheme of $\mathbb{L}$. The fiber of $\boldsymbol{\alpha}_{\mathcal{L},a}$ over a point $x \in S$ is isomorphic to the group scheme $\boldsymbol{\alpha}_{p,a(x)}$. In particular, using this notation, we have $\boldsymbol{\alpha}_{\mathcal{O}_S,0} \cong \boldsymbol{\alpha}_p$ and $\boldsymbol{\alpha}_{\mathcal{O}_S,1} \cong \mathbb{Z}/p\mathbb{Z}$.

Example 0.1.8 We now proceed to subgroup schemes of $\mathbb{G}_m$. For every integer $n \geq 1$, we define

$$\boldsymbol{\mu}_n := \mathrm{Spec}\,\mathbb{Z}[u]/(u^n - 1) \cong \mathrm{Spec}\,\mathbb{Z}[\zeta], \quad \text{where} \quad \zeta^n = 1,$$

and these subschemes inherit comultiplication, inverse, and counit from $\mathbb{G}_m$. The group scheme $\boldsymbol{\mu}_{n,S} \to S$ is finite and flat of length n over S. It is smooth over S if and only if it is étale over S if and only if the characteristic of the residue field of every point of S is coprime to n. If S is the spectrum of a field $\Bbbk$ containing n distinct n-th roots of unity (in particular, $(\mathrm{char}(\Bbbk), n) = 1$), then $\boldsymbol{\mu}_n$ is isomorphic to the constant group scheme $\mathbb{Z}/n\mathbb{Z}$. In any case, we denote by $[n] : \mathbb{G}_m \to \mathbb{G}_m$ the homomorphism of group schemes defined by $\mathbb{Z}[u, u^{-1}] \overset{u\mapsto u^n}{\longrightarrow} \mathbb{Z}[u, u^{-1}]$, or, equivalently, by $x \mapsto x^n$ for all $x \in \mathbb{G}_m(T)$ and all schemes T. Identifying commutative group schemes with their associated abelian sheaves in the flat topology, we obtain the *Kummer exact sequence*

$$0 \to \boldsymbol{\mu}_{n,S} \to \mathbb{G}_{m,S} \xrightarrow{[n]} \mathbb{G}_{m,S} \to 0, \tag{0.1.6}$$

which is an exact sequence of sheaves in the flat topology. The long exact sequence of flat cohomology gives an exact sequence

$$0 \to \operatorname{Coker}\left(H^0(S, \mathcal{O}_S^\times) \xrightarrow{[n]} H^0(S, \mathcal{O}_S^\times)\right) \to H^1_{\mathrm{fl}}(S, \boldsymbol{\mu}_n) \to {}_n\operatorname{Pic}(S) \to 0, \tag{0.1.7}$$

where ${}_n\operatorname{Pic}(S)$ denotes the subgroup of n-torsion elements.

Remark 0.1.9 Using these examples, we obtain a description and the classification of $\boldsymbol{\alpha}_p$-, $\mathbb{Z}/p\mathbb{Z}$, and $\boldsymbol{\mu}_n$-torsors over proper varieties over algebraically closed fields, see Proposition 0.2.29.

We already saw that if a field $\Bbbk$ of characteristic $p \geq 0$ contains the n-th roots of unity and $p \nmid n$, then every choice ζ_n of a primitive n-th root of unity gives rise to an isomorphism of group schemes $\mathbb{Z}/n\mathbb{Z} \cong \boldsymbol{\mu}_n$ over $\operatorname{Spec}\Bbbk$. On the other hand, if $n = p > 0$, then $\boldsymbol{\mu}_p$ is a non-reduced scheme over $\operatorname{Spec}\Bbbk$, whereas $\mathbb{Z}/p\mathbb{Z}$ is étale over $\operatorname{Spec}\Bbbk$. In particular, they are not isomorphic, not even as schemes. Next, if $\Bbbk$ is a field of characteristic $p > 0$, then the schemes underlying $\boldsymbol{\mu}_p$ and $\boldsymbol{\alpha}_p$ are both isomorphic to $\operatorname{Spec}\Bbbk[t]/(t^p)$, which is not reduced. However, they are not isomorphic as group schemes. Put differently, the associated Hopf algebras are isomorphic as $\Bbbk$-algebras but have non-isomorphic coalgebra structures. The just-discussed group schemes comprise all group schemes that are of prime length over an algebraically closed field, which is a theorem of Tate and Oort. We refer to [584] or [733] for proofs or details of this fundamental result.

Theorem 0.1.10 *Let $\Bbbk$ be an algebraically closed field of characteristic $p \geq 0$, let ℓ be a prime, and let G be a finite and flat group scheme of length ℓ over* $\operatorname{Spec}\Bbbk$.

1. *If $\ell \neq p$, then G is isomorphic to $\mathbb{Z}/\ell\mathbb{Z}$, which is isomorphic to $\boldsymbol{\mu}_\ell$.*
2. *If $\ell = p$, then G is isomorphic to either $\mathbb{Z}/p\mathbb{Z}$, or to $\boldsymbol{\mu}_p$, or to $\boldsymbol{\alpha}_p$.*

We refer to Example 1.6.6 for explicit equations if $\ell = 2$.

Example 0.1.11 Let E be an elliptic curve over an algebraically closed field $\Bbbk$ of characteristic $p > 0$. Let $\mathbf{F}_\Bbbk : E \to E^{(p)}$ be the $\Bbbk$-linear Frobenius morphism, which is a homomorphism of group schemes over $\operatorname{Spec}\Bbbk$. Then, $E[\mathbf{F}] := \operatorname{Ker}(\mathbf{F})$, the kernel of Frobenius, is a finite, flat, commutative, and non-reduced group scheme of length p over $\Bbbk$. If $E[\mathbf{F}] \cong \boldsymbol{\mu}_p$, then E is called *ordinary* and, if $E[\mathbf{F}] \cong \boldsymbol{\alpha}_p$, then E is called *supersingular*.

The group schemes $\boldsymbol{\mu}_n$ and $\mathbb{G}_m$ can be generalized as follows: if M is a finitely generated abelian group, then the group algebra $\mathbb{Z}[M]$ can be turned quite naturally into a Hopf algebra over $\mathbb{Z}$ (for example, the coalgebra structure is given by $m \mapsto m \otimes m$), which turns $\mathsf{D}(M) := \operatorname{Spec} \mathbb{Z}[M]$ into an affine and flat group scheme over $\operatorname{Spec} \mathbb{Z}$. Below, we will see that it is the Cartier dual of the constant group scheme M. If S is an arbitrary base scheme, then we obtain a relative group scheme $\mathsf{D}(M) \times_{\operatorname{Spec} \mathbb{Z}} S \to S$ by base change. Group schemes that arise this way are called *diagonalizable* or *of multiplicative type*. For example, we have $D(\mathbb{Z}) \cong \mathbb{G}_m$ and $D(\mathbb{Z}/n\mathbb{Z}) \cong \boldsymbol{\mu}_n$. The name comes from the fact that an affine group scheme G over a field $\Bbbk$ is diagonalizable if and only if every representation $G \to \mathrm{GL}(V)$, where V is a finite–dimensional $\Bbbk$-vector space, is diagonalizable. In the special case, where $G \cong D(\mathbb{Z}^n) \cong \mathbb{G}_m^n$ for some n, the group scheme G is called a *(split) torus*. We note that some authors define diagonalizable group schemes over a field $\Bbbk$ to be group schemes that become diagonalizable in the sense above over an algebraic closure $\overline{\Bbbk}$.

We have seen that the group schemes $\boldsymbol{\mu}_p$ and $\boldsymbol{\alpha}_p$ over fields of characteristic $p > 0$ are non-reduced schemes. Moreover, in Section 0.9, we will discuss the phenomenon that the Picard scheme of a smooth and projective variety may be non-reduced. However, this phenomenon of non-reduced group schemes can only occur in characteristic $p > 0$ by a theorem of Cartier. More precisely, we have the following structure results, see, for example, [392], Lemma 9.5.1 and [538], Lecture 25 for details and proofs.

Theorem 0.1.12 *Let G be a group scheme that is locally of finite type over a field $\Bbbk$ of characteristic $p \geq 0$. Then, G is separated over $\Bbbk$. Moreover:*

1. *If $p = 0$, then G is smooth over $\Bbbk$.*
2. *If $p > 0$, then G is smooth over $\Bbbk$ if and only if it is geometrically reduced.*

If G° denotes the connected component containing $e \in G$, then G° is an open and closed subgroup scheme of G, which is geometrically irreducible and of finite type over $\Bbbk$.

The subgroup scheme G° is called the *connected component* or the *identity component* of G, and we refer to Section 0.9 for a discussion of the connected component for group schemes over more general base schemes. Group schemes with $G^\circ = G$ are called *connected*. We note that if G° is smooth over $\Bbbk$, then the *Barsotti–Chevalley–Rosenlicht structure theorem* states that G° is an extension of an abelian variety over $\Bbbk$ by an affine group scheme over $\Bbbk$, see, for example, [510, Theorem 8.24].

Remark 0.1.13 By Cartier's Theorem 0.1.12, group schemes over fields that are non-reduced can only exist in positive characteristic. However, there are restrictions: for example, if G is a finite and connected group scheme over an algebraically closed field $\Bbbk$ of characteristic $p > 0$, then there exists an isomorphism of schemes over $\Bbbk$ (discarding the group structure)

$$G \;\cong\; \Bbbk[t_1, \ldots, t_r] \,/\, (t_1^{p^{n_1}}, \ldots, t_r^{p^{n_r}})$$

for some integers $r \geq 0$, $n_1 \geq 1, \ldots, n_r \geq 1$, see [160], Exposé VIIB, 5.4.

Next, let G be a finite group scheme over $\operatorname{Spec} \Bbbk$. Since $\Bbbk$ is a field, G is automatically flat over $\operatorname{Spec} \Bbbk$. Although not obvious, it is true that quotients by normal subgroup schemes exist for finite flat group schemes over fields [733]. We set $G^{\text{ét}} := G/G^\circ$ and obtain a short exact sequence

$$1 \to G^\circ \to G \to G^{\text{ét}} \to 1, \tag{0.1.8}$$

of finite and flat group schemes over $\operatorname{Spec} \Bbbk$, the *connected-étale exact sequence*. The group scheme G° is a *local* group scheme, that is, the spectrum of a local $\Bbbk$-algebra, and thus, a non-reduced group scheme over $\Bbbk$. It has only one geometric point, namely the neutral element. On the other hand, $G^{\text{ét}}$ is a smooth group scheme over $\Bbbk$, and since it is finite, it is étale over $\Bbbk$. If $\Bbbk$ is perfect, then the reduction G_{red} (as a scheme) is a subgroup scheme of G and provides us with a canonical splitting of the sequence (0.1.8). In this case, we obtain a canonical decomposition of G as a semi-direct product $G \cong G^\circ \rtimes G^{\text{ét}}$. However, if $\Bbbk$ is not perfect, then a splitting may not exist and G_{red} may not be a subgroup scheme of G, see [733], Exercises 9 and 10 on page 53 for counter-examples.

Next, let G be a finite, flat, and commutative group scheme of length n over some base scheme S, which we assume, as usual, to be noetherian and separated. Then, the sheaf $\mathcal{H}om(G, \mathbb{G}_m)$ of homomorphisms of group schemes from G to $\mathbb{G}_m$ in the flat topology S_{fl} is representable by a finite, flat, and commutative group scheme of length n over S, which is called the *Cartier dual* of G and which is denoted by G^* or G^D. Moreover, there exists a canonical isomorphism $(G^D)^D \cong G$ of group schemes over S, which justifies the name duality. We note that sections of G^D over S are morphisms of group schemes $G \to \mathbb{G}_m$ over S, that is, *characters* of the group scheme G. In particular,

$$\operatorname{Hom}_S(G, \mathbb{G}_m) \cong H^0(S, \mathcal{H}om(G, \mathbb{G}_m)) \cong H^0(S, G^D)$$

is called the *character group* of G. If $S = \operatorname{Spec} \Bbbk$ for some field $\Bbbk$, then the $\Bbbk$-algebra $A := H^0(G, \mathcal{O}_G)$ carries the structure of a Hopf algebra over $\Bbbk$. Next, $A^\vee := \operatorname{Hom}_{\Bbbk}(A, \Bbbk)$ carries a Hopf algebra structure, called the *dual Hopf algebra*: the dual of the comultiplication of A is the multiplication of $A^\vee$, the dual of the coinverse of A is the inverse of $A^\vee$, etc. In this case, $\operatorname{Spec} A^\vee \cong G^D$, that is, Cartier duality of finite, flat, and commutative group schemes over a field is given by the dual Hopf algebra.

Example 0.1.14 For every integer $n \geq 1$ and every base scheme S, we have Cartier duals

$$\boldsymbol{\mu}_{n,S}^D \cong (\mathbb{Z}/n\mathbb{Z})_S \quad \text{and} \quad (\mathbb{Z}/n\mathbb{Z})_S^D \cong \boldsymbol{\mu}_{n,S}.$$

If S is a scheme of characteristic $p > 0$, then

$$\boldsymbol{\alpha}_{p,S}^D \cong \boldsymbol{\alpha}_{p,S}.$$

If $S = \operatorname{Spec} \Bbbk$ for some algebraically closed field $\Bbbk$ and G is a finite, flat, and commutative group scheme of length prime to $p = \operatorname{char}(\Bbbk)$ over S, then G is étale and there exists a non-canonical isomorphism $G^D \cong G$ of S-group schemes.

If G is a finite, flat, and commutative group scheme over $S = \operatorname{Spec} \Bbbk$, where $\Bbbk$ is a perfect field, then the connected-étale exact sequences (0.1.8) for G and G^D are both split and can be combined. It follows that every finite, flat, and commutative group scheme G over $\Bbbk$ possesses a canonical decomposition

$$G \cong G^{\mathrm{loc,loc}} \times G^{\mathrm{loc,\acute{e}t}} \times G^{\mathrm{\acute{e}t,loc}} \times G^{\mathrm{\acute{e}t,\acute{e}t}},$$

such that $G^{\mathrm{loc,loc}}$ is local with local Cartier dual, $G^{\mathrm{loc,\acute{e}t}}$ is local with étale Cartier dual, etc. We refer to [733] for details and to [583] for the classification of finite, flat, and commutative group schemes. For example, (ét, loc)-group schemes are étale group schemes, whose lengths are p-powers, (loc, ét)-group schemes are Cartier dual group schemes of (ét, loc)-group schemes, and (ét, ét)-group schemes are étale group schemes, whose lengths are prime to p. We come back to the classification of group schemes of (loc, loc)-type below.

An important class of finite group schemes, which are not necessarily commutative, is the following: a finite and flat group scheme G over a field $\Bbbk$ is called *linearly reductive* if every finite–dimensional representation of $G \to \mathrm{GL}_{n,\Bbbk}$, is semi-simple. If $p = \operatorname{char}(\Bbbk) = 0$, then all finite group schemes over $\Bbbk$ are étale and linearly reductive. However, if $\operatorname{char}(\Bbbk) > 0$ and $\Bbbk$ is perfect, then a theorem that is usually attributed to Nagata [547] (but see also the discussion in [**?**, Section 2.2]) states that a finite and flat group scheme G over $\Bbbk$ is linearly reductive if and only if it is an extension of a finite and étale group scheme, whose length is prime to p, by a diagonalizable group scheme. Thus, if $\Bbbk$ is algebraically closed, then G is linearly reductive if and only if there exists an isomorphism of group schemes over $\Bbbk$

$$G = G^\circ \rtimes G^{\mathrm{\acute{e}t}}, \qquad \text{where} \qquad G^\circ \cong \prod_{i=1}^{s} \boldsymbol{\mu}_{p^{n_i}}$$

for some integers $s \geq 0$ and $n_i \geq 1$, and where $G^{\mathrm{\acute{e}t}}$ is finite and étale of length prime to p. Quotients of schemes by actions by linearly reductive group schemes are well-behaved, also in positive characteristic, and we will come back to them when discussing rational double points as quotient singularities in Section 0.4 below.

Next, we briefly digress on Witt vectors, which are not only important for the classification of (loc, loc)-group schemes, but also for the discussion of crystalline cohomology in Section 0.10. We start by recalling the construction, and refer to [660], Chapitre II.2.6 for details and to [462] for another survey. First, we define the *Witt polynomials* (with respect to a fixed prime p) to be the following polynomials with integer coefficients:

$$\begin{aligned}
W_0(x_0) \quad &:= x_0, \\
W_1(x_0, x_1) \quad &:= x_0^p + px_1, \\
&\ldots \\
W_n(x_0, ..., x_n) &:= \textstyle\sum_{i=0}^n p^i x_i^{p^{n-i}} \;=\; x_0^{p^n} \;+\; px_1^{p^{n-1}} \;+\; \ldots \;+\; p^n x_n \,.
\end{aligned}$$

Then, there exist unique polynomials S_n and P_n in $(2n + 2)$ variables with integer coefficients such that

$$\begin{aligned}
W_n(x_0, ..., x_n) + W_n(y_0, ..., y_n) &= W_n(S_0(x_0, y_0), ..., S_n(x_0, \ldots, x_n, y_0, \ldots, y_n)), \\
W_n(x_0, ..., x_n) \;\cdot\; W_n(y_0, ..., y_n) &= W_n(P_n(x_0, y_0), ..., P_n(x_0, \ldots, x_n, y_0, \ldots, y_n))
\end{aligned}$$

for all n. For an arbitrary commutative ring R, we define the *(truncated) Witt ring*, or *(truncated) ring of Witt vectors*, $W_n(R)$ to be set R^n together with operations

$$\begin{aligned}
(x_0, ..., x_{n-1}) \;&\oplus\; (y_0, ..., y_{n-1}) \\
&:= (S_0(x_0, y_0), ..., S_{n-1}(x_0, ..., x_{n-1}, y_0, ..., y_{n-1}))\,, \\
(x_0, ..., x_{n-1}) \;&\odot\; (y_0, ..., y_{n-1}) \\
&:= (P_0(x_0, y_0), ..., P_{n-1}(x_0, ..., x_{n-1}, y_0, ..., y_{n-1}))\,.
\end{aligned}$$

These turn $W_n(R)$ into a commutative ring with zero $0 = (0, ..., 0)$ and one $1 = (1, 0, ..., 0)$. Since we have $S_0(x_0, y_0) = x_0 + y_0$ and $P_0(x_0, y_0) = x_0 \cdot y_0$, it follows that $W_1(R)$ is isomorphic to R with its usual addition and multiplication. If R is characteristic $p > 0$, that is $pR = 0$, then we define two operators on $W_n(R)$

$$\begin{aligned}
V \,&:\, (x_0, ..., x_{n-1}) \mapsto (0, x_0, ..., x_{n-2}), \\
\sigma \,&:\, (x_0, ..., x_{n-1}) \mapsto (x_0^p, ..., x_{n-1}^p).
\end{aligned}$$

The map V is additive and is called *Verschiebung* (German for “shift”), whereas σ is a ring homomorphism, called *Frobenius*. (It is customary to call this map σ rather than F in order to avoid clashes of notations with other Frobenius maps.) The maps σ and V are related by the formula

$$\sigma \circ V \;=\; V \circ \sigma \;=\; p \cdot \mathrm{id}.$$

The map $R \to W_n(R)$ that sends $x \mapsto [x] := (x, 0, ..., 0)$ is multiplicative and is called the *Teichmüller lift*. Next, projection onto the first $(n-1)$ components induces surjective ring homomorphisms $W_n(R) \to W_{n-1}(R)$ for all $n \geq 2$. By definition, the *Witt ring*, or *ring of Witt vectors*, $W(R)$ is the projective limit

$$W(R) \;:=\; \varprojlim_{n} \, W_n(R),$$

where the limit is taken with respect to the projection homomorphisms. The maps V, σ, $[-]$ extend to the limit and are compatible with all the projection homomorphisms $W(R) \to W_n(R)$.

Example 0.1.15 In the case of fields, we collect the following examples, remarks and properties.

1. For the field $\mathbb{F}_p$ with p elements we obtain $W_n(\mathbb{F}_p) \cong \mathbb{Z}/p^n\mathbb{Z}$ and the ring of Witt vectors $W(\mathbb{F}_p)$ is isomorphic to the ring $\mathbb{Z}_p$ of p-adic integers. In this case, σ is the identity and V is multiplication by p.
2. Let $q = p^m$ for some prime p and some integer $m \geq 1$. Then, $W(\mathbb{F}_q)$ is the ring of integers in the unique unramified extension $\mathbb{Q}_p \subseteq \mathbb{Q}_q$ that is of degree m over $\mathbb{Q}_p$. Moreover, $W(\overline{\mathbb{F}}_p)$ is the ring of integers of $\widehat{\mathbb{Q}}_p^{\mathrm{nr}}$, the p-adic completion of the maximal unramified extension $\mathbb{Q}_p^{\mathrm{nr}}$ of $\mathbb{Q}_p$.
3. If $\Bbbk$ is a perfect field, then $W(\Bbbk)$ is a discrete valuation ring of characteristic zero with residue field $\Bbbk$. The unique maximal ideal of $W(\Bbbk)$ is the principal ideal generated by p and $W(\Bbbk)$ is complete with respect to the p-adic topology. Moreover, if $(S, \mathfrak{m})$ is an $\mathfrak{m}$-adically complete discrete valuation ring of characteristic zero with residue field $\Bbbk$, then it contains $W(\Bbbk)$ as a subring.
4. If $\Bbbk$ is a field of characteristic p that is not perfect, that is, the map $x \mapsto x^p$ is not surjective, then the kernel of $W(\Bbbk) \to \Bbbk$ still contains the ideal generated by p, but it is not equal to it. In this case, this kernel is not a finitely generated ideal and the ring $W(\Bbbk)$ is not noetherian.

Coming back to group schemes, we now discuss *Witt group schemes*: these are affine and commutative group schemes $\mathbb{W}_n \to \operatorname{Spec} \mathbb{F}_p$ that are isomorphic to $\operatorname{Spec} \mathbb{F}_p[x_0, ..., x_{n-1}]$ as schemes and whose coalgebra structure is defined via the Witt polynomials. We refer to [583], Section II.9 for details. Then, we have $\mathbb{W}_1 \cong \mathbb{G}_a$ and $\mathbb{W}_n$ is a successive extension of $\mathbb{G}_a$'s. Passing to the projective limit, we obtain a group scheme $\mathbb{W}$, which is not of finite type over $\operatorname{Spec} \mathbb{F}_p$. Moreover, Frobenius and Verschiebung give rise to morphisms of group schemes $\mathbf{F} : \mathbb{W}_n \to \mathbb{W}_n$ and $V : \mathbb{W}_n \to \mathbb{W}_{n+1}$. We use Frobenius to define for integers $m, n \geq 1$ the group scheme

$$\mathbb{L}_{n,m} := \operatorname{Ker}(\mathbf{F}^m : \mathbb{W}_n \to \mathbb{W}_n),$$

which is finite, flat, commutative, and of length p^{mn} over $\operatorname{Spec} \mathbb{F}_p$. Moreover, it is of (loc, loc)-type and Cartier duality interchanges the indices: $\mathbb{L}_{n,m}^D \cong \mathbb{L}_{m,n}$. We have $\mathbb{L}_{1,1} \cong \boldsymbol{\alpha}_p$ and each $\mathbb{L}_{n,m}$ is a successive extension of $\boldsymbol{\alpha}_p$'s.

Next, an affine group scheme G over a field $\Bbbk$ is called *unipotent* if every representation $G \to \mathrm{GL}(V)$, where V is a finite–dimensional $\Bbbk$-vector space, possesses a filtration $0 = V_0 \subseteq V_1 \subseteq ... \subseteq V_n = V$ into G-stable subspaces such that the induced G-representation on V_i/V_{i-1} is trivial for all i. One can show that this is equivalent to $G \times_{\operatorname{Spec} \Bbbk} \operatorname{Spec} \overline{\Bbbk}$ having a composition series, in which all composition factors are isomorphic to $\mathbb{G}_a$, $\boldsymbol{\alpha}_p$, or $\mathbb{Z}/p\mathbb{Z}$. For example, $\mathbb{L}_{n,m}$ and $\mathbb{Z}/p^n\mathbb{Z}$ are examples of finite, commutative, and unipotent group schemes over $\operatorname{Spec} \mathbb{F}_p$ – the former are of (loc, loc)-type and the latter is of (ét, loc)-type. To classify unipotent group schemes over a perfect field $\Bbbk$ of characteristic $p > 0$, we let $W = W(\Bbbk)$ be the ring of Witt vectors over $\Bbbk$, and define the *Dieudonné ring* A of $\Bbbk$ to be the non-commutative polynomial ring $W\langle F, V\rangle$ over W with the relations $Fr = \sigma(r)F$, $rV = V\sigma(r)$, and $FV = VF = p$ for all $r \in W$. Then, for a finite group scheme G over $\Bbbk$, one defines the *(contravariant) Cartier–Dieudonné module* of G to be

$$\mathbb{D}(G) := \varinjlim_n \operatorname{Hom}(G, \mathbb{W}_n),$$

which carries the structure of a left A-module. For example, we have

$$\mathbb{D}(\mathbb{L}_{n,m}) \;\cong\; A/(F^m, V^n).$$

Moreover, Cartier and Gabriel proved that $\mathbb{D}$ induces an anti-equivalence between the category of finitely generated left A-modules that are annihilated by some power of V and the category of unipotent algebraic group schemes over the perfect field $\mathbb{k}$. We note that there exist several variants of this theory: for example, there are covariant rather than contravariant versions. Also, the functor $\mathbb{D}$ can be modified so to induce an anti-equivalence between the category of left A-modules that are finitely generated as W-modules and the category of finite and commutative group schemes over $\mathbb{k}$, whose length is a p-power. We refer to [583], Section II.(15.3) for details, as well as the discussion of the Cartier ring of formal group laws and the discussion of **F**-crystals in Section 0.10 for related topics. Putting all the previous discussions together, we obtain a good overview over finite and commutative group schemes.

Let G be a group scheme that is locally of finite type over some field $\mathbb{k}$. Let $e \in G(\mathbb{k})$ be its neutral element and let $\mathfrak{m} \subseteq \mathcal{O}_{G,e}$ be the local ring at e. We define the *Lie algebra* of G to be the Zariski tangent space of $e \in G$,

$$\mathrm{Lie}(G) \;:=\; (\mathfrak{m}/\mathfrak{m}^2)^\vee \;:=\; \mathrm{Hom}_{\mathbb{k}}(\mathfrak{m}/\mathfrak{m}^2, \mathbb{k}).$$

This is a finite–dimensional $\mathbb{k}$-vector space, which is naturally isomorphic to the space of $\mathbb{k}$-linear derivations from $\mathcal{O}_{G,e}$ to $\mathbb{k}$. The latter is isomorphic to the space of left invariant derivations on G. If η, ξ are two such derivations, then also $[\eta, \xi] := \eta \circ \xi - \xi \circ \eta$ is one, and thus, $\mathrm{Lie}(G)$ carries the structure of a Lie algebra over $\mathbb{k}$. Moreover, if $\mathbb{k}$ is of characteristic $p > 0$ and if η is a left invariant derivation, then so is $\eta^{[p]} := \eta \circ \ldots \circ \eta$ (p-fold composition with itself). In this case, the p-power operation $-^{[p]}$ together with the Lie bracket $[-,-]$ turn $\mathrm{Lie}(G)$ into a *restricted Lie algebra* or *p-Lie algebra* over $\mathbb{k}$. We refer to [342], Chapter V.7 for the precise definition of restricted Lie algebras, as well to Section 0.3 for the relation to purely inseparable morphisms. We finally note that the Lie algebra of G only depends on the connected component G° of the identity, that is, $\mathrm{Lie}(G^\circ) \cong \mathrm{Lie}(G)$.

Example 0.1.16 If $\mathbb{k}$ is a field of characteristic $p > 0$, then there exists an isomorphism of restricted Lie algebras

$$\mathrm{Lie}(\boldsymbol{\mu}_{p,\mathbb{k}}) \;\cong\; \mathbb{k} \cdot x,$$

where the Lie bracket on the right–hand side is zero and $x^{[p]} = x$. If x is replaced by $x' := c \cdot x$ for some $c \in \mathbb{k}^\times$, then $x'^{[p]} = c^{p-1} \cdot x'$. Next, there exists an isomorphism of restricted Lie algebras

$$\mathrm{Lie}(\boldsymbol{\alpha}_{p,\mathbb{k}}) \;\cong\; \mathbb{k} \cdot y,$$

where the Lie bracket on the right–hand side is zero and $y^{[p]} = 0$. A derivation ξ with $\xi^{[p]} = \xi$ (resp. $\xi^{[p]} = 0$) is called *multiplicative* (resp. *additive*) and we come back to this in Section 0.3.

Next, we recall the definition of a formal group law. Given a group scheme G that is locally of finite type over some field $\mathbb{k}$ with neutral element $e \in G(\mathbb{k})$, one can also study the formal completion $\widehat{\mathcal{O}}_{G,e}$ of the local ring of G at e, which usually carries more information than the Zariski tangent space $\mathrm{Lie}(G)$. More precisely, the multiplication $\mu : G \times_{\mathrm{Spec}\,\mathbb{k}} G \to G$ gives rise to a morphism of complete $\mathbb{k}$-algebras

$$\hat{\mu}^{\#} \,:\, \widehat{\mathcal{O}}_{G,e} \;\to\; \widehat{\mathcal{O}}_{G,e} \,\hat{\otimes}_{\mathbb{k}}\, \widehat{\mathcal{O}}_{G,e},$$

where $\hat{\otimes}$ denotes the completed tensor product. Moreover, $\widehat{G} := \mathrm{Spf}\,\mathcal{O}_{G,e}$ is the completion of G along the zero section e and $\hat{\mu}^{\#}$ turns $\widehat{G}$ into a group object in the category of formal schemes over $\mathrm{Spf}\,\mathbb{k}$. If G is smooth over $\mathbb{k}$, then there exists an isomorphism $\widehat{\mathcal{O}}_{G,e} \cong \mathbb{k}[[t_1, ..., t_m]]$ of complete $\mathbb{k}$-algebras. In this case, $\hat{\mu}^{\#}$ becomes a morphism of complete $\mathbb{k}$-algebras

$$\psi \,:\, \mathbb{k}[[t_1, ..., t_m]] \;\to\; \mathbb{k}[[u_1, ..., u_m, w_1, ..., w_m]],$$

which is completely determined by the m formal power series $\psi(t_i)$, $i = 1, ..., m$. This is formalized in the notion of a *formal group law*, or *formal Lie group*, of dimension m over $\mathbb{k}$.
indexformal group By definition, this is a set of m formal power series $\mathbf{F} := (F_1, ..., F_m)$ with $F_i = F_i(\mathbf{u}, \mathbf{w}) \in \mathbb{k}[[\mathbf{u}, \mathbf{w}]]$, $i = 1, ..., m$ and with $\mathbf{u} = (u_1, ..., u_m)$ and $\mathbf{w} = (w_1, ..., w_m)$, such that:

1. $F_i(\mathbf{u}, \mathbf{w})$ is equal to $u_i + w_i$ plus terms of degree ≥ 2 for all i, and
2. $F_i(\mathbf{F}(\mathbf{u}, \mathbf{w}), \mathbf{y}) = F_i(\mathbf{u}, \mathbf{F}(\mathbf{w}, \mathbf{y}))$ for all i.

The formal group law is said to be *commutative* if $\mathbf{F}(\mathbf{u}, \mathbf{w}) = \mathbf{F}(\mathbf{w}, \mathbf{u})$. We refer to [300], Chapter II for details, as well as the notion of (iso-)morphisms between formal group laws. We note that one-dimensional formal group laws over fields are automatically commutative and refer to [300], Chapter I.6 for details and proof.

Example 0.1.17 If G is smooth group scheme of dimension m over a field $\mathbb{k}$, then $\widehat{\mathcal{O}}_{G,e} \cong \mathbb{k}[[t_1, ..., t_m]]$ and $\hat{\mu}^{\#}$ turns it into a formal group law $\widehat{G}$ of dimension m over $\mathbb{k}$.

1. If $G = \mathbb{G}_a$ is the additive group, then the *formal additive group law* $\widehat{\mathbb{G}}_a$ is given by $\hat{\mu}^{\#} : t_1 \mapsto u_1 + w_1$.
   indexformal group!$\widehat{\mathbb{G}}_a$
2. If $G = \mathbb{G}_m$ is the multiplicative group, then the *formal multiplicative group law* $\widehat{\mathbb{G}}_m$ is given by $\hat{\mu}^{\#} : t_1 \mapsto u_1 + w_1 + u_1 w_1$.

If the field $\mathbb{k}$ is of characteristic zero, then every m-dimensional commutative formal group law over $\mathbb{k}$ is isomorphic to $\widehat{\mathbb{G}}_a^m$ via the *formal logarithm*. On the other hand, this is not true if $\mathbb{k}$ is of positive characteristic $p > 0$: as in the case of group schemes, we have the *Frobenius morphism*

$$\mathbf{F} \,:\, \widehat{G} \;\to\; \widehat{G}^{(p)},$$

which is a homomorphism of formal group laws over $\Bbbk$. This can be used to obtain a discrete invariant of commutative formal group laws, namely their *height*, which we now explain in the one-dimensional case: if $\widehat{G}$ is a one-dimensional formal group law, that is, $\widehat{G} \cong \operatorname{Spf} \Bbbk[[t_1]]$, then multiplication by p is a homomorphism $[p] : \widehat{G} \to \widehat{G}$ and one can compare it to the Frobenius morphism $\mathbf{F}$. The height is the largest integer h such that there exists a factorization $\widehat{G} \to \widehat{G}^{(p^h)} \to \widehat{G}$ of $[p]$. In the case $[p] = 0$, one defines $h := \infty$.

Example 0.1.18 Let $\Bbbk$ be a field of characteristic $p > 0$. Then,

$$h(\widehat{\mathbb{G}}_m) \;=\; 1 \quad \text{and} \quad h(\widehat{\mathbb{G}}_a) \;=\; \infty.$$

In particular, these two formal group laws are not isomorphic over $\Bbbk$. If E is an elliptic curve over $\Bbbk$, then the formal completion $\widehat{E}$ of E at the neutral element is a one-dimensional and commutative formal group law, whose height satisfies

$$h(\widehat{E}) \;=\; \begin{cases} 1 & \text{if } E \text{ is ordinary} \\ 2 & \text{if } E \text{ is supersingular,} \end{cases}$$

see also Example 0.1.11.

If $\Bbbk$ is an algebraically closed field of characteristic $p > 0$, then a theorem of Lazard states that two one-dimensional formal group laws over $\Bbbk$ are isomorphic if and only if they have the same height [300, Theorem 19.4.1]. Moreover, there exists a formal group law for every given height $\in \{1, 2, \dots, \infty\}$. Again, we refer to [300] for details and proofs, as well as for the definition of heights for higher–dimensional commutative formal group laws. Unfortunately, the height does not suffice to classify higher–dimensional formal group laws. As in the case of commutative group schemes above, let $\Bbbk$ be a perfect field of characteristic $p > 0$ and then, the *Cartier ring* $\operatorname{Cart}(\Bbbk)$ is defined to be the non-commutative ring $W(\Bbbk)\langle\langle V\rangle\rangle\langle F\rangle$ (formal power series in V, polynomials in F) with relations $FV = p$, $VrF = V(r)$, $Fr = \sigma(r)F$, and $rV = V\sigma(r)$ for all $r \in W(\Bbbk)$. For every commutative formal group law $\widehat{G}$ over $\Bbbk$, there exists a left $\operatorname{Cart}(\Bbbk)$-module $\mathbb{D}(\widehat{G})$, the *Cartier–Dieudonné module* of $\widehat{G}$. The functor $\mathbb{D}$ induces an (anti-)equivalence between the category of commutative formal group laws over $\Bbbk$ and a certain subcategory of the category of left $\operatorname{Cart}(\Bbbk)$-modules. Again, we warn the reader that there exists a covariant and a contravariant version of this theory, and that the rôles of F and V are exchanged in this theory, which easily leads to confusion. Apart from [300], we refer to [462] for a more detailed survey and further references.

Example 0.1.19 Let $\Bbbk$ be an algebraically closed field of characteristic $p > 0$ and let $\widehat{G}_h$ be the unique one-dimensional formal group law of height h over $\Bbbk$. Then, the (covariant) Cartier–Dieudonné module of $\widehat{G}_h$ is

$$\mathbb{D}(\widehat{G}_h) \;\cong\; \begin{cases} \operatorname{Cart}(\Bbbk)/(F - V^{h-1}) & & \text{if } h < \infty, \\ \Bbbk[[x]], & F = 0,\ Vx^n = x^{n+1} & \text{if } h = \infty. \end{cases}$$

In particular, if $h < \infty$, then $\mathbb{D}(\widehat{G}_h)$ is a free $W(\Bbbk)$-module of rank h, which can be interpreted as an $\mathbf{F}$-crystal of slope $1 - \frac{1}{h}$, see Section 0.10.

Next, we recall the definition of the *Weil restriction functor* [86, 7.6]. Let $f : S' \to S$ be a morphism of schemes. then, for any S'-scheme X', the functor

$$\mathfrak{R}_{S'/S}(X') \;:\; (\text{Schemes}/S)^{\circ} \;\to\; (\text{Sets}), \quad T \;\to\; \text{Hom}(T \times_S S', X')$$

is equal to the Zariski sheaf $f_*(h_{X'})$, where $h_{X'}$ is the Yoneda sheaf in the Zariski topology represented by X'. If $\mathfrak{R}_{S'/S}$ is representable by an S-scheme, then it (and its representing scheme) is called the *Weil restriction* of X'. By definition, there is a canonical bijection

$$\text{Hom}_S(T, \mathfrak{R}_{S'/S}(X')) \xrightarrow{\sim} \text{Hom}_{S'}(T \times_S S', X'),$$

functorial in T. A condition for representability of $\mathfrak{R}_{S'/S}(X')$ is given in Theorem 4 from [86, 7.6]. For example, if $X' \to S'$ is quasi-projective and $S' \to S$ is affine, then $\mathfrak{R}_{S'/S}(X')$ is representable. Also, if $S' \to S$ is finite and flat, and $X' \to S'$ is smooth, then $\mathfrak{R}_{S'/S''}(X')$ is smooth [203, Lemma 2.2].

It follows from the definition of the adjoint functor f^* that there is a canonical morphism of S-schemes

$$\iota_{S'/S} \;:\; X' \;\to\; \mathfrak{R}_{S'/S}(X') \times_S S'.$$

Many applications of Weil restriction arise in theory of group schemes, where X is a S'-group scheme G' and where $S' \to S$ is a finite flat morphism. In this case, $G = \mathfrak{R}_{S'/S}(X')$ is an S-group scheme.

Example 0.1.20 The following is the most notorious example of Weil restriction: we take $S = \operatorname{Spec} \mathbb{R}$ and $S' = \operatorname{Spec} \mathbb{C}$ with the natural morphism $S' \to S$. We take $G' = \mathbb{G}_{m,\mathbb{C}}$ and then, G is an algebraic group over $\mathbb{R}$ isomorphic to $\operatorname{Spec} \mathbb{R}[U, V]/(U^2+V^2-1)$ with $G(R) = (R \otimes_{\mathbb{R}} \mathbb{C})^{\times}$ for any $\mathbb{R}$-algebra R. In particular, we have $G(\mathbb{R}) = \mathbb{C}^{\times}$ and $G(\mathbb{C}) = \mathbb{C}^{\times} \times \mathbb{C}^{\times}$. The group G is an anisotropic 2-dimensional torus. The morphism $\iota_{\mathbb{C}/\mathbb{R}}$ is the embedding

$$\mathbb{C}^{\times} \hookrightarrow \mathbb{C}^{\times} \times \mathbb{C}^{\times}, \quad z \mapsto (z, \bar{z}).$$

This G is the *Deligne torus* and it plays an important role in Hodge theory because it gives a very elegant way to define and deal with Hodge structures.

We end our discussion of group schemes with perfect group schemes, which we will need to study the flat cohomology in Section 0.10. First, a *perfect scheme* of characteristic $p > 0$ is a scheme X of characteristic $p > 0$, such that the absolute Frobenius morphism $\mathbf{F} : X \to X$ is an isomorphism. For example, if $\Bbbk$ is a perfect field of characteristic $p > 0$, then $X = \operatorname{Spec} \Bbbk$ is a perfect scheme. Moreover, if X is a scheme of characteristic $p > 0$, then its *perfection* is defined to be

$$X^{\text{pf}} := \varprojlim_n X,$$

where the projective limit is taken over all integers $n \geq 0$ with respect to the absolute Frobenius morphism $\mathbf{F} : X \longrightarrow X$. The perfection is a perfect scheme. Moreover, if X is a perfect scheme, then $X_{\mathrm{pf}} \cong X$ and if $X = \operatorname{Spec} \mathbb{k}$ for some field $\mathbb{k}$ of characteristic $p > 0$, then $X^{\mathrm{pf}} = \operatorname{Spec} \mathbb{k}^{p^{-\infty}}$, the spectrum of the *perfect closure* of $\mathbb{k}$. Moreover, if X is a possibly non-reduced scheme with reduction X_{red}, then the canonical morphism $X_{\mathrm{red}} \longrightarrow X$ induces an isomorphism of perfections $X_{\mathrm{red}}^{\mathrm{pf}} \cong X^{\mathrm{pf}}$.

Next, let S be a scheme, let $S_{\text{ét}}$ be the category of S-schemes equipped with the étale topology, and let S_{perf} be the full subcategory of perfect S-schemes equipped with the étale topology. Then, there exists a canonical morphism $\pi : S_{\text{ét}} \longrightarrow S_{\mathrm{pf}}$ that associates to a sheaf $\mathcal{F}$ of abelian groups on $S_{\text{ét}}$ a sheaf $\mathcal{F}^{\mathrm{pf}} := \pi_* \mathcal{F}$ of abelian groups on S_{pf}. For example, if $\mathcal{F}$ is represented by a commutative group scheme G over S, then $\mathcal{F}^{\mathrm{pf}}$ is determined by its values on perfect schemes. In this case, one can show that $\mathcal{F}^{\mathrm{pf}}$ is represented by the perfection G^{pf} of the group scheme G, which coincides with the perfection of G_{red}.

A *perfect group scheme* G over S is an object in the category S_{pf} that is of the form $\underline{G}^{\mathrm{pf}}$ for some object $\underline{G}$ of $S_{\text{ét}}$ that is represented by an affine group scheme G that is of finite type over S. Perfect group schemes over S form an abelian category that is equivalent to the category of *quasi-algebraic groups* in the sense of Serre, see [657, Proposition I.10]. This category admits projective limits, and, by definition, a *pro-algebraic group* is a projective limit of quasi-algebraic groups. Note that there is a more general definition of a pro-algebraic group, which we will not use.

Example 0.1.21 Let $\mathbb{k}$ be a perfect field of characteristic $p > 0$ and $S := \operatorname{Spec} \mathbb{k}$. As in the case of formal group laws, the affine group schemes $\mathbb{G}_a$ and $\mathbb{G}_m$ give rise to perfect group schemes over S_{pf}.

1. For $\mathbb{G}_a \cong \operatorname{Spec} \mathbb{k}[t]$ we have $\mathbb{G}_a^{\mathrm{pf}} \cong \operatorname{Spec} \mathbb{k}[t, \ldots, t^{1/p^n}, \ldots]$. In particular, if A is a $\mathbb{k}$-algebra, then $\mathbb{G}_a^{\mathrm{pf}}(A) = A^{p^{-\infty}}$, whose group structure is given by addition.
2. For $\mathbb{G}_m \cong \operatorname{Spec} \mathbb{k}[t, t^{-1}]$ we have $\mathbb{G}_m^{\mathrm{pf}} = \operatorname{Spec} \mathbb{k}[t, t^{-1}, \ldots, t^{1/p^n}, t^{-1/p^n} \ldots]$. Then, we have $\mathbb{G}_m^{\mathrm{pf}}(A) = (A^{p^{-\infty}})^\times$ for every $\mathbb{k}$-algebra A and the group structure is given by multiplication. We note that multiplication by p (rather exponentiation by p) $[p] : \mathbb{G}_m \longrightarrow \mathbb{G}_m$ defines an automorphism of $\mathbb{G}_m^{\mathrm{pf}}$.

In both cases, the Frobenius morphism induces an automorphism of perfect group schemes.

Concerning their structure: every commutative and perfect group scheme G over an algebraically closed field $\mathbb{k}$ of characteristic $p > 0$ possesses a composition series, whose composition factors are $\mathbb{G}_a^{\mathrm{pf}}$, $\mathbb{G}_m^{\mathrm{pf}}$, as well as perfections of abelian varieties and finite groups, see [659, Section I.3]. Moreover, if G is killed by some power of p, then it fits into an extension

$$0 \longrightarrow \mathsf{U} \longrightarrow \mathsf{G} \longrightarrow \mathsf{D} \longrightarrow 0, \tag{0.1.9}$$

where $\mathsf{U} = U^{\mathrm{pf}}$ is perfect group scheme that is obtained from a unipotent, smooth, connected, and commutative group scheme U, and where $\mathsf{D} = D^{\mathrm{pf}}$ is a perfect group scheme that is obtained from a finite, étale, and commutative group scheme D that can be identified with D^{pf}. Moreover, the perfect group U admits a composition series, whose composition factors are isomorphic to $\mathbb{G}_a^{\mathrm{pf}}$. We call U the *connected component of the identity* of G and denote it by G°. Therefore, (0.1.9) is an analog of the connected-étale exact sequence (0.1.8). Finally, we denote by $\mathcal{P}(p^n)$ the category of perfect S-group schemes that are killed by p^n. For example, for every perfect ring R of characteristic $p > 0$, the ring $W_n(R)$ of Witt vectors of length n has natural structure of a unipotent group killed by p^n and thus, defines an object W_n^{pf} of $\mathcal{P}(p^n)$. The functor $\mathsf{G}^\cdot \mapsto \mathsf{G}^{\cdot\vee} := R\mathcal{H}om_{\mathcal{P}(p^n)}(\mathsf{G}^\cdot, \mathbb{Z}/p^n\mathbb{Z})$ defines a duality in the derived category $\mathcal{D}^b(\mathcal{P}(p^n))$. Next, we define $\mathcal{P}(p^\infty)$ to be the union of all objects of $\mathcal{P}(p^n)$ for all n and we define duality in this category by setting $\mathsf{G}^{\cdot\vee} := \varinjlim_n R\mathcal{H}om_{\mathcal{P}(p^n)}(\mathsf{G}^\cdot, \mathbb{Z}/p^n\mathbb{Z})$. We note that the canonical morphism $\mathsf{G}^\cdot \to (\mathsf{G}^{\cdot\vee})^\vee$ is an isomorphism in the derived category, which justifies to call it a duality. We refer to [657] for more about perfect and quasi-algebraic group schemes, as well as to [507] and [509] for applications to flat cohomology.

The category of quasi-algebraic groups is abelian. In particular, one can use derived functors to define the higher homotopy groups π_i as the derived functors of the functor π_0, which assigns to a group its largest étale quotient, which is a constant group in our case. Furthermore, by passing to a projective systems of quasi-algebraic groups, one obtains the category of pro-algebraic groups. The homotopy groups of a pro-algebraic group are pro-finite commutative groups.

Example 0.1.22 Let G be a connected (that is, $\pi_0(G) = 0$) commutative group scheme over a field $\Bbbk$ of characteristic $p \geq 0$, such that the homomorphism $[\ell]$ of multiplication by a prime number ℓ is surjective. Assume that ${}_\ell G = (\mathbb{Z}/\ell\mathbb{Z})^I$ for some set I. Then, $\pi_1(G) \cong (\mathbb{Z}/\ell\mathbb{Z})^I$, see [657, p. 45].

1. If $G = \mathbb{G}_{m,\Bbbk}$, then we get étale covers defined by the Kummer exact sequence (0.1.6). Using this, one can show that
$$\pi_1(\mathbb{G}_{m,\Bbbk})(\ell) \cong \mathbb{Z}_\ell \quad \text{if } \ell \neq p \quad \text{and} \quad \pi_1(\mathbb{G}_{m,\Bbbk})(p) = 0.$$
Here, $A(\ell)$ denotes the ℓ-primary component of a pro-finite abelian group.
2. If $G = \mathbb{G}_{a,\Bbbk}$, then we get étale covers using the Artin–Schreier exact sequence (0.1.3)
$$0 \to \mathbb{Z}/p\mathbb{Z} \to \mathbb{G}_a \xrightarrow{\mathbf{F}-a\,\mathrm{id}} \mathbb{G}_a \to 0,$$
where $a \neq 0$. It is shown in [657, p. 53] that
$$\pi_1(\mathbb{G}_{a,\Bbbk}) \cong \mathrm{Hom}(\Bbbk, \mathbb{Z}/p\mathbb{Z})$$
and in particular,
$$\pi_1(\mathbb{G}_{a,\Bbbk})(\ell) = \{0\} \quad \text{if } \ell \neq p.$$
3. If G is an elliptic curve E over $\Bbbk$, then
$$\pi_1(E)(\ell) \cong \mathbb{Z}_\ell^2 \quad \text{if } \ell \neq p \text{ and} \quad \pi_1(E)(p) = \mathbb{Z}_p^e,$$

where $e = 1$ if E is ordinary and $e = 0$ if E is supersingular.

The previous examples dealt with quasi-algebraic groups arising from one-dimensional commutative group schemes over a field $\Bbbk$. Next, we consider pro-algebraic groups defined by $(\underline{G}_i)$, where G is a one-dimensional group scheme of finite type over the ring of formal power series $R = \Bbbk[[t]]$ over some field $\Bbbk$ of characteristic $p \geq 0$. We denote by $K = \Bbbk((t))$ the field of fractions of R.

Example 0.1.23 Let $G = \mathbb{G}_{m,R}$ with $R = \Bbbk[[t]]$ and for $n \geq 0$, we define

$$U_K^n := \{x \in \Bbbk[[t]] : x \equiv 1 \mod \mathfrak{m}^n\}.$$

Then, we obtain a composition series

$$\mathbb{G}_m(R) = U_K^0 \supset U_K^1 \supset \ldots \supset U_K^n \supset \ldots$$

with quotients

$$U_K/U_K^n = \mathbb{G}_m(R/\mathfrak{m}^n) \quad \text{and} \quad U_K^n/U_K^{n+1} \cong \mathbb{G}_a(\Bbbk) \quad \text{for } n \geq 1.$$

Therefore, the group $\underline{G}_n := U_K^0/U_K^n$ is an extension of a unipotent commutative group of dimension $(n-1)$ associated to the group U_K^1/U_K^n by $\mathbb{G}_{m,\Bbbk} = U_K^0/U_K^1$. The associated pro-algebraic group $\underline{G}$ defines a structure of a pro-algebraic group on $U_K = U_K^0 = \varprojlim U_K/U_K^n$. Next, for any finite Galois extension L/K, the kernel of the norm homomorphism $N_{L/K} : U_L^* \to U_K^*$ is isomorphic to the abelianization $\mathrm{Gal}(L/K)^{\mathrm{ab}}$ of the Galois group $\mathrm{Gal}(L/K)$. Now, if π is a uniformizer of the integral closure of R in L, then we define a homomorphism $\mathrm{Gal}(L/K) \to \mathrm{Ker}(N_{L/K})$ via $\sigma \mapsto \sigma(\pi)/\pi$. This gives rise to an exact sequence of pro-algebraic groups

$$0 \to \mathrm{Gal}(L/K)^{\mathrm{ab}} \to U_L \to U_K \to 0.$$

The homotopy exact sequence defines a homomorphism

$$\theta : \pi_1(U_K) \to \pi_0\left(\mathrm{Gal}(L/K)^{\mathrm{ab}}\right) = \mathrm{Gal}(L/K)^{\mathrm{ab}}.$$

Passing to the projective limit, we obtain an isomorphism of pro-algebraic groups

$$\pi_1(U_K) \to \mathrm{Gal}(K^{\mathrm{ab}}/K) = \varprojlim{}_{L/K} \mathrm{Gal}(L/K)^{\mathrm{ab}},$$

where K^{ab}/K denotes the maximal abelian Galois extension of K, see [658, §4, Théorème 1]. Using the composition series for U_K and the computation of the fundamental groups of $\mathbb{G}_{a,\Bbbk}$ and $\mathbb{G}_{m,\Bbbk}$ from Example 0.1.22, we conclude that

$$\pi_1(\mathbb{G}_{m,R})(\ell) \cong \mathbb{Z}_\ell \quad \text{if } \ell \neq p,$$

and that $\pi_1(\mathbb{G}_{m,R})(p)$ is a successive extension of the groups $\pi_1(\mathbb{G}_{a,\Bbbk}) \cong \mathrm{Hom}(\Bbbk, \mathbb{Z}/p\mathbb{Z})$.

Finally, assume that $\mathrm{char}(K) = p \neq 0$ and recall that a non-trivial element of $\mathrm{Hom}(\mathrm{Gal}(K^{\mathrm{ab}}/K), \mathbb{Z}/p\mathbb{Z})$ defines a cyclic and separable extension L/K of degree

p. This is an Artin–Schreier extension defined by some equation $x^p - x = a$ with $a \in K$. This comes from passing to exact Galois cohomology sequence for the exact sequence

$$0 \to \mathbb{Z}/p\mathbb{Z} \to K \xrightarrow{\wp} K \to 0,$$

where $\wp : x \mapsto x^p - x$. In this way, we obtain an isomorphism

$$\mathrm{Hom}\left(\pi_1(\mathbb{G}_{m,R}), \mathbb{Z}/p\mathbb{Z}\right) \cong \mathbb{G}_a(K). \tag{0.1.10}$$

The previous discussion also applies to projective system of groups associated to a group scheme G over a complete local Noetherian ring R with maximal ideal $\mathfrak{m}$. For our applications later on, we will assume that R is a $\mathbb{k}$-algebra. Let $R_i = R/\mathfrak{m}^i, i \geq 1$ and $G_i = G \otimes_R R_i$ be the base change with respect to the natural morphisms $q_i : \operatorname{Spec} R_i \longrightarrow \operatorname{Spec} R$. This is a group scheme over the artinian $\mathbb{k}$-algebra R_i. Let

$$\mathcal{G}_i(G) := \mathfrak{R}_{R_i/\mathbb{k}}(G_i)$$

be the Weil restriction of G_i to $\operatorname{Spec} \mathbb{k}$ with respect the structure of a $\mathbb{k}$-algebra on R_i. The functor $G \longrightarrow \mathcal{G}_i(G)$ preserves many good properties of G: for example, $\mathcal{G}_i(G)$ is affine (resp. smooth) if G is affine (resp. smooth), see [57, §7]. It follows from the definition of the Weil restriction that

$$\mathcal{G}_i(G)(\mathbb{k}) = G(R_i).$$

The natural truncation homomorphisms $R_{i+1} \longrightarrow R_i$ define, by functoriality, a homomorphism of group schemes $\alpha_i : \mathcal{G}_i(G) \longrightarrow \mathcal{G}_{i-1}(G)$. The projective system $\{\mathcal{G}_i(G), \alpha_i\}_{i \geq 1}$ in the category of commutative group schemes over $\mathbb{k}$ is called the *Greenberg realization* of G.

Let $\mathcal{G}_i(G)^{\mathrm{pf}}$ be the perfection of the group scheme $\mathcal{G}_i(G)$. It defines a functor from the category of group schemes over R to the category of perfect group schemes over $\mathbb{k}$. The natural truncation homomorphisms $R_{i+1} \longrightarrow R_i$ define, by functoriality, a projective system $\mathcal{G}_i(G)^{\mathrm{pf}}$ in the abelian category of quasi-algebraic groups over $\mathbb{k}$. The pro-algebraic group

$$\mathcal{G}(G) := \varprojlim_i \mathcal{G}_i(G)$$

is called the *perfect Greenberg realization* of G.

Example 0.1.24 If $G = \mathbb{G}_{a,R}$, then

$$\mathcal{G}_i(G) = \mathfrak{R}_{R_i/\mathbb{k}}(\mathbb{G}_{a,R_i}) \cong \mathbb{G}_{a,\mathbb{k}}^i$$

and $\mathcal{G}_i(G)(\mathbb{k}) = R_i$. We have an isomorphism of the additive group of the ring R_i with $\mathbb{k}^{\oplus i} \cong \mathbb{G}_{a,\mathbb{k}}^i(\mathbb{k})$. Since $\mathcal{G}_i(G)^{\mathrm{pf}} \cong (\mathbb{G}_{a,\mathbb{k}}^{\mathrm{pf}})^i$, we have

$$\mathcal{G}(\mathbb{G}_a) = \operatorname{Spec} \mathbb{k}[(t_i), (t_i^{1/p}), \ldots, (t_i)^{1/p^n})].$$

Its value on a perfect $\mathbb{k}$-algebra A is equal to $A[[t]]$.

0.2 Cyclic Covers

In this section, we discuss cyclic covers of a variety over an algebraically closed field $\mathbb{k}$, where we pay special attention to the case where the characteristic $p = \mathrm{char}(\mathbb{k})$ divides the degree of the cover.

To motivate our discussion, let us recall some well-known facts from field theory: let L/K be a finite Galois extension of fields, whose Galois group G is cyclic of order n. Then, if n is prime to $p = \mathrm{char}(K)$ and if K contains the n-th roots of unity, then L/K is a *Kummer extension*, that is, of the form $L = K(a^{1/n})$ for some $a \in K$. On the other hand, if $n = p > 0$, then L/K is an *Artin–Schreier extension*, that is, of the form $L = K(\eta)$, where η is a root of an equation of the form $x^p - x + a = 0$. Finally, if L/K is a finite and *inseparable* field extension of degree $p > 0$, then $L = K(a^{1/p})$ for some $a \in K$. There is no group acting in the latter case, but it turns out that there acts an infinitesimal group scheme that plays the role of a Galois group. In order to globalize these three types of extensions to finite morphisms between schemes, we start by recalling a couple of facts about locally free and reflexive sheaves, which are interesting in their own right.

Let $\mathcal{F}$ be a coherent $\mathcal{O}_X$-module on an integral, separated, and noetherian scheme X. Then, the *dual* of $\mathcal{F}$ is defined to be the $\mathcal{O}_X$-module $\mathcal{F}^\vee := \mathcal{H}om(\mathcal{F}, \mathcal{O}_X)$ and we let $\phi : \mathcal{F} \to (\mathcal{F}^\vee)^\vee$ be the natural morphism of $\mathcal{O}_X$-modules. Since X is an integral scheme, the kernel and cokernel of ϕ are torsion $\mathcal{O}_X$-modules. By definition, $\mathcal{F}^{\vee\vee}$ is called the *reflexive hull* of $\mathcal{F}$ and $\mathcal{F}$ is called *reflexive* if ϕ is an isomorphism. In particular, a reflexive coherent $\mathcal{O}_X$-module is torsion-free. For example, locally free $\mathcal{O}_X$-modules of finite rank are reflexive. Moreover, a coherent $\mathcal{O}_X$-module $\mathcal{F}$ is reflexive if and only if there exists an open affine cover $\mathfrak{U} = \{U_i\}_i$ of X such that each $\mathcal{F}|_{U_i}$ is the kernel of some homomorphism of coherent $\mathcal{O}_{U_i}$-modules $\psi : \mathcal{E}_i \to \mathcal{G}_i$, where $\mathcal{E}_i$ is locally free and $\mathcal{G}_i$ is torsion-free, see [295, Proposition 1.1]. In particular, if $\mathcal{F}$ is an arbitrary coherent $\mathcal{O}_X$-module, then $\mathcal{F}^\vee$ is a reflexive $\mathcal{O}_X$-module. For a coherent $\mathcal{O}_X$-module $\mathcal{M}$, we define its *rank* to be the rank of the $\mathcal{O}_{X,\eta}$-module $\mathcal{M}_\eta$, where $\eta \in X$ denotes the generic point of X. The following extension result is an important characterization of reflexive sheaves [295, Proposition 1.6].

Proposition 0.2.1 *Let X be an integral, separated, and noetherian scheme, which is also normal. Then, a coherent $\mathcal{O}_X$-module $\mathcal{F}$ is reflexive if and only if for every open subset $U \subseteq X$ and every closed subset $Z \subseteq U$ of codimension ≥ 2 the natural restriction map*

$$H^0(U, \mathcal{F}|_U) \to H^0(U\backslash Z, \mathcal{F}|_{U\backslash Z})$$

is an isomorphism.

Let us give a first application of this extension result: if a Cohen–Macaulay scheme X is of finite type over a field, then it possesses a *dualizing sheaf* ω_X°, which is a coherent $\mathcal{O}_X$-module, see [294, Section III.7]. Moreover, X is called *Gorenstein* if ω_X° is an invertible $\mathcal{O}_X$-module. For example, smooth varieties over fields and local complete intersections in them are Gorenstein. For a scheme X of finite type over a field $\Bbbk$, we let $\Omega_{X/\Bbbk}$ be the coherent $\mathcal{O}_X$-module of *Kähler differentials*.

Proposition 0.2.2 *Let X be a variety over a field $\Bbbk$.*

1. *If X is smooth over $\Bbbk$, then there exists an isomorphism of invertible $\mathcal{O}_X$-modules $\omega_X^\circ \cong \Lambda^{\dim(X)}\Omega_{X/\Bbbk}$.*
2. *If X is normal, then there exists an isomorphism of reflexive and coherent $\mathcal{O}_X$-modules $(\omega_X^\circ)^{\vee\vee} \cong (\Lambda^{\dim(X)}\Omega_{X/\Bbbk})^{\vee\vee}$.*

Proof Assertion (1) is well-known, see, for example [294, Corollary III.7.12]. To prove Assertion (2), let $X_{\mathrm{sm}} \subseteq X$ be the smooth locus of X. Since X is normal, the complement $X\backslash X_{\mathrm{sm}}$ is closed and of codimension ≥ 2 by Serre's normality criterion, see [497, Theorem 23.8], for example. By (1), we already have the desired isomorphism over X_{sm}. Passing to reflexive hulls, we obtain an isomorphism of reflexive $\mathcal{O}_X$-modules over X_{sm} that extends to X by reflexivity. □

For a normal variety X over a field $\Bbbk$, the $\mathcal{O}_X$-module $(\Lambda^{\dim(X)}\Omega_{X/\Bbbk})^{\vee\vee}$ is sometimes called the *canonical sheaf*. Another result that can be proved using the extension property of reflexive sheaves is the following, see [295, Corollary 1.7] for a proof.

Proposition 0.2.3 *Let $f : X \to Y$ be a dominant and proper morphism between normal, integral, and separated schemes, such that all fibers are of the same dimension. If $\mathcal{F}$ is a reflexive and coherent $\mathcal{O}_X$-module, then $f_*\mathcal{F}$ is a reflexive and coherent $\mathcal{O}_Y$-module.*

Now, we link reflexive $\mathcal{O}_X$-modules of rank 1 to Weil divisors and the class group: let X be an integral, separated, normal, and noetherian scheme. Then, a *prime divisor* on X is an integral subscheme of codimension one, and a *Weil divisor* is a finite formal sum $D = \sum_i a_i D_i$ with $a_i \in \mathbb{Z}$ and prime divisors D_i. The divisor is called *effective* if $a_i \geq 0$ for all i. A Weil divisor that arises as the divisor $\mathrm{div}(f)$ of some rational function f on X is called a *principal divisor*. Two Weil divisors are said to be *linearly equivalent* if their difference is a principal divisor. The abelian group of Weil divisors modulo principal divisors, or, equivalently, the abelian group of Weil divisors modulo linear linear equivalence, is called the *Weil divisor class group*, and we refer to [294, Chapter II.6] for details. Associated to a Weil divisor, there is an associated reflexive $\mathcal{O}_X$-module $\mathcal{O}_X(D)$ of rank 1, whose sections over some open set $U \subseteq X$ are those rational functions f on X with $\mathrm{div}(f)|_U + D|_U \geq 0$. Then, we have $\mathcal{O}_X(-D) \cong \mathcal{O}_X(D)^\vee$ and if D is effective, then $\mathcal{O}_X(-D)$ is the ideal sheaf of D. Next, if D_1, D_2 are two Weil divisors on X, then we have $\mathcal{O}_X(D_1 + D_2) \cong (\mathcal{O}_X(D_1) \otimes \mathcal{O}_X(D_2))^{\vee\vee}$.

The set $\mathrm{Cl}(X)$ of reflexive $\mathcal{O}_X$-modules of rank 1 on X is an abelian group with product $\mathcal{F}\cdot\mathcal{G} := (\mathcal{F}\otimes\mathcal{G})^{\vee\vee}$ and inverse $\mathcal{F}^\vee$ is an abelian group, called the *class group* of X. The association $D \mapsto \mathcal{O}_X(D)$ induces an isomorphism of the Weil divisor class group with the class group. Finally, a *Cartier divisor* is a Weil divisor D that is locally principal, that is, there exists an open affine cover $\mathfrak{U} = \{U_i\}_i$ of X such that for all i, the restriction $D|_{U_i}$ is of the form $\mathrm{div}(f_i)$ for some rational function f_i on U_i. The group of Cartier divisors modulo linear equivalence is called the *Cartier divisor class group* and the reflexive $\mathcal{O}_X$-module $\mathcal{O}_X(D)$ associated to a Cartier divisor is an invertible $\mathcal{O}_X$-module. The set $\mathrm{Pic}(X)$ of invertible $\mathcal{O}_X$-modules with product $\otimes_{\mathcal{O}_X}$ and inverse $-^\vee$ is the *Picard group* and here, the association $D \mapsto \mathcal{O}_X(D)$ induces an isomorphism of the Cartier divisor class group with the Picard group. Since every invertible $\mathcal{O}_X$-module is reflexive, we obtain a homomorphism of abelian groups

$$\mathrm{Pic}(X) \to \mathrm{Cl}(X),$$

which is injective, but not necessarily surjective, and we refer to Proposition 0.4.19 for examples. If X is locally factorial, that is, all local rings of X are unique factorization domains, then this homomorphism is also surjective. This holds, for example, if X is a regular scheme. Again, we refer to [294, Chapter II.6] for details and proofs.

Let X be an integral, separated, regular, and noetherian scheme and let $\mathcal{F}$ be a coherent $\mathcal{O}_X$-module. By definition, the *singular locus* of $\mathcal{F}$ is the subset of X

$$\mathrm{Sing}(\mathcal{F}) := \{x \in X : \mathcal{F}_x \text{is not a locally free } \mathcal{O}_{X,x}\text{-module}\},$$

which is closed set of codimension ≥ 1. In the following cases, the singular locus is even smaller.

1. If $\mathcal{F}$ is a torsion-free $\mathcal{O}_X$-module, then $\operatorname{codim} \mathrm{Sing}(\mathcal{F}) \geq 2$. In particular, if $\dim(X) \leq 1$, then torsion-free and coherent $\mathcal{O}_X$-modules are locally free.
2. If $\mathcal{F}$ is a reflexive $\mathcal{O}_X$-module, then $\operatorname{codim} \mathrm{Sing}(\mathcal{F}) \geq 3$. In particular, if $\dim(X) \leq 2$, then reflexive and coherent $\mathcal{O}_X$-modules are locally free.
3. If $\mathcal{F}$ is a reflexive $\mathcal{O}_X$-module of rank 1, then $\mathcal{F}$ is locally free, that is, an invertible $\mathcal{O}_X$-module. Put differently, Weil and Cartier divisors coincide on X.

We refer to [240, Section 2], as well as to [295] for details, proofs, and further results. We end our discussion of sheaves and their singularities with the following useful flatness result, see, for example [497, Theorem 23.1], the corollary to [497, Theorem 23.3], or [204, Corollary 18.17].

Proposition 0.2.4 *Let $f : Y \to X$ be a finite morphism between noetherian schemes, where X is regular. Then, f is flat if and only if Y is Cohen–Macaulay.*

Remark 0.2.5 If Y is a normal scheme of dimension ≤ 2, then Y is Cohen–Macaulay by Serre's criterion for normality (see [497, Theorem 23.8], for example). Thus, a finite morphism from a normal curve (resp. surface) to a regular curve (resp. surface) is automatically flat.

Now, we come to $\boldsymbol{\mu}_n$-covers of schemes. These generalize Kummer extensions of fields to schemes and we refer to Proposition 0.2.29 for the more special case of $\boldsymbol{\mu}_n$-torsors.

First, following [294] we recall the definition of a vector bundle and its transition functions. Let $\mathcal{E}$ be a locally free $\mathcal{O}_X$-module of rank r on a separated and connected scheme X, which is sometimes also called a rank r *vector bundle* over X. It defines an associated *geometric vector bundle* or the *total space* $\mathbb{V}(\mathcal{E}) := \operatorname{Spec} S^\bullet(\mathcal{E}) \to X$, where $S^\bullet(\mathcal{E})$ denotes the symmetric algebra of $\mathcal{E}$. We refer to [294, Chapter II, Exercise 5.18] for details and note that the sheaf of sections of $\mathbb{V}(\mathcal{E})$ is isomorphic to the dual $\mathcal{O}_X$-Module $\mathcal{E}^\vee := \mathcal{H}om(\mathcal{E}, \mathcal{O}_X)$.

By abuse of terminology, we will often identify locally free sheaves with their total spaces. Let $\mathfrak{U} = \{U_\alpha\}_{\alpha\in I}$ be an open affine cover of X trivializing $\mathcal{E}$, that is, for every $\alpha \in I$, there exists an isomorphism $\phi_\alpha : \mathcal{E}_{|U_\alpha} \to \mathcal{O}^r_{U_\alpha}$, where r is the rank of $\mathcal{E}$. By definition, the *transition functions* of $\mathcal{E}$ with respect to the trivializing cover $\mathfrak{U}$ are the isomorphisms $g_{\alpha\beta} = \phi_\alpha \circ \phi_\beta^{-1}$ of $\mathcal{O}^r_{U_\alpha\cap U_\beta}$, where we denote the restriction of ϕ_α to $U_\alpha \cap U_\beta$ again by ϕ_α. In the standard basis $\underline{e} = (e_1, \ldots, e_r)$ of $\mathcal{O}^r_{U_\alpha\cap U_\beta}$, $g_{\alpha\beta}$ is given by an $r \times r$-matrix (a_{ij}) with entries in $\mathcal{O}_X(U_\alpha \cap U_\beta)$. The sheaf of sections $\mathcal{E}^\vee$ has the dual base $(u_1, \ldots, u_r)$ with transition functions ${}^t g_{\alpha\beta}^{-1}$.

Let $\underline{e}^{(\alpha)} = (e_1^{(\alpha)}, \ldots, e_r^{(\alpha)})$ and $\underline{e}^{(\beta)} = (e_1^{(\beta)}, \ldots, e_r^{(\beta)})$ be the images of the basis $\underline{e}$ under ϕ_α and ϕ_β. Then, we find

$$e_j^{(\beta)} \;=\; \sum_{i=1}^{r} a_{ij}\, e_i^{(\alpha)}.$$

Thus, the matrix (a_{ij}) can be viewed as the transition matrix from the basis $\underline{e}^{(\alpha)}$ to the basis $\underline{e}^{(\beta)}$. In particular, the coordinate vectors $(u_1^{(\alpha)}, \ldots, u_r^{(\alpha)})$, $(u_1^{(\beta)}, \ldots, u_r^{(\beta)})$ of a vector in $\mathcal{E}^\vee_{|U_\alpha\cap U_\beta}$ are transformed as follows

$$\begin{pmatrix} u_1^{(\alpha)} \\ \vdots \\ u_r^{(\alpha)} \end{pmatrix} \;=\; g_{\alpha\beta} \begin{pmatrix} u_1^{(\beta)} \\ \vdots \\ u_r^{(\beta)} \end{pmatrix}, \tag{0.2.1}$$

where we identify $g_{\alpha\beta}$ with the matrix (a_{ij}).

Let us now have a closer look at the case of rank 1: let $\mathcal{L}$ be an invertible sheaf, that is, a locally free $\mathcal{O}_X$-module of rank 1. In this case, the associated vector bundle $\mathbb{L} := \mathbb{V}(\mathcal{L})$ is called a *line bundle*. The identification of $\mathcal{O}_X$ with $S^0(\mathcal{L}) \subseteq S^\bullet(\mathcal{L})$ turns $\mathcal{O}_X$ into a subsheaf and a direct summand of $S^\bullet(\mathcal{L})$. Passing to relative spectra, we obtain a canonical morphism $\pi : \mathbb{V}(\mathcal{L}) \to X$. We have

$$\pi_*\mathcal{O}_{\mathbb{L}} \;=\; \bigoplus_{i\geq 0} \mathcal{L}^{\otimes i} \;=\; S^\bullet(\mathcal{L}) \tag{0.2.2}$$

and then, for every integer k, the projection formula gives

$$\pi_*\pi^*\mathcal{L}^{\otimes k} \;=\; \pi_*\mathcal{O}_{\mathbb{L}} \otimes \mathcal{L}^{\otimes k} \;=\; \bigoplus_{i\geq 0} \mathcal{L}^{\otimes(k+i)}.$$

Using the natural isomorphism $H^0(X, \pi_*\pi^*\mathcal{L}^{\otimes k}) = H^0(\mathbb{L}, \pi^*\mathcal{L}^{\otimes k})$, we obtain, for $k \geq 0$, the section of the invertible sheaf $\pi^*\mathcal{L}^{\otimes -k}$ on $\mathbb{L}$ corresponding to the section 1 in the direct summand $\mathcal{O}_X$ of $S^\bullet(\mathcal{L})$. It is called the *tautological section* of $\pi^*\mathcal{L}^{\otimes -k}$.

We now discuss cyclic covers of schemes and start with the main example of simple $\boldsymbol{\mu}_n$-covers associated to the data $(\mathcal{L}, s)$ and their singularities and after that we discuss more general $\boldsymbol{\mu}_n$-covers. Here $\mathcal{L}$ is an invertible sheaf and, following the customary convention, we will construct such covers as closed subschemes of the geometric vector bundle $\mathbb{L} := \mathbb{V}(\mathcal{L}^\vee)$ associated to $\mathcal{L}^\vee$.

Let X be an integral and separated scheme, let $\mathcal{L}$ be an invertible $\mathcal{O}_X$-module, let $n \geq 1$ be an integer, and let s be a nonzero section of $\mathcal{L}^{\otimes n}$. Let t be the tautological section of $\pi^*\mathcal{L}$ on the line bundle $\pi : \mathbb{L} = \mathbb{V}(\mathcal{L}^\vee) \to X$ and let

$$\begin{tikzcd} Y := \{t^n - \pi^*(s) = 0\} \arrow[r] \arrow[dr, "f"'] & \mathbb{L} \arrow[d, "\pi"] \\ & X \end{tikzcd}$$

be the zero scheme of $t^n - \pi^*(s)$. We denote by $i : Y \hookrightarrow \mathbb{L}$ the inclusion, set $f := \pi \circ i$, and call $f : Y \to X$ the *simple* $\boldsymbol{\mu}_n$*-cover* associated to the data $(\mathcal{L}, s)$. Then, Y is an effective Cartier divisor in $\mathbb{L}$ and we have $\mathcal{O}_{\mathbb{L}}(Y) \cong \pi^*\mathcal{L}^{\otimes n}$, that is, its sheaf of ideals is equal to $\mathcal{O}_{\mathbb{L}}(-Y) = \pi^*\mathcal{L}^{\otimes(-n)}$. Moreover, the inclusion $\pi^*\mathcal{L}^{\otimes(-n)} \subset \mathcal{O}_{\mathbb{L}}$ corresponds to the inclusion $\mathcal{L}^{\otimes(-n)} \subset S^\bullet(\mathcal{L}^{-1})$ in the decomposition (0.2.2). In particular, we obtain an isomorphism of $\mathcal{O}_X$-modules

$$f_*\mathcal{O}_Y \cong \mathcal{A} := \mathcal{O}_X \oplus \mathcal{L}^{-1} \oplus \ldots \oplus \mathcal{L}^{\otimes -(n-1)} \qquad (0.2.3)$$

and thus, $Y \cong \operatorname{Spec} f_*\mathcal{O}_Y$, where the $\mathcal{O}_X$-algebra structure on (0.2.3) is determined by the map $\mathcal{L}^{\otimes(-n)} \to \mathcal{O}_X$ corresponding to the section $s : \mathcal{O}_X \to \mathcal{L}^{\otimes n}$. Locally over an open affine subset $U \subseteq X$ with $\mathcal{L}|_U \cong \mathcal{O}_U$, the simple $\boldsymbol{\mu}_n$-cover is isomorphic to

$$\operatorname{Spec} \mathcal{O}_X(U)[T]/(T^n - s_U) \to \operatorname{Spec} \mathcal{O}_X(U) \cong U, \qquad (0.2.4)$$

where s_U is a local equation of the section s over U. Moreover, if U and V are two open affine subsets of X, then the local sections s_U and s_V satisfy $s_U = g_{UV}^n \cdot s_V$ on the intersection $U \cap V$, where $g_{UV} \in \mathcal{O}_X(U \cap V)^\times$ is the transition function of the invertible sheaf $\mathcal{L}$ with respect to $\{U, V\}$. If $X = \operatorname{Spec} K$ for some field K, then every global section of an invertible $\mathcal{O}_X$-module $\mathcal{L}$ can be identified with an element $s \in K$. Then, the local description (0.2.4) shows that the simple $\boldsymbol{\mu}_n$-cover of X associated to $(\mathcal{L}, s)$ is $Y = \operatorname{Spec} L \to X = \operatorname{Spec} K$, where $L = K(\sqrt[n]{s})$. Thus, if n is coprime to $p = \operatorname{char}(K)$, then L/K is a separable Kummer extension. Next, we remind the reader of the finite, flat, and commutative group scheme $\boldsymbol{\mu}_n$ of length n introduced in Example 0.1.8. The following result justifies calling $f : Y \to X$ a $\boldsymbol{\mu}_n$-cover.

Proposition 0.2.6 *Let X be an integral and separated scheme and let $f : Y \to X$ be the simple $\boldsymbol{\mu}_n$-cover associated to $(\mathcal{L}, s)$. Then, f is a finite and flat morphism of degree n and there exists a $\boldsymbol{\mu}_n$-action on Y such that f is the quotient by this action*

$$Y \xrightarrow{f} X \cong Y/\boldsymbol{\mu}_n.$$

Moreover,

1. *if X is regular, then Y is Cohen–Macaulay.*
2. *If X is a smooth variety over an algebraically closed field $\Bbbk$ of characteristic $p \geq 0$, if $p \nmid n$, and if the zero locus $Z(s) \subseteq X$ of the section s is an integral divisor, then Y is normal. In this case, we have an equality of singular (non-smooth) loci*

$$\mathrm{Sing}(Y) \;=\; f^{-1}(\mathrm{Sing}(Z(s)))$$

(discarding scheme structures).

Proof By (0.2.3), $f_*\mathcal{O}_Y$ is a locally free $\mathcal{O}_X$-module of rank n, and thus, f is a finite and flat morphism of degree n. If X is regular, then Y is Cohen–Macaulay by Proposition 0.2.4. Moreover, if X is as in (2), then the local description (0.2.4) and the Jacobian criterion for smoothness shows that the singular locus of Y is of codimension ≥ 2 and in fact equal to the pre-image of the singular locus of $Z(s)$ via f. Being R_1 and S_2, it follows that Y is normal by Serre's criterion, see [497, Theorem 23.8], for example.

In any case, the group scheme $\boldsymbol{\mu}_n$ (see Example 0.1.8) acts on Y via the coaction that is locally defined by

$$\mathcal{O}_U[T]/(T^n - s_U) \;\to\; \mathcal{O}_U[T]/(T^n - s_U) \otimes_{\mathcal{O}_U} \mathcal{O}_U[\zeta], \quad T \mapsto T \otimes \zeta,$$

where $U \subseteq X$ is an open and affine subset with $\mathcal{L}|_U \cong \mathcal{O}_U$ and s_U is as in (0.2.4). It is easy to see that the ring of invariants $\mathcal{O}_Y(f^{-1}(U))^{\boldsymbol{\mu}_n}$ is equal to the subring $\mathcal{O}_X(U)$ of $\mathcal{O}_Y(f^{-1}(U))$, from which we conclude that f is the quotient morphism with quotient $X \cong Y/\boldsymbol{\mu}_n$. □

Let us mention a special case: assume that $n = km$ for some integers $k, m \geq 1$ and let $s \in H^0(X, \mathcal{L}^{\otimes n})$. Then, s is a global section of $(\mathcal{L}^{\otimes m})^{\otimes k} = \mathcal{L}^{\otimes n}$ and the pair $(\mathcal{L}^{\otimes m}, s)$ defines a simple $\boldsymbol{\mu}_k$-cover $g : Z \to X$. We leave to the reader to check that the simple $\boldsymbol{\mu}_n$-cover $f : Y \to X$ defined by $(\mathcal{L}, s)$ factors through g and that this corresponds to the quotients $Y \to Z = Y/\boldsymbol{\mu}_m \to X = Z/\boldsymbol{\mu}_k = Y/\boldsymbol{\mu}_{km}$. In the case of a Kummer extension $K \subseteq K(\sqrt[n]{s})$ of a field K, this factorization corresponds to the inclusions $K \subseteq K(\sqrt[k]{s}) \subseteq K(\sqrt[n]{s})$.

We now turn to branch and ramification loci of a simple $\boldsymbol{\mu}_n$-cover $f : Y \to X$ associated to $(\mathcal{L}, s)$. Let us also assume that X is a smooth variety over an algebraically closed field $\Bbbk$ of characteristic $p \geq 0$ and that Y is a normal variety. First, assume that $(n, p) = 1$. In this case, f is *separable*, that is, generically étale. There is a maximal closed and proper subset $\mathsf{B}(f) \subset X$, possibly empty, over which f is not étale. Before proceeding, let us recall Zariski's theorem on the purity of the branch locus from [739].

Theorem 0.2.7 *Let $f : Y \to X$ be a finite morphism from a normal variety to a smooth variety. Assume that f is separable, that is, the induced extension of function fields $\Bbbk(X) \subset \Bbbk(Y)$ is separable. Let $\mathsf{B}(f) \subset X$ be the set where f is not étale. Then, $\mathsf{B}(f)$ is empty or a divisor.*

For a simple $\boldsymbol{\mu}_n$-cover associated to $(\mathcal{L}, s)$ the branch divisor is the zero locus $Z(s)$ of the section s. The reduced inverse image of $\mathsf{B}(f)$ is called the *(reduced) ramification divisor* $\mathsf{R}_{\mathrm{red}}(f)$ of f. For a simple $\boldsymbol{\mu}_n$-cover $f : Y \to X$ with X smooth and Y normal, the branch divisor $\mathsf{B}(f)$ is reduced and a local computation shows that we have an equality of Cartier divisors on Y

$$f^*(\mathsf{B}(f)) \;=\; n \cdot \mathsf{R}_{\mathrm{red}}(f).$$

By Proposition 0.2.6, the singular locus of Y is contained in the ramification divisor of f.

Proposition 0.2.8 *Let $f : Y \to X$ be a simple $\boldsymbol{\mu}_n$-cover associated to the data $(\mathcal{L}, s)$. Then, the sheaf of relative differentials $\Omega^1_{Y/X}$ admits the following projective resolution:*

$$0 \to f^*\mathcal{L}^{\otimes(-n)} \to f^*\mathcal{L}^{-1} \to \Omega^1_{Y/X} \to 0. \tag{0.2.5}$$

Proof We know that Y is a closed subscheme of $\mathbb{V}(\mathcal{L}^{-1})$ given by the ideal sheaf $\mathcal{O}_{\mathbb{L}}(-Y) = \pi^*\mathcal{L}^{\otimes(-n)}$. It is easy to see that

$$\Omega^1_{\mathbb{L}/X} \;\cong\; \pi^*\mathcal{L}^{-1}.$$

We apply the usual exact sequence from [294, Chapter II, Proposition 8.12]

$$0 \to \mathcal{O}_{\mathbb{L}}(-Y)/\mathcal{O}_{\mathbb{L}}(-2Y) \xrightarrow{d} \Omega^1_{\mathbb{L}/X} \otimes \mathcal{O}_Y \to \Omega^1_{Y/X} \to 0,$$

where we use that $Y \hookrightarrow \mathbb{L}$ is a regular embedding, which implies that the homomorphism d is injective. Since $\mathcal{L}^{\otimes(-n)}/\mathcal{L}^{\otimes(-2n)} \cong \mathcal{L}^{\otimes(-n)} \otimes \mathcal{O}_Y$, the claim follows. □

Corollary 0.2.9 *Under the assumptions from the previous proposition assume moreover that $(p, n) = 1$. Then, the relative tangent sheaf has the following projective resolution:*

$$0 \to f^*\mathcal{L} \to f^*\mathcal{L}^{\otimes n} \to \Theta_{Y/X} \to 0.$$

Proof The morphism $f : Y \to X$ is finite and separable, which implies that the sheaf $\Omega^1_{Y/X}$ is supported on a closed proper closed subset of Y. Taking the dual of the exact sequence

$$0 \to f^*\Omega^1_{X/\mathbb{k}} \to \Omega^1_{Y/\mathbb{k}} \to \Omega^1_{Y/X} \to 0,$$

we obtain an exact sequence

$$0 \to \Theta_{Y/\mathbb{k}} \to f^*\Theta_{X/\mathbb{k}} \to \Theta_{Y/X} \to 0, \tag{0.2.6}$$

where

$$\Theta_{Y/X} \;\cong\; \mathcal{E}xt^1(\Omega^1_{Y/X}, \mathcal{O}_Y)$$

is the *relative tangent sheaf*. This implies the claimed exact sequence. □

Next, assume that $p|n$. In this case, the cover $f : Y \to X$ is *inseparable*. In analogy to the above, we can still set $\mathsf{B}(f) := Z(s)$ and let $\mathsf{R}_{\mathrm{red}}(f)$ be the reduced inverse image of $\mathsf{B}(f)$. Since f is inseparable, we do not have well-defined branch and ramification loci, since there is no point over which f is étale, that is, the branch locus (resp. ramification locus) of f is equal to X (resp. Y). On the other hand, it follows from (0.2.10) below that the Cartier divisors $\mathsf{R}_{\mathrm{red}}(f)$ and $\mathsf{B}(f)$ are at least well-defined invariants of f up to numerical equivalence. To illustrate that the divisor $\mathsf{B}(f)$ is not an invariant of f, we note that the data $(\mathcal{L}, s)$ and $(\mathcal{L}, s + u^p)$ with $u \in H^0(X, \mathcal{L}^{\otimes n/p})$ define the same simple $\boldsymbol{\mu}_n$-cover.

Before coming to the singularities of $\boldsymbol{\mu}_n$-covers in the inseparable case, we make a small detour on connections: let X be a scheme and let $\mathcal{E}$ be a $\mathcal{O}_X$-module. Then, a *connection* on $\mathcal{E}$ is a map of sheaves of abelian groups

$$\nabla : \mathcal{E} \to \Omega^1_{X/\Bbbk} \otimes_{\mathcal{O}_X} \mathcal{E},$$

such that for every open subset $U \subseteq X$ and local sections $f \in \mathcal{O}_X(U)$, $\xi \in \mathcal{E}(U)$, Leibniz's rule $\nabla(f \cdot \xi) = df \cdot \xi + f \cdot \nabla(\xi)$ holds true. For example, the classical differential $d : \mathcal{O}_X \to \Omega^1_{X/\Bbbk}$ is a connection, but in general, there is no canonical choice of connection on a given $\mathcal{O}_X$-module $\mathcal{E}$. However, if X is an integral and separated scheme in characteristic $p > 0$ with absolute Frobenius morphism $\mathbf{F} : X \to X$ and if $\mathcal{E}$ is a coherent $\mathcal{O}_X$-module, then $\mathbf{F}^*\mathcal{E}$ carries a distinguished connection, the *Cartier connection* or *canonical connection*, which is denoted by ∇_{can}. We refer to [376, Section 5] for the general case and only discuss the case where $\mathcal{E}$ is an invertible $\mathcal{O}_X$-module: let $U \subseteq X$ be an open subset and let $t \in \mathcal{E}(U)$ be a section such that t generates $\mathcal{E}$ over U. Then t^p generates $\mathbf{F}^*\mathcal{E}$ over U. For $f \in \mathcal{O}_X(U)$, we define $\nabla_{\mathrm{can}}(f \cdot t^p) := df \cdot t^p$. A different choice of generator t' of $\mathcal{E}$ over U differs by some invertible section $s \in \mathcal{O}_X(U)^\times$ and thus, $t^p = s^p \cdot t'^p$. It follows from $ds = 0$ that ∇_{can} is well-defined, that is, does not depend on the choice of generator of $\mathcal{E}$ over U. In particular, these affine local definitions glue to a well-defined connection ∇_{can} on $\mathbf{F}^*\mathcal{E}$. Now, since $\mathcal{E}$ is an invertible $\mathcal{O}_X$-module, we have $\mathbf{F}^*\mathcal{E} \cong \mathcal{E}^{\otimes p}$. In particular, every invertible sheaf on X that is divisible by p in the Picard group of X carries a Cartier connection. After these preparations, we have the following result.

Proposition 0.2.10 *Let X be smooth variety over an algebraically closed field $\Bbbk$ of characteristic $p > 0$ and let $f : Y \to X$ be an inseparable and simple cyclic $\boldsymbol{\mu}_n$-cover associated to $(\mathcal{L}, s)$. Let ∇_{can} be the Cartier connection of $\mathcal{L}^{\otimes n}$. Then, the singular locus of Y is equal to the pre-image of the zero set of*

$$\alpha_f := \nabla_{\mathrm{can}}(s) \in H^0(X, \Omega^1_{X/\Bbbk} \otimes_{\mathcal{O}_X} \mathcal{L}^{\otimes n}), \tag{0.2.7}$$

If Y is reduced, then the section α_f is not identically zero.

Proof If $U \subseteq X$ is an open affine subset and T_U is a local section that generates $\mathcal{L}$ over U, then Y is given locally over U by $T_U^n - s_U$, see the local description in (0.2.4). Since f is inseparable, we have $p|n$, that is, $\nabla_{\mathrm{can}}(s)$ is well-defined and equal to $d(s|_U)$ over U. Using $p|n$ and the Jacobian criterion of smoothness, the zeros of

$d(s_U)$ lie under the singular locus of Y. Moreover, if $\alpha_f = 0$, then $d(s_U) = 0$ for all U, and hence, Y is singular everywhere, that is, Y it not reduced. □

Remark 0.2.11 Let X be a smooth variety over an algebraically closed field $\Bbbk$ of characteristic $p \geq 0$ and let $f : Y \to X$ be a simple $\boldsymbol{\mu}_n$-cover associated to data $(\mathcal{L}, s)$. To describe the singularities of $\boldsymbol{\mu}_n$-covers in the generic case, assume that that $\mathcal{L}^{\otimes n}$ is very ample and that s is a sufficiently general section.

1. If $p \nmid n$, that is, f is separable, then it follows from Bertini's theorem that $Z(s)$ is a smooth divisor, and thus, Y is smooth by Proposition 0.2.6.
2. If $p \mid n$, that is, f is inseparable, and $\dim(X) \geq 2$, then Y is usually normal but not smooth. For example, if $p = n$ and $\dim(X) = 2$, then one expects that Y has at most rational double points of type A_{p-1}.

We refer to [459, Section 2] for details and more results about generic inseparable covers and Cartier's connection with a view toward inseparable covers. We also refer to Example 0.2.22 below for an example.

Let $f : Y \to X$ be a simple $\boldsymbol{\mu}_n$-cover of some regular and separated scheme X associated to $(\mathcal{L}, s)$. Then, Y is a hypersurface inside the line bundle $\pi : \mathbb{L} = \mathbb{V}(\mathcal{L}^{-1}) \to X$, the latter of which is also regular. In particular, Y is a complete intersection in a regular scheme, which implies that Y is Gorenstein, that is, the dualizing sheaf ω_Y is an invertible $\mathcal{O}_Y$-module. The corresponding divisor class (the equivalence is linear equivalence) is usually denoted by K_Y.

Proposition 0.2.12 *Let X be a regular and separated scheme and let $f : Y \to X$ be a simple $\boldsymbol{\mu}_n$-cover associated to $(\mathcal{L}, s)$. Then Y is Gorenstein and its dualizing sheaf is given by*

$$\omega_Y \cong f^*(\omega_X \otimes \mathcal{L}^{\otimes(n-1)}). \tag{0.2.8}$$

Proof We denote by ι the embedding of Y as a hypersurface into the line bundle $\pi : \mathbb{L} \to X$ and we have already seen that Y is defined by an equation of the form $t^n - \pi^*(s)$. By Grothendieck's duality theorem, the dualizing sheaf of Y is given by the *adjunction formula*

$$\omega_Y \cong \mathcal{E}xt^1(\mathcal{O}_Y, \omega_{\mathbb{L}}).$$

The determinant of the cotangent sequence together with the isomorphism $\omega_{\mathbb{L}/X} \cong \pi^*(\mathcal{L}^{-1})$ give

$$\omega_{\mathbb{L}} \cong \pi^*(\omega_X) \otimes \omega_{\mathbb{L}/X} \cong \pi^*(\omega_X \otimes \mathcal{L}^{-1}).$$

Applying $\mathcal{H}om(-, \omega_{\mathbb{L}})$ to the exact sequence $0 \to \mathcal{O}_{\mathbb{L}}(-Y) \to \mathcal{O}_{\mathbb{L}} \to \mathcal{O}_Y \to 0$, we obtain a short exact sequence

$$0 \to \mathcal{H}om(\mathcal{O}_{\mathbb{L}}, \omega_{\mathbb{L}}) \to \mathcal{H}om(\mathcal{O}_{\mathbb{L}}(-Y), \omega_{\mathbb{L}}) \to \mathcal{E}xt^1(\mathcal{O}_Y, \omega_{\mathbb{L}}) \to 0,$$

from which we deduce isomorphisms

$$\mathcal{E}xt^1(\mathcal{O}_Y, \omega_{\mathbb{L}}) \cong \iota^*(\omega_{\mathbb{L}} \otimes \mathcal{O}_{\mathbb{L}}(Y)) \cong \iota^*(\pi^*(\omega_X \otimes \mathcal{L}^{\otimes(n-1)})) = f^*(\omega_X \otimes \mathcal{L}^{\otimes(n-1)}),$$

which establishes the claim. □

Remark 0.2.13 If $f : Y \to X$ is a simple $\boldsymbol{\mu}_n$-cover of smooth varieties over a field $\Bbbk$, then formula (0.2.8) also follows immediately from Proposition 0.2.8 by taking first Chern classes in the exact sequences (0.2.5) and

$$0 \to f^*\Omega^1_{X/\Bbbk} \to \Omega^1_{Y/\Bbbk} \to \Omega^1_{Y/X} \to 0.$$

Namely, we compute

$$\begin{aligned} K_Y &= c_1(\Omega^1_{Y/\Bbbk}) = c_1(f^*\Omega^1_{X/\Bbbk}) + c_1(\Omega^1_{Y/X}) \\ &= c_1(f^*\Omega^1_{X/\Bbbk}) + c_1(f^*\mathcal{L}^{-1}) - c_1(f^*\mathcal{L}^{\otimes -n}) = f^*(\omega_X \otimes \mathcal{L}^{\otimes(n-1)}). \end{aligned} \tag{0.2.9}$$

Thus, for a simple $\boldsymbol{\mu}_n$-cover $f : Y \to X$, where Y is normal and X is a smooth variety over an algebraically closed field $\Bbbk$, we can rewrite the dualizing sheaf in terms of the ramification divisor class as

$$K_Y \equiv f^*(K_X) + \frac{(n-1)}{n} f^*(\mathsf{B}(f)) \equiv f^*(K_X) + (n-1)\mathsf{R}_{\mathrm{red}}(f), \tag{0.2.10}$$

where "≡" denotes equality of invertible sheaves (or, Cartier divisors) modulo numerical equivalence. In particular, (0.2.10) shows that the numerical equivalence class of $R_{\mathrm{red}}(f)$ can be recovered from f. Let us stress again that we only have well-defined branch and ramification loci (rather than Cartier divisor classes modulo numerical equivalence) if the cover is separable. Thus, in order to describe the dualizing sheaf of Y, the safest thing is to use Proposition 0.2.12. Before turning to more general $\boldsymbol{\mu}_n$-covers, let us give the following interesting application of the above discussion, which we will use in the next chapters.

Proposition 0.2.14 *Let $f : Y \to X$ be a finite morphism of proper varieties over an algebraically closed field $\Bbbk$. Let $\mathcal{L}$ be an invertible $\mathcal{O}_X$-module that lies in the kernel of the homomorphism*

$$f^* : \mathrm{Pic}(X) \to \mathrm{Pic}(Y).$$

Then, there exists an integer $n \geq 1$ and an isomorphism $s : \mathcal{L}^{\otimes n} \cong \mathcal{O}_X$ such that f factors through a $\boldsymbol{\mu}_n$-torsor, which is a simple $\boldsymbol{\mu}_n$-cover defined by $(\mathcal{L}, s)$.

Proof Let $\mathcal{A} := f_*\mathcal{O}_Y$, which is a finite $\mathcal{O}_X$-algebra. Let $\mathcal{L} \in \mathrm{Pic}(X)$ be such that $f^*\mathcal{L} \cong \mathcal{O}_Y$. Using the projection formula, we find $\mathcal{A} \cong f_*f^*\mathcal{L} \cong \mathcal{A} \otimes \mathcal{L}$. Using the inclusion $\mathcal{O}_X \to \mathcal{A}$, we obtain an inclusion $\phi_1 : \mathcal{L} \to \mathcal{A}$ and by induction, inclusions $\phi_i : \mathcal{L}^{\otimes i} \to \mathcal{A}$ for all $i \geq 1$. Using properness, we find $\mathrm{Aut}(\mathcal{O}_Y) = H^0(Y, \mathcal{O}_Y)^\times \cong \Bbbk^\times$. Thus, we may assume that the ϕ_i's satisfy $\phi_{i+j} = \phi_i \cdot \phi_j$ with respect to the product in $\mathcal{A}$ for all $i, j \geq 1$. Let $\phi : S^\bullet\mathcal{L} \to \mathcal{A}$ be the homomorphism of $\mathcal{O}_X$-algebras constructed from these ϕ_i's. If we denote by $\mathcal{B} \subseteq \mathcal{A}$ the image of ϕ, then this is a finite $\mathcal{O}_X$-algebra, and thus, there exists an $i \geq 1$ with an injective homomorphism $\mathcal{L}^{\otimes i} \to \mathcal{O}_X$. Replacing $\mathcal{L}$ by $\mathcal{L}^\vee$ and running through the above discussion, we find some $j \geq 1$ and an injective homomorphism $\mathcal{L}^{\otimes(-j)} \to \mathcal{O}_X$. Thus, there exists some $n \geq 1$ such that $\mathcal{L}^{\otimes n}$ and $\mathcal{L}^{\otimes(-n)}$ admit injective homomorphisms to $\mathcal{O}_X$. Let $n \geq 1$ be minimal integer with this property. Since X is integral and proper over

$\mathbb{k}$, this implies that we have an isomorphism $s : \mathcal{O}_X \to \mathcal{L}^{\otimes n}$ and think of s as a global section of $\mathcal{L}^{\otimes n}$. From this, we deduce that $\mathcal{A}$ contains the $\mathcal{O}_X$-subalgebra $\mathcal{O}_X \oplus \mathcal{L}^{-1} \oplus \ldots \oplus \mathcal{L}^{\otimes -(n-1)}$, whose algebra structure is given by the homomorphism s. By (0.2.3), this corresponds to a simple $\boldsymbol{\mu}_n$-cover associated to $(\mathcal{L}, s)$ and the morphism f factors through this simple $\boldsymbol{\mu}_n$-cover. Since $\mathcal{L}^{\otimes n} \cong \mathcal{O}_X$, it follows from Example 0.1.8 that this simple $\boldsymbol{\mu}_n$-cover is in fact a $\boldsymbol{\mu}_n$-torsor, see also Proposition 0.2.29 below. □

Let us now discuss more general classes of $\boldsymbol{\mu}_n$-covers: let X be a regular and separated scheme. We define a *(general) $\boldsymbol{\mu}_n$-cover* of X to be a finite morphism $f : Y \to X$ of degree n together with a $\boldsymbol{\mu}_n$-action on Y such that $Y/\boldsymbol{\mu}_n \cong X$. Since f is finite, $\mathcal{A} := f_*\mathcal{O}_Y$ is a coherent $\mathcal{O}_X$-module, which carries a $\boldsymbol{\mu}_n$-action. For a character $\chi : \boldsymbol{\mu}_n \to \mathbb{G}_m$, we denote by $\mathcal{A}^\chi$ the subsheaf of $\mathcal{A}$, whose local sections s satisfy $\sigma \cdot s = \chi(\sigma)s$ for all σ in $\boldsymbol{\mu}_n$ (if $p \mid n$, then this has to be read scheme-theoretically). From this, we obtain a direct sum decomposition

$$f_*\mathcal{O}_Y \;=\; \mathcal{A} \;\cong\; \bigoplus_\chi \mathcal{A}^\chi, \tag{0.2.11}$$

where χ runs through the character group $\mathrm{Hom}(\boldsymbol{\mu}_n, \mathbb{G}_m) \cong \mathbb{Z}/n\mathbb{Z}$. If χ_0 denotes the trivial character of $\boldsymbol{\mu}_n$, then we have $\mathcal{A}^{\chi_0} \cong \mathcal{O}_X$.

Proposition 0.2.15 *Let $f : Y \to X$ be a $\boldsymbol{\mu}_n$-cover of a regular scheme. Assume that Y is Cohen–Macaulay or that Y is normal or that f is flat. Then,*

1. *f is flat and Y is Cohen–Macaulay, and*
2. *each $\mathcal{A}^\chi$ in* (0.2.11) *is an invertible $\mathcal{O}_X$-module.*

Proof If f is flat or Y is Cohen–Macaulay, then Claim (1) follows from Proposition 0.2.4. In this case, $\mathcal{A}$ is a coherent and flat $\mathcal{O}_X$-module, that is, locally free. If Y is normal, then $\mathcal{A}$ is a coherent and reflexive $\mathcal{O}_X$-module by Proposition 0.2.3. Thus, in any case, $\mathcal{A}$ is a reflexive $\mathcal{O}_X$-module. Being direct summands of a reflexive and coherent $\mathcal{O}_X$-module, each $\mathcal{A}^\chi$ is a reflexive and coherent $\mathcal{O}_X$-module. It is easy to see that each $\mathcal{A}^\chi$ is of rank 1 and since X is a regular scheme, each $\mathcal{A}^\chi$ is an invertible $\mathcal{O}_X$-module. In particular, $\mathcal{A}$ is locally free $\mathcal{O}_X$-module, which implies that f is flat in any case, and then, Proposition 0.2.4 implies that Y is Cohen–Macaulay in any case. □

Next, we come to the $\mathcal{O}_X$-algebra structure on $\mathcal{A}$: it is uniquely determined by multiplication maps $\mathcal{A}^\chi \otimes \mathcal{A}^{\chi'} \to \mathcal{A}^{\chi\chi'}$ for all characters χ, χ' of $\boldsymbol{\mu}_n$. These maps correspond to global sections of $\mathcal{A}^{\chi\chi'} \otimes (\mathcal{A}^\chi \otimes \mathcal{A}^{\chi'})^{-1}$. The following result shows that normal $\boldsymbol{\mu}_n$-covers of regular schemes are always simple $\boldsymbol{\mu}_n$-covers over some open subset. However, it is not true that every $\boldsymbol{\mu}_n$-cover of a regular scheme is a simple $\boldsymbol{\mu}_n$-cover, not even Zariski locally.

Proposition 0.2.16 *Let X be a separated and regular scheme, let $f : Y \to X$ be a $\boldsymbol{\mu}_n$-cover, and assume that Y is normal. Then, there exists a simple $\boldsymbol{\mu}_n$-cover $Z \to X$*

associated to some data $(\mathcal{L}, s)$, where Z is integral and Cohen–Macaulay but not necessarily normal, and such that f factors as

$$Y \to Z \to X$$

compatible with the $\boldsymbol{\mu}_n$-actions on Y and Z, and such that $Y \to Z$ is the normalization morphism. In particular, there exists an open and dense subset $U \subseteq X$, such that $f^{-1}(U) \to U$ is the simple $\boldsymbol{\mu}_n$-cover associated to $(\mathcal{L}|_U, s|_U)$.

Proof First, we choose a generator χ_1 of the cyclic group $\mathrm{Hom}(\boldsymbol{\mu}_n, \mathbb{G}_m) \cong \mathbb{Z}/n\mathbb{Z}$ and set $\mathcal{L} := (\mathcal{A}^{\chi_1})^{\vee}$. Next, we set $\mathcal{A}' := \mathcal{O}_X \oplus \mathcal{L}^{-1} \oplus \ldots \oplus \mathcal{L}^{\otimes(-n-1)}$ and using the $\mathcal{O}_X$-algebra structure on $\mathcal{A}$, we find a sub-$\mathcal{O}_X$-algebra $\mathcal{A}' \subseteq \mathcal{A}$. More precisely, the $\mathcal{O}_X$-algebra structure is given by a morphism $\mathcal{L}^{\otimes(-n)} \to \mathcal{O}_X$ of $\mathcal{O}_X$-modules, that is, a section s of $\mathcal{L}^{\otimes n}$. Thus, $Z := \operatorname{Spec} \mathcal{A}' \to X$ is a simple $\boldsymbol{\mu}_n$-cover associated to $(\mathcal{L}, s)$, and we obtain a factorization $Y \to Z \to X$, compatible with the $\boldsymbol{\mu}_n$-actions on Y and Z. By Proposition 0.2.4, Z is Cohen–Macaulay. Since $\mathcal{A}'$ and $\mathcal{A}$ are both locally free $\mathcal{O}_X$-modules of rank n and $\mathcal{A}'$ is contained in $\mathcal{A}$, they are isomorphic at the generic point of Y. Thus, there exists an open and dense subset $U \subseteq X$, such that $\mathcal{A}'|_U$ and $\mathcal{A}|_U$ are isomorphic as $\mathcal{O}_U$-algebras. Thus, $Y \to Z$ is an isomorphism over $f^{-1}(U)$, which also implies that Z is integral. In particular, their normalizations are isomorphic and since Y was assumed to be normal, $Y \to Z$ is in fact the normalization morphism. □

Let X be a smooth variety over an algebraically closed field $\Bbbk$ of characteristic $p \geq 0$. For separable $\boldsymbol{\mu}_n$-covers $f : Y \to X$, that is $p \nmid n$, with Y is normal, we even have a structure result. To state it, we first decompose the branch locus $\mathsf{B}(f)$, which is a divisor in this case, as follows: let R be an integral component of $\mathsf{R}(f)$, which is also a divisor. Then, the subgroup scheme $H := \{\sigma \in \boldsymbol{\mu}_n | \sigma(R) = R\}$ is a cyclic subgroup scheme of $\boldsymbol{\mu}_n$, called the *inertia subgroup scheme*. The local ring $\mathcal{O}_{X,R}$ is a DVR and by [588, Lemma 1.2], there exists a uniformizer $t \in \mathcal{O}_{X,R}$ and a character $\psi : H \to \mathbb{G}_m$, such that $\sigma(t) = \psi(\sigma) \cdot t$ for all $\sigma \in \boldsymbol{\mu}_n$ (in fact, ψ generates the cyclic group $\mathrm{Hom}(H, \mathbb{G}_m)$). The pair (H, ψ) is the same for every component of $f^{-1}f(R)$, and thus, it is an invariant of the component $f(R)$ of B. From this, we obtain a decomposition

$$\mathsf{B}(f) = \sum_{(H,\psi)} D_{H,\psi} \qquad (0.2.12)$$

as a sum of divisors, where the sum runs over all cyclic subgroup schemes H of $\boldsymbol{\mu}_n$ and over all generators ψ of $\mathrm{Hom}(H, \mathbb{G}_m)$. Given a pair of characters χ, χ' of $\boldsymbol{\mu}_n$, a cyclic subgroup scheme H of $\boldsymbol{\mu}_n$, and a generator ψ of $\mathrm{Hom}(H, \boldsymbol{\mu}_n)$, there exist integers ι_χ and $\iota_{\chi'}$ such that

$$\chi|_H = \psi^{\iota_H} \quad \text{and} \quad \chi|_{H'} = \psi^{\iota_{H'}} \quad \text{with} \quad \iota_\chi, \iota_{\chi'} \in \{0, \ldots, n-1\}.$$

We use these integers to define

$$\varepsilon_{\chi,\chi'}^{H,\psi} := \begin{cases} 0 & \text{if } \iota_\chi + \iota_{\chi'} < n, \\ 1 & \text{else.} \end{cases}$$

Using this notation, we have the following classification of separable $\boldsymbol{\mu}_n$-covers.

Proposition 0.2.17 *Let $\Bbbk$ be an algebraically closed field of characteristic $p \geq 0$ and let $n \geq 1$ be an integer with $p \nmid n$. Let X be a smooth variety over $\Bbbk$.*

1. *Let $f : Y \to X$ be a $\boldsymbol{\mu}_n$-cover with Y normal. Then,*
$$\mathcal{A}^{\chi} \otimes \mathcal{A}^{\chi'} \;\cong\; \mathcal{A}^{\chi \cdot \chi'} \otimes \mathcal{O}_X\left(\sum_{(H,\psi)} \varepsilon_{\chi,\chi'}^{H,\psi} \cdot D_{H,\psi}\right), \tag{0.2.13}$$
and these isomorphisms determine the multiplication maps $\mathcal{A}^{\chi} \otimes \mathcal{A}^{\chi'} \to \mathcal{A}^{\chi\cdot\chi'}$ of the $\mathcal{O}_X$-algebra $f_*\mathcal{O}_Y \cong \bigoplus_{\chi} \mathcal{A}^{\chi}$.
2. *Conversely, given invertible $\mathcal{O}_X$-modules $\mathcal{A}^{\chi}$ for all $\chi \in \operatorname{Hom}(\boldsymbol{\mu}_n, \mathbb{G}_m)$ and effective divisors $D_{H,\psi}$ on X that satisfy* (0.2.13)*, then there exists a $\boldsymbol{\mu}_n$-cover*
$$Y \;:=\; \operatorname{Spec} \bigoplus_{\chi} \mathcal{A}^{\chi} \;\to\; X$$
such that $\mathcal{A}^{\chi} = (f_*\mathcal{O}_Y)^{\chi}$ and such that the $D_{H,\psi}$ are as in the decomposition (0.2.12) *of the branch divisor $\mathsf{B}(f)$.*

Proof We refer to [588, Theorem 2.1], where everything is stated for $\Bbbk = \mathbb{C}$, but the proof also works in characteristic $p > 0$ if $p \nmid n$. ◻

In fact, one can simplify the linear equivalences (0.2.13): given a character $\chi : \boldsymbol{\mu}_n \to \mathbb{G}_m$, let d^{χ} be its order in the character group and given (H, ψ) as above, let $r^{\chi}_{H,\psi} \in \{0, ..., |H| - 1\}$ such that $\chi|_H = \psi^{r^{\chi}_{H,\psi}}$. Then, by [588, Proposition 2.1], we have
$$(\mathcal{A}^{\chi})^{\otimes d_{\chi}} \;\cong\; \mathcal{O}_X\left(\sum_{(H,\psi)} \frac{d_{\chi} \cdot r^{\chi}_{H,\psi}}{|H|} \cdot D_{H,\psi}\right) \tag{0.2.14}$$
and note that each $(d_{\chi} \cdot r^{\chi}_{H,\psi})/|H|$ is an integer. Moreover, if χ_1 is a character that generates the cyclic group $\operatorname{Hom}(\boldsymbol{\mu}_n, \mathbb{G}_m)$, then there are isomorphisms of $\mathcal{O}_X$-modules
$$\mathcal{A}^{i\cdot\chi_1} \;\cong\; (\mathcal{A}^{\chi_1})^{\otimes i} \otimes \mathcal{O}_X\left(-\sum_{(H,\psi)} \left[\frac{i \,\cdot\, r^{i\cdot\chi_1}_{H,\psi}}{|H|}\right] D_{H,\psi}\right),$$
where $[-]$ denotes the integral part of a real number. In particular, the invertible sheaf $\mathcal{A}^{\chi_1}$ and the effective divisors $D_{H,\psi}$ determine the remaining invertible sheaves $\mathcal{A}^{\chi}$, up to isomorphism, and satisfy the isomorphisms (0.2.13). Thus, by Proposition 0.2.17, given an invertible sheaf $\mathcal{A}^{\chi_1}$ and effective divisors $D_{H,\psi}$ satisfying (0.2.14), there exists an associated $\boldsymbol{\mu}_n$-cover of X. By [588, Corollary 3.1], the so-constructed cover is normal if and only if every prime divisor of X occurs in $\sum_{(H,\psi)} D_{H,\psi}$ with multiplicity at most 1. For a smoothness criterion of this cover, we refer to [588, Proposition 3.1].

Let us also link the case of general $\boldsymbol{\mu}_n$-covers back to the case of simple $\boldsymbol{\mu}_n$-covers: If $f : Y \to X$ is a normal $\boldsymbol{\mu}_n$-cover of a smooth variety X over an algebraically closed fiel $\Bbbk$, then Proposition 0.2.17 applies. Let χ_1 be a character that generates the cyclic group $\mathrm{Hom}(\boldsymbol{\mu}_n, \mathbb{G}_m)$. Then, we may consider the simple $\boldsymbol{\mu}_n$-cover $f' : Y' \to X$ associated to $(\mathcal{A}^{\chi_1}, s)$, where s is the section of $\mathcal{A}^{n \cdot \chi_1}$ corresponding to $\sum_{(H,\psi)} \frac{n \cdot r^{\chi_1}_{H,\psi}}{|H|} \cdot D_{H,\psi}$. A local computation (or looking at the generic point) shows that we have a factorization $Y \to Y' \to X$, where the map $Y \to Y'$ is birational. In particular, since Y was assumed to be normal, $Y \to Y'$ is the normalization of Y', see also Proposition 0.2.16. Since simple $\boldsymbol{\mu}_n$-covers of smooth varieties are always Gorenstein by Proposition 0.2.12, and since there exist normal but non-Gorenstein $\boldsymbol{\mu}_n$-covers of smooth varieties, there do exist examples of $\boldsymbol{\mu}_n$-covers that are not simple, and where the normalization map $Y \to Y'$ is non-trivial.

If $\Bbbk = \mathbb{C}$, then we refer to [588] for more details and proofs, as well as a description of more general abelian covers and their singularities. In loc. cit., there is also an algorithm that reduces the singularities of abelian covers of surfaces to cyclic quotient singularities (Hirzebruch–Jung singularities), see also Proposition 0.4.20. For a more detailed studies of these singularities, we refer to [113] or [455]. For another approach to $\boldsymbol{\mu}_n$-covers, we refer the interested reader to [223, Section 3.5].

Having discussed $\boldsymbol{\mu}_n$-covers, which are analogs of Kummer extensions of fields, we now discuss analogs of Artin–Schreier extensions of degree p, as well as inseparable extensions of degree p. Let $\mathcal{L}$ be an invertible sheaf on a separated and integral scheme X of characteristic $p > 0$. Given a global section a of $\mathcal{L}^{\otimes(p-1)}$, we defined in Example 0.1.7 the finite and flat group scheme $\boldsymbol{\alpha}_{\mathcal{L},a}$ of length p over X. By construction, it sits in a short exact sequence

$$0 \to \boldsymbol{\alpha}_{\mathcal{L},a} \to \mathcal{L} \xrightarrow{\mathbf{F}-a} \mathcal{L}^{\otimes p} \to 0$$

of group schemes in the flat topology on X. Consider $\boldsymbol{\alpha}_{\mathcal{L},a}$ as an abelian sheaf in the flat topology on X. By Theorem 0.1.3, the cohomology group $H^1_{\mathrm{fl}}(X, \boldsymbol{\alpha}_{\mathcal{L},a})$ classifies isomorphism classes of $\boldsymbol{\alpha}_{\mathcal{L},a}$-torsors in the flat topology and we will refer to such torsors as *$\boldsymbol{\alpha}_{\mathcal{L},a}$-torsors*. There are two cases (see also Example 0.1.7):

1. If $a \neq 0$, we call such an $\boldsymbol{\alpha}_{\mathcal{L},a}$-torsor an *Artin–Schreier torsor* of degree p. Moreover, in the specal case where $\mathcal{L} \cong \mathcal{O}_X$ and $a = 1$, we have $\boldsymbol{\alpha}_{\mathcal{L}} \cong \mathbb{Z}/p\mathbb{Z}$ and then, f is a $\mathbb{Z}/p\mathbb{Z}$-torsor. If $f : Y \to X$ is a non-trivial $\boldsymbol{\alpha}_{\mathcal{L},a}$-torsor of X, then f is generically étale (separable) and if X and Y are varieties over a field $\Bbbk$, we will see below that the induced extension $\Bbbk(X) \subset \Bbbk(Y)$ of function fields is a separable Artin–Schreier extension of degree p, whence the name.
2. If $a = 0$, we call such an $\boldsymbol{\alpha}_{\mathcal{L},a}$-torsor an *$\boldsymbol{\alpha}_{\mathcal{L}}$-torsor* and drop the a. Moreover, in the specal case where $\mathcal{L} \cong \mathcal{O}_X$ and $a = 0$, we have $\boldsymbol{\alpha}_{\mathcal{L}} \cong \boldsymbol{\alpha}_p$ and then, f is an $\boldsymbol{\alpha}_p$-torsor. If $f : Y \to X$ is a non-trivial $\boldsymbol{\alpha}_{\mathcal{L}}$-torsor of an integral scheme X, then f is not generically étale (*inseparable*) and if X and Y are varieties over a field $\Bbbk$, we will see below that the induced extension $\Bbbk(X) \subset \Bbbk(Y)$ of function fields is purely inseparable of degree p.

For an explicit classification of $\boldsymbol{\alpha}_p$- and $\mathbb{Z}/p\mathbb{Z}$-torsors of proper varieties, we refer to Proposition 0.2.29 below.

Remark 0.2.18 Let X and Y be varieties over a field $\Bbbk$ of characteristic $p > 0$ and let $Y \to X$ be an $\boldsymbol{\alpha}_{\mathcal{L}}$- or $\boldsymbol{\alpha}_p$-torsor. Then, the morphism $Y \to X$ is purely inseparable and it factors over the $\Bbbk$-linear Frobenius morphism $\mathbf{F} : X^{1/p} \to Y \to X$.

We now give a more explicit description of an $\boldsymbol{\alpha}_{\mathcal{L},a}$-torsor $f : Y \to X$, where X is an integral and separated scheme of characteristic $p > 0$. First, let $\mathfrak{U} = \{U_i\}_i$ be an open affine cover of X trivializing the invertible sheaf $\mathcal{L}$, that is, $\mathcal{L}|_{U_i} \cong \mathcal{O}_{U_i}$ for all i. Over each U_i, the scheme that represents the sheaf $\boldsymbol{\alpha}_{\mathcal{L},a}$ is isomorphic to

$$\boldsymbol{\alpha}_{\mathcal{L},a}|_{U_i} \;\cong\; \operatorname{Spec} \mathcal{O}_X(U_i)[t_i]/(t_i^p - a_i t_i) \;\to\; U_i,$$

where $a_i \in \mathcal{O}_X(U_i)$ corresponds to the global section a via restriction to U_i and the trivialization of $\mathcal{L}$ over U_i. On the intersection $U_i \cap U_j$, we have $t_j = g_{ij} \cdot t_i$, where the $g_{ij} \in \mathcal{O}_X(U_i \cap U_j)^\times$ are transition functions of the invertible sheaf $\mathcal{L}$. Since a is a global section of $\mathcal{L}^{\otimes(p-1)}$, we find $a_i = g_{ij}^{p-1} \cdot a_j$, that is, conversely, the a_i glue together to the global section a. From the short exact sequence (0.1.5), we infer that an $\boldsymbol{\alpha}_{\mathcal{L},a}$-torsor is locally over U_i given by

$$Y|_{f^{-1}(U_i)} \;\cong\; \operatorname{Spec} \mathcal{O}_X(U_i)[s_i]/(s_i^p - a_i s_i + b_i) \;\to\; U_i \tag{0.2.15}$$

for some local sections $b_i \in \mathcal{L}^{\otimes p}(U_i)$. More explicitly, we can solve the equation $b_i = c_i^p - a_i c_i$ locally in the flat topology, so that $(s_i + c_i)^p - a_i(s_i + c_i) = 0$, which gives an explicit trivialization of the $\boldsymbol{\alpha}_{\mathcal{L},a}$-torsor in the flat topology. Next, we have $s_i = g_{ij} s_j + h_{ij}$ over $U_i \cap U_j$ for some $h_{ij} \in \mathcal{O}_X(U_i \cap U_j)$ that satisfy

$$(s_i^p - a_i s_i + b_i) \;=\; g_{ij}^p \cdot (s_j^p - a_j s_j + b_j).$$

From this, we find

$$a_i \;=\; g_{ij}^{p-1} a_j \quad \text{and} \quad b_i \;=\; g_{ij}^p b_j + a_j g_{ij}^{p-1} h_{ij} - h_{ij}^p. \tag{0.2.16}$$

Moreover, the functions (h_{ij}) form an $\mathcal{L}$-valued 1-cocycle with respect to the open affine cover $\mathfrak{U}$, whose image under $\mathbf{F} - s$ is equal to the trivial 1-cocycle $((g_{ij}^p b_j)|_{U_i \cap U_j} - b_i|_{U_i \cap U_j})$. In terms of the exact sequence (0.1.5), the previous computations give an explicit description of the map $H^1_{\mathrm{fl}}(X, \boldsymbol{\alpha}_{\mathcal{L},a}) \to H^1(X, \mathcal{L})$. Finally, the group scheme $\boldsymbol{\alpha}_{\mathcal{L},a}$ acts on Y via the coaction that is locally defined by

$$\begin{array}{rccc} \sigma : & \mathcal{O}_{U_i}[s_i]/(s_i^p - a_i s_i + b_i) & \to & \mathcal{O}_{U_i}[s_i]/(s_i^p - a_i s_i + b_i) \otimes_{\mathcal{O}_{U_i}} \mathcal{O}_{U_i}[t_i]/(t_i^p - a_i t_i) \\ & s_i & \mapsto & s_i \otimes 1 + 1 \otimes t_i, \end{array}$$

and we find $X \cong Y/\boldsymbol{\alpha}_{\mathcal{L},a}$. If X and Y are varieties over some field $\Bbbk$ of characteristic $p > 0$, then it follows from the local description above that the induced extension of function fields $\Bbbk(X) \subset \Bbbk(Y)$ is an Artin–Schreier extension if $a \neq 0$ and that it is purely inseparable of degree p if $a = 0$.

We say that an $\boldsymbol{\alpha}_{\mathcal{L},a}$-torsor $f : Y \to X$ *splits* if the image of the cohomology class associated to the torsor maps to zero in the map $H^1(X, \boldsymbol{\alpha}_{\mathcal{L},a}) \to H^1(X, \mathcal{L})$, see also exact sequence (0.1.5). In this case, the 1-cocycle (h_{ij}) is a 1-coboundary, and we can find local sections $d_i \in \mathcal{L}(U_i)$ such that $h_{ij} = d_i|_{U_i \cap U_j} - d_j|_{U_i \cap U_j}$. One checks that the $(d_i^p - a_i d_i)$ form a global section of $\mathcal{L}^{\otimes p}$, and after replacing b_i by $b_i + d_i^p - a_i d_i$, we may assume that $h_{ij} = 0$ and that the (b_i) glue to a global section b of $\mathcal{L}^{\otimes p}$. Thus, we obtain global sections

$$a = (a_i) \in H^0(X, \mathcal{L}^{\otimes(p-1)}), \quad b = (b_i) \in H^0(X, \mathcal{L}^{\otimes p}),$$

and note that the global section b is only well-defined up to replacing b by $b + (d^p - ad)$ for some global section d of $\mathcal{L}$. In terms of the short exact sequence (0.1.5), the previous computations give an explicit description of the map $H^0(X, \mathcal{L}^{\otimes p}) \to H^1_{\mathrm{fl}}(X, \boldsymbol{\alpha}_{\mathcal{L},a})$. We have the following explicit description of split $\boldsymbol{\alpha}_{\mathcal{L},a}$-torsors: let X be an integral and separated scheme of characteristic $p > 0$, let $\mathcal{L}$ be an invertible $\mathcal{O}_X$-module, let $\pi : \mathbb{L} := \mathbb{V}(\mathcal{L}^{-1}) \to X$ be the line bundle associated to $\mathcal{L}$, and let t be the tautological section of $\pi^*\mathcal{L}$. Then, the split $\boldsymbol{\alpha}_{\mathcal{L},a}$-torsor associated to the global sections $a \in H^0(X, \mathcal{L}^{\otimes(p-1)})$ and $b \in H^0(X, \mathcal{L}^{\otimes p})$ is given as a hypersurface inside $\mathbb{L}$

$$\begin{tikzcd} Y := \{t^p - \pi^*(a) \cdot t + \pi^*(b) = 0\} \arrow[r] \arrow[dr, "f"'] & \mathbb{L} = \mathbb{V}(\mathcal{L}^{-1}) \arrow[d, "\pi"] \\ & X \end{tikzcd}$$

Thus, split $\boldsymbol{\alpha}_{\mathcal{L},a}$-torsors are similar to simple $\boldsymbol{\mu}_n$-covers. As in the latter case, we denote by $i : Y \hookrightarrow \mathbb{L}$ the inclusion and set $f := \pi \circ i$.

Remark 0.2.19 From this explicit description, it also follows that a split $\boldsymbol{\alpha}_{\mathcal{L}}$-torsor (that is, $a = 0$) is the same as a simple $\boldsymbol{\mu}_p$-cover associated to $(\mathcal{L}, -b)$. In this case, $f : Y \to X$ is inseparable of degree p and Y admits both, a $\boldsymbol{\mu}_p$- and an $\boldsymbol{\alpha}_{\mathcal{L}}$-action with quotient isomorphic to X.

Let us now return to arbitrary $\boldsymbol{\alpha}_{\mathcal{L},a}$-torsors $f : Y \to X$, where X is an integral and separated scheme of characteristic $p > 0$. For simple $\boldsymbol{\mu}_n$-covers, we have seen in (0.2.3) that $f_*\mathcal{O}_Y$ is a direct sum of invertible sheaves. For $\boldsymbol{\alpha}_{\mathcal{L},a}$-torsors, it follows from the local description above that $f_*\mathcal{O}_Y$ has a basis $1, x_i, \ldots, x_i^{p-1}$ over U_i that extends a global filtration of $\mathcal{O}_X$-modules

$$0 = \mathcal{F}_0 \subset \mathcal{O}_X = \mathcal{F}_1 \subset \mathcal{F}_2 \subset \cdots \subset \mathcal{F}_p = f_*\mathcal{O}_Y \tag{0.2.17}$$

with quotients $\mathcal{F}_{i+1}/\mathcal{F}_i$ isomorphic to $\mathcal{L}^{\otimes(-i)}$. Moreover, the $\boldsymbol{\alpha}_{\mathcal{L},a}$-torsor is split if and only if the filtration splits, that is,

$$f_*\mathcal{O}_Y = \bigoplus_{i=0}^{p-1} \mathcal{L}^{\otimes(-i)}.$$

For $a = 0$ we see again that a split $\boldsymbol{\alpha}_{\mathcal{L}}$-torsor is the same as a simple $\boldsymbol{\mu}_p$-cover. However, if $a \neq 0$, then the multiplication law of the $\mathcal{O}_X$-algebra $f_*\mathcal{O}_Y$ is different from the multiplication law of the $\mathcal{O}_X$-algebra of a $\boldsymbol{\mu}_p$-cover (one $\mathcal{O}_X$-algebra is generically étale, whereas the other one is not). From the point of view of representation theory of group schemes, the reason for obtaining a direct sum decomposition into invertible sheaves for $\boldsymbol{\mu}_n$-covers as in (0.2.3) is that $\boldsymbol{\mu}_n$ is a linearly reductive and commutative group scheme. On the other hand, $\boldsymbol{\alpha}_{\mathcal{L},a}$ is a unipotent group scheme, which explains the existence of a filtration with invertible subquotients as in (0.2.17) for $\boldsymbol{\alpha}_{\mathcal{L},a}$-torsors.

Using (0.2.16), we see that the $(1, a_i, b_i)$ glue to a section of a locally free $\mathcal{O}_X$-module $\mathcal{E}$ of rank 3 on X, whose transition functions are given by

$$\begin{pmatrix} 1 \\ a_i \\ b_i \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & g_{ij}^{p-1} & 0 \\ -h_{ij}^p & h_{ij} g_{ij}^{p-1} & g_{ij}^p \end{pmatrix} \begin{pmatrix} 1 \\ a_j \\ b_j \end{pmatrix}. \tag{0.2.18}$$

From this transition matrix we infer that $\mathcal{E}$ sits in a short exact sequence

$$0 \to \mathcal{L}^{\otimes p} \to \mathcal{E} \to \mathcal{O}_X \oplus \mathcal{L}^{\otimes (p-1)} \to 0. \tag{0.2.19}$$

Then, the global section of $\mathcal{E}$ corresponding to $(1, a_i, b_i)$ maps to $(1, a)$ in $\mathcal{O}_X \oplus \mathcal{L}^{\otimes (p-1)}$, where 1 is the constant section of $\mathcal{O}_X$ and a is the global section of $\mathcal{L}^{\otimes (p-1)}$ corresponding to a_i. Moreover, if the $\boldsymbol{\alpha}_{\mathcal{L},a}$-cover splits, then there exists a choice of the b_i such that all h_{ij} are zero, and then, the b_i glue to a global section b of $\mathcal{L}^{\otimes p}$, the short exact sequence (0.2.19) splits, and $\mathcal{E}$ is isomorphic to $\mathcal{O}_X \oplus \mathcal{L}^{\otimes (p-1)} \oplus \mathcal{L}^{\otimes p}$.

We now describe the branch and ramification loci of an $\boldsymbol{\alpha}_{\mathcal{L},a}$-torsor $f : Y \to X$, where X is a smooth variety over an algebraically closed field $\Bbbk$ of characteristic $p > 0$. First, let us assume that $a \neq 0$, in which case f is separable, that is, generically étale. Then, the zeros of the section a define a divisor $\mathsf{B}(f)$ on X, the *branch divisor*, and this is precisely the branch locus of f, that is, the locus over which f is not étale. As in the case of $\boldsymbol{\mu}_n$-covers, we call the reduced pre-image $\mathsf{R}_{\mathrm{red}}(f) := f^*(\mathsf{B}(f))_{\mathrm{red}}$ the *(reduced) ramification divisor*. Note however, that f is *wildly ramified*, and thus, the contribution of the ramification divisor to the canonical sheaf of Y is more complicated than the Riemann-Hurwitz formula in the tame case. We refer to [660, Chapter IV.1] for details. We have seen above that for a separable and simple $\boldsymbol{\mu}_n$-cover associated to $(\mathcal{L}, s)$, the branch divisor is a global section of $\mathcal{L}^{\otimes n}$. On the other hand, for a separable $\boldsymbol{\alpha}_{\mathcal{L},a}$-torsor, the branch divisor is a global section of $\mathcal{L}^{\otimes (p-1)}$. Second, assume that $a = 0$, in which case f is inseparable. In particular, f is everywhere ramified and as in the case of inseparable $\boldsymbol{\mu}_n$-covers, we do not have a well-defined branch divisor. In any case, it follows from (0.2.15) that Zariski locally, Y is a hypersurface inside the line bundle $\pi : \mathbb{L} := \mathbb{V}(\mathcal{L}^{-1}) \to X$. In particular, Y is Gorenstein and we have the following analog of Proposition 0.2.12, whose proof we leave to the reader.

Proposition 0.2.20 *Let X be a smooth variety over the algebraically closed field $\Bbbk$ of characteristic $p > 0$ and let $f : Y \to X$ be an $\boldsymbol{\alpha}_{\mathcal{L},a}$-torsor. Then, Y is Gorenstein* *and its dualizing sheaf is given by*

$$\omega_Y \;\cong\; f^*(\omega_X \otimes \mathcal{L}^{\otimes(p-1)}).$$

In particular, we find

$$K_Y \;\equiv\; f^*(K_X) \;+\; f^*(\mathsf{B}(f))$$

(where "$\equiv$" denotes numerical equivalence of Cartier divisor classes) for $\boldsymbol{\alpha}_{\mathcal{L},a}$-torsors, which is different from the analogous formula (0.2.10) for simple $\boldsymbol{\mu}_n$-covers.

Concerning the singular locus of $\boldsymbol{\alpha}_{\mathcal{L},a}$-torsors of smooth varieties, we have the following analog of Proposition 0.2.6 and Proposition 0.2.10. We refer to Proposition 0.4.16 for more about singularities of $\boldsymbol{\alpha}_{\mathcal{L},a}$-torsors in characteristic $p = 2$.

Proposition 0.2.21 *Let X be a smooth variety over the algebraically closed field $\Bbbk$ of characteristic $p > 0$ and let $f : Y \to X$ be an $\boldsymbol{\alpha}_{\mathcal{L},a}$-torsor. Then, the singular locus of Y is contained in the pre-image of the scheme of zeros $Z(a)$ of the global section a of $\mathcal{L}^{\otimes(p-1)}$. More precisely and using the local description* (0.2.15)*, the differentials $(s_i da_i - db_i)$ restricted to $Z(a)$ glue to a global section*

$$\alpha_f \;\in\; H^0\left(Z(a),\, \Omega^1_{X/\Bbbk} \otimes \mathcal{L}^{\otimes p} \otimes \mathcal{O}_{Z(a)}\right).$$

The pre-image of the zero set of this section is equal to the singular locus of Y.

Proof The proof is analogous to the proof of Proposition 0.2.10. By the Jacobian criterion for smoothness, the singular locus of Y over U_i is equal to the pre-image of $Z(a_i) \cap Z(s_i da_i - db_i)$. Using (0.2.16), we compute

$$\begin{aligned} s_i da_i - db_i &= (g_{ij}s_j + h_{ij})\, d(g_{ij}^{p-1}a_j) \;-\; d\left(g_{ij}^p b_j + a_j g_{ij}^{p-1} h_{ij} - h_{ij}^p\right) \\ &= g_{ij}^p(s_j da_j - db_j) \;+\; a_j(g_{ij}s_j dg_{ij}^{p-1} - g_{ij}^{p-1} dh_{ij}). \end{aligned}$$

Thus, after restricting to $Z(a)$ (we note that we have $Z(a) = X$ in the case $a = 0$), we find $(s_i da_i - db_i) = g_{ij}^p \cdot (s_j da_j - db_j)$. Thus, the differentials $(s_i da_i - db_i)$ glue to a global section of $\Omega^1_{X/\Bbbk} \otimes \mathcal{L}^{\otimes p} \otimes \mathcal{O}_{Z(a)}$. $\square$

Example 0.2.22 Let $X = \mathbb{P}^n = \operatorname{Proj} \Bbbk[t_0, ..., t_n]$ be projective n-space over an algebraically closed field $\Bbbk$ of characteristic $p > 0$. Assume $n \geq 2$, let $k \geq 1$ be an integer, and set $\mathcal{L} := \mathcal{O}_{\mathbb{P}^n}(k)$. Since $H^1(\mathbb{P}^n, \mathcal{L}) = 0$, every $\boldsymbol{\alpha}_{\mathcal{L}}$-torsor over $\mathbb{P}^n$ is automatically split. Thus, every $\boldsymbol{\alpha}_{\mathcal{L}}$-torsor $f : Y \to \mathbb{P}^n$ is isomorphic to a hypersurface in weighted projective space $\mathbb{P}(1^{n+1}, k)$ given by an equation

$$t_{n+1}^p \;+\; F(t_0, \ldots, t_n) \;=\; 0,$$

where $F = F(t_0, ..., t_n)$ is a global section of $\mathcal{L}^{\otimes p} \;\cong\; \mathcal{O}_{\mathbb{P}^n}(kp)$. Note that this $\boldsymbol{\alpha}_{\mathcal{L}}$-torsor is a simple $\boldsymbol{\mu}_p$-cover associated to $(\mathcal{L}, F)$. The singular locus of Y lies over the zero locus of the section α_f from Proposition 0.2.10. For a generic global section F of $\mathcal{L}^{\otimes(pk)}$, this singular locus consist of N ordinary double points, where $N = c_n(\Omega^1_{\mathbb{P}^n}(pk))$. Using the formula for the Chern classes of tensor products (see [242, Example 3.2.2], for example), we find

$$N(n, p, k) = \sum_{i=0}^{n} \binom{n+1}{i} \cdot (pk)^{n-i}. \tag{0.2.20}$$

For example, we have

$$\begin{aligned} N(2, p, k) &= p^2k^2 - 3pk + 3, \\ N(3, p, k) &= p^3k^3 - 4p^2k^2 + 6pk - 4, \end{aligned}$$

and in particular,

1. If $p = 2$, $n = 2$, and $k = 1$, we obtain $N(2, 2, 1) = 1$ and then, the unique singular point lies over the point in the plane equal to the intersection of all tangents to the conic $Z(F)$, and Y is isomorphic to a quadric cone in $\mathbb{P}^3$.
2. If $p = 2$, $n = 2d + 1$, and $k = 1$, we obtain $N(2d + 1, 2, 1) = 0$ and then, Y is isomorphic to a smooth quadric hypersurface in $\mathbb{P}^{2d+2}$. Moreover, Bloch and Ekedahl proved that the only finite and inseparable morphisms $Y \to \mathbb{P}^n$ of degree p, such that Y is smooth, are the $\boldsymbol{\alpha}_{\mathcal{L}}$-torsors with $p = 2$, $n = 2d + 1$, and $k = 1$, see [210, Proposition 2.5].

Having discussed $\boldsymbol{\alpha}_{\mathcal{L},a}$-torsors, we now turn to more general covers than torsors, at least in the case where $\mathcal{L} \cong \mathcal{O}_X$ and $a \in \{0, 1\}$. Let X be an integral and separated scheme of characteristic $p > 0$. We define a *$\mathbb{Z}/p\mathbb{Z}$-cover* (resp. *$\boldsymbol{\alpha}_p$-cover*) to be a finite morphism $f : Y \to X$ of degree p together with a $\mathbb{Z}/p\mathbb{Z}$-action on Y such that $Y/(\mathbb{Z}/p\mathbb{Z}) \cong X$ (resp. with an $\boldsymbol{\alpha}_p$-action on Y such that $Y/\boldsymbol{\alpha}_p \cong X$) via f. Since f is a finite morphism, $\mathcal{A} := f_*\mathcal{O}_Y$ is a coherent $\mathcal{O}_X$-module and it carries a $\mathbb{Z}/p\mathbb{Z}$-action (resp. an $\boldsymbol{\alpha}_p$-action). In this situation, we have the following analog of Proposition 0.2.15 and Proposition 0.2.16.

Proposition 0.2.23 *Let X be a regular and separated scheme of characteristic $p > 0$. Let $f : Y \to X$ be a $\mathbb{Z}/p\mathbb{Z}$-cover or an $\boldsymbol{\alpha}_p$-cover. Assume that Y is normal. Then:*

1. *f is a flat morphism, Y is Cohen–Macaulay, and there exists a global filtration of $\mathcal{O}_X$-modules*

$$0 = \mathcal{F}_0 \subset \mathcal{O}_X = \mathcal{F}_1 \subset \mathcal{F}_2 \subset \ldots \subset \mathcal{F}_p = \mathcal{O}_Y$$

such that each $\mathcal{F}_i$ is a locally free $\mathcal{O}_X$-module of rank i, and each quotient $\mathcal{F}_{i+1}/\mathcal{F}_i$ is an invertible $\mathcal{O}_X$-module.

2. *There exists an invertible sheaf $\mathcal{L}$ on X, a global section $a \in H^0(X, \mathcal{L}^{\otimes(p-1)})$, and an $\boldsymbol{\alpha}_{\mathcal{L},a}$-torsor $Z \to X$, where Z is integral and Cohen–Macaulay but not necessarily normal, and such that f factors as*

$$Y \to Z \to X,$$

such that $Y \to Z$ is the normalization morphism. In particular, there exists an open and dense subset $U \subseteq X$, such that $f^{-1}(U) \to U$ is an $\boldsymbol{\alpha}_{\mathcal{L}|_U, a|_U}$-torsor.

Proof Let $G = \mathbb{Z}/p\mathbb{Z}$ or $G = \boldsymbol{\alpha}_p$ be the group scheme in question. We have an induced G-action on $\mathcal{A} := f_*\mathcal{O}_Y$. We set $\mathcal{F}_p := \mathcal{A}$ and it is easy to see that the subsheaf $\mathcal{F}_1$ of G-invariants inside $\mathcal{F}_p$ is isomorphic to $\mathcal{O}_X$. Having constructed an $\mathcal{O}_X$-submodule $\mathcal{F}_i$ of $\mathcal{F}_p$ with $G(\mathcal{F}_i) \subseteq \mathcal{F}_i$ inductively, we obtain an induced G-action on $\mathcal{F}/\mathcal{F}_i$ and let $\mathcal{F}_{i+1}$ be the pre-image of the G-invariant subsheaf of this quotient. This gives the desired global filtration. By Proposition 0.2.3, $\mathcal{A}$ is a reflexive $\mathcal{O}_X$-module. To show the remaining assertions of Claim (1), we replace X and Y by G-stable open affine subsets, say $X = \operatorname{Spec} A$ and $Y = \operatorname{Spec} B$. Thus, B is a reflexive A-module and the G-action on Y induces a coaction

$$\delta \;:\; B \;\to\; B \otimes_A \Bbbk[t]/(t^p - \varepsilon t) \;\cong\; B[t]/(t^p - \varepsilon t)$$

with $\varepsilon = 1$ (resp. $\varepsilon = 0$) if $G = \mathbb{Z}/p\mathbb{Z}$ (resp. $G = \boldsymbol{\alpha}_p$). For every $b \in B$, we define $b_i \in B$ for $i = 0, \ldots, p-1$ by requiring

$$\delta(b) \;=\; b_0 \;+\; b_1 \cdot t \;+\; \ldots \;+\; b_{p-1} \cdot t^{p-1} \;\in\; B[t]/(t^p - \varepsilon t),$$

and then, we set $\delta_i(b) := b_i$. It follows from the axioms of an action that $\delta_0 = \mathrm{id}_B$, as well as $i!\delta_i = \delta_1^i$. In particular, we find

$$\delta(b) \;=\; b \;+\; \sum_{i=1}^{p-1} \frac{\delta_1^i(b)}{i!} \cdot t^i.$$

If $F_i \subseteq B$ denotes the A-submodule corresponding to $\mathcal{F}_i \subseteq f_*\mathcal{O}_Y$, then the previous computation shows that $F_i = \operatorname{Ker}(\delta_1^i)$. Moreover, F_i is the inverse image of $A + At + \ldots + At^i$ of $A[t]$ under δ, which shows that $F_{i+1}/F_i \cong At^{i+1}$. In particular, F_i is a free A-module of rank i and the quotient F_{i+1}/F_i is a free A-module of rank 1. This also shows that f is flat and that B is Cohen–Macaulay. This establishes the first claim and we refer the interested reader to [698, Proposition 1.1] for further details.

Since $\mathcal{F}_2/\mathcal{F}_1$ is an invertible $\mathcal{O}_X$-module, also its dual $\mathcal{L}$ is an invertible $\mathcal{O}_X$-module. In this notation, $\mathcal{F}_2$ is an extension of $\mathcal{L}^\vee$ by $\mathcal{O}_X$. Using the multiplication map, we obtain a morphism of $\mathcal{O}_X$-modules $\psi : S^\bullet(\mathcal{F}_2) \to \mathcal{A}$ and denote by $\mathcal{A}' \subseteq \mathcal{A}$ the image of ψ, where, $S^\bullet$ denotes the symmetric algebra. Then, $S^i(\mathcal{F}_2)$ maps to $\mathcal{F}_{i+1}$ under ψ. From the surjection $\mathcal{F}_2 \to \mathcal{L}^\vee$ we obtain a surjection $S^i(\mathcal{F}_2) \to \mathcal{L}^{\otimes(-i)}$, from which we see that the $\mathcal{O}_X$-module $\mathcal{A}'$ is a successive extension of $\mathcal{L}^{\otimes(-i)}$'s for $i = 0, \ldots, p-1$. To describe $\mathcal{A}'$ better, we choose an open affine cover $\mathfrak{U} = \{U_i\}_i$ of X such that $\mathcal{L}|_{U_i} \cong \mathcal{O}_{U_i}$. Next, we choose $s_i \in \mathcal{F}_2(U_i)$ such that its image in $\mathcal{L}^\vee$ generates the invertible sheaf over U_i. Then, we have $\delta(s_i) = s_i + c_i t$ for some $c_i \in \mathcal{O}_X(U_i)$. Using the equality $\delta(s_i^p) = s_i^p + c_i t^p = s_i^p + \varepsilon c_i^p t$, we conclude that $s_i^p \in \mathcal{F}_2(U_i)$ and thus, there exists an equation of the form

$$s_i^p \;+\; a_i s_i \;+\; b_i \;= 0 \tag{0.2.21}$$

for some $a_i, b_i \in \mathcal{O}_X(U_i)$. On the overlap $U_i \cap U_j$, we have $s_i = g_{ij}s_j + h_{ij}$ for some $g_{ij} \in \mathcal{O}_X(U_i \cap U_j)^\times$ and $h_{ij} \in \mathcal{O}_X(U_i \cap U_j)$. Comparing coefficients in

$$0 = s_i^p + a_i s_i + b_i = g_{ij}^p s_j^p + h_{ij}^p + a_i g_{ij} s_j + a_i g_{ij} h_{ij} + b_i$$

with $0 = g_{ij}^p(s_j^p + a_j s_j + b_j)$, we find

$$a_i = g_{ij}^{p-1} a_j \quad \text{and} \quad b_i = g_{ij}^p b_j - a_j g_{ij}^p h_{ij} - h_{ij}^p.$$

It follows that the a_i glue to a global section $a \in H^0(X, \mathcal{L}^{\otimes(p-1)})$. From (0.2.15) and (0.2.16), we infer that $Z := \operatorname{Spec} \mathcal{A}' \to X$ carries the structure of an $\boldsymbol{\alpha}_{\mathcal{L},a}$-torsor and we obtain a factorization $Y \to Z \to X$. From here, we argue as in the proof of Proposition 0.2.16. We also refer to [698, Lemma 1.2 and Lemma 1.3] for further details. □

Remark 0.2.24 The local sections (h_{ij}) constructed in the proof give rise to a 1-cocycle with values in $\mathcal{L}$ and thus, to a cohomology class in $H^1(X, \mathcal{L})$. Using the isomorphism $H^1(X, \mathcal{L}) \cong \operatorname{Ext}^1(\mathcal{L}^\vee, \mathcal{O}_X)$, this cohomology class corresponds to the extension class of $0 \to \mathcal{O}_X \to \mathcal{F}_2 \to \mathcal{L}^\vee \to 0$. Moreover, if this class happens to be zero, then the $\boldsymbol{\alpha}_{\mathcal{L},a}$-torsor $Z \to X$ is split. In this case, the b_i glue to a global section $b \in H^0(X, \mathcal{L}^{\otimes p})$ and the cover $Z \to X$ is also globally of the form (0.2.21).

Corollary 0.2.25 *Under the assumptions of the proposition assume moreover that $p = 2$. Then, $f : Y \to X$ carries the structure of an $\boldsymbol{\alpha}_{\mathcal{L},a}$-torsor with respect to some $a \in H^0(X, \mathcal{L})$.*

Proof Let $Y = \operatorname{Spec} \mathcal{A} \to Z = \operatorname{Spec} \mathcal{A}' \to X$ be as in the proof of Proposition 0.2.23. Since $p = 2$, we have $\mathcal{F}_2 = \mathcal{A}$ and from the construction of $\mathcal{A}'$ it follows that we have $\mathcal{A}' = \mathcal{A}$ in this case. In particular, we may choose $U = X$ and the statement follows. □

Remark 0.2.26 If $p > 2$ then it is not true in general that $f : Y \to X$ is an $\boldsymbol{\alpha}_{\mathcal{L},a}$-torsor, that is, we cannot choose $U = X$ in general. We refer to [698, Example 1.5] for explicit counter-examples in dimension two.

As an application of and supplement to the above discussions, we have the following result, which generalizes the fact that a finite and separable field extension of degree 2 is automatically a Galois extension with group $\mathbb{Z}/2\mathbb{Z}$.

Proposition 0.2.27 *Let X be a smooth variety over an algebraically closed field $\Bbbk$ of characteristic $p \geq 0$. Let $f : Y \to X$ be a finite morphism of degree 2 and assume that f is flat or that Y is Cohen–Macaulay.*

1. *If $p \neq 2$, then f is a simple $\boldsymbol{\mu}_2$-cover.*
2. *If $p = 2$, then f is an $\boldsymbol{\alpha}_{\mathcal{L},a}$-torsor for an invertible sheaf $\mathcal{L}$ and a global section $a \in H^0(X, \mathcal{L})$.*
3. *If $p = 2$ and $f : Y \to X$ is a $\boldsymbol{\mu}_2$-cover, then f is a simple $\boldsymbol{\mu}_2$-cover.*

Proof By Proposition 0.2.4, f is flat in any case, and thus, $\mathcal{A} := f_*\mathcal{O}_Y$ is locally free of rank 2 and contains $\mathcal{O}_X$ as an $\mathcal{O}_X$-submodule.

First, assume that $p \neq 2$. Using the trace map $\mathrm{tr} : f_*\mathcal{O}_Y \to \mathcal{O}_X$, which is equal to multiplication by 2 on $\mathcal{O}_X$ inside $f_*\mathcal{O}_Y$, we obtain a direct sum decomposition of $f_*\mathcal{O}_Y$ into $\mathcal{O}_X$ and $\mathcal{N} := \mathrm{Ker}(\mathrm{tr})$. It is easy to see that $\mathcal{N}$, being a direct summand of a locally free $\mathcal{O}_X$-module of rank 2, is reflexive and of rank 1 and thus, an invertible $\mathcal{O}_X$-module. We set $\mathcal{L} := \mathcal{N}^\vee$. The $\mathcal{O}_X$-algebra structure on $f_*\mathcal{O}_Y \cong \mathcal{O}_X \oplus \mathcal{L}^\vee$ is given by a map of $\mathcal{O}_X$-modules $\mathcal{L}^\vee \otimes \mathcal{L}^\vee \to \mathcal{O}_X$, which we regard as a global section $s \in H^0(X, \mathcal{L}^{\otimes 2})$. From this, it is easy to see that $f : Y \to X$ is isomorphic to the simple $\boldsymbol{\mu}_2$-cover defined by $(\mathcal{L}, s)$, which establishes Claim (1).

Now, assume that $p = 2$ and assume that f is a $\boldsymbol{\mu}_2$-cover. Again, $f_*\mathcal{O}_X$ is locally free of rank 2, and Proposition 0.2.15 yields a direct sum decomposition $f_*\mathcal{O}_X \cong \mathcal{O}_X \oplus \mathcal{N}$ into invertible sheaves as in (0.2.11). We set $\mathcal{L} := \mathcal{N}^\vee$ and argue as in the $p \neq 2$-case, from which we obtain Claim (3).

Finally, assume that $p = 2$. As before, we obtain a short exact sequence of locally free $\mathcal{O}_X$-modules

$$\iota \,:\, 0 \to \mathcal{O}_X \to \mathcal{A} \to \mathcal{N} \to 0.$$

Since f is of degree p, the trace map $\mathrm{tr} : \mathcal{A} \to \mathcal{O}_X$ is zero. Thus, it induces a morphism $\mathcal{N} \to \mathcal{O}_X$, which we identify with a section $a \in H^0(X, \mathcal{L})$, where $\mathcal{L} := \mathcal{N}^\vee$. If t is a local section of $\mathcal{A}$, then $t^2 - \mathrm{tr}(t)$ lies in $\mathcal{O}_X$. Thus, the map $t \mapsto t^2 - \mathrm{tr}(t)$ gives a splitting of $(\mathbf{F} - \mathrm{id})^*(\iota)$. From this, we obtain the structure of an $\boldsymbol{\alpha}_{\mathcal{L},a}$-torsor on $f : Y = \mathrm{Spec}\,\mathcal{A} \to X$, which establishes Claim (2). We refer to [211, Proposition 1.11] for details. □

We refer to Proposition 0.4.16 for results about the singularities occurring on such double covers $Y \to X$ in the case where X is a smooth surface.

We now briefly discuss the restriction of covers and torsors to closed subsets: let $f : Y \to X$ be a $\boldsymbol{\mu}_n$-cover or an $\boldsymbol{\alpha}_{\mathcal{L},a}$-torsor, where Y is a normal and X is a smooth variety over an algebraically closed field $\Bbbk$ of characteristic $p \geq 0$. Let G be the group scheme acting on Y. For an irreducible and closed subset $Z \subseteq X$, we will say that $f^{-1}(Z) \to Z$ is a *trivial cover* if there exists a G-equivariant isomorphism $f^{-1}(Z) \to Z \times G$ over Z.

1. First, assume that f is a simple $\boldsymbol{\mu}_n$-cover associated to $(\mathcal{L}, s)$, where $\mathcal{L}$ is an invertible $\mathcal{O}_X$-module and $s \in H^0(X, \mathcal{L}^{\otimes n})$ is a global section. Let $\mathcal{L}_Z := \mathcal{L} \otimes_{\mathcal{O}_X} \mathcal{O}_Z$ be the restriction of $\mathcal{L}$ to Z and let s_Z be the image of s under the natural homomorphism $H^0(X, \mathcal{L}) \to H^0(Z, \mathcal{L}_Z)$, that is, the restriction of s to Z. Then, the restriction $f^{-1}(Z) \to Z$ is a simple $\boldsymbol{\mu}_n$-cover associated to $(\mathcal{L}_Z, s_Z)$. The section s corresponds to an injective homomorphism $\mathcal{O}_X \to \mathcal{L}^{\otimes n}$ of $\mathcal{O}_X$-modules, whose cokernel is supported on an effective divisor $B \subseteq X$ such that $\mathcal{L}^{\otimes n} \cong \mathcal{O}_X(B)$. Let $j : Z \to X$ be the canonical inclusion, set $B_Z := j^*(B)$, and identify s_Z with a rational function on Z. Then, the $\boldsymbol{\mu}_n$-cover trivializes over Z if and only if s_Z is the n-th power of some rational function on Z. These equivalences hold, for example, if Z is disjoint from B.
2. Second, assume that f is an $\boldsymbol{\alpha}_{\mathcal{L},a}$-torsor for some invertible sheaf $\mathcal{L}$ and a global section $a \in H^0(X, \mathcal{L}^{\otimes(p-1)})$. Let $\beta \in H^1(X, \boldsymbol{\alpha}_{\mathcal{L},a})$ be the associated cohomology

class. We have a natural restriction map $H^1(X, \boldsymbol{\alpha}_{\mathcal{L},a}) \to H^1(Z, \boldsymbol{\alpha}_{\mathcal{L}_Z,a_Z})$ and let β_Z be the image of β under this map. The cover f trivializes over Z if and only if $\beta_Z = 0$. Next, assume that the $\boldsymbol{\alpha}_{\mathcal{L},a}$-torsor is split, that is, it is globally given by $f : Y = \operatorname{Spec} \mathcal{O}_X[s]/(s^p + as + b) \to X$ for some global section $b \in H^0(X, \mathcal{L}^{\otimes p})$. If Z is contained in the zero locus of b, then $b_Z = 0$ and the cover trivializes over Z. Another example, where f trivializes over Z, is if $\mathcal{L}_Z \cong \mathcal{O}_Z$ and $H^0(Z, \mathcal{O}_Z) = \Bbbk$. For example, this condition is fulfilled if Z is disjoint from the zero locus $Z(a)$ of a and $H^0(Z, \mathcal{O}_Z) = \Bbbk$.

Let us, moreover assume that $Z \subset X$ is an integral and effective Weil divisor, that is, a prime divisor.

1. First, assume that f is a simple $\boldsymbol{\mu}_n$-cover. If $p \nmid n$, then $f^*(Z) = pA$ for some Weil divisor A on Y if and only if Z is contained in the branch locus $\mathsf{B}(f)$ of f. If $n = p$, then $f^*(Z) = pA$ for some Weil divisor A on Y if and only if f trivializes over Z.
2. Second, assume that f is a split $\boldsymbol{\alpha}_{\mathcal{L},a}$-torsor defined by some global section $b \in H^0(X, \mathcal{L}^{\otimes p})$. Then, $f^*(Z) = pA$ for some Weil divisor A on Y if and only if Z is contained in the intersection of the zero loci $Z(a) \cap Z(b)$. In particular, it is somewhat confusing to call $Z(a)$ the branch locus of f since the restriction of f to an irreducible component $Z \subseteq Z(a)$ could be a non-trivial inseparable cover of degree p.

For example, let $f : Y \to X$ be a simple $\boldsymbol{\mu}_n$-cover, where X is a smooth surface over an algebraically closed field $\Bbbk$ of characteristic $p \geq 0$. We assume that $(p, n) = 1$ and in particular, f is a separable morphism, that is, generically étale. (For the purely inseparable case of degree p, see Proposition 0.3.19.) Let $(\mathcal{L}, s)$ be data defining the $\boldsymbol{\mu}_n$-cover.

Let $C \subset X$ be an integral curve, which is not a component of the branch divisor $B = \operatorname{div}(s)$. We now briefly address the question when C is *split* under this cover, that is, whether $f^*(C)$ is a reducible curve. We assume moreover that Y is normal, which implies that B is a reduced divisor. Assume that C is given by local equations $\phi_\alpha = 0$ in some affine cover $\mathfrak{U} = \{U_\alpha\}_{\alpha \in I}$ of X and assume that the restriction $V_\alpha := f^{-1}(U_\alpha) \to U_\alpha$ of the cover is given locally by $y_\alpha^n + a_\alpha = 0$. A necessary condition for the splitting of $C \cap U_\alpha$ in the cover $V_\alpha \to U_\alpha$ is that the image of a_i in $\mathcal{O}(C \cap U_\alpha) = \mathcal{O}(U_\alpha)/(\phi_\alpha)$ is a k-th power of some element with $(n, k) > 1$. Globalizing, we conclude that a necessary condition is that the restriction s_C of s to a section of $\mathcal{L}^{\otimes n} \otimes \mathcal{O}_C$ is equal to $u^{\otimes k}$, where u is a global section of some invertible sheaf $\mathcal{M}$ on C with $\mathcal{M}^{\otimes k} \cong \mathcal{L}^{\otimes n} \otimes \mathcal{O}_C$.

Write $n = dn'$, $k = dk'$, and let $\mathcal{N} := \mathcal{M}^{\otimes k'} \otimes \mathcal{L}^{\otimes(-n')}$, so that $\mathcal{N}^{\otimes d} \cong \mathcal{O}_C$. The curve C splits in the cover f if and only if $\mathcal{N} \cong \mathcal{O}_C$. In particular, we see that C always splits if $\operatorname{Pic}(C)$ has no torsion elements. For example, this is the case if $C \cong \mathbb{P}^1$ or if C is a rational cuspidal curve.

Now, suppose that C does not split in the cover f. Then, $f^{-1}(C)$ is an integral curve on X and the composition of the normalization map $\tilde{C} \to f^{-1}(C)$ with the map $f_C : f^{-1}(C) \to C$ is an étale cover $\pi : \tilde{C} \to C$ of degree d given by the invertible sheaf $\mathcal{N}$. We know that the ramification curve of the cover is given by a section t of $f^*(\mathcal{L})$. Since $\pi^*(\mathcal{N}) \cong \mathcal{O}_{\tilde{C}}$, we see that the restriction of t to $f^{-1}(C)$ defines a section of $f_C^*(\mathcal{M})$.

Example 0.2.28 Let $X = \mathbb{P}^2$ and $\mathcal{L} = \mathcal{O}_{\mathbb{P}^2}(1)$ in characteristic $p \neq 2$. Let $B \subset X$ be a smooth conic defined by a section s of $\mathcal{L}^{\otimes 2}$ and let $f : Y \to X$ be the simple $\boldsymbol{\mu}_2$-cover associated to the data $(\mathcal{L}, s)$. Then, Y is isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$. Let $C \subset X$ be an integral curve of degree m. Since the pre-image of a general line is a curve of bidegree $(1, 1)$, the pre-image $f^{-1}(C)$ is a curve of bidegree (m, m) on Y. The covering involution σ sends the divisor class of bidegree (a, b) to the divisor class of bidegree (b, a). Next, let $\nu : \tilde{C} \to C \subset \mathbb{P}^2$ be the normalization. Then, $\nu^*(s)$ defines a section of $\nu^*(\mathcal{L}^{\otimes 2})$ equal to u^2, where u is a section of $\mathcal{M}$ with $\mathcal{M}^{\otimes 2} \cong \nu^*(\mathcal{L}^{\otimes 2})$. We assume from now on, for simplicity, that C is smooth at the intersection points with B.

Suppose that C splits in the cover, that is, $f^{-1}(C) = C_1 + C_2$. Then, $\sigma(C_1) = C_2$, so the curves $C_1 \cong C_2$ are both isomorphic to C. Let C_1 (resp. C_2) be of bidegree (a, b) (resp. (b, a)) with $a + b = m$. The arithmetic genus of a curve of bidegree (a, b) is equal to $(a - 1)(b - 1)$, whereas the arithmetic genus of C is equal to $\frac{1}{2}((a + b - 1)(a + b - 2)$. We see that these numbers are equal if and only if $m = 1$ or $m = 2$, that is, if and only if C is a line or a conic. In the former case, C_1 and C_2 are lines from different rulings and in the latter case, C_1 and C_2 are two conics in the Segre embedding intersecting at two points.

In particular, a cuspidal cubic does not split although its Picard group does not have torsion elements. This shows that no smooth conic is tangent to such curve at three points.

We end our discussion of covers by classifying $\boldsymbol{\mu}_n$-torsors, $\boldsymbol{\alpha}_p$-torsors, and $\mathbb{Z}/p\mathbb{Z}$-torsors over proper varieties. We will also refer to torsors under $\boldsymbol{\mu}_n$, $\boldsymbol{\alpha}_p$, and $\mathbb{Z}/p\mathbb{Z}$ as *cyclic torsors* or *principal cyclic torsors*.

Proposition 0.2.29 *Let X be a proper variety over an algebraically closed field $\Bbbk$ of characteristic $p \geq 0$. Then, there exist natural bijections between isomorphism classes of G-torsors and the following objects.*

- *$G = \boldsymbol{\mu}_n$:*
  1. *$\boldsymbol{\mu}_n$-torsors $f : Y \to X$ in the flat topology (in the étale topology if $p \nmid n$),*
  2. *invertible sheaves $\mathcal{L}$ on X such that $\mathcal{L}^{\otimes n} \cong \mathcal{O}_X$,*
  3. *simple $\boldsymbol{\mu}_n$-covers associated to data $(\mathcal{L}, s)$, where s is a section of $\mathcal{L}^{\otimes n}$ that induces an isomorphism $\mathcal{O}_X \cong \mathcal{L}^{\otimes n}$.*
- *$p > 0$ and $G = \mathbb{Z}/p\mathbb{Z}$:*
  1. *$\mathbb{Z}/p\mathbb{Z}$-torsors $f : Y \to X$ in the flat (or étale) topology,*
  2. *elements of the kernel of $(F - \mathrm{id}) : H^1(X, \mathcal{O}_X) \to H^1(X, \mathcal{O}_X)$.*
- *$p > 0$ and $G = \boldsymbol{\alpha}_p$:*
  1. *$\boldsymbol{\alpha}_p$-torsors $f : Y \to X$ in the flat topology,*
  2. *elements of the kernel of $F : H^1(X, \mathcal{O}_X) \to H^1(X, \mathcal{O}_X)$.*

Proof By Theorem 0.1.3, $\boldsymbol{\mu}_n$-torsors over X are in bijection with elements in $H^1(X, \boldsymbol{\mu}_n)$. Taking cohomology in the Kummer exact sequence (0.1.6) and using that $H^0(X, \mathbb{G}_m) \cong \Bbbk^\times$ as well as that $\Bbbk$ is an algebraically closed field, we find that $H^1(X, \boldsymbol{\mu}_n)$ is isomorphic to the n-torsion subgroup of $H^1(X, \mathbb{G}_m)$ and thus, isomorphic to the n-torsion subgroup of $\mathrm{Pic}(X)$, see (0.1.7). Explicitly, the $\boldsymbol{\mu}_n$-torsor associated to an n-torsion element $\mathcal{L} \in \mathrm{Pic}(X)$ is a simple $\boldsymbol{\mu}_n$-cover associated to $(\mathcal{L}, s)$ where s is a global section of $\mathcal{L}^{\otimes n}$ inducing an isomorphism with $\mathcal{O}_X$, see the local descriptions (0.2.3) and (0.2.4). Finally, given a simple $\boldsymbol{\mu}_n$-cover associated to $(\mathcal{L}, s)$, this cover is a $\boldsymbol{\mu}_n$-torsor outside the zero locus of the global section s. Thus, if s induces an isomorphism $\mathcal{O}_X \cong \mathcal{L}^{\otimes n}$, then it has no zeros, and the $\boldsymbol{\mu}_n$-cover is a $\boldsymbol{\mu}_n$-torsor. This establishes the claimed bijections for $G = \boldsymbol{\mu}_n$. We leave the cases $G = \mathbb{Z}/p\mathbb{Z}$ and $G = \boldsymbol{\alpha}_p$, which follow from similar arguments applied to the Artin–Schreier sequence (0.1.3), to the reader. □

Example 0.2.30 Let E be an elliptic curve over an algebraically closed field $\Bbbk$ of characteristic $p > 0$ and let $\mathbf{F} : E \to E^{(p)}$ be the $\Bbbk$-linear Frobenius morphism. In Example 0.1.11, we saw that $\mathbf{F}$ divides elliptic curves into two types: ordinary and supersingular ones.

1. If E is ordinary, then $\mathbf{F}$ is a $\boldsymbol{\mu}_p$-torsor. In fact, $\mathbf{F}$ is a simple $\boldsymbol{\mu}_p$-torsor associated to $(\mathcal{L}, 1)$, where $\mathcal{L}$ is a non-trivial invertible sheaf with $\mathcal{L}^{\otimes p} \cong \mathcal{O}_X$.
2. If E is supersingular, then $\mathbf{F}$ is an $\boldsymbol{\alpha}_p$-torsor. Note that $\mathbf{F}$ is not a split $\boldsymbol{\alpha}_p$-torsor, for otherwise, (0.2.17) would split, that is, we would have $\mathbf{F}_*\mathcal{O}_E \cong \mathcal{O}_{E^{(p)}}^{\oplus p}$, and taking cohomology we would find $h^1(E, \mathcal{O}_E) = p \cdot h^1(E^{(p)}, \mathcal{O}_{E^{(p)}}) = p \geq 2$, a contradiction.

Remark 0.2.31 By Proposition 0.2.29, the set of $\boldsymbol{\mu}_n$-torsors over X is bijective to $H^1_{\text{ét}}(X, \boldsymbol{\mu}_n) \cong {}_n\mathrm{Pic}(X)$. If $p = n$, then we have an alternative classification of $\boldsymbol{\mu}_n$-torsors: namely, we have an exact sequence of abelian sheaves in the flat topology

$$0 \to \mathcal{O}_X^\times \xrightarrow{[p]} \mathcal{O}_X^\times \xrightarrow{\mathrm{dlog}} \Omega_{X/\Bbbk},$$

where dlog is the map $f \mapsto \mathrm{dlog}\, f = f^{-1}df$ (we come back to this map in Section 0.10). By definition, the image of dlog is a subsheaf $\Omega_{X/\Bbbk,\log}$ of $\Omega_{X/\Bbbk}$, the *sheaf of log differentials*. Passing to the exact sequence of cohomology, we find an isomorphism of abelian groups

$${}_p\mathrm{Pic}(X) \cong H^0(X, \Omega_{X/\Bbbk,\log}). \tag{0.2.22}$$

Thus, the set of $\boldsymbol{\mu}_p$-torsors over X is also in bijection with $H^0(X, \Omega_{X/\Bbbk,\log})$. Note however, that this latter group of sections is not a vector space over $\Bbbk$.

Remark 0.2.32 By Proposition 0.2.29, the set of $\boldsymbol{\alpha}_p$-torsors over X is bijective to the kernel of $\mathbf{F} : H^1(X, \mathcal{O}_X) \to H^1(X, \mathcal{O}_X)$. To give an alternative classification of $\boldsymbol{\alpha}_p$-torsors, we use the exact sequence

$$0 \to \mathcal{O}_X \xrightarrow{\mathbf{F}} \mathcal{O}_X \xrightarrow{d} \Omega_{X/\Bbbk}$$

of sheaves in the Zariski topology, where d denotes the differential. By definition, the image of d is a subsheaf $B\Omega_{X/\Bbbk}$ of the sheaf $\Omega_{X/\Bbbk}$ of Kähler differentials, the *sheaf of exact differential* 1*-forms*. Passing to the exact sequence of cohomology, we find an isomorphism of abelian groups

$$H^0(X, B\Omega_{X/\Bbbk}) \;\cong\; \mathrm{Ker}\Big(\mathbf{F} : H^1(X, \mathcal{O}_X) \to H^1(X, \mathcal{O}_X)\Big). \tag{0.2.23}$$

Thus, the set of $\boldsymbol{\alpha}_p$-torsors over X is also bijective to $H^0(X, B\Omega_{X/\Bbbk})$.

As a consequence of these remarks, we obtain the following result, see also Proposition 0.2.21.

Proposition 0.2.33 *Let X be a smooth variety over an algebraically closed field $\Bbbk$ of characteristic $p \geq 0$. Let $f : Y \to X$ be a non-trivial $\boldsymbol{\mu}_p$- or $\boldsymbol{\alpha}_p$-torsor. Then,*

$$H^0(X, \Omega_{X/\Bbbk}) \;\neq\; 0$$

and Y is reduced and irreducible, that is, a variety over $\Bbbk$.

Proof By Theorem 0.1.4, the torsor f is also non-trivial at the generic point, that is, induces a non-trivial extension of the function field $\Bbbk(X)$, which implies that Y is irreducible and generically reduced. By Proposition 0.2.4, Y is Cohen–Macaulay and thus, Y is an integral scheme. If f is a $\boldsymbol{\mu}_p$-torsor (resp. $\boldsymbol{\alpha}_p$-torsor), then we find $H^0(\Omega_{X/\Bbbk,\log}) \neq 0$ (resp. $H^0(Z\Omega_{X/\Bbbk}) \neq 0$) by Remarks 0.2.31 and 0.2.32. In both cases, this implies $H^0(\Omega_{X/\Bbbk}) \neq 0$. □

If X is a proper variety over an algebraically closed field $\Bbbk$, then its Picard functor is representable by a scheme, the Picard scheme $\mathrm{Pic}_{X/\Bbbk}$ of X, see Theorem 0.9.4 below. Anticipating this result, the first part of Proposition 0.2.29 can be rephrased by saying that $\boldsymbol{\mu}_n$-torsors of X are in bijection with homomorphisms of group schemes $(\mathbb{Z}/n\mathbb{Z})_\Bbbk \to \mathrm{Pic}_{X/\Bbbk}$. Next, $H^1(X, \mathcal{O}_X)$ is isomorphic to the Lie algebra of $\mathrm{Pic}_{X/\Bbbk}$, see Proposition 0.9.7. Then, the remaining parts of Proposition 0.2.29 can be rephrased by saying that $\mathbb{Z}/p\mathbb{Z}$-torsors (resp. $\boldsymbol{\alpha}_p$-torsors) of X are in bijection with homomorphisms of group schemes $\boldsymbol{\mu}_{p,\Bbbk} \to \mathrm{Pic}_{X/\Bbbk}$ (resp. $\boldsymbol{\alpha}_{p,\Bbbk} \to \mathrm{Pic}_{X/\Bbbk}$). After these preparations, we have the following theorem of Raynaud, which is a generalization of Proposition 0.2.29.

Theorem 0.2.34 *Let X be proper variety over an algebraically closed field $\Bbbk$ and let $\mathrm{Pic}_{X/\Bbbk}$ be the Picard scheme of X. Let G be a finite and commutative group scheme over $\Bbbk$ with Cartier dual $G^D = \mathcal{H}om(G, \mathbb{G}_m)$. Then, there exists a bijection between*

1. *isomorphism classes of G-torsors over X, and*
2. *homomorphisms $G^D \to \mathrm{Pic}_{X/\Bbbk}$ of group schemes over $\Bbbk$.*

Proof This is a special case of [606, Proposition 6.2.1]. □

We end this section with a useful lemma of Cartan, which was originally stated over the complex numbers, and its application to fixed loci and ramification loci.

Lemma 0.2.35 *Let $A = \Bbbk[[t_1, \ldots, t_n]]$ and let σ be a $\Bbbk$-automorphism of A that is of some finite order prime to $p :=$ char($\Bbbk$). Then, there exists an $\Bbbk$-automorphism τ of A such that $\sigma' = \tau^{-1} \circ \sigma \circ \tau$ acts via*

$$\sigma'(t_i) = \xi_u t_i, \quad i = 1, \ldots, n,$$

where $\xi_i \in \Bbbk^*$ are roots of unity.

Proof Let $\mathfrak{m} \subset A$ be the maximal ideal. Since the order of σ is prime to $p =$ char($\Bbbk$), the action of σ on the linear $\Bbbk$-space $T = \mathfrak{m}/\mathfrak{m}^2$ can be decomposed into a direct sum of eigensubspaces with eigenvalues $\xi_1, \ldots, \xi_n$. After a linear change of parameters, we find a conjugate automorphism σ' that acts on T by sending $\bar{t}_i = t_i + \mathfrak{m}^2$ to $\xi_i \bar{t}_i$. It acts on the parameters t_i by

$$t_i \mapsto \xi_i t_i + f_i, \quad i = 1, \ldots, n,$$

where $f_i \in \mathfrak{m}^2$. Composing it with the linear automorphism $t_i \mapsto \xi_i^{-1} t_i$, we may assume that σ acts as identity on $\mathfrak{m}/\mathfrak{m}^2$.

Let n be the order of σ. We define an automorphism

$$\phi = \frac{1}{n}(\sum_{k=0}^{n-1} \sigma^k)$$

and here, we use that n is coprime to p. Since each σ^i acts on T as identity, ϕ acts on T also as the identity. In particular, ϕ is an automorphism of A. Since $\sigma \circ \phi = \frac{1}{n}(\sum_{k=0}^{n-1} \sigma^{k+1} = \phi$, we obtain that σ is the identity. □

This automorphism τ is called a *linearization* of the σ-action. If linearizations exist, then local properties of actions, their ramification loci, and their quotient morphisms can be studied using methods from linear algebra. The previous lemma shows that automorphisms of order prime to $p =$ char($\Bbbk$) admit such linearizations around fixed points and after passing to completions. In fact, the action of a finite group scheme G can be linearized around fixed points after passing to completions if and only if G is linearly reductive, see [235, 236, 465].

Corollary 0.2.36 *Let X be a smooth variety over $\Bbbk$. Let σ be an automorphism of X that is of some finite order prime to $p =$ char($\Bbbk$). Then, the locus X^σ of fixed points of σ is smooth. Moreover, for any point $x \in X^\sigma$, the action of the cyclic group $\langle\sigma\rangle$ on the tangent space $T_x(X)$ is faithful.*

Proof Let $x \in X$ be a fixed point of σ, that is, let $x \in X^\sigma$. The local ring $\mathcal{O}_{X,x}$ is a subring of its formal completion $\hat{\mathcal{O}}_{X,x}$, which is isomorphic to $\Bbbk[[t_1, \ldots, t_n]]$, where we may assume that the t_i are the images of a system u_i of local parameters of $\mathcal{O}_{X,x}$. By the previous lemma, we may assume that σ acts by $u_i \mapsto \xi_i u_i$. The local equations of X^σ at x are given by the vanishing of the subset of those local parameters t_j with $\xi_j = 1$. In particular, X^σ is smooth at x.

If some power σ^i acts the identity on the tangent space $T_x(X)$, then it acts as the identity on $\hat{O}_{X,x}$ and thus, it acts as the identity on $O_{X,x}$. The field of fractions of $O_{X,x}$ coincides with the field of rational functions of X and then, σ^i acts as identity on this field and hence, on X. This implies that the action of $\langle\sigma\rangle$ on $T_x(X)$ is faithful. □

In fact, smoothness of fixed point schemes holds more generally to actions of linearly reductive group schemes, see [235, 236].

This corollary shows in particular that the ramification locus of a $\boldsymbol{\mu}_n$-cover $f : Y \to X$ with X and Y smooth and with $p \nmid n$ is either a smooth divisor or empty, see also Proposition 0.2.6 and the discussion after it.

0.3 Inseparable Morphisms and Vector Fields

In this section, we discuss the relationship between vector fields and purely inseparable morphisms between normal varieties in positive characteristic. In this context, we discuss the relationship between $\boldsymbol{\mu}_p$- (resp. $\boldsymbol{\alpha}_p$-) actions and multiplicative (resp. additive) vector fields, as well as the singularities of quotients of smooth varieties by these vector fields. In our discussion, we put an emphasis on dimension two, the case of surfaces. The proofs, when omitted, can be found in [210] and [626]. Moreover, we refer to [458, Section 10] and [520, Lecture III] for two overviews.

Let R be a commutative ring and let M be an R-module. Then, a *derivation* $\partial : R \to M$ is an additive map that satisfies Leibniz's rule $\partial(fg) = f\partial(g) + g\partial(f)$ for all $f, g \in R$. The set $\mathrm{Der}(R, M)$ of all derivations from R to M forms an R-module, and we shall simply write $\mathrm{Der}(R)$ for $\mathrm{Der}(R, R)$. Moreover, if R is a S-algebra for some ring S, then a derivation $\partial : R \to M$ is called S-linear if $\partial(sf) = s\partial(f)$ for all $s \in S$ and all $f \in R$. Since Leibniz' rule implies $\partial(1) = 0$, it follows that $\partial(s) = 0$ for all $s \in S$ if ∂ is S-linear. We denote by $\mathrm{Der}_S(R, M)$ the set of all S-linear derivations from R to M. Moreover, there exists an R-module $\Omega_{R/S}$, the module of *Kähler differentials*, together with a universal S-linear derivation $d : R \to \Omega_{R/S}$, such that there exists a canonical and functorial isomorphism of R-modules $\mathrm{Der}_S(R, M) \cong \mathrm{Hom}_R(\Omega_{R/S}, M)$ for every R-module M. In particular, we have $\Theta_{R/S} := \mathrm{Der}_S(R) := \mathrm{Der}_S(R, R) \cong \mathrm{Hom}_S(\Omega_{R/S}, R)$. For every $\partial \in \mathrm{Der}(R)$, the subset

$$R^\partial := \{r \in R : \partial(r) = 0\}$$

is, in fact, a subring of R. Moreover, assume that R is an integral domain and let $Q(R)$ be its field of fractions. We note that the behavior of these subrings very much depends on the characteristic of $Q(R)$:

1. If $Q(R)$ is of characteristic zero, then $Q(R)$ is a purely transcendental extension of the field of fractions $Q(R^\partial)$. This follows from the fact that every K-derivation of a separable algebraic extension L/K is trivial, see [88, Chapter V].
2. If $Q(R)$ is of characteristic $p > 0$, then R^∂ contains $\mathbf{F}(R) = R^p$ as subring, where $\mathbf{F}$ denotes the absolute Frobenius morphism $r \mapsto r^p$. In particular, every

derivation $\partial : R \to R$ is automatically R^p-linear. Thus, if R is moreover a finitely generated algebra over a perfect field $\Bbbk$ of characteristic p, then every derivation $\partial : R \to R$ is automatically $\Bbbk$-linear (since $\Bbbk$ is perfect) and from the inclusions $R^p \subseteq R^\partial \subseteq R$, we see that both field extensions $Q(R^p) \subseteq Q(R^\partial)$ and $Q(R^\partial) \subseteq Q(R)$ are finite.

More generally, if $\mathfrak{g} \subseteq \mathrm{Der}(R)$ is an arbitrary subset, we set

$$R^{\mathfrak{g}} := \{r \in R | \partial(r) = 0\ \forall \partial \in \mathfrak{g}\},$$

which is a subring of R. If R is of characteristic $p > 0$, then it satisfies $R^p \subseteq R^{\mathfrak{g}} \subseteq R$.

Let X be a variety over a perfect field $\Bbbk$ of characteristic $p > 0$. A *rational vector field* on X is a derivation $\partial : \Bbbk(X) \to \Bbbk(X)$, where $\Bbbk(X)$ denotes the field of rational functions of X. By the above, ∂ is $\Bbbk$-linear and $\Bbbk(X)^\partial$ contains $\Bbbk$. For any open affine subset $U = \operatorname{Spec} R$ of X, we let R^∂ be as above, set $U^\partial := \operatorname{Spec} R^\partial$, and obtain a morphism $U \to U^\partial$. Next, we choose an open affine cover $\mathfrak{U} = \{U_i\}_i$ of X and since derivations extend uniquely to localizations, we can glue the $\{U_i^\partial\}_i$ to a scheme X^∂, which is again a variety over $\Bbbk$. It is easy to see that X^∂ does not depend on the choice of cover $\mathfrak{U}$ and it is called the *quotient of X by the rational vector field ∂*. Moreover, it comes with a dominant and $\Bbbk$-linear morphism

$$\pi^\partial : X \to X^\partial.$$

More generally, if $\mathfrak{g}$ is a subset of $\mathrm{Der}(\Bbbk(X))$ and $U = \operatorname{Spec} R$ is an open affine subset of X, we set $R^{\mathfrak{g}} := \{r \in R | \partial(r) = 0\ \forall \partial \in \mathfrak{g}\}$ and $U^{\mathfrak{g}} := \operatorname{Spec} R^{\mathfrak{g}}$. As before, these glue to a scheme $X^{\mathfrak{g}}$ and we obtain a dominant $\Bbbk$-linear morphism $\pi^{\mathfrak{g}} : X \to X^{\mathfrak{g}}$. If X is normal, then it is easy to see that also X^∂ and $X^{\mathfrak{g}}$ are normal.

Given a finite and purely inseparable field extension $L \subseteq K$, there exists an integer $n \geq 1$ such that $K^{p^n} \subseteq L \subseteq K$. The minimal such n is called the *height* of the field extension. We define the height of a finite and purely inseparable morphism $\pi : X \to Y$ of varieties over a perfect field $\Bbbk$ to be the height of the induced field extension $\Bbbk(Y) \subseteq \Bbbk(X)$. For example, the n-fold $\Bbbk$-linear Frobenius morphism $\mathbf{F}^n : X \to X^{(p^n)}$ is a finite and purely inseparable morphism of degree $p^{n\cdot\dim(X)}$ and height n.

Lemma 0.3.1 *Let $\pi : X \to Y$ be a finite morphism between normal varieties over a perfect field $\Bbbk$ of characteristic $p > 0$ and assume that the induced extension $\Bbbk(Y) \subseteq \Bbbk(X)$ of function fields is purely inseparable of height n. Then, there exists a factorization*

$$\mathbf{F}^n : X \xrightarrow{\pi} Y \to X^{(p^n)},$$

where $\mathbf{F}$ denotes the $\Bbbk$-linear Frobenius morphism.

Proof If $U = \operatorname{Spec} B \subseteq Y$ is an open affine subset, then also $V = \pi^{-1}(U) \subseteq X$ is open and affine, say $V = \operatorname{Spec} A$. Also, $\Bbbk(X)$ and $\Bbbk(Y)$ are the fields of fractions of A and B, respectively. Next, the ring extension $B \subseteq \Bbbk(Y) \cap A$ is finite as a B-module and since both rings are normal and have the same field of fractions, they are equal. Using the equality $B^{p^n} = \mathbb{k}(X)^{p^n} \cap B$, we obtain inclusions $A^{p^n} \subseteq B \subseteq A$ and passing to spectra, we obtain a factorization $\mathbf{F}^n : V \to U \to V^{(p^n)}$ of $\pi|_V$. Globalizing these observations, the lemma follows. □

For example, if X is a normal variety over $\mathbb{k}$, then the quotient morphisms π^∂ and $\pi^\mathfrak{g}$, if non-trivial, are finite and purely inseparable morphisms of height 1 and degree p^i for some $1 \leq i \leq \dim X$.

In order to classify finite morphisms of height 1 between normal varieties, we need more structure. We recall that given a commutative ring R and two derivations $\delta, \eta : R \to R$, their composition $\delta \circ \eta$ is usually not a derivation. However, the *Lie bracket* $[\delta, \eta] := \delta \circ \eta - \eta \circ \delta$ is again a derivation, which turns $\mathrm{Der}(R)$ into a *Lie algebra*. Moreover, if R is of characteristic $p > 0$, then also the p-fold composition $\partial^p := \partial \circ \cdots \circ \partial$, sometimes also denoted by $\partial^{[p]}$, is a derivation, which turns $\mathrm{Der}(R)$ into a *p-Lie algebra* or *restricted Lie algebra*. We encountered this structure already in Section 0.1 and refer to [342, Chapter V.7] for details and precise definitions.

Lemma 0.3.2 *If X be a d-dimensional variety over a perfect field $\mathbb{k}$ of characteristic $p > 0$, then* $\mathrm{Der}(\mathbb{k}(X))$ *is a d-dimensional $\mathbb{k}(X)$-vector space.*

Proof If $K = \mathbb{k}(t_1, ..., t_d)$, then $\mathrm{Der}(K)$ is a d-dimensional K-vector space generated by the derivations $\frac{\partial}{\partial t_i}$, $i = 1, ..., d$. Since X is d-dimensional, there exists a finite and separable field extension $K \subseteq \mathbb{k}(X)$. Using $\Omega_{\mathbb{k}(X)/K} = 0$ and the relative cotangent sequence, we find $\Omega_{\mathbb{k}(X)/\mathbb{k}} \cong \Omega_{K/\mathbb{k}} \otimes_K \mathbb{k}(X)$, from which the assertion follows. □

Given a p-Lie algebra $\mathfrak{g}$ over a field K of characteristic $p > 0$, a *sub-p-Lie algebra* of $\mathfrak{g}$ is a K-subvector space that is closed under Lie brackets and p-powers. It is easy to see that every derivation $\partial \in \mathfrak{g}$ is contained in a unique smallest sub-p-Lie algebra of $\mathfrak{g}$, namely the K-vector space generated by the ∂^{p^i}, $i = 0, 1, 2, ...$ If this smallest sub-p-Lie algebra is one-dimensional, then the derivation ∂ is called *p-closed*, which is equivalent to saying that there exists a $f \in K$ such that $\partial^p = f \cdot \partial$. In the special case where $f = 1$ (resp. $f = 0$), we say that the p-closed vector field ∂ is *of multiplicative type* (resp. *of additive type*). The terminology will become clear from Example 0.3.6 below. Now, let us turn to the special case where $\mathfrak{g} = \mathrm{Der}(K) = \mathrm{Der}_{K^p}(K)$. Then, associated to a subset $V \subseteq \mathfrak{g}$, we have the associated height 1 extension $K^p \subseteq K^V \subseteq K$. Conversely, given a height 1 extension $K^p \subseteq L \subseteq K$, then the set $\{\partial | \partial(x) = 0 \ \forall x \in L\} \subseteq \mathfrak{g}$ turns out to be a sub-p-Lie algebra of $\mathfrak{g}$. By Jacobson's theory of field extensions that are purely inseparable and of height 1 [343], which is an analog of Galois theory for purely inseparable field extensions, this establishes a bijection between height 1 extensions of $K^p \subseteq L \subseteq K$ and sub-p-Lie algebras of $\mathfrak{g} = \mathrm{Der}(K)$. Let us note one major difference to Galois theory: given a finite and separable field extension L/K, it follows from Galois theory that there is only a finite number of intermediate fields $K \subseteq F \subseteq L$. This is not true for inseparable height one extensions, as the following example shows.

Example 0.3.3 Let $\mathbb{k}$ be a perfect field of characteristic $p > 0$ and let $K := \mathbb{k}(t_1, t_2)$, which is the function field of $\mathbb{A}^2$ and $\mathbb{P}^2$ over $\mathbb{k}$. Then, $K^p = \mathbb{k}(t_1^p, t_2^p)$ and the extension $K^p \subseteq K$ is finite and purely inseparable of degree p^2. For every $\lambda \in K^p$, we set

$$\partial_\lambda := \frac{d}{dt_1} - \lambda \cdot \frac{d}{dt_2} \in \mathfrak{g} := \mathrm{Der}(K),$$

which is an additive vector field, that is, $\partial^p = 0$. Thus, $\mathfrak{h}_\lambda := \langle \partial_\lambda \rangle$ is a one-dimensional sub-p-Lie algebra of the two-dimensional Lie algebra $\mathfrak{g}$, and $L_\lambda := K^{\partial_\lambda} \cong K^p(\lambda t_1 + t_2)$. In particular, we find infinitely many distinct intermediate fields between K^p and K that are parametrized by $\lambda \in K^p$.

If X is a smooth variety over a perfect field $\Bbbk$ of characteristic $p > 0$, then the previous discussion globalizes as follows: let $\Theta_X := \Theta_{X/\Bbbk}$ be the tangent sheaf of X. Then, a *p-closed foliation* or an *integrable foliation*, on X is a saturated O_X-submodule $\mathcal{F}$ of Θ_X that is closed under Lie brackets and under the p-power operation. We recall that being saturated means that $\Theta_X/\mathcal{F}$ is a torsion-free O_X-module. Arguing as in the proof of Lemma 0.3.1, we obtain the following correspondence and refer to [343], [210], and [626] for details.

Proposition 0.3.4 *Let X be a normal variety over a perfect field $\Bbbk$ of characteristic $p > 0$. Then, there exists a bijection*

$$\begin{array}{ccc} \{\textit{sub-p-Lie-algebras of } \mathrm{Der}(\Bbbk(X))\} & \leftrightarrow & \left\{ \begin{array}{c} \textit{height } 1 \textit{ morphisms} \\ \textit{of normal varieties} \end{array} \right\} \\ \mathfrak{g} & \mapsto & X \to X^{\mathfrak{g}} \to X^{(p)}. \end{array}$$

Moreover, if X is smooth over $\Bbbk$, then there exists a bijection

$$\{\textit{p-closed foliations on } X\} \leftrightarrow, \{\textit{sub-p-Lie algebras of } \mathrm{Der}(\Bbbk(X))\}.$$

Under this correspondence, the identity morphism of X corresponds to the zero sub-p-Lie algebra of $\mathrm{Der}(\Bbbk(X))$ and to the zero subsheaf of Θ_X if X is smooth. On the other extreme, the $\Bbbk$-linear Frobenius morphism $\mathbf{F} : X \to X^{(p)}$ corresponds to $\mathrm{Der}(\Bbbk(X))$, considered as p-Lie subalgebra of itself, and to Θ_X, considered as a p-closed foliation of itself, if X is smooth. We refer to [210] for a description of finite and purely inseparable morphisms of height n in terms of higher order differential operators.

In general, it is difficult to write down nonzero p-closed rational vector fields explicitly. We will do it in some examples later in Volume II, where we will construct Enriques surfaces as quotients of K3 surfaces by rational vector fields.

If we identify global vector fields on a smooth variety X with their associated derivations of $\Bbbk(X)$, then we have the following useful source of p-closed rational vector fields due to Rudakov and Shafarevich, see [626, Lemma 1].

Lemma 0.3.5 *Let X be a smooth and proper variety over an algebraically closed field $\Bbbk$ of characteristic $p > 0$. If $H^0(X, \Theta_{X/\Bbbk}) \neq 0$, then there exists a $0 \neq \partial \in H^0(X, \Theta_{X/\Bbbk})$ with $\partial^p = 0$ or with $\partial^p = \partial$. In particular, there exists a non-trivial and p-closed vector field.*

Proof Let $\mathfrak{g} := \langle \partial^{p^i}, i = 0, 1, \ldots \rangle$ be the smallest sub-p-Lie algebra of $H^0(X, \Theta_{X/\Bbbk})$ containing ∂, which is a finite–dimensional $\Bbbk$-vector space since the latter is. Then, $\mathfrak{g}$ is an abelian p-Lie algebra and we set $\mathfrak{g}_0 := \mathrm{Ker}(x \mapsto x^p)$, which is a p-Lie subalgebra of $\mathfrak{g}$. If $\mathfrak{g}_0 \neq \{0\}$, then there exists a $0 \neq \eta \in \mathfrak{g}_0$ with $\eta^p = 0$ and we are done. Otherwise, the p-power map on $\mathfrak{g}$ is injective and thus, bijective since $\Bbbk$ is perfect. By [343, Chapter V, Theorem 13], there exists a basis $\eta_1, \ldots, \eta_n$ of $\mathfrak{g}$ such that $\eta_i^p = \eta_i$ for all i. From this, the assertion follows. □

Let us recall that a vector field ∂ with $\partial^p = \partial$ (resp. $\partial^p = 0$) is said to be of multiplicative type (resp. of additive type). Moreover, if $L \subseteq K$ is a purely inseparable field extension of degree p, then there exists an $x \in K$ such that $K = L(x)$. It is easy to see that every derivation $\partial \in \mathrm{Der}_L(K)$ is determined by $\partial(x) \in K$. Moreover, if $\partial(x) = a \in K^\times$, then $\partial' := a^{-1}\partial \in \mathrm{Der}_L(K)$ satisfies $\partial'(x) = 1$, which implies $\partial'^p = 0$. Thus, every non-zero p-closed derivation of $\mathrm{Der}_L(K)$ generates a one-dimensional sub-p-Lie algebra, which also contains a non-zero derivation of additive type. However, note that if $\partial \in H^0(X, \Theta_X)$ is as in the above lemma, then the just-constructed additive rational vector field ∂' of $\mathrm{Der}(\Bbbk(X))$ need not be regular, that is, it need not lie in $H^0(X, \Theta_X)$. The following example is taken from [517] and connects additive and multiplicative vector fields to actions of infinitesimal group schemes.

Example 0.3.6 Let $\Bbbk$ be a perfect field of characteristic $p > 0$ and let $c \in \{0, 1\}$. Consider the non-reduced scheme of length p over $\Bbbk$

$$G_c := \mathrm{Spec}\, \Bbbk[\varepsilon]/(\varepsilon^p) \cong \mathrm{Spec}\, \Bbbk[t]/(t^p - c) \quad \text{via } \varepsilon \mapsto t - c,$$

which becomes a group scheme over $\Bbbk$ via the comultiplication $t \mapsto t \otimes t$ and coinverse $t \mapsto -t$ if $c = 0$ and $t \mapsto t^{-1}$ if $c = 1$. In fact, we have isomorphisms $G_0 \cong \boldsymbol{\alpha}_p$ and $G_1 \cong \boldsymbol{\mu}_p$ of group schemes over $\Bbbk$, see Example 0.1.7 and Example 0.1.8. Next, let R be a $\Bbbk$-algebra and set $X := \mathrm{Spec}\, R$. To give an action $G_c \times X \to X$ is equivalent to giving a homomorphism of $\Bbbk$-algebras

$$\gamma : R \to R \otimes_\Bbbk \Bbbk[t]/(t^p - c), \quad r \mapsto \sum_{i=0}^{p-1} \gamma_i(r)\varepsilon^i,$$

where the $\gamma_i : R \to R$ are $\Bbbk$-linear maps satisfying certain axioms that we will now describe depending on c.

1. First, suppose that $c = 0$, that is, $G_c \cong \boldsymbol{\alpha}_p$. Then, the axioms of the action imply

$$\sum_{i=0}^{p-1} \sum_{j=0}^{p-1} \partial_j(\partial_i(r))\varepsilon^j \otimes \varepsilon^i = \sum_{i=0}^{p-1} \partial_i(r)(\varepsilon^i \otimes 1 + 1 \otimes \varepsilon^i)$$

and then, we obtain the conditions

$$\partial_i = \frac{1}{i!}\partial_1^i, \quad i = 0, \ldots, p-1, \quad \text{and} \quad \partial_1^p = 0.$$

Conversely, a collection of $\Bbbk$-linear functions $\partial_i : R \to R$ satisfying the previous conditions defines an action $\boldsymbol{\alpha}_p \times X \to X$. Note that the conditions imply that ∂_1 determines all ∂_i with $i \geq 2$.

2. Second, suppose that $c = 1$, that is, $G_c \cong \boldsymbol{\mu}_p$. In this case, the axioms of the action imply
$$\sum_{i=0}^{p-1} \sum_{j=0}^{p-1} \gamma_j(\gamma_i(r))\varepsilon^j \otimes t^i \;=\; \sum_{i=0}^{p-1} \gamma_i(r) t^i \otimes t^i,$$
and then, comparing the coefficients at $t^j \otimes t^i$, we obtain that
$$\gamma_i \circ \gamma_i \;=\; \gamma_i, \quad \gamma_i \circ \gamma_j = 0 \text{ if } i \neq j, \quad \text{and} \quad \sum_{i=0}^{p-1} \partial_i = \mathrm{id}_R\,.$$
Conversely, a collection of $\Bbbk$-linear functions $\gamma_i : R \to R$ satisfying the previous conditions defines an action $\boldsymbol{\mu}_p \times X \to X$. If we want to compute the ∂_i, which are functions in ε from the functions γ_i, which are functions in t, then we need to substitute $\varepsilon = t - 1$ and we find
$$\partial_0 \;=\; \mathrm{id}, \quad \partial_1 \;=\; \gamma_1 + 2\gamma_2 + \ldots + (p-1)\gamma_{p-1}, \ldots \quad, \partial_{p-1} \;=\; \gamma_{p-1}.$$
We refer to [338, page 113] for details, where one also finds the computation that ∂_1 satisfies $\partial_1^p = \partial_1$, as well as the verification of the fact that ∂_1 determines all ∂_i with $i \geq 2$.

Next, we compute
$$\partial(rs) \;=\; \partial(r)\cdot\partial(s) \;=\; \left(\sum_{i=0}^{p-1} \partial_i(r)\varepsilon^i\right)\cdot\left(\sum_{j=0}^{p-1} \partial_j(s)\varepsilon^j\right) \;=\; rs + (r\partial_1(s) + s\partial_1(r))\,\varepsilon + \cdots,$$
which implies that ∂_1 is a $\Bbbk$-linear map satisfying $\partial_1(rs) = r\partial_1(s) + s\partial_1(r)$. Thus, ∂_1 is a derivation of R and we have seen above that $\partial_1^p = c \cdot \partial_1$. Thus, we obtain bijections
$$\begin{aligned} &\{\,\boldsymbol{\mu}_p\text{-actions on } X\} \leftrightarrow \{\text{derivations in } \mathrm{Der}(R) \text{ of multiplicative type}\}, \\ &\{\boldsymbol{\alpha}_p\text{-actions on } X\} \;\leftrightarrow \{\text{derivations in } \mathrm{Der}(R) \text{ of additive type}\}. \end{aligned}$$
This result also fits with the restricted Lie algebras of $\boldsymbol{\mu}_p$ and $\boldsymbol{\alpha}_p$, which we have seen in Example 0.1.16. We refer to [517] and [710] for further details.

Globalizing this affine example, we obtain the following result, whose proof we leave it the reader.

Proposition 0.3.7 *Let X be a scheme over a perfect field $\Bbbk$ of characteristic $p > 0$ and let G be the group scheme $\boldsymbol{\mu}_p$ (resp. $\boldsymbol{\alpha}_p$) over $\Bbbk$.*

1. Given an action $G \times X \to X$, then the quotient $f : X \to Y := X/G$ exists and there exists a regular vector field ∂ of multiplicative type (resp. additive type) such that f coincides with the quotient π^{∂} of X by ∂.
2. Conversely, given a regular p-closed vector field ∂ of multiplicative type (resp. additive type), on X, there exists a G-action $G \times X \to X$, such that the quotient of X by this action coincides with the quotient π^{∂} of X by ∂.

Note that the usual assumption that any orbit of G must be contained in an open affine subset is not needed in this case because G, being an infinitesimal group scheme, leaves invariant any affine open subset.

Example 0.3.8 Let $\mathbb{k}$ be a perfect field of characteristic $p > 0$.

1. If $X = \mathbb{P}^1_{\mathbb{k}}$, then $\Theta_X \cong \mathcal{O}_{\mathbb{P}^1}(2)$ and $h^0(X, \Theta_X) = 3$. If ∂ is a regular vector field of multiplicative type, then it can be written in appropriate coordinates as $x\frac{d}{dx}$ and it has two distinct zeros, namely, $x = 0$ and $x = \infty$. If ∂ is of additive type, then it can be written in approriate coordinates as $\frac{d}{dx}$ and it has a double zero at $x = \infty$. More precisely, we have an isomorphism of abelian restricted Lie algebras
$$H^0(X, \Theta_X) \;\cong\; \mathfrak{g}_0 \oplus \mathfrak{g}_1 \quad \text{with} \quad \mathfrak{g}_0 \;:=\; \left\langle \frac{d}{dx}, x^2\frac{d}{dx} \right\rangle \text{ and } \mathfrak{g}_1 \;:=\; \left\langle x\frac{d}{dx} \right\rangle,$$
where the p-power map on $\mathfrak{g}_0$ is zero. Since X admits both multiplicative as well as additive vector fields, the $\mathbb{k}$-linear Frobenius morphism $\mathbf{F} : X \to X^{(p)}$ can be written as a quotient by $\boldsymbol{\mu}_p$-actions, as well as by $\boldsymbol{\alpha}_p$-actions.
2. If E is an elliptic curve over $\mathbb{k}$, then $\Theta_E \cong \mathcal{O}_E$ and $h^0(E, \Theta_E) = 1$. Thus, up to scaling by $\mathbb{k}$, there exists precisely one regular vector field, and it is automatically p-closed. In particular, $H^0(E, \Theta_E)$ can be generated by a vector field ∂ that is either of multiplicative type or of additive type. In the first case, the $\mathbb{k}$-linear Frobenius morphism $\mathbf{F} : E \to E^{(p)}$ is the quotient by a $\boldsymbol{\mu}_p$-action and E is ordinary, whereas $\mathbf{F}$ is the quotient by an $\boldsymbol{\alpha}_p$-action and E is supersingular in the second case. See also Example 0.1.11 and Example 0.2.30.
3. If X is a smooth and proper curve of genus $g \geq 2$ over $\mathbb{k}$, then $H^0(X, \Theta_X) = 0$ and thus, there exist neither $\boldsymbol{\mu}_p$-actions nor $\boldsymbol{\alpha}_p$-actions on X that are non-trivial. On the other hand, a simple $\boldsymbol{\mu}_p$-cover $f : Y \to X$ is given by an invertible sheaf $\mathcal{L}$ of some degree d and a section $s \in H^0(X, \mathcal{L}^{\otimes p})$. By Proposition 0.2.12, we have $\omega_Y = f^*(\omega_X \otimes \mathcal{L}^{\otimes(p-1)})$, which is an invertible sheaf of degree $2p_a(Y) - 2 = p(2g - 2 + (p-1)d)$ on Y, where $p_a(Y)$ denotes the arithmetic genus of Y. The singular points of Y lie over the zeros of s, and thus, for $d \gg 0$ and a generic choice of s, one expects Y to have $(pd + 1 - g)$ ordinary nodes. The normalization $\nu : \widetilde{Y} \to Y$ is a smooth curve of genus $g(Y) = g$, since the composition $f \circ \nu$ coincides with the $\mathbb{k}$-linear Frobenius morphism $\mathbf{F} : \widetilde{Y} \to \widetilde{Y}^{(p)} = X$. Thus, if $g \geq 2$, then we obtain a $\boldsymbol{\mu}_p$-action on the non-normal curve Y with quotient X, but this action does not extend to the normalization $\widetilde{Y}$, since the latter has no nonzero global vector fields.

Next, we turn to singularities of vector fields and their quotients. Let ∂ be a rational vector field on a smooth and n-dimensional variety X over a perfect field $\Bbbk$ of characteristic $p > 0$. If $x \in X$ is a closed point and $t_1, \ldots, t_n \in \mathcal{O}_{X,x}$ are local coordinates at x, then ∂ can be written locally around x in the form

$$\partial = \psi_x \cdot \sum_{i=1}^{n} \phi_{i,x} \frac{d}{dt_i} \quad \text{with} \quad \psi_x \in \Bbbk(X) \quad \text{and} \quad \phi_{i,x} \in \mathcal{O}_{X,x} \text{ for } i = 1, ..., n, \tag{0.3.1}$$

such that $(\phi_{1,x}, ..., \phi_{n,x}) \subseteq \mathcal{O}_{X,x}$ is an ideal of height ≥ 2. The functions ψ_x are local equations of a Cartier divisor $D = \mathrm{div}(\partial)$ of X, called the *divisor* of ∂. The ideals $(\phi_{1,x}, \ldots, \phi_{n,x})$ define a closed subscheme Z of X of codimension ≥ 2, the scheme of *non-divisorial zeros* of ∂. If $\dim_x Z = 0$, we say that x is an *isolated zero* of ∂ and, if $\dim Z = 0$, then Z is called the scheme of isolated zeros. For example, if X is a surface, then Z is empty or zero-dimensional. For an isolated zero $x \in X$ of ∂, the dimension

$$\mathrm{mult}_x\, \partial \;:=\; \dim_{\Bbbk} \mathcal{O}_{X,x}/(\phi_{1,x}, \ldots, \phi_{n,x})$$

is called the *multiplicity* of the isolated zero. The following theorem relates the zeros of ∂ to the singularities of the quotient X^{∂}. It is due to Rudakov and Shafarevich and we refer to [626, Theorem 1 and Theorem 2] for details and proof.

Theorem 0.3.9 *Let X be a smooth variety over an algebraically closed field $\Bbbk$ of characteristic $p > 0$. Let $0 \neq \partial \in H^0(X, \Theta_{X/\Bbbk})$ be a p-closed vector field, let $Z \subset X$ be its scheme of non-divisorial zeros, let $\pi^{\partial} : X \to X^{\partial}$ be the quotient map, and let $x \in X$ be a closed point.*

1. *If $x \notin Z$, then $\pi^{\partial}(x) \in X^{\partial}$ is a smooth point on the quotient.*
2. *Let ∂ be of multiplicative type. Then, Z is a smooth subscheme of X. More precisely, if $x \in Z$, then there exist local parameters $t_1, ..., t_n$ and a function ψ in the completion $\widehat{\mathcal{O}}_{X,x}$, such that*

$$\partial = \psi \left(\sum_{i=1}^{n} \alpha_i \cdot t_i \frac{d}{dt_i} \right) \quad \text{with } \alpha_i \in \mathbb{F}_p \text{ for all } i.$$

In particular, all isolated zeros of ∂ are of multiplicity 1.

The following result slightly extends a result of Hirokado [304, Theorem 2.3] and describes the singularities of quotients of smooth varieties by multiplicative vector fields. These singularities are examples of cyclic quotient singularities, and we will come back to them in Proposition 0.4.20.

Proposition 0.3.10 *Let $\Bbbk$ be a perfect field of characteristic $p > 0$ and consider the multiplicative vector field $\partial = \sum_{i=1}^{n} \alpha_i \cdot t_i \frac{d}{dt_i}$ with $\alpha_i \in \{1, ..., p-1\}$ on $\Bbbk[[t_1, ..., t_n]]$. Then,*

$$\Bbbk[[t_1, ..., t_n]]^{\partial} = \Bbbk[[t_1^{\beta_1} \cdots t_n^{\beta_n} \,|\, \beta_1 \geq 0, ..., \beta_n \geq 0, \sum_{i=1}^{n} \alpha_i \beta_i \equiv 0 \mod p]],$$

which is a description in terms of toric geometry of the cyclic quotient singularity of type $\frac{1}{p}(\alpha_1, ..., \alpha_n)$.

Proof It is easy to see that the right-hand side is contained in the left-hand side. Both sides are normal rings and their fields of fractions are of degree p over the field of fractions of $\Bbbk[[t_1, ..., t_n]]$. From this, the assertion follows, see also the proof of Proposition 0.4.20. □

Example 0.3.11 In dimension $n = 2$ the quotient of $R = \Bbbk[[t_1, t_2]]$ by the multiplicative vector field $\partial := t_1 \frac{d}{dt_1} - t_2 \frac{d}{dt_2}$ is a singularity of type $\frac{1}{p}(1, p-1)$, which is a rational double point singularity of type A_{p-1}, see Proposition 0.4.20.

Remark 0.3.12 The notion of a cyclic quotient singularity has to be taken with a grain of salt in positive characteristic:

1. Quotients of smooth varieties by the cyclic group schemes $\mathbb{Z}/p\mathbb{Z}$ or $\boldsymbol{\alpha}_p$ may lead to very complicated singularities that are not of the above type, and that are not even rational. The problem is that these two group schemes are not linearly reductive, and we refer to [468] and Section 0.4 for details. We refer to [480] and [481] for the study of wild actions of groups $\mathbb{Z}/p\mathbb{Z}$ and, in particular, for some information on a resolution of the quotient singularities.
2. Quotients of smooth varieties by multiplicative vector fields correspond to quotients by $\boldsymbol{\mu}_p$-actions, see Example 0.3.6. Since $\boldsymbol{\mu}_p$ is linearly reductive, the quotients behave much better and Proposition 0.3.10 classifies the occurring singularities.

Example 0.3.13 Not much is known about singularities occurring on quotients by vector fields of additive type, even for smooth surfaces. Let $R = \Bbbk[[t_1, t_2]]$, where $\Bbbk$ is algebraically closed of characteristic $p = 2$, and let $\partial = f(t_1)\frac{d}{dt_1} + g(t_2)\frac{d}{dt_2}$ be a p-closed vector field. Then, we know from [456] that the singularity of $\Bbbk[[t_1, t_2]]^\partial$ is:

1. a rational double point of type A_1 if f, g both have a simple zero,
2. a rational double point of type D_4 if f, g both have a zero of order 2,
3. a rational double point of type D_8 if f has a zero of order 2 and g has a zero of order 4,
4. an elliptic singularity if f, g both have a zero of order 4. In fact, this singularity is of type $(19)_0$ with respect to Wagreich's classification [729], see also [456].

In particular, the quotient may not have rational singularities. On the other hand, if $\Bbbk$ is algebraically closed of characteristic $\neq 2$ and if G is a finite flat group scheme of length 2 acting on $R = \Bbbk[[t_1, t_2]]$, then $G \cong \boldsymbol{\mu}_{2,\Bbbk} \cong (\mathbb{Z}/2\mathbb{Z})_\Bbbk$, the G-action can be linearized, and the quotient R^G is is either smooth (if the fixed locus of G is not isolated) or a rational double point of type A_1 (if the fixed locus of G is isolated). We refer to Section 0.4 for more about rational double points, especially Remark 0.4.33, as well as to [11] and [10] for some general results on quotients by vector fields.

Having studied p-closed vector fields and some of their singularities locally, let us now globalize our discussion. Let X be a smooth and n-dimensional variety over a perfect field $\Bbbk$ of characteristic $p > 0$ and let ∂ be a global regular vector field on X with divisor $D = \mathrm{div}(\partial)$. As above, we choose for every closed point $x \in X$ local coordinates $t_1, ..., t_n \in \mathcal{O}_{X,x}$ and write ∂ locally as $\psi_x \cdot \sum_{i=1}^n \phi_{i,x} \frac{d}{dt_i}$ as in (0.3.1). We use this to define a homomorphism of $\mathcal{O}_X$-modules

$$\partial \,:\, \mathcal{O}_X(D) \,\to\, \Theta_X$$

by locally defining it as $1 \mapsto \sum_{i=1}^n \phi_{i,x} \frac{d}{dt_i}$. We set $\mathcal{F} := \partial(\mathcal{O}_X(D))$. For every $x \in X$, the ideal $(\phi_{1,x}, ..., \phi_{n,x}) \subseteq \mathcal{O}_{X,x}$ is of height ≥ 2, which implies that the quotient $\Theta_X/\mathcal{F}$ has no torsion in codimension 1 and is of projective dimension ≤ 1. Therefore, this quotient is torsion-free, that is, $\mathcal{F} \subseteq \Theta_X$ is a saturated subsheaf. Being of rank 1, it is automatically closed under the Lie bracket. Thus, if D is p-closed, then $\mathcal{F}$ is a p-closed foliation of rank one.

Next, let $f := \pi^\partial : X \to Y := X^\partial$ be the quotient by the rational vector field D. We have seen above that Y is a normal variety over $\Bbbk$. It is easy to see that the image of $\mathcal{F} := \partial(\mathcal{O}_X(D))$ in $f^*(\Theta_Y)$ is zero, that is, $\mathcal{F}$ is contained in $\Theta_{X/Y} := \mathrm{Ker}(\Theta_X \to f^*(\Theta_Y))$. (If Y is not smooth over $\Bbbk$, then Θ_Y denotes $\mathcal{H}om(\Omega_{Y/\Bbbk}, \mathcal{O}_Y)$, which may not be a locally free $\mathcal{O}_Y$-module.) If ∂ is p-closed, then we have just seen that $\mathcal{F} \subseteq \Theta_X$ is a p-closed foliation of rank one. In this case, the degree of f is equal to p and $\Omega_{X/Y}$ and $\Theta_{X/Y}$ both are $\mathcal{O}_X$-modules of rank 1. Moreover, since $\mathcal{F}$ is contained in $\Theta_{X/Y}$, and both are saturated subsheaves of Θ_X of rank 1, they are equal, and we conclude that there is an isomorphism of $\mathcal{O}_X$-modules

$$\mathcal{F} \;=\; \mathcal{O}_X(D) \;\cong\; \Theta_{X/Y}.$$

Since the $\Bbbk$-linear Frobenius morphism $\mathbf{F} : X \to X^{(p)}$ factors over f, there exists a morphism $g : Y \to X^{(p)}$ such that $\mathbf{F} = g \circ f$, see also Lemma 0.3.1. Moreover, the inclusion $\mathcal{F} \subseteq \Theta_X$ extends to an exact sequence of $\mathcal{O}_X$-modules

$$0 \to \mathcal{F} \to \Theta_{X/\Bbbk} \to f^*\Theta_{Y/\Bbbk} \to f^*(g^*(\Theta_{X^{(p)}/\Bbbk})) = \mathbf{F}^*\Theta_{X^{(p)}/\Bbbk}.$$

In the case where the quotient Y is also smooth over $\Bbbk$, one can show that the image of the last homomorphism is equal to $\mathbf{F}^*\sigma^*\mathcal{F} = \mathcal{O}_X(pD)$, where $\sigma : X^{(p)} \to X$ is the canonical (non $\Bbbk$-linear) isomorphism. In this case, we obtain an exact sequence

$$0 \to \mathcal{O}_X(D) \to \Theta_{X/\Bbbk} \to f^*\Theta_{Y/\Bbbk} \to \mathcal{O}_X(pD) \to 0. \tag{0.3.2}$$

We refer to [210, Corollary 3.4] for details. Taking determinants, we obtain the following formula for the canonical sheaf from [626, Corollary 1].

Proposition 0.3.14 *Let ∂ be a rational and p-closed vector field on a smooth variety X over a perfect field $\Bbbk$ of characteristic $p > 0$. Let $f := \pi^\partial : X \to Y := X^\partial$ be the quotient morphism and assume that Y is smooth over $\Bbbk$. Then, f is a finite morphism of degree p and*

$$K_X = f^*(K_Y) + (p-1)D,$$

where $D := \operatorname{div}(\partial)$ *denotes the divisor of zeros of* ∂.

Remark 0.3.15 One should compare this formula with the formula for the canonical sheaf of simple $\boldsymbol{\mu}_p$-covers and $\boldsymbol{\alpha}_{\mathcal{L},a}$-torsors, see Proposition 0.2.12 and Proposition 0.2.20. It follows that $\mathcal{O}_X(D)$ is isomorphic to $f^*(\mathcal{L})$, where $\mathcal{L}$ is the invertible sheaf discussed in connection with these covers. Moreover, one should also compare this formula with the case of cyclic covers that are generically étale: then, the invertible sheaf $\mathcal{O}_X(D)$ looks like the class of some ramification divisor. However, we stress that in the purely inseparable case there is no distinguished section, that is, something like a well-defined ramification divisor - inseparable morphisms are everywhere ramified. (See the discussion after Proposition 0.2.6).

Let us give an application of this formula, which we will need in the next chapter. We refer to [626, §2, Corollary 3] for further details and to Section 1.1 for the definition of the Kodaira dimension $\kappa(X)$ of a smooth and proper variety.

Corollary 0.3.16 *Let* X *be a smooth and proper variety over a perfect field* $\Bbbk$ *of characteristic* $p > 0$ *and let* $0 \neq \partial \in H^0(X, \Theta_X)$. *Let* $Y := X^{\partial}$ *be the quotient of* X *by* ∂ *and assume that there exists a resolution of singularities* $g : \widetilde{Y} \to Y$ *(for example,* $\dim(Y) \leq 2$*). Then, we have*

$$h^0(\widetilde{Y}, \mathcal{O}_{\widetilde{Y}}(nK_{\widetilde{Y}})) \;\leq\; h^0(X, \mathcal{O}_X(nK_X))$$

for all $n \geq 0$. *In particular, we have* $\kappa(\widetilde{Y}) \leq \kappa(X)$.

Proof Let $Z \subset X$ be the closed subscheme of non-divisorial schemes of ∂, which is of codimension ≥ 2, and let $U := X \backslash Z \subseteq X$. Then, $V := U^{\partial} \subseteq Y$ is a smooth, open, and dense subset. In particular, $\widetilde{V} := g^{-1}(V) \to V$ is a birational morphism between smooth varieties, from which we deduce inclusions and equalities for all $n \geq 0$

$$H^0(\widetilde{Y}, \mathcal{O}_{\widetilde{Y}}(nK_{\widetilde{Y}})) \;\subseteq\; H^0(\widetilde{V}, \mathcal{O}_{\widetilde{V}}(nK_{\widetilde{V}})) \;=\; H^0(V, \mathcal{O}_V(nK_V)).$$

Applying the previous proposition, we find the inclusions and isomorphisms

$$(\pi^{\partial})^* \;:\; H^0(V, \mathcal{O}_V(nK_V))) \;\hookrightarrow\; H^0(U, \mathcal{O}_U(nK_U)) \;\cong\; H^0(X, \mathcal{O}_X(nK_X)),$$

where $\pi^{\partial} : X \to Y$ denotes the quotient projection and where we use the fact that X is smooth and the complement of U in X is of codimension ≥ 2. □

Remark 0.3.17 The following examples show that the assumption that ∂ is a regular vector field in Corollary 0.3.16 is crucial: let $\Bbbk$ be an algebraically closed field of characteristic $p > 0$, let $X := \mathbb{P}^2_{\Bbbk}$, and let f be a generic global section of $\mathcal{L} := \mathcal{O}_X(n)$ for some $n \geq 1$. Let $Y \to X$ be the simple $\boldsymbol{\mu}_p$-cover of X associated to $(\mathcal{L}, f)$, see also Example 0.2.22. Then, $\Bbbk(X) \subset \Bbbk(Y)$ is a purely inseparable field extension of degree p and $\Bbbk(Y) \subset \Bbbk(X)^{(1/p)}$. These field extensions correspond to finite and purely inseparable morphisms of degree p

$$X^{(1/p)} \cong \mathbb{P}^2 \;\to\; Y \;\to\; X = \mathbb{P}^2,$$

whose composition is the $\Bbbk$-linear Frobenius morphism $\mathbf{F} : X^{(1/p)} \to X$. The surface Y arising this way is called a *Zariski surface*. This class of surfaces was introduced by Zariski in [740] in connection with Castelnuovo's rationality criterion and in order to construct unirational surfaces that are not rational. If $n \gg 0$ and since f is assumed to be generic, then one can show that Y is a normal surface and that the minimal resolution of singularities of Y is a surface of general type, see also [66], [68], and [304]. In particular, we have the Kodaira dimension $\kappa(X) = \kappa(X^{(1/p)}) = -\infty$ and $\kappa(Y) = 2$. On the other hand, by Proposition 0.3.4, there exists a rational vector field ∂ on $X^{(1/p)}$ such that the morphism $X^{(1/p)} \to Y$ is the quotient morphism by ∂. This shows that Corollary 0.3.16 is wrong for rational vector fields that are not regular. For example, we will meet Enriques surfaces that are Zariski surfaces in Proposition 1.2.9.

For the remainder of this section, let X be a smooth surface over a perfect field $\Bbbk$ of characteristic $p > 0$ and let ∂ be a p-closed rational vector field. Let $D := \mathrm{div}(\partial)$ be its divisor of zeros and let $\mathcal{F} := \mathcal{O}_X(D) \subset \Theta_X$ be the associated p-closed foliation. Since X is two-dimensional, the closed subscheme Z of non-divisorial zeros of ∂ is zero-dimensional, that is, all these zeros are isolated. Next, the quotient sheaf $\mathcal{Q} := \Theta_X/\mathcal{F}$ is torsion-free and generically of rank one. Thus, its dual $\mathcal{Q}^\vee := \mathcal{H}om(\mathcal{Q}, \mathcal{O}_X)$ is reflexive and generically of rank one. Since all local rings of X are regular local rings, it follows that $\mathcal{Q}^\vee$ is in fact an invertible sheaf. Then, also $\mathcal{M} := (\mathcal{Q}^\vee)^\vee$ is an invertible sheaf, and we have a canonical injective morphism $\mathcal{Q} \to \mathcal{M}$ of $\mathcal{O}_X$-modules. In fact, a local computation reveals that $\mathcal{Q} \otimes \mathcal{M}^\vee \subseteq \mathcal{O}_X$ is the ideal sheaf $\mathcal{I}_Z$ of the scheme Z of non-divisorial zeros of ∂. Thus, we find

$$\Theta_X/\mathcal{F} = \mathcal{Q} \cong \mathcal{I}_Z(\mathcal{M}) = \mathcal{I}_Z \otimes_{\mathcal{O}_X} \mathcal{M}$$

and refer to [304] or [373] for details, computations, and different approaches.

Proposition 0.3.18 *Let ∂ be a rational vector field on a smooth and proper surface X over a perfect field $\Bbbk$ of characteristic $p > 0$, let $D := \mathrm{div}(\partial)$ be its divisor of zeros, and let Z be its scheme of isolated zeros. Then, there exists a short exact sequence*

$$0 \to \mathcal{O}_X(D) \xrightarrow{\partial} \Theta_{X/\Bbbk} \to \mathcal{I}_Z(-K_X - D) \to 0. \tag{0.3.3}$$

Moreover,

$$\mathrm{lg}(Z) := \dim_\Bbbk H^0(X, \mathcal{O}_Z) = c_2(X) + D \cdot K_X + D^2, \tag{0.3.4}$$

where $c_2(X)$ denotes the second Chern number, that is, the ℓ-adic Euler characteristic of X.

Proof By the above discussion, we find a short exact sequence like (0.3.3), but with quotient $\mathcal{I}_Z(\mathcal{M})$. Taking determinants, we find an isomorphism $\mathcal{M} \cong \mathcal{I}_Z(-K_X - D)$ of $\mathcal{O}_X$-modules. The formula for $\mathrm{lg}(Z)$ follows from standard properties of Chern classes of coherent sheaves and we refer to [373, Proposition 2.1] for details. □

We keep the notations and assumptions from the previous proposition and assume moreover that $\Bbbk$ is algebraically closed. If C is an integral curve on X, then C is said to be an *integral curve with respect to* ∂ if C is tangent to ∂ at the generic point of C. More explicitly: for a closed point $x \in C \subset X$ we choose local coordinates $t_1, t_2 \in \mathcal{O}_{X,x}$. Then, locally around x, we have

$$\partial \;=\; \psi_x \cdot \left(\phi_{1,x} \frac{d}{dt_1} + \phi_{2,x} \frac{d}{dt_2} \right) \qquad \text{and} \qquad C = \{f = 0\},$$

where $\phi_{1,x}, \phi_{2,x}, f \in \mathcal{O}_{X,x}$, $\psi_x \in \Bbbk(X)$, and where the ideal $(\phi_{1,x}, \phi_{2,x}) \subseteq \mathcal{O}_{X,x}$ is of height ≥ 2, see (0.3.1). We set $\partial' := \psi_x^{-1} \cdot \partial = \phi_{1,x} \frac{d}{dt_1} + \phi_{2,x} \frac{d}{dt_2}$. Then, C is integral with respect to ∂ if and only if $\partial'(f)$ is zero in $\mathcal{O}_{X,x}/(f)$. If we take an open subset U containing smooth points of C if we choose f to be a local parameter in X at $x \in C$, then, after dualizing the exact sequence

$$0 \to \mathcal{I}_C/\mathcal{I}_C^2 \xrightarrow{df} \Omega^1_{U/\Bbbk} \otimes \mathcal{O}_C \to \Omega^1_{C/\Bbbk} \to 0,$$

we see that C is an integral curve if and only the projection of ∂ to the normal sheaf $\mathcal{N}_C = (\mathcal{I}_C/\mathcal{I}_C^2)^\vee$ is equal to zero.

After these preparations, we have the following result of Rudakov and Shafarevich from [626, Proposition 1] that concerns intersection numbers, pull-backs, and push forwards.

Proposition 0.3.19 *Let ∂ be a rational and p-closed vector field on a smooth and proper surface X over an algebraically closed field $\Bbbk$ of characteristic $p > 0$. Let $f := \pi^\partial : X \to Y := X^\partial$ be the quotient morphism, which is purely inseparable of degree p, and assume that Y is smooth over $\Bbbk$. Let $C \subset X$ be an integral curve and set $C' := f(C)$. Then,*

$$\begin{array}{lll} f^*(C') = C, & f_*(C) = pC',\ C^2 = pC'^2 & \textit{if } C \textit{ is integral with respect to } \partial, \\ f^*(C') = pC, & f_*(C) = C', \quad pC^2 = C'^2 & \textit{else.} \end{array}$$

Next, we study the effect of blow-ups on the isolated zeros of a p-closed vector field on a smooth surface X over an algebraically closed field $\Bbbk$ of characteristic $p > 0$, and we refer to [304, Remarks 2.4] for further information. As before, let $D = \operatorname{div}(\partial)$ be the divisor of zeros and Z be the scheme of isolated zeros of ∂. We fix a closed point $x \in Z$, let $g : \widetilde{X} \to X$ be the blow-up in x, let $E \subset \widetilde{X}$ be the exceptional divisor of g, and let $\widetilde{\partial}$ be the rational vector field induced by ∂ on $\widetilde{X}$. Let $\widetilde{Z}$ be the scheme of isolated zeros of $\widetilde{\partial}$ and since g is an isomorphism outside E, there exists some integer $r \geq 0$ such that $\operatorname{div}(\widetilde{\partial}) = g^*D + rE$. Dualizing the cotangent sequence, we obtain an inclusion $\Theta_{\widetilde{X}} \to g^*\Theta_X$, whose cokernel is supported on E. In particular, the sequence (0.3.3) for $\widetilde{\partial}$ and the pull-back of the sequence (0.3.3) for ∂ via g^* agree outside E. From this, we obtain a short exact sequence

$$0 \;\to\; \mathcal{O}_X(g^*(D) + rE) \;\to\; \Theta_{\widetilde{X}} \;\to\; \mathcal{I}_{\widetilde{Z}}(-K_{\widetilde{X}} - g^*(D) - rE) \to 0. \qquad (0.3.5)$$

Computing the Chern classes, and comparing with formula (0.3.4), we obtain

$$\lg(\widetilde{Z}) = \lg(Z) - (r^2 + r - 1). \tag{0.3.6}$$

Let us now study the situation locally: as seen above, we can write the vector field ∂ around $x \in X$ as $\partial = \psi_x \cdot \partial'$, where ψ_x is a local equation for D and where $\partial' = \phi_{1,x}\frac{d}{dt_1} + \phi_{2,x}\frac{d}{dt_2}$, such that $(\phi_{1,x}, \phi_{2,x}) \subseteq \mathcal{O}_{X,x}$ is a height 2 ideal, see (0.3.1). By definition, we have $\mathrm{mult}_x\,\partial = \dim_{\Bbbk} \mathcal{O}_{X,x}/(\phi_{1,x}, \phi_{2,x})$. Let $\widetilde{\partial}'$ and $\widetilde{\partial}$ be the rational vector fields induced by ∂' and ∂ on $\widetilde{X}$. One chart of the blow-up is given by $t_1 = uw$ and $t_2 = w$. In it, we compute

$$\frac{d}{dt_1} = \frac{1}{w}\frac{d}{du}, \qquad \frac{d}{dt_2} = -\frac{u}{w}\frac{d}{du} + \frac{d}{dw}$$

and find

$$\widetilde{\partial}' = \frac{\phi_{1,x}(uw, u) - u \cdot \phi_{2,x}(uw, w)}{w}\frac{d}{du} + \phi_{2,x}(uw, w)\frac{d}{dw}.$$

Next, the integer $r \geq 0$ from the above is the multiplicity of $\widetilde{\partial}'$ along E, that is, the maximal w-power dividing $\widetilde{\partial}'$. If ν_E denotes the valuation of $\Bbbk(X) = \Bbbk(\widetilde{X})$ associated to E, then the local description of $\widetilde{\partial}'$ shows that $r \geq \min\{\nu_E(\phi_{1,x}), \nu_E(\phi_{2,x})\} - 1$. Together with the other chart of the blow-up we conclude

$$\sum_{x' \in E} \mathrm{mult}_{x'}\,\widetilde{\partial} = \mathrm{mult}_x\,\partial - (r^2 + r - 1), \tag{0.3.7}$$

which is a local and explicit version of (0.3.6).

Example 0.3.20 Let $x \in X$ be an isolated zero of a multiplicative vector field ∂ on a smooth surface. By Theorem 0.3.9, there exist local parameters $t_1, t_2 \in \widehat{\mathcal{O}}_{X,x}$, such that $\partial = \alpha_1 t_1 \frac{d}{dt_1} + \alpha_2 t_2 \frac{d}{dt_2}$ for some $\alpha_1, \alpha_2 \in \mathbb{F}_p\backslash\{0\}$. Let $g : \widetilde{X} \to X$ be the blow-up in $x \in X$ with exceptional curve E. In the chart of the blow-up given by $t_1 = uw$ and $t_2 = w$, we find

$$\widetilde{\partial} = (\alpha_1 - \alpha_2)u\frac{d}{du} + \alpha_2 w\frac{d}{dw},$$

and similarly for the other chart. Thus, if $p = 2$, then $\alpha_1 = \alpha_2 = 1$ and the induced vector field $\widetilde{\partial}$ on $\widetilde{X}$ has no isolated zeros on E.

In general, if $p \geq 3$, one cannot resolve the isolated zeros of a p-closed vector field on a smooth surface by successively blowing up its isolated zeros. The situation is different if $p = 2$, where we have the following result of Hirkoado that shows the existence of canonical resolutions of singularities of p-closed foliations [304].

Proposition 0.3.21 *Let X be a smooth surface over an algebraically closed field $\Bbbk$ of characteristic $p = 2$. Let ∂ be a rational and p-closed vector field on X. Then, after repeatedly blowing up the isolated zeros of the induced vector fields on the blow-ups, one obtains a birational morphism $g : \widetilde{X} \to X$ between smooth surfaces and a commutative diagram*

$$\begin{tikzcd} X & \widetilde{X} \arrow[l, "g"'] \arrow[d, "\pi^{\widetilde{\partial}}"'] \\ X^{\partial} \arrow[from=u, "\pi^{\partial}"'] & \widetilde{X}^{\widetilde{\partial}} \arrow[l, "h"'] \end{tikzcd}$$

where $\widetilde{\partial}$ denotes the rational p-closed vector field induced by ∂ on $\widetilde{X}$, where $\widetilde{X}^{\widetilde{\partial}}$ is a smooth surface over $\Bbbk$, and where π^{∂} and $\pi^{\widetilde{\partial}}$ denote the quotient morphisms by ∂ and $\widetilde{\partial}$, respectively. Moreover, $h : \widetilde{X}^{\widetilde{\partial}} \to X^{\partial}$ is a resolution of singularities (but not necessarily the minimal one).

We refer the interested reader to Remark 0.4.33 for an example, and to [459] and [710] for more about canonical resolutions of p-closed foliations in characteristic $p = 2$, as well as to [11], [10], and [710] for more about singularities of quotients by vector fields.

0.4 Rational Double Point Singularities

In this section, we first discuss singularities of normal surfaces, their resolutions, and then, turn to rational double points. They are classified according to their dual resolution graphs, which turn out to be Dynkin diagrams of type A, D, and E. In characteristic zero, these singularities coincide with quotient singularities by finite subgroups of SL_2, and in positive characteristic there is a close connection to quotients by linearly reductive group schemes. Finally, we discuss rational double points in positive characteristic and their local cohomology groups. The proofs, when omitted, can be found in [19], [20], [28] or [597], and we refer to [38, Chapters 3 and 4] for another overview. Over the complex numbers, rational double points have various different characterizations, and we refer to [199] for details.

If X is a scheme of finite type over a perfect field $\Bbbk$, we will say that a closed point $x \in X$ is a *singular point* if the local ring $\mathcal{O}_{X,x}$ is not a regular local ring. Since $\Bbbk$ is a perfect field, this is equivalent to saying that X is not smooth over $\Bbbk$ at x, see [497, Theorem 28.7] . We will not discuss the distinction between regularity and smoothness in the case where the field $\Bbbk$ is not perfect, but defer this discussion to Section 4.1 when discussing quasi-elliptic fibrations. Next, two singular points on two schemes are said to be *analytically isomorphic* or *formally isomorphic* if the formal completions of their local rings are isomorphic. By the structure theorems for complete local rings (see [497, Chapter 29], for example), the formal isomorphism class of a singular point can be represented by an ideal in a formal power series ring $\Bbbk[[t_1, \ldots, t_n]]$. A *resolution of a singular point* $x \in X$ is a proper birational morphism $\pi : Y \to X$ with Y is non-singular. Often, we will also require π to be an isomorphism over $X \setminus \{x\}$. We will say that a closed and positive-dimensional subscheme of Y is *π-exceptional* if it is contained in the fibers of π.

Now, let X be a normal surface over an algebraically closed field $\Bbbk$ of characteristic $p \geq 0$, that is, X is an integral, separated, and two-dimensional scheme of finite type over $\Bbbk$ such that $\mathcal{O}_{X,x}$ is a normal ring for every $x \in X$. Since X is normal, it follows from Serre's criterion for normality (see [497, Theorem 23.8], for example) that the singular locus of X is zero-dimensional, that is, all singularities of X are isolated closed points. By the following classical theorem in theory of surfaces, there always exists a resolution of singularities, and even a distinguished one.

Theorem 0.4.1 *Let X be a normal surface over an algebraically closed field $\Bbbk$. Then, there exists a unique resolution of singularities $\pi : Y \to X$ that is characterized by either of the following properties:*

1. *every resolution of singularities $Y' \to X$ can be factored as $Y' \to Y \xrightarrow{\pi} X$,*
2. *K_Y is π-nef, that is, $K_Y \cdot C \geq 0$ for every π-exceptional curve $C \subset Y$.*

Proof See [407, Theorem 2.16]. □

In particular, for normal surface singularities, there exists a unique resolution that cannot be non-trivially factored through another resolution. By definition, it is called the *minimal* resolution of singularities. Now, if $x \in X$ is a normal surface singularity and $\pi : Y \to X$ is an arbitrary resolution of singularities, then there exists an open affine neighborhood $U \subseteq X$ of x such that π is an isomorphism over $U \setminus \{x\}$. By Zariski's Main Theorem (see, [294, Corollary III.11.4], for example), the set-theoretical fiber $E = \pi^{-1}(x)_{\mathrm{red}}$ is a connected curve on Y. It is called the *exceptional curve* of the resolution π. The following result on intersection numbers is central to the resolution of singularities and contractions for surfaces.

Proposition 0.4.2 *Let $\pi : Y \to X$ be a birational morphism from a smooth surface Y to a normal surface X over an algebraically closed field $\Bbbk$. For a singular point $x \in X$, let $R_1, \ldots, R_n$ be some irreducible components of the exceptional curve $\pi^{-1}(x)_{\mathrm{red}}$. Then, the* intersection matrix *$(R_i \cdot R_j)_{1 \leq i,j \leq n}$ is negative definite.*

Proof This is a direct consequence of (a suitable version of) the Hodge index theorem, see [38, Corollary 2.7], or [407, Theorem 2.12]. □

This result allows us to give a numerical characterization of minimal resolutions: if $\pi : Y \to X$ is a resolution of a normal surface singularity that is not minimal, then by Theorem 0.4.1, there exists an integral and π-exceptional curve $E \subset Y$ with $K_Y \cdot E < 0$. By Proposition 0.4.2, we also have $E^2 < 0$. From the adjunction formula $2p_a(E) - 2 = E^2 + K_Y \cdot E$, we thus infer $K_Y^2 = E^2 = -1$ and $E \cong \mathbb{P}^1_{\Bbbk}$. A curve E with these properties is called an *exceptional curve of the first kind* or a (-1)*-curve*. Thus, we obtain the following result.

Corollary 0.4.3 *Let X be a normal surface over an algebraically closed field $\Bbbk$ and let $\pi : Y \to X$ be a resolution of singularities. Then, π is the minimal resolution if and only if there is no (-1)-curve in the fibers of π.*

Next, let $x \in X$ be a normal surface singularity and let $\pi : Y \to X$ be the minimal resolution of singularities. Then, $\dim_{\Bbbk}(R^1\pi_*\mathcal{O}_Y)_x$ is called the *genus* of the singularity. Moreover, if $\pi' : Y' \to X$ is an arbitrary resolution of $x \in X$, then it factors as $Y' \to Y \to X$ by Theorem 0.4.1. Since $f : Y' \to Y$ is a birational morphism between smooth surfaces, it is a sequence of blow-ups in closed and smooth points, which implies $R^1 f_* \mathcal{O}_{Y'} = 0$ and then, a Grothendieck–Leray spectral sequence argument implies an isomorphism $(R^1\pi'_*\mathcal{O}_{Y'})_x \cong (R^1\pi_*\mathcal{O}_Y)_x$. In particular, the genus is equal to $\dim_{\Bbbk}(R^1\pi'_*\mathcal{O}_Y)_x$ for any resolution π'. Moreover, if X is affine and π' is an isomorphism outside $\pi'^{-1}(x)$, then the genus is equal to $\dim_{\Bbbk} H^1(Y', \mathcal{O}_{Y'})$. We refer to [38, Chapter 3] or [407, Chapter 2.2], for proofs and further details.

A normal surface singularity of genus zero (resp. one) is called a *rational* (resp. *elliptic*) singularity. Some authors include nonsingular points into the class of rational singularities. In any case, we will say that a normal surface has *at worst rational singularities* if all its singular points have rational singularities.

Remark 0.4.4 Let X be a normal variety over an algebraically closed field $\Bbbk$ of any dimension, let $x \in X$ be a point, and let $\pi : Y \to X$ be a resolution of singularities. If $\mathrm{char}(\Bbbk) = 0$, then the Grauert–Riemenschneider vanishing theorem states that $R^i\pi_*\omega_Y = 0$ for all $i \geq 1$. Moreover, we have $R^1\pi_*\omega_Y = 0$ if X is a surface and $\mathrm{char}(\Bbbk)$ is arbitrary. In general, one says that $x \in X$ is a *rational singularity* if it is normal and satisfies $R^i\pi_*\mathcal{O}_Y = 0$ as well as $R^i\pi_*\omega_Y = 0$ for all $i \geq 1$ and every resolution of singularities $\pi : Y \to X$. By the previous remarks, the second set of conditions can be dropped if $\dim(X) \leq 2$ or $\mathrm{char}(\Bbbk) = 0$. In particular, the general definition coincides with our definition in dimension two above. We refer to [407, Chapter 2.2] for proofs in dimension two, as well as further references.

Before proceeding, let us mention the following useful base-change property of rational surface singularities, see [255, Proposition 2.4] for details.

Proposition 0.4.5 *Let $f : Y \to X$ be a finite surjective morphism of normal surfaces over an algebraically closed field $\Bbbk$. Assume that X has at worst rational singularities and let $\pi : X' \to X$ be a resolution of singularities. Let $Y' := (Y \times_X X')_{\mathrm{red}}$, and let $\pi' : Y' \to Y$ and $f' : Y' \to X'$ be the induced morphisms. Then,*

1. *f' is a finite and flat morphism, that is, $f'_*\mathcal{O}_{Y'}$ is a locally free $\mathcal{O}_{X'}$-module,*
2. *$R^1\pi'_*\mathcal{O}_{Y'} = 0$, $\pi'_*\mathcal{O}_{Y'} \cong \mathcal{O}_Y$, and Y' has at worst rational singularities.*

The next result gives various characterizations of rational surface singularities. To state it, we need one more definition: let $x \in X$ be a normal surface singularity and let $\pi : Y \to X$ be a resolution of singularities. Then, any effective divisor supported on the exceptional curve of π is called an *exceptional cycle*. The following results are due to Artin, see [19] and [20].

Proposition 0.4.6 *Let $x \in X$ be a normal surface singularity over an algebraically closed field $\Bbbk$ and let $\pi : Y \to X$ be a resolution of singularities. Then, the following properties are equivalent:*

1. *$x \in X$ is a rational singularity.*
2. *$H^1(Z, \mathcal{O}_Z) = 0$ for every exceptional cycle Z.*
3. *$p_a(Z) := 1 + \frac{1}{2}(Z \cdot (Z + K_Y)) \leq 0$ for every exceptional cycle Z.*

4. *Let $R_i \subset Y$, $i = 1, \ldots, n$ be the integral curves contained in $E = \pi^{-1}(x)_{\mathrm{red}}$. Then, for every exceptional cycle Z, whose support contains all the R_i, the homomorphism* $\mathrm{Pic}(Z) = H^1(Z, \mathcal{O}_Z^*) \to \mathbb{Z}^n$ *that is defined by $\mathcal{L} \to (\ldots, \deg(\mathcal{L} \otimes \mathcal{O}_{R_i}), \ldots)$ is an isomorphism.*
5. *The canonical maps $H^i(X, \mathcal{O}_X) \to H^i(Y, \mathcal{O}_Y)$ are isomorphisms.*

Keeping the notations from the previous proposition, there is a minimal and positive exceptional cycle for every resolution of a normal surface singularity $x \in X$: first, we define a partial ordering on exceptional cycles by saying that we have an inequality $Z_1 \leq Z_2$ if $Z_j = \sum_{i=1}^n m_i^{(j)} R_i$, $j = 1, 2$ with $m_i^{(1)} \leq m_i^{(2)}$ for all $i = 1, \ldots, n$. An exceptional cycle $Z = \sum_{i=1}^n m_i R_i$ with $m_i > 0$ for all $i = 1, \ldots, n$ is called a *fundamental cycle* if the following two conditions are satisfied:

1. $Z \cdot R_i \leq 0$ for all $i = 1, \ldots, n$ and
2. Z is minimal among all exceptional cycles satisfying (1).

Before proceeding, we define the *multiplicity* of $x \in X$ to be the multiplicity of the local ring $\mathcal{O}_{X,x}$. That is, if $\mathfrak{m} \subset \mathcal{O}_{X,x}$ denotes the unique maximal ideal, then the Hilbert–Samuel function $n \mapsto \dim_{\Bbbk}(\mathfrak{m}^n/\mathfrak{m}^{n+1})$ becomes for $n \gg 0$ a polynomial of degree $(d-1)$ with leading coefficient $e/(d-1)!$, where d is the dimension of $\mathcal{O}_{X,x}$ ($d = 2$ in our case) and e is the multiplicity. We refer to [204, Chapter 12] or [497, Section 14] for details. Moreover, the dimension of the Zariski tangent space of $x \in X$, that is, the $\Bbbk$-dimension of $\mathfrak{m}/\mathfrak{m}^2$, is called the *embedding dimension* of the singularity. The name comes from the fact that the embedding dimension is the smallest integer d such that the $\mathfrak{m}$-adic completion $\widehat{\mathcal{O}}_{X,x}$ is isomorphic to a quotient of $\Bbbk[[t_1, ..., t_d]]$. The following results are due to Artin [20].

Proposition 0.4.7 *Let $x \in X$ be a normal surface singularity over an algebraically closed field $\Bbbk$, and let $\pi : Y \to X$ be a resolution singularities. Then:*

1. *a fundamental cycle Z exists, is unique, and satisfies $p_a(Z) \geq 0$,*
2. *$x \in X$ is a rational singularity if and only if $p_a(Z) = 0$.*

Moreover, if $x \in X$ is a rational singularity, then we have for all $n \geq 1$

$$\dim_{\Bbbk}(\mathfrak{m}^n/\mathfrak{m}^{n+1}) = -nZ^2 + 1 \quad \text{and} \quad nZ = Y \times_X \mathrm{Spec}(\mathcal{O}_{X,x}/\mathfrak{m}^n).$$

In this case, we also have:

3. *the multiplicity of $x \in X$ is equal to $-Z^2$,*
4. *the embedding dimension of $x \in X$ is equal to $-Z^2 + 1$.*

For a normal surface singularity, one can define further one-dimensional cycles that are supported on the exceptional locus of a resolution of singularities: the *canonical cycle*, the *cohomological cycle*, and the *fiber cycle*. We refer to [612] for definitions, properties, and their relation to the fundamental cycle, which is called the *numerical cycle* in loc. cit.

A normal surface singularity $x \in X$ that is a rational singularity of multiplicity two is called a *rational double point*. We proceed with a characterization of rational double points via the irreducible components of the exceptional curve of the minimal resolution.

Proposition 0.4.8 *Let $x \in X$ be a normal surface singularity over an algebraically closed field $\mathbb{k}$, let $\pi : Y \to X$ be the minimal resolution singularities, let $E = \pi^{-1}(x)_{\text{red}}$ be its exceptional curve, and let $R_1, \ldots, R_n$ be the irreducible components of E. Then, $x \in X$ is a rational double point if and only if*

$$R_i \cong \mathbb{P}^1_{\mathbb{k}} \quad \text{and} \quad R_i^2 = -2 \quad \text{for all} \quad i = 1, \ldots, n.$$

Proof First, suppose that $x \in X$ is a rational double point. Let $Z = \sum_{i=1}^n m_i R_i$ be its fundamental cycle. By Proposition 0.4.7, we have $p_a(Z) = 0$ and $Z^2 = -2$. From the adjunction formula we infer

$$Z \cdot K_Y = \sum_{i=1}^{n} m_i (R_i \cdot K_Y) = 2p_a(Z) - 2 - Z^2 = 0. \tag{0.4.1}$$

We have $p_a(R_i) = 0$ by Proposition 0.4.6, which implies $R_i \cong \mathbb{P}^1_{\mathbb{k}}$ for all i. Since π is the minimal resolution, we have $R_i \cdot K_Y \geq 0$ for all i. Together with (0.4.1), we conclude $R_i \cdot K_Y = 0$ for all i. Together with $p_a(R_i) = 0$, the adjunction formula implies $R_i^2 = -2$ for all i.

Conversely, if $R_i \cong \mathbb{P}^1_{\mathbb{k}}$ and $R_i^2 = -2$ for all i, then the adjunction formula yields $R_i \cdot K_Y = 0$ for all i. This implies $Z \cdot K_Y = 0$ for every exceptional cycle Z. Moreover, $Z^2 < 0$ by Proposition 0.4.2, and since Z^2 is even, we find $Z^2 \leq -2$. Thus, $p_a(Z) = 1 + \frac{1}{2}(Z(Z + K_Y)) \leq 0$, and thus, $x \in X$ is a rational singularity by Proposition 0.4.6. In particular, if Z is the fundamental cycle, then Proposition 0.4.7 gives $p_a(Z) = 0$ and then, $Z \cdot K_Y = 0$ implies $Z^2 = -2$. But then, $x \in X$ is of multiplicity two by Proposition 0.4.7, whence a rational double point. □

Proposition 0.4.9 *Let $x \in X$ be a rational double point over an algebraically closed field $\mathbb{k}$, let $\pi : Y \to X$ be the minimal resolution of singularities, let $E = \pi^{-1}(x)_{\text{red}}$ be its exceptional curve, and let $R_1, \ldots, R_n$ be the irreducible components of E. Then,*

$$R_i \cdot R_j \leq 1 \text{ if } i \neq j \quad \text{and} \quad R_i \cap R_j \cap R_k = \emptyset \text{ for pairwise distinct } i, j, k.$$

Let Γ to be the graph obtained by assigning a vertex to each R_i and joining two of them by an edge if $R_i \cdot R_j = 1$. Then, Γ is one of the graphs in Figure 0.1.

Proof By Proposition 0.4.2, the matrix $(R_i \cdot R_j)_{1 \leq i,j \leq n}$ is negative definite and by Proposition 0.4.8, all diagonal entries are equal to -2. Since E is connected, this matrix is indecomposable, that is, it cannot be written as a non-trivial block-matrix. Such matrices are classified in the theory of Lie algebras, more precisely, these are the Coxeter matrices of simple Lie algebras of finite type. The corresponding graph is the *Coxeter–Dynkin diagram* associated to the Coxeter matrix, see Section 0.8 or [88]. □

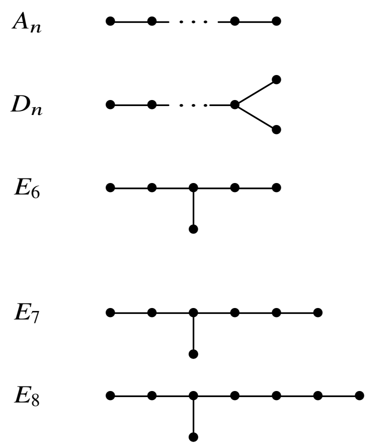


Fig. 0.1 Dual resolution graphs of rational double points

In view of this proposition, a rational double point is said to be *a rational double point of type* A_n, D_n, E_6, E_7 or E_8 if the associated graph Γ has this type. The graph Γ itself is called the *dual resolution graph* of the minimal resolution of the surface singularity. Sometimes, it can be useful to consider a nonsingular point on a surface as a rational double point of type A_0. A rational double point of type A_1 is also called an *ordinary double point* or an *ordinary node*). The exceptional curve of its minimal resolution consists of a smooth rational curve R with $R \cong \mathbb{P}^1_{\Bbbk}$ and $R^2 = -2$. Such a curve is called a (-2)-*curve*. Quite generally, a smooth rational curve on a smooth surface with self-intersection number $-n$ is called a $(-n)$-*curve*. Also, the effective cycle E in Proposition 0.4.9 is called the *nodal cycle* and its dual graph is also of type Γ. It follows from Corollary 0.4.12 below that the nodal cycle is equal to Z_{red}, where Z is the fundamental cycle of the rational double point.

Remark 0.4.10 If X is a geometrically normal surface over a field $\Bbbk$ that is not necessarily algebraically closed and $x \in X$ is a non-smooth point, then one can still define minimal resolutions of singularities, rational singularities, and rational double points, and we note that degrees of residue field extensions have to be taken into account when defining intersection numbers. We refer to [469] for details. In this case, the classification of dual resolution graphs is more complicated: for example, it could happen that the exceptional curve $E = \pi^{-1}(x)_{\mathrm{red}}$ of the minimal resolution $\pi : Y \to X$ is a union of two curves R_1 and R_2, such that $R_1 \cong \mathbb{P}^1$ and $R_1^2 = -2$, whereas R_2 is a smooth curve of genus zero with $R_2^2 = -6$ and $R_1 \cdot R_2 = 3$. In this case, the curve R_2 has no $\Bbbk$-rational point and is not geometrically reducible: over $\overline{\Bbbk}$, it splits into a disjoint union of three $\mathbb{P}^1$'s, each meeting R_1 once. Thus, over $\overline{\Bbbk}$, this is a singularity of type D_4, whereas the dual resolution graph of π over $\Bbbk$ is a Coxeter–Dynkin diagram of type G_2. In this way, also the Coxeter–Dynkin diagrams of the remaining, non–simply–laced, simple Lie algebras of finite type arise, see [469, Section 24 and Remark 25.3].

Already Proposition 0.4.9, its proof, and the previous remark show a close connection between rational double points and the theory of Lie algebras. Keeping the assumptions and notations of this proposition, let $M \subseteq \mathrm{Pic}(Y)$ be the subgroup generated by the classes $\alpha_i := [R_i]$. Next, the intersection form on $\mathrm{Pic}(Y)$ turns M into a negative definite quadratic lattice and the α_i form a root basis of finite type inside M, see Section 0.8 for definitions. This root basis defines a partial ordering on M by declaring $\alpha \geq 0$ if and only if $\alpha = \sum_{i=1}^{n} m_i \alpha_i$ with $m_i \geq 0$ for all i. Then, there exists a unique highest root $\alpha_{\max}$ with respect to this root basis, that is, for every positive root $\alpha \in M$ we have $\alpha_{\max} \geq \alpha$, see, for example, [88, Chapter VI.8]. Coming back to rational double points, the highest root has the following interpretation, see [597, Section 10].

Proposition 0.4.11 *The class $[Z]$ of the fundamental cycle Z in M is the highest root with respect to the root basis $\{[R_i]\}_{i=1,\ldots,n}$.*

Proof From $Z^2 = -2$ it follows that $[Z]$ is a root in M and since Z is effective, it follows that it is positive with respect to the root basis $\{[R_i]\}_i$. Let $\beta \in M$ be a root with $\beta \geq [Z]$ and let $\gamma := \beta - [Z]$. Being roots, we have $\beta^2 = Z^2 = -2$ and since Z is a fundamental cycle, we have $[Z]\gamma \leq 0$. Plugging these (in)equalities into $\beta^2 = ([Z] + \gamma)^2 = Z^2 + 2[Z]\gamma + \gamma^2$, we find $\gamma^2 \geq 0$. Thus, $\gamma = 0$ by Proposition 0.4.2 and it follows that $[Z]$ is a highest root, that is, equal to $\alpha_{\max}$. We refer to [597, Section 10] for details. □

Using this, Lie theory gives us the fundamental cycles of rational double points.

Corollary 0.4.12 *Let $x \in X$ be a rational double point over an algebraically closed field $\Bbbk$, let $\pi : Y \to X$ be the minimal resolution singularities, let $E = \pi^{-1}(x)_{\mathrm{red}}$ be its exceptional curve, and let $R_1, \ldots, R_n$ be the irreducible components of E. Then, the fundamental cycle Z is as follows:*

$$\begin{array}{ll}
R_1 + \cdots + R_n & \text{if } \Gamma \text{ is of type } A_n, \\
R_1 + 2R_2 + \cdots + 2R_{n-2} + R_{n-1} + R_n & \text{if } \Gamma \text{ is of type } D_n, \\
R_1 + 2R_2 + 2R_3 + 3R_4 + 2R_5 + R_6 & \text{if } \Gamma \text{ is of type } E_6, \\
2R_1 + 2R_2 + 3R_3 + 4R_4 + 3R_5 + 2R_6 + R_7 & \text{if } \Gamma \text{ is of type } E_7, \\
2R_1 + 3R_2 + 4R_3 + 6R_4 + 5R_5 + 4R_6 + 3R_7 + 2R_8 & \text{if } \Gamma \text{ is of type } E_8.
\end{array}$$

Here, our numbering of the vertices in Figure 0.1 is as in the plates in the appendix of [88]: if $\Gamma = A_n$, then we number the vertices R_i from the left to the right. If $\Gamma = D_n$, then we let $R_1, \ldots, R_{n-2}$ be the vertices from the left to the right and let R_{n-1} and R_n be the two right-most vertices. If $\Gamma = E_n$, then we let R_2 be the lower vertex, and we number the upper row of vertices $R_1, \ldots, R_n$ from the left to right (omitting R_2, of course).

We now give explicit equations: let $x \in X$ be a rational double point over an algebraically closed field $\Bbbk$. Then, the embedding dimension is equal to 3 by Proposition 0.4.7, which implies that there exists an isomorphism after completion

$$\widehat{\mathcal{O}}_{X,x} \cong \Bbbk[[x, y, z]]/(f)$$

for some non-zero power series $f \in \Bbbk[[x, y, z]]$. If $\mathfrak{m} \subset \widehat{\mathcal{O}}_{X,x}$ denotes the maximal ideal, then we have $\dim_{\Bbbk}(\mathfrak{m}/\mathfrak{m}^2) = 3$ and $\dim_{\Bbbk}(\mathfrak{m}^2/\mathfrak{m}^3) = 5$ by Proposition 0.4.7, which implies that f lies in $(x, y, z)^2 \backslash (x, y, z)^3$. More precisely, we have the following explicit classification in arbitrary characteristic, which is due to Artin [28].

Theorem 0.4.13 *Let $x \in X$ be a rational double point over an algebraically closed field $\Bbbk$. Then, the completion $\widehat{\mathcal{O}}_{X,x}$ is isomorphic to $\Bbbk[[x, y, z]]/(f)$ with f as follows.*

1. *In characteristic $\neq 2, 3, 5$, the classical forms:*

A_n	$z^{n+1} + xy$	$n \geq 1$
D_n	$z^2 + x^2y + y^{n-1}$	$n \geq 4$
E_6	$z^2 + x^3 + y^4$	
E_7	$z^2 + x^3 + xy^3$	
E_8	$z^2 + x^3 + y^5$	

2. *In characteristic* 2

A_n	*classical forms*	
D_{2n}^0	$z^2 + x^2y + xy^n$	$n \geq 2$
D_{2n}^r	$z^2 + x^2y + xy^n + xy^{n-r}z$	$r = 1, \ldots, n-1$
D_{2n+1}^0	$z^2 + x^2y + y^nz$	$n \geq 2$
D_{2n+1}^r	$z^2 + x^2y + y^nz + xy^{n-r}z$	$r = 1, \ldots, n-1$
E_6^0	$z^2 + x^3 + y^2z$	
E_6^1	$z^2 + x^3 + y^2z + xyz$	
E_7^0	$z^2 + x^3 + xy^3$	
E_7^1	$z^2 + x^3 + xy^3 + x^2yz$	
E_7^2	$z^2 + x^3 + xy^3 + y^3z$	
E_7^3	$z^2 + x^3 + xy^3 + xyz$	
E_8^0	$z^2 + x^3 + y^5$	
E_8^1	$z^2 + x^3 + y^5 + xy^3z$	
E_8^2	$z^2 + x^3 + y^5 + xy^2z$	
E_8^3	$z^2 + x^3 + y^5 + y^3z$	
E_8^4	$z^2 + x^3 + y^5 + xyz$	

3. *In characteristic* 3

A_n, D_n	*classical forms*
E_6^0	$z^2 + x^3 + y^4$
E_6^1	$z^2 + x^3 + y^4 + x^2y^2$
E_7^0	$z^2 + x^3 + xy^3$
E_7^1	$z^2 + x^3 + xy^3 + x^2y^2$
E_8^0	$z^2 + x^3 + y^5$
E_8^1	$z^2 + x^3 + y^5 + x^2y^3$
E_8^2	$z^2 + x^3 + y^5 + x^2y^2$

4. *In characteristic* 5

A_n, D_n, E_6, E_7	*classical forms*
E_8^0	$z^2 + x^3 + y^5$
E_8^1	$z^2 + x^3 + y^5 + xy^4$

,

Remark 0.4.14 Let $x \in X$ be a normal surface singularity over an algebraically closed field $\Bbbk$.

1. By inspection the explicit classification, we see that the equation $f = 0$ of a rational double point is *semi-quasihomogeneous* in the sense that $f = f_1 + f_2$, where $f_1(x, y, z)$ is a quasi-homogeneous polynomial of degree d with weights q_1, q_2, q_3 and $f_2(x, y, z)$ has degree $> d$ in the same weights. The quasi-homogeneous parts with the weights $(q_1, q_2, q_3; d)$ and the degrees in our case are

$$\begin{aligned}
A_n &: (n+1, n+1, 2; 2n+2), \quad f_1 = xy + z^{n+1}, \\
D_n &: (n-2, 2, n-1; 2n-2), \quad f_1 = z^2 + x^2y + y^{n-1} \text{ or } f_1 = z^2 + x^2y, \\
E_6 &: (4, 3, 6; 12), \quad f_1 = z^2 + x^3 + y^4 \text{ or } f_1 = z^2 + x^3 + y^2z, \\
E_7 &: (4, 6, 9; 18); \quad f_1 = z^2 + x^3 + xy^3, \\
E_8 &: (10, 6, 15; 30), \quad f_1 = z^2 + x^3 + y^5.
\end{aligned}$$

   It is known that any isolated singularity defined by a semi-quasi-homogeneous polynomial with the quasi-homogeneous part of type A_n, D_n, E_n is formally isomorphic to a rational double point [621, Corollary 3.3].
2. The singularity is said to be *taut* if its formal isomorphism class is determined by the dual resolution graph Γ (together with the self-intersection numbers) of the minimal resolution. It follows from the explicit classification that rational double points in characteristic $\neq 2, 3, 5$ are taut. The dual resolution graphs of normal surface singularities over $\Bbbk = \mathbb{C}$ that are taut were classified by Laufer [443, Section 2.2] and Tyurina [709]. For more about the tautness of rational double points in positive characteristic, we refer to [641].
3. By the explicit classification, non-taut rational double point singularities exist only in characteristic $p \in \{2, 3, 5\}$. We note that the index in a family of singularities of type Y_n^r with $Y \in \{D, E\}$ is upper semi-continuous, while the co-index r is lower semi-continuous. We refer to [28] for details and the dimensions of the deformation spaces of these singularities.

For an interpretation of the number of vertices of Γ, which coincides with the index n for a singularity of type Y_n^r, we recall the *Milnor number* of a hypersurface (singularity): let $f \in \Bbbk[[t_1, ..., t_n]]$ be a formal power series and set $f_i := \frac{\partial f}{\partial t_i}$. Then, the *Jacobian algebra* of $R := \Bbbk[[t_1, ..., t_n]]/(f)$ is defined to be $\Bbbk[[t_1, ..., t_n]]/(f_1, ..., f_n)$. in the case where the Jacobian algebra of R is a finite–dimensional $\Bbbk$-vector space, its dimension is called the *Milnor number* $\mu = \mu(R)$. Now, if $x \in X$ is a normal surface singularity of embedding dimension 3, then $\widehat{\mathcal{O}}_{X,x}$ is a hypersurface singularity, and thus, the Jacobian algebra of $\widehat{\mathcal{O}}_{X,x}$ is defined. If this is finite–dimensional as $\Bbbk$-vector space, then we have a Milnor number $\mu_x := \mu(\widehat{\mathcal{O}}_{X,x})$ that only depends on the formal isomorphism class of the singularity. For details and proofs in the case where $\Bbbk = \mathbb{C}$, we refer to [162]. In particular, if $x \in X$ is a rational double point singularity of type Γ in characteristic zero, then it follows from the explicit classification in Theorem 0.4.13 that the Milnor number μ_x is defined and equal to the number n of vertices of Γ. On the other hand, the Jacobian algebra of an A_n-singularity in characteristic

p with $p|(n+1)$ is not finite–dimensional as $\mathbb{k}$-vector space, that is, we do not have a well-defined Milnor number.

Remark 0.4.15 It follows from the explicit classification that all rational double points can be realized as singular points of double covers of smooth surfaces. This is obvious from the equations except for the A_n-singularities. However, these singularities are formally isomorphic to

$$\begin{aligned} z^2 + x^2 + y^{n+1} &= 0 \quad \text{if } \operatorname{char}(\mathbb{k}) \neq 2, \\ z^2 + xz + y^{n+1} &= 0 \quad \text{if } \operatorname{char}(\mathbb{k}) = p \geq 0, \end{aligned}$$

and then, it is also clear in these cases. Note that the equation in the $\operatorname{char}(\mathbb{k}) = 2$-case is a double cover branched along a smooth curve.

In fact, there is a close connection between double covers of smooth surfaces and rational double points, which we will now discuss in some detail: let $f(x,y) \in \mathbb{k}[[x,y]]$ with $\operatorname{char}(\mathbb{k}) \neq 2$ and assume that $f(x,y) = 0$ represents the formal isomorphism class of a one-dimensional singular point. We will call it a *simple curve singularity* if $z^2 + f(x,y) = 0$ is a rational double point. Moreover, this singularity of f is said to be *of type* a_n, d_n, e_n, e_n^r if the singularity $z^2 + f(x,y) = 0$ is a rational double point of type A_n, D_n, E_n, E_n^r, respectively. Note that this equation can be interpreted as defining a finite and flat double cover of $\operatorname{Spec} \mathbb{k}[[x,y]]$ branched over the curve $\{f = 0\}$. This said, we have the following.

Proposition 0.4.16 *Let $\pi : X \to S$ be a finite flat double cover, where S is a smooth surface over an algebraically closed field $\mathbb{k}$ of characteristic $p \geq 0$.*

1. *If π is generically étale, then X is smooth outside the ramification divisor.*
   a. *Moreover, if $p \neq 2$ then $x \in X$ is a rational double point singularity if and only if $\pi(x)$ is a simple curve singularity of the branch curve.*
   b. *Moreover, if $p = 2$ and $\pi(x)$ is a smooth point of the branch curve, then $x \in X$ is a smooth point or a rational double point of type A_n.*
2. *If π is purely inseparable, then $p = 2$ and π carries the structure of an $\boldsymbol{\alpha}_{\mathcal{L},a}$-torsor. Let α_π be the global section of $\Omega^1_S \otimes \mathcal{L}^{\otimes 2}$ from Proposition 0.2.21. If α_π has only simple isolated zeros, then each singular point of X is a rational double point of type A_1.*

Proof If $p \neq 2$, then this follows from the above discussion. If $p = 2$ and π is generically étale, then this follows from a formal local computation and Remark 0.4.15, but see also [698].

If $p = 2$ and π is purely inseparable, then π carries the structure of an $\boldsymbol{\alpha}_{\mathcal{L},a}$-torsor by Proposition 0.2.27. The remaining assertions follows from a formal local computation: we may assume that X is given by an equation $z^2 + f(x,y) = 0$. Then, it follows from the definition of α_π that a point $x \in X$ is singular if and only if $\pi(x)$ is a zero of the differential form $df = f_x dx + f_y dy$. Adding a constant to z, we may assume that $\pi(x) = (0,0)$. The zero of α_π at $\pi(x)$ is simple if the ideal generated by

f_x and f_y is equal to the maximal ideal $\mathfrak{m}_{S,\pi(x)}$. After a change of local parameters at $\pi(x)$, we may assume that $f_x = y$ and $f_y = x$, which implies that $f = xy + g$ for some $g \in \mathfrak{m}^3_{S,\pi(x)}$. This shows that the singularity $x \in X$ is formally isomorphic to the singularity $z^2 + xy = 0$. □

We refer to [43, Chapter III.7] for more details about simple singularities of curves, double covers, and rational double points in the case where $\Bbbk = \mathbb{C}$.

Let S be a smooth surface over an algebraically closed field $\Bbbk$ of characteristic $p \geq 0$. If $\mathcal{L}$ is a sufficiently ample invertible $\mathcal{O}_S$-module and $\pi : X \to S$ is a generically finite flat double cover with $\mathcal{L}^\vee \cong \pi_*\mathcal{O}_X/\mathcal{O}_S$, then X will be smooth if $p \neq 2$, or it will have at worst ordinary double point singularities if $p = 2$. This follows from Bertini's theorem if $p \neq 2$, and if $p = 2$, then we refer to [459], Section 2 for details.

In order to relate properties of singularities of double covers and singularities of branch curves in characteristic $\neq 2$ further, we have to recall a couple of facts on singularities of curves: let C be a reduced and connected curve over an algebraically closed field $\Bbbk$ and let $\pi : \widetilde{C} \to C$ be its normalization. For a closed point $x \in C$, we define δ_x to be the length of the $\mathcal{O}_{C,x}$-module $\widetilde{\mathcal{O}}_{\widetilde{C},\pi^{-1}(x)}/\mathcal{O}_{C,x}$. Then, $\delta_x = 0$ if and only if $x \in C$ is a non-singular point, see [656], Chapter IV.1. Next, if C has h irreducible components and $g_1, ..., g_h$ are the genera of the components of $\widetilde{C}$, then the arithmetic genus $p_a(C) = h^1(\mathcal{O}_C) = 1 - \chi(\mathcal{O}_C)$ is given by

$$p_a(C) \;=\; \sum_{i=1}^{h}(g_i - 1) + \sum_{x\in C}\delta_x + 1. \tag{0.4.2}$$

Moreover, if $\mathrm{char}(\Bbbk) = 0$, then the Milnor number μ_x of f satisfies the *Jung–Milnor formula*

$$\mu_x \;=\; 2\delta_x - r_x + 1,$$

where r_x is the number of formal local branches through x, that is, the cardinality of $\pi^{-1}(x)$, see see [358] or [511], Chapter 10. For example, for simple curve singularities in characteristic zero, we find

$$\begin{array}{llllll} & & \mu & r & \delta & \\ a_k & x^2+y^{k+1} & k & 1 & k/2 & \text{if } k \text{ is even} \\ & & k & 2 & (k+1)/2 & \text{if } k \text{ is odd} \\ d_k & x^2y+y^{k-1} & k & 3 & (k+2)/2 & \text{if } k \text{ is even} \\ & & k & 2 & (k+1)/2 & \text{if } k \text{ is odd} \\ e_6 & x^3+y^4 & 6 & 1 & 3 & \\ e_7 & x^3+xy^3 & 7 & 2 & 4 & \\ e_8 & x^3+y^5 & 8 & 1 & 4 & \end{array} \tag{0.4.3}$$

In positive characteristic, the entries for r, δ are still true, whereas μ may not even be defined.

We will now give yet another characterization of rational double points: let us recall that every Cohen–Macaulay ring possesses a dualizing module (at least, if it is the quotient of a polynomial ring or power series rings over a field), and then, the ring is said to be Gorenstein if its dualizing module is locally free of rank one. A singular point $x \in X$ is said to be *Cohen–Macaulay* (resp. *Gorenstein*) if its local ring $\mathcal{O}_{X,x}$ is Cohen–Macaulay (resp. Gorenstein), which is equivalent to its completion $\widehat{\mathcal{O}}_{X,x}$ being Cohen–Macaulay (resp. Gorenstein). We refer to [101] and [294], Chapter III.7 for background. By Serre's normality criterion, a normal surface singularity $x \in X$ is automatically Cohen–Macaulay. Since complete intersection rings are Gorenstein, it follows from Theorem 0.4.13 that rational double point singularities are Gorenstein. Interestingly, also the converse is true, see [19], Theorem 2.7.

Proposition 0.4.17 *Let $x \in X$ be a normal surface singularity over an algebraically closed field $\Bbbk$ and let $\pi : Y \to X$ be its minimal resolution of singularities. Then, $x \in X$ is a rational Gorenstein singularity if and only if it is a rational double point singularity. In this case, we have*

$$\pi_* \omega_Y \;\cong\; \omega_X \quad \textit{and} \quad \omega_Y \;\cong\; \pi^* \omega_X,$$

where ω_X and ω_Y denote the respective dualizing sheaves.

Another characterization of rational double point singularities comes from the minimal model program: there, so-called *terminal* and *canonical* singularities play an important role, and we refer to [408] for definitions as well as a proof of the following result.

Proposition 0.4.18 *Let X be a normal surface over an algebraically closed field $\Bbbk$. Then, a closed point $x \in X$:*

1. *has a terminal singularity if and only if it is non-singular,*
2. *has a canonical singularity if and only if it is non-singular or a rational double point singularity.*

Next, we introduce two important groups associated to a singularity: let $x \in X$ be a normal surface singularity over an algebraically closed field $\Bbbk$, let $R := \widehat{\mathcal{O}}_{X,x}$ be the completion (or Henselization) of $\mathcal{O}_{X,x}$, set $U := \operatorname{Spec} R \backslash \{x\}$, and let $j : U \hookrightarrow \operatorname{Spec} R$ be the natural open embedding. Since X was assumed to be normal and of dimension two, $x \in X$ is an isolated singularity and U is a regular and connected scheme.

1. First, the *local Picard group* or *class group* of the singularity $x \in X$ is defined to be the Picard group $\operatorname{Pic}(U)$. We note that every invertible sheaf $\mathcal{L}$ on U has a unique extension $(j_* \mathcal{L})^{\vee\vee}$ to a reflexive sheaf of rank 1 on $\operatorname{Spec} R$ and conversely, every reflexive sheaf of rank 1 on $\operatorname{Spec} R$ restricts to an invertible sheaf on U. Thus, $\operatorname{Pic}(U)$ is isomorphic to the class group $\operatorname{Cl}(\operatorname{Spec} R)$ of $\operatorname{Spec} R$ as discussed in Section 0.3.
2. Second, the *local fundamental group* of $x \in X$ is defined to be the étale fundamental group $\pi_1^{\text{ét}}(U)$. By construction, the local fundamental group classifies (limits of) finite and étale covers of U and thus, torsors under (limits of) finite, flat, and étale $\Bbbk$-group schemes over U. Moreover, by theorems of Mumford [537] and Flenner [234], a closed point on a normal surface over $\mathbb{C}$ is nonsingular if and only

if its local fundamental group is trivial, which shows that the local fundamental group is an important invariant of a singularity.

In positive characteristic, one may also be interested in torsors under arbitrary finite and flat $\Bbbk$-group schemes over U, which is accomplished by the *local Nori fundamental group scheme*, see [224]. Unfortunately, it is not well-behaved, even for rational double points, see [466] for some interesting phenomena. What is still true is that a closed point on a normal surface over an algebraically closed field is nonsingular if and only if the local Nori fundamental group scheme is trivial, see [224] and [466].

Coming back to rational double point singularities: their local Picard groups have been determined by Lipman in [469], Section 24, see also Corollary 0.4.24 below for an easy proof over the complex numbers.

Proposition 0.4.19 *Let $x \in X$ be a rational double point singularity over an algebraically closed field $\Bbbk$. Then, the local Picard group depends on the type only:*

	A_n	D_n		E_6	E_7	E_8
$\mathrm{Pic}(U)$	$\mathbb{Z}/(n+1)\mathbb{Z}$	$(\mathbb{Z}/2\mathbb{Z})^2$	*if n is even*	$\mathbb{Z}/3\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}$	$\{0\}$
		$\mathbb{Z}/4\mathbb{Z}$	*if n is odd*			

This result also describes reflexive modules of rank one on these singularities, and we refer to [33] for the classification of reflexive modules of arbitrary rank. The computation of local fundamental groups of the rational double points in characteristic zero follows easily from Proposition 0.4.21 below, see Corollary 0.4.24. From this result, it follows that the local fundamental groups even detect the type of the rational double points. On the other hand, the local fundamental groups of the rational double points in positive characteristic are more complicated: they depend not only on the type, but also on the characteristic and the co-index, see [28]. For example, by Proposition 0.4.20 below, the local fundamental group of a rational double point of type A_{p^n-1} with $n \geq 1$ in characteristic $p > 0$ is trivial, and thus, neither can rational double points be distinguished by their local fundamental groups, nor is Mumford's theorem true in positive characteristic. On the other hand, Mumford's theorem also holds in characteristic $p \geq 5$ if instead of the local fundamental group, the local Nori fundamental group scheme is considered, see [224] and [466].

Yet another characterization of rational double points is in terms of quotient singularities, at least in characteristic $\neq 2, 3, 5$: let V be a finite–dimensional vector space over an algebraically closed field $\Bbbk$ of characteristic $p \geq 0$ and let $G \subset \mathrm{GL}(V)$ be a finite subgroup, or, more generally, a finite $\Bbbk$-subgroup scheme. The quotient $V/G \cong \mathrm{Spec}(S^{\bullet}V)^G$ is normal and it is smooth outside the point lying under $0 \in V$. If $p = 0$, then, by a theorem of Chevalley, Shephard, and Todd, V/G is smooth if and only if G is generated by pseudo-reflections, and we refer to [98] and [635] for extensions of this result to positive characteristic. Next, a singularity that is formally isomorphic to a singularity of the form V/G is called a *(finite) quotient singularity*. If G is moreover a linearly reductive group scheme, then the singularity is called a *linearly reductive quotient singularity*. For example, the group scheme $\boldsymbol{\mu}_n$ is linearly reductive and the corresponding linearly reductive quotient singularities are called *cyclic quotient singularities*. We remind the reader that all finite group schemes in characteristic zero are linearly reductive, but that $\mathbb{Z}/p\mathbb{Z}$ and $\boldsymbol{\alpha}_p$ are examples of finite group schemes in characteristic $p > 0$ that are not linearly reductive. By a theorem of Hochster [307], linearly reductive quotient singularities are rational. On the other hand, we have already encountered quotient singularities in characteristic $p > 0$ that are not rational in Remark 0.3.12 and Example 0.3.13. (Note however, that unlike in our definition of quotient singularity above, the actions by the group schemes there are not linear.) Before returning to rational double point singularities, let us classify cyclic quotient singularities in dimension two.

Proposition 0.4.20 *Let $\Bbbk$ be an algebraically closed field of characteristic $p \geq 0$, let $\varphi : \boldsymbol{\mu}_n \to \mathrm{GL}(V)$ be a homomorphism of group schemes with $\dim_{\Bbbk} V = 2$, and assume that the quotient $V/\boldsymbol{\mu}_n$ is singular. Then:*

1. *There exists an integer $m \geq 2$ and an integer q with $1 \leq q \leq m-1$ and coprime to m, such that the singularity $V/\boldsymbol{\mu}_n$ is formally isomorphic to the quotient $V/\boldsymbol{\mu}_m$ with respect to the injective group homomorphism*

$$\begin{array}{rcl} \psi_q : \boldsymbol{\mu}_m & \to & \mathrm{GL}_{2,\Bbbk} \\ \eta & \mapsto & \begin{pmatrix} \zeta_m & 0 \\ 0 & \zeta_m^q \end{pmatrix}. \end{array}$$

   Here, η is a generator of $\boldsymbol{\mu}_m$ and ζ_m is a primitive m-th root of unity in $\Bbbk$ (to be taken with a grain of salt if p divides m). We denote this singularity by $\frac{1}{m}(1,q)$.
2. *The singularity $\frac{1}{m}(1,q)$ is a rational singularity and a toric surface singularity. Moreover, it is Gorenstein if and only if $q = m-1$, and then, $\frac{1}{m}(1,m-1)$ is formally isomorphic to the rational double point A_{m-1}.*
3. *The exceptional locus of the minimal resolution of singularities of $\frac{1}{m}(1,q)$ is a chain of $\mathbb{P}^1$'s. In particular, the dual resolution graph is the diagram*

$$\tfrac{1}{m}(1,q) \qquad \bullet\!\!-\!\!\!-\!\!\bullet\!\!-\, \cdots \,-\!\!\bullet\!\!-\!\!\!-\!\!\bullet$$

   The self-intersection numbers $(-n_1, \ldots, -n_k)$ of the $\mathbb{P}^1$'s can be computed via the continued fractions expansion

$$\frac{m}{q} = n_1 - \frac{1}{n_2 - \frac{1}{n_3 - \ldots}}.$$

4. *The class group of $\frac{1}{m}(1,q)$ is isomorphic to $\mathbb{Z}/m\mathbb{Z}$ and the local fundamental group is isomorphic to $\mathbb{Z}/m'\mathbb{Z}$, where $m = p^n \cdot m'$ with $p \nmid m'$ and $n \geq 0$.*

Proof Over the complex numbers, all this is classical: for example, Claim 1 is shown in [43, Proposition III.5.3] and Claim 3 is discussed in [43, Chapter III.5]. The description of these singularities in terms of toric geometry, as well as Claim 3, are discussed in [243, Chapters 2.2 and 2.6]. Since $\boldsymbol{\mu}_n$ is a linearly reductive group scheme, the representation φ splits into a direct sum of one-dimensional representations in arbitrary characteristic. From this, the just mentioned results carry over to algebraically closed fields of arbitrary characteristic, which we leave to the reader. As already mentioned, linearly reductive quotient singularities are rational in every characteristic. Using Claim 3, it is easy to see that $\frac{1}{m}(1, m-1)$ is a rational double point singularity of type A_{m-1}, which is Gorenstein. To show the converse, let t_1, t_2 be the coordinates of $\mathbb{A}^2_{\Bbbk}$ such that $\psi_q(\eta)$ sends $t_1 \mapsto t_1$ and $t_2 \mapsto \zeta_m^q t_2$. We set $\omega := dt_1 \wedge dt_2$ and compute $\psi_q(\eta)(\omega) = \zeta_m^{q+1} \cdot \omega$. If $\frac{1}{m}(1, q)$ is Gorenstein, then the pull-back of the dualizing sheaf to $\mathbb{A}^2_{\Bbbk}$ is a $\psi_q(\eta)$-invariant two-form, which must be of the form $f \cdot \omega$ for some $f \in \Bbbk[[t_1, t_2]]$ with $f(0,0) \neq 0$. From this, it is easy to see that $q = m-1$, which establishes Claim 2.

To compute the local fundamental group of $\frac{1}{m}(1, q)$, we let $m = p^n \cdot m'$ with $p \nmid m'$ and note that we can factor the quotient as $V \to V/\boldsymbol{\mu}_{p^n} \to V/\boldsymbol{\mu}_m$. The first morphism is purely inseparable, which implies that the local fundamental group of $V/\boldsymbol{\mu}_{p^n}$ is trivial, and since $\boldsymbol{\mu}_{m'}$ acts freely outside the image of the point $(0,0)$ in $V/\boldsymbol{\mu}_{p^n}$, it follows that the local fundamental group of $V/\boldsymbol{\mu}_m$ is isomorphic to $\mathbb{Z}/m'\mathbb{Z}$. As explained in [469, Section IV], the class group of $\frac{1}{m}(1, q)$ is finite and can be computed from the dual resolution graph of the minimal resolution of singularities. From this, one could compute the class group – in any case, since the dual resolution graph is independent of the characteristic, we may assume $\mathrm{char}(\Bbbk) = 0$. Next, we set $R := \widehat{\mathcal{O}}_{X,x}$ and $U := \operatorname{Spec} R\backslash\{x\}$. Since R is complete and its residue field is algebraically closed of characteristic zero, Hensel's Lemma implies that the map $R \to R$, $r \mapsto r^n$ is surjective. Using this and (0.1.7), we conclude that finite cyclic subgroups of order n of the Picard group of U correspond to $\mathbb{Z}/n\mathbb{Z}$-torsors of U. The latter set is bijective to the set of quotients of the local fundamental group that are cyclic of order n. From this, it is easy to deduce that the class group of $\frac{1}{m}(1, q)$ is cyclic of order m. □

We note that two-dimensional cyclic quotient singularities are also called *Hirzebruch–Jung singularities*, that the dual resolution graphs in Part 3 are called *Hirzebruch–Jung strings*, and that the continued fractions in Part 4 are called *Hirzebruch–Jung continued fractions*. Moreover, since the cyclic quotient singularities $\frac{1}{p}(1, q)$ in characteristic $p > 0$ are quotients by $\boldsymbol{\mu}_p$-actions, they can also be described as quotients by multiplicative vector fields, which we already discussed in Theorem 0.3.9 and Proposition 0.3.10.

Coming back to rational double points, let $\Bbbk$ be an algebraically closed field of characteristic $p \geq 0$, let V be a two-dimensional $\Bbbk$-vector space, and let $G \subset \mathrm{SL}(V)$ be a finite $\Bbbk$-subgroup scheme. Then, G automatically contains no pseudo-reflections and the dualizing sheaf on $V \cong \operatorname{Spec} S^\bullet V$ descends to V/G, which implies that V/G is Gorenstein. If G is moreover linearly reductive, then the quotient V/G is a normal, rational, and a Gorenstein surface singularity, whence a rational double point singularity by Proposition 0.4.17. Thus, finite and linearly reductive subgroup schemes of $\mathrm{SL}(V)$ are a source of rational double points. The classification of finite subgroups of $\mathrm{SL}_2(\mathbb{C})$ and their quotients is classical, see, for example, [197], Section

26, [199], or [427]. The classification of finite linearly reductive subgroup schemes of SL_2 in positive characteristic is due to Hashimoto [297], see also [468].

Theorem 0.4.21 *Let $\Bbbk$ be an algebraically closed field of characteristic $p \geq 0$ and let $G \subset \mathrm{SL}_{2,\Bbbk}$ be a finite and linearly reductive $\Bbbk$-subgroup scheme. Then, up to conjugation, G is one of the following:*

A_n *($n \geq 1$) The group scheme $\boldsymbol{\mu}_{n+1}$ of length $(n+1)$ generated by*

$$\begin{pmatrix} \zeta_{n+1} & 0 \\ 0 & \zeta_{n+1}^{-1} \end{pmatrix}$$

inside $\mathrm{SL}_{2,\Bbbk}$, where ζ_{n+1} denotes a primitive $(n+1)$-th root of unity (to be taken with a grain of salt if p divides $(n+1)$). This group scheme is étale if and only if $p \nmid (n+1)$, in which case it is cyclic. In any case, the associated quotient singularity is a rational double point of type A_n.

D_n *($n \geq 4$ and $p \neq 2$) The group scheme of length $4(n-2)$ generated by A_{2n-5} and*

$$\begin{pmatrix} 0 & \zeta_4 \\ \zeta_4 & 0 \end{pmatrix}$$

inside $\mathrm{SL}_{2,\Bbbk}$. This group scheme is étale if and only if $p \nmid (n-2)$, in which case it is the binary dihedral group. In any case, the associated quotient singularity is a rational double point of type D_n.

E_6 *($p \neq 2, 3$) The binary tetrahedral group scheme generated by D_4 and*

$$\frac{1}{\sqrt{2}} \begin{pmatrix} \zeta_8^7 & \zeta_8^7 \\ \zeta_8^5 & \zeta_8 \end{pmatrix}$$

inside $\mathrm{SL}_{2,\Bbbk}$. This group scheme is étale of length 24 over $\Bbbk$. The associated quotient singularity is a rational double point singularity of type E_6.

E_7 *($p \neq 2, 3$) The binary octahedral group scheme generated by E_6 and A_7. This group scheme is étale of length 48 over $\Bbbk$. The associated quotient singularity is a rational double point singularity of type E_7.*

E_8 *($p \neq 2, 3, 5$) The binary icosehedral group scheme generated by A_9,*

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \text{and} \quad \frac{1}{\zeta_5^2 - \zeta_5^3} \begin{pmatrix} \zeta_5 + \zeta_5^{-1} & 1 \\ 1 & -(\zeta_5 + \zeta_5^{-1}) \end{pmatrix},$$

inside $\mathrm{SL}_{2,\Bbbk}$. This group scheme is étale of length 120 over $\Bbbk$. The associated quotient singularity is a rational double point singularity of type E_8.

Remark 0.4.22 The class of quotient singularities by finite and linearly reductive group schemes is particularly nice. Many results that are classically known for finite quotient singularities over $\mathbb{C}$ carry over to this setting in positive characteristic. For rational double point singularities, we will see this already below, but we refer to [465] for details and a thorough treatment of these singularities.

As an application, we obtain the following characterization of rational double points as certain quotient singularities. Again, this is well-known over the complex numbers, and the case of positive characteristic was independently established in [297] and [468].

Corollary 0.4.23 *Let $\Bbbk$ be an algebraically closed field of characteristic $p \notin \{2,3,5\}$. Then, the rational double point singularities over $\Bbbk$ coincide with the quotient singularities by finite and linearly reductive subgroup schemes of* $\mathrm{SL}_{2,\Bbbk}$.

It is easy to compute class groups and local fundamental groups of quotient singularities over algebraically closed fields of characteristic zero. Using the characterization of rational double points from the previous corollary, we obtain the following result.

Corollary 0.4.24 *Let $x \in X$ be a rational double point over an algebraically closed field $\Bbbk$ of characteristic zero. If $G \subset \mathrm{SL}_2(\Bbbk)$ is the finite subgroup such that $x \in X$ is formally isomorphic to the quotient singularity by G, then:*

1. *the local fundamental group of $x \in X$ is isomorphic to G,*
2. *the local Picard group of $x \in X$ is isomorphic to G^{ab}, the abelianization of G.*

Proof We set $V := \Bbbk^2 \backslash \{(0,0)\}$ and note that $\pi_1^{\text{ét}}(V) \cong \{1\}$, which follows, for example, by using comparision theorems between étale and topological fundamental groups. Since G acts freely on V, we find $\pi_1^{\text{ét}}(V/G) \cong G$, and since V/G is an open neighborhood of the pointed singularity $x \in X$, it is not difficult to deduce the first claim. The second claim can be shown along the lines of the computation of the class group in the proof of Proposition 0.4.20, which we leave to the reader. □

Theorem 0.4.21 has the following application to torsors over the smooth locus of some rational double point singularities: let $x \in X$ be a normal surface singularity over an algebraically closed field $\Bbbk$ of characteristic $p \geq 0$, let $R := \widehat{\mathcal{O}}_{X,x}$ (or Henselization), and let $j : U := \operatorname{Spec} R \backslash \{x\} \hookrightarrow \overline{U} := \operatorname{Spec} R$ be as above. Let $\pi : V \to U$ be a finite morphism, where V is an integral and normal scheme, and let $\overline{V} = \operatorname{Spec} S$ be the integral closure of $\operatorname{Spec} R$ inside the field of fractions L of $H^0(V, \mathcal{O}_V)$. Thus, S is the integral closure of R inside L and it is a local and complete (or Henselian) $\Bbbk$-algebra. We will say that $\overline{\pi} : \overline{V} \to \overline{U}$ is a *local principal G-cover*, if G is a finite and flat $\Bbbk$-group scheme acting on $\overline{V}$ such that $\overline{U} \cong \overline{V}/G$ and such that the restriction $\pi : V \to U$ is a G-torsor. As explained in [224], Section 2.2, it suffices to construct $\pi : V \to U$ and then, the extension to $\overline{V} \to \overline{U}$ is automatic. Moreover, if $x \in X$ is a non-singular point, then it follows from purity results for torsors over regular schemes that also the extension $\overline{V} \to \overline{U}$ is a G-torsor, see Theorem 0.1.4. After these preparations, Theorem 0.4.21 implies the existence of local principal covers of rational double points of type A_n and D_n, see also [468], Proposition 4.2.

Corollary 0.4.25 *Let $x \in X$ be a rational double point over an algebraically closed field $\Bbbk$ of characteristic $p \geq 0$. As before, we set $R := \widehat{\mathcal{O}}_{X,x}$, $U :=$* $\operatorname{Spec} R\backslash\{x\}$*, and* $\overline{U} := \operatorname{Spec} R$.

1. *Let $x \in X$ be of type A_n. Then, there exists a $\boldsymbol{\mu}_{n+1}$-torsor $V \to U$ that extends to a local principal $\boldsymbol{\mu}_{n+1}$-cover $\overline{V} \to \overline{U}$, where $\overline{V}$ is nonsingular.*
2. *Let $x \in X$ be of type D_n and assume $p \neq 2$. Let $\mathbf{BD}_{n-2}$ be the binary dihedral group scheme of length $4(n-2)$ over $\Bbbk$ from Theorem 0.4.21. Then, there exists a $\mathbf{BD}_{n-2}$-torsor $V \to U$ that extends to a local principal $\mathbf{BD}_{n-2}$-cover $\overline{V} \to \overline{U}$, where $\overline{V}$ is nonsingular.*

We end this section by briefly discussing rational double points in positive characteristic, local cohomology groups, and the role of Frobenius in this context. Quite generally, let R be a noetherian integral domain of characteristic $p > 0$. A famous theorem of Kunz [423] states that R is regular if and only if the Frobenius morphism $\mathbf{F} : R \to R$, $x \mapsto x^p$ is flat. Thus, $\mathbf{F}$ detects regularity. Next, for an ideal $I \subseteq R$ and a p-power $q = p^e$, we let $I^{[q]}$ be the ideal generated by all a^q with $a \in I$, or, equivalently, the ideal generated by $\mathbf{F}^e(I)$. Then, the *tight closure* I^* of I is defined to be the set of all elements $x \in R$ for which there exists an $0 \neq c \in R$ with $cx^q \in I^{[q]}$ for $q \gg 0$. We always have $I \subseteq I^*$ and $I^{**} = I^*$. Moreover, an ideal $I \subseteq R$ is said to be *tightly closed* if $I = I^*$. We refer to [101, III.10] for details and different characterizations. Using tight closure, we can define interesting classes of singularities in positive characteristic: a noetherian integral domain R of characteristic $p > 0$ is called *weakly F-regular* if all its ideals are tightly closed and it is called *F-regular* if all its localizations are weakly F-regular. Moreover, R is called *F-rational* if all ideals of principal class, that is, all ideals I generated by height(I) elements, are tightly closed. For example, regular rings are F-regular, F-regular rings are weakly F-regular, weakly F-regular rings are F-rational, and F-rational rings are normal. Again, we refer to [101], Chapter III.10 for details and proofs. This recalled, we will say that a singularity $x \in X$ in positive characteristic is (weakly) F-regular and F-rational, etc. if $\widehat{\mathcal{O}}_{X,x}$ has the respective property.

Using the classification of F-regular surface singularities by Hara [283] and the classification results discussed above, we obtain the following equivalences.

Proposition 0.4.26 *Let $x \in X$ be a rational double point over an algebraically closed field $\Bbbk$ of characteristic $p > 0$. Then, the following are equivalent:*

1. *$x \in X$ is an F-regular singularity,*
2. *$x \in X$ is an F-rational singularity,*
3. *$x \in X$ is a taut singularity,*
4. *$x \in X$ is a linearly reductive quotient singularity.*

Proof For Gorenstein rings, F-regularity and F-rationality coincide, see, for example, [101, Proposition III.10.3.7], and thus, Proposition 0.4.17 gives $(1) \Leftrightarrow (2)$. The equivalence $(1) \Leftrightarrow (3)$ follows from comparing Hara's classification [283, Theorem 1.1], with the explicit classification list in Theorem 0.4.13. Moreover, comparing this list with Corollary 0.4.23, we find $(1) \Rightarrow (4)$. Finally, if $S = \Bbbk[[u_1, .., u_n]]$ and G is a linearly reductive group scheme over $\Bbbk$ acting linearly on S, then its ring of invariants $R := S^G$ is a direct summand of S. Thus, R is F-regular, see [101], Chapter III.10.1, which shows that linearly reductive quotient singularities are F-regular, and we find $(4) \Rightarrow (1)$. $\square$

Remark 0.4.27 It would be very interesting to have a proof of this result without using explicit classification lists.

Next, let us recall that a noetherian integral domain R of characteristic $p > 0$ is said to be *F-finite* if R, considered as a module over itself via Frobenius, is finite. Moreover, R is called *F-split* if the Frobenius map splits as a map of modules. For reduced and F-finite rings, being F-split is equivalent to being *F-pure*, and we refer to [101] for details. Finally, a local and noetherian integral domain $(R, \mathfrak{m})$ of characteristic $p > 0$ is said to be F-injective if the map $\mathbf{F} : H^i_{\mathfrak{m}}(R) \to H^i_{\mathfrak{m}}(R)$ induced by Frobenius on local cohomology is injective for all $i = 0, ..., \dim(R)$. If R is Cohen–Macaulay, then $H^i_{\mathfrak{m}}(R) = 0$ for $i < \dim(R)$ and in this case, to decide whether R is F-injective, it suffices to check $i = \dim(R)$ only. We will say that a singularity $x \in X$ in positive characteristic is *F-split*, *F-injective*, etc. if $\widehat{\mathcal{O}}_{X,x}$ has the respective property.

In some cases, there is a useful way to compute the kernel of Frobenius on the local cohomology group $H^2_x(X, \mathcal{O}_X)$, and, in particular, to decide F-injectivity, of a normal two-dimensional singularity $x \in X$ over an algebraically closed field $\Bbbk$ of characteristic $p > 0$: namely, assume that there exists a *graded* commutative ring $R = \bigoplus_{n \geq 0} R_n$ of finite type over $\Bbbk$ with $R_0 = \Bbbk$ and $\mathfrak{m} = \bigoplus_{n \geq 1} R_n$, such that $\widehat{\mathcal{O}}_{X,x} \cong \widehat{R}$. By the *Pinkham–Demazure construction*, there exists an ample $\mathbb{Q}$-divisor $D = \sum_i a_i P_i$ on the smooth and projective curve $C := \operatorname{Proj} R$ over $\Bbbk$ and an isomorphism of graded rings

$$R \;\cong\; \bigoplus_{n \geq 0} H^0(C, \mathcal{O}_C([nD])), \tag{0.4.4}$$

where $[-]$ denotes the integral part of a $\mathbb{Q}$-divisor. (If $B = \sum_i b_i Q_i$ is a $\mathbb{Q}$-divisor on C, that is, $b_i \in \mathbb{Q}$ and the $Q_i \in C$ are closed points, then $[B] := \sum_i [b_i] Q_i$, where $[b_i]$ denotes the largest integer, which is less or equal to b_i.) Using this isomorphism, one can compute the kernel of Frobenius on $H^2_{\mathfrak{m}}(R)$ as

$$H^2_{\mathfrak{m}}(R)[\mathbf{F}] \;\cong\; \bigoplus_{n \in \mathbb{Z}} \operatorname{Ker}(\mathbf{F} \,:\, H^1(C, \mathcal{O}_C([nD])) \;\to\; H^1(C, \mathcal{O}_C([pnD]))), \tag{0.4.5}$$

see [284, Section 2]. Let us assume moreover that $x \in X$ is a rational double point, in which case D is easy to compute. Namely, since the singularity is rational and the dual resolution graph of the minimal resolution of singularities is star shaped, the results of [284, Section 4.4] apply: namely, we have $C = \operatorname{Proj} R \cong \mathbb{P}^1_{\Bbbk}$ in this case and

$$D \;=\; -K_{\mathbb{P}^1} - \sum_{i=1}^{r} \frac{e_i - 1}{e_i} P_i,$$

for some $r \leq 3$ and some pairwise distinct closed points P_i. Moreover, the indices e_i are given by the following table

$$(e_1, \ldots, e_r) = \begin{cases} (e) & \text{type } A_e, \\ (2,2,e) & \text{type } D_{e+2}, \\ (2,3,3) & \text{type } E_6, \\ (2,3,4) & \text{type } E_7, \\ (2,3,5) & \text{type } E_8, \end{cases}$$

see [284, Section 4]. Since $[nD]$ has non-negative degree for all $n \geq 0$ and $C \cong \mathbb{P}^1$, we find $H^1(C, \mathcal{O}_C([nD])) = 0$ for all $n \geq 0$ and thus, only summands of negative degree in (0.4.5) have to be taken into account. Applying the condition in [284], Corollary 2.6, it follows that the singularity $x \in X$ is F-injective if p does not divide any of the e_i.

Example 0.4.28 Let $\Bbbk$ be an algebraically closed field of characteristic $p > 0$.

1. Let $C = \mathbb{P}^1_{\Bbbk}$ and $D = \frac{n+1}{n} \cdot (\infty)$. In this case, (0.4.4) yields $R \cong \Bbbk[x, y, z]/(xy + z^{n+1})$ with $\deg(x) = \deg(y) = n + 1$ and $\deg(z) = 1$, which has a rational double point of type A_n at the closed point $x = y = z = 0$ by Theorem 0.4.13. The criterion just mentioned shows that the singularity is F-injective if $p \nmid n$. In fact, we will see in Proposition 0.4.29 that A_n-singularities are F-injective in every positive characteristic.
2. Let $C = \mathbb{P}^1_{\Bbbk}$ and $D = \frac{1}{2} P_1 + \frac{1}{2} P_2 - (1 - \frac{1}{e}) P_3$ for three distinct points P_1, P_2, P_3 on C. In this case, (0.4.4) yields a graded ring R, which is a rational double point of type D_n with $n = e + 2$. In characteristic $p = 2$, it is of type D_n^0. The criterion just mentioned shows that the singularity is F-injective if $p \geq 3$ and $p \nmid (n - 2)$. In fact, we will see in Proposition 0.4.29 that D_n-singularities are F-injective in every positive characteristic $p \geq 3$. Moreover, using [284], Theorem 2.3, lengthy computations show that the only possibly non-zero summands in (0.4.5) are in degree (-1). From this, we deduce

$$\dim_{\Bbbk} H^2_x(X, \mathcal{O}_X)[\mathbf{F}] \;\leq\; 1,$$

   if $x \in X$ is a rational double point singularity of type D_n^0 in characteristic 2. By Proposition 0.4.29, these singularities are not F-injective, which shows that we have equality. On the other hand, rational double point singularities of type D_4^1 and D_5^1 in characteristic 2 are F-injective.

We refer to [465] for further details and results. Concerning the F-injectivity of rational double point singularities, we have the following.

Proposition 0.4.29 *Let $x \in X$ be a rational double point singularity over an algebraically closed field $\Bbbk$ of characteristic $p > 0$. Then, the following are equivalent:*

1. *$x \in X$ is F-split,*
2. *$x \in X$ is F-injective,*
3. *$x \in X$ is a rational double point with $p \geq 7$ or*

$p = 5$ *and of type different from* E_8^0,
$p = 3$ *and of type different from* $E_6^0, E_7^0, E_8^0, E_8^1$,
$p = 2$ *and of type* $A_n, D_{2n}^{n-1}, D_{2n+1}^{n-1}, E_6^1, E_7^3, E_8^4$.

In particular, if $x \in X$ is F-regular then it is F-split, but the converse need not hold.

Proof Being reduced and F-finite singularities, being F-split is equivalent to being F-pure. Next, F-pure singularities are F-injective, and F-injective Gorenstein singularities are F-pure, see [231, Lemma 3.3]. This establishes (1) $\Leftrightarrow$ (2).

In characteristic $p \geq 7$, rational double point singularities are F-regular by Proposition 0.4.26, and thus, F-split, see for example [233]. To check F-splitting in characteristic $p \leq 5$, we apply Fedder's criterion: namely, a hypersurface singularity $\Bbbk[[t_1, ..., t_n]]/(f)$ is F-pure if and only if $f^{p-1} \notin (t_1^p, ..., t_n^p)$, see [231, Proposition 2.2]. Using the explicit equations of Theorem 0.4.13, the assertion follows, and we leave the computations to the reader. □

Let $x \in X$ be a normal and isolated singularity over an algebraically closed field $\Bbbk$ in characteristic $p > 0$, let $R := \widehat{\mathcal{O}}_{X,x}$ (or Henselization), $U := \operatorname{Spec} R\backslash\{x\}$, and let $j : U \hookrightarrow \operatorname{Spec} R$ be as above. Next, let G be a finite and flat $\Bbbk$-group scheme. By Theorem 0.1.3, G-torsors over U are classified by the flat cohomology group $H^1_{\mathrm{fl}}(U, G)$. Moreover, if G is étale, then this group is isomorphic to $H^1_{\text{ét}}(U, G)$, and reduced and irreducible G-torsors over U are in bijection with surjective homomorphisms from the local fundamental group of $x \in X$ onto G. The local fundamental groups of rational double points can be found in [28], which allows us to determine all G-torsors over U, where G is an étale $\Bbbk$-group scheme, and thus, we also classify local principal G-covers of $x \in X$. Let us now turn to local principal $\boldsymbol{\alpha}_p$- and $\boldsymbol{\mu}_p$-covers of a normal surface singularity $x \in X$. These cannot be detected or classified by the local fundamental group. From (0.1.4) and (0.1.7), and using $\operatorname{Pic}(R) = 0$ and $H^1(\operatorname{Spec} R, \mathcal{O}_{\operatorname{Spec} R}) = 0$, we obtain isomorphisms $R^\times/R^{\times p} \cong H^1_{\mathrm{fl}}(\operatorname{Spec} R, \boldsymbol{\mu}_p)$ and $R/R^p \cong H^1_{\mathrm{fl}}(\operatorname{Spec} R, \boldsymbol{\alpha}_p)$. Then, applying (0.1.4) and (0.1.7) again, it follows that $\boldsymbol{\mu}_p$- and $\boldsymbol{\alpha}_p$-torsors over U modulo those torsors that extend to $\operatorname{Spec} R$ are classified by

$$_p\operatorname{Pic}(U) = \operatorname{Pic}(U)[p] \quad \text{and} \quad H^1(U, \mathcal{O}_U)[\mathbf{F}],$$

the kernels of multiplication by p and of Frobenius, respectively. From the long exact sequence of local cohomology, we have $H^1(U, \mathcal{O}_U) \cong H^2_{\mathfrak{m}}(R)$. In particular, if $x \in X$ is F-injective, then $H^1(U, \mathcal{O}_U)[\mathbf{F}] = 0$, and thus, does not admit local principal $\boldsymbol{\alpha}_p$-covers.

For the description and partial classification of torsors under finite group schemes over the rational double points, we refer to [465]. For example, also the Frobenius-action on the Witt vector valued local cohomology groups $H^2_x(X, W_n\mathcal{O}_X)$ for all n is computed. This is related to the classification of local principal G-covers over $x \in X$, where G is a finite and commutative group scheme that is local with local Cartier dual. Instead of giving the full classifcation, we only give a couple of examples: we end this section by giving some rational double points in characteristic $p = 2$ that are not linearly reductive quotient singularities, but quotient singularities by $\boldsymbol{\alpha}_p$- or

$\mathbb{Z}/p\mathbb{Z}$-actions. We note that these group schemes are not linearly reductive and that the actions are not linear or linearizable.

Proposition 0.4.30 *Let $\Bbbk$ be an algebraically closed field of characteristic 2 and set $S := \Bbbk[[u, w]]$.*

1. *For every even integer $n \geq 2$, there exists an $\boldsymbol{\alpha}_2$-action on* Spec S *such that the quotient is a rational double point of type D_{2n}^0.*
2. *For every even integer $n \geq 2$, there exist $(\mathbb{Z}/2\mathbb{Z})$-actions on* Spec S *such that the quotients are rational double point of type $D_{2n}^{n/2}$ and $D_{2n+1}^{n/2}$, respectively.*
3. *There exist an $(\mathbb{Z}/2\mathbb{Z})$-action on* Spec S *such that the quotient is a rational double point of type E_8^2.*

Proof If n is even, then the vector field $u^2\frac{d}{du} + w^n\frac{d}{dw}$ on S is additive with ring of invariants equal to $\Bbbk[[u^2, w^2, u^2w + uw^n]]$. Setting $x := u^2$, $y := w^2$, and $z := u^2w + uw^n$, we see that it is isomorphic to $\Bbbk[[x, y, z]]/(z^2 + x^2y + xy^n)$. By Theorem 0.4.13, this is a rational double point singularity of type D_{2n}^0. Moreover, the additive vector field corresponds to an $\boldsymbol{\alpha}_2$-action on Spec S by Example 0.3.6, and establishes Claim 1.

By Artin's classification [27], a surface singularity in characteristic 2 is a quotient singularity by $\mathbb{Z}/2\mathbb{Z}$ if and only if it formally isomorphic to a hypersurface singularity of the form $\Bbbk[[x, y, z]]/(f)$ with

$$f = z^2 + abz + a^2y + b^2x,$$

where $a, b \in \Bbbk[[x, y]]$ are nonunits that are relatively prime. Using the explicit equations from Theorem 0.4.13, the remaining assertions follow. □

For the application to symmetroid quartic surfaces in $\mathbb{P}^4$ in characteristic $p = 2$ below, see Remark 0.6.23, let us note the following explicit descriptions of rational double points of type D_4^0, D_4^1, D_5^0, and D_5^1 as quotients by $\boldsymbol{\alpha}_2$- and $\mathbb{Z}/2\mathbb{Z}$-actions. This makes some of the results from the previous proposition a little bit more explicit.

Proposition 0.4.31 *Let $\Bbbk$ be an algebraically closed field of characteristic 2. If $x \in X$ is a rational double point of type D_4^0 (resp. D_4^1) over $\Bbbk$, then there exists a local principal G-cover $\overline{V} \to$ Spec $\widehat{\mathcal{O}}_{X,x}$ with $G = \boldsymbol{\alpha}_2$ (resp. $G = \mathbb{Z}/2\mathbb{Z}$) such that $\overline{V}$ is nonsingular.*

Proof For $a \in \{0, 1\}$, the scheme

$$\boldsymbol{\alpha}_{2,a} := \operatorname{Spec} \Bbbk[\varepsilon]/(\varepsilon^2 - a\varepsilon)$$

becomes a finite $\Bbbk$-group scheme of length 2 with respect to the comultiplication $\varepsilon \mapsto \varepsilon \otimes 1 + 1 \otimes \varepsilon$. More precisely, we have $\boldsymbol{\alpha}_{2,0} \cong \boldsymbol{\alpha}_2$ and $\boldsymbol{\alpha}_{2,1} \cong \mathbb{Z}/2\mathbb{Z}$, see Example 0.1.7. We now define an $\boldsymbol{\alpha}_{2,a}$-action on $S := \Bbbk[[u, v]]$ via

$$\delta(u) = u + \delta_1(u)\varepsilon := u + \frac{u^2}{1 + au}\varepsilon, \quad \delta(v) = v + \delta_1(v)\varepsilon := v + \frac{v^2}{1 + av}\varepsilon.$$

Since δ_1 is a derivation (see also the computations in Example 0.3.6), we have

$$\delta(uv) \;=\; uv + (u\delta_1(v) + v\delta(u))\varepsilon \;=\; uv + \frac{uv(u+v)}{(1+au)(1+av)}\varepsilon.$$

It is easy to see that the formal power series

$$x \;:=\; \frac{u^2}{1+au}, \quad y \;:=\; \frac{v^2}{1+av}, \quad z \;:=\; \frac{uv(u+v)}{(1+au)(1+av)}$$

belong to the ring of $\boldsymbol{\alpha}_{2,a}$-invariants S and that they satisfy the relation

$$z^2 + axyz + xy(x+y) = 0,$$

which is the local equation of a rational double point of type D_4^a by Theorem 0.4.13. Moreover, since the completion of the $\Bbbk$-subalgebra R of S generated by x, y, and z is normal and since the extension of fields of fractions $Q(R) \subset Q(S)$ is of degree $p = 2$, it follows that R is not only contained in, but also equal to $S^{\alpha_{2,a}}$. It remains to show that the finite morphism $\pi : \overline{V} := \operatorname{Spec} S \to \operatorname{Spec} R \cong \operatorname{Spec} \widehat{\mathcal{O}}_{X,x}$ is a local principal $\boldsymbol{\alpha}_{2,a}$-cover, that is, the restriction to $\pi : V \to U := \operatorname{Spec} R \setminus \{\mathfrak{m}\}$ is an $\boldsymbol{\alpha}_{2,a}$-torsor. It is not difficult to see that the fixed locus of the $\boldsymbol{\alpha}_{2,a}$-action on $\overline{V}$ consists of the unique closed point only, which we leave to the reader.

Let us also give a cohomological approach, which is interesting in its own right: we set $U_1 := U \setminus \{x = 0\}$, $U_2 := U \setminus \{y = 0\}$ and consider the 1-cocycle $c = \{c_{12}\} = \{z/xy\}$ with respect to the Zariski-open cover $\{U_i\}_i$ of U. We compute

$$c^2 + ac + x^{-1} + y^{-1} \;=\; \left(\frac{z}{xy}\right)^2 + a\left(\frac{z}{xy}\right) + x^{-1} + y^{-1} \;=\; \frac{1}{(xy)^2}\Big(z^2 + axyz + xy(x+y)\Big) \;=\; 0,$$

which shows that $c^2 + ac = -x^{-1} - y^{-1}$, that is, a 1-coboundary. Thus, c defines a cohomology class in $H^1(U, \mathcal{O}_U)$ that lies in the kernel of $\mathbf{F} - a \cdot \mathrm{id}$. In particular, it shows the existence of an $\boldsymbol{\alpha}_{p,a}$-torsor over U using (0.1.4). We set $t_1 = u^{-1}$ and $t_2 = v^{-1}$, and then, over U_1, this torsor is given by the equation $t_1^2 + at_1 + x^{-1} = 0$, whereas over U_2, it is given by the equation $t_2^2 + at_2 + y^{-1} = 0$. We leave to the reader to show that this recovers S together with the $\boldsymbol{\alpha}_{2,a}$-action defined above. □

We have a similar description of D_5-singularities in characteristic $p = 2$ as quotients of A_1-singularities by $\boldsymbol{\alpha}_2$- and $\mathbb{Z}/2\mathbb{Z}$-actions.

Proposition 0.4.32 *Let $\Bbbk$ be an algebraically closed field of characteristic 2. If $x \in X$ is a rational double point of type D_5^0 (resp. D_5^1) over $\Bbbk$, then there exists a local principal G-cover $\overline{V} \to \operatorname{Spec} \widehat{\mathcal{O}}_{X,x}$ with $G = \boldsymbol{\alpha}_2$ (resp. $G = \mathbb{Z}/2\mathbb{Z}$) such that $\overline{W}$ is a rational double point of type A_1.*

Proof Let $T := \Bbbk[[u, v, w]]/(w^2 + uv)$, which is a rational double point of type A_1 by Theorem 0.4.13. First, we define an action of $\mathbb{Z}/2\mathbb{Z}$ on T via

$$u \;\mapsto\; \frac{u}{1+w+v}, \quad v \;\mapsto\; \frac{u+v}{1+w+v}, \quad \text{and} \quad w \;\mapsto\; \frac{w+u}{1+w+v}.$$

It is easy to see that the fixed locus of this action is given by $u = w + v = 0$, that is, the action is free outside the closed point of $\operatorname{Spec} T$. Next, it is easy to see that

$$x := \frac{u + vw + uv}{1 + w + v}, \quad y := \frac{v^2 + w^2}{1 + w + v}, \quad \text{and} \quad z := \frac{u(v + w)}{1 + w + v}$$

lie in the invariant subring $T^{(\mathbb{Z}/2\mathbb{Z})}$ and that these invariants satisfy

$$z^2 + xyz + y^2 z + x^2 y = 0.$$

By Theorem 0.4.13, this is the equation of the rational double point of type D_5^1. Arguing as in the proof of Proposition 0.4.31, we find that the quotient of $\overline{W} := \operatorname{Spec} T$ by the $\mathbb{Z}/2\mathbb{Z}$-action is in fact equal to the complete $\mathbb{k}$-subalgebra of T generated by x, y, and z, and that the quotient morphism is a local principal $\mathbb{Z}/2\mathbb{Z}$-cover.

Second, we define an action of $\boldsymbol{\alpha}_2 = \operatorname{Spec} \mathbb{k}[\varepsilon]/(\varepsilon^2)$ on T by

$$u \mapsto u + uv\varepsilon, \quad v \mapsto v + v^2\varepsilon, \quad \text{and} \quad w \mapsto w + (wv + u)\varepsilon.$$

It is easy to see that

$$x' := u + vw, \quad y' := v^2, \quad \text{and} \quad z' := w^2 = uv$$

lie in the invariant subring $T^{\boldsymbol{\alpha}_2}$ and that these invariants satisfy

$$z'^2 + z'y'^2 + x'^2 y' = 0.$$

By Theorem 0.4.13, this is an equation of the rational double point of type D_5^0. From here, we argue as above that the quotient of $\overline{W} := \operatorname{Spec} T$ by the $\boldsymbol{\alpha}_2$-action is in fact equal to the complete $\mathbb{k}$-subalgebra of T generated by x, y, and z, and that quotient morphism is a local principal $\boldsymbol{\alpha}_2$-cover. Alternatively, we note that the pre-image of the open subset $U := D(x) \cup D(y) \subset \operatorname{Spec} T^{\boldsymbol{\alpha}_2}$ is equal to the open subset $W := D(u) \cup D(v) \subseteq \overline{W} := \operatorname{Spec} T$. The base change $W \times_U W \to W$ is isomorphic to $\mathcal{O}_W[x]/(x^2) \cong W \times \boldsymbol{\alpha}_2$, which shows that $W \to U$ is a nontrivial $\boldsymbol{\alpha}_2$-torsor. □

Remark 0.4.33 By Example 0.3.6, actions by $\boldsymbol{\mu}_p$ or $\boldsymbol{\alpha}_p$ correspond to p-closed rational vector fields of multiplicative or additive type. We classified quotients by $\boldsymbol{\mu}_p$-actions of smooth varieties in Theorem 0.3.9 and Proposition 0.3.10, see also Proposition 0.4.20. In particular, the quotient of a smooth surface by a $\boldsymbol{\mu}_2$-action in characteristic $p = 2$ is either smooth or a rational double point of type A_1. The latter corresponds to the quotient by the multiplicative vector field

$$\partial_1 := u\frac{d}{du} + v\frac{d}{dv}$$

on $\overline{V} := \operatorname{Spec} \mathbb{k}[[u, v]]$. Using Example 0.3.6, it is not difficult to see that the $\boldsymbol{\alpha}_2$-action on $\overline{V}$ from Proposition 0.4.31 with quotient a rational double point of type D_4^0 corresponds to the additive vector field

$$\partial_2 \ := \ u^2 \frac{d}{du} \ + \ v^2 \frac{d}{dv},$$

see also Example 0.3.13 and the proof of Proposition 0.4.30. Moreover, we leave to the reader to check that the quotient of $\overline{W} := \operatorname{Spec} \Bbbk[[u, v, w]]/(w^2 + uv)$, which has a rational double point of type A_1, by the additive vector field

$$\partial_3 \ := \ uv \frac{d}{du} \ + \ v^2 \frac{d}{dv} \ + \ (u + vw) \frac{d}{dw}$$

acquires a rational double point singularity of type D_5^0. Concerning the resolution of singularities, let us recall that p-closed foliations in characteristic $p = 2$ admit resolutions of singularities, see Proposition 0.3.21.

1. For ∂_1, the singularity of the quotient $\overline{V}^{\partial_1}$ is a rational double point singularity of type A_1. The blow-up of the closed point of $\overline{V}^{\partial_1}$ resolves the singularity, Moreover, the induced vector field $\widetilde{\partial}_1$ on the blow-up $\widetilde{\overline{V}}$ has no isolated zeros, see Example 0.3.20.
2. For ∂_2, the singularity of the quotient $\overline{V}^{\partial_2}$ is a rational double point of type D_4^0. The exceptional divisor of the blow-up of the closed point of $\overline{V}^{\partial_2}$ is isomorphic to $\mathbb{P}^1$ and contains three rational double points of type A_1. Moreover, for the induced vector field $\widetilde{\partial}_2$ on the blow-up $\widetilde{\overline{V}}$, we get $r = 1$ in formula (0.3.6) and the sum of the multiplicities of isolated zeros of $\widetilde{\partial}_2$ is equal to $4 - 1 = 3$. Explicit computations show that $\widetilde{\partial}_2$ has in fact three isolated zeros of multiplicity 1, and thus, the quotient of $\widetilde{\overline{V}}$ by $\widetilde{\partial}_2$ has three rational double points of type A_1:

$$\begin{tikzcd}
\widetilde{\overline{V}} \arrow[r] \arrow[d, "\pi^{\widetilde{\partial}_2}"] & \overline{V} \arrow[d, "\pi^{\partial_2}"] \\
\widetilde{\overline{V}}^{\widetilde{\partial}_2} \arrow[r] & \overline{V}^{\partial_2}.
\end{tikzcd}$$

   The singularities of $\widetilde{\partial}_2$ and the quotient $\widetilde{\overline{V}}^{\widetilde{\partial}_2}$ can be resolved as explained in (1).
3. The blow-up $\widetilde{\overline{W}} \to \overline{W}$ resolves that rational double point of type A_1. The rational vector field $\widetilde{\partial}$ induced by ∂ on the blow-up has an isolated zero like ∂_2. From here, we resolve the isolated zeros of $\widetilde{\partial}$ as in (2), which yields an explicit resolution of the singularity of the quotient $\overline{W}^{\partial_3}$, which is a rational double point of type D_5^0.

0.5 Del Pezzo Surfaces and Surfaces of Small Degree

In this section, we discuss the classification of non-degenerate surfaces of degree $(n-1)$ and n in $\mathbb{P}^n$. Since rational normal scrolls and anti-canonical models of weak del Pezzo surfaces play an important role in this classification, we will discuss them as well. These results are important for explicit constructions of Enriques surfaces later on, see, for example Proposition 3.1.1. We refer to [132], [160], [177, Chapter 8], [486, 7.2], and [47, Chapter IV] for more results on del Pezzo surfaces.

We will work over an algebraically closed field $\Bbbk$ of arbitrary characteristic $p \geq 0$ in this section. We recall that a subvariety, that is, a reduced and irreducible subscheme $X \subseteq \mathbb{P}^n$ is said to be *non-degenerate* if it is not contained in a proper linear subspace of $\mathbb{P}^n$. Then, we have the following classical result.

Proposition 0.5.1 *Let X be a non-degenerate subvariety of $\mathbb{P}^n$ over an algebraically closed field $\Bbbk$. Then, it satisfies the inequality*

$$\deg(X) \ \geq \ \operatorname{codim}(X) \ + \ 1.$$

Proof See, for example [205], [259, page 173], or [546]. □

If equality holds, then the subvariety is said to be of *minimal degree*. Surfaces of minimal degree, that is, non-degenerate surfaces of degree $(n-1)$ in $\mathbb{P}^n$, have been classified by del Pezzo, whereas minimal subvarieties of arbitrary dimension have been classified by Bertini, and we refer to [205] for overview.

Before proceeding, let us recall that a *rational normal scroll* $S_{a,n}$ with $a \leq b$ and $a+b = n-1$, which is also denoted by $S_{a,n-1-a;n}$, is a surface of degree $(n-1)$ in $\mathbb{P}^n$ that is equal to the linear join of two Veronese curves of degrees a and $b = n-1-a$ lying in complementary linear subspaces of dimensions a and b of $\mathbb{P}^n$, respectively. Recall that the linear join of two projective subvarieties V_1 and V_2 of a projective space lying in complementary linear subspaces is the union of lines joining a point in V_1 with a point in V_2. Then, $S_{a,n} = S_{a,n-1-a;n}$ is a non-degenerate surface of degree $(n-1)$ in $\mathbb{P}^n$. In particular, it is a surface of minimal degree. We do not exclude the case $a = 0$, where one of the curves becomes a point and thus, $S_{0,n} = S_{0,n-1;n} \subset \mathbb{P}^n$ is the cone over a Veronese curve of degree $(n-1)$. For example, $S_{1,1;3}$ is a nonsingular quadric and $S_{0,2;3}$ is an irreducible quadric cone in $\mathbb{P}^3$. We refer to [177, 8.1] and [289, Lecture 8] for more examples and details.

Next, for an integer $n \geq 0$ we consider the $\mathbb{P}^1$-bundle

$$\pi : \mathbf{F}_n := \mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-n)) \to \mathbb{P}^1,$$

which is also known as a *rational minimal ruled surface*, or *Hirzebruch surface*, or a *Segre surface*. Here and in the sequel we follow [294, Chapter V,§2]. Then,

$$\operatorname{Pic}(\mathbf{F}_n) \ \cong \ \mathbb{Z}\mathfrak{f} \ \oplus \ \mathbb{Z}\mathfrak{e},$$

where $\mathfrak{f}$ is the class of a fiber of π and $\mathfrak{e}$ is the class of a section of π with $\mathfrak{e}^2 = -n$. Such a section always exists and if $n > 0$, then there is only one such section. The canonical divisor class $K_{\mathbf{F}_n}$ is linearly equivalent to $-(n+2)\mathfrak{f} - 2\mathfrak{e}$ and the intersection form on $\operatorname{Pic}(\mathbf{F}_n)$ is determined by the intersection numbers

$$\mathfrak{f}^2 \ = \ 0, \qquad \mathfrak{e} \cdot f \ = \ 1, \qquad \text{and} \qquad \mathfrak{e}^2 \ = \ -n.$$

Next, the linear system $|d\mathfrak{f} + \mathfrak{e}|$ is base-point free (resp. very ample) if $d \geq n$ (resp. $d > n$). Moreover, if $d > n$, then $|d\mathfrak{f} + \mathfrak{e}|$ embeds $\mathbf{F}_n$ as a rational normal scroll $S_{d-n,2d-n+1}$ of degree $(2d - n)$ into $\mathbb{P}^{2d-n+1}$. We refer to [294, Chapter V, §2] and [612, Chapter 2] for details and more results.

Theorem 0.5.2 *Let X be a surface of minimal degree, that is, a non-degenerate surface of degree $(n - 1)$ in $\mathbb{P}^n$ over an algebraically closed field $\Bbbk$. Then, X is isomorphic to one of the following:*

1. $\mathbb{P}^2$*,*
2. *a Veronese surface $\nu_2(\mathbb{P}^2)$ in $\mathbb{P}^5$,*
3. *a rational normal scroll $S_{a,n-1-a;n}$ in $\mathbb{P}^n$ for some $a \geq 0$.*

Proof See, for example [205], [259, p. 525], or [546]. □

We now proceed to the classification of non-degenerate surfaces of degree n in $\mathbb{P}^n$, that is, to the next-to minimal degree case. We will achieve this goal in Theorem 0.5.5 below. To state it, we first recall weak del Pezzo surfaces, their classification, and their anti-canonical models.

Theorem 0.5.3 *Let X be a smooth and proper surface over an algebraically closed field $\Bbbk$. Then, the anti-canonical sheaf ω_X^{-1} is big and nef if and only if one of the following cases holds:*

1. *X is isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$ or $\mathbf{F}_2$. In the first case, ω_X^{-1} is ample, whereas ω_X^{-1} is big and nef but not ample in the second case.*
2. *There exists a birational morphism $X \to \mathbb{P}^2$ that is the blow-up in a set $\Sigma \subset \mathbb{P}^2$ of $(9 - d)$ points (possibly infinitely near) with $d = K_X^2$, satisfying the following conditions:*

   a. *no more than 3 points lie on a line,*
   b. *no more than 6 points lie on a conic.*

   Then, ω_X^{-1} is ample if moreover:

   c. *no more than 2 points lie on a line,*
   d. *no more than 5 points lie on a conic,*
   e. *there is no cubic through Σ and has a double point at some point of Σ.*

   In any case, d satisfies $1 \leq d \leq 9$.

In particular, these surfaces are rational and we will come back to blowing up possibly infinitely near points in the next section. In the case where In the case where$-K_X$ is ample (resp. big and nef), the surface X is called a *del Pezzo surface* (resp. *weak del Pezzo surface*) and $d = K_X^2$ is called the *degree* of the weak del Pezzo surface. We note that some authors exclude the surfaces from Case 1 of the theorem from the list of (weak) del Pezzo surfaces. One can show that there is precisely one del Pezzo surface, up to isomorphism, in each degree $d \in \{9, 7, 6, 5\}$. One can also show that there are two del Pezzo surfaces of degree 8, up to isomorphism, namely $\mathbb{P}^1 \times \mathbb{P}^1$ and the blow-up of $\mathbb{P}^2$ in one point.

We will discuss the blow-up $X \to \mathbb{P}^2$ with applications to lines and pencils of conics these surfaces at the end of this section. First, we discuss anti-canonical models in greater detail. For a weak del Pezzo surface X, the graded algebra

$$\mathcal{A}_X := \bigoplus_{m=0}^{\infty} H^0\left(X, O_X(-mK_X)\right)$$

is called the *anti-canonical algebra* of X. Since ω_X^{-1} is big and nef, the natural and a priori only rational map

$$\phi_{\mathrm{can}} : X \dashrightarrow X_{\mathrm{can}} := \operatorname{Proj} \mathcal{A}_X$$

is a birational morphism. More precisely, ϕ_{can} blows down all (-2)-curves on X to rational double point singularities on X_{can} and it is an isomorphism outside the (-2)-curves. The surface X_{can} is called the *anti-canonical model* of X. It is a surface with at worst rational double point singularities and the anti-canonical sheaf $\omega_{X_{\mathrm{can}}}^{-1}$ is ample. In terms of the minimal model program, X_{can} is a *Fano surface*, which is possibly of Picard number greater than one and which has at worst canonical singularities. Moreover, X is a del Pezzo surface if and only if X_{can} is smooth, that is, has terminal surface singularities, in which case ϕ_{can} is an isomorphism, see also Proposition 0.4.18. Since the terminology Fano surface may be confused with some other surfaces that bear Fano's name, we will call the anti-canonical model of a weak del Pezzo surface an *anti-canonical del Pezzo surface*.

Concerning the anti-canonical models and algebras, we have the following results. The proofs can be found in [160] or, in characteristic $p \neq 2, 3$, in [406, Chapter III.3] or [177, Section 8.3.1].

Proposition 0.5.4 *Let X be a weak del Pezzo surface of degree d over an algebraically closed field $\Bbbk$ with anti-canonical model $X_{can} = \operatorname{Proj} \mathcal{A}_X$.*

1. *If X is from Case (1) of Theorem 0.5.3, then K_X is uniquely divisible by 2 in $\operatorname{Pic}(X)$ and $|-\frac{1}{2}K_X|$ defines a morphism $\phi_{half\,can}$ to $\mathbb{P}^3$ that is birational onto its image. If $X \cong \mathbb{P}^1 \times \mathbb{P}^1$ (resp. $X \cong \mathbf{F}_2$), then this image is a rational normal scroll $S_{1,1;3}$ (resp. $S_{0,2;3}$). The anti-canonical morphism $|-K_X|$ is equal to $\phi_{half\,can}$ followed by the second Veronese morphism ν_2. Its image is a projectively normal surface of degree 8 in $\mathbb{P}^8$.*
   In particular, the half-anti-canonical algebra $\mathcal{B}_X := \bigoplus_{m \geq 0} H^0(X, O_X(-\frac{m}{2}K_X))$ is generated by 4 elements in degree 1 with one relation in degree 2. The algebra $\mathcal{A}_X$ is the second Veronese subalgebra of $\mathcal{B}_X$.
2. *If X is from Case (2) of Theorem 0.5.3, then:*
   a. *if $d \geq 3$, then the algebra $\mathcal{A}_X$ is generated by $(d+1)$ elements of degree 1. This gives rise to a closed embedding $X_{can} \hookrightarrow \mathbb{P}^d$ that is an isomorphism onto a projectively normal surface of degree d.*
   b. *If $d = 2$, then the algebra $\mathcal{A}_X$ is generated by elements t_0, t_1, t_2 of degree 1 and an element t_3 of degree 2. The kernel of the surjective homomorphism*

$k[T_0, T_1, T_2, T_3] \to \mathcal{A}_X$, $T_i \mapsto t_i$ is a principal ideal generated by a polynomial of the form

$$\begin{array}{rcll} F & = & T_3^2 + B(T_0, T_1, T_2) & \text{if} \quad p \neq 2, \\ F & = & T_3^2 + A(T_0, T_1, T_2)\, T_3 + B(T_0, T_1, T_2) & \text{if} \quad p = 2, \end{array}$$

where A is a homogeneous polynomial of degree 2 and B is a homogeneous polynomial of degree 4*. This gives rise to a closed embedding X_{can} into weighted projective space* $\mathbb{P}(1, 1, 1, 2)$ *that is an isomorphism onto a degree* 4 *hypersurface with at worst rational double points as singularities.*

c. *If $d = 1$, then the algebra $\mathcal{A}_X$ is generated by elements t_0, t_1 of degree* 1*, an element t_2 of degree* 2*, and an element t_3 of degree 3. The kernel of the surjective homomorphism $k[T_0, T_1, T_2, T_3] \to \mathcal{A}_X, T_i \mapsto t_i$ is a principal ideal generated by a polynomial of the form*

$$\begin{array}{rcll} F & = & T_3^2 + T_2^3 + A(T_0, T_1) & \text{if} \quad p \neq 2, 3, \\ F & = & T_3^2 + T_2^3 + B(T_0, T_1) T_2^2 + C(T_0, T_1) T_2 + A(T_0, T_1) & \text{if} \quad p = 3, \\ F & = & T_3^2 + D(T_0, T_1) T_2 T_3 + E(T_0, T_1) T_3 + T_2^3 + A(T_0, T_1) & \text{if} \quad p = 2, \end{array}$$

where A, B, C, D, E are homogeneous polynomials of degree 6, 2, 4, 1, *and* 3, *respectively. This gives rise to a closed embedding of X_{can} into weighted projective space* $\mathbb{P}(1, 1, 2, 3)$ *that is an isomorphism onto a degree* 6 *hypersurface with, at worst, rational double points as singularities.*

Proof See [132, Section 8.5], [160], [177, Section 8.3.1], or [406, Chapter III.3]. □

Let us give a couple of applications and corollaries of this proposition and refer to [177], Chapter 8.3 for more details and proofs. For example,

1. If $d \geq 3$, then $\mathcal{A}_X$ is generated in degree 1, all ϕ_m with $m \geq 1$ are birational morphisms, and $\phi_1(X)$ is equal to $X_{\text{can}} \subset \mathbb{P}^d$ with $\omega_{X_{\text{can}}} \cong \mathcal{O}_{\mathbb{P}^d}(-1)$.
2. If $d = 2$, then $\mathcal{A}_X$ is generated in degree 2 and all ϕ_m with $m \geq 2$ are birational morphisms. The projection onto the first three coordinates of $X_{\text{can}} \subset \mathbb{P}(1, 1, 1, 2)$ defines finite degree 2 morphism $f : X_{\text{can}} \to \mathbb{P}^2$ and ϕ_1 is the composition of f with the contraction morphism $X \to X_{\text{can}}$. It follows from Example 0.2.22 that X_{can} is always singular if $p = 2$ and f is inseparable (that is, $p = 2$ and $A = 0$ in Proposition 0.5.4). By [459, Theorem 2.4], a generic such X_{can} has 7 rational double points of type A_1. In particular, if $p = 2$ and X is a del Pezzo surface, then f is separable (that is, $p = 2$ and $A \neq 0$ in Proposition 0.5.4).
3. If $d = 1$, then $\mathcal{A}_X$ is generated in degree three and all ϕ_m with $m \geq 3$ are birational morphisms. The projection onto the first three coordinates of $X_{\text{can}} \subset \mathbb{P}(1, 1, 2, 3)$ defines a finite degree 2 morphism $f : X_{\text{can}} \to \mathbb{P}(1, 1, 2)$, which is a quadric cone in $\mathbb{P}^3$, and ϕ_2 is the composition of f with the contraction morphism $X \to X_{\text{can}}$. As in the $d = 2$-case, if $p = 2$ and f is inseparable, then X_{can} is always singular, and a generic such X_{can} has 8 rational double points of type A_1. Finally, the intersection of $X_{\text{can}} \subset \mathbb{P}(1, 1, 2, 3)$ with $V(a_0 T_0 + a_1 T_1)$ defines a pencil of curves of arithmetic genus one, that is, a rational map $g : X_{\text{can}} \dashrightarrow \mathbb{P}^1$. The map ϕ_1 is the

composition of g with the contraction $X \to X_{\mathrm{can}}$, and since $|-K_X|$ has a unique base point, the rational map ϕ_1 is not a morphism.

It follows from Proposition 0.5.4 that every weak del Pezzo surface of degree $d \geq 3$ arises as the minimal resolution of a non-degenerate surface of degree d in $\mathbb{P}^d$ with at most rational double point singularities. Conversely, every normal and non-degenerate surface of degree d in $\mathbb{P}^d$ with at most rational double points as singularities is isomorphic to the anti-canonical del Pezzo surface. The following result classifies non-degenerate and normal surfaces of degree d in $\mathbb{P}^d$.

Theorem 0.5.5 *Let X be a non-degenerate and normal surface of degree n in $\mathbb{P}^n$ over an algebraically closed field $\Bbbk$. Then, X is isomorphic to one of the following:*

1. *a projection of a surface $X' \subset \mathbb{P}^{n+1}$ of degree n with center outside X',*
2. *the cone over an elliptic normal curve that is contained in a hyperplane of $\mathbb{P}^n$,*
3. *the anti-canonical del Pezzo surface of degree n,*

Proof See [546, Theorem 8 and Proposition 11] and [177, 8.1]. □

Remark 0.5.6 If the surfaces from Case 1 of Theorem 0.5.3 are excluded from the list of weak del Pezzo surfaces, then one has to add their anti-canonical models by hand to the previous list: namely, surfaces that are the second Veronese embedding $\nu_2(Q) \subset \mathbb{P}^8$ of an irreducible quadric surface $Q \subset \mathbb{P}^3$ (see also the first case of Proposition 0.5.4). We mention this in order to explain the discrepancy to other classification lists of surfaces of degree n in $\mathbb{P}^n$ that can be found in the literature.

We end the section by briefly discussing lines and pencils of conics on surfaces of degree $d \geq 3$ in $\mathbb{P}^d$, at least if they are anti-canonical del Pezzo surfaces: to do so, we first introduce some notations and definitions that we will use in the sequel and refer to [177, Chapter 8] for further details.

Let $\pi : X \to S$ be a birational morphism between smooth surfaces. Then, we can factor it as

$$\pi \ : \ X =: X_N \xrightarrow{\pi_N} X_{N-1} \xrightarrow{\pi_{N-1}} \ldots \xrightarrow{\pi_2} X_1 \xrightarrow{\pi_1} X_0 := S, \tag{0.5.1}$$

where each $\pi_i : X_i \to X_{i-1}$ is the blow-up in a closed point $x_i \in X_{i-1}$. For $N \geq k > i \geq 0$, we set

$$\pi_{ki} \ := \ \pi_{i+1} \circ \ldots \circ \pi_k \ : \ X_k \ \to \ X_i,$$

as well as $\pi_{ki} :=$ id if $k = i$. We say that X is obtained from S by blowing up the points $x_1, \ldots, x_N$, and note that these points may lie on different surfaces. For all i, we define effective divisors

$$E_i \ := \ \pi_i^{-1}(x_i) \quad \text{ and } \quad \mathcal{E}_i \ := \ \pi_{Ni}^*(E_i), \tag{0.5.2}$$

on X_i and $X = X_N$, respectively, and note that the $\mathcal{E}_i$ need be neither irreducible nor reduced divisors. The divisors $\mathcal{E}_i$ are called the *exceptional configuration* of the birational morphism $\pi : X \to S$. Moreover, if $\pi_{ki}(x_{k+1}) = x_i$, then the point x_{k+1} is said to be *infinitely near* to x_i of order $s = k + 1 - i$. In this case, we write $x_{k+1} \succ_s x_i$ and simply $x_{k+1} \succ x_i$ if $k = i$. Points that are not infinitely near to any other will be identified with their images on S. Assume $S = \mathbb{P}^2$. The divisor classes

$$e_0 = c_1(\pi^* \mathcal{O}_{\mathbb{P}^2}(1)),\ e_1 = [\mathcal{E}_1], \ldots, e_N = [\mathcal{E}_N].$$

form a basis $(e_0, e_1, \ldots, e_N)$ in $\mathrm{Pic}(X)$. It is called a *geometric basis* of X. Of course, it depends on the blowing-down morphism $\pi : X \to \mathbb{P}^2$. It satisfies

$$e_0^2 = 1, \quad e_i^2 = -1,\ i = 1, \ldots, N, \quad e_i \cdot e_j = 0,\ i, j = 0, \ldots, N. \tag{0.5.3}$$

The known formula for the behavior of the canonical class under a blow-up gives:

$$K_X = -3e_0 + e_1 + \cdots + e_N. \tag{0.5.4}$$

Let D be a non-zero and effective divisor on S. Then, we define the *multiplicity* $\mathrm{mult}_{x_i} D$ in $x_i \in X_{i-1}$ inductively as follows: we set $\mathrm{mult}_{x_1} D$ to be the usual multiplicity of D at x_1, that is, it is defined as the largest integer m such that the local equation of D around x_1 belongs to the m-th power of the maximal ideal of the local ring $\mathcal{O}_{X,x_1}$. We now assume that the multiplicity $\mathrm{mult}_{x_i} D$ is defined. Then, we take the proper inverse transform $\pi_i^{-1}(D)$ of D in X_i and define $\mathrm{mult}_{x_{i+1}}(D) := \mathrm{mult}_{x_{i+1}} \pi_i^{-1}(D)$. It follows from the definition that

$$\pi^{-1}(D) = \pi^*(D) - \sum_{i=1}^{N} m_i \mathcal{E}_i, \quad \text{where} \quad m_i = \mathrm{mult}_{x_i} D.$$

If $x_k \succ x_i$, then it also follows from the definition that $\mathrm{mult}_{x_k} D \geq \mathrm{mult}_{x_i} D$. We will say that a divisor D has x_i as a point of multiplicity mult_{x_i} or just passes through x_i if the multiplicity is positive. For an invertible sheaf $\mathcal{L}$ on S we denote by $|\mathcal{L} - \sum m_i x_i|$ the linear system of divisors $D \in |\mathcal{L}|$ on S such that $\mathrm{mult}_{x_i} D \geq m_i$. The fixed part of the full transform of $|\mathcal{L} - \sum m_i x_i|$ on X contains the divisor $\sum m_i \mathcal{E}_i$. The linear system $|\pi^*(\mathcal{L})(-m_1\mathcal{E}_1 - \cdots - m_N\mathcal{E}_N)|$ is called the *proper transform* of the linear system $|\pi^*(\mathcal{L}) - \sum m_i x_i|$.

Now, let X be a weak del Pezzo surface of degree $d = 9 - N$ over an algebraically closed field $\Bbbk$ that is not isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$ or $\mathbf{F}_2$. By Theorem 0.5.3, there exists a birational morphism $\sigma : X \to \mathbb{P}^2$ that is a composition

$$\sigma : X = X_N \xrightarrow{\sigma_N} X_{N-1} \xrightarrow{\sigma_{N-1}} \ldots \xrightarrow{\sigma_2} X_1 \xrightarrow{\sigma_1} X_0 = \mathbb{P}^2, \tag{0.5.5}$$

where $\sigma_i : X_i \to X_{i-1}$ is the blow-up of a closed point $x_i \in X_{i-1}$. Thus, X is obtained by blowing up the ordered set $(x_1, \ldots, x_N)$, which may include infinitely near points. The geometric basis defined by a choice of the blowing-down morphism σ is a basis of $\mathrm{Pic}(X)$. The formula for the canonical class (0.5.4) gives

$$K_X^2 = 9 - N, \tag{0.5.6}$$

showing again that $N \leq 8$. We will discuss the intersection pairing on $\mathrm{Pic}(X)$ in connection with the Enriques lattice later in Section 1.5.

On X, there are many smooth rational curves. Those with $K_X \cdot C = 0$ satisfy $C^2 = -2$ by the adjunction formula – these are called (-2)-*curves* or *nodal curves*. We note that (-2)-curves are precisely those curves that get contracted to rational double point singularities on the anti-canonical model X_{can} under the anti-canonical map ϕ_{can}. In particular, if $-K_X$ is ample, that is, if X is a del Pezzo surface, then ϕ_{can} is an isomorphism onto its image and X contains no (-2)-curves. The (-2)-curves contained in the exceptional locus of the birational morphism $\sigma : X \to \mathbb{P}^2$ are said to be *vertical*. Vertical (-2)-curves are irreducible components of some reducible divisor $\mathcal{E}_i$. In particular, such curves exist if and only if some $\mathcal{E}_i$ is reducible, which exist if and only if there are infinitely near points among the blown up points $\{x_1, ..., x_N\}$. Using the formula for the canonical class (0.5.4) and the fact that $-K_X$ is always nef, it is not difficult to prove the following two propositions, and we refer to [160] for details and proof.

Proposition 0.5.7 *Let X be a weak del Pezzo surface of degree $d \geq 1$ over an algebraically closed field $\Bbbk$ and from the second case of Theorem 0.5.3. Let $N = 9-d$, $\Sigma := \{x_1, \ldots, x_N\}$, and $\sigma : X \to \mathbb{P}^2$ be as above. Then, the non-vertical (-2)-curves on X are precisely the following:*

1. *the proper transform of a line in $\mathbb{P}^2$ that passes through at least 3 points of Σ,*
2. *the proper transform of an irreducible conic that passes through at least 6 points of Σ,*
3. *the proper transform of an irreducible cubic that passes through at least 7 points of Σ and that passes through one more point of Σ with multiplicity ≥ 2.*

If $d \geq 3$, then the anti-canonical linear system $|\omega_X^{-1}|$ defines a birational morphism ϕ_{can} by Proposition 0.5.4. Next, we study lines and conics on the anti-canonical model X_{can}.

Proposition 0.5.8 *Let X be as in Proposition 0.5.7 and assume moreover $d \geq 3$. Let $f : X \to X_{can} \subset \mathbb{P}^d$ be the anti-canonical birational morphism. Then, the lines on X_{can} are precisely the images under f of the proper transform under σ of:*

1. *either a non-nodal component of a curve $\mathcal{E}_i$,*
2. *or a line that passes through exactly 2 points of Σ,*
3. *or a conic that passes through exactly 5 points of Σ.*

The conics on X_{can} are precisely the images under f of the proper transform under σ of:

1. *either a line that passes through exactly 1 point of Σ,*
2. *or a conic that passes through exactly 4 points of Σ.*

We note that every conic on X_{can} moves in a pencil. For the anti-canonical model del Pezzo surface of degree $d \geq 3$ in $\mathbb{P}^d$ that is not isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$, we have the following classic and well-known list.

Degree	3	4	5	6	7	8	9
Lines	27	16	10	6	3	1	0
Pencils of conics	21	10	5	3	2	1	0

For an anti-canonical del Pezzo surface X of degree $d \geq 3$, the number of lines and pencils of conics depends on the set of (-2)-curves on X and thus, on the singularities of X. In degree $d = 4$, we will explore this phenomenon in Proposition 0.6.2 in the next section.

Fix a geometric basis $(e_0, e_1, \ldots, e_N)$ of a weak del Pezzo surface X. Let

$$\alpha_0 = e_0 - e_1 - e_2 - e_3, \quad \alpha_1 = e_1 - e_2, \quad \ldots, \quad \alpha_{N-1} = e_{N-1} - e_N.$$

We see that each $\alpha_i \in \mathrm{Pic}(X)$ satisfies $\alpha_i^2 = -2$. For $N \geq 3$, we can express the intersection matrix $(\alpha_i \cdot \alpha_j)$ by the graphs in Figure 0.2.

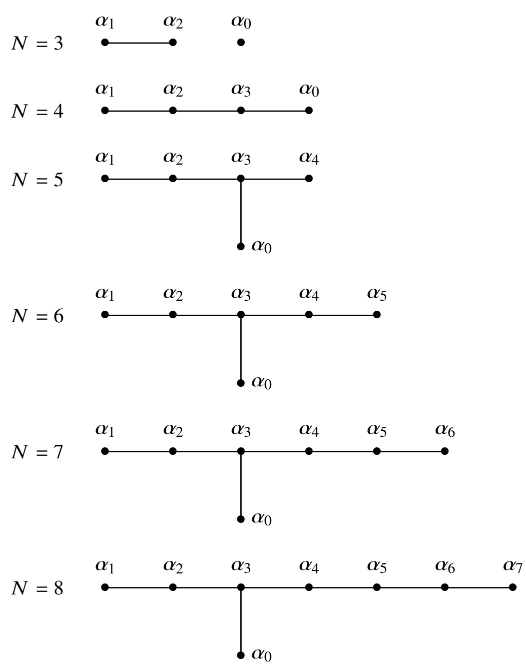


Fig. 0.2 Root diagrams of weak del Pezzo surfaces

The α_i form the vertices of these graphs. If $i \neq j$, then there is an edge (resp. no edge) from α_i to α_j if and only if $\alpha_i \cdot \alpha_j = 1$ (resp. $\alpha_i \cdot \alpha_j = 0$). We call this graph the *root diagram* of the weak del Pezzo surface.

We see that the diagrams coincide with the diagrams of the fundamental cycles of rational double points of types A_4 if $N = 4$, type D_5 of $N = 5$, type E_6 if $N = 6$, type E_7 if $N = 7$, and type E_8 if $N = 8$.

Since the discriminant $K_X^2 = 9 - N$ of the sublattice $\mathbb{Z}K_X$ in the unimodular lattice $\mathrm{Pic}(X)$ coincides with the discriminant of the root lattice generated by $\alpha_0, \ldots, \alpha_{N-1}$, we obtain that $(\alpha_0, \ldots, \alpha_{N-1})$ is a basis of $(\mathbb{Z}K_X)^\perp$.

Proposition 0.5.9 *Let X be an anti-canonical del Pezzo surface of degree $d \le 6$. Let $\{x_1, \ldots, x_k\}$ be the rational double points of X. Let $\Gamma(x_i)$ be the dual resolution graph of $x_i \in X$ and let $\alpha_{\max}$ be the fundamental cycle of $x_i \in X$. Let Γ_N^* be the graph obtained from the root diagram Γ_N by adding one vertex that is incident to the vertex α_1 and α_0 (resp. α_2 if $N = 4$, resp. α_0 if $N = 6$, resp. α_1 if $N = 7$, resp. α_7 if $N = 8$). Then, $\Gamma(x_1), \ldots, \Gamma(x_k)$ are disjoint subdiagrams of Γ_N^*.*

Using this proposition, we can classify all possible types of exceptional curves in the minimal resolution of singularities of an anti-canonical del Pezzo surface. We omit the proof; but it will become clear after we have introduced more results on root lattices, their Weyl groups, and if apply the Borel–de Siebenthal–Dynkin algorithm for describing root sublattices of a root lattice (see Section 6.4 in Volume II).

0.6 Symmetroid Quartic Surfaces in $\mathbb{P}^4$

In this section, we give a more detailed classification of non-degenerate normal quartic surfaces in $\mathbb{P}^4$ than the one already obtained in Theorem 0.5.5. We refer to [177, Chapter 8.6], for background and more information. In particular, we classify and describe anti-canonical del Pezzo surfaces of degree 4 that are symmetroid (in the sense of Definition 0.6.4), which are quartic surfaces in $\mathbb{P}^4$ that admit torsors under group schemes of length 2 over their smooth locus. Not surprisingly, the classification is more complicated in characteristic $p = 2$. We note that symmetroid quartic surfaces will be important for the explicit construction of Enriques surfaces later on.

As in the previous section, we will be working over an algebraically closed field $\Bbbk$ of characteristic $p \ge 0$. If $X \subset \mathbb{P}^4$ is an non-degenerate and normal surface of degree 4, then, by Theorem 0.5.2 and Theorem 0.5.5, X is one of the following:

1. an anti-canonical del Pezzo surface of degree 4,
2. a cone over a elliptic normal curve in $\mathbb{P}^3$,
3. a projection of a Veronese surface $v_2(\mathbb{P}^2)$ in $\mathbb{P}^5$,
4. a projection of a scroll $S_{0,4;5}$, $S_{1,3;5}$, or $S_{2,2;5}$ in $\mathbb{P}^5$.

The first case can be characterized in terms of linear systems of quadrics as follows:

Proposition 0.6.1 *Let $X \subset \mathbb{P}^4$ be a non-degenerate and normal surface of degree 4 over an algebraically closed field $\Bbbk$. Let Q be the linear system of quadrics in $\mathbb{P}^4$ containing X. Then, its projective dimension satisfies*

$$\dim_{\Bbbk} \mathrm{Ker}\left(H^0(\mathbb{P}^4, \mathcal{O}_{\mathbb{P}^4}(2)) \xrightarrow{r} H^0(X, \mathcal{O}_X(2))\right) - 1 = \dim Q \le 1.$$

Moreover, equality holds if and only if

1. *either X is an anti-canonical del Pezzo surface, in which case X has at worst rational double point singularities and a generic member of $\mathcal{Q}$ is a smooth quadric,*
2. *or X is the cone over an elliptic normal curve, in which case X has a unique elliptic Gorenstein singularity and no member of $\mathcal{Q}$ is smooth.*

Proof Suppose $\dim \mathcal{Q} \geq 1$. Then, X is contained in a quartic surface $Q_1 \cap Q_2$, where Q_1 and Q_2 are two linear independent quadrics from $\mathcal{Q}$. Since $\deg(X) = 4$, it follows that we have, in fact, an equality $X = Q_1 \cap Q_2$ and thus, $\mathcal{Q}$ is spanned by Q_1 and Q_2. This proves the inequality. Moreover, if equality holds, then X is a complete intersection of two quadrics in $\mathbb{P}^4$, and thus, a Gorenstein surface with very ample anti-canonical sheaf ω_X^{-1}. In this case, it is also easy to see that $h^1(X, \mathcal{O}_X) = h^2(X, \mathcal{O}_X) = 0$. We assumed X to be normal and know it is Gorenstein. Thus, if X has rational singularities, then the singularities are rational double point singularities by Proposition 0.4.17 and then, X is an anti-canonical del Pezzo surface in this case. If X does not have rational singularities, then its minimal resolution of singularities is not a rational surface, and thus, it follows from the classification of quartic surfaces in $\mathbb{P}^4$ (see Theorem 0.5.2 and Theorem 0.5.5) that X must be the cone over an elliptic normal curve of degree 4 in $\mathbb{P}^3$.

Conversely, an elliptic normal curve of degree 4 in $\mathbb{P}^3$ is the complete intersection of two quadrics, and thus, also the cone over it in $\mathbb{P}^4$ is a complete intersection of two quadrics. From this description, one also sees that every member of $\mathcal{Q}$ is singular. Finally, suppose that X is the anti-canonical model Y_{can} of a weak del Pezzo surface Y of degree 4. Let $\sigma : Y \to \mathbb{P}^2$ be the blow-up of $\mathbb{P}^2$ at a set of points Σ, some of which may be infinitely near to each other. Restriction gives a map $r : H^0(\mathbb{P}^4, \mathcal{O}_{\mathbb{P}^4}(2)) \to H^0(X, \mathcal{O}_X(2))$, and thus, we have $\mathcal{Q} = |\operatorname{Ker}(r)|$. By Proposition 0.5.4, X is projectively normal, and thus, the map r is surjective. We know that $|\mathcal{O}_X(2)| \cong |-2K_Y|$ and the latter is equal (via σ) to the linear system of plane sextics in $\mathbb{P}^2$ passing through the points of Σ with multiplicities ≥ 2. Counting constants (or applying Riemann–Roch), we see that it is of dimension 12. Since $\dim |\mathcal{O}_{\mathbb{P}^4}(2)| = 14$, we find $\dim \mathcal{Q} = 1$. Moreover, since X is not a cone, it follows from Bertini's theorem that there is a smooth quadric in $\mathcal{Q}$ and thus, the generic quadric in $\mathcal{Q}$ is smooth. □

We continue with our analysis of anti-canonical del Pezzo surfaces of degree 4 by classifying lines and pencils of conics on them. We already discussed lines and pencils of conics on anti-canonical del Pezzo surfaces of degree $d \geq 3$ in $\mathbb{P}^d$ at the end of Section 0.5. For the anti-canonical model $X_{\text{can}} \subset \mathbb{P}^d$ of an anti-canonical Pezzo surface X of degree $d \geq 3$, the number of lines and pencils of conics depends on the singularities of X_{can} and thus, on the (-2)-curves on the minimal resolution of singularities, which is X. In degree 4, we have the following classification, which is a corollary of Proposition 0.5.8.

Proposition 0.6.2 *Let X be an anti-canonical del Pezzo surface of degree* 4 *over an algebraically closed field $\Bbbk$ and let $X_{can} \subset \mathbb{P}^4$ be its anti-canonical model. Then, the number of lines, pencils of conics, depending on the type of the singularities on X_{can} is given by the following Table:*

Lines	Pencils of conics	Singularities of X_{can}
16	10	$\emptyset$
12	8	A_1
9	6	$A_1 + A_1$
8	7	$A_1 + A_1$
8	6	A_2
6	5	$A_1 + A_1 + A_1$
6	4	$A_2 + A_1$
5	4	A_3
4	5	A_3
4	4	$A_1 + A_1 + A_1 + A_1$
4	3	$A_1 + A_1 + A_2$
3	3	$A_1 + A_3$
3	2	A_4
2	3	D_4
2	2	$A_1 + A_1 + A_3$
1	1	D_5

Table 0.1 Anti-canonical quartic del Pezzo surfaces

Next, we study anti-canonical quartic del Pezzo surfaces that admit non-trivial torsors of length 2 over their smooth locus– this will be important for the analysis of Enriques surfaces later on. First, let us recall a special case of Theorem 0.1.10: let G be a finite $\Bbbk$-group scheme of length 2 over an algebraically closed field $\Bbbk$ of characteristic $p \geq 0$. If $p \neq 2$, then $G \cong \boldsymbol{\mu}_2 \cong \mathbb{Z}/2\mathbb{Z}$, which is étale. On the other hand, by Theorem 0.1.10, there are three isomorphism classes if $p = 2$, namely $\boldsymbol{\mu}_2$, $\boldsymbol{\alpha}_2$, and $\mathbb{Z}/2\mathbb{Z}$. Here, only the last one is étale, whereas the first two are nonreduced and infinitesimal group schemes over $\Bbbk$.

Lemma 0.6.3 *Let X be a smooth and rational surface over an algebraically closed field $\Bbbk$ and let G be a finite and flat $\Bbbk$-group scheme of length* 2*. Then, there exist no nontrivial G-torsors over X.*

Proof Since X is a smooth and rational surface, the Picard group $\mathrm{Pic}(X)$ is torsion-free, which implies that X does not admit nontrivial $\boldsymbol{\mu}_2$-torsors by Proposition 0.2.29. Moreover, since X is a rational surface, it satisfies $H^1(\mathcal{O}_X) = 0$. Thus, by Proposition 0.2.29, it also does not admit non-trivial $\mathbb{Z}/2\mathbb{Z}$-torsors nor $\boldsymbol{\alpha}_2$-torsors in characteristic $p = 2$. □

Thus, since del Pezzo surfaces are smooth rational surfaces, there are no such torsors over them. This leads us to the following definition.

Definition 0.6.4 Let X be a normal and proper surface over an algebraically closed field $\Bbbk$ with at worst rational double point singularities and let G be a finite flat group scheme of length 2 over $\Bbbk$. Then, X is a *symmetroid surface* of type G if there exists a non-trivial G-torsor $Y \to X^{\mathrm{sm}} = X \setminus \mathrm{Sing}(X)$ that defines a local principal G-cover over each singular point of X.

We refer to Corollary 0.4.25 for some examples of local principal covers over rational double point singularities, which will become relevant below. In Proposition

0.7.6, we will give a characterization of cubic surfaces in $\mathbb{P}^3$ that are symmetroid in terms of determinantal equations. More generally, an example of a symmetroid surface of degree $(2n+1)$ in $\mathbb{P}^3$ in characteristic $p \neq 2$ is given by the determinant of a symmetric square matrix of size $(2n+1)$, all of whose entries are linear forms (plus some regularity condition), see [177], Section 4.2.6. Another example, still assuming $p \neq 2$, is a quartic surface in $\mathbb{P}^3$ that is the Kummer surface associated to an abelian surface, and we refer to [177, Theorem 10.3.18], for explicit equations. Other examples of symmetroid surfaces are discussed in [110]. Also higher–dimensional examples are provided by EPW-sextics in $\mathbb{P}^5$ [208] and closures of some nilpotent orbits.

In the remainder of the section, we will classify non-degenerate quartic surfaces in $\mathbb{P}^4$ that are symmetroid. First, we establish the following connection to weak del Pezzo surfaces.

Proposition 0.6.5 *A non-degenerate symmetroid quartic surface $X \subset \mathbb{P}^4$ over an algebraically closed field $\Bbbk$ of characteristic $p \geq 0$ is an anti-canonical quartic del Pezzo surface. Depending on type and characteristic, there are the possible singularities of X:*

	Type	*Singularities*
$p \neq 2$	$\boldsymbol{\mu}_2$	$4A_1$ *or* $2A_1 + A_3$
$p = 2$	$\boldsymbol{\mu}_2$	$4A_1$ *or* $2A_1 + A_3$
	$\boldsymbol{\alpha}_2$	D_4^0 *or* D_5^0
	$\mathbb{Z}/2\mathbb{Z}$	D_4^1 *or* D_5^1

Proof Let $X \subset \mathbb{P}^4$ be a normal and non-degenerate quartic surface with at worst rational double points as singularities and let H be a general hyperplane section. Let $\pi : Y \to X^{\mathrm{sm}}$ be a non-trivial G-torsor as in Definition 0.6.4. Seeking a contradiction, assume that X is not an anti-canonical del Pezzo surface of degree 4. It follows from the classification of non-degenerate quartic surfaces in $\mathbb{P}^4$ in Theorem 0.5.5 that H is a smooth rational curve. Since smooth rational curves do not admit non-trivial G-torsors, it follows that the pre-image of H on the non-trivial torsor π is disconnected or non-reduced (the latter can only happen if $p = 2$). But this implies that Y is disconnected or non-reduced, contradicting non-triviality of the torsor π.

Let $f : \widetilde{X} \to X$ be the minimal resolution of singularities, which is a weak del Pezzo surface of degree $d = 4$. In particular, the Picard number of $\widetilde{X}$ is equal to $9 - d = 5$ by Theorem 0.5.3. From Proposition 0.4.2 and Proposition 0.4.9 it follows that the only singularities on X can only be rational double point singularities of type A_n with $2 \leq n \leq 5$ or of type D_m with $4 \leq m \leq 5$.

First, assume that $p = 2$ and that the torsor $Y \to X^{\mathrm{sm}}$ is of type $\boldsymbol{\alpha}_2$ or $\mathbb{Z}/2\mathbb{Z}$. As explained at the end of Section 0.6, a rational double point of type A_n does not admit local principal covers of this type. By Proposition 0.4.31 and Proposition 0.4.32, there do exist local principal $\mathbb{Z}/2\mathbb{Z}$-covers (resp. $\boldsymbol{\alpha}_2$-covers) over rational double points of type D_4^1 and D_5^1 (resp. D_4^0 and D_5^0). On the other hand, there are no local principal $\mathbb{Z}/2\mathbb{Z}$-covers of D_4^0 or D_5^0 by Artin's computation of the local fundamental groups of these singularities [28]. By Proposition 0.4.29, the singularities D_4^1 and

D_5^1 are F-injective and thus, do not admit local principal $\boldsymbol{\alpha}_2$-covers. From this and Proposition 0.6.2, the assertions on symmetroid quartic surfaces of type $\boldsymbol{\alpha}_2$ and $\mathbb{Z}/2\mathbb{Z}$ in characteristic 2 follow.

Next, assume that the torsor $Y \to X^{\mathrm{sm}}$ is of type $\boldsymbol{\mu}_2$ in characteristic $p \geq 0$. Let $j : X^{\mathrm{sm}} \to X$ be the inclusion and let $\overline{\pi} : \overline{Y} := \operatorname{Spec} j_*\pi_*\mathcal{O}_Y \to X$ be the integral closure of X, which is a $\boldsymbol{\mu}_2$-cover. Let E be the exceptional divisor of f and then, f induces an isomorphism of $U := \widetilde{X} \backslash E$ with X_{sm}. Pulling back to $\widetilde{X}$, we obtain a $\boldsymbol{\mu}_2$-cover of $\widetilde{X}$ that is a torsor over U. By Proposition 0.2.27, this is a simple $\boldsymbol{\mu}_2$-cover associated to some invertible sheaf $\widetilde{\mathcal{L}}$ and a global section thereof. Next, we claim that $H^0(U, \mathcal{O}_U^\times) = \Bbbk^\times$: in fact, the divisor of any rational function ϕ that is invertible on U is a linear combination of irreducible components of E. Since the intersection matrix of E is negative definite, this divisor must be equal to zero, hence ϕ is constant. Thus, the Kummer sequence (0.1.7) shows that $H^1_{\mathrm{fl}}(U, \boldsymbol{\mu}_2) \cong {}_2\operatorname{Pic}(U)$. In fact, the restriction $\mathcal{L}|_U$ must be the 2-torsion element of $\operatorname{Pic}(U)$ defining the $\boldsymbol{\mu}_2$-torsor over U. Since $\mathcal{L}|_U^{\otimes 2} \cong \mathcal{O}_U$, we conclude that $\mathcal{L} \cong \mathcal{O}_{\widetilde{X}}(D)$ for some effective divisor D on $\widetilde{X}$ such that $2D \sim W$ for some divisor W on $\tilde{X}$ that is supported on $\widetilde{X} \backslash U = E$. Let Q_i be the root lattice associated to a connected component E_i of E and let R be an irreducible component of E_i (see also Section 0.8 for definitions). From $W \cdot R = 2D \cdot R$, we see that this number is an even integer. Thus, $\frac{1}{2}[W]$ belongs to the dual lattice Q_i. In particular, the discriminant group of Q_i is of even order. Inspecting Table 0.1 from Proposition 0.6.2, we see that the singularities of X must be rational double points of type A_1, A_3, D_4, or D_5. Moreover, since $D \cdot K_{\widetilde{X}} = W \cdot K_{\widetilde{X}} = 0$, the adjunction formula shows that D^2 is even and hence, $W^2 = 4D^2$ is divisible by 8. Replacing W by $W + 2A$ for some divisor A supported on E, we may assume that W is a sum of irreducible components of E taken with multiplicity 1. In order for W^2 to be divisible by 8, X must have 4 singularities of type A_1 or two singularities of type A_1 and one singularity of type A_3. □

Remark 0.6.6 If $X \subset \mathbb{P}^4$ is a symmetroid quartic surface in characteristic $p \neq 2$, then we have $G = \boldsymbol{\mu}_2$.

First, assume that X has four rational double points of type A_1 and let X_{sm} be the smooth locus of X. Then, we have a homomorphism of the local fundamental groups of the four singularities, that is, $(\mathbb{Z}/2\mathbb{Z})^4$ to $\pi_1(X_{\mathrm{sm}})$. The image of $(1,1,1,1) \in (\mathbb{Z}/2\mathbb{Z})^{\oplus 4}$ gives a (possibly trivial) G-torsor over X_{sm}, which then extends to a local principal G torsor over X. In particular, this shows that there is at most one local principal G-torsor over X. Second, a similar argument shows that there is at most one such local G-torsor if X has one rational double point of type A_1 and one of type A_3.

Using class groups instead of local fundamental groups, and the description of $\boldsymbol{\mu}_2$-torsors in characteristic $p = 2$, one can show, along the above lines, that if $G = \boldsymbol{\mu}_2$, then there exists at most one local principal G-cover over a symmetroid quartic surface $X \subset \mathbb{P}^4$ also in characteristic $p = 2$.

In view of this proposition, we will now study anti-canonical quartic del Pezzo surfaces admitting rational double points of type $4A_1$, $2A_1 + A_3$, D_4, or D_5 in any characteristic. We will see that there is only one isomorphism class for each type in characteristic $p \neq 2$. In characteristic $p = 2$, there is still only one isomorphism class for type $4A_1$ and for type $2A_1 + A_3$. However, for such surfaces in characteristic $p = 2$ with a singular point of type D_4 or type D_5, there are two isomorphism classes that are distinguished by the type of the singularity D_4^0 or D_4^1 (resp. D_5^0 or D_5^1). After establishing this classification, we will explicitly construct local principal G-covers over the smooth loci of these surfaces. These will be our quartic symmetroid anti-canonical del Pezzo surfaces.

We will start with finding a possible set of points $\{x_1, x_2, x_3, x_4, x_5\}$ in $\mathbb{P}^2$, such that their blow-up is the minimal resolution of a quartic anti-canonical del Pezzo surface X of type $4A_1$, $2A_1 + A_3$, D_4, and D_5, respectively.

Proposition 0.6.7 *Let X be a quartic del Pezzo surface X that has singularities of type $4A_1$ or $2A_1 + A_3$ or D_4 or D_5 over an algebraically closed field $\Bbbk$. Then, its minimal resolution of singularities X' is isomorphic to the blow-up of one of the following sets of of points $\{p_1, p_2, p_3, p_4, p_5\}$ in $\mathbb{P}^2$.*

$$\begin{aligned}
4A_1 &: p_3 \succ p_2, p_5 \succ p_4,\ p_1, p_2, p_3 \textit{ and } p_1, p_4, p_5 \textit{ are on a line},\\
2A_1 + A_3 &: p_3 \succ p_2 \succ p_1, p_5 \succ p_4,\ p_1, p_2, p_3 \textit{ and } p_1, p_4, p_5 \textit{ are on a line},\\
D_4 &: p_4 \succ p_3 \succ p_2 \succ p_1, p_5,\ p_5, p_1, p_2 \textit{ are on a line},\\
D_5 &: p_5 \succ p_4 \succ p_3 \succ p_2 \succ p_1, p_5,\ p_1, p_2, p_3 \textit{ are on a line}.
\end{aligned}$$

Proof We use Proposition 0.5.9, which implies that the divisor classes of the irreducible components of the exceptional curve of the resolution of singularities $X' \to X$ can be expressed in terms of a geometric basis of $\mathrm{Pic}(X')$ as follows.

If X has singularities of type $4A_1$, then the divisor classes are

$$\alpha_{\max} = e_0 - e_3 - e_4 - e_5, \quad \alpha_0 = e_0 - e_1 - e_2 - e_3, \quad \alpha_1 = e_1 - e_2, \quad \alpha_3 = e_4 - e_5.$$

Using Proposition 0.5.7, we obtain that $p_5 \succ p_4$, $p_2 \succ p_1$ and that the points p_3, p_4, p_5 and p_1, p_2, p_3 are on a line. It remains to renumber them, to obtain the assertion of the proposition.

If X has singularities of type $2A_1 + A_3$, then the divisor classes are

$$\alpha_{\max} = e_0 - e_3 - e_4 - e_5,\ \alpha_0 = e_0 - e_1 - e_2 - e_3,\ \alpha_1 = e_1 - e_2,\ \alpha_3 = e_3 - e_4,\ \alpha_4 = e_4 - e_5.$$

Using Proposition 0.5.7 we obtain that $p_5 \succ p_4 \succ p_3$, $p_2 \succ p_1$ and that the points p_3, p_4, p_5 and p_1, p_2, p_3 are on a line.

If X has singularities of type D_4, then the divisor classes are

$$\alpha_0 = e_0 - e_1 - e_2 - e_3, \quad \alpha_2 = e_2 - e_3, \quad \alpha_3 = e_3 - e_4, \quad \alpha_4 = e_4 - e_5.$$

Using Proposition 0.5.7 we obtain that $p_5 \succ p_4 \succ p_3 \succ p_2$ and that the points p_1, p_2, p_3 are on line.

Finally, if X has singularities of type D_5, then the divisor classes are

$$\alpha_0 = e_0 - e_1 - e_2 - e_3, \quad \alpha_1 = e_1 = e_2, \quad \alpha_2 = e_2 - e_3, \quad \alpha_3 = e_4 - e_5, \quad \alpha_4 = e_4 - e_5.$$

Using Proposition 0.5.7 we obtain that $p_5 \succ p_4 \succ p_3 \succ p_2 \succ p_1$ and that the points p_1, p_2, p_3 are on line. □

Corollary 0.6.8 *Let X be a quartic anti-canonical del Pezzo surface with singularities of type $4A_1, 2A_1 + A_3, D_4$, or D_5 over an algebraically closed field $\Bbbk$ of characteristic $p \geq 0$. Then, it is a complete intersection of two quadrics in $\mathbb{P}^4$ given by the following equations*

$$\begin{aligned}
4A_1 &: x_0^2 + x_1x_2 = x_0^2 + x_3x_4 = 0,\\
2A_1 + A_3 &: x_0^2 + x_1x_2 = x_4^2 + x_0x_3 = 0,\\
D_4^0 &: x_0^2 + x_1x_2 = x_1x_3 + x_4(x_4 + x_2) = 0,\\
D_4^1 &: x_0^2 + x_1x_2 = x_1x_3 + x_4(x_4 + x_2 + x_0) = 0,\\
D_5^0 &: x_0^2 + x_1x_2 = x_1x_3 + x_0x_2 + x_4^2 = 0,\\
D_5^1 &: x_0^2 + x_1x_2 = x_1x_3 + x_0(x_4 + x_2) + x_4^2 = 0.
\end{aligned}$$

If $p \neq 2$, then the surfaces D_4^0 and D_4^1 (resp. D_5^0 and D_5^1) are isomorphic. If $p = 2$, then the surfaces have singularities of the indicated type and in particular, they are not isomorphic.

Proof Suppose X is of type $4A_1$. We choose projective coordinates in the plane such that

$$p_1 = [0, 0, 1], \quad p_2 = [0, 1, 0], \quad p_4 = [1, 0, 0].$$

The infinitely near point $p_3 \succ p_2$ (resp. $p_5 \succ p_4$) corresponds to the tangent direction of the line $x = 0$ (resp. $y = 0$) at the point p_2 (resp. p_4). The plane cubics passing through $\{p_1, \ldots, p_5\}$ are given by equation

$$x^2(a_1y + a_2z) + y^2(a_3x + a_4z) + z^2(a_5x + a_6y) + a_7xyz = 0.$$

The condition that the cubics are tangent to the line $V(x =)$ and p_2 and tangent to the line $V(y)$ at p_4 imply that $a_2 = a_4 = 0$. This shows that the linear system on of cubics through $p_1, \ldots, p_5$ is generated by monomials

$$x_0 = xyz, \quad x_1 = -x^2y, \quad x_2 = yz^2, \quad x_3 = xy^2, \quad x_4 = -xz^2. \tag{0.6.1}$$

The obvious quadratic relations between these monomials give us the asserted equation of the surface.

Suppose X is of type $2A_1 + A_3$. Choose coordinates such that

$$p_1 = [0, 0, 1], \quad p_4 = [0, 1, 0].$$

A plane cubic C passing through p_1, p_4 is given by an equation

$$x^2(a_1y + a_2z) + y^2(a_3x + a_4z) + z^2(a_5x + a_6y) + a_7xyz + a_8x^3 = 0.$$

The condition that p_1, p_2, p_3 (resp. p_1, p_4, p_5) lie on a line implies that the line $y = 0$ is tangent to C at the point p_1 with multiplicity 3 and that the line $x = 0$ is tangent to C at p_4 with multiplicity 2. This gives $a_2 = a_5 = a_6 = 0$. Thus, we can choose a basis in the anti-canonical linear system represented by sections

$$-x^2y, \quad -x^3, \quad xy^2, \quad z^2y, \quad xyz. \tag{0.6.2}$$

From this, it is clear that the equation of X in $\mathbb{P}^4$ is given by two quadratic forms:

$$x_0^2 + x_1x_2 = x_4^2 + x_0x_3 = 0. \tag{0.6.3}$$

Suppose X is of type D_4. Choose coordinates such that

$$p_1 = [0,0,1], \quad p_5 = [0,1,0].$$

The point p_2 corresponds to the tangent direction of the line $x = 0$ at p_1. A plane cubic C passing through $\{p_1, p_2, p_5\}$ is given by an equation

$$x^2(a_1y + a_2z) + y^2(a_3x + a_4z) + a_5xz^2 + a_6xyz + a_7x^3 = 0. \tag{0.6.4}$$

The coefficients a_4, a_5 are non-zero for a general member of the linear system because otherwise it would be singular. We have a choice for p_3 and p_4. By blowing up p_1 and then p_2, we see that the choice for p_4 is determined by the ratio a_5/a_4. By applying a projective transformation of the form

$$T : [x,y,z] \mapsto [x, Ay + Cx, Bz + Dx] \tag{0.6.5}$$

that leaves invariant the linear system of cubics, we find that a_4/a_5 changes to Aa_4/Ba_4. Thus, we may fix p_4 to assume that $a_4 = a_5$. Next, we blow up p_4 and obtain that the choice for p_5 is determined by a_6. The transformation T changes a_6 to $ABa_6 + 2ABCa_4$. If $p \neq 2$, we may assume that it is equal to 0. If $p = 2$, we may assume that either $a_6 = 0$ or else $a_6 = a_4 = a_5$.

So, we see that if $p \neq 2$, there is only one projective equivalence class of linear systems of cubics through the sets of points $\{p_1, \dots, p_5\}$, which is represented by the linear systems

$$x^2(a_1y + a_2z) + y^2(a_3x + a_4z) + a_4xz^2 + a_7x^3 = 0. \tag{0.6.6}$$

If $p = 2$, then we have two projective equivalence classes represented by a linear system of cubics

$$\begin{aligned} &x^2(a_1y + a_2z) + y^2(a_3x + a_4z) + a_4xz^2 + a_4xyz + a_7x^3 = 0,\\ &x^2(a_1y + a_2z) + y^2(a_3x + a_4z) + a_4xz^2 + a_7x^3 = 0. \end{aligned} \tag{0.6.7}$$

To obtain the equations of the surfaces, we use the following bases

$$x_0 = x^2y, \quad x_1 = x^3, \quad x_2 = -xy^2, \quad x_3 = y^2z + xz^2 + \epsilon xyz,\ x_4 = x^2z, \quad \epsilon = 0,1. \tag{0.6.8}$$

The relations

$$(x^2y)^2 + (x^3)(xy^2) = 0, \quad (y^2z + xz^2 + \epsilon xyz)x^3 + (x^2z + xy^2 + \epsilon x^2y)x^2z = 0$$

give the following quadratic equations for X

$$x_0^2 + x_1x_2 = x_1x_3 + x_4(x_4 + x_2 + \epsilon x_0) = 0. \tag{0.6.9}$$

The singular point of the surface is $[0,0,0,1,0]$. Setting $x_3 = 1$ and eliminating x_1 using the second equation, we find the equation of the singular point. Comparing it with Artin's equations of rational double point singularities, we find that $\epsilon = 1$ corresponds to the rational double point of type $D_4^{(1)}$.

Suppose that X is of type D_5. We may assume that $p_1 = [0,0,1]$ and that the line $x = 0$ passes through the points p_1, p_2, p_3. The linear system of cubics passing through $\{p_1, p_2, p_3\}$ is given by an equation

$$a_1x^2y + a_2x^2z + a_3xy^2 + a_4xz^2 + a_5xyz + a_6x^3 + a_7y^3 = 0. \tag{0.6.10}$$

We can transform the linear system using the projective transformations

$$T : [x, y, z] \mapsto [x, Ay + Cx, Bz + Dx + Ey]. \tag{0.6.11}$$

As in the previous case, we can choose T to assume that $a_4 = a_7$ and $a_4 = a_5$ or $a_5 = 0$. This gives a basis of the anti-canonical linear system

$$x_0 = x^2y, \quad x_1 = x^3, \quad x_2 = xy^2, \quad x_3 = xz^2 + y^3 + \epsilon xyz, \; x_4 = x^2z. \tag{0.6.12}$$

As in the previous case, we may assume that $\epsilon = 0$ if $p \neq 2$. Then, we find that the relations between the monomials give the asserted equations for X. If $p = 2$, then we check that $\epsilon = 0$ leads to a rational double point of type D_5^0 on X and for $\epsilon = 1$ one obtains a rational double point of type D_5^1 on X. □

Let Q be the pencil of quadrics in $\mathbb{P}^4$, whose base locus is equal to a quartic anti-canonical del Pezzo surface with singularities as described by the equations from the previous proposition. If X is of type $4A_1$ (resp. $2A_1 + A_3$), then one can check that Q contains two quadrics of corank 2 and one quadric of corank 1 (resp. no more singular quadrics). The following proposition proves the converse.

Proposition 0.6.9 *Let X be a quartic anti-canonical del Pezzo surface defined by a pencil that contains two quadrics of corank 2 and one quadric of corank 1 (resp. no more singular quadrics). Then, X is of type $4A_1$ (resp. $2A_1 + A_3$).*

Proof Applying a projective transformation, we may assume that one of the quadrics Q_1 of corank 2 is given by equation $x_0^2 + x_1x_2 = 0$. Let

$$x_3L_1(x_0, x_1, x_2) + x_4L_2(x_0, x_1, x_2) + q(x_0, x_1, x_2) + ax_3^2 + bx_3x_4 + cx_4^2 = 0$$

be the equation of the second quadric Q_2 of corank 2. The singular line $x_0 = x_1 = x_2 = 0$ of Q_1 intersects Q_2 at the points $[0,0,0,\alpha,\beta]$, where $a\alpha^2 + b\alpha\beta_4 + c\beta^2 = 0$.

Changing the coordinates x_3, x_4, we may assume that $ax_3^2 + bx_3x_4 + cx_4^2 = x_3x_4$ or x_4^2.

In the first case, after replacing x_3 with $x_3 + L_2(x_0, x_1, x_2)$ and x_4 with $x_4 + L_1(x_0, x_1, x_2)$, we may assume that $L_1 = L_2 = 0$. The quadric $V(q(x_0, x_1, x_2)+x_3+x_4)$ is of corank 2 if and only if $q(x_0, x_1, x_2) = l(x_0, x_1, x_2)^2$ is of rank 1. The line $V(l)$ intersects the conic $V(x_0^2+x_1x_2)$ at two points or it is tangent to it. After an orthogonal transformation of the conic, we may assume that $l = x_0$ in the first case, giving us the asserted equations, or $l = x_1$ in the second case. In the latter case, the pencil $\mathcal{Q}$ does not contain quadrics of corank 1.

In the second case, the point $[0, 0, 0, 1, 0]$ cannot be a singular point of Q_2 since otherwise X is a cone. Thus, $L_1 \neq 0$, and, as above, we may assume that $L_1 = x_0$ or $L_1 = x_1$.

If $L_1 = x_0$, then we replace x_3 with $x_3 + ax_0 + bx_1 + cx_2$ in order to assume that q does not depend on x_0. This implies that the singular locus of Q_2 is contained in the line $x_0 = x_3 = x_4 = 0$. Since Q_2 is a corank 2 quadric, we see that the singular locus coincides with this line and hence, taking the partial derivatives in x_1, x_2, we conclude that $q = 0$. This gives us the asserted equations. If $L_1 = x_1$, then by a similar argument we obtain equations

$$x_0^2 + x_1x_2 \;=\; x_4^2 + x_1x_3 \;=\; 0.$$

It is immediate to see that the pencil $\mathcal{Q}$ contains a quadric of corank 1. □

Remark 0.6.10 If X is of type D_4 (in any characteristic), then the pencil of quadrics $\mathcal{Q}$ contains one quadric of corank 2 and one quadric of corank 1. However, this is not enough to characterize these surfaces. Indeed, the surface

$$x_0^2 + x_1x_2 \;=\; x_0x_3 + x_2x_4 \;=\; 0$$

satisfies these conditions, but it has 2 singular points $[0, 0, 0, 1, 0]$ and $[0, 1, 0, 0, 0]$ that are rational double points of type A_1 and A_3, respectively. Similarly, surfaces of type D_5 are not characterized by their singularities alone.

Our goal is to establish Corollary 0.6.14 and to describe the local principal covers over the symmetroid quartic surfaces explicitly. This will we done achieved in a case-by-case analysis and anticipating this result, we will denote the resulting surfaces D_1, D_1', D_2, D_2', D_3, D_3', see also Definition 0.6.15 below.

Proposition 0.6.11 *Let $\mathbb{k}$ be a field of characteristic $p \geq 0$ and let the $\mathbb{k}$-group scheme $\boldsymbol{\mu}_2 = \operatorname{Spec} \mathbb{k}[\varepsilon]/(\varepsilon^2 - 1)$ act on $\mathbb{P}^3$ via*

$$[t_0, t_1, t_2, t_3] \;\to\; [t_0 \otimes \varepsilon, t_1 \otimes \varepsilon, t_2, t_3]$$

and consider the two quadrics in $\mathbb{P}^3$ defined by

$$Q \;:\; t_0t_1 + t_2t_3 = 0 \qquad \textit{and} \qquad Q' \;:\; t_0t_1 + t_2^2 = 0.$$

Then, Q is non-singular, whereas Q' is singular, and both are $\boldsymbol{\mu}_2$-invariant. Next, $\mathsf{D} := Q/\boldsymbol{\mu}_2$ is isomorphic to the surface D_1 and $\mathsf{D}' := Q'/\boldsymbol{\mu}_2$ is isomorphic to the surface D_1'. Moreover, the restriction of the quotient maps to $\mathsf{D} \setminus \mathrm{Sing}(\mathsf{D}_1)$ and $\mathsf{D}' \setminus \mathrm{Sing}(\mathsf{D}_1')$ are principal $\boldsymbol{\mu}_2$-covers. In particular, D_1 and D_1' are symmetroid surfaces.

Proof It is easy to see that Q is non-singular, Q' is singular with unique singular point $[0,0,0,1]$, and that both are $\boldsymbol{\mu}_2$-invariant. Let us first consider the non-singular quadric Q. The following quadratic forms in $t_0, \dots, t_4$ belong to the ring of invariants $\Bbbk[Q]^{\boldsymbol{\mu}_2}$:

$$(q_0, q_1, q_2, q_3, q_4) = (t_0t_1, -t_0^2, t_1^2, t_2^2, -t_3^2) \tag{0.6.13}$$

and it is easy to see that they define a base-point-free linear subsystem of dimension 4 in $|\mathcal{O}_Q(2)|$. Thus, the map π defined by this linear system is a morphism and it factors through $Q/\boldsymbol{\mu}_2$. We immediately check that $x_i = q_i$, restricted to Q satisfy the relations $x_0^2 + x_1x_2 = 0$ and $x_0^2 + x_3x_4 = 0$ (here, we have to take into account the equation of Q). By Proposition 0.6.16, the surface X defined by these equations is the surface D_1. From this, it is easy to see that π is the quotient map by the $\boldsymbol{\mu}_2$-action. Moreover, the $\boldsymbol{\mu}_2$-action on Q is free outside the four points $[1,0,0,0]$, $[0,1,0,0]$, $[0,0,1,0]$, $[0,0,0,1]$, and π maps these to the four nodes of X, which proves that the restriction of π over $X \setminus \mathrm{Sing}(X)$ is a $\boldsymbol{\mu}_2$-torsor.

Next, consider the singular quadric Q'. Here, we use the invariant quadratic forms

$$(q_0, q_1, q_2, q_3, q_4) = (t_2^2, -t_0^2, t_1^2, -t_3^2, t_2t_3).$$

This time the relations satisfied by the $x_i = q_i$ are $x_1x_2 + x_0^2 = 0$ and $x_0x_3 + x_4^2 = 0$, which defines the surface D_1'. From here, we conclude as in the previous case. □

Proposition 0.6.12 *Let $\Bbbk$ be a field of characteristic $p = 2$ and let the $\Bbbk$-group schemes $(\mathbb{Z}/2\mathbb{Z})$ and $\boldsymbol{\alpha}_2 = \mathrm{Spec}\, \Bbbk[\varepsilon]/(\varepsilon^2)$ act on $\mathbb{P}^3$ via*

$$\begin{array}{lll} \mathbb{Z}/2\mathbb{Z}: & [t_0,t_1,t_2,t_3] & \mapsto [t_1,t_0,t_3,t_2] \\ \boldsymbol{\alpha}_2 \ : & t_0 & \mapsto t_0 \otimes 1 + (t_2+t_3) \otimes \epsilon, \\ & t_1 & \mapsto t_1 \otimes 1 + (t_2+t_3) \otimes \epsilon, \\ & t_2 & \mapsto t_2 \otimes 1 + (t_0+t_1) \otimes \epsilon, \\ & t_3 & \mapsto t_3 \otimes 1 + (t_0+t_1) \otimes \epsilon, \end{array}$$

and consider the nonsingular quadric in $\mathbb{P}^3$ defined by

$$Q \ : \ t_0t_1 + t_2t_3 = 0\,.$$

Then, Q is invariant under both actions and the quotient $Q/(\mathbb{Z}/2\mathbb{Z})$ (resp. $Q/\boldsymbol{\alpha}_2$) is isomorphic to the surface D_2 (resp. D_3). Moreover, the restriction of the quotient maps to the non-singular locus of the quotient is a principal $(\mathbb{Z}/2\mathbb{Z})$-cover (resp. $\boldsymbol{\alpha}_2$-cover). In particular, D_2 and D_3 are symmetroid surfaces.

Proof First, we treat the case $G = \mathbb{Z}/2\mathbb{Z}$. Then, we consider the linear system of invariant quadrics on Q generated by the quadrics $V(q_i)$, where

$$(q_0, q_1, q_2, q_3, q_4) = (t_0t_1, t_0^2 + t_1^2, t_2^2 + t_3^2, t_0t_2 + t_1t_3, t_0t_3 + t_1t_2).$$

We argue as in the proof of Proposition 0.6.11 and check that the $x_i = q_i$ satisfy the relations $x_1x_2 + (x_3 + x_4)^2 = 0$ and $x_3x_4 + x_0(x_1 + x_2) = 0$. After a linear change of coordinates $(x_0, x_1, x_2, x_3, x_4) = (y_3, y_1 + y_2, y_2, y_0 + y_2 + y_4, y_4)$, we obtain the equation of the surface D_2.

Next, we treat the case $G = \boldsymbol{\alpha}_2$. Here, we consider the linear system of invariant quadrics on Q generated by the quadrics $V(q_i)$, where

$$(q_0, q_1, q_2, q_3, q_4) = \Big((t_0 + t_1)(t_2 + t_3), (t_0 + t_1)^2, (t_2 + t_3)^2, t_1^2, t_3^2\Big).$$

We argue as in the proof of Proposition 0.6.11 and check that the $x_i = q_i$ satisfy the relations $(x_1 + x_3)x_3 + (x_2 + x_4)x_4 = 0$ and $x_1x_2 + x_0^2 = 0$. After a linear change of coordinates $(x_0, x_1, x_2, x_3, x_4) = (y_0 + y_1, y_1 + y_2, y_2, y_3, y_3 + y_4)$, we obtain the equation of the surface D_3.

The remaining assertions are shown as in the proof of Proposition 0.6.11. $\square$

Proposition 0.6.13 *Let $\Bbbk$ be a field of characteristic $p = 2$ and let the $\Bbbk$-group schemes $\mathbb{Z}/2\mathbb{Z}$ and $\boldsymbol{\alpha}_2 = \operatorname{Spec} \Bbbk[\varepsilon]/(\varepsilon^2)$ act on $\mathbb{P}^3$ via*

$$\begin{aligned} \mathbb{Z}/2\mathbb{Z} &: [t_0, t_1, t_2, t_3] \mapsto [t_0, t_0 + t_1, t_0 + t_2, t_3 + t_1 + t_2] \\ \boldsymbol{\alpha}_2 &: [t_0, t_1, t_2, t_3] \mapsto [t_0 \otimes 1, t_1 \otimes 1, t_2 \otimes 1 + t_0 \otimes \epsilon, t_3 \otimes 1 + t_2 \otimes \epsilon] \end{aligned}$$

and consider the singular quadric in $\mathbb{P}^3$ defined by

$$Q' \ : \ t_0t_1 + t_2^2 = 0 .$$

Then, Q' is invariant under both actions and the quotient $Q'/(\mathbb{Z}/2\mathbb{Z})$ (resp. $Q'/\boldsymbol{\alpha}_2$) is isomorphic to the surface D_2' (resp. D_3'). Moreover, the restriction of the quotient maps to the non-singular locus of the quotient is a principal $\mathbb{Z}/2\mathbb{Z}$-cover (resp. $\boldsymbol{\alpha}_2$-cover). In particular, D_2' and D_3' are symmetroid surfaces.

Proof First, we treat the case $G = \mathbb{Z}/2\mathbb{Z}$. We consider the linear system of invariant quadrics on Q' generated by the quadrics $V(q_i)$, where

$$(q_0, q_1, q_2, q_3, q_4) = \Big(t_0(t_1 + t_2), t_1^2 + t_2^2, t_0^2, t_3(t_1 + t_2 + t_3), t_0t_3 + t_1t_2 + t_0t_1\Big).$$

We argue as in the proof of Proposition 0.6.11 and check that the $x_i = q_i$ satisfy the relations $x_1x_2 + x_0^2 = 0$ and $x_1x_3 + x_0(x_2 + x_4) + x_4^2 = 0$, which is the equation of the surface D_2'.

Next, we treat the case $G = \boldsymbol{\alpha}_2$. Here, we consider the linear system of invariant quadrics on Q' generated by the quadrics $V(q_i)$, where

$$(q_0, q_1, q_2, q_3, q_4) = (t_2^2, t_0^2, t_1^2, t_3^2, t_1t_2 + t_0t_3).$$

We argue as in the proof of Proposition 0.6.11 and check that the $x_i = q_i$ satisfy the relations $x_1x_2 + x_0^2 = 0$ and $x_1x_3 + x_0x_2 + x_4^2 = 0$, which is the equation of the surface D_3'.

The remaining assertions are shown as in the proof of Proposition 0.6.11. □

Corollary 0.6.14 *Let $X \subset \mathbb{P}^4$ be a non-degenerate quartic surface over an algebraically closed field $\Bbbk$ of characteristic $p \geq 0$ that is symmetroid. Then, all possibilities of Proposition 0.6.5 do exist and the surfaces are isomorphic to the surfaces given by the following equations:*

	equations	*singularities*	*characteristic*
D_1	$x_0^2 + x_1x_2 = x_0^2 + x_3x_4 = 0$	$4A_1$	*all*
D_1'	$x_0^2 + x_1x_2 = x_3x_0 + x_4^2 = 0$	$2A_1 + A_3$	*all*
D_2	$x_0^2 + x_1x_2 = x_3x_1 + x_4(x_0 + x_2 + x_4) = 0$	D_4^1	$p = 2$
D_3	$x_0^2 + x_1x_2 = x_3x_1 + x_4(x_2 + x_4) = 0$	D_4^0	$p = 2$
D_2'	$x_0^2 + x_1x_2 = x_3x_1 + x_0(x_2 + x_4) + x_4^2 = 0$	D_5^1	$p = 2$
D_3'	$x_0^2 + x_1x_2 = x_3x_1 + x_0x_2 + x_4^2 = 0$	D_5^0	$p = 2$

Definition 0.6.15 We will use D_i and D_i' with $i = 1, 2, 3$ to denote an anti-canonical del Pezzo surface of degree 4 isomorphic to a surface given in the corresponding row of the previous table. A surface isomorphic to the surfaces D_1 (resp. D_1') is called a *four-nodal quartic del Pezzo surface* (resp. *degenerate four-nodal quartic del Pezzo surface*).

To summarize what we have found in this section, let us state the following propositions.

Proposition 0.6.16 *Let X be a non-degenerate irreducible quartic surface in $\mathbb{P}^4$ over an algebraically closed field $\Bbbk$ of characteristic $p \geq 0$ and let $Y \to X$ be its minimal resolution of singularities. Let Q be the linear system of quadrics containing X. Then, the following are equivalent:*

1. *X is the anti-canonical quartic del Pezzo surface with 4 singular points of type A_1,*
2. *Q is a pencil spanned by two quadrics of rank 3 and contains a quadric of rank 4,*
3. *X contains exactly 4 lines and 4 pencils of conics,*
4. *X is a weak del Pezzo surface of degree 4 obtained from $\mathbb{P}^2$ by blowing up 5 points $p_1, p_3 \succ p_2, p_5 \succ p_4$ such that the points p_1, p_2, p_3 and p_1, p_4, p_5 are collinear,*
5. *X is isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$ blown up at four points that are the vertices of a quadrangle of lines formed by the rulings,*
6. *X is isomorphic to the surface D_1.*

Proposition 0.6.17 *Let X be a non-degenerate irreducible quartic surface in $\mathbb{P}^4$ over an algebraically closed field $\Bbbk$ of characteristic $p \geq 0$ and let $Y \to X$ be its minimal resolution of singularities. Let Q be the linear system of quadrics containing X. Then, the following are equivalent:*

1. *X is the anti-canonical quartic del Pezzo surface with 2 singular points of type A_1 and one singular point of type A_3,*
2. *Q is a pencil spanned by two quadrics of rank 3 that does not contain any quadric of rank 4,*
3. *X contains exactly 2 lines and 2 pencils of conics,*
4. *X is a weak del Pezzo surface of degree 4 obtained from $\mathbb{P}^2$ by blowing up 5 points $p_3 \succ p_2 \succ p_1, p_5 \succ p_4$ such that the points p_1, p_2, p_3 and p_1, p_4, p_5 are collinear,*
5. *Y is isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$ blown up at four points $q_1, q_2, q_3 \succ q_1, q_4 \succ q_2$, where q_1, q_2 lie on the same ruling, and q_3, q_4 correspond to the tangent directions defined by the lines from the different rulings passing through q_1 and q_2,*
6. *X is isomorphic to the surface D_1'.*

Remark 0.6.18 Using the explicit equations in Corollary 0.6.14, it is easy to study the geometry of these surfaces in detail:

1. Let $X = \mathsf{D}_1$ be the four-nodal quartic del Pezzo surface. First, the singular points are the points $[0,1,0,0,0]$, $[0,0,1,0,0]$, $[0,0,0,1,0]$, and $[0,0,0,0,1]$. In terms of Proposition 0.6.16, three lines come from the blow-up of the points p_1, p_3, and p_5 and the fourth is the line joining p_2 and p_4. In terms of the equation in Corollary 0.6.14, these lie in the hyperplane $x_0 = 0$ and are cut out by hyperplanes as follows:
$$x_0 = x_1 = x_3 = 0, \qquad x_0 = x_1 = x_4 = 0,$$
$$x_0 = x_2 = x_3 = 0, \qquad x_0 = x_2 = x_4 = 0.$$
Three of the pencils of conics come from the pencils of lines in $\mathbb{P}^2$ passing through the points p_1, p_2 and p_4, respectively. The fourth pencil comes from the pencil of conics that passes through the points p_2, p_3, p_4, and p_5.
In particular, the union of its four lines is a quadrangle with vertices at the nodes. We will refer to this quadrangle of lines as the *quadrangle* of D_1 and to the vertices as the *vertices* of D_1. One also easily checks the following simple properties:
   a. Each of the diagonals of the quadrangle of D_1 is the singular line of one of the two quadrics of rank 3 containing D_1.
   b. A pair of intersecting sides of the quadrangle and the opposite pair of sides are members of the same pencil of conics. The two pencils obtained in this way are cut out by the pencils of planes (with respect to the equation from Corollary 0.6.14)
$$\lambda x_0 - \mu x_2 = 0, \qquad \lambda x_1 + \mu x_0 = 0,$$
$$\lambda x_0 - \mu x_3 = 0, \qquad \lambda x_4 + \mu x_0 = 0.$$
   c. Each side of the quadrangle, taken with multiplicity 2, belongs to a pencil of conics on D_1. The same pencil contains the opposite side, taken with multiplicity 2. The two pencils obtained in this way are cut out by the pencils of planes (with respect to the equation from Corollary 0.6.14)

$$\lambda x_2 + \mu x_3 = \lambda x_4 + \mu x_1 = 0,$$
$$\lambda x_2 + \mu x_4 = \lambda x_3 + \mu x_1 = 0.$$

2. Let D_1' be the degenerate four-nodal quartic del Pezzo surface. First, the two singularities of type A_1 are the points $[0, 1, 0, 0, 0]$ and $[0, 0, 1, 0, 0]$ and the A_3-singularity is $[0, 0, 0, 1, 0]$. In terms of Proposition 0.6.17, the two lines come from the blow-up of the points p_1 and p_2 (the last blown-up component). In terms of the equation from Corollary 0.6.14, the lines lie in the plane $x_0 = x_4 = 0$ and are cut out by an additional equation $x_2 = 0$ and $x_1 = 0$.
   The two pencils of conics come from the pencils of lines in $\mathbb{P}^2$ passing through the points p_1 and p_2.
   In analogy to the previous case, the union of these two lines the *degenerate quadrangle* of D_1'. The point of intersection of the two lines is the singular point of type A_3, and we will call it the A_3-*vertex*. Each line also passes through one node, and we will call these nodes the *simple vertices* of D_1'.
   As before, one also easily checks the following simple properties:

   a. The line joining the two simple vertices of the degenerate quadrangle is the double line of one of the two quadrics of rank 3 containing D_1'. The double line of the other quadric of rank 3 is tangent to the first one at the A_3-vertex.
   b. The planes passing through the simple vertices of D_1' cut out a pencil of conics on D_1'. Its equation is (with respect to the equation from Corollary 0.6.14)

   $$\lambda x_0 = 0, \qquad \lambda x_3 + \mu x_4 = 0.$$

   c. Each line of the degenerate quadrangle, taken with multiplicity 2, belongs to a pencil of conics on D_1'. The same pencil contains the other line taken with multiplicity 2. Its equation is (with respect to the equation from Corollary 0.6.14)

   $$\lambda x_2 + \mu x_0 = 0, \qquad \lambda x_0 - \mu x_1 = 0.$$

Proposition 0.6.19 *Let X be a non-degenerate irreducible quartic surface in $\mathbb{P}^4$ over an algebraically closed field $\Bbbk$ of characteristic $p = 2$ and let $Y \to X$ be its minimal resolution of singularities. Let Q be the linear system of quadrics containing X. Then, the following are equivalent:*

1. *X is the anti-canonical quartic del Pezzo surface with a singular point of type D_4^1 (resp. D_4^0),*
2. *X has exactly 2 lines and 3 pencils of conics,*
3. *Y is a weak del Pezzo surface of degree 4 obtained from $\mathbb{P}^2$ by blowing up 5 points $p_4 \succ p_3 \succ p_2 \succ p_1, p_5$ such that the points p_1, p_2, p_5 are collinear,*
4. *X is isomorphic to the surface D_2 (resp. D_3).*

Remark 0.6.20 One can show that $\mathrm{Aut}(\mathbb{P}^2)$ acts on the sets of points $(p_1, \ldots, p_5)$ as in (4) with two orbits. One orbit is represented by a surface that admits a $\mathbb{G}_m$-action of projective transformations, and the other orbit is represented by a surface that does not admit such an action. However, if $p \neq 2$, then there exists a quadratic Cremona transformation defined by the linear system of conics passing through the points p_2, p_3, p_4 that maps the first projective orbit to the second one. If $p = 2$, then the transformation does not change the isomorphism class of the surfaces.

Finally, we characterize the quartic symmetroid surfaces D_2' and D_3'.

Proposition 0.6.21 *Let X be a non-degenerate irreducible quartic surface in $\mathbb{P}^4$ over an algebraically closed field $\Bbbk$ of characteristic $p = 2$ and let $Y \to X$ be its minimal resolution of singularities. Let Q be the linear system of quadrics containing X. Then, the following are equivalent:*

1. *X is an anti-canonical quartic del Pezzo surface with a singular point of type D_5^1 (resp. D_5^0),*
2. *X has exactly one line and one pencil of conics,*
3. *Y is a weak del Pezzo surface of degree 4 obtained from $\mathbb{P}^2$ by blowing up 5 points $p_5 \succ p_4 \succ p_3 \succ p_2 \succ p_1$, such that the points p_1, p_2, p_3 are collinear,*
4. *X is isomorphic to the surface D_2' (resp. D_3').*

Remark 0.6.22 Remark 0.6.20 also applies to anti-canonical quartic del Pezzo surfaces with one singularity of type D_5^0 or D_5^1.

Remark 0.6.23 It is easy to check that the induced local action at the point $[1,1,1,1] \in Q$ (resp. $[0,0,0,1] \in Q'$) in Proposition 0.6.12 (resp. Proposition 0.6.13) is isomorphic to the action from Proposition 0.4.31 (resp. Proposition 0.4.32).

Remark 0.6.24 Let D be the anti-canonical quartic del Pezzo surface X of type D_4 or D_5 in characteristic $p = 2$. In Proposition 0.6.12 and Proposition 0.6.13 we established the existence of a principal G-cover $\pi : V \to \mathsf{D}^{\mathrm{sm}} := \mathsf{D} \setminus \mathrm{Sing}(\mathsf{D})$, where G isomorphic to $\mathbb{Z}/2\mathbb{Z}$ or $\boldsymbol{\alpha}_2$. This G-action on π extends to $\overline{\pi} : \overline{V} \to \mathsf{D}$ and finally, $\overline{V}$ is isomorphic to an irreducible quadric surface in $\mathbb{P}^3$.

In fact, the principal G-covers over the smooth locus of D are unique up to isomorphism: let $\overline{\pi} : \overline{V} \to \mathsf{D}$ be such a cover. Then, the formula for the canonical class (0.2.10) shows that $\omega_V \cong \pi^*(\omega_{\mathsf{D}^{\mathrm{sm}}})$, which implies that $\overline{\pi}^*(\omega_{\mathsf{D}}) \cong \omega_{\overline{V}}$. Since $\overline{\pi}$ is a finite morphisms and $\omega_{\mathsf{D}}^{-1} \cong \mathcal{O}_{\mathsf{D}}(1)$ is ample, it follows that $\omega_{\overline{V}}^{-1}$ is ample. Let $\sigma : \widetilde{V} \to \overline{V}$ be the minimal resolution of singularities and thus, $\widetilde{V}$ is a weak del Pezzo surface with anti-canonical model $\overline{V}$. Then, $(\overline{\pi} \circ \sigma)^*(\omega_{\mathsf{D}}) \cong \omega_{\widetilde{V}}$, hence $K_{\widetilde{V}} = 2K_{\mathsf{D}}^2 = 8$, where we have used the intersection theory of Cartier divisors.

1. Assume that $\overline{V}$ is nonsingular, that is, D has a singularity of type D_4, see Proposition 0.4.31 and Proposition 0.4.32. This implies that $\overline{V}$ is a quadric or the Hirzebruch surface $\mathbf{F}_1$ (the blow-up of $\mathbb{P}^2$ in one point) in $\mathbb{P}^3$. Seeking a contradiction, assume that $\overline{V}$ is isomorphic to $\mathbf{F}_1$. Then, the group scheme G acts on $\overline{V}$ and leaves invariant the unique (-1)-curve E of $\mathbf{F}_1$. Since $-K_{\overline{V}} \cdot E = 1$, the curve E is mapped one-to-one onto a line on D, hence the quotient map is ramified on E, a contradiction. Thus, $\overline{V}$ is a nonsingular quadric in $\mathbb{P}^3$. It is easy to see that the G-action on $\overline{V}$ with one fixed point is isomorphic to the action defined in Proposition 0.6.12.

2. Now, assume that $\overline{V}$ is singular, that is, D has a singularity of type D_5, see Proposition 0.4.31 and Proposition 0.4.32. Since $K^2_{\widetilde{V}} = 8$ and $\widetilde{V}$ contains a (-2)-curve, we conclude that $\widetilde{V}$ is a minimal ruled surface of type $\mathbf{F}_2$ and that $\overline{V}$ is a singular irreducible quadric in $\mathbb{P}^3$. Again, we check that the G-action on $\overline{V}$ with one fixed point is isomorphic to the action defined in Proposition 0.6.13.

   Finally, let us show that the principal G-cover $\pi : V \to \mathsf{D}^{\mathrm{sm}}$ is unique:

3. If the singularity of D is of type D_4^0 or D_5^0, then uniqueness of the $\boldsymbol{\alpha}_2$-cover follows from Example 0.4.28, which shows that $\dim H^1_{\mathrm{fl}}(\mathsf{D}^{\mathrm{sm}}, \boldsymbol{\alpha}_2) = 1$. The computation in loc. cit. also shows that the Frobenius map $F : H^2_{\mathfrak{m}}(A) \to H^2_{\mathfrak{m}}(A)$ cannot be the identity map. Thus, singularities of type D_4^0 or D_5^0 do not admit local principal $\mathbb{Z}/2\mathbb{Z}$-covers, which also follows from Artin's computation of local fundamental groups of these singularities in characteristic 2 from [28].
4. If the singularity of D is of type D_4^1 or D_5^1, then $\overline{\pi}$ is a finite morphism of degree 2 of complete surfaces. By [28], the local fundamental group of a singularity of type D_4^1 or D_5^1 is of order 2. Thus, a local principal $\mathbb{Z}/2\mathbb{Z}$-cover of the singular point x_0 of the surface D is isomorphic to the cover that we constructed in Propositions 0.4.31 and 0.4.32.

In particular, $\overline{V}$ is nonsingular at the point $y_0 = \overline{\pi}^{-1}(x_0)$ if x_0 is of type D_4 and $\overline{V}$ has a singular point of type A_1 if x_0 is of type D_5.

Finally, we turn to the automorphism groups of symmetroid quartic del Pezzo surfaces. First, we note that an anti-canonical del Pezzo surface has a finite automorphism group isomorphic to a subgroup of $(\mathbb{Z}/2\mathbb{Z})^4 \rtimes \mathfrak{S}_5$, where $\mathfrak{S}_5$ denotes the symmetric group on 5 elements, see [177, 8.6.4] and [184] (for arbitrary characteristic). On the other hand, automorphism groups of symmetroid quartic del Pezzo surfaces are rather large, which also gives a partial explanation for the name *symmetroid* in Definition 0.6.4, see also Proposition 0.7.6 below. More precisely, concerning the automorphism group schemes of these surfaces, we have the following.

Theorem 0.6.25 *Let X be a quartic symmetroid surface D_i or D'_i with $i = 1, 2, 3$. Then, the automorphism group of X is given by the following table:*

Name	$\mathrm{Aut}(X)^\circ$	G	*Name*	$\mathrm{Aut}(X)^\circ$	G
D_1	$\mathbb{G}_m^2$	D_8	D'_1	$(\mathbb{G}_a \rtimes \mathbb{G}_m) \rtimes \mathbb{G}_m$	$\mathbb{Z}/2\mathbb{Z}$
D_2	$\mathbb{G}_a^2$	$\mathbb{Z}/2\mathbb{Z}$	D'_2	$\mathbb{G}_a^2 \rtimes \mathbb{G}_a$	$\{1\}$
D_3	$\mathbb{G}_a^2 \rtimes \mathbb{G}_m$	$\mathbb{Z}/2\mathbb{Z}$	D'_3	$(\mathbb{G}_a^2 \rtimes \mathbb{G}_a) \rtimes \mathbb{G}_m$	$\{1\}$

Here, D_8 denotes the dihedral group of order 8. As usual, $\mathrm{Aut}(X)^\circ$ *denotes the connected component of the automorphism group scheme* $\mathrm{Aut}(X)$ *and we set* $G := \mathrm{Aut}(X)/\mathrm{Aut}(X)^\circ$.

Proof Let X be an anti-canonical del Pezzo surface and let $\mathrm{Aut}(X)$ be its automorphism group scheme. Since $|-K_X|$ is ample, $\mathrm{Aut}(X)$ is a closed subgroup of the algebraic group of projective automorphisms of X. It is also isomorphic to the group of automorphisms of its minimal resolution $\bar{X}$ of X. Its connected component of identity $\mathrm{Aut}(X)^\circ$ acts trivially on $\mathrm{Pic}(\bar{X})$ and hence, fixes any geometric basis $(e_0, e_1, \dots, e_n)$ of $\mathrm{Pic}(\bar{X})$. In particular, it fixes the linear systet $|e_0|$ that defines an isomorphism from $\bar{X}$ to the blow-up of $\mathbb{P}^2$ at a set of points $p_1, \dots, p_n$. This implies that $\mathrm{Aut}(X)^\circ$ is isomorphic to the subgroup of $\mathrm{Aut}(\mathbb{P}^2)$ that fixes this set of points.

It is known that the natural homomorphism $\mathrm{Aut}(X) \to \mathrm{O}(\mathrm{Pic}(X))$ is injective for all weak del Pezzo surfaces of degree ≤ 5 [177, Proposition 8.2.39]. It is easy to see that in our cases, this implies that an automorphism that acts trivially on the set of lines on X belongs to $\mathrm{Aut}(X)^\circ$.

After these general remarks, we now compute $\mathrm{Aut}(X)$.

First, let $X = \mathsf{D}_1$. Then, $\mathrm{Aut}(X)^\circ$ is a subgroup of $\mathrm{Aut}(\mathbb{P}^2)$ that preserves the coordinate triangle. It consists of scaling transformations $[x, y, z] \mapsto [\lambda x, \mu y, \gamma z]$. It is isomorphic to the torus $\mathbb{G}_m^2$. In projective coordinates given in (0.6.1), the action is given by

$$[x_0, x_1, x_2, x_3, x_4] \ \mapsto \ [x_0, \lambda x_1, \lambda^{-1} x_2, \mu x_3, \mu^{-1} x_4], \tag{0.6.14}$$

The group of connected components $G = \mathrm{Aut}(X)/\mathrm{Aut}(X)^\circ$ acts on X by leaving invariant the quadrangle of lines given by equations from Remark 0.6.18. It is immediate to see that it is generated by permutations of the coordinates

$$\begin{aligned} \sigma_{1234} : x_i \mapsto x_{\sigma(i)}, \quad & \sigma = (1324) \in \mathfrak{S}_5, \\ \sigma_{12} : x_i \mapsto x_{\sigma(i)}, \quad & \sigma = (12) \in \mathfrak{S}_5. \end{aligned}$$

The group is isomorphic to the dihedral group D_8 of order 8. In the plane coordinates $[x, y, z]$, the group is generated by the following birational transformations of the plane:

$$\begin{aligned} \sigma_{1234} : [x, y, z] &\mapsto [xy, -z^2, xz], \\ \sigma_{12} : [x, y, z] &\mapsto [-z^2, xy, xz]. \end{aligned}$$

Second, let $X = \mathsf{D}_1'$. In this case, $\mathrm{Aut}(X)^\circ$ consists of projective transformations that fix the points $[0, 0, 1]$ and $[0, 1, 0]$ and also fixes the line $y = 0$. The group is isomorphic to the group $(\mathbb{G}_a \rtimes \mathbb{G}_a) \times \mathbb{G}_m$ of transformations

$$[x, y, z] \ \mapsto \ [x, ay, bz + cx].$$

In projective coordinates $[x_0, x_1, x_2, x_3, x_4]$ given in (0.6.2), the action is given by the formula

$$[x_0, x_1, x_2, x_3, x_4] \ \mapsto \ [ax_0, x_1, a^2 x_2, -ac^2 x_0 + ab^2 x_3 + 2abcx_4, -acx_0 + abx_4]. \tag{0.6.15}$$

The group of connected components $\mathrm{Aut}(X)/\mathrm{Aut}(X)^\circ$ is of order 2 and acts by switching the two lines on X. In plane coordinates, it is defined by the birational transformation

$$[x, y, z] \ \mapsto \ [xy, x^2, -yz].$$

In the coordinates $[x_0,\dots,x_4]$ it acts by switching x_1 with x_2.

Third, let X have a singular point of type D_4^1. It follows from the proof of Corollary 0.6.8 that $\mathrm{Aut}(X)^\circ \cong \mathbb{G}_a^2$. In the plane coordinates, the transformations are given by formula (0.6.5).

$$[x,y,z] \to [x, y+ax, z+bx].$$

In projective coordinates in $\mathbb{P}^4$ given by (0.6.8) with $\epsilon = 1$, this gives transformations

$$[x_0,x_1,x_2,x_3,x_4] \mapsto [x_0{+}ax_1, x_1, bx_0{+}b(a^2{+}a{+}b)x_1{+}bx_2{+}x_3{+}a(a{+}1)x_4, ax_1{+}x_4].$$

Assume X has a singular point of type D_4^0. Then, the group $\mathrm{Aut}(X)^\circ$ is larger and isomorphic to $\mathbb{G}_a^2 \rtimes \mathbb{G}_m$. More precisely, it is isomorphic to the group of projective transformations of $\mathbb{P}^2$ given by formula

$$[x,y,z]\, to\, [x, cy+ax, c^2z+bx].$$

In the projective coordinates in $\mathbb{P}^4$ given by (0.6.8) with $\epsilon = 1$, this gives transformations

$$[x_0,x_1,x_2,x_3,x_4] \mapsto [cx_0{+}ax_1, x_1, a^2x_1{+}c^2x_2, b(a^2{+}b)x_1{+}bc^2x_2{+}c^4x_3{+}a^2x_4, bx_1{+}c^2x_4].$$

In both cases, the surface has two lines and the group of connected components $G = \mathrm{Aut}(X)/\mathrm{Aut}(X)^\circ$ is of order 2 that switches the lines. In coordinates $[x_0,\dots,x_4]$ it acts in both cases by

$$[x_0,x_1,x_2,x_3,x_4] \mapsto [x_0,x_1,x_2,x_3,x_1+x_4].$$

In the plane coordinates, it is given by a projective involution

$$[x,y,z] \mapsto [x,y,z+x].$$

Thus, we find $\mathrm{Aut}(X) \cong \mathrm{Aut}(X)^\circ \rtimes (\mathbb{Z}/2\mathbb{Z})$.

Assume that X has a singular point of type D_5^1. Since X contains only one line, we conclude $\mathrm{Aut}(X) = \mathrm{Aut}(X)^\circ$. It follows from the proof of Corollary 0.6.8 that $\mathrm{Aut}(X)^\circ$ is isomorphic to the group of projective transformations of $\mathbb{P}^2$

$$[x,y,z] \mapsto [x, y+ax, z+by+cx].$$

This shows that $\mathrm{Aut}(X)^\circ \cong \mathbb{G}_a^2 \rtimes \mathbb{G}_a$, as asserted. In coordinates $(x_0,\dots,x_4)$, it acts as follows:

$$\begin{aligned}&[x_0,\dots,x_4] \mapsto [x_0+ax_1, x_1, a^2x_1+x_2,\\ &(a^2+ab+c)x_0+(c^2+a^3+ac)x_1+(a+b+b^2)x_2+x_3+ax_4, bx_0+cx_1+x_4]\end{aligned} \tag{0.6.16}$$

Finally, if X has a singular point of type D_5^1, we have again $\mathrm{Aut}(X) = \mathrm{Aut}(X)^\circ$ and this group is isomorphic to projective transformations of the plane

$$[x,y,z] \mapsto [x, ay+bx, a^2z+cy+d].$$

The group is isomorphic to $(\mathbb{G}_a^2 \rtimes \mathbb{G}_a) \rtimes \mathbb{G}_m$, as asserted. It acts in $\mathbb{P}^4$ by formula

$$[x_0,\dots,x_4] \mapsto [ax_0+bx_1, x_1, b^2x_1+a^4x_2,$$
$$(a^2b^2+ab+c)x_0+(d^2+b^3)x_1+(a^4b+c^2)x_2+x_3+a^6x_4, cx_0+dx_1+a^3x_4]. \tag{0.6.17}$$

0.7 Symmetroid Cubic Surfaces in $\mathbb{P}^3$

In this section, we continue our analysis of symmetroid surfaces that began in the previous section. We will classify and describe cubic surfaces in $\mathbb{P}^3$ that are symmetroid in the sense of Definition 0.6.4. As with symmetroid quartic surfaces in $\mathbb{P}^4$, the classification in characteristic $p = 2$ is more subtle and more difficult.

By Theorem 0.5.5, a normal cubic surface in $\mathbb{P}^3$ is either the cone over a smooth plane cubic curve or an anti-canonical del Pezzo surface of degree 3, see also the discussion in [177, Section 9.2.2]. In particular, if not a cone, a normal cubic surface has at worst rational double point singularities. For the classification of non-normal cubic surfaces, we refer to [177, Theorem 9.2.1]. The next proposition is the key to classifying symmetroid cubic surfaces using the classification of symmetroid quartic surfaces from the previous section.

Proposition 0.7.1 *Let X be a normal cubic surface in $\mathbb{P}^3$ with at worst rational double point singularities over an algebraically closed field $\Bbbk$ of characteristic $p \geq 0$. Then, X is a symmetroid surface if and only if it is the projection of a symmetroid quartic del Pezzo surface in $\mathbb{P}^4$ from its nonsingular point.*

Proof First of all, we claim that every cubic surface in $\mathbb{P}^3$ is the projection of an anti-canonical model of an anti-canonical quartic del Pezzo surface from its nonsingular point: Indeed, let $Y \to X$ be the minimal resolution of singularities of X. By Theorem 0.5.3, Y is the blow-up of $\mathbb{P}^2$ in 6 points, and we let $\sigma : Y = Y_6 \to \dots \to Y_0 = \mathbb{P}^2$ be the composition of blow-ups as in (0.5.5). Since Y is a weak del Pezzo surface of degree 3, it follows that Y_5 is a weak del Pezzo surface of degree 4. Let $Y_{5,\mathrm{can}}$ be the anti-canonical model of Y_5, which is a quartic surface in $\mathbb{P}^4$ via its anti-canonical map $|-K_{Y_5}|$. Let $x \in Y_{5,\mathrm{can}}$ be the image of the point $x_5 \in Y_5$. It is easy to see that the projection of $Y_{5,\mathrm{can}}$ from x is projectively isomorphic to the surface X. We refer to [586, Section 2.3], for explicit equations. Conversely, the projection of an anti-canonical quartic del Pezzo surface from a nonsingular point is a cubic surface.

Now, if X is a symmetroid del Pezzo surface, then the pull-back of the principal G-cover over $X \setminus \mathrm{Sing}(X)$ as in Definition 0.6.4 gives rise to a principal G-cover over the smooth locus of $X_{5,\mathrm{can}}$. Conversely, a principal G-cover over the smooth locus of $X_{5,\mathrm{can}}$ descends to the smooth locus of X. □

As in the case of symmetroid quartic surfaces in $\mathbb{P}^4$, there are different types of symmetroid cubic surfaces in $\mathbb{P}^3$ depending on the group scheme of the principal cover. More precisely, by Definition 0.6.4 and Theorem 0.1.10, there is one type in characteristic $p \neq 2$, namely $\boldsymbol{\mu}_2 \cong \mathbb{Z}/2\mathbb{Z}$, whereas there are 3 types in characteristic $p = 2$, namely $\boldsymbol{\mu}_2$, $\boldsymbol{\alpha}_2$, and $\mathbb{Z}/2\mathbb{Z}$. We start with the classification type $\boldsymbol{\mu}_2$ in every characteristic.

As always we use the notation from the previous section that applies to anti-canonical symmetroid quartic del Pezzo surfaces.

Proposition 0.7.2 *Let X be a symmetroid cubic surface of type $\boldsymbol{\mu}_2$ over an algebraically closed field $\Bbbk$ of characteristic $p \geq 0$. Let $Y \to X$ be its minimal resolution of singularities. Then,*

1. *X is isomorphic to one of the following surfaces:*

   (C_1) $x_0x_1x_2 + x_0x_1x_3 + x_0x_2x_3 + x_0x_2x_3 = 0$,
   (C_1') $x_0x_1x_2 + x_1x_3^2 + x_2x_3^2 = 0$,
   (C_1'') $x_0x_1x_2 + x_0x_3^2 + x_2^3 = 0$.

2. (C_1) *C_1 is the projection of D_1 from a point not lying on any line.*
   (C_1') *C_1' is the projection of D_1 from a nonsingular point lying on any line or the projection of D_1' from a nonsingular point not lying on a line.*
   (C_1'') *C_1'' is the projection of D_1' from a nonsingular point lying on a line.*
3. *X has only rational double singularities, and more precisely:*

   (C_1) *has four singular points of type A_1,*
   (C_1') *has two singular points of type A_1 and one singular point of type A_3,*
   (C_1'') *has one singular point of type A_1 and one singular point of type A_5.*

4. *Y is a weak del Pezzo surface obtained by blowing up 6 points $\{p_1, \ldots, p_6\}$ in $\mathbb{P}^2$ that are in special position:*

   (C_1) *$p_1, \ldots, p_5$ are as in Proposition 0.6.16, and p_6 is a point lying outside of the triangle of lines $\ell_1 = \langle p_1, p_2\rangle$, $\ell_2 = \langle p_1, p_3\rangle$ and $\ell_3 = \langle p_2, p_4\rangle$.*
   (C_1') *$p_1, \ldots, p_5$ are as in Proposition 0.6.16, and p_6 is a point that lies on the line ℓ_3 or infinitely near to p_1, or p_3, or p_5.*
   (C_1'') *$p_1, \ldots, p_5$ are as in Proposition 0.6.17, and p_6 is a point that is infinitely near to p_3 or p_5.*

5. (C_1) *C_1 is isomorphic to the quotient of a del Pezzo surface Q of degree 6 by the group scheme $\boldsymbol{\mu}_2$ (Q is isomorphic to the blow-up of a nonsingular quadric at two points, infinitely near if $p = 2$)),*
   (C_1') *C_1' is isomorphic to the quotient of an anti-canonical del Pezzo surface Q' of degree 6 with one singular point of type A_1 by the group scheme $\boldsymbol{\mu}_2$ (Q' is isomorphic to the blow-up 2 points on a singular quadric).*
   (C_1'') *C_1'' is isomorphic to the quotient of an anti-canonical weak del Pezzo surface Q'' of degree 6 with one singular point of type A_2 by the group scheme $\boldsymbol{\mu}_2$.*

Proof See [138, Proposition 0.5.2] and [177, Section 9.3.3]. □

We leave the proofs to the reader, who may consult [138, Chapter 0.5]. We note that symmetroid cubic surfaces are cubic symmetroid hypersurfaces in the sense that they are projectively isomorphic to a hypersurface of the form $\det(A) = 0$ in $\mathbb{P}^n$, where A is a symmetric (3×3)-matrix, whose entries are linear forms in the $(n+1)$ coordinates, see [177, Section 9.3.3], and Proposition 0.7.6 below.

Proposition 0.7.3 *Let X be a symmetroid cubic surface of type $\mathbb{Z}/2\mathbb{Z}$ over an algebraically closed field $\Bbbk$ of characteristic $p = 2$. Let $Y \to X$ be its minimal resolution of singularities. Then,*

1. X is isomorphic to one of the following surfaces:

(C_2) $x_1x_2x_3 + x_0x_3^2 + x_1^2x_2 + x_1x_2^2 = 0$,
(C_2') $x_1x_2x_3 + x_0x_3^2 + x_1^2x_2 + x_2^2x_3 = 0$,
(C_2'') $x_1x_2x_3 + x_0x_3^2 + x_0x_2^2 + x_1^3 = 0$.

(C_2) C_1 *is the projection of* D_2 *from a point not lying on any line.*
(C_2') C_1' *is the projection of* D_2 *from a nonsingular point lying on any line or the projection of* D_1' *from a nonsingular point not lying on a line.*
(C_2'') C_1'' *is the projection of* D_2' *from a nonsingular point lying on a line.*

2. X has one rational double point of the following type:

(C_2) *one point of type* D_4^1,
(C_2') *one point of type* D_5^1,
(C_2'') *one point of type* E_6^1.

3. Y is a weak del Pezzo surface obtained by blowing up 6 points $\{p_1, \dots, p_6\}$ in $\mathbb{P}^2$ that are in special position:

(C_2) $p_1, \dots, p_5$ *as in Proposition 0.6.19 and* p_6 *is disjoint from the points* $p_1, \dots, p_5$,
(C_2') $p_1, \dots, p_5$ *as in Proposition 0.6.21 and* p_6 *is disjoint from the points* $p_1, \dots, p_5$,
(C_2'') $p_1, \dots, p_5$ *as in Proposition 0.6.21 and* p_6 *is infinitely near to* p_5.

4. (C_2) C_2 *is isomorphic to the quotient of a del Pezzo surface* Q *of degree 6 by* $\mathbb{Z}/2\mathbb{Z}$ *(*Q *is isomorphic to the blow-up of a nonsingular quadric at two points),*
(C_2') C_2' *is isomorphic to the quotient of an anti-canonical weak del Pezzo surface* Q' *of degree 6 with one singular point of type* A_1 *by* $\mathbb{Z}/2\mathbb{Z}$ *(*Q' *is isomorphic to the blow-up 2 points on a singular quadric).*
(C_2'') C_2'' *is isomorphic to the quotient of an anti-canonical weak del Pezzo surface* Q'' *of degree 6 with one singular point of type* A_2 *by* $\mathbb{Z}/2\mathbb{Z}$.

Proposition 0.7.4 *Let X be a symmetroid cubic surface of type α_2 over an algebraically closed field $\Bbbk$ of characteristic $p = 2$. Let $Y \to X$ be its minimal resolution of singularities. Then,*

1. X is isomorphic to one of the following surfaces:

(C_3) $x_0x_3^2 + x_1^2x_2 + x_1x_2^2 = 0$,
(C_3') $x_0x_3^2 + x_1^2x_2 + x_3x_2^2 = 0$,
(C_3'') $x_0x_3^2 + x_1^3 + x_0x_2^2 = 0$.

(C_3) C_1 *is the projection of* D_3 *from a point not lying on any line.*
(C_3') C_1' *is the projection of* D_3 *from a nonsingular point lying on any line or the projection of* D_1' *from a nonsingular point not lying on a line.*
(C_3'') C_1'' *is the projection of* D_3' *from a nonsingular point lying on a line.*

2. X has one rational double point of the following types:

(C_3) *one point of type* D_4^0,
(C_3') *one point of type* D_5^0,
(C_3'') *one point of type* E_6^0.

3. Y is a weak del Pezzo surface obtained by blowing up 6 points $\{p_1, \dots, p_6\}$ *in* $\mathbb{P}^2$ *that are in special position:*

(C_3) $p_1, \dots, p_5$ *as in Proposition 0.6.19 and* p_6 *is disjoint from the points* $p_1, \dots, p_5$,
(C_3') $p_1, \dots, p_5$ *as in Proposition 0.6.21 and* p_6 *is disjoint from the points* $p_1, \dots, p_5$,
(C_3'') $p_1, \dots, p_5$ *as in Proposition 0.6.21 and* p_6 *is infinitely near to* p_5.

4. (C_3) C_3 *is isomorphic to the quotient of a del Pezzo surface* Q *of degree 6 by the constant group scheme* $\boldsymbol{\alpha}_2$ *(*Q *is isomorphic to the blow-up of a nonsingular quadric at two points),*
(C_3') C_3' *is isomorphic to the quotient of the anti-canonical weak del Pezzo surface* Q' *of degree 6 with one singular point of type* A_1 *by the group scheme* $\boldsymbol{\alpha}_2$ *(*Q' *is isomorphic to the blow-up of two points on a singular quadric).*
(C_3'') C_3'' *is isomorphic to the quotient of an anti-canonical weak del Pezzo surface* Q'' *of degree 6 with one singular point of type* A_2 *by the group scheme* $\boldsymbol{\alpha}_2$.

Remark 0.7.5 The surface C_1 is the famous *Cayley cubic surface*. It is obtained as the projectivization of the affine surface:

$$\frac{1}{x} + \frac{1}{y} + \frac{1}{z} = 1.$$

It can also be characterized as the unique normal cubic surface with the maximal possible number of nodes, which is unique up to projective automorphism of the ambient $\mathbb{P}^3$. The minimal resolution $\widetilde{\mathsf{C}}_1$ of the singularities of C_1 is isomorphic to the blow-up of six points $p_1, \dots, p_6$ in $\mathbb{P}^2$ that are the vertices of a complete quadrilateral.

If $p \neq 2$, then the surface C_1 is isomorphic to the quotient of $\mathbb{P}^2$ by the standard Cremona involution with fundamental points $p_1 = [1, 0, 0]$, $p_2 = [0, 1, 0]$, $p_3 = [0, 0, 1]$. Its fixed points $[1, 1, 1], [1, -1, 1], [1, 1, -1], [1, -1, -1]$ are mapped to the singular points of the cubic. The del Pezzo surface Q of degree 6 from Part 4 of

Proposition 0.7.2 is isomorphic to the blow-up of the fundamental points and so, we also obtain the surface as the quotient of a biregular involution of a del Pezzo surface of degree 6.

If $p = 2$, then the Cremona involution has only one fixed point $[1, 1, 1]$ and the quotient surface becomes isomorphic to the cubic surface C_2.

The other surfaces C_1', C_1'' (resp. C_2', C_2'') are obtained when we degenerate the standard Cremona involution by allowing infinitely near fundamental points $p_2 \succ p_1, p_3$ of $p_3 \succ p_2 \succ p_1$, see [177, Example 7.1.9].

Next, we give another explanation for the name symmetroid cubic surface.

Proposition 0.7.6 *Let X be a normal cubic surface in $\mathbb{P}^3$ with at worst rational double point singularities over an algebraically closed field $\Bbbk$ of characteristic $p \neq 2$. Then, X is a symmetroid surface if and only if its equation can be written as the determinant of a symmetric 3×3-matrix, whose entries are linear forms in homogeneous coordinates. In other terms, X is the discriminant hypersurface of a web of conics.*

Proof Using Proposition 0.7.2, we verify directly that each surface $\mathsf{C}_1, \mathsf{C}_1', \mathsf{C}_1''$ is isomorphic to a hypersurface in $\mathbb{P}^3$ that is given by the determinant of the following 3×3 symmetric matrices:

$$\mathsf{C}_1 \,:\, \begin{pmatrix} x_0 + x_3 & x_3 & x_3 \\ x_3 & x_1 + x_3 & x_3 \\ x_3 & x_3 & x_2 + x_3 \end{pmatrix}, \quad \mathsf{C}_1' \,:\, \begin{pmatrix} x_0 & x_3 & -x_3 \\ x_3 & -x_1 & 0 \\ -x_3 & 0 & -x_2 \end{pmatrix}, \quad \mathsf{C}_1'' \,:\, \begin{pmatrix} -x_0 & x_2 & 0 \\ x_2 & x_1 & x_3 \\ 0 & x_3 & -x_2 \end{pmatrix}.$$

Conversely, if C is given by a determinantal equation, then there exists a 3-dimensional linear system (a *web*) $W \cong \mathbb{P}^3$ of conics in $\mathbb{P}^2$ such that

$$X \;\cong\; \{Q \in W \,:\, \operatorname{rank}(Q) < 3\}.$$

Webs of conics can be classified, see Chapter 6. In fact, they correspond bijectively to pencils of conics and the latter are classified by analyzing all possible configurations of their base points. Doing this, we easily find that there are only three projective classes of webs that give rise to a normal surface. Each of them is isomorphic to one of the above surfaces. □

Remark 0.7.7 In Section 7.3 in Volume II, we will introduce the notion of the *half-discriminant* of a quadratic form in odd number n of variables over a field of characteristic $p = 2$. For example, if $n = 3$, then the half-discriminant of a quadratic form $q = \sum_{1 \le i \le j \le 3} a_{ij} x_i x_j$ is equal to $D_3 = a_{11}a_{23}^2 + a_{22}a_{13}^2 + a_{33}a_{12}^2 + a_{12}a_{13}a_{23}$. The equation $D = 0$ defines a cubic hypersurface in $\mathbb{P}^5$, whose subscheme of non-smooth points equal to $V(a_{12}^2, a_{23}^2, a_{13}^2)$. A web of conics W defines a linear section $W \cap V(D_3)$ isomorphic to a symmetroid cubic surface C_2, C_2', or C_2''. The equations of the surfaces C_2, C_2', C_2'' show that C_2 (resp. C_2', resp. C_3') is obtained as the intersection of $V(D_3)$ with the linear subspace $a_{13} - a_{33} = a_{12} - a_{22} = 0$ (resp. $a_{33} - a_{13} = a_{22} - a_{13} = 0$, resp. $a_{12} - a_{33} = a_{11} - a_{22} = 0$).

0.8 Quadratic Lattices and Root Bases

In this section, we briefly survey the theory of quadratic lattices, which plays an important role in the study of Enriques surfaces and K3 surfaces. We start with definitions, invariants, and some general results. Then, we turn to root bases, reflection groups, and Coxeter–Dynkin diagrams, and use these to study root bases of finite type, of affine type, of crystallographic type, as well as those of hyperbolic type. The important tools are an action of orthogonal groups and reflection subgroups, as well as their polyhedral fundamental chambers. We end this section by discussing k-reflective lattices. We refer to [661] for an introduction and to [169] and [556] for more advanced results, proofs, background, and further references.

A *quadratic lattice* or simply, a *lattice*, is a free abelian group M of finite rank $\operatorname{rank} M$ together with a symmetric bilinear form $b : M \times M \to \mathbb{Z}$. To simplify notation, we set $x \cdot y := b(x, y)$ and $x^2 := x \cdot x = b(x, x)$ for all $x, y \in M$. The function $q : x \mapsto x^2$ is a *quadratic form* $q : M \to \mathbb{Z}$ satisfying

$$q(x + y) - q(x) - q(y) = 2 \cdot b(x, y) = 2(x \cdot y).$$

A lattice M is called *even* if the quadratic form q is even, that is, it takes values in $2\mathbb{Z}$. It is called *odd* otherwise. Quite generally, we recall that for a commutative ring R and an R-module M, which is not necessarily free, a map $q : M \to R$ is called a *quadratic form* if $q(rm) = r^2 q(m)$ for all $r \in R$ and all $m \in M$ and if the map $(x, y) \mapsto q(x + y) - q(x) - q(y)$ is bilinear, see [394, Section I.2] or [661, Chapter IV]. In this case, the pair (M, q) is called a *quadratic module*. We note that there is a natural bijection between even lattices and integral quadratic forms on free $\mathbb{Z}$-modules: given a lattice (M, b), it becomes a quadratic $\mathbb{Z}$-module by setting $q(x) := b(x, x)$ for all $x \in M$. Conversely, given a quadratic and free $\mathbb{Z}$-module (M, q), it becomes a lattice by setting $b(x, y) := \frac{1}{2}(q(x + y) - q(x) - q(y))$ for all $x, y \in M$.

A lattice (M, b) induces a symmetric bilinear form on the real vector space $M_{\mathbb{R}} := M \otimes_{\mathbb{Z}} \mathbb{R}$, whose dimension over $\mathbb{R}$ is equal to $\operatorname{rank} M$. By definition, the *signature* $\operatorname{sign}(M)$ of M is defined to be the Sylvester signature (t_+, t_-, t_0) of the real quadratic form $q_{M_{\mathbb{R}}} : x \mapsto x^2$ on $M_{\mathbb{R}}$. Thus, we can speak about *(positive, negative) definite*, *semi-definite*, and *indefinite* lattices. A lattice is called *non-degenerate* if $t_0 = 0$, in which case we shall drop the last component t_0 from its signature. A lattice M of rank $\operatorname{rank} M \geq 2$ and signature $(1, \operatorname{rank} M - 1)$ is called *hyperbolic*.

A homomorphism of lattices $f : M \to M'$ is a homomorphism of abelian groups that respects the bilinear forms on both sides, that is, $f(x) \cdot f(y) = x \cdot y$ for all $x, y \in M$. An injective (resp. bijective) homomorphism of lattices is called an *embedding* (resp. *isometry*). Two lattices are called *isomorphic* or *isometric* if there exists an isometry from one to the other. The set of all isometries $\sigma : M \to M$ is a group with respect to composition of maps. It is called the *orthogonal group* of M and is denoted by $\mathrm{O}(M)$.

A *sublattice* of a lattice M is an abelian subgroup $N \subseteq M$ equipped with the induced bilinear form. A sublattice $N \subseteq M$ is said to be *primitive* (resp. *of finite* *index m*) if the quotient group M/N is a free abelian group (resp. a finite group of order m). An element $x \in M$ is called *primitive* if the sublattice $\mathbb{Z} \cdot x$ spanned by x inside M is primitive. An embedding of lattices is called *primitive embedding* if its image is a primitive sublattice.

Given two sublattices M_1 and M_2 of some lattice M, the *sum* $M_1 + M_2$ is defined to be the minimal sublattice of M containing both M_1 and M_2. Moreover, if $x \cdot y = 0$ for all $x \in M_1$ and $y \in M_2$, then this sum is said to be an *orthogonal sum* and it is denoted by $M_1 \perp M_2$ or, sometimes, $M_1 \oplus M_2$. Next, the *orthogonal sum of two lattices* M_1 and M_2 is the abelian group $M_1 \oplus M_2$ together with the bilinear form $(x_1, x_2) \cdot (y_1, y_2) := x_1 \cdot y_1 + x_2 \cdot y_2$. Similarly, one defines the orthogonal sum of any finite number of lattices. The *orthogonal complement* of a sublattice N of a lattice M is the sublattice of M that is defined to be

$$N^\perp \ := \ \{x \in M : x \cdot y = 0 \text{ for all } y \in N\} \ \subseteq \ M.$$

Next, if $\underline{e} = \{e_1, \ldots, e_r\}$ is a basis of M as $\mathbb{Z}$-module, then the matrix $G(\underline{e}) := (e_i \cdot e_j)$ is called the *Gram matrix* of M with respect to the basis $\underline{e}$. Any symmetric matrix with integer entries is the Gram matrix of some lattice and determines the lattice structure uniquely. We note that the determinant of the Gram matrix $\det(G(\underline{e}))$ does not depend on the choice of a basis. It is called the *discriminant* of the lattice and it is denoted by $\operatorname{discr}(M)$. If it is not zero, then its sign is equal to $(-1)^{t_-}$, where (t_+, t_-, t_0) is the signature of M. Set $M^\vee := \operatorname{Hom}_{\mathbb{Z}}(M, \mathbb{Z})$, which, when M is nondegenerate, is a free abelian group of the same rank as M. Let $i_M : M \to M^\vee$ be the homomorphism of abelian groups that assigns to $x \in M$ the linear function $y \mapsto x \cdot y$. If we choose a basis $\underline{e}$ of M and let $\underline{e}^\vee$ be the dual basis of $M^\vee$, then the homomorphism i_M is given by the Gram matrix $G(\underline{e})$ of M.

The lattice M is called *unimodular* if the homomorphism i_M is an isomorphism. This is equivalent to the Gram matrix having determinant ± 1. In general, the kernel of i_M is denoted by $\operatorname{Rad}(M)$ and it is called the *radical* of M. Note that the lattice M is non-degenerate if and only if $\operatorname{Rad}(M) = \{0\}$, which is equivalent to the Gram matrix being invertible over $\mathbb{Q}$. Thus, M is non-degenerate (resp. unimodular) if and only if $\operatorname{discr}(M) \neq 0$ (resp. $\operatorname{discr}(M) = \pm 1$). Next, the cokernel

$$D(M) \ := \ M^\vee / i_M(M)$$

is called the *discriminant group* of the lattice M. If M is non-degenerate, then $D(M)$ is a finite abelian group and its order is equal to the absolute value of the discriminant of M. Moreover, M is unimodular if and only if $D(M)$ is trivial. If $D(M)$ is a finite abelian and p-elementary group for some prime p, then M is called a *p-elementary* lattice. Quite generally, if M is non-degenerate, then $M^\vee$ can be identified with the abelian group $\{x \in M \otimes_{\mathbb{Z}} \mathbb{Q} : x \cdot y \in \mathbb{Z} \text{ for all } y \in M\}$, which equips $M^\vee$ with a rational–valued quadratic form inherited from the rational quadratic form on $M \otimes_{\mathbb{Z}} \mathbb{Q}$. By abuse of terminology, $M^\vee$ is called the *dual lattice* of M.

Let (M, b) be a non-degenerate lattice and let $M^\vee$ be its dual lattice, both considered as $\mathbb{Z}$-submodules of $M \otimes_{\mathbb{Z}} \mathbb{Q}$. Then, we have an induced symmetric bilinear form on the discriminant group of M

$$b_{D(M)} \,:\, D(M)\times D(M) \;\to\; \mathbb{Q}/\mathbb{Z}, \quad (x+M,\, y+M) \;\mapsto\; b(x,y) \mod \mathbb{Z},$$

for all $x, y \in M^\vee$. Let us assume moreover that M is an even lattice with associated quadratic form q. Then, we define the *discriminant quadratic form* on $D(M)$ by setting

$$q_{D(M)} \,:\, D(M) \;\to\; \mathbb{Q}/2\mathbb{Z}, \quad x + M \;\mapsto\; x^2 \mod 2\mathbb{Z},$$

for all $x \in M^\vee$. More explicitly, let r be a positive integer such that $rM^\vee \subseteq M$ – for example, we could choose r to be largest elementary divisor of the Gram matrix of M with respect to some choice of basis. Then, the just-defined quadratic form on $D(M)$ can also be computed as $q_{D(M)}(rx) = \frac{1}{r^2} q_M(rx)$ for all $x \in M^\vee$, where we note that $rx \in M$ for all $x \in M^\vee$. We note that the function $q_{D(M)}$ on $D(M)$ is a quadratic form, not in the sense of the above definition, but in the sense that $q_{D(M)}(nx) = n^2 q_{D(M)}(x)$ and $q(x+y) - q(x) - q(y) = 2b_{D(M)}(x,y) \mod 2\mathbb{Z}$ for all $x, y \in D(M)$ and all $n \in \mathbb{Z}$. We denote by $\mathrm{O}(D(M))$ the group of those automorphisms of the abelian group $D(M)$ that preserve the values of the quadratic form $q_{D(M)}$. If M_1, M_2 are two lattices, then we have

$$q_{D(M_1 \perp M_2)} = q_{D(M_1)} + q_{D(M_2)} \quad \text{and} \quad b_{D(M_1 \perp M_2)} = b_{D(M_1)} + b_{D(M_2)}.$$

Finally, if we have two non-degenerate lattices M_1 and M_2, then there exists an isomorphism $D(M_1) \cong D(M_2)$ of discriminant groups together with bilinear forms if and only if there exist unimodular lattices L_1 and L_2 and an isomorphism of lattices $M_1 \perp L_1 \cong M_2 \perp L_2$. Moreover, if we have two non-degenerate and even lattices M_1 and M_2, then there exists an isomorphism $D(M_1) \cong D(M_2)$ of discriminant groups together with quadratic forms if and only if there exist even and unimodular lattices L_1 and L_2 and an isomorphism of lattices $M_1 \perp L_1 \cong M_2 \perp L_2$. We refer to [556] for details and further results.

Two lattices (M_1, b_1) and (M_2, b_2) are said to belong to the same *genus* if they have the same signature, and if for every prime p, there exists an isomorphism of $\mathbb{Z}_p$-modules $M_1 \otimes_{\mathbb{Z}} \mathbb{Z}_p \cong M_2 \otimes_{\mathbb{Z}} \mathbb{Z}_p$ that is compatible with the bilinear forms $b_i \otimes \mathbb{Z}_p$, $i = 1, 2$. Here, $\mathbb{Z}_p$ denotes the p-adic numbers. We recall that M_1 and M_2 have the same signature if and only if there exists an isomorphism of $\mathbb{R}$-vector spaces $M_1 \otimes_{\mathbb{Z}} \mathbb{R} \cong M_2 \otimes_{\mathbb{Z}} \mathbb{R}$ that is compatible with the bilinear forms on both sides. It is known that there exist only finitely many isomorphism classes of lattices in each genus and that the genus of a lattice determines the isomorphism class of $M \otimes_{\mathbb{Z}} \mathbb{Q}$. Moreover, if M is an even lattice, then the signature and the discriminant $D(M)$ together with its quadratic form determine the genus of M.

We will also need the classification of finite discriminant forms. Let p be an odd prime, k a positive integer, and $\epsilon \in \{\pm\}$. We define $\mathsf{w}^{\epsilon}_{p,k}$ to be the abelian group $\mathbb{Z}/p^k\mathbb{Z}$ together with the quadratic form, whose value on the generator 1 is equal to $ap^{-k} \mod 2\mathbb{Z}$. Here, a is the smallest positive odd number that is a quadratic residue (resp. not a quadratic residue) if $\epsilon = 1$ (resp. $\epsilon = -1$). We also define $\mathsf{w}^{\epsilon}_{2,k}$ to be the abelian group $\mathbb{Z}/2^k\mathbb{Z}$ together with the quadratic form, whose value on the generator 1 is equal to $\frac{\epsilon}{2}$ if $k = 1$ and $\epsilon = \pm 1$ and which is it is equal to $\frac{\epsilon}{2^k}$ if $k > 1$ and $\epsilon = \pm 1, \pm 5$.

On the abelian group $(\mathbb{Z}/2^k\mathbb{Z})^{\oplus 2}$, we define quadratic forms u_k and v_k via the matrices

$$\mathsf{u}_k = \begin{pmatrix} 0 & 2^{-k} \\ 2^{-k} & 0 \end{pmatrix}, \quad \mathsf{v}_k = \begin{pmatrix} 2^{-k+1} & 2^{-k} \\ 2^{-k} & 2^{-k+1} \end{pmatrix}.$$

The following theorem of Nikulin classifies discriminant quadratic forms and we refer to [556] for proof.

Theorem 0.8.1 *The discriminant quadratic form on a non-degenerate quadratic lattice is isomorphic to the orthogonal sum of the quadratic forms*

$$\mathsf{w}^{\epsilon}_{p,k}, \quad \mathsf{w}^{\epsilon}_{2,k}, \quad \mathsf{u}_k, \quad \mathsf{v}_k.$$

Next, if M is a sublattice of finite index m of a non-degenerate lattice M', then we have inclusions of lattices

$$M \subseteq M' \subseteq M'^{\vee} \subseteq M^{\vee}, \tag{0.8.1}$$

from which it is easy to deduce that

$$\mathrm{discr}(M) = m^2 \cdot \mathrm{discr}(M'). \tag{0.8.2}$$

Now, assume that M is an even lattice. Then, the restriction of the quadratic form $q_{D(M)}$ to the subgroup M'/M of $M^{\vee}/M$ is identically zero. We say that it is an *isotropic subgroup*.

The next proposition [556, Propositions 1.4.1 and 1.4.2] follows immediately from the tower of lattices (0.8.1).

Proposition 0.8.2 *The assignment $M' \mapsto H_{M'}$ establishes a bijective correspondence between*

1. *the set of isotropic subgroups of order m of the discriminant group $D(M)$ and*
2. *the set even lattices M' containing M as a sublattice of index m.*

Moreover, the orthogonal complement $H^{\perp}_{M'}$ of $H_{M'}$ in $D(M)$ coincides with $M'^{\vee}/M$ and the restriction of $q_{D(M)}$ to $H^{\perp}_{M'}$ defines a quadratic form on $H^{\perp}_{M'}/H_{M'}$ that coincides with $q_{M'}$. Two such over-lattices M' and M'' are isomorphic if and only if the subgroups $H_{M'} \subset D(M')$ and $H_{M''} \subset D(M'')$ are conjugate by an automorphism of M.

For example, let $M \to M'$ be a primitive embedding of even and non-degenerate lattices and $K := M'^{\perp}$ be the orthogonal complement of M' in M. Then, $M \oplus K$ embeds into M' as a sublattice of finite index. Applying the previous proposition, we obtain the following.

Corollary 0.8.3 *A primitive embedding of even and non-degenerate lattices $M \to M'$ with orthogonal complement isomorphic to a fixed lattice K is determined by the subgroup H of $D(M)$ and an injective homomorphisms $\gamma : H \to D(K)$ satisfying $q_K \circ \gamma = -q_M|H$ and whose graph $\Gamma_\gamma \subset H \times D(K)$ satisfies $(\Gamma^{\perp}_{\gamma}/\Gamma_\gamma, q_M \oplus q_K) \cong$*

$(D(M'), q_{M'})$. Two such pairs (H, γ) and (H', γ') define isomorphic embeddings if and only if $H = H'$ and $\gamma' = \gamma \circ \bar{\phi}$ for some $\phi \in \mathrm{O}(K)$.

As a corollary of this, we obtain a classification of primitive sublattices of unimodular lattices:

Corollary 0.8.4 *Let M' be an unimodular lattice. Then, primitive embeddings of $M \to M'$ with orthogonal complement isomorphic to a fixed lattice K are in bijection with isomorphisms $\gamma : D(M) \to D(K)$, such that*

$$q_{D(M)}(x) = -q_{D(K)}(\gamma(x)). \tag{0.8.3}$$

For a lattice M, let

$$\rho_M : \mathrm{O}(M) \to \mathrm{O}(D(M)) \tag{0.8.4}$$

be the natural homomorphism of orthogonal groups. We will often denote the image $\rho_M(\sigma)$ by $\bar{\sigma}$.

Suppose we have $\phi \in \mathrm{O}(M)$ and $\psi \in \mathrm{O}(K)$ such that $\bar{\phi} = \gamma(\bar{\psi})$, where γ is as in (0.8.3). Then, Corollary 0.8.3 implies that the pair (ϕ, ψ) extends to an isometry of M'. In particular, taking $\psi = \mathrm{id}_K$, we obtain that any isometry from

$$\mathrm{O}(M)^\sharp := \mathrm{Ker}(\rho_M)$$

extends to an isometry of M', which is the identity on K.

The following, a little more general proposition, will be used often in our discussion of moduli spaces and automorphism groups of Enriques surfaces.

Proposition 0.8.5 *Let M_1, M_2 be two primitive sublattices of M' and let $K_i := M_i^\perp$ be their orthogonal complements. Then an isometry $\phi : M_1 \to M_2$ extends to an isometry of M' if and only if there exists an isometry of $\psi : K_1 \to K_2$ with $\gamma_1 \circ \bar{\phi} = \bar{\psi} \circ \gamma_2$.*

The previous discussion about the discriminant group $D(M)$ and its quadratic form $q_{D(M)}$ plays a prominent role in Nikulin's work on primitive embeddings of lattices [556].

For a finite abelian group A, we denote by $l(A)$ the minimal number of generators of A. We now come to the following useful result that characterizes many indefinite lattices and their embeddings into unimodular lattices.

Theorem 0.8.6 *Let M be an even indefinite nondegenerate quadratic lattice of signature (t_+, t_-).*

1. *Suppose that M satisfies the following conditions*
   - *For every prime $p \neq 2$, we have $\mathrm{rank}(M) \geq l(D(M \otimes \mathbb{Z}_p)) + 2$,*
   - *if $\mathrm{rank}(M) \neq l(D(M \otimes \mathbb{Z}_2))$, then $(D(M), q_{D(M)}) \cong \mathsf{u}_2 \oplus q'$ or $(D(M), q_{D(M)}) \cong \mathsf{v}_2 \oplus q'$.*

Then the signature, the rank and the discriminant quadratic form determine uniquely the isomorphism class of M and the homomorphism (0.8.4) *is surjective, see [556, Theorem 1.14.2].*

2. *Let L be an unimodular lattice of signature (l_+, l_-) and M be a lattice as in 1 satisfying both conditions. Suppose*
   - $a_+ := l_+ > t_+$ *and* $a_- := l_- - t_- > 0$,
   - *for every prime $p \neq 2$, we have* $\operatorname{rank} L - \operatorname{rank} M \geq l(D(M \otimes \mathbb{Z}_p)) + 2$,
   - *if* $\operatorname{rank} L - \operatorname{rank} M = l(D(M \otimes \mathbb{Z}_2))$, *then the discriminant group of M satisfies the second of condition 1.*

   Then M admits a primitive embedding into L and all such embeddings differ by an automorphism of $\mathrm{O}(L)$, *see [556, Corollary 1.12.4 and Theorem 1.14.4].*

We now discuss some lattices that will be important later on explicitly. For an integer $n \in \mathbb{Z}$ and a lattice (M, b), we define the following lattices and subsets:

- $\langle n \rangle$: a lattice of rank one generated by an element e with $e^2 = n$.
- $M(n)$: the lattice with underlying abelian group M and bilinear form $(x, y) \mapsto n \cdot b(x, y)$.
- $M_n = \{x \in M : x^2 = n\}$. Elements of M_0 are called *isotropic vectors*.
- M'_n: the subset of primitive vectors of M_n.
- $M^{\oplus n}$: the lattice of rank $n \cdot \operatorname{rank} M$ that is the orthogonal sum of n copies of M.

Example 0.8.7 We will be dealing with the following series of lattices in the sequel.

1. For integers $m, n \geq 0$,
$$\mathsf{I}^{m,n} := \langle 1 \rangle^m \perp \langle -1 \rangle^n$$
is an unimodular lattice of rank $(m+n)$, but this lattice is not even.
2. For integers $1 \leq p \leq q \leq r$, we define the graph $T_{p,q,r}$ as in Figure 0.3.

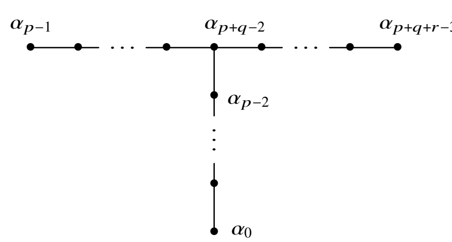


Fig. 0.3 The graph $T_{p,q,r}$

Associated to $T_{p,q,r}$, we define a lattice of rank $(p+q+r-2)$ by setting
$$\mathsf{E}_{p,q,r} := \mathbb{Z}\alpha_0 \oplus \ldots \oplus \mathbb{Z}\alpha_{p+q+r-3}$$
with $\alpha_i^2 = -2$ and with $\alpha_i \cdot \alpha_j = 1$ or 0 for $i \neq j$ depending on whether or not α_i is joined to α_j in the graph $T_{p,q,r}$. We note that the $\mathsf{E}_{p,q,r}$ are even lattices,

whose discriminants satisfy

$$\mathrm{discr}(\mathsf{E}_{p,q,r}) = |pqr - pq - qr - pr|. \tag{0.8.5}$$

3.
$$\mathrm{sign}(\mathsf{E}_{p,q,r}) = \begin{cases} (1, p+q+r-3) & \text{if } \frac{1}{p}+\frac{1}{q}+\frac{1}{r} < 1, \\ (0, p+q+r-2) & \text{if } \frac{1}{p}+\frac{1}{q}+\frac{1}{r} > 1, \\ (0, p+q+r-3, 1) & \text{otherwise.} \end{cases}$$

4. The non-hyperbolic lattices $\mathsf{E}_{p,q,r}$ are as follows, and they have their own notation:

$\mathsf{A}_n := \mathsf{E}_{1,1,n}$, $\mathsf{D}_n := \mathsf{E}_{2,2,n-2} (n \geq 4)$, $\mathsf{E}_6 := \mathsf{E}_{2,3,3}$, $\mathsf{E}_7 := \mathsf{E}_{2,3,4}$, $\mathsf{E}_8 := \mathsf{E}_{2,3,5}$,
$\widetilde{\mathsf{E}}_6 := \mathsf{E}_{3,3,3}$, $\widetilde{\mathsf{E}}_7 := \mathsf{E}_{2,4,4}$, $\widetilde{\mathsf{E}}_8 := \mathsf{E}_{2,3,6}$.

The lattices $\widetilde{\mathsf{E}}_6, \widetilde{\mathsf{E}}_7, \widetilde{\mathsf{E}}_8$ are negative semi-definite and their radicals are of rank 1.
The lattices $\mathsf{A}_n, \mathsf{D}_n, \mathsf{E}_6, \mathsf{E}_7, \mathsf{E}_8$ are even negative definite.
not $\mathsf{A}_n, \mathsf{D}_n, \mathsf{E}_6, \mathsf{E}_7, \mathsf{E}_8$
5. For an integer $n \geq 2$, we define the lattice

$$\mathsf{U}_{[n]} := \mathbb{Z}f_1 \oplus \cdots \oplus \mathbb{Z}f_n$$

with $f_i \cdot f_j = 1 - \delta_{ij}$, where δ_{ij} is the Kronecker symbol. We note that these lattices are even. When $n = 2$, this lattice is called the *standard hyperbolic plane* and it is also denoted by U or H. We note that the lattices $\mathsf{U}_{[n]}$ are all hyperbolic and that they contain a negative definite sublattice spanned by the vectors $f_i - f_{i+1}$, which is isomorphic to the lattice A_{n-1}, and whose orthogonal complement is spanned by $v := f_1 + \cdots + f_n$ and satisfies $v^2 = n(n-1)$.

The discriminant quadratic forms on the lattice $\mathsf{A}_n, \mathsf{D}_n, \mathsf{E}_n$ are given in Table 0.2 below.

Here, we denote by $\langle a \rangle_n$ the quadratic form on $\mathbb{Z}/n\mathbb{Z}$ defined by $q(1) = a \mod 2\mathbb{Z}$.

We refer to [95] and [514] for the computation of the discriminant quadratic forms of the hyperbolic lattices $\mathsf{E}_{p,q,r}$.

The next theorem is usually referred to as the *Milnor Theorem*. indexlattice!Milnor Theorem

Theorem 0.8.8 *Let M be a unimodular and indefinite lattice of signature (a, b) with $a \leq b$. Then, there exists an isometry*

$$M \cong \begin{cases} \mathsf{U}^a \perp \mathsf{E}_8^{b-a} & \textit{if } M \textit{ is an even lattice, and} \\ \langle 1 \rangle^a \perp \langle -1 \rangle^b & \textit{otherwise.} \end{cases}$$

Proof See [661], Chapter 5, Theorem 5. □

Let M be a lattice. Then, a primitive vector $\alpha \in M$ with $\alpha^2 \neq 0$ is called a *root vector* or just a *root* if

Lattice	Discriminant group	Discriminant form
A_n	$\mathbb{Z}/(n+1)\mathbb{Z}$	$\langle -n\rangle_{n+1}$
D_{8k}	$(\mathbb{Z}/2\mathbb{Z})^{\oplus 2}$	u_1
D_{8k+1}	$\mathbb{Z}/4\mathbb{Z}$	$\mathsf{w}_{2,2}^{-1}$
D_{8k+2}	$(\mathbb{Z}/2\mathbb{Z})^{\oplus 2}$	$(\mathsf{w}_{2,1}^{-1})^{\oplus 2}$
D_{8k+3}	$\mathbb{Z}/4\mathbb{Z}$	$\mathsf{w}_{2,2}^{-3}$
D_{8k+4}	$(\mathbb{Z}/2\mathbb{Z})^{\oplus 2}$	v_1
D_{8k+5}	$\mathbb{Z}/4\mathbb{Z}$	$\mathsf{w}_{2,2}^{-5}$
D_{8k+6}	$(\mathbb{Z}/2\mathbb{Z})^{\oplus 2}$	$(\mathsf{w}_{2,1}^{1})^{\oplus 2}$
D_{8k+7}	$\mathbb{Z}/4\mathbb{Z}$	$\mathsf{w}_{2,2}^{1}$
E_6	$\mathbb{Z}/3\mathbb{Z}$	$\mathsf{w}_{3,1}^{-1}$
E_7	$\mathbb{Z}/2\mathbb{Z}$	$\mathsf{w}_{1,1}^{1}$
E_8	0	0

Table 0.2 Discriminant quadratic forms of lattices $\mathsf{A}_n, \mathsf{D}_n, \mathsf{E}_n$

$$\frac{2\alpha \cdot x}{\alpha^2} \in \mathbb{Z}. \tag{0.8.6}$$

for every $x \in M$. A root vector defines an isometry of M via

$$r_\alpha : x \mapsto x - \frac{2x \cdot \alpha}{\alpha^2}\alpha, \tag{0.8.7}$$

which is called the *reflection* in the root vector α. It follows from (0.8.6) that

$$\check{\alpha} := \frac{2}{\alpha^2}\alpha \in M^\vee.$$

Since $\check{\alpha}^2 = \frac{4}{\alpha^2}$, we conclude that $\check{\alpha} \in M$ if and only if $|\alpha^2| \in \{1, 2, 4\}$. We note that $|\alpha^2| = 1$ cannot occur if M is an even lattice. Moreover, if $\check{\alpha} \in M$, then $\frac{1}{2}\alpha^2$ must divide the discriminant of M. This implies, for example, that in a unimodular lattice, all root vectors satisfy $|\alpha^2| \in \{1, 2\}$.

Assume that M is either a negative definite lattice, or a negative semi-definite lattice whose radical of rank 1, or a hyperbolic lattice of rank $n + 1$. Let

$$V := M_\mathbb{R} = M \otimes_\mathbb{Z} \mathbb{R}.$$

If M is definite, then the orthogonal group $\mathrm{O}(V)$ of V is isomorphic to the orthogonal group $\mathrm{O}(n+1)$ of the standard inner-product space $\mathbb{R}^{n+1}$. In this case, $\mathrm{O}(V)$ is realized as the group of isometries or *motions* of the n-dimensional sphere $\mathbb{S}^n \subset \mathbb{R}^{n+1}$, considered as a Riemannian manifold of constant positive curvature. In the semi-definite case, $\mathrm{O}(V)$ is equal to the semi-direct product $(V_0^\perp/V_0)^\vee \rtimes \mathrm{O}(V_0^\perp/V_0)$, where V_0 denotes the radical of V. More precisely, the subgroup $(V_0^\perp/V_0)^\vee$ of $\mathrm{O}(V)$ is equal to the image of the homomorphism $\iota : (V_0^\perp/V_0)^\vee \to \mathrm{O}(V)$ that is defined by the formula

$$l \mapsto \imath(l) : v \mapsto v + l(\bar{v}) \cdot f, \tag{0.8.8}$$

for $v \in V$ and $l \in (V_0^\perp/V_0)^\vee$, where $V_0 := \mathbb{R}f$ and where $\bar{v}$ denotes image of v under the projection of $V_0^\perp \to (V_0^\perp/V_0)$. In particular, this group is isomorphic to the *affine orthogonal group* $\mathrm{AO}(n-1) = \mathbb{R}^{n-1} \rtimes \mathrm{O}(n-1)$ of the Euclidean space $\mathbb{E}^{n-1}$. It is isomorphic to the group of motions of $\mathbb{E}^{n-1}$, considered as a Riemannian manifold of constant zero curvature.

Let us finally assume that M is hyperbolic. Then, the orthogonal group $\mathrm{O}(V)$ contains a natural subgroup $\mathrm{O}(V)'$ of index 2 that consists of those isometries that leave invariant one of the two connected components of the *positive cone*

$$V^+ := \{v \in V : v^2 > 0\}.$$

More precisely, let $(e_0, e_1, \ldots, e_n)$ be an orthonormal basis of V so that the quadratic form of V is given by $x_0^2 - x_1^2 - \cdots - x_n^2$ with respect to this basis. Next, let V_0^+ be the connected component of V^+ that is determined by $x_0 > 0$. Then, a reflection $r_v \in \mathrm{O}(V)$ belongs to $\mathrm{O}(V)'$ if and only if $v^2 < 0$. We note that the orthogonal group $\mathrm{O}(V)$ of any non-degenerate quadratic vector space is generated by reflections in vectors $v \in V$, see, for example, [279].

Let $V \backslash \{0\} \to \mathbb{P}(V)$ be the natural projection onto the real projective space $\mathbb{P}(V)$ of lines in V, and let $\mathbb{H}(V)$ be the image of V^+ in $\mathbb{P}(V)$, that is,

$$\mathbb{H}(V) := \left\{v \in V : v^2 > 0\right\} / \mathbb{R}^* \subset \mathbb{P}(V).$$

Thus, we can represent points of $\mathbb{H}(V)$ by vectors $v \in V$ such that $v^2 = 1$ and such that v belongs to the fixed connected component V_0^+ of V^+. The hyperbolic inner product in V induces a structure of a Riemannian manifold of constant negative curvature on $\mathbb{H}(V)$. Equipped with this Riemannian metric, $\mathbb{H}(V)$ is called *hyperbolic space* or *Lobachevsky space*. After fixing an orthonormal basis in V as above, we may identify $\mathbb{H}(V)$ with the space

$$\mathbb{H}^n := \left\{ x = (x_0, x_1, \ldots, x_n) \in \mathbb{R}^{n+1} : x_0^2 - \sum_{i=1}^{n} x_i^2 = 1,\, x_0 > 0 \right\} \subseteq \mathbb{R}^{n+1}.$$

The hyperbolic distance $d(x, y)$ is given by the formula

$$\cosh\, d(x, y) = x \cdot y. \tag{0.8.9}$$

Another model of $\mathbb{H}^n$ is given by passing to affine coordinates $y_i = x_i/x_0$ in $\mathbb{P}^n(\mathbb{R})$. Then, we find

$$\mathbb{H}^n := \left\{ y = (y_1, \ldots, y_n) \in \mathbb{R}^n : y_1^2 + \cdots + y_n^2 < 1 \right\} \subseteq \mathbb{R}^n.$$

The hyperbolic distance in this model is different, namely $d(y, y') = \log |R(a, x, y, b)|$, where $R(a, x, y, b)$ denotes the cross-ratio of four points on the line joining the points y, y', where a, b are the points where the line intersects the *absolute*

$$\partial\mathbb{H}^n := \left\{y = (y_1, \dots, y_n) \in \mathbb{R}^n : y_1^2 + \cdots + y_n^2 = 1\right\} = \mathbb{S}^{n-1} \subseteq \mathbb{R}^n.$$

The group of isometries of the hyperbolic space $\mathbb{H}^n$ is isomorphic to $\mathrm{O}(V)'$. It is also isomorphic to the group $\mathrm{PO}(V)$ of projective transformations of $\mathbb{P}(V)$ that preserve the quadric $q_V = 0$. We note that this group is a Lie group that is not connected. Its connected component of the identity is a subgroup $\mathrm{O}(V)_0'$ of $\mathrm{O}(V)'$ of index 2 that consists of orientation–preserving isometries of $\mathbb{H}^n$. It consists of those orthogonal transformations of V that can be written as the product of an even number of rotations in vectors v with $v^2 < 0$. For these basic facts about hyperbolic spaces, we refer to [4].

Let M a quadratic lattice of signature (t_+, t_-, t_0) as above and set $V := M_{\mathbb{R}} = M \otimes_{\mathbb{Z}} \mathbb{R}$. If M is hyperbolic, then we set

$$\mathrm{O}(M)' := \mathrm{O}(M) \cap \mathrm{O}(V)',$$

and otherwise, we set $\mathrm{O}(M)' := \mathrm{O}(M)$. Then, after a suitable choice of orthonormal basis, the subgroup $\mathrm{O}(M)'$ of $\mathrm{O}(V)$ is isomorphic to a *discrete group* of motions of $X = \mathbb{S}^n, \mathbb{E}^{n-1}, \mathbb{H}^n$, that is, each orbit of the group is a discrete subset of X and each stabilizer subgroup is finite. Quite generally, let Γ be a discrete subgroup of motions of X that is generated by a set I (not necessary finite) of reflections $r_{v_i}, i \in I$, in vectors v_i with $v_i^2 < 0$. Let H_i be the hyperplane in V of vectors that are orthogonal to v_i. Since $v_i^2 < 0$, the intersection of H_i with any of the Riemannian manifolds $X = \mathbb{S}^n, \mathbb{E}^{n-1}, \mathbb{H}^n$ is non-empty. By abuse of terminology, we call them *hyperplanes* in X and keep the notation. The set of hyperplanes $H_{v_i}, i \in I$, is locally finite and invariant with respect to Γ. The closure of a connected component P of the complement of the union of the hyperplanes H_{v_i} is called a *chamber* of Γ. For every chamber P there exists a smallest subset $I(P) \subseteq I$ (in particular, no vector in $I(P)$ is a positive linear combination of other vectors in $I(P)$) such that

$$P = \bigcap_{i \in I(P)} H_{v_i}^+, \tag{0.8.10}$$

where $H_{v_i}^+ = \{x \in V : x \cdot v_i \geq 0\} \cap X$. The group Γ permutes the chambers and the interior of each chamber is a (closed) fundamental domain for the action of Γ on X. It is clear that Γ is generated by the reflections $s_i := r_{v_i}$ with $i \in I(P)$. For $i \in I(P)$, we let $e_i = \frac{1}{\sqrt{-v_i^2}} v_i$ be the normalized vectors of norm -1. It follows from the discreteness of Γ that we have either $e_i \cdot e_j > 1$ or

$$e_i \cdot e_j = \cos \frac{\pi}{m_{ij}}, \tag{0.8.11}$$

where $m_{ij} \in \mathbb{Z}_{>0} \cup \{\infty\}$. If $e_i \cdot e_j \geq 1$, then we set $m_{ij} := \infty$, see [724]. The group Γ is generated by reflections $s_i, i \in I(P)$ and these satisfy the relation $(s_i s_j)^{m_{ij}} = 1$, that is, Γ is isomorphic to a Coxeter group.

Let us briefly recall this definition: an abstract *Coxeter group* is a pair (G, S) that consists of a group G and its set of generators S subject to defining relations

$(ss')^{m_{s,s'}} = 1$ for all $s, s' \in S$. Here, we have $m_{s,s'} \in \mathbb{Z}_{>0} \cup \{\infty\}$, such that $m_{s,s} = 1$ and $m_{s,s'} = m_{s',s}$ for all $s, s' \in S$, and finally, $m_{s,s'} = \infty$ means that there is no relation between s and s'. Note that $m_{s,s} = 1$ implies $s^2 = 1$ for all $s \in S$. We refer to [88] for details and proofs. Associated to (G, S), we have its *Coxeter–Dynkin diagram*, which is defined to be the unoriented graph, whose vertices are given by the set S. Two vertices s, s' are joined by an edge labeled by $(m_{s,s'} - 2)$ if $m(s, s') > 2$ and it is customary to omit the label if $m(s, s') = 3$. If $m_{s,s'} \in \{4, 5\}$, then one doubles or triples the edges instead of labeling them. Conversely, a Coxeter–Dynkin diagram uniquely determines the data (G, S) of a Coxeter group.

Note that for any discrete group Γ of motions of X, we have

$$\Gamma = \Gamma_r(\mathcal{R}) \rtimes \mathrm{Sym}(P), \tag{0.8.12}$$

where $\Gamma_r(\mathcal{R})$ is the subgroup of Γ generated by a set $\mathcal{R}$ of reflections from Γ, which is invariant with respect to inner automorphisms of Γ, and where $\mathrm{Sym}(P)$ is the group of motions in Γ that leaves invariant a chamber P of $\Gamma_r(\mathcal{R})$, see [721], 1.5.

Now, let $V = M_{\mathbb{R}} = M \otimes_{\mathbb{Z}} \mathbb{R}$ and let Γ be a subgroup of $\mathrm{O}(M)'$ that is generated by reflections r_{α_i}, where the α_i belong to a set I of root vectors in M. Since $\mathrm{O}(M)'$ is a discrete group of motions of X, it follows that Γ is a discrete reflection group of motions. Let P be a chamber defined by the subset $I(P) \subseteq I$ of root vectors. It follows from (0.8.11) that

$$\frac{\alpha \cdot \beta}{\sqrt{-\alpha^2}\sqrt{-\beta^2}} = \cos \frac{\pi}{m_{\alpha,\beta}}, \tag{0.8.13}$$

for every $\alpha, \beta \in I(P)$ for some $m_{\alpha,\beta} \in \mathbb{Z}_{>0} \cup \{\infty\}$. In particular, let $\mathrm{Ref}(M)$ be the subgroup of $\mathrm{O}(M)'$ generated by reflections in all root vectors in M and let $\mathrm{Ref}_k(M)$ be its normal subgroup generated by reflections in root vectors $\alpha \in M$ with $\alpha^2 = -k$. We call $\mathrm{Ref}(M)$ (resp. $\mathrm{Ref}_k(M)$) the *reflection group* (resp. *k-reflection group*) of the lattice M. We have

$$\cos \frac{\pi}{m_{\alpha,\beta}} \in \frac{1}{k}\mathbb{Z}$$

for every $r_\alpha, r_\beta \in \mathrm{Ref}_k(M)$. This gives strong restrictions on the Coxeter group. For example, if $k = 2$, then we have $2\cos\frac{\pi}{m_{\alpha,\beta}} \in \mathbb{Z}$, which implies that $m_{\alpha,\beta}$ is one of the following

$$m_{\alpha,\beta} = \begin{cases} 1 & \text{if } \alpha = \beta, \\ 2 & \text{if } \alpha \cdot \beta = 0, \\ 3 & \text{if } \alpha \cdot \beta = 1, \\ \infty & \text{if } \alpha \cdot \beta = 2. \end{cases}$$

In particular, the only possible label occurring in its Coxeter–Dynkin graph is ∞. We call the reflection group $\mathrm{Ref}_2(M)$ the *Weyl group* of the lattice M and denote it by $W(M)$.

Let P be a chamber of a reflection group Γ of M. We denote the set of vectors defining the reflections from $I(P)$ by B_P or just B, if no confusion arises, and call it a *root basis* of M. Note that we do not assume that the set B spans the $\mathbb{R}$-vector space $M_{\mathbb{R}}$. For two distinct roots $\alpha, \beta \in \mathsf{B}$ that span a negative definite sublattice of M, we have

$$\check{\alpha}(\beta) = 2 \cdot \sqrt{\frac{\beta^2}{\alpha^2}} \cdot \cos \frac{\pi}{m_{\alpha,\beta}} \in \mathbb{Z}_{>0},$$

and no vector in B is a positive linear combination of other vectors in B. In particular, if $\alpha, \beta \in M_{-2}$, then this implies $\alpha \cdot \beta \in \{0, 1, 2\}$ or $|\alpha \cdot \beta| > 2$. Moreover, the reflections r_α, $\alpha \in \mathsf{B}$ give rise to a root basis of the reflection group Γ_{B}. Its chamber $P(\mathsf{B})$ in X is the image of the subset

$$C(\mathsf{B}) = \{x \in M_{\mathbb{R}} : x \cdot \alpha \geq 0,\ \alpha \in \mathsf{B}\} \subseteq M_{\mathbb{R}} = V \qquad (0.8.14)$$

in X. We call the subset $C(\mathsf{B})$ the *fundamental chamber* of the root basis B in $M_{\mathbb{R}}$.

Example 0.8.9 For the lattice $M = \mathsf{E}_{p,q,r}$, the reflection group generated by the reflections $r_{\alpha_i}, i = 0, \ldots, N = p + q + r - 3$, is a Weyl group, which is denoted by $W_{p,q,r}$, and we can take the set $\{\alpha_0, \ldots, \alpha_N\}$ as a root basis. Its Coxeter–Dynkin diagram is the graph $T_{p,q,r}$. A root basis in $\mathsf{E}_{p,q,r}$ with Coxeter–Dynkin diagram of form $T_{p,q,r}$ is called a *canonical root basis* of $\mathsf{E}_{p,q,r}$. It is known that $W_{p,q,r}$ is a proper subgroup of the Weyl group $W(\mathsf{E}_{p,q,r})$ if $\mathsf{E}_{p,q,r}$ is a hyperbolic lattice and $(p, q, r) \neq (2, 3, 7), (2, 4, 6), (3, 3, 4)$, see also below.

Next, the Coxeter–Dynkin diagram of a reflection group Γ in $\mathrm{O}(V)'$ is not connected if and only if for a chamber P, the set $I(P)$ is equal to the union of two subsets I_1 and I_2 such that $e_i \cdot e_j = 0$ if $i \in I_1$, $j \in I_2$. In this case, we have $\Gamma = \Gamma_1 \times \Gamma_2$, where Γ_i is the reflection group generated by reflections r_{e_j} with $j \in I_i$.

Given a Coxeter group (G, S) and a non-empty subset $S' \subseteq S$, the subgroup $G' = G(S')$ of G generated by S' is again a Coxeter group and, in fact, equal to (G', S'), see [88, Chapter IV, §1, Theorem 2]. If S is the disjoint union of the two subsets S_1 and S_2 and we have $[s_1, s_2] = (s_1 s_2)^2 = 1$ for all $s_1 \in S_1$ and all $s_2 \in S_2$, then $G = G(S_1) \times G(S_2)$. Moreover, such a disjoint decomposition $S = S_1 \cup S_2$ exists if and only if the Coxeter–Dynkin diagram of (G, S) is not connected. A Coxeter group is called *irreducible* if its Coxeter–Dynkin diagram is connected. In the sequel, we will also need the following generalization of this observation.

Proposition 0.8.10 *Let (G, S) be a Coxeter group and assume that there exists a non-trivial disjoint decomposition $S = S_1 \cup S_2$ such that $m(s_1, s_2)$ is even or infinite for all $s_1 \in S_1$ and all $s_2 \in S_2$. Then,*

$$G = N \rtimes G_1,$$

where G_1 is the subgroup generated by S_1 and where N is the smallest normal subgroup of G containing S_2.

Proof See [721, Proposition, p. 2]. □

Let Γ be a reflection group of M and let $\mathcal{R}(\Gamma)$ be the set of root vectors $\alpha \in M$ with $r_\alpha \in \Gamma$. Elements of $\mathcal{R}(\Gamma)$ are called *root vectors* of Γ. If B is a root basis of

Γ, then every root vector $\alpha \in \mathcal{R}(\Gamma)$ belongs to the orbit of a root vector from B. In particular, if B is a finite set, then there are only finitely many orbits of Γ in $\mathcal{R}(\Gamma)$.

Corollary 0.8.11 *Let W_{B} be the Weyl group of a root basis $\mathsf{B} \subseteq M_{-2}$ and let $\mathcal{G}(\mathsf{B})$ be the associated Coxeter–Dynkin diagram. If $\mathcal{G}(\mathsf{B})'$ denotes the subgraph of $\mathcal{G}(\mathsf{B})$ that is obtained by removing all labelled edges from $\mathcal{G}(\mathsf{B})$, then there exists a bijection*

$$\{ \textit{ orbits of } W_{\mathsf{B}} \textit{ on the set } \mathcal{R}(W_{\mathsf{B}}), \mathcal{G}(\mathsf{B}), \Gamma(\mathsf{B})\} \leftrightarrow \{ \textit{ connected components of } \mathcal{G}(\mathsf{B})'\}.$$

Proof As we observed before, every root vector $\alpha \in \mathcal{R}(W_{\mathsf{B}})$ is equal to $w(\alpha_i)$ for some $w \in W_{\mathsf{B}}$ and some $\alpha_i \in \mathsf{B}$. Thus, the reflection r_α in α is conjugate to the reflection r_{α_i}. Conversely, the conjugate $w \circ s_{\alpha_i} \circ w^{-1}$ of a reflection in $\alpha_i \in \mathsf{B}$ is the simple reflection r_α in the root $\alpha = w(\alpha_i)$. Thus, the set of W_{B}-orbits of roots is bijective to the set of conjugacy classes of generators of the Coxeter group $(W_{\mathsf{B}}, \{r_\alpha, \alpha \in \mathsf{B}\})$. Now, let $\Gamma(\mathsf{B})'^{o}$ be a connected component of $\Gamma(\mathsf{B})'$ and set $S_2 := \{r_\alpha : \alpha \in \Gamma(\mathsf{B})'^{o}\}$ as well as $S_1 = S \setminus S_2$. It follows from the proposition that none of the roots in $\Gamma(\mathsf{B})'^{o}$ is W_{B}-conjugate to a root in $\Gamma(\mathsf{B}) \setminus \Gamma(\mathsf{B})'^{o}$. On the other hand, if $\alpha, \beta \in \mathsf{B}$ with $\alpha \cdot \beta = 1$, then $r_\beta \circ r_\alpha(\beta) = r_\beta(\beta + \alpha) = \alpha$. From this, it follows that the vertices of every connected component of $\Gamma(\mathsf{B})'$ are W_{B}-conjugate. This proves the corollary. □

Let (G, S) be a Coxeter group. Let $L := \mathbb{R}^{|S|}$ be the $\mathbb{R}$-vector space with basis $\{e_s\}_{s\in S}$ and let b_L be the symmetric bilinear form on L defined by

$$b_L(e_s, e_{s'}) = \cos \frac{\pi}{m_{s,s'}}.$$

The *geometric representation* or *Tits representation* is the homomorphism $\rho : G \to \mathrm{O}(L)$ defined on the basis $\{e_s\}_{s\in S}$ by

$$\rho(s)(v) = v + 2\, b_L(v, e_s) \cdot e_s,$$

for $s \in S \subseteq G$ and $v \in L$, see [88, Chapter V, §4]. (Here, we intentionally multiplied the bilinear form by -1 compared to the original definition.) This representation is faithful and hence, realizes G as a reflection group in L. We say that (G, S) is *spherical*, *Euclidean*, or *hyperbolic* if the symmetric bilinear form (x, y) in L is positive definite, positive semi-definite with one-dimensional radical, or of signature $(n, 1)$ with $n \geq 1$. In these cases, we can consider G as a discrete reflection group of isometries of $X = \mathbb{S}^n$, $\mathbb{E}^{n-1}$, or $\mathbb{H}^n$. Its chambers are simplicial convex polyhedra. One of the chambers is given by inequalities $b_L(x, e_s) \geq 0$ for all $s \in S$. Conversely, given a discrete reflection group of isometries of X with simplicial chambers, its representation in the corresponding linear space is isomorphic to the Tits representation.

Example 0.8.12 The Weyl groups $W_{p,q,r}$ from Example 0.8.9 together with generators defined by reflections in a canonical root basis acting in the space $(\mathsf{E}_{p,q,r})_{\mathbb{R}}$ is an example of a Tits representation. It is spherical, Euclidean, hyperbolic depending on whether $\frac{1}{p} + \frac{1}{q} + \frac{1}{r} > 1, 0, < 1$, respectively.

Given a Coxeter group (G, S), one defines the *length function* $l : G \to \mathbb{Z}_{\geq 0}$ of G with respect to S as follows: given $g \in G$, we define $l(g)$ to be the smallest non-negative number r such that g can be written as the product of r elements from S. By definition, we have $l(1) = 0$ and we have $l(g) = 1$ if and only if $g \in S$.

Proposition 0.8.13 *Let Γ be a discrete reflection group of isometries of $X = \mathbb{S}^n$, $\mathbb{E}^{n-1}$, or $\mathbb{H}^n$. Let P be a chamber and $I(P)$ be the set of vectors e_i of norm -1 such that $P = \cap H^+_{e_i}$ as in* (0.8.10). *We consider Γ as a Coxeter group with generators the reflection $s_i = r_{e_i}$ in vectors e_i. Then, the following property holds:*

(T) For every $\gamma \in \Gamma$ and $i \in I$, either $\gamma(P) \subseteq H^+_i$ or $\gamma(P) \subseteq s_i(H^+_i)$ holds true.

Moreover, we have $l(s_i\gamma) = l(\gamma) - 1$ in the second case.

Proof If P is simplicial, then Γ acts via a Tits representation and then, property (T) is proven in [88, Chapter V, §4.8]. The proof is based on a lemma that says that it is enough to check property (T) for a subgroup generated by two reflections (this is referred to as Tits' Lemma in [148, 4.8]). In our situation, let $\langle e_i, e_j \rangle$ be the subspace spanned by two vectors e_i, e_j and $(s_i s_j)^{m_{ij}} = 1$ is a Coxeter relation. After normalizing, we may assume that $e_i \cdot e_j = -1$. Then, the Gram matrix of the basis (e_i, e_j) is equal to

$$\begin{pmatrix} -1 & \cos\frac{\pi}{m_{ij}} \\ \cos\frac{\pi}{m_{ij}} & -1 \end{pmatrix}.$$

The action of Γ in this space is a Tits representation of dimension 2 and for this, property (T) holds. □

Corollary 0.8.14 *Let Γ be a reflection group of a lattice M that is negative definite, semi-negative definite, or hyperbolic. Let B be a root basis of Γ and let $C(\mathsf{B})$ be its fundamental chamber. For every $x \in M_{\mathbb{R}}$, there exists a $\gamma \in \Gamma$ such that*

$$\gamma(x) \in C(\mathsf{B}) \qquad \textit{and} \qquad x = \gamma(x) + \sum_{\alpha \in \mathsf{B}} m_\alpha \alpha \quad \textit{with } m_\alpha \in \mathbb{Z}_{\geq 0}.$$

Proof Let $\gamma \in \Gamma$ be such that $\gamma(x) \in C(\mathsf{B})$. Let $k := l(\gamma)$ and note that the case $k = 0$ is trivial. If $k \geq 1$, then there exist $\alpha_1, \ldots, \alpha_k \in \mathsf{B}$ such that $\gamma = r_{\alpha_1} \circ \ldots \circ r_{\alpha_k}$. Since $l(r_{\alpha_k} \circ \gamma^{-1}) = l(\gamma^{-1}) - 1$, the previous proposition implies

$$r_{\alpha_k}(x) = r_{\alpha_k}\left(\gamma^{-1}(\gamma(x))\right) \in r_{\alpha_k}\left(r_{\alpha_k}(H^+_{\alpha_k})\right) = H^+_{\alpha_k},$$

where

$$H^+_{\alpha_k} = \{x \in M_{\mathbb{R}} \,:\, x \cdot \alpha_k \geq 0\}. \tag{0.8.15}$$

Thus, $r_{\alpha_k}(x) \cdot \alpha_k > 0$ and hence, $x \cdot \alpha_k < 0$ and $r_{\alpha_k}(x) = x - m_k \alpha_k$ for some $m_k \geq 0$. The statement now follows from induction on k. □

Given a reflection group Γ of a lattice M with root basis B and fundamental chamber $C(\mathsf{B})$, we will say that a root $\alpha \in \mathcal{R}(\Gamma)$ is *positive* with respect to B if $\alpha \cdot x \geq 0$ for all $x \in C(\mathsf{B})$. Using the notation from (0.8.15), a root α is positive if and only if $C(\mathsf{B}) \subseteq H_\alpha^+$. It is clear that $r_\alpha(H_\alpha^+) = H_{-\alpha}^+ = H_\alpha^-$. It then follows from Proposition 0.8.13 applied to $\gamma = s_i = r_\alpha$ that either α or $-\alpha$ is a positive root. Thus, denoting by $\mathcal{R}(\Gamma)^+$ the set of positive roots, we obtain a disjoint union of the roots

$$\mathcal{R}(\Gamma) = \mathcal{R}(\Gamma)^+ \coprod \mathcal{R}(\Gamma)^-,$$

where $\mathcal{R}(\Gamma)^- := \{-\alpha : \alpha \in \mathcal{R}(\mathsf{B})^+\}$ are the *negative* roots. In the special case, where $\mathsf{B} = \{\alpha_1, \dots, \alpha_r\}$ is a basis of the $\mathbb{R}$-vector space $M_\mathbb{R}$, we denote by $\{\alpha_1^*, \dots, \alpha_r^*\}$ the dual basis of $(M_\mathbb{R})^*$. Using the isomorphism $M_\mathbb{R} \cong M_\mathbb{R}^*$ that is given by $v \mapsto \langle -, v \rangle$, we find

$$C(\mathsf{B}) = \mathbb{R}_{\geq 0}\, \alpha_1^* + \cdots + \mathbb{R}_{\geq 0}\, \alpha_r^*,$$

In particular, we find

$$\mathcal{R}(\Gamma)^+ = \mathcal{R}(\Gamma) \cap (\mathbb{Z}_{\geq 0}\alpha_1 + \cdots + \mathbb{Z}_{\geq 0}\alpha_r).$$

Let M be a lattice and let $\mathsf{B} \subseteq M$ be a root basis of its reflection group Γ. We denote by M_B the sublattice of M spanned by the subset B. A root basis B is said to be of *finite type* (resp. *affine type*) if the sublattice M_B is negative definite (resp. negative semi-definite). The following two results give a complete classification of root bases of finite type and of affine type, respectively.

Proposition 0.8.15 *Let Γ be an irreducible reflection group of an even lattice M and let B be a root basis of Γ. Then, the following statements are equivalent:*

1. *B is of finite type.*
2. *Γ is finite and coincides with the reflection group of M_B.*
3. *$M_\mathsf{B} \cong \mathsf{E}_{p,q,r}(k)$ for some $k \geq 1$, where $\frac{1}{p} + \frac{1}{q} + \frac{1}{r} > 1$.*
4. *As a set, B is finite, and its Gram matrix $(\alpha \cdot \beta)_{\alpha,\beta \in \mathsf{B}}$ is negative definite.*
5. *The Coxeter–Dynkin diagram $\mathcal{G}(\mathsf{B})$ is equal to A_n or D_n for some n, E_6, E_7, or E_8.*

Proof Let B be of finite type and set $V := (M_\mathsf{B})_\mathbb{R}$. Then, the orthogonal group $\mathrm{O}(V)$ is a compact Lie group. A discrete subgroup of such a group must be a finite group. Now, assertion (2) follows from the classification of finite reflection groups in a real inner product vector space V, see [88, Chapter VI, §4], [317, Chapter 1] or [724, Chapter V, §1]. Each such group is a reflection group of isometries of the unit sphere $\mathbb{S}^n$ and for this reason, it is called a *spherical reflection group*. Its Coxeter–Dynkin diagram is either of type $T_{p,q,r}$ with $\frac{1}{p} + \frac{1}{q} + \frac{1}{r} > 1$ or it is a spherical group of type B_n, C_n, F_4, $\mathsf{I}_2(m)$, H_3, or H_4. In the latter cases, the group does not preserve any sublattice M of V. It follows that a chamber of Γ is bounded by hyperplanes H_i in V orthogonal to vectors $e_i \in V$ with $e_i^2 = -1$ and $e_i \cdot e_j = \cos\frac{\pi}{m_{ij}}$, where $m_{ij} = 1$ (if $i = j$), 2, or 3. The $\mathbb{Z}$-sublattice spanned by the vectors $\alpha_i := \sqrt{2}e_i$ has the Gram matrix of the lattice $\mathsf{E}_{p,q,r}$. From this, it is easy to deduce the remaining implications. □

A lattice M isomorphic to the orthogonal sum of lattices $\mathsf{A}_n, \mathsf{D}_n, \mathsf{E}_6, \mathsf{E}_7, \mathsf{E}_8$ is called a *root lattice*.

Proposition 0.8.16 *Let* B *be an irreducible root basis of an even lattice* M*. Then, the following statements are equivalent:*

1. *The root basis* B *is of affine type.*
2. *There exists a unique nonzero vector* $\mathfrak{f} = \sum_i m_i\alpha_i$ *with* $m_i \in \mathbb{Z}_{\geq 0}$*, such that the radical satisfies* $Rad(M_{\mathsf{B}}) = \mathbb{Z}\mathfrak{f}$*, and then, the bilinear form on* M *defines a non-degenerate bilinear form on* $\overline{M} = (\mathbb{Z}\mathfrak{f})^{\perp}/\mathbb{Z}\mathfrak{f}$ *equipping it with a structure of a negative definite even lattice.*
3. *The Cartan matrix of* B *is not invertible and every proper and connected subdiagram of the Coxeter–Dynkin diagram* $\Gamma(\mathsf{B})$ *is of finite type.*
4. *The Coxeter–Dynkin diagram of* Γ *is of type* $\tilde{\mathsf{A}}_n$ *or* $\tilde{\mathsf{D}}_n$ *for some* n *or* $\tilde{\mathsf{E}}_6$*,* $\tilde{\mathsf{E}}_7$ *or* $\tilde{\mathsf{E}}_8$*:*

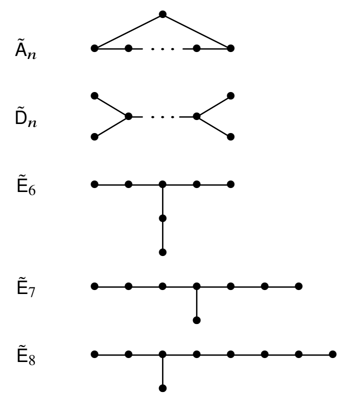


Fig. 0.4 Affine Dynkin diagrams

Note that n *is equal to the number of vertices minus 1.*

Proof Let B be of affine type, in which case the lattice $\mathrm{N} = M_{\mathsf{B}}$ is semi-definite with radical N_0 of rank one. Hence, Γ is isomorphic to a discrete reflection group of the Euclidean space $(N_0^{\perp}/N_0)_{\mathbb{R}}$. The classification of such groups, called *Euclidean reflection groups* is well-known, see see [88, Chapter VI, §4, Theorem 4], [317, Chapter 5], or [724, Chapter V, §1]. It follows from this classification that Γ is the Weyl group of N with Dynkin diagram as in (4) and that $N_0^{\perp}/N_0$ is a lattice isomorphic to A_n, D_n, E_6, E_7, or E_8. All Euclidean reflection groups are realized as *affine Weyl groups* of the simple Lie algebras of types A_n, D_n, E_6, E_7, E_8 and B_n, C_n, F_4, G_2. In the latter cases, the group does not preserve any lattice M with $M_{\mathbb{R}} = V$. Properties (2) and (3) follow from the classification. It is also clear that (2) implies (1), and it is easy to see that (3) is equivalent to (4). □

Remark 0.8.17 Let M be an even and negative definite or negative semi-definite lattice.

1. Let B be a root basis of the k-reflection group $\mathrm{Ref}_k(M)$ and let Γ_{B} be its reflection group. Applying (0.8.12), we obtain

$$\mathrm{O}(M)' = \mathrm{Ref}_k(M) \rtimes A(\mathsf{B}), \tag{0.8.16}$$

where $A(\mathsf{B})$ is the subgroup of $\mathrm{O}(M)'$ that leaves the root basis B invariant. For example, if B is of finite type, then the symmetry group $\mathrm{Sym}(\mathcal{G}(\mathsf{B}))$ of the Coxeter–Dynkin diagram $\mathcal{G}(\mathsf{B})$ is trivial if B is of type E_7 or E_8. In these cases, we obtain

$$\mathrm{O}(\mathsf{E}_n) \cong W_{2,3,n}, \quad n = 4, 5.$$

On the other hand, the groups of symmetries of the Coxeter–Dynkin diagram of types D_n and A_n for $n \geq 2$ are of order 2. It is easy to see that the non-trivial symmetry defines an isometry of M and we find that the Weyl group of M is a subgroup of index 2 in all other cases. We have (see [88, Tables])

$$\begin{array}{lll} W(\mathsf{A}_n) \cong \mathfrak{S}_{n+1} & \mathrm{O}(\mathsf{A}_n) = W(\mathsf{A}_n) \times \{\pm \mathrm{id}_{\mathsf{A}_n}\} & \text{if } n \geq 2, \\ W(\mathsf{D}_4) \cong (\mathbb{Z}/2\mathbb{Z})^3 \rtimes \mathfrak{S}_4 & \mathrm{O}(\mathsf{D}_4) = W(\mathsf{D}_4) \times \mathfrak{S}_3, & \\ W(\mathsf{D}_n) \cong (\mathbb{Z}/2\mathbb{Z})^{n-1} \rtimes \mathfrak{S}_n & \mathrm{O}(\mathsf{D}_n) = W(\mathsf{D}_n) \times \mathbb{Z}/2\mathbb{Z} & \text{if } n \geq 5. \end{array}$$

The quotient of the lattice E_8 by $2\mathsf{E}_8$ is isomorphic to the vector space $\mathbb{F}_2^8$ equipped with a non-degenerate quadratic form of even type. This defines an isomorphism

$$\mathrm{O}(\mathsf{E}_8) \cong W_{2,3,5} \cong \mathrm{O}^+(8, \mathbb{F}_2).$$

This is a finite group of order $2^{14} \cdot 3^5 \cdot 5^2 \cdot 7$ and it contains a simple subgroup of index 2. Similarly, we have $W(\mathsf{E}_7) = W(\mathsf{E}_7)' \times \{\pm \mathrm{id}_{\mathsf{E}_7}\}$ and the quotient of E_7 by $2\mathsf{E}_7$ defines an isomorphism

$$W(\mathsf{E}_7)' \cong \mathrm{Sp}(6, \mathbb{F}_2),$$

where $\mathrm{Sp}(6, \mathbb{F}_2)$ is the group of automorphisms of $\mathbb{F}_2^6$ that preserve a non-degenerate symplectic form. The order of $W(\mathsf{E}_7)$ is equal to $2^{10} \cdot 3^{14} \cdot 5 \cdot 7$ and the group $\mathrm{Sp}(6, \mathbb{F}_2)$ is simple. Finally, the group $W(\mathsf{E}_6)$ contains a simple subgroup $W(\mathsf{E}_6)'$ of index 2 such that the reduction of E_6 by $3\mathsf{E}_6$ gives rise to an isomorphism

$$W(\mathsf{E}_6)' \cong \mathrm{Sp}(4, \mathbb{F}_3).$$

The order of $W(\mathsf{E}_6)$ is equal to $72 \cdot 6!$.

2. Assume now that B is of affine type. Then the symmetry group $\mathrm{Sym}(\mathcal{G}(\mathsf{B}))$ of the Coxeter–Dynkin diagram $\mathcal{G}(\mathsf{B})$ is trivial only in the case where B is of type $\tilde{E}_8$.

   On the other hand,

$$\mathrm{Sym}(\mathcal{G}(\mathsf{B})) \cong \begin{cases} D_{2n+2} & \text{if } \mathsf{B} \text{ is of type } \tilde{\mathsf{A}}_n, \\ D_8 & \text{if } \mathsf{B} \text{ is of type } \tilde{\mathsf{D}}_n, n \neq 4, \\ \mathfrak{S}_4 & \text{if } \mathsf{B} \text{ is of type } \tilde{\mathsf{D}}_4, \\ \mathfrak{S}_3 & \text{if } \mathsf{B} \text{ is of type } \tilde{\mathsf{E}}_6, \\ \mathbb{Z}/2\mathbb{Z} & \text{if } \mathsf{B} \text{ is of type } \tilde{\mathsf{E}}_7. \end{cases}$$

Here, D_{2n} denotes the dihedral group $(\mathbb{Z}/n\mathbb{Z}) \rtimes (\mathbb{Z}/2\mathbb{Z})$ of order $2n$.
We also have

$$\mathrm{O}(M)' \cong \bar{M}^\vee \rtimes \mathrm{O}(\bar{M})', \tag{0.8.17}$$

where the radical of M is generated by $\mathfrak{f}$ and $\bar{M} = (\mathbb{Z}\mathfrak{f})^\perp/\mathbb{Z}\mathfrak{f}$ and the inclusion of $\bar{M}^\vee$ into $\mathrm{O}(M)'$ is given by formula (0.8.8). In particular, we have

$$\mathrm{O}(\tilde{\mathsf{E}}_8)' \cong W_{2,3,6} \cong \mathsf{E}_8 \rtimes W_{2,3,5}.$$

Next, we turn to hyperbolic lattices. A root basis B in a lattice M is said to be of *hyperbolic type* if the sublattice M_B of M is a hyperbolic lattice. We assume that $M_\mathsf{B} = M$. Let $C(\mathsf{B})$ be the fundamental chamber of B. If B is a finite set, then it follows from (0.8.12) that W_B is of finite index in $\mathrm{O}(M)$. We say that a discrete reflection subgroup Γ of $\mathrm{O}(V)'$ with a chamber P is of *finite covolume* if P is the convex hull of a finite set of points $[v] \in \mathbb{P}(V)$ lying in $\mathbb{H}(V)$ in its boundary. The reason for this name is that this happens if and only if the volume of P in the hyperbolic metric is finite. We say that Γ is *cocompact* or *uniform* if none of these points lies in the boundary. It follows from this definition that the set of bounding hyperplanes of P is a finite set. Of course, the converse is not true.

A root basis B (or the Weyl group W_B) in a hyperbolic lattice M with $M = M_\mathsf{B}$ is said to be *crystallographic* if the reflection group Γ_B is of finite covolume.

Proposition 0.8.18 *Let* B *be a root basis of a hyperbolic lattice* M*. Then,* B *is crystallographic if and only if* Γ_B *is a subgroup of finite index in* $\mathrm{O}(M)'$.

Proof Suppose that Γ_B is of finite index in $\mathrm{O}(M)'$. Then, $\mathrm{O}(M)'$ acts in $\mathbb{H}(V)$ with a fundamental domain of finite volume, see [681]. Since $C(\mathsf{B})$ is a fundamental domain for Γ_B, its volume must be also finite. The converse statement follows from (0.8.12), because B is finite and thus, $A(C(\mathsf{B}))$ is a finite group. □

An even and hyperbolic lattice M is called a *k-reflective lattice* (resp. *reflective lattice*) if $\mathrm{Ref}_k(M)$ (resp. $\mathrm{Ref}(M)$) is of finite index in $\mathrm{O}(M)'$. This is equivalent to M_k containing a finite crystallographic basis. The classification of 2-reflective lattices can be found in [557] and [722] (for lattices of rank 4), see also the survey [169]. By the previous remarks, if W_B is crystallographic for some root basis B in M, then M is a 2-reflective lattice. In this case, it follows from the proof of Proposition 0.8.20 that $\mathsf{B} = \mathsf{B}(M)$ and $W_\mathsf{B} = W(M)$.

Example 0.8.19 The lattice $\mathsf{E}_{p,q,r}$ is 2-reflective if and only if (p,q,r) is one of the following 9 triples

$$(2,3,7),\ (2,3,8),\ (2,3,9),\ (2,3,10),\ (2,4,5),\ (2,4,6),\ (3,3,4),\ (3,3,5),\ (3,3,6).$$

Only three of them, namely $\mathsf{E}_{2,3,7}, \mathsf{E}_{2,4,5}$, and $\mathsf{E}_{3,3,4}$, have a crystallographic canonical root basis.

Proposition 0.8.20 *Let* B *be a hyperbolic root basis in some lattice. Suppose that* B' *is crystallographic for some subset* $\mathsf{B}' \subseteq \mathsf{B}$. *Then,* B *is crystallographic and* $\mathsf{B}' = \mathsf{B}$.

Proof Clearly, $W_{\mathsf{B}'} \subseteq W_{\mathsf{B}}$. By Proposition 0.8.18, $W_{\mathsf{B}'}$ is of finite index in $\mathrm{O}(M)$ and thus, W_{B} is of finite index, too. Thus, also B is crystallographic. If $\alpha \in \mathsf{B} \setminus \mathsf{B}'$, then $\alpha \cdot \beta \geq 0$ for all $\beta \in \mathsf{B}'$. This implies that α belongs to $C(\mathsf{B}')$ and since B' is crystallographic, we find $C(\mathsf{B}') \subseteq V^+$. Hence, $\alpha^2 \geq 0$, a contradiction. □

A discrete reflection subgroup of finite covolume (resp. uniform) of $\mathrm{O}(V)'$ with simplicial chambers is called a *quasi-Lanner group* (resp. *Lanner group*), see [724]. The Coxeter–Dynkin diagrams of quasi-Lanner and Lanner groups have been classified, see [724] or [317, Section 6.9]. Irreducible Lanner (resp. quasi-Lanner) group exists only in $\mathbb{H}^n$ with $n \leq 4$ (resp. $n \leq 9$). The classification list can be found in loc. cit. or [724]. A Lanner (resp. quasi-Lanner) group is characterized by the property that each proper subdiagram of its Coxeter–Dynkin diagram is the Coxeter–Dynkin diagram of a finite (resp. finite or Euclidean) reflection group.

Proposition 0.8.21 *Let* Γ *be a crystallographic reflection group of a hyperbolic lattice* M. *Then, possible labels in its Coxeter–Dynkin diagram of* Γ *are equal to* 2 *or* 4. *If* Γ *is a Weyl group, then the Coxeter–Dynkin diagram is simply-laced.*

Proof Let B be a root basis of a Γ. It follows from (0.8.13) that if $\alpha, \beta \in \mathsf{B}$, then $\cos \frac{\pi}{m_{\alpha,\beta}}$ belongs to $\mathbb{Q}$ or to a quadratic extension $K/\mathbb{Q}$. Quite generally, if m is a positive integer, then $\cos \frac{\pi}{m}$ generates the real subfield of the cyclotomic field $\mathbb{Q}(\zeta_{2m})$, which is an extension of $\mathbb{Q}$ of degree $\frac{1}{2}\phi(2m)$, where ϕ denotes Euler's ϕ-function. Thus, we find $\phi(2m) \in \{2,4\}$. If we write $2m = 2^a p_1^{a_1} \cdots p_k^{a_k}$ for distinct and odd primes $p_1, ..., p_k$, then we have $\phi(2m) = 2^{a-1} p_1^{a_1-1} \cdots p_k^{a_k-1}(p_1 - 1) \cdots (p_k - 1)$. From this, we deduce $m \in \{1,2,3,4,6\}$. Thus, the possible labels in the Coxeter–Dynkin diagram of a crystallographic group are 2 or 4. In particular, if the crystallographic group is a Weyl group, then the associated Coxeter–Dynkin diagram is simply-laced, that is, there are no labels. □

The following corollary follows from the list of Coxeter–Dynkin diagrams of Lanner type.

Corollary 0.8.22 *A Weyl group of a lattice is never a Lanner group.*

We note that there are hyperbolic lattices with reflection group of Lanner type in all possible ranks. There also exist hyperbolic lattices with Weyl groups of quasi-Lanner type in all ranks $r = 4, \ldots, 10$. We will see those of rank 10 later in Chapter 5.

Next, we want to determine whether a root basis inside a hyperbolic lattice is crystallographic. Quite generally, let B be a root basis inside some lattice M with

$M = M_{\mathsf{B}}$ and let $\mathcal{G}(\mathsf{B})$ be the Coxeter–Dynkin diagram of B. Obviously, there is a natural bijection between the following sets:

1. Non-empty subsets of B.
2. Full subgraphs of $\mathcal{G}(\mathsf{B})$.
3. Principal submatrices of the Gram matrix of B (sometimes also called the *Cartan matrix*).

We will say that a subgraph of the graph $\mathcal{G}(\mathsf{B})$ is of *finite type* (resp. *parabolic* or *affine* type) if the corresponding subset of B is a root basis of finite (resp. affine) type. Its *rank* is defined to be the rank of the corresponding Cartan matrix or, more geometrically, the number of vertices minus the number of connected components. Finally, faces (vertices) of $P(\mathsf{B})$ correspond to principal negative definite submatrices of rank n. 's The next theorem is known as the *Vinberg criterion*.

Theorem 0.8.23 *Let* B *be a root basis inside a hyperbolic lattice* M *of rank* $n+1$ *with* $M = M_{\mathsf{B}}$.

1. *If* $(\alpha\cdot\beta)^2 > \alpha^2\cdot\beta^2$ *for some* $\alpha,\beta\in\mathsf{B}$, *then* B *is not crystallographic.*
2. *Assume that* $\alpha\cdot\beta \le 2$ *for all* $\alpha,\beta\in\mathsf{B}$. *Then,* B *is crystallographic if and only if every connected subgraph of affine type of* $\Gamma(\mathsf{B})$ *is contained in a subgraph of affine type of maximal rank (=* $n-1$*).*

Proof A proof can be found in [719, Theorem 2.6 bis]. However, in view of the importance of this result for our applications to automorphisms of Enriques surfaces, we include the proof.

The chamber $P = P(\mathsf{B})$ is of finite volume if and only if its closure $\bar{P}$ in $\mathbb{P}(V)$ lies in $\bar{\mathbb{H}}(V)$. Suppose there exists a point $x_0 = [v_0] \in \bar{P}\setminus P$ for some $v_0 \in V = M_{\mathbb{R}}$. Let $I \subseteq \mathsf{B}$ be the subset of those $\alpha\in\mathsf{B}$ such that $v_0\cdot\alpha = 0$. Since the linear subspace V_{B} is indefinite, $I \ne \mathsf{B}$ and V_I is not negative definite, but for any proper subset $J \subset I$, the subspace V_J is negative definite. Suppose V_I is indefinite. Then B contains a root subbase B' such that $\Gamma_{\mathsf{B}'}$ is of Lanner group. However, as we observed before, none of them occurs as a reflection group of a lattice. Thus, we may assume that the Coxeter–Dynkin diagram does not contain subdiagrams of Lanner type. In particular, no two vectors in B span a hyperbolic lattice (it will define a Lanner subdiagram with two vertices). Obviously, $\bar{P}$ is compact if and only if it is contained in $\mathbb{H}(V)$ and this happens if and only if its Coxeter–Dynkin diagram does not contain parabolic subdiagrams. So the assertion is true in this case.

Assume that P is not compact and then, $x_0 = [v_0] \in \partial\mathbb{H}(V)$. A neighborhood $U(x_0)$ of $x_0 \in \bar{\mathbb{H}}(V)$ of the form $(\mathbb{R}v_0)^\perp/\mathbb{R}^\times$ is called an *orisphere* with center x_0. It intersects $\mathbb{H}(V)$ in an open subset isomorphic (as a Riemannian manifold) to the Euclidean space $\mathbb{E}^{n-1}$. In order for $P(\mathsf{B})$ to be of finite volume, it is necessary and sufficient that $P(\mathsf{B})$ intersects each such orisphere $U(x_0)$ along a bounded subset of the Euclidean space. This happens if and only if x_0 is a vertex of the closure of $P(\mathsf{B})$ in $\bar{\mathbb{H}}(V)$, hence I is contained in a subset J such that the rank of the corresponding principal submatrix is equal to $(n-1)$. This proves the assertion. $\square$

0.9 Picard Schemes and Albanese Varieties

In this section, we discuss the Picard group, Picard functors, and the Picard scheme of a given scheme. We also discuss Picard lattices, Néron–Severi groups, algebraic, linear and numerical equivalence of divisors, as well as infinitesimal properties of Picard schemes. Many of these results are classical, but were established by Grothendieck [266] in the generality presented here. We refer the reader to [86, Chapter 8] and [392] for surveys and more details than those presented here. We end this section by discussing the Albanese variety of a normal variety.

For a scheme X, we denote by $\mathrm{Pic}(X)$ its Picard group, that is, the set of isomorphism classes of invertible sheaves on X, which becomes an abelian group with composition given by $\otimes$, with neutral element $\mathcal{O}_X$, and inverse given by taking duals. If X is an integral scheme, then the Picard group is naturally isomorphic to the group of Cartier divisors modulo linear equivalence, see [294, Proposition II.6.15]. One can also interpret elements of $\mathrm{Pic}(X)$ as torsors under the multiplicative group scheme $\mathbb{G}_m$ over X with respect to different Grothendieck topologies (Zariski, étale, flat)

$$\mathrm{Pic}(X) \;\cong\; H^1(X, \mathcal{O}_X^\times) \;\cong\; H^1_{\mathrm{Zar}}(X, \mathbb{G}_m) \;\cong\; H^1_{\text{ét}}(X, \mathbb{G}_m) \;\cong\; H^1_{\mathrm{fl}}(X, \mathbb{G}_m). \qquad (0.9.1)$$

We already explained the first two isomorphisms in Example 0.1.6. The latter two isomorphisms are deeper and rely on Grothendieck's generalization of Hilbert's Theorem 90, see, for example, [508, Proposition III.4.9], For the last isomorphism, we also refer to Theorem 0.1.3.

Let $f : X \to S$ be a separated morphism of finite type between locally noetherian schemes. Then, the *relative Picard functor* of X over S is the functor $\mathrm{Pic}_{X/S}$ that associates to every morphism of schemes $S' \to S$ the abelian group $\mathrm{Pic}(X \times_S S')/\mathrm{Pic}(S')$. Then, we have associated sheaves in the Zariski, étale, and flat topologies

$$\underline{\mathrm{Pic}}_{(X/S)(\mathrm{Zar})}, \quad \underline{\mathrm{Pic}}_{(X/S)(\text{ét})}, \quad \text{and} \quad \underline{\mathrm{Pic}}_{(X/S)(\mathrm{fl})}.$$

Using (0.9.1), it is not difficult to see that for every morphism $S' \to S$ of schemes there is an isomorphism

$$\underline{\mathrm{Pic}}_{(X/S)(-)}(S') \;\cong\; H^0(S', R^1_- f_{S'*}\mathbb{G}_m),$$

where $f_{S'}$ denotes the base-change $f \times_S S' : X \times_S S' \to S'$. Here, $-$ denotes the Zariski, étale, or flat topology, and $R^1_- f_{S'*}$ denotes the higher direct image with respect to this topology. In general, these sheaves depend on the choice of Grothendieck topology. However, Grothendieck established the following comparison theorem, see [266] or [392, Theorem 9.2.5].

Proposition 0.9.1 *Let $f : X \to S$ be a separated morphism of finite type between locally noetherian schemes. Assume that the natural map $\mathcal{O}_S \to f_*\mathcal{O}_X$ is a universal isomorphism, that is, remains an isomorphism after every base-change $S' \to S$. Then, the natural homomorphisms of abelian groups*

$$\mathrm{Pic}_{X/S}(S') \to \underline{\mathrm{Pic}}_{(X/S)(Zar)}(S') \to \underline{\mathrm{Pic}}_{(X/S)(\acute{e}t)}(S') \to \underline{\mathrm{Pic}}_{(X/S)(fl)}(S')$$

are injective. If f has a section, then all maps are isomorphisms. If f has a section locally in the Zariski topology, then the latter two maps are isomorphisms. If f has a section locally in the étale topology, then the last map is an isomorphism.

There is a Grothendieck–Leray spectral sequence

$$E_2^{i,j} = H^i(S', R^j_- f_{S'*}\mathbb{G}_m) \Rightarrow H^{i+j}_{(-)}(X_{S'}, \mathbb{G}_m),$$

where $-$ denotes the étale or flat topology. From the exact sequence in low degrees and the previous proposition, we obtain the following result.

Proposition 0.9.2 *Let $f : X \to S$ be a separated morphism of finite type between locally noetherian schemes. Assume that $\mathcal{O}_S \to f_*\mathcal{O}_X$ is a universal isomorphism. Let $S' \to S$ be a morphism of schemes and let $f' := f_{S'} : X_{S'} := X \times_S S' \to S'$ be the base-change. Then, there exists an exact sequence*

$$0 \to \mathrm{Pic}(X_{S'})/\mathrm{Pic}(S') \xrightarrow{\alpha} \underline{\mathrm{Pic}}_{(X/S)(\acute{e}t)}(S') \xrightarrow{\delta} H^2_{\acute{e}t}(S', \mathbb{G}_m).$$

The homomorphism α is bijective if f' has a section or if $H^2_{\acute{e}t}(S', \mathbb{G}_m) = 0$.

For a scheme Z, the group $H^2_{\text{ét}}(Z, \mathbb{G}_m)$ is called the *cohomological Brauer group* of Z and it is denoted by $\mathrm{Br}_{\mathrm{ct}}(Z)$. We refer to [249], [271], and [508] for background and the theory of these groups. If $Z = \operatorname{Spec} K$, where K is a field, then this group coincides with the usual Brauer group $\mathrm{Br}(K)$ of the field K that is defined as the set of central simple K-algebras modulo Brauer or Morita equivalence, which becomes an abelian group with composition given by $\otimes$, with neutral element K, and inverse given by the opposite algebra [89, §15]. Moreover, if the field K is finite, or algebraically closed, or the field of rational functions of an algebraic curve over an algebraically closed field, then $\mathrm{Br}(K) = 0$, and we refer to [249] for details and proofs. We will discuss the Brauer groups in greater detail in the next chapter.

Example 0.9.3 Let $f : X \to S$ be a proper and geometrically integral scheme over $S = \operatorname{Spec} \Bbbk$, where $\Bbbk$ is a field, that is, X is a proper variety over $\Bbbk$. Then, f has sections locally in the étale topology (see, for example, [249, Appendix A]) and thus, $\underline{\mathrm{Pic}}_{(X/\Bbbk)(\text{ét})} = \underline{\mathrm{Pic}}_{(X/\Bbbk)(\text{fl})}$ by Proposition 0.9.1. Moreover, if $\Bbbk^{\text{sep}}$ denotes the separable closure of $\Bbbk$ and if $G_\Bbbk := \mathrm{Gal}(\Bbbk^{\text{sep}}/\Bbbk)$ denotes its absolute Galois group, then we set $\overline{X} := X \times_\Bbbk \Bbbk^{\text{sep}}$ and obtain isomorphisms of abelian groups

$$\mathrm{Pic}(\overline{X})^{G_\Bbbk} \cong \underline{\mathrm{Pic}}_{(X/\Bbbk)(\text{ét})}(\Bbbk^{\text{sep}}) \cong \underline{\mathrm{Pic}}_{(X/\Bbbk)(\text{fl})}(\Bbbk^{\text{sep}}),$$

where $-^{G_\Bbbk}$ denotes Galois-invariants. Then, the exact sequence of Proposition 0.9.2 becomes

$$0 \to \mathrm{Pic}(X) \to \mathrm{Pic}(\overline{X})^{G_\Bbbk} \xrightarrow{\delta} \mathrm{Br}(\Bbbk).$$

We note that δ is zero if X has a $\mathbb{k}$-rational point or if $\mathrm{Br}(\mathbb{k}) = 0$. We refer to [463] for details and how to deal with the case where δ is non-zero and a connection to Brauer–Severi varieties over $\mathbb{k}$.

The next theorem, which combines results of Grothendieck, Mumford, Murre, and Oort, gives sufficient conditions for the relative Picard functor to be representable by a group scheme.

Theorem 0.9.4 *Let $f : X \to S$ be a proper and flat morphism of finite type between noetherian schemes. If f is projective with geometrically integral fibers or if S is the spectrum of a field, then* $\mathrm{Pic}_{(X/S)(fl)}$ *is representable by a group scheme $\mathbf{Pic}_{X/S}$, which is separated and locally of finite type over S.*

Proof The first case is due to Grothendieck [266], and the second case is due to Murre [545] and Oort [582]. □

We refer to [21] and [86], Chapter 8.3, for representability results by algebraic spaces. In any case, Theorem 0.9.4 is sufficient for our applications later on and we refer to [86, Chapter 8] and [392] for details, proofs, (counter-)examples, and further results. The group scheme $\mathbf{Pic}_{X/S}$ is called the *Picard scheme* of X over S.

Remark 0.9.5 This representability theorem is rather sharp: Mumford gave an example of a flat and projective morphism $f : X \to S = \operatorname{Spec} \mathbb{R}[[t]]$, whose fibers are curves that are geometrically reduced, but whose special fiber is not geometrically irreducible, and where $\mathrm{Pic}_{(X/S)(\mathrm{fl})}$ is not representable by a group scheme over S. We refer to [86, Chapter 8.2] for details.

Let $f : X \to S$ be a morphism of schemes such that the Picard functor is representable by a group scheme $\mathbf{Pic}_{X/S}$ over S. Since the Picard scheme represents the Picard functor, there exists an invertible sheaf $\mathcal{P}$ on $X \times_S \mathbf{Pic}_{X/S}$ with the following universal property: for every morphism of schemes $T \to S$ and every invertible sheaf $\mathcal{L}$ on $X \times_S T$, there exists a unique morphism of schemes $\psi : T \to \mathbf{Pic}_{X/S}$ over S and an invertible sheaf $\mathcal{N}$ on T such that

$$\mathcal{L} \;\cong\; (\mathrm{id}_X \times \psi)^* \mathcal{P} \otimes f_T^* \mathcal{N}.$$

This universal invertible sheaf is called the *Poincaré sheaf*.

Let G be a commutative group scheme that is separated and locally of finite type over a field $\mathbb{k}$. Let G° be its identity component defined in Section 0.1. It is a commutative group scheme that is separated and locally of finite type over $\mathbb{k}$, and the quotient G/G° exists and is a group scheme, which is étale and locally of finite type over $\mathbb{k}$. If $\mathbb{k}$ is algebraically closed, then G/G° is a constant group scheme over $\mathbb{k}$. Next, we consider the inverse image of the torsion of G/G° in G, that is,

$$G^\tau \;:=\; \bigcup_{n>0} n^{-1}(G^\circ),$$

where $n : G \to G$ denotes multiplication by n. From this, we obtain morphisms $G^\circ \subseteq G^\tau \subseteq G$ of group schemes over $\mathbb{k}$, and each one is an open subscheme of the next one. Next, let S be an arbitrary scheme and let G be a group scheme that is separated and locally of finite type over S. For a point $s \in S$, let $\kappa(s)$ be its residue field and set $G_s := G \times_S \operatorname{Spec} \kappa(s)$, which is a group scheme that is locally of finite type and separated over $\kappa(s)$. Then, we define subfunctors G° (resp. G^τ) as follows: for every morphism $S' \to S$, we define $G^\circ(S')$ (resp. $G^\tau(S')$) to be the subgroup of $G(S')$ of elements that induce an element of $G_s^\circ(s')$ (resp. $G_s^\tau(s')$) for every point $s' \in S'$ lying over a point $s \in S$. For example, if G is smooth over S along the unit section, then G° is representable by an open subscheme of G, and we refer to [86], Chapter 8.4 for details and further representability results for G° and G^τ. The group scheme G° is called the *identity component* of G.

If X is a scheme that is proper over a field $\Bbbk$, then, by Theorem 0.9.4, the functor $\operatorname{Pic}_{(X/\Bbbk)(\mathrm{fl})}$ is representable by a group scheme $\mathbf{Pic}_{X/\Bbbk}$, which is separated and locally of finite type over $\Bbbk$. Applying the previous notions to this group scheme, we have the following result.

Theorem 0.9.6 *Let X be a scheme that is proper over a field $\Bbbk$.*

1. *$\mathbf{Pic}^\circ_{X/\Bbbk}$ and $\mathbf{Pic}^\tau_{X/\Bbbk}$ are group schemes, which are separated and of finite type over $\Bbbk$.*
2. *$\mathbf{Pic}^\tau_{X/\Bbbk}/\mathbf{Pic}^\circ_{X/\Bbbk}$ is a group scheme, which is finite over $\Bbbk$.*
3. *$(\mathbf{Pic}_{X/\Bbbk}/\mathbf{Pic}^\circ_{X/\Bbbk})(\overline{\Bbbk})$ is a finitely generated abelian group.*

Proof See [266] or [392, Section 9.5]. □

Next, we turn to the infinitesimal study of Picard schemes. Quite generally, for a group–valued functor $\underline{G}$ on the category of schemes over S, one can define the *Lie algebra functor* $\underline{\mathrm{Lie}}(\underline{G}/S)$ that associates to every morphism $S' \to S$ the abelian group $\operatorname{Ker}(\underline{G}(S'[\varepsilon]) \to \underline{G}(S'))$, where $S'[\varepsilon]$ denotes the scheme of dual numbers over S', that is, $S'[\varepsilon] \cong S' \times_{\operatorname{Spec}\mathbb{Z}} \operatorname{Spec}\mathbb{Z}[\varepsilon]/(\varepsilon^2)$. The restriction of $\underline{\mathrm{Lie}}(\underline{G}/S)$ to the Zariski open subsets of S yields a quasi-coherent sheaf $\mathcal{L}ie(\underline{G}/S)$ on S.

Proposition 0.9.7 *Let $f : X \to S$ be a proper and flat morphism of finite type between noetherian schemes. Then, there exists a canonical isomorphism*

$$\mathcal{L}ie(\underline{\mathrm{Pic}}_{(X/S)(fl)}/S) \;\cong\; R^1 f_* \mathcal{O}_X .$$

Proof See [86, Chapter 8.4] or [266]. □

Assume now that $S = \operatorname{Spec}\Bbbk$ for some field $\Bbbk$ and that X is a proper variety over $\Bbbk$. By Theorem 0.9.4, the Picard scheme $\mathbf{Pic}^\circ_{X/\Bbbk}$ exists as a scheme and by Theorem 0.9.6, the group scheme $\mathbf{Pic}^\circ_{X/\Bbbk}$ is separated and of finite type over $\Bbbk$. Moreover, the Zariski tangent space of $\mathbf{Pic}^\circ_{X/\Bbbk}$ at the origin is isomorphic to $H^1(X, \mathcal{O}_X)$ by Proposition 0.9.7. Since $\mathbf{Pic}^\circ_{X/\Bbbk}$ may not be a reduced scheme, we let $(\mathbf{Pic}^\circ_{X/\Bbbk})_{\mathrm{red}}$ be its reduction. If $\Bbbk$ is algebraically closed, then $(\mathbf{Pic}^\circ_{X/\Bbbk})_{\mathrm{red}}$ is a group scheme that is separated and smooth over $\Bbbk$, see also the discussion in connection with the connected-étale sequence (0.1.8) and its splitting. As already noted there, the reduction of a group scheme need not be a group scheme, see [733, Chapter 6, Exercises 9 and 10], and the discussion in [392, Remark 9.5.2].

Proposition 0.9.8 *Let X be a normal and projective variety over an algebraically closed field $\Bbbk$. Then, $\boldsymbol{Pic}^{\circ}_{X/\Bbbk}$ and $\boldsymbol{Pic}^{\tau}_{X/\Bbbk}$ are group schemes that are projective over $\Bbbk$. Moreover:*

1. *$(\boldsymbol{Pic}^{\circ}_{X/\Bbbk})_{\mathrm{red}}$ is an abelian variety of dimension at most $h^1(\mathcal{O}_X)$ over $\Bbbk$.*
2. *If $\mathrm{char}(\Bbbk) = 0$ or if $h^2(\mathcal{O}_X) = 0$, then $\boldsymbol{Pic}^{\circ}_{X/\Bbbk} = (\boldsymbol{Pic}^{\circ}_{X/\Bbbk})_{\mathrm{red}}$ and it is an abelian variety of dimension $h^1(\mathcal{O}_X)$ over $\Bbbk$.*

Proof For the first statement, see [266], and see [538, Lecture 27], for the second statement, as well as [392, Section 9.6], for overview and background. □

Remark 0.9.9 The second statement can be made more precise. By Proposition 0.9.7, the Zariski tangent space of $\mathbf{Pic}^{\circ}_{X/\Bbbk}$ at the origin is isomorphic to $H^1(X, \mathcal{O}_X)$. If $\mathrm{char}(\Bbbk) = p > 0$, then, by [538, Lecture 27], the Zariski tangent space of $(\mathbf{Pic}^{\circ}_{X/\Bbbk})_{\mathrm{red}}$ at the origin is isomorphic to the subspace

$$H^1_0(X, \mathcal{O}_X) \ := \bigcap_{r \geq 1} \mathrm{Ker}(\beta_r) \ \subseteq \ H^1(X, \mathcal{O}_X),$$

where $\beta_{r+1} : \mathrm{Ker}(\beta_r) \to H^2(X, \mathcal{O}_X)/\mathrm{Im}(\beta_r)$ with $\mathrm{Ker}(\beta_r) \subseteq H^1(X, \mathcal{O}_X)$ are certain *Bockstein operators*. In particular, if $h^2(\mathcal{O}_X) = 0$, then $H^1_0(X, \mathcal{O}_X) = H^1(X, \mathcal{O}_X)$, which implies that $\mathbf{Pic}^{\circ}_{X/\Bbbk}$ is reduced, and thus, an abelian variety of dimension $h^1(\mathcal{O}_X)$ over $\Bbbk$.

Now, if X is proper variety over an algebraically closed field $\Bbbk$, then $\mathbf{Pic}^{\circ}_{X/\Bbbk}$ is a commutative group scheme that is connected, separated, and of finite type over $\Bbbk$ by Theorem 0.9.6. Moreover, $G := (\mathbf{Pic}^{\circ}_{X/\Bbbk})_{\mathrm{red}}$ is even reduced, and thus, smooth over $\Bbbk$. Thus, by the structure theorem of Chevalley and Rosenlicht, G is an extension of an abelian variety A over $\Bbbk$ by a group scheme that is smooth and affine over $\Bbbk$. The latter group scheme is an extension of a unipotent group scheme U by a torus T, and we refer to the discussion in Section 0.3, to [86, Chapter 9.2], or to [656] for details. For an integer $n \geq 1$ and an abelian group H, we denote by ${}_nH := \{h \in H : nh = 0\}$ the n-torsion subgroup of H. If n is coprime to $\mathrm{char}(\Bbbk)$, then

$$\begin{aligned} {}_nA(\Bbbk) &\cong (\mathbb{Z}/n\mathbb{Z})^{2g}, \quad \text{where } g = \dim A, \\ {}_nT(\Bbbk) &\cong (\mathbb{Z}/n\mathbb{Z})^{t}, \quad \text{where } t = \dim T, \\ {}_nU(\Bbbk) &= \{0\}. \end{aligned}$$

We still assume X to be a proper variety over an algebraically closed field $\Bbbk$. Then, taking étale cohomology in the Kummer exact sequence (0.1.6) in the étale topology, still assuming n to be coprime to the characteristic of $\Bbbk$,

$$0 \to \boldsymbol{\mu}_n \to \mathbb{G}_m \xrightarrow{\times n} \mathbb{G}_m \to 0$$

and using (0.9.1), we conclude

$$ {}_n\mathbf{Pic}^{\tau}_{X/\Bbbk}(\Bbbk) \;\cong\; H^1_{\text{ét}}(X, \boldsymbol{\mu}_n), \tag{0.9.2} $$

see also (0.1.7). From Theorem 0.9.6 and the previous discussion, it follows that if n is only divisible by sufficiently large primes, then $H^1_{\text{ét}}(X, \boldsymbol{\mu}_n)$ is a free $(\mathbb{Z}/n\mathbb{Z})$-module of rank $(2g + t)$. If X is moreover geometrically normal, then $t = 0$ by Proposition 0.9.8. In any case, this rank is denoted by $b_1(X)$ and is called the *first Betti number* of X. We come back to this in the next section. We summarize the previous discussion as follows.

Proposition 0.9.10 *Let X be a normal and projective variety over an algebraically closed field $\Bbbk$. Then:*

1. *$(\mathbf{Pic}^{\circ}_{X/\Bbbk})_{\text{red}}$ is an abelian variety of dimension $\frac{1}{2}b_1(X)$ and*
2. *the Zariski tangent space of $\mathbf{Pic}^{\circ}_{X/\Bbbk}$ is isomorphic to $H^1(X, \mathcal{O}_X)$.*

In particular,

$$ \Delta(X) \;:=\; 2h^1(\mathcal{O}_X) \,-\, b_1(X) $$

is an even integer that satisfies $0 \leq \Delta(X) \leq 2h^2(\mathcal{O}_X)$. Moreover, $\Delta(X) = 0$ if and only if $\mathbf{Pic}^{\circ}_{X/\Bbbk}$ is reduced. If $\mathrm{char}(\Bbbk) = 0$, then $\Delta(X) = 0$.

We refer to Remark 0.9.23 for an analytic construction of the Picard scheme and the Picard variety for smooth and projective varieties over the complex numbers.

Remark 0.9.11 Since a smooth and projective curve X over an algebraically closed field $\Bbbk$ satisfies $H^2(X, \mathcal{O}_X) = 0$, its Picard scheme $\mathbf{Pic}^{\circ}_{X/\Bbbk}$ is reduced by Remark 0.9.9. The first examples of smooth and projective varieties over algebraically closed fields of positive characteristic with non-reduced Picard schemes are due to Igusa [325] and we refer to [457] for an analysis of non-reduced Picard schemes for smooth and projective surfaces. In the next chapter, we will encounter Enriques surfaces with non-reduced Picard schemes in characteristic 2 and refer to [458, Section 3.3] and Section 0.10 for further information.

Example 0.9.12 Let A be an abelian variety over an algebraically closed field $\Bbbk$. Then, $A^{\vee} := \mathbf{Pic}^{\circ}_{A/\Bbbk}$ is reduced, that is, $A^{\vee}$ is an abelian variety, which is called the *dual abelian variety* of A. It is an abelian variety of the same dimension as A and the name is justified by the fact that there exists a canonical isomorphism $A \cong (A^{\vee})^{\vee}$ of abelian varieties over $\Bbbk$. We refer to [541, Section 13] for details and proofs.

Next, let X be a smooth and proper over an algebraically closed field $\Bbbk$. Let $\mathcal{L}_i$, $i = 1, 2$ be two invertible sheaves on X. Then, $\mathcal{L}_1$ and $\mathcal{L}_2$ are said to be *linearly equivalent* if $\mathcal{L}_1 \cong \mathcal{L}_2$. They are said to be *algebraically equivalent* if there exists a connected scheme T of finite type over $\Bbbk$, an invertible sheaf $\mathcal{M}$ on $X \times T$, and two closed points $t_i \in T$, $i = 1, 2$ such that $\mathcal{L}_i \cong \mathcal{M}|_{X \times \{t_i\}}$ for $i = 1, 2$. Finally, $\mathcal{L}_1$ and $\mathcal{L}_2$ are said to be *numerically equivalent* if $\deg \mathcal{L}_1|_C = \deg \mathcal{L}_2|_C$ for every integral curve C on X. (For an invertible sheaf $\mathcal{N}$ on a proper curve C over $\Bbbk$, the degree is defined to be $\deg \mathcal{N} := \chi(\mathcal{N}) - \chi(\mathcal{O}_C)$. If C is smooth, then this coincides with the usual definition of degree.) Two linearly equivalent invertible sheaves are algebraically equivalent, and two algebraically invertible sheaves are numerically equivalent. An invertible sheaf is said to be linearly (resp. algebraically, numerically) equivalent to zero if it is linearly (resp. algebraically, numerically) equivalent to $\mathcal{O}_X$. We will also use the same terminology for (Cartier) divisors. We refer to [242] or [392] for definitions and details.

Proposition 0.9.13 *Let X be a smooth and projective variety over an algebraically closed field $\Bbbk$.*

1. $\mathbf{Pic}^{\circ}_{X/\Bbbk}(\Bbbk)$ *is the group of divisor classes that are algebraically equivalent to zero, and*
2. $\mathbf{Pic}^{\tau}_{X/\Bbbk}(\Bbbk)$ *is the group of divisor classes that are numerically equivalent to zero.*

Proof See [266] or [392]. □

If X is smooth and projective over an algebraically closed field $\Bbbk$, then $(\mathbf{Pic}^{\circ}_{X/\Bbbk})_{\mathrm{red}}$ is called the *Picard variety* of X, which is an abelian variety over $\Bbbk$, see Proposition 0.9.13. Let

$$\begin{aligned} \mathrm{NS}(X) \quad &:= \mathbf{Pic}_{X/\Bbbk}(\Bbbk)/\mathbf{Pic}^{\circ}_{X/\Bbbk}(\Bbbk), \\ \mathrm{Num}(X) &:= \mathbf{Pic}_{X/\Bbbk}(\Bbbk)/\mathbf{Pic}^{\tau}_{X/\Bbbk}(\Bbbk) \end{aligned}$$

be the quotient groups of divisor classes on X modulo algebraic and numerical) equivalence, respectively. The group $\mathrm{NS}(X)$ is called the *Néron–Severi group* of X.

In the case where X is a smooth projective surface over $\Bbbk$, then we have an *intersection form* on $\mathrm{Pic}(X)$, which can be defined as

$$\mathrm{Pic}(X)\times\mathrm{Pic}(X) \;\to\; \mathbb{Z},\ (\mathcal{L}_1, \mathcal{L}_2) \;\mapsto\; \chi(\mathcal{O}_X)-\chi(\mathcal{L}_1^{-1})-\chi(\mathcal{L}_2^{\otimes -1})+\chi(\mathcal{L}_1^{-\otimes 1}\otimes\mathcal{L}_2^{\otimes -1}).$$

If $\mathcal{L}_1 = \mathcal{O}_X(D_1)$ and $\mathcal{L}_2 = \mathcal{O}_X(D_2)$ for some Cartier divisors D_1 and D_2, then we denote the value of the intersection form on $(\mathcal{L}_1, \mathcal{L}_2)$ by $\mathcal{L}_1 \cdot \mathcal{L}_2$ (resp. $D_1 \cdot D_2$). We set $\mathcal{L}^2 := \mathcal{L} \cdot \mathcal{L}$ (resp. $D^2 := D \cdot D$). We refer to [538, Lecture 12] and [294, Chapter 5] for the above definition, as well as for different expressions of the value of the intersection form in terms of the intersection theory of Cartier divisors. The intersection form on $\mathrm{Pic}(X)$ is trivial on $\mathbf{Pic}^{\tau}_{X/\Bbbk}$, and thus, intersection forms

$$\mathrm{NS}(X) \times \mathrm{NS}(X) \;\to\; \mathbb{Z} \quad \mathrm{Num}(X) \otimes \mathrm{Num}(X) \;\to\; \mathbb{Z}. \tag{0.9.3}$$

Since $\Bbbk$ is a field, we have $\mathbf{Pic}_{X/\Bbbk}(\Bbbk) = (\mathbf{Pic}_{X/\Bbbk})_{\mathrm{red}}(\Bbbk)$ and similarly for $\mathbf{Pic}^{\circ}_{X/\Bbbk}$ and $\mathbf{Pic}^{\tau}_{X/\Bbbk}$. It follows from Theorem 0.9.6 that the torsion subgroup Tors of $\mathrm{NS}(X)$ is finitely generated and that there exists an isomorphism

$$\mathrm{Num}(X) \;\cong\; \mathrm{NS}(X)/\mathrm{Tors}\,.$$

It is a deep theorem that the abelian group $\mathrm{Num}(X)$ is finitely generated, see [86, Chapter 8] or [392] for details and further references. Its rank is denoted by $\rho(X)$ and it is called the *Picard number* of X.

For surfaces, the intersection form (0.9.3) defines a structure of a quadratic lattice on $\mathrm{Num}(X)$

$$\mathrm{Num}(X) \times \mathrm{Num}(X) \to \mathbb{Z}. \tag{0.9.4}$$

It is called the *Picard lattice* of X.

In connection with this discussion, we also have the following result concerning Picard ranks in families, which is due to Ekedahl, Hyland, and Shepherd-Barron, see [214, Proposition 4.2].

Proposition 0.9.14 *Let $f : \mathcal{X} \to S$ be a smooth and projective morphism such that S is Noetherian, $f_*\mathcal{O}_\mathcal{X} \cong \mathcal{O}_S$, and such that*

$$\frac{1}{2} b_1(\mathcal{X}_{\bar{s}}) = h^1(\mathcal{O}_{\mathcal{X}_{\bar{s}}}) - h^2(\mathcal{O}_{\mathcal{X}_{\bar{s}}})$$

for every geometric point $\bar{s} \to S$. Then, the Picard number of $\mathcal{X}_{\bar{s}}$ in this family is locally constant.

Example 0.9.15 Let X be a proper curve over an algebraically closed field $\Bbbk$. Then, $\mathbf{Pic}_{X/\Bbbk}$ exists as a reduced group scheme that is locally of finite type over $\Bbbk$ and $J(X) := \mathbf{Pic}^\circ_{X/\Bbbk}$ is called the (generalized) *Jacobian variety* of X, see also Remark 0.9.20. Let $X = X_1 \cup \ldots \cup X_r$ be the decomposition of X into irreducible components. Let m_i be the multiplicity of X_i, which is defined to be the length of the local artinian ring $\mathcal{O}_{X,\eta_i}$ where η_i denotes the generic point of X_i. Then, $\mathrm{NS}(X)$ is a free abelian group of rank r. More precisely, the homomorphism

$$\begin{array}{ccc} \mathrm{NS}(X) & \to & \mathbb{Z}^r \\ \mathcal{L} & \mapsto & \left(\deg \mathcal{L}|_{X_1}, \ldots, \deg \mathcal{L}|_{X_r}\right) \end{array}$$

is injective and has finite cokernel. Let X_{red} be the largest reduced subscheme of X and let $\widetilde{X}$ be the normalization of X_{red}, which is the product of the normalizations $\widetilde{X}_i$ of the X_i. Moreover, let $X' \to X_{\mathrm{red}}$ be the largest curve between $\widetilde{X}$ and X, which is homeomorphic to X. Then, there are canonical surjective homomorphisms

$$\mathbf{Pic}^\circ_{X/\Bbbk} \xrightarrow{\alpha} \mathbf{Pic}^\circ_{X_{\mathrm{red}}/\Bbbk} \xrightarrow{\beta} \mathbf{Pic}^\circ_{X'/\Bbbk} \xrightarrow{\gamma} \mathbf{Pic}^\circ_{\widetilde{X}/\Bbbk},$$

where:

1. $\mathrm{Ker}(\alpha)$ is a smooth, connected, and unipotent group scheme, which is a successive extension of additive group schemes of type $\mathbb{G}_a$,
2. $\mathrm{Ker}(\beta)$ is a connected and unipotent group scheme, which is trivial if and only if the canonical morphism $X' \to X_{\mathrm{red}}$ is an isomorphism,
3. $\mathrm{Ker}(\gamma)$ is a torus, which is trivial if and only if every morphism $\widetilde{X}_i \to X_i$ is a homeomorphism and if the configuration of the components of X is tree-like.

In particular, this relates the Chevalley–Rosenlicht decomposition of the group scheme $\mathbf{Pic}^\circ_{X/\Bbbk}$ to the geometry of X. We refer to [86, Chapter 9.2], for details, proofs, and further results.

As an application of the results of this section, we briefly discuss Albanese varieties: given a variety X over an algebraically closed field $\mathbb{k}$, one can ask whether there exists a morphism $f : X \to A$ to a commutative group variety over $\mathbb{k}$ that is universal with respect to morphisms to commutative group varieties. More precisely, let C be a category of commutative group varieties over $\mathbb{k}$ such that:

1. if A_1, A_2 are in C, then so is $A_1 \times_{\operatorname{Spec}\mathbb{k}} A_2$,
2. if $f : A \to B$ is morphism of commutative group schemes over $\mathbb{k}$ with $B \in C$ and $\operatorname{Ker}(f)$ is a finite group scheme over $\mathbb{k}$, then $A \in C$.

For example, C could be the category of abelian varieties over $\mathbb{k}$, which is the classical setup, but one could also consider the category of tori over $\mathbb{k}$ or the category of semi-abelian varieties over $\mathbb{k}$, whose objects are extensions of abelian varieties by tori. Next, we fix a base point $x_0 \in X$. Then, an *Albanese variety of* (X, x_0) *with respect to* C is a morphism $f : X \to A$ with $A \in C$ and $f(x_0) = 0_A$, where 0_A denotes the neutral element of A, such that whenever $g : X \to B$ is a morphism with $B \in C$ and $g(x_0) = 0_B$, then there exists a unique morphism $\psi : A \to B$ of group schemes over $\mathbb{k}$ such that $g = \psi \circ f$. Being characterized by a universal property, it is clear that if an Albanese variety of (X, x_0) with respect to C exists, then it is unique up to unique isomorphism. In the following cases, Serre [654] established the existence.

Theorem 0.9.16 *Let X be a variety over an algebraically closed field $\mathbb{k}$ with base point $x_0 \in X$.*

1. *There exists an Albanese variety of (X, x_0) with respect to the category C of abelian varieties.*
2. *Let C be a category as above such that the additive group $\mathbb{G}_a$ is not an object of C. Then, there exists an Albanese variety of (X, x_0) with respect to C.*
3. *Let C be a category as above and let X be proper over $\mathbb{k}$. Then, there exists an Albanese variety of (X, x_0) with respect to C.*

We note that if X is a proper variety over $\mathbb{k}$, then any morphism to an affine scheme over $\mathbb{k}$ is constant. Thus, when discussing Albanese varieties for proper varieties over $\mathbb{k}$, it usually suffices to restrict to the category C of abelian varieties over $\mathbb{k}$. If X is moreover normal, then we can be very specific about its Albanese variety.

Theorem 0.9.17 *Let X be a normal and projective variety over an algebraically closed field $\mathbb{k}$ with base point $x_0 \in X$. Then, the Albanese variety of (X, x_0) with respect to the category C of abelian varieties over $\mathbb{k}$ exists and it is isomorphic to the dual of the Picard variety. In particular, it is of dimension $\frac{1}{2}b_1(X)$.*

Proof Existence follows from Theorem 0.9.16, but it also follows directly from the construction that we now give: the Picard variety $P(X) := (\mathbf{Pic}^\circ_{X/\mathbb{k}})_{\mathrm{red}}$ is an abelian variety over $\mathbb{k}$ by Proposition 0.9.10 and let $\operatorname{Alb}(X) := P(X)^\vee$ be the dual abelian variety as defined in Example 0.9.12. Let $\mathcal{P}$ be the Poincaré sheaf on $X \times_{\operatorname{Spec}\mathbb{k}} \mathbf{Pic}^\circ_{X/\mathbb{k}}$, let $\mathcal{P}_{\mathrm{red}}$ be its restriction to $X \times_{\operatorname{Spec}\mathbb{k}} P(X)$, and let $\mathcal{F}$ be the Poincaré sheaf on $P(X) \times_{\operatorname{Spec}\mathbb{k}} \operatorname{Alb}(X)$. Next, let $\sigma : P(X) \times_{\operatorname{Spec}\mathbb{k}} X \to X \times_{\operatorname{Spec}\mathbb{k}} P(X)$ be the isomorphism that is defined by $(a, b) \mapsto (b, a)$. By the universal property of the dual abelian variety $P(X)^\vee$, there exists a unique and canonical morphism

$$\mathrm{alb}_X \,:\, X \,\to\, \mathrm{Alb}(X),$$

such that $\sigma^*(\mathcal{P}_{\mathrm{red}}) = (\mathrm{id}_{P(X)} \times \mathrm{alb}_X)^*\mathcal{F}$ and $\mathrm{alb}_X(x_0) = 0_{\mathrm{Alb}(X)}$.

To show that this is the Albanese variety with respect to the category C of abelian varieties, let $g : X \to B$ be a morphism to an abelian variety over $\Bbbk$ with $g(x_0) = 0_B$. Then, the pull-back of invertible sheaves induces a morphism of group schemes from $B^\vee = \mathbf{Pic}^\circ_{B/\Bbbk}$ to $\mathbf{Pic}^\circ_{X/\Bbbk}$. Passing to the reduction $P(X)$ of $\mathbf{Pic}^\circ_{X/\Bbbk}$ and then, to dual abelian varieties, we obtain a morphism of abelian varieties $\psi : \mathrm{Alb}(X) = P(X)^\vee \to (B^\vee)^\vee \cong B$. It is easy to see that $g = \psi \circ \mathrm{alb}_X$. Using various universal properties, we leave to the reader to check that $(\mathrm{Alb}(X), \mathrm{alb}_X)$ is the Albanese variety of (X, x_0) with respect to C, see also the discussion in [38, Chapter 5]. The statement about the dimension of $\mathrm{Alb}(X)$ follows from Proposition 0.9.10. □

Under the assumptions of the previous theorem, the pair $(\mathrm{Alb}(X), \mathrm{alb}_X)$ is simply called the *Albanese variety* of the pair (X, x_0), the map alb_X is called the *Albanese morphism*, and we will drop the category C in the future and oftentimes, even the base point $x_0 \in X$.

Let us mention another interesting feature of the Albanese map concerning the pull-back of global differential 1-forms from the Albanese variety, which is due to Igusa [324]. What makes it interesting is that it is even true if the ground field $\Bbbk$ is of positive characteristic, where the Albanese morphism could be inseparable.

Proposition 0.9.18 *Let X be a smooth and projective variety over an algebraically closed field $\Bbbk$ with Albanese morphism $\mathrm{alb}_X : X \to \mathrm{Alb}(X)$. Then, the natural map*

$$\mathrm{alb}_X^* \,:\, H^0\left(\mathrm{Alb}(X),\, \Omega^1_{\mathrm{Alb}(X)/\Bbbk}\right) \,\to\, H^0\left(X,\, \Omega^1_{X/\Bbbk}\right)$$

is injective.

If $\Bbbk$ is of characteristic zero, then this map is even an isomorphism (see below), whereas this need not be the case if $\Bbbk$ is of positive characteristic, see Remark 0.9.22. The following two classes of varieties behave well in any characteristic.

Theorem 0.9.19 *Let X be a smooth and projective variety over an algebraically closed field $\Bbbk$. Assume that:*

1. *X is a curve of genus $g(X) := h^{1,0}(X) = h^0(X, \omega_X)$ or that*
2. *X is an abelian variety of dimension $g(X)$.*

Then, the Picard scheme $\mathbf{Pic}^\circ_{X/\Bbbk}$ is an abelian variety of dimension $g(X)$, we have the equalities

$$g(X) = \frac{1}{2} b_1(X) = h^{0,1}(X) = h^{1,0}(X),$$

and the map alb_X^* from Proposition 0.9.18 is an isomorphism.

Proof First, assume that X is a curve. Since $H^2(X, \mathcal{O}_X) = 0$, it follows that $\mathbf{Pic}^\circ_{X/\Bbbk}$ is reduced by Proposition 0.9.10 and thus, we find $\frac{1}{2}b_1(X) = h^{0,1}(X)$. Since X is a curve, Serre duality gives an isomorphism $H^0(\omega_X) \cong H^1(\mathcal{O}_X)^\vee$ and thus, implies $h^{1,0}(X) = h^{0,1}(X)$. By Proposition 0.9.18, the map alb_X^* is injective, we have $h^{1,0}(X) = g(X)$ and since $h^0(\Omega^1_{\mathrm{Alb}(X)/\Bbbk}) \geq \dim \mathrm{Alb}(X) = \frac{1}{2}b_1(X) = g(X)$, it follows that alb_X^* is an isomorphism.

Second, assume that X is an abelian variety. Then, $\mathbf{Pic}^\circ_{X/\Bbbk}$ is reduced, that is, an abelian variety, see [541, Section 13]. By loc.cit., we have

$$h^i(X, \mathcal{O}_X) = \binom{g(X)}{i}, \tag{0.9.5}$$

which, together with Serre duality, gives the assertion on $h^{0,1}(X) = h^{1,0}(X) = g(X)$. By Proposition 0.9.18, the map alb_X^* is injective, and since both both vector spaces are of dimension $g(X)$, this map is an isomorphism. $\square$

We refer to Example 0.10.15 below or [541] for more on the cohomology of abelian varieties.

Remark 0.9.20 Let X be a smooth and projective curve over an algebraically closed field $\Bbbk$. Then, the Picard variety $\mathbf{Pic}^\circ_{X/\Bbbk}$ comes with a natural principal polarization, and thus, it is naturally isomorphic to the Albanese variety of X. In this case, these two abelian varieties are called the *Jacobian variety* of X. For higher–dimensional varieties, the Picard variety and the Albanese variety are in general not isomorphic.

Combining these results with Proposition 0.9.10 and Proposition 0.9.18, we conclude the following.

Corollary 0.9.21 *Let X be a smooth and projective variety over an algebraically closed field $\Bbbk$. Then, the inequalities*

$$\begin{array}{l} \frac{1}{2}b_1(X) \leq h^{0,1}(X) = \dim_\Bbbk H^1(X, \mathcal{O}_X) \quad \textit{and} \\ \frac{1}{2}b_1(X) \leq h^{1,0}(X) = \dim_\Bbbk H^0(X, \Omega^1_{X/\Bbbk}) \end{array}$$

hold true.

Remark 0.9.22 If $\Bbbk$ is of characteristic zero, then both inequalities are equalities and we obtain the *Hodge symmetry* $h^{0,1}(X) = h^{1,0}(X)$. In fact, we established the equality $h^{0,1}(X) = \frac{1}{2}b_1(X)$ in characteristic zero in Proposition 0.9.10 and mentioned in Remark 0.9.11 that it may fail in positive characteristic. The first examples of smooth and projective varieties with $\frac{1}{2}b_1(X) \neq h^{1,0}(X)$ are due to Igusa [325] and Mumford [536] constructed Enriques surfaces in characteristic 2 with this property. We refer to [456] for examples of smooth and projective surfaces in characteristic 2, where the differences $(h^{0,1} - \frac{1}{2}b_1)$ and $(h^{1,0} - \frac{1}{2}b_1)$ get arbitrarily large. We refer to [458], Sections 3.3 and 3.4 and Section 0.10 for further discussion.

By the Lefschetz principle, the just–mentioned Hodge symmetry in characteristic zero follows from the following remark or the results of the next section.

Remark 0.9.23 Let X be a smooth and projective variety over $\mathbb{C}$, which we may also consider as a complex manifold. Then,

1. the Picard scheme $\mathbf{Pic}^{\circ}_{X/\mathbb{C}}$ is reduced, that is, an abelian variety by Proposition 0.9.10. Using analytic methods, one can show that there exists an isomorphism of abelian groups
$$\mathbf{Pic}^{\circ}_{X/\mathbb{C}} \cong H^1(X, \mathcal{O}_X)/H^1(X, \mathbb{Z}),$$
and that the right–hand side carries the structure of an abelian variety. Here, the map $H^1(X, \mathbb{Z}) \to H^1(X, \mathcal{O}_X)$ comes from taking cohomology in the exponential sequence, see (0.10.9) below.
2. Similarly, the Albanese variety can be constructed analytically as
$$\mathrm{Alb}(X) \cong H^0(X, \Omega^1_{X/\mathbb{C}})^\vee / H_1(X, \mathbb{Z}).$$
Next, after choosing a base point $x_0 \in X$ and a $\mathbb{C}$-basis $\omega_1, ..., \omega_g$ of $H^0(X, \Omega^1_{X/\mathbb{C}})$, there is a well-defined and analytic map
$$\begin{array}{rcl} \mathrm{alb}_X : X & \to & \mathrm{Alb}(X) \\ x & \mapsto & \left(\int_{x_0}^{x} \omega_1, ..., \int_{x_0}^{x} \omega_g \right), \end{array}$$
which gives an analytic construction of the Albanese morphism.

By Hodge theory (see Section 0.10), we have equalities
$$h^1(X, \mathcal{O}_X) = h^0(X, \Omega^1_{X/\mathbb{C}}) = \frac{1}{2} b_1(X),$$
Rahm–Witt which shows Hodge symmetry $h^{0,1}(X) = h^{1,0}(X)$ and gives an analytic proof of the fact that $\dim \mathbf{Pic}^{\circ}_{X/\mathbb{C}} = \dim \mathrm{Alb}(X) = \frac{1}{2} b_1(X)$, see Proposition 0.9.10 and Theorem 0.9.17. In this case, the map alb_X^* from Proposition 0.9.18 is an isomorphism. We refer to [259, Chapter 2.6] for details and proofs.

0.10 Cohomology of Algebraic Surfaces

In this section, we will briefly recall several cohomology theories (singular, de Rham, ℓ-adic, crystalline, de Rham–Witt, and flat) for smooth and projective varieties and discuss their interplay. We have chosen the material with a view toward algebraic surfaces and we have included some discussion of cohomology groups of small degree.

First, we consider varieties over the complex numbers from the topological point of view: let X be a smooth and projective variety of dimension d over the complex numbers. We can view X as a topological manifold (with respect to the classical topology) of real dimension $2d$ and we may study singular simplices that is, continuous maps $\sigma : \Delta^i \to X$, where Δ^i denotes the standard i-simplex. We denote by $C_i(X)$ the free abelian group generated by these singular simplices, and taking boundaries ∂ (with the correct sign convention) gives rise to a complex $C_\bullet(X)$, the *singular chain complex*. Dually, we have singular cochains $\mathrm{Hom}(C_i(X), \mathbb{Z})$ and a dual chain complex $C^\bullet(X) := \mathrm{Hom}(C_\bullet, \mathbb{Z})$. For an abelian group A, we define the *i-th (singular) homology group of X with values in A*, denoted by $H_i(X, A)$, to be the i-th homology group of the chain complex $C_\bullet(X) \otimes_\mathbb{Z} A$. Similarly, we define the *i-th (singular) cohomology of X with values in A* to be the cohomology of the cochain complex $C^\bullet(X) \otimes_\mathbb{Z} A$. Since X is of real dimension $2d$, we have $H_i(X, A) = 0$ and $H^i(X, A) = 0$ for all abelian groups A if $i < 0$ or if $i > 2d$. The relation between homology and cohomology, as well as the relation of homology with coefficients in $\mathbb{Z}$ and A, is given by the following *universal coefficient formulas*:

$$\begin{array}{llllll} 0 \to & \mathrm{Ext}^1\left(H_{i-1}(X, \mathbb{Z}), A\right) & \to H^i(X, A) \to & \mathrm{Hom}\left(H_i(X, \mathbb{Z}), A\right) & \to 0, \\ 0 \to & H_i(X, \mathbb{Z}) \otimes A & \to H_i(X, A) \to & \mathrm{Tor}_1\left(H_{i-1}(X, \mathbb{Z}), A\right) & \to 0. \end{array} \tag{0.10.1}$$

In particular, if the torsion subgroup $H_{i-1}(X, \mathbb{Z})$ is zero, then we find $H^i(X, \mathbb{Z}) \cong \mathrm{Hom}(H_i(X, \mathbb{Z}), \mathbb{Z})$. Concerning the torsion subgroups, we also have the following notable shift:

$$\begin{aligned} \mathrm{Tors}\ H^i(X, \mathbb{Z}) &\cong \mathrm{Hom}\left(\mathrm{Tors}\ H_{i-1}(X, \mathbb{Z}), \mathbb{Q}/\mathbb{Z}\right), \\ \mathrm{Tors}\ H_i(X, \mathbb{Z}) &\cong \mathrm{Hom}\left(\mathrm{Tors}\ H^{i+1}(X, \mathbb{Z}), \mathbb{Q}/\mathbb{Z}\right). \end{aligned}$$

Next, there is the *cap-product* $\cap : C_k(X) \times C^m(X) \to C_{k-m}(X)$, $k \geq m$, which is defined via

$$\sigma \cap \phi := \phi\left(\sigma|[t_0, \ldots, t_m]\right) \sigma|[t_m, \ldots, t_k], \quad \sigma \in C_k(X), \phi \in C^m(X).$$

For all i, the cap-product induces an isomorphism

$$H_i(X, \mathbb{Z}) \cong H^{2d-i}(X, \mathbb{Z}), \tag{0.10.2}$$

called *Poincaré duality*. Combining Poincaré duality with the homomorphism $H^i(X, \mathbb{Z}) \to \mathrm{Hom}(H_i(X, \mathbb{Z}), \mathbb{Z})$ from (0.10.1), we obtain the *cup-product pairing*

$$H^i(X, \mathbb{Z}) \times H^{2d-i}(X, \mathbb{Z}) \to \mathbb{Z} \tag{0.10.3}$$

Let $H^i(X, \mathbb{Z})' := H^i(X, \mathbb{Z})/\mathrm{Tors}$. The cup-product pairing defines a perfect pairing

$$H^i(X, \mathbb{Z})' \times H^{2d-i}(X, \mathbb{Z})' \to \mathbb{Z}. \tag{0.10.4}$$

Concerning torsion subgroups, we find isomorphisms

$$\mathrm{Tors}\ H^i(X, \mathbb{Z}) \cong \mathrm{Tors}\ H_{2d-i}(X, \mathbb{Z}) \cong \mathrm{Hom}\left(\mathrm{Tors}\ H^{2d-i+1}(X, \mathbb{Z}), \mathbb{Q}/\mathbb{Z}\right). \tag{0.10.5}$$

Since X is a compact manifold, all singular (co-)homology groups $H_i(X, A)$ and $H^i(X, A)$ are finitely generated A-modules. The rank of $H^i(X, \mathbb{Z})$, which coincides with the ranks of $H_i(X,\mathbb{Z})$, $H_{2d-i}(X,\mathbb{Z})$, and $H^{2d-i}(X,\mathbb{Z})$ by (0.10.2) and (0.10.4), is called the i-th *Betti number* of X and denoted by $b_i(X)$. Moreover, the alternating sum

$$e(X) \;:=\; \sum_{i=0}^{2d} (-1)^i\, b_i(X)$$

is called the *Euler–Poincaré characteristic* of X. All of this can be found in any good textbook on algebraic topology, see, for example, [299] or [544].

Second, we still assume X to be a smooth and complex projective variety of dimension d, but now, we consider it as a differentiable manifold of real dimension $2d$. Then, we have the *de Rham complex*

$$0 \;\to\; \mathcal{A}_X^0 \;\xrightarrow{d}\; \mathcal{A}_X^1 \;\xrightarrow{d}\; \cdots \xrightarrow{d} \mathcal{A}_X^{2d} \;\to\; 0,$$

where $\mathcal{A}_X^i$ denotes the sheaf (with respect to the classical topology) of real-valued smooth differential i-forms. We let $A^i(X) := H^0(X, \mathcal{A}_X^i)$ be the real vector space of global smooth differential i-forms on X. Then, de Rham's Theorem states that, for every i, there exists a natural isomorphism

$$H^i(X,\mathbb{R}) \;\cong\; \operatorname{Ker}(A^i(X) \xrightarrow{d} A^{i+1}(X))/\operatorname{Im}(A^{i-1}(X) \xrightarrow{d} A^i(X)), \qquad (0.10.6)$$

that is, *de Rham cohomology* is isomorphic to singular cohomology with coefficients in $\mathbb{R}$. Moreover, under this isomorphism the isomorphism $H^i(X,\mathbb{R})\times H^{n-i}(X,\mathbb{R}) \to \mathbb{R}$ coming from Poincaré duality (0.10.4) coincides with the one coming from the pairing arising from integrating differential forms over X:

$$\langle [\omega], [\tau] \rangle \;=\; \int_X \omega \wedge \tau.$$

Again, this is classic and can be found, for example, in [732].

Third, let us consider X as a complex manifold. Then there are three differentials: d (real differentiation), as well as ∂ and $\overline{\partial}$ (holomorphic and anti-holomorphic differentiation, respectively). Also, we obtain a decomposition of the complexified space of global differential i-forms

$$A^i_{\mathbb{C}}(X) \;:=\; A^i(X) \otimes_{\mathbb{R}} \mathbb{C} \;\cong\; \bigoplus_{p+q=i} A^{p,q}_{\mathbb{C}}(X)$$

into (p,q)-forms, that is, differential forms that are locally (in the classical topology) sums of the form $f \cdot dz_{i_1} \wedge \ldots \wedge dz_{i_p} \wedge d\overline{z}_{j_1} \wedge \ldots \wedge d\overline{z}_{j_q}$ for some complex-valued C^∞-function f and in some holomorphic coordinates $z_1, \ldots, z_d$ on X. We also assumed that X is projective, that is, it admits a holomorphic embedding into $\mathbb{P}^N_{\mathbb{C}}$ for some N. Since a projective space is a *Kähler manifold*, for example, via the Fubini-Study metric, we can restrict a Kähler metric from $\mathbb{P}^N_{\mathbb{C}}$ to X, which implies that X is also a Kähler manifold. With respect to the choice of a Kähler metric on X, there exists a unique Hodge star operator $*$, which allows us to construct adjoint operators

$d^* = -*d*$, and similarly ∂^* and $\overline{\partial}^*$. This gives rise to three Laplace operators on X, namely $\Delta_d := dd^* + d^*d$, and similarly Δ_∂ and $\Delta_{\overline{\partial}}$. By definition, forms $\omega \in A^i(X)$ with $\Delta_d(\omega) = 0$ are called *harmonic* and by a fundamental theorem in Hodge theory, there exist isomorphisms

$$\operatorname{Ker}\Delta_d|_{A^i_{\mathbb{C}}(X)} \;\cong\; \operatorname{Ker}(A^i_{\mathbb{C}}(X) \xrightarrow{d} A^{i+1}_{\mathbb{C}}(X))/\operatorname{Im}(A^{i-1}_{\mathbb{C}}(X) \xrightarrow{d} A^i_{\mathbb{C}}(X)),$$

that is, every cohomology class in de Rham cohomology has a unique harmonic representative. By (0.10.6), this cohomology group is isomorphic to the singular cohomology group $H^i(X, \mathbb{C})$. Next, we have sheaves (with respect to the classical topology) Ω^i_X of holomorphic differential forms and we define the *Hodge cohomology* groups to be

$$H^{p,q}(X) \;:=\; H^q(X, \Omega^p_X).$$

Since X is a compact complex manifold, these are finite–dimensional complex vector spaces and we define $h^{p,q}(X)$ to be the dimension of $H^{p,q}(X)$. Since X is projective, it follows from Serre's GAGA theorems [652] that these cohomology groups coincide with the cohomology groups of the same name (this is a deliberate abuse of notation) if X is considered as an algebraic variety, if Ω^p_X is considered as a sheaf (with respect to the Zariski topology) that is a coherent $\mathcal{O}_X$-module, and if H^q is meant in the sense of cohomology of coherent sheaves. Similar to the above, Hodge theory provides us with isomorphisms

$$\operatorname{Ker}\Delta_{\overline{\partial}}|_{A^{p,q}_{\mathbb{C}}(X)} \;\cong\; H^{p,q}(X).$$

It follows from the fact that the metric used to define the Hodge star operator $*$ is Kähler, and that the three Laplacians are related by the formula $\Delta_d = 2\Delta_\partial = 2\Delta_{\overline{\partial}}$, which is the key to relating de Rham cohomology to Hodge cohomology. More precisely, the *Hodge decomposition* theorem for Kähler manifolds states an isomorphism

$$H^i(X, \mathbb{C}) \;\cong\; \bigoplus_{p+q=i} H^{p,q}(X). \tag{0.10.7}$$

Moreover, complex conjugation induces the *Hodge symmetry* isomorphisms

$$H^{p,q}(X) \;\cong\; \overline{H^{q,p}(X)}.$$

As a direct consequence, this implies that $h^{p,q}(X) = h^{q,p}(X)$ for all $p, q \geq 0$ and that the Betti number $b_i(X)$ is even if i is odd. Let us also note that Poincaré duality, Serre duality, Hodge symmetry, and the Hodge decomposition imply the following equalities for a Kähler manifold of complex dimension d

$$\begin{array}{l} b_i \quad\;\; = \sum_{p+q=i} h^{p,q}, \\ b_{2d-i} = b_i, \\ h^{2d-q,2d-p} \;=\; h^{p,q} \;=\; h^{q,p} \;=\; h^{2d-p,2d-q}. \end{array}$$

Again, all this is classical and we refer to [259], [320], or [725] for details, proofs, and further background. For $i = d$, Poincaré duality defines a non-degenerate bilinear form on $H^d(X,\mathbb{R})$, which is symmetric if d is even and skew-symmetric if d is odd. Assume that d is even. Then, the Sylvester signature (t_+, t_-) gives rise to the *signature of* X, which is defined to be $I(X) := t_+ - t_-$. Using the Lefschetz decomposition, one can show that

$$I(X) \;=\; \sum_{p \equiv q \mod 2} (-1)^p\, h^{p,q} \qquad \text{and thus,} \qquad t_\pm \;=\; \frac{1}{2}(b_d(X) \pm I(X)), \quad (0.10.8)$$

see [259, Chapter 0.7]. By *Hirzebruch's signature theorem*, the signature coincides with the *L-genus* of the underlying differentiable manifold. For example, if X is a complex projective surface, this becomes

$$I(X) \;=\; \frac{1}{3} p_1(X) \;=\; \frac{1}{3}(c_1(X)^2 \;-\; 2c_2(X)),$$

where the p_i denote the Pontryagin classes and the c_i denote the Chern classes of the tangent bundle of X. We refer to [306] or [446] for details, background, and proofs.

Before continuing, we will introduce some notations for abelian groups, which we will use often in the sequel. Let ℓ be a prime number. For any abelian group A, we denote by $[n] : A \to A$ multiplication by $n \in \mathbb{Z}$ (or raising to the n-th power in the case where the group is written multiplicatively). Then, we set

$$\begin{array}{ll} {}_nA & := \mathrm{Ker}([n] : A \to A), \\ A^{(n)} & := \mathrm{Coker}([n]) \;=\; A/nA \;\cong\; A \otimes_{\mathbb{Z}} (\mathbb{Z}/n\mathbb{Z}), \\ {}_{\ell^\infty}A & := \varinjlim_n \mathrm{Ker}([l^n] : A \to A), \\ T_\ell(A) & := \varprojlim_n \mathrm{Ker}([\ell^n] : A \to A), \end{array}$$

and $T_\ell(A)$ is called the *ℓ-adic Tate module* of A. For example, we have $T_\ell(\mathbb{Q}/\mathbb{Z}) \cong \mathbb{Z}_\ell$. Let us also recall that an abelian group A is called *divisible* (resp. *uniquely divisible*) if the equation $n \cdot x = a$ can be solved (resp. uniquely solved) in A for every $n \in \mathbb{Z}$ and every $a \in A$. For example, the additive group of a linear space is a divisible group (uniquely divisible if the characteristic is zero), and, since every quotient of a divisible group is divisible, $\mathbb{Q}/\mathbb{Z}$ and $\mathbb{Q}_\ell/\mathbb{Z}_\ell$ are divisible groups. A divisible group is an injective object in the category of abelian groups. In particular, every divisible abelian group inside some abelian group is automatically a direct summand. Therefore, every abelian group is the direct sum of a divisible group and an abelian group that does not contain any non-trivial divisible subgroups. The main result in the theory of divisible groups asserts that every divisible group is isomorphic to a direct sum of groups isomorphic to $\mathbb{Q}$ and to groups isomorphic to $\mathbb{Q}_\ell/\mathbb{Z}_\ell$ for various primes ℓ, see [361]. Let us also recall the following result from loc.cit., Theorem 10.

Proposition 0.10.1 *Let ℓ be a prime, let A be an abelian ℓ-group, that is, $A = {}_{\ell^\infty}A$, and assume that ${}_\ell A$ is a finite group. Then*

$$A \cong (\mathbb{Q}_\ell/\mathbb{Z}_\ell)^n \oplus F,$$

where F is a finite abelian ℓ-group and $n \leq \dim_{\mathbb{F}_\ell} {}_\ell A$. In particular, we have

$$T_\ell(A) \cong \mathbb{Z}_\ell^n.$$

We continue with various *Chern class maps* for invertible sheaves. First, assume that X is a complex projective variety. Then, there exists a Chern class homomorphism

$$c_1 : \mathrm{Pic}(X) \to H^2(X, \mathbb{Z}),$$

which is induced from the natural map from the group of divisors that assigns to a divisor $D = \sum n_i E_i$, where the E_i are prime divisors, its fundamental class $[D] := \sum n_i [E_i]$. Alternatively, one can start from the *exponential sequence*

$$0 \to \mathbb{Z} \to \mathcal{O}_X \xrightarrow{\exp} \mathcal{O}_X^\times \to 0, \tag{0.10.9}$$

where X is equipped with the classical topology in order for this sequence to be exact. After identifying $\mathrm{Pic}(X)$ with $H^1(X, \mathcal{O}_X^\times)$ as in (0.9.1), it follows from the long exact sequence in cohomology that there is an exact sequence

$$H^1(X, \mathcal{O}_X) \to \mathrm{Pic}(X) \xrightarrow{\delta} H^2(X, \mathbb{Z}).$$

Then, one can identify δ with c_1 and the tangent space of $\mathrm{Pic}^\circ(X)$ at zero with $H^1(X, \mathcal{O}_X)$ (see Proposition 0.9.7), which eventually shows that the kernel of δ is equal to $\mathrm{Pic}^\circ(X)$. Thus, we obtain an injective homomorphism

$$c_1^{\mathrm{NS}} : \mathrm{NS}(X) \to H^2(X, \mathbb{Z}),$$

which induces an isomorphism of torsion subgroups and which is compatible with the intersection form (0.9.3) on $\mathrm{NS}(X)$ and the cup-product pairing (0.10.3) on the cohomology groups.

Another approach to the first Chern class is via the map $\mathrm{dlog} : \mathcal{O}_X^\times \to \Omega^1_{X/\mathbb{C}}$ that is locally defined by $f \mapsto \frac{df}{f}$. Taking cohomology and identifying again $\mathrm{Pic}(X)$ with $H^1(X, \mathcal{O}_X^\times)$, we obtain a homomorphism

$$\mathrm{dlog} : \mathrm{Pic}(X) \to H^1(X, \Omega^1_{X/\mathbb{C}}).$$

By our discussions above, we have inclusions of $H^2(X, \mathbb{Z})/\mathrm{Tors}$ and $H^{1,1} = H^1(X, \Omega^1_{X/\mathbb{C}})$ into $H^2(X, \mathbb{C})$ and by definition, the intersection $H^{1,1} \cap H^2(X, \mathbb{Z})$ of the two inside $H^2(X, \mathbb{C})$ is called the space of *integral* $(1,1)$*-classes*. By the Lefschetz theorem on $(1,1)$-classes, the image of c_1 inside $H^2(X, \mathbb{C})$ is equal to the space of integral $(1,1)$-classes. In particular, if ρ denotes the *Picard number* of X, that is, the rank of the Néron–Severi group $\mathrm{NS}(X)$ as an abelian group, then we obtain inequalities

$$\rho(X) \leq h^{1,1}(X) \leq b_2(X)$$

for complex projective varieties. We refer to [259, Chapter 1.2] for details and proofs. Next, by the Hodge decomposition and Hodge symmetries, we have $b_2 = h^{1,1} + 2h^{2,0}$ and we find

$$\mathrm{t}(X) \;:=\; b_2(X) - \rho(X) \;\geq\; b_2(X) - h^{1,1}(X) \;=\; 2h^{2,0}(X). \tag{0.10.10}$$

Now, assume that X is a surface, still over the complex numbers. Then, the Poincaré duality pairing turns $H^2(X,\mathbb{Z})/\mathrm{Tors}$ into an unimodular lattice and the induced map

$$c_1^{\mathrm{Num}} \;:\; \mathrm{Num}(X) \;\to\; H^2(X,\mathbb{Z})/\mathrm{Tors}$$

is an embedding of lattices. Concerning signatures, we have

lattice	rank	signature
$H^2(X,\mathbb{Z})/\mathrm{Tors}$	b_2	$(2h^{2,0}+1,\ h^{1,1}-1)$
$\mathrm{Num}(X)$	ρ	$(1,\ \rho-1)$

where the signature of the second lattice is as stated by the *Hodge index theorem*, see, for example, [294, Theorem V.1.9]. We define the *transcendental lattice* $T(X) := c_1(\mathrm{Num}(X))^{\perp}$, which is a sublattice of $H^2(X,\mathbb{Z})$. Since the restriction of the pairing on $H^2(X,\mathbb{Z})/\mathrm{Tors}$ to $c_1(\mathrm{Num}(X))$ is non-degenerate, it follows that $c_1(\mathrm{Num}(X)) \cap T(X) = \{0\}$ and thus, the projection

$$T(X) \;=\; \mathrm{Num}(X)^{\perp} \;\to\; \left(H^2(X,\mathbb{Z})/\mathrm{Tors}\right)/\mathrm{Num}(X)$$

induces an injective homomorphism of free abelian groups of rank $\mathrm{t}(X)$. Since the pairing on $H^2(X,\mathbb{Z})' = H^2(X,\mathbb{Z})/\mathrm{Tors}$ is unimodular, we can also identify the transcendental lattice $T(X)$ with the dual of the lattice $H^2(X,\mathbb{Z})'/\mathrm{Num}(X)$. Using the Lefschetz theorem on $(1,1)$-classes, and the fact that c_1^{NS} identifies the torsion subgroups of $\mathrm{NS}(X)$ and $H^2(X,\mathbb{Z})$, we obtain a short exact sequence

$$0 \;\to\; \mathrm{NS}(X) \;\xrightarrow{c_1^{\mathrm{NS}}}\; H^2(X,\mathbb{Z}) \;\to\; T_X^{\vee} \;\to\; 0. \tag{0.10.11}$$

To determine the parity of the lattice $H^2(X,\mathbb{Z})/\mathrm{Tors}$, we consider the composition

$$H^2(X,\mathbb{Z}) \;\to\; H^2(X,\mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{F}_2 \;\to\; H^2(X,\mathbb{F}_2).$$

Poincaré duality induces a pairing $H^2(X,\mathbb{F}_2) \times H^2(X,\mathbb{F}_2) \to \mathbb{F}_2$ of $\mathbb{F}_2$-vector spaces and we have $(a,b) \mod 2 = (\overline{a},\overline{b})$. Next, we define $w_2(X) := \overline{c_1(X)}$, where $c_1(X) := c_1(TX)$ denotes the first Chern class of the tangent bundle. We note that $c_1(X) = -c_1(K_X)$, where K_X denotes the canonical line bundle of X. Then, *Wu's formula* states

$$(w_2, \overline{c}) \;=\; (\overline{c}, \overline{c}), \tag{0.10.12}$$

for all $c \in H^2(X,\mathbb{Z})$, see [512, page 132]. In particular, the lattice $H^2(X,\mathbb{Z})/\mathrm{Tors}$ is even if and only if $w_2 = 0$. To understand torsion in cohomology better and to link the transcendental lattice to the cohomological Brauer group, we consider for every integer $n \geq 1$ the short exact sequence (with respect to the classical topology)

$$0 \to \mathbb{Z}/n\mathbb{Z} \to \mathcal{O}_X^\times \xrightarrow{x \mapsto x^n} \mathcal{O}_X^\times \to 0. \tag{0.10.13}$$

Using the identification $\mathrm{Pic}(X) \cong H^1(X, \mathcal{O}_X^\times)$ and taking cohomology again, we obtain

$$0 \to \mathrm{Pic}(X)^{(n)} \to H^2(X, \mathbb{Z}/n\mathbb{Z}) \to {}_nH^2(X, \mathcal{O}_X^\times) \to 0. \tag{0.10.14}$$

In particular, using the short exact sequence (0.10.11), we find

$$(T_X^\vee)^{(n)} \cong {}_nH^2(X, \mathcal{O}_X^\times)\,. \tag{0.10.15}$$

Moreover, multiplication by n gives rise to a short exact sequence $0 \to \mathbb{Z} \to \mathbb{Z} \to \mathbb{Z}/n\mathbb{Z} \to 0$ and taking cohomology, we obtain a short exact sequence

$$0 \to H^2(X, \mathbb{Z})^{(n)} \to H^2(X, \mathbb{Z}/n\mathbb{Z}) \to {}_nH^3(X, \mathbb{Z}) \to 0\,, \tag{0.10.16}$$

which can also be deduced from the universal coefficient formulas of singular homology or cohomology (0.10.1). Comparing the short exact sequences (0.10.14) and (0.10.16) and using the isomorphism (0.10.15), we obtain a short exact sequence

$$0 \to \mathrm{Hom}(T_X, \mathbb{Z}/n\mathbb{Z}) \to {}_nH^2(X, \mathcal{O}_X^\times) \to {}_nH^3(X, \mathbb{Z}) \to 0. \tag{0.10.17}$$

We note that the torsion subgroup of $H^2(X, \mathcal{O}_X^\times)$ of a scheme X is called the *cohomological Brauer group* $\mathrm{Br}(X)$ of X, see also the discussion below. If X is a complex manifold or more generally, a complex space, then $H^2(X, \mathcal{O}_X^\times)$ (where this group is defined in the analytic category) is called the *analytic Brauer group* and it need not be a torsion group.

Next, we discuss an algebraic analog of singular cohomology, namely *ℓ-adic cohomology*. This cohomology theory was envisioned by Grothendieck with a view toward proving the Weil conjectures, and we refer to [294, Appendix C] for history and overview. Let X be a smooth and proper variety over a field $\Bbbk$ of characteristic $p \geq 0$. Since the Zariski topology is too coarse to give a good cohomology theory for locally constant sheaves, one equips X with its *étale topology* $X_{\text{ét}}$. In this Grothendieck topology, the objects are étale morphisms of finite type $V \to U$ over X. Then, a covering of $U \to X$ consists of a collection of finite étale morphism of finite type $\{V_i \to U\}_i$ such that the union of the images (all of which are Zariski-open subsets of U) is equal to U. For example, if G is a finite and étale group scheme over X – for example, $G = \boldsymbol{\mu}_n$ with $p \nmid n$ – or if $\mathcal{F}$ is a finite local system that is locally constant for the étale topology – for example $G = \mathbb{Z}/n\mathbb{Z}$ – then the cohomology groups $H^i_{\text{ét}}(X, G) := H^i(X_{\text{ét}}, G)$ are defined and satisfy the usual properties, such as long exact sequences in cohomology, etc. We refer to [154] and [508] for the precise definition and details. For a prime ℓ, we define ℓ-adic cohomology to be

$$H^i_{\text{ét}}(X, \mathbb{Z}_\ell) := \varprojlim_n H^i_{\text{ét}}(X, \mathbb{Z}/\ell^n\mathbb{Z}),$$
$$H^i_{\text{ét}}(X, \mathbb{Q}_\ell) := H^i_{\text{ét}}(X, \mathbb{Z}_\ell) \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell.$$

(The technical reason for this definition is that étale cohomology works best for local systems that are finite. If one wants to work directly with infinite local systems that are pro-finite, one can use the pro-étale site of Bhatt and Scholze [64].) Next, for a prime number $\ell \neq p$, we define the *ℓ-adic Tate module* to be

$$\mathbb{Z}_\ell(1) := \varprojlim_n \boldsymbol{\mu}_{\ell^n}(\overline{\mathbb{k}}),$$

together with the action of the absolute Galois group $\mathrm{Gal}(\overline{\mathbb{k}}/\mathbb{k})$. Taking duals and tensor products with itself, we obtain $\mathbb{Z}_\ell(k)$ for arbitrary $k \in \mathbb{Z}$. This leads us to define ℓ-adic cohomology with a *Tate-twist* via

$$H^i_{\text{ét}}(X, \mathbb{Z}_\ell(k)) := \varprojlim_n H^i_{\text{ét}}(X, \boldsymbol{\mu}_{\ell^n}^{\otimes k})$$
$$H^i_{\text{ét}}(X, \mathbb{Q}_\ell(k)) := H^i_{\text{ét}}(X, \mathbb{Z}_\ell(k)) \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell$$

Since X is proper over $\mathbb{k}$, all these cohomology groups are finitely generated modules over $\mathbb{Z}_\ell$ and $\mathbb{Q}_\ell$, respectively. Moreover, they are zero if $i < 0$ or $i > 2\dim(X)$. If $\mathbb{k}$ is a field of characteristic $p > 0$ and $\ell \neq p$, then the $\mathbb{Q}_\ell$-dimension of $H^i_{\text{ét}}(X, \mathbb{Q}_\ell(k))$ is independent of ℓ and k, and we define the i-th *(ℓ-adic) Betti number* of X to be

$$b_i(X) := \dim_{\mathbb{Q}_\ell} H^i_{\text{ét}}(X, \mathbb{Q}_\ell).$$

We refer to Example 0.10.15 below to get an impression of what happens if $\ell = p$. On the other hand, if $\mathbb{k} = \mathbb{C}$ and one chooses an embedding $\mathbb{Q}_\ell \subset \mathbb{C}$ (such embeddings exist by the axiom of choice, but they are neither unique, nor natural), there exist *comparison isomorphisms*

$$H^i_{\text{ét}}(X, \mathbb{Q}_\ell) \otimes_{\mathbb{Q}_\ell} \mathbb{C} \cong H^i(X, \mathbb{C}),$$

where the right–hand side denotes singular cohomology of the topological manifold X with coefficients in $\mathbb{C}$ as discussed above. Moreover, there is an analog of Poincaré duality that is obtained from a perfect duality of $(\mathbb{Z}/\ell^n\mathbb{Z})$-modules

$$H^i_{\text{ét}}(X, \boldsymbol{\mu}_{\ell^n}^{\otimes j}) \times H^{2d-i}_{\text{ét}}(X, \boldsymbol{\mu}_{\ell^n}^{\otimes (d-j)}) \to H^{2d}_{\text{ét}}(X, \boldsymbol{\mu}_{\ell^n}^{\otimes d}) \cong \mathbb{Z}/\ell^n\mathbb{Z}, \tag{0.10.18}$$

where the tensor product is taken in the category of locally constant sheaves of $(\mathbb{Z}/\ell^n\mathbb{Z})$-modules, see [508, Chapter 6.11]. Passing to the projective limit, we obtain a perfect duality of $\mathbb{Z}_\ell$-modules modulo torsion subgroups

$$H^i_{\text{ét}}(X, \mathbb{Z}_\ell(j)) \times H^{2d-i}_{\text{ét}}(X, \mathbb{Z}_\ell(2d-j)) \to \mathbb{Z}_\ell. \tag{0.10.19}$$

In particular, we find

$$b_i(X) = b_{2d-i}(X).$$

As in topology, we define the *ℓ-adic Euler–Poincaré characteristic* of X to be

$$e(X) := \sum_{i=0}^{2d} (-1)^i \, b_i(X).$$

We will be using a more general Poincaré duality that applies to any constructible sheaf of $\mathbb{Z}/\ell^n\mathbb{Z}$-modules [508, Chapter VI, Theorem 11.1]. It asserts that there is a nondegenerate pairing

$$H^i(X, \mathcal{F}) \times \mathrm{Ext}^{2d-i}(\mathcal{F}, \boldsymbol{\mu}_{\ell^n}^{\otimes d}) \to H^{2d}(X, \boldsymbol{\mu}_{\ell^n}^{\otimes d}) \cong \mathbb{Z}/\ell^n \tag{0.10.20}$$

from which (0.10.18) follows by taking $F = \boldsymbol{\mu}_{\ell^n}^{\otimes j}$. Moreover, for every coherent sheaf of $\mathcal{O}_X$-modules $\mathcal{E}$ on X, there exist *Chern classes* $c_i(\mathcal{E}) \in A^i(X)$ in the Chow groups of X, see [242]. Using the degree map $\deg : A^d(X) \to \mathbb{Z}$, we have

$$e(X) = c_d(X) := \deg\left(c_d(\Theta_{X/\Bbbk})\right), \tag{0.10.21}$$

where $\Theta_{X/\Bbbk} = \Omega^\vee_{X/\Bbbk}$ denotes the tangent sheaf of X. By the comparison theorem with singular cohomology, the Betti numbers and the Euler–Poincaré characteristic defined in terms of ℓ-adic cohomology and singular cohomology coincide if $\Bbbk = \mathbb{C}$. Finally, if X is a smooth and proper surface over a field, we have *Noether's formula*

$$12\,\chi(X, \mathcal{O}_X) = K_X^2 + c_2(X). \tag{0.10.22}$$

In fact, in [508, Chapter 5, Theorem 3.12], Noether's formula is used to prove (0.10.21) for surfaces. For later use, we also mention the *Riemann-Roch theorem* for an invertible sheaf $\mathcal{L}$ on a surface X

$$\chi(X, \mathcal{L}) := \sum_{i=0}^{2} (-1)^i \, h^i(X, \mathcal{L}) = \chi(X, \mathcal{O}_X) + \frac{1}{2}\mathcal{L} \cdot (\mathcal{L} \otimes \omega_X^{-1}), \tag{0.10.23}$$

see, for example, [294, Theorem V.1.6] or the discussion in [38, Chapter 5].

Recall that for any field F, the natural exact sequence of algebraic groups

$$1 \to \mathbb{G}_{m,F} \to \mathrm{GL}_{n+1,F} \to \mathrm{PGL}_{n,F} \to 1$$

induces a bijection of the Galois cohomology pointed sets $H^1(F, \mathrm{PGL}_{n,F}) \to H^2(F, \mathbb{G}_{m,F})$, see [89, VIII, §10, Prop. 7]. The set $H^1(F, \mathrm{PGL}_{n,F})$ is equal to the set of isomorphism classes of F-forms of the matrix algebra $M_n(F)$, that is, central simple algebras over F that become isomorphic to the matrix algebra $M_n(F')$ after some finite and separable extension F' of F. The group $H^2(F, \mathbb{G}_{m,F})$ is denoted by $\mathrm{Br}(F)$ and it is called the Brauer group of the field F. The group law corresponds to the tensor product on the set of central simple algebras modulo Morita equivalence, see [89, §15]. Grothendieck extended this definition to any quasi-compact scheme X by introducing the notion of an *Azumaya algebra* over X as a form in the étale topology of the endomorphism algebra of a free sheaf of some rank over X [264, I]. Then, he defined a commutative group law on such $\mathcal{O}_X$-algebras modulo an analogue of Morita equivalence. The resulting commutative group is denoted by $\mathrm{Br}(X)$

and it comes with a natural injective homomorphism $\delta : \mathrm{Br}(X) \to H^2(X, \mathbb{G}_m)$. In contrast to the case of fields, this homomorphism need not be surjective. In fact, the group $\mathrm{Br}(X)$ is always a torsion group since an Azumaya algebra can be trivialized after a finite cover of X. On the other hand, the group $H^2(X, \mathbb{G}_m)$ is not always a torsion group. However, in the case of curves and regular surfaces over $\Bbbk$, the group $H^2(X, \mathbb{G}_m)$ is a torsion group and δ is an isomorphism, see [271, II,Théoréme 2.1]. The torsion subgroup of $H^2(X, \mathbb{G}_m)$ is called the *cohomological Brauer group* and this definition agrees with our previous definition in the complex analytical case.

We now extend the computation of the cohomological Brauer group given by (0.10.17) by computing the ℓ-torsion ${}_{\ell^\infty}H^2(X, \mathbb{G}_m)$ for any smooth projective surface X over $\Bbbk$ of characteristic $p \neq \ell$. We have already seen that this group coincides with the l-torsion part of the Brauer group $\mathrm{Br}(X)$. To deal with the p-torsion part, we will need more techniques which we will discuss later in this section.

The analog of the exact sequence (0.10.13) in the étale topology is the Kummer exact sequence (0.1.6)

$$0 \to \boldsymbol{\mu}_n \to \mathbb{G}_m \xrightarrow{\times n} \mathbb{G}_m \to 0, \tag{0.10.24}$$

which is exact if $(n, p) = 1$. (To obtain an exact sequence if p divides n, one has to work in the flat topology.) Taking cohomology, we obtain the following algebraic analog of (0.10.14):

$$0 \to \mathrm{Pic}(X)^{(n)} \to H^2_{\text{ét}}(X, \boldsymbol{\mu}_n) \to {}_nH^2_{\text{ét}}(X, \mathbb{G}_m) \to 0 \tag{0.10.25}$$

and passing to powers of a prime ℓ, we find

$$0 \to \mathrm{Pic}(X) \otimes (\mathbb{Q}_\ell/\mathbb{Z}_\ell) \to H^2_{\text{ét}}(X, \boldsymbol{\mu}_{\ell^\infty}) \to {}_{\ell^\infty}H^2_{\text{ét}}(X, \mathbb{G}_m) \to 0.$$

Now, since $\mathbf{Pic}^\circ_{X/\Bbbk}(\Bbbk)$ is n-divisible for all n, we find $\mathrm{Pic}(X)^{(n)} = \mathrm{NS}(X)^{(n)}$. Thus, applying this to ℓ^n for all n and passing to the projective limit, we obtain an exact sequence

$$0 \to \mathrm{NS}(X) \otimes \mathbb{Z}_\ell \to H^2_{\text{ét}}(X, \mathbb{Z}_\ell(1)) \to T_\ell\,(\mathrm{Br}(X)) \to 0. \tag{0.10.26}$$

Note that the map $\mathrm{NS}(X) \otimes \mathbb{Z}_\ell \to H^2_{\text{ét}}(X, \mathbb{Z}_\ell(1))$ is the ℓ-adic analog of the first Chern class map in exact sequence (0.10.11). As in the case of complex surfaces it is the map factored from the map

$$c_1 : \mathrm{Pic}(X) \to H^2_{\text{ét}}(X, \mathbb{Z}_\ell). \tag{0.10.27}$$

Since the cohomology groups $H^i_{\text{ét}}(X, \boldsymbol{\mu}_n)$ are finite abelian groups, it follows that ${}_n\mathrm{Br}(X)$ is finite for all n. Thus, by Proposition 0.10.1, there exists a finite abelian group A and isomorphisms

$${}_{\ell^\infty}\mathrm{Br}(X) \cong (\mathbb{Q}_\ell/\mathbb{Z}_\ell)^{\mathrm{t}_\ell} \oplus A \quad \text{and} \quad T_\ell\,(\mathrm{Br}(X)) \cong \mathbb{Z}_\ell^{\mathrm{t}_\ell}. \tag{0.10.28}$$

Together with the short exact sequence (0.10.26), we obtain the following analog of (0.10.10)

$$\mathrm{t}_\ell \ = \ b_2(X) \ - \ \rho(X) \qquad (0.10.29)$$

, for all $\ell \neq p$. In particular, we obtain *Igusa's inequality* [326]

$$\rho(X) \ \leq \ b_2(X),$$

which holds regardless of the characteristic of $\mathbb{k}$.

Multiplication by ℓ^m in $\mathbb{G}_m$ defines an embedding of $\boldsymbol{\mu}_{\ell^k}$ into $\boldsymbol{\mu}_{\ell^{k+m}}$ with quotient $\boldsymbol{\mu}_{\ell^m}$. This gives an exact sequence

$$0 \ \to \ \boldsymbol{\mu}_{\ell^k} \ \to \ \boldsymbol{\mu}_{\ell^{k+m}} \ \to \ \boldsymbol{\mu}_{\ell^m} \ \to \ 0 \qquad (0.10.30)$$

that leads to exact sequences

$$0 \ \to \ H^i_{\text{ét}}(X, \mathbb{Z}_\ell(1))^{\ell^m} \ \to \ H^i_{\text{ét}}(X, \boldsymbol{\mu}_{\ell^m}) \ \to \ {}_{\ell^m}H^{i+1}_{\text{ét}}(X, \mathbb{Z}_\ell(1)) \ \to \ 0, \quad (0.10.31)$$

see [508, Chapter V, Lemma 1.11]. It is an analogue of (0.10.16). The composition

$$H^2_{\text{ét}}(X, \mathbb{Z}_\ell(1))^{(\ell^m)} \ \to \ H^i_{\text{ét}}(X, \boldsymbol{\mu}_{\ell^m}) \ \to \ {}_{\ell^m}\operatorname{Br}(X)$$

defined by the exact sequence (0.10.25) has kernel $\operatorname{Pic}(X)^{(\ell^m)}$ and hence, we obtain an exact sequence

$$0 \ \to \ (\mathbb{Z}/\ell^m\mathbb{Z})^{\mathrm{t}_\ell} \ \to \ {}_{\ell^m}\operatorname{Br}(X) \ \to \ {}_{\ell^m}H^3_{\text{ét}}(X, \mathbb{Z}_\ell(1)) \ \to \ 0 \qquad (0.10.32)$$

which is the ℓ-adic analog of (0.10.17) and (0.10.16). Passing inductive limits, we get an exact sequence

$$0 \ \to \ (\mathbb{Q}_\ell/\mathbb{Z}_\ell)^{\mathrm{t}_\ell} \ \to \ {}_{\ell^\infty}\operatorname{Br}(X) \ \to \ {}_{\ell^\infty}H^3_{\text{ét}}(X, \mathbb{Z}_\ell(1)) \ \to \ 0 \qquad (0.10.33)$$

Using the universal coefficient formula and the fact that a maximal ℓ-divisible subgroup must be a direct factor, we get:

Theorem 0.10.2 *For any prime $\ell \neq p$, there exists an isomorphism*

$${}_{\ell^\infty}\operatorname{Br}(X) \ \cong \ (\mathbb{Q}_\ell/\mathbb{Z}_\ell)^{\mathrm{t}_\ell} \oplus \operatorname{Hom}({}_{\ell^\infty}\operatorname{NS}(X), \mathbb{Q}_\ell/\mathbb{Z}_\ell).$$

Next, we will briefly discuss *algebraic de Rham cohomology*: let $X \to S$ be a smooth and proper morphism of schemes of relative dimension d over some noetherian base scheme S. We will mostly be dealing with the case where $S = \operatorname{Spec}\mathbb{k}$, where $\mathbb{k}$ is an algebraically closed field or where S is the spectrum of a discrete valuation ring with an algebraically closed residue field $\mathbb{k}$. Then, the relative Kähler differentials $\Omega_{X/S}$ and the exterior differential $d = d_{X/S}$ give rise to the *relative de Rham complex* $(\Omega^\bullet_{X/S}, d)$, where $\Omega^i_{X/S} := \bigwedge^i \Omega_{X/S}$. These sheaves usually have non-trivial higher cohomology groups (unlike the case of C^∞-differential forms, where it follows from the Poincaré lemma that the sheaves $\mathcal{A}^i$ that are sheaves with respect to the classical topology have no higher cohomology, we are dealing with sheaves in the Zariski topology), which is why algebraic de Rham cohomology is defined to be the hypercohomology of the de Rham complex

$$H^*_{\mathrm{DR}}(X/S) \;:=\; \mathbf{H}^*(\Omega^\bullet_{X/S}).$$

Since X is proper over S, these cohomology groups are coherent $\mathcal{O}_S$-modules. If $S = \operatorname{Spec} \Bbbk$ is a field, then this says that they are finite–dimensional $\Bbbk$-vector spaces and we define the *de Rham Betti numbers* to be

$$b_i^{\mathrm{DR}}(X) \;:=\; \dim_{\Bbbk} H^i_{\mathrm{DR}}(X/\Bbbk). \tag{0.10.34}$$

It is a non-trivial result of Grothendieck [270] that if X is a smooth and proper variety over $\mathbb{C}$, then $H^*_{\mathrm{DR}}(X/\mathbb{C})$ is isomorphic to the de Rham cohomology $H^*(X, \mathbb{C})$ discussed above. Since algebraic de Rham cohomology arises as hypercohomology of a complex of sheaves, it comes with a spectral sequence

$$E_1^{p,q} \;:=\; H^{p,q}(X) \;=\; H^q(X, \Omega^p_{X/S}) \;\Longrightarrow\; H^{p+q}_{\mathrm{DR}}(X/S), \tag{0.10.35}$$

the *Hodge versus de Rham spectral sequence* or *Frölicher spectral sequence*. This gives rise to a filtration $\mathcal{F}^\bullet$ on $H^*_{\mathrm{DR}}(X/S)$, the *Hodge filtration*. If this spectral sequence degenerates at E_1, then we have $E_1^{p,q} = \mathcal{F}^p/\mathcal{F}^{p+1}$. If X is a complex projective variety, then the Hodge decomposition (0.10.7) implies that the Hodge versus de Rham spectral sequence degenerates at E_1. Using the Lefschetz principle, we even obtain the following.

Theorem 0.10.3 *Let X be a smooth projective variety over a field $\Bbbk$ of characteristic zero. Then, the Hodge versus de Rham spectral sequence degenerates at E_1 and gives*

$$b_i^{\mathrm{DR}}(X) = \sum_{p+q=i} h^{p,q}(X), \tag{0.10.36}$$

where $h^{p,q}(X) = \dim_{\Bbbk} H^q(X, \Omega^p_{X/\Bbbk})$.

Remark 0.10.4 The first proof of this theorem uses the Hodge theory of Kähler manifolds as sketched above. The first algebraic proof was given by Faltings [229] using p-adic Hodge theory. Mumford [536] gave explicit examples of smooth and projective surfaces X in positive characteristic, where the exterior differential $d : H^0(\Omega^1_X) \to H^0(\Omega^2_X)$ is non-zero - in particular, the Hodge versus de Rham spectral sequences of these surfaces do not degenerate at E_1. By a fundamental result of Deligne and Illusie [156] (see also [331] or [568]), the Hodge versus de Rham spectral sequence of a smooth and projective variety X over a perfect field $\Bbbk$ of characteristic $p > 0$ degenerates at E_1 if $\dim(X) \leq p$ and X lifts to the truncated Witt ring $W_2(\Bbbk)$. In contrast to the case of characteristic zero, the Hodge symmetry $h^{p,q}(X) = h^{q,p}(X)$ may not hold in positive characteristic, even if the Hodge versus de Rham spectral sequence of X degenerates at E_1. We discussed the relation between $h^{1,0}$, $h^{0,1}$, and $b_1(X)$ in the previous section.

In the case where X is a complex projective variety, complex conjugation and the Hodge symmetry give rise to a second filtration $\overline{\mathcal{F}}^{\bullet}$ in de Rham cohomology, the *complex conjugate Hodge filtration*, which satisfies

$$H^{p,n-p}(X) \;\cong\; \mathcal{F}^p \cap \overline{\mathcal{F}}^{n-p} \;\subseteq\; H^n_{\mathrm{DR}}(X/\mathbb{C}) \quad \text{and} \quad \mathcal{F}^p \oplus \overline{\mathcal{F}}^{n-p+1} \;=\; H^n_{\mathrm{DR}}(X/\mathbb{C})\,. \tag{0.10.37}$$

Let us put this into a larger, more general, and algebraic perspective: let $X \to S$ be a morphism of schemes. Then, we have the complex $(\Omega^{\bullet}_{X/S}, d = d_{X/S})$ and the associated abelian subsheaves of $\Omega^i_{X/S}$

$$B\Omega^i_{X/S} \;:=\; d\Omega^{i-1}_{X/S} \quad \text{and} \quad Z\Omega^i_{X/S} \;:=\; \mathrm{Ker}\left(\Omega^i_{X/S} \stackrel{d}{\longrightarrow} \Omega^{i+1}_{X/S}\right),$$

the *boundaries* and *cycles*, respectively. Since $d \circ d = 0$, we have $B\Omega^i_{X/S} \subseteq Z\Omega^i_{X/S}$ and set

$$\mathcal{H}^i(\Omega^{\bullet}_{X/S}) \;:=\; Z\Omega^i_{X/S}/B\Omega^i_{X/S}.$$

Then, there exists a *second spectral sequence of hypercohomology*

$$E_2^{p,q} \;:=\; H^p(X, \mathcal{H}^q(\Omega^{\bullet}_{X/S})) \;\Longrightarrow\; H^{p+q}_{\mathrm{DR}}(X/S), \tag{0.10.38}$$

that gives rise to a filtration on the right–hand side. If X is a smooth and projective variety over $\mathbb{C}$ and we equip it with the classical topology, then the Poincaré Lemma implies that the cohomology sheaves $\mathcal{H}^i(\mathcal{A}^{\bullet}_X)$ are zero for $i \geq 1$. Similarly, the cohomology sheaves $\mathcal{H}^i(\Omega^{\bullet}_{X/\mathbb{C}})$ are zero for $i \geq 1$ if $\Omega^{\bullet}_X$ denotes the complex of sheaves of locally holomorphic differential forms with respect to the classical topology. In these two cases, the filtration in de Rham cohomology arising from (0.10.38) is trivial. On the other hand, the cohomology sheaves $\mathcal{H}^i(\Omega^{\bullet}_X)$ may be non-zero for $i \geq 1$ if $\Omega^{\bullet}_X$ is considered as a complex of $\mathcal{O}_X$-modules with respect to the Zariski topology. In characteristic zero, the filtration in de Rham cohomology arising from (0.10.38) is the *coniveau filtration*, see [71]. Now, let us study this second filtration in positive characteristic: assume that X and S are schemes of characteristic $p > 0$ and let $\mathbf{F} = \mathbf{F}_{X/S} : X \to X^{(p)}$ be the S-linear Frobenius morphism. Then, the abelian sheaves $\mathbf{F}_*(B\Omega^i_{X/S})$ and $\mathbf{F}_*(Z\Omega^i_{X/S})$ are $\mathcal{O}_{X^{(p)}}$-modules. Moreover, if X is smooth over S, then there exists a unique family of additive maps, the *Cartier operators*

$$C_{X/S} \;:\; \mathbf{F}_*(Z\Omega^i_{X/S}) \;\to\; \Omega^i_{X^{(p)}/S}$$

that satisfy the following properties:

1. $C(1) = 1$,
2. $C(f^p\omega) = f \cdot C(\omega)$ for local sections $f \in \mathcal{O}_X$ and $\omega \in Z\Omega^i_{X/S}$,
3. $C(\omega \wedge \omega') = C(\omega) \wedge C(\omega')$ for local sections $\omega \in Z\Omega^i_{X/S}$ and $\omega' \in Z\Omega^{i'}_{X/S}$,
4. $C(\omega) = 0$ if and only if $\omega \in B\Omega^i_{X/S}$,
5. $C(f^{p-1}df) = df$.

Property (2) shows that the Cartier operator is not a homomorphism of $\mathcal{O}_X$-modules, but that it can be considered as a homomorphism of $\mathcal{O}_{X^{(p)}}$-modules. Next, property (4) and a theorem of Cartier show that $C_{X/S}$ induces an exact sequence of $\mathcal{O}_{X^{(p)}}$-modules

$$0 \to \mathbf{F}_*(B\Omega^i_{X/S}) \to \mathbf{F}_*(Z\Omega^i_{X/S}) \xrightarrow{C_{X/S}} \Omega^i_{X^{(p)}/S} \to 0$$

and thus, an isomorphism

$$C_{X/S} : \mathcal{H}^i(\mathbf{F}_*\Omega^\bullet_{X/S}) \to \Omega^i_{X^{(p)}/S}, \tag{0.10.39}$$

the *Cartier isomorphism*. If $i = 0$, then its inverse $C^{-1}_{X/S}$ is the natural isomorphism $\mathcal{O}_{X^{(p)}} \to \mathbf{F}_*\mathcal{O}_X$. If $i = 1$, then $C^{-1}_{X/S}$ is defined locally by sending $1 \otimes ds$ to the class of $s^{p-1}ds$ in $\mathcal{H}^1(\mathbf{F}_*\Omega^\bullet_{X/S})$. We refer to [331, Section 3], [329, Section 0.2], and [376, Theorem 7.2] for proofs and further details. The following illustrates the Cartier operator and Cartier isomorphism with an example.

Example 0.10.5 Let $\Bbbk$ be a perfect field of characteristic $p > 0$ and let $X = \mathbb{A}^1_\Bbbk = \operatorname{Spec} \Bbbk[t]]$. We can identify $X^{(p)}$ with X via the isomorphism of rings $a_i t^i \mapsto a_i^p t^i$ and consider the relative Frobenius map $\mathbf{F}_{X/\Bbbk} : X \to X^{(p)} = X, t \mapsto t^p$. We have $C(t^{pk+i}dt) = t^{pk}C(t^i dt)$. If $i \neq p-1$, then $t^i dt = \frac{dt^{i+1}}{i+1}$ and hence $C(t^{pk+i}dt) = 0$. If $i = p-1$, then we have $C(t^{pk+i}dt) = t^{pk}dt$. This shows that $C(\Omega^1_{\Bbbk[t]/\Bbbk}) = \Bbbk[t^p]dt$, and hence,

$$C_{X/\Bbbk} : Z^1\Omega_{X/\Bbbk}/B^1\Omega_{X/\Bbbk} = \Omega^1_{X/\Bbbk}/d\Omega^1_{X/\Bbbk} \to \Omega^1_{X/\Bbbk}$$

is an isomorphism of $\mathcal{O}_{X^{(p)}}$-modules.

We can also globalize this by replacing $\mathbb{A}^1_\Bbbk$ with $X = \mathbb{P}^1_\Bbbk$. Set $f = \mathbf{F}_{X/\Bbbk} : X \to X^{(p)}$ and let $B^1 = B^1\Omega^1_{X/\Bbbk}$ considered as an $\mathcal{O}_{X^{(p)}}$-Module. We have two exact sequences of locally free sheaves on $X^{(p)}$,

$$0 \to \mathcal{O}_{X^{(p)}} \to f_*\mathcal{O}_X \to B^1 \to 0,$$

and

$$0 \to B^1 \to f_*\Omega^1_{X/\Bbbk} \to \Omega^1_{X^{(p)}/\Bbbk} \to 0.$$

The sheaf $f_*\Omega_{X/\Bbbk}$ is of rank p with $\chi(f_*\Omega^1_{X/\Bbbk}) = \chi(\Omega^1_{X/\Bbbk}) = -1$. It follows that $\chi(B^1) = 0$ and using $H^1(X^{(p)}, \mathcal{O}_{X^{(p)}}) = 0$ and $\dim H^0(X^{(p)}, f_*\mathcal{O}_X) = 1$, we find that $H^0(X^{(p)}, B^1) = 0$. Since any locally free sheaf on $\mathbb{P}^1$ splits into a direct sum of invertible sheaves and an invertible sheaf $\mathcal{L}$ of negative degree with $\chi(\mathcal{L}) = 0$ is isomorphic to $\mathcal{O}_{\mathbb{P}^1}(-1)$, we see that $B^1 \cong \mathcal{O}_{\mathbb{P}^1}^{\oplus p-1}$. The second exact sequence gives us an isomorphism

$$f_*\Omega^1_{X/\Bbbk} \cong \mathcal{O}_{\mathbb{P}^1}(-1)^{\oplus p-1} \oplus \mathcal{O}_{\mathbb{P}^1}(-2).$$

If X is a smooth and proper variety over a field $\Bbbk$ of characteristic $p > 0$, then combining the second spectral sequence (0.10.38) and the Cartier isomorphism (0.10.39) gives

$$E_2^{p,q} := H^p(X, \Omega^q_{X/\Bbbk}) \Longrightarrow H^{p+q}_{\mathrm{DR}}(X/\Bbbk), \qquad (0.10.40)$$

the *conjugate spectral sequence*. This name is merely chosen in analogy to the two filtrations in de Rham cohomology that one has in complex geometry (0.10.37) - however, there is no complex conjugation in positive characteristic. (Some people view Frobenius as an analog of complex conjugation, which justifies this terminology a little bit.) Next, the data of the $\Bbbk$-vector space $H^{p+q}_{\mathrm{DR}}(X/\Bbbk)$ together with the two filtrations coming from the Hodge versus de Rham and the conjugate spectral sequence is captured in the notion of an *F-zip*. Moreover, the two filtrations $\mathcal{F}^\bullet$ and $\overline{\mathcal{F}}^\bullet$ in the complex case are "as disjoint as possible" in a sense made precise by (0.10.37). On the other hand, the relative position of these two filtrations in positive characteristic is a very interesting discrete invariant of smooth and projective varieties in positive characteristic, which is related to ordinarity and supersingularity of varieties. We refer to [462] or [735] for details and further information.

The "correct" cohomology with p-adic coefficients in characteristic p is crystalline cohomology. For example, we will see in Example 0.10.15 below that $H^*_{\text{ét}}(X, \mathbb{Q}_p)$ does not give the "right" answer. Crystalline cohomology takes values in *Witt vectors*, which we introduced and discussed in Section 0.3. As a first approximation to crystalline cohomology, let us introduce Witt vector cohomology: if $(X, \mathcal{O}_X)$ is a scheme, then the assignment $U \mapsto W_n(\mathcal{O}_X(U))$ for every Zariski-open subset $U \subseteq X$ defines a sheaf of rings $W_n(\mathcal{O}_X)$ on X for all $n \geq 1$. If X is a proper scheme of finite type over a perfect field $\Bbbk$ of characteristic $p > 0$, then the cohomology groups $H^i(X, W_n(\mathcal{O}_X))$ are finitely generated $W_n(\Bbbk)$-modules for all n and i. Clearly, they are zero for $i < 0$ or $i > \dim(X)$. These cohomology groups are called *Serre's Witt vector cohomology groups* and were first studied by Serre in [653]. Clearly, the maps σ and V and the projection maps $W_n(\mathcal{O}_X) \to W_m(\mathcal{O}_X)$ for $m \leq n$ induce maps on Witt vector cohomology groups. Finally, one can also consider the cohomology groups

$$H^i(X, W(\mathcal{O}_X)) := \varprojlim_n H^i(X, W_n(\mathcal{O}_X)),$$

which are $W(\Bbbk)$-modules, but may fail to be finitely generated.

Before turning to the construction of crystalline cohomology, let us briefly digress on *divided power structures* or *PD-structures* (because of the French puissances divisées): for a commutative ring R and an ideal $I \subseteq R$, a PD-structure consists of maps $\gamma_n : I \to R$ for $n \geq 0$ satisfying the following axioms:

1. $\gamma_0(x) = 1$ and $\gamma_1(x) = x$ for all $x \in I$,
2. $\gamma_n(I) \subseteq I$ for all $n \geq 1$,
3. $\gamma_n(x + y) = \sum_{i+j=n} \gamma_i(x)\gamma_j(x)$,
4. $\gamma_n(rx) = r^n \gamma_n(x)$ for all $x \in I$ and $r \in R$,
5. $\gamma_n(x)\gamma_m(x) = \binom{m+n}{n}\gamma_{m+n}(x)$ for all $x \in R$ and all $m, n \in \mathbb{Z}_{\geq 0}$,
6. $\gamma_m(\gamma_n(x)) = \frac{(mn)!}{m!n!}\gamma_{mn}(x)$ for all $x \in I$ and all $m, n \in \mathbb{Z}_{\geq 0}$.

If R is a $\mathbb{Q}$-algebra, then it easily follows from the axioms that every ideal $I \subseteq R$ possesses a unique PD-structure, which is given by $\gamma_n(x) := x^n/n!$. In fact, PD-structures were introduced in order to have an analog of the operators $x \mapsto x^n/n!$ in rings where $n!$ may not be invertible. Such operators are needed in order to define analogs of exponential functions. Next, since the a priori rational number $p^n/n!$ is in fact a p-adic integer divisible by p for all integers $n \geq 1$, it follows that for every ring R of characteristic $p > 0$, the principal ideal (p) of the Witt ring $W(R)$ carries a PD-structure. Similarly, one can define a PD-structure on the ideal $(p) \subseteq W_n(R)$ for all $n \geq 1$. In the case where R is a ring of characteristic $p > 0$ with PD-structure on the ideal $I \subseteq R$, it follows from the axioms that $x^p = 0$ for every $x \in I$ and thus, the ideal I is nilpotent. In particular, non-zero ideals in integral domains of positive characteristic do not possess PD-structures.

Example 0.10.6 Let R be a commutative ring. Then, we have the following divided power version of a polynomial ring in r variables over R: we define $R\langle x_1, ..., x_r\rangle$ to be the commutative and graded R-algebra, whose degree d part is the free R-module generated by the symbols $x_1^{[k_1]} \cdots x_r^{[k_r]}$ with $k_i \in \mathbb{Z}_{\geq 0}$ satisfying $k_1 + \cdots + k_r = d$. The algebra structure is defined by the relations $x_i^{[a]} \cdot x_i^{[b]} = \binom{a+b}{a} x_i^{[a+b]}$. The ideal $I \subseteq R\langle x_1, ..., x_r\rangle$ that is generated by all elements of positive degree carries a unique PD-structure via $\gamma_n(x_i) := x_i^{[n]}$.

Now, let X be a scheme of finite type over a perfect field $\Bbbk$ of characteristic $p > 0$. Then, there exists a Grothendieck topology $(X/W_n)_{\mathrm{crys}}$ on X, whose objects are commutative diagrams

$$\begin{tikzcd} U \arrow[r, "i"] \arrow[d] & V \arrow[d] \\ \operatorname{Spec} \Bbbk \arrow[r] & \operatorname{Spec} W_n(\Bbbk), \end{tikzcd}$$

where $U \subseteq X$ is a Zariski open subset and $i : U \to V$ is a closed embedding of schemes over $W_n(\Bbbk)$ such that the ideal sheaf $\mathcal{I}_U \subseteq \mathcal{O}_V$ is equipped with a PD-structure γ that is compatible with the PD-structure on the ideal $(p) \subseteq W_n(\Bbbk)$. A morphism $(U, V, \gamma) \to (U', V', \gamma')$ in $(X/W_n)_{\mathrm{crys}}$ is a commutative diagram given by an open embedding $U \to U'$ and a morphism $V \to V'$ that is compatible with the PD-structures. Finally, one defines a covering $\{(U_i, V_i, \gamma_i)\}_i$ of (U, V, γ) to be a collection of morphisms $(U_i, V_i, \gamma_i) \to (U, V, \gamma)$ as before such that the morphisms $V_i \to V$ are open embeddings with $V = \bigcup_i V_i$.

Next, an abelian sheaf $\mathcal{F}$ on $(X/W_n)_{\mathrm{crys}}$ consists of a collection of $\mathcal{O}_V$-modules $\mathcal{F}_V$ on every V and every object (U, V, γ) of $(X/W_n)_{\mathrm{crys}}$. These $\mathcal{F}_V$ must be compatible with respect to morphisms $g : (U, V, \gamma) \to (U', V', \gamma')$ in the sense that $g^*(\mathcal{F}_{V'})$ is isomorphic to $\mathcal{F}_V$ if $V \to V'$ is an open embedding plus some transitivity condition. The main example of an abelian sheaf on $(X/W_n)_{\mathrm{crys}}$ is the sheaf $\mathcal{O}_{X/W_n}$ that is defined by assigning to each (U, V, γ) in $(X/W_n)_{\mathrm{crys}}$ the structure sheaf $\mathcal{O}_V$.

After these preparations, we define *crystalline cohomology* for a smooth and proper variety X of dimension d over a perfect field $\Bbbk$ by setting

$$\begin{aligned} H^i(X/W_n) &:= H^i((X/W_n)_{\mathrm{crys}}, \mathcal{O}_{X/W_n}), \\ H^i(X/W) &:= \varprojlim{}_n H^i((X/W_n)_{\mathrm{crys}}, \mathcal{O}_{X/W_n}). \end{aligned}$$

Since X is proper over $\Bbbk$, the cohomology groups $H^i(X/W_n)$ and $H^i(X/W)$ are finitely generated $W_n(\Bbbk)$-modules and $W(\Bbbk)$-modules, respectively. As in the case of ℓ-adic cohomology, they are zero if $i < 0$ or $i > 2\dim(X)$. We will denote by K the field of fractions of $W(\Bbbk)$ and set $H^i(X/W)_K := H^i(X/W) \otimes_W K$. Then, $H^i(X/W)_K$ comes with properties and structures expected from singular cohomology: it is a contravariant functor, there is a Poincaré duality pairing

$$\langle -, - \rangle \; : \; H^i(X/W) \times H^{2d-i}(X/W) \to H^{2d}(X/W) \xrightarrow{\mathrm{Tr}} W(\Bbbk), \qquad (0.10.41)$$

that is perfect modulo torsion, there exist cycle class maps, there exists a Lefschetz fixed–point formula for endomorphisms,... One of the main properties of crystalline cohomology is given by the following theorem of Berthelot and Grothendieck [60].

Theorem 0.10.7 *Let X be a smooth and proper scheme over a perfect field $\Bbbk$ of positive characteristic p. Suppose X lifts to $W(\Bbbk)$, that is, there exists a proper and flat scheme $\mathcal{X} \to \operatorname{Spec} W(\Bbbk)$ such that $\mathcal{X} \times_{\operatorname{Spec} W(\Bbbk)} \operatorname{Spec} \Bbbk \cong X$. Then,*

$$H^i(X/W) \;\cong\; H^i_{\mathrm{DR}}(\mathcal{X}/\operatorname{Spec} W(\Bbbk)),$$

where the right–hand side denotes algebraic de Rham cohomology.

In fact, it was Grothendieck's insight that the algebraic de Rham cohomology of a lift of X does not depend on the choice of lift, provided that a lift exists. Even more: this cohomology is so canonical that actually no lift is required – the above construction and the previous theorem make this precise.

Remark 0.10.8 If one drops the divided power structures in the definition of the crystalline site, then the same result holds in characteristic zero, but not in positive characteristic. The reason is that the Poincaré lemma fails in characteristic $p > 0$ even for such a simple ring as $\Bbbk[t]$. In fact, $t^{p-1}dt \in \Omega^1_{\Bbbk[t]/\Bbbk}$ lies in the kernel of d, but $t^{p-1}dt \notin d(\Bbbk[t])$. (We note that this is closely related to the Cartier isomorphism (0.10.39).) However, the Poincaré lemma is true for the ring $\Bbbk\langle t\rangle$ from Remark 0.10.6 in any characteristic since we have $d(\sum a_k t^{[k]}) = (\sum a_k t^{[k-1]})dt$. This observation is an essential ingredient in the proof of the previous theorem and gives a hint of why PD-structures are important.

We refer to [60], [62], or [329] for details, definitions, and background on crystalline cohomology. For surveys, we refer to [327] or [116]. Finally, crystalline cohomology works best for smooth and proper varieties over $\Bbbk$. For smooth and affine schemes over $\Bbbk$, Monsky and Washnitzer constructed a cohomology theory that is well suited for computations. Berthelot generalized crystalline and Monsky–Washnitzer cohomology to *rigid cohomology*, and we refer to [448] for details.

We now come back to the problem of defining the “correct” Betti numbers for p-adic cohomology theories in characteristic p, see also Example 0.10.15 below. By a result of Katz and Messing [378], we have

$$\mathrm{rank}_W\ H^i(X/W)\ =\ b_i(X). \tag{0.10.42}$$

In particular, crystalline cohomology yields the expected Betti numbers. An important feature of crystalline cohomology is that the W-modules $H^i(X/W)$ may have torsion, which sometimes provides striking insights into “pathologies” (to quote from the title of [536]) in characteristic p. In view of Theorem 0.10.7, it is not surprising that crystalline and de Rham cohomology of a smooth and proper variety X over $\Bbbk$ are related by a *universal coefficient formula*

$$0\ \to\ H^i(X/W)\otimes_{W(\Bbbk)}\Bbbk\ \to\ H^i_{\mathrm{DR}}(X/\Bbbk)\ \to\ \mathrm{Tor}_1^{W(\Bbbk)}\left(H^{i+1}(X/W),\Bbbk\right)\ \to\ 0\,. \tag{0.10.43}$$

As an application of our discussion, we obtain the following identities and (in-)equalities, which are analogous to the case of Kähler manifolds discussed above.

Proposition 0.10.9 *Let X be a smooth and proper variety over a perfect field $\Bbbk$ of characteristic $p > 0$.*

1. *For all i, there are inequalities*

$$\sum_{p+q=i} h^{p,q}(X)\ \geq\ b_i^{\mathrm{DR}}(X/\Bbbk)\,.$$

   Equality for all i is equivalent to the degeneration of the Hodge versus de Rham spectral sequence at E_1.
2. *For all i, there are inqualities*

$$b_i(X)\ \geq\ b_i^{\mathrm{DR}}(X/\Bbbk)\,.$$

   Equality for all i is equivalent to all crystalline cohomology groups $H^i(X/W)$ being W-torsion free.

Proof The first statement is a general fact about spectral sequences. The second statement follows immediately from (0.10.43). □

As before, let $\Bbbk$ be a perfect field of characteristic $p > 0$, let $W := W(\Bbbk)$ be the ring of Witt vectors, let $\sigma : W \to W$ be the Frobenius morphism, and let K be the field of fractions of W. Then, a map of W-modules (or K-vector spaces) $\varphi : M \to N$ is called *σ-semilinear* if it is additive and if it satisfies $\varphi(rm) = \sigma(r)\varphi(m)$ for all $m \in M$ and all $r \in W$ (resp. all $r \in K$). Next, an **F**-*crystal* (resp. **F**-*isocrystal*) is a finitely generated and free W-module (resp. finite–dimensional K vector space) M together with an injective and σ-semilinear map $\varphi : M \to M$. A *morphism* of **F**-crystals $f : (M,\varphi) \to (N,\psi)$ (resp. **F**-isocrystals) is a morphism of W-modules (resp. K-vector spaces) such that $f\circ\varphi = \psi\circ f$. A morphism f of **F**-crystals is called an *isogeny* if $f\otimes K$ is an isomorphism of **F**-isocrystals.

Example 0.10.10 Here are two important sources of examples of **F**-crystals: the first one arises from geometry and the second one is purely algebraic.

1. Let X be a smooth and proper variety of dimension d over $\Bbbk$. Then, the absolute Frobenius map $\mathbf{F} : X \to X$ induces, for any i, a self-map $\varphi_i := \mathbf{F}^*$ of $H^i(X/W)$. Since $\mathbf{F}$ is not a morphism over $\operatorname{Spec} \Bbbk$ (unless $\Bbbk = \mathbb{F}_p$), the maps φ_i are σ-linear. Next, set $M^i := H^i(X/W)/\mathrm{Tors}$. The Poincaré duality pairing (0.10.41) satisfies the following compatibility with Frobenius:
$$\langle \varphi_i(x), \varphi_{2d-i}(y) \rangle \;=\; p^d \cdot \sigma(\langle x, y \rangle) \quad \text{for all } x \in M^i, y \in M^{2d-i}.$$
Since σ is injective on $W = W(\Bbbk)$, it follows that all φ_i's are injective. In particular, (M^i, φ_i) is an **F**-crystal. Moreover, the finite–dimensional K-vector space $M_i \otimes_W K$ together with $\varphi_i \otimes K$ defines an F-isocrystal.
2. Let $W_\sigma\langle T\rangle$ be the non-commutative polynomial ring in the variable T over $W = W(\Bbbk)$ subject to the relations $T \cdot x = \sigma(x) \cdot T$ for all $x \in W$. Next, let $\alpha = r/s \in \mathbb{Q}$, where r, s are coprime integers with $s \geq 1$. Then,
$$M_\alpha \;:=\; W_\sigma\langle T\rangle/(T^s - p^r)$$
together with $\varphi : m \mapsto T \cdot m$ defines an **F**-crystal (M_α, φ), which is of rank s as a W-module. The rational number α is called the *slope* of (M_α, φ). If $r \geq 0$, that is, if $\alpha \geq 0$, then the **F**-crystal (M_α, φ) is called *effective*.

The importance of the **F**-crystals (M_α, φ) lies in the following fundamental result due to Dieudonné and Manin [484].

Theorem 0.10.11 *Let $\Bbbk$ be an algebraically closed field of characteristic $p > 0$. Then, the category of* **F***-isocrystals over $\Bbbk$ is semi-simple and its simple objects are isomorphic to the* **F***-crystals (M_α, φ) from Example 0.10.10.*

Thus, every **F**-crystal (M, φ) is isogenous to a **F**-crystal of the form
$$M \;\sim\; \bigoplus_{\alpha \in \mathbb{Q}_{\geq 0}} M_\alpha^{\oplus n_\alpha},$$
where $\sim$ denotes isogeny of **F**-crystals. The numbers α that occur in this direct sum are called the *slopes* of M and the integers $\lambda_\alpha := n_\alpha \cdot \operatorname{rank} M_\alpha$ is called the *multiplicity* of the slope α. If all slopes of M are non-negative, the **F**-crystal is called *effective*.

We note that all **F**-crystals arising as $H^i(X/W)/\mathrm{Tors}$ from geometry as in Example 0.10.10 are effective, which is why we will only discuss effective **F**-crystals in the sequel.

The slopes and multiplicities of an effective **F**-crystal are encoded in the *Newton polygon*: let $\alpha_1 \leq \ldots \leq \alpha_r$ be the slopes of M and let $\lambda_1, \ldots, \lambda_s$ be the corresponding multiplicities. Then, we have $r := \operatorname{rank}_W M = \lambda_1 + \cdots + \lambda_s$. We define a piecewise linear function $\mathrm{Nw}_M : [0, r] \to \mathbb{R}$ via

$$\mathrm{Nw}_M(t) = \begin{cases} 0 & 0 \le t \le \lambda_1, \\ \alpha_k(t - \sum_{i=1}^k \lambda_i) + \sum_{i=1}^{k-1} \alpha_i\lambda_{i+1} & \sum_{i=1}^{k-1} \lambda_i \le t \le \sum_{i=1}^k \lambda_i. \end{cases}$$

Its graph is called the *Newton polygon*. It is a convex graph of a function on the interval $[0, b_n(X)]$ starting in $(0,0)$ that is of slope α_1 in the interval $[0,\lambda_1]$, of slope α_2 in the interval $[\lambda_1, \lambda_1 + \lambda_2]$, etc. By definition, the Newton polygon of an **F**-crystal depends only on its isogeny class.

Associated to an effective **F**-crystal (M,φ), there is a second convex polygon, the *Hodge polygon*, which is not invariant under isogenies: here, the collection (α_i,λ_i) is replaced with the collection (i, h_i), where

$$M/\varphi(M) \;\cong\; \bigoplus_{i\ge 1} \left(W/p^iW\right)^{h_i}$$

and $h_0 := \mathrm{rank}_W\, M - \sum_{i\ge 1} h_i$. As in the case of the Newton polygon, we use this collection to define a piecewise linear function $\mathrm{Hdg}_M : [0,r] \to \mathbb{R}$ on the interval $[0, b_n(X)]$, whose graph is a convex polygon, the *Hodge polygon*. Then,

$$\mathrm{Nw}_M(t) \;\ge\; \mathrm{Hdg}_M(t), \quad t \in [0, \mathrm{rank}_W\, M],$$

that is, the Newton polygon lies on or above the Hodge polygon.

Note that one can define the first Chern class with values in crystalline cohomology

$$c_1 \;:\; \mathrm{Pic}(X) \;\to\; H^2(X/W). \tag{0.10.44}$$

As in the cases of classical and étale cohomology, this map factors through a map

$$c_1^{\mathrm{NS}} \;:\; \mathrm{NS}(X) \;\to\; H^2(X/W),$$

which is injective modulo torsion. The absolute Frobenius $\mathbf{F} : X \to X$ acts like $\mathcal{L} \mapsto \mathcal{L}^{\otimes p}$ on $\mathrm{NS}(X)$. Thus, the image $c_1(\mathrm{NS}(X))$ lies in the *Tate module*

$$T_H \;:=\; \{x \in H^2(X/W) \;:\; \varphi(x) = px\} \;\subseteq\; H^2(X/W),$$

where φ denotes the Frobenius on $H^2(X/W)$. We note that T_H is a $\mathbb{Z}_p$-module (rather than a W-module) and that $T_H \otimes_{\mathbb{Z}_p} W$ is a sub-**F**-crystal of slope 1 inside $H^2(X/W)$.

The connection to geometry is as follows: let X be a smooth and proper variety over a perfect field $\Bbbk$ of characteristic p, and let $(H^n(X/W)/\mathrm{Tors}, \varphi)$ be the **F**-crystal associated to the n-th crystalline cohomology group as explained in Example 0.10.10. Now, we have three convex polygons: the Newton polygon, the Hodge polygon, and the *geometric Hodge polygon*. The latter is graph associated to the piecewise linear function $\widetilde{\mathrm{Hdg}}$ defined on the interval $[0, b_n(X)]$ with slope 0 over the interval $[0, h'_0]$ and slope i on the interval $[h'_0 + \cdots + h'_i, h'_0 + \cdots + h'_{i+1}]$, where

$$h'_i \;:=\; h^{i,n-i}(X) \;=\; \dim_{\Bbbk} H^{n-i}(X, \Omega^i_{X/\Bbbk}), \quad \text{for } i = 0, \ldots, n.$$

Thus, the geometric Hodge polygons encodes the Hodge numbers of X in a convex polygon. After these preparations, we have the following fundamental result of Mazur [501], Nygaard [564], and Ogus [62], which links these three polygons.

Theorem 0.10.12 *Let X be a smooth and proper variety over a perfect field $\Bbbk$ of characteristic $p > 0$. Let (M, φ) be the* **F***-crystal associated to* $H^n(X/W)/\mathrm{Tors}$. *Then,*

$$\mathrm{Nw}_M(t) \ \geq \ \widetilde{\mathrm{Hdg}}_M(t), \quad t \in [0, \mathrm{rank}_W M].$$

Moreover, if $H^n(X/W)$ is a torsion-free W-module and if the Hodge versus de Rham spectral sequence of X degenerates at E_1, then

$$\mathrm{Hdg}_M(t) \ = \ \widetilde{\mathrm{Hdg}}_M(t), \quad t \in [0, \mathrm{rank}_W M],$$

that is, the **F***-crystal M computes the Hodge numbers $h^{i,n-i}$.*

Remark 0.10.13 We note that **F**-crystals arising from geometry as in Example 0.10.10 satisfy further constraints: if X is smooth and proper of dimension d over $\Bbbk$, then the slopes of the **F**-crystal associated to $H^n(X/W)/\mathrm{Tors}$ lie in the interval $[0, d]$ by Poincaré duality. Moreover, if X is projective over $\Bbbk$, then the Hard Lefschetz theorem together with Poincaré duality imply that the slopes of $H^n(X/W)/\mathrm{Tors}$ lie in the interval $[0, n]$ if $0 \leq n \leq d$ and in the interval $[n-d, d]$ if $d \leq n \leq 2d$.

Example 0.10.14 Let X be a K3 surface over an algebraically closed field $\Bbbk$ of characteristic $p > 0$. The geometric Hodge polygon associated to $H^2(X/W)$ is determined by $h^{0,2} = h^{2,0} = 1$ and $h^{1,1} = 20$, see also Section 1.4. The Hodge versus de Rham spectral sequence of X degenerates at E_1 and all crystalline cohomology groups are torsion free (again, see Section 1.4) and thus, Theorem 0.10.12 implies that the Hodge polygon is equal to the geometric polygon.

There is an integer h, called the *height* of X such that the slopes of $H^2(X/W)$ are equal to $1 - \frac{1}{h}$, 1 and $1 + \frac{1}{h}$.

This implies that (geometric) Hodge polygon and the possible Newton polygons associated to $H^2(X/W)$ are given in Figure 0.5.

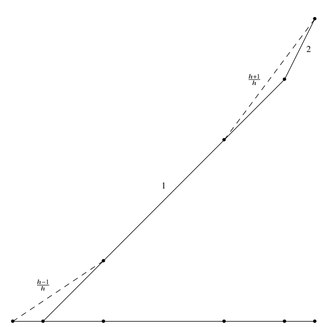


Fig. 0.5 Newton and Hodge polygons of a K3 surface

If $h = 1$, then the K3 surface is said to be *ordinary*, which is equivalent to saying that the Hodge polygon is equal to the Newton polygon. The other extreme is $h = \infty$, in which case the Newton polygon coincides with the graph of the identity function and then, the K3 surface is said to be *supersingular*. This lattter case happens if and only if we have $H^2(X/W) = T_H \otimes_{\mathbb{Z}_p} W$. We refer to [329, Section II.7.2] or [462] for details and proofs. We will encounter supersingular K3 surfaces often in Volume II. For K3 surfaces, there are other notions of being supersingular, see also Proposition below.

Example 0.10.15 Let us compute the different cohomology groups discussed so far for an abelian variety A of dimension g over an algebraically closed field $\Bbbk$.

1. If $\Bbbk = \mathbb{C}$, then, as a differentiable manifold, A is diffeomorphic to $(\mathbb{S}^1)^{2g}$, where $\mathbb{S}^1$ denotes the one-sphere. From this, it is easy to see

$$H^i(A, \mathbb{Z}) \cong \Lambda^i H^1(A, \mathbb{Z}) \quad \text{and} \quad H^1(A, \mathbb{Z}) \cong \mathbb{Z}^{2g},$$

and similarly for cohomology with coefficients $\mathbb{R}$ and $\mathbb{C}$. Moreover, one has

$$H^1(A, \mathbb{C}) \cong H^{0,1} \oplus H^{0,1} \quad \text{with} \quad H^{0,1} \cong \mathbb{C}^g \cong H^{1,0}.$$

2. If $\Bbbk$ is a perfect field of characteristic $p > 0$, then

$$\begin{array}{lll} H^i_{\mathrm{DR}}(A/\Bbbk) \cong \Lambda^i H^1_{\mathrm{DR}}(A/\Bbbk) & \text{with} & H^1_{\mathrm{DR}}(A/\Bbbk) \cong \Bbbk^{2g} \\ H^i(A/W) \cong \Lambda^i H^1(A/W) & \text{with} & H^1(A/W) \cong W(\Bbbk)^{2g} \end{array}$$

In particular, all crystalline cohomology groups are torsion-free $W(\Bbbk)$-modules. Moreover, the Hodge versus de Rham spectral sequence degenerates at E_1. To describe the ℓ-adic cohomology groups, let ${}_nA$ be the kernel of multiplication by n on A. Then, ${}_nA$ is a finite flat group scheme of length n^{2g} over $\Bbbk$, and for every prime ℓ, its $\Bbbk$-rational points satisfy

$${}_\ell A(\Bbbk) \cong \begin{cases} (\mathbb{Z}/\ell\mathbb{Z})^{2g} & \text{if } \ell \neq p, \text{ and} \\ (\mathbb{Z}/p\mathbb{Z})^r & \text{for some } 0 \leq r \leq g \text{ if } \ell = p. \end{cases}$$

The integer r is called the *p-rank* of the abelian variety A. Moreover, there exists an isomorphism

$$H^1_{\text{ét}}(A, \mathbb{Z}_\ell)^\vee \cong \varprojlim_n {}_{\ell^n}A(\Bbbk),$$

and one obtains

$$\dim_{\mathbb{Q}_\ell} H^1_{\text{ét}}(A, \mathbb{Q}_\ell) = \begin{cases} 2g & \text{if } \ell \neq p, \text{ and} \\ r & \text{where } r \text{ is the } p\text{-rank of } A. \end{cases}$$

Thus, $b_1(A) = 2g$ can be computed from ℓ-adic cohomology for all $\ell \neq p$, but not if $\ell = p$. One can also show

$$H^1_{\text{ét}}(A, \mathbb{Q}_p) \otimes_{\mathbb{Q}_p} K \cong \left(H^1(A/W) \otimes_W K\right)_{[0]} \subseteq H^1(A/W) \otimes_W K,$$

where the subscript [0] denotes the sub-**F**-isocrystal of slope 0. Thus, the crystalline cohomology group $H^1(A/W)$ has the expected rank $2g$ and the rank of $H^1_{\text{ét}}(A, \mathbb{Q}_p)$ can be explained using slopes.

We refer to [329, Section II.7.1].

As seen above, there exists a spectral sequence (0.10.35) from Hodge cohomology to de Rham cohomology. Also, we have seen above that crystalline cohomology is closely related to de Rham cohomology, see Theorem 0.10.7, for example. This motivated the construction of the *de Rham–Witt complex*, introduced by Illusie [329]: let X be a proper and smooth scheme over a perfect field $\Bbbk$ of characteristic $p > 0$. Then, the de Rham–Witt complex $W\Omega^\bullet_{X\Bbbk}$ is the limit of a projective system $(W_n\Omega^\bullet_{X/\Bbbk})_{n\geq 1}$ of complexes of abelian sheaves on the étale topology of X. For $n = 1$ it specializes to the usual de Rham complex of regular differential forms, that is, $W_1\Omega^i_{X/\Bbbk} = \Omega^i_{X/\Bbbk}$. Moreover, in degree 0 it specializes as $W\Omega^0_{X/\Bbbk} = WO_X$ and $W_n\Omega^0_{X/\Bbbk} = W_nO_X$, discussed above in connection with Serre's Witt vector cohomology. The construction of the de Rham–Witt complex is rather involved and we note that $W_n\Omega^\bullet_{X/\Bbbk}$ does not coincide with $\Omega^\bullet_{W_nX/\Bbbk}$, where W_nX denotes the topological space X together with the sheaf of rings W_nO_X. Then, the *de Rham–Witt cohomology* groups $H^i(X, W\Omega^j_{X/\Bbbk})$ are $W = W(\Bbbk)$-modules that are finitely generated modulo torsion. However, the torsion subgroups may not be finitely generated W-modules. If the cohomology groups $H^i(X, W\Omega^j_{X/\Bbbk})$ are finitely generated for all i and j, then X is said to be *Hodge–Witt*. In any case, there exists a spectral sequence

$$E_1^{i,j} := H^j(X, W\Omega^i_{X/\Bbbk}) \implies H^{i+j}(X/W) \tag{0.10.45}$$

from de Rham–Witt to crystalline cohomology, the *slope spectral sequence*. The canonical filtration on $H^n(X/W)$ arising from this spectral sequence is denoted by $P^iH^n(X/W)$, see [329, Definition II.(3.1.2)]. The following degeneracy result is due to Illusie [329] and in it, K denotes the field of fractions of $W = W(\Bbbk)$.

Theorem 0.10.16 *Let X be a smooth and proper variety over a perfect field $\Bbbk$ of characteristic $p > 0$. Then:*

1. *the slope spectral sequence degenerates modulo torsion at E_1. In particular, there exists a decomposition*

$$H^n(X/W) \otimes_W K \cong \bigoplus_{i+j=n} H^j(X, W\Omega^i_{X/\Bbbk}) \otimes_W K,$$

which is compatible with the slope decomposition of the **F**-*isocrystal $H^n(X/W)$.*

2. *The slope spectral sequence degenerates at E_1 if and only if $H^i(X, W\Omega^j_{X/\Bbbk})$ is a finitely generated W-module for all i and j, that is, if and only if X is Hodge–Witt.*

Degeneration modulo torsion is closely related to Theorem 0.10.11: namely, the groups $E_1^{p,q}/\mathrm{Tors}$ carry natural structures of **F**-crystals, and then, the differentials on the E_2-page of the slope spectral sequence are morphisms between **F**-crystals of different slopes, and thus, must be zero. On the other hand, the slope spectral sequence may not degenerate at E_1 even for some K3 surfaces and some abelian varieties, see [329, Section II.7].

In any case, one can show that the Hodge–Witt cohomology groups

$$H^0(X, W\Omega^i_{X/\Bbbk}), \quad H^1(X, W\mathcal{O}_X), \quad \text{and} \quad H^i(X, \Omega^{\dim X}_{X/\Bbbk}) \tag{0.10.46}$$

are finitely generated W-modules for all i that are even free in the first two cases, see [329, Corollaire II.2.17, Corollaire II.2.18, and Proposition II.2.19]. From this, it follows that one always has a short exact sequence

$$0 \to H^0(X, W\Omega^1_{X/\Bbbk}) \to H^1(X/W) \to H^1(X, W\mathcal{O}_X) \to 0 \tag{0.10.47}$$

of free W-modules, see [329, Proposition II.3.11], which is a partial degeneration result.

Remark 0.10.17 Let us mention two useful results in connection with degeneration of the slope spectral sequence of a smooth and proper variety X over $\Bbbk$.

1. If X is curve, then the slope spectral sequence degenerates at E_1. This follows from the discussion above and in particular, (0.10.46) and (0.10.47).
2. If X is a surface, then it follows from the discussion above, in particular from (0.10.46) and (0.10.47) that the only differential that could be non-zero is

$$d_1 : H^2(X, W\mathcal{O}_X) \to H^2(X, W\Omega^1_{X/\Bbbk}).$$

From this, one can deduce that the slope spectral sequence of a surface degenerates at E_1 if and only if $H^2(W\mathcal{O}_X)$ is a finitely generated W-module, see [329, Corollary II.3.14] or [563] for details. Moreover, if the slope spectral sequence of a surface degenerates at E_1, then d_1 is zero and since $H^2(W\mathcal{O}_X) \to H^2(\mathcal{O}_X)$ is surjective, it follows that in this case also the differentials

$$H^2(X, \mathcal{O}_X) \to H^2(X, \Omega^1_{X/\Bbbk}) \quad \text{and} \quad H^0(X, \Omega^1_{X/\Bbbk}) \to H^2(X, \Omega^1_{X/\Bbbk})$$

in the Frölicher spectral sequence (0.10.35) are zero (the second assertion follows from Serre duality). In particular, all global 1-forms on X are d-closed. We refer to [329, Remarks II.5.17.(2)].

Let us briefly digress on the sub-**F**-isocrystals of $H^n(X/W) \otimes K$ of slope < 1 and their relation to Serre's Witt vector cohomology group $H^n(X, W\mathcal{O}_X)$, as well as to the Artin–Mazur functors $\Phi^n_{X/\Bbbk}$ introduced in [29]: given a variety X that is proper over a field $\Bbbk$, Artin and Mazur [29] studied the functor

$$\begin{array}{rcl} \Phi^n_{X/\Bbbk} : \begin{pmatrix} \text{artinian } \Bbbk\text{-algebras} \\ \text{with residue field } \Bbbk \end{pmatrix} & \to & (\text{abelian groups}) \\ S & \mapsto & \mathrm{Ker}(H^n_{\text{ét}}(X \times_{\Bbbk} S, \mathbb{G}_m) \to H^n_{\text{ét}}(X, \mathbb{G}_m)). \end{array}$$

This functor possesses a deformation-obstruction theory with obstruction space $H^{n+1}(X, \mathcal{O}_X)$ and tangent space $H^n(X, \mathcal{O}_X)$. Moreover, if $H^{n-1}(X, \mathcal{O}_X) = 0$ or

$n = 1$, then $\Phi^n_{X/\Bbbk}$ is pro-representable, that is, there exists a local and complete $\Bbbk$-algebra R such that $\Phi^n_{X/\Bbbk}$ is isomorphic to $\mathrm{Hom}(-, \mathrm{Spf}\, R)$. In this case, $\mathrm{Spf}\, R$ is a group object in the category of formal schemes, that is, there exists a morphism $\mu : \mathrm{Spf}\, R \times \mathrm{Spf}\, R \to \mathrm{Spf}\, R$ and $e : \mathrm{Spf}\, \Bbbk \to \mathrm{Spf}\, R$ satisfying the axioms of a group. If moreover $H^{n+1}(X, \mathcal{O}_X) = 0$ holds true, then the functor $\Phi^n_{X/\Bbbk}$ is formally smooth over $\Bbbk$, which implies $R \cong \Bbbk[[t_1, ..., t_m]]$, where $m = h^n(\mathcal{O}_X)$. In this latter case, $\Phi^n_{X/\Bbbk}$ is a commutative formal group law of dimension m as discussed in Section 0.3.

Example 0.10.18 For $n \leq 2$, we have the following description on $\Phi^n_{X/\Bbbk}$.

1. If $n = 1$, then $H^1_{\text{ét}}(X, \mathbb{G}_m)$ classifies $\mathbb{G}_m$-torsors over X, that is, invertible sheaves and we have $H^1_{\text{ét}}(X, \mathbb{G}_m) \cong \mathrm{Pic}(X)$, see Example 0.1.6 and (0.9.1). From this, we deduce that $\Phi^1_{X/\Bbbk}(S)$ is the set of invertible sheaves on $X \times_{\Bbbk} \mathrm{Spec}\, S$ that restrict to $\mathcal{O}_X$ in the special fiber $X \times \mathrm{Spec}\, \Bbbk$, that is, $\Phi^1_{X/\Bbbk}$ describes the infinitesimal neighborhoods of $\mathrm{Pic}^\circ_{X/\Bbbk}$ at $[\mathcal{O}_X]$. Thus, $\Phi^1_{X/\Bbbk}$ is the formal completion of the Picard scheme $\mathrm{Pic}_{X/\Bbbk}$ along its zero-section, which is why it is called the *formal Picard group* and denoted $\widehat{\mathrm{Pic}}_{X/\Bbbk}$. We refer to Section 0.3 for such formal completions and especially, Example 0.1.17 and Example 0.1.18. If $H^2(X, \mathcal{O}_X) = 0$, then $\mathrm{Pic}^\circ_{X/\Bbbk}$ is smooth, that is, an abelian variety, and $\widehat{\mathrm{Pic}}_{X/\Bbbk}$ is a formal group law.
2. Recall from Section 0.9 that $H^2_{\text{ét}}(X, \mathbb{G}_m)$ is the cohomological Brauer group of X, which is why $\Phi^2_{X/\Bbbk}$ is called the *formal Brauer group* and denoted by $\widehat{\mathrm{Br}}_{X/\Bbbk}$. We note that, in general, there exists no Brauer group scheme of X such that $\widehat{\mathrm{Br}}_{X/\Bbbk}$ is the formal completion of this group scheme along its zero-section.

Now, assume that X is a smooth and proper variety over a perfect field $\Bbbk$ of characteristic $p > 0$ and assume that $\Phi^n_{X/\Bbbk}$ is pro-representable by a formal group law. Then, there exist isomorphisms of **F**-(iso-)crystals

$$\mathbb{D}(\Phi^n_{X/\Bbbk}) \cong H^n(X, W\mathcal{O}_X) \quad \text{and} \quad H^n(X, W\mathcal{O}_X) \otimes K \cong (H^n(X/W) \otimes_W K)_{<1},$$

where $\mathbb{D}(-)$ denotes the Cartier–Dieudonné module of a formal group law and where the subscript < 1 denotes the sum of all sub-**F**-isocrystals of slope < 1 in $(H^n(X/W) \otimes K)$. For example, if $\Phi^n_{X/\Bbbk} \cong \widehat{\mathbb{G}}_a$, then $\mathbb{D}(\Phi^n_{X/\Bbbk})$ is not a finitely generated W-module by Example 0.1.19, which also explains why $H^n(X, W\mathcal{O}_X)$ may fail to be finitely generated. We refer to [462] for a more detailed survey, as well as further references.

Let us now discuss various cohomology groups in low degree: let X be a smooth and proper variety over a perfect field of characteristic $p \geq 0$. Let $\mathrm{alb}_X : X \to \mathrm{Alb}(X)$ be the *Albanese morphism*, that is, every morphism from X to an abelian variety factors through alb_X, see Section 0.9. There, we showed that there exists an isomorphism $\mathrm{Alb}(X) \cong (\mathrm{Pic}^\circ_{X/\Bbbk,\mathrm{red}})^\vee$, where red denotes the reduction of $\mathrm{Pic}^\circ_{X/\Bbbk}$, which is an abelian variety, and $^\vee$ denotes the dual abelian variety. Then, the Albanese morphism induces isomorphisms

$$\begin{aligned} H^1_{\text{ét}}(X, \mathbb{Z}_\ell) &\cong H^1_{\text{ét}}(\text{Alb}(X), \mathbb{Z}_\ell), \\ H^1(X/W) &\cong H^1(\text{Alb}(X)/W) \quad \text{if } p > 0, \end{aligned}$$

which can be computed using Example 0.10.15. Moreover, setting $g := b_1(X)/2$, we find

$$\begin{array}{llll} H^0(X, W\Omega^1_{X/\Bbbk}) & \cong H^0(\text{Alb}(X), W\Omega^1_{\text{Alb}(X)/\Bbbk}) & & \cong W^g, \\ H^1(X, W\mathcal{O}_X) & \cong H^0(\text{Alb}(X), W\mathcal{O}_{\text{Alb}(X)}) & & \cong W^g, \\ H^1(X/W) & \cong H^0(X, W\Omega^1_{X/\Bbbk}) \oplus H^1(X, W\mathcal{O}_X) & & \cong W^{2g}. \end{array}$$

In particular, these W-modules are torsion-free. It remains to treat the algebraic de Rham cohomology group $H^1_{\text{DR}}(X/\Bbbk)$, whose $\Bbbk$-dimension may be strictly larger that that of $H^1_{\text{DR}}(\text{Alb}(X)/\Bbbk)$, which is equal to $2g$. By the universal coefficient formula (0.10.43) this happens if and only if $H^2(X/W)$ has non-trivial W-torsion. In order to understand the source of this torsion, the following two results are crucial: first, by a theorem of Igusa [324], see Proposition 0.9.18, the pull-back of global 1-forms induces an injective $\Bbbk$-linear map

$$\text{alb}_X^* \,:\, H^0(\text{Alb}(X), \Omega^1_{\text{Alb}(X)/\Bbbk}) \;\to\; H^0(X, \Omega^1_{X/\Bbbk}),$$

whose image is g-dimensional and all global regular 1-forms in the image are closed under the exterior differential d. Thus, the image of alb_X^* lies in the subspace $H^0(Z\Omega^1_{X/\Bbbk})$ of $H^0(\Omega^1_{X/\Bbbk})$, where $Z\Omega^1_{X/\Bbbk} = \text{Ker}(d : \Omega^1_{X/\Bbbk} \to \Omega^2_{X/\Bbbk})$ is the sheaf of cycles. Second, we have seen in Section 0.9 that the Picard scheme $\text{Pic}^\circ_{X/\Bbbk}$ may not be reduced, which is the case if and only if g is strictly smaller than $h^1(\mathcal{O}_X)$. The following result is due to Illusie [329, Proposition II.5.6] and explains the W- torsion of $H^2(X/W)$.

Proposition 0.10.19 *Let X be a smooth and proper variety over a perfect field $\Bbbk$ of characteristic $p > 0$ with Albanese morphism $\text{alb}_X : X \to \text{Alb}(X)$. Then, the following are equivalent:*

1. *$H^2(X/W)$ is W-torsion free, and*
2. *$\text{Pic}^\circ_{X/\Bbbk}$ is reduced and $\text{alb}_X^*\left(H^0(\text{Alb}(X), \Omega^1_{\text{Alb}(X)/\Bbbk})\right) = H^0(X, Z\Omega^1_{X/\Bbbk})$.*

Concerning torsion in $H^2(X/W)$, we have injective homomorphisms

$$\text{NS}(X) \otimes_{\mathbb{Z}} \mathbb{Z}_p \;\to\; H^2_{\text{fl}}(X, \mathbb{Z}_p(1)) \;\to\; H^2(X, W\Omega^{\geq 1}_{X/\Bbbk}) \;=:\; P^1H^2(X/W),$$

where $P^1H^2(X/W)$ denotes the indicated filtration W-submodule coming from the slope spectral sequence, see [329, Proposition II.6.8]. This induces an isomorphism of torsion groups of W-modules

$$\text{Tors}(\text{NS}(X) \otimes_{\mathbb{Z}} W) \;\cong\; \text{Tors}(P^1H^2(X/W)). \tag{0.10.48}$$

However, unlike the singular cohomology in the complex case or the case of ℓ-adic cohomology, the torsion in $H^2(X/W)$ can be more complicated than the torsion coming from $\mathrm{NS}(X)$. (However, we note that the torsion of $H^2_{\mathrm{fl}}(X, \mathbb{Z}_p(1))$ coincides with the torsion of $\mathrm{NS}(X) \otimes \mathbb{Z}_p$, see (0.10.63) below.) To get a grip on the torsion of $H^2(X/W)$, Illusie defined in [329, Section II.6.7] the *divisorial torsion* $H^2(X/W)_d$. We refer to loc. cit. for the definition and note that it sits in a short exact sequence

$$0 \to (\mathrm{NS}(X) \otimes_{\mathbb{Z}} W)_{p-\mathrm{Tors}} \to H^2(X/W)_d \to H^2(X, WO_X)_{V-\mathrm{Tors}} \to 0\,. \tag{0.10.49}$$

Using the identification of $H^2(WO_X)$ with the Cartier–Dieudonné module of the formal Brauer group, also the V-torsion can be reasonably explained. By definition, the quotient of the torsion submodule of $H^2(X/W)$ by $H^2(X/W)_d$ is called the *exotic torsion*. If X is an Enriques surface, then we compute the torsion of $H^2(X/W)$ in Theorem 1.4.13 and Corollary 1.4.14. Finally, we refer the interested reader to [329, Proposition II. 6.9] for the connection of torsion in $H^2(X/W)$ and Oda's subspace of $H^1_{\mathrm{DR}}(X/\Bbbk)$ from [567].

Example 0.10.20 Let X be a smooth and proper curve of genus g over an algebraically closed field $\Bbbk$ of characteristic $p > 0$. Let $\mathrm{Jac}(X)$ be the Jacobian of X. Then, the Hodge versus de Rham spectral sequence of X degenerates at E_1 and the crystalline cohomology groups $H^n(X/W)$ are torsion-free. Moreover, the Albanese morphism $\mathrm{alb}_X : X \to \mathrm{Jac}(X)$ induces isomorphisms of ℓ-adic and crystalline cohomology

$$H^1_{\text{ét}}(X, \mathbb{Z}_\ell) \cong H^1(\mathrm{Jac}(X), \mathbb{Z}_\ell) \quad \text{and} \quad H^1(X/W) \cong H^1(\mathrm{Jac}(X)/W),$$

as well as isomorphisms

$$H^1_{\mathrm{DR}}(X/\Bbbk) \cong H^1_{\mathrm{DR}}(\mathrm{Jac}(X)/\Bbbk)$$

of algebraic de Rham cohomology.

Concerning torsion of $H^2(WO_X)$, the slopes of $H^2(X/W) \otimes_W K$, and its relation to the Picard scheme $\mathbf{Pic}_{X/\Bbbk}$, we also have the following result, see also Proposition 0.9.14. It is a straightforward generalization of a result of Illusie [329, Proposition II.7.3.2], see also [464, Proposition 2.2].

Proposition 0.10.21 *Let X be a smooth and projective variety over an algebraically closed field $\Bbbk$ of positive characteristic that satisfies*

$$\frac{1}{2} b_1(X) = h^1(X, O_X) - h^2(X, O_X).$$

Then, the F-isocrystal $H^2(X/W) \otimes_W K$ is of slope one and

$$H^2(X, WO_X) = \mathrm{Tors}\left(H^2(X, WO_X)\right) \cong \mathbb{D}(\boldsymbol{Pic}^\circ_{X/\Bbbk}/\boldsymbol{Pic}^\circ_{X/\Bbbk,\mathrm{red}}),$$

where Tors *denotes torsion as W-module.*

Proof By [329, Remarque II.6.4], the V-torsion $H^2_{V-\mathrm{tors}}$ of $H^2(WO_X)$ is isomorphic to the Cartier–Dieudonné module $\mathbb{D}(\mathbf{Pic}^\circ_{X/\Bbbk}/\mathbf{Pic}^\circ_{X/\Bbbk,\mathrm{red}})$. Thus, by Dieudonné theory, the $\Bbbk$-dimension of $H^2_{V-\mathrm{tors}}/VH^2_{V-\mathrm{tors}}$ is equal to the dimension of the Zariski tangent space of $\mathbf{Pic}^\circ_{X/\Bbbk}/\mathbf{Pic}^\circ_{X/\Bbbk,\mathrm{red}}$, which is equal to $h^1(\mathcal{O}_X) - \frac{1}{2}b_1(X)$. Thus, in the exact sequence

$$\ldots \to H^1(\mathcal{O}_X) \to H^2(W\mathcal{O}_X) \xrightarrow{V} H^2(W\mathcal{O}_X) \xrightarrow{\alpha} H^2(\mathcal{O}_X) \to \ldots,$$

the restriction $\alpha|_{H^2_{V-\mathrm{tors}}} : H^2_{V-\mathrm{tors}} \to H^2(\mathcal{O}_X)$ is surjective by our assumptions. Next, we set $L := H^2(W\mathcal{O}_X)/H^2_{V-\mathrm{tors}}$ and denote the map induced by V on L again by V. Using the snake lemma, we conclude $L/VL = 0$. As explained in the proof of [329, Proposition II.7.3.2], L is V-adically separated, which implies $L = 0$. Thus, $H^2(W\mathcal{O}_X) = H^2_{V-\mathrm{tors}}$ and this W-module is torsion.

Since the slope spectral sequence of X degenerates up to torsion by Theorem 0.10.16, we conclude

$$0 = H^2(W\mathcal{O}_X) \otimes_W K = (H^2(X/W) \otimes_W K)_{[0,1[}.$$

Since X is projective over $\Bbbk$, the Hard Lefschetz Theorem implies that also the part of slope $]1, 2]$ is zero. Thus, $H^2(X/W) \otimes_W K$ is of slope one. □

Definition 0.10.22 A smooth and proper variety over $\Bbbk$ is called *ordinary in degree n* (in the sense of Bloch, Kato, Illusie, and Raynaud) if any of the following equivalent properties is satisfied.

1. $H^j(X, B\Omega^i_{X/\Bbbk}) = 0$, for any i, j with $i + j = n$.
2. $\mathbf{F} : H^j(X, W\Omega^i_{X/\Bbbk}) \to H^j(X, W\Omega^i_{X/\Bbbk})$ is bijective for all i, j with $i + j = n$.
3. The Hodge versus de Rham spectral sequence, as well as its conjugate spectral sequence degenerate on the E_1-page and the two induced filtrations on $H^n_{\mathrm{DR}}(X)$ are as disjoint as possible.
4. X is Hodge–Witt in degree n (that is, for all $i, j \geq 0$ with $i + j = n$, the W-module $H^i(X, W\Omega^j_{X/\Bbbk})$ is finitely generated) and $H^i(X, BW\Omega^{n+1-i}_{X/\Bbbk}) = 0$ for all $i \geq 0$.

If $H^n(X/W)$ has no torsion, then these properties are equivalent to the property

5. the Newton and the Hodge polygon of $H^n(X/W)$ coincide.

If $H^j(X, W\Omega^i_{X/\Bbbk})$ has no torsion for all i, j with $i + j = n$, then these properties are equivalent to

6. the slopes of the Frobenius on $H^n(X/W)$ are integers.

The variety X is *ordinary* if it is ordinary in all degrees.

We refer to [70], [332] and [116] for details and the proof of the equivalences. One usually expects that varieties that are ordinary and that have torsion-free crystalline cohomology groups should behave as nicely as varieties over the complex numbers. Here are some properties of ordinary varieties.

1. The slope spectral sequence degenerates modulo torsion at E_1. This follows from Theorem 0.10.16.

2. If X' is the blow-up of an ordinary variety X with smooth center then X' is ordinary [330, Proposition 1.6]. In particular, if X' is an ordinary surface birationally isomorphic to a surface X, then X' is ordinary.
3. A projective bundle over an ordinary variety is ordinary, see Proposition 1.4 from loc. cit..
4. The product $X \times Y$ is ordinary if and only if X and Y are ordinary, see [332, IV, Corollaire 4.14].
5. If $f : X \to C$ is over a local $C = \operatorname{Spec} R$ with ordinary closed fiber, then the geometric generic fiber is ordinary, see Proposition 1.9 from loc.cit.. The converse is usually not true.
6. If $H^i(X/W)$ and $H^{\dim X - i}(X/W)$ are both torsion-free, then X is ordinary in degree i if and only if it is ordinary in degree $\dim X - i$. This follows from Property (4).

Before discussing these notions for curves, we need a lemma on *p-linear* maps, see [653, pp. 38–39], or [3], or [278, Expose XII, Corollaire 1.1.10], or [508, III, §4, Lemma 4.13].

Lemma 0.10.23 *Let V be a finite–dimensional linear space over an algebraically closed field $\Bbbk$ of characteristic $p > 0$. Let $\phi : V \to V$ be a p^k-linear map, that is, $\phi(\lambda x) = \lambda^{p^k}\phi(x)$ for any $\lambda \in \Bbbk$. Then, there exists a canonical decomposition of $\Bbbk$-vector spaces*

$$V = V_{ss} \oplus V_{nil},$$

such that ϕ is bijective on V_{ss} and ϕ is nilpotent on V_{nil}. Moreover, $\phi - \mathrm{id}$ is surjective on V and the kernel of $\phi - \mathrm{id}$ is a vector space over the finite field $\mathbb{F}_{p^k}$ of dimension equal to $\dim V_{ss}$.

Note that the dimension of V_{ss} from the lemma can be computed as follows: choose a $\Bbbk$-basis $\underline{e}$ of V and let $\phi(\underline{e}) = A\underline{e}$ for some matrix A, sometimes called the *Hasse–Witt matrix*. Then

$$\dim_{\Bbbk} V_{\mathrm{ss}} = \operatorname{rank}(A \cdot A^{(p)} \cdot \dots \cdot A^{(p^{n-1})}), \tag{0.10.50}$$

where $n = \dim_{\Bbbk} V$, and where $A^{(p^k)}$ denotes the matrix obtained from A by raising the entries of the matrix A to the p^k-th power. The rank of the matrix $(A \cdot A^{(p)} \cdot \dots \cdot A^{(p^{n-1})})$ is called the *stable rank* of A. For more details on computations with Hasse-Witt matrices and some subtleties, we refer to [3].

Example 0.10.24 Let X be a smooth and projective curve of genus $g > 0$ over an algebraically closed field $\Bbbk$ of characteristic $p > 0$. Let $\mathrm{Jac}(X) = \mathbf{Pic}^{\circ}_{X/\Bbbk}$ be its Jacobian variety of X. Then we have an exact sequence

$$0 \to \mathcal{O}_X \xrightarrow{x \mapsto x^p} \mathcal{O}_X \xrightarrow{d} B\Omega^1_{X/\Bbbk} \to 0 \tag{0.10.51}$$

that gives an exact sequence

$$0 \to H^0(X, B\Omega^1_{X/\Bbbk}) \to H^1(X, \mathcal{O}_X) \xrightarrow{\mathbf{F}} H^1(X, \mathcal{O}_X) \to H^1(X, B\Omega^1_{X/\Bbbk}) \to 0.$$

This shows that X is ordinary if and only if the $\mathbf{F}$-action on $H^1(X, \mathcal{O}_X)$ is bijective. We note that the Frobenius map $\mathbb{F}$ on $H^1(X, \mathcal{O}_X)$ is p-linear and we now apply Lemma 0.10.23.

1. If $\mathbf{F}$ is injective on $H^1(X, \mathcal{O}_X) = H^1(X, \mathcal{O}_X)_{\mathrm{ss}}$, then $\mathrm{Ker}(\mathrm{id} - \mathbf{F}) = (\mathbb{Z}/p\mathbb{Z})^g$. It follows from the Artin–Schreier exact sequence that this happens if and only if
$$H^1\left(X, (\mathbb{Z}/p\mathbb{Z})_X\right) = \mathrm{Ker}\left(\mathrm{id} - \mathbf{F} : H^1(X, \mathcal{O}_X \to H^1(X, \mathcal{O}_X)\right) = (\mathbb{Z}/p\mathbb{Z})^g.$$
Applying Theorem (0.2.34), we obtain that $\mathrm{Hom}(\boldsymbol{\mu}_p, \mathrm{Jac}(X)) \cong (\mathbb{Z}/p\mathbb{Z})^g$. This implies that $\boldsymbol{\mu}_p^g$ is a closed subscheme of ${}_p\,\mathrm{Jac}(X)$ and applying Cartier duality, we obtain
$$(\mathbb{Z}/p\mathbb{Z})^g \subseteq {}_p\,\mathrm{Jac}(X)(\Bbbk) = {}_p\,\mathrm{Pic}(X).$$
Thus, X is ordinary if and only if the p-rank of $\mathrm{Jac}(X)$ (see Example 0.10.15) is maximal, that is, equal to $g = \dim \mathrm{Jac}(X)$.
2. On the other extreme, X is called *supersingular* if its p-rank is equal to 0. The part $H^1(X/W)^{\mathbf{F}=\mathrm{id}}$ is of dimension g and its dual part with respect to the Poincaré duality is equal to $H^1(X/W)^{\mathbf{F}=p\,\mathrm{id}}$.

The same argument shows that, if X is an ordinary variety of arbitrary dimension, then the p-ranks of its Albanese variety and its Picard variety are maximal possible, that is, equal to their dimension.

Example 0.10.25 Let A be an abelian variety of dimension $g > 0$ over an algebraically closed field $\Bbbk$ of characteristic $p > 0$. We know from Example 0.10.15 that $H^n(A/W) \cong \Lambda^i H^1(A/W)$ and that $H^1(A/W) \cong W^{2g}$ has no torsion. Thus, we can apply Property (4) to deduce that A is ordinary in all degrees if and only if it is ordinary in degree 1.

Example 0.10.26 Let X be a smooth and projective surface over an algebraically closed field $\Bbbk$ of characteristic $p > 0$.

If X is ordinary in degree 2, then Property (2) implies that $\mathbf{F}$ is bijective on $H^2(X, W\mathcal{O}_X)$, on $H^1(X, W\Omega^1_{X/\Bbbk})$, and on $H^0(X, W\Omega^2_{X/\Bbbk})$. We know that $H^2(X, W\mathcal{O}_X)$ is isomorphic to the Cartier–Dieudonné module $\mathbb{D}(\Phi^2_{X/\Bbbk})$. Since it is finitely generated over W when X is ordinary, we conclude that the formal Brauer group $\widehat{\mathrm{Br}}(X)$ is of finite height. Moreover, since $H^2(X, W\mathcal{O}_X)_W \otimes K = (H^2(X/W) \otimes_W K\mathcal{O}_X))_{<1}$, the Newton and Hodge polygons coincide if and only if the height $h(X)$ of $\widehat{\mathrm{Br}}(X)$ is equal to $h^{0,2} = p_g(X)$. The Igusa–Artin–Mazur inequality now becomes
$$b_2(X) - \rho(X) \geq 2p_g(X),$$
as familiar from complex algebraic geometry.

Also, if $b_1^{\mathrm{DR}} = 0$, for example, if $H^1(X, \mathcal{O}_X) = 0$ and $d_1 : H^0(X, \Omega^1_{X/\Bbbk}) \to H^2(X, \mathcal{O}_X)$ is injective, then $H^2(X/W)$ is torsion free, as follows from the universal coefficient formula (0.10.43). Moreover, we have $H^1(X/W) = 0$ in this case, which shows that X is automatically ordinary in degree 1. Thus, the equality $h(X) = p_g(X)$ implies that X is ordinary in this case.

Finally, we use the definition and exact sequence (0.10.51) to get an exact sequence

$$H^1(X, B\Omega^1_{X/\Bbbk}) \to \operatorname{Ker}\left(\mathbf{F} : H^2(X, \mathcal{O}_X) \to H^2(X, \mathcal{O}_X)\right) \to H^2(X, B\Omega^1_{X/\Bbbk}) \to 0$$

to show that, if X is ordinary in degree 2, then the $\mathbf{F}$-action on $H^2(X, \mathcal{O}_X)$ is a bijection.

In particular, let X be a surface with $p_g(X) = 0$: It is ordinary in degree 1 if and only if its Albanese variety is ordinary. By Proposition 0.9.8, the Picard scheme is reduced. It follows from Proposition 0.10.21 that $H^2(X, W\mathcal{O}_X) = 0$. Since $H^1(X, W\mathcal{O}_X)$ and $H^0(X, \Omega^1_{\Bbbk})$ have no torsion, the slope spectral sequence shows that $H^2(X/W)$ is torsion-free. We know that X is ordinary in degree 2 if and only if the Newton and Hodge polygons of $H^2(X/W)$ coincide. But the latter polygon is the graph of the identity function on the interval $[0, b_2]$ and since the Newton polygon lies above it, it must coincide with it. Thus, X is automatically ordinary in degree 2.

We now turn to the case where $\ell = p$ and note already here that the flat cohomology groups $H^2_{\mathrm{fl}}(X, \boldsymbol{\mu}_{p^n})$ are more difficult to handle – we will come back to these groups below. First, we note that the absolute Frobenius morphism $\mathbf{F} : X \to X$ acts like $\mathcal{L} \mapsto \mathbf{F}^*\mathcal{L} \cong \mathcal{L}^{\otimes p}$ on $\mathrm{NS}(X)$. We set

$$\begin{aligned} h(X) &= \dim_K \left(H^2(X, W\mathcal{O}_X) \otimes_W K\right) && (0.10.52)\\ &= \dim_K \left(H^2(X/W) \otimes_W K\right)_{[0,1)} = \dim_K \left(H^2(X/W) \otimes_W K\right)_{(1,2]}, \end{aligned}$$

where the last equality comes from the slope spectral sequence (0.10.45) and Theorem 0.10.16. If X is projective, then the Hard Lefschetz theorem implies that $h(X)$ is equal to the K-dimension of the slope > 1 part of the $\mathbf{F}$-isocrystal $H^2(X/W) \otimes K$ and using the slope decomposition of $H^2(X/W) \otimes K$, we find the *Igusa–Artin–Mazur inequality*

$$\rho(X) \leq b_2(X) - 2h(X). \tag{0.10.53}$$

For example, we see that in the case of a K3 surface, the number $h(X)$ coincides with the number h in the picture of the Newton diagram in Example 0.10.14.

Remark 0.10.27 Since $H^2(X, W\mathcal{O}_X)$ is the Cartier–Dieudonné module of the formal Brauer group $\widehat{\mathrm{Br}}_{X/\Bbbk}$ of X, it follows that $h(X)$ is equal to the height of $\widehat{\mathrm{Br}}_{X/\Bbbk}$ - if the latter is finite. A surface X is called *supersingular* if the height of its formal Brauer group is infinite. In this case, the formal Brauer group is isomorphic to the formal group associated to a unipotent algebraic group of dimension equal to $p_g(X) = \dim H^2(X, \mathcal{O}_X)$, where $H^2(X, \mathcal{O}_X)$ is naturally isomorphic to the tangent space of the formal group. In this context, the crystalline version of the *Tate conjecture* asks whether the inclusion $c_1(\mathrm{NS}(X)) \otimes \mathbb{Z}_p \subseteq T_H$ is in fact an equality, or at least, equal up to torsion or finite index. If true, this would be a characteristic p analog of the Lefschetz theorem on $(1,1)$-classes. For example, it would also imply that the **F**-isocrystal $H^2(X/W) \otimes K$ is of slope 1 (the **F**-crystal is *supersingular*) if and only if $\rho = b_2$ (the variety is *supersingular in the sense of Shioda*). It follows from (0.10.53) that a surface with $p_g(X) > 0$ that is supersingular in the sense of Shioda is supersingular. Since the Tate conjecture holds for K3 surfaces over finite fields by [500], [483] and [390], Proposition 0.10.28 implies that K3 surfaces that are supersingular are supersingular in the sense of Shioda.

There is also an ℓ-adic version of the Tate conjecture for smooth and proper varieties X over fields $\Bbbk$ that are finitely generated over their prime field that conjecturally describes the image $c_1(\mathrm{NS}(X))$ inside $H^2_{\text{ét}}(X_{\bar{\Bbbk}}, \mathbb{Z}_\ell(1))$ in terms of the action of the absolute Galois group $\mathrm{Gal}(\bar{\Bbbk}/\Bbbk)$. So far, this conjecture has been established for curves, products of curves, K3 surfaces, Enriques surfaces, and a couple of more classes of varieties, see [700], [704], and [55] for survey.

The following result is well-known to the experts and follows from some usual spreading out and specialization techniques, but somehow hard to find stated explicitly in the literature.

Proposition 0.10.28 *Let X be a supersingular (in the sense of Remark 0.10.27) K3 surface over an algebraically closed field $\Bbbk$ of characteristic $p > 0$. Assume that the Tate conjecture holds for K3 surfaces over finite fields of characterstic p. Then, X is supersingular in the sense of Shioda.*

Proof First, assume that $\Bbbk$ is not an algebraic closure of $\mathbb{F}_p$. Then, there exists a field K that is finitely generated over $\mathbb{F}_p$ and contained in $\Bbbk$, such that X can be defined over K. Replacing K by a finite field extension, K is still finitely generated over $\mathbb{F}_p$ and we may assume that the Néron–Severi group of X is isomorphic to the Néron–Severi group over $\Bbbk$. Now, there exists an integral scheme of finite type T over $\mathbb{F}_p$ with the residue field at the generic point equal to K. After possibly replacing T by an open and dense subset, we may assume that there exists a smooth and proper morphism $\mathcal{X} \to T$ over $\operatorname{Spec} \mathbb{F}_p$, whose generic fiber is X (this is called *spreading out*). If $t \in T$ is a closed point of T, then the fiber $\mathcal{X}_t$ is a K3 surface over the residue field $\kappa(t)$, which is a finite field of characteristic p. Moreover, since X is supersingular, so is $\mathcal{X}_t$ by [25], Corollary (1.3). But then, [25, Theorem (1.1)] implies that the ranks of the Néron–Severi groups of X and $\mathcal{X}_t$ coincide. Thus, it suffices to prove the proposition for $\mathcal{X}_t$.

This reduces to the case where $\Bbbk$ is an algebraic closure of $\mathbb{F}_p$, which we will now assume. If X is a supersingular K3 surface over $\Bbbk$, then it can be defined over some finite field $\mathbb{F}_q$ and by replacing the latter by a finite extension, we may assume that the Néron–Severi group of X is isomorphic to the Néron–Severi group of $X_{\Bbbk}$. Since X is supersingular, the **F**-isocrystal $H^2(X_{\Bbbk}/W) \otimes_{\mathbb{Z}} \mathbb{Q}$ is of slope 1. After possibly replacing $\mathbb{F}_q$ by another finite extension, we may assume that $H^2(X/W) \otimes_{\mathbb{Z}} \mathbb{Q}$ is a direct sum of one-dimensional **F**-isocrystals of slope 1. In particular, the **F**-invariants of $H^2(X/W)(1) \otimes \mathbb{Z}$ form a $\mathbb{Q}_p$-vector space of rank equal to $b_2(X) = 22$. Since we assumed the Tate conjecture for K3 surfaces over finite fields, this implies that the image of $\mathrm{NS}(X) \otimes \mathbb{Q}_p$ inside $H^2(X/W)(1) \otimes \mathbb{Q}$, which actually lies in the **F**-invariants of the latter, is a $\mathbb{Q}_p$-vector space of dimension $b_2(X)$. Thus, $\rho(X) = b_2(X)$, that is, X is supersingular in the sense of Shioda. $\square$

For a survey on the Tate conjecture for K3 surfaces we refer to [706].

In order to refine the Igusa–Artin–Mazur inequality, we use the Kummer sequence (0.1.6) and (0.10.25).

$$0 \to \boldsymbol{\mu}_{p^n} \to \mathbb{G}_m \xrightarrow{x \mapsto x^{p^n}} \mathbb{G}_m \to 0. \tag{0.10.54}$$

In analogy to the ℓ-adic case, we set

$$H^i_{\mathrm{fl}}(X, \mathbb{Z}_p) := \varprojlim_n H^i_{\mathrm{fl}}(X, \mathbb{Z}/p^n\mathbb{Z}), \tag{0.10.55}$$

$$H^i_{\mathrm{fl}}(X, \mathbb{Z}_p(1)) := \varprojlim_n H^i_{\mathrm{fl}}(X, \boldsymbol{\mu}_{p^n}). \tag{0.10.56}$$

Note, however, that if $\Bbbk$ is of characteristic $p > 0$, then the finite group schemes $(\mathbb{Z}/p^n\mathbb{Z})$ are étale and the finite group schemes $\boldsymbol{\mu}_{p^n}$ are not étale. In particular, the cohomology groups $H^i_{\mathrm{fl}}(X, \mathbb{Z}_p)$ and $H^i_{\mathrm{fl}}(X, \mathbb{Z}_p(1))$ need not be isomorphic, not even non-canonically as $\mathbb{Z}_p$-modules. In fact, the former is related to the slope zero part of $H^i(X/W)$, whereas the latter is related to the slope one part.

Let us now study the groups $H^i_{\mathrm{fl}}(X, \mathbb{Z}_p(1))$. Since the $\boldsymbol{\mu}_{p^n}$ are flat but not étale, étale cohomology is not suited to computation of these cohomology groups, which makes the discussion of these cohomology groups technically more demanding. It was shown by Artin in [25] that the group $H^i(X, \boldsymbol{\mu}_{p^n})$, considered as a sheaf $Rf^i_*\boldsymbol{\mu}_{p^n,X}$, where $f : X \to \operatorname{Spec} \Bbbk$ is the $\Bbbk$-scheme structure morphism, admits a structure of a quasi-algebraic group, possibly of positive dimension. For example, if X is a supersingular K3 surface, then its formal Brauer group is isomorphic to $\widehat{\mathbb{G}}_a(\Bbbk)$ and the Kummer exact sequence shows that $H^2(X, \boldsymbol{\mu}_p) \cong {}_p\operatorname{Br}(X) \cong \Bbbk$, which is not a finite group. We will come back to this later in this section. The Kummer sequence in the flat topology gives

$$0 \to H^{i-1}(X, \mathbb{G}_m)^{(p^n)} \to H^i(X, \boldsymbol{\mu}_{p^n}) \to {}_{p^n}H^i(X, \mathbb{G}_m) \to 0 \tag{0.10.57}$$

that, after taking the limits, gives us an exact sequence

$$0 \to \varprojlim {}_n H^{i-1}_{\mathrm{fl}}(X, \mathbb{G}_m)^{(p^n)} \to H^i_{\mathrm{fl}}(X, \mathbb{Z}_p(1)) \to T_p H^i_{\mathrm{fl}}(X, \mathbb{G}_m) \to 0. \tag{0.10.58}$$

Since $\mathbb{G}_m$ is a smooth group scheme, it follows that $H^i_{\mathrm{fl}}(X, \mathbb{G}_m) \cong H^i_{\text{ét}}(X, \mathbb{G}_m)$, that is, flat and étale cohomology coincide in this case, see Theorem 0.1.3. For example, if $i \leq 1$, we obtain isomorphisms

$$H^0_{\mathrm{fl}}(X, \mathbb{Z}_p(1)) = 0, \tag{0.10.59}$$

$$H^1_{\mathrm{fl}}(X, \mathbb{Z}_p(1)) \cong T_p \operatorname{Pic}(X), \tag{0.10.60}$$

see also Example 0.10.15 in the case where X is an abelian variety. For $i = 2$, we obtain a short exact sequence

$$0 \to \operatorname{NS}(X) \otimes_{\mathbb{Z}} \mathbb{Z}_p \to H^2_{\mathrm{fl}}(X, \mathbb{Z}_p(1)) \to T_p H^2(X, \mathbb{G}_m) \to 0. \tag{0.10.61}$$

By [329, Proposition II.5.9] (we will give a proof of this important fact later), all terms in this sequence are finitely generated $\mathbb{Z}_p$-modules. Moreover, $T_p H^2(X, \mathbb{G}_m)$ is even a free $\mathbb{Z}_p$-module, and we denote its rank by t_p. We thus find

$$T_p(\mathrm{Br}(X)) \cong \mathbb{Z}_p^{\mathrm{t}_p} \tag{0.10.62}$$

and

$$\mathrm{Tors}\, H^2_{\mathrm{fl}}(X, \mathbb{Z}_p(1)) = \mathrm{Tors}\left(\mathrm{NS}(X) \otimes \mathbb{Z}_p\right) = {}_{p^\infty}\mathrm{NS}(X)\,. \tag{0.10.63}$$

As an application of (0.10.61), we conclude

$$\mathrm{rank}_{\mathbb{Z}_p} H^2_{\mathrm{fl}}(X, \mathbb{Z}_p(1)) = \rho(X) + \mathrm{t}_p.$$

However, the group ${}_p\mathrm{Br}(X)$ may not be a finite group and so we cannot apply Proposition 0.10.1 to deduce (0.10.28), where ℓ is replaced with p. In fact, the exact sequence (0.10.57) shows that ${}_p H^2_{\mathrm{fl}}(X, \mathbb{G}_m)$ is the image of the group $H^2_{\mathrm{fl}}(X, \boldsymbol{\mu}_p)$. As we have remarked earlier and will explain later in this section, the group $H^2_{\mathrm{fl}}(X, \boldsymbol{\mu}_p)$ admits a structure of a commutative quasi-algebraic group, whose connected component of identity, if not trivial, is a connected quasi-algebraic unipotent group of positive dimension whose p-torsion group is infinite.

The $\mathbb{Z}_p$-rank of $H^i_{\mathrm{fl}}(X, \mathbb{Z}_p(1))$ is equal to the $\mathbb{Q}_p$-dimension of $H^i_{\mathrm{fl}}(X, \mathbb{Q}_p(1))$ and the latter is connected to crystalline cohomology as follows: by [329, Theorem II.5.5], there exist short exact sequences for all i

$$0 \to H^i_{\mathrm{fl}}\left(X, \mathbb{Q}_p(1)\right) \to H^i(X/W) \otimes K \xrightarrow{p-\mathbf{F}} H^i(X/W) \otimes K \to 0, \tag{0.10.64}$$

from which it follows that $H^i_{\mathrm{fl}}(X, \mathbb{Q}_p(1)) \otimes K$ is equal to the sub-$\mathbf{F}$-isocrystal of slope 1 inside $H^i(X/W) \otimes K$. In particular, $H^i_{\mathrm{fl}}(X, \mathbb{Z}_p(1))$ modulo torsion (resp. $H^i_{\mathrm{fl}}(X, \mathbb{Q}_p(1))$) is a $\mathbb{Z}_p$-module of finite rank (resp. $\mathbb{Q}_p$-vector space of finite dimension) for all i. Using the slope-decomposition of $H^2(X/W) \otimes K$ into the sub-$\mathbf{F}$-isocrystals of slopes < 1, $= 1$, and > 1, as well as the Hard Lefschetz theorem (here, projectivity of X is needed), we find the following improvement of the Igusa–Artin–Mazur inequality (0.10.53) to an equality

$$\rho(X) = b_2(X) - 2h(X) - \mathrm{t}_p\,. \tag{0.10.65}$$

We refer to [329, Proposition II.5.12] for details. Finally, let us mention the following long exact sequences:

$$\ldots \to H^{n+1}_{\mathrm{fl}}(X,\, \mathbb{Z}_p(1)) \to H^n(X,\, W\Omega^1_{X/\Bbbk}) \xrightarrow{1-\mathbf{F}} H^n(X,\, W\Omega^1_{X/\Bbbk}) \to \ldots \tag{0.10.66}$$

$$\ldots \to H^n_{\mathrm{fl}}(X,\, \mathbb{Z}_p(1)) \to H^n(X,\, W\Omega^{\geq 1}_{X/\Bbbk}) \xrightarrow{1-\mathbf{F}} H^n(X,\, W\Omega^{\geq 1}_{X/\Bbbk}) \to \ldots \tag{0.10.67}$$

that relate the cohomology of $\mathbb{Z}_p(1)$ to the cohomology of the de Rham–Witt complex, see [329, Theorem II.5.5]. Tensoring with $\mathbb{Q}$, and using that the slope spectral sequence degenerates modulo torsion, we obtain (0.10.64).

In order to generalize the relation between ρ and $h^{1,1}$ in the complex case from (0.10.10), we note that there is an algebraic dlog map $\mathcal{O}_X^\times \to \Omega^1_{X/\Bbbk}$ via $f \mapsto \frac{df}{f}$. This gives rise to a homomorphism of abelian groups

$$\mathrm{dlog} \, : \, \mathrm{Pic}(X) \, \to \, H^1(X, \Omega^1_{X/\Bbbk}), \tag{0.10.68}$$

which is an algebraic analog of homomorphism (0.10.68). Via the Hodge versus de Rham spectral sequence (0.10.35), $H^1(X, \Omega^1_X)$ contributes to $H^2_{\mathrm{DR}}(X/\Bbbk)$. However, if $p > 0$, then this spectral sequence may not degenerate at E_1 and thus, the former may not be a subquotient of the latter. Moreover, the $\Bbbk$-vector spaces $H^1(X, \Omega^1_X)$ and $H^2_{\mathrm{DR}}(X/\Bbbk)$ are p-torsion when considered as abelian groups, which implies that ${}_p\,\mathrm{Pic}(X)$ lies in the kernel of c_1 and dlog. We note that even the induced map $\mathrm{NS}(X) \otimes_{\mathbb{Z}} \Bbbk \to H^1(X, \Omega^1_X)$ may fail to be injective. In particular, the inequality $\rho \le h^{1,1}$, which holds if $\Bbbk$ is of characteristic zero, may fail if $\Bbbk$ is of positive characteristic – for example, supersingular K3 surfaces satisfy $\rho = b_2 = 22$ and $h^{1,1} = 20$. To further extend the relation between the Picard group, $H^1(X, \Omega^1_X)$, and $H^2_{\mathrm{DR}}(X/\Bbbk)$ via Chern class and dlog-maps to the p-adic setting, we use de Rham–Witt cohomology $H^1(X, W\Omega^1_X)$ and crystalline cohomology $H^2(X/W)$, rather than étale cohomology with $\mathbb{Z}_p$-coefficients.

Let

$$\epsilon \, : \, X_{\mathrm{fl}} \, \to \, X_{\mathrm{\acute{e}t}}$$

be the natural morphism from the flat site to the étale site of X. Then, applying the functor $R^1\epsilon_*$ to the Kummer exact sequence (0.10.54) on X_{fl} yields an exact sequence on $X_{\mathrm{\acute{e}t}}$

$$0 \, \to \, \mathbb{G}_m \, \xrightarrow{\times p^n} \, \mathbb{G}_m \, \to \, R^1\epsilon_* \boldsymbol{\mu}_{p^n} \, \to \, 0. \tag{0.10.69}$$

Here, we used that $R^q\epsilon_*$ is zero for all $q > 0$ and any abelian sheaf that is represented by a smooth group scheme. Thus, if we define

$$\nu_n(1) \, := \, R^1\pi_* \boldsymbol{\mu}_{p^n},$$

then we obtain a sheaf on the étale site of X. Applying the Leray spectral sequence for π and comparing the cohomology of (0.10.54) and (0.10.69), we find isomorphisms for all i

$$H^i_{\mathrm{fl}}(X, \boldsymbol{\mu}_{p^n}) \, \cong \, H^{i-1}_{\mathrm{\acute{e}t}}(X, \nu_n(1)). \tag{0.10.70}$$

In particular, this allows us to compute the flat cohomology of $\boldsymbol{\mu}_{p^n}$ via the étale cohomology of $\nu_n(1)$.

For any scheme S, let S_{pf} be the category of perfect S-schemes equipped with the étale topology, let $S_{\mathrm{\acute{e}t}}$ be the category of S-schemes equipped with the étale topology, and let $S_{\mathrm{\acute{e}t}} \to S_{\mathrm{fl}}$ be the perfection functor $X \mapsto X^{\mathrm{pf}}$ introduced in Section 0.1. We have a natural morphism

$$\epsilon \, : \, S_{\mathrm{pf}} \, \to \, S_{\mathrm{\acute{e}t}}.$$

For any abelian sheaf $\mathcal{A}$, we denote by A^{pf} the restriction of $\mathcal{A}$ to S_{fl}. If $\mathcal{A}$ is representable by a commutative group scheme G, then $\mathcal{A}^{\mathrm{pf}}$ is representable by its perfection G^{pf}. This defines the functor

$$\epsilon_{S,*} \,:\, \tilde{S}_{\text{ét}} \,\rightarrow\, \tilde{S}_{\text{fl}}$$

from the category of abelian sheaves on $S_{\text{ét}}$ to the category of abelian sheaves $\tilde{S}_{\text{fl}}$ on S_{fl}. It admits a left adjoint functor $\epsilon_S^* : \tilde{S}_{\text{fl}} \rightarrow \tilde{S}_{\text{ét}}$. For example, if $\mathcal{A}$ is represented by a commutative group S-scheme G, then $\mathcal{A}^{\mathrm{pf}}$ is represented by its perfection group scheme G^{pf}. In particular, for every purely inseparable homomorphism ϕ of group S-schemes, the morphism $(\epsilon_{S,*}$ is an isomorphism.

We take $S = \operatorname{Spec} \Bbbk$ and denote ϵ_S by ϵ. For any commutative algebraic group scheme G over $\Bbbk$, we have $\epsilon_* G = G^{\mathrm{pf}}$ and $R^i \epsilon_* G = 0$ for all $i > 0$. If G is smooth, then this follows from vanishing of étale cohomology of abelian sheaves represented by smooth group schemes. If G is not smooth, then it admits a composition series of sheaves that are either smooth or isomorphic to $\boldsymbol{\mu}_{p,\Bbbk}$ or $\boldsymbol{\alpha}_{p,\Bbbk}$ for which the assertion is proved using the Kummer or Artin–Schreier exact sequences in the étale topology, see [61, Lemma 2.3].

For any perfect affine commutative group scheme G annihilated by p^n for some n, there exists a maximal connected subgroup U with quotient an étale group D. The group U is a unipotent quasi-algebraic group and as such admits a composition series whose subquotients are isomorphic $\mathbb{G}_a^{\mathrm{pf}}$.

The category of commutative algebraic groups over a field of positive characteristic is not abelian because of the existence of inseparable isogenies. Thus, in order to use homological algebra and in particular, the have the higher derived functors $R^i \epsilon_*$, we pass to the category of perfect schemes and consider quasi-algebraic groups. For example, to justify the use of a Leray spectral sequence to obtain the isomorphism (0.10.70), we have to consider the $H^i_{\text{fl}}(X, \boldsymbol{\mu}_{p^m})$ as quasi-algebraic groups, which we denote by $\underline{H}^i_{\text{fl}}(X, \boldsymbol{\mu}_{p^m})$. Of course, $\underline{H}^i_{\text{fl}}(X, \boldsymbol{\mu}_{p^m})(\Bbbk) = H^i_{\text{fl}}(X, \boldsymbol{\mu}_{p^m})$.

We note that the kernel of dlog in the étale topology is equal to $\mathcal{O}_X^{\times p}$, which implies that there is an injective map of abelian sheaves on $X_{\text{ét}}$ from $\nu_n(1)$ into $Z\Omega^1_X$. In order to explain the relation between the latter and the dlog-map, we recall that we introduced the sheaves $B\Omega^i_{X/\Bbbk}$ of boundaries and the sheaves $Z\Omega^i_{X/\Bbbk}$ of cycles, as well as the Cartier operators $C = C_{X/\Bbbk} : \mathbf{F}_*(Z\Omega^i_{X/\Bbbk}) \rightarrow \Omega^i_{X^{(p)}/\Bbbk}$. Using Properties (2) and (5) of Cartier operators, we find

$$C(\operatorname{dlog} f) \;=\; C\left(\frac{df}{f}\right) \;=\; f^{-1} \cdot C\left(f^p \cdot \frac{df}{f}\right) \;=\; f^{-1} \cdot C(f^{p-1} df) \;=\; \frac{df}{f}.$$

In particular, the image of dlog is contained in the kernel of $(1-C)$. More precisely, one can even show that this generalizes to exact sequences in the étale topology of X

$$0 \,\rightarrow\, \nu_1(i) \,\rightarrow\, \mathbf{F}_*(Z\Omega^i_{X/\Bbbk}) \,\xrightarrow{1-C}\, \Omega^i_{X^{(p)}/\Bbbk} \,\rightarrow\, 0, \tag{0.10.71}$$

see, for example, [507]. In the case $i = 1$, this identifies the kernel of $(1 - C)$ with $\operatorname{dlog}(\nu_1(1))$. In the case $i = 0$, this exact sequence coincides with the Artin–Schreier sequence (0.1.3) and identifies $\nu_1(0)$ with the locally constant sheaf $\mathbb{Z}/p\mathbb{Z}$ on X.

Finally, by a theorem of Bloch, the sheaf $\nu_1(i)$ is the abelian subsheaf of $Z\Omega^i_{X/\Bbbk}$ that is generated (locally in the étale topology) by differentials $\mathrm{dlog}\, f_1 \wedge \cdots \wedge \mathrm{dlog}\, f_i$, where the $f_j \in \mathcal{O}_X^\times$ are local sections, see [329, Section 0.2.4]. Next, let $C^{-1} : \Omega^i_{X^{(p)}/\Bbbk} \to \mathcal{H}^i(\Omega^\bullet_{X/\Bbbk}) = Z\Omega^i_{X/\Bbbk}/B\Omega^i_{X/\Bbbk}$ be the inverse of the Cartier operator. We have the following commutative diagram with exact rows

$$\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & \\
 & & B\Omega^i_{X/\Bbbk} \arrow[d] & B\Omega^i_{X^{(p)}/\Bbbk} \arrow[d] & \\
0 \arrow[r] & \nu_1(i) \arrow[r] \arrow[d, equal] & Z\Omega^i_{X/\Bbbk} \arrow[r, "1-C"] \arrow[d] & \Omega^i_{X^{(p)}/\Bbbk} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \nu_1(i) \arrow[r] & Z\Omega^i_{X/\Bbbk}/B\Omega^i_{X/\Bbbk} \arrow[r, "1-C"] \arrow[d] & \Omega^i_{X^{(p)}/\Bbbk}/B\Omega^i_{X^{(p)}/\Bbbk} \arrow[r] \arrow[d] & 0. \\
 & & 0 & 0 &
\end{tikzcd}$$

Applying the Cartier operator (0.10.71) to the bottom row, we deduce the exact sequences

$$0 \to \nu_1(i) \to \Omega^i_{X^{(p)}/\Bbbk} \to \Omega^i_{X^{(p)}/\Bbbk}/B\Omega^i_{X^{(p)}/\Bbbk} \to 0. \tag{0.10.72}$$

This exact sequence can be generalized using the de Rham–Witt complex as follows, see [329]: for all i and k, there exist Frobenius morphism $\mathbf{F} : W_{k+1}\Omega^i_{X/\Bbbk} \to W_k\Omega^i_{X/\Bbbk}$ (note the difference in the lower indices), which gives rise to morphisms $\mathbf{F} : W_k\Omega^i_X \to W_k\Omega^i_X/dV^{k-1}\Omega^{i-1}_X$, and this latter coincides with C^{-1} if $k = 1$, see [329, Proposition I.3.3]. Then, (0.10.72) is just the beginning of a series of short exact sequences

$$0 \to \nu_k(i) \to W_k\Omega^i_{X/\Bbbk} \xrightarrow{\mathbf{F}-1} W_k\Omega^i_{X/\Bbbk}/dW_k\Omega^{i-1}_{X/\Bbbk} \to 0. \tag{0.10.73}$$

Let us briefly discuss the $i = 1$-case, which is important for our discussion of Picard groups and Chern class maps: the kernel of $\mathbf{F} - 1 : W_{k+1}\Omega^1_X \to W_k\Omega^1_X$ is contained in $\mathrm{dlog}(\mathcal{O}_X^\times) + V(W_k\Omega^1_X) + dV(W_k\Omega^1_X)$ by [329, Proposition I.3.24]. Moreover, as k tends to infinity, it follows from [329, Corollaire I.3.27] that there exists a short exact sequence

$$0 \to \mathcal{O}_X^\times/\mathcal{O}_X^{\times p} \xrightarrow{\mathrm{dlog}} W\Omega^1_{X/\Bbbk} \xrightarrow{\mathbf{F}-1} W\Omega^1_{X/\Bbbk} \to 0 \tag{0.10.74}$$

of abelian sheaves in the étale toplogy. Let us now come back to the Chern class and dlog-maps and assume that X is a smooth and projective surface over $\Bbbk$. Using (0.10.56) and (0.10.70), we find a short exact sequence

$$0 \to H^2_{\mathrm{fl}}(X, \mathbb{Z}_p(1)) \to H^1(X, W\Omega^1_{X/\Bbbk}) \xrightarrow{\mathbf{F}-1} H^1(X, W\Omega^1_{X/\Bbbk}) \to 0,$$

see also (0.10.67). Next, there is a commutative diagram with exact rows (see [329, Section II.5.22])

$$\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & \\
 & & H^0(X, W\Omega^2_{X/\Bbbk}) \arrow[r, "1-p\mathbf{F}"] \arrow[d] & H^0(X, W\Omega^2_{X/\Bbbk}) \arrow[d] & \\
0 \arrow[r] & H^2_{\mathrm{fl}}(X, \mathbb{Z}_p(1)) \arrow[r] \arrow[d, equal] & H^2(X, W\Omega^{\geq 1}_{X/\Bbbk}) \arrow[r, "1-\mathbf{F}"] \arrow[d] & H^2(X, W\Omega^{\geq 1}_{X/\Bbbk}) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & H^2_{\mathrm{fl}}(X, \mathbb{Z}_p(1)) \arrow[r] & H^1(X, W\Omega^1_{X/\Bbbk}) \arrow[r] \arrow[d] & H^1(X, W\Omega^1_{X/\Bbbk}) \arrow[r] \arrow[d] & 0 \\
 & & 0 & 0 &
\end{tikzcd},$$

see also (0.10.67). Then, the first Chern class map factorizes as follows:

$$c_1 \,:\, \mathrm{NS}(X) \otimes \mathbb{Z}_p \to H^2_{\mathrm{fl}}(X, \mathbb{Z}_p(1)) \to H^2(X, W\Omega^{\geq 1}_{X/\Bbbk}) \to H^2(X/W), \tag{0.10.75}$$

where all maps are injective, see [329, Remarque II.5.21.4]. The first map is (0.10.61), and the composite of the first two maps $\mathrm{NS}(X) \otimes \mathbb{Z}_p \to H^2(X, W\Omega^{\geq 1}_X)$ is a de Rham–Witt version of the dlog-map via the previous diagram. Finally, the map $H^2(X, W\Omega^{\geq 1}_X) \to H^2(X/W)$ comes from the slope spectral sequence (0.10.45).

Now, let X be a scheme that is smooth and proper over a perfect scheme S with geometrically connected fibers of dimension d. Let $(X/S)_{\mathrm{pf}}$ be the category whose objects are pairs (Y, T), where T is a perfect scheme over S and where Y is an étale scheme over $X \times T$ with the obvious morphisms. We equip it with the étale topology and let $\pi : (X/S)_{\mathrm{pf}} \to S_{\mathrm{pf}}$, $(Y, T) \mapsto T$ be the natural morphism of the categories. For an abelian sheaf $\mathcal{A}$ on $(X/S)_{\mathrm{pf}}$ that is killed by p^n the sheaves $R^i\pi_*\mathcal{A}$ are objects of the category $\mathcal{P}(p^n)$. Moreover, it follows from (0.10.69) that the étale cohomology of the sheaves $\nu_n(1)$ carries the structure of a commutative group scheme that is killed by p^n. Using (0.10.70), we equip the flat cohomology of the sheaves $\boldsymbol{\mu}_{p^n}$ with the same structure. Thus, the sheaf $R^i\pi_*\nu_n(1)$ is the perfect group scheme associated to the flat cohomology group $H^{i+1}_{\mathrm{fl}}(X, \boldsymbol{\mu}_{p^n})$ and we shall denote it by $\underline{H}^{i+1}_{\mathrm{fl}}(X, \boldsymbol{\mu}_{p^n})$. For example, if $S = \operatorname{Spec} \Bbbk$ for some algebraically closed field $\Bbbk$, then the group of

$\Bbbk$-rational points of the perfect group scheme $\underline{H}^{i+1}_{\mathrm{fl}}(X, \boldsymbol{\mu}_{p^n})$ is equal to the abelian group $H^{i+1}_{\mathrm{fl}}(X, \boldsymbol{\mu}_{p^n})$ (functorially in X). By the structure result (0.1.9) of perfect group schemes, we obtain short exact sequence of perfect group schemes for all i

$$0 \to \mathsf{U}^i(X, \boldsymbol{\mu}_{p^n}) \to \underline{H}^i_{\mathrm{fl}}(X, \boldsymbol{\mu}_{p^n}) \to \mathsf{D}^i(X, \boldsymbol{\mu}_{p^n}) \to 0,$$

whose kernel U^i is smooth, connected, and unipotent, and whose cokernel D^i is finite and étale. For example, using (0.10.54), we find the following equalities and isomorphisms in small degrees

$$\begin{aligned}&\underline{H}^0_{\mathrm{fl}}(X, \boldsymbol{\mu}_{p^n}) = 0,\\ &\underline{H}^1_{\mathrm{fl}}(X, \boldsymbol{\mu}_{p^n}) \cong \mathsf{D}^1(X, \boldsymbol{\mu}_{p^n}) \cong {}_{p^n}\operatorname{Pic}(X) \quad \text{and} \quad \mathsf{U}^1(X, \boldsymbol{\mu}_{p^n}) = 0.\end{aligned}$$

Passing to the limit over n, we find $H^0_{\mathrm{fl}}(X, \mathbb{Z}_p(1)) = 0$ and $H^1_{\mathrm{fl}}(X, \mathbb{Z}_p(1)) = T_p \operatorname{Pic}(X)$, which we already established in (0.10.59) and (0.10.60).

Passing to the formal groups in the Kummer exact sequences

$$0 \to H^{i-1}(X, \mathbb{G}_m)^{(p^n)} \to \underline{H}^i_{\mathrm{fl}}(X, \boldsymbol{\mu}_{p^n}) \to {}_{p^n}H^i(X, \mathbb{G}_m) \to 0 \qquad (0.10.76)$$

and taking tangent spaces gives a bound for the dimension of $\underline{H}^i_{\mathrm{fl}}(X, \boldsymbol{\mu}_{p^n})$

$$\dim \underline{H}^i_{\mathrm{fl}}(X, \boldsymbol{\mu}_{p^n}) \le h^{i-1}(\mathcal{O}_X) + h^i(\mathcal{O}_X), \qquad (0.10.77)$$

see [25, p. 554].

Next, we consider exact sequence (0.10.30) where ℓ is replaced with p. It follows from $\mathsf{U}^1(X, \boldsymbol{\mu}_{p^n}) = 0$ that $\mathsf{U}^2(X, \boldsymbol{\mu}_{p^n})$ embeds into $\mathsf{U}^2(X, \boldsymbol{\mu}_{p^{n+1}}$ and since the dimensions of these groups are bounded, this embedding is a bijection for $n \gg 0$. Therefore, the pro-object $\mathsf{U}^2(X, \boldsymbol{\mu}_{p^n})$ is essentially zero and we obtain after passing to the projective limit an isomorphism:

$$H^2_{\mathrm{fl}}(X, \mathbb{Z}_p(1)) \cong \varprojlim_n \mathsf{D}^2(X, \boldsymbol{\mu}_{p^n}). \qquad (0.10.78)$$

The morphism of the projective systems $\{\boldsymbol{\mu}_{p^{n-1}}\} \to \{\boldsymbol{\mu}_{p^n}\}$ defines the multiplication by p homomorphism in $H^2_{\mathrm{fl}}(X, \mathbb{Z}_p(1))$, whose cokernel is isomorphic to a subgroup of a étale finite group. Since any complete p-adic abelian group A (this is the dual to the notion of a p-divisible group) is finitely generated if A/pA is finite (dual to the assertion of Proposition 0.10.1), we obtain that $T_p H^2_{\mathrm{fl}}(X, \mathbb{Z}_p(1))$ is free of rank t_p and via the Kummer exact sequence $T_p H^2(X, \mathbb{G}_m) \cong \mathbb{Z}_p^{t_p}$ (as promised earlier).

Now, we are ready to extend Theorem 0.10.2 to the p-torsion subgroup $\operatorname{Br}(X)$. We have that $\operatorname{Br}(X) \cong H^2(X, \mathbb{G}_m)$ is a torsion group if X is a smooth and projective surface over $\Bbbk$ (see [271, II,Théoréme 2.1]), which allows us to complete the computation of the Brauer group in this case.

Theorem 0.10.29 *Let X be a smooth projective algebraic surface over an algebraically closed field $\Bbbk$ of characteristic $p > 0$. If X is not supersingular, then*

$$_{p^\infty}\operatorname{Br}(X) \;\cong\; (\mathbb{Q}_p/\mathbb{Z}_p)^{t_p} \oplus {}_{p^\infty}\operatorname{NS}(X),$$

whereas if X is supersingular, we have

$$_{p^\infty}\operatorname{Br}(X) \;\cong\; \Bbbk^{p_g(X)} \oplus (\mathbb{Q}_p/\mathbb{Z}_p)^{t_p} \oplus {}_{p^\infty}\operatorname{NS}(X).$$

Proof The proof follows the proof of Theorem 0.10.2, where we replace ℓ by p and use the exact sequences (0.10.31), (0.10.32) and (0.10.33). We also use the universal coefficient formula from [506] that gives an isomorphism

$$_{p^\infty}H^3_{\mathrm{fl}}(X, \mathbb{Z}_p(1)) \;\cong\; {}_{p^\infty}\operatorname{NS}(X). \tag{0.10.79}$$

The inductive limit $\varinjlim H^2_{\mathrm{fl}}(X, \boldsymbol{\mu}_{p^n})$ is equal to the direct sum of $\mathsf{D}^2(X, \boldsymbol{\mu}_{p^\infty}) = \varinjlim \mathsf{D}^2_{\mathrm{fl}}(X, \boldsymbol{\mu}_{p^n})$ and $\mathsf{U}^2(X, \boldsymbol{\mu}_{p^\infty}) = \varinjlim \mathsf{U}^2(X, \boldsymbol{\mu}_{p^n})$. It follows from the previous discussion that $\mathsf{U}^2(X, \boldsymbol{\mu}_{p^\infty}) \cong \mathsf{U}^2(X, \boldsymbol{\mu}_{p^n})$ for some sufficiently large n. This group has the structure of a connected unipotent quasi-algebraic group and its formal completion at the origin is isomorphic to the formal Brauer group $\Phi^2(X) = \widehat{\operatorname{Br}}(X)$, which must be isomorphic to $\widehat{\mathbb{G}}_a^r$. Its tangent space is $H^2(X, \mathcal{O}_X)$, which implies $r = p_g(X)$. By definition, it is trivial only if X is not supersingular or $p_g(X) = 0$. □

After these preparations, we now discuss the duality theorem for flat cohomology of the sheaves $\boldsymbol{\mu}_{p^n}$ on a smooth and projective surface X over a perfect field $\Bbbk$ of characteristic $p > 0$, which is due to Milne [507] for $n = 1$ and to Berthelot [61] for arbitrary n. To state it, we first consider the pairing of étale sheaves on a smooth and projective variety $\pi : X \longrightarrow \operatorname{Spec}\Bbbk$ of dimension d

$$(\mathbb{G}_m/[p^n]\mathbb{G}_m) \times (\mathbb{G}_m/[p^n]\mathbb{G}_m) \;=\; \nu_n(1) \times \nu_n(1) \;\longrightarrow\; \nu_n(2)$$

of sheaves on $X_{\text{ét}}$ that is defined by $(f, g) \mapsto \operatorname{dlog} f \wedge \operatorname{dlog} g \in W_n\Omega^2_{X/\Bbbk}$ at the level of local sections. Next, there exists a trace isomorphism $R^d\pi_* W_n\Omega^d_{X/\Bbbk} \to W_n(\Bbbk)$, as well as surjective morphisms $R^d\pi_* W_n\Omega^d_X / dW_n\Omega^{d-1}_X \to W_n(\Bbbk)$ for all $n \geq 1$, see [61, Corollary 1.7]. Moreover, for all n, there exist surjective morphisms

$$\eta_n \;:\; R^d\pi_*\nu_n(d) \;\longrightarrow\; \mathbb{Z}/p^n\mathbb{Z}, \tag{0.10.80}$$

such that the following diagrams commute for all n:

$$\begin{tikzcd}
0 \arrow[r] & R^d\pi_*\nu_n(d) \arrow[r] \arrow[d, "\eta_n"] & R^d\pi_* W_n\Omega^d_{X/\Bbbk} \arrow[r, "\mathbf{F}-1"] \arrow[d, "\cong"] & R^d\pi_*\left(\frac{W_n\Omega^d_{X/\Bbbk}}{d(W_n\Omega^{d-1}_{X/\Bbbk})}\right) \arrow[d] \\
0 \arrow[r] & \mathbb{Z}/p^n\mathbb{Z} \arrow[r] & W_n(\Bbbk) \arrow[r, "\sigma-1"] & W_n(\Bbbk),
\end{tikzcd}$$

where $\sigma : W_n(\Bbbk) \to W_n(\Bbbk)$ denotes the Frobenius map. Then, we have the following duality theorem.

Theorem 0.10.30 *Let X be a smooth and proper variety of dimension d over a perfect field $\Bbbk$ of characteristic $p > 0$. Then, the pairings $\nu_n(r) \times \nu_n(d-r) \to \nu_n(d)$ together with the maps η_n give rise to isomorphisms*

$$R\pi_* \nu_n(r) \ \to \ R\mathcal{H}om_{\mathcal{P}(p^n)}\left(R\pi_* \nu_n(d-r), \mathbb{Z}/p^n\mathbb{Z}[-d]\right) \tag{0.10.81}$$

in the derived category $\mathcal{D}^b(\mathcal{P}(p^n))$.

In view of (0.10.70), to derive the duality for flat cohomology $\underline{H}^i(X, \boldsymbol{\mu}_{p^n})$ we must take $r = 1$ in the theorem, where we use the trace isomorphism (0.10.80) and hence, may assume that $d = 2$. So, we assume now that X is a smooth and proper surface over a perfect field $\Bbbk$ of characteristic $p > 0$.

Let $\mathcal{U}_\Bbbk$ be the category of commutative unipotent algebraic groups over $\Bbbk$ and let $Q\mathcal{U}_\Bbbk$ be the abelian category of quasi-algebraic unipotent groups, see Section 0.1. There is a *duality functor*

$$G^\bullet \to (G^\bullet)^\vee \ := \ R\underline{\mathrm{Hom}}\left(G^\bullet, \mathbb{Q}_p/\mathbb{Z}_p\right)$$

on the category of complexes in $Q\mathcal{U}_\Bbbk$. For example,

$$(\mathbb{Z}/p\mathbb{Z})^\vee \ = \ \mathcal{H}om(\mathbb{Z}/p\mathbb{Z}, \mathbb{Q}/\mathbb{Z}) \ = \ \mathcal{H}om(\mathbb{Z}/p\mathbb{Z}, \mathbb{Q}_p/\mathbb{Z}_p) \ = \ \mathbb{Z}/p\mathbb{Z},$$

and

$$(\mathbb{G}_a^{\mathrm{pf}})^\vee \ = \ \mathcal{E}xt^1(\mathbb{G}_a^{\mathrm{pf}}, \mathbb{Q}/\mathbb{Z}) \ = \ \mathcal{E}xt^1(\mathbb{G}_a^{\mathrm{pf}}, \mathbb{Q}_p/\mathbb{Z}_p) \ = \ \mathbb{G}_a^{\mathrm{pf}}.$$

Here, the final $\mathbb{G}_a^{\mathrm{pf}}$ is considered as a complex shifted by one. To prove this isomorphism, we use that the Artin–Schreier exact sequence defines a quasi-isomorphism of complexes

$$(0 \to \mathbb{Z}/p\mathbb{Z}) \ \to \ (\mathbb{G}_{a,\Bbbk} \xrightarrow{\mathbf{F}-\mathrm{id}} \mathbb{G}_{a,\Bbbk}).$$

The sheaves $R^i\pi_*\nu_n(r)$ are representable by a perfect unipotent group scheme$\mathsf{G}_n^i(r)$. Let U_n^i be its connected part and let $\mathsf{D}_n^i(r) = G_n^i(r)/U_n^i$ be its étale quotient. Then, there is an isomorphism of perfect group scheme

$$\mathsf{U}_n^i(r)^\vee \ \cong \ U_n^{d+1-i}(r)^\vee, \quad \mathsf{D}_n^i(r) \ \cong \ \mathsf{D}_n^{d-i}(r)^\vee.$$

As explained in [61, Corollaire 3.8], Theorem 0.10.30 implies isomorphisms for all i:

$$\begin{aligned} \mathsf{U}^i(X, \boldsymbol{\mu}_{p^n}) &\cong \left(\mathsf{U}^{5-i}(X, \boldsymbol{\mu}_{p^n})\right)^\vee, \\ \mathsf{D}^i(X, \boldsymbol{\mu}_{p^n}) &\cong \left(\mathsf{D}^{4-i}(X, \boldsymbol{\mu}_{p^n})\right)^\vee = \mathcal{H}om(\mathsf{D}^{4-i}(X, \boldsymbol{\mu}_{p^n}), \mathbb{Q}_p/\mathbb{Z}_p). \end{aligned} \tag{0.10.82}$$

In particular, if $\Bbbk$ is algebraically closed, then the $\Bbbk$-valued points of $\mathsf{D}^i(X, \boldsymbol{\mu}_{p^n})$ and $\mathsf{D}^{4-i}(X, \boldsymbol{\mu}_{p^n})$ are finite and dual groups. Since $\mathbb{G}_a^\vee \cong \mathbb{G}_a$ and since every smooth, connected, commutative, and unipotent algebraic group has a composition series with factors isomorphic to $\mathbb{G}_a$, we also find

$$\dim \mathsf{U}^i(X, \boldsymbol{\mu}_{p^n}) \;=\; \dim \mathsf{U}^{5-i}(X, \boldsymbol{\mu}_{p^n}).$$

Using $\mathcal{H}om(\mathsf{U}^i(X, \boldsymbol{\mu}_{p^n}), \mathbb{Q}_p/\mathbb{Z}_p) = 0$, Theorem 0.10.30 implies the existence of a pairing in the category of perfect group schemes for all n

$$\underline{H}^i_{\mathrm{fl}}(X, \boldsymbol{\mu}_{p^n}) \times \underline{H}^{4-i}_{\mathrm{fl}}(X, \boldsymbol{\mu}_{p^n}) \;\to\; \mathbb{Z}/p^n\mathbb{Z}. \tag{0.10.83}$$

Note however, that in general, this pairing is not perfect: its left kernel is isomorphic to $\mathsf{U}^i(X, \boldsymbol{\mu}_{p^n})$. On the other hand, we have already seen above that $\mathsf{U}^i(X, \boldsymbol{\mu}_{p^n})$ is zero for $i \le 1$ and that the pro-object $\mathsf{U}^2(X, \boldsymbol{\mu}_{p^n})$ is essentially zero. Therefore, when passing to the limit, we obtain a pairing

$$H^i_{\mathrm{fl}}(X, \mathbb{Z}_p(1)) \times H^{4-i}_{\mathrm{fl}}(X, \mathbb{Z}_p(1)) \;\to\; \mathbb{Z}_p \tag{0.10.84}$$

that is non-degenerate modulo torsion groups. This is a p-adic analog of Poincaré duality in the classical topplogy. Note however, that we do not claim that this pairing is perfect: Remark 0.10.31 gives counter-examples. The case $i = 2$ is particularly interesting for surfaces: from the construction that the duality pairing (0.10.84), it follows that it is compatible with the pairing on $\mathrm{NS}(X) \otimes \mathbb{Z}_p$ induced by (0.10.61). Therefore, we obtain the following commutative diagram:

$$\begin{tikzcd}
0 \arrow[d] & \\
\big(\mathrm{NS}(X) \otimes \mathbb{Z}_p\big) \times \big(\mathrm{NS}(X) \otimes \mathbb{Z}_p\big) \arrow[r] \arrow[d, "c_1 \times c_1"] & \mathbb{Z}_p \arrow[d, equal] \\
H^2_{\mathrm{fl}}(X, \mathbb{Z}_p(1)) \times H^2_{\mathrm{fl}}(X, \mathbb{Z}_p(1)) \arrow[r] \arrow[d] & \mathbb{Z}_p \arrow[d, equal] \\
T_p \operatorname{Br}(X) \times T_p \operatorname{Br}(X) \arrow[r] \arrow[d] & \mathbb{Z}_p \\
0 & .
\end{tikzcd}$$

In particular, the $\mathbb{Z}_p$-module $T_p \operatorname{Br}(X)$ plays the role of the dual of the trancendental lattice.

Remark 0.10.31 Suppose that X is a smooth and proper surface over an algebraically closed field $\Bbbk$ and assume furthermore that $b_2(X) = \rho(X)$, that is, X is supersingular in the sense of Shioda. By (0.10.26), (0.10.61), and Proposition 0.10.1, there exists an isomorphism $\mathrm{NS}(X) \otimes \mathbb{Z}_\ell \cong H^2_{\text{ét}}(X, \mathbb{Z}_\ell(1))$ that is compatible with intersection pairings on both sides for all primes ℓ (including $\ell = p$). It follows from Poincaré duality in ℓ-adic cohomology that the induced pairing on $\mathrm{Num}(X) \otimes \mathbb{Z}_\ell$ is perfect for all $\ell \neq p$. From this, it follows that the discriminant of the lattice $\mathrm{Num}(X)$ is power of p. One can also prove with much more effort that the discriminant group is a p-group, see Theorem 10.1.6 in Volume II. For example, if X is a Shioda-supersingular K3 surface, that is, we have $\rho(X) = b_2(X) = 22$, then

$$\mathrm{discr}(\mathrm{Num}(X)) \;=\; -p^{2\sigma_0} \quad \text{and} \quad D(\mathrm{Num}(X)) \;\cong\; (\mathbb{Z}/p\mathbb{Z})^{2\sigma_0}$$

for an integer $\sigma_0 = \sigma_0(X)$ that satisfies $1 \leq \sigma_0 \leq 10$. This integer is called the *Artin invariant* of X and we refer to [25] or [462] for details, proofs, and further references. Since $\sigma_0 \neq 0$, the pairings on $H^2_{\mathrm{fl}}(X, \mathbb{Z}_p(1))$ and $\mathrm{Num}(X)$ are *not* unimodular.

We already mentioned above that the torsion of $H^3_{\mathrm{fl}}(X, \mathbb{Z}_p(1))$ may be not finitely generated as a $\mathbb{Z}_p$-module, see [329], Section II.5.D. More precisely, and still assuming X to be a surface, it follows from [329, Section II.(5.22.5)], that there exists a short exact sequence

$$0 \;\to\; H^3_{\mathrm{fl}}(X, \mathbb{Z}_p(1)) \;\to\; H^2(X, W\Omega^1_{X/\Bbbk}) \;\xrightarrow{\mathbf{F}-\mathrm{id}}\; H^2(X, W\Omega^1_{X/\Bbbk}) \;\to\; 0\,, \qquad (0.10.85)$$

which is a special case of [329], Theorem II.5.5. Finally, still assuming X to be a surface, it follows from Poincaré duality that we also have for all $m \geq 1$

$$H^4_{\mathrm{fl}}(X, \boldsymbol{\mu}_{p^m}) \;=\; 0 \quad \text{and thus,} \quad H^4_{\mathrm{fl}}(X, \mathbb{Z}_p(1)) \;=\; 0.$$

Putting all these results together, we obtain a fairly good grip on the flat cohomology groups of $\boldsymbol{\mu}_{p^m}$ and $\mathbb{Z}_p(1)$, at least for smooth and proper surfaces.

Remark 0.10.32 One could also study flat cohomology of the locally constant sheaves $\mathbb{Z}/p^n\mathbb{Z}$ on X. But then, the Artin–Schreier exact sequence

$$0 \;\to\; \mathbb{Z}/p^n\mathbb{Z} \;\to\; W_nO_X \;\xrightarrow{\mathbf{F}-1}\; W_nO_X \;\to\; 0$$

implies that $H^i_{\text{ét}}(X, \mathbb{Z}/p^n\mathbb{Z}) = 0$ for all $i > \dim X + 1$. In particular, there is no perfect Poincaré duality type pairing for these cohomology groups.

Bibliographical Notes

Most of the material from this chapter is more or less well-known, although sometimes, it is hard to find references. For example, the finite group schemes from Section 0.1 are discussed in [733]. We also refer to Milne's book [510] for basic facts about algebraic groups over arbitrary fields.

The general theory of cyclic covers discussed in Section 0.2 can be found, for example, in [43], at least in characteristic zero. We know of no textbook that discusses the theory of cyclic covers in positive characteristic. We have cited some papers that deal with it.

The study of inseparable morphisms of varieties via vector fields, as discussed in Section 0.3, was first exploited in the work of Rudakov and Shafarevich [626]. It relies on Jacobson's Galois correspondence for purely inseparable field extensions of height one via p-Lie algebras [343] and we note that Ekedahl [210] gave a scheme-theoretic framework. The classification of the occurring singularities is still rudimentary.

The theory of rational double points from Section 0.6 goes back to du Val [200] and Salmon [631]. The modern treatments in characteristic zero are due to Artin [20], Brieskorn [94] and Tyurina [708]. An exposition of this theory can be found in [596] and [597], see also [199] or [427]. We refer to [552] for a comprehensive exposition of the theory of normal surface singularities. In positive characteristic (especially in small characteristics), the theory is still rather rudimentary. Artin's explicit classification [28] is still one of the most important contributions to this subject.

The material on varieties of minimal degree and del Pezzo surfaces of Section 0.5 is classical and well-known. It is almost as old as algebraic geometry itself. We refer to [160] and [177] for the most complete exposition of the theory. The classification of complete intersections of two quadrics in $\mathbb{P}^4$ ($p \neq 2$) discussed in Section 0.6 is due to Segre [650]. Smooth intersections of two quadrics in $\mathbb{P}^4$ in characteristic 2 are studied in [184]. The general definition and the classification of symmetroid quartic surfaces in all characteristics seem to be new. The classification of symmetroid cubic surfaces discussed in Section 0.5 is classical and well-known in characteristic $p \neq 2$. One of them, the Cayley cubic, occurs in many situations in algebraic geometry. A modern treatments of this classification can be found in Catanese's article [111]. The extension of this classification to the case $p = 2$ seems to be new.

The theory of quadratic lattices discussed in Section 0.8 can be found in many articles and textbooks. The best source for this theory, with a view toward K3 surfaces and their automorphisms, are Nikulin's article [556] and the survey [169].

The theory of Picard functors and Picard schemes from Section 0.8 in the generality presented here is due to Grothendieck [266] and we refer to [86, Chapter 8] and [392] for surveys.

A very good survey of different cohomology theories of algebraic varieties can be found in Danilov's article [145]. For complex projective varieties, we followed the textbook of Griffiths and Harris [259] and for the discussion of ℓ-adic cohomology, we followed Milne's book [508]. Various duality theorems are discussed in [508] as well as [237]. The computation of the l-torsion part of the Brauer group of a surface is taken from [271, III]. The computation of the p-torsion seems new.

For surveys on crystalline cohomology (which is only briefly mentioned on [145]) with an emphasis on geometry, we refer to [116] or [462]. Flat cohomology is discussed in [509] and further references are given in the text of this section. In our exposition of the flat duality for curves and surfaces we followed closely [30], [507] and [61].

Chapter 1
Enriques surfaces: generalities

In this chapter, we introduce Enriques surfaces. We briefly recall the Kodaira–Enriques classification of algebraic surfaces and place Enriques surfaces in this classification. We determine the basic invariants of Enriques surfaces, and introduce the K3-cover, which is more involved in characteristic $p = 2$. We finally give explicit examples and equations of Enriques surfaces.

1.1 Classification of Algebraic Surfaces

Let $\Bbbk$ be an algebraically closed field of arbitrary characteristic $p \geq 0$. In this section, we recall the fundamental results of the classification of smooth projective surfaces over $\Bbbk$. For characteristic zero, we refer to the textbooks [43], [47], [259], [612] and the references given there. For positive characteristic, we refer to the original articles [539], [78], [77], to the textbooks [38] and [424], as well as to the survey [458].

For any invertible sheaf $\mathcal{L}$ or a divisor D, we denote by $|\mathcal{L}|$ or $|D|$ the complete linear system of effective divisors C with $\mathcal{O}_X(C) \cong \mathcal{L}$ or linearly equivalent to D. It is clear that $|D|$ depends only on the linear equivalence class of D.

First, we define the *Kodaira–Iitaka dimension* $\kappa(X, \mathcal{L})$ of an invertible sheaf $\mathcal{L}$ on a normal and projective variety X to be $-\infty$ if $h^0(X, \mathcal{L}^{\otimes m}) = 0$ for all $m \geq 1$. Otherwise, we define it by the following equivalent properties:

1. The function $m \mapsto h^0(X, \mathcal{L}^{\otimes m})$ grows like $m^{\kappa(X,\mathcal{L})}$ as m tends to infinity.
2. The maximal dimension of the image of the rational map defined by $|\mathcal{L}^{\otimes m}|$ with $H^0(X, \mathcal{L}^{\otimes m}) \neq \{0\}$ is equal to $\kappa(X, \mathcal{L})$.
3. The *section ring* of $\mathcal{L}$, that is, the graded $\Bbbk$-algebra

$$R(X, \mathcal{L}) := \bigoplus_{m \geq 0} H^0(X, \mathcal{L}^{\otimes m})$$

is an integral domain and its field of homogeneous fractions is of transcendence degree $\kappa(X, \mathcal{L})$ over $\Bbbk$.

In particular, it follows from the second characterization that $\kappa(X, \mathcal{L})$ is either equal to $-\infty$ or it is an integer lying in between 0 and $\dim(X)$. By definition, an invertible sheaf $\mathcal{L}$ is called *big* if $\kappa(X, \mathcal{L}) = \dim(X)$. For example, if $\mathcal{L}$ is ample, then it is big, but the converse need not be true if $\dim(X) \geq 2$. We refer to [447, Definition 2.1.3] and [447, Corollary 2.1.38] for details and proofs in arbitrary dimensions, as well as to [38, Section 14] for these results if X is of dimension at most two. Let us also note that even if X is a surface, then the section ring $R(X, \mathcal{L})$ need not be a finitely generated $\Bbbk$-algebra and that Zariski [741] (see also [38, Theorem 14.19]) settled when finite generation holds and when it fails.

If X is a smooth and projective variety with canonical invertible sheaf ω_X, then $\kappa(X) := \kappa(X, \omega_X)$ is called the *Kodaira dimension* or *canonical dimension* of X. This is the main invariant in higher-dimensional algebraic geometry. The Kodaira dimension is a birational invariant of smooth and proper varieties. Moreover, the canonical section ring $R(X) := R(X, \omega_X)$ is called the *canonical algebra*. It is expected that $R(X)$ is always a finitely generated $\Bbbk$-algebra, but at the moment this is only known to be true if $\dim(X) \leq 2$ or if $\mathrm{char}(\Bbbk) = 0$ – we refer the interested reader to [38, Section 14.31] for a proof in dimension two and to [380] for a survey of the higher-dimensional case.

Example 1.1.1 If X is a smooth and proper curve over $\Bbbk$, then its genus $g = g(X)$ controls the Kodaira dimension as follows:

g	0	1	≥ 2
κ	$-\infty$	0	1

Next, we turn to dimension two: first, a note on the category we are working in, which is classical in dimension one and due to Zariski and Goodman in dimension two. The extension to algebraic spaces can be found, for example, in [399, Theorem V.4.9 and Section V. 4.10].

Theorem 1.1.2 *Let X be an algebraic space that is smooth, proper, and of dimension at most two over an algebraically closed field $\Bbbk$. Then, X is a scheme that is projective over $\Bbbk$.*

In particular, when talking about curves and surfaces that are smooth and proper over $\Bbbk$, there is no difference between working with algebraic spaces (as one has to do when constructing moduli spaces) or with projective varieties (as one usually does in classical algebraic geometry).

Let X be a smooth and proper surface over $\Bbbk$. Then, X is called *minimal* if every birational morphism $f : X \to X'$ onto a smooth and proper surface X' is an isomorphism. Equivalently, this means that X does not contain (-1)-curves. Moreover, if $\kappa(X) \geq 0$, then X is minimal if and only if K_X is a *nef divisor class*, that is, $K_X \cdot C \geq 0$ for every effective curve C. (We will come back to nef invertible sheaves in Section 2.1.) Let us denote by $\equiv$ (resp. $\sim$) the numerical (resp. linear) equivalence of divisors. Then, we have the following fundamental result about minimal models and the Kodaira dimension of surfaces.

Theorem 1.1.3 *Let X be a smooth and proper surface over an algebraically closed field $\Bbbk$. Then, there exists a birational morphism $f : X \to X'$ onto a minimal surface X' that satisfies precisely one of the following properties:*

1. $\kappa(X') = 2$, $K_{X'}^2 > 0$,
2. $\kappa(X') = 1$, $K_{X'}^2 = 0$, $K_{X'} \not\equiv 0$,
3. $\kappa(X') = 0$, $K_{X'}^2 = 0$, $K_{X'} \equiv 0$,
4. $\kappa(X') = -\infty$, *$X' \cong \mathbb{P}^2$ or X' is a* minimal ruled surface*, that is, there exists a smooth morphism $f : X' \to C$ onto a smooth projective curve C such that all geometric fibers are isomorphic to $\mathbb{P}^1$.*

Remark 1.1.4 To complete the picture, let us mention the following additional results.

1. If $\kappa(X) \geq 0$, then the surface X' from Theorem 1.1.3 is unique, that is, X has a unique minimal model, called *the minimal model* of X, which is unique up to isomorphism.
2. If $\kappa(X) = -\infty$, then minimal models are not unique. For example, $\mathbb{P}^2$ and $\mathbb{P}^1 \times \mathbb{P}^1$ are minimal and birationally equivalent surfaces that are not isomorphic. By a theorem of Tsen (see [47, Theorem III.4] or [38, Theorem 11.3]), a minimal ruled surface $f : X \to C$, that is, a smooth a smooth fibration f, where C is a curve and where all geometric fibers are isomorphic to $\mathbb{P}^1$ has a section, which implies that $X \to C$ is a $\mathbb{P}^1$-bundle, and thus, isomorphic to $\mathbb{P}(\mathcal{E}) \to C$, where $\mathcal{E}$ is a locally free sheaf of rank 2 on C. In particular, X is birationally equivalent to $C \times \mathbb{P}^1$.
3. A surface X with $\kappa(X) = -\infty$ and $q = 0$ is birationally isomorphic to $\mathbb{P}^2$, and hence, it is a *rational surface*. *Castelnuovo's Rationality Criterion* characterizes rational surfaces as those surfaces whose numerical invariants q, p_g, and $P_2 := \dim H^0(X, \mathcal{O}_X(2K_X))$ vanish (see [740] or [38, Chapter 13] for a characteristic free proof).
4. If $\kappa(X) = 2$, then X is called a *surface of general type*. By fundamental results of Bombieri [76] and Ekedahl [211], we have that if $m \geq 5$, then $|mK_X|$ defines a morphism to projective space, and X is birational onto its image.
5. In Chapter 4, we will study surfaces admitting genus one fibrations, that is, fibrations, whose generic fiber is an integral curve of arithmetic genus one. In the case where the generic fiber is smooth, such a fibration is called *elliptic*, and *quasi-elliptic* otherwise. The latter type exists in characteristic $p = 2, 3$ only (see Theorem 4.1.3). A (quasi-)elliptic surface X satisfies $\kappa(X) \leq 1$. If $\kappa(X) = 1$, then X carries a unique genus one fibration, and in fact, this fibration arises from the morphism associated to $|mK_X|$ for $m \geq 14$ [375] ($m \geq 5$ if $p = 3$ [365]). We will see later that all Enriques surfaces admit genus one fibrations, but that these fibrations are not unique.

Since Enriques surfaces, the objective of this book, have Kodaira dimension zero, let us have a closer look at this class of surfaces. First, let us recall the fundamental numerical invariants of surfaces: we denote by $b_i(X)$ the *Betti numbers* computed with respect to the classical or étale topology as explained in Section 0.10. We denote by $e(X) = \sum_i (-1)^i b_i(X)$ the *Euler–Poincaré characteristic*. We also have $\chi(\mathcal{O}_X) = \sum_i (-1)^i h^i(\mathcal{O}_X)$ and the *Hodge numbers* $h^{i,j}(X) = h^j(\Omega_X^i)$. Let us recall from Section 0.10 that b_1(X) is twice the dimension of the Picard scheme $\mathbf{Pic}^{\circ}_{X/\Bbbk}$ and that the Zariski tangent space of $\mathbf{Pic}^{\circ}_{X/\Bbbk}$ at $\mathcal{O}_X$ is isomorphic to $H^1(\mathcal{O}_X)$. In Proposition 0.9.10, we defined

$$\Delta(X) = 2h^1(\mathcal{O}_X) - b_1(X)$$

and showed that this is a non-negative and even integer, which is zero if and only if $\mathbf{Pic}^{0}_{X/\Bbbk}$ is reduced, that is, an abelian variety. Moreover, if $p = \mathrm{char}(\Bbbk) = 0$, then $\Delta(X) = 0$, whereas if $p > 0$, then we have the bound $0 \le \Delta(X) \le 2h^2(\mathcal{O}_X)$. Finally, we denote by $p_g(X) = h^0(K_X)$ the *geometric genus* of X, and if X is a surface, then Serre duality gives $h^{0,2}(X) = h^2(\mathcal{O}_X) = h^0(K_X) = p_g(X)$. The next result determines the possible values of these invariants for minimal surfaces of Kodaira dimension zero, that is, the third case of Theorem 1.1.3.

Proposition 1.1.5 *Let X be a smooth and proper surface over an algebraically closed field $\Bbbk$ that is a minimal surface of Kodaira dimension zero. Then, $K_X^2 = 0$, and we have the following possible invariants:*

b_2	b_1	e	χ	$h^{0,1}$	p_g	Δ
22	0	24	2	0	1	0
14	2	12	1	1	1	0
10	0	12	1	0	0	0
				1	1	2
6	4	0	0	2	1	0
2	2	0	0	1	0	0
				2	1	2

$\underbrace{b_2,\ b_1,\ e,\ \chi}_{\text{invariants under deformation}}$ $\quad$ $\underbrace{h^{0,1},\ p_g,\ \Delta}_{\text{invariants, which are in general only upper-semicontinuous under deformation}}$

Table 1.1 Possible invariants for minimal surfaces of Kodaira dimension zero

We will see in Proposition 1.1.9 that the case $b_2(X) = 14$ does not occur.

Proof From the third case of Theorem 1.1.3, we obtain $K_X^2 = 0$, as well as $p_g(X) \le 1$. In particular, $\Delta(X) = 0$ or $\Delta(X) = 2$ and by Proposition 0.9.10, the latter is possible only if $p_g(X) = 1$. By Serre duality, we have $h^2(\mathcal{O}_X) = p_g(X)$ and deduce $\chi(\mathcal{O}_X) \le 2$ from this. Next, Noether's formula (0.10.22) becomes $12\chi(\mathcal{O}_X) = e(X) = 2 - 2b_1(X) + b_2(X)$, which we expand and arrange as follows:

$$10 + 12p_g(X) = 8h^1(\mathcal{O}_X) + 2\Delta(X) + b_2(X).$$

Then, each term and both sides of this formula are non-negative. Using $p_g(X) \le 1$ and the other just-established inequalities, we obtain the stated list. $\square$

Now, we define the main object of this book.

Definition 1.1.6 An *Enriques surface* is a smooth and proper surface over an algebraically closed field $\Bbbk$ that is of Kodaira dimension zero and satisfies $b_2(X) = 10$.

Note that it follows from Table **??** that an Enriques surface additionally satisfies the following properties:

$$b_1(X) = 0, \quad \chi(\mathcal{O}_X) = 1. \tag{1.1.1}$$

From now on, we denote Enriques surfaces by S to distinguish them from other surfaces, which we will usually denote by X. By Table **??**, we have $h^{0,1}(S) = p_g(S) \leq 1$ and this inequality is an equality if and only if $\Delta(S) = 2$, which can only happen in positive characteristic. In particular, if $h^{0,1}(S) \neq 0$, then the absolute Frobenius morphism of S induces a semi-linear map of one-dimensional $\Bbbk$-vector spaces

$$\mathbf{F} \,:\, H^1(S, \mathcal{O}_S) \,\to\, H^1(S, \mathcal{O}_S). \tag{1.1.2}$$

Here, *semi-linear* means that we have $\mathbf{F}(\lambda \cdot s) = \lambda^p \cdot \mathbf{F}(s)$ for all $\lambda \in \Bbbk$ and all $s \in H^1(S, \mathcal{O}_S)$. Since $\Bbbk$ is algebraically closed, whence perfect, this map is either zero or bijective. Following the traditional terminology from [77], we make the following definition.

Definition 1.1.7 An Enriques surface is called:

1. *classical* if $h^{0,1}(S) = p_g(S) = 0$,
2. *ordinary* (or, *singular*) if $h^{0,1}(S) = p_g(S) = 1$ and $\mathbf{F}$ is bijective on $H^1(\mathcal{O}_S)$, and
3. *supersingular* if $h^{0,1}(S) = p_g(S) = 1$ and $\mathbf{F}$ is zero on $H^1(\mathcal{O}_S)$.

For reasons that will become clear in Theorem 1.2.1 below, ordinary Enriques surface are also called $\boldsymbol{\mu}_2$*-surfaces* and supersingular Enriques surfaces are also called $\boldsymbol{\alpha}_2$*-surfaces* – see also Definition 1.2.6 As already mentioned, non-classical Enriques surfaces can only exist in positive characteristic. The following result shows that even there, they can only exist in characteristic 2.

Theorem 1.1.8 *An Enriques surface S in characteristic $p \neq 2$ is classical.*

Proof If $p = 0$, then $\Delta(S) = 0$, and thus, S is classical.

Suppose that S is not classical. Then, $p > 0$ and from $h^2(\mathcal{O}_S) = h^0(K_S) = 1$, we deduce $K_S = 0$. Since $h^1(\mathcal{O}_S) = 1$, any $0 \neq e \in H^1(\mathcal{O}_S)$ is a basis. Then, the absolute Frobenius $\mathbf{F}$ induces a semi-linear action on $H^1(S, \mathcal{O}_S)$, say with $e \mapsto \lambda e$ for some $\lambda \in \Bbbk$. Let $\boldsymbol{\alpha}_\lambda$ be the kernel of the map $(\mathbf{F} - \lambda) : \mathcal{O}_S \to \mathcal{O}_S$, where we consider $\mathcal{O}_S$ as the group scheme $\mathbb{G}_a$ over S in the flat topology. Then, $G \cong \mathbb{Z}/p\mathbb{Z}$ if $\lambda \neq 0$ and $G \cong \boldsymbol{\alpha}_p$ if $\lambda = 0$, see also Example 0.1.7. Taking the flat cohomology in the exact sequence

$$0 \,\to\, \boldsymbol{\alpha}_\lambda \,\to\, \mathbb{G}_{a,S} \xrightarrow{\mathbf{F}-\lambda} \mathbb{G}_{a,S} \,\to\, 0,$$

we see that $H^1(S, \boldsymbol{\alpha}_\lambda)$ is non-zero, and thus, there exists a non-trivial $\boldsymbol{\alpha}_\lambda$-torsor $\pi : X \to S$. Then, π is a finite and flat morphism of degree p, which is a principal Artin–Schreier cover if $\lambda \neq 0$ and purely inseparable if $\lambda = 0$. It follows from Proposition

0.2.20 that X is an integral Gorenstein surface with $\omega_X \cong \pi^*(\omega_S) \cong \pi^*(\mathcal{O}_S) \cong \mathcal{O}_X$ and hence $\chi(\mathcal{O}_X) \leq 2$. On the other hand, $\pi_* \mathcal{O}_Y$ has a composition series of length p, all of whose simple factors are isomorphic to $\mathcal{O}_S$, see (0.2.17). Since π is a finite morphism, we obtain

$$2 \ \geq \ \chi(\mathcal{O}_X) \ = \ \chi(\pi_* \mathcal{O}_X) \ = \ p \cdot \chi(\mathcal{O}_S) \ = \ p,$$

which proves the theorem. □

For the sake of completeness, we end this section by briefly discussing the remaining surfaces from Table 1.1. We start with the following two cases.

Proposition 1.1.9 *Let X be a smooth, proper, and minimal surface of Kodaira dimension zero over an algebraically closed field $\Bbbk$.*

1. *Surfaces with $b_2(X) = 14$ do not exist.*
2. *If $b_2(X) = 22$, then:*

   a. *$K_X = 0$,*
   b. *there exists an integer $1 \leq \rho \leq 22$ and $\rho \neq 21$ with*

   $$\boldsymbol{Pic}_{X/\Bbbk} \ \cong \ \mathrm{NS}(X)_{\Bbbk} \ \cong \ \mathrm{Num}(X)_{\Bbbk} \ \cong \ (\mathbb{Z}^{\rho})_{\Bbbk},$$

   c. *X is* algebraically simply connected*, that is, X does not admit any non-trivial finite étale covers.*

Proof By Table 1.1, a hypothetical surface with $b_2 = 14$ satisfies $b_1 = 2$ and $\chi(\mathcal{O}_X) = 1$, and thus, $(\mathbf{Pic}^0_{X/\Bbbk})_{\mathrm{red}}$ is of dimension 1. Thus, there exists a numerically trivial invertible sheaf $\mathcal{L} \not\cong \mathcal{O}_X$. Since $p_g > 0$ and $K_X \equiv 0$, we have $K_X = 0$. Thus, using Riemann–Roch and Serre duality, we find

$$h^0(\mathcal{L}) \ + \ h^0(\mathcal{L}^{-1}) \ = \ h^0(\mathcal{L}) \ + \ h^0(\omega_X \otimes \mathcal{L}^{-1}) \ \geq \ \chi(\mathcal{O}_X) \ = \ 1,$$

which implies that $h^0(\mathcal{L}) \geq 1$ or $h^0(\mathcal{L}^{-1}) \geq 1$. But since $\mathcal{L}$ and $\mathcal{L}^{-1}$ are both non-trivial and numerically trivial, neither of them can be represented by an effective divisor. This contradiction excludes the possibility of surfaces with $b_2 = 14$.

Let us now assume that $b_2 = 22$. As before, $p_g > 0$ and $K_X \equiv 0$ imply $K_X = 0$. Next, by Table 1.1, we have $H^1(\mathcal{O}_X) = 0$, which implies that the tangent space of $\mathbf{Pic}_{X/\Bbbk}$ at the origin is trivial, and thus, $\mathbf{Pic}^0_{X/\Bbbk} = 0$. Let $\mathcal{L} \in \mathbf{Pic}^{\tau}_{X/\Bbbk}(\Bbbk)$. As in the previous discussion, $K_X = 0$, Riemann–Roch, and the Serre duality imply $h^0(\mathcal{L}) + h^0(\mathcal{L}^{\vee}) \geq 2$. This shows that $\mathcal{L}$ or $\mathcal{L}^{\vee}$ has a non-trivial section and thus, $\mathcal{L} \cong \mathcal{O}_X$. Thus, $\mathrm{NS}(X)$ has no non-trivial torsion. Since S is a projective surface, we have $\rho \geq 1$, and using the Igusa–Artin-Mazur inequality (0.10.53), we deduce $\rho \leq b_2 = 22$ and $\rho \neq 21$. It remains to prove the assertion on the algebraic fundamental group: let $f : Y \to X$ be a finite, connected, and étale cover of degree $n \geq 2$. Then $f^*(\omega_X) \cong \omega_Y$ and thus, $K_Y = 0$. Since $\chi(Y, \mathcal{O}_Y) = 2 - q(Y) \leq 2 = \chi(X, \mathcal{O}_X)$ and $e(Y) > e(X)$, we get a contradiction with Noether's formula. □

In Section 1.3, we will see that surfaces with $b_2 = 22$ as in the previous proposition are closely linked to Enriques surfaces. Let us give the classical definition of this type of surfaces.

Definition 1.1.10 A *K3 surface* is a smooth and proper surface X over a field satisfying $K_X = 0$ and $h^1(\mathcal{O}_X) = 0$.

From the point of view of Enriques surfaces, the following examples of K3 surfaces will play an important role later on. Moreover, we refer to Example 1.6.10 for another construction of K3 surfaces that uses abelian surfaces.

Lemma 1.1.11 *Let X be an integral surface over an algebraically closed field $\Bbbk$ that is:*

1. *a hypersurface of degree* 4 *in* $\mathbb{P}^3$*, or*
2. *a complete intersection of a quadric and a cubic in* $\mathbb{P}^4$*, or*
3. *a complete intersection of three quadrics in* $\mathbb{P}^5$.

Then, X is an integral Gorenstein surface with $\omega_X \cong \mathcal{O}_X$ and $H^1(X, \mathcal{O}_X) = 0$. If X is smooth, then it is a K3 surface.

Proof In all cases, we have $\omega_X \cong \mathcal{O}_X$ by the adjunction formula. Quite generally, a complete intersection Z of dimension d in some projective space satisfies $H^i(Z, \mathcal{O}_Z(k)) = 0$ for all $1 \leq i \leq d-1$ and all $k \in \mathbb{Z}$, see, for example, [294, Exercise III.5.5]. Finally, if X is smooth, then $K_X = 0$ and $h^{0,1} = 0$, which identifies X as a K3 surface. □

It remains to deal with the remaining cases of Table 1.1. In both cases, we have $b_1 \neq 0$, and thus, the Albanese morphism $\mathrm{alb}_X : X \to \mathrm{Alb}(X)$ is non-trivial. This is the key to their classification.

Proposition 1.1.12 *Let X be a smooth, proper, and minimal surface of Kodaira dimension zero over an algebraically closed field $\Bbbk$.*

1. *If $b_1(X) = 2$, then the Albanese morphism gives rise to a fibration $X \to E$ onto an elliptic curve E, all of whose fibers are integral curves of arithmetic genus one.*
2. *If $b_1(X) = 4$, then the Albanese morphism of X is an isomorphism. In particular, X is an abelian surface, that is, an abelian variety of dimension two.*

Proof If $b_1 = 2$, then the Stein factorization of the Albanese morphism is a fibration $X \to E$, where E is an elliptic curve. If F is a fiber, then $F^2 = 0$ (being a fiber) and $K_X \cdot F = 0$ (since $K_X \equiv 0$), imply that F is of arithmetic genus one. Moreover, by the Shioda-Tate formula (4.3.2), the Picard number of X is equal to $\rho = 2 + \sum_x (m_x - 1)$, where m_x denotes the number of irreducible components of the fiber over $x \in E$. Since $\rho \leq b_2 = 2$, we find $\rho = 2$, and thus, every fiber of $X \to E$ is irreducible.

If $b_1 = 4$, then the image of alb_X is an abelian variety of dimension 2. We refer to [78, Section 5] or [38, Theorem 10.19] for a proof that alb_X is an isomorphism in this case. □

Using the previous result, one can actually classify surfaces with $b_1 = 2$ in characteristic $p \geq 0$ very explicitly. Let us sketch the idea and refer to [78] and [77] for details: if $p \neq 2, 3$, then the geometric fibers of $X \to E$ are all smooth genus one curves, that is, elliptic curves. Therefore, the fibration is isotrivial, and all geometric fibers are isomorphic to one elliptic curve C. Then, there exists a finite and étale Galois-cover, such that this isotrivial fibration becomes a trivial product family. This eventually exhibits X as the quotient of the product of two elliptic curves by a fixed-point-free action of some finite group K. In characteristic $p = 2, 3$ there is the additional possibility that all fibers $X \to E$ are isomorphic to one curve C, that is a rational curve with an ordinary cusp. In this case, the smooth locus of C carries a group scheme structure that is isomorphic to $\mathbb{G}_a$. This is an example of a quasi-elliptic fibration, see Section 4.1. In all cases, there exists an elliptic curve E' over $\Bbbk$, a finite subgroup scheme $K \subset \mathrm{Aut}(E')$, an injective homomorphism $\alpha : K \to \mathrm{Aut}(C)$, and an isomorphism

$$X \cong (E' \times C)/K,$$

where $k \in K$ acts via $(e, c) \mapsto (e + k, \alpha(k)(c))$ on $E' \times C$. Then, the Albanese morphism coincides with the projection onto the first factor and $E \cong E'/K$. Moreover, the projection onto the second factor induces an elliptic fibration $X \to \mathbb{P}^1$. Since these surfaces come with two genus one fibrations, one makes the following definition.

Definition 1.1.13 Let X be a smooth, proper, and minimal surface of Kodaira dimension zero with $b_1 = 2$. Let $X \to E$ be the Albanese fibration. In the case where the generic fiber is smooth, X is called a *bielliptic surface* or *hyperelliptic surface*. Otherwise, X is called a *quasi-bielliptic surface* or a *quasi-hyperelliptic surface*.

One can explicitly classify all possible K, $K \subset E'$ and $\alpha : K \to \mathrm{Aut}(C)$: in characteristic zero, this leads to the classical list of Bagnera and de Franchis, see [47, List VI.20]. In positive characteristic, a similar list has been worked out by Bombieri and Mumford for bielliptic surfaces in [78] and for quasi-bielliptic surfaces in [77]. As already mentioned, quasi-bielliptic surfaces exist in characteristic $p = 2, 3$ only.

Let us mention the following application of these classification lists: we have $K_X = 0$ if X is an abelian surface or a K3 surface and we will establish $2K_X = 0$ for Enriques surfaces in Corollary 1.2.3 below. When combining these results with the lists of Bagnera-de Franchis and Bombieri–Mumford just mentioned, we obtain the following.

Theorem 1.1.14 *A smooth, proper, and minimal surface X of Kodaira dimension zero satisfies* $12K_X = 0$.

1.2 The Picard Scheme and the Brauer Group

In this section, we will compute the Picard number, the connected component of the Picard scheme and its torsion, and the Brauer group of an Enriques surface. This leads to another characterization of the three types of Enriques surfaces introduced in Definition 1.1.7 via their Picard schemes. The most difficult result of this section is that all Enriques surfaces have Picard number $\rho = b_2 = 10$. Whereas this result is easy to prove in characteristic zero, it requires more effort in positive characteristic.

We start with the connected component of the Picard scheme and its torsion, which we introduced and discussed in Section 0.9.

Theorem 1.2.1 *Let S be an Enriques surface over an algebraically closed field $\Bbbk$ of characteristic $p \geq 0$. Then,*

$$(\boldsymbol{Pic}^0_{S/\Bbbk})_{\mathrm{red}} = 0,$$

and $\boldsymbol{Pic}^\tau_{S/\Bbbk}$ is a finite and flat group scheme of length 2 over $\Bbbk$. More precisely,

$$\boldsymbol{Pic}^\tau_{S/\Bbbk} \cong \begin{cases} (\mathbb{Z}/2\mathbb{Z})_\Bbbk & \textit{if } S \textit{ is classical,} \\ \boldsymbol{\mu}_{2,\Bbbk} & \textit{if } S \textit{ is ordinary,} \\ \boldsymbol{\alpha}_{2,\Bbbk} & \textit{if } S \textit{ is supersingular.} \end{cases}$$

In particular, if $p \neq 2$, then S is classical and $\boldsymbol{Pic}^\tau_{S/\Bbbk}$ is étale.

Remark 1.2.2 If S is a supersingular or a classical Enriques surface in characteristic 2, then the group scheme $\mathbf{Pic}^\tau_{S/\Bbbk}$ is unipotent. For this reason, these two classes of surfaces are also called *unipotent Enriques surfaces*. We will see in Section 1.2 and Section 1.3 that they are the classes of Enriques surfaces that are the most difficult to handle. In Theorem 1.4.13, we will see that the class of unipotent Enriques surfaces coincides with the class of algebraically simply connected Enriques surfaces.

Proof By Table 1.1, we have $b_1(S) = 0$. Therefore, $(\mathbf{Pic}^0_S)_{\mathrm{red}}$ is an abelian variety of dimension zero and thus, trivial.

First, assume that S is classical. Then $H^1(\mathcal{O}_S) = 0$, which implies that $\mathbf{Pic}_S$ is a discrete and reduced group scheme. In particular, we have $\mathbf{Pic}^\tau_S = (\mathrm{Tors}(\mathrm{Pic}(S)))_\Bbbk$ in this case. Let $\mathcal{L} \in \mathrm{Tors}(\mathrm{Pic}(S))$. By Riemann–Roch and Serre duality, we find $h^0(\mathcal{L}) + h^2(\mathcal{L}) \geq 1$, which implies that $\mathcal{L} \cong \mathcal{O}_S$ or $\mathcal{L} \cong \omega_S$. Since S is classical, we have $h^0(\mathcal{O}_S) = 1$ and $h^0(\omega_S) = 0$ and thus, $\mathcal{O}_S \not\cong \omega_S$. Hence, $\mathbf{Pic}^\tau_S \cong (\mathbb{Z}/2\mathbb{Z})_\Bbbk$ and this group is generated by ω_S.

Now, assume that S is non-classical. Then, we have $p = 2$ by Theorem 1.1.8. Arguing as before, we find $(\mathbf{Pic}^\tau_S)_{\mathrm{red}} = 0$ and thus, $G := \mathbf{Pic}^\tau_S$ consists of one point with a one-dimensional Zariski tangent space. As a scheme, a finite flat group scheme G over $\Bbbk$ with $G(\Bbbk) = \{1\}$ is the spectrum of a ring of the form $\Bbbk[t_1, ..., t_r]/(t_1^{p^{n_i}}, ..., t_r^{p^{n_r}})$, see [160, Exposé VIIB, 5.4] and Remark 0.1.13. In our case, the Zariski-tangent space of G is one-dimensional and thus, $G \cong \operatorname{Spec} \Bbbk[t]/(t^{p^n})$ for some $n \geq 1$. Seeking a contradiction, assume that $n \geq 2$. Then, the morphism $\operatorname{Spec} \Bbbk[t]/(t^2) \to \mathbf{Pic}_S$ defined by a nonzero tangent vector can be extended to a morphism $\mathrm{Spec}\,\Bbbk[t]/(t^3) \to G \hookrightarrow \mathbf{Pic}_S$. By [538, Lecture 27], this implies that the first Bockstein operation $\beta_1 : H^1(O_S) \to H^2(O_S)$ is not bijective (see also Remark 0.9.9). However, this is impossible: indeed, in characteristic 2, the map β_1 is equal to the composition of the map

$$H^1(S, O_S) \to H^1(S, O_S) \otimes H^1(S, O_S), \quad x \mapsto x \otimes x$$

followed by the cup-product

$$H^1(S, O_S) \otimes H^1(S, O_S) \to H^2(S, O_S).$$

Since S is non-classical, we have $K_S = 0$ and then, the cup-product map coincides with the map

$$H^1(S, O_S) \otimes H^1(S, \Omega_S^2) \to H^2(S, \Omega_S^2),$$

which is an isomorphism by Serre duality. In particular, β_1 is bijective.

Thus, still assuming that S is non-classical, we know that $\mathbf{Pic}_S^\tau$ is a non-reduced group scheme of length $p = 2$, and thus, isomorphic as a scheme to $\mathrm{Spec}\,\Bbbk[t]/(t^2)$. Thus, by Theorem 0.1.10, this group scheme is isomorphic to $\boldsymbol{\mu}_2$ or to $\boldsymbol{\alpha}_2$. By Example 0.1.16, the Frobenius map on Zariski tangent spaces is bijective in the first case and it is zero in the second case. □

Corollary 1.2.3 *Let S be an Enriques surface. Then, we have*

$$2K_S = 0.$$

Moreover, we have $K_S = 0$ if and only if S is non-classical.

Proof By Theorem 1.1.3, we have $K_S \equiv 0$, where $\equiv$ denotes numerical equivalence. If S is non-classical, then $h^0(K_S) = 1$ and we conclude $K_S = 0$. If S is classical, then $h^0(K_S) = 0$, which implies $K_S \neq 0$. But since K_S is an element of $\mathbf{Pic}_S^\tau(\Bbbk)$, which is of length 2, we find $2K_S = 0$. □

Remark 1.2.4 Another way of proving $2K_S = 0$ goes as follows: Enriques surfaces satisfy $b_1(S) = 0$ and, being of Kodaira dimension zero, they are not rational. Thus, we must have $h^0(2K_S) \neq 0$ for otherwise we would obtain a contradiction to Castelnuovo's Rationality Criterion. Since $K_S \equiv 0$, we conclude $2K_S = 0$.

Corollary 1.2.5 *Let S be an Enriques surface over an algebraically closed field $\Bbbk$ of characteristic $p > 0$. Then, $H^1(WO_S) = 0$,*

$$H^2(WO_S) \cong \begin{cases} 0 & \textit{if } S \textit{ is classical,} \\ \Bbbk \cdot x \textit{ with } \mathbf{F}x = x^p \textit{ and } Vx = 0 & \textit{if } S \textit{ is ordinary,} \\ \Bbbk \cdot x \textit{ with } \mathbf{F}x = Vx = 0 & \textit{if } S \textit{ is supersingular,} \end{cases}$$

and the $\mathbf{F}$-isocrystal $H^2(S/W) \otimes_W K$ is of slope one.

Proof Since $\mathbf{Pic}_S^\circ$ is zero-dimensional and $H^1(WO_S)$ is always without p-torsion, we conclude that $H^1(WO_S) = 0$, see also Section 0.10.

Next, it follows from Proposition 0.10.21 that $H^2(WO_X) \cong \mathbb{D}(\mathbf{Pic}^\circ_S/\mathbf{Pic}^\circ_{S,\mathrm{red}})$ and that the slope of $H^2(S/W) \otimes_W K$ is equal to one. From this, the assertion follows, see also the computation of Dieudonné modules in Section 0.3. □

In view of Theorem 1.2.1, we will also use the following terminology.

Definition 1.2.6 An ordinary (resp. supersingular) Enriques surface is also called a $\boldsymbol{\mu}_2$*-surface* (resp. an $\boldsymbol{\alpha}_2$*-surface*).

Having determined the torsion and infinitesimal structure $\mathbf{Pic}^\tau_{S/\Bbbk}$ of an Enriques surface, let us now determine the Picard number $\rho(S)$, which is the main result of this section.

Theorem 1.2.7 *Let S be an Enriques surface over an algebraically closed field* $\Bbbk$*. Then, its Picard number satisfies*

$$\rho(S) = b_2(S) = 10.$$

The proof will require some work. In characteristic zero, it is actually not so difficult.

Lemma 1.2.8 *Theorem 1.2.7 is true if* $\Bbbk$ *is of characteristic zero.*

Proof By the Lefschetz principle, we may assume $\Bbbk = \mathbb{C}$. Then, taking cohomology in the exponential sequence (0.10.9)

$$0 \to \mathbb{Z} \to O_X \xrightarrow{\exp} O_X^\times \to 0, \tag{1.2.1}$$

and using $h^1(O_S) = h^2(O_S) = 0$, we find $\mathrm{Pic}(S) = H^1(S, O_S^\times) \cong H^2(S,\mathbb{Z})$, which implies $\rho(S) = b_2(S) = 10$. □

Since Theorem 1.2.7 is a fundamental result for Enriques surfaces, let us note that there are currently three proofs available in positive characteristic:

1. The first proof is due to Bombieri and Mumford [77]: first, they show that every Enriques surface carries a genus one fibration. The associated Jacobian fibration is a rational surface, which is easily seen to satisfy $\rho = b_2 = 10$, from which it follows that Enriques surfaces also satisfy this equality. We will come back to this in Section 4.1.
2. In the case where an Enriques surface in positive characteristic lifts to characteristic zero, one obtains Theorem 1.2.7 from Lemma 1.2.8 using Corollary 1.2.12. But although lifting of Enriques surfaces to characteristic zero is true by [458] (see also Remark 1.4.11 and Volume II), the proof there requires Theorem 1.2.7 at some point.
3. The second proof is due to Lang [433]: it uses a mixture of the previous lifting argument and unirationality results In the case where the lifting of the surface in question is not so obvious. We will present it below.
4. The third proof is due to Liedtke [464]: first, it reduces to the case of Enriques surfaces over $\overline{\mathbb{F}}_p$ and then, it uses the Tate conjecture. We will present it below.

We start with Lang's proof of Theorem 1.2.7 from [433], which also contains some results that are interesting in their own right: we start with the easy case of unirational Enriques surfaces. Recall that a variety X over an algebraically closed field $\Bbbk$ is said to be *unirational* if there exists a dominant rational map $f : \mathbb{P}^n \dashrightarrow X$. By restricting f to a generic linear subspace of dimension $\dim(X)$, there is no loss in generality in assuming $n = \dim(X)$ in this definition. An example of a unirational surface of non-negative Kodaira dimension is a Zariski surface (see Example 0.3.17). We will come back to the unirationality of Enriques surfaces in Theorem 1.3.11 and to classical Enriques surfaces with nontrivial global vector fields in Corollary 1.4.9 below.

Proposition 1.2.9 *Let S be an Enriques surface over an algebraically closed field $\Bbbk$ of characteristic $p > 0$.*

1. *If S is unirational, then it satisfies $\rho(S) = b_2(S) = 10$.*
2. *If S is classical and $H^0(S, \Theta_S) \neq 0$, then S is a Zariski surface. (These surfaces are very rare, see Theorem 1.4.10.)*

Proof Smooth rational surfaces satisfy $\rho = b_2$, since this holds for $\mathbb{P}^2$ and since this equality is preserved under blow-ups and blow-downs. Moreover, the property $\rho = b_2$ also holds for images of dominant and generically finite morphisms by a theorem of Shioda [675]. In particular, it holds for unirational surfaces, and we obtain the first assertion.

To prove the second assertion, we choose a non-zero vector field $\partial \in H^0(S, \Theta_S)$, which we may assume to be p-closed by Lemma 0.3.5. Let $\pi = \pi^\partial : S \to S^\partial$ be the corresponding quotient map, see Section 0.3. Let $\sigma : Y \to S^\partial$ be a resolution of singularities and $\pi' : S' \to Y \times_{S^\partial} S$ be a resolution of singularities of the base change. Thus, $S' \to S$ is a birational morphism of smooth surfaces, and we obtain a commutative diagram

$$\begin{tikzcd} S' \arrow[r, "\pi'"] \arrow[d, "\tau"] & Y \arrow[d, "\sigma"] \\ S \arrow[r, "\pi"] & S^\partial. \end{tikzcd}$$

Then, $h^0(nK_S) \geq h^0(nK_Y)$ for all $n \geq 1$ and $0 = \kappa(S) \geq \kappa(Y)$ by Corollary 0.3.16. If the Albanese morphism of Y were non-trivial, then so would be that of S', which contradicts $b_1(S') = b_1(S) = 0$. This contradiction shows that $b_1(Y) = 0$. In particular, if $\kappa(Y) = -\infty$, then Y is a rational surface. Thus, S' is a rational surface and the map $S' \to S$ is an inseparable cover of S, whence S is a Zariski surface and we are done.

Thus, we may assume now that $\kappa(Y) = 0$, and we want to show that this case does not occur. From $b_1(Y) = 0$ and Table **??** it follows that Y is birational to an Enriques surface or to a K3 surface. Since we assumed S to be a classical Enriques surface, it follows from $0 = h^0(K_S) \geq h^0(K_Y)$ that Y cannot be a K3 surface.

Thus, Y is birational to an Enriques surface and we let $f : Y \to Y'$ be a birational morphism to the unique minimal model Y' of Y. Since every (-1)-curve contributes positively to the canonical class of Y and K_S is numerically trivial, it follows that the exceptional locus of f is contained in the exceptional locus of σ. In particular, if σ is the minimal resolution of singularities, then there are no (-1)-curves in the exceptional locus of σ and f is an isomorphism, that is, we may assume that Y is a minimal surface. Thus, we assume that Y is an Enriques surface.

Since π is a finite and purely inseparable morphism, it is a homeomorphism in the étale topology, from which we conclude $b_2(S^\partial) = b_2(S) = 10$. Being an Enriques surface, we have $b_2(Y) = 10$. Since every exceptional divisor of $Y \to S^\partial$ would contribute positively to the difference $b_2(Y) - b_2(S^\partial)$, which is zero, it follows that $Y \to S^\partial$ is an isomorphism. Thus, the quotient of S by ∂ is Y, which is a smooth surface and thus, the vector field ∂ has no isolated zeros, see Theorem 0.3.9. On the other hand, we have $K_S = \pi^* K_Y + (p-1)R$ by Proposition 0.3.14, where R denotes the divisor of ∂. This implies that $R = 0$, which shows that ∂ has no singular points at all. However, in view of equality (0.3.4), this contradicts $c_2(S) = 12 \neq 0$. □

Next, we want to show that Enriques surfaces that lift to characteristic zero satisfy $\rho = b_2 = 10$. To do so, we start with the following result of Katsura and Ueno [375], which is interesting in its own right. It shows that some of the most important invariants of surfaces do not change in smooth families. Note that for families in positive or mixed characteristic, the Hodge numbers $h^{i,j}$ only satisfy semi-continuity and may jump - in fact, families of Enriques surfaces provide examples (see also Table **??**).

Theorem 1.2.10 *Let $f : \mathcal{X} \to \operatorname{Spec} R$ be a smooth morphism of relative dimension 2 over a Dedekind domain R. Let $\mathcal{X}_{\bar{\eta}}$ be the geometric generic fiber and let $\mathcal{X}_{\bar{0}}$ be a geometric special fiber. Then,*

$$\begin{array}{llll} b_i(\mathcal{X}_{\bar{\eta}}) & = b_i(\mathcal{X}_{\bar{0}}), & e(\mathcal{X}_{\bar{\eta}}) & = e(\mathcal{X}_{\bar{0}}), \\ \chi(\mathcal{O}_{\mathcal{X}_{\bar{\eta}}}) & = \chi(\mathcal{O}_{\mathcal{X}_{\bar{0}}}), & K^2_{\mathcal{X}_{\bar{\eta}}} & = K^2_{\mathcal{X}_{\bar{0}}}, \\ \kappa(\mathcal{X}_{\bar{\eta}}) & = \kappa(\mathcal{X}_{\bar{0}}), & \rho(\mathcal{X}_{\bar{\eta}}) & \leq \rho(\mathcal{X}_{\bar{0}}). \end{array}$$

If $\mathcal{X}_{\bar{\eta}}$ is a minimal surface, then also $\mathcal{X}_{\bar{0}}$ is minimal. If $\mathcal{X}_{\bar{0}}$ is a minimal surface and $\kappa(\mathcal{X}_{\bar{0}}) \geq 0$, then also $\mathcal{X}_{\bar{\eta}}$ is minimal.

Proof After localizing at the maximal ideal corresponding to the special point $\bar{0}$, and passing to unramified extensions and completions, we may assume that R is a local and complete DVR with algebraically closed residue field.

We start with the assertion on Picard numbers. Let $j : \mathcal{X}_\eta \to \mathcal{X}$ and $i : \mathcal{X}_0 \to \mathcal{X}$ be the inclusion morphisms, which give rise to homomorphisms of Picard groups $i^* : \operatorname{Pic}(\mathcal{X}) \to \operatorname{Pic}(\mathcal{X}_0)$ and $j^* : \operatorname{Pic}(\mathcal{X}) \to \operatorname{Pic}(\mathcal{X}_\eta)$. Replacing R by a finite extension if necessary, we may assume $\operatorname{Pic}(\mathcal{X}_\eta) = \operatorname{Pic}(\mathcal{X}_{\bar{\eta}})$ from now on. By projectivity of $\mathcal{X}_\eta$, every Cartier divisor on $\mathcal{X}_\eta$ can be written as a difference of two effective Cartier divisors. The closure of these two divisors in $\mathcal{X}$ are Weil divisors, which are, moreover, Cartier divisors, since $\mathcal{X}$ is regular. This defines a section of j^*, and since j^* is injective, it follows that j^* is an isomorphism of Picard groups. Thus, we obtain a specialization homomorphism

$$\mathrm{sp} : \mathrm{Pic}(\mathcal{X}_{\bar{\eta}}) \to \mathrm{Pic}(\mathcal{X}_{\bar{0}}) \tag{1.2.2}$$

and refer to [274, Section 7.8] for further details. By loc. cit, this homomorphism is compatible with the intersection forms on both sides and thus, it induces an injective homomorphism

$$\mathrm{Num}(\mathcal{X}_{\bar{\eta}}) \to \mathrm{Num}(\mathcal{X}_{\bar{0}}).$$

From this, we obtain $\rho(\mathcal{X}_{\bar{\eta}}) \leq \rho(\mathcal{X}_{\bar{0}})$.

The equality of Betti numbers follows from the base change theorem in étale cohomology (see [508, Chapter 6, Theorem 4.1], or [375, Section 9]). By definition, this implies the equality of Euler–Poincaré characteristics. Since Euler characteristics of coherent sheaves are constant in flat families, and intersection numbers of Cartier divisors can be defined using such Euler characteristics, we deduce $\chi(\mathcal{O}_{\mathcal{X}_{\bar{\eta}}}) = \chi(\mathcal{O}_{\mathcal{X}_{\bar{0}}})$ and $K^2_{\mathcal{X}_{\bar{\eta}}} = K^2_{\mathcal{X}_{\bar{0}}}$ (for the latter, one could also use Noether's formula (0.10.22)).

Let us denote by $p_n(Y)$ the n-th plurigenus $h^0(Y, \omega_Y^{\otimes n})$. By the semi-continuity theorem, we have $p_n(\mathcal{X}_{\bar{\eta}}) \leq p_n(\mathcal{X}_{\bar{0}})$ for all $n \geq 1$, which implies $\kappa(\mathcal{X}_{\bar{\eta}}) \leq \kappa(\mathcal{X}_{\bar{0}})$.

If $\mathcal{X}_{\bar{\eta}}$ is not minimal, then there exists a curve D with $K_{\bar{\eta}} \cdot D < 0$, and after specializing, we find that also $K_{\mathcal{X}_{\bar{0}}}$ is not nef. If moreover $\kappa(\mathcal{X}_{\bar{0}}) \geq 0$, then $K_{\mathcal{X}_{\bar{0}}}$ not being nef implies that $\mathcal{X}_{\bar{0}}$ is not minimal. Conversely, if $\mathcal{X}_{\bar{0}}$ is not minimal, then it contains a (-1)-curve and this curve lifts to $\mathcal{X}_{\bar{\eta}}$ by a deformation argument, see [375, Lemma 9.4]. This implies both minimality assertions.

By lifting (-1)-curves as in loc. cit. and contracting them in families, we may assume that $\mathcal{X}_{\bar{0}}$ is minimal. Now, let $\kappa(\mathcal{X}_{\bar{\eta}}) = -\infty$. Then, $K_{\mathcal{X}_{\bar{\eta}}}$ is not nef, which implies that $K_{\mathcal{X}_{\bar{0}}}$ is not nef (by the same argument as above) and minimal, whence $\kappa(\mathcal{X}_{\bar{0}}) = -\infty$.

To show equality of Kodaira dimensions in the remaining cases, we may assume that $\mathcal{X}_{\bar{0}}$ and $\mathcal{X}_{\bar{\eta}}$ are both minimal surfaces of non-negative Kodaira dimension. If $\kappa(\mathcal{X}_{\bar{\eta}}) = 2$, then $K^2_{\mathcal{X}_{\bar{0}}} = K^2_{\mathcal{X}_{\bar{\eta}}} > 0$ since intersection numbers are preserved under specialization. Applying Theorem 1.1.3 to $\mathcal{X}_{\bar{0}}$, we find $\kappa(\mathcal{X}_{\bar{0}}) = 2$. Also, if $\kappa(\mathcal{X}_{\bar{\eta}}) = 0$, then $12K_{\mathcal{X}_{\bar{\eta}}} = 0$ by Theorem 1.1.14, which implies that $12K_{\mathcal{X}_{\bar{0}}} = 0$, whence $\kappa(\mathcal{X}_{\bar{0}}) = 0$. But then, also $\kappa(\mathcal{X}_{\bar{\eta}}) = 1$ must imply $\kappa(\mathcal{X}_{\bar{0}}) = 1$. □

By inspecting Table **??**, we see that the type of surfaces of Kodaira dimension zero does not change in smooth families.

Corollary 1.2.11 *In the situation of the theorem, $\mathcal{X}_{\bar{0}}$ is an Enriques surface (resp. K3 surface, abelian surface, (quasi-)bielliptic surface) if and only if $\mathcal{X}_{\bar{\eta}}$ is of the same type.*

We will study the degenerations of Enriques surfaces varying in not necessary smooth families in Chapter 5 and Chapter 9 of Volume II.

Another application is that Enriques surfaces that are liftable to characteristic zero satisfy $\rho = b_2 = 10$. To be precise, we have the following.

Corollary 1.2.12 *In the situation of the theorem, if $\mathcal{X}_{\bar{0}}$ is an Enriques surface and R is of characteristic zero, then $\rho(\mathcal{X}_{\bar{0}}) = b_2(\mathcal{X}_{\bar{0}}) = 10$.*

Proof If R is of characteristic zero and $\mathcal{X}_{\bar{0}}$ is an Enriques surface, then $\mathcal{X}_{\bar{\eta}}$ is an Enriques surface in characteristic zero and we have $\rho(\mathcal{X}_{\bar{\eta}}) = b_2(\mathcal{X}_{\bar{\eta}}) = 10$ by Lemma 1.2.8. The assertion then follows from the inequalities $10 = b_2(\mathcal{X}_{\bar{\eta}}) = \rho(\mathcal{X}_{\bar{\eta}}) \le \rho(\mathcal{X}_{\bar{0}}) \le b_2(\mathcal{X}_{\bar{0}}) = 10$. □

We are now in the position to sketch Lang's proof of Theorem 1.2.7 from [433]:

Proof By Lemma 1.2.8, the assertion is true if $\Bbbk$ is of characteristic zero. Thus, we will assume that $\Bbbk$ is of characteristic $p > 0$. We start with the following observation: since $\bigwedge^2 \Omega^1_{S/\Bbbk} \cong \omega_S$, we have $\Theta_{S/\Bbbk} \cong \Omega^1_{S/\Bbbk} \otimes \omega_S$, from which we obtain isomorphisms

$$H^2(S, \Theta_{S/\Bbbk}) \cong H^2(S, \Omega^1_{S/\Bbbk} \otimes \omega_S) \cong H^0(S, \Theta_{S/\Bbbk})^\vee, \tag{1.2.3}$$

where the second one is Serre duality.

First, let us assume that S is classical and that $h^0(\Theta_S) = 0$. By the previous observation, we find $h^2(\Theta_S) = 0$ and thus, deformation theory implies that S admits a formal lift over the Witt ring $W = W(\Bbbk)$. Moreover, since we have $h^2(\mathcal{O}_S) = 0$, deformation theory implies that every invertible sheaf $\mathcal{L}$ on S extends to such a formal lift. In particular, if we choose an ample $\mathcal{L}$, this shows that every formal lift of S is algebraizable by Grothendieck's existence theorem. From this, we deduce that S admits projective algebraic lifts to W (see the details and the references in Volume II). Thus, Corollary 1.2.12 gives the assertion.

Second, assume that S is classical and that $h^0(\Theta_S) \neq 0$. Then, S is unirational by Proposition 1.2.9 and thus, $\rho(S) = b_2(S)$ holds true by loc. cit.

Thus, we may assume that S is non-classical, in which case we have $p = 2$ by Theorem 1.1.8. From here on, we will only sketch the proof: if S is a $\boldsymbol{\mu}_2$-surface, then there exists an algebraic lift by [433, Theorem 1.3] (see also Volume II) and then, the assertion follows from Corollary 1.2.12. If S is an $\boldsymbol{\alpha}_2$-surface, then S is unirational by the analysis in [433] (see also Theorem 1.3.11) and the assertion follows from Proposition 1.2.9. □

We now come to Liedtke's proof of Theorem 1.2.7 from [464]: let us recall that we discussed the crystalline version of the Tate conjecture for divisors in Remark 0.10.27. Its connection to 1.2.7 is as follows.

Proposition 1.2.13 *Let S be an Enriques surface over a finite field $\mathbb{F}_q$. If S satisfies the Tate conjecture, then Theorem 1.2.7 holds for $S \times_{\mathbb{F}_q} \overline{\mathbb{F}}_q$.*

Proof We set $\overline{S} := S \times_{\mathbb{F}_q} \overline{\mathbb{F}}_q$. By Proposition 0.10.21 or Corollary 1.2.5, we find that the $\mathbf{F}$-isocrystal $H^2(\overline{S}/W) \otimes_W K$ is of slope one. Thus, after possibly replacing $\mathbb{F}_q$ by a finite extension, there exists a K-basis $\{e_i\}$ of $H^2(S/W) \otimes_W K$ such that Frobenius acts as $\mathbf{F}(e_i) = p \cdot e_i$ for all i. In particular, the Tate module $T_H \subseteq H^2(S/W)$ is a $\mathbb{Z}_p$-module of rank $b_2(S) = 10$. Since we assumed that the Tate conjecture holds for S, we conclude that $\overline{S}$ satisfies $\rho = b_2$. □

Interestingly, this special case is sufficient to deal with the general case.

Proposition 1.2.14 *In order to prove Theorem 1.2.7, it suffices to establish it for Enriques surfaces over* $\overline{\mathbb{F}}_p$.

Proof Let S be an Enriques surface over an algebraically closed field $\Bbbk$. Then, there exists a sub-$\mathbb{Z}$-algebra R of $\Bbbk$ that is of finite type over $\mathbb{Z}$ and a smooth and projective morphism $\mathcal{S} \to B := \operatorname{Spec} R$ with $\mathcal{S} \times_B \operatorname{Spec} \Bbbk \cong S$. Moreover, if $s \in B$ is a closed point, then the residue field $\kappa(s)$ is a finite field. In particular, the geometric fiber $\mathcal{S}_{\bar{s}}$ is an Enriques surface over $\overline{\kappa(s)}$ and we have $\rho(\mathcal{S}_{\bar{s}}) = b_2(\mathcal{S}_{\bar{s}})$ by assumption. Using Table **??** and Proposition 0.9.14, the first assertion follows. □

Corollary 1.2.15 *If the Tate conjecture holds for Enriques surfaces over finite fields, then Theorem 1.2.7 is true.*

Remark 1.2.16 This corollary reduces Theorem 1.2.7 to a fundamental conjecture in arithmetic geometry, namely, the Tate conjecture. However, at the moment, the Tate conjecture is still wide open. For Enriques surfaces, it can be rather easily deduced from the already-established cases, see [464, Section 3], and we obtain an unconditional proof of Theorem 1.2.7. Since the Tate conjecture is an arithmetic version of the Lefschetz theorem on $(1,1)$-classes, this proof of Theorem 1.2.7 is very close to the proof of Lemma 1.2.8.

Having dealt with Picard schemes and Picard groups of Enriques surfaces, let us end this section by computing their cohomological Brauer groups, which we introduced in Chapter 0.10.

Theorem 1.2.17 *Let S be an Enriques surface over an algebraically closed field $\Bbbk$. Then,*

$$\operatorname{Br}(S) \;\cong\; \begin{cases} \mathbb{Z}/2\mathbb{Z} & \text{if } S \text{ is classical}, \\ \{0\} & \text{if } S \text{ is non-classical.} \end{cases}$$

Proof First, we prove the result in characteristic zero. By the Lefschetz principle, we may assume $\Bbbk = \mathbb{C}$ and then, we may use analytic methods: taking cohomology in the exponential sequence (0.10.9) and using $h^1(\mathcal{O}_S) = h^2(\mathcal{O}_2) = 0$, we obtain $H^2(S,\mathbb{Z}) \cong \operatorname{Pic}(S) \cong \mathbb{Z}^{10} \oplus (\mathbb{Z}/2\mathbb{Z})$. Thus, the universal coefficient formula 0.10.1 and Poincaré duality yield

$$\operatorname{Tors} H^3(S,\mathbb{Z}) \;\cong\; \operatorname{Tors} H_2(S,\mathbb{Z}) \;\cong\; \operatorname{Tors} H^2(S,\mathbb{Z}) \;\cong\; (\mathbb{Z}/2\mathbb{Z}).$$

Since the transcendental lattice of S is zero, exact sequence (0.10.17) yields the statement.

Essentially the same proof also works in positive characteristic $p > 0$, but we have to treat the p-torsion and the prime-to-p-torsion separately: it follows from (0.10.26) or (0.10.61) and Theorem 1.2.7 that there exists isomorphisms $\operatorname{NS}(S) \otimes \mathbb{Z}_\ell \cong H^2_{\text{ét}}(S, \mathbb{Z}_\ell(1))$ and that we have $\mathrm{t}_\ell = 0$ for all primes ℓ (including $\ell = p$). From this, it already follows that $\operatorname{Br}(S)$ is a finite abelian group. Using that S is a surface, Poincaré duality in ℓ-adic cohomology, and (0.10.32), we find

$$_{\ell^\infty}\operatorname{Br}(S) \;\cong\; _{\ell^\infty}H^3_{\text{ét}}(S,\mathbb{Z}_\ell(1)) \;\cong\; _{\ell^\infty}H^2_{\text{ét}}(S,\mathbb{Z}_\ell(1))$$

for all primes $\ell \neq p$ from which Theorem 1.2.1 yields the ℓ-power torsion of $\operatorname{Br}(S)$. Moreover, using (0.10.79) and (0.10.85), we find

$$_{p^\infty}\operatorname{Br}(S) \;\cong\; _{p^\infty}H^3_{\text{ét}}(S,\mathbb{Z}_p(1)) \;\cong\; _{p^\infty}\operatorname{Ker}(\mathbf{F}-1 \,:\, H^2(S,\, W\Omega^1_{S/\Bbbk}) \;\to\; H^2(S,\, W\Omega^1_{S/\Bbbk})).$$

Using Theorem 1.2.1 and the computation of the Hodge–Witt cohomology groups of Enriques surfaces that we will establish in Proposition 1.4.16 below, the result follows. □

We already mentioned that the proof of Theorem 1.2.7 in [77] uses genus one fibrations. Similarly, Theorem 1.2.17 can be established using genus one fibrations by applying Theorem 4.10.3 to Theorem 4.3.13. Concerning Brauer groups of Enriques surfaces, we refer the interested reader to [51] for more information, as well as to [271, Section 8] for a more algebraic point of view.

1.3 The K3-cover

In this section, we discuss the K3-cover $\pi : X \to S$ of an Enriques surface S. In characteristic $p \neq 2$, the surface X is a K3 surface and π is an étale morphism of degree 2, which links the theory of Enriques surfaces to the theory of K3 surfaces. However, in characteristic 2, the situation is much more complicated: for example, X may even be a non-normal and rational surface, although it will always be "K3 like" in a certain sense. Then, we discuss the unirationality of Enriques surfaces in positive characteristic and relate it to the unirationality of the K3-cover. We end the section by studying the pull-back of the Picard group and the Brauer group from an Enriques surface to its K3-cover.

Theorem 1.3.1 *Let S be an Enriques surface over an algebraically closed field $\Bbbk$ of characteristic p. Then, there exists a non-trivial $(\boldsymbol{Pic}^\tau_{S/\Bbbk})^D$-torsor*

$$\pi \;:\; X \;\to\; S.$$

In particular, π is a finite and flat morphism of degree 2.

Proof This follows by applying Theorem 0.2.34 to Theorem 1.2.1. □

Before proceeding, let us be a little bit more explicit about these torsors, see also Proposition 0.2.29.

1. First, let S be a classical Enriques surface, in which case we have $\mathbf{Pic}^\tau_{S/\Bbbk} \cong (\mathbb{Z}/2\mathbb{Z})$ and $(\mathbf{Pic}^\tau_{S/\Bbbk})^D \cong \boldsymbol{\mu}_2$. By Corollary 1.2.3, the canonical sheaf $\omega_S = \mathcal{O}_S(K_S)$ is a non-trivial 2-torsion element of $\operatorname{Pic}(S)$. As explained in (0.2.4), a choice of isomorphism $\omega_S^{\otimes 2} \cong \mathcal{O}_S$ defines an $\mathcal{O}_S$-algebra structure on $(\mathcal{O}_S \oplus \omega_S)$, and thus, a finite flat double cover

$$\pi \,:\, X \,:=\, \mathrm{Spec}(\mathcal{O}_S \oplus \omega_S) \,\to\, S,$$

which is a $\boldsymbol{\mu}_2$-torsor. In particular, π is étale if $p \neq 2$ and purely inseparable if $p = 2$.

2. Next, let S be a $\boldsymbol{\mu}_2$-surface, in which case we have $p = 2$, as well as $\mathbf{Pic}^{\tau}_{S/\Bbbk} \cong \boldsymbol{\mu}_2$ and $(\mathbf{Pic}^{\tau}_{S/\Bbbk})^D \cong (\mathbb{Z}/2\mathbb{Z})$. Then, the Artin–Schreier exact sequence in étale topology
$$0 \,\to\, (\mathbb{Z}/2\mathbb{Z})_S \,\to\, \mathbb{G}_{a,S} \xrightarrow{\mathbf{F}-\mathrm{id}} \mathbb{G}_{a,S} \,\to\, 0$$
gives rise to an isomorphism
$$H^1_{\text{ét}}(S, \mathbb{Z}/2\mathbb{Z}) \,\cong\, \mathrm{Ker}\left(H^1(S, \mathcal{O}_S) \xrightarrow{\mathbf{F}-\mathrm{id}} H^1(S, \mathcal{O}_S)\right) \,\neq\, 0.$$
As explained in Section 0.3, a non-zero element η of this cohomology group defines a non-trivial étale double cover $\pi : X \to S$, that is, a $(\mathbb{Z}/2\mathbb{Z})$-torsor. It is easy to see that the isomorphism class of this cover does not depend on the choice of η.

3. Finally, let S be an $\boldsymbol{\alpha}_2$-surface, in which case we have $p = 2$, as well as $\mathbf{Pic}^{\tau}_{S/\Bbbk} \cong (\mathbf{Pic}^{\tau}_{S/\Bbbk})^D \cong \boldsymbol{\alpha}_2$. Then, the exact sequence in flat topology
$$0 \,\to\, \boldsymbol{\alpha}_{2,S} \,\to\, \mathbb{G}_{a,S} \xrightarrow{\mathbf{F}} \mathbb{G}_{a,S} \,\to\, 0$$
gives rise to an isomorphism
$$H^1_{\mathrm{fl}}(S, \boldsymbol{\alpha}_2) \,\cong\, \mathrm{Ker}\left(H^1(S, \mathcal{O}_S) \xrightarrow{\mathbf{F}} H^1(S, \mathcal{O}_S)\right) \,\neq\, 0.$$
Again, as explained in Section 0.3, a non-zero element η of this cohomology group defines a flat double cover $\pi : X \to S$, which is an $\boldsymbol{\alpha}_2$-torsor. In particular, π is purely inseparable. It is easy to see that the isomorphism class of this cover does not depend on the choice of η.

Definition 1.3.2 The flat double cover $\pi : X \to S$ is called the *K3-cover* of S (or the *canonical cover*).

Since it is a non-trivial torsor under a finite flat group scheme of prime order, X is reduced and irreducible, that is, an integral scheme. Moreover, the name is justified by the fact that it is always "K3-like" in the following sense.

Proposition 1.3.3 *Let $\pi : X \to S$ be the K3-cover of an Enriques surface S. Then, X is an integral Gorenstein surface (not necessary normal) satisfying*
$$H^1(X, \mathcal{O}_X) = 0 \quad \textit{and} \quad \omega_X \cong \mathcal{O}_X.$$
Moreover:

1. if $p \neq 2$ or S is a $\boldsymbol{\mu}_2$-surface, then X is smooth and a K3 surface, and

2. if $p = 2$ and S is classical or an $\boldsymbol{\alpha}_2$-surface, then X is not a smooth surface.

Proof As seen in Section 0.3 or via the above case-by-case analysis, X is locally a hypersurface in a line bundle over S. In particular, X is Gorenstein and we already mentioned above that X is an integral variety. Moreover, we have an exact sequence

$$0 \to \mathcal{O}_S \to \pi_*\mathcal{O}_X \to \mathcal{L}^{-1} \to 0 \tag{1.3.1}$$

with $\mathcal{L} = \omega_S$. From this, we conclude $\chi(X, \mathcal{O}_X) = \chi(S, \pi_*\mathcal{O}_X) = 2\chi(S, \mathcal{O}_S) = 2$. Next, for the dualizing sheaf of X we have $\omega_X = \pi^*(\omega_S \otimes \mathcal{L}^{-1}) \cong \pi^*(\mathcal{O}_S) \cong \mathcal{O}_X$, see Proposition 0.2.12 and Proposition 0.2.20. Then, Serre duality implies $h^2(\mathcal{O}_X) = h^0(\omega_X) = 1$. Combining this with $\chi(\mathcal{O}_X) = 2$, we find $h^1(\mathcal{O}_X) = 0$.

If $p \neq 2$ or if S is a $\boldsymbol{\mu}_2$-surface, then π is étale, hence X is smooth. Since $\omega_X \cong \mathcal{O}_X$ and $h^1(\mathcal{O}_X) = 0$, it follows from the very definition that X is a K3 surface.

In the remaining cases, we have $p = 2$ and π is purely inseparable. Seeking a contradiction, assume that X is smooth. Since π is a homeomorphism in the étale topology, we find $c_2(X) = c_2(S) = 12$. On the other hand, we have $\chi(\mathcal{O}_X) = 2$, and $\omega_X \cong \mathcal{O}_X$ implies $K_X^2 = 0$, which contradicts Noether's formula (0.10.22). □

Now, we study the K3-cover $\pi : X \to S$ in the case where X is not smooth, that is, if $p = 2$ and S is not a $\boldsymbol{\mu}_2$-surface, or, equivalently, S is a unipotent Enriques surface. Since X may not be normal, let

$$\nu : Y \to X$$

be the normalization of X. Then, the composition $\tilde{\pi} = \pi \circ \nu : Y \to S$ is a finite and inseparable morphism of degree 2 from a normal surface onto a smooth surface. Since Y is Cohen–Macaulay (it is a normal surface) and X is regular, $\tilde{\sigma}$ is flat by Proposition 0.2.4. Thus, by Proposition 0.2.27, also $\tilde{\pi}$ is a torsor under a finite flat group scheme of length 2. In particular, also Y is a Gorenstein scheme. Also, we have an exact sequence

$$0 \to \mathcal{O}_S \to \tilde{\pi}_*\mathcal{O}_Y \to \tilde{\mathcal{L}}^{-1} \to 0 \tag{1.3.2}$$

for some invertible sheaf $\tilde{\mathcal{L}}$ on S. If this is a $\boldsymbol{\mu}_2$-torsor, then the exact sequence splits. We also have

$$\omega_Y \cong \tilde{\pi}^*\left(\omega_S \otimes \tilde{\mathcal{L}}\right) \cong \tilde{\pi}^*(\tilde{\mathcal{L}}). \tag{1.3.3}$$

Here, we have used that $\tilde{\pi}^*\omega_S = \nu^*\pi^*\omega_S = \nu^*\mathcal{O}_X \cong \mathcal{O}_Y$. To understand the normalization ν, we consider the *conductor ideal*

$$\mathfrak{C} := \mathrm{Ann}(\nu_*\mathcal{O}_Y/\mathcal{O}_X) \subseteq \mathcal{O}_X$$

of the normalization ν, which can be defined equivalently as $\mathcal{H}om_{\mathcal{O}_X}(\nu_*\mathcal{O}_Y, \mathcal{O}_X)$. We note that $\mathfrak{C}$ can be considered as an ideal sheaf of $\mathcal{O}_Y$ and that it is the largest ideal sheaf of $\mathcal{O}_X$ that is also an ideal sheaf of $\mathcal{O}_Y$. We refer to [611] for details. Let $D \subset X$ and $C \subset Y$ be the closed subschemes defined by these ideal sheaves, both of which are of pure codimension one (if nonempty). The duality theorem for finite morphisms (see [611, Proposition 2.3] in this situation and [292] for the general machinery) gives isomorphisms

$$\begin{array}{lll} \nu_*\omega_Y \cong \mathcal{H}om_{\mathcal{O}_X}(\nu_*\mathcal{O}_Y,\omega_X) & & \cong \mathfrak{C}\cdot\omega_X, \\ \nu^*\omega_X \cong \nu^*\mathcal{O}_X \cong \mathcal{O}_Y \cong \mathcal{H}om_{\mathcal{O}_Y}(\mathfrak{C},\omega_Y) & & \cong \omega_Y(C), \end{array} \tag{1.3.4}$$

where $\omega_Y(C)$ denotes the reflexive saturation of of $\omega_Y\otimes\mathcal{O}_Y(C)$, that is, the reflexive $\mathcal{O}_Y$-module of rank one, whose local sections are rational sections of ω_Y with at worst a single pole along C. Taking into account (1.3.3), we obtain

$$\mathcal{O}_Y(C) \cong \omega_Y^{-1} \cong \tilde{\pi}^*(\tilde{\mathcal{L}}^{-1}).$$

In particular, if non-empty, then $C\subseteq Y$ is an effective Cartier divisor. From the commutative diagram with exact rows (1.3.1) and (1.3.2)

$$\begin{array}{ccccccccc} 0 & \to & \mathcal{O}_S & \to & \pi_*\mathcal{O}_X & \to & \omega_S^{-1} & \to & 0 \\ & & || & & \downarrow & & \downarrow & & \\ 0 & \to & \mathcal{O}_S & \to & \tilde{\pi}_*\mathcal{O}_Y & \to & \tilde{\mathcal{L}}^{-1} & \to & 0 \end{array}$$

we obtain an injective homomorphism of invertible sheaves $\omega_S^{-1}\to\tilde{\mathcal{L}}^{-1}$ on S, which gives rise to an effective Cartier divisor A on S, and we conclude

$$\tilde{\mathcal{L}} \cong \omega_S(-A). \tag{1.3.5}$$

Since $\nu|_{Y-D}:(Y-C)\to(X-D)$ is an isomorphism, it follows that A lies below $C\subseteq Y$, as well as below $D\subseteq X$. Following [213], we make the following definition.

Definition 1.3.4 The effective Cartier divisor A of S is called the *conductrix* and $B:=2A$ is called the *bi-conductrix*.

The relevance of the bi-conductrix will become clear in Proposition 1.3.8 and when discussing exceptional Enriques surfaces in Section 6.2 from Volume II, see also [213]. After these preparations, we have the following result that describes the geometry of the K3-cover X In the case where it is not smooth.

Theorem 1.3.5 *Let S be an Enriques surface over an algebraically closed field of characteristic* 2 *that is classical or an $\boldsymbol{\alpha}_2$-surface. Let $\pi:X\to S$ be its K3-cover.*

1. *If X is normal with at worst rational singularities, then it has only rational double point singularities. The minimal resolution X' of singularities is a K3 surface that satisfies $b_2=\rho=22$ (i.e. X' is supersingular in the Shioda sense).*
2. *If X is normal with non-rational singularities, then it is a rational surface with one elliptic Gorenstein singularity.*
3. *If X is non-normal, then its normalization is a rational surface with at worst rational double point singularities.*

Proof By Proposition 1.3.3, X is Cohen–Macaulay. Thus, if X has only isolated singularities, then it is normal by Serre's normality criterion. We now do a case-by-case analysis.

Case 1: X has only isolated singularities that are, at worst, rational.

Then, let $\tau : X' \to X$ be the minimal resolution of singularities. Since X is Gorenstein, Proposition 0.4.17 implies that all singularities of X are rational double points and we compute $\omega_{X'} \cong \tau^*\omega_X \cong \tau^*\mathcal{O}_X \cong \mathcal{O}_{X'}$. Using Serre duality, we find $h^2(\mathcal{O}_{X'}) = h^0(\omega_{X'}) = 1$. From $\chi(\mathcal{O}_{X'}) = \chi(\mathcal{O}_X) = 2$, we find $h^1(\mathcal{O}_{X'}) = 0$ and thus, X' is a K3 surface.

Moreover, the composition $X' \to X \to S$ and the Frobenius morphism give rise to a dominant and rational map $S^{(1/p)} \dashrightarrow X'$, which extends to a generically finite morphism $\tilde{S}^{(1/p)} \to X$ after a suitable blow-up of $S^{(1/p)}$. Since $b_2 = \rho$ holds for Enriques surfaces, it also holds for their blow-ups, and thus, in particular, for $\tilde{S}^{(1/p)}$. Then, this also holds for X by [675], that is, X is supersingular in the Shioda sense.

Case 2: X has only isolated singularities, at least one of which is not rational.

Again, let $\tau : X' \to X$ be the minimal resolution of singularities. Then, the five-term exact sequence of the Grothendieck-Leray spectral sequence

$$E_2^{i,j} := H^j(X,\, R^i\tau_*\mathcal{O}_{X'}) \Longrightarrow H^{i+j}(X', \mathcal{O}_{X'})$$

is the long exact sequence

$$0 \to H^1(X, \mathcal{O}_X) \to H^1(X', \mathcal{O}_{X'}) \to H^0(X, R^1\tau_*\mathcal{O}_{X'}) \to H^2(X, \mathcal{O}_X) \to H^2(X', \mathcal{O}_{X'}).$$

By Serre duality and Proposition 1.3.3, we find $h^1(X, \mathcal{O}_X) = 0$ and $h^2(X, \mathcal{O}_X) = 1$.

Since $\omega_X \cong \mathcal{O}_X$, the canonical divisor class $K_{X'}$ has a representative D that is supported on the exceptional curve of the resolution τ. If we had $|nK_{X'}| \neq \emptyset$ for some $n \geq 1$, then we could find a rational function f on X' such that $(f) + nD \geq 0$. Consider f as a rational function on X, then it is regular outside isolated normal singularities, thus, it has to be regular everywhere, and thus, constant. Hence, $nD \geq 0$ for all $n \geq 0$. If we had $nD = 0$ for some $n \geq 1$, then $\omega_{X'}^{\otimes n} = \tau^*\omega_X^{\otimes n}$. But then, every integral curve R inside the exceptional divisor of τ would satisfy $R \cdot K_{X'} = 0$, as well as $R^2 < 0$ (being exceptional). The adjunction formula then yields $R^2 = -2$ and $R \cong \mathbb{P}^1$, which implies that all singularities of X are rational double points by Proposition 0.4.8. Since we assumed to have at least one non-rational singularity, we conclude that $nD > 0$ for at least one $n \geq 1$, which contradicts the non-emptyness of $|nK_{X'}|$. This contradiction implies $|nK_{X'}| = \emptyset$ for all $n \geq 1$.

Since the morphism $\pi : X \to S$ is purely inseparable, there exists a dominant rational map $S^{(1/p)} \dashrightarrow X'$, and thus, if X' had a non-trivial Albanese morphism, then so would S, contradicting $b_1(S) = 0$. This contradiction implies $b_1(X') = 0$. Since $|2K_{X'}| = \emptyset$ by the above, Castelnuovo's Rationality Criterion shows that X' is a rational surface.

Being a rational surface, we have $h^1(\mathcal{O}_{X'}) = h^2(\mathcal{O}_{X'}) = 0$. Together with $h^2(\mathcal{O}_X) = 1$, we find $h^0(R^1\tau_*\mathcal{O}_{X'}) = 1$. In particular, there is precisely one singular point that is not rational, and it is elliptic. Since X is Gorenstein by Proposition 1.3.3, this is an elliptic Gorenstein singularity. This shows that there is a unique elliptic Gorenstein singularity on X.

Case 3: X is not normal.

Let $\nu : Y \to X$ be the normalization, let A be the conductrix, and let $\tau : Y' \to Y$ be the minimal resolution of singularities. We have $\omega_Y \cong \tilde{\pi}^*(\tilde{\mathcal{L}}) \cong \tilde{\pi}^* \mathcal{O}_S(-A)$ by (1.3.3) and (1.3.5). In particular, $K_Y < 0$, which implies $K_{Y'} < 0$, and we obtain $|nK_{Y'}| = \emptyset$ for all $n \geq 1$. As In case 2, we find $b_1(Y') = b_1(S) = 0$, and conclude that Y' is a rational surface by Castelnuovo's Rationality Criterion.

Since Y' is rational, we have $h^1(Y', \mathcal{O}_{Y'}) = 0$, and since Y is a Gorenstein surface with $K_Y < 0$, we have $h^2(Y, \mathcal{O}_Y) = h^0(Y, \omega_Y) = 0$. Applying the Grothendieck-Leray spectral sequence to its minimal resolution $\tau : Y' \to Y$ as In case 1, the long exact sequence in low terms yields $h^0(Y, R^1\tau_*\mathcal{O}_{Y'}) = 0$. This implies that all singularities of Y are rational, and since Y is Gorenstein, its singularities are rational double points. □

Remark 1.3.6 In [639, Theorem 14.1], it is shown in the second case X is smooth outside of its elliptic singularity. We will return to this case later in Volume II.

Corollary 1.3.7 *Let S be an Enriques surface over an algebraically closed field of characteristic 2 that is classical or an α_2-surface. Then, S is algebraically simply connected.*

Proof Let $\pi : X \to S$ be the K3-cover. Then, the minimal resolution of the singularities of X is a rational surface or a K3 surface, both of which are algebraically simply connected, see [272, Corollaire XI.1.2] and Proposition 1.1.9. Since π is purely inseparable, it is a homeomorphism in the étale topology, and so, S is algebraically simply connected. □

We note that the first case of Theorem 1.3.5 is the generic case, see Remark 1.3.9 below. Let us now continue with a more detailed analysis of the singularities of the K3-cover X of an Enriques surface S in characteristic 2. By Proposition 0.2.21, the singularities of X lie above the zeros of a section of the sheaf $\Omega^1_{S/\Bbbk}$. This already implies $h^{1,0}(S) = h^0(\Omega^1_{S/\Bbbk}) \neq 0$ for such surfaces, that is, there exists a non-zero and regular 1-form ω. Let $Z = Z(\omega)$ be its scheme of isolated zeros and let $D = D(\omega)$ be its divisorial part, that is, the largest effective divisor such that $\mathcal{O}_S(D)$ is a subsheaf of $\Omega^1_{S/\Bbbk}$ containing the image of ω. Thus, we obtain a short exact sequence

$$0 \to \mathcal{O}_S(D) \to \Omega^1_{S/\Bbbk} \to \mathcal{I}_Z(D') \to 0, \tag{1.3.6}$$

where $\mathcal{I}_Z$ is the ideal sheaf of the 0-dimensional closed subscheme Z and D' is a divisor that is linearly equivalent to $-K_S - D$, see Proposition 0.3.18 and (0.3.3) and use that $\Omega^1_{S/\Bbbk} \cong \Theta_{S/\Bbbk}$ for classical and α_2-surfaces (see also the proof of Theorem 1.4.4). We note that the scheme of zeros of ω is the union of the supports of D and Z. Here, Z may intersect the support of D and there could be non-reduced scheme structures on both closed subsets.

Proposition 1.3.8 *Let S be an Enriques surface over an algebraically closed field of characteristic 2 that is classical or an α_2-surface. Let $\pi : X \to S$ be its K3-cover. Then,*

$$H^0(S, \Omega^1_{S/\Bbbk}) \neq 0.$$

More precisely, let ω be a non-zero and regular 1-form and let D and Z be its schemes of divisorial and isolated zeros as above. Then, $D = 2A$, where A denotes the conductrix of S and we have

$$-D^2 + h^0(\mathcal{O}_Z) = 12. \tag{1.3.7}$$

Moreover, if X is non-normal, then:

1. *the divisor A is effective, supported on (-2)-curves, numerically connected and satisfies $A^2 = -2$, as well as $h^0(S, \mathcal{O}_S(2A)) = 1$, and*
2. *the normalization of X is a rational surface with four rational double points of type A_1 or one rational double point of type $D_4^{(0)}$.*

Proof We have established $h^0(\Omega_S^1) \neq 0$ already in the above discussion. Next, taking Chern classes in (1.3.6) and using $K_S = c_1(\Omega_S^1) = D + D'$, we find $12 = c_2(S) = c_2(\Omega_S^1) = -D^2 + h^0(\mathcal{O}_Z)$, see also Proposition 0.3.18.

Let $\nu : Y \to X$ be the normalization and thus, ν is an isomorphism if and only if X is normal. We set $\tilde{\pi} := \pi \circ \nu : Y \to S$, which is an $\alpha_{\tilde{\mathcal{L}}}$-torsor with respect to an invertible $\mathcal{O}_S$-module $\tilde{\mathcal{L}}$ as explained in (1.3.2). By Proposition 0.2.21 there exists an injection $\tilde{\mathcal{L}}^{\otimes(-p)} \to \Omega_S^1$, whose zero set lies below the singularities of Y (note that, since the cover is inseparable, $a = 0$, in the notation of the proposition). Since Y is normal, its singularities are isolated, and this injection is saturated. We thus obtain a short exact sequence

$$0 \to \tilde{\mathcal{L}}^{\otimes(-2)} \to \Omega^1_{S/\Bbbk} \to \mathcal{I}_{Z'}(D'') \to 0, \tag{1.3.8}$$

where $\mathcal{I}_{Z'}$ is the ideal sheaf of some 0-dimensional closed subscheme Z' of S and D'' is a divisor on S. Inspecting the proof of Proposition 0.2.21, we see that the two short exact sequences (1.3.8) and (1.3.6) coincide. Using (1.3.5), we obtain $D = 2A$.

Finally, assume that X is not normal, that is, $A \neq 0$. Then, we have

$$0 < h^0(\mathcal{O}_S(2A)) \leq h^0(\Omega^1_{S/\Bbbk}) = h^{1,0}(S).$$

In Corollary 1.4.9 below, we will see that $h^{1,0}(S) = 1$ for classical and $\boldsymbol{\alpha}_2$-surfaces in characteristic 2, from which we conclude $h^0(\mathcal{O}_S(2A)) = 1$. Now, if we had $A^2 \geq 0$, then Riemann–Roch would imply the $h^0(\mathcal{O}_S(2A)) \geq 2$, a contradiction. Thus, $A^2 < 0$ and after applying (1.3.7), we conclude $A^2 = -2$. Moreover, if $A = A_1 + A_2$ for some $A_1 > 0$, $A_2 > 0$, then $A_1^2 < 0$ and $A_2^2 < 0$, because otherwise $2A_1$ or $2A_2$ would move. This gives $A^2 = A_1^2 + 2A_1 \cdot A_2 + A_2^2 = -2$, hence $A_1 \cdot A_2 > 0$. Thus, A is 1-connected. Moreover, we also see that every irreducible component of A is a (-2)-curve.

Now, consider $\tilde{\pi} : Y \to S$. Using Proposition 0.2.10 and Proposition 0.2.21, we see that the singular points of Y are defined by the zeros of a section of $\Omega_S^1 \otimes \mathcal{O}_S(2A)$. A computation with Chern classes shows that $c_2(\Omega_S^1 \otimes \mathcal{O}_S(2A)) = 4$. Moreover, by Theorem 1.3.5, all singular points of Y are rational double points. Since Y is a purely inseparable double cover of a smooth surface, these singular points have local equations of the form $z^2 + f(x, y) = 0$ and we know that the colength of (f_x, f_y) is at most 4. Using the list of rational double points from Proposition 0.4.13, we find that the only possibilities are singularities of type A_1, which are of colength 1, or of type $D_4^{(0)}$, which are of colength 4. □

Remark 1.3.9 To complete the picture, let us also mention the following results concerning K3-covers of Enriques surfaces.

1. In every characteristic and even when non-smooth or when non-normal, the K3-cover is always birationally equivalent to the complete intersection of three quadrics in $\mathbb{P}^5$ (see Corollary 3.4.2 and Lemma 1.1.11, Case 3). This shows some relation to the theory of K3 surfaces in all cases.
2. In characteristic 2, the K3-cover of an Enriques surface S is non-normal if and only if S admits a quasi-elliptic fibration, see [489].
3. In characteristic 2 and for a quantitative comparison of the three cases of the K3-cover from Theorem 1.3.5, we have to use moduli spaces, which we will construct in Chapter 5.

   a. Inside moduli spaces for classical and α_2-surfaces in characteristic 2, there exist an open and dense subsets, such that the K3-covers of the corresponding surfaces are normal with 12 rational double points of type A_1. Moreover, for a surface S on these open subsets, every non-zero regular 1-form $\omega \in H^0(\Omega^1_{S/\Bbbk})$ has no divisorial part and 12 isolated zeros. In particular, the first case of Theorem 1.3.5 is generic.
   b. The locus of classical and α_2-surfaces, whose K3-cover is not a K3 surface with rational double points, that is, cases 2 and 3 of Theorem 1.3.5, is closed and everywhere of codimension at least 3.

   We also refer to Remark 1.6.9 for some illustrating examples and refer to [214] for details.

We will say more about singularities of the K3-cover in Volume II.

We have already seen in Proposition 1.2.9 that Enriques surfaces with non-zero regular vector fields are Zariski surfaces. In characteristic zero, unirational varieties are of Kodaira dimension $-\infty$, and in particular, K3 surfaces and Enriques surfaces in characteristic zero are never unirational. However, K3 surfaces in positive characteristic can be unirational, and Shioda [675] showed that then, the K3 surface is Shioda-supersingular. For a Shioda-supersingular K3 surface, the discriminant of the Néron-Severi group is of the form $p^{2\sigma_0}$ for some integer $1 \leq \sigma_0 \leq 10$, called the *Artin invariant*, see also Remark 0.10.31. The following nontrivial results are due to Rudakov–Shafarevich [627] if $p = 2, 3$ to Pho–Shimada [594] if $p = 5$ and to Shioda [676] for $p \geq 3$.

Theorem 1.3.10 *Let X be a Shioda-supersingular K3 surface in characteristic $p >$ 0.*

1. *If $p = 2$ or else $p = 3$ and $\sigma_0 \leq 6$ or else $p = 5$ and $\sigma_0 \leq 3$, then X is a Zariski surface, and thus, unirational.*

2. *If $p \geq 5$ and $\sigma_0 \leq 2$, then X is unirational.*

In [655], Serre showed that the étale fundamental group of a smooth, projective, and unirational variety is finite, and over the complex numbers, the étale and topological fundamental groups are even trivial. Moreover, by loc. cit., a smooth and projective variety is unirational if and only if some (and hence, every) finite étale cover is unirational. This allows us to determine which Enriques surfaces are unirational.

Theorem 1.3.11 *Let S be an Enriques surface in characteristic p.*

1. *If $p = 2$, then S is unirational if and only if it is not a $\boldsymbol{\mu}_2$-surface. Moreover, if the minimal resolution of singularities of the K3-cover is not a K3 surface, then S is a Zariski surface.*
2. *If $p \geq 3$, then S is unirational if and only if its K3-cover is a unirational K3 surface.*

Proof Let us first assume that p is an odd prime. Then, an Enriques surface is unirational if and only if its K3-cover is unirational by [655] or [676, Lemma 3.1].

We will now assume that $p = 2$. If S is a $\boldsymbol{\mu}_2$-surface, then its K3-cover is an étale double cover. On the other hand, by [655] and [143], the étale fundamental group of a smooth, projective, and unirational variety in characteristic p is finite of order prime to p. Therefore, S is not unirational.

Finally, assume that S is not a $\boldsymbol{\mu}_2$-surface. Let $\pi : X \to S$ be its K3-cover, which is purely inseparable of degree $p = 2$. If X is rational, then S is a Zariski surface by definition, and we are done. Thus, we may assume that X is not a rational surface. But then, its minimal resolution of singularities X' is a Shioda-supersingular K3 surface by Theorem 1.3.5, which is unirational by Theorem 1.3.10. In particular, S is also unirational. □

Remark 1.3.12 By a result of Crew [143, Theorem 2.7], the K3-cover of a $\boldsymbol{\mu}_2$-surface is an *ordinary K3 surface*, that is, the height of the formal Brauer group of X is equal to one, see Section 0.10 and Theorem 1.4.21. We refer to Remark 1.6.12 and Example 1.6.13 for families and examples of unirational as well as non-unirational Enriques surfaces in characteristic $p \geq 3$.

For the remainder of this section, let S be an Enriques surface over an algebraically closed field $\Bbbk$ of characteristic $p \geq 0$ and assume that the K3-cover $\pi : X \to S$ is étale, that is, $p \neq 2$ or that $p = 2$ and S is a $\boldsymbol{\mu}_2$-surface. In particular, X is a K3 surface. We want to compare the Picard groups and the Brauer groups of S and X. To do so, we have pull-backs

$$\pi^* : \mathrm{Pic}(S) \to \mathrm{Pic}(X) \qquad \text{and} \qquad \pi^* : \mathrm{Br}(S) \to \mathrm{Br}(X),$$

which are homomorphisms of abelian groups. Since $\pi^*(\omega_S) \cong \mathcal{O}_X$, the first homomorphism factors through an injective homomorphism which we continue to denote by π^*

$$\pi^* : \mathrm{Num}(S) \hookrightarrow \mathrm{Pic}(X) = \mathrm{Num}(X).$$

Let G be the group of deck transformations of X, which is isomorphic to $\mathbb{Z}/2\mathbb{Z}$. If we denote the subgroup of G-invariant invertible sheaves of X by $\mathrm{Pic}(X)^G$, then the following result describes kernel and image of π^* for Picard groups.

Proposition 1.3.13 *Let S be an Enriques surface over an algebraically closed field $\Bbbk$ of characteristic $p \neq 2$ and let $\pi : X \to S$ be its K3-cover. Then, there exists a short exact sequence*

$$0 \to \mathbb{Z}/2\mathbb{Z} \to \mathrm{Pic}(S) \xrightarrow{\pi^*} \mathrm{Pic}(X)^G \to 0,$$

and the kernel of π^* is generated by ω_S. Moreover, the lattice embedding $\pi^* : \mathrm{Num}(S)(2) \to \mathrm{Pic}(X)$ is primitive.

Proof Consider the *Hochschild–Serre spectral sequence* in étale cohomology

$$E_2^{i,j} := H^i\left(G, H^j_{\text{ét}}(X, \mathbb{G}_m)\right) \Rightarrow H^{i+j}_{\text{ét}}(S, \mathbb{G}_m). \tag{1.3.9}$$

Using $E_2^{1,0} = H^1(G, \mathbb{G}_m) = \mathrm{Hom}(G, \mathbb{G}_m)$ and $E_2^{2,0} = H^2(G, \mathbb{G}_m) = 0$, the five-term exact sequence yields a short exact sequence

$$0 \to \mathrm{Hom}(G, \mathbb{G}_m) \to \mathrm{Pic}(S) \xrightarrow{\pi^*} \mathrm{Pic}(X)^G \to 0,$$

see also [351]. We have $\mathrm{Hom}(G, \mathbb{G}_m) \cong \boldsymbol{\mu}_2 \cong \mathbb{Z}/2\mathbb{Z}$ and since $\pi^*\omega_S \cong \omega_X \cong \mathcal{O}_X$, we see that ω_S is a non-trivial element of $\mathrm{Ker}\,\pi^*$, and thus, a generator.

Finally, every torsion class of $\mathrm{Pic}(X)/\mathrm{Pic}(X)^G$ can be lifted to an $\mathcal{L} \in \mathrm{Pic}(X)$ with $\mathcal{L}^{\otimes n} \in \mathrm{Pic}(X)^G$ for some $n \geq 1$. Since $\mathcal{L}^{\otimes n}$ is G-invariant and $\mathrm{Pic}(X)$ is torsion-free by Proposition 1.1.9, also $\mathcal{L}$ is G-invariant, that is, $\mathcal{L} \in \mathrm{Pic}(X)^G$. This shows that the quotient $\mathrm{Pic}(X)/\mathrm{Pic}(X)^G$ is torsion-free, that is, the embedding $\mathrm{Pic}(X)^G \to \mathrm{Pic}(X)$ is primitive. □

The behavior of π^* on cohomological Brauer groups is more complicated. To understand it, let us recall the *norm homomorphism*

$$\mathrm{Nm} : \mathrm{Pic}(X) \to \mathrm{Pic}(S), \tag{1.3.10}$$

which is defined as follows: if $\mathcal{L} \in \mathrm{Pic}(X)$, then $\pi_*\mathcal{L}$ is a locally free $\mathcal{O}_S$-module of rank 2, and then, $\mathrm{Nm}(\mathcal{L}) := \det(\pi_*\mathcal{O}_X)^\vee \otimes \det(\pi_*\mathcal{L}) \cong \omega_S \otimes \det(\pi_*\mathcal{L})$, where the last isomorphism follows from taking the determinants in (1.3.1). Next, let $\sigma \in G$ be the generator, which we will refer to as the *Enriques involution* on X. Being a homomorphism and G-invariant, we compute $\mathrm{Nm}(\mathcal{L}^{-1} \otimes \sigma^*\mathcal{L}) = \mathrm{Nm}(\mathcal{L})^{-1} \otimes \mathrm{Nm}(\mathcal{L}) \cong \mathcal{O}_S$ and conclude

$$(\mathrm{id} - \sigma^*)\,\mathrm{Pic}(X) \subseteq \mathrm{Ker}(\mathrm{Nm}).$$

By the following result of Beauville [51], the quotient of these groups controls the kernel of π^* of Brauer groups.

Theorem 1.3.14 *Let S be an Enriques surface over an algebraically closed field $\Bbbk$ of characteristic $p \neq 2$ and let $\pi : X \to S$ be its K3-cover. Then,*

$$\operatorname{Ker}\left(\operatorname{Br}(S) \xrightarrow{\pi^*} \operatorname{Br}(X)\right) \cong \operatorname{Ker}(\mathrm{Nm})/(\mathrm{id} - \sigma^*)\operatorname{Pic}(X). \tag{1.3.11}$$

Proof Let us only sketch the proof and refer to [51] for details. As in the proof of the previous proposition, the starting point is the Hochschild–Serre spectral sequence (1.3.9). From $E_2^{2,0} = H^2(G, \mathbb{G}_m) = 0$, we obtain an isomorphism

$$\operatorname{Ker}\left(\operatorname{Br}(S) \xrightarrow{\pi^*} \operatorname{Br}(X)\right) \cong E_\infty^{1,1} = \operatorname{Ker}(E_2^{1,1} \xrightarrow{d_2} E_2^{3,0}).$$

Using periodicity of the group cohomology of G, we find that $E_2^{3,0} = H^3(G, \mathbb{G}_m)$ is isomorphic to $H^1(G, \mathbb{G}_m) = \operatorname{Hom}(G, \mathbb{G}_m) = G^D$, the Cartier dual group scheme. If we denote by ψ the endomorphism of $\operatorname{Pic}(X)$ that is defined by $\mathcal{L} \mapsto \mathcal{L} \otimes \sigma^*(\mathcal{L})$, then $E_2^{1,1} = H^1(G, \operatorname{Pic}(X))$ is isomorphic to $\operatorname{Ker}(\psi)/\operatorname{Im}(\mathrm{id} - \sigma^*)$. Since we have $\pi^* \operatorname{Nm}(\mathcal{L}) = \psi(\mathcal{L})$ for all $\mathcal{L} \in \operatorname{Pic}(X)$, we find that the norm homomorphism maps $\operatorname{Ker}(\psi)$ to $\operatorname{Ker}(\pi^* : \operatorname{Pic}(S) \to \operatorname{Pic}(X))$, and that the latter is canonically isomorphic to $G^D \cong \boldsymbol{\mu}_2$ by (the proof of) Proposition 1.3.13. Since $\mathrm{Nm} \circ (\mathrm{id} - \sigma^*) = 0$, the norm induces a homomorphism $H^1(G, \operatorname{Pic}(X)) \to \operatorname{Ker}(\pi^* : \operatorname{Pic}(S) \to \operatorname{Pic}(X)) \cong G^*$. By [51, Lemma 4.2], which is a non-trivial computation, this homomorphism coincides with the differential d_2. $\square$

Remark 1.3.15 Let S be an Enriques surface over the complex numbers. In this case, Beauville [51] showed that the kernel (1.3.11) is non-trivial if and only if there exists an invertible sheaf $\mathcal{L} \in \operatorname{Pic}(X)$ with $\sigma^*\mathcal{L} \cong \mathcal{L}^{-1}$, whose self-intersection satisfies $\mathcal{L}^2 \equiv 2 \mod 4$. From this, he deduced that over the complex numbers, the locus of Enriques surfaces inside their moduli space, where the kernel (1.3.11) is non-trivial, forms an infinite and countable union of non-empty hypersurfaces, whereas this kernel is trivial for a very general Enriques surface. We refer to [247] for more information in the case where the K3-cover is a Kummer surface.

Remark 1.3.16 Given a K3 surface X, one may ask in how many ways it can be realized as the canonical cover of an Enriques surface. In other words, we ask how many conjugacy classes of fixed-point-free involutions there are in its automorphism group. We will come back to this in Section 10.7 from Volume II.

1.4 Cohomological Invariants

In this section, we will determine the following fundamental invariants of an Enriques surface: the Betti and Hodge numbers, the fundamental group, the de Rham, Hodge–Witt, and crystalline cohomology groups, as well as the cohomology of the tangent sheaf. On our way, we will also study the degeneration behavior of the Frölicher spectral sequence from Hodge to de Rham cohomology, as well as the slope spectral sequence from Hodge–Witt to crystalline cohomology.

Over the complex numbers, an Enriques surface S can also be considered merely as a topological or differentiable 4-manifold, and we may consider its singular cohomology and homology groups.

Theorem 1.4.1 *Let S be an Enriques surface over $\Bbbk = \mathbb{C}$. Then, as a topological 4-manifold, S has the following singular cohomology and homology groups:*

i	0	1	2	3	4
$H_i(S, \mathbb{Z})$	$\mathbb{Z}$	$(\mathbb{Z}/2\mathbb{Z})$	$\mathbb{Z}^{10} \oplus (\mathbb{Z}/2\mathbb{Z})$	0	$\mathbb{Z}$
$H^i(S, \mathbb{Z})$	$\mathbb{Z}$	0	$\mathbb{Z}^{10} \oplus (\mathbb{Z}/2\mathbb{Z})$	$(\mathbb{Z}/2\mathbb{Z})$	$\mathbb{Z}$

Moreover, the topological fundamental group is isomorphic to $(\mathbb{Z}/2\mathbb{Z})$ and the universal cover of S is the K3-cover.

Proof The assertions about singular (co-)homology for $i = 0, 4$ follow from the fact that S is a compact, orientable, and topological 4-manifold. Using $h^1(\mathcal{O}_S) = h^2(\mathcal{O}_S) = 0$ and taking cohomology in the exponential sequence (0.10.9), we find $H^2(S, \mathbb{Z}) \cong \mathrm{Pic}(S)$. From this, we obtain the statement about H^2 using Theorem 1.2.1 and Theorem 1.2.7. Since $b_1(S) = 0$, we obtain $H^1(S, \mathbb{Z}) = 0$, and then, the universal coefficient formula yields $H_1(S, \mathbb{Z}) \cong \mathrm{Tors}(H^2(S, \mathbb{Z})) \cong \mathbb{Z}/2\mathbb{Z}$. The remaining (co-)homology groups can be computed using the isomorphisms $H^i(S, \mathbb{Z}) \cong H_{4-i}(S, \mathbb{Z})$ induced from Poincaré duality.

Let $\pi : X \to S$ be the K3-cover of S. We already know from Proposition 1.1.9 that a K3 surface is algebraically simply connected. The fact that it is a simply connected 4-manifold if $\Bbbk = \mathbb{C}$ is much harder to prove (see, for example, [43, Corollary (8.6)] or [320, Chapter 7, Theorem 1.1]). It follows that X is the universal cover of S, and since π is of degree 2, we find $\pi_1(S) \cong \mathbb{Z}/2\mathbb{Z}$. □

Remark 1.4.2 We will construct in Section 5.3 a moduli space for complex Enriques surfaces that is connected. Therefore, by a theorem of Ehresmann, all Enriques surfaces are diffeomorphic as differentiable 4-manifolds, and in particular, homeomorphic as topological spaces, see Corollary 5.3.10. We refer to [580] for the homotopy type, homeomorphism type, and the smooth structures of the manifold underlying an Enriques surface - for example, there exist infinitely many distinct smooth structures on the topological manifold underlying an Enriques surface.

Let us now work again over arbitrary algebraically closed ground fields. We start with a result that holds for all smooth and proper surfaces.

Proposition 1.4.3 *Let X be a smooth and proper surface over a field $\Bbbk$. Then,*

1. $\sum_j (-1)^j h^{i,j}(X) = \chi(\mathcal{O}_X)$ *for* $i = 0, 2$,
2. $\sum_j (-1)^j h^{1,j}(X) = 2\chi(\mathcal{O}_X) - c_2(X)$,
3. $\sum_{i,j} (-1)^{i+j} h^{i,j}(X) = c_2(X)$,
4. $\sum_i (-1)^i h^i_{DR}(X) = c_2(X)$.

Proof Assertion (1) follows from the definition and Serre duality and (2) follows from Riemann–Roch theorem applied to the sheaf Ω_X^1. Assertion (3) follows from (1) and (2). Finally, (4) follows from (3) and the existence of the Frölicher spectral sequence (degeneracy at E_1 is not needed) from Hodge to de Rham cohomology (0.10.35). ◻

Before turning to Enriques surfaces, we need a couple of results about K3 surfaces.

Theorem 1.4.4 *Let X be a K3-surface over an algebraically closed field $\Bbbk$. Then,*

$$\Theta_{X/\Bbbk} \cong \Omega^1_{X/\Bbbk}$$

and

$$h^{i,j}(X) = h^{j,i}(X) = \begin{cases} 1 & \text{if } (i,j) = (0,0), (2,0), (0,2), (2,2), \\ 20 & \text{if } (i,j) = (1,1), \\ 0 & \text{otherwise.} \end{cases}$$

Moreover, the étale fundamental group of X is trivial, that is, X is algebraically simply connected.

Proof The natural multiplication map $\bigwedge^r \Omega_X^1 \otimes \bigwedge^{2-r} \Omega_X^1 \to \bigwedge^2 \Omega_X^1$ is a perfect pairing for any r by [294, Exercise II.5.16]. We thus obtain isomorphisms $\Omega_X^1 \cong \Theta_X \otimes \omega_X \cong \Theta_X$, where the last isomorphism uses that $\omega_X \cong \mathcal{O}_X$. This implies that for all j

$$h^j(X, \Omega_X^1) = h^{2-j}(X, \omega_X \otimes (\Omega_X^1)^\vee) = h^{2-j}(X, \Omega_X^1).$$

By a highly non-trivial result of Rudakov and Shafarevich [626] (see also [440], [563] and [491] for different proofs), we have

$$h^0(X, \Theta_X) = 0.$$

(In characteristic zero, this is an easy consequence of the isomorphism $\Omega_X^1 \cong \Theta_X$ and the Hodge symmetry $h^{0,1} = h^{1,0}$, but this symmetry is known to fail in general for surfaces in positive characteristic.) Thus, $h^j(\Omega_X^1) = 0$ if $j \neq 1$. Using Serre duality and the definition of a K3 surface, we also have $h^j(\Omega_X^0) = h^j(\Omega_X^2) = 1$ for $j = 1, 2$ and $h^1(\Omega_X^0) = h^1(\Omega_X^2) = 0$. Using $e(X) = c_2(X) = 24$ (see Table A.2.5) and Proposition 1.4.3, the remaining Hodge numbers follow. The assertion on the étale fundamental group was already shown in Proposition 1.1.9. ◻

The previous result allows us to compute the cohomology of tangent and cotangent bundle of those Enriques surfaces, whose K3-cover is a smooth K3 surface.

Corollary 1.4.5 *Let S be an Enriques surface over an algebraically closed field $\Bbbk$ of characteristic $p \geq 0$. Assume that $p \neq 2$ or that S is a $\boldsymbol{\mu}_2$-surface. Then,*

$$h^{1,j}(S) = h^j(\Omega^1_{S/\Bbbk}) = h^j(\Theta_{S/\Bbbk}) = \begin{cases} 10 & \text{if } j = 1, \text{ and} \\ 0 & \text{otherwise.} \end{cases}$$

Moreover, the étale fundamental group of S is isomorphic to $(\mathbb{Z}/2\mathbb{Z})$.

Proof Let $\pi : X \to S$ be the K3-cover of S, which is étale of degree 2. Then, X is a K3 surface by Proposition 1.3.3, and since K3 surfaces are algebraically simply connected by Proposition 1.1.9, the assertion on étale fundamental groups follows.

Since π is étale, we have $\pi^*\Omega^i_S \cong \Omega^i_X$ for all i, and then, the projection formula yields

$$\pi_*(\Omega^i_X) = \pi_*\left(\pi^*\Omega^i_S\right) = \Omega^i_S \otimes \pi_*(O_X).$$

From this, we obtain a short exact sequence

$$0 \to \Omega^i_S \to \pi_*\Omega^i_X \to \Omega^i_S \otimes \omega_S \to 0. \tag{1.4.1}$$

If $p \neq 2$, then this sequence splits. Then, taking cohomology and using Theorem 1.4.4, we find $h^0(\Omega^1_S) = h^2(\Omega^1_S) = 0$. Using $c_2(S) = 12$, $\chi(O_S) = 1$ (see Table **??**), and Proposition 1.4.3, we find $h^1(\Omega^1_S) = 10$. Taking cohomology in (1.4.1) again allows us to compute $h^j(\Omega^1_S \otimes \omega_S)$, which, by Serre duality, is equal to $h^{2-j}(\Theta_S)$.

If S is a $\boldsymbol{\mu}_2$-surface, then $\omega_S \cong O_S$. As in the proof of Theorem 1.4.4, we find $\Omega^1_S \cong \Theta_S$. Taking cohomology in (1.4.1), we find $h^0(\Omega^1_S) = h^2(\Omega^1_S) = 0$. From this, we compute $h^1(\Omega^1_S) = \frac{1}{2}h^1(\Omega^1_X) = 10$. □

The analogous results for Enriques surfaces in characteristic 2 that are not $\boldsymbol{\mu}_2$-surfaces are more complicated. We start with the following result, which is independently due to Illusie [329] and Lang [432].

Theorem 1.4.6 *Let S be an Enriques surface over an algebraically closed field $\Bbbk$ of characteristic p. Then, all regular 1-forms are d-closed, that is, $d\alpha = 0$ for all $\alpha \in H^0(S, \Omega^1_{S/\Bbbk})$.*

Proof We have to show that the differential

$$d_1 : E_1^{1,0} = H^0(S, \Omega^1_S) \to E_1^{2,0} = H^0(S, \Omega^2_S)$$

in the Frölicher spectral sequence is zero and we shall follow [432]. If S is a classical Enriques surface, then $h^0(\Omega^2_S) = 0$, and if S is a $\boldsymbol{\mu}_2$-surface, then $h^0(\Omega^1_S) = 0$. In these cases, the assertion is trivially true, and we may assume that $p = 2$ and that S is an $\boldsymbol{\alpha}_2$-surface. Then, we have $\omega_S \cong O_S$ and $\Omega^1_S \cong \Theta_S$. Seeking a contradiction, we assume that d_1 is non-zero. Then, it follows from $h^0(\Omega^2_S) = 1$ that d_1 is surjective. Thus, by Poincaré duality in de Rham cohomology, the transpose

$$d_1^\vee : E_1^{0,2} = H^2(S, O_S) \to E_1^{1,2} = H^2(S, \Omega^1_S)$$

of d_1 is injective.

Let $a \in H^1(S, O_S)$ be a non-zero element. We have already seen in the proof of Theorem 1.2.1 that the cup-product $a \cup a \in H^2(S, O_S)$ is non-zero. Using $p = 2$, we compute $d_1^\vee(a \cup a) = a \cup d_1^\vee(a) + d_1^\vee(a) \cup a = 2(a \cup d_1^\vee a) = 0$, which contradicts the injectivity of $d_1^\vee$. Thus, d_1 is not surjective, whence, the zero-map, and so, all regular 1-forms are d-closed, also for $\boldsymbol{\alpha}_2$-surfaces. □

Remark 1.4.7 We will see in Proposition 1.4.12 below that the slope spectral sequence of an Enriques surface degenerates at E_1, which also implies Theorem 1.4.6, see Remark 0.10.17. Over the complex numbers, it follows from Stokes' theorem that holomorphic 1-forms on a compact complex surface are d-closed (even without the Kähler assumption), see [43, Lemma IV.2.1]. On the other hand, Mumford [536] gave examples of smooth projective surfaces in positive characteristic with regular 1-forms that are not closed.

Corollary 1.4.8 *Let S be an Enriques surface over an algebraically closed field $\mathbb{k}$ of characteristic $p \geq 0$. Then,*

$$h^1_{\mathrm{DR}}(S) \;=\; \begin{cases} 0 & \text{if } p \neq 2, \\ 1 & \text{otherwise.} \end{cases}$$

Proof Since all 1-forms are d-closed by Theorem 1.4.6, Oda's results [567] imply that $H^1_{\mathrm{DR}}(S/\mathbb{k})$ is isomorphic to the Dieudonne module of the p-torsion subgroup scheme of $\mathbf{Pic}^{\tau}_{S/\mathbb{k}}$, see also the proof of [432, Theorem 2]. By Theorem 1.2.1, this module is zero if $p \neq 2$ and a 1-dimensional $\mathbb{k}$-vector space if $p = 2$. □

Corollary 1.4.9 *Let S be an Enriques surface over an algebraically closed field $\mathbb{k}$ of characteristic $p = 2$ that is not a $\boldsymbol{\mu}_2$-surface. Then,*

$$h^{1,j}(S) \;=\; h^j(S,\,\Omega^1_{S/\mathbb{k}}) \;=\; \begin{cases} 12 & \text{if } j = 1, \text{ and} \\ 1 & \text{otherwise.} \end{cases}$$

Moreover:

1. *if S is an $\boldsymbol{\alpha}_2$-surface, then $h^j(S, \Theta_{S/\mathbb{k}}) = h^j(S, \Omega^1_{S/\mathbb{k}})$ for all j, and*
2. *if S is a classical Enriques surface, then*

$$h^j(S,\,\Theta_{S/\mathbb{k}}) \;=\; \begin{cases} 10 + 2a & \text{if } j = 1, \text{ and} \\ a & \text{otherwise,} \end{cases}$$

for some integer $0 \leq a \leq 1$. If $a \neq 0$, then the K3-cover $X \to S$ is a non-normal and rational surface. Thus, in this case, S is a Zariski surface via its K3-cover.

Proof Passing to E_2-terms of the Frölicher spectral sequence (0.10.38), we obtain an inclusion of $\mathrm{Ker}(d : H^0(\Omega^1_S) \to H^0(\Omega^2_S))$ into $H^1_{\mathrm{DR}}(S)$. The former coincides with $H^0(\Omega^1_S)$ by Theorem 1.4.6 and it is non-zero by Proposition 1.3.8. Since $H^1_{\mathrm{DR}}(S)$ is one-dimensional by the previous corollary, we find $h^0(\Omega_S) = 1$. Next, Serre duality gives $h^2(\Omega^1_S) = 1$, and using Proposition 1.4.3, we find $h^1(\Omega^1_S) = 12$.

If S is an $\boldsymbol{\alpha}_2$-surface, then $\omega_S \cong \mathcal{O}_S$, which gives $\Omega^1_S \cong \Theta_S$ as in the proof of Theorem 1.4.4. In particular, we obtain $h^j(\Omega^1_S) = h^j(\Theta_S)$ for all j.

If S is classical, then we set $a := h^0(\Theta_S)$. Using Serre duality and Proposition 1.4.3, we conclude $h^1(\Theta_S) = 10 + a$ and $h^2(\Theta_S) = a$. It remains to show that $a \leq 1$. Let $\pi : X \to S$ be the K3-cover and let A be the conductrix, which could be zero. Dualizing the short exact sequence (1.3.8) and using (1.3.5), we find a closed subscheme Z (empty or zero-dimensional) with ideal sheaf $\mathcal{I}_Z$ and a short exact sequence

$$0 \to \omega_S(2A) \to \Theta_S \to \mathcal{I}_Z(-2A) \to 0. \tag{1.4.2}$$

The singular locus of X lies over $Z \cup A$, and thus, since X is not smooth by Proposition 1.3.3, Z or A cannot be both empty. From this, we deduce $a = h^0(\Theta_S) = h^0(\omega_S(2A))$. In particular, if $a \neq 0$, then $A \neq 0$ and X is not normal. In this case, X is a rational surface by Theorem 1.3.5 and since $X \to S$ is purely inseparable of degree p, it follows that S is a Zariski surface via X. Let $B = 2A$ be the biconductrix, and then, taking cohomology in the short exact sequence

$$0 \to \omega_S \to \omega_S(B) \to \omega_B \to 0$$

and, using that S is classical, we find $h^0(\omega_S(B)) = h^0(B, \omega_B)$. We will show in Proposition 4.10.6 that A is a proper part of reducible fibers of a genus one fibration on S. This then implies that $|2A + K_S|$ does not have a moving part varying in a linear system of positive dimension. Thus, we find $a = h^0(\omega_S(B)) \leq 1$. □

Enriques surfaces in characteristic $\neq 2$ have no nonzero global vector fields. On the other hand, $\boldsymbol{\alpha}_2$-surfaces always have nonzero global vector fields, whereas $\boldsymbol{\mu}_2$-surfaces have no nonzero global vector fields. Concerning vector fields on classical Enriques surfaces in characteristic 2, we have just established $h^0(\Theta_S) \leq 1$ and note that we encountered classical Enriques surfaces with vector fields in Proposition 1.2.9 in connection with Lang's proof of Theorem 1.2.7. The following result is due to Ekedahl and Shepherd–Barron [213].

Theorem 1.4.10 *Let S be a classical Enriques surface over an algebraically closed field of characteristic* 2. *Then,* $H^0(S, \Theta_S) \neq 0$ *if and only if S contains a configuration of smooth rational curves that defines a root basis of type* $T_{3,3,4}$, $T_{2,4,5}$, *or* $T_{2,3,7}$ *inside* $\mathrm{Num}(S)$.

Proof We will only mention the ingredients of the proof: let $\pi : X \to S$ be the K3-cover. If X is normal, then we have $h^0(\Theta_S) = 0$ by Corollary 1.4.9. Thus, if $h^0(\Theta_S) \neq 0$, then X is non-normal and thus, there is a non-zero conductrix A. More precisely, A is equal to the divisorial part of the scheme of zeros of a regular 1-form on S. Using (1.4.2), we conclude that $h^0(\Theta_S) \neq 0$ if and only if $h^0(2A + K_S) \neq 0$. From here, the proof consists of a careful analysis of possible divisors A and finding all possible A for which this condition is satisfied. We will give more details in Section **??** when discussing *exceptional Enriques surfaces*. □

The following table summarizes our findings. As usual, we set $h^{i,j}(S) = h^j(\Omega^i_{S/\Bbbk})$ and Serre duality gives $h^{i,j}(S) = h^{2-i,2-j}(S)$. Moreover, we set $t^j := h^j(\Theta_{S/\Bbbk})$. The integer a is the one from Corollary 1.4.9, where we also established $0 \leq a \leq 1$.

Remark 1.4.11 Let S be an Enriques surface over an algebraically closed $\Bbbk$ of characteristic $p > 0$. If $t^2(S) = 0$, then the deformation theory implies the existence of a

$\mathbf{Pic}^{\tau}_{S/\Bbbk}$	$h^{1,0}$	$h^{0,1}$	$h^{2,0}$	$h^{1,1}$	$h^{0,2}$	t^0	t^1	t^2	h^1_{DR}	π_1
characteristic $p \neq 2$:										
$\boldsymbol{\mu}_2 \cong \mathbb{Z}/2\mathbb{Z}$	0	0	0	10	0	0	10	0	0	$\mathbb{Z}/2\mathbb{Z}$
characteristic $p = 2$:										
$\mathbb{Z}/2\mathbb{Z}$	1	0	0	12	0	a	$10+2a$	a	1	$\{e\}$
$\boldsymbol{\mu}_2$	0	1	1	10	1	0	10	0	1	$\mathbb{Z}/2\mathbb{Z}$
$\boldsymbol{\alpha}_2$	1	1	1	12	1	1	12	1	1	$\{e\}$

Table 1.2 Differential invariants of Enriques surfaces

formal lifting of S over the Witt ring $W = W(\Bbbk)$. If moreover $h^{2,0}(S) = 0$ holds true, then even algebraic lifts over W are easy to establish. We note that both conditions are satisfied if $p \neq 2$. On the other hand, $\boldsymbol{\alpha}_2$-surfaces do not even lift over W_2, but there always exist algebraic lifts over ramified extensions of W. We will come back to this in Theorem 5.11.5 and Volume II.

Next, in order to compute the crystalline and Hodge–Witt cohomology groups of an Enriques surface, let us remind the reader that we computed Serre's Witt vector cohomology groups $H^i(W\mathcal{O}_S)$ already in Corollary 1.2.5 using results on $\mathbf{Pic}^{\tau}_{S/\Bbbk}$. This is already sufficient the degeneration behavior of the slope spectral sequence (0.10.45).

Proposition 1.4.12 *The slope spectral sequence of an Enriques surface over an algebraically closed field of positive characteristic degenerates at E_1.*

Proof By Theorem 0.10.16, we have to show that $H^j(W\Omega^i_S)$ is a finitely generated W-module for all i, j. Moreover, by Remark 0.10.17, it suffices to check that $H^2(W\mathcal{O}_S)$ is finitely generated, which is true by Corollary 1.2.5. □

Next, we compute the crystalline cohomology groups, which should be compared to the singular cohomology groups in the complex case in Theorem 1.4.1.

Theorem 1.4.13 *, Let S be an Enriques surface over an algebraically closed field $\Bbbk$ of characteristic $p > 0$ and let $W = W(\Bbbk)$ be the ring of Witt vectors. Then, S has the following crystalline and de Rham cohomology groups:*

		0	1	2	3	4
$p \neq 2$	$H^i(S/W)$	W	0	W^{10}	0	W
	$H^i_{\mathrm{DR}}(S)$	$\Bbbk$	0	$\Bbbk^{10}$	0	$\Bbbk$
$p = 2$	$H^i(S/W)$	W	0	$W^{10} \oplus \Bbbk$	$\Bbbk$	W
	$H^i_{\mathrm{DR}}(S)$	$\Bbbk$	$\Bbbk$	$\Bbbk^{11}$	0	$\Bbbk$

Table 1.3 Crystalline and de Rham cohomology of Enriques surfaces

Moreover, if $p \neq 2$ or S is a $\boldsymbol{\mu}_2$-surface, then the étale fundamental group is isomorphic to $\mathbb{Z}/2\mathbb{Z}$. In the remaining cases, S is algebraically simply connected.

Proof The assertions about $H^i(S/W)$ for $i = 0, 4$ follow from the fact that S is a smooth and proper surface. Also, since $\mathbf{Pic}^\circ_S$ is zero-dimensional, we find $H^1(S/W) = 0$.

First, assume that $p \neq 2$. Since $\mathrm{NS}(S)$ has no p-torsion, also $P^1H^1(S/W)$ has no p-torsion by (0.10.48). And since $H^2(WO_S) = 0$ by Corollary 1.2.5, we have $H^2(S/W) = P^1H^2(S/W)$, and conclude that $H^2(S/W)$ has no p-torsion. Since it is of rank $b_2 = 10$, we find $H^2(S/W) \cong W^{10}$. From this, Poincaré duality gives $H^3(S/W) = 0$.

Next, assume that $p = 2$. If S is classical, then we have $H^2(WO_S) = 0$ by Corollary 1.2.5 and conclude $H^2(S/W) = P^1H^2(S/W)$ as before. The p-torsion of $P^1H^2(S/W)$ is computed from that of $\mathrm{NS}(S)$, and we find $P^1H^2(S/W)_{\mathrm{tors}} \cong \Bbbk$. Again, the rank of $H^2(S/W)$ is equal to $b_2 = 10$, and we conclude $H^2(S/W) \cong W^{10} \oplus \Bbbk$. Thus, we obtain $H^3(S/W) \cong \Bbbk$ by Poincaré duality.

Finally, assume that S is non-classical. Since $\mathrm{NS}(S)$ has no p-torsion, neither has $P^1H^2(S/W)$ by (0.10.48). By Corollary 1.2.5, we have $H^2(WO_S) = \Bbbk$. Now, the V-torsion of $H^2(WO_S)$ injects into the torsion of $H^2(W/S)$, and the torsion of $H^2(W/S)$ surjects onto the torsion of $H^2(WO_S)$, see also [329, Section II.(6.7.2)]. This implies that the torsion of $H^2(S/W)$ is $\Bbbk$, and since $b_2 = 10$, we find $H^2(S/W) \cong W^{10} \oplus \Bbbk$. Using Poincaré duality, we find $H^3(S/W) = \Bbbk$.

In all cases, the de Rham cohomology groups can easily be computed from the crystalline cohomology groups using the universal coefficient formula (0.10.43). □

Corollary 1.4.14 *Let S be an Enriques surface over an algebraically closed field $\Bbbk$. Then, the exotic torsion of S is zero.*

Proof Applying the theorem and Corollary 1.2.5 to (0.10.49), it follows that the torsion of $H^2(S/W)$ is divisorial and thus, the exotic torsion is zero. □

Corollary 1.4.15 *Let S be an Enriques surface over an algebraically closed field $\Bbbk$.*

1. *The Frölicher spectral sequence of S from Hodge to de Rham cohomology degenerates at E_1 if and only if S not an $\boldsymbol{\alpha}_2$-surface.*
2. *If S is an $\boldsymbol{\alpha}_2$-surface, then the Frölicher spectral sequence degenerates at E_2 and the non-zero differentials on the E_1-page are*

$$d_1^{0,1} : H^1(S, O_S) \to H^1(S, \Omega^1_{S/\Bbbk}) \quad \textit{and} \quad d_1^{1,1} : H^1(S, \Omega^1_{S/\Bbbk}) \to H^1(S, \Omega^2_{S/\Bbbk}).$$

Proof The Frölicher spectral sequence of a smooth and proper variety degenerates at E_1 if and only if $h^n_{\mathrm{DR}} = \sum_{i+j=n} h^{i,j}$ holds for all n, see Proposition 0.10.9. Using this, the first claims follows from inspecting Table 1.2 and Table 1.3.

Let S be an $\boldsymbol{\alpha}_2$-surface. Since global 1-forms on S are d-closed by Theorem 1.4.6, it follows that the only differentials on the E_1-page of the Frölicher spectral sequence that can be non-zero are $d_1^{0,1}$ and $d_1^{1,1}$. By the established first claim, at least one of them must be non-zero. Since $d_1^{0,1}$ and $d_1^{1,1}$ are dual maps via Serre duality, it follows that if one is non-zero, then so is the other. Therefore, both maps must be non-zero. □

For more details about the Frölicher spectral sequence of an Enriques surface and its differentials, we refer the interested reader to [329, Proposition II.7.3.8] and its proof. We end this section by computing the Hodge–Witt cohomology groups

$$H_W^{i,j}(S) := H^j(S, W\Omega^i_{S/\Bbbk}),$$

that is, the cohomology groups of the de Rham-Witt complex, which we discussed in Section 0.10.

Proposition 1.4.16 *Let S be an Enriques surface over an algebraically closed field $\Bbbk$ of characteristic $p > 0$ and let $W = W(\Bbbk)$ be the ring of Witt vectors. Then, we have $H_W^{0,0} \cong H_W^{2,2} \cong W$, as well as*

$\mathbf{Pic}^\tau_{S/\Bbbk}$	$H_W^{1,0}$	$H_W^{0,1}$	$H_W^{2,0}$	$H_W^{1,1}$	$H_W^{0,2}$	$H_W^{2,1}$	$H_W^{1,2}$
characteristic $p \neq 2$:							
$\boldsymbol{\mu}_2 \cong \mathbb{Z}/2\mathbb{Z}$	0	0	0	W^{10}	0	0	0
characteristic $p = 2$:							
$\mathbb{Z}/2\mathbb{Z}$	0	0	0	$W^{10} \oplus \Bbbk$	0	0	$\Bbbk$
$\boldsymbol{\mu}_2$	0	0	0	W^{10}	$\Bbbk$	$\Bbbk$	0
$\boldsymbol{\alpha}_2$	0	0	0	W^{10}	$\Bbbk$	0	$\Bbbk$

Table 1.4 Hodge–Witt cohomology of Enriques surfaces

Proof Since S is a smooth and proper surface, we have $H_W^{0,0} \cong H_W^{2,2} \cong W$. Next, by partial degeneration of the slope spectral sequence (0.10.45) we obtain a short exact sequence of free W-modules

$$0 \to H^0(S, W\Omega^1_S) \to H^1(S/W) \to H^1(S, W\mathcal{O}_S) \to 0,$$

see (0.10.47). Since $H^1(S/W) = 0$ by Table 1.3, we conclude $H_W^{1,0} = H_W^{0,1} = 0$. Next, the $\mathbf{F}$-isocrystal $H^2(S/W) \otimes K$ is of slope 1 by Corollary 1.2.5, which implies that $H^0(W\Omega^2_S) \otimes K$ is zero, since the latter is isomorphic to the sub-$\mathbf{F}$-isocrystal of $H^2(S/W) \otimes K$ of slope 2. All $H_W^{i,0}$ are free W-modules of finite rank by (0.10.46), from which we conclude that $H_W^{2,0} = 0$. Moreover, we already computed $H_W^{0,2} = H^2(W\mathcal{O}_S)$ in Corollary 1.2.5.

By Proposition 1.4.12, the slope spectral sequence of S degenerates at E_1, and thus, looking up $H^2(S/W)$ in the Table 1.3, as well as using the already computed groups $H_W^{2,0}$ and $H_W^{0,2}$, this enables us to compute $H_W^{1,1}$.

It remains to compute $H_W^{2,1}$ and $H_W^{1,2}$. Again, using that the slope spectral sequence degenerates at E_1, we obtain a short exact sequence

$$0 \to H^1(S, W\Omega^2_S) \to H^3(S/W) \to H^2(S, W\Omega^1_S) \to 0$$

and $H^3(S/W)$ is given by Table 1.3. If $p \neq 2$, then this immediately implies $H_W^{2,1} = H_W^{1,2} = 0$. We may thus assume that $p = 2$. From here, we sketch only the proof and refer to [329, Section II.7.3] for details. Since $p = 2$, we have $H^3(S/W) \cong \Bbbk$, which implies that one of the two desired cohomology groups is isomorphic to $\Bbbk$, whereas the other one is zero. Therefore, it suffices to compute $H^1(W\Omega_S^2)$. Now, Verschiebung V induces a short exact sequence $0 \to W\Omega_S^2 \to W\Omega_S^2 \to W\Omega_S^2/VW\Omega_S^2 \to 0$. Taking cohomology, using $H^2(W\Omega_S^2) \cong W$ and using the Cartier operator C in Hodge–Witt cohomology, we obtain an isomorphism

$$H^1(S, W\Omega_S^2)/VH^1(S, W\Omega_S^2) \cong \varprojlim_C H^1(S, \Omega_S^2).$$

Next, Frobenius $\mathbf{F}$ is an automorphism of $H_W^{2,1}$ and since $V = p\mathbf{F}^{-1}$, the term on the left is isomorphic to $H_W^{2,1}/pH_W^{2,1}$. Thus, if S is classical, then $H^1(\Omega_S^2) = 0$, which implies $H_W^{2,1}/pH_W^{2,1} = 0$ and we conclude $H_W^{2,1} = 0$. In the case where S is not classical the C-limit over $H^1(\Omega_S^2)$ is dual to the $\mathbf{F}$-limit over $H^1(\mathcal{O}_S)$. This is isomorphic to $\Bbbk$ if S is a $\boldsymbol{\mu}_2$-surface and it is zero if S is an $\boldsymbol{\alpha}_2$-surface. Thus, we find $H_W^{2,1} \cong \Bbbk$ (resp. = 0) In the case where S is a $\boldsymbol{\mu}_2$-surface (resp. an $\boldsymbol{\alpha}_2$-surface). □

We end this section by discussing F-split and ordinary Enriques surfaces in positive characteristic – these behave particularly nicely when it comes to degeneration of the Frölicher spectral sequence from Hodge to de Rham cohomology, in view of their crystalline cohomology, and in view of the vanishing theorems that we will discuss in Section 2.1.

Let X be a variety over a perfect field $\Bbbk$ of characteristic $p > 0$ and let $\mathbf{F} : X \to X$ be the absolute Frobenius morphism. Then, X is called *Frobenius split*, or *F-split* for short, if the injective homomorphism $\mathcal{O}_X \to \mathbf{F}_*\mathcal{O}_X$ splits as a homomorphism of $\mathcal{O}_X$-modules. We remind the reader that we already discussed F-split singularities in Section 0.4 and refer the reader to [97] for an introduction to F-split varieties.

Proposition 1.4.17 *Let X be a smooth and proper variety over an algebraically closed field $\Bbbk$ of characteristic $p > 0$ that is F-split. Then, $h^0(X, \omega_X^{\otimes(1-p)}) \neq 0$. In particular, the Kodaira dimension of X satisfies $\kappa(X) \leq 0$.*

Proof We only sketch the proof and refer to [97, Section 1.3] for details: by definition of $-^!$, there is an isomorphism

$$\mathcal{H}om(\mathbf{F}_*\mathcal{O}_X, \mathcal{O}_X) \cong \mathbf{F}_*(\mathbf{F}^!\mathcal{O}_X).$$

Using duality for finite and flat morphisms and suitable trace maps, one can show that there exists an isomorphism

$$\mathbf{F}^!\mathcal{O}_X \cong \omega_X^{\otimes(1-p)}.$$

Thus, if X is F-split, then a choice of splitting $\varphi : \mathbf{F}_*\mathcal{O}_X \to \mathcal{O}_X$ yields a non-zero section of $\mathbf{F}_*(\mathbf{F}^!\mathcal{O}_X)$, which implies that $h^0(\mathbf{F}^!\mathcal{O}_X) = h^0(\omega_X^{\otimes(1-p)}) \neq 0$. □

In particular, surfaces of Kodaira dimension zero as classified in Section 1.1 have a chance of being F-split. We refer to Section 0.10 for a discussion of ordinary varieties.

Theorem 1.4.18 *Let X and S be smooth and proper varieties in characteristic $p > 0$.*

1. *If X is an abelian variety or a K3 surface, then X is F-split if and only if X is ordinary.*
2. *If S is an Enriques surface, then:*
   a. *If $p \neq 2$, then S is F-split if and only if its K3-cover is F-split.*
   b. *If $p = 2$, then S is F-split if and only if S is a $\boldsymbol{\mu}_2$-surface. In this case, the K3-cover of S is a K3 surface that is ordinary or, equivalently, F-split.*

Proof If X is an abelian variety, then the equivalence of being ordinary and being F-split is well-known, see, for example, [505, Lemma 1.1].

Next, it follows from [97, Remarks 1.3.9] that a smooth, proper, and n-dimensional variety X is F-split if and only if the map

$$\mathbf{F}^* \;:\; H^n(X, \omega_X) \;\to\; H^n(X, \omega_X^{\otimes p})$$

induced by Frobenius on cohomology is non-zero.

Thus, if X is a K3 surface, then X is F-split if and only if $\mathbf{F}^* : H^2(\mathcal{O}_X) \to H^2(\mathcal{O}_X)$ is non-zero. We showed in Example 0.10.26 that this means that it is ordinary in degree 2. Since $H^1(\mathcal{O}_X) = 0$, it is ordinary in degree 1. We refer to this example for other characterizations of an ordinary K3 surface.

Now, let S be an Enriques surface in characteristic $p > 0$, say with K3-cover $\pi : X \to S$. First, assume that $p \neq 2$. Since π is a finite and étale morphism, we have $\pi^*\omega_S \cong \omega_X$ and thus, we obtain a commutative diagram

$$\begin{array}{ccc} H^2(X, \omega_X) & \xrightarrow{\mathbf{F}} & H^2(X, \omega_X^{\otimes p}) \\ \uparrow{\scriptstyle \pi^*} & & \uparrow{\scriptstyle \pi^*} \\ H^2(S, \omega_S) & \xrightarrow{\mathbf{F}} & H^2(S, \omega_S^{\otimes p}). \end{array}$$

Since π is a finite of degree prime to p, the vertical maps are split injections (via trace maps). On the other hand, the vector spaces on the left are one-dimensional, from which it is easy to see that the upper horizontal map is injective if and only if the lower horizontal map is. Using the above criterion for F-splitting, assertion (a) follows.

If $p = 2$ and S is an F-split Enriques surface, then Proposition 1.4.17 implies that $h^0(\omega_S^{-1}) \neq 0$, that is, S is a non-classical Enriques surface. Moreover, since $\mathcal{O}_S \to \mathbf{F}_*\mathcal{O}_S$ is a split injection, so is $\mathbf{F} : H^1(\mathcal{O}_S) \to H^1(\mathbf{F}_*\mathcal{O}_S)$, which implies that the Frobenius action on $H^1(\mathcal{O}_S)$ is injective. Thus, S is a $\boldsymbol{\mu}_2$-surface. Conversely, let S be a $\boldsymbol{\mu}_2$-surface. Then, the K3-cover $\pi : X \to S$ is an ordinary K3 surface by [143], see also Remark 1.3.12. In particular, $\mathbf{F}^* : H^2(X, \mathcal{O}_X) \to H^2(X, \mathcal{O}_X)$ is injective. Since π is a $\mathbb{Z}/2\mathbb{Z}$-torsor, we have a short exact sequence

$$0 \to \mathcal{O}_S \to \pi_* \mathcal{O}_X \to \mathcal{O}_S \to 0,$$

see Section 0.3. From this, it is easy to see that the injective $\mathbf{F}$-action on $H^2(X, \mathcal{O}_X)$ forces also the $\mathbf{F}$-action on $H^2(S, \mathcal{O}_S)$ to be injective. Since $\omega_S \cong \mathcal{O}_S$, it follows that S is F-split. □

Theorem 1.4.19 *Let S be an Enriques surface in characteristic $p > 0$. Then:*

1. *If $p \neq 2$, then S is ordinary (in the sense of Definition 0.10.22).*
2. *If $p = 2$, then S is ordinary if and only if S is not an $\boldsymbol{\alpha}_2$-surface.*

In particular, F-split Enriques surfaces are Bloch–Kato–Illusie–Raynaud ordinary, but the converse is not true in general.

Proof First, assume that $p \neq 2$ or that $p = 2$ and that S is classical. Taking cohomology in the short exact sequence

$$0 \to \mathcal{O}_S \to \mathbf{F}_* \mathcal{O}_S \xrightarrow{d} B\Omega^1_{S/\Bbbk} \to 0 \tag{1.4.3}$$

and using $h^1(\mathcal{O}_S) = h^2(\mathcal{O}_S) = 0$, we find $H^i(B\Omega^1_{S/\Bbbk}) = 0$ for all $i \geq 0$. If $p = 2$ and S is a $\boldsymbol{\mu}_2$-surface, then it is F-split by Theorem 1.4.18 and thus, (1.4.3) is split. Therefore, the maps $H^i(\mathcal{O}_S) \to H^i(\mathbf{F}_* \mathcal{O}_S)$ are injective for all $i \geq 0$, which implies that $H^i(B\Omega^1_S) = 0$ for all $i \geq 0$.

Since S is a surface, the Cartier operator induces a perfect pairing

$$\mathbf{F}_*(\mathcal{O}_S) \otimes \mathbf{F}_*(\Omega^2_{S/\Bbbk}) \to \Omega^2_{S/\Bbbk} = \omega_S$$

that is given by $(f, \omega) \mapsto C(f\omega)$. This induces a perfect pairing $B\Omega^1_S \otimes B\Omega^2_S \to \omega_S$, see the proofs of [354, Theorem 2.4.1]. and [505, Lemma 1.1]. Thus, a surface S that satisfies $H^i(B\Omega^1_S) = 0$ for all $i \geq 0$ automatically satisfies $H^i(B\Omega^2_S) = 0$ for all $i \geq 0$ and is thus ordinary in the sense of Bloch, Kato, Illusie, and Raynaud.

It remains to show if S is an $\boldsymbol{\alpha}_2$-surface over an algebraically closed field $\Bbbk$ of characteristic 2, then it is not ordinary: taking cohomology in (1.4.3) and using that the map $H^1(\mathcal{O}_S) \to H^1(\mathbf{F}_* \mathcal{O}_S)$ is zero, we obtain an isomorphism and an injection $\Bbbk \cong H^1(\mathbf{F}_* \mathcal{O}_S) \to H^1(B\Omega^1_S)$, which implies that S is not Bloch–Kato–Illusie–Raynaud ordinary. □

Remark 1.4.20 For varieties that are ordinary in the sense of Bloch, Kato, Illusie, and Raynaud, the Frölicher spectral sequence from Hodge to de Rham cohomology degenerates at E_1 by [332, Théorème IV.4.13]. Therefore, we obtain another proof that the Frölicher spectral sequences degenerates at E_1 for all Enriques surfaces that are not $\boldsymbol{\alpha}_2$-surfaces. By Corollary 1.4.15, we know that it does not degenerate at E_1 for $\boldsymbol{\alpha}_2$-surfaces.

If S is an Enriques surface in characteristic $p > 0$, then $H^1(S/W)$ and $H^3(S/W)$ are zero and torsion, respectively, see Theorem 1.4.13. Moreover, the Newton polygon and the Hodge polygon arising from $H^2(S/W)$ coincide and are straight lines. In particular, these crystalline cohomology groups do not give rise to interesting invariants. However, if $p \neq 2$ or $p = 2$ and S is a $\boldsymbol{\mu}_2$-surface, then the K3-cover $\pi : X \to S$ is a K3 surface and we can look at the height $h(X) := h(\widehat{\mathrm{Br}}_{X/\Bbbk})$ of the formal Brauer group. If $h(X) = 1$, then X is an ordinary K3 surface. Moreover, if $h(X) = \infty$, that is, if X is supersingular, then we may consider the Artin invariant $\sigma_0(X)$.

Theorem 1.4.21 *Let S be an Enriques surfaces in characteristic $p > 0$ and let $\pi : X \to S$ be its K3-cover:*

1. *If $p = 2$ and S is a $\boldsymbol{\mu}_2$-surface, then $h(X) = 1$, that is, X is an ordinary K3 surface.*
2. *If $p \geq 3$, then:*
   a. *either $h(X) \leq 6$,*
   b. *or else $h(X) = \infty$, that is, X is supersingular, and then, $\sigma_0(X) \leq 5$.*
3. *Conversely, if $p \geq 3$, then:*
   a. *for every $1 \leq h \leq 6$, there exists a K3 surface X with $h(X) = h$ in characteristic p that is the K3-cover of an Enriques surface.*
   b. *Every supersingular K3 surface X in characteristic p with $\sigma_0(X) \leq 5$ is the K3-cover of an Enriques surface.*

Proof The assertion for $p = 2$ is due to Crew [143], see also Theorem 1.4.18, and thus, we may assume $p \geq 3$.

We have $b_2(X) = 22$ since X is a K3 surface. Next, we have $\rho(S) = 10$ by Theorem 1.2.7, and the map $\pi^* : \mathrm{Pic}(S) \to \mathrm{Pic}(X)$ is injective up to torsion (see also Proposition 1.3.13) and we find $\rho(X) \geq 10$. Thus, if $h(X) < \infty$, then the Igusa–Artin–Mazur inequality (0.10.53) yields $h(X) \leq \frac{1}{2}(b_2(X) - \rho(X)) \leq 6$.

If $h(X) = \infty$, then the assertion $\sigma_0(X) \leq 5$ is due to Jang, see [347, Corollary 3.4].

The converse result (3) is also due to Jang, see [347, Theorem 2.3] and Corollary 2.4. □

Remark 1.4.22 The formal moduli space of K3 surfaces (resp. Enriques surfaces) in characteristic p of height $\geq h$ is of dimension $21 - h$ (resp. $11 - h$), see [25] and [347]. Moreover, if $p \geq 3$, then every K3 surface can be the K3-cover of at most finitely many Enriques surfaces. Thus, for dimensional reasons, a general K3 surface of height ≤ 6 is not the K3-cover of an Enriques surface.

For example, let A be an abelian surface in characteristic $p \geq 3$, and X be a minimal resolution of the associated Kummer surface $\mathrm{Kum}(A)$. Then the quotient $S := X/\langle \iota \rangle$ by a fixed–point–free involution ι is an Enriques surface with the K3-cover X. Since X is a Kummer surface, it is well-known that we have either $h(X) \leq 2$ or else $h(X) = \infty$ and $\sigma_0(X) \leq 2$. We refer to Example 1.6.13 for explicit examples.

1.5 The Enriques Lattice

In this section, we turn to the Néron-Severi lattice of an Enriques surface S. We will study root bases, Weyl groups, fundamental weights, and orbits of primitive vectors. We start with the isometry class of this lattice; it is the same for all Enriques surfaces.

Proposition 1.5.1 *Let S be an Enriques surface over an algebraically closed field $\Bbbk$. Then,*

$$\mathrm{Num}(S) \;=\; \mathrm{NS}(S)/\mathrm{Tors} \;=\; \mathrm{NS}(S)/(K_S) \;\cong\; \mathsf{U} \oplus \mathsf{E}_8,$$

which is an even and unimodular lattice of rank 10 *and signature* $(1, 9)$.

Proof By Theorem 1.2.7, the rank of $\mathrm{Num}(S)$ is equal to 10. Let us first treat the case where $\Bbbk$ is of characteristic zero. By the Lefschetz principle, we may assume that $\Bbbk = \mathbb{C}$. Then, Lefschetz's Theorem on $(1, 1)$-classes implies that the first Chern class $c_1 : \mathrm{NS}(S) \to H^2(S, \mathbb{Z})$ gives rise to an isomorphism of abelian groups up to torsion subgroups, that is,

$$\mathrm{Num}(S) \;\cong\; H^2(S, \mathbb{Z})/\mathrm{Tors} \;\cong\; \mathbb{Z}^{10}.$$

Since this isomorphism is compatible with intersection pairings on both sides, Poincaré duality for $H^2(S, \mathbb{Z})$ implies that $\mathrm{Num}(S)$ is a unimodular lattice. By the Hodge Index Theorem, it is of signature $(1, 9)$. Given an effective class $D \in \mathrm{Num}(S)$, the adjunction formula yields $D^2 = D^2 + D \cdot K_S = -2\chi(\mathcal{O}_D)$, which is an even integer. Since S is projective, every divisor class $D \in \mathrm{Num}(S)$ can be written as the difference of two effective divisor classes, from which it follows that D^2 is in general an even integer. This shows that $\mathrm{Num}(S)$ is an even lattice. By Proposition 0.8.8, there exists only one isomorphism class of even and unimodular lattices of signature $(1, 9)$ and it is represented by $\mathsf{U} \oplus \mathsf{E}_8$.

Let us now treat the case, where $\Bbbk$ is of characteristic $p > 0$. For all primes $\ell \neq p$, it follows from (0.10.29) that $T_\ell(\mathrm{Br}(S)) = 0$, which shows that there is an isomorphism

$$\mathrm{Num}(S) \otimes_{\mathbb{Z}} \mathbb{Z}_\ell \;\cong\; H^2_{\text{ét}}(S, \mathbb{Z}_\ell(1))/\mathrm{Tors}$$

that is compatible with intersection pairings on both sides. By Poincaré duality for ℓ-adic cohomology of algebraic surfaces, we find that $\mathrm{Num}(S) \otimes \mathbb{Z}_\ell$ is a unimodular lattice over $\mathbb{Z}_\ell$ for all $\ell \neq p$. Next, the Artin–azur formula (0.10.65) yields $h(S) = \mathrm{t}_p = 0$, and then, (0.10.61) implies that there is an isomorphism

$$\mathrm{Num}(S) \otimes_{\mathbb{Z}} \mathbb{Z}_p \;\cong\; H^2_{\text{fl}}(S, \mathbb{Z}_p(1))/\mathrm{Tors}$$

that is compatible with intersection pairings on both sides. We note that even if a surface satisfies $\rho = b_2$, then this does not imply that the pairing on the right–hand side is unimodular, see Remark 0.10.31. In our case, it follows from (the proof of) Theorem 1.4.13 that there exist isomorphisms

$$\mathrm{Num}(S) \otimes_{\mathbb{Z}} W \;\cong\; (H^2_{\text{ét}}(S, \mathbb{Z}_p(1)) \otimes_{\mathbb{Z}_p} W)/\mathrm{Tors} \;\cong\; H^2(X/W)/\mathrm{Tors}$$

that are compatible with intersection pairings. By Poincaré duality in crystalline cohomology, the pairing on the right–hand side is perfect, which implies that the pairing on $\mathrm{Num}(S) \otimes \mathbb{Z}_p$ is perfect. Putting all these computations together, we find that $\mathrm{Num}(S) \otimes \mathbb{Z}_\ell$ is a unimodular lattice over $\mathbb{Z}_\ell$ for all primes ℓ (including $\ell = p$), which implies that $\mathrm{Num}(S)$ is a unimodular lattice over $\mathbb{Z}$. From here, we proceed as in the characteristic zero case. □

Definition 1.5.2 The lattice

$$\mathsf{E}_{10} := \mathsf{U} \oplus \mathsf{E}_8 \cong \mathsf{E}_{2,3,7}$$

is called the *Enriques lattice*.

In fact, this lattice is part of the following series of lattices: for every integer $n \geq 4$, we define

$$\mathsf{E}_n := \mathsf{E}_{2,3,n-3},$$

where the lattices $\mathsf{E}_{p,q,r}$ are the ones introduced in Example 0.8.7. In this notation, the Enriques lattice is E_{10}, which is consistent with the previous definition. Moreover, we note that there are isomorphisms of lattices for $n \geq 11$

$$\mathsf{E}_n \cong \mathsf{U} \oplus \mathsf{E}_8 \oplus \mathsf{A}_{n-10}.$$

These lattices can be realized as follows using rational surfaces, see also the discussion around (0.5.1): consider

$$\pi : X = X_n \xrightarrow{\pi_n} X_{n-1} \xrightarrow{\pi_{n-1}} \ldots \xrightarrow{\pi_2} X_1 \xrightarrow{\pi_1} X_0 := \mathbb{P}^2, \tag{1.5.1}$$

where each $\pi_i : X_i \to X_{i-1}$ is the blow-up of a point $x_i \in X_{i-1}$. If e_0 denotes the pull–back of the divisor class of a line in $\mathbb{P}^2$ and the e_i, $i = 1, \ldots, n$ denote the classes of the exceptional configuration $\mathcal{E}_i = (\pi \circ \cdots \circ \pi_i)^{-1}(x_i)$, then we have already seen in the discussion around (0.5.5) that the $e_0, e_1, \ldots, e_n$ form a basis of $\mathrm{Pic}(X)$.

For an integer $n \geq 3$, let $\mathsf{I}^{1,n} = \langle 1 \rangle \oplus \langle -1 \rangle^n$ be the standard hyperbolic lattice with its standard basis $\mathbf{e}_0, \ldots, \mathbf{e}_n$, see Example 0.8.7. Then, the map that sends $\mathbf{e}_i \mapsto e_i$ for all i defines an isomorphism of lattices $\mathsf{I}^{1,n} \cong \mathrm{Pic}(X)$. Next, we define

$$\mathbf{k}_n := 3\mathbf{e}_0 - \mathbf{e}_1 - \cdots - \mathbf{e}_n,$$

which corresponds to the anti-canonical divisor class $-K_X$ in $\mathrm{Pic}(X)$. It is easy to check that

$$\mathsf{E}_n \cong (\mathbb{Z}\mathbf{k}_n)^\perp \subset \mathsf{I}^{1,n}. \tag{1.5.2}$$

Under this identification, the vectors

$$\boldsymbol{\alpha}_0 := \mathbf{e}_0 - \mathbf{e}_1 - \mathbf{e}_2 - \mathbf{e}_3, \quad \boldsymbol{\alpha}_i := \mathbf{e}_i - \mathbf{e}_{i+1},\ i = 1, \ldots, n-1,$$

form a root basis of E_n. In particular, if $n = 10$, that is, for the Enriques lattice E_{10}, then the Coxeter–Dynkin diagram of this root basis is of type $T_{2,3,7}$, see also Example 0.8.7 for the definition of the graphs $T_{p,q,r}$:

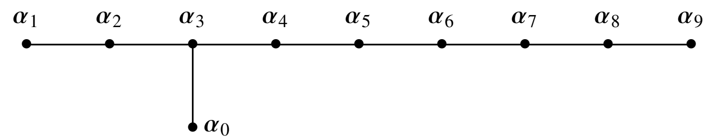


Fig. 1.1 The Enriques lattice E_{10}

With respect to this root basis, the splitting $\mathsf{E}_{10} \cong \mathsf{U} \oplus \mathsf{E}_8$ can be seen as follows: first, the $\alpha_0, \ldots, \alpha_7$ span a lattice of type E_8. Next, we define

$$\mathfrak{f} := 3\alpha_0 + 2\alpha_1 + 4\alpha_2 + 6\alpha_3 + 5\alpha_4 + 4\alpha_5 + 3\alpha_6 + 2\alpha_7 + \alpha_8, \tag{1.5.3}$$

and then, we have $\mathfrak{f} \cdot \alpha_i = 0$ for $i = 0, \ldots, 7$. Moreover, $\mathfrak{f}$ and α_9 span a sublattice isomorphic to U and orthogonal to E_8, which yields the asserted splitting.

The Dynkin diagram in Figure (1.1) has only one parabolic subdiagram, namely the one that is obtained by deleting the vertex α_9. Since its rank is equal to 8, Theorem 0.8.23 implies that $\mathsf{B} = (\alpha_0, \ldots, \alpha_9)$ is a crystallographic basis. By Proposition 0.8.20, we have $W_{\mathsf{B}}(\mathsf{E}_{10}) = W(\mathsf{E}_{10})$. Since the graph $T_{2,3,7}$ has no non-trivial symmetries, (0.8.16) implies that there is an isomorphism

$$\mathrm{O}(\mathsf{E}_{10})' \cong W(\mathsf{E}_{10}). \tag{1.5.4}$$

As explained in Section 0.3, reflections in the α_i give rise to an action of $W(\mathsf{E}_{10})$ on $\mathsf{I}^{1,n}$ and it is easy to see that it leaves vector $\mathbf{k}_{10}$ fixed. Conversely, using (1.5.2), we see that an isometry $\sigma \in \mathrm{O}(\mathsf{I}^{1,10})$ that fixes $\mathbf{k}_{10}$ defines an isometry of E_{10}. Thus, by (1.5.4) we find

$$\mathrm{O}(\mathsf{E}_{10})' \cong \{\sigma \in \mathrm{O}(\mathsf{I}^{1,10}) : \sigma(\mathbf{k}_{10}) = \mathbf{k}_{10}\}.$$

Using the isomorphism $\mathrm{Num}(S) \cong \mathsf{E}_{10}$ for an Enriques surface S, the pre-image $W(\mathrm{Num}(S))$ of $W(\mathsf{E}_{10})$ becomes a subgroup of index 2 of the orthogonal group $\mathrm{O}(\mathrm{Num}(S))$. The group $W(\mathrm{Num}(S))$ is called the *Weyl group* of S.

Since E_{10} is unimodular, the intersection form induces a canonical isomorphism with its dual lattice $\mathsf{E}_{10}^\vee$ and we denote by $\mathsf{B}^* = (\omega_0, \ldots, \omega_9)$ the dual basis of B. Its elements are called the *fundamental weights* of E_{10}. We note that this is compatible with Lie theory: for E_{10}, the set of roots B coincides with the set of coroots, and thus, the fundamental weights are dual to the coroots, as defined, for example, in [88, Chapter VI, Section 10].

Proposition 1.5.3 *The vectors*

$$\mathbf{f}_i := \mathbf{k}_{10} + \mathbf{e}_i, \quad i = 1, \ldots, 10, \quad \textit{and} \quad \boldsymbol{\Delta} := \frac{1}{3}(\mathbf{f}_1 + \cdots + \mathbf{f}_{10})$$

belong to E_{10} *and satisfy*

$$\mathbf{f}_i \cdot \mathbf{f}_j = 1 - \delta_{ij}.$$

Moreover, the corresponding fundamental weights are as follows:

$$\begin{aligned}
\boldsymbol{\omega}_0 &= \boldsymbol{\Delta} \\
\boldsymbol{\omega}_1 &= \boldsymbol{\Delta} - \mathbf{f}_1, \\
\boldsymbol{\omega}_2 &= 2\boldsymbol{\Delta} - \mathbf{f}_1 - \mathbf{f}_2, \\
\boldsymbol{\omega}_i &= 3\boldsymbol{\Delta} - \mathbf{f}_1 - \cdots - \mathbf{f}_i = \mathbf{f}_{i+1} + \cdots + \mathbf{f}_{10},\ i \geq 3.
\end{aligned}$$

The Gram matrix with respect to the basis B^* *is the following:*

$$\begin{pmatrix}
10 & 7 & 14 & 21 & 18 & 15 & 12 & 9 & 6 & 3 \\
7 & 4 & 9 & 14 & 12 & 10 & 8 & 6 & 4 & 2 \\
14 & 9 & 18 & 28 & 24 & 20 & 16 & 12 & 8 & 4 \\
21 & 14 & 28 & 42 & 36 & 30 & 24 & 18 & 12 & 6 \\
18 & 12 & 24 & 36 & 30 & 25 & 20 & 15 & 10 & 5 \\
15 & 10 & 20 & 30 & 25 & 20 & 16 & 12 & 8 & 4 \\
12 & 8 & 16 & 24 & 20 & 16 & 12 & 9 & 6 & 3 \\
9 & 6 & 12 & 18 & 15 & 12 & 9 & 6 & 4 & 2 \\
6 & 4 & 8 & 12 & 10 & 8 & 6 & 4 & 2 & 1 \\
3 & 2 & 4 & 6 & 5 & 4 & 3 & 2 & 1 & 0
\end{pmatrix}$$

Proof Since $\mathbf{k}_{10}^2 = -1$ and $\mathbf{e}_i \cdot \mathbf{k}_{10} = 1$ for all $i \geq 1$, we find $\mathbf{f}_i \in \mathbf{k}_{10}^{\perp} = \mathsf{E}_{10}$ using(1.5.2). Next, we have

$$\Delta = \frac{1}{3}\left(10\mathbf{k}_{10} + \sum_{i=1}^{10} \mathbf{e}_i\right) = \frac{1}{3}\left(30\mathbf{e}_0 - 9\sum_{i=1}^{10} \mathbf{e}_i\right) = 10\mathbf{e}_0 - 3\sum_{i=1}^{10} \mathbf{e}_i,$$

which thus lies in $\mathsf{I}^{1,10}$. It is easy to see that it is perpendicular to $\mathbf{k}_{10}$, which shows that Δ lies in E_{10}. The remaining assertions are easily verified by straightforward computations. □

Explicitly, one can express the isotropic vectors $\mathbf{f}_i$ in terms of the root basis B as follows:

$$\mathbf{f}_i = \mathfrak{f} + \sum_{j=i}^{9} \boldsymbol{\alpha}_j, \quad \text{for } i = 1, \ldots, 9, \quad \text{and} \quad \mathbf{f}_{10} = \mathfrak{f}. \tag{1.5.5}$$

Let us recall the following notations that we introduced in Section 0.8: if M is a lattice, then M_d denotes the set of vectors $v \in M$ that satisfy $v^2 = d$. Moreover, we denote by M_d' the subset of M_d that consists of primitive vectors. As an application of the previous proposition, we have the following.

Corollary 1.5.4 *The* $\mathsf{O}(\mathsf{E}_{10})$*-orbits of primitive vectors of* E_{10} *of self-intersection at most* 10 *are as follows:*

$$
\begin{aligned}
(\mathsf{E}_{10})'_0 &= \mathrm{O}(\mathsf{E}_{10})\cdot\boldsymbol{\omega}_9\\
(\mathsf{E}_{10})'_2 &= \mathrm{O}(\mathsf{E}_{10})\cdot\boldsymbol{\omega}_8\\
(\mathsf{E}_{10})'_4 &= \mathrm{O}(\mathsf{E}_{10})\cdot\boldsymbol{\omega}_1 \bigsqcup \mathrm{O}(\mathsf{E}_{10})\cdot(\boldsymbol{\omega}_8+\boldsymbol{\omega}_9)\\
(\mathsf{E}_{10})'_6 &= \mathrm{O}(\mathsf{E}_{10})\cdot\boldsymbol{\omega}_7 \bigsqcup \mathrm{O}(\mathsf{E}_{10})\cdot(\boldsymbol{\omega}_8+2\boldsymbol{\omega}_9)\\
(\mathsf{E}_{10})'_8 &= \mathrm{O}(\mathsf{E}_{10})\cdot(2\boldsymbol{\omega}_8) \bigsqcup \mathrm{O}(\mathsf{E}_{10})\cdot(\boldsymbol{\omega}_8+3\boldsymbol{\omega}_9) \bigsqcup \mathrm{O}(\mathsf{E}_{10})\cdot(\boldsymbol{\omega}_1+\boldsymbol{\omega}_9)\\
(\mathsf{E}_{10})'_{10} &= \mathrm{O}(\mathsf{E}_{10})\cdot\boldsymbol{\omega}_0 \bigsqcup \mathrm{O}(\mathsf{E}_{10})\cdot(\boldsymbol{\omega}_7+3\boldsymbol{\omega}_9) \bigsqcup \mathrm{O}(\mathsf{E}_{10})\cdot(\boldsymbol{\omega}_8+4\boldsymbol{\omega}_9).
\end{aligned}
\tag{1.5.6}
$$

Proof Since $C(\mathsf{B}) = \mathbb{R}_{\geq 0}\boldsymbol{\omega}_0 + \cdots + \mathbb{R}_{\geq 0}\boldsymbol{\omega}_9$ is a fundamental domain for the W_{B}-action on V^+ and since $\mathrm{O}(\mathsf{E}_{10}) = W_{\mathsf{B}} \times \{\pm 1\}$, each vector in E_{10} belongs to the orbit of a vector that is a positive sum of $\boldsymbol{\omega}_i$'s. From this observation, the assertion follows from straightforward computations with the Gram matrix of B^*. □

Remark 1.5.5 All fundamental weights can be written as positive sums of isotropic vectors: by the previous proposition, this is obvious for all $\boldsymbol{\omega}_i$ with $i \geq 3$. In the remaining cases, we define

$$\mathbf{f}_{i,j} \ := \ \boldsymbol{\Delta} - \mathbf{f}_i - \mathbf{f}_j, \quad 1 \leq i < j \leq 10,$$

which are isotropic vectors and then, we have

$$\boldsymbol{\omega}_1 \ := \ \mathbf{f}_{1,2} + \mathbf{f}_2, \quad \boldsymbol{\omega}_2 \ := \ 2\mathbf{f}_{1,2} + \mathbf{f}_1 + \mathbf{f}_2.$$

Definition 1.5.6 An *ordered isotropic k-sequence* in a lattice is a set of vectors $f_1, \ldots, f_k$ with $f_i \cdot f_j = 1 - \delta_{ij}$ for all $1 \leq i, j \leq k$.

In particular, the vectors of an ordered isotropic k-sequence are isotropic and $(\mathbf{f}_1, \ldots, \mathbf{f}_{10})$ is an example of an ordered isotropic 10-sequence in the Enriques lattice E_{10}. In fact, we have the following observation.

Lemma 1.5.7 *To give an ordered isotropic* 10*-sequence in the Enriques lattice* E_{10} *is equivalent to giving a primitive embedding* $\mathsf{E}_{10} \hookrightarrow \mathsf{I}^{1,10}$. *Both determine a root basis in* E_{10}.

Proof In Proposition 1.5.3, we saw that a primitive embedding $\mathsf{E}_{10} \hookrightarrow \mathsf{I}^{1,10}$ gives rise to a root basis in E_{10} and an ordered isotropic 10-sequence. Conversely, an ordered isotropic 10-sequence $(f_1, \ldots f_{10})$ defines a primitive embedding $\mathsf{E}_{10} \hookrightarrow \mathsf{I}^{1,10}$ and hence, a root basis in E_{10}: to see this, we consider the sublattice L of E_{10} spanned by $f_1, \ldots, f_{10}$. A direct computation shows that its discriminant is equal to 9, and thus, it is a sublattice of index 3 in E_{10}. Moreover, the vector $\delta = \frac{1}{3}(f_1 + \cdots + f_{10})$ has integer intersection with each f_i and hence, it defines an element in the dual lattice L^* with $3\delta \in L$. This implies that $\delta \in \mathsf{E}_{10}$ and we may set $\omega_0 = \delta$, $\omega_1 = \delta - f_1$, etc. as in Proposition 1.5.3. As there, the vectors ω_i represent the dual of a root basis $\alpha_0, \ldots, \alpha_{10}$. □

Let

$$C := \mathbb{R}_+\omega_0 + \cdots + \mathbb{R}_+\omega_9$$

be the fundamental domain of the Weyl group $W(\mathsf{E}_{10})$ in $(\mathsf{E}_{10})_{\mathbb{R}}$. Any vector $v \in \mathsf{E}_{10}$ belongs to the orbit of a unique vector $\sum m_i\omega_i \in C$.

We can also write any $v \in \mathsf{E}_{10}$ uniquely in the form

$$v = m\mathbf{\Delta} - \sum_{i=1}^{10} m_i\mathbf{f}_i, \tag{1.5.7}$$

where $3m - \sum_{i=1}^{10} m_i = 0$.

Lemma 1.5.8 *Assume that $v \in C$. Then:*

- $m - m_1 - m_2 - m_3 \geq 0$;
- $m_1 \geq m_2 \geq \cdots \geq m_{10}$;
- $v \cdot \mathbf{\Delta} = m$;
- $v \cdot \mathbf{f}_i = m_i$;
- $v^2 = m^2 - \sum_{i=1} m_i^2$.

Proof Since $v \in C$, we have $v \cdot \boldsymbol{\alpha}_i \geq 0, i = 0, \ldots, 9$. This verifies the first two assertions. We have $v \cdot \mathbf{\Delta} = 10m - 3\sum_{j=1}^{10} m_j = m$. Also, we get $v \cdot \mathbf{f}_i = 3m - \sum_{j=1, j\neq i}^{10} m_j = m_i$. The last assertion is immediate. □

This is reminiscent of the well-known representation of divisor classes of the blow-up of a set of points in $\mathbb{P}^2$ in terms of the geometric basis. In fact, using Proposition 1.5.3, we can write v in the form

$$v = m\mathbf{e}_0 - \sum_{i=1}^{10} m_i\mathbf{e}_i,$$

where $(\mathbf{e}_0, \ldots, \mathbf{e}_{10})$ is the orthonormal basis of $\mathsf{I}^{1,10}$ that corresponds to a geometric basis of the blow-up X of 10 points in $\mathbb{P}^2$ under a geometric marking $\mathrm{Pic}(X) \to \mathsf{I}^{1,10}$. The condition $3m = \sum_{i=1}^{10} m_i$ means that the divisor class is orthogonal to the canonical class of X. As we will see later in section 5.4 and Chapter 9 in Volume II, this analogy acquires a geometric meaning when we degenerate S to a Coble surface.

1.6 Examples

In this section, we present several constructions to obtain some more or less explicit families of Enriques surfaces in arbitrary characteristic. We start with three classical constructions, due to Castelnuovo and Enriques. Then, we settle existence of all types of Enriques surfaces in every characteristic using a construction of Bombieri and Mumford. To make this latter construction even more explicit, we use Kummer surfaces associated to Jacobians of genus two curves. We start with the following well-known observation.

Lemma 1.6.1 *Let X be a smooth and projective variety of dimension $d \geq 2$ over an algebraically closed field $\Bbbk$ such that:*

1. *X is the complete intersection of $(N - d)$ smooth hypersurfaces in $\mathbb{P}^N_{\Bbbk}$, or*
2. *X is a simple $\boldsymbol{\mu}_n$-cover of $\mathbb{P}^d_{\Bbbk}$ associated to some data $(\mathcal{L}, s)$.*

Then, X is algebraically simply connected and satisfies $h^1(X, \mathcal{O}_X) = 0$.

Proof To prove the first assertion, we note that we have shown $h^1(X, \mathcal{O}_X) = 0$ already in the proof of Lemma 1.1.11. To show that the étale fundamental group $\pi_1(X)$ is trivial, we proceed by induction: let $X = Y \cap H$, where Y is a smooth complete intersection of smooth hypersurfaces, and H is a smooth hypersurface. By the Lefschetz hyperplane theorem, the homomorphism of étale fundamental groups $\pi_1(Y) \to \pi_1(X)$ induced by restriction is an isomorphism if $\dim(X) \geq 2$, see [273, Théorème X.3.10].

To prove the second assertion, let $X \to \mathbb{P}^d$ be a simple $\boldsymbol{\mu}_n$-cover associated to data $(\mathcal{L}, s)$. Since $d \geq 2$, we have $H^1(\mathbb{P}^2, \mathcal{L}^{\otimes i}) = 0$ for all $i \in \mathbb{Z}$. Therefore, (0.2.3) implies $h^1(X, \mathcal{O}_X) = 0$. Next, X is a hypersurface in the total space of the line bundle $\mathbb{L} = \mathbb{V}(\mathcal{L}^{-1}) \to \mathbb{P}^d$, see the discussion in Section 0.3. We can embed $\mathbb{L}$ into a $\mathbb{P}^1$-bundle $B \to \mathbb{P}^d$. Since B is a proper, rational, and smooth variety, $\pi_1(B)$ is trivial by [272, Corollaire XI.1.2]. Since X is an ample divisor inside B, the restriction homomorphism $\pi_1(B) \to \pi_1(X)$ is an isomorphism by Lefschetz's theorem. □

Since Enriques surfaces are not algebraically simply connected in characteristic $\neq 2$ and those that are algebraically simply connected in characteristic 2 satisfy $h^{0,1} \neq 0$, it follows that the standard constructions from Lemma 1.6.1 never give Enriques surfaces. Thus, it should not be surprising that the examples below arise as resolutions of singularities of non-smooth hypersurfaces in projective space and of non-smooth and rational double covers.

We start with Enriques' original construction [109], [218] of the first Enriques surface over the complex numbers, see also the bibliographical notes at the end of this section, [179], and Theorem 3.5.1. We note that the generic Enriques surface over the complex numbers arise this way, which was shown by Enriques [219]. Moreover, Mumford [536] observed that this construction actually works in any characteristic.

Example 1.6.2 Let $\Bbbk$ be an algebraically closed field of arbitrary characteristic and consider the sextic surface in $\mathbb{P}^3_{\Bbbk}$ that is given by

$$F_6(Q) \;:\; t_0t_1t_2t_3Q(t_0, t_1, t_2, t_3) + t_1^2t_2^2t_3^2 + t_0^2t_2^2t_3^2 + t_0^2t_1^2t_3^2 + t_0^2t_1^2t_2^2 \;=\; 0,$$

where $Q = Q(t_0, t_1, t_2, t_3)$ is a quadratic form. Then, the surface $F_6(Q)$ is singular along the tetrahedron given by the six lines $\{t_i = t_j = 0\}_{i \neq j}$. In particular, $F_6(Q)$ is not normal.

More precisely, for a generic choice of Q (for example being nondegenerate and not vanishing on the edges of the coordinate tetrahedron), the surface $F_6(Q)$ has very classical types of singularities: quite generally, an integral, but possible non-normal surface X over an algebraically closed field $\Bbbk$ has *ordinary singularities* if the locus of non-smooth points consists of a *double curve* Γ that is smooth outside *triple points* and *pinch points*, singular points of the surface. More precisely, the completion of a local ring at a general point of Γ is isomorphic to $\Bbbk[[z_1, z_2, z_3]]/(z_1 z_2)$, it is isomorphic to $\Bbbk[[z_1, z_2, z_3]]/(z_1 z_2 z_3)$ at *triple points*, and to $\Bbbk[[z_1, z_2, z_3]]/(z_1^2 + z_2^2 z_3)$ at *pinch points*. Let $\nu : X' \to X$ be the normalization morphism, which is an isomorphism outside Γ, and then, an easy computation shows that X' is even smooth over $\Bbbk$. Moreover, the pre-image of a general point (resp. triple point, resp. pinch point) of Γ under ν consists of two points (resp. three points, resp. one point). The importance of these singularities comes from the fact that every smooth and projective surface over the complex numbers is birationally equivalent to a surface in $\mathbb{P}^3$ with ordinary singularities, see [259, Chapter 4.6] for details, pictures, and proofs.

Proposition 1.6.3 *Let $\Bbbk$ be an algebraically closed field of characteristic $p \geq 0$ and let $Q = Q(t_0, t_1, t_2, t_3) \in \Bbbk[t_0, \ldots, t_3]$ be a general quadratic form. Then, $F_6(Q)$ has ordinary singularities and its normalization is an Enriques surface. If $p = 2$, then this Enriques surface is classical.*

Proof We leave it to the reader to check that if $Q = Q(t_0, t_1, t_2, t_3) \in \Bbbk[t_0, \ldots, t_3]$ is a generic quadratic form, then $F_6(Q)$ has at worst ordinary singularities. Thus, let Q be generic and let $\nu : S \to F := F_6(Q)$ be the normalization morphism. The conductor ideal $\mathfrak{C}$ of ν gives rise to closed subschemes $\Delta \subset S$ and $\Gamma \subset F$, respectively. By the above discussion, S is a smooth surface over $\Bbbk$ and the singular locus of F is equal to Γ, which is equal to six lines forming a tetrahedron as explained in Example 1.6.2. Also, F has no pinch points and the triple points of F correspond the intersection points of three lines of the tetrahedron.

Being a hypersurface in $\mathbb{P}^3$ and a smooth surface, respectively, both surfaces F and S are Gorenstein and we denote by ω_F and ω_S their dualizing sheaves. Moreover, the adjunction formula for the degree 6 surface F in $\mathbb{P}^3$ yields $\omega_F \cong\cong \omega_{\mathbb{P}^3} \otimes \mathcal{O}_F(F) \cong \mathcal{O}_F(2)$. duality for the finite morphism ν yields

$$\omega_S \;\cong\; \nu^* \omega_F(-\Delta) \;\cong\; \nu^*(\mathcal{O}_F(2))(-\Delta), \tag{1.6.1}$$

see also (1.3.4). From this, the projection formula yields $\nu_* \omega_S \cong \mathcal{O}_F(2) \otimes \mathcal{I}_\Gamma$. Taking global sections and using that Γ is a curve of degree 6 in $\mathbb{P}^3$, we find $h^0(S, \omega_S) = 0$.

It is known that the curve Γ is an arithmetically Cohen–Macaulay (ACM) scheme [177, 7.2.2] (where the exposition is characteristic-free). It implies that the natural restriction morphism $H^0(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(n)) \to H^0(\Gamma, \mathcal{O}_\Gamma(n))$ is surjective for all $n \geq 0$ and Γ does not lie on a quadric. Using this and taking cohomology in the short exact sequence

$$0 \to \mathcal{I}_\Gamma(2) \to \mathcal{O}_{\mathbb{P}^2}(2) \to \mathcal{O}_\Gamma(2) \to 0,$$

we find $H^1(S, \omega_S) \cong H^1(F, \mathcal{I}_\Gamma(2)) \cong H^1(\mathbb{P}^3, \mathcal{I}_\Gamma(2)) = 0$. and thus, $H^1(S, \mathcal{O}_S) = 0$.

Next, using (1.3.4) or (1.6.1), we find

$$\omega_S^{\otimes 2} \;\cong\; \pi^* \left(\omega_F^{\otimes 2}(-2\Gamma) \right) \;\cong\; \pi^* \left(\mathcal{O}_F(4)(-2\Gamma) \right) \;\cong\; \pi^*(\mathcal{O}_F(4) \otimes \mathcal{I}_\Gamma^{(2)}),$$

where $J^{(n)}$ denotes the n-th symbolic power of an ideal (sheaf) J. We leave it to the reader to deduce $\omega_S^{\otimes 2} \cong O_S$ from this. Thus, S is a minimal surface of Kodaira dimension zero with $h^{0,1} = h^{0,2} = 0$. Thus, Table **??** identifies S as an Enriques surface with $\Delta = 0$. In particular, if $p = 2$, then S is classical. We refer to [179] for more details and note that if $p = 2$, then Mumford [536] explicitly showed that, if S is non-classical, then $h^{0,1}(S) \neq 0$ and $h^{1,0}(S) = 0$. □

Let us mention the following variant of Example 1.6.2, which is due to Castelnuovo and yields a birational model that is birationally equivalent to $F_6(Q)$.

Example 1.6.4 Let $\Bbbk$ be an algebraically closed field of characteristic $p \geq 0$ and consider the rational and birational self-map

$$\begin{array}{cccc} T: & \mathbb{P}^3_{\Bbbk} & \dashrightarrow & \mathbb{P}^3_{\Bbbk} \\ & [y_0, \ldots, y_3] & \mapsto & [y_2 y_3, y_0 y_1, y_0 y_2, y_0 y_3]. \end{array}$$

(This is a *Cremona transformation* of degree 2.) Let $F_6(Q)$ be as in Example 1.6.2 and, after plugging in T into $F_6(Q)$ and dividing by $y_0^3 y_2^2 y_3^2$, we find

$$G_5(Q) \; : \; y_1 Q(y_2 y_3, y_0 y_1, y_0 y_2, y_0 y_3) \; + \; y_0^3 y_1^2 \; + \; y_0 y_2^2 y_3^2 \; + \; y_0 y_1^2 y_3^2 \; + \; y_0 y_1^2 y_2^2 \; = \; 0.$$

If the quadric Q is general, then this defines a normal surface of degree 5 in $\mathbb{P}^3$ that has four singular points: namely the points $[1,0,0,0]$, $[0,1,0,0]$, $[0,0,1,0]$, and $[0,0,0,1]$, all of which are *simple elliptic singularities*, the first two are of degree 2, the last two are of degree 3.[1] In classical terminology, the first two singularities are *ordinary tacnodes* and the last two are *ordinary triple points*. The minimal resolution of singularities is an Enriques surface.

In fact, T induces a birational map between $F_6(Q)$ and $G_5(Q)$, which implies that the Enriques surfaces constructed from Q in Example 1.6.2 and Example 1.6.4 are in fact isomorphic. We refer to [389], [686], and [711] for modern treatments and the study of quintic models of Enriques surfaces. Let us also mention the following presentation of Example 1.6.4 as a rational double cover of the plane, which is due to Enriques.

Example 1.6.5 Let $\Bbbk$ be an algebraically closed field of characteristic $p \geq 0$ and let $G_5(Q)$ be the quintic surface from Example 1.6.4. Since $[0,0,0,1]$ is a triple point, we can write this surface as

$$G_5(Q) \; : \; y_3^2 A(y_0, y_1, y_2) + y_3 B(y_0, y_1, y_2) + C(y_0, y_1, y_2) \; = \; 0,$$

where A, B, and C are homogeneous forms in the variables y_0, y_1, y_2 that are of degree 3, 4, and 5, respectively. In this form, we see that projection away from $[0,0,0,1]$ induces a rational map

[1] A simple elliptic singularity of degree d is the singularity of the affine spectrum of the section ring of an invertible sheaf of degree d over an elliptic curve.

$$f : G_5(Q) \dashrightarrow \mathbb{P}^2$$

that is generically finite of degree 2. If $p \neq 2$, then this map is generically étale and the branch curve is given by $B^2 - 4AC = 0$, which is of degree 8. More precisely, this curve is the union of two lines $\ell_1 = V(y_0)$ and $\ell_2 = V(y_1)$ with a degree 6 curve W. The sextic curve W has a double point at the intersection of these lines and two simple points of type a_3 (classically called *tacnodes*) such that the lines ℓ_1 and ℓ_2 intersect W at these points with multiplicity 4. This is an *Enriques octic* and we refer to Example 3.3.20 for explicit equations.

If $p = 2$, the map f could be separable or inseparable. We will study these covers in Section 3.3.

The next construction is due to Bombieri and Mumford [77], who attribute it to Miles Reid. It gives examples of Enriques surfaces in arbitrary characteristic. In particular, it also gives examples of all three types of Enriques surfaces in characteristic 2 as in Definition 1.1.7. In fact, by a result of Liedtke [460], all Enriques surfaces in arbitrary characteristic arise this way (when also allowing rational double point singularities). We will construct our surfaces as quotients of complete intersections of 3 quadrics in $\mathbb{P}^5$ by a finite flat group scheme of length 2. If X is such a complete intersection, then $h^1(\mathcal{O}_X) = 0$ and $\omega_X \cong \mathcal{O}_X$, and if X is smooth, then it is a K3 surface, see Lemma 1.1.11. By Theorem 0.1.10, there exist three group schemes of length 2 over an algebraically closed field $\Bbbk$ of characteristic $p = 2$, namely, $(\mathbb{Z}/2\mathbb{Z})_{\Bbbk}$, $\boldsymbol{\mu}_{2,\Bbbk}$, and $\boldsymbol{\alpha}_{2,\Bbbk}$. On the other hand, if $p \neq 2$, then there is only one, namely $(\mathbb{Z}/2\mathbb{Z})_{\Bbbk} \cong \boldsymbol{\mu}_2$. Let us slightly extend our setup for later use.

Example 1.6.6 Let R be a local, complete, and noetherian ring with residue field $\Bbbk$ of characteristic $p \geq 0$. Fix $a, b \in R$ with $ab = 2$. Then, we define

$$\mathcal{G}_{a,b} := \operatorname{Spec} R[t]/(t^2 - at),$$

which is a finite and flat scheme of length 2 over R. It becomes a group scheme over R via the comultiplication map

$$R[t]/(t^2 - at) \to R[t]/(t^2 - at) \otimes_R R[t]/(t^2 - at), \quad t \mapsto t \otimes 1 + 1 \otimes t - bt \otimes t.$$

By the classification results of Oort and Tate [584] (see also Theorem 0.1.10), every finite group scheme of length 2 over R is isomorphic to $\mathcal{G}_{a,b}$ for some a, b as above. Let us make the following remarks:

1. If $u \in R$ is invertible, then we have $\mathcal{G}_{a,b} \cong \mathcal{G}_{au,bu^{-1}}$.
2. In particular, if $p \neq 2$, then $2 \in R$ is invertible, and all finite flat group schemes of length 2 over R are isomorphic to $\mathcal{G}_{1,2} \cong \mathcal{G}_{2,1} \cong \boldsymbol{\mu}_{2,R} \cong (\mathbb{Z}/2\mathbb{Z})_R$.
3. If $p = 2$, then there are three possibilities over $\Bbbk$:

$$\mathcal{G}_{0,1,\Bbbk} \cong \boldsymbol{\mu}_{2,\Bbbk}, \qquad \mathcal{G}_{0,0,\Bbbk} \cong \boldsymbol{\alpha}_{2,\Bbbk}, \qquad \mathcal{G}_{1,0,\Bbbk} \cong (\mathbb{Z}/2\mathbb{Z})_{\Bbbk},$$

   see also Theorem 0.1.10. In the case where R is a local and complete DVR with residue field $\Bbbk$ of characteristic 2 and $\mathcal{G}_{a,b}$ is as before, then the Oort–Tate classification implies that if the special fiber over $\Bbbk$ is isomorphic to $\boldsymbol{\mu}_2$ or $\mathbb{Z}/2\mathbb{Z}$, respectively, then so is the generic fiber. On the other hand, if the special fiber is isomorphic to $\boldsymbol{\alpha}_2$, then it is easy to construct examples over $R = \Bbbk[[u]]$, where the generic fiber is isomorphic to $\boldsymbol{\alpha}_2$ (e.g. $a = b = 0$), or to $\boldsymbol{\mu}_2$ (e.g. $a = 0$, $b = u$), or to $\mathbb{Z}/2\mathbb{Z}$ (e.g. $a = u$, $b = 0$). This observation will become important when discussing moduli of Enriques surfaces in characteristic 2.

4. Cartier duality takes the following form:

$$\mathcal{G}_{a,b}^D = \mathcal{H}om(\mathcal{G}_{a,b}, \mathbb{G}_m) \cong \mathcal{G}_{b,a}.$$

For the construction of our examples, we need an action of $\mathcal{G}_{a,b}$ on $\mathbb{P}^5$ over R. First, we consider the two-dimensional representation of $\mathcal{G}_{a,b}$ that is given by associating to every R-algebra S the homomorphism

$$\mathcal{G}_{a,b}(S) \to \mathrm{GL}_{2,R}(S), \quad s \mapsto \left(\begin{smallmatrix} 1 & s \\ 0 & 1-bs \end{smallmatrix}\right).$$

This defines an isomorphism from $\mathcal{G}_{a,b}$ onto a closed subgroup scheme of $\mathrm{GL}_{2,R}$. We extend this to a 6-dimensional representation

$$\rho : \mathcal{G}_{a,b} \to \mathrm{GL}_{6,R}$$

by simply taking the direct sum of three copies of the former representation. More precisely, consider the variables $x_0, x_1, x_2, y_0, y_1, y_2$, and then, define a $\mathcal{G}_{a,b}$-action by setting

$$x_i \mapsto x_i, \quad y_i \mapsto sx_i + (1-bs)y_i \quad \text{for } i = 0, 1, 2 \text{ and } s \in \mathcal{G}_{a,b}(S) = \{s \in S | s^2 = as\}.$$

After these preparations, we have the following lemma, due to Bombieri and Mumford [77, page 222].

Lemma 1.6.7 *The following 12 elements of degree 2 in $R[x_0, x_1, x_2, y_0, y_1, y_2]$ span the R-module of $\mathcal{G}_{a,b}$-invariants of degree 2:*

$$x_i x_j, \quad y_i^2 - a x_i y_i, \quad x_i y_j + x_j y_i + b y_i y_j.$$

Moreover, they also generate the R-algebra of $\mathcal{G}_{a,b}$-invariants of even degree.

Consider the induced $\mathcal{G}_{a,b}$-action on $\mathbb{P}^5 = \operatorname{Proj} R[x_0, x_1, x_2, y_0, y_1, y_2]$. Let us also specialize to the case $R = \Bbbk$. Then, the examples of Bombieri–Mumford–Reid from [77] are the following.

Example 1.6.8 Let $\Bbbk$ be an algebraically closed field of characterstic $p \geq 0$, let $\mathcal{G}_{a,b}$ with $a, b \in \Bbbk$ and $ab = 2$ be the finite flat group scheme of length 2 over $\Bbbk$ as in Example 1.6.6, and consider $\mathbb{P}^5$ with the $\mathcal{G}_{a,b}$-action introduced above. Let $X \subset \mathbb{P}^5$ be the complete intersection of three generic $\mathcal{G}_{a,b}$-invariant quadrics. By Lemma 1.1.11, X is an integral Gorenstein surface with $\omega_X \cong \mathcal{O}_X$ and $h^1(\mathcal{O}_X) = 0$, and by construction, it carries a $\mathcal{G}_{a,b}$-action. Since X is generic, the quotient

$$S := X/\mathcal{G}_{a,b}$$

is an Enriques surface with K3-cover $X \to S$ and $\mathbf{Pic}^{\tau}_S \cong \mathcal{G}^D_{a,b} \cong \mathcal{G}_{b,a}$. In particular, this shows the existence of Enriques surfaces in every characteristic, as well as the existence of the three possible types in characteristic 2.

We will see in Corollary 3.4.3 that all Enriques surfaces arise in this way, when also allowing rational double point singularities. This will be done in Section 3.4, where we discuss in greater detail quotients of degree 8 polarized K3 surfaces that lead to polarizations of degree 4 on Enriques surfaces.

Remark 1.6.9 Let us make the following remarks concerning moduli and deformations of the three types of Enriques surfaces in characteristic 2.

1. Let $\mathcal{S} \to B$ be a flat family of Enriques surfaces, where B is a base scheme of characteristic 2. Table 1.2 and the semi-continuity theorem applied to $h^{0,1}$ and to $h^{1,0}$ show that the locus of classical, as well as the locus of $\boldsymbol{\mu}_2$-surfaces, is open in B. The locus of $\boldsymbol{\alpha}_2$-surfaces is closed in B. In particular, if B is irreducible, then this family cannot contain both classical and $\boldsymbol{\mu}_2$-surfaces.
2. Let $S = X/\mathcal{G}_{0,0}$ be an $\boldsymbol{\alpha}_2$-surface over an algebraically closed field $\mathbb{k}$ of characteristic 2 as in Example 1.6.8. As explained in Example 1.6.6, there exists finite and flat group schemes $\mathcal{G}_{a,b}$ of length 2 over $R := \mathbb{k}[[u]]$ with special fiber $\mathcal{G}_{0,0} \cong \boldsymbol{\alpha}_2$ and generic fiber $\mathcal{G}_{0,1} \cong \boldsymbol{\mu}_2$ or $\mathcal{G}_{1,0} \cong \mathbb{Z}/2\mathbb{Z}$. It is not difficult to see that one can deform the three $\mathcal{G}_{0,0}$-invariant quadrics cutting out $X \subset \mathbb{P}^5$ to three $\mathcal{G}_{a,b}$-invariant families over R. The resulting complete intersection $\mathcal{X}$ in $\mathbb{P}^5$ over R is $\mathcal{G}_{a,b}$-invariant, and the quotient $\mathcal{S} = \mathcal{X}/\mathcal{G}_{a,b}$ yields families of Enriques surfaces over R, whose special fiber is an $\boldsymbol{\alpha}_2$-surface, and whose generic fiber is a classical or a $\boldsymbol{\mu}_2$-surface, respectively.

This shows that moduli spaces of Enriques surfaces in characteristic 2 cannot be irreducible, since classical and $\boldsymbol{\mu}_2$-surfaces have to lie on different irreducible components. On the other hand, some $\boldsymbol{\alpha}_2$-surfaces can be deformed into classical, as well as into $\boldsymbol{\mu}_2$-surfaces (in fact, this is true for all $\boldsymbol{\alpha}_2$-surfaces). We refer to Section 5.9 for details and examples.

Finally, we give explicit constructions of Enriques surfaces in characteristic $\neq 2$ as quotients of complete intersections of three quadrics in $\mathbb{P}^5$ by free $\mathbb{Z}/2\mathbb{Z}$-actions. This makes Example 1.6.8 more explicit. In fact, these complete intersections are Kummer K3 surfaces associated to Jacobians of genus two curves.

Example 1.6.10 Let A be an abelian surface over an algebraically closed field $\mathbb{k}$ of characteristic $p \neq 2$. Then, the sign involution (with respect to the group structure) $\iota_A : x \mapsto -x$ is an involution, that is, it generates a cyclic group $\langle \iota_A \rangle \cong \mathbb{Z}/2\mathbb{Z}$. The quotient $\mathrm{Kum}(A) := A/\langle \iota_A \rangle$ is a normal surface, which has 16 rational double point singularities of type A_1. Its minimal resolution $\widetilde{\mathrm{Kum}}(A)$ of singularities is a K3 surface, called the *Kummer K3 surface* associated to A.

To obtain explicit equations, let C be a smooth and projective curve of genus two over an algebraically closed field $\mathbb{k}$ of characteristic $p \neq 2$. Since C is a hyperelliptic curve, there exists a finite morphism $C \to \mathbb{P}^1$ of degree 2 that is branched over 6 points, say $[1 : a_i]$, $i = 0, \ldots, 5$ for pairwise distinct $a_i \in \Bbbk$. The Jacobian variety $\mathrm{Jac}(C) = \mathbf{Pic}^{\circ}_{C/\Bbbk}$ of C is an abelian surface, see Example 0.9.15. In this case, it is well-known that the Kummer K3 surface $X = \widetilde{\mathrm{Kum}}(\mathrm{Jac}(C))$ is isomorphic to the complete intersection of three quadrics in $\mathbb{P}^5$ with equations

$$\sum_{i=0}^{5} t_i^2 \;=\; \sum_{i=0}^{5} a_i t_i^2 \;=\; \sum_{i=0}^{5} a_i^2 t_i^2 \;=\; 0, \tag{1.6.2}$$

see, for example, [177, Chapter 10.3.3]. Next, let $I \subset \{0, 1, 2, 3, 4, 5, 6\}$ be a subset with 3 elements and let σ_I be the involution of X that is defined by $t_i \mapsto -t_i$ if $i \in I$ and $t_i \mapsto t_i$ otherwise. It has no fixed points on X and thus, the quotient $\pi : X \to S$ by $\langle \sigma_I \rangle$ is an Enriques surface and X is its K3-cover.

As an application, we find examples of unirational and non-unirational Enriques surfaces in every characteristic $p \geq 3$. We have already seen in Theorem 1.3.10 that an Enriques surface in odd characteristic is unirational if and only if its K3-cover is. In the case of Kummer K3 surfaces, unirationality can be decided using the Picard number ρ by the following result of Shioda [676, Theorem 1.1].

Theorem 1.6.11 *Let A be an abelian surface over an algebraically closed field $\Bbbk$ of characteristic $p \geq 3$ and $X = \widetilde{\mathrm{Kum}}(A)$ be a Kummer K3 surface. Then, the following are equivalent:*

1. *X is a unirational K3 surface,*
2. *$\rho(X) = 22$, that is, X is a Shioda-supersingular K3 surface,*
3. *$\rho(A) = 6$,*
4. *A is a supersingular abelian surface,*
5. *A is isogenous to the product of two supersingular elliptic curves.*

Remark 1.6.12 Both, the moduli space $\mathcal{M}_2$ of genus two curves, as well the moduli space of principally polarized abelian surfaces $\mathcal{A}_2$, are 3-dimensional and the map $C \mapsto \mathrm{Jac}(C)$ defines a morphism $\mathcal{M}_2 \to \mathcal{A}_2$ that induces a bijection on geometric points by the Torelli theorem. Moreover, the moduli space of supersingular abelian surfaces $\mathcal{A}_2^{\mathrm{ss}}$ is a 1-dimensional subspace of $\mathcal{A}_2$.

It follows from Theorem 1.6.11 that, for a generic curve C of genus two in characteristic $p \geq 3$, the Enriques quotients of $\widetilde{\mathrm{Kum}}(\mathrm{Jac}(C))$ constructed in Example 1.6.10 are not unirational. On the other hand, this construction also yields 1-dimensional families of unirational Enriques surfaces in every characteristic $p \geq 3$. It follows from Example 1.6.2 that isomorphism classes of Enriques surfaces depend on 10 parameters, which implies that the family of Enriques Kummer quotients is of large codimension. We will come back to this in Chapter 5 and in Volume II.

Even more explicitly, the following two examples are due to Shioda [676, Section 1], and yield explicit examples of unirational Enriques surfaces in infinitely many positive characteristics.

Example 1.6.13 We keep the assumptions and notations of Example 1.6.10.

1. Assume $p = 3$. If we represent a point in $\mathcal{M}_2$ by a hyperelliptic curve $x_2^2 + f_6(x_0, x_1) = 0$, where $f_6 = \sum_{i=0}^{6} c_i x_0^i x_1^{6-i}$ is a binary form of degree 6, then the pre-image of $\mathcal{A}_2^{\mathrm{ss}}$ in $\mathcal{M}_2$ consists of the isomorphism classes of hyperelliptic curves satisfying the conditions
$$c_2 c_1^2 + c_4^3 = c_1^3 c_5 + c_4^4 = 0.$$
(see Example 0.10.24).
2. Assume that $p \neq 2, 5$. Then, there exists a unique curve C of genus two that admits an automorphism of order 5 and it is known that the Kummer surface $X = \widetilde{\mathrm{Kum}}(\mathrm{Jac}(C))$ is isomorphic to the complete intersection from (1.6.2) with
$$(a_0, \ldots, a_5) \;=\; (0, 1, \zeta_5, \zeta_5^2, \zeta_5^3, \zeta_5^4),$$
where ζ_5 is a primitive 5-th root of unity. By a direct computation, Shioda [676] shows that X is unirational if $p \not\equiv 1 \mod 5$. In particular, the construction of Example 1.6.10 yields unirational Enriques surfaces.
3. Similarly, if $p \neq 2, 3$ and if C is the unique curve that has an automorphism of order 6, then $\widetilde{\mathrm{Kum}}(\mathrm{Jac}(C))$ is unirational if $p \equiv 2 \mod 3$. Again, we obtain unirational Enriques surfaces in these characteristics via Example 1.6.10.

Bibliographical Notes

The classification of algebraic surfaces over fields of characteristic zero is mostly due to Italian algebraic geometers, see, for example, Enriques' book [221]. In the sixties, this classification was revived by Shafarevich and his students in Moscow, Zariski and his students, and Kodaira in the USA. It was Shafarevich [5] who introduced the notion of what is nowadays called the Kodaira dimension and who arranged the classification according to this invariant. Since then, the place of Enriques surfaces in the classification of algebraic surfaces has become much clearer, namely, as one of the four classes of surfaces of Kodaira dimension zero. On the other hand, Enriques' original motivation to construct the first Enriques surface in [218] was to give a surface with $p_g = q = 0$ that is not rational and thus, to show that Castelnuovo's rationality criterion $p_2 = q = 0$ cannot be improved. A modern exposition of Enriques's constructions was given in the theses of Averbuch [36], [37] in Moscow and Artin [18] at Harvard. We refer for more history to the survey [179].

At the beginning of Section 1.1, we gave references to modern expositions of the theory of algebraic surfaces. In particular, we refer the interested reader to [43], where most results of this chapter are discussed over the field of complex numbers. The classification of algebraic surfaces was extended to the case of positive characteristic by Bombieri and Mumford in [539], [78], and [77] and was reviewed in the books of Badescu [38] and Kurke [424]. The definition of Enriques surfaces in all characteristics and the theory of their Picard schemes in characteristic two presented in Section 1.2 is taken from [78]. On the other hand, the proof of the fundamental result Theorem 1.2.7 that $\rho = b_2 = 10$ is different from ours. Here, we have presented the proofs of Lang [432] and

Liedtke [464] and defer the original proof of Bombieri and Mumford [78] to later when we discuss (quasi-)elliptic fibrations.

The results on the singularities of the K3-cover in Section 1.3 in characteristic 2 are due to W. Lang, Ekedahl, and Shepherd–Barron, and we refer to [214], [489], and [640] for more information on K3-covers of Enriques surfaces. The unirationality results in Section 1.2 are due to Blass [67] in characteristic two. The fact that a $\boldsymbol{\mu}_2$-surface is not unirational was first proved by Crew [143]. In the case where $\Bbbk$ is the algebraic closure of a finite field, this was first noticed by Katsura [363]. The results on pull-backs of Brauer classes to the K3-cover are due to Beauville [51]. The computation of the differential invariants in Section 1.4 is due mostly to Lang [433] and the computation of crystalline and Hodge–Witt cohomology is due mostly to Illusie [329]. Moreover, we refer to the articles of Ekedahl and Shepherd–Barron [212] and [213] for more information about vector fields on classical Enriques surfaces.

The results on the Enriques lattice in Section 1.5 are essentially due to Fano [227]. In particular, he introduced the isotropic vectors $\mathbf{f}_i$ that we discuss in Proposition 1.5.3. Their important role in the geometry of Enriques surfaces became evident in the articles of Barth and Peters [42] and Cossec [135]. The Gram matrix of the fundamental weights was first shown to one of the authors by Mukai in 1983.

The first example of an Enriques surface is due to Enriques himself. Enriques's example of a non-normal sextic model was included in Castelnuovo's paper [109] and only briefly mentioned by Enriques himself in [218]. In his article, Castelnuovo also gave the birationally equivalent normal quintic model. In [219], Enriques showed that the generic Enriques surface over the complex numbers arises this way, and there, he also gave the construction of Enriques surfaces as double covers of $\mathbb{P}^2$ branched over Enriques octics. The observation that the non-normal sextic model also works in positive characteristic, where it yields Enriques surfaces with $h^{1,0} \neq h^{0,1}$ in characteristic 2 is due to Mumford [536]. The uniform examples of all types of Enriques surfaces in every characteristic are due to Mumford and Bombieri [78], who attribute this construction to Reid. Another example of an Enriques surface in characteristic two was given by Blass [66].

Chapter 2
Linear Systems on Enriques Surfaces

2.1 Vanishing Theorems

After stating the Riemann–Roch theorem and Serre duality for Enriques surfaces and making some elementary, yet useful observations, we turn to the vanishing theorems of cohomology on Enriques surfaces. On our way, we discuss ample, big, and nef invertible sheaves, as well as criteria to check whether a given invertible sheaf has these properties. Then, we turn to the Zariski decomposition, as well as general vanishing theorems. We refer to [612] for more on these notions for surfaces, and to [447] for a general background on positivity questions and applications. For vanishing theorems for surfaces over the complex numbers, we refer the interested reader to [43, Chapter IV.12].

Let S be an Enriques surface over an algebraically closed field $\Bbbk$. Since the canonical divisor class K_S is numerically trivial (see Theorem 1.1.14 or Corollary 1.2.3) and we have $\chi(\mathcal{O}_S) = 1$ by Table **??**, the Riemann–Roch theorem (0.10.23) for an invertible sheaf $\mathcal{L}$ on S becomes

$$h^0(S,\mathcal{L}) - h^1(S,\mathcal{L}) + h^2(S,\mathcal{L}) = \tfrac{1}{2}\mathcal{L}^2 + 1. \tag{2.1.1}$$

Moreover, Serre duality for S becomes

$$H^i(S,\mathcal{L}) \cong H^{2-i}(S,\omega_S \otimes \mathcal{L}^{-1})^\vee \quad \text{and thus,} \quad h^i(S,\mathcal{L}) = h^{2-i}(S,\omega_S \otimes \mathcal{L}^{-1}) \tag{2.1.2}$$

for $0 \le i \le 2$. Next, if D is an effective divisor on S, then the adjunction formula for the arithmetic genus $p_a(D)$ of D becomes

$$p_a(D) = \tfrac{1}{2}D^2 + h^0(D,\mathcal{O}_D). \tag{2.1.3}$$

Combined with (2.1.1), we obtain the following relation:

$$h^0(S,\mathcal{O}_S(D)) - h^1(S,\mathcal{O}_S(D)) + h^0(S,\mathcal{O}_S(K_S - D)) = p_a(D).$$

Concerning the vanishing of h^2, we have the following elementary observation.

Lemma 2.1.1 *Let S be an Enriques surface in characteristic $p \geq 0$. Let $\mathcal{L}$ be an invertible sheaf on S with $h^0(S, \mathcal{L}) \neq 0$. If $p = 2$ and S is non-classical, assume moreover that $\mathcal{L} \not\cong \mathcal{O}_S$. Then,*

$$h^2(S, \mathcal{L}) = h^0(S, \omega_S \otimes \mathcal{L}^{-1}) = 0$$

holds true.

Proof The first equality is Serre duality (2.1.2). If $p \neq 2$ or $p = 2$ and S is classical, then we have $h^2(\mathcal{O}_S) = 0$ by Serre duality and Definition 1.1.7, that is, we may assume $\mathcal{L} \not\cong \mathcal{O}_S$ in any case. Being projective, we may choose an ample divisor H on S. Since $\mathcal{L}$ is effective and non-trivial, we have $H \cdot \mathcal{L} > 0$. On the other hand, if we had $h^0(\omega_S \otimes \mathcal{L}^{-1}) \neq 0$, then there would exist a divisor $D \in |\omega_S \otimes \mathcal{L}^{-1}|$. This divisor would satisfy $H \cdot D \geq 0$, as well as $H \cdot D = H \cdot (\omega_S \otimes \mathcal{L}^{-1}) = -H \cdot \mathcal{L} < 0$, a contradiction. □

We now want to give conditions under which h^1 is zero. To do so, we introduce a couple of notions, which are meaningful and important for much more general varieties than Enriques surfaces and we refer to [447] for the general background. An invertible sheaf $\mathcal{L}$ on a smooth and proper variety X over some algebraically closed field $\Bbbk$ is called *nef* (short for *numerically effective* or *numerically eventually free*) if we have

$$\mathcal{L} \cdot C \geq 0$$

for every curve C on X. Clearly, it suffices to check $\mathcal{L} \cdot C \geq 0$ for every integral curve C on X, that is, every curve that is reduced and irreducible. If D is a divisor on X, then it is called nef if the associated invertible sheaf $\mathcal{O}_X(D)$ is. A related notion is the following: an invertible sheaf $\mathcal{L}$ on a smooth and projective surface X is called *pseudo-effective* if $\mathcal{L} \cdot \mathcal{M} \geq 0$ for all ample invertible sheaves on $\mathcal{M}$. The name is justified by the fact that if $h^0(X, \mathcal{L}^{\otimes n}) \neq 0$ for some $n \geq 1$, then $\mathcal{L}$ is pseudo-effective. For Enriques surfaces, we have the following elementary, but useful observations.

Lemma 2.1.2 *Let S be an Enriques surface. Let $\mathcal{L}$ be an invertible sheaf on S with $\mathcal{L}^2 \geq 0$. Then:*

1. *$\mathcal{L}$ or $\mathcal{L}^\vee$ is pseudo-effective.*
2. *If $\mathcal{L}$ is nef, then $\mathcal{L}$ is pseudo-effective.*
3. *If $\mathcal{L}^2 > 0$ and $\mathcal{L}$ is nef, then $\mathcal{L}$ is effective and we have $h^0(S, \mathcal{L}) \geq 1 + \frac{1}{2}\mathcal{L}^2 \geq 2$.*

Proof By Riemann–Roch and Serre duality, we have

$$h^0(\mathcal{L}) + h^0(\omega_S \otimes \mathcal{L}^\vee) \geq \chi(\mathcal{O}_S) + \frac{1}{2}\mathcal{L}^2 \geq 1.$$

Since the order of ω_S in $\mathrm{Pic}(S)$ is at most two, it follows that $h^0(\mathcal{L}^{\otimes 2}) \neq 0$ or $h^0(\mathcal{L}^{\otimes -2}) \neq 0$. In particular, $\mathcal{L}$ or $\mathcal{L}^\vee$ is pseudo-effective. Assume furthermore that $\mathcal{L}$ is nef. If $h^0(\mathcal{L}^{\otimes 2}) \neq 0$, then $\mathcal{L}$ is pseudo-effective and we are done. If

$h^0(\mathcal{L}^{\otimes 2}) = 0$, then $h^0(\mathcal{L}^{\otimes(-2)}) \neq 0$ and thus, we have $\mathcal{L} \cdot \mathcal{M} \leq 0$ for all ample invertible sheaves $\mathcal{M}$ on S. Since $\mathcal{L}$ was nef, it follows that $\mathcal{L}$ must be numerically trivial in this case, and thus, $\mathcal{L}$ is again pseudo-effective. Finally, assume that $\mathcal{L}^2 > 0$ and that $\mathcal{L}$ is nef. Then, $\mathcal{L}$ is not numerically trivial and the previous analysis shows that $h^0(\mathcal{L}^{\otimes -2}) = 0$, which implies $h^0(\omega_S \otimes \mathcal{L}^{\otimes -1}) = 0$. In particular, Serre duality yields $h^2(\mathcal{L}) = 0$ and then, Riemann–Roch implies $h^0(\mathcal{L}) \geq 1 + \frac{1}{2}\mathcal{L}^2 \geq 2$. □

Let us recall that we introduced the Kodaira–Iitaka dimension $\kappa(X, \mathcal{L})$ of an invertible sheaf $\mathcal{L}$ on X in Section 1.1. Moreover, we defined $\mathcal{L}$ to be big if $\kappa(X, \mathcal{L}) = \dim(X)$. If D is a divisor on X, then it is called *big* if the associated invertible sheaf $\mathcal{O}_X(D)$ is. Clearly, if $\mathcal{L}$ is an ample invertible sheaf on X, then it is big and nef. Using Riemann–Roch for $\mathcal{L}^{\otimes m}$ for all $m > 0$, we have the following useful result and refer to [38, Lemma 14.7] for proof.

Lemma 2.1.3 *Let X be a smooth and proper surface and let $\mathcal{L}$ be an invertible sheaf on X that is nef. Then, $\mathcal{L}$ is big if and only if $\mathcal{L}^2 > 0$.*

Next, we have the following fundamental theorem, which characterizes ample invertible sheaves in terms of intersection theory.

Theorem 2.1.4 (Nakai–Moishezon criterion) *Let X be a smooth and proper surface and let $\mathcal{L}$ be an invertible sheaf on X. Then, $\mathcal{L}$ is ample if and only if the following two conditions are fulfilled:*

1. *$\mathcal{L}^2 > 0$ and*
2. *$\mathcal{L} \cdot C > 0$ for every curve C on X.*

Remark 2.1.5 Mumford gave an example of a smooth and complex projective surface X together with an invertible sheaf $\mathcal{L}$ that is not ample but that satisfies $\mathcal{L} \cdot C > 0$ for every curve C on X. In particular, the assumption $\mathcal{L}^2 > 0$ cannot be dropped. For proofs, history, examples, and further details, we refer to [38, Theorem 1.22] or [447, I:Section 1.2.B].

For K3 surfaces and Enriques surfaces, we have the following interesting result that describes the difference between invertible sheaves that are ample and those that are merely big and nef. This result also establishes a connection to the negative definite lattices A_n, D_n, E_6, E_7, and E_8 introduced in Example 0.8.7. Finally, it also shows a connection to the theory of rational double point singularities, see Proposition 0.4.8 and Proposition 0.4.9. Despite its simplicity, this is a key observation.

Proposition 2.1.6 *Let X be a K3 surface or an Enriques surface.*

1. *Let C be an integral curve on X.*

   a. *If $C^2 < 0$, then $K_X \cdot C = 0$, $C^2 = -2$, and $C \cong \mathbb{P}^1$.*
   b. *If $C \cong \mathbb{P}^1$, then $K_X \cdot C = 0$ and $C^2 = -2$*

   Such a curve C is called a (-2)-curve *or a* nodal curve.

2. Let $\mathcal{L}$ be a big and nef invertible sheaf on X. Then, the set

$$\{C|\ C \text{ is an integral curve on } X \text{ with } \mathcal{L} \cdot C = 0\}$$

is finite and consists of (-2)-curves only. These span an even and negative definite sublattice M inside $\mathrm{Num}(X)$ *(not necessary primitive) that is the orthogonal sum of negative definite lattices of type A_n, D_n, E_6, E_7, and E_8. Moreover, these (-2)-curves form a root basis of finite type inside M.*

Proof Since K_X is numerically trivial, we have $K_X \cdot C = 0$ for every integral curve C. Moreover, the adjunction formula gives $p_a(C) = 1 + \frac{1}{2}C^2$. If we have $C^2 < 0$, then $p_a(C) \geq 0$ and the adjunction formula imply $p_a(C) = 0$ and $C^2 = -2$. In particular, we find $C \cong \mathbb{P}^1$. Conversely, if $C \cong \mathbb{P}^1$, then $K_X \cdot C = 0$ and $p_a(C) = 0$ imply $C^2 = -2$. This establishes the first claim.

Now, let $\mathcal{L}$ be a big and nef invertible sheaf. If C is an irreducible curve with $\mathcal{L} \cdot C = 0$, then the Hodge index theorem implies $C^2 < 0$, and thus, C is a (-2)-curve by the first claim. By definition, the classes of these curves lie in the orthogonal complement $M' := \langle \mathcal{L} \rangle^{\perp}$ inside $\mathrm{Num}(X)$. Since $\mathrm{Num}(X)$ is of signature $(1, \rho - 1)$ and $\mathcal{L}^2 > 0$, it follows that M' is a negative definite primitive sublattice equal to the primitive closure of M. It follows from Riemann–Roch and the fact that K_X is numerically trivial that $\mathrm{Num}(X)$ is an even lattice and thus, also the sublattice M is even.

Since $C^2 = -2$ for every integral curve with $\mathcal{L} \cdot C = 0$, the C are primitive vectors in M and it is easy to verify (0.8.6), that is, these curves are roots. The remaining assertions can be shown along the lines of Proposition 0.4.9 or follow from Proposition 0.8.15, which we leave to the reader. □

A fundamental result for pseudo-effective divisors is the *Zariski decomposition*. Before stating it, we note that this is a result for which we have to pass to $\mathbb{Q}$-divisors. Let us briefly digress on this: if $\mathrm{Div}(X)$ denotes the abelian group of divisors on the smooth and proper surface X, then a $\mathbb{Q}$*-divisor* is an element of $\mathrm{Div}(X) \otimes_{\mathbb{Z}} \mathbb{Q}$. Being defined in terms of intersection numbers, the notions *nef* and *pseudo-effective* also make sense for $\mathbb{Q}$-divisors. Moreover, if $D = \sum_i a_i P_i$ for some prime divisors P_i and some $a_i \in \mathbb{Q}$, we define $[D] := \sum_i [a_i] P_i$, where $[x]$ denotes the largest integer $\leq x$ for all $x \in \mathbb{Q}$. Moreover, we will say that D is *effective*, denoted by $D \geq 0$, if we have $a_i \geq 0$ for all i. Also, if D is a $\mathbb{Q}$-divisor on X, one can still define a sheaf $\mathcal{O}_X(D)$ by defining its global sections over a Zariski open subset $U \subseteq X$ to be

$$H^0(U, \mathcal{O}_X(D)) \ := \ \{f \in \Bbbk(X) \,|\, [(f) + D]|_U \ \geq \ 0\} \ \cup \ \{0\}.$$

Since X is a smooth surface, this is an invertible sheaf.

Theorem 2.1.7 (Zariski–Fujita) *Let X be a smooth and proper surface and let $D \in \mathrm{Pic}(X) \otimes_{\mathbb{Z}} \mathbb{Q}$ be a pseudo-effective $\mathbb{Q}$-divisor. Then, there exists a unique decomposition of $\mathbb{Q}$-divisors*

$$D \ = \ P + N \ \in \ \mathrm{Div}(X) \otimes_{\mathbb{Z}} \mathbb{Q},$$

such that P is a nef $\mathbb{Q}$-divisor class and N is effective (not necessary nef). Moreover, if $N > 0$, say $N = \sum_{i=1}^{n} a_i C_i$, with pairwise distinct integral curves C_i and $a_i \in \mathbb{Q}_{>0}$, then $P \cdot C_i = 0$ for all i and the intersection matrix

$$\left(C_i \cdot C_j\right)_{1 \le i,j \le n}$$

is negative definite.

In this decomposition, P is called the *positive part* and N is called the *negative part* of D. For example, we have the following useful corollary, which generalizes Lemma 2.1.3. We refer to [38, Corollary 14.18] for proof.

Corollary 2.1.8 *Let $D \in \mathrm{Div}(X) \otimes_{\mathbb{Z}} \mathbb{Q}$ be a pseudo-effective $\mathbb{Q}$-divisor on X with Zariski decomposition $D = P + N$. Then, D is big if and only if $P^2 > 0$.*

For K3 surfaces and Enriques surfaces, we have the following result, whose proof we leave to the reader since it can be done along the lines of the proof of Proposition 2.1.6. This shows again the connection between the geometry of K3 surfaces and Enriques surfaces, the theory of even and negative definite lattices of finite type, and the theory of rational double point singularities.

Proposition 2.1.9 *, Let X be a K3 surface or an Enriques surface and let $D \in \mathrm{Div}(X) \otimes_{\mathbb{Z}} \mathbb{Q}$ be a pseudo-effective $\mathbb{Q}$-divisor on X with Zariski decomposition $D = P + N$, say with $N = \sum_i a_i C_i$ for pairwise distinct integral curves C_i and $a_i \in \mathbb{Q}_{>0}$. Let $M \subseteq \mathrm{Num}(X)$ be the sublattice spanned by the C_i.*

1. *The lattice M is even and negative definite and the C_i form a root basis of finite type.*
2. *Moreover, M is the orthogonal sum of negative definite lattices of type A_n, D_n, E_6, E_7, and E_8.*

Next, if X is a smooth and proper surface and D is an effective divisor on X, then D is called *numerically connected* if whenever we have two non-zero and effective divisors D_1 and D_2 on X with $D = D_1 + D_2$, then

$$D_1 \cdot D_2 > 0$$

holds true. In Section 2.5, we will generalize and refine this notion. We start with a couple of general remarks from [612, Lemma 3.11] and include the proofs for the reader's convenience.

Lemma 2.1.10 *Let X be a smooth and proper surface.*

1. *Let D be an irreducible curve (but not necessarily reduced) on X with $D^2 > 0$. Then, D is nef and numerically connected.*
2. *Let D be an effective and numerically connected divisor on X. Then, we have*

$$h^0(D, \mathcal{O}_D) = 1.$$

3. *Let $\mathcal{L}$ be a big and nef invertible sheaf on X with $h^0(X, \mathcal{L}) \neq 0$. Then, every effective divisor $0 \neq D \in |\mathcal{L}|$ is numerically connected.*
4. *Assume that $h^1(X, \mathcal{O}_X) = 0$ and let $\mathcal{L}$ be a big and nef invertible sheaf on X with $h^0(X, \mathcal{L}) \neq 0$. Then, we have*

$$h^1(X, \mathcal{L}^{-1}) \;=\; h^1(X, \omega_X \otimes \mathcal{L}) \;=\; 0.$$

Proof First, let D be an irreducible curve on X with $D^2 > 0$. Numerical connectivity is trivially true. To prove that D is nef, let C be a reduced and irreducible curve on X. If the supports of C and D coincide, then we find $C \cdot D > 0$ using $D^2 > 0$. Otherwise, $C \cdot D$ is equal to the number of intersection points of C and D, counted with multiplicities, which is a non-negative number. This proves the first assertion.

To prove assertion (2), let $D_1 \subseteq D$ be an effective divisor with $h^0(D_1, \mathcal{O}_{D_1}) = 1$ (such divisors exist, for example, one could choose D_1 to be reduced and irreducible). Now, if $D_1 \neq D$, then $D_1 \cdot (D - D_1) \geq 1$ by numerical connectivity. Thus, there exists a reduced and irreducible divisor $\Gamma \subseteq (D - D_1)$ with $D_1 \cdot \Gamma \geq 1$. Clearly, we have an exact sequence

$$H^0(\Gamma, \mathcal{O}_\Gamma(-D_1)) \;\to\; H^0(D_1 + \Gamma, \mathcal{O}_{D_1+\Gamma}) \;\to\; H^0(D_1, \mathcal{O}_{D_1})\,.$$

Since the $\mathbb{k}$-vector space on the left is zero and the one on the right is one-dimensional, we find $h^0(\mathcal{O}_{D_1+\Gamma}) = 1$. From this, the claim follows by induction.

To prove Assertion (3), let $0 \neq D \in |\mathcal{L}|$. Let D_1 and D_2 be two effective and non-zero divisors with $D = D_1 + D_2$. Since D is nef, we find

$$\begin{aligned} D_1^2 \,+\, D_1 \cdot D_2 &= D \cdot D_1 \geq 0, \\ D_1 \cdot D_2 \,+\, D_2^2 &= D \cdot D_2 \geq 0. \end{aligned}$$

Now, if we had $D_1 \cdot D_2 \leq 0$, then we would find $D_1^2 \cdot D_2^2 \geq (D_1 \cdot D_2)^2 \geq 0$, which contradicts the Hodge index theorem and establishes the assertion.

To show Claim (4), let $\mathcal{L}$ be a big and nef invertible sheaf on X with $h^0(X, \mathcal{L}) \neq 0$. The first equality is Serre duality (2.1.2). Consider the short exact sequence

$$0 \;\to\; \mathcal{O}_X(-D) \;\to\; \mathcal{O}_X \;\to\; \mathcal{O}_D \;\to\; 0.$$

By assumption, we have $H^1(\mathcal{O}_X) = 0$ and it follows from assertion (2) that the restriction map $H^0(\mathcal{O}_X) \to H^0(\mathcal{O}_D)$ is surjective. Thus, taking cohomology, we find $h^1(\mathcal{L}^{-1}) = h^1(\mathcal{O}_X(-D)) = 0$. □

For Enriques surfaces, we obtain the following effectivity and vanishing result.

Corollary 2.1.11 *Let S be an Enriques surface in characteristic $p \geq 0$ and let $\mathcal{L}$ be a big and nef invertible sheaf on S. Then:*

1. *$h^0(S, \mathcal{L}) \neq 0$ and every divisor $D \in |\mathcal{L}|$ is numerically connected.*
2. *If $p \neq 2$ or if $p = 2$ and S is classical, then $h^1(S, \mathcal{L}) = h^1(S, \mathcal{L}^{-1}) = 0$.*

Proof By Lemma 2.1.2, we have $h^0(\mathcal{L}) \neq 0$ and then, Assertion (1) follows from Lemma 2.1.10.(3). If $p \neq 2$ or $p = 2$ and S is classical, then we have $h^1(\mathcal{O}_S) = 0$ and thus, $h^1(\mathcal{L}^{-1}) = 0$ follows from Lemma 2.1.10.(4). Since also $\omega_S \otimes \mathcal{L}$ is big and nef and $\omega_S \cong \omega_S^{-1}$ by Corollary 1.2.3, we find $0 = h^1(\omega_S \otimes \mathcal{L}^{-1}) = h^1(\mathcal{L})$, where the first equality follows from what we have just established. □

For more results on curves on surfaces, we refer the interested reader to [538].

Before establishing vanishing results also for non-classical Enriques surfaces in characteristic 2 in Theorem 2.1.16 below, let us put vanishing of H^1 into a larger perspective. Namely, we have the following classical and famous results for surfaces, which are much more sophisticated than the results of the above discussion and much more difficult to prove.

Theorem 2.1.12 *Let X be a smooth and proper surface over an algebraically closed field $\Bbbk$ of characteristic $p \geq 0$. Let $\mathcal{L}$ be a big and nef invertible sheaf on X. Then,*

$$h^i(X, \mathcal{L}^{-1}) = h^{2-i}(X, \omega_X \otimes \mathcal{L}) = 0$$

for all $i \leq 1$ holds true in the following cases:

1. *$p = 0$,*
2. *$p > 0$ and X admits a flat lift to $W_2(\Bbbk)$,*
3. *$p > 0$, $\mathcal{L}$ is effective, and Frobenius $\mathbf{F}$ acts injectively on $H^1(\mathcal{O}_X)$.*

Proof In all cases, the first equality is Serre duality (2.1.2).

If $\Bbbk = \mathbb{C}$ and $\mathcal{L}$ is ample, then the stated vanishing is the famous *Kodaira vanishing theorem* [400], see also, for example, [259, Chapter 1.2]. If $\Bbbk = \mathbb{C}$ and $\mathcal{L}$ is merely big and nef, then the stated vanishing is the *Kawamata–Viehweg vanishing theorem* [379], [717], see also [447, I:Chapter 4.3]. By the Lefschetz principle, these results hold more generally if $\Bbbk$ is of characteristic zero. Moreover, we refer the interested reader to [223] for an introduction to vanishing theorems.

If $p > 0$ and X lifts to $W_2(\Bbbk)$, then these vanishing results are due to Deligne and Illusie [156], see also [331] and [568] for overviews and further information.

Finally, $h^0(\mathcal{L}^{-1}) = 0$ follows easily from the fact that $\mathcal{L}$ is big and nef. Moreover, if $p > 0$, $\mathcal{L}$ is effective, and $\mathbf{F}$ acts injectively on $H^1(\mathcal{O}_X)$, then the assertion $h^1(\mathcal{L}^{-1}) = 0$ is the *Ramanujam vanishing theorem* [602], see also [612, Theorem 3.13]. □

Remark 2.1.13 Raynaud [608] gave examples of smooth and projective surfaces and ample invertible sheaves in positive characteristic, for which these vanishing results do not hold.

Let us recall that we introduced Frobenius-split, or, F-split, varieties in Section 1.4. For this class of varieties in positive characteristic, vanishing theorems are easy to establish:

Theorem 2.1.14 *Let X be a proper variety over an algebraically closed field $\Bbbk$ of characteristic $p > 0$ that is F-split. Let $\mathcal{L}$ be an invertible sheaf on X.*

1. If $h^i(X, \mathcal{L}^{\otimes \nu}) = 0$ for a fixed index i and $\nu \gg 0$, then $h^i(X, \mathcal{L}) = 0$.
2. If $\mathcal{L}$ is ample, then $h^i(X, \mathcal{L}) = 0$ for all $i > 0$.

Proof We only sketch the proofs and refer to [97] for details: first of all, we have $\mathbf{F}^*\mathcal{L} \cong \mathcal{L}^{\otimes p}$ and $\mathbf{F}_*\mathcal{L} = \mathcal{L} \otimes_{\mathcal{O}_X} \mathbf{F}_*\mathcal{O}_X$, see [97, Lemma 1.2.6]. Since X is F-split, it follows that also the induced map $\mathcal{L} \to \mathbf{F}_*\mathcal{L} \cong \mathcal{L} \otimes \mathbf{F}_*\mathcal{O}_X$ is split, which implies that the induced map on cohomology $H^i(X, \mathcal{L}) \to H^i(\mathcal{L} \otimes \mathbf{F}_*\mathcal{O}_X)$ is split. Since $\mathcal{L} \otimes \mathbf{F}_*\mathcal{O}_X \cong \mathbf{F}_*(\mathbf{F}^*\mathcal{L}) \cong \mathbf{F}_*(\mathcal{L}^{\otimes p})$, we find an isomorphism $H^i(\mathcal{L} \otimes \mathbf{F}_*\mathcal{O}_X) \cong H^i(\mathcal{L}^{\otimes p})$ and thus, we obtain a split injection

$$H^i(X, \mathcal{L}) \to H^i(X, \mathcal{L}^{\otimes p}).$$

In particular, if $H^i(\mathcal{L}^{\otimes p^j}) = 0$ for some $j \geq 0$, we find $H^i(\mathcal{L}) = 0$. From this, assertion (1) follows, see also [97, Lemma 1.2.7] and Theorem 1.2.8. Assertion (2) is a consequence of (1) and Serre vanishing [294, Proposition III.5.3]. □

In Theorem 1.4.18, we have determined, which Enriques surfaces are F-split. In terms of moduli spaces, "most" Enriques surfaces in characteristic $p \geq 3$ are F-split, and the previous theorem gives an elementary proof of Kodaira vanishing for them. However, one can prove Kodaira vanishing for Enriques surfaces in characteristic zero using Frobenius-splittings in characteristic p - of course, we know this already and we already have easier proofs, but the arguments of this proof may be nevertheless interesting.

Proposition 2.1.15 *Let S be an Enriques surface in characteristic zero and let $\mathcal{L}$ be an ample invertible sheaf on S. Then, $h^i(S, \mathcal{L}) = 0$ for all $i \geq 1$.*

Proof Being a variety in characteristic zero, S and $\mathcal{L}$ can be defined over a field $\Bbbk$ that is finitely generated over $\mathbb{Q}$. Thus, $\Bbbk$ is the function field of some variety B over a number field K, that is, a finite extension of $\mathbb{Q}$. Spreading out, we obtain a family $f : \mathcal{S} \to B$ over K, whose generic fiber is isomorphic S. Replacing B by an open and dense subset B if necessary, we may assume that f is smooth and (by openness of ampleness) that the restriction of $\mathcal{L}$ to the fiber $\mathcal{S}_b$ is still ample for all $b \in B$. By semi-continuity of cohomology it suffices to find a $b \in B$ such that $H^i(\mathcal{S}_b, \mathcal{L}|_{\mathcal{S}_b}) = 0$ for all $i \geq 1$. Thus, we may assume that $\Bbbk$ is a number field.

Let $\pi : X \to S$ be the K3-cover of the Enriques surface S that is defined over a number field K. Let $\mathcal{O}_K \subset K$ be the ring of integers and for a prime $\mathfrak{p} \in \mathcal{O}_K$ we denote by $X_{\mathfrak{p}}$ (resp. $S_{\mathfrak{p}}$) a reduction modulo $\mathfrak{p}$, which is unique up to isomorphism if it is smooth. After possibly replacing K by a finite extension, the set of primes

$$P := \{\mathfrak{p} \in \operatorname{Spec} \mathcal{O}_K \ : \ X_{\mathfrak{p}} \text{ is a smooth and ordinary K3 surface}\}$$

is of density one, see [74] and the preprint version of [354]. In particular, this set is infinite. For all but finitely many primes $\mathfrak{p} \in P$, also the reduction $S_{\mathfrak{p}}$ is smooth and $X_{\mathfrak{p}} \to S_{\mathfrak{p}}$ is the K3-cover. For all but finitely many primes $\mathfrak{p} \in P$, the reduction of $\pi^*(\mathcal{L})$ modulo $\mathfrak{p}$ is still ample. In particular, there exists a $\mathfrak{p} \in \mathcal{O}_K$ such that $X_{\mathfrak{p}}$ is an ordinary K3 surface, such that $\pi^*(\mathcal{L})_{\mathfrak{p}}$ is ample, and such that $\pi_{\mathfrak{p}} : X_{\mathfrak{p}} \to S_{\mathfrak{p}}$

is the K3-cover. Being ordinary, $X_{\mathfrak{p}}$ is F-split by Theorem 1.4.18 and thus, we have $H^i(X_{\mathfrak{p}}, \pi^*(\mathcal{L})_{\mathfrak{p}}) = 0$ for all $i \geq 1$ by Theorem 2.1.14. By semi-continuity, this implies $H^i(X, \pi^*\mathcal{L}) = 0$ for all $i \geq 1$. Since π is a simple $\boldsymbol{\mu}_2$-cover associated to ω_S, we obtain a short exact and split sequence

$$0 \to \mathcal{L} \to \pi_*\pi^*\mathcal{L} \to \mathcal{L} \otimes \omega_S \to 0.$$

In particular, $H^i(S, \mathcal{L})$ is a direct summand of $H^i(X, \pi^*\mathcal{L})$, which implies the desired vanishing. □

We now come to the main result of this section, namely, a vanishing theorem that holds for every Enriques surface in every characteristic. In view of Raynaud's counter-examples mentioned in Remark 2.1.13, it is interesting that Enriques surfaces do satisfy the desired vanishing, even for $\boldsymbol{\alpha}_2$-surfaces in characteristic 2: for these surfaces, $\mathbf{F}$ does not act injectively on $H^{0,1}$ by definition, they are not F-split, and they do not admit flat lifts to $W_2(\Bbbk)$ as mentioned in Remark 1.4.11. In particular, none of the vanishing results established above applies to $\boldsymbol{\alpha}_2$-surfaces.

Theorem 2.1.16 (Vanishing Theorem) *Let S be an Enriques surface and let $\mathcal{L}$ be a big and nef invertible sheaf on S. Then,*

$$h^1(S, \mathcal{L}) = h^1(S, \mathcal{L}^{-1}) = 0$$

holds true.

Proof Let S be an Enriques surface over an algebraically closed field $\Bbbk$ of characteristic $p \geq 0$. If $p \neq 2$ or if $p = 2$ and S is classical, then we established the assertions on h^1 already in Corollary 2.1.11.

Thus, we may assume that $p = 2$ and that S is non-classical. In particular, we have $\omega_S \cong \mathcal{O}_S$ and then, the first equality follows from Serre duality. By Lemma 2.1.2, there exists non-zero section $0 \neq s \in H^0(\mathcal{L})$, say with divisor of zeros C, and then, we obtain a short exact sequence

$$0 \to \mathcal{O}_S \xrightarrow{\times s} \mathcal{L} \to \mathcal{L}|_C \to 0. \tag{2.1.4}$$

Since $\omega_S \cong \mathcal{O}_S$, the adjunction formula on S yields

$$\mathcal{L}|_C \cong \mathcal{O}_S(C)|_C \cong \omega_S(C)|_C \cong \omega_C,$$

and thus, Serre duality for the Gorenstein curve C implies

$$h^1(C, \mathcal{L}|_C) = h^1(C, \omega_C) = h^0(C, \mathcal{O}_C).$$

By Lemma 2.1.10, the divisor C is numerically connected and thus, we find $h^0(C, \mathcal{O}_C) = 1$. We have $h^2(\mathcal{L}) = 0$ by Lemma 2.1.1 and thus, taking cohomology in (2.1.4), we obtain a long exact sequence

$$\ldots \to H^1(S, \mathcal{O}_S) \to H^1(S, \mathcal{L}) \to H^1(C, \mathcal{L}|_C) \to H^2(S, \mathcal{O}_S) \to 0 \tag{2.1.5}$$

We have $h^1(S, \mathcal{O}_S) = h^1(C, \mathcal{L}|_C) = h^2(S, \mathcal{O}_S) = 1$, from which we conclude that $H^1(S, \mathcal{O}_S) \to H^1(S, \mathcal{L})$ is surjective and that $h^1(S, \mathcal{L}) \leq 1$.

Now, every section $s \in H^0(S, \mathcal{L})$ gives rise to a map $\mathcal{O}_S \to \mathcal{L}$, which induces a map $H^1(S, \mathcal{O}_S) \to H^1(S, \mathcal{L})$ and gives n a linear map

$$H^0(S, \mathcal{L}) \;\to\; \mathrm{Hom}(H^1(S, \mathcal{O}_S),\, H^1(S, \mathcal{L})). \tag{2.1.6}$$

We denote by $H^0(S, \mathcal{L})_\circ$ its kernel, that is, the subspace that consists of those sections $s \in H^0(S, \mathcal{L})$ such that the corresponding map $H^1(S, \mathcal{O}_S) \to H^1(S, \mathcal{L})$ in (2.1.5) is trivial. Since $h^1(S, \mathcal{O}_S) = 1$ and $h^1(S, \mathcal{L}) \leq 1$, we find that the right-hand side of (2.1.6) is at most one-dimensional. This implies that $\operatorname{codim} H^0(S, \mathcal{L})_\circ \leq 1$.

On the other hand, we have $h^0(S, \mathcal{L}) \geq 2$ by Lemma 2.1.2, which implies that $H^0(S, \mathcal{L})_\circ \neq \{0\}$. In particular, there does exist a non-zero section $0 \neq s \in H^0(S, \mathcal{L})_\circ$. By definition of this latter vector space, the corresponding map $H^1(S, \mathcal{O}_S) \to H^1(S, \mathcal{L})$ is zero, and thus, the surjectivity of this map established after (2.1.5) implies that $H^1(S, \mathcal{L}) = 0$. □

Corollary 2.1.17 *Let S be an Enriques surface and $\mathcal{L}$ an invertible sheaf that is big and nef. Then,*

$$\dim |\mathcal{L}| \;=\; \dim |\omega_S \otimes \mathcal{L}| \;=\; \tfrac{1}{2}\mathcal{L}^2,$$

or, equivalently, $h^0(S, \mathcal{L}) = h^0(S, \omega_S \otimes \mathcal{L}) = 1 + \frac{1}{2}\mathcal{L}^2$.

Proof The assertion on h^0 follows immediately from the just–established vanishing of h^1, Riemann–Roch (2.1.1), Lemma 2.1.1, and Lemma 2.1.2. □

In Corollary 2.4.4, we will give another proof of this fundamental vanishing result using Bogomolov instability of rank two vector bundles.

2.2 Nef Divisors and Genus One Pencils

In this section, we study nef divisors and their associated linear systems on Enriques surfaces. First, we make a couple of general remarks concerning the cones of effective, nef, and ample divisors on an Enriques surface and relate these cones to root systems and Weyl groups. Then, we classify nef divisor classes of self-intersection number zero, which leads to genus one pencils, their (half-)fibers, and Kodaira's classification of degenerate fibers. This will take up most of this section. We end this section by treating linear systems associated to big and nef divisors, their fixed components, and whether they contain irreducible divisors.

Let X be a smooth and proper surface over an algebraically closed field $\Bbbk$ of characteristic $p \geq 0$. In Section 0.9, we defined the group $\mathrm{Num}(X)$ of divisor classes of X modulo numerical equivalence. It follows from Theorem 0.9.6 that $\mathrm{Num}(X)$ is a finitely generated abelian group, whose rank $\rho(X)$ is called the Picard number of X. Thus, $\mathrm{Num}(X)_\mathbb{R} := \mathrm{Num}(X) \otimes_\mathbb{Z} \mathbb{R}$ is a finite-dimensional real vector space. Since X is a surface, $\mathrm{Num}(X)$ is equipped with an intersection pairing, and by the

Hodge index theorem, this pairing is non-degenerate of signature $(1, \rho - 1)$. As explained in Chapter 0.8, the non-degeneracy implies that we may view $\mathrm{Num}(X)$ as a sublattice of finite index of the dual lattice $\mathrm{Num}(X)^\vee$. Clearly, this inclusion induces an isomorphism $\mathrm{Num}(X)_\mathbb{R} \cong \mathrm{Num}(X)^\vee_\mathbb{R}$ that is compatible with intersection forms.

The classes of effective divisors of X generate a cone

$$\mathrm{Eff}(X) \subseteq \mathrm{Num}(X)_\mathbb{R},$$

the *cone of effective divisors*, or, the *effective cone*, of X. We remind the reader that a *cone* is a subset $C \subseteq V$ of a real vector space, such that for all $c \in C$ and all $\lambda \in \mathbb{R}_{>0}$ we have $\lambda c \in V$. Moreover, a cone C is said to be *convex* if for all $c_1, c_2 \in C$ we also have $c_1 + c_2 \in C$. It is easy to see that $\mathrm{Eff}(X)$ is a convex cone.

As defined in the previous section, a divisor D is nef, if we have $D \cdot C \geq 0$ for every effective curve C on X. Thus, in terms of the effective cone, a nef divisor class is an element of $\mathrm{Num}(X) \subseteq \mathrm{Num}(X)^\vee$ that is non-negative on the effective cone $\mathrm{Eff}(X)$. Thus, the *cone of nef divisors*, or, the *nef cone*, is the dual cone to $\mathrm{Eff}(X)$

$$\mathrm{Nef}(X) = \mathrm{Eff}(X)^\vee := \{x \in \mathrm{Num}(X)_\mathbb{R} \mid x \cdot y \geq 0 \text{ for all } y \in \mathrm{Eff}(X)\}.$$

It is easy to see that $\mathrm{Nef}(X)$ is a convex cone inside $\mathrm{Num}(X)_\mathbb{R}$ and that the set of nef divisor classes is equal to the intersection $\mathrm{Nef}(X) \cap \mathrm{Num}(X)$ inside $\mathrm{Num}(X)_\mathbb{R}$.

We note that a nef divisor D satisfies $D^2 \geq 0$, see, for example, [612, Chapter D.2.3]. By Theorem 2.1.4, a nef divisor D is ample if and only if $D^2 > 0$. Thus, the *cone of ample divisors*, or, the *ample cone*, denoted by $\mathrm{Amp}(X)$, is the cone generated by ample divisor classes inside $\mathrm{Num}(X)_\mathbb{R}$. Again, it is easy to see that this is a convex cone, and that we have an inclusion of convex cones

$$\mathrm{Amp}(X) \subseteq \mathrm{Nef}(X) \subseteq \mathrm{Num}(X)_\mathbb{R}.$$

More precisely, $\mathrm{Amp}(X)$ is the interior (with respect to the classical topology) of $\mathrm{Nef}(X)$ inside $\mathrm{Num}(X)_\mathbb{R}$.

Next, let

$$V_X := \{x \in \mathrm{Num}(X)_\mathbb{R} : x^2 \geq 0\},$$

Then, $V_X \setminus \{0\}$ has two connected components, since the lattice $\mathrm{Num}(X)$ is hyperbolic, see also Section 0.8. We set $V_X \setminus \{0\}^+$ to be the connected component that contains the ample cone. We denote the interior of the cone V_X^+ by $\mathrm{Big}(X)$. This the *cone of big divisor classes*. The intersection $\mathrm{Big}(X) \cap \mathrm{Num}(X)$ consists of numerical classes of effective divisors with positive self-intersection. We have

$$\mathrm{Amp}(X) \subseteq \mathrm{Big}(X) \subseteq \mathrm{Eff}(X) \subseteq \mathrm{Num}(X)_\mathbb{R}.$$

We recall from Section 0.8 that the image of the interior of V_X in the real projective space $|\mathrm{Num}(X)_\mathbb{R})| = \mathrm{Num}(X)_\mathbb{R} \setminus \{0\}/\mathbb{R}^\times$ is the hyperbolic space associated to $\mathrm{Num}(X)_\mathbb{R}$. The following result applies to K3 surfaces and Enriques surfaces and should be compared with Proposition 2.1.6 and Proposition 2.1.9, where we already observed a connection to the theory of lattices and their root bases.

Proposition 2.2.1 *Let X be a K3 surface or an Enriques surface. Let $\mathcal{R} := \mathcal{R}(X) \subset$ $\mathrm{Num}(X)$ be the set of classes of (-2)-curves. Then:*

1. *$\mathcal{R}$ is a root basis in* $\mathrm{Num}(X)$,
2. *the nef cone* $\mathrm{Nef}(X)$ *is the fundamental chamber $C(\mathcal{R})$ of $\mathcal{R}$, and*
3. *the ample cone* $\mathrm{Amp}(X)$ *is the interior of the fundamental chamber.*

Proof Let D be a divisor class with $D^2 \geq 0$. By Riemann–Roch, we have $h^0(D) + h^0(K_X - D) \geq \chi(X, \mathcal{O}_X) > 0$ and thus, D or $(K_X - D)$ is effective. Passing to their classes in $\mathrm{Num}(X)$, we find that $[D]$ or $-[D]$ lies in $\mathrm{Eff}(X)$. Thus, we may represent $[D]$ by a divisor $D = \pm \sum m_i [D_i]$, where $m_i > 0$ and where the D_i are integral curves. Intersecting with the class of an ample divisor, we conclude that $[D] \in V_X^+$ if and only if $[D]$ is effective. If D is effective and C is an integral curve, then the intersection number $D \cdot C = \sum m_i D_i \cdot C$ can only be negative if C coincides with a component D_i with $D_i^2 < 0$. Since K_X is numerically trivial, the adjunction formula implies that such a D_i must be a (-2)-curve, see also Proposition 2.1.6. Thus, $\mathrm{Nef}(X)$ consists of classes of divisors D that satisfy $D \cdot R \geq 0$ for every $R \in \mathcal{R} = \mathcal{R}(X)$. Since we have $R \cdot R' \geq 0$ if $R \neq R'$, we see that $\mathcal{R}$ satisfies the conditions for a root basis in $\mathrm{Num}(X)$. By definition, its fundamental chamber coincides with the nef cone. In our discussion above, we have already seen that the ample cone is the interior of the nef cone. □

We denote by W_X^{nod} the Weyl group defined by the root basis $\mathcal{R} = \mathcal{R}(X)$. This is called the *nodal Weyl group* of X and it is a subgroup of the Weyl group $W(\mathrm{Num}(X))$. If X is a K3 surface, then the group W_X^{nod} coincides with the Weyl group $W(\mathrm{Num}(X))$, see [175, Proposition 5.10], or [549, Remark 3.5]. We will discuss nodal Weyl groups of Enriques surfaces in Section 2.3.

We now turn to Enriques surfaces and classify nef divisors on them. Thus, let S be an Enriques surface over an algebraically closed field and let D be a nef divisor.

First, we treat the case where $D^2 = 0$. These divisors are interesting because they occur as fibers of genus one fibrations, see Lemma 2.2.2 below. We write $D = \sum n_i R_i$ as a sum of its irreducible components with multiplicities n_i. Since $K_S \equiv 0$, we have $K_S \cdot R_i = 0$. If $R_i^2 < 0$, then R_i is a (-2)-curve by Proposition 2.1.6. On the other hand, for every component R_i with $R_i^2 \geq 0$, we compute

$$0 = D^2 \geq D \cdot R_i = n_i R_i^2 + \sum_{j \neq i} n_j R_j \cdot R_i,$$

which implies that $R_i^2 = 0$, as well as $R_i \cdot R_j = 0$ for all j. In particular, if D is connected, then it is either supported on an integral curve with self-intersection number zero or else it is supported on a union of (-2)-curves. We will refine this observation in Proposition 2.2.5 below.

Let us remind the reader that we defined the notion of numerically connected divisors in the previous section. We note that if E is an irreducible curve with $E^2 = 0$

and $m \geq 1$ is an integer, then mE is connected for all $m \geq 1$, whereas mE is numerically connected if and only if $m = 1$.

We now define (indecomposable) divisors of canonical type on smooth and proper surfaces, which were introduced by Mumford [539]. Divisors of canonical type should be thought of as sums of fibers of genus one fibrations, and indecomposable divisors of canonical type should be thought of as non-multiple and connected fibers of genus one fibrations. If X is a smooth and proper surface, then an effective divisor $D = \sum n_i R_i$ with $n_i > 0$ for all i that satisfies $K_X \cdot R_i = 0$ and $D \cdot R_i = 0$ for all i is said to be a *of canonical type*. In particular, divisors of canonical type are nef. If a divisor of canonical type D is moreover connected and if the greatest common divisor of the n_i's is equal to 1, then it is said to be *indecomposable*, see [38, Section 7] or [539, Section 2]. Clearly, every divisor of canonical type is the sum of indecomposable divisors of canonical type.

In [612], Reid defines a divisor D to be *of elliptic fiber type* if it is nef, not numerically trivial, and if it satisfies $D^2 = K_X \cdot D = 0$. Using the Hodge index theorem it is easy to see that an effective divisor is of elliptic fiber type if and only if it is of canonical type. Moreover, in [612], Reid defines a 0-*curve* to be an effective divisor of elliptic fiber type $D = \sum n_i R_i$ with $n_i > 0$ for all i, such that the greatest common divisor of all n_i is equal to 1. Thus, a 0-curve is the same as an indecomposable divisor of elliptic fiber type. We note that such a curve need not be irreducible, that is, this definition is not compatible with the notion of $(-n)$-curves previously introduced. Before coming to the classification of indecomposable divisors of canonical type in Proposition 2.2.5 below, we motivate this notion.

Lemma 2.2.2 *Let X be a K3 surface or an Enriques surface. Let $f : X \dashrightarrow B$ be a dominant and rational map to a smooth and proper curve. Then:*

1. *$B \cong \mathbb{P}^1$.*
2. *If f is a fibration, that is, a morphism with $f_* \mathcal{O}_X = \mathcal{O}_B$, then all fibers of f are divisors of canonical type. In particular, all irreducible fibers are curves of arithmetic genus one.*

Proof By resolution of indeterminacies, there exists a smooth blow-up $\pi : \widetilde{X} \to X$ such that f extends to a morphism $f \circ \pi : \widetilde{X} \to B$. Since $b_1(\widetilde{X}) = b_1(X) = 0$, the Albanese variety of X is trivial. Thus, by the universal property defining the Albanese map, also the Jacobian of B must be trivial, that is, $B \cong \mathbb{P}^1$. This establishes claim (1).

Next, assume that f is a fibration and let $F = \sum n_i R_i$ be a fiber of f. Since K_X is numerically trivial, we have $K_X \cdot R_i = 0$. Let F' be another fiber of f, distinct of F. Then, F' is disjoint from every component R_i of F and thus, $F' \cdot R_i = 0$ for all i. Since F is linearly equivalent to F', we find $F \cdot R_i = 0$ for all i, which identifies F as a curve of canonical type and establishes claim (2). □

To avoid confusion of terminology, we will call a linear system $|D|$ a *pencil* if it has no fixed components and if its dimension is equal to one. If $D^2 = 0$ and D is a pencil, then $|D|$ defines a morphism to $\mathbb{P}^1$, whose connected fibers equal to the members of $|D|$. If $|D|$ is a positive-dimensional linear system with $D^2 = 0$ and without fixed components, then $|D| = |kP|$, where P is a pencil. If $k > 1$, then D is said to be *composite with a pencil*, see [294, Chapter III, Exercise 11.3]. In this case, the linear system $|D|$ has dimension k and defines a morphism, whose Stein factorization is of the form $f : S \to B \to \mathbb{P}^1$, where $B \to \mathbb{P}^1$ is a finite cover of degree k, see [294, Corollary 11.5].

Next, we establish some elementary properties of (indecomposable) divisors of canonical type and relate them to some previously defined notions.

Proposition 2.2.3 *Let X be a smooth and proper surface with K_X nef.*

1. *If D is an effective and nef divisor with $K_X \cdot D = D^2 = 0$, then D is a divisor of canonical type. If D is moreover numerically effective, then D is indecomposable.*
2. *If $D = \sum_i n_i R_i$ is an indecomposable divisor of canonical type, then D is numerically connected. Moreover, if $\mathcal{L}$ is an invertible $\mathcal{O}_D$-module with* $\deg \mathcal{L}|_{R_i} = 0$ *for all i, then*
$$H^0(D, \mathcal{L}) \neq 0 \quad \textit{if and only if} \quad \mathcal{L} \cong \mathcal{O}_D.$$

Proof First, let $D = \sum_i n_i R_i$ be as in the first part of Assertion (1). Since K_X is nef and $K_X \cdot D = 0$, we conclude $K_X \cdot R_i = 0$ for all i. Similarly, since D is nef and $D^2 = 0$, we conclude $D \cdot R_i = 0$ for all i. Thus, D is of canonical type. Since D is of canonical type, it is a sum $\sum a_i D_i$ of indecomposable divisors D_i of canonical type, where D_i and D_j are disjoint for $i \neq j$. Thus, if D is numerically connected, then D is indecomposable.

Now, let $D = \sum_{i=1}^s n_i R_i$ be an indecomposable divisor of canonical type. If D is irreducible, then being indecomposable implies that it is also numerically connected and thus, we may assume $s \geq 2$. Since K_X is nef and $K_X \cdot D = 0$, we conclude $K_X \cdot R_i = 0$ for all i. Since $s \geq 2$, the Hodge index theorem implies $R_i^2 < 0$ for all i, and thus, the adjunction formula actually implies $R_i^2 = -2$. If $M \subseteq \mathrm{Num}(X)$ denotes the sublattice spanned by the R_i, then $\mathsf{B} := \{R_i\}_i$ is a root basis of M. Since $D \cdot R_i = 0$, the R_i lie in the orthogonal complement of the class $[D]$ and thus, $[D]$ lies in the radical of M. Since $D^2 = 0$, it follows from the Hodge index theorem that M is negative semi-definite and D is connected, the root system B is irreducible. But then, the radical of M is actually spanned by $[D]$, which implies that for every decomposition $D = A + B$, where A, B are effective and non-zero, we have $A^2 < 0$ and $B^2 < 0$. Thus, from $0 = A \cdot D = A^2 + AB$, we conclude $A \cdot B \geq 1$, that is, D is numerically connected. For the remaining assertions of Claim (2), we refer to [38, Theorem 7.8] or [539, Section 2]. □

In particular, if K_X is numerically trivial, then we obtain the following corollary.

Corollary 2.2.4 *Let X be a K3 surface or an Enriques surface. For a divisor D on X, the following are equivalent:*

1. *D is effective, nef, and numerically connected with $D^2 = 0$, and*
2. *D is indecomposable of canonical type.*

We now come to the classification of indecomposable divisors of canonical type: quite generally, for any effective divisor $D = \sum n_i R_i$ with $n_i \geq 1$ and R_i integral curves for all i on a smooth projective surface, we denote by $\Gamma(D)$ the graph whose vertices are the irreducible components R_i of D and two vertices are joined by an edge if the components intersect. Moreover, we label the edge with intersection number if it is greater than 1.

It follows from the above discussion or the proof of the previous Proposition that an indecomposable divisor of canonical type D is either of the form R, where R is an irreducible curve with $R^2 = 0$, or it is of the form $D = \sum n_i R_i$, where the R_i are (-2)-curves, the greatest common divisor of the n_i's is equal to 1, and the graph $\Gamma(D)$ is connected. In case R is irreducible with $R^2 = 0$, the adjunction formula implies that R is of arithmetic genus one, and thus, one of the following: a smooth elliptic curve, or a rational curve that is singular with one ordinary node singularity, or a rational curve that is singular with one ordinary cusp singularity. In terms of simple curve singularities as discussed in Chapter 0.4 and if the characteristic of the ground field is different from 2, an ordinary node (resp. cusp) is a singularity of type a_1 (resp. a_2). On the other hand, if D is a reducible divisor, then have already seen in the proof of the previous Proposition that the set $\mathsf{B} := \{[R_i]\}_i$ is an irreducible root system of affine type of the lattice M_{B} spanned by the classes $[R_i]$ inside $\mathrm{Num}(X)$. Moreover, the class $[D]$ belongs to the radical of M_{B}. Applying the classification of irreducible root systems of affine type from Proposition 0.8.16, we thus obtain the following classification.

Proposition 2.2.5 *Let $D = \sum n_i R_i$ be an indecomposable divisor of canonical type on a smooth and proper surface over an algebraically closed field. Then, either D is irreducible and it is one of the following*

I_0 *a smooth elliptic curve,*
I_1 *a singular rational curve with an ordinary node,*
II *a singular rational curve with a cusp,*

or else D is reducible, all R_i are (-2)-curves, and it is one of the following

I_2 *: $\Gamma(D)$ is of type $\widetilde{A}_1$, that is,*
 $D = R_1 + R_2$*, where R_1 and R_2 intersect transversally in two points,*
III*: $\Gamma(D)$ is of type $\widetilde{A}_1$, that is,*
 $D = R_1 + R_2$*, where R_1 is tangent to R_2 at one point and $R_1 \cdot R_2 = 2$,*
I_3 *: $\Gamma(D)$ is of type $\widetilde{A}_3$, that is,*
 $D = R_1 + R_2 + R_3$ *with $R_i \cdot R_j = 1$ if $i \neq j$ and $R_1 \cap R_2 \cap R_3 = \emptyset$,*
IV*: $\Gamma(D)$ is of type $\widetilde{A}_3$, that is,*
 $D = R_1 + R_2 + R_3$ *with $R_i \cdot R_j = 1$ if $i \neq j$ and $R_1 \cap R_2 \cap R_3 \neq \emptyset$,*
I_n *: $\Gamma(D)$ is of type $\widetilde{A}_n$, and $n \geq 4$, that is,*
 $D = R_1 + \cdots + R_n$*, with $R_1 \cdot R_2 = R_2 \cdot R_3 = \ldots = R_{n-1} \cdot R_n = R_n \cdot R_1 = 1$ and $R_i \cdot R_j = 0$ otherwise,*
I_{n+4}^**: $\Gamma(D)$ is of type $\widetilde{D}_{n+4}$, that is,*
 $D = R_0 + R_1 + R_2 + R_3 + 2R_4 + \cdots + 2R_{4+n}$ *with $R_1 \cdot R_4 = R_2 \cdot R_4 = R_4 \cdot R_5 = \ldots = R_{3+n} \cdot R_{4+n} = R_{4+n} \cdot R_2 = R_{4+n} \cdot R_3 = 1$ and $R_i \cdot R_j = 0$ otherwise,*

IV^*: $\Gamma(D)$ *is of type* $\widetilde{E}_6$, *that is,*
$D = R_0 + 2R_1 + R_2 + 2R_3 + 3R_4 + 2R_5 + R_6$ *with* $R_0 \cdot R_1 = R_1 \cdot R_2 = R_1 \cdot R_4 = R_2 \cdot R_3 = R_3 \cdot R_4 = R_4 \cdot R_5 = R_5 \cdot R_6 = 1$ *and* $R_i \cdot R_j = 0$ *otherwise,*

III^*: $\Gamma(D)$ *is of type* $\widetilde{E}_7$, *that is,*
$D = 2R_0 + R_1 + 2R_2 + 3R_3 + 4R_4 + 3R_5 + 2R_6 + R_7$ *with* $R_0 \cdot R_4 = R_1 \cdot R_2 = R_2 \cdot R_3 = \ldots = R_6 \cdot R_7 = 1$ *and* $R_i \cdot R_j = 0$ *otherwise,*

II^*: $\Gamma(D)$ *is of type* $\widetilde{E}_8$, *that is,*
$D = 2R_0 + 2R_1 + 4R_2 + 6R_3 + 5R_4 + 4R_5 + 3R_6 + 2R_7 + R_8$, *where* $R_0 \cdot R_3 = R_1 \cdot R_2 = R_2 \cdot R_3 = \ldots = R_7 \cdot R_8 = 1$ *and* $R_i \cdot R_j = 0$ *otherwise.*

The multiplicities should be compared to the fundamental cycles associated to root systems of finite type, see Corollary 0.4.12.

Remark 2.2.6 Here, we have used Kodaira's notation from [401], but there is also Néron's notation from [553], which we include here for the reader's convenience:

Kodaira	I_0	I_1	$I_n, n > 1,$	II	III	IV	I_0^*	I_n^*	IV^*	III^*	II^*
Néron	a	b_1	b_n	$c1$	$c2$	$c3$	$c4$	$c5_n$	$c6$	$c7$	$c8$
root lattice	$\widetilde{A}_0$	$\widetilde{A}_0^*$	$\widetilde{A}_{n-1}$	$\widetilde{A}_0^{**}$	$\widetilde{A}_1^*$	$\widetilde{A}_2^*$	$\widetilde{D}_4$	$\widetilde{D}_{n+4}$	$\widetilde{E}_6$	$\widetilde{E}_7$	$\widetilde{E}_8$

We will use the notation for the corresponding type of the affine root basis. The only cases where the type of the root basis does not determine the Kodaira type are the pairs (I_2, III) and (I_3, IV). Also, the types I_0, I_1, and II do not have an associated root basis. We will see later in Section 4.8 that all the above types with at most 9 irreducible components do occur on Enriques surfaces.

We end our discussion of indecomposable divisors of canonical type with the following useful remarks and computations, which were already more or less implicit in the above discussion.

Lemma 2.2.7 *Let X be a smooth and proper surface over an algebraically closed field $\Bbbk$ with K_X nef. Let D be an indecomposable divisor of canonical type. Then:*

1. $h^i(D, \mathcal{O}_D) = 1$ *for* $i = 0, 1$.
2. $\omega_D \cong \mathcal{O}_D$.
3. $\mathcal{O}_D(D) \cong \mathcal{O}_D(K_S)$.
4. *If E is an effective divisor on X with $E \cdot R_i = 0$ for all $i \in I$, then $E = nD + A$ for some integer $n \geq 0$ and an effective divisor A that is disjoint from D.*
5. *There exists $a \in \mathbb{Q}$ with $a \geq 0$ such that $K_X = a \cdot D$ in* $\mathrm{Num}(X) \otimes \mathbb{Q}$.

Proof (1) It suffices to consider the case where D is reducible. Then, it follows from the previous classification that there exists an $i \in I$ such that $D = R_i + D'$, where D' does not contain R_i and coincides with the fundamental cycle Z of a rational double point, see Corollary 0.4.12. Thus, we have $h^0(D', \mathcal{O}_{D'}) = h^1(D', \mathcal{O}_{D'}) = 1$. Now, the assertion follows easily from taking cohomology in the exact sequence

$$0 \to \mathcal{O}_{R_i} \otimes \mathcal{O}_X(-D') \to \mathcal{O}_D \to \mathcal{O}_{D'} \to 0 \qquad (2.2.1)$$

and the observation that $\deg(\mathcal{O}_{R_i} \otimes \mathcal{O}_X(-D')) = -R_i \cdot D' = -2$.

Alternatively, we know from Proposition 2.2.3 that D is numerically connected and thus, we have $h^0(\mathcal{O}_D) = 1$ by Lemma 2.1.10. Moreover, the adjunction formula yields $\chi(\mathcal{O}_D) = 0$, which implies $h^1(\mathcal{O}_D) = h^0(\mathcal{O}_D) = 1$.

(2) Tensoring exact sequence (2.2.1) with $\mathcal{O}_X(K_X + D)$, we obtain the exact sequence

$$0 \to \omega_{R_i} \to \omega_D \to \mathcal{O}_{D'} \otimes \mathcal{O}_X(K_X + D) \to 0. \tag{2.2.2}$$

There exists a non-zero section $s \in H^0(D, \omega_D) \cong H^1(D, \mathcal{O}_D)^\vee \cong \Bbbk$. Restricting s to D', we obtain a non-zero section of $\mathcal{L} := \mathcal{O}_{D'} \otimes \mathcal{O}_X(K_X + D) \cong \mathcal{O}_{D'} \otimes \omega_D$. Since $\deg(\mathcal{L} \otimes \mathcal{O}_{R_j}) = D \cdot R_j = 0$ for all components R_j of D', we deduce $\mathcal{L} \cong \mathcal{O}_{D'}$ from Proposition 0.4.6. Therefore, s generates ω_D at each point of D'. Since $\deg(\omega_D \otimes \mathcal{O}_{R_j}) = 0$, the section s generates $\omega_D \otimes \mathcal{O}_{R_j}$ or it is identically zero. Since R_i intersects D', the latter case does not occur. Thus, s generates ω_D everywhere.

Alternatively, we have $\omega_D \cong \mathcal{O}_X(K_X + D)|_D$ and thus, we have $\deg \omega_X|_{R_i} = (K_X + D) \cdot R_i = 0$ for all i. Since $h^0(\omega_D) = h^1(\mathcal{O}_D) = 1$ by Serre duality, Proposition 2.2.3 implies $\omega_D \cong \mathcal{O}_D$.

(3) By the adjunction formula, we have $\mathcal{O}_D \cong \omega_D \cong \mathcal{O}_D(D + K_D)$ and from this, the assertion follows.

(4) Write $E = A + B$ as a sum of effective divisors, where A is chosen such that it does not have a common irreducible component with D. For every $i \in I$, we have $0 = E \cdot R_i = A \cdot R_i + B \cdot R_i = B \cdot R_i$. Since $\Gamma(D)$ is connected, it follows that the support of B is equal to the support of D. Since D is numerically connected, it also follows that $B = nD$ for some integer $n \geq 0$.

(5) is a consequence of the Hodge index theorem, see also [612, Section E.6]. □

Proposition 2.2.8 *Let S be an Enriques surface. Let D be an effective, numerically connected, and nef divisor with $D^2 = 0$ or, equivalently, an indecomposable divisor of canonical type. Then:*

1. *$|D|$ or $|2D|$ is a pencil without base points.*
2. $\dim |D| = 0$ *if and only if the class $[D] \in \mathrm{Num}(S)$ is a primitive isotropic vector.*

Moreover, if S is classical then $\dim |D + K_S| = 0$.

Proof By assumption, we have $h^0(\mathcal{O}_S(D)) \geq 1$. It follows from Riemann–Roch that $h^0(\mathcal{O}_S(D)) = 1$ if and only if $h^1(\mathcal{O}_S(D)) = 0$. Assume that this is the case. Then, taking cohomology of the exact sequence

$$0 \to \mathcal{O}_S(D) \to \mathcal{O}_S(2D) \to \mathcal{O}_D(2D) \to 0 \tag{2.2.3}$$

shows that $h^0(S, \mathcal{O}_S(2D)) = h^0(S, \mathcal{O}_S(D)) + h^0(D, \mathcal{O}_D(2D))$. By Lemma 2.2.7 and the adjunction formula, we have

$$\mathcal{O}_D(2D) \cong \mathcal{O}_D. \tag{2.2.4}$$

Since $h^0(D, \mathcal{O}_D) = 1$ and $h^1(S, \mathcal{O}_S(D)) = 0$, we find $h^0(S, \mathcal{O}_S(2D)) = 2$. Let $|2D| = T + |M|$, where T denotes the fixed part of the linear system $|2D|$. Since every proper subdivisor of D has negative self-intersection, we get $M^2 = (2D - T)^2 = T^2 < 0$, which contradicts the fact that M is movable. Thus, $|2D|$ has no fixed part and since $D^2 = 0$, it must be an irreducible pencil without base points.

Next, assume that $h^1(S, \mathcal{O}_S(D)) \neq 0$. The adjunction formula and Lemma 2.2.7 give $\mathcal{O}_D \cong \omega_D \cong \mathcal{O}_D(K_S + D)$. Using that $2K_S = 0$, we find $\mathcal{O}_D(D) = \mathcal{O}_D(K_S)$, as well as

$$0 \neq h^0(\mathcal{O}_D(D)) = h^0(\mathcal{O}_D(K_S)) \leq h^0(\mathcal{O}_D(2K_S)) = h^0(\mathcal{O}_D) = 1,$$

and thus, $h^0(\mathcal{O}_D(D)) = h^0(\mathcal{O}_D(K_S)) = 1$. We have the standard exact sequence

$$0 \to \mathcal{O}_S \to \mathcal{O}_S(D) \to \mathcal{O}_D(D) \to 0. \tag{2.2.5}$$

If S is classical, then taking cohomology, shows $h^0(\mathcal{O}_S(D)) = 2$. If S is non-classical, then we have $K_S = 0$ and thus, $\mathcal{O}_D(D) \cong \mathcal{O}_D$. Inspecting the long exact sequence in cohomology, we find $h^0(\mathcal{O}_S(D)) = 2$. Thus, we find $\dim |D| = 1$ in any case. Arguing as above, we conclude again that $|D|$ is an irreducible pencil without base points. This establishes Claim 1.

To prove Claim 2, let us first assume that $\dim |D| \geq 1$. Since $h^0(S, \mathcal{O}_S(D)) \geq 2$, it follows from Riemann–Roch that $h^1(S, \mathcal{O}_S(D)) \neq 0$. Moreover, it follows from Riemann–Roch that $h^0(S, \mathcal{O}_S(K_S + D)) \geq 1$, that is, there exists an effective divisor $D' \in |K_S + D|$, which is of canonical type. Seeking a contradiction, we assume that D' is indecomposable. In particular, D' is numerically connected and thus, satisfies $h^0(\mathcal{O}_{D'}) = 1$. Taking cohomology in the standard exact sequence

$$0 \to \mathcal{O}_S(-D') \to \mathcal{O}_S \to \mathcal{O}_{D'} \to 0$$

and using Lemma 2.2.7, we obtain a long exact sequence

$$\begin{aligned} 0 &\to H^1(S, \mathcal{O}_S(-D')) \to H^1(S, \mathcal{O}_S) \to H^1(D', \mathcal{O}_{D'}) \\ &\to H^2(S, \mathcal{O}_S(-D')) \to H^2(S, \mathcal{O}_S) \to 0. \end{aligned} \tag{2.2.6}$$

If S is classical, this and Serre duality give

$$0 = h^1(S, \mathcal{O}_S(-D')) = h^1(S, \mathcal{O}_S(D)),$$

contradicting what we have already seen. If S is non-classical, that is, $K_S = 0$, then we have $\mathcal{O}_D(D) \cong \omega_D \cong \mathcal{O}_D$ by Lemma 2.2.7. Exact sequence (2.2.3) shows that the map

$$H^0(S, \mathcal{O}_S(D)) \to H^0(D, \mathcal{O}_D(D))$$

is surjective. Hence, we have a long exact sequence

$$\begin{aligned} 0 &\to H^1(S, \mathcal{O}_S) \to H^1(S, \mathcal{O}_S(D)) \to H^1(D, \mathcal{O}_D) \\ &\to H^2(S, \mathcal{O}_S(D)) = H^0(S, \mathcal{O}_S(-D)) = 0. \end{aligned} \tag{2.2.7}$$

Since $h^1(\mathcal{O}_S(D)) = 1$, we get a contradiction, also in the non-classical case. These contradictions show that D' is not indecomposable. We write $D' = \sum_j a_j A_j$ as sum of indecomposable divisors of canonical type. Since the lattice $\mathrm{Num}(S)$ is hyperbolic, each A_j, as well as D' all span the same line inside the vector space $\mathrm{Num}(S) \otimes \mathbb{Q}$. Since D' is not indecomposable, this shows that $[D']$ is not primitive in $\mathrm{Num}(S)$.

Conversely, assume that the class $[D]$ is not primitive. Then, we either have $D = nD'$ or $D = nD' + K_S$ for some divisor of canonical type D' and some $n \geq 2$. In the first case, we have $\dim |D| \geq 1$ using the already established first claim. In the second case, we note that, $h^0(D' + K_S) + h^0(-D') \geq \frac{1}{2}D'^2 + 1 \geq 1$ implies that there exists an effective divisor $D'' \in |D' + K_S|$. Obviously, D and $(n-1)D' + D''$ are distinct divisors in $|D|$ and thus, $\dim |D| \geq 1$. This establishes the Claim 2.

To prove the last assertion, we take cohomology in the exact sequence

$$0 \to \mathcal{O}_S(-D) \to \mathcal{O}_S \to \mathcal{O}_D \to 0 \tag{2.2.8}$$

and using Lemma 2.2.7 as well as $h^1(\mathcal{O}_S) = 0$, we find $h^1(\mathcal{O}_S(-D)) = 0$. By Serre duality, we find $h^1(\mathcal{O}_S(D + K_S)) = h^1(\mathcal{O}_S(-D)) = 0$ and thus, Riemann–Roch yields $\dim |D + K_S| = 0$. $\square$

If D is as in the previous proposition, then we call the pencil $|D|$ or $|2D|$ a *genus one pencil*. Moreover, if D is as in the second assertion of the proposition, then it is called a *half-fiber*. Every genus one fibration on an Enriques surface has at least one and at most two half-fibers, as the next result shows.

Corollary 2.2.9 *Let S be an Enriques surface.*

1. *If S is classical (resp. non-classical), then every genus one pencil on S has precisely two half-fibers (resp. one half-fiber).*
2. *If D is an effective and nef divisor on S with $D^2 = 0$, then:*
   a. *either* $\dim |D| = 0$ *and D is the sum of $n \leq 2$ half-fibers of some genus one pencil $|P|$.*
   b. *or else* $\dim |D| > 0$ *and there exists a genus one pencil $|P|$ such that $|D| = |kP + eF|$ for some $k > 0$ and $e = 0, 1$, where F is a half-fiber of $|P|$. Moreover,* $\dim |D| = \dim |kP + eF| = k + 1$.

In particular, a member of a linear system $|D|$ without fixed part has even intersection number with any effective divisor on S.

Proof Let $[D]$ be the class of D in $\mathrm{Num}(S)$. We have $[D] = mf$, where m is a positive integer and f is a primitive isotropic vector represented by some indecomposable effective nef divisor F. If S is classical (resp. non-classical), then the torsion of $\mathrm{Pic}(S)$ is equal to $\mathbb{Z}/2\mathbb{Z}$ (resp. trivial) and thus, there exist precisely two (resp. one) representative(s) of f. If S is not classical, we obtain that $D \in |mF|$ and if S is classical, we have $D \in |mF|$ or $D \in |mF + F'|$, where $F' \in |F + K_S|$. If $m = 1$, then we find $D = F$ or $D = F + F'$ and in both cases, we have $\dim |D| = 0$. If $m > 1$, then we write $m = 2k + e$ with $e = 0$or $e = 1$. If $e = 0$, then $|2F| = |P|$ for some pencil

P by the previous proposition and $\dim|mF| = \dim|kP| = k+1 > 0$. If $e = 1$, then $|D| = |kP + F|$ or $|kP + F'|$. Let G denote F or F'. The exact sequence

$$0 \to \mathcal{O}_S(kP) \to \mathcal{O}_S(kP+G) \to \mathcal{O}_G(kP+G) \to 0$$

together with the fact that $\mathcal{O}_G(kP+G) \cong \mathcal{O}_G(G)$, and hence $h^0(\mathcal{O}_G(kP+G)) = h^0(\mathcal{O}_G(G)) = 0$ (because $\dim|G| = 0$) shows that $\dim|kP| = \dim|kP+G|$. Thus, F or F' is the fixed component of the linear system $|D|$ and $\dim|D| > 0$. □

Remark 2.2.10 Half-fibers taken with multiplicity 2 are examples of *multiple fibers* of genus one pencils. Moreover, the unique half-fibers taken with multiplicity 2 of genus one pencils on non-classical Enriques surfaces are examples of multiple fibers that are *wild*. The generic member of a genus one fibration of an Enriques surface in characteristic $\neq 2$ is in fact a *smooth* curve, that is, the pencil is an *elliptic pencil*. However, in characteristic 2, it may happen that the generic member of a genus one pencil on an Enriques surface is a rational curve with a cusp singularity, in which case the pencil is *quasi-elliptic*. We will come back to these topics when discussing genus one fibrations on arbitrary surfaces in Chapter 4.

We end this section by describing linear systems associated to big and nef invertible sheaves. Compared to the previous analysis of genus one fibrations and divisors of canonical type, this is relatively easy.

Proposition 2.2.11 *Let S be an Enriques surface and let $\mathcal{L}$ be a big and nef invertible sheaf on S. Then, every effective divisor in $|\mathcal{L}|$ is numerically connected and:*

1. *either $|\mathcal{L}|$ has no fixed components,*
2. *or else $\mathcal{L}^2 = 2$ and $|\mathcal{L}| = |P| + R$, where the moving part $|P|$ is a genus one pencil, and where the fixed component R is a smooth rational curve with $P \cdot R = 2$.*

Proof Numerical connectivity of effective divisors in $|\mathcal{L}|$ follows from Lemma 2.1.10.(3) or Corollary 2.1.11.

By Riemann–Roch, we have $h^0(S, \mathcal{L}) \geq 2$. We write $|\mathcal{L}| = |M| + Z$, where Z is the fixed part and $|M|$ is the moving part and we will now assume that $Z \neq 0$.

Seeking a contradiction, we suppose that $M^2 > 0$. Then, since M is nef, we can apply the Vanishing Theorem (Theorem 2.1.16) to conclude that $h^0(S, \mathcal{O}_S(M)) = \frac{1}{2}M^2 + 1$. Since every divisor in $|\mathcal{L}|$ is numerically connected, we find the strict inequality $\mathcal{L}^2 \geq \mathcal{L} \cdot M = M^2 + Z \cdot M > M^2$. On the other hand, we have $\frac{1}{2}M^2 + 1 = h^0(S, \mathcal{O}_S(M)) = h^0(S, \mathcal{L}) \geq \frac{1}{2}\mathcal{L}^2 + 1$, a contradiction. This implies $M^2 = 0$.

Thus, by Corollary 2.2.9, the linear system $|M| = |kP|$ is composed of a genus one pencil. Since $\mathcal{L}^2 = Z^2 + Z \cdot M > 0$ and $Z \cdot M > 0$ (by numerical connectivity of $Z + P$), we obtain $Z^2 < 0$ by the Hodge index theorem. Let R be an irreducible component of Z with $Z \cdot R < 0$, which exists since we have $Z^2 < 0$. Since $\mathcal{L}$ is nef, we obtain $\mathcal{L} \cdot R = Z \cdot R + M \cdot R \geq 0$, which implies $R \cdot P > 0$. We have $r := R \cdot P \geq 2$ by Corollary 2.2.9 and compute

$$(R+P)^2 = -2 + 2R \cdot P = 2r - 2 \geq 2 \quad \text{and} \quad R \cdot (R+P) = -2 + r \geq 0.$$

Thus, $R+P$ is a nef divisor with $(R+P)^2 > 0$. By Riemann–Roch and the Vanishing Theorem, we have $h^0(R+P) = r \geq 2$. Since R is the fixed part of $|R+P|$ and $h^0(P) = h^0(R+P) = r$, we get $r = 2$. This implies that $k = 2$ and $1 = \dim|R+P| = \dim|P|$. Since $2 = h^0(P) = h^0(\mathcal{L}) = \frac{1}{2}\mathcal{L}^2 + 1$, we conclude $\mathcal{L}^2 = 2$.

Next, we compute

$$2 = \mathcal{L}^2 = \mathcal{L} \cdot (P+Z) = R \cdot P + (Z-R) \cdot P + \mathcal{L} \cdot Z = 2 + (Z-R) \cdot P + \mathcal{L} \cdot Z,$$

which gives $\mathcal{L} \cdot Z = 0$ and hence, $\mathcal{L} \cdot R = 0$ and $\mathcal{L}^2 = \mathcal{L} \cdot P = 2$. Thus, we find

$$\begin{aligned} &\det\begin{pmatrix} \mathcal{L}^2 & \mathcal{L} \cdot (P+R) \\ \mathcal{L} \cdot (P+R) & (P+R)^2 \end{pmatrix} \\ &= \mathcal{L}^2 \cdot (P+R)^2 - (\mathcal{L} \cdot (P+R))^2 = 2(2P \cdot R - 2) - 4 = 0. \end{aligned}$$

By the Hodge index theorem, the divisor classes of $P+R$ and $\mathcal{L}$ are proportional in $\mathrm{Num}(S)_{\mathbb{R}}$ and since $(P+R)^2 = \mathcal{L}^2$, they are equal. Finally, since $(P+R) \cdot R = 0$ and P is nef, we conclude that $P+R$ is nef. By Riemann–Roch and the Vanishing Theorem, we find $h^0(P+R) = \frac{1}{2}(P+R)^2 + 1 = 2 = h^0(P)$ and thus, R is the fixed part of $|P+R|$. □

We refer to Section 2.4 and in particular, to Corollary 2.4.6, for base points of linear systems on Enriques surfaces. Moreover, as an application of Proposition 2.2.11, we have the following Bertini–type theorems for linear systems on Enriques surfaces.

Corollary 2.2.12 *Let S be an Enriques surface and let $\mathcal{L}$ be a big and nef invertible sheaf on S.*

1. *If $\mathcal{L}^2 > 2$, then the linear system $|\mathcal{L}|$ contains an integral divisor, that is, a reduced and irreducible curve.*
2. *If $\mathcal{L}^2 = 2$ and S is classical, then $|\mathcal{L}|$ or $|\mathcal{L} \otimes \omega_S|$ contains an integral divisor.*

Proof If $\mathcal{L}^2 > 2$, then $|\mathcal{L}|$ has no fixed components by the previous proposition . In characteristic zero, Bertini's theorem states that a general member of $|\mathcal{L}|$ is even a smooth divisor away from the base locus. In positive characteristic, it is at least still true that a general member of $|\mathcal{L}|$ is an integral divisor. We refer to [355, Théorème 5.1] for proofs and details. For further details, see also [433, Lemma 3.3].

Next, suppose that $\mathcal{L}^2 = 2$ and that S is classical. If the linear system $|\mathcal{L}|$ contains no irreducible curve, then $|\mathcal{L}| = |P| + R$, where $|P|$ is a genus one pencil and R is a smooth rational curve with $R \cdot P = 2$. Let us show that in this case $|P+R+K_S|$ contains an irreducible curve. If not, then we can write $|P+R+K_S| = |P'| + R'$ for some genus one pencil $|P'|$ and a smooth and rational curve R' with $R' \cdot P' = 2$. From $2 = P \cdot (P+R) = P \cdot (P'+R') \geq P \cdot P' \geq 0$ and the fact that $|P| = |2F|$, $|P'| = |2F'|$ for two half-fibers F, F', we conclude $P \cdot P' = 0$. This implies $P = P'$ and $R + K_S \sim R'$. The latter equality gives $2R \sim 2R'$, and since $(2R)^2 = -8 < 0$, we find $R = R'$ (see also Lemma 2.3.2) and thus, $K_S = 0$, a contradiction. □

Remark 2.2.13 If $\mathcal{L}$ is a big and nef invertible sheaf, then it follows from the Enriques Reducibility Lemma (see Corollary 2.3.5 below) that the linear system $|\mathcal{L}|$ also contains at least one *reducible* curve.

2.3 The Nodal Weyl Group and the Enriques Reducibility Lemma

In this section, we study effective divisors on Enriques surfaces that are not necessarily nef. We introduce the nodal Weyl group and prove the Enriques Reducibility Lemma, which states that every effective divisor is linearly equivalent to a sum of reduced and irreducible curves of arithmetic genus zero and one. As an application, we will see that every Enriques surface carries at least one genus pencil.

Let S be an Enriques surface and let R be a smooth and rational curve. By the adjunction formula, this is equivalent to saying that R is an irreducible curve with $R^2 = -2$, see also Proposition 2.1.6. We denote by $\mathcal{R}(S)$ the set of such (-2)-curves on S.

Definition 2.3.1 An Enriques surface S is called *unnodal* if $\mathcal{R}(S) = \emptyset$, that is, if S does not contain any (-2)-curves. Otherwise, S is called *nodal*.

In Chapter 5, we will see that inside the 10-dimensional moduli spaces of (polarized) Enriques surfaces, the sets of unnodal Enriques surfaces are open and dense, whereas nodal Enriques surfaces form divisors inside these moduli spaces. We will study nodal Enriques surfaces in detail in Volume II.

The following observation shows that we may identify a (-2)-curve with its class in $\mathrm{Num}(S)$ or $\mathrm{NS}(S)$.

Lemma 2.3.2 *Let R_1 and R_2 be (-2)-curves on an Enriques surface S. Then,*

$$R_1 \equiv R_2 \quad \Leftrightarrow \quad R_1 \sim R_2 \quad \Leftrightarrow \quad R_1 = R_2,$$

where $\equiv$ and $\sim$ denote numerical and linear equivalence of divisors, respectively.

Proof The implications from the right to the left are trivial. Since $\dim |R_1| = \dim |R_2| = 0$, we see that $R_1 \sim R_2$ implies $R_1 = R_2$. Finally, $R_1 \equiv R_2$ implies that either $R_1 \sim R_2$ holds and we are done or else, we must have $R_1 \sim R_2 + K_S$ with $K_S \neq 0$. However, the second case cannot occur, since then, we would deduce $2R_1 \sim 2R_2$ and from $\dim |2R_1| = \dim |2R_2| = 0$, we infer $R_1 = R_2$, a contradiction. □

In the previous section, we identified the nef cone $\mathrm{Nef}(S)$ of an Enriques surface S with the fundamental chamber of the root basis in $\mathrm{Num}(S)$ formed by the classes of (-2)-curves. Let W_S^{nod} be the Weyl group of this basis, which we call the *nodal Weyl group* of S. It is a subgroup of the Weyl group $W(\mathrm{Num}(S))$. In particular, every effective divisor of S can be moved into the nef cone by an element of W_S^{nod}. Since the numerical class of any (-2)-curve has a unique representative in $\mathrm{Pic}(S)$, the group W_S^{nod} also acts on $\mathrm{Pic}(S)$. More precisely, we have the following result.

Theorem 2.3.3 *Let D be an effective divisor on an Enriques surface S with $D^2 \geq 0$. Then,*

$$D \sim D' + \sum_i m_i R_i, \quad m_i \geq 0, \tag{2.3.1}$$

where D' belongs to the unique nef divisor class in the W_S^{nod}-orbit of the divisor class of D and where the R_i are (-2)-curves. Moreover, one of the following cases occurs:

1. *$D^2 > 0$ and $|D'|$ contains an integral curve,*
2. *$D^2 = 2$ and $|D'|$ or $|D' + K_S|$ contain an integral curve,*
3. *$D^2 = 2$ and $D' \in |P + R|$, where $|P|$ is a genus one pencil and R is a (-2)-curve with $R \cdot P = 2$, or*
4. *$D^2 = 0$ and D' is nef.*

Proof The numerical class $[D] \in \mathrm{Num}(S)$ belongs to the closure of a connected component of $V_S^+ = \{x \in \mathrm{Num}(S)_{\mathbb{R}} : x^2 > 0\}$. The nef cone is a fundamental chamber of W_S^{nod} by Proposition 2.2.1. By Corollary 0.8.14, we can write a representative D of its numerical class in the asserted form (2.3.1) with $[D'] \in \mathrm{Nef}(S)$. The remaining assertions follow from Proposition 2.2.11 and Corollary 2.2.12. □

Corollary 2.3.4 *Every Enriques surface carries at least one genus one pencil.*

Proof Let S be an Enriques surface. Since $\rho(S) = 10$ and $\mathrm{Num}(S)$ is a unimodular and indefinite lattice, it contains an isotropic vector, see, for example, [661, Chapter 5]. Alternatively, we have seen and classified (orbits) of isotropic vectors in the Enriques lattice in Corollary 1.5.4. Applying Theorem 2.3.3 and Proposition 2.2.8, the assertion follows. □

The next assertion is known as *Enriques' Reducibility Lemma* or simply as the *Reducibility Lemma*, which is a fundamental result for the study of linear systems on Enriques surfaces.

Theorem 2.3.5 (Enriques Reducibility Lemma) *Let D be a divisor on an Enriques surface S.*

1. *If D is effective, then it is linearly equivalent to a sum of integral curves of arithmetic genus zero or one.*
2. *If D is big and nef, then the linear system $|D|$ contains a reducible divisor. Moreover, one can even find $D_1 + D_2 \in |D|$, where the D_i, $i = 1, 2$, are effective divisors with $D_i^2 \geq 0$.*

Proof Let D be an effective divisor on S. Since D is a sum of integral divisors, we are reduced to the case where D is integral with $D^2 > 0$. By Theorem 2.3.3, we may reduce to the case where D is nef. Applying a suitable element $w \in W(\mathrm{Num}(S))$, we can write $w([D])$ as a linear combination of fundamental weights. Thus, by Remark 1.5.5, we can write the class $w([D])$ in $\mathrm{Num}(S)$ as a non-negative linear combination of isotropic vectors f_i such that $f_i \cdot f_j > 0$ for all $i \neq j$. Applying w^{-1} to $w([D])$, we obtain a numerical equivalence $D \equiv \sum n_i F_i$ with $F_i^2 = 0$ and

$F_i \cdot F_j > 0$ for $i \neq j$. Since D is nef and $D^2 > 0$, we have $D \cdot F_i > 0$. Thus, we may assume each F_i to be an effective divisor. If $K_S = 0$, then numerical equivalence coincides with linear equivalence and thus, D is in fact linearly equivalent to the sum of effective divisors $D_i = n_i F_i$ with $D_i^2 = 0$. If $K_S \neq 0$, then we replace D_1 with $D_1' \in |D_1 + K_S|$ if necessary to conclude the same. In particular, we see that any big and nef divisor D can be written as a sum $D_1 + (D - D_1)$ of two divisors with non-negative self-intersection.

Applying Theorem 2.3.3, we can write each D_i as a sum $P_i + \sum_{i \in I} R_i$, where P_i is a nef divisor with $P_i^2 = 0$. By Corollary 2.2.9, each P_i is linearly equivalent to a sum of half-fibers. Moreover, each half-fiber is either an irreducible curve of arithmetic genus one or a sum of (-2)-curves. □

Remark 2.3.6 If S is a classical Enriques surface and D is a big and nef divisor, then Enriques gave the following beautiful and geometric argument for the existence of a reducible divisor in $|D|$: inside the projective space $|2D| = \mathbb{P}(H^0(S, \mathcal{O}_S(2D)))$, we have the two closed subschemes V and V' formed by divisors of the form $D_1 + D_2$ and $D_1' + D_2'$ with $D_i \in |D|$ and $D_i' \in |D + K_S|$, respectively. Since $n := \dim |D| = \dim |D + K_S| = \frac{1}{2} D^2$ by Corollary 2.1.17, we conclude that $\dim V = \dim V' = 2n$. Similarly, we have $\dim |2D| = 4n$ and $|2D|$ contains integral divisors by Corollary 2.2.12. Thus, $V \cap V' \neq \emptyset$ and there exists a divisor $C \in |2D|$ that can be written as

$$C = D_1 + D_2 = D_1' + D_2',$$

with $D_i \in |D|$ and $D_i' \in |D + K_S|$. However, since $|D| \cap |D + K_S| = \emptyset$, this can only happen if D_1 and D_2 are reducible divisors. This establishes a reducible divisor in $|D|$.

In particular, if an Enriques surface is unnodal, then it does not contain curves of arithmetic genus zero, and then, we have the following.

Corollary 2.3.7 *If S is an unnodal Enriques surface, then every effective divisor on S is linearly equivalent to a sum of integral curves of arithmetic genus one.*

Finally, since the canonical class K_S is trivial or equal to the class of the difference of two half-fibers of a genus one pencil, we obtain the following.

Corollary 2.3.8 *If S is an Enriques surface, then* $\mathrm{Pic}(S)$ *is generated by the classes of curves of arithmetic genus zero and one.*

Remark 2.3.9 Note that we can prove this corollary without using the Enriques Reducibility Lemma. In fact, we may use Proposition 1.5.3 to conclude that the Enriques lattice E_{10} is generated by isotropic primitive vectors $f_1, \ldots, f_{10}$ and $\mathbf{\Delta} - f_1 - f_2$. This corollary should also be compared with the fact that the Picard group of a rational surface is generated by classes of smooth rational curves.

2.4 Base Points and the Φ-function

In this section, we discuss Bogomolov instability for rank two vector bundles and Reider's theorem for surfaces. As an application, we not only obtain another proof of the fundamental Vanishing Theorem for Enriques surfaces (Theorem 2.1.16), but also results on base points of linear systems and Fujita's conjecture. Then, we introduce the function Φ for invertible sheaves, which is fundamental for the study of big and nef linear systems on Enriques surfaces. Finally, we briefly discuss k-ampleness and Seshadri constants.

Definition 2.4.1 A rank 2 vector bundle $\mathcal{E}$ on a smooth and proper surface X is called *Bogomolov unstable* if there exists a short exact sequence

$$0 \to \mathcal{O}_X(A) \to \mathcal{E} \to \mathcal{I}_Z(B) \to 0, \qquad (2.4.1)$$

where $\mathcal{I}_Z$ is the ideal sheaf of a 0-dimensional closed subscheme $Z \subset X$ and the divisor $A - B$ satisfies $(A-B)^2 > 0$ and $(A-B)\cdot H > 0$ for some ample divisor H.

The following theorem, due to Bogomolov [72] over the complex numbers, gives a very useful criterion for a rank two vector bundle to be Bogomolov unstable.

Theorem 2.4.2 *Let X be a smooth and proper surface in characteristic $p \geq 0$ and let $\mathcal{E}$ be a rank two vector bundle on X. Assume that $p = 0$ or that $p > 0$ and that X is of Kodaira dimension $\kappa(X) \leq 0$. If $c_1^2(\mathcal{E}) > 4c_2(\mathcal{E})$, then $\mathcal{E}$ is Bogomolov unstable.*

Proof Over the complex numbers, we refer to [72], as well as [43, Chapter IV, Sections 10 and 12] and [610]. In positive characteristic, this theorem is a special case of [668, Theorem 7]. □

Remark 2.4.3 In positive characteristic p, more is known: this theorem still holds if $\kappa(X) = 1$ with some very explicit exceptions if $p \leq 3$, see [211, Theorem 1.6]. For surfaces of general type, that is, if $\kappa(X) = 2$, then we refer the reader to [211] and [668]. Finally, we refer the reader to [441] and [442] for further results and with a view towards the Bogomolov–Miyaoka–Yau inequality in positive characteristic.

Before applying this result and Reider's theorem below to the study of base points of linear systems on Enriques surfaces, we give another proof of the Vanishing Theorem (Theorem 2.1.16). Let us recall some results from Section 2.2: if X is a smooth and proper surface, then we defined the cone $\mathrm{Big}(X)$ of big numerical divisor classes. If we fix some ample class $h \in \mathrm{NS}(X)$, then the set of big divisors

$$\left\{x \in \mathrm{NS}(X) \,:\, x^2 > 0, x \cdot h > 0\right\}$$

does not depend on the choice of h and generates the cone $\mathrm{Big}(X)$. If $D \in \mathrm{NS}(X)$ is a numerically effective class, then the Hodge index theorem implies that we have $[D] \cdot x > 0$ for all $x \in \mathrm{Big}(X)$. After these preparations, we give another proof of Theorem 2.1.16.

Corollary 2.4.4 *Let X be a smooth and proper surface that satisfies the assumptions of Theorem 2.4.2, for example, an Enriques surface. If D is an effective and nef divisor with $D^2 > 0$ on X, then*

$$H^1(S, \mathcal{O}_X(-D)) = 0.$$

Proof Set $\mathcal{L} := \mathcal{O}_X(D)$ and assume that $H^1(X, \mathcal{L}^{-1}) \neq 0$. Using the isomorphism $\mathrm{Ext}^1(\mathcal{L}, \mathcal{O}_X) \cong H^1(X, \mathcal{L}^{-1})$, we deduce that there exists a non-trivial extension of invertible sheaves

$$0 \to \mathcal{O}_X \to \mathcal{E} \to \mathcal{L} \to 0. \tag{2.4.2}$$

Computing the Chern classes, we find $c_1(\mathcal{E}) = [D]$ and $c_2(\mathcal{E}) = 0$. By Theorem 2.4.2, the vector bundle $\mathcal{E}$ is Bogomolov–unstable. Thus, there exists an exact sequence like (2.4.1) for $\mathcal{E}$ and we let $\phi : \mathcal{O}_X(A) \to \mathcal{L}$ be the composition of the inclusion map $\mathcal{O}_X(A) \to \mathcal{E}$ followed by the projection $\mathcal{E} \to \mathcal{L}$.

Seeking a contradiction, let us assume $\phi = 0$. Then, the image of $\mathcal{O}_X(A)$ in $\mathcal{E}$ is contained in $\mathcal{O}_X$ and hence, $A \leq 0$. Moreover, if $A < 0$, then, after taking global sections in (2.4.1), we find that $B \geq 0$ and hence $(A-B)\cdot H < 0$ for all ample invertible sheaves H on X, which contradicts the properties of (2.4.1). This contradiction implies $A = 0$. Thus, the exact sequence (2.4.1) gives $c_1(\mathcal{E}) = [A] + [B] = [D]$. Hence, $[B] = [D]$ and again $(A - B) \cdot H = -D \cdot H \leq 0$ for all ample H. This contradiction shows that $\phi \neq 0$.

Using $\phi \neq 0$, we conclude $h^0(\mathcal{L}(-A)) = h^0(\mathcal{O}_X(B)) \neq 0$. Now, since $[A - B] \in \mathrm{Big}(X)$, we conclude $D \cdot (D - 2B) = D \cdot (A - B) > 0$. Also, we have $0 = c_2(\mathcal{E}) = A \cdot B + h^0(\mathcal{O}_Z)$, which implies $A \cdot B = (D - B) \cdot B \leq 0$. Thus, we obtain $B^2 \geq D \cdot B$ and $D^2 > 2D \cdot B \geq D \cdot B$, which gives

$$D^2 > 0 \quad \text{and} \quad B^2 \cdot D^2 > (D \cdot B)^2.$$

By the Hodge index theorem, this implies $[B] = 0$ and hence, $B = 0$. Thus, $\mathcal{L} = \mathcal{O}_X(A)$ and ϕ defines a splitting of the exact sequence (2.4.2), a contradiction. This final contradiction shows that $h^1(X, \mathcal{L}^{-1}) = 0$. □

We now come to the main application of Theorem 2.4.2, which is the analysis of base points of adjoint linear systems, due to Reider [613].

Theorem 2.4.5 *Let X be a smooth and proper surface that satisfies the assumptions of Theorem 2.4.2. Let $\mathcal{L}$ be a big, nef, and effective invertible sheaf.*

1. *Suppose that $\mathcal{L}^2 \geq 5$ and that $|\mathcal{L} \otimes \omega_X|$ has a base point $x \in X$. Then, there exists an effective divisor E that contains x, such that either*

   a. *$E^2 = 0$ and $\mathcal{L} \cdot E = 1$, or*
   b. *$E^2 = -1$ and $\mathcal{L} \cdot E = 0$.*

2. *Suppose that $\mathcal{L}^2 \geq 9$ and that $|\mathcal{L} \otimes \omega_X|$ does not separate two points $x, y \in X$ (possibly infinitely near). Then, there exists an effective divisor E that contains x and y, such that*

a. $E^2 = 0$ *and* $\mathcal{L} \cdot E \leq 2$*, or*
b. $E^2 = -1$ *and* $\mathcal{L} \cdot E \leq 1$*, or*
c. $E^2 = -2$ *and* $\mathcal{L} \cdot E = 0$*, or*
d. $\mathcal{L}^2 = 9$*,* $E^2 = 1$*, and* $\mathcal{L} \equiv 3E$ *in* $\mathrm{Num}(X)$.

Proof A point $x \in X$ is a base point of the linear system $|\mathcal{L} \otimes \omega_X|$ if and only if $h^0(\mathcal{I}_x \otimes \mathcal{L} \otimes \omega_X) = h^0(\mathcal{L} \otimes \omega_X)$, where $\mathcal{I}_x \subseteq \mathcal{O}_X$ denotes the ideal sheaf of $x \in X$. Taking cohomology in the short exact sequence

$$0 \to \mathcal{I}_x \otimes \mathcal{L} \otimes \omega_X \to \mathcal{L} \otimes \omega_X \to \mathcal{O}_x \to 0$$

and applying Corollary 2.4.4, we conclude that x is a base point if and only if

$$0 \neq H^1(X, \mathcal{I}_x \otimes \mathcal{L} \otimes \omega_X) \cong \mathrm{Ext}^1(\mathcal{I}_x \otimes \mathcal{L} \otimes \omega_X, \omega_X) \cong \mathrm{Ext}^1(\mathcal{I}_x \otimes \mathcal{L}, \mathcal{O}_X),$$

where the first isomorphism is Serre duality. Thus, a non-zero element in the latter Ext-group gives rise to a non-split exact sequence

$$0 \to \mathcal{O}_X \to \mathcal{E} \to \mathcal{I}_x \otimes \mathcal{L} \to 0. \tag{2.4.3}$$

One can show that $\mathcal{E}$ is locally free and thus, a vector bundle of rank 2. Computing Chern classes, we find $c_1(\mathcal{E}) = [\mathcal{L}]$ and $c_2(\mathcal{E}) = [x]$. Since $\mathcal{L}^2 \geq 5$, Theorem 2.4.2 implies that $\mathcal{E}$ is Bogomolov unstable, that is, there exists a short exact sequence

$$0 \to \mathcal{O}_X(A) \to \mathcal{E} \to \mathcal{I}_Z(B) \to 0 \tag{2.4.4}$$

as in (2.4.1). Moreover, we have $2h \cdot A > h \cdot \mathcal{L} > 0$ for every ample class h. Since $\mathcal{L}$ is big and nef, it is a limit of ample divisors, from which we conclude $2\mathcal{L} \cdot A \geq A^2$. Combining (2.4.3) and (2.4.4), we claim that the composition

$$\mathcal{O}_X(A) \to \mathcal{E} \to \mathcal{I}_x \otimes \mathcal{L}$$

is non-zero: otherwise, we obtain $A = -E$ for some effective divisor E, and thus, $2h \cdot A \leq 0$ for all ample classes, which contradicts the above. From this, we conclude that there exists an effective divisor E on X such that

$$\mathcal{L} \cong \mathcal{O}_X(A + E) \quad \text{and} \quad x \in E.$$

In particular, E is not numerically trivial and thus, the inequality $2\mathcal{L} \cdot A \geq A^2$ becomes $\mathcal{L}^2 \geq 2\mathcal{L} \cdot E$. We rewrite (2.4.4) as the short exact sequence

$$0 \to \mathcal{L}(-E) \to \mathcal{E} \to \mathcal{I}_Z \otimes \mathcal{O}_X(E) \to 0$$

and computing Chern classes, we find $1 = c_2(\mathcal{E}) = \mathcal{L} \cdot E - E^2 + h^0(\mathcal{O}_Z)$. This implies $\mathcal{L} \cdot E - E^2 \leq 1$ and we conclude the following inequalities

$$\begin{array}{llll} 2\mathcal{L} \cdot E \leq \mathcal{L}^2 & & \mathcal{L} \cdot E - E^2 \leq 1 & \\ \mathcal{L} \cdot E \geq 0 & & \mathcal{L}^2 \cdot E^2 & \leq (\mathcal{L} \cdot E)^2 \,. \end{array}$$

The third inequality follows from the fact that $\mathcal{L}$ is nef and E is effective and the fourth inequality follows from the Hodge index theorem. Putting these together, we conclude

$$\mathcal{L} \cdot E \;\leq\; 1 + E^2 \;\leq\; 1 + \frac{(\mathcal{L} \cdot E)^2}{\mathcal{L}^2} \;\leq\; 1 + \frac{1}{2}(\mathcal{L} \cdot E).$$

From this, we obtain $\mathcal{L} \cdot E \leq 2$ and we have equality if and only if $\mathcal{L}$ and E generate the same line in $\mathrm{Num}(X)$. In this latter case, we find $E^2 \geq 1$ and $\mathcal{L}^2 \leq 4$ contradicting our assumptions. Thus, we have $\mathcal{L} \cdot E = 0$ or $\mathcal{L} \cdot E = 1$. In the first case, we have $E^2 \leq 0$ by the Hodge index theorem and we have $E^2 = 0$ if and only if E is numerically trivial, which is not the case. Since $-E^2 \leq 1$, we conclude $E^2 = -1$. In the second case, that is, if $\mathcal{L} \cdot E = 1$, then we find $E^2 \leq (\mathcal{L}^2)^{-1} \leq \frac{1}{5}$ and thus, $E^2 \leq 0$. On the other hand, we have $E^2 \geq 1 - \mathcal{L} \cdot E = 0$ and we find $E^2 = 0$. This establishes the first claim.

Similarly, a pair x, y of points on X (possibly infinitely near), which are not separated by $|\mathcal{L} \otimes \omega_X|$ gives rise to a non-split extension

$$0 \to \mathcal{O}_X \to \mathcal{E} \to \mathcal{I}_{x,y} \otimes \mathcal{L} \to 0, \tag{2.4.5}$$

and we note that we have $h^0(\mathcal{O}_{x,y}) = 2$. Again, $\mathcal{E}$ turns out to be a vector bundle of rank two and computing Chern classes, we find $c_1(\mathcal{E}) = [\mathcal{L}]$ and $c_2(\mathcal{E}) = [x + y]$. Thus, if $\mathcal{L}^2 > 4h^0(\mathcal{O}_{x,y}) = 8$, then $\mathcal{E}$ is Bogomolov unstable by Theorem 2.4.2. From there, an analysis similar to the above yields the second claim. We refer the interested reader to [43, Chapter IV, Theorem 11.4], or [240, Chapter 9, Theorem 6] or [613] for details. □

Next, we apply Reider's theorem to Enriques surfaces, which allows us to analyze base points of linear systems rather than their adjoint linear systems.

Corollary 2.4.6 *Let S be an Enriques surface and let $\mathcal{L}$ be a big and nef invertible sheaf.*

1. *If $\mathcal{L}^2 \geq 6$ and $|\mathcal{L}|$ has a base point x, then there exists an effective divisor E containing x such that $E^2 = 0$ and $\mathcal{L} \cdot E = 1$.*
2. *If $\mathcal{L}^2 \geq 10$ and $|\mathcal{L}|$ does not separate two points x, y (possibly infinitely near), then there exists an effective divisor E containing x, y such that*
   a. *$E^2 = 0$ and $\mathcal{L} \cdot E = 2$, or*
   b. *$E^2 = -2$ and $\mathcal{L} \cdot E = 0$.*

Proof Since ω_S is numerically trivial, also $\mathcal{L}' := \mathcal{L} \otimes \omega_S$ is big and nef with $\mathcal{L}^2 = \mathcal{L}'^2$ and then, we can apply Theorem 2.4.5 to $\mathcal{L}'$. By Proposition 1.5.1, the intersection pairing on $\mathrm{Num}(S)$ is even, which implies that the cases 1 (a), 2 (b), and 2 (d) of Theorem 2.4.5 cannot occur. □

Remark 2.4.7 We can even say a little bit more about the effective divisors E occuring in Corollary 2.4.6:

1. In Case 2 (b), the Hodge index theorem implies that we may assume E to be a sum of (-2)-curves.

2. In the remaining cases, we have $E^2 = 0$ and then, Theorem 2.3.3 shows that E is numerically equivalent to the sum of a multiple of some half-fiber F and a non-negative sum of (-2)-curves. Again, by the Hodge index theorem, we must have $\mathcal{L} \cdot F > 0$, and hence
   a. either $\mathcal{L} \cdot F = 1$ and $|\mathcal{L}|$ has a fixed point on F,
   b. or else $\mathcal{L} \cdot F = 2$ and $|\mathcal{L}|$ does not separate two points. These two points either lie on a F or $F' \in |F + K_S|$ or on both F and F'.

We will say more about base points of linear systems in Theorem 2.4.14, as well as in Section 2.6 below.

Next, we apply these results to establish Fujita's conjecture for K3 surfaces and Enriques surfaces: given an ample invertible sheaf $\mathcal{L}$ on a smooth and projective variety X of dimension d over an algebraically closed field, it follows from the very definition that $\mathcal{L}^{\otimes N}$, as well as $\mathcal{L}^{\otimes N} \otimes \omega_X$, will be base point free and even very ample for $N \gg 0$. More precisely, Fujita [241] conjectured that $\mathcal{L}^{\otimes N} \otimes \omega_X$ should be base point free (resp. very ample) if $N \geq (d+1)$ (resp. $N \geq (d+2)$). Reider's theorem can be used to prove this conjecture for surfaces and we refer the interested reader to [447] for more about this conjecture. For K3 surfaces and Enriques surfaces, we have the following.

Corollary 2.4.8 *Let X be a K3 surface or an Enriques surface and let $\mathcal{L}$ be an ample invertible sheaf.*

1. *Both, $\mathcal{L}^{\otimes 2}$ and $\mathcal{L}^{\otimes 2} \otimes \omega_X$, are globally generated, that is, the associated linear systems have no base points.*
2. *Both, $\mathcal{L}^{\otimes 3}$ and $\mathcal{L}^{\otimes 3} \otimes \omega_X$, are very ample.*

Proof Since $\mathcal{L}$ is ample and the intersection form on $\mathrm{Num}(X)$ is even, we have $\mathcal{L}^2 \geq 2$.

Thus, we have $(\mathcal{L}^{\otimes 2})^2 \geq 8$ and every effective divisor E satisfies $\mathcal{L}^{\otimes 2} \cdot E \geq 2$. Thus, Theorem 2.4.5.(1) implies that $\mathcal{L}^{\otimes 2} \otimes \omega_X$ has not base points. If X is a K3 surface, then $\omega_X \cong \mathcal{O}_X$ and thus, also $\mathcal{L}^{\otimes 2}$ has no base points. If X is an Enriques surface, then Corollary 2.4.6.(1) shows that also $\mathcal{L}^{\otimes 2}$ has no base points.

The proof of the second claim follows similarly and we leave to the reader. □

In view of the Enriques Reducibility Lemma (Theorem 2.3.5), as well as the previous analysis, it is clear that genus one fibrations and isotropic vectors in $\mathrm{Num}(S)$ play an important role for linear systems and their base points on an Enriques surface S. This leads to introducing the function Φ. In Section 0.9, we introduced the notation $\mathrm{Num}(S)_0$ for the set of isotropic vectors in $\mathrm{Num}(S)$, that is, classes of self-intersection zero. Moreover, in Corollary 1.5.4, we classified isotropic vectors of $\mathrm{Num}(S)$. Then, we define

$$\begin{array}{rcl} \Phi : \mathrm{Num}(S) & \to & \mathbb{Z}_{\geq 0} \\ x & \mapsto & \inf \{ |x \cdot f| \mid f \in \mathrm{Num}(S)_0 \} \ . \end{array} \tag{2.4.6}$$

We extend it by linearity to a function on $\mathrm{Num}(S)_{\mathbb{R}}$. To simplify notation, if D is a divisor or if $\mathcal{L}$ is an invertible sheaf on S, then we denote by $\Phi(D)$ and $\Phi(\mathcal{L})$ the just–defined function Φ evaluated on the corresponding class in $\mathrm{Num}(S)$. If $\mathcal{L}$ is a big and nef invertible sheaf, then $\Phi(\mathcal{L})$ should be thought of as a sort of positivity measure or ampleness measure – we will make this precise below.

Remark 2.4.9 Clearly, Φ can be defined for every lattice M that contains isotropic vectors (see [138, Chapter III, §7], where this function is studied for hyperbolic lattices). Note that Φ makes sense only for very few lattices beside the hyperbolic ones. First, the need for isotropic vectors implies that M cannot be positive or negative definite. Moreover, if M is an indefinite lattice of rank ≥ 6 and signature (t_+, t_-) with $t_+ \geq 2$ and $t_- \geq 2$, then for every $0 \neq x \in M$, the sublattice $(\mathbb{Z} \cdot x)^\perp \subseteq M$ is indefinite of rank ≥ 5, thus, contains isotropic vectors, which implies $\Phi(x) = 0$, which makes Φ a trivial function.

Next, we recall some notation. Let $\mathrm{Big}(S) \subset \mathrm{Num}(S)$ be the cone of big divisors; we set

$$\mathrm{Num}(S)^+ := \mathrm{Big}(S) \cap \mathrm{Num}(S).$$

Then, we have the following useful properties of Φ.

Lemma 2.4.10 *Let S be an Enriques surface and let $x \in \mathrm{Num}(S)$.*

1. *For every isometry $\psi \in \mathrm{O}(\mathrm{Num}(S))$ we have*

$$\Phi\left(\psi(x)\right) = \Phi(x).$$

In particular, this applies to elements of the Weyl group $W(\mathrm{Num}(S))$ and the nodal Weyl group W_S^{nod}.

2. *If $x \in \mathrm{Num}(S)^+$, then $\Phi(x)$ can be achieved on the class of a half-fiber. In particular, we have*

$$\Phi(x) = \frac{1}{2} \inf\{x \cdot P, \text{ where } |P| \text{ is a genus one pencil}\} \qquad (2.4.7)$$

and $\Phi(x) \geq 1$ in this case.

Proof We have

$$\Phi(\psi(x)) = \inf\{|\psi(x)\cdot f| : f \in \mathrm{Num}(S)_0\} = \inf\{|x\cdot\psi^{-1}(f)| : f \in \mathrm{Num}(S)_0\} \geq \Phi(x).$$

Replacing ψ with ψ^{-1}, we get the opposite inequality, which establishes the first claim.

In particular, to compute $\Phi(x)$, we may replace x with a nef class in the same $W(\mathrm{Num}(S))$-orbit.

To prove the second assertion, we may assume that $x \in \mathrm{Num}(S)^+$ is nef by the above. By Theorem 2.3.3, we can write $f = f_0 + R$, where f_0 is nef and R is a non-negative sum of the classes of (-2)-curves. Since $x \cdot f_0 \leq x \cdot f$ and $x \cdot f$ is minimal, we conclude that $\Phi(x) = x \cdot f_0$. Thus, f_0 is a divisor of canonical type. Since $x \cdot f_0$ is minimal and x is nef, it follows that f_0 is indecomposable. Moreover, again by minimality, f_0 must be a half-fiber. Clearly, if $|P|$ is a genus one pencil, then

$[P] \in \mathrm{Num}(S)_0$ and it is divisible by 2 in $\mathrm{Num}(S)$ and thus, we have the inequality "$\leq$" in (2.4.7). Since $|2f_0|$ is a genus one pencil and $\Phi(x) = x \cdot f_0$, this shows that we have equality in (2.4.7). The positivity of $\Phi(x)$ follows from the Hodge index theorem. □

Next, we have the following inequalities.

Proposition 2.4.11 *Let S be an Enriques surface and $x \in \mathrm{Num}(S)^+$. Then, the inequality*

$$\Phi(x) \ \leq \ \sqrt{x^2}$$

holds true. Moreover, equality holds if and only if x is in the orbit of the fundamental weight $\boldsymbol{\omega}_1$ and thus satisfies $x^2 = 4$, $\Phi(x) = 2$.

Proof Consider the function $\psi(x) = \frac{\Phi(x)^2}{x^2}$. We have to show that $\psi(X) \leq 1$ and have determine when equality holds.

First, we show that it is a convex function on the fundamental chamber C of $W(\mathsf{E}_{10})$ in $(\mathsf{E}_{10})_{\mathbb{R}}$, the convex cone of the fundamental weights $\boldsymbol{\omega}_0, \ldots, \boldsymbol{\omega}_9$. Since any two non-proportional vectors in C span a hyperbolic plane, we have $(x \cdot y)^2 > x^2 \cdot y^2$. For any two positive $\lambda, \mu \in \mathbb{R}$, we compute

$$\begin{aligned}
&\psi(\lambda x + \mu y)(\lambda x + \mu y)^2 \\
\leq\ & (\lambda x \cdot \boldsymbol{\omega}_9 + \mu y \cdot \boldsymbol{\omega}_9)^2 \\
=\ & \lambda^2 (x \cdot \boldsymbol{\omega}_9)^2 + 2\lambda\mu (x \cdot \boldsymbol{\omega}_9)(y \cdot \boldsymbol{\omega}_9) + \mu^2 (y \cdot \boldsymbol{\omega}_9)^2 \\
<\ & \lambda^2 \Phi(x)^2 x^2 + 2\lambda\mu \Phi(x)\Phi(y)\sqrt{x^2 y^2} + \mu^2 \Phi(y)^2 y^2 \\
\leq\ & \max\{\psi(x), \psi(y)\}(\lambda x + \mu y)^2.
\end{aligned}$$

Thus, $\psi(\lambda x + \mu y) < \max\{\psi(x), \psi(y)\}$, which shows that ψ is convex.

Using the convexity of ψ, it now suffices to check $\psi(x) \leq 1$ on the fundamental weights of $\boldsymbol{\omega}_i, i \neq 10$. Using Proposition 1.5.3, we compute that $\psi(\boldsymbol{\omega}_0) = 9/10$, $\psi(\boldsymbol{\omega}_1) = 1$, $\psi(\boldsymbol{\omega}_2) = 8/9$, $\psi(\boldsymbol{\omega}_3) = 6/7$, $\psi(\boldsymbol{\omega}_4) = 5/6$, $\psi(\boldsymbol{\omega}_5) = 4/5$, $\psi(\boldsymbol{\omega}_6) = 3/4$, $\psi(\boldsymbol{\omega}_7) = 2/3$, and $\psi(\boldsymbol{\omega}_8) = 1/2$. From these computations, it is also easy to see that we have $\psi(x) = 1$ if and only if x is in the orbit of $\boldsymbol{\omega}_1$. □

Remark 2.4.12 As remarked already in Remark 2.4.9, we can define the function Φ for any hyperbolic lattice M with isotropic vectors. In [138, Theorem 2.7.1], it is shown that $\Phi(x)/x^2$ is always bounded by some constant, which is an invariant of the lattice. For example, it is known that $\Phi(x)/x^2 \leq 2$ if $M \cong \mathsf{E}_{2,4,5}, \mathsf{E}_{2,4,6}$ and $\Phi(x)/x^2 \leq \frac{3}{2}$ if $M = \mathsf{E}_{3,3,4}$.

Remark 2.4.13 It is shown in [396, Proposition 4.1] that the function Φ does not take all possible positive values. In fact, it is proven in loc. cit. that

$$x^2 \ \leq \ \Phi(x)^2 + \Phi(x) - 2.$$

As a first result that shows the usefulness of the Φ-function, we show that it does detect whether a linear system on an Enriques surface has base points. The proof uses some results that we will establish in Section 2.6 below, but it is useful to already state it at this point.

Theorem 2.4.14 *Let S be an Enriques surface and let $\mathcal{L}$ be a big and nef invertible sheaf on S. Then, the following are equivalent:*

1. $\Phi(\mathcal{L}) = 1$.
2. *The linear system $|\mathcal{L}|$ has at least one base point.*

Proof First, assume that $\Phi(\mathcal{L}) = 1$. Using Lemma 2.4.10 (3), we see that there exists a half-fiber F of some genus one pencil such that $\mathcal{L} \cdot F = 1$. By Lemma 2.2.7, we have $\omega_F \cong \mathcal{O}_F$. It thus follows from Riemann–Roch that $h^0(F, \mathcal{L}|_F) = 1$. Thus, there exists a unique point $x \in F$ such that every member of $|\mathcal{L}|$ passes through x, that is, x is a base point of $|\mathcal{L}|$.

Conversely, if $\Phi(\mathcal{L}) \geq 2$, then $|\mathcal{L}|$ has no base points by Corollary 2.6.8. (In the case where $\mathcal{L}^2 \geq 6$, it already follows from Corollary 2.4.6 that $|\mathcal{L}|$ has no base points.) □

More precisely, we will see in Proposition 2.6.4 below that if $\Phi(\mathcal{L}) = 1$, then the linear system $|\mathcal{L}|$ has exactly two simple base points (if S is not classical, then one base point is infinitely near).

Corollary 2.4.15 *Let S be an Enriques surface and let $\mathcal{L}$ be a big and nef invertible sheaf with $\Phi(\mathcal{L}) \geq 2$ and $\mathcal{L}^2 = 2d$. Then, the linear system $|\mathcal{L}|$ has no base points and the associated morphism*

$$\varphi \,:\, S \,\to\, \mathbb{P}^d$$

is generically finite onto its image, which is a surface. Moreover, we have the estimate for the generic degree $\deg \varphi$ *of* φ

$$1 \,\leq\, \deg \varphi \,\leq\, 2 \cdot \left(1 + \frac{1}{d-1}\right) \,\leq\, 4\,.$$

In particular, if $d \geq 4$, then $\deg \varphi \leq 2$.

Proof Since $\Phi(\mathcal{L}) \geq 2$, the linear system $|\mathcal{L}|$ has no base points, and since $\mathcal{L}^2 > 0$, the image of φ must be a surface. Next, we have $h^0(S, \mathcal{L}) = 1 + \frac{1}{2}\mathcal{L}^2$ by Corollary 2.1.17, and thus, φ is a morphism to projective space of dimension $\frac{1}{2}\mathcal{L}^2$. By Proposition 0.5.1, we have $\deg \varphi(S) \geq d - 1$ and thus, we find

$$\mathcal{L}^2 \,=\, \deg \varphi \cdot \deg \varphi(S) \,\geq\, \deg \varphi \cdot (d-1).$$

From this and the fact that $2d \geq \Phi(\mathcal{L})^2 \geq 4$, the claimed estimates follow. □

The following theorem says that linear systems $|\mathcal{L}|$ arising from big and nef invertible sheaves $\mathcal{L}$ with $\Phi(\mathcal{L}) \geq 3$ always give rise to birational morphisms. We will study linear systems $|\mathcal{L}|$ and their base points with $\Phi(\mathcal{L}) \leq 2$ in detail in Section 2.6 below.

Theorem 2.4.16 *Let S be an Enriques surface and let $\mathcal{L}$ be a big and nef invertible sheaf with $\mathcal{L}^2 = 2d$. Then, the following are equivalent:*

1. $\Phi(\mathcal{L}) \geq 3$.
2. *The linear system $|\mathcal{L}|$ has no base points and the associated morphism $\varphi : S \to \mathbb{P}^d$ is birational onto its image $S' = \varphi(S)$, which is a normal surface.*

In this case, S' has at worst rational double point singularities.

Proof (2) $\Rightarrow$ (1) Since $|\mathcal{L}|$ has no base points, we have $\Phi(\mathcal{L}) \geq 2$ by Theorem 2.4.14. Seeking a contradiction, we assume $\Phi(\mathcal{L}) = 2$. Using Lemma 2.4.10.(3), we see that there exists a genus one pencil $|2F|$ with $\mathcal{L} \cdot F = 2$. More precisely, by Lemma 2.2.7, we have $\omega_F \cong \mathcal{O}_F$. By Riemann–Roch, we have $h^0(F, \mathcal{L}|_F) = 2$, which implies that either $|\mathcal{L}|$ has at least one base point on F, which contradicts our assumptions (or Theorem 2.4.14), or else $|\mathcal{L}|$ induces a morphism of degree 2 from F to $\mathbb{P}^1$. However, since φ was assumed to be birational with normal image S', this contradicts Zariski's Main Theorem (see [294, Corollary III.11.4]).

(1) $\Rightarrow$ (2) First, we have $d \geq 5$ by Proposition 2.4.11 and we may apply Corollary 2.4.6. First of all, this implies that $|\mathcal{L}|$ has no base points and we let $\varphi : S \to \mathbb{P}^N$ be the associated morphism. In fact, we have $N = d$ by Corollary 2.1.17. If φ does not separate two points x, y (possibly infinitely near), then Corollary 2.4.6 implies that there exists a (-2)-curve R with $\mathcal{L} \cdot R = 0$ that contains x and y (the other possibility does not occur because we have $\Phi(\mathcal{L}) \geq 3$). Obviously, φ blows down R to a point in the image. Since $[\mathcal{L}]^\perp$ is a negative lattice inside $\mathrm{Num}(S)$, the number of (-2)-curves R such that $\mathcal{L} \cdot R = 0$ is finite, see also Proposition 2.1.6. This shows that φ is an isomorphism outside the union of such curves. Moreover, all these curves are blown down to rational double points of $S' = \varphi(S)$ by Proposition 0.4.8 and Proposition 2.1.6. □

As an application, we obtain the following generalization of Corollary 2.4.8 and Fujita's conjecture.

Corollary 2.4.17 . *Let S be an Enriques surface and let $\mathcal{L}$ be a big and nef invertible sheaf.*

1. *The invertible sheaves $\mathcal{L}^{\otimes 2}$ and $\mathcal{L}^{\otimes 2} \otimes \omega_S$ are globally generated, that is, their associated linear systems have no base points.*
2. *The rational maps associated to the linear systems $|\mathcal{L}^{\otimes 3}|$ and $|\mathcal{L}^{\otimes 3} \otimes \omega_S|$ are morphisms and S is birational to the image.*

Proof First, we have $\Phi(\mathcal{L}) \geq 1$ by Lemma 2.4.10.(3). Thus, we have $\Phi(\mathcal{L}^{\otimes 2}) = \Phi(\mathcal{L}^{\otimes 2} \otimes \omega_S) \geq 2$ and then, the first claim follows from Theorem 2.4.14. Similarly, we have $\Phi(\mathcal{L}^{\otimes 3}) = \Phi(\mathcal{L}^{\otimes 3} \otimes \omega_S) \geq 3$ and then, the second claim follows from Theorem 2.4.16. □

Interestingly, the Φ-function also controls the behavior of the linear system $|\pi^*\mathcal{L}|$ on the K3-cover $\pi : X \to S$ of an Enriques surface S, see Section 3.1.

We end this section by relating Φ to two classical positivity measures for invertible sheaves. First, if $\mathcal{L}$ is an invertible sheaf on some proper variety X, then $\mathcal{L}$ is called

k-very ample if for every closed and zero-dimensional subscheme $Z \subset X$ with $h^0(Z, \mathcal{O}_Z) \leq (k+1)$ the restriction map

$$H^0(X, \mathcal{L}) \to H^0(Z, \mathcal{L} \otimes \mathcal{O}_Z)$$

is surjective. Clearly, $\mathcal{L}$ is 0-very ample if and only if $\mathcal{L}$ is globally generated. Moreover, $\mathcal{L}$ is 1-very ample if and only if $\mathcal{L}$ is very ample in the usual sense. Geometrically, k-very ampleness for $k \geq 1$ means that under the embedding defined by $|\mathcal{L}|$ there are no $(m+1)$-secant $(m-1)$-planes to X for $m \leq k$. Equivalently, any zero-dimensional subscheme $Z \subseteq X$ of length m with $m \leq (k+1)$ imposes independent conditions on global sections of $\mathcal{L}$, that is,

$$h^0(X, \mathcal{I}_Z \otimes \mathcal{L}) = h^0(X, \mathcal{L}) - \lg(Z),$$

where $\mathcal{I}_Z \subseteq \mathcal{O}_X$ denotes the ideal sheaf of Z and $\lg(Z) = h^0(\mathcal{O}_Z)$ as it was defined in (0.3.3). A relation between this notion and the Φ-function is given by the following theorem, which is due to Szemberg [696] – it seems likely that these results also hold in positive characteristic.

Theorem 2.4.18 *Let S be an Enriques surface over an algebraically closed field of characteristic zero and let $\mathcal{L}$ be a big and nef invertible sheaf.*

1. *If $\mathcal{L}$ is k-very ample, then $\Phi(\mathcal{L}) \geq (k+2)$.*
2. *If S is unnodal and $\Phi(\mathcal{L}) \geq (k+2)$ for some integer $k \geq 1$, then $\mathcal{L}$ is k-very ample.*

Let us also mention the following theorem, due to Knutsen [396] and Szemberg [696] - again, it seems likely that it also holds in positive characteristic.

Theorem 2.4.19 *Let S be an Enriques surface over an algebraically closed field of characteristic zero and let $\mathcal{L}$ be a big and nef invertible sheaf.*

1. *$\mathcal{L}$ is k-very ample if and only if there exists no effective divisor E on S with*
   a. *$E^2 = -2$ and $\mathcal{L} \cdot E \leq (k-1)$ or*
   b. *$E^2 = 0$ and $\mathcal{L} \cdot E \leq (k+1)$.*
2. *If $\mathcal{L}$ is ample and $n \geq (k+2)$, then $\mathcal{L}^{\otimes n}$ is k-very ample.*

Remark 2.4.20 In the following two cases, we already established the first statement and even in arbitrary characteristic:

1. If $k = 0$, then $\mathcal{L}$ is 0-very ample, that is, $\mathcal{L}$ is globally generated, if and only if $\Phi(\mathcal{L}) \geq 2$. We have seen this in Theorem 2.4.14.
2. If $k = 1$, then $\mathcal{L}$ is 1-very ample, that is, $\mathcal{L}$ is very ample, if and only if $\Phi(\mathcal{L}) \geq 3$ and there exists no (-2)-curve E with $\mathcal{L} \cdot E = 0$. This easily follows from Theorem 2.4.16.

Moreover, the first statement can be thought of as a generalization of Corollary 2.4.6 and the second statement can be thought of as a generalization of Corollary 2.4.8.

The next positivity measures are the following and we refer to [45] or [447, Part I] for introduction and background: if $\mathcal{L}$ is an ample invertible sheaf on a smooth and proper variety X over some algebraically closed field, then we define the *Seshadri constant* of $\mathcal{L}$ in the closed point $x \in X$ to be the real number

$$\epsilon(\mathcal{L}, x) \ := \ \inf_{x \in C} \frac{\mathcal{L} \cdot C}{\mathrm{mult}_x C},$$

where the infimum is taken over all curves C passing through x. Moreover, the *Seshadri constant* of $\mathcal{L}$ is defined to be

$$\epsilon(\mathcal{L}) \ := \ \inf_{x \in X} \epsilon(\mathcal{L}, x).$$

A priori, Seshadri constants are real numbers and on surfaces, we have the estimate

$$0 \ \leq \ \epsilon(\mathcal{L}) \ \leq \ \sqrt{\mathcal{L}^2},$$

see, for example, [447, II:Proposition 5.1.9]. In general, not much is known about Seshadri constants. For Enriques surfaces, we have the following result, which slightly extends a theorem of Szemberg.

Theorem 2.4.21 *Let $\mathcal{L}$ be an ample invertible sheaf on an Enriques surface S. Then, $\epsilon(\mathcal{L})$ is a rational number and satisfies*

$$\frac{1}{2} \ \leq \ \epsilon(\mathcal{L}) \ \leq \ \Phi(\mathcal{L}) \ \leq \ \sqrt{\mathcal{L}^2}. \tag{2.4.8}$$

Moreover:

1. *If $\Phi(\mathcal{L}) \geq 2$, or, equivalently, if $\mathcal{L}$ is globally generated, then $\epsilon(\mathcal{L}) \geq 1$.*
2. *If $\epsilon(\mathcal{L}) < 1$, then $\Phi(\mathcal{L}) = 1$ and $\epsilon(\mathcal{L}) \in \{\frac{1}{2}, \frac{2}{3}\}$.*

Proof The inequality $\Phi(\mathcal{L}) \leq \sqrt{\mathcal{L}^2}$ is Proposition 2.4.11. Moreover, by definition of Φ and Lemma 2.4.10, there exists a half-fiber F of S such that $\mathcal{L} \cdot F = \Phi(\mathcal{L})$ and thus, if $x \in F$, we find

$$\epsilon(\mathcal{L}) \ \leq \ \epsilon(\mathcal{L}, x) \ \leq \ \frac{\mathcal{L} \cdot F}{\mathrm{mult}_x F} \ \leq \ \mathcal{L} \cdot F \ = \ \Phi(\mathcal{L}).$$

Over the complex numbers, $\varepsilon(\mathcal{L}) \in \mathbb{Q}$ is shown in [696, Theorem 3.3]. We leave it the reader to check that the proof works in arbitrary characteristic. If $\mathcal{L}$ is ample, then $\mathcal{L}^{\otimes 2}$ is globally generated by Corollary 2.4.17. Since $\mathcal{L}$ is ample and globally generated, we have $\varepsilon(\mathcal{L}^{\otimes 2}) \geq 1$ (see, for example [447, I:Example 5.1.18]), which implies $\varepsilon(\mathcal{L}) \geq \frac{1}{2}$. This establishes all inequalities in (2.4.8).

An ample invertible sheaf $\mathcal{L}$ is globally generated if and only if $\Phi(\mathcal{L}) \geq 2$ by Theorem 2.4.14 and for ample invertible sheaves that are globally generated, we already mentioned that we have $\epsilon(\mathcal{L}) \geq 1$.

If $\epsilon(\mathcal{L}) < 1$, then $\Phi(\mathcal{L}) < 2$ by the just established result and thus, $\Phi(\mathcal{L}) = 1$ by Lemma 2.4.10. In particular, there exists a half-fiber F such that $\mathcal{L} \cdot F = 1$. Moreover, if $x \in S$ is a point such that $\epsilon(\mathcal{L}, x) < 1$, then there exists a unique non-multiple $E(x) \in |2F|$ passing through x. As explained in the proof of [696, Proposition 3.5], the infimum $\epsilon(\mathcal{L}, x)$ is achieved on a component of $E(x)$. This implies that

$$\frac{\mathcal{L} \cdot E(x)}{\mathrm{mult}_x E(x)} = \frac{2}{\mathrm{mult}_x E(x)} = \epsilon(\mathcal{L}, x) < 1.$$

Since $\epsilon(\mathcal{L}) \geq \frac{1}{2}$, the classification of irreducible divisors of canonical type shows that $E(x)$ must be a double curve and x must be a non-smooth point of its reduction or $E(x)$ is a triple point of a fiber of type $\tilde{A}_2^*$. From this, we infer $\epsilon(\mathcal{L}, x) = \frac{1}{2}$ or $\frac{2}{3}$ and the remaining assertion follows. □

Remark 2.4.22 In the case where there exists a half-fiber F on S with $\Phi(\mathcal{L}) = \mathcal{L} \cdot F$ and such that F is a singular curve, then the proof shows that $\epsilon(\mathcal{L}) \leq \frac{1}{2}\Phi(\mathcal{L})$ holds true in this case.

We refer to [696] and the more recent article [246] for more about Seshadri constants on Enriques surfaces, further bounds, and the relation to generation of s-jets. It would be interesting to extend these results to positive characteristic.

2.5 Numerically Connected Divisors

In Section 2.1, we introduced numerically connected divisors. In this section, we briefly discuss a generalization and some useful applications.

Let D be an effective divisor on a smooth and proper surface X. We say that D is called *numerically m-connected* if for every decomposition $D = D_1 + D_2$ as a sum of two nonzero effective divisors D_1 and D_2, we have $D_1 \cdot D_2 \geq m$. Clearly, an effective divisor is numerically 1-connected if and only if it is numerically connected in the sense of Section 2.1. In particular, we refer to Lemma 2.1.10 for some easy results on 1-connected divisors. Next, we define a linear system $|D|$ (resp. an invertible sheaf $\mathcal{L}$) on X to be m-connected if every effective divisor in $|D|$ (resp. every effective divisor in $|\mathcal{L}|$) is m-connected.

Proposition 2.5.1 *Let C be an irreducible curve on an Enriques surface S with $C^2 > 0$. Then:*

1. *$\Phi(C) \geq 1$ and $|C|$ is 1-connected.*
2. *If $|C|$ is m-connected, then $C^2 \geq 2m$.*
3. *$|C|$ is m-connected if and only if $|C + K_S|$ is m-connected.*
4. *$|C|$ is 2-connected if and only if $\Phi(C) \geq 2$.*

Proof Since C is irreducible with $C^2 > 0$, it follows that $\mathcal{O}_X(C)$ is big and nef. Thus, $|C|$ is 1-connected by Lemma 2.1.10.(3) and we have $\Phi(C) \geq 1$ by Lemma 2.4.10. From this, the first claim follows.

Assume that $|C|$ is m-connected. By the Reducibility Lemma (Theorem 2.3.5), $|C|$ contains a sum of non-zero effective divisors $D_1 + D_2$ with non-negative self-intersection. Thus, $C^2 = (D_1 + D_2)^2 = D_1^2 + D_2^2 + 2D_1 \cdot D_2 \geq 2m$, which establishes the second claim.

To show the third claim, assume that $|C|$ is m-connected. Seeking a contradiction, suppose that $D \in |C + K_S|$ is not m-connected. Thus, there exist two non-zero and effective divisors D_1 and D_2 with $D = D_1 + D_2$ and $D_1 \cdot D_2 < m$. By the previous assertion, we have $D^2 = C^2 \geq 2m$, hence one of the divisors D_1, D_2, say D_1, has positive self-intersection. Thus, by Riemann–Roch, there exists an effective divisor D_1' that is linearly equivalent to $D_1 + K_2$. From this we find $D_1' + D_2 \in |C|$ with $D_1' \cdot D_2 < m$, contradicting the assumption that $|C|$ is m-connected. This contradiction establishes the third claim.

To show the fourth claim, let us first assume $\Phi(C) \leq 1$ and thus, $\Phi(C) = 1$ by the first claim. Let E be a genus one curve with $C \cdot E = 1$ and let $2n = C^2$. Then, $(C - nE)^2 = 0$ and $|C - nE| \neq \emptyset$. Thus, we can write $C = C_1 + C_2$ with $C_1 = E$ and $C_2 = (n-1)E + E'$ for some $E' \in |C - nE|$. Since $C_1 \cdot C_2 = E \cdot (C - E) = 1$, it follows that $|C|$ is not 2-connected. Conversely, suppose that $|C|$ is not 2-connected. Then, we can find effective divisors D_1, D_2 with $D_1 + D_2 \in |C|$ and $D_1 \cdot D_2 = 1$. First, suppose that we have $D_1^2 \leq 0$ and $D_2^2 \leq 0$. Since $0 < (D_1 + D_2)^2 = D_1^2 + D_2^2 = 0$, we find $D_1^2 = D_2^2 = 0$ and $D_1 \cdot D_2 = 1$, from which we infer $\Phi(C) = \Phi(D) = 1$ in this case. Thus, we may now suppose that $D_1^2 > 0$ or $D_2^2 > 0$, say $D_1^2 > 0$. Then, by the Hodge index theorem, we find $D_1^2 \cdot D_2^2 \leq (D_1 \cdot D_2)^2 = 1$, which implies $D_2^2 \leq 0$. If $D_2^2 = 0$, then $D \cdot D_2 = 1$, and we find $\Phi(C) = \Phi(D) = 1$. Finally, if $D_2^2 < 0$, then we find $D_1^2 \geq D^2$ and $h^0(D_1) \geq h^0(D)$. However, since $|D| = |C|$ has no fixed components, we get a contradiction and thus, this case does not exist. This establishes the fourth claim. ◻

Proposition 2.5.2 *If F is an indecomposable divisor of canonical type, then it is 2-connected.*

Proof If we write F as a sum of two proper and effective divisors $F = D_1 + D_2$, then $D_1^2 \leq -2$ and $D_2^2 \leq -2$. Hence, $0 = F^2 = D_1^2 + D_2^2 + 2D_1 \cdot D_2$ implies that $D_1 \cdot D_2 \geq 2$. ◻

Let D be nef divisor with $D^2 > 0$ on an Enriques surface S, that is, D is big and nef. By the Hodge index theorem, see also Proposition 2.1.6, the orthogonal complement of $[D]$ in $\mathrm{Num}(S)$

$$[D]^\perp := \{x \in \mathrm{Num}(S) \mid x \cdot [D] = 0\} \subseteq \mathrm{Num}(S)$$

contains the orthogonal sum of negative definite lattices that are spanned by the (-2)-curves

$$\mathcal{R}_D := \{R \in \mathcal{R}(S) \mid R \cdot D = 0\},$$

see also Proposition 2.1.6. More precisely, for every such orthogonal summand, there is a root basis of finite type formed by (-2)-curves. In this situation, we established in Proposition 0.4.7 a unique fundamental cycle. Moreover, by Proposition 0.4.11, this fundamental cycle can be identified with the highest root. This motivates to define the *fundamental cycle* of D to be the sum of the fundamental cycles of these orthogonal summands. By the results of Section 0.4, the fundamental cycle of D is the unique effective divisor Z of S such that $Z^2 = -2$ and $Z \cdot R \leq 0$ for every (-2)-curve R with $R \cdot D = 0$. Since D is big and nef, the linear system $|D|$ is 1-connected by Lemma 2.1.10. Moreover, in most cases also $|D - Z|$ is 1-connected:

Proposition 2.5.3 *Let D be a big and nef divisor on an Enriques surface S and let Z be its fundamental cycle. If $|D - Z|$ is not 1-connected, then $D^2 = 2$ and at least one of $|D|$ and $|D + K_S|$ has a fixed component.*

Proof Let D_1, D_2 be effective divisors with $D_1 + D_2 \in |D - Z|$. Since $|D|$ is 1-connected by Lemma 2.1.10, we find $D_1 \cdot (D_2 + Z) \geq 1$ and $D_2 \cdot (D_1 + Z) \geq 1$. This gives

$$2D_1 \cdot D_2 + (D - Z) \cdot Z = 2D_1 \cdot D_2 - Z^2 = 2D_1 \cdot D_2 + 2 \geq 2.$$

Therefore, we have $D_1 \cdot D_2 \geq 1$, unless $D_1 \cdot D_2 = 0$ and $D_1 \cdot Z = D_2 \cdot Z = 1$. Now, assume the latter. Since $0 \leq D \cdot D_1 = (D_1 + D_2 + Z) \cdot D_1 = D_1^2 + 1$, we conclude $D_1^2 \geq 0$, and similarly, we find $D_2^2 \geq 0$. Using the Hodge index theorem, we conclude $D_1 \equiv D_2$, hence $D_1^2 = D_2^2 = 0$ and $D^2 = 2$. Then, we can write $D_1 = E_1 + Z_1$ and $D_2 = E_2 + Z_2$, where $|2F_1| = |2F_2|$ is a genus one pencil and Z_1, Z_2 are nodal cycles. This gives $|D| = |E_1 + E_2 + Z_1 + Z_2 + Z|$. If $E_1 = E_2$, then we find $\dim |D| = \dim |2E_1| = 1$ and $|D|$ has a fixed component. If $E_1 \neq E_2$, then we find $\dim |D + K_S| = \dim |2E_1| = 1$, and $|D + K_S|$ has a fixed component. It follows from Proposition 2.2.11 that in each of these cases Z is a (-2)-curve. □

2.6 Big and Nef Divisors with $\Phi \leq 2$

In Section 2.4, we began our study of the linear system associated to a big and nef invertible sheaf $\mathcal{L}$ on an Enriques surface S. If $\Phi(\mathcal{L}) \geq 2$, then $|\mathcal{L}|$ has no base points, see Theorem 2.4.14. If $\Phi(\mathcal{L}) \geq 3$, then $|\mathcal{L}|$ defines a morphism to projective space such that S is birational onto its image, see Theorem 2.4.16. This motivates to study divisors and linear systems with $\Phi \leq 2$ in greater detail, which is the subject of this section.

We start with linear systems associated to big and nef divisors with $\Phi = 1$.

Proposition 2.6.1 *Let D be a big and nef divisor on an Enriques surface S with $\Phi(D) = 1$ and $D^2 = 2n \geq 2$. Then, one of the following cases occurs:*

1. *$|D| = |nF_1 + F_2|$, where $|2F_1|$ and $|2F_2|$ are genus one pencils with $F_1 \cdot F_2 = 1$,*
2. *$|D| = |(n + 1)F_1 + R|$, where $|2F_1|$ is a genus one pencil and R is a (-2)-curve with $R \cdot F_1 = 1$,*
3. *$|D| = |(n+1)F_1 + R + K_S|$, where $|2F_1|$ is a genus one pencil and R is a (-2)-curve with $R \cdot F_1 = 1$.*

If the linear system $|D|$ has a fixed component, then $n = 1$ and D is from the second case. More precisely, we then have $|D| = |2F_1| + R$ for a genus one pencil $|2F_1|$ with $D \cdot F_1 = 1$.

Proof Let D be as in the assumptions. Let $|2F|$ be a genus one pencil with $D \cdot F = 1$. Then, $(D - nF)^2 = 0$, hence $D \sim nF + A$, where A is a divisor with $A^2 = 0$ and $A \cdot F = 1$. This shows that $[A] \in \mathrm{Num}(S)$ is a primitive isotropic vector. Replacing A by an effective divisor and applying Proposition 2.3.3, we find $A \sim F' + Z$, where F' is a nef divisor with $F'^2 = 0$. This implies that $|2F'|$ is a genus one pencil and that Z is a nodal cycle. We have:

1. either $F \cdot F' = 1$ and $F' \cdot Z = 0$,
2. or else $F \cdot F' = 0$ and $F \cdot Z = 1$.

In the first case, we find $|D| = |nF + F' + Z|$ and $D^2 = (nF + F')^2$ implies that $|D| = |nF + F'|$ because $|nF + F'|$ has no fixed components. This leads to the first case claimed in the proposition. In the second case, we have $F' = F$ or $F' \sim F + K_S$ and then, we find $|D| = |(n + 1)F + Z|$ or $|D| = |(n + 1)F + Z + K_S|$. Let R be the unique irreducible component of Z such that $R \cdot F = 1$. If $n \geq 2$, then, since $((n + 1)F + R)^2 = 2n = D^2$ and $|D|$ has no fixed components, we obtain that $Z = R$. If $n = 1$, we come to the same conclusion by applying Proposition 2.2.11.

The assertion on the fixed components follows applying the just established classification to Proposition 2.2.11 and Corollary 2.2.12. □

All three cases of the previous Proposition do exist. Of course, if S is unnodal, Cases (2) and (3) cannot occur on S.

Corollary 2.6.2 *Let D a nef divisor with $\Phi(D) = 1$ and $D^2 \geq 4$. Then, there exists a unique genus one pencil $|2F|$ such that $D \cdot F = 1$.*

Proof If $|D| = |nF_1 + F_2|$ as in the first case of the Proposition, then F_1 satisfies $D \cdot F_1 = 1$. Moreover, if $|E|$ is a genus one pencil on S different from $|2F_1|$, then $F_1 \cdot E \geq 1$ and thus, $D \cdot E = (nF_1 + F_2) \cdot E \geq nF_1 \cdot E \geq n \geq 2$ (here, we use $D^2 = 2n \geq 4$). Thus, $|2F_1|$ is the unique genus one pencil with $D \cdot F_1 = 1$. We leave the remaining cases to the reader. □

Lemma 2.6.3 *Let $|2F_1|$ and $|2F_2|$ be two genus one pencils on an Enriques surface with $F_1 \cdot F_2 = 1$. Then, F_1 and F_2 have no common irreducible components. In particular, $F_1 \cap F_2$ consists of one point.*

Proof By Proposition 2.5.2, the divisors F_1 and F_2 are 2-connected. Let D_1 be the maximal effective divisor with $D_1 \leq F_1$ and $D_1 \leq F_2$. If we let $F_1 = D_1 + D_2$ and $F_2 = D_1 + D_2'$ be the decompositions into effective divisors, then we have $D_2 \cdot D_2' \geq 0$. Therefore, $1 = F_1 \cdot F_2 = (D_1 + D_2) \cdot F_2 = D_2 \cdot F_2 = D_2 \cdot D_1 + D_2 \cdot D_2' \geq D_2 \cdot D_1$, where we use that $D_1 \cdot F_2 = 0$. Hence, $D_1 = 0$. □

We now describe the base points of linear systems $|D|$ if D is big and nef with $\Phi(D) = 1$.

Proposition 2.6.4 *Let D be a big and nef divisor on an Enriques surface S with $\Phi(D) = 1$ and $D^2 = 2n$. Assume that $|D|$ has no fixed components. Then, $|D|$ has two simple base points; one of them is infinitely near if S is not a classical Enriques surface.*

Proof Since $\Phi(D) = 1$ and using Lemma 2.4.10.(3), we see that there exists a genus one pencil $|2F|$ with $D \cdot F = 1$. More precisely, F is a half-fiber and by Lemma 2.2.7, we have $\omega_F \cong \mathcal{O}_F$. It thus follows from Riemann–Roch that $h^0(F, \mathcal{O}_S(D)|_F) = 1$. Thus, there exists a unique point $x \in F$ such that every member of $|D|$ passes through x, that is, x is a base point of $|D|$. If S is classical, then $|2F|$ has another half-fiber by Corollary 2.2.9 and we find a second base point of $|D|$.

Next, let us show that the number of base points, counted with multiplicities is even. If $D^2 = 2$,then this is obvious since $|D|$ is a pencil, hence the number is equal to $D^2 = 2$. So, we may assume that $D^2 \geq 4$. Let P be a general member of $|2F|$. The short exact sequence

$$0 \to \mathcal{O}_S(D - P) \to \mathcal{O}_S(D) \to \mathcal{O}_S(D)|_P \to 0,$$

together with the Vanishing Theorem 2.1.16 applied to $\mathcal{O}_S(D - P)$ (here, we use that $D^2 \geq 4$) shows that the restriction of $|D|$ to P is a complete linear system of degree 2 on P, which defines a morphism $P \to \mathbb{P}^1$ of degree 2. In particular, the rational map defined by $|D|$ on S is generically finite of degree 2. Thus, the number of base points of $|D|$ is even, when counted with multiplicities.

Now, suppose n is odd, say $n = 2k + 1$. We use the notation from Proposition 2.6.1. Then, $|D|$ contains $|2kF_1| + |D - 2kF_1|$, and since $|2kF_1|$ is composite with k genus one pencils $|2F_1|$, the base points of $|D|$ are contained in the base points of $|D - 2kF_1|$.

In Case (1) of Proposition 2.6.1, we have $|D - 2kF_1| = |F_1 + F_2|$. This is an irreducible pencil with $(F_1 + F_2)^2 = 2$. Thus, the number of base points is equal to $(F_1 + F_2)^2 = 2$. We have

$$\mathcal{O}_{F_1}(F_1 + F_2) \cong \mathcal{O}_{F_1}(F_1) \otimes \mathcal{O}_{F_1}(F_2).$$

If S is classical, that is, $K_S \neq 0$, then $\mathcal{O}_{F_1}(F_1)$ is the non-trivial normal sheaf of F_1 and hence, the base point on F_1 is different from the intersection point $F_1 \cap F_2$. Since $\mathcal{O}_{F_1'}(F_1 + F_2) \cong \mathcal{O}_{F_1}(F_1 \cap F_2)$, the second base point is the intersection point $F_1' \cap F_2$.

If S is classical, then we find two base points lying on half-fibers of $|2F_1|$. If S is non-classical, that is, $K_S = 0$, then we find one base point $F_1 \cap F_2$ and the infinitely near point to it corresponding to the tangent direction different to that of F_1 and F_2.

In Case (2) of Proposition 2.6.1, we have $|D - 2kF_1| = |2F_1 + R|$. This linear system has a fixed component, which is equal to R, see also Proposition 2.2.11. More preicsely, all base points of $|2F_1 + R| = |2F_1| + R$ lie on R. Writing $D = (2k - 2)F_1 + F_1 + (3F_1 + R)$, we see that the base points of $|D|$ outside F_1 are base points of $|3F_1 + R|$. Since $(3F_1 + R) \cdot R = 1$, we find one base point on R outside $F_1 \cap R$. In fact, this must be equal to $F_1' \cap R$, where $F_1' \sim K_S + F_1$. The other base point is $F_1 \cap R$. If $K_S \neq 0$, then $|D|$ contains a divisor $kF + F_1' + R$, where $F \neq 2F_1$ belongs to $|2F_1|$. This divisor is smooth at the point $F_1 \cap R$ and its direction at this point is equal to R. Since $(3F_1 + R) \cdot R = 1$, this shows that the base point $F_1 \cap R$ is simple. Similarly, we see that the base point $F_1' \cap R$ is simple. If $K_S = 0$, then $|3F_1 + R|$ is a double point with two branches tangent to F_1 and R. Again, a general member at this point is smooth. After blowing up this point, we find a simple base point of the proper transform of the linear system. This shows that $F_1 \cap R$ is a point of multiplicity 2.

In Case (3) of Proposition 2.6.1, we use the decomposition $D = 2kF_1 + (2F_1 + R + K_S)$. We may assume that $K_S \neq 0$ (otherwise we are in the previous case). Then, the linear system $|2F_1 + R + K_S|$ is irreducible and $(2F_1 + R + K_S)^2 = 2$. We argue as in first case above and find two simple base points on F_1 and F_1'.

Next, suppose that n is even, say $n = 2k$.

In Case (1) of Proposition 2.6.1, we use the decomposition $D = (2k-2)F_1 + (2F_1 + F_2)$. The base points are base points of $|2F_1 + F_2|$. Suppose $K_S \neq 0$. Then the restriction of $|2F_1 + F_2|$ to F_1 (resp. F_1') has one base point equal to $F_1 \cap F_2$ (resp. $F_1' \cap F_2$). Since $(2F_1 + F_2) \cdot F_2 = 2$, there are no more base points on F_2. Thus, we conclude that there are only two base points. To compute their multiplicities, we argue as in the previous case: we use that the divisor $F + F_2$, where F is a general member of $|2F_1|$ is nonsingular at $F_1 \cap F_2$ and $F_1' \cap F_2$. Its tangent direction is F_2. Since $(2F_1 + F_2) \cdot F_2 = 2$, we conclude that the base points are simple. If $K_S = 0$, then a similar argument shows that all members of $|2F_1 + F_2|$ are tangent to F_2 at the unique base point.

We leave the remaining cases to the reader. □

If S is a classical Enriques surface and D is a big and nef divisor with $\Phi(D) = 1$ and $D^2 = 2n \geq 2$, then the following pictures show the positions of the two base points of $|D|$. The three columns correspond to the three cases of Proposition 2.6.1 and the two rows distinguish, whether n is even or odd - we refer to the proof of Proposition 2.6.4 for details.

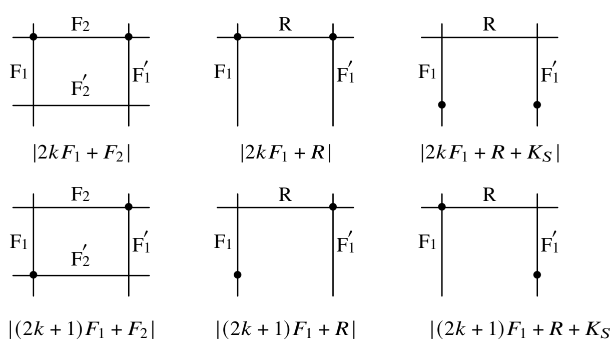


Fig. 2.1 Base points of $|D|$ for big and nef divisors D with $\Phi(D) = 1$

As another corollary of the proof of Proposition 2.6.4, we describe the rational map associated to the linear system $|D|$ of a big and nef divisor with $\Phi(D) = 1$. If $|D|$ has a fixed component, then $|D| = |2F| + R$ for some genus one fibration $|2F|$ and a (-2)-curve R with $F \cdot R = 1$ by Proposition 2.6.1. Thus, the map associated to $|D|$ is the genus one fibration $|2F|$. In the remaining cases, we have the following result. We continue our study of these rational maps in Section 3.2.

Corollary 2.6.5 . *Let D be a big and nef divisor on an Enriques surface S with $\Phi(D) = 1$ and $D^2 = 2n \geq 2$. Assume that $|D|$ has no fixed components. Let $\bar{S} \to S$ be the blow-up in the two base points (possibly infinitely near) of $|D|$. Then, the rational map $\phi_{|D|}$ defined by $|D|$ extends to a morphism*

$$\bar{\phi}_{|D|} \, : \, \bar{S} \, \to \, \mathbb{P}^n$$

with the following properties:

1. *If $n = 1$, then ψ is a fibration, whose generic fiber is an integral curve of genus two. If the ground field is of characteristic $p \notin \{2, 3, 5\}$, then the generic fiber of $\bar{\phi}_{|D|}$ is smooth.*
2. *If $n \geq 2$, then $\bar{\phi}_{|D|}$ is generically finite of degree 2 onto a surface of minimal degree $(n-1)$ in $\mathbb{P}^n$, see Theorem 0.5.2.*

Proof If $n \geq 2$, then we saw already in the proof of Proposition 2.6.4 that the rational map defined by $|D|$ is generically finite of degree 2 onto its image. If $\bar{D}$ denotes the strict transform of D on $\bar{S}$, then $\bar{D}^2 = D^2 - 2 = 2(n-1)$. Thus, $\bar{\phi}_{|D|}(\bar{S})$ is a surface of degree $(n-1)$ in $\mathbb{P}^n$, that is, a surface of minimal degree.

If $n = 1$, then either $|D| = |F_1 + F_2|$ or $|D| = |2F + K_S + R|$ and a generic member of $|D|$ is an integral curve of arithmetic genus two by the adjunction formula, see also the proof of Proposition 2.6.4 In these cases, $\bar{\phi}_{|D|}$ is a genus two fibration. It follows from Theorem 4.1.3 that the generic fiber of $\bar{\phi}_{|D|}$ is smooth if the ground field is of characteristic $p \notin \{2, 3, 5\}$. □

Next, we study big and nef divisors D with $\Phi(D) = 2$ modulo numerical equivalence. By Proposition 2.4.11, such divisors satisfy $D^2 \geq 4$. We distinguish between self-intersection numbers D^2 of the form $4k$ and $4k + 2$.

Proposition 2.6.6 *Let D be a big and nef divisor on an Enriques surface S with $\Phi(D) = 2$ and $D^2 = 4k > 0$. Then, $D \equiv C$, where C is one of the following curves:*

1. *$kF_1 + 2F_2$, where $F_1 \cdot F_2 = 1$ and $k \geq 2$,*
2. *$(k+2)F_1 + 2R_1$, where $F_1 \cdot R_1 = 1$ and $k \geq 2$,*
3. *$kF_1 + F_2$, where $F_1 \cdot F_2 = 2$,*
4. *$(k+1)F_1 + R_1 + R_2$, where $F_1 \cdot R_1 = F_1 \cdot R_2 = 1$ and $R_1 \cdot R_2 = 0$,*
5. *$kF_1 + F_2 + R$, where $R = R_1 + \cdots + R_n$ is a fundamental cycle of type A_n with $R \cdot R_1 = R \cdot R_n = -1$ and $F_1 \cdot R = F_1 \cdot R_1 = F_2 \cdot R = F_2 \cdot R_n = F_1 \cdot F_2 = 1$,*
6. *$(k+1)F_1 + R$, where $R = 2R_1 + \cdots + 2R_n + R_{n+1} + R_{n+2}$ with $F_1 \cdot R_1 = R_i \cdot R_{i+1} = 1$, $i = 1, \ldots, n$, $R_n \cdot R_{n+1} = R_n \cdot R_{n+2} = 1$ and all other intersection indices of different components are zeros.*

Here, the $|2F_i|$ are genus one pencils and the R_i are (-2)-curves.

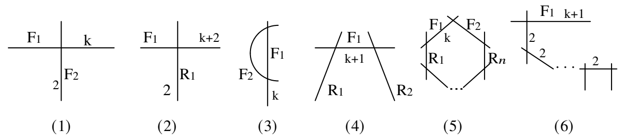


Fig. 2.2 Big and nef divisors D with $\Phi(D) = 2$ and $D^2 = 4k$

Proof The proof is similar to the proof of Proposition 2.6.1. We only sketch the proof and refer the interested reader to [133] for details. Let $|2F_1|$ be a genus one pencil with $D \cdot F_1 = 2$. Then, $(D - kF_1)^2 = 0$, hence $D \sim kF_1 + A$, where $A^2 = 0$ and $F_1 \cdot A = 2$. We have either $A \sim 2F_2 + R$, where R is a nodal cycle with $F_2 \cdot A = 0$, or else $A \sim F_2 + R$, where R is a nodal cycle with $F_2 \cdot R = 1$. The first possibility leads to cases (1) and (2). The second possibility leads to the remaining cases. □

Proposition 2.6.7 *Let D be a big and nef divisor on an Enriques surface S with $\Phi(D) = 2$ and $D^2 = 4k + 2 > 0$. Then, $D \equiv C$, where C is one of the following curves:*

1. *$kF_1 + F_2 + F_3$, where $F_i \cdot F_j = 1$ for all $i \neq j$,*
2. *$kF_1 + 2F_2 + R_1$, where $F_1 \cdot F_2 = F_2 \cdot R_1 = 1$ and $F_1 \cdot R_1 = 0$,*
3. *$(k+1)F_1 + F_2 + R_1$, where $F_1 \cdot F_2 = F_1 \cdot R_1 = 1$ and $F_2 \cdot R_1 = 0$,*
4. *$(k+2)F_1 + 2R_1 + R_2$, where $R_1 \cdot R_2 = F_1 \cdot R_1 = 1$ and $F_1 \cdot R_2 = 0$.*

Here, the $|2F_i|$ are genus one pencils and the R_i are (-2)-curves.

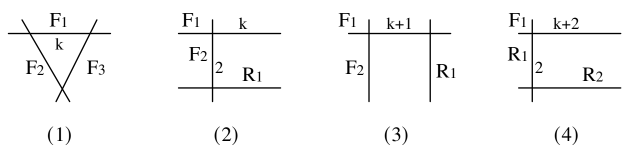


Fig. 2.3 Big and nef divisors D with $\Phi(D) = 2$ and $D^2 = 4k + 2$

As an important application, we obtain the following result on base-point freeness of linear systems on Enriques surfaces.

Corollary 2.6.8 *Let D be a big and nef divisor on an Enriques surface S with $\Phi(D) \geq 2$. Then, the linear system $|D|$ has no base points.*

Proof Since $\Phi(D) \geq 2$, we have $D^2 \geq 4$ by Proposition 2.4.11. By Corollary 2.4.6, the assertion is true if $D^2 \geq 6$. Thus, we may assume $D^2 = 4$. By the Proposition 2.6.6, D is numerically equivalent to a curve of one of the types (3) – (6) with $k = 1$.

First, assume that D is of type (3), that is, $D \equiv F_1 + F_2$, where $|2F_i|$ are genus one pencils with $F_1 \cdot F_2 = 2$. Replacing F_2 by $F_2 + K_S$ if necessary, we may assume that $D \sim F_1 + F_2$. Taking cohomology in the exact sequence

$$0 \to \mathcal{O}_S(F_1) \to \mathcal{O}_S(D) \to \mathcal{O}_S(D)|_{F_2} \to 0$$

together with the fact that $H^1(S, \mathcal{O}_S(F_1)) = 0$, it follows that the restriction homomorphism $H^0(S, \mathcal{O}_S(D)) \to H^0(F_2, \mathcal{O}_S(D)|_{F_2})$ is surjective. Since $\mathcal{O}_S(D)|_{F_2}$ is an invertible sheaf of degree 2 on a curve of arithmetic genus one curve, the linear system $|D|_{F_2}|$ has no base points. This shows that $|D|$ has no base points on F_2. Similarly, it follws that $|D|$ has no base points on F_1. Hence, divisor $F_1 + F_2 \in |D|$ does not pass through base points, hence $|D|$ has no base points.

The remaining cases can be treated similarly, and we leave to the reader. Assume $D^2 \leq 4$. It follows from Corollary 1.5.4 that, for any vector $v \in \mathsf{E}_{10}$ with $v^2 \leq 4$, $\Phi(v) \leq 2$ and the equality takes place only if $v^2 = 4$. Thus, we may assume that $D^2 = 4$. By Proposition 2.6.6, D is of types (3)-(6i) with $k = 1$. Assume D is of type (3). Then, the exact sequence

$$0 \to \mathcal{O}_S(F_1) \to \mathcal{O}_S(D) \to \mathcal{O}_{F_2}(D) \to 0,$$

together with vanishing of $H^1(S, \mathcal{O}_S(F_1))$, shows that the restriction homomorphism $H^0(S, \mathcal{O}_S(D)) \to H^0(F_2, \mathcal{O}_{F_2}(D))$ is surjective. Since $\mathcal{O}_{F_2}(D)$ is a degree 2 invertible sheaf on a curve of arithmetic genus one curve, the linear system $|\mathcal{O}_{F_2}(D)|$ has no base points. This shows that $|D|$ has no base points on F_2. Similarly, we show that it has no base points on F_1. Hence, the divisor $F_1 + F_2 \in |D|$ does not pass through base points, hence $|D|$ has no base points. The remaining cases can be treated similarly, and we leave it to the reader. □

Bibliographical Notes

The standard material about properties of invertible sheaves or divisor classes can be found now in many textbooks, for example [38], [447], [612]. The vanishing theorem $H^1(\mathcal{O}_X(-D)) = 0$ for a big and nef divisor D over ground fields of characteristic zero is due to Ramanujam [602]. The first counter-examples in positive characteristic were given by Lang [431] and Raynaud [608]. The fact that Ramanujam's proof applies to an Enriques surface S over a field of arbitrary characteristic except when S is supersingular was observed by Lang [433, Theorem 0.8]. The first proof of the Vanishing Theorem for Enriques surfaces in characteristic 2 is given in [138, Theorem 1.5.1].

The relationship between the nef cone and a fundamental chamber of a reflection group was first exploited by Looijenga [475] in his work on rational surfaces. The idea that a primitive isotropic numerical divisor class gives rise to a genus one pencil is borrowed from the work of Piatetsky-Shapiro and Shafarevich on the Torelli Theorem for K3 surfaces [599].

The notion of a divisor of canonical type was first introduced by Mumford [538]. The classification of such divisors was first given by Kodaira [401] and, independently, by Néron [553]. The fact that the classification immediately follows from the classification of affine root systems of types A, D, and E was noticed later by many people [43].

Reider's Theorem was first proven in characteristic zero by Reider in [613], who deduced it from Bogomolov's Unstability Theorem 2.4.2. Shepherd-Barron extended Bogomolov's Theorem to positive characteristic with the possible exceptions of surfaces or general type or quasi-elliptic surfaces of Kodaira dimension one. He also deduced Ramananujam's Vanishing Theorem from it. Both of these results can be applied to Enriques surfaces. They greatly simplify many arguments about the properties of linear systems on Enriques surfaces, which were previously used by Cossec [133] and reproduced in [138].

The notion of m-connectedness of a divisor comes from the work of Bombieri [76] on surfaces of general type.

The usefulness of the function Φ for the classification of linear systems was first exploited in the work of Saint-Donat [630] for K3 surfaces and Cossec [133] for Enriques surfaces.

The proof of the Enriques Reducibility Lemma in the case of a classical Enriques surface follows almost without change from the classical proof of Enriques [221]. It was also reproduced in Artin's thesis [18], as well as in [5, Chapter 10] and in [433, Lemma 3.2]. W. Lang also deduces the theorem for non-classical Enriques surfaces by using a lattice-theoretical argument and the non-trivial fact that there exist unnodal Enriques surfaces in any characteristic. Our proof also uses a lattice-theoretical result but does not rely on the latter fact. Another proof that works in any characteristic was given by Bombieri and Mumford [77, Proposition 10]. It is used to show that an Enriques surface contains a genus one pencil. Our proof in the non-classical case uses this fact that we proved earlier in Chapter 1.

The classification of linear systems $|D|$ with $\Phi(D) \leq 2$ was first given in Cossec's thesis and it is reproduced in [138].

Chapter 3
Projective Models of Enriques Surfaces

In this chapter, we study (rational) maps from Enriques surfaces that arise from big and nef divisor classes of small self-intersection and with small Φ. This leads to explicit models and explicit covers of Enriques surfaces. We find birational models of Enriques surfaces as surfaces of small degree in low-dimensional projective spaces. We also find presentations of Enriques surfaces as covers of small degree of explicit rational surfaces. On our way, we also study (rational) maps from K3-covers of Enriques surfaces.

3.1 Preliminaries

In this section, we collect some general facts about maps of K3 surfaces and Enriques surfaces defined by a complete linear system.

Let S be an Enriques surface and let D be a nef divisor with $D^2 = 2d > 0$. In this section, we will study rational maps

$$\phi_{|D|} \;:\; S \dashrightarrow |D|^\vee \;\cong\; \mathbb{P}^d$$

defined by the complete linear system $|D|$. We assume that $d \geq 2$ and then, by Corollary 2.2.12, the linear system $|D|$ has no fixed components and it contains an irreducible curve. In particular, it is not composed of a pencil and hence, its image $S' := \phi_{|D|}(S)$ is a surface, and we refer to Proposition 3.1.1 for the list of possible cases. A refined analysis in the next sections leads to explicit birational models of Enriques surfaces (Section 3.5) and to descriptions of Enriques surfaces as branched double covers of rational surfaces. On our way, we also establish a similar analysis for K3-covers of Enriques surfaces, see Theorem 3.1.7 and Section 3.4.

In Section 2.3, we introduced the set $\mathcal{R}(S)$ of (-2)-curves on S and in Section 2.5, we introduced

$$\mathcal{R}_D \;:=\; \{R \in \mathcal{R}(S)\,|\,R\cdot D \;=\; 0\}\,.$$

Since $D^2 > 0$, by Theorem 2.1.6, the (-2)-curves of $\mathcal{R}_D$ form a root system of finite type (not necessarily irreducible) inside the lattice $\mathrm{Num}(S)$ and thus, they span a negative definite lattice that is orthogonal to the class of D. We will identify $\mathcal{R}_D$ with the union of its members and call it the *nodal cycle* of $|D|$. If no base points of the linear system $|D|$ lie on $\mathcal{R}_D$ then every (-2)-curve of $\mathcal{R}_D$ is contracted to a point under $\phi_{|D|}$. More precisely, if we factor $\phi_{|D|}$ through the normalization of $\phi_{|D|}(S)$, then the image of $\mathcal{R}_D$ on this normalization consists of rational double points, see also Theorem 2.4.16.

Let us also recall that we introduced in Section 2.4 the function $\Phi : \mathrm{Num}(S) \to \mathbb{Z}_{\geq 0}$, which satisfies the estimates

$$1 \leq \Phi(D) \leq \sqrt{D^2}$$

by Lemma 2.4.10 and Proposition 2.4.11. Moreover, the linear system $|D|$ has base-points if and only if $\Phi(D) = 1$ by Theorem 2.4.14 and Corollary 2.6.8. Moreover, if $\Phi(D) = 1$, then we gave a complete description of the base points of $|D|$ and the geometry of $\phi_{|D|}$ in Section 2.6. If $\Phi(D) \geq 2$, then $\phi_{|D|}$ is generically finite onto its image and the generic degree satisfies the estimate

$$1 \leq \deg \phi_{|D|} \leq 2 \cdot \left(1 + \frac{2}{D^2 - 2}\right) \leq 4$$

by Corollary 2.4.15. Putting all these observations together, we obtain the following list of possibilities. In it, the classification of surfaces of small degree from Section 0.5 will be very useful in describing the image $S' := \phi_{|D|}(S)$ if $\deg \phi_{|D|} \geq 2$, that is, if S is not birational to S'.

Proposition 3.1.1 *Let S be an Enriques surface, let D be a nef divisor with $D^2 = 2d > 0$, let*

$$\phi_{|D|} : S \dashrightarrow \mathbb{P}^d$$

be the rational map associated to $|D|$ and let $S' := \phi_{|D|}(S)$. Then, one of the following cases occurs.

1. *$\Phi(D) = 1$ and $D^2 = 2$. Then,*
   a. *either $|D|$ has a fixed component and $\phi_{|D|}$ rationally defines a genus one fibration,*
   b. *or $|D|$ has no fixed components and $\phi_{|D|}$ rationally defines a genus two fibration.*

   These are the only cases, where S' is not a surface.
2. *$\Phi(D) = 1$ and $D^2 \geq 4$. Then, $|D|$ has two base points, maybe infinitely near, and*

   $$\deg \phi_{|D|} = 2 \quad \textit{and} \quad \deg S' = \operatorname{codim} S' + 1.$$

   In particular, the possible images S' are classified in Theorem 0.5.2.
3. *$\Phi(D) = 2$, $D^2 \geq 4$, and $|D|$ has no base points. Then, we have the following subcases:*

a. $\deg \phi_{|D|} = 1$, *that is,* $S \to S'$ *is a birational morphism.*
b. $\deg \phi_{|D|} = 2$ *and* $\deg S' = \operatorname{codim} S' + 2$. *In particular, the possible images* S' *are classified in Theorem 0.5.5.*
c. $D^2 = 4$, $\deg \phi_{|D|} = 4$, *and* $S' = \mathbb{P}^2$.
d. $D^2 = 6$, $\deg \phi_{|D|} = 2$, *and* $S' \subseteq \mathbb{P}^3$ *is a cubic surface.*

4. $\Phi(D) \geq 3$, $D^2 \geq 10$, *and* $|D|$ *has no base points. Then,* $\deg \phi_{|D|} = 1$, *that is,* $S \to S'$ *is a birational morphism, and* S' *has at worst rational double point singularities.*

Remark 3.1.2 We will show in Proposition 3.3.1 that $D^2 = 8$ in case (3b).

Proof If $|D|$ has a fixed component, then $D^2 = 2$ and $|D| = |M| + R$ for a genus one fibration $|M|$ with $M \cdot R = 2$, see Proposition 2.2.11. In this case, $\Phi(D) = 1$ and $|D|$ rationally defines a genus one fibration. If $|D|$ has no fixed components and $D^2 = 2$, then $|D|$ has two base points and rationally defines a genus two fibration by Corollary 2.6.5. This establishes Case (1).

If $\Phi(D) = 1$ and $2d = D^2 \geq 4$, then $|D|$ has two base points, counted with multiplicity, see Theorem 2.4.14 and Corollary 2.6.8. The analysis leading to Case (2) has been carried out in Corollary 2.6.5.

Third, if $\Phi(D) \geq 2$, then $|D|$ has no basepoints and together with $D^2 > 0$, we conclude that S' is a surface. Moreover, we have $D^2 \geq \Phi(D)^2 \geq 4$ by Proposition 2.4.11. The generic degree $\deg \phi_{|D|}$ can be estimated using Corollary 2.4.15. From this, the subcases (a) and (b) of Case (3) follow immediately. If $D^2 = 4$, then the image $S' = \phi_{|D|}(X)$ is $\mathbb{P}^2$, which implies $\deg \phi_{|D|} = 4$, and we obtain subcase (c). If $D^2 = 6$, then we have $1 \leq \deg \phi_{|D|} \leq 3$. Since $\Phi(D) = 2$, there exists an elliptic fibration $|E|$ with $D \cdot E = 2$ and so, the restriction of $\phi_{|D|}$ to a general fiber of $|E|$ is of degree 2, which implies $\deg \phi_{|D|} = 2$. This gives subcase (d).

Finally, if $\Phi(D) \geq 3$, then $2d = D^2 \geq 9$ by Proposition 2.4.11 and S' has at worst rational double point singularities by Theorem 2.4.16. This establishes Case (4). □

In the next sections, we will analyze Cases (2) and (3) in greater detail - they will lead to some classical models and presentations of Enriques surfaces. To do so, we collect a couple of results and facts that we will use in this chapter.

Proposition 3.1.3 *Let* S *be a classical Enriques surface, that is,* $K_S \neq 0$. *Let* D *be an effective divisor on* S *with* $H^1(S, \mathcal{O}_S(D)) = 0$. *Then,*

$$H^0(D, \mathcal{O}_D(K_S)) = 0.$$

Proof Consider the long exact sequence in cohomology associated to

$$0 \to \mathcal{O}_S(K_S - D) \to \mathcal{O}_S(K_S) \to \mathcal{O}_D(K_S) \to 0.$$

Then $h^0(S, \mathcal{O}_S(K_S)) = 0$ and $h^1(S, \mathcal{O}_S(K_S - D)) = h^1(S, \mathcal{O}_S(D)) = 0$, which follows from Serre duality, imply $h^0(D, \mathcal{O}_D(K_S)) = 0$. □

Applying the adjunction formula, we obtain the following useful result.

Corollary 3.1.4 *Let S be an Enriques surface and let D be a nef and effective divisor. Assume that $D^2 > 0$ or that D is the half-fiber of a genus one pencil. Then,*

$$\mathcal{O}_D(D) \cong \omega_D(\varepsilon),$$

where ε is an element of order at most 2 in $\mathrm{Pic}(D)$*. If S is classical (resp. non-classical), that is, $K_S \neq 0$ (resp. $K_S = 0$), then ε is non-trivial (resp. trivial).*

Proof If we set $\varepsilon := \mathcal{O}_D(K_S)$, then the adjunction formula yields $\mathcal{O}_D(D) \cong \omega_D(\varepsilon)$, where ω_D denotes the dualizing sheaf of the Gorenstein curve D. If $K_S = 0$, then ε is trivial, whereas if $K_S \neq 0$, then ε is a 2-torsion element of $\mathrm{Pic}(D)$ since K_S is a 2-torsion element of $\mathrm{Pic}(S)$. Moreover, if $K_S \neq 0$, then $h^0(D, \varepsilon) = 0$ by Proposition 3.1.3 and thus, ε is a non-trivial 2-torsion element. □

If C is a smooth and proper curve with canonical sheaf ω_C and $\varepsilon \in \mathrm{Pic}(C)$ is a non-trivial 2-torsion element, then the map associated to the complete linear system $|\omega_C|$ (resp. $|\omega_C(\varepsilon)|$) is called the *canonical map* (resp. a *Prym canonical map*). Thus, if S is a classical (resp. non-classical) Enriques surface and D is as in the previous corollary, then the restriction of $|D|$ to D gives rise to a Prym canonical map (resp. to the canonical map) of D. Moreover, if S is classical, then the restriction of $|D + K_S|$ to D gives rise to the canonical map of D and if $D' \in |K_S + D|$, then the restriction of $|D|$ to D' gives rise to the canonical map of D'.

Finally, one should compare Proposition 3.1.1, as well as some of the results of Chapter 2, with the analogous results for K3 surfaces, which are due to Saint-Donat [630], see also [612, Sections 3.8 and 3.15].

Theorem 3.1.5 *Let X be a K3 surface and let D be an effective divisor on X.*

1. *The linear system $|D|$ contains a divisor of the form $M + \sum n_i R_i$, where M is nef and where the R_i are (-2)-curves.*
2. *Assume that D is nef.*
   a. *If $D^2 = 0$, then $D = kF$, where $|F|$ is a genus one pencil.*
   b. *If $D^2 > 0$, then $H^1(X, \mathcal{O}_X(D)) = 0$ and thus,* $\dim |D| = \frac{1}{2}D^2 + 1$.
   c. *If $D^2 > 0$ and $|D|$ has a fixed component, then $D \sim kF + R$, where $|F|$ is a genus one pencil and R is a (-2)-curve with $F \cdot R = 1$.*
3. *Assume that D is big and nef and that $|D|$ has no fixed components. Then, $|D|$ has no base points and for the associated morphism $\phi_{|D|}|D| : X \to \mathbb{P}^{\frac{1}{2}D^2+1}$ one of the following cases occurs:*
   a. *$\phi_{|D|}$ is of degree 1, that is, birational, onto a normal surface $X' \subseteq \mathbb{P}^{\frac{1}{2}D^2+1}$ of degree D^2 that has at worst rational double points as singularities.*
   b. *$\phi_{|D|}$ is a morphism of degree 2 onto a surface $X' \subseteq \mathbb{P}^{\frac{1}{2}D^2+1}$ of degree $\frac{1}{2}D^2$. In particular, X' is a surface of minimal degree, as classified in Theorem 0.5.2. Moreover, $\phi_{|D|}$ can be factored as composition $\psi \circ \phi_{|D'|}$, where $\phi_{|D'|} : X \to \bar{X}'$ is a birational morphism onto a normal surface with at worst rational double points as singularities and $\psi : \bar{X}' \to X'$ is a finite morphism of degree 2.*

Case (b) happens if and only if there exists a genus one pencil $|P|$ on X such that $D \cdot P = 2$.

Concerning the equations defining a subvariety $X \subseteq \mathbb{P}^N$, we recall that Mumford [540] showed that if the embedding of X is given by a "sufficiently ample" invertible sheaf, then the homogeneous ideal of X is generated by quadrics. For example, if C is a smooth and proper curve of genus g and if $\mathcal{L}$ is an invertible sheaf of degree $\geq 2g+1$ on C, then $\mathcal{L}$ is very ample and the image of the map defined by $|\mathcal{L}|$ is cut out by quadrics, see for example, [540, Section 2].

When studying linear systems on surfaces, it is a general technique to restrict these linear systems to general divisors and to study the induced linear system on these restrictions. Thus, from Corollary 3.1.4 it already becomes clear that canonical maps and Prym canonical maps of curves play an important role in the study of linear systems on K3 surfaces and Enriques surfaces. For canonical maps of curves, we have classical results that go back to Babbage, Enriques, Noether, and Petri, see, for example, [12, Chapter III.3]. For example, if $g \geq 3$ and if C is not hyperelliptic, then ω_C is very ample, the canonical map $|\omega_C|$ embeds C as a curve of degree $2g-2$ into $\mathbb{P}^{g-1}$, and the homogeneous ideal is generated by quadrics and cubics. Cubics are only needed if the curve is not trigonal, that is, if there does not exist a morphism $C \to \mathbb{P}^1$ of degree 3.

For K3 surfaces, we have the following theorem of Saint-Donat [630] concerning the equations defining their projective models.

Theorem 3.1.6 *Let X be a K3 surface and let D be a big and nef divisor with $D^2 \geq 8$ such that $|D|$ has no base points. Let $\phi_{|D|} : X \to X' \subseteq \mathbb{P}^{\frac{1}{2}D^2+1}$ be the associated morphism.*

1. *If $\deg \phi_{|D|} = 1$, then the homogeneous ideal of X' is generated by quadrics and cubics. Moreover, cubics are needed if and only if*
   a. *$|D| = |2C + R|$, where C is an irreducible curve with $C^2 = 2$ and R is a (-2)-curve with $R \cdot C = 1$, or*
   b. *there exists a genus one pencil $|P|$ on X such that $D \cdot P = 3$.*
2. *If $\deg \phi_{|D|} = 2$, then the image X' is a surface of minimal degree, whose homogeneous ideal is generated by quadrics.*

Let $\pi : X \to S$ be the K3-cover of an Enriques surface S. In view of Saint-Donat's results, we obtain the following analog of Proposition 3.1.1 for the K3-cover. What makes this result a little bit tricky (and not a straightforward corollary of Theorem 3.1.6), is that the K3-cover in characteristic 2 may not be a K3 surface, in fact, it may not even be normal. Over the complex numbers, the following result is due to Cossec [133] and its extension to positive characteristic (including characteristic 2) is due to Liedtke [460].

Theorem 3.1.7 *Let S be an Enriques surface, let $\pi : X \to S$ be its K3-cover, and let D be a nef divisor with $D^2 = 2d > 0$. Then, the a priori rational map*

$$\widetilde{\phi}_{|D|} \ : \ X \ \dashrightarrow \ \mathbb{P}^{1+2d}$$

associated to the invertible sheaf $\pi^* \mathcal{O}_S(D)$ *is a morphism, its image* $X' := \widetilde{\phi}_{|D|}(X)$ *is a surface, and it is generically finite of degree* $\deg \widetilde{\phi}_{|D|} \leq 2$.

1. *If* $\Phi(D) = 1$*, then* $\deg \widetilde{\phi}_{|D|} = 2$ *and* X' *is a surface of minimal degree* $2d$*. In particular, the possible images* X' *are classified in Theorem 0.5.2).*
2. *If* $\Phi(D) \geq 2$*, then* $D^2 \geq 4$ *and* $\deg \widetilde{\phi}_{|D|} = 1$*, that is,* $X \to X'$ *is a birational morphism, and* X' *is a surface of degree* $4d$.

Proof We work over an algebraically closed field $\Bbbk$ of characteristic $p \geq 0$.

We start with the second assertion and follow [460, Theorem 2.2]: since $\Phi(D) \geq 2$, the linear system $|D|$ has no base points on S by Theorem 2.4.14 and thus, $\pi^* \mathcal{O}_S(D)$ is a globally generated invertible sheaf on X. (Since X may not even be normal, we will use the language of invertible sheaves, global generation, etc. rather than divisors and linear systems.) By Proposition 2.4.11, we have $D^2 \geq \Phi(D)^2 \geq 4$. Next, we set $\mathcal{L} := \mathcal{O}_S(D)$ and consider the short exact sequence

$$0 \to \mathcal{L} \to \pi_* \pi^* \mathcal{L} \to \mathcal{L} \otimes \omega_S \to 0. \tag{3.1.1}$$

We have $h^0(S, \mathcal{L}) = h^0(S, \mathcal{L} \otimes \omega_S) = 1 + \frac{1}{2}\mathcal{L}^2$ and $h^1(S, \mathcal{L}) = 0$ by Theorem 2.1.16 and Corollary 2.1.17. Thus, we find $h^0(X, \pi^* \mathcal{L}) = 2 + \mathcal{L}^2$. Thus, $\pi^* \mathcal{L}$ gives rise to a morphism $\widetilde{\phi}_{|D|}$ from X to projective space of dimension $1 + \mathcal{L}^2 = 1 + 2d$. Also, since the image of the map associated to $|\mathcal{L}|$ is a surface by Corollary 2.4.15, the same is true for $\widetilde{\phi}_{|D|}$. Moreover, $X' = \widetilde{\phi}_{|D|}(X)$ is an integral surface, that is, reduced and irreducible, since X is.

If $p \neq 2$ or if S is a $\boldsymbol{\mu}_2$-surface, then X is a smooth K3 surface and we compute $(\pi^* \mathcal{L})^2 = 2\mathcal{L}^2$. Since $\pi^* \mathcal{L}$ is globally generated, we find $4d = 2\mathcal{L}^2 = \deg \widetilde{\phi}_{|D|} \cdot \deg(X')$. By Proposition 0.5.1, a non-degenerate and integral surface in $\mathbb{P}^N$ has degree at least $(N-1)$, and thus, we conclude $\deg \widetilde{\phi}_{|D|} \leq 2$.

If $p = 2$ and S is classical or supersingular, then π is a torsor under $\boldsymbol{\mu}_2$ or $\boldsymbol{\alpha}_2$. In particular, π is purely inseparable of degree 2, and the extension $\Bbbk(S) \subset \Bbbk(X)$ of function fields is obtained by adjoining a square root. If we denote by $\Bbbk(S)^{1/2}$ the field that is obtained by adjoining all square roots of $\Bbbk(S)$, then the resulting field extension $\Bbbk(S) \subset \Bbbk(S)^{1/2}$ is purely inseparable. Moreover, we have an inclusion of fields $\Bbbk(S) \subset \Bbbk(X) \subset \Bbbk(S)^{1/2}$. If we denote by $S^{(1/2)}$ the normalization of S inside $\Bbbk(S)^{1/2}$, then $S^{(1/2)}$ is abstractly isomorphic to S, and the field extension $\Bbbk(S) \subset \Bbbk(S)^{1/2}$ induces a purely inseparable and finite morphism $\mathbf{F} : S^{(1/2)} \to S$ of degree 4, the $\Bbbk$-linear Frobenius morphism. Similarly, $\Bbbk(X) \subset \Bbbk(S)^{1/2}$ induces a purely inseparable and finite morphism $\varpi : S^{(1/2)} \to X$ of degree 2 such that $\mathbf{F} = \pi \circ \varpi$. Thus, we obtain the following diagram:

$$
\begin{tikzcd}
S^{(1/2)} \arrow[dr, "\varpi"] \arrow[ddr, "\mathbf{F}"'] & & \\
& X \arrow[r, "\widetilde{\phi}_{|D|}"'] \arrow[d, "\pi"] & \mathbb{P}^{1+2d} \\
& S &
\end{tikzcd}
\tag{3.1.2}
$$

The composition $\varphi \circ \varpi$ corresponds to a linear subsystem of $|2D|$ (here, we identify S with $S^{(1/2)}$). Both, $\widetilde{\phi}_{|D|}$ and ϖ are morphisms, and we have $2 \deg \widetilde{\phi}_{|D|} = \deg(\widetilde{\phi}_{|D|} \circ \varpi)$, as well as $(\mathcal{L}^{\otimes 2})^2 = 4\mathcal{L}^2$. As before, we find $\deg \widetilde{\phi}_{|D|} \leq 2$, this time by arguing on $S^{(1/2)}$.

In order to show $\deg \widetilde{\phi}_{|D|} = 1$ (now again, for arbitrary π and p), we assume $\deg \widetilde{\phi}_{|D|} \neq 1$ and seek a contradiction. Then, $\deg \widetilde{\phi}_{|D|} = 2$ and the image X' is an integral surface of degree $2d$ in $\mathbb{P}^{1+2d}$, that is, a surface of minimal degree. These surfaces have been explicitly classified in Theorem 0.5.2.

Now, the morphism π is a torsor under a finite flat group scheme G, which is of length 2 over $\Bbbk$. Since the quotient of X by G is isomorphic to S and not isomorphic to $\widetilde{\phi}_{|D|}(X)$, it follows that the G-action on X induces a non-trivial G-action on $\mathbb{P}(H^0(X, \pi^*\mathcal{L}))$ and $\widetilde{\phi}_{|D|}(X)$. As already seen above, we can write global sections of $\pi^*\mathcal{L}$ as

$$
0 \to H^0(S, \mathcal{L}) \to H^0(X, \pi^*\mathcal{L}) \xrightarrow{\text{pr}} H^0(S, \mathcal{L} \otimes \omega_S) \to 0 \tag{3.1.3}
$$

and consider it as a sequence of G-modules. It is not difficult to see that $H^0(S, \mathcal{L})$ is the id-eigenspace for the G-action on $H^0(X, \pi^*\mathcal{L})$ and that G acts via the determinant of its regular representation on $H^0(S, \mathcal{L} \otimes \omega_S)$.

We set $\mathbb{P}_+ := \mathbb{P}(H^0(S, \mathcal{L}))$. In the case G where is linearly reductive, that is, if $p \neq 2$ or if $p = 2$ and $G \cong \boldsymbol{\mu}_2$, then the G-action has a second eigenspace, providing us with a splitting of (3.1.3), and which we can identify with $H^0(S, \mathcal{L} \otimes \omega_S)$. We denote by $\mathbb{P}_-$ its projectivization and set $\mathbb{P}_- := \emptyset$ in the case G where is not linearly reductive. Clearly, if a point in $\mathbb{P}(H^0(X, \pi^*\mathcal{L}))$ is fixed under the G-action (in the scheme-theoretic sense) then it lies in $\mathbb{P}_+$ or in $\mathbb{P}_-$.

For $v \in H^0(S, \mathcal{L} \otimes \omega_S)$, the hyperplane $\mathbb{P}_v := \mathbb{P}(\text{pr}^{-1}(v))$ is G-stable and contains $\mathbb{P}_+$. For generic v, the intersection $\Delta := \mathbb{P}_v \cap \varphi(X)$ is an irreducible and non-degenerate curve inside $\mathbb{P}_v \cong \mathbb{P}^{\mathcal{L}^2}$. Since Δ is of degree $(\mathcal{L}^2 - 1)$ in a $\mathcal{L}^2$-dimensional projective space, it is a rational normal curve and in particular, smooth and rational. Since Δ is isomorphic to $\mathbb{P}^1$ and equipped with a non-trivial G-action, its fixedpoint scheme has length 2.

In particular, $\widetilde{\phi}_{|D|}(X)$ contains points that are fixed under G and so, its intersection with $\mathbb{P}_+$ or $\mathbb{P}_-$ is non-empty. On the other hand,

$$
\mathbb{P}_+ \cap \widetilde{\phi}_{|D|}(X) = \bigcap_{s \in \pi^* H^0(S, \mathcal{L})^\vee} \{s = 0\} \cap \widetilde{\phi}_{|D|}(X),
$$

and similarly for $\mathbb{P}_- \cap \widetilde{\phi}_{|D|}(X)$ and $s \in \pi^* H^0(S, \mathcal{L} \otimes \omega_S)^\vee$. This implies that $\mathcal{L}$ or $\mathcal{L} \otimes \omega_S$ is not globally generated, a contradiction. Thus, $\deg \widetilde{\phi}_{|D|} = 1$, which establishes Assertion (2).

We now establish the first assertion and follow [460, Theorem 2.5]: As in the proof of the second assertion, we find $h^0(\widetilde{X}, \pi^* \mathcal{O}_S(D)) = 2 + 2d$. Let us first assume $D^2 \geq 4$. In this case, $|D|$ has no fixed component, but two base points, and $X' = \widetilde{\phi}_{|D|}(X)$ is a surface since already the image of the rational map associated to $|D|$ on S is a surface, see Proposition 3.1.1.

Seeking a contradiction, we assume that $\widetilde{\phi}_{|D|}$ is birational. As in the proof of the second assertion, we find that a generic Cartier divisor $\widetilde{D} \in |\pi^* \mathcal{O}_X(D)|$ is an integral Gorenstein curve. Since $\Phi(D) = 1$, there exists a genus one half-pencil E' on S such that $D \cdot E' = 1$. Then $\mathcal{M} := \pi^* \mathcal{O}_S(E')|_{\widetilde{D}}$ satisfies $\deg \mathcal{M} = 2$ and taking cohomology in

$$0 \to \pi^* \mathcal{O}_S(E' - D) \to \pi^* \mathcal{O}_S(E') \to \mathcal{M} \to 0$$

we find $h^0(\widetilde{D}, \mathcal{M}) \geq 2$. Since $p_a(\widetilde{D}) \geq 5$, Riemann–Roch implies $h^1(\widetilde{D}, \mathcal{M}) \neq 0$. But then, Clifford's inequality $h^0(\widetilde{D}, \mathcal{M}) \leq 2$ is in fact an equality, which implies that $\widetilde{D}$ is hyperelliptic. In the proof of the second assertion, we have seen that $\widetilde{\phi}_{|D|}$ restricted to $\widetilde{D}$ induces $|\omega_{\widetilde{D}}|$, which contradicts the fact that $\widetilde{\phi}_{|D|}$ is birational. Thus, $\deg \widetilde{\phi}_{|D|} \geq 2$ and since X' is a non-degenerate integral surface in $\mathbb{P}^{1+2d}$, we conclude

$$2D^2 \leq \deg \widetilde{\phi}_{|D|} \cdot D^2 \leq \deg \widetilde{\phi}_{|D|} \cdot \deg(X').$$

On the other hand, $\pi^* \mathcal{O}_X(D)$ is globally generated outside a finite set of points and thus, we find

$$\deg \widetilde{\phi}_{|D|} \cdot \deg(X') \leq 2D^2$$

with equality if and only if $\pi^* \mathcal{O}_X(D)$ is globally generated: this is clear if π is étale, because then $\widetilde{X}$ is smooth. If π is inseparable, we consider $\widetilde{\phi}_{|D|} \circ \varpi$ in the diagram (3.1.2) and obtain the same result by arguing on $S^{(1/2)}$.

Putting these inequalities together, we find that $\pi^* \mathcal{O}_S(D)$ is globally generated, $\deg \widetilde{\phi}_{|D|} = 2$ and $\deg(X') = D^2$. In particular, X' is a surface of minimal degree.

It remains to deal with the case $\Phi(D) = 1$ and $D^2 = 2$. Then, $|D|$ may have fixed components, see Proposition 3.1.1 (and Proposition 2.2.11). Assume first that $|D|$ has no fixed part. Since $\Phi(D) = 1$, we may choose a genus one half-pencil E' with $D \cdot E' = 1$. Moreover, $\widetilde{\phi}_{|D|}$ is a possibly rational map to $\mathbb{P}^3$. By contradiction, assume that $X' = \widetilde{\phi}_{|D|}(X)$ is a curve. A generic $G \in |\pi^* \mathcal{O}_S(F')|$, where F' is a half-fiber with $D \cdot F' = 1$, is an integral curve with $p_a = 1$. We find $\deg \pi^* \mathcal{O}_S(D)|_G = 2$, which implies $h^0(G, \pi^* \mathcal{O}_S(D)|_G) = 2$ by Riemann–Roch and Clifford's inequality. This implies that $\widetilde{\phi}_{|D|}(G)$ is a linearly embedded $\mathbb{P}^1 \subset \mathbb{P}^3$. But then, X' is equal to this $\mathbb{P}^1$, contradicting the fact that X' linearly spans the ambient $\mathbb{P}^3$. Thus, X' is a surface and we conclude as before.

It remains to consider the case where $|D|$ has a fixed part. We write $|D| = A + |M|$, where A (resp. M) is the fixed (resp. movable) part. Let F be a half-fiber with

$D \cdot F = A \cdot F + M \cdot F = 1$. If $M \cdot F = 0$, then $|M|$ is a genus one pencil and $D^2 = A^2 + 2A \cdot M = 2$ together with $0 \leq D \cdot A = A^2 + A \cdot M$ imply that $A \cdot M = 2$, i.e., $A \cong \mathbb{P}^1$. Since the K3-cover splits over A and M, we obtain $\tilde{D} \sim A_1 + A_2 + 2\tilde{M}$, where A_1, A_2 are (-2)-curves (equal if π is inseparable), and $|\tilde{M}|$ is a genus one pencil. Since $\dim |2\tilde{M}| = 2$ and $\dim |\tilde{D}| = 3$, the linear system $|\tilde{D}|$ has no fixed components and we conclude as before. On the other hand, if $M \cdot F = 1$, then $A \cdot F = 0$ and hence, A is a component of a member of $|2F|$. Thus, $M \cdot A \leq 2M \cdot F = 2$. A before, we conclude that $|M|$ is a pencil and A is a (-2)-curve with $M \cdot A = 2$. We finish as before by showing that $|\tilde{D}|$ has no fixed component. □

We end this section by a result on the equations defining Enriques surfaces in projective space. Assume that D is a nef divisor with $D^2 = 2d > 0$ on an Enriques surface S such that the complete linear system $|D|$ defines an embedding $\phi_{|D|} : S \hookrightarrow \mathbb{P}^d$. By Proposition 3.1.1, we have $\Phi(D) \geq 2$ and $d \geq 3$. We will see in Section 3.5 below that if $\Phi(D) = 2$ and $d \leq 4$, then the map $\phi_{|D|}$ cannot be an embedding. Since $\Phi(D) \geq 3$ implies $d \geq 5$, this shows that in order for $\phi_{|D|}$ to define an embedding, we must have $d \geq 5$. The question of projective normality and the homogeneous ideal of an Enriques surface in projective space was addressed in [250] and [398]. The following is the main result of these articles.

Theorem 3.1.8 *Let S be an Enriques surface in characteristic zero, let D be a nef divisor on S with $D^2 = 2d > 0$, and assume that the linear system $|D|$ defines an embedding $S \hookrightarrow \mathbb{P}^d$. Then, we have $d \geq 5$.*

1. *If $d \geq 6$ or if $d = 5$ and S does not lie on a quadric, then S is projectively normal in $\mathbb{P}^d$ and its homogeneous ideal is generated by quadrics and cubics.*
2. *We have $\Phi(D) \geq 4$ if and only if the homogeneous ideal of S is generated by quadrics only.*

We will see in Section 8.9 of Volume II that, if $d = 5$ and $S \subset \mathbb{P}^d$ lies on a quadric, then S has a smooth rational curve, that is, S is nodal. Conversely, if S has an ample linear system with $D^2 = 10$ and S contains a smooth rational curve, then $|D + K_S|$ or $|D|$ embeds S into a smooth quadric in $\mathbb{P}^5$. Note that a general Enriques surface is unnodal, that is, it does not contain smooth rational curves.

3.2 Hyperelliptic Maps

In this section, we study linear systems $|D|$ on an Enriques surface S such that D is big and nef and such that $|D|$ has base points. This is Case (2) of Proposition 3.1.1, that is, D is a nef divisor with $D^2 = 2n \geq 4$ and $\Phi(D) = 1$. We remind the reader that we classified such linear systems in Section 2.6 in terms of genus one fibrations and nodal curves. In particular, $|D|$ has two base points (counted with multiplicity) and the rational map $\phi_{|D|}$ is generically of degree $\deg \phi_{|D|} = 2$. Thus, the image $S' := \phi_{|D|}(S)$ of

$$\phi_{|D|} : S \dashrightarrow \mathbb{P}^n$$

is a nondegenerate surface of degree $(n-1)$ in $\mathbb{P}^n$, that is, S' is a surface of minimal degree as classified in Theorem 0.5.2. In this case, we say that $|D|$ or $\phi_{|D|}$ is *hyperelliptic*. In this section, we will classify the images of hyperelliptic maps, as well as their branch and ramification loci. The name is justified by Proposition 3.2.8 below.

By Theorem 0.5.2, the surface S' is either $\mathbb{P}^2$, or a Veronese surface of degree 4 in $\mathbb{P}^5$, or a rational normal scroll $S_{a;n} = S_{a,b;n} \subseteq \mathbb{P}^n$, where $a+b = n-1$. It follows from Section 0.5 that we have the following isomorphisms and special cases:

$S_{0;2}$	$\cong \mathbb{P}^2$,	which arises in this context as the contraction of the unique (-1)-curve on $\mathbf{F}_1$,
$S_{0;3}$	$\cong \mathbb{P}(1,1,2)$	the quadric cone in $\mathbb{P}^3$, which is isomorphic to the contraction of the unique (-2)-curve on $\mathbf{F}_2$,
$S_{k;2k+1}$	$\cong \mathbf{F}_0$,	embedded via $\lvert k\mathfrak{f}+\mathfrak{e}\rvert$ and $k \geq 1$,
$S_{k-1;2k}$	$\cong \mathbf{F}_1$,	embedded via $\lvert k\mathfrak{f}+\mathfrak{e}\rvert$ and $k \geq 2$,
$S_{k-1;2k+1}$	$\cong \mathbf{F}_2$,	embedded via $\lvert (k+1)\mathfrak{f}+\mathfrak{e}\rvert$ and $k \geq 2$.

After these preparations, we determine the images of hyperelliptic maps in terms of their classification via genus one pencils, as established in Proposition 2.6.1. It turns out that only the just mentioned examples occur as images of hyperelliptic maps.

Proposition 3.2.1 *Let $|D|$ be a hyperelliptic linear system on an Enriques surface S with $D^2 = 2n \geq 4$. Let $\phi_{|D|} : S \dashrightarrow \mathbb{P}^n$ be the associated rational map and $S' = \phi_{|D|}(S)$ be its image. Then, one of the following cases occurs:*

1. *If $n = 2k$ is even, then $S' = S_{k-1;n} \subseteq \mathbb{P}^n$.*
2. *If $n = 2k+1$ is odd and*
   a. *$|D| = |nF_1 + F_2|$ or*
   b. *$|D| = |(n+1)F_1 + R + K_S|$ with $K_S \neq 0$,*

   then $S' = S_{k;n} \subseteq \mathbb{P}^n$.
3. *If $n = 2k+1 \geq 3$ is odd and $|D| = |(n+1)F_1 + R|$, then $S' = S_{k-1;n} \subseteq \mathbb{P}^n$.*

Here, we use the classification from Proposition 2.6.1, that is, $|2F_1|$ and $|2F_2|$ are genus one pencils on S with $F_1 \cdot F_2 = 1$ and R is a (-2)-curve with $F_1 \cdot R = 1$.

Proof Let $\bar{S} \to S$ be the blow-up of S at the two base points of $|D|$ and let $|\bar{D}|$ be the proper transform of $|D|$ on $\bar{S}$, which is without base points. Thus, $|\bar{D}|$ gives rise to a morphism $\bar{\phi}_{|D|} : \bar{S} \to \mathbb{P}^n$, which resolves the indeterminacy of $\phi_{|D|}$ and which is generically finite onto its image $S' = \phi_{|D|}(S)$, which is a surface of minimal degree $(n-1)$. We already established this in Proposition 2.6.4 and Corollary 2.6.5.

Next, let $Q_1 + Q_2$ be the exceptional divisor of the blow-up $\bar{S} \to S$, where Q_2 is a (-1)-curve. Since $\bar{D} \cdot Q_2 = 1$, the map $\bar{\phi}_{|D|}$ maps Q_2 to a line in $\mathbb{P}^n$. Since the Veronese surface of degree 4 in $\mathbb{P}^5$ does not contain lines, it cannot be the image of $\bar{\phi}_{|D|}$. Thus, by Theorem 0.5.2 the image S' of $\bar{\phi}_{|D|}$ is a rational normal scroll $S_{a;n} \subseteq \mathbb{P}^n$.

First, suppose we are in Case (1) of Proposition 2.6.1, that is, $|D| = |nF_1 + F_2|$ for two genus pencils $|2F_i|$, $i = 1, 2$ with $F_1 \cdot F_2 = 1$ and $n \geq 2$. Then, the ruling of the scroll $S_{a;n}$ is the image of the pencil $|2F_1|$.

Next, let $\bar{F}_2$ be the proper transform of F_2 in $\bar{S}$. The restriction of the proper transform of $|D|$ to $\bar{F}_2$ is a linear system of degree $2k$ if $n = 2k + 1$ and of degree $2k - 2$ if $n = 2k$. Since $F_2 \cdot 2F_1 = 2$ and the since map $\phi_{|D|}$ defines a degree 2 map of a general member of the pencil $|2F_1|$, we see that the image of F_2 is a curve of degree $(k - 1)$ if $n = 2k$ (resp. of degree k if $n = 2k + 1$). (Here, a curve of degree 0 means a point.) It spans a linear subspace of $\mathbb{P}^n$ of codimension equal to $\dim |D - F_2| = \dim |nF_1|$. Moreover, we know that $|nF_1|$ is composed of k pencils if $n = 2k$, or composed of k pencils and one fixed component F_1 if $n = 2k + 1$. From this, we conclude that we have $\dim |D - F_2| = k$ in both cases. Thus, the image of F_2 spans a subspace of dimension equal to its degree (if F_2 is reducible, then all its components except one belong to $\mathcal{R}_D$, and are hence blown down to points). It is a Veronese curve of degree $(k - 1)$ or k. A rational normal scroll in $\mathbb{P}^n$ containing such a curve is generated by this curve and another Veronese curve of degree $(k - 1)$ (resp. k). Thus, it must be the scroll $S_{k-1;n}$ if $n = 2k$ (resp. $S_{k;n}$ if $n = 2k + 1$).

Second, suppose we are in Case (2) of Proposition 2.6.1, that is, $|D| = |(n+1)F_1 + R|$ for a genus one pencil $|2F_1|$ and a (-2)-curve R with $F_1 \cdot R = 1$. We argue as in the previous case, replacing F_2 by R. If $n = 2k$, then R contains one base point and its proper transform intersects the proper transform of D with multiplicity $(2k - 2)$. The image $\phi_{|D|}(R)$ is a Veronese curve of degree $(k - 1)$, hence $S' = S_{k-1;2k} \subseteq \mathbb{P}^{2k}$. If $n = 2k + 1$, then R contains two base points, and its proper transform intersects the proper transform of D with multiplicity $(2k - 2)$. The image $\phi_{|D|}(R)$ is a again a Veronese curve of degree $(k - 1)$, hence $S' = S_{k-1;2k+1} \subseteq \mathbb{P}^{2k+1}$.

Finally, suppose we are in Case (3) of Proposition 2.6.1, that is, $|D| = |(n+1)F_1 + R + K_S|$ for a genus one pencil $|2F_1|$, a (-2)-curve R with $F_1 \cdot R = 1$, and $K_S \neq 0$. Then, $D \sim (n + 1)F_1 + R + K_S \sim nF_1 + F_1' + R$ and R contains one base point on F_1 if $n = 2k$ (resp. no base points if $n = 2k + 1$). The image of R is a Veronese curve of degree $(k - 1)$ in the former case and degree k in the latter case. This shows that $S' = S_{k-1;n} \subseteq \mathbb{P}^n$ if $n = 2k$ (resp. $S' = S_{k;n} \subseteq \mathbb{P}^n$ if $n = 2k + 1$). □

Next, we will describe the branch divisor of the double cover $\bar{\phi}_{|D|} : \bar{S} \to S'$, where $|D|$ is a hyperelliptic linear system and where $\bar{S} \to S$ is the blow-up in the base points of $|D|$. If a hyperelliptic linear system is of the form $|nF_1 + F_2|$, that is, as in Case (1) of Proposition 2.6.1, then it is called *non-special*, otherwise, the linear system is called *special*. For example, if S is unnodal (which is the generic case), then every hyperelliptic linear system on S is non-special. To describe the branch locus of $\bar{\phi}_{|D|}$, we will use the notation from Proposition 3.2.1. We start with the non-special case.

Theorem 3.2.2 *Let S be an Enriques surface over an algebraically closed field of characteristic $p \neq 2$ and let $|D| = |nF_1 + F_2|$ be a non-special hyperelliptic linear system. Let $\bar{S} \to S$ be the blow-up at the two base points of $|D|$ and let $\bar{\phi}_{|D|} : \bar{S} \to S'$ be the associated generically finite morphism of degree 2. Then, the Stein factorization of $\bar{\phi}_{|D|}$ is equal to*

$$\bar{S} \xrightarrow{f} \bar{S}' \xrightarrow{g} \bar{\phi}_{|D|}(S) = S' \subseteq \mathbb{P}^n,$$

where

1. *f is the birational morphism that blows down the nodal cycle $\mathcal{R}_D$ and the proper transforms of the curves F_1 and F_1'. If $n = 2$, then f also blows down the proper transform of F_2.*
2. *g is a finite morphism of degree 2, whose branch divisor $B \subset S'$ is given by one of the following three cases*
   a. *$S' = S_{0;2} \cong \mathbb{P}^2$ and $|D|$ is from Case (1) of Proposition 3.2.1:*
   B is a curve of degree 8 equal to the union of two lines ℓ_1, ℓ_2 and a curve of degree 6 that has an ordinary double point at $p_1 := \ell_1 \cap \ell_2$, as well as four additional double points $p_2, p_3 \in \ell_1$ and $p_4, p_5 \in \ell_2$, where $p_3 \succ p_2$, $p_5 \succ p_4$ are infinitely near points.
   b. *$n = 2k \geq 4$, $S' = S_{k-1;n} \cong \mathbf{F}_1$ and $|D|$ is from Case (1) of Proposition 3.2.1:*
   $\mathbf{F}_1$ can be identified with the blow-up of $\mathbb{P}^2$ at p_1 and then, B is the proper transform of the branch curve from Case (1).
   c. *$n = 2k + 1 \geq 3$ is odd and $S' = S_{k;n} \cong \mathbf{F}_0$ and $|D|$ is from Case (2) of Proposition 3.2.1:*
   $\mathbf{F}_0$ is isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$ and B is the union of two fibers ℓ_1, ℓ_2 of one of the projection maps $\mathbf{F}_0 \to \mathbb{P}^1$ and a curve of bidegree $(4, 4)$ that has double points $p_1, p_1' \in \ell_1$ and $p_2, p_2' \in \ell_2$, where $p_1' \succ p_1$, $p_2' \succ p_2$ are infinitely near points.

The remaining singularities (if any) of B are simple curve singularities.

Proof We only sketch the main points of the proof and leave the details to the reader. The assertion about f is clear since the proper transform of $|D|$ does not intersect the proper transforms of F_1, F_1' and $\mathcal{R}_D$. (As usual, F_1' denotes the half-fiber that is the unique effective divisor in $|F_1 + K_S|$, that is, F_1 and F_1' are the half-fibers of the genus one pencil $|2F_1|$.) Moreover, if $n = 2$, then the proper transform of $|D|$ also does not intersect the proper transform of F_2.

Since S is a minimal surface of non-negative Kodaira dimension, the rational deck transformation of $\bar{\phi}_{|D|}$ extends to an automorphism σ of order 2 of S. The locus of fixed points of σ consists of a curve W (nonsingular because $p \neq 2$) and some isolated fixed points.

Since $D \cdot F_1 = 1$, the restriction of $\bar{\phi}_{|D|}$ to a general member of $|2F_1|$ is a morphism of degree 2 onto $\mathbb{P}^1$, see also the proof of Proposition 2.6.4. Thus, the restriction of σ to a general member F of $|2F_1|$ is an involution of an elliptic curve with quotient isomorphic to $\mathbb{P}^1$. Since this map has 4 fixed points, we conclude $W \cdot F = 4$. Next, let C be a general member of the pencil $|F_1 + F_2|$. It is spanned by the curves $F_1 + F_2$ and $F_1' + F_2'$.

If $n = 2k + 1 \geq 3$, then $S' = S_{k;n}$ is isomorphic to $\mathbf{F}_0 \cong \mathbb{P}^1 \times \mathbb{P}^1$ embedded into $\mathbb{P}^n$ via the complete linear system $|k\mathfrak{f} + \mathfrak{e}|$. In this case, the set of base points of $|C|$ is equal to the set of base points of $|D|$. The proper transform $\bar{C}$ of C on $\bar{S}$ has self-intersection zero and its image C' on S' is a curve of the same self-intersection.

Therefore, C' must belong to the ruling $|\mathfrak{e}|$. The other ruling $|\mathfrak{f}|$ is the image of the pencil $|2F_1|$. Since C is of genus two, we find that the branch curve B intersects C' at 4 points. The other two branch points come from the intersections with the images λ_1, λ_2 of Q_1, Q_2, where the Q_i are the exceptional divisor of $\bar{S} \to S$ as in the proof of Proposition 3.2.1. This implies that B is a curve of bidegree $(4, 4)$. It contains two singular points $p_1 \in \ell_1$, $p_2 \in \ell_2$, which are the images of F_1 and F_1'. Since the base points of $|D|$ are isolated fixed points of the deck transformation, the curve B intersects ℓ_1 and ℓ_2 with multiplicity 4 at its singular points p_1, p_2. The member $F_1 + F_2$ (resp. $F_1' + F_2'$) of $|C|$ has the image equal to a section s (resp. s') from $|\mathfrak{e}|$ passing through the singular point p_1 (resp. p_2) of B. Since $W \cdot F_2 = W \cdot F_2' = 2$, we conclude that B intersects s (resp. s') with multiplicity 2 at p_1 (resp. at p_2). This shows that p_1 and p_2 are double points of B and it shows that also B has two infinitely near double points $p_1' \succ p_1$ and $p_2' \succ p_2$.

If $n = 2k \geq 4$, then $S' = S_{k-1;n}$ is isomorphic to $\mathbf{F}_1$ embedded into $\mathbb{P}^n$ via the complete linear system $|k\mathfrak{f} + \mathfrak{e}|$. In this case, only one base point of $|D|$ is a base point of $|F_1 + F_2|$ and it is equal to $F_1' \cap F_2$. The proper transform $|C|$ of $|F_1 + F_2|$ on $\bar{S}$ has one base point, which is equal to the pre-image of the point $F_1 \cap F_2'$. The image of $|C|$ is a pencil, which is contained in the linear system $|\mathfrak{f} + \mathfrak{e}|$ on $S' \cong \mathbf{F}_1$. The branch curve consists of the union of two members ℓ_1, ℓ_2 from $|\mathfrak{f}|$ and a curve B from $|6\mathfrak{f} + 4\mathfrak{e}|$. It has two double points $p_2 \in \ell_1$, $p_4 \in \ell_2$ and two infinitely near points $p_3 \succ p_2$, $p_5 \succ p_4$ with tangent directions ℓ_1, ℓ_2.

If $n = 2$, then the branch curve on $S' \cong \mathbb{P}^2$ is the image of the branch curve from the previous curve under the blow-up morphism $\mathbf{F}_1 \to \mathbb{P}^2$. $\square$

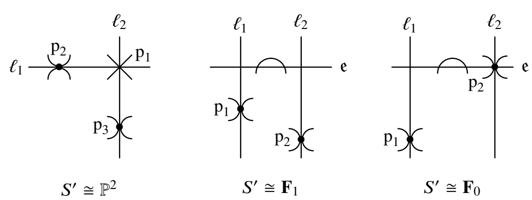


Fig. 3.1 Branch curve of a non-special hyperelliptic map

Remark 3.2.3 We note that the branch curve B may have additional infinitely near points at the points p_1, p_2, p_3. This happens if one of the half-fibers of $|2F_1|$ or $|2F_2|$ is reducible.

The next theorem describes the branch locus of $\bar{\phi}_{|D|}$ in case $|D|$ is a special hyperelliptic linear system. Again, we will use the notation from Proposition 2.6.1 and Proposition 3.2.1.

Theorem 3.2.4 *Let S be an Enriques surface over an algebraically closed field of characteristic $p \neq 2$ and let $|D|$ be a special hyperelliptic linear system with $D^2 = 2n \geq 4$. Let $\bar{S} \to S$ be the blow-up in the two base points of $|D|$ and let*

$\bar{\phi}_{|D|} : \bar{S} \to S'$ be the associated generically finite morphism of degree 2. Then, the Stein factorization of $\bar{\phi}_{|D|}$ is equal to

$$\bar{S} \xrightarrow{f} \bar{S}' \xrightarrow{g} \bar{\phi}_{|D|}(S) = S' \subseteq \mathbb{P}^n,$$

where:

1. *f is the birational morphism that blows down the nodal cycle $\mathcal{R}_D$ and the proper transforms of the curves F_1 and F_1'. If $S' \cong \mathbb{P}^2$ or $S' \cong \mathbb{P}(1,1,2)$, then f also blows down the proper transform of R.*
2. *g is a finite morphism of degree 2, whose branch divisor $B \subset S'$ is given by one of the following five cases:*

   a. *$S' = S_{0;2} \cong \mathbb{P}^2$ and $|D|$ is from Case (1) of Proposition 3.2.1:*
   B is a curve of degree 8 equal to the union of two lines ℓ_1, ℓ_2 and a curve B' of degree 6. The curve B' has double points in $p_1 = \ell_1 \cap \ell_2$ and another point $p_4 \in \ell_2$. It also has two infinitely near points $p_3 \succ p_2 \succ p_1$, such that the line ℓ_1 passes through them and an infinitely near point $p_5 \succ p_4$, such that the line ℓ_2 passes through p_1, p_4, and p_5.
   b. *$n = 2k \geq 4$, $S' = S_{k-1;n} \cong \mathbf{F}_1$ and $|D|$ is from Case (1) of Proposition 3.2.1:*
   B is the union of two members ℓ_1 and ℓ_2 of $|\mathfrak{e} + \mathfrak{f}|$ and a curve $B' \in |4\mathfrak{e} + 6\mathfrak{f}|$. The curve B' has double points $p_1 \in \ell_1$ and $p_2 \in \ell_2$ and one of them lies on the exceptional section $\mathfrak{e}$. It also has two infinitely near points $p_1' \succ p_1$ and $p_2' \succ p_2$ with tangent directions ℓ_1 and ℓ_2. The image of R is equal to $\mathfrak{e}$.
   c. *$n = 2k + 1 \geq 3$, $S' = S_{k;n} \cong \mathbf{F}_0$ and $|D|$ is from Case (2) of Proposition 3.2.1*
   $\mathbf{F}_0$ is isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$ and B consists of the union of two members ℓ_1 and ℓ_2 of $|\mathfrak{f}|$ and a curve B' of bidegree $(4,4)$. The curve B' has two double points $p_1 \in \ell_1$ and $p_2 \in \ell_2$. It also has two infinitely near points $p_1' \succ p_1$ and $p_2' \succ p_2$ with tangent directions ℓ_1 and ℓ_2. The image of R is a section from $|\mathfrak{e}|$, which passes through p_1 and p_2.
   d. *$n = 2k + 1 \geq 5$, $S' = S_{k-1;n} \cong \mathbf{F}_2$ and $|D|$ is from Case (3) of Proposition 3.2.1:*
   B is the union of two members ℓ_1 and ℓ_2 of $|\mathfrak{f}|$ and a curve $B' \in |8\mathfrak{f} + 4\mathfrak{e}|$. The curve B' has two double points $p_1 \in \ell_1$ and $p_2 \in \ell_2$, neither of which lies on the exceptional section $\mathfrak{e}$. It also has two infinitely near points $p_1' \succ p_1$ and $p_2' \succ p_2$ with tangent directions ℓ_1 and ℓ_2. The image of R is equal to $\mathfrak{e}$.
   e. *$n = 3$, $S' = S_{0;3} \cong \mathbb{P}(1,1,2)$ and $|D|$ is from Case (3) of Proposition 3.2.1:*
   B is the image the branch curve in the previous case under the contraction $\mathbf{F}_2 \to \mathbb{P}(1,1,2)$ of $\mathfrak{e}$.

   The remaining singularities (if any) of B are simple curve singularities.

Proof We only indicate and sketch the main points of the proof and leave the details to the reader. The assertion about f is clear since the proper transform of $|D|$ does not intersect the proper transforms of F_1, F_1' and $\mathcal{R}_D$. (As usual, F_1' denotes the half-fiber that is the unique effective divisor in $|F_1 + K_S|$, that is, F_1 and F_1' are the half-fibers of the genus one pencil $|2F_1|$.) Moreover, if $n = 2$, then the proper transform of $|D|$ also does not intersect the proper transform of F_2.

Since S is a minimal surface of non-negative Kodaira dimension, the rational deck transformation of $\bar{\phi}_{|D|}$ extends to an automorphism σ of order 2 of S. The locus of fixed points of σ consists of a curve W (nonsingular because $p \neq 2$) and some isolated fixed points.

Since $D \cdot F_1 = 1$, the restriction of $\bar{\phi}_{|D|}$ to a general member of $|2F_1|$ is a morphism of degree 2 onto $\mathbb{P}^1$, see also the proof of Proposition 2.6.4. Thus, the restriction of σ to a general member F of $|2F_1|$ is an involution of an elliptic curve with quotient isomorphic to $\mathbb{P}^1$. Since this map has 4 fixed points, we conclude $W \cdot F = 4$. Next, let C be a general member of the pencil $|F_1 + F_2|$. It is spanned by the curves $F_1 + F_2$ and $F_1' + F_2'$.

If $n = 2k + 1 \geq 3$, then $S' = S_{k;n}$ is isomorphic to $\mathbf{F}_0 \cong \mathbb{P}^1 \times \mathbb{P}^1$ embedded into $\mathbb{P}^n$ via the complete linear system $|k\mathfrak{f} + \mathfrak{e}|$. In this case, the set of base points of $|C|$ is equal to the set of base points of $|D|$. The proper transform $\bar{C}$ of C on $\bar{S}$ has self-intersection zero and its image C' on S' is a curve of the same self-intersection. Therefore, C' must belong to the ruling $|\mathfrak{e}|$. The other ruling $|\mathfrak{f}|$ is the image of the pencil $|2F_1|$. Since C is of genus 2, we find that the branch curve B intersects C' at 4 points. The other two branch points come from the intersections with the images λ_1, λ_2 of Q_1, Q_2, where the Q_i are the exceptional divisor of $\bar{S} \to S$ as in the proof of Proposition 3.2.1. This implies that B is a curve of bidegree $(4, 4)$. It contains two singular points $p_1 \in \ell_1$, $p_2 \in \ell_2$, which are the images of F_1 and F_1'. Since the base points of $|D|$ are isolated fixed points of the deck transformation, the curve B intersects ℓ_1 and ℓ_2 with multiplicity 4 at its singular points p_1, p_2. The member $F_1 + F_2$ (resp. $F_1' + F_2'$) of $|C|$ has the image equal to a section s (resp. s') from $|\mathfrak{e}|$ passing through the singular point p_1 (resp. p_2) of B. Since $W \cdot F_2 = W \cdot F_2' = 2$, we conclude that B intersects s (resp. s') with multiplicity 2 at p_1 (resp. at p_2). This shows that p_1 and p_2 are double points of B and it shows that also B has two infinitely near double points $p_1' \succ p_1$ and $p_2' \succ p_2$.

If $n = 2k \geq 4$, then $S' = S_{k-1;n}$ is isomorphic to $\mathbf{F}_1$ embedded into $\mathbb{P}^n$ via the complete linear system $|k\mathfrak{f} + \mathfrak{e}|$. In this case, only one base point of $|D|$ is a base point of $|F_1 + F_2|$ and it is equal to $F_1' \cap F_2$. The proper transform $|C|$ of $|F_1 + F_2|$ on $\bar{S}$ has one base point, which is equal to the pre-image of the point $F_1 \cap F_2'$. The image of $|C|$ is a pencil, which is contained in the linear system $|\mathfrak{f} + \mathfrak{e}|$ on $S' \cong \mathbf{F}_1$. The branch curve consists of the union of two members ℓ_1, ℓ_2 from $|\mathfrak{f}|$ and a curve B from $|6\mathfrak{f} + 4\mathfrak{e}|$. It has two double points $p_2 \in \ell_1$, $p_4 \in \ell_2$ and two infinitely near points $p_3 \succ p_2$, $p_5 \succ p_4$ with tangent directions ℓ_1, ℓ_2.

If $n = 2$, then the branch curve on $S' \cong \mathbb{P}^2$ is the image of the branch curve from the previous curve under the blow-up morphism $\mathbf{F}_1 \to \mathbb{P}^2$. □

We will deal with branch divisors of hyperelliptic maps in characteristic $p = 2$ in the next section.

Remark 3.2.5 These results admit converses, that is, the double cover of S' branched over a curve $B = B' + \ell_1 + \ell_2$ as described above is birationally equivalent to an

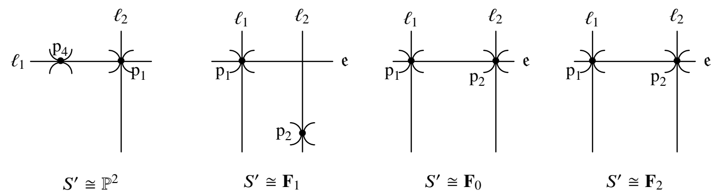


Fig. 3.2 Branch curve of a special hyperelliptic map

Enriques surface. This is an exercise, which can be done using the formula for the canonical class of a nonsingular model of a double cover as in Section 0.3.

Let us sketch this claim for Case (2a) of Theorem 3.2.2, that is, where the Enriques surface S is birational to a double cover of $S' = \mathbb{P}^2$ associated to a non-special hyperelliptic linear system $|D|$ with $D^2 = 4$. We leave the remaining cases to the reader. For this construction, we will use the notion of a geometric basis of the blow-up of points in $\mathbb{P}^2$ as introduced in (0.5.5) in Section 0.3.

Example 3.2.6 Choose five points $p_1, ..., p_5$ in $\mathbb{P}^2$ such that p_1, p_2, p_4 are not collinear and such that $p_3 \succ p_2$ and $p_5 \succ p_4$. Next, choose a reduced plane curve of degree 6 in the linear system

$$B' \in |\mathcal{O}_{\mathbb{P}^2}(6) - 2(p_1 + \cdots + p_5)|,$$

and let

$$\ell_1 \in |\mathcal{O}_{\mathbb{P}^2}(1) - p_1 - p_2 - p_3| \quad \text{and} \quad \ell_2 \in |\mathcal{O}_{\mathbb{P}^2}(1) - p_1 - p_4 - p_5|,$$

that is, ℓ_1 (resp. ℓ_2) is the unique line passing through p_1, p_2, p_3 (resp. p_1, p_4, p_5). Let B be the union of B' and the two lines ℓ_1 and ℓ_2.

Claim: The minimal resolution $\bar{S}$ of the double cover of $\mathbb{P}^2$ branched along B is isomorphic to the blow-up of the two base points of a hyperelliptic linear system $|2F_1 + F_2|$ on an Enriques surface S.

Proof (of the claim) Let $\pi : Y \to \mathbb{P}^2$ be the blow-up in five points $p_1, ..., p_5$. Let e_0 be the divisor class of $\pi^*(\mathcal{O}_{\mathbb{P}^2}(1))$ and let $e_i = [\mathcal{E}_i]$ be the divisor classes of the exceptional curve of π over p_i. Then, we have

$$\pi^{-1}(B) \sim 8e_0 - 4e_1 - 3e_2 - 3e_3 - 3e_4 - 3e_5.$$

Next, $e_2 - e_3$ (resp. $e_4 - e_5$) is the class of the unique component R_2 (resp. R_3) of $\mathcal{E}_2$ (resp. $\mathcal{E}_3$) with self-intersection -2. On Y, we consider the curve

$$\overline{B}' := \overline{B} + R_2 + R_3 \sim 8e_0 - 4e_1 - 2e_2 - 4e_3 - 2e_4 - 4e_5,$$

where $\bar{B}$ denotes the proper transform of B on Y. Then, the canonical divisor class of Y is linearly equivalent to

$$K_Y \sim -3e_0 + e_1 + \cdots + e_5.$$

Let $\tau : X \to Y$ be the double cover of Y branched over $\overline{B}$. By formula (0.2.10), we have

$$K_X \sim \tau^*(e_0 - e_1 - e_3 - e_5).$$

A member of the linear system $|e_0 - e_1 - e_3 - e_5|$ is a line passing through p_1, p_3, p_5. This implies that the line also passes through p_1, p_2, p_3 and by assumption on the position of these points, we conclude $|e_0 - e_1 - e_3 - e_5| = \emptyset$. Using the projection formula, we compute

$$h^0(X, \mathcal{O}_X(K_X)) = h^0(Y, \tau_*(\mathcal{O}_X(K_X))) = h^0(\mathcal{O}_Y(K_Y)) + h^0\left(\mathcal{O}_Y\left(K_Y - \frac{1}{2}\bar{B}\right)\right) = 0.$$

On the other hand, we have

$$\begin{aligned} 2K_X &\sim \tau^*(2e_0 - 2e_1 - 2e_3 - 2e_5) \\ &= \tau^*((e_0 - e_1 - e_2 - e_4) + (e_0 - e_1 - e_3 - e_5) + (e_2 - e_3) + (e_4 - e_5)). \end{aligned}$$

Using the projection formula again, we find that the only effective divisor in $|2K_X|$ is the curve $\tau^*(\bar{\ell}_1 + \bar{\ell}_2 + R_2 + R_3)$. Since the curves $\bar{\ell}_i$ and R_i are components of the branch locus of τ, we conclude that

$$2K_X \sim 2(Q_1 + Q_2 + Q_2' + Q_3'),$$

where $\tau^*(\bar{\ell}_i) = 2Q_i$ and $\tau^*(R_i) = 2Q_i'$. It is not difficult to see that the Q_i and Q_i' are (-1)-curves. After blowing down these four curves, we obtain a surface S with $2K_S = 0$ and $K_S \neq 0$ and thus, S is an Enriques surface. The composition $\pi \circ \tau : X \to \mathbb{P}^2$ corresponds to the complete linear system associated to the invertible sheaf

$$\tau^*\pi^*\mathcal{O}_{\mathbb{P}^2}(1) \sim \tau^*((e_0 - e_1 - e_2 - e_3) + e_1 + (e_2 - e_3) + 2e_3) = 2Q_1 + E_1 + 2Q_2' + 2E_3,$$

where $E_1 = \tau^*(e_1)$ and $E_3 = \tau^*(e_3)$. Let F_1 (resp. F_2) be the image of E_3 (resp. E_1) on S. Then, the images of the (-1)-curves Q_1 and Q_2' on S are the two base points of the hyperelliptic linear system $|2F_1 + F_2|$. The surface X' obtained from X by blowing down the curves Q_2 and Q_1' is the blow-up of S at the base points. The rational map $S \dashrightarrow \mathbb{P}^2$ defined by linear system $|2F_1 + F_2|$ induces the morphism $\pi \circ \tau : X' \to \mathbb{P}^2$. □

We end this section by describing the general member of a hyperelliptic linear system on an Enriques surface. Extending the usual definition, we will say that an integral curve C that is proper over an algebraically closed field is *hyperelliptic* if there exists an invertible sheaf $\mathcal{L}$ of degree 2 on C such that $h^0(C, \mathcal{L}) = 2$. It follows that the linear system $|\mathcal{L}|$ has no base points and that the curve C is Gorenstein, see [308]. We mention that if C is smooth and of genus ≥ 2, then C is hyperelliptic in the classical sense and the linear system $|\mathcal{L}|$ is the unique g_2^1 on C, see, for example, [294, Proposition IV.5.3]. A smooth rational or an elliptic curve is hyperelliptic with respect to our generalization.

The linear system $|\mathcal{L}|$ defines a finite morphism $\pi : C \to \mathbb{P}^1$ of degree 2. It is a separable map if C is smooth. The formula for the canonical sheaf of a double cover shows that $\omega_C \cong \pi^* \mathcal{O}_{\mathbb{P}^1}(n-2) = \mathcal{L}^{\otimes(n-2)}$, where $\pi_* \mathcal{O}_C = \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-n)$. Applying the projection formula, we obtain $p_a(C) = h^0(\omega_C) = h^0(\mathcal{O}_{\mathbb{P}^1}(n-2)) = n-1$. This gives the familiar formula

$$\omega_C \cong \mathcal{L}^{\otimes(p_a(C)-1)}. \tag{3.2.1}$$

Assume that $p_a(C) \geq 2$. Since $\dim S^{p_a(C)-1}(H^0(C,\mathcal{L})) = p_a(C) = \dim H^0(C,\omega_C)$, we conclude that the canonical map $f : C \to \mathbb{P}^{p_a(C)-1}$ is a degree 2 map onto $\mathbb{P}^1$, which is embedded into $\mathbb{P}^{p_a(C)-1}$ as a Veronese curve. Since the canonical map is independent of a choice of $\mathcal{L}$ defining a hyperelliptic curve, it follows that $\mathcal{L} = f^*(\mathcal{O}_{\mathbb{P}^1}(1))$ is uniquely defined. We keep the classical notation g_2^1 for the linear system $|\mathcal{L}|$. We refer the interested reader to [112], [114], [308], and [692] for some extensions of some standard results about basepoints of linear systems and pluricanonical maps from smooth to integral Gorenstein curves with a view towards embeddings of curves and surfaces.

Remark 3.2.7 The hyperelliptic linear system $|D|$ with $D^2 = 4$ was originally used by Enriques in order to define a *double plane* model of an Enriques surface S, that is, a degree 2 cover of $\mathbb{P}^2$ branched over the octic curve B that is birationally equivalent to S. He also distinguished special and non-special models corresponding to special or non-special hyperelliptic linear systems.

If $|D|$ is a hyperelliptic linear system on an Enriques surface S, then $\Phi(D) = 1$, see Case (2) of Proposition 3.1.1 or the beginning of Section 3.2.

Proposition 3.2.8 *Let S be an Enriques surface over an algebraically closed field of characteristic $p \geq 0$.*

1. *If D is a big and nef divisor on S with $\Phi(D) = 1$, then a general member of $|D|$ is a hyperelliptic curve. If moreover*
   a. *$D^2 = 2$ and $p \notin \{2,3,5\}$ or*
   b. *$D^2 \geq 4$ and $p \neq 2$,*

   then a general member of $|D|$ is a smooth curve.
2. *Conversely, let $C \subset S$ be a smooth and hyperelliptic curve with $C^2 \geq 4$. If S is classical or $C^2 \geq 6$, then $|C|$ is a hyperelliptic linear system.*

Proof First, let D be a big and nef divisor with $D^2 = 2n > 2$ and $\Phi(D) = 1$. Let $\bar{S} \to S$ be the blow-up in the base points of $|D|$ and $\bar{\phi}_{|D|} : \bar{S} \to \mathbb{P}^n$ be the map that resolves the indeterminacy of the rational map $\phi_{|D|}$ associated to $|D|$. Being of codimension one in a smooth variety, every divisor $C \in |D|$ is a Gorenstein curve. If $n \geq 2$, then the pre-image of a general hyperplane section of $\mathbb{P}^n$ is an integral curve in $|D|$. Similarly, if $n = 1$, then again a general member of $|D|$ is integral by

Corollary 2.6.5. Since $\Phi(D) = 1$, there exists a genus one pencil $|2F|$ with $D \cdot F = 1$. Thus, a general member $C \in |D|$ is an integral Gorenstein curve. The restriction of $\mathcal{O}_S(2F)$ to C is an invertible sheaf of degree 2. Moreover, taking cohomology in the short exact sequence

$$0 \to \mathcal{O}_S(2F - C) \to \mathcal{O}_S(2F) \to \mathcal{O}_S(2F)|_C \to 0$$

we find $h^0(C, \mathcal{O}_S(2F)|_C) = 2$ and thus, C is a hyperelliptic curve (in the generalized sense above).

Finally, if $D^2 = 2$, then $\bar{\phi}_{|D|}$ is a fibration. Moreover, if $p \notin \{2, 3, 5\}$, then a general fiber $\bar{C}$ of $\bar{\phi}_{|D|}$ is smooth by Corollary 2.6.5. If $D^2 = 2n \geq 4$, then $\bar{\phi}_{|D|}$ is a morphism that is generically finite of degree 2 onto its image $S' \subseteq \mathbb{P}^n$. If $p \neq 2$, then this map is separable and thus, the inverse image $\bar{C}$ of a general hyperplane section under $\bar{\phi}_{|D|}$ is smooth by Bertini's theorem, see [630, Lemma 5.8.2]. Since the linear system $|D|$ has two simple base points by Proposition 2.6.4, it follows in both cases that the two (-1)-curves of $\bar{S}$ intersect the smooth curve $\bar{C}$ transversally. But then, the image of $\bar{C}$ on S is a smooth curve, which shows that a general member of $|D|$ is smooth.

To prove the converse, we follow the argument from [714, Lemma 1.2]: let $C \subset S$ be a smooth curve with $C^2 \geq 4$ that is hyperelliptic.

First, assume that S is classical, that is, $K_S \neq 0$. The adjunction formula and the short exact sequence

$$0 \to \mathcal{O}_S(K_S) \to \mathcal{O}_S(C + K_S) \to \mathcal{O}_S(C + K_S)|_C \to 0 \tag{3.2.2}$$

show that the restriction of the linear system $|C + K_S|$ to C is the canonical linear system $|\omega_C|$. Since C is hyperelliptic, it follows from (3.2.1) that it defines a degree 2 map onto $\mathbb{P}^1$ embedded as a Veronese curve in $\mathbb{P}^{p_a(C)-1}$. This shows that the (possibly rational) map ϕ_{C+K_S} associated $|C + K_S|$ cannot be birational onto its image. From Theorem 2.4.16, we see $\Phi(C) = \Phi(C+K_S) \leq 2$. Seeking a contradiction, we assume that the linear system $|C|$ is not hyperelliptic, that is, $\Phi(C) = \Phi(C + K_S) = 2$. By Theorem 2.4.14, the linear system $|C + K_S|$ has no base points.

Thus, for every point $x \in S$, the linear system $|C - x|$ of divisors of $|C|$ passing through x is of dimension $\dim |C| - 1$. Let $|\mathcal{L}|$ be the unique g^1_2 on C.

Assume that $|C|$ is not hyperelliptic, that is, $\Phi(C) = 2$ and $|C|$ has no base points. Let x be a general point on S and $|C-x|$ be the linear system of dimension $\dim |C| - 1$ of divisors in $|C|$ containing the point x. Since $|\omega_C|$ is composed of the pencil g^1_2, for any $D \in |C - x|$, there exists a unique point $x_D \in D$ such that $x + x_D \in g^1_2$. This implies that any divisor from $|C+K_S|$ passing through x contains x_D, too. Thus, x_D is a base point of the linear system $|C+K_S-x|$. Since $\Phi(C+K_S) = \Phi(C) = 2$, the linear system $|C + K_S|$ has no base points. Hence, the codimension one linear subsystem $|C+K_S-x|$ has only finitely many base points. This implies that x_D does not depend on D and it must be a base point of $|C - x|$. Thus, $h^0(\mathcal{O}_C(C)) = h^0(\mathcal{O}_S(C)) - 1 = n$ and $h^0(\mathcal{O}_C(C - x - x_D)) = h^0(\mathcal{O}_C(C - x)) = n - 1$. This gives

$$h^0(\mathcal{O}_C(C-x-x_D)) = h^1(\mathcal{O}_C(K_C-C+x+x_D)) = h^1(\mathcal{O}_C(K_S+x+x_D)) = n-1.$$

By Riemann–Roch on C, we find $h^0(\mathcal{O}_C(K_S + x + x_D)) = 2 + n - 1 + 1 - p_a(C) = 2 + n - (n + 1) = 1$. Let $a + b \in |\mathcal{O}_C(K_S + x + x_D)|$. Since

$$h^0(\mathcal{O}_C(C-a-b)) = h^0(\mathcal{O}_C(C+K_S-x-x_D)) = h^0(\mathcal{O}_C(C'-x))-1 = n = h^0(\mathcal{O}_C(C)),$$

we obtain that a and b are base points of $|\mathcal{O}_C(C)|$ and hence, of $|\mathcal{O}_S(C)|$. This contradiction shows that the linear system $|C|$ is hyperelliptic.

Finally, assume that $C^2 \geq 6$ and that S is not necessarily classical. Using the exact sequence (3.2.2) and the fact that $h^1(\mathcal{O}_S) = 1$, we conclude that the restriction of $|C|$ to C is a codimension 1 linear subsystem of the canonical linear system $|\omega_C|$. If it has a base point, then $|C|$ is hyperelliptic. Since the canonical linear system maps C two-to-one onto a Veronese curve R of degree $n := p_a(C) - 1$ in $\mathbb{P}^n$, we obtain that the restriction of $\phi_C := \phi_{|C|}$ to C is equal to the composition of this map with the projection from a point outside of R. Since $C^2 \geq 6$, we find $p_a(C) \geq 4$ and thus, $n \geq 3$. Since $n > 2$, the projection is a degree 1 map onto a rational curve of degree n in $\mathbb{P}^{n-1}$. This implies that ϕ_C is a degree 2 map onto a surface of degree n, whose hyperplane sections are rational curves of degree n. By Proposition 0.5.5, the image S' of ϕ_C is a projection of a rational normal scroll. A line from the ruling of the scroll intersects a general hyperplane section at one point. This implies that the pre-image of this line is a divisor P such that $P \cdot C = 2$. By the Hodge Index Theorem, $C^2 \cdot P^2 - (P \cdot C)^2 = 2nP^2 - 4 < 0$. This implies $P^2 = 0$ and we find $\Phi(C) = 1$. In particular, $|C|$ is hyperelliptic. □

3.3 Bielliptic Maps

In this section, we study linear systems $|D|$ on an Enriques surface S that give rise to double covers, that is, Cases (3b) and (3d) of Proposition 3.1.1. Thus, we study nef divisors D with $D^2 = 2d \geq 4$, $\Phi(D) = 2$, the linear system $|D|$ has no base points, and the map $\phi_{|D|}$ associated to $|D|$ is generically finite of degree 2 onto its image $\phi_{|D|}(S)$. Thus, the image S' of

$$\phi_{|D|} : S \to \mathbb{P}^d$$

is a non-degenerate surface of degree d in $\mathbb{P}^d$, that is, one of the surfaces classified in Theorem 0.5.5. In this case, we say that the linear system $|D|$ is *bielliptic*. We note that such linear systems were called *superelliptic* in [138]. We remind the reader that we classified such linear systems in Section 2.6 in terms of genus one fibrations and nodal curves. Proposition 3.3.1 explains the terminology.

In the course of our analysis, we will see that $d \in \{3, 4\}$ and that $\mathsf{D} := \phi_{|D|}(S) \subset \mathbb{P}^d$ is an anti-canonical del Pezzo surface of degree d. We will analyze the branch locus of $\phi_{|D|}$ and we will see that D is a symmetroid surface in the sense of Definition 0.6.4. If $d = 4$, then D is a symmetroid quartic surface as classified in Section 0.6. Composing this map with some birational map to a minimal ruled surface, we obtain various birational models of S that exhibit it as a double cover of a rational surface.

Among them, there is the *Horikawa model*, which is frequently used in the literature, see Remark 3.3.21.

We start with the classification of the possible images of a bielliptic morphism. Let us recall that we defined an anti-canonical del Pezzo surface to be the anti-canonical model of a weak del Pezzo surface.

Proposition 3.3.1 *Let $|D|$ be a bielliptic linear system on an Enriques surface S. Then, $D^2 = 6$ or $D^2 = 8$, and the image $\phi_{|D|}(S)$ is an anti-canonical Pezzo surface of degree $\frac{1}{2}D^2$. The restriction of $\phi_{|D|}$ to a general member of $|D|$ is a double cover of an elliptic curve.*

1. *If $D^2 = 8$, then $|D|$ is one of the following linear systems from Proposition 2.6.6:*
   - a. *$|2F_1 + 2F_2|$ and $F_1 \cdot F_2 = 1$ (Case (1)),*
   - b. *$|4F_1 + 2R|$ and $F_1 \cdot R = 1$ (Case (2)).*
2. *If $D^2 = 6$, then $|D|$ is one of the following linear systems from Proposition 2.6.7:*
   - a. *$|F_1 + F_2 + F_3|$ and $|F_1 + F_2 - F_3| \neq \emptyset$ (Case (1)),*
   - b. *$|F_1 + 2F_2 + R_1|$ and $|F_1 - R_1| \neq \emptyset$ (Case (2)),*
   - c. *$|3F_1 + 2R_1 + R_2|$ and $|F_1 - R_2| \neq \emptyset$ (Case (3)).*

Proof We know that $\phi_{|D|}$ has no base points and that $Y := \phi_{|D|}(S)$ is a surface of degree d in $\mathbb{P}^d$, see Theorem 0.5.5 for a classification of these surfaces. The argument from the last paragraph of the proof of Proposition 3.2.8 shows that Y is not the projection of a normal rational scroll. It also cannot be a cone over a normal elliptic curve, since there are no maps from S to a curve of positive genus. Thus, Y is an anti-canonical del Pezzo surface of degree d. In particular, we have $d \leq 9$.

The case $d = 9$ cannot occur: in that case, Y would be a Veronese surface of degree 9, hence $D = 2A$, where $|A|$ defines a degree 2 map onto $\mathbb{P}^2$. But then, we compute $A^2 = 2$ and $D^2 = 8$, a contradiction.

Since $\Phi(D) = 2$, there exists a genus one pencil $|P|$ on S with $P \cdot D = 4$. Its image on Y is a pencil of conics. Since a conic is contained in a plane, we have $h^0(D - P) = h^0(\mathcal{O}_S(D)) - 3 = d - 2$. Thus, $|D - P|$ is a linear system of dimension $d - 3$ without fixed components. Since $D^2 = 2d \leq 16$, we obtain $(D - P)^2 = 2d - 8 \leq 8$. If $D - P$ is big and nef or $(D - P)^2 = 0$ and $h^1(D - P) = 0$, then $\dim |D - P| = d - 4 < d - 3$ and we obtain a contradiction. This shows that if $(D - P)^2 > 0$, then $D - P$ is not nef.

Next, we use the description of linear systems $|D|$ with $\Phi(D) = 2$. First, assume $2d = D^2 = 4k$ and that we are in Case (1) of Proposition 2.6.6. Then, $|P|$ is equal to the pencil $|2F_1|$ or $|2F_2|$ and $k = 2$. In the first case $|D - P| = |(k - 2)F_1 + 2F_2|$. If $k > 2$, then the divisor $D - P$ is big and nef, hence $\dim |D - P| = d - 4 < d - 3$. This shows that $k = 2$, $d = 4$, and $|D| = |2F_1 + 2F_2|$. In both cases, we find $|D| = |2F_1 + 2F_2|$. Let B be a general member of the pencil $|F_1 + F_2|$. It is a curve of arithmetic genus 2, and

$$\omega_B^{\otimes 2} \cong \mathcal{O}_B(2F_1 + F_2 + 2K_S) \cong \mathcal{O}_B(2F_1 + 2F_2).$$

The short exact sequence

$$0 \to \mathcal{O}_S(F_1 + F_2) \to \mathcal{O}_S(2F_1 + 2F_2) \to \mathcal{O}_B(2F_1 + 2F_2) \to 0$$

implies that the restriction of $|D|$ to B is the complete bicanonical linear system. Since B is hyperelliptic, the restriction of $\phi_{|D|}$ is of degree 2, hence $\phi_{|D|}$ is of degree 2, and $|D|$ is bielliptic.

In Case (2) of Proposition 2.6.6, we must have $|P| = |2F_1|$. If $k \geq 3$, then $D - P = kF_1 + 2R$ is big and nef and hence, $|D|$ is not bielliptic. If $k = 2$, then $P = 2F_1$ and $D - P = (k-2)F_1 + F_2$. On the other hand, if $k > 2$, then $D - P$ is big and nef, which shows $k = 2$. Restricting the map $\phi_{|D|}$ to a general member of the pencil $|2F_1 + R_1|$, we obtain, as above, that $|D| = |4F_1 + 2R_1|$ is bielliptic. We leave it to the reader to check that in the remaining cases of Proposition 2.6.6, the linear system $|D|$ is not bielliptic.

Assume we are in Case (1) of Proposition 2.6.7. If $k \geq 2$, then $|P| = |2F_i|$ and $|D - P| = |(k-2)F_1 + F_2 + F_3|$. Moreover, if $k \geq 2$, then the divisor $D - P$ is big and nef or $h^1(D - P) = 0$. Thus, $|D|$ is not bielliptic. If $k = 1$, then we may assume that $P = F_1$ and $(D - P)^2 = (F_2 + F_3 - F_1)^2 = -2$. We have $\dim |D - P| = n - 3 = 0$ if and only if $F_2 + F_3 - F_1$ is effective. Similar to the previous case we can show that the restriction of $\phi_{|D|}$ to a general member of the pencil $|F_1 + F_2|$ is a degree 2 map. Thus, $|D|$ is bielliptic.

In Case (2), we obtain with similar arguments that $|P| = |2F_1|$ or $|2F_2|$ and $k = 1$. In the first case $|D - P| = |2F_2 + R_1 - F_1|$ must be of dimension $n - 3 = 0$ and hence, $|2F_2 + R_1 - F_1| \neq \emptyset$. Since the unique effective divisor from this linear system must be connected, we must have $F_1 - R_1 > 0$. If $|P| = |2F_2|$, then $|D - P| = |F_1 + R_1|$. In this case, the image of $|P|$ is a pencil of conics on a cubic surface, so R_1 must be again a component of F_1.

In Case (3), we must have $k = 1$, and this case is identical to the previous one.

In Case (4), we get $|P| = |2F_1|$ and $k = 1$ with similar arguments as above. Then, $|D - P| = |F_1 + 2R_1 + R_2|$ is not empty. As above, the unique effective divisor linearly equivalent to $D - P$ must be connected, hence $F_1 - R_2 > 0$. □

Remark 3.3.2 One can show (see [138, Corollary 4.7.1]) that a bielliptic linear system $|D|$ of degree 6 and in Case (1a) is equal to a linear system

$$|D| = |Z_1 + Z_2 + Z_3|,$$

where the Z_i are (-2)-curves with $Z_i \cdot Z_j = 2$ for $i \neq j$ and where the $|2F_i| = |Z_i + Z_j|$ are genus one pencils.

We will call a bielliptic linear system of the form $|2F_1 + 2F_2|$ if $D^2 = 8$ or of the form $|F_1 + F_2 + F_3|$ if $D^2 = 6$ a *non-special* bielliptic linear system. These correspond to Case (1) in the previous proposition. Bielliptic linear systems of the remaining types will be called *special*.

Moreover, we have just established that the image $\phi_{|D|}(S) \subset \mathbb{P}^d$ is the anti-canonical model of a weak del Pezzo surface with $d = \frac{1}{2}D^2 \in \{3, 4\}$. We remind the reader that we studied such surfaces of minimal degree in Section 0.5. Now, the morphism $S \to \phi_{|D|}(S)$ is generically finite of degree 2. Thus, after passing to its Stein factorization, we obtain a finite degree 2 cover of $\phi_{|D|}(S)$. By Proposition 0.2.27, the restriction of this cover to the smooth locus of $\phi_{|D|}(S)$ is a torsor under a finite flat group scheme of length 2. Thus, $\phi_{|D|}(S)$ is a symmetroid surface in the sense of Definition 0.6.4 and we refer to Section 0.6 and Section 0.7 for their classification.

We start with the case $d = \frac{1}{2}D^2 = 4$, in which case $\phi_{|D|}(S) \subset \mathbb{P}^4$ is a symmetroid quartic surface. We will be frequently using such bielliptic systems.

Definition 3.3.3 A pair of genus one pencils $|2F_1|$ and $|2F_2|$ with $F_1 \cdot F_2 = 1$ (resp. a genus one pencil $|2F|$ and a special bisection R) is called a *U-pair* (resp. *degenerate U-pair*).

The reason for this terminology is that a U-pair of genus one fibrations corresponds to a canonical pair of isotropic vectors $f_1 = [F_1]$ and $f_2 = [F_2]$ (resp. $f_2 = [F + R]$) that in its turn corresponds to a primitive embedding $\mathsf{U} \hookrightarrow \mathsf{E}_{10}$ of the hyperbolic plane U into the Enriques lattice E_{10}.

We will use the notations and classification results from Section 0.6.

Theorem 3.3.4 *Let $\phi_{|D|} : S \to \mathsf{D} \subset \mathbb{P}^4$ be a bielliptic map defined by a bielliptic linear system $|D|$ with $D^2 = 8$.*

1. *Assume that S is classical, that is, $K_S \neq 0$.*
   a. *If $|D|$ is non-special, then* $\mathsf{D} = \mathsf{D}_1$.
   b. *If $|D|$ is special, then* $\mathsf{D} = \mathsf{D}_1'$.
2. *Assume that S is a $\boldsymbol{\mu}_2$-surface.*
   a. *If $|D|$ is non-special, then* $\mathsf{D} = \mathsf{D}_2$.
   b. *If $|D|$ is special, then* $\mathsf{D} = \mathsf{D}_2'$.
3. *Assume that S is an $\boldsymbol{\alpha}_2$-surface.*
   a. *If $|D|$ is non-special, then* $\mathsf{D} = \mathsf{D}_3$.
   b. *If $|D|$ is special, then* $\mathsf{D} = \mathsf{D}_3'$.

Proof We start with Case (1). First, assume that $|D|$ is non-special, that is, $|D| = |2F_1 + 2F_2|$. The restriction of $\phi_{|D|}$ to the half-fibers F_1, F_1', F_2, F_2' is a degree 2 map. Since $h^0(\mathcal{O}_S(D - F_i)) = h^0(\mathcal{O}_S(D - F_i')) = 3$, the image of each of these four half-fibers is a line on D. Conversely, if an irreducible curve C is mapped onto a line, then $C \cdot D = 2(C \cdot F_1 + C \cdot F_2) = 2$ and $\dim |D - C| = 2$. The first condition gives $C \cdot F_1$ or $C \cdot F_2 = 0$. Thus, we may assume that $C \cdot F_1 = 0$ and $C \cdot F_2 = 1$. The first condition implies that C is a component of F_1 or F_1' or that C is a component of a member of $|2F_1|$ that is not equal to $2F_1$ or $2F_1'$. Since $C \cdot F_2 = 1$, we see that in the first case C is mapped to the line $\phi_{|D|}(F_1)$ or $\phi_{|D|}(F_1')$. In the second case, we have $\dim |D - C| = \dim |2F_2| = 1$. Thus, all lines are accounted for: we have exactly four lines on D and we note that they span a hyperplane. Next, we count the number of conic pencils on D. We have obvious pencils coming from the four pencils in $|D|$, which are given by $|2F_1|$, $|2F_2|$, $|F_1+F_2|$, and $|F_1+F_2+K_S|$. Let us show that there are no more: the pre-image of a conic is a movable nef divisor G with $G^2 \geq 0$ and $G\cdot(2F_1+2F_2) = 4$. By the Hodge Index Theorem, we have $G \sim 2F_i$ or $G\cdot F_1 = G\cdot F_2 = 1$. In the latter case, we find $\Phi(G) = 1$ and applying Proposition 2.6.1, we obtain that $G \equiv F_1 + F_2$. This proves the claim. It remains to apply Proposition 0.6.16 to conclude that D is isomorphic to D_1.

Next, assume that $|D|$ is special, that is, $|D| = |4F+2R|$. Then, we have two lines equal to the images of F and $F' \in |F+K_S|$ and two pencils of conics equal to the images of the pencils $|2F|$ and $|2F+R|$. As before, applying Proposition 2.6.1 and the Hodge Index Theorem, we see that there are no more lines on D and no more pencils of conics. We also see that Proposition 0.6.17 shows that $\mathsf{D} = \mathsf{D}_1'$.

We continue with Case (2). First, assume that $|D| = |2F_1+2F_2|$ is non-special. The images of F_1 and F_2 are lines ℓ_1 and ℓ_2 on D. As in Case (1), one can show that there are no more lines on D. We also have three pencils of conics $|2F_1|$, $|2F_2|$, and $|F_1+F_2|$. As in the previous cases, one can show that there are no more. Applying Proposition 0.6.19, we obtain that $\mathsf{D} = \mathsf{D}_2$ or $\mathsf{D} = \mathsf{D}_3$.

Suppose that $\mathsf{D} = \mathsf{D}_3$. By Proposition 0.6.12, there exists a principal $\boldsymbol{\alpha}_2$-cover $Q' \to \mathsf{D}^{\mathrm{sm}}$, where Q' is the complement of one point on a nonsingular quadric. Its pull-back on $S \setminus \{s_0\}$ is a principal $\boldsymbol{\alpha}_2$-cover. By Theorem 0.1.4, it extends to a principal $\boldsymbol{\alpha}_2$-cover of S. We claim that this cover is non-trivial. Then D is an $\boldsymbol{\alpha}_2$-surface and we are done. This claim is true if $\phi_{|D|}$ is a separable map because in this case the base change $S\times_{\mathsf{D}} X$ is reduced. Thus, we may assume that $\phi_{|D|}$ defines an inseparable cover. Let $\mathbf{F} : S \to S^{(2)}$ be the relative Frobenius morphism. The surface D is sandwiched between S and $S^{(2)}$, thus there exists a degree 2 inseparable cover $\psi : \mathsf{D} \to S^{(2)}$ and we identify $S^{(2)}$ with S. Let $\pi : X \to S$ be the K3-cover of S, which is a non-trivial and principal $(\mathbb{Z}/2\mathbb{Z})$-cover. Since ψ is an homeomorphism in the étale topology over a complement of finitely many points (over which ψ is not finite), the pre-image of the K3-cover is a non-trivial principal cover of D^{sm}. This implies that $\mathsf{D} \cong \mathsf{D}_2$.

If $|D| = |4F+2R|$ is a special bielliptic linear system, then we use similar arguments as above to show that D contains only one line and one pencil of conics and hence, by Proposition 0.6.21, it must be either D_2' or D_3'. Using the previous argument, we conclude that $\mathsf{D} = \mathsf{D}_2'$.

In Case (3), the arguments from Case (2) show that the assumptions $\mathsf{D} = \mathsf{D}_2$ or D_2' imply that S admits a nontrivial $(\mathbb{Z}/2\mathbb{Z})$-cover, which contradicts to the assumption that S is a $\boldsymbol{\alpha}_2$-surface. □

We continue with the case $d = \frac{1}{2}D^2 = 3$ and then, $\phi_{|D|}(S) \subset \mathbb{P}^3$ is a symmetroid cubic surface. We will use the notations and classification results from Section 0.7.

Theorem 3.3.5 *Let $\phi_{|D|} : S \to \mathsf{C} \subset \mathbb{P}^3$ be a bielliptic map defined by a bielliptic linear system $|D|$ with $D^2 = 6$.*

1. Assume that $K_S \neq 0$

a. If $|D|$ is non-special, then $\mathsf{C} = \mathsf{C}_1$.

b. If $|D|$ is special, then $\mathsf{C} = \mathsf{C}_1'$ *(resp.* $\mathsf{C} = \mathsf{C}_1''$*) if $|D|$ is of type (b) (resp. of type (c)).*

2. Assume that S is a $\boldsymbol{\mu}_2$-surface.

a. If $|D|$ is non-special, then $\mathsf{C} = \mathsf{C}_2$.
b. If $|D|$ is special, then $\mathsf{C} = \mathsf{C}_2'$ *(resp.* $\mathsf{C} = \mathsf{C}_2''$*) if $|D|$ is of type (b) (resp. of type (c)).*

3. Assume that S is an $\boldsymbol{\alpha}_2$-surface.

a. If $|D|$ is non-special, then $\mathsf{C} = \mathsf{C}_3$.
b. If $|D|$ is special, then $\mathsf{C} = \mathsf{C}_3'$ *(resp.* $\mathsf{C} = \mathsf{C}_3''$*) if $|D|$ is of type (b) (resp. of type (c)).*

Proof Applying Proposition 3.3.1, we can write the divisor class D in the form

$$D = F_1 + F_2 + F_3 = (2F_1 + 2F_2) - (F_1 + F_2 - F_3), \text{ in case (a)},$$
$$D = F_1 + 2F_2 + R_1 = (4F_2 + 2R_1) - (2F_2 - F_1 + R_1), \text{ in case (b)},$$
$$D = 3F_1 + 2R_1 + R_2 = (4F_1 + 2R_1) - (F_1 - R_2), \text{ in case (c)}.$$

Here, $Z = F_1 + F_2 - F_3$ (resp. $(2F_2 - F_1 + R_1)$, resp. $(F_1 - R_2)$) is an effective divisor with $Z^2 = -2$ and $F_1 \cdot Z = F_2 \cdot Z = 0$ (resp. $Z \cdot F_2 = 0$, resp. $Z \cdot F_1 = 0$). This implies that Z is a connected nodal cycle contained in a fiber of $|2F_1$ and in a fiber of $|2F_2|$. Under the bielliptic map $\phi : S \to \mathsf{D}$ given by the linear system $|2F_1 + 2F_2|$ or $|4F_1 + 2R_1|$, this nodal cycle Z is mapped to a point x_0 of an anti-canonical quartic del Pezzo surface D . In case (1), we have $\mathsf{D} = \mathsf{D}_1, \mathsf{D}_2$ or D_3 and the point x_0 lies on the intersection of conics from different pencils on D = and hence, does not lie on a line. Composing ϕ with the projection map with center at x_0, we map to the cubic anti-canonical del Pezzo surface $\mathsf{C} = \mathsf{C}_1, \mathsf{C}_2$ or C_3 depending on the type of S. In case (2), we have $\mathsf{D} = \mathsf{D}_1', \mathsf{D}_2'$ or D_3' and x_0 does not lie on the line $\phi(F_2)$. Projecting from x_0, we map to $\mathsf{C} = \mathsf{C}_1', \mathsf{C}_2'$ or C_3'. In case (3), we find that Z is a component of a half-fiber F_1 of the pencil $|2F_1|$, so its image lies on a line and projection from x_0 gives us the surface $\mathsf{C}_3, \mathsf{C}_3'$ or C_3''. □

Next, we describe the branch locus of a bielliptic map $\phi : S \to \phi_{|D|}(S)$. We restrict ourselves to the case $D^2 = 8$ and leave the case $D^2 = 6$ as an exercise to the reader. First, we introduce some notations that will be used in the statements and in their proofs.

If $\mathsf{D} \subset \mathbb{P}^4$ is a symmetroid quartic surface, then we let $j : \mathsf{D}^{\mathrm{sm}} \hookrightarrow \mathsf{D}$ be the open immersion of the smooth locus of D. Let $\phi := \phi_{|D|} : S \to \mathsf{D}$ be a bielliptic map defined by a bielliptic linear system $|D|$ with $D^2 = 8$ and let

$$S \xrightarrow{\sigma} S' \xrightarrow{\phi'} \mathsf{D} \subset \mathbb{P}^4 \tag{3.3.1}$$

be the Stein factorization of ϕ. Then, ϕ' is a finite morphism of degree 2 and σ is a birational morphism from S onto a normal surface with at worst rational double point singularities. More precisely, σ contracts all (-2)-curves $C \subset S$ with $C \cdot D = 0$ and nothing else.

Lemma 3.3.6 *We keep the notations and assumptions. Let $y \in \mathsf{D}$ be a singular point and let $s = \phi^{-1}(y)$ and $s' = \phi'^{-1}(y)$.*

1. *If ϕ is a non-special bielliptic map, then $s' \in S'$ is a nonsingular point and the morphism ϕ is finite at s.*
2. *If S is a special bielliptic map, if y is a rational double point of type A_1, and if $\mathsf{D} = \mathsf{D}_1'$, then $s' \in S'$ is a nonsingular point and the morphism ϕ is finite at s. In the remaining cases, $s' \in S'$ is a rational double point of type A_1 and the birational morphism $\sigma : S \to S'$ contracts a (-2)-curve R to the point s'.*

Proof First, assume D is not special, that is, $\mathsf{D} = \mathsf{D}_1, \mathsf{D}_2$, or D_3. Then, every singular point of D is equal to the intersection of two lines. The pre-image of a line on S is a half-fiber of $|2F_1|$ or $|2F_2|$. If $K_S \neq 0$, then we obtain that $\phi^{-1}(\mathrm{Sing}(\mathsf{D}))$ consists of four points $s_{ij} = F_1^{(i)} \cap F_2^{(j)}$, where $F_i^{(1)}, F_i^{(2)}$ are the irreducible components of the half-fibers of $|2F_i|$ containing a point from $\phi^{-1}(\mathrm{Sing}(\mathsf{D}))$. If $K_S = 0$, then $\mathrm{Sing}(\mathsf{D}) = \{y_0\}$ and $s_0 = \pi^{-1}(y_0)$ is equal to $F_1 \cap F_2$, where F_i are irreducible components of the unique half-fibers of $|2F_i|$. Any (-2)-curve R' passing through s_0 intersects positively both components $F_1^{(i)}$ and $F_2^{(j)}$ (resp. F_1 and F_2). Hence, it cannot be contracted by the map σ. Thus, every point $s' \in \phi'^{-1}(y_0)$ is nonsingular and ϕ' is a finite morphism near s_0'.

Next, assume that D is special, that is, $\mathsf{D} = \mathsf{D}_1', \mathsf{D}_2'$, or D_3'. In this case, the linear system $|D| = |4F + 2R|$ is special and the curve R is contracted by ϕ to a singular point $s_0' \in S'$.

First, assume that $K_S \neq 0$. Then, the images of the half-fibers F and F' of $|2F|$ are the lines on D_1'. Thus, the image of R is the intersection point y_0 of the two lines ℓ_1 and ℓ_2. It follows from the description of the surface D_1' that this point is a rational double point of type A_3. The other two singular points y_1 and y_2 lie on the lines ℓ_1 and ℓ_2. Any (-2)-curve $R' \neq R$ on S passing through one of the points $s_0 = \phi^{-1}(y_0)$, $s_1 = \phi^{-1}(y_1)$, $s_2 = \phi^{-1}(y_2)$ must intersect R or the irreducible components of F or F' passing through these points. Hence, such an R' is not contracted by σ. This shows that $s_0' = \sigma(s_0)$ is a rational double point of type A_1 and that the points $s_1' = \phi(s_1)$ and $s_2' = \phi(s_2)$ are nonsingular points of S'. The map ϕ' is a finite morphism near s_0', s_1', and s_2'.

Finally, we assume that $K_S = 0$, that is, $\mathsf{D} = \mathsf{D}_2'$ or D_3'. In this case, the image of the unique half-fiber of $|2F|$ is the unique line ℓ on D. The unique singular point y_0 of D lies on ℓ. The curve R is contracted to a point $y \in \ell$. The pre-image of the 2-dimensional linear system of hyperplane sections of D containing the line ℓ consists of divisors of the form $E + R + D' \in |D|$. They are all singular at the point $s_0 = R \cap F$, where F is an irreducible component of E intersecting R. This implies that $y = y_0$. Any other (-2)-curve R' passing through s_0 intersects F and R, hence $R' \cdot D \neq 0$, and R' is not contracted to a point on S'. This shows that $s_0' = \sigma(s_0)$ is a rational double point of type A_1 and that ϕ' is a finite morphism near s_0'. ◻

Now, let $\tau : \widetilde{\mathsf{D}} \to \mathsf{D}$ be the minimal resolution of singularities of D and let $\tilde{S}'$ be the normalization of $\widetilde{\mathsf{D}}$ in the field of rational functions on S. A proof of the next lemma can be found in [28, Proposition 1.5] in the case of a principal $(\mathbb{Z}/2\mathbb{Z})$-cover, but it also works for principal $\boldsymbol{\mu}_2$-covers or $\boldsymbol{\alpha}_2$-covers.

Lemma 3.3.7 *Let $X = \operatorname{Spec} A$ be the henselization of the local ring of a rational double point singularity on a normal algebraic surface. Let $X' \to X$ be its minimal resolution of singularities and let $Y \to X$ be a finite degree 2 map that is a non-trivial principal double cover outside the closed point $x_0 \in X$. Assume that the closed point $y_0 \in Y$ over x_0 is again a rational double point singularity. Let $Y' \to X'$ be the normalization of X' in the field of fractions of Y. Then, there is a morphism $Y' \to Y$ that factors through a minimal resolution Y'' of Y. A complete integral curve $C \subset Y'$ gets contracted to a point on Y'' if and only if $2C$ is equal to the pre-image of an irreducible component of a resolution of singularities $X' \to X$.*

Let $\tilde{S}'$ be the normalization of $\widetilde{\mathsf{D}}$ in the field of rational functions of S'. It comes with a birational morphism $\tau' : \tilde{S}' \to S'$ and a degree 2 cover $\tilde{\phi}' : \tilde{S}' \to \widetilde{\mathsf{D}}$. We set $U = S' \setminus \phi'^{-1}(\mathrm{Sing}(\mathsf{D}))$ and note that its complement consists of finitely many points $s_1, \ldots, s_k$. We can identify U with an open subset of $\tilde{S}'$ that lies over the complement of the exceptional divisor of $\widetilde{\mathsf{D}} \to \mathsf{D}$. The morphism $\sigma^{-1}(U) \to U$ is a resolution of singularities. It defines a rational map $S \dashrightarrow \tilde{S}'$, which we extend to a birational morphism $\sigma' : \tilde{S} \to \tilde{S}'$, where $\tau'' : \tilde{S} \to S$ is a suitable composition of blow-ups with centers at nonsingular points. Summing up, we obtain the following commutative diagram:

$$\begin{tikzcd} \tilde{S} \arrow[r, "\sigma'"] \arrow[d, "\tau''"] & \tilde{S}' \arrow[r, "\tilde{\phi}'"] \arrow[d, "\tau'"] & \widetilde{\mathsf{D}} \arrow[d, "\tau"] \\ S \arrow[r, "\sigma"] & S' \arrow[r, "\phi'"] & \mathsf{D} \end{tikzcd} \tag{3.3.2}$$

and we define

$$\tilde{\phi} := \tilde{\phi}' \circ \sigma' : \tilde{S} \to \widetilde{\mathsf{D}}.$$

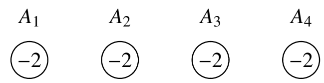
index[not]$\tau : \widetilde{\mathsf{D}} \to \mathsf{D}$

Applying Lemma 3.3.7, we conclude that τ' factors through the minimal resolution of singular points s_i - in fact, there can be only one such point and this happens only if $\mathsf{D} = \mathsf{D}_1', \mathsf{D}_2'$, or D_3'. This implies that we may assume σ' to be an isomorphism over $\tau'^{-1}(s_i)$ and hence, $\tilde{\phi}$ is a finite morphism of degree 2 over a neighborhood of A. In particular, the exceptional divisor $\tilde{A}$ of τ'' is a finite cover of the exceptional divisor A of τ.

Let us describe the exceptional divisor $A = \sum_{i=1}^{k} A_i$, where $k = 4$ if $\mathsf{D} = \mathsf{D}_i$ and $k = 5$, otherwise, in terms of a geometric basis $(e_0, \ldots, e_5)$ of $\mathrm{Pic}(\widetilde{\mathsf{D}})$ given in the Propositions 0.6.16, 0.6.17, 0.6.19, and 0.6.21.

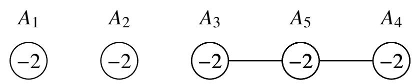
- $\mathsf{D} = \mathsf{D}_1$:

$$A_1 = e_0 - e_1 - e_2 - e_3, \quad A_2 = e_2 - e_3, \quad A_3 = e_4 - e_5, \quad A_4 = e_0 - e_1 - e_4 - e_5$$



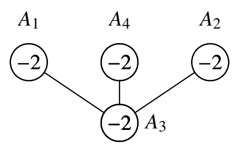
- $\mathsf{D} = \mathsf{D}_1'$:

$$A_1 = e_0 - e_1 - e_2 - e_3, \quad A_2 = e_4 - e_5, \quad A_3 = e_0 - e_1 - e_4 - e_5, \quad A_4 = e_2 - e_3, \quad A_5 = e_1 - e_2.$$



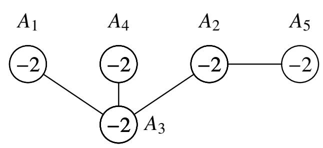
- $\mathsf{D} = \mathsf{D}_2, \mathsf{D}_3$:

$$A_1 = e_0 - e_1 - e_2 - e_3, \quad A_2 = e_2 - e_3, \quad A_3 = e_3 - e_4, \quad A_4 = e_4 - e_5.$$



- $\mathsf{D} = \mathsf{D}_2', \mathsf{D}_3'$:

$$A_1 = e_0 - e_1 - e_2 - e_3, \quad A_2 = e_2 - e_3, \quad A_3 = e_3 - e_4, \quad A_4 = e_4 - e_5, \quad A_5 = e_1 - e_2.$$



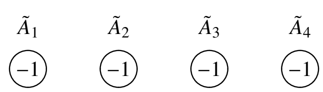
Observe that

$$A_1 + \cdots + A_4 = \begin{cases} 2(e_0 - e_1 - e_3 - e_5) & \text{if } \mathsf{D} = \mathsf{D}_1, \mathsf{D}_1', \\ e_0 - e_1 - e_3 - e_5 & \text{otherwise.} \end{cases}$$

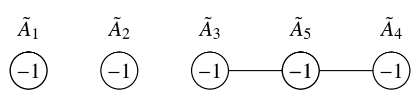
Set $\tilde{A}_i := \tilde{\phi}^{-1}(A_i)_{\mathrm{red}}$. Observe that the image of the component $\tilde{A}_5$ in S is the special bisection R that is mapped under σ to the ordinary double point $s_i \in S'$ lying over a singular point of D.

Lemma 3.3.8 *The dual graph of the exceptional divisor $\tilde{A}$ of τ'' is as follows:*

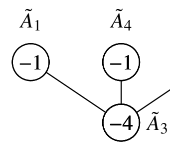
- $\mathsf{D} = \mathsf{D}_1$*:*



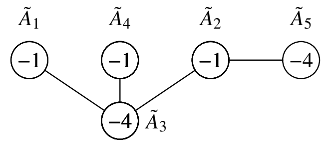
- $\mathsf{D} = \mathsf{D}_1'$*:*



- $\mathsf{D} = \mathsf{D}_2, \mathsf{D}_3$*:*



- $\mathsf{D} = \mathsf{D}_2', \mathsf{D}_3'$:



Proof Since $\tilde{S}$ is smooth, the pre-image $\tilde{A}$ of the exceptional divisor A of τ is a divisor on $\tilde{S}$ that can be blown down to nonsingular points. For each component A_i of A, the pre-image $\tilde{A}_i$ is either a (-1)-curve or a (-4)-curve. The former case happens if and only if the pullback of A_i is equal to $2\tilde{A}_i$.

If $\mathsf{D} = \mathsf{D}_1$, then A consists of four disjoint (-2)-curves, hence $\tilde{A}$ consists of four disjoint smooth rational curves that can be blown down to nonsingular points. They have to be (-1)-curves.

If $\mathsf{D} = \mathsf{D}_1'$, then A consists of two disjoint (-2)-curves and a chain of three (-2)-curves, hence $\tilde{A}$ consists of two disjoint smooth rational curves and a chain of three smooth rational curves. We know that the middle component gets mapped to a special bisection R on S. Thus, the extreme components get mapped to nonsingular points and this leaves us with the only possibility given by the picture.

Assume that $\mathsf{D} = \mathsf{D}_2$ or D_3. Then $\tilde{A}$ is the union of three disjoint smooth rational curves $\tilde{A}_1, \tilde{A}_2, \tilde{A}_4$ and a smooth rational curve $\tilde{A}_3$ that intersects each $\tilde{A}_i, i \neq 3$, with multiplicity 1. Let τ'' be the composition of the blow-downs of one (-1)-curve. It is immediate to see that the first curve that is blown down cannot be $\tilde{A}_3$. Thus, we may assume that it is $\tilde{A}_1$. The image of $\tilde{A}_3$ after the first blow-down is a chain of three smooth rational curves. The only way that this gets blow them down to a nonsingular point, is to assume that E_0 is a (-4)-curve and that all other $\tilde{A}_i$ are (-1)-curves. This gives the asserted picture.

Finally, assume that $\mathsf{D} = \mathsf{D}_2'$ or D_3'. In this case $\tilde{A}_5$ gets mapped to a special bisection with self-intersection -2, and $\tilde{A} - \tilde{A}_3$ gets mapped to a nonsingular point on S. Similar arguments to the above give us the asserted picture. □

We now continue with our analysis of the double cover $\tilde{\phi} : \tilde{S} \to \widetilde{\mathsf{D}}$. We have a short exact sequence

$$0 \to \mathcal{O}_{\widetilde{\mathsf{D}}} \to \tilde{\phi}'_* \mathcal{O}_{S'} \to \mathcal{L}^{-1} \to 0, \tag{3.3.3}$$

where $\mathcal{L}$ is an invertible sheaf. If $p \neq 2$, then ϕ' this exact sequence splits, then the cover $\tilde{\phi}'$ is given by a global section s of $\mathcal{L}^{\otimes 2}$, whose zero divisor $\mathsf{B}(\tilde{\phi}')$ is equal to the branch divisor of the cover. If $p = 2$ and if ϕ' is separable, then the cover is an Artin–Schreier cover defined by data $(\mathcal{L}, a)$, where a is a section of $\mathcal{L}$, whose zero divisor $\mathsf{B}(\tilde{\phi}')$ is the branch divisor of the cover. If $p = 2$, if ϕ' is inseparable, and if the exact sequence splits, then the cover is defined by a section of $\mathcal{L}^{\otimes 2}$ modulo a square of a section of $\mathcal{L}$. If the exact sequence does not split, then we get a cover of type $\boldsymbol{\alpha}_{\mathcal{L}}$.

Since D has rational singularities, we have $R^1\tau_*\mathcal{O}_{\widetilde{\mathsf{D}}} = 0$ and thus, after applying τ_* to the exact sequence (3.3.3), we obtain a short exact sequence

$$0 \to \mathcal{O}_{\mathsf{D}} \to \phi'_*\mathcal{O}_{\tilde{S}'} \to \tau_*\mathcal{L}^{-1} \to 0. \tag{3.3.4}$$

We know that the direct image of a reflexive sheaf is reflexive and thus, $\mathcal{Q} := \tau_*\mathcal{L}^{-1}$ is a reflexive sheaf. If ϕ' is separable and $p \neq 2$ (resp. $p = 2$), then $\mathcal{Q}^{[2]} \cong \mathcal{O}_{\mathsf{D}}(-\mathsf{B}(\phi))$ (resp. $\mathcal{Q} \cong \mathcal{O}_{\mathsf{D}}(-\mathsf{B}(\phi))$), where $\mathsf{B}(\phi) = \tau_*(\tilde{B})$ is the Weil divisor that is defined to be the image of the branch divisor of $\tilde{\phi}$. We call $\mathsf{B}(\phi)$ the branch divisor of ϕ. Its pre-image in $\tilde{D}$ is the branch divisor $\mathsf{B}(\tilde{\phi})$ of $\tilde{\phi}'$.

Proposition 3.3.9 *Let $\mathcal{L}$ be the invertible sheaf from the data defining the double cover $\tilde{\phi} : \tilde{S} \to \widetilde{\mathsf{D}}$. Then:*

1.
$$\mathcal{L} \cong \begin{cases} \omega_{\widetilde{\mathsf{D}}}^{-1} \otimes \mathcal{O}_{\widetilde{\mathsf{D}}}(\frac{1}{2}A) & \textit{if } \mathsf{D} = \mathsf{D}_1, \mathsf{D}_1', \\ \omega_{\widetilde{\mathsf{D}}}^{-1} \otimes \mathcal{O}_{\widetilde{\mathsf{D}}}(A) & \textit{otherwise.} \end{cases}$$

2.
$$H^1(\widetilde{\mathsf{D}}, \mathcal{L}) \cong \begin{cases} 0 & \textit{if } \mathsf{D} = \mathsf{D}_1, \mathsf{D}_1', \\ \Bbbk & \textit{otherwise.} \end{cases}$$

Proof (1) It follows from Lemma 3.3.8 that

$$K_{\tilde{S}} = \begin{cases} \tau''^*(K_S) + \tilde{A}_1 + \tilde{A}_2 + \tilde{A}_3 + \tilde{A}_4 & \text{if } \mathsf{D} = \mathsf{D}_1, \mathsf{D}_1', \\ \tau''^*(K_S) + 2\tilde{A}_1 + 2\tilde{A}_2 + \tilde{A}_3 + 2\tilde{A}_4 & \text{otherwise.} \end{cases}$$

By Proposition 0.2.10, we have

$$\omega_{\tilde{S}'} \cong \tilde{\phi}'^*(\omega_{\widetilde{\mathsf{D}}} \otimes \mathcal{L}).$$

Since all singularities of $\tilde{S}'$ are rational double points, we have

$$\omega_{\tilde{S}} \cong \tilde{\phi}^*(\omega_{\widetilde{\mathsf{D}}} \otimes \mathcal{L}).$$

Applying Proposition 0.2.14, we obtain that $\omega_{\mathsf{D}} \otimes \mathcal{L}$ is a 2-torsion element of $\mathrm{Pic}(\mathsf{D})$ and it is a non-trivial 2-torsion element only if $K_S \neq 0$, that is, $\mathsf{D} = \mathsf{D}_1, \mathsf{D}_1'$. By Lemma 3.3.8, we have $\tilde{\phi}^*(A_i) = 2\tilde{A}_i$ for $i = 1, \ldots, 4$ if $\mathsf{D} = \mathsf{D}_1, \mathsf{D}_1'$, and otherwise, we have $\tilde{\phi}^*(A_i) = 2\tilde{A}_i, i = 1, 2, 4$ and $\tilde{\phi}^*(A_3) = \tilde{A}_3$ otherwise. This proves the first assertion.

(2) Assume that $\mathsf{D} = \mathsf{D}_1, \mathsf{D}_1'$. We have $-K_{\widetilde{\mathsf{D}}} + \frac{1}{2}A \sim 4e_0 - 2e_1 - e_2 - 2e_3 - e_4 - 2e_5$. It is easy to see that the moving part of the linear system $|-K_{\widetilde{\mathsf{D}}} + \frac{1}{2}A|$ is equal to $|2e_0 - e_2 - e_4|$ (resp. $|2e_0 - e_1 - e_4|$) if $\mathsf{D} = \mathsf{D}_1$ (resp. $\mathsf{D} = \mathsf{D}_1'$). It corresponds to

the linear system of conics through the points p_2, p_4 (resp. p_1, p_4). Thus, we find $h^0(\mathcal{L}) = 4$. On the other hand, we obtain by Riemann–Roch that

$$h^0(\mathcal{L}) \;=\; \frac{1}{2}(2K_{\widetilde{\mathsf{D}}}^2 + \tfrac{1}{4}A^2) + 1 + h^1(\mathcal{L}) \;=\; 4 + h^1(\mathcal{L}).$$

This shows that $h^1(\mathcal{L}) = 0$.

Finally, assume that $\mathsf{D} \neq \mathsf{D}_1, \mathsf{D}_1'$. Let a be a section of $\omega_{\widetilde{\mathsf{D}}}^{-1}$ with zero divisor F disjoint from $A + A_5$. This defines a section of $\mathcal{L}$ and the corresponding short exact sequence

$$0 \;\to\; \mathcal{O}_{\widetilde{\mathsf{D}}} \;\to\; \mathcal{L} \to \mathcal{O}_{F+A}(F+A) \;\to\; 0. \tag{3.3.5}$$

We compute

$$H^1(\mathcal{O}_{F+A}(F+A)) \;\cong\; H^1(\mathcal{O}_F(F)) \oplus H^1(\mathcal{O}_A(A)) \;\cong\; H^1(\mathcal{O}_A(A)) \;\cong\; H^1(\omega_A) \;\cong\; \Bbbk.$$

Taking cohomology in (3.3.5), we obtain the remaining assertion. □

Remark 3.3.10 Let a be a global section of $\mathcal{L}$. It follows from the proof that the map

$$\mathbf{F} + a \;:\; H^1(\widetilde{\mathsf{D}}, \mathcal{L}) \;\to\; H^1(\widetilde{\mathsf{D}}, \mathcal{L}^{\otimes 2}) \tag{3.3.6}$$

can be identified with the map $\mathbf{F} + a : H^1(A, \mathcal{O}_A(A)) \to H^1(A, \mathcal{O}_A(2A))$. Both maps $\mathbf{F}$ and a come from the same map $\mathcal{O}_A(A) \to \mathcal{O}_A(2A)$ that is multiplication by the section of $\mathcal{O}_{\tilde{D}}(A)$, whose scheme of zeros is equal to A. Thus, the map $\mathbf{F} + a$ is zero and we conclude that

$$\mathrm{Ker}(\mathbf{F} + a) \;\cong\; H^1(\widetilde{\mathsf{D}}, \mathcal{L}) \;\cong\; \Bbbk \tag{3.3.7}$$

if $H^1(\widetilde{\mathsf{D}}, \mathcal{L}) \neq 0$.

Theorem 3.3.11 *Concerning the double cover $\tilde{\phi} : \tilde{S} \to \widetilde{\mathsf{D}}$, we have the following:*

1. *If $p \neq 2$, then:*

   (i) $\mathsf{D} = \mathsf{D}_1$ *or* D_1'.
   (ii) is a separable $\boldsymbol{\mu}_2$ cover with branch divisor

   $$\mathsf{B}(\tilde{\phi}) \;=\; W + A,$$

   where $W \in |-2K_{\widetilde{\mathsf{D}}}|$ is a reduced curve disjoint from A.
   (iii) The elliptic fibrations on S are the pre-images of the pencils $|e_0 - e_1|$ and $|2e_0 - e_2 - e_3 - e_4 - e_5|$ if $\mathsf{D} = D_1$ and $|e_0 - e_1|$ if $\mathsf{D} = \mathsf{D}_1'$.

2. *If $p = 2$ and $K_S \neq 0$, then:*

   (i) $\mathsf{D} = \mathsf{D}_1$ *or* D_1'.
   (ii) $\tilde{\phi}$ is an inseparable $\boldsymbol{\mu}_2$-cover defined by a section b of $\mathcal{L}$ with

   $$Z(b) \;=\; W + A,$$

where $W \in |-2K_{\tilde{\mathsf{D}}}|$ *is a reduced curve disjoint from* A, *or*

(ii') $\tilde{\phi}$ is a separable split Artin–Schreier cover defined by data $(\mathcal{L}, a, b)$ *with*

$$Z(a) = A + A_5 + Z(a)_0, \quad Z(b) = A + Z(b)_0,$$

where $Z(a)_0 \in |2e_0 - e_2 - e_4|$ *(resp.* $|2e_0 - e_1 - e_4|$*) if* $\mathsf{D} = \mathsf{D}_1$ *(resp.* $\mathsf{D} = \mathsf{D}_1'$*) and* $Z(b)_0 \in |-2K_{\tilde{\mathsf{D}}}|$.

(iii) The genus one fibrations on S are the pre-images of the pencils $|e_0 - e_1|$ *and* $|2e_0 - e_2 - e_3 - e_4 - e_5|$ *if* $\mathsf{D} = \mathsf{D}_1$ *and* $|e_0 - e_1|$ *if* $\mathsf{D} = \mathsf{D}_1'$.

3. *If* $p = 2$ *and S is a* $\boldsymbol{\mu}_2$*-surface, then:*

(i) $\mathsf{D} = \mathsf{D}_2$ *or* D_2'.

(ii) $\tilde{\phi}$ is a non-split and separable Artin–Schreier cover defined by data $(\mathcal{L}, a)$ *with*

$$Z(a) = A + Z(a)_0,$$

where $Z(a)_0 \in |-K_{\tilde{\mathsf{D}}}|$ *is disjoint from* A.

(iii) The genus one fibrations on S are the pre-images of the pencils $|e_0 - e_1|$ *and* $|2e_0 - e_2 - e_3 - e_4 - e_5|$ *if* $\mathsf{D} = \mathsf{D}_2$ *and* $|e_0 - e_1|$ *if* $\mathsf{D} = \mathsf{D}_2'$.

4. *If* $p = 2$ *and S is an* $\boldsymbol{\alpha}_2$*-surface, then:*

(i) $\mathsf{D} = \mathsf{D}_3$ *or* D_3'.

(ii) $\tilde{\phi}$ is a non-split separable Artin–Schreier cover defined by data $(\mathcal{L}, a)$ *with*

$$Z(a) = 2A + 2A_5 + A_3 + Z(a)_0,$$

where $Z(a)_0 \in |2e_0 - e_2 - e_3|$ *(resp.* $|2e_0 - e_1 - e_2|$*) if* $\mathsf{D} = \mathsf{D}_3$ *(resp.* $\mathsf{D} = \mathsf{D}_3'$*).*

(ii') $\tilde{\phi}$ is a non-split and inseparable $\boldsymbol{\alpha}_2$*-cover.*

(iii) The genus one fibrations on S are the pre-images of the pencils $|e_0 - e_1|$ *and* $|2e_0 - e_2 - e_3 - e_4 - e_5|$ *if* $\mathsf{D} = \mathsf{D}_3$ *and* $|e_0 - e_1|$ *if* $\mathsf{D} = \mathsf{D}_3'$.

Proof In each of the four cases Assertion (i) follows from Theorem 3.3.4 and we listed it only for convenience of the reader.

Assume that $p \neq 2$. In this case, we have $\mathsf{D} = \mathsf{D}_1$ or D_2. Applying Proposition 3.3.9, we find that $\tilde{\phi}$ is a separable $\boldsymbol{\mu}_2$-cover, whose branch divisor is given by a section b of $\mathcal{O}_S(-2K_{\tilde{D}} + A)$. Since ϕ defines a local $\boldsymbol{\mu}_2$-cover of singular points, $Z(b)$ consists of a reduced curve $W \in |-2K_{\tilde{\mathsf{D}}}|$ and the curve A that is disjoint from W.

Assume that $p = 2$, $K_S \neq 0$, and $\tilde{\phi}$ is separable. Applying Proposition 3.3.9, we obtain that its branch divisor $Z(a)$ belongs to the linear system $|-K_{\tilde{D}} + \frac{1}{2}A|$. By Lemma 3.3.8, the curves A_i with $i \neq 5$ are contained in the branch curve. The residual curve belongs to

$$|-K_{\tilde{D}} - \frac{1}{2}A| = |(3e_0 - e_1 - \cdots - e_5) - (e_0 - e_1 - e_3 - e_5)| = |2e_0 - e_2 - e_4|.$$

If $\mathsf{D} = \mathsf{D}_1'$, then $A_5 \cdot (2e_0 - e_2 - e_4) = (e_1 - e_2) \cdot (2e_0 - e_2 - e_4) = -1$ and thus, A_5 is an irreducible component of $Z(a)$, whose residual component is linearly equivalent to $(2e_0 - e_2 - e_4) - (e_1 - e_2) = 2e_0 - e_1 - e_4$. If $\mathsf{D} = \mathsf{D}_1$, then the linear system $|2e_0 - e_2 - e_4|$ is irreducible and consists of the inverse transform of conics passing through the points p_2, p_3, p_4. If $\mathsf{D} = \mathsf{D}_1'$, then the linear system $|2e_0 - e_1 - e_4|$ is irreducible and consists of the inverse transform of conics passing through the points p_1, p_4. Thus, we can write

$$Z(a) = Z(a)_0 + A + A_5,$$

where we ignore A_5 if $\mathsf{D} = \mathsf{D}_1$ and $Z(a)_0$ is as in the assertion of the theorem.

Assume $p = 2$, $K_S \neq 0$, and $\tilde{\phi}$ is inseparable. By Theorem 3.3.9, we have $H^1(\tilde{D}, \mathcal{L}) = 0$. Thus, the cover is a split $\boldsymbol{\mu}_2$-cover defined by data $(\mathcal{L}, b)$, where $Z(b) \in |-2K_{\tilde{D}} + A|$. Using Lemma 3.3.8 again, we see that A enters in $Z(b)$. Thus, $Z(b) = W + A$ for some reduced curve $W \in |-2K_{\tilde{D}}|$.

Assume that $p = 2$, S is a $\boldsymbol{\mu}_2$-surface, and $\tilde{\phi}$ is inseparable. Then, the pre-image of the pencil of conics on D is a pencil of genus one curves on S, whose general fiber is isomorphic to an inseparable cover of $\mathbb{P}^1$. This curve is isomorphic to a cuspidal cubic. According to the terminology of Section 4.1, we obtain a quasi-elliptic fibration on S. We will prove later in Theorem 4.10.3 that a $\boldsymbol{\mu}_2$-surface does not admit quasi-elliptic fibrations and hence, $\tilde{\phi}$ must be separable. Applying Proposition 3.3.9, we obtain that it is a separable Artin–Schreier cover defined by data $(\mathcal{L}, a)$ with $Z(a) \in |-K_{\tilde{D}} + A - A_5|$.

The unique singular point of D admits a local principal $\mathbb{Z}/2\mathbb{Z}$-cover by a nonsingular point if $\mathsf{D} = \mathsf{D}_2$ and a rational double point of type A_1 if $\mathsf{D} = \mathsf{D}_2'$. Since such a point does not admit a ramified local principal cover by a nonsingular (resp. singular point of type A_1), the branch curve $\mathsf{B}(\phi)$ does not pass through the singular point of D. This implies that $Z(a) = Z(a)_0 + R$, where R is supported on the exceptional divisor and $Z(a)_0$ is disjoint from R. Applying Lemma 3.3.8, we see that the curves A_1, A_2, A_4 enter in R. This gives

$$\begin{aligned}
&Z(a) - A_1 - A_2 - A_4 \\
&\sim (3e_0 - e_1 - \cdots - e_5) + (e_0 - e_1 - e_3 - e_5) - (e_2 - e_3) - (e_0 - e_1 - e_2 - e_3) - (e_4 - e_5) \\
&\sim 3e_0 - e_1 - e_2 - 2e_4 - e_5.
\end{aligned}$$

Intersecting with $A_3 \sim e_3 - e_4$, we obtain that A_3 is a component of $Z(a) - A_1 - A_2 - A_4$ and that $Z(a) - A_1 - A_2 - A_3 - A_4 = Z(a) - (A - A_5) \sim 3e_0 - e_1 - e_2 - e_3 - e_4 - e_5 = -K_{\widetilde{\mathsf{D}}}$. Thus, we can write $Z(a) = Z(a)_0 + A - A_5$ with $Z(a)_0 \in |-K_{\widetilde{\mathsf{D}}}|$. Applying Lemma 3.3.8, we see that A_3 enters in the branch divisor but its pre-image under the cover is a reduced curve.

Finally, assume that $p = 2$, that S is an $\boldsymbol{\alpha}_2$-surface, and that $\tilde{\phi}$ is separable. Since $H^0(\widetilde{\mathsf{D}}, \mathcal{L}) \neq 0$, we obtain that $\tilde{\phi}$ is split or a non-split Artin-Schreier cover defined by some data $(\mathcal{L}, a)$. The singular point of D is of type $D_4^{(0)}$ or $D_5^{(0)}$. It has no local principal covers with Galois group $\mathbb{Z}/2\mathbb{Z}$. This shows that $Z(a)$ contains a curve $Z(a)_0$, which has no exceptional components but intersects the exceptional locus. As in the previous case, we obtain that $Z(a) \sim Z(a)' + A$ with $Z(a)' \in |-K_{\tilde{\mathsf{D}}}|$. Thus, $Z(a)' = Z(a)_0 + A'$, where $A' \neq 0$ is supported on the exceptional divisor. We may assume that A' is the largest divisor with this property. Thus, $(-K_{\tilde{\mathsf{D}}} - A') \cdot A_i = -A' \cdot A_i \geq 0$ for each exceptional component A_i. This implies that A' is the fundamental cycle of the singularity, hence equal to $A_1 + A_2 + 2A_3 + A_4$ (resp. $A_1 + 2A_2 + 2A_3 + A_4 + A_5$) if $\mathsf{D} = \mathsf{D}_3$ (resp. $\mathsf{D}_3 = \mathsf{D}_3'$). Thus, we get

$$Z(a) = 2A + A_3 + Z(a)_0, \quad \text{(resp.} \quad Z(a) = 2A + A_2 + A_3 - A_5 + Z(a)_0),$$

where $Z(a)_0 \in |2e_0 - e_2 - e_3|$ (resp. $|2e_0 - e_1 - e_2|$). If $\mathsf{D} = \mathsf{D}_3$, then we compute

$$\begin{aligned} & 2A + A_3 + Z(a)_0 \\ &= 2(e_0 - e_1 - e_3 - e_5) + (e_3 - e_4) + (2e_0 - e_2 - e_3) \\ &= 4e_0 - 2e_1 - e_2 - 2e_3 - e_4 - 2e_5 \\ &= (3e_0 - e_1 - e_2 - e_3 - e_4 - e_5) + (e_0 - e_1 - e_3 - e_5) \\ &= -K_{\tilde{\mathsf{D}}} + A = Z(a). \end{aligned}$$

We restrict the sequence to the punctured local ring of the singular point of D and then, the cover splits. However, when the cover is an $\boldsymbol{\alpha}_2$-cover of a $\mathbb{Z}/2\mathbb{Z}$-cover in characteristic 2, the local cover of the punctured local ring does not split. This follows from exact sequence (0.1.5) because $\mathbf{F} - a$ is surjective on global sections. □

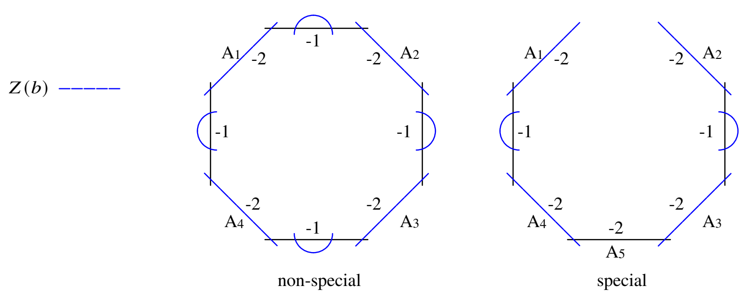


Fig. 3.3 Branch curve of a bielliptic map ($p \neq 2$)

If $p = 2$, then a bielliptic map $\phi_{|D|} : S \to \mathsf{D}$ could be inseparable. However, the next proposition shows that in this case the surface S is a nodal Enriques surface, that is, it contains at least one smooth rational curve. In the next chapter, we will reprove this result by using the theory of genus one fibrations on S.

Remark 3.3.12 In the case $p = 2$ and $K_S \neq 0$, the cover splits over the complement of the union of the lines. In fact, the blow-up $\widetilde{\mathsf{D}} \to \mathbb{P}^2$ makes it isomorphic to the complement of one line on the plane and hence, it is affine. Over this open subset the cover is given by data $(\mathcal{L}, a, b)$, where a is a section of $\mathcal{L}$ and b is a section of $\mathcal{L}^{\otimes 2}$ restricted to this open subset. So, one can construct a birational model of S by using this data.

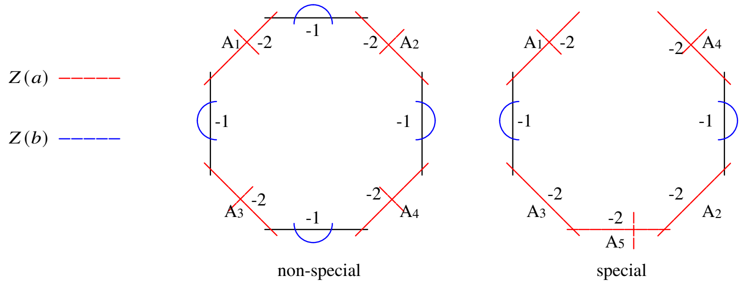


Fig. 3.4 Branch curve of a separable bielliptic map ($p = 2, K_S \neq 0$)

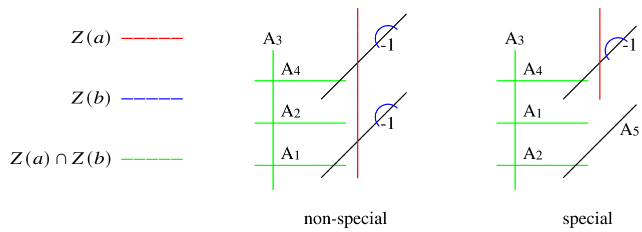


Fig. 3.5 Branch curve of a separable bielliptic map ($\boldsymbol{\mu}_2$-surface)

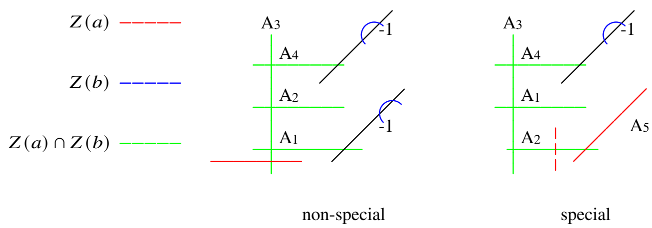


Fig. 3.6 Branch curve of a separable bielliptic map ($\boldsymbol{\alpha}_2$-surface)

Proposition 3.3.13 *Assume that $p = 2$, that $|D|$ be a bielliptic linear system with $D^2 = 8$, and that the associated bielliptic map $\phi : S \to \mathsf{D}$ is inseparable. If $|D|$ is non-special (resp. special), then ϕ contracts a set of eight (resp. nine) (-2)-curves. In particular, S contains smooth rational curves and thus, is a nodal Enriques surface.*

Proof The maps $\sigma' : \tilde{S} \to \tilde{S}'$ and $\tilde{\phi}' : \tilde{S}' \to \widetilde{\mathsf{D}}$ from (3.3.2) are the minimal resolution of singularities and an inseparable finite map of degree 2, respectively. We have $e(\tilde{S}') = e(\widetilde{\mathsf{D}}) = e(\mathbb{P}^2) + 5 = 8$ and $e(\tilde{S}) = e(S) + 4 = 16$. Being the minimal resolution of rational double points, we have $e(\tilde{S}) = e(\tilde{S}') + n$, where n is the number of smooth rational curves blown down to singular points. If $|D|$ is non-special, then the exceptional divisor of σ' is isomorphic to the exceptional divisor of σ, so we get that σ blows down $n = 8$ smooth rational curves. If $|D|$ is special, then the exceptional divisor of σ consists of the exceptional divisor of σ' and a special bisection R that is blown down to the singular point of S' lying over the singular point of D. Thus, σ' blows down 9 smooth rational curves. □

Remark 3.3.14 We know that the singular locus of the inseparable cover $\tilde{\phi}' : \tilde{S}' \to \widetilde{\mathsf{D}}$ lies over the zeros of a section α of $\Omega^1_{\widetilde{\mathsf{D}}} \otimes \mathcal{L}^{\otimes 2}$. We have

$$c_2(\Omega^1_X \otimes \mathcal{L}^{\otimes 2}) = c_2(\Omega^1_X) + c_1(\Omega^1_X) \cdot c_1(\mathcal{L}^{\otimes 2}) + c_1(\mathcal{L}^{\otimes 2})^2. \tag{3.3.8}$$

We know that $c_2(\Omega^1_{\widetilde{\mathsf{D}}}) = e(\widetilde{\mathsf{D}}) = 8$ and

$$c_1(\Omega^1_{\widetilde{\mathsf{D}}}) \cdot c_1(\mathcal{L}^{\otimes 2}) = K_{\widetilde{\mathsf{D}}} \cdot (L_1 + \cdots + L_4 + W) = K_{\widetilde{\mathsf{D}}} \cdot W = K_{\mathsf{D}} \cdot W = -8,$$
$$c_1(\mathcal{L}^{\otimes 2})^2 = (L_1 + \cdots + L_4 + W)^2 = -8 + 16 = 8.$$

Adding up, we find $c_2(\Omega^1_{\widetilde{\mathsf{D}}} \otimes \mathcal{L}^{\otimes 2}) = 8$. In the case where the zero cycle Z of the section α is reduced, this shows that $\mathrm{Sing}(\tilde{S}')$ consists of 8 ordinary double points and this agrees with the previous proposition. It also suggests that, for any singular point x of type A_n, D_n, E_n, we have $h^0(\mathcal{O}_{Z,x}) = n$.

Let $\mathbf{F} : \widetilde{\mathsf{D}} \to \tilde{S}' \to \widetilde{\mathsf{D}}$ be the factorization of the Frobenius map. Comparing formula (3.3.8) with formula (0.3.4) and using Remark 0.3.15 we find that $h^0(\mathcal{O}_{\tilde{Z}}) = h^0(Z)$, where $\tilde{Z}$ is the scheme of zeros of the rational vector field ∂ on $\widetilde{\mathsf{D}}$ such that $\widetilde{\mathsf{D}}^{\partial} \cong \tilde{S}'$.

Proposition 3.3.15 *Let $|D|$ be a bielliptic linear system with $D^2 = 8$ and let $\phi : S \to \mathsf{D}$ be the associated bielliptic map. Let $\rho : Q \to \mathsf{D}$ be the degree 2 cover by a quadric surface described in Propositions 0.6.11, 0.6.12, 0.6.13. Then, the first projection $\pi : S \times_{\mathsf{D}} Q \to S$ is isomorphic to the K3-cover of S.*

Proof Set $Q' := \rho^{-1}(\mathsf{D}^{\mathrm{sm}})$. If $\mathsf{D} = \mathsf{D}_1, \mathsf{D}_2, \mathsf{D}_3$, then the Q is a nonsingular quadric and $Q \setminus Q'$ consists of 4 points if $\mathsf{D} = \mathsf{D}_1$ and one point otherwise. If $\mathsf{D} = \mathsf{D}_1', \mathsf{D}_2', \mathsf{D}_3'$, then Q is a quadric cone and $Q \setminus Q'$ consists of the singular point of Q and two more points if $\mathsf{D} = \mathsf{D}_1$ and consists of only the singular point otherwise. The cover $\rho' : Q' \to \mathsf{D} \setminus \mathsf{D}^{\mathrm{sm}}$ is a principal G-cover, where $G = \boldsymbol{\mu}_2$ (resp. $(\mathbb{Z}/2\mathbb{Z})$, resp. $\boldsymbol{\alpha}_2$) if $\mathsf{D} = \mathsf{D}_1$ (resp. D_2, resp. D_3). The base change $S \times_{\mathsf{D}} Q' \to S^{\dagger} := \phi^{-1}(\mathsf{D}^{\mathrm{sm}})$ is a nontrivial principal G-cover isomorphic to the pre-image of the torsor $Q' \to \mathsf{D}^{\mathrm{sm}}$ by the morphism ϕ. By Theorem 0.1.4, it extends to a nontrivial principal cover of S isomorphic to the K3-cover. ◻

Now, consider the second projection $\bar{\phi} : X = S \times_{\mathsf{D}} Q \to Q$. We can factor both ϕ and $\bar{\phi}$ as birational morphisms $\sigma, \bar{\sigma}$ followed by a finite morphism $\phi', \bar{\phi}'$ of degree 2 and obtain a commutative diagram

$$\begin{tikzcd}
X \arrow[r, "\bar{\sigma}"] \arrow[d, "\pi"] & X' \arrow[r, "\bar{\phi}'"] \arrow[d, "\pi'"] & Q \arrow[d, "\rho"] \\
S \arrow[r, "\sigma"] & S' \arrow[r, "\phi'"] & \mathsf{D}.
\end{tikzcd}$$

It follows that the double cover $\bar{\phi}' : X' \to Q$ is given by the pre-image to Q of the data defining the cover $\phi' : S' \to \mathsf{D}$.

1. If $p \neq 2$, then the cover $\bar{\phi}'$ is a $\boldsymbol{\mu}_2$-cover, whose branch curve satisfies
$$\mathsf{B}(\bar{\phi}') \in |\rho^*(-2K_{\mathsf{D}})| = |-2K_Q|$$
and is contained in Q'.
2. If $p = 2$, if $K_S \neq 0$, and if ϕ is separable, then $\bar{\phi}'$ is a split Artin–Schreier cover associated to data (ω_Q^{-1}, a, b), where
$$\mathsf{B}(\bar{\phi}') = Z(a) \in |-2K_Q|, \quad Z(b) \in |-4K_Q|.$$
In this case, $\mathsf{B}(\bar{\phi}')$ is contained in $Q \setminus Q'$.
3. If $p = 2$ and if $K_S = 0$, then $\bar{\phi}$ is a separable and non-split Artin–Schreier cover, whose branch curve $\mathsf{B}(\bar{\phi}') \in |-2K_Q|$ is contained in Q' if S is a $\boldsymbol{\mu}_2$-surface.

Theorem 3.3.11 admits a converse, whose proof we leave to the reader. It gives an explicit recipe for constructing birational models of Enriques surfaces as double covers of symmetroid quartic surfaces:

Theorem 3.3.16 *Let $\mathsf{D} \subset \mathbb{P}^4$ be a non-degenerate symmetroid quartic surface and let A and $\mathcal{L}$ be as in Theorem 3.3.9.*

1. *Assume that $p \neq 2$. Let $\mathsf{D} = \mathsf{D}_1$ (resp. D_1') and $W \in |-2K_{\widetilde{\mathsf{D}}}|$ be a reduced divisor disjoint from the exceptional curve A. Then, a minimal resolution of the double cover $\tilde{S}' \to \widetilde{\mathsf{D}}$ branched along $W + A_1 + \cdots + A_4$ is isomorphic to the blow-up of an Enriques surface S at four points $x_1, \ldots, x_4$. It descends to a bielliptic map $\phi : S \to \mathsf{D}$ defined by a non-special (resp. special) bielliptic linear system $|D|$ of degree 8. If $|D| = |2F_1 + 2F_2|$ (resp. $|4F + 2R|$), then the points x_i are the intersection points of the irreducible components $F_1^{(i)} \cap F_2^{(j)}$ of the two half-fibres $F_1^{(i)}$ of $|2F_1|$ and two half-fibers $F_2^{(i)}$ of $|2F_2|$. If $|D| = |4F_1 + 2R|$, then $x_i = R \cap F_1^{(i)}$, $i = 1, 2$ and $x_3 \in F_1^{(1)}$, $x_4 \in F_1^{(2)}$.*
2. *Assume $p = 2$ and let $\mathsf{D} = \mathsf{D}_1$ (resp. D_1'). Let a be a section of $\mathcal{L}$ with $Z(a) = A + \mathsf{B}$ with $\mathsf{B} \in |2e_0 - e_2 - e_4|$ (resp. $|2e_0 - e_1 - e_4|$) and let b be a section of $\mathcal{L}^{\otimes 2}$ with $Z(b) = A + W$ and $W \in |-2K_{\widetilde{\mathsf{D}}}|$. Then, a minimal resolution of the normalization of the split Artin–Schreier double cover of $S' \to \mathsf{D}$ defined by the data $(\mathcal{L}, a, b)$ is isomorphic to a classical Enriques surface S blown up at four points as in (1).*
3. *Assume $p = 2$ and let $\mathsf{D} = \mathsf{D}_2$ (resp. D_2'). Let a be a section of $\mathcal{L}$ with $Z(a) = A + \mathsf{B}$, where $\mathsf{B} \in |-K_{\widetilde{\mathsf{D}}}|$ is disjoint from A, and let S' be a non-trivial $\boldsymbol{\alpha}_{\mathcal{L},a}$-torsor, whose cohomology class is mapped to a nonzero element of* $\mathrm{Ker}(\mathbf{F} + a : H^1(\mathsf{D}, \mathcal{L})) \to H^1(\mathsf{D}, \mathcal{L}^{\otimes 2}))$. *Then, a minimal resolution of the normalization of S' is isomorphic to the blow-up of a $\boldsymbol{\mu}_2$-Enriques surface S blown up at four points $x_2 \succ x_1, x_3 \succ x_1, x_4 \succ x_1$. The cover $S' \to \widetilde{\mathsf{D}}$ descends to a bielliptic map defined by the linear system $|D|$. If $|D = |2F_1 + 2F_2|$ (resp. $|4F_1 + 2R|$), the point x_1 is the intersection point of irreducible components of F_1 and F_2 (resp. R).*
4. *Assume $p = 2$ and let $\mathsf{D} = \mathsf{D}_3$ (resp. D_3'). Let a be a section of $\mathcal{L}$ with $Z(a) = 2A + A_3 + B$ with $B \in |2e_0 - e_2 - e_4|$ (resp. $|2e_0 - e_1 - e_4|$). Then, a minimal resolution of the normalization of the non-split Artin–Schreier double cover of*

$S' \to \mathsf{D}$ defined by an $\boldsymbol{\alpha}_{\mathcal{L},a}$-torsor is isomorphic to an $\boldsymbol{\alpha}_2$-Enriques surface S blown up at four points $x_2 \succ x_1$, $x_3 \succ x_1$, $x_4 \succ x_1$ as in (3).

5. *Assume $p = 2$ and let $\mathsf{D} = \mathsf{D}_1$ (resp. D_1'). Let $W \in |-2K_{\widetilde{\mathsf{D}}}|$ be a divisor disjoint from the exceptional curve A such that the inseparable $\boldsymbol{\mu}_2$-cover defined by $Z(b) = W + A - A_5$ is normal. Then, its minimal resolution is isomorphic to the blow-up of an Enriques surface S at four points $x_1, \ldots, x_4$ as in (1).*
6. *Assume $p = 2$ and let $\mathsf{D} = \mathsf{D}_3$ (resp. D_3') and let S' be a non-trivial $\boldsymbol{\alpha}_{2,\mathcal{L}}$-torsor. Assume that S' is normal. Then, its minimal resolution is isomorphic to the blow-up of an $\boldsymbol{\alpha}_2$-Enriques surface S at four points $x_1, \ldots, x_4$ as in (1).*

Remark 3.3.17 It follows from Proposition 3.3.23 below and Corollary 6.2.14 in Volume II that if the K3-cover is a normal surface, then every bielliptic map is separable.

Remark 3.3.18 Let $\phi : S \to \mathsf{D}$ be a bielliptic map onto an anti-canonical quartic del Pezzo surface. If $\mathsf{D} = \mathsf{D}_i$ (resp. $\mathsf{D} = \mathsf{D}_i'$), then one can also construct a rational map of degree 2 from S onto an irreducible and nonsingular (resp. singular) quadric:

1. If $\mathsf{D} = \mathsf{D}_1$ (resp. D_1'), then this map is defined to be the composition of the induced map $\tilde{S} \to \widetilde{\mathsf{D}}$ followed by a birational morphism from $\tilde{S}$ to a nonsingular (resp. singular) quadric D_Q (not to be confused with the double cover Q of D) given by the linear system $|2e_0 - e_2 - e_4|$ (resp. $|2e_0 - e_1 - e_4|$). It blows down the four (resp. two) lines on D to points on the quadric D_Q.
   If $p \neq 2$, then the branch curve is the union of a curve W of bidegree $(4, 4)$ with four double points at the vertices of a quadrangle of lines on D_Q and the four sides of the quadrangle (resp. the union of two lines of the ruling and a curve W of degree 4 with two tacnodes with tangent directions along the lines). The linear system defining the rational map $S \dashrightarrow \mathsf{D}_Q$ is given by the linear subsystem of the linear system $|D + K_S|$ spanned by the two pencils $F_1 + F_1' + |2F_2|$ and $F_2 + F_2' + |2F_1|$ (resp. by the linear subsystem of $|4F + R + K_S|$ spanned by the pencils $R + 2F + |2F + R + K_S|$ and $2R + 2F' + |2F|$).
2. If $\mathsf{D} = \mathsf{D}_2, \mathsf{D}_3$ (resp. $\mathsf{D}_2', \mathsf{D}_3'$), then the map is the composition of the induced map $\tilde{S} \to \widetilde{\mathsf{D}}$ and a birational morphism from $\tilde{S}$ onto a singular quadric D_Q given by the linear system $|2e_0 - e_2 - e_3|$ (resp. $|2e_0 - e_1 - e_2|$). It blows down the two lines (resp. one line) on D to points on the same line on the quadric D_Q. It is given by the 3-dimensional linear subsystem of $|2F_1 + 2F_2|$ that is spanned by the pencils $F_1 + F_2 + |F_1 + F_2|$, $2F_2 + |2F_1|$, and $2F_1 + |2F_2|$ with one base point of multiplicity 2 (resp. the 3-dimensional linear subsystem of $|4F + 2R|$ spanned by the plane $2R + |4F|$ and the pencil $2F + R + |2F + R|$).

Remark 3.3.19 Suppose that we have a non-degenerate canonical isotropic sequence (F_1, F_2), such that $|2F_1 + 2F_2|$ defines an inseparable bielliptic map $\phi : S \to \mathsf{D}$. Then, both genus one fibrations $|2F_1|$ and $|2F_2|$ are quasi-elliptic. On the other hand, if ϕ is separable, then we will see in Proposition 3.3.23 that at most one of the genus one fibrations can be quasi-elliptic. In fact, one would expect that both fibrations are elliptic, but in special cases one of them can be quasi-elliptic, see Example 3.3.24.

Example 3.3.20 In the case where $\mathsf{D} = \mathsf{D}_1, \mathsf{D}_1'$, let us give equations of the image $V(F_6)$ of the curve $W' \in |6e_0 - 2(e_1 + \cdots + e_5)|$ in the plane under the blow-down $\mathsf{D} \to \mathbb{P}^2$ defined by the geometric basis $e_0, e_1, \ldots, e_5$ of $\mathrm{Pic}(\mathsf{D})$. In this example, we give an explicit formula for the double plane model (see Remark 3.2.7) of S.

1. $\mathsf{D} = \mathsf{D}_1$. We choose the coordinates such that D_1 such that $p_1 = [1,0,0]$, $p_2 = [0,1,0]$, and $p_4 = [0,0,1]$ and D is the blow-up of these points followed by the blow-up the infinitely near points $p_3 \succ p_2$ and $p_5 \succ p_4$ corresponding to the directions defined by $t_1 = 0$ and $t_2 = 0$.
   If $p \neq 2$ or $p = 2$ and the cover $S \to \mathsf{D}_1$ is inseparable, then the double plane model of S from Remark 3.2.7 is a birational model equal to the double cover of $\mathbb{P}^2$ branched along the curve $V(t_1t_2F_6)$ of degree 8 passing through the points $p_1, \ldots, p_5$ with multiplicity 2. By straightforward computations we find that
   $$F_6 \;=\; t_0^4A_1(t_1,t_2)+t_0^3t_1t_2A_2(t_1,t_2)+t_0^2t_1t_2A_3(t_1,t_2)+t_0t_1^2t_2^2A_4(t_1,t_2)+t_1^2t_2^2A_5(t_1,t_2) \;=\; 0. \tag{3.3.9}$$
   Thus, the equation of the double plane in the weighted projective space $\mathbb{P}(1,1,1,2)$ is
   $$w^2 + t_1t_2F_6(t_0,t_1,t_2) \;=\; 0. \tag{3.3.10}$$
   If $p = 2$, and the cover $S \to \mathsf{D}_1$ is separable, then the equation of the double plane is
   $$w^2 + wt_1t_2(at_0^2 + bt_0t_1 + ct_0t_2 + dt_1t_2) + t_1t_2F_6(t_0,t_1,t_2) \;=\; 0. \tag{3.3.11}$$
2. $\mathsf{D} = \mathsf{D}_2$. We choose the coordinates such that D_2 is the blow-up of points $p_1, \ldots, p_5$, where $p_1 = [1,0,0]$, $p_2 = [0,0,1]$ and $p_5 \succ p_4 \succ p_2 \succ p_2$ with the line $t_1 = 0$ passing through p_1, p_2, p_3. The equation of the double plane in affine coordinates $x = t_0/t_1$, $y = t_2/t_1$) outside the line $t_1 = 0$ is
   $$w^2 + wF_3(x,1,y) + F_6(x,1,y) \;=\; 0, \tag{3.3.12}$$
   where
   $$F_3 \;=\; a_1t_1t_2^2 + (a_2t_0^2 + a_3t_1^2)t_2 + t_1(a_4t_0^2 + a_5t_0t_1 + a_6t_1^2) = 0$$
   is the equation of a cubic curve passing through the points $p_1, \ldots, p_5$.
3. $\mathsf{D} = \mathsf{D}_3$. If the cover is inseparable, then the affine equation over the complement of the line $t_1 = 0$ is
   $$w^2 + F_6(x,1,y) \;=\; 0, \tag{3.3.13}$$
   If the cover is separable, then the affine equation is
   $$w^2 + wF_2(x,1,y) + F_6(x,1,y) \;=\; 0, \tag{3.3.14}$$
   where $F_2 = a_1t_0^2 + a_2t_0t_1 + a_3t_1t_2 + a_4t_1^2 = 0$ is the equation of a conic passing through the points p_2, p_3.

4. $\mathsf{D} = \mathsf{D}_1'$. In this case, $p_1 = [1,0,0]$, $p_4 = [0,0,1]$ and $p_3 \succ p_2 \succ p_1$ and $p_5 \succ p_4$ are infinitely near points and we find
$$F_6 = t_0^3 t_2^2 A_1(t_0,t_1) + t_0^2 t_1 t_2 A_2(t_1,t_2) + t_0 t_1^2 t_2 A_3(t_1,t_2) + t_1^2 A_4(t_1,t_2) = 0. \tag{3.3.15}$$
If $p \neq 2$ or $p = 2$ and the cover $S \to \mathsf{D}_1$ is inseparable, then the double plane model of S is given by equation (3.3.10). If $p = 2$ and the cover is separable, then the double plane model is given by the equation (3.3.11).
5. $\mathsf{D} = \mathsf{D}_2'$. The surface D is the blow-up of $p_5 \succ p_4 \succ p_3 \succ p_2 \succ p_1$, where $p_1 = [1,0,0]$ and p_2 is defined by the tangent direction $t_1 = 0$. The cover must be separable and the equation of the double plane over the complement of the
$$w^2 + wF_3(x,1,y) + F_6(x,1,y) = 0, \tag{3.3.16}$$
where $F_3(t_0,t_1,t_2) = 0$ is the equation of a cubic passing through the points $p_1,\ldots,p_5$.
6. $\mathsf{D} = \mathsf{D}_3'$. If the cover is inseparable, then the affine equation of the double plane over the complement of the line is (3.3.13). If the cover is separable, then the equation is (3.3.14), where $F_2(t_0,t_1,t_2) = 0$.

The octic curve $V(t_1t_2F_6)$ is called an *Enriques octic*. It is called *non-degenerate* (resp. *degenerate* if its equation is given by (3.3.9) (resp. (3.3.15)). The equation (3.3.10) (resp. (3.3.15)) is called the *Eniriques double plane*. (resp. *Enriques degenerate double plane*)

Remark 3.3.21 One can derive various birational models of a bielliptic map by composing it with a birational map from $\mathcal{D}$ onto a ruled surface. For example, let $c : \widetilde{\mathsf{D}}_1' \to \mathbf{F}_2$ be the birational morphism from the minimal resolution of a degenerate 4-nodal anti-canonical quartic del Pezzo surface D_1' onto the minimal ruled surface $\mathbf{F}_2$ (see Section 0.5) that blows down the curves from the divisor classes $e_4 - e_5$, e_5, $e_2 - e_3$, and e_3. The image of the (-2)-curve A_5 is the unique section of $\mathbf{F}_2 \to \mathbb{P}^1$ with self-intersection number -2 and the images of the curves A_1 and A_3 are fibers. We can define similar birational morphisms from $\widetilde{\mathsf{D}}_2'$ and $\widetilde{\mathsf{D}}_3'$ onto $\mathbf{F}_2$, as well as birational morphisms from $\widetilde{\mathsf{D}}_1$, $\widetilde{\mathsf{D}}_2$, $\widetilde{\mathsf{D}}_3$ onto $\mathbf{F}_0 = \mathbb{P}^1 \times \mathbb{P}^1$. These models are often called *Horikawa models* of an Enriques surface, see [309].

Let $\phi : S \to \mathsf{D}$ be a bielliptic map to an anti-canonical quartic del Pezzo surface. If $\mathsf{D} = \mathsf{D}_1, \mathsf{D}_2, \mathsf{D}_3$, then ϕ gives rise to two genus one fibrations $|2F_1|$, $|2F_2|$ on S with $F_1 \cdot F_2 = 1$, see Proposition 3.3.1. Otherwise, ϕ gives rise to a genus one fibration $|2F_1|$ with a bisection $R \cong \mathbb{P}^1$. Moreover, if ϕ is an inseparable map, then it is easy to see that both, $|F_1|$ and $|F_2|$ are quasi-elliptic fibrations.

Lemma 3.3.22 *Let $f : C \to \mathbb{P}^1$ be a separable degree two cover in characteristic 2, where C is a cuspidal curve of arithmetic genus one. Then, C is given by an Artin–Schreier equation*
$$F = y^2 + u^2 y + A(u,v) = 0,$$

where $A(u,v)$ is a binary form in u,v of degree 4, such that u^2 divides dA.

Proof Let $|D|$ be the linear system of degree 2 on C that defines f.

Let us first assume that there exists a nonsingular point $p \in C$ such that $2p \in |D|$. We claim that then, all divisors of $|D|$ are of the form $2q$. There exists a group law on the smooth locus of C, isomorphic to the additive group $\mathbb{G}_a$, and we may assume that p is zero with respect to this group law. In this case, any smooth point q with $2q \in |D|$ is a 2-torsion point with respect to this group law and conversely, every 2-torsion point q of the group law satisfies $2q \in |D|$. Since we are dealing with $\mathbb{G}_a$ in characteristic 2, there is a one-dimensional linear subsystem of $|D|$ consisting of divisors of the form $2q$. By Riemann–Roch, $\dim |2p| = 1$, and thus, this subsystem is actually equal to $|2p|$. This proves that every divisor in $|D|$ is of the form $2q$ and thus, the map f is inseparable, contradicting our assumptions.

So, we may assume that $|D|$ contains a divisor of the form $p+q$ with $p \neq q$ and hence the map is separable. Since $|D|$ does not contain divisors of the form $2p$, the only ramification point is the cusp. In particular, we see that the equation of C must be of the form $F = y^2 + u^2 y + A(u,v) = 0$, where $A(u,v) = \sum_{i=0}^{4} a_i u^i v^{4-i}$ is a binary form of degree 4. By taking partial derivatives, we obtain that $dA = (a_1 u^2 + a_3 v^2)(udu + vdv)$ vanishes at the point $[0,1]$. This happens if and only if $a_3 = 0$.

The condition $a_3 = 0$ is equivalent to the condition in the assertion of the lemma. □

Proposition 3.3.23 *Let $\phi : S \to \mathsf{D}$ be a separable bielliptic map. Then, at most one of the two genus one fibrations $|2F_1|$ and $|2F_2|$ is quasi-elliptic.*

Proof Assume that ϕ is separable and that the genus one fibration $|2F_1|$ is quasi-elliptic. We use the double plane model of S from Example 3.3.20. We will prove later in Theorem 4.10.3 that a $\boldsymbol{\mu}_2$-Enriques surface does not admit a quasi-elliptic pencil. So, we may assume that $\mathsf{D} = \mathsf{D}_1$ or $\mathsf{D} = \mathsf{D}_3$.

First, assume that $\mathsf{D} = \mathsf{D}_1$. The genus one pencils on S are the pre-images of the pencil of lines through the point p_1 on the plane defining the linear system $|e_0 - e_1|$ on $\widetilde{\mathsf{D}}$ and the pencil of conics through p_2, p_3, p_4, p_5 defining the linear system $\mathcal{P} = |2e_0 - e_2 - e_3 - e_4 - e_5|$. Its pre-image on S is the pencil $|2F_2|$. We will often identify these linear systems. Changing the blow-down morphism $\mathsf{D} \to \mathbb{P}^2$, we may assume that the pencil $|2F_1|$ defined by $|e_0 - e_1|$ is quasi-elliptic. By Lemma 3.3.22, a general line through p_1 is tangent to the conic $\bar{B}$ through p_2, p_4 representing on the plane the curve $B \in |2e_0 - e_2 - e_4|$ (or the conic is the double line $\langle p_2, p_4 \rangle$). Note that this condition implies that either $|2F_1|$ is quasi-elliptic or its general fiber is a supersingular elliptic curve. This means that p_1 is the *strange point*, that is, a point contained in all tangent lines of the conic $\bar{B}$. It implies that the conic is given by the equation $t_0^2 + \lambda t_1 t_2 = 0$ and hence, belongs to the pencil $\mathcal{P}$. It follows that the curve B is equal to the union of a conic from this pencil and the lines l_3, l_5 with the divisor classes e_3, e_5. Their pre-images on S are the half-fibers F_1, F_1' of $|2F_1|$. Thus, the ramification divisor of a general member F of $|F_2|$ consists of two distinct points of intersection of F with F_1, F_1'. By Lemma 3.3.22, it must be an elliptic curve.

Assume $\mathsf{D} = \mathsf{D}_3$. In this case, $Z(a)_0$ is a conic from $|2e_0 - e_2 - e_3|$. The argument is similar to the previous case. We may assume that the quasi-elliptic pencil is the pre-image of the pencil $|e_0 - e_1|$ passing through p_1. The part B of the branch curve is represented by a conic $\bar{B}$ in the plane passing through the points e_2 and e_3, that is, a conic passing through p_1 and tangent to the line $\langle p_1, p_2\rangle$. The point p_1 must be the strange point, and hence, $\bar{B}$ must be defined by the equation $t_0^2+\lambda t_1t_2+\mu t_1^2$, where we assume that $p_1 = [1,0,0]$ and $p_2 = [0,0,1]$. Let a general member C of the pencil $\mathcal{P} = |2e_0 - e_2 - e_3 - e_4 - e_5|$ be given by the equation $t_0^2 + at_0t_2 + \lambda t_1t_2 + \mu t_1^2$, which expresses the condition that it passes through p_2 and is tangent to the line $t_1 = 0$. Assume that $\bar{B}$ does not belong to this pencil, that is, $a \neq 0$. The pencil generated by the conics C and $\bar{B}$ contains the reducible conic $t_1(at_0+\alpha t_1+\beta t_2) = 0$, where $a \neq 0$. We see that $\bar{B}$ intersects this conic at three points, it is tangent to the component $t_1 = 0$ and it intersects the other component in two distinct points. This shows that $a_0 = 0$ and $\bar{B}$ belongs to the pencil $\mathcal{P}$. Thus, B intersects a general member of the pencil at two distinct points on the lines l_1 and l_2, and we finish as in the previous case. ◻

Example 3.3.24 Assume that $\mathsf{D} = \mathsf{D}_1$ and choose equation (3.3.9) to be

$$\Phi \ = \ t_3^2 + t_1t_2(t_1t_2 + t_0^2)(t_3 + q_4(t_0,t_1,t_2)) \ = \ 0,$$

where $V(q_4)$ is a quartic curve with a double point at $p_1 = [1,0,0]$ and tangent to $t_1 = 0$ and $t_2 = 0$ at the points $p_2 = [0,1,0]$ and $p_4 = [0,0,1]$. To satisfy these conditions, q_4 must be of the form

$$q_4 \ = \ t_0^2A_2(t_1,t_2) + t_0t_1t_2A_1(t_1,t_2) + A_0t_1^2t_2^2,$$

where A_i is a binary form of degree i.

In the open subset $t_1 \neq 0$, we use affine coordinates $z = t_3/t_1^4$, $x = t_2/t_1$, and $y = t_0/t_1$, to rewrite the equation of the surface in the form

$$z^2 + x(x+y^2)z + x(x+y^2)(y^2A_2(1,x) + yxA_1(1,x) + A_0x^2) \ = \ 0.$$

The pre-image of a general line $\ell = V(x+\lambda)$ through p_1 is given by the equation

$$z^2 + \lambda(\lambda+y^2)z + \lambda(\lambda+y^2)(y^2A_2(1,\lambda) + \lambda yA_1(1,\lambda) + A_0\lambda^2) \ = \ 0.$$

We see that the equation satisfies the condition of Lemma 3.3.22 and hence, the pre-image of the line is a quasi-elliptic curve.

3.4 Degree 4 Covers of the Plane

In this section, we study linear systems $|D|$ on an Enriques surface S such that D is nef with $D^2 = 4$ and $\Phi(D) = 2$. This is case (3d) of Proposition 3.1.1. In this case, the linear system $|D|$ has no base points and the associated morphism

$$\phi_{|D|} \ : \ S \ \to \ S' \ := \ \phi_{|D|}(S) \ = \ \mathbb{P}^2$$

is generically finite of degree 4, which is also denoted by ϕ_+ below. We note that in Proposition 2.6.6 we classified such linear systems in terms of genus one fibrations and nodal curves. More precisely, they are types (3)–(6) with $k = 1$. A linear system of type (3), that is, $D \sim F_1 + F_2$ with $F_1 \cdot F_2 = 2$, where the $|2F_i|$ are genus one pencils, is called *non-special* otherwise *special*. In [460], the associated invertible sheaf $\mathcal{O}_S(D)$ was called a *Cossec–Verra polarization*.

To analyze the morphism $\phi_{|D|} : S \to \mathbb{P}^2$, we first study the morphism associated to $\pi^*\mathcal{O}_S(D)$ on the K3-cover $\pi : X \to S$ in more detail and refer to Theorem 3.1.7 for the first general results. This gives also explicit equations for the K3-cover of an Enriques surfaces, which we already encountered in Example 1.6.8. If the K3-cover is smooth, then this is shown in [133] and in [460] in the general case. It refines Theorem 3.1.6 and Theorem 3.1.7 for K3-covers of Enriques surfaces.

Theorem 3.4.1 *Let S be an Enriques surface and let D be a nef divisor with $D^2 = 4$ and $\Phi(D) = 2$. Let $\pi : X \to S$ be the K3-cover of S, let*

$$\widetilde{\phi}_{|D|} \;:\; X \;\to\; \mathbb{P}^5 = \mathbb{P}(H^0(S, \mathcal{O}_S(D)))$$

be the morphism associated to $\pi^*\mathcal{O}_S(D)$ and let $X' := \widetilde{\phi}_{|D|}(X)$ be the image. Then, $X \to X'$ is a birational morphism and X' is a surface of degree 8 in $\mathbb{P}^5$ that is a complete intersection of three quadrics.

More precisely, let $G := (\mathrm{Pic}^{\tau}_{S/\Bbbk})^D$ be the Cartier dual of the torsion subgroup scheme of $\mathrm{Pic}_{S/\Bbbk}$. Then, there exists a Cartesian diagram

$$\begin{tikzcd} X \arrow[r, "\widetilde{\phi}_{|D|}"] \arrow[d, "\pi"'] & X' \arrow[r, "\subset"] \arrow[d, "\pi'"'] & \mathbb{P}^5 \\ S \arrow[r, "\psi"] & S' & \end{tikzcd}$$

such that:

1. *π and π' are G-torsors and $\widetilde{\phi}_{|D|}$ and ψ are birational morphisms.*
2. *The morphism ψ contracts precisely those curves on S that have zero-intersection with D, all of which are (-2)-curves. In particular, S' has at worst rational double point singularities.*
3. *The G-action on X extends to X' and to a linear action on the ambient $\mathbb{P}^5$. The three quadrics cutting out X' can be chosen to be individually G-invariant.*

Proof By Theorem 3.1.7, the map $\widetilde{\phi}_{|D|}$ is a morphism to $\mathbb{P}^5$, the induced morphism $X \to X' := \widetilde{\phi}_{|D|}(X)$ is birational, and X' is a surface of degree 8. By Theorem 1.3.1, the morphism π is a G-torsor. Next, we define

$$\psi \;:\; S \;\to\; S' \;:=\; \mathrm{Proj} \bigoplus_{n \geq 0} H^0(S, \mathcal{O}_S(nD)).$$

Since D is big and nef, ψ is a proper birational morphism onto a normal surface. It contracts precisely those curves that have zero-intersection with D. Such curves are (-2)-curves and thus, S' has at worst rational double point singularities. Then, $\mathcal{O}_{S'}(1)$ is an ample invertible sheaf on S' with $\psi^*\mathcal{O}_{S'}(1) \cong \mathcal{O}_S(D)$.

Since $\mathrm{Pic}^\tau_{S/\Bbbk} \cong \mathrm{Pic}^\tau_{S'/\Bbbk}$, there exists a G-torsor $\pi'' : X'' \to S'$, such that the G-torsor $X \to S$ arises as pull-back from $X'' \to S'$ via $S \to S'$. Moreover, $\pi''^*\mathcal{O}_{S'}(1)$ is an ample invertible sheaf and $\widetilde{\phi}_{|D|}$ factors as $X \to X'' \to X' \to \mathbb{P}^5$. Clearly, $X \to X'' \to X'$ are birational morphisms, and it is not difficult to see that $X'' \to X'$ is actually an isomorphism.

Since $\mathcal{O}_S(D)$ is an invertible sheaf on S, there is a G-action on $\pi^*\mathcal{O}_S(D)$ and thus, a linear G-action on $H^0(X, \pi^*\mathcal{O}_S(D))$. Therefore, the G-action on X extends to X' and to a linear G-action on the ambient $\mathbb{P}^5$.

As in (3.1.3) in the proof of Theorem 3.1.7, we have a short exact sequence

$$0 \to H^0(S, \mathcal{O}_S(D)) \to H^0(X, \pi_*\pi^*\mathcal{O}_S(D)) \to H^0(S, \omega_S(D)) \to 0\,. \quad (3.4.1)$$

We have an induced G-action on the function field $\Bbbk(X)$ with invariants $\Bbbk(S)$ and consider the space $H^0(X, \pi_*\pi^*\mathcal{O}_S(D))$ as a $\Bbbk$-sub-vector space of $\Bbbk(X)$. From this, we conclude that the G-action on $H^0(X, \pi_*\pi^*\mathcal{O}_S(D))$ is a direct sum of three copies of the 2-dimensional regular representation ρ of G over $\Bbbk$. The G-action restricts to a trivial representation on $H^0(S, \mathcal{O}_S(D))$. In characteristic $p \neq 2$, the G-action on $H^0(S, \omega_S(D))$ is via the sign-involution.

From this, we obtain a short exact sequence of G-representations

$$0 \to \mathrm{Ker}\,\mu \to S^2H^0(X, \pi^*\mathcal{O}_S(D)) \xrightarrow{\mu} H^0(X, \pi^*\mathcal{O}_S(D)^{\otimes 2}) \to 0\,. \quad (3.4.2)$$

It is easy to compute that $\mathrm{Ker}\,\mu$ is 3-dimensional. Working out the G-representations, it is not difficult to see that the G-action on $\mathrm{Ker}\,\mu$ is trivial. This implies that the quadrics of $\mathbb{P}^5$ that contain X' form a 3-dimensional space and that all these quadrics are G-invariant.

It remains to show that X' is actually a complete intersection of three quadrics. Here, we will only treat the case where $p \neq 2$ or where $p = 2$ and that X is a $\boldsymbol{\mu}_2$-surface. We refer to [460, Proposition 2.4] for the two remaining cases in characteristic 2. By Theorem 3.1.6, $X' \subset \mathbb{P}^5$ is cut out by quadrics and cubics. Moreover, if cubics are needed, then there are two possible cases. We set $\widetilde{D} := \pi^*D$ and note that we have $\widetilde{D}^2 = 8$. Case (1) is impossible, since we have $|\widetilde{D}| = |2C + R|$ with $C^2 = 2$, $R^2 = -2$, and $C \cdot R = 1$, which implies $\widetilde{D}^2 = 10$. In Case (2), there exists a genus one fibration $|P|$ on X with $P \cdot \widetilde{D} = 3$. Let $\tau : X \to X$ be the covering involution of $\pi : X \to S$. Using the Hodge Index Theorem, we find $\widetilde{D}^2 \cdot (P \cdot \tau(P)) \leq 9$. This implies $|P| = |\tau(P)|$ or $P \cdot \tau(P) = 2$. If $|P| = |\tau(P)|$, then there exists a genus one fibration $|F| = |2F'|$ on S such that $|P| = |\pi^{-1}(D)|$ and we find $2F' \cdot D = F \cdot D = 3$, which is impossible. Thus, we may assume $P \cdot \tau(P) = 2$. We compute $(\widetilde{D} - P - \tau(P))^2 = 0$ and so, Riemann–Roch shows that the moving part $|M|$ of $|\widetilde{D} - P - \tau(P)|$ is non-empty. Since $M \cdot \widetilde{D} \leq 2$, the Hodge Index Theorem implies that $|M|$ is a genus one pencil. If we had $|M| = |P|$, then we would find $|\widetilde{D}| = |2P + \tau(P)|$ since $\dim|\widetilde{D}| = \dim|2P + \tau(P)|$, hence $P \cdot \widetilde{D} = P \cdot (2P + \tau(P)) = 2$, a contradiction. Similarly, one shows that $|\widetilde{D}|$ is distinct from $|\tau(P)|$. From this, we obtain

$$5 \;=\; \dim|\widetilde{D}| \;\geq\; \dim|M+P+\tau(P)| \;\geq\; 1 \;+\; \frac{1}{2}(M+P+\tau(P))^2 \;=\; 7,$$

a contradiction. Thus, also Case (2) does not occur and thus, X' is cut out by quadrics in $\mathbb{P}^5$. Since $\operatorname{Ker}\mu$ is 3-dimensional, it follows that X' is a complete intersection of quadrics. □

As an application, we obtain the following result:

Corollary 3.4.2 *Let S be an Enriques surface with K3-cover $\pi : X \to S$. Then, S admits a Cossec–Verra polarization. In particular, X is birationally equivalent to a complete intersection of three quadrics in $\mathbb{P}^5$.*

Proof It suffices to show that S admits a Cossec–Verra polarization: in fact, these correspond to the fundamental weight $\boldsymbol{\omega}_1$ in the Enriques lattice $\mathsf{E}_{10} \cong \operatorname{Num}(S)$. Applying Proposition 2.2.1, we may assume that it is represented by a nef divisor class. We have seen in the proof of Proposition 2.4.11 that $\Phi(\boldsymbol{\omega}_1) = 2$, that is, every invertible sheaf with class $\boldsymbol{\omega}_1$ corresponds to a Cossec–Verra polarization. □

In fact, one can be very explicit about the G-action on $\mathbb{P}^5$ and the G-invariant quadrics: the group scheme G in Theorem 3.4.1 is of length two, and thus, described by Example 1.6.6. Moreover, in the same lemma, we described the regular representation $\rho : G \to \mathrm{GL}_2$, which yields a G-action on the polynomial ring $\Bbbk[x, y]$. Taking three copies of ρ, we obtain a G-action on $\Bbbk[x_0, x_1, x_2, y_0, y_1, y_2]$, and thus, on $\mathbb{P}^5$. Now, we set

$$E_+ \;:=\; H^0(S, \mathcal{O}_S(D)), \quad E_- \;:=\; H^0(S, \omega_S(D)), \quad \text{and} \quad E \;:=\; H^0(X, \pi^*\mathcal{O}_S(D)),$$

which are $\Bbbk$-vector spaces of dimension 3, 3, and 6, respectively. By (3.4.1), we obtain an extension of $\Bbbk$-vector spaces

$$0 \;\to\; E_+ \;\to\; E \;\to\; E_- \;\to\; 0$$

The G-action on E restricts to a trivial G-action on E_+ and there is an induced G-action on the quotient $E_- \cong E/E_+$. In characteristic $p \neq 2$, the non-trivial element of G acts as $-\mathrm{id}$, so one can canonically split this sequence as $E \cong E_+ \oplus E_-$ in such a way that is compatible with the G-action – we note that this is not possible if $p = 2$. In any case, the proof shows that the G-action on $\mathbb{P}^5$ stated in Theorem 3.4.1 is of this form. The space of G-invariant quadrics have been computed by Bombieri in Mumford [77, page 222], see also Lemma 1.6.7. Thus, the previous corollary shows that the examples given in Example 1.6.8 yield in fact all Enriques surfaces:

Corollary 3.4.3 *Let S be an Enriques surface. Then, after possibly contracting some (-2)-curves to rational double points $S \to S'$, the surface S' arises via the Bombieri–Mumford–Reid construction given in Example 1.6.8.*

Remark 3.4.4 Assume that the characteristic is $p \neq 2$.

1. First, the Bombieri–Mumford–Reid construction simplifies as follows: in this case, $G \cong \mathbb{Z}/2\mathbb{Z} \cong \boldsymbol{\mu}_2$, the G-action on $\Bbbk[x_0, x_1, x_2, y_0, y_1, y_2]$ is given by the involution
$$\tau : x_i \mapsto x_i \quad \text{and} \quad y_i \mapsto -y_i,$$
and the τ-invariant quadrics are
$$x_i x_j \quad \text{and} \quad y_i y_j,$$
for $0 \leq i \leq j \leq 2$. In particular, the image $X' \subset \mathbb{P}^5$ of the K3-cover is the complete intersection of three quadrics $\tilde{A}_0, \tilde{A}_1, \tilde{A}_2$ that are of the form
$$\tilde{A}_r = \left\{ \sum_{i,j} a_{ij}^{(r)} x_i x_j + \sum_{i,j} b_{ij}^{(r)} y_i y_j = 0 \right\} \subset \mathbb{P}^5 .$$
Without loss of generality, we may assume that the $A_r := (a_{ij}^r)$ and $B_r := (b_{ij}^r)$ are symmetric 3×3 matrices.
2. Consider the three reducible quadrics
$$\tilde{A}_i := \{x_i^2 - y_i^2 = 0\} = \{x_i - y_i = 0\} \cup \{x_i + y_i = 0\} \subset \mathbb{P}^5,$$
for $0 \leq i \leq 2$, each of which is the union of two hyperplanes. We note that τ interchanges all these pairs of connected components. The complete intersection $X_\infty := Q_1 \cap Q_2 \cap Q_3$ is a reducible surface of degree 8 in $\mathbb{P}^5$ that is a union of 8 planes, which intersect along 12 lines, and the 12 lines intersect in 6 points. Thus, these planes form an octahedron upon which τ acts.
Given a smooth complete intersection X of three τ-invariant quadrics in $\mathbb{P}^5$, one can easily find a one-parameter family deforming them into X_∞ keeping the involution τ. This is an example of a *type III degeneration* of a K3 surface. The polyhedron associated to this degeneration is a topological 2-sphere $\mathbb{S}^2$ and these types of degenerations of K3 surfaces are characterized by maximal order of nilpotence of the monodromy operator on the second cohomology. We will discuss all of this in Section 5.10. By assumption, we can form the quotient by τ in this family and obtain an example of a type III degeneration of an Enriques surface. We will discuss this later, in Section 5.10. Here, the polyhedron associated to the degeneration is the real projective plane $\mathbb{RP}^2$, which arises as quotient of $\mathbb{S}^2$ by the involution induced by τ. We refer to [421, 591] for details.
After a linear change of variables, one may assume that the Q_i are of the form $z_i w_i$ for some coordinates z_i, w_i of the projective space $\mathbb{P}^5$, that is, the above degeneration X_∞ is an example of a monomial degeneration that is compatible with the structure of $\mathbb{P}^5$ as a toric variety. Being a degeneration of maximal unipotent monodromy, such types of degenerations play an important role in some aspects of mirror symmetry [139]. From these, one can pass to the tropicalizations of K3 surfaces and Enriques surfaces, see [75].

We now turn to the morphisms

$$\phi_+ : S \to \mathbb{P}(E_+) = \mathbb{P}(H^0(S, \mathcal{O}_S(D))),$$
$$\phi_- : S \to \mathbb{P}(E_-) = \mathbb{P}(H^0(S, \omega_S(D))),$$

both of which are generically finite of degree 4 onto $\mathbb{P}^2$. In the terminology introduced earlier, we have $\phi_+ = \phi_{|D|}$ and $\phi_- = \phi_{|K_S+D|}$ and it turns out to be useful to study both morphisms at the same time via the morphism $\widetilde{\phi}_{|D|}$ from the K3-cover X. We note that both morphisms $\phi_\pm$ factor over $S \to S'$, where S' is the contraction of the nodal cycle Z_D formed by all (-2)-curves that have zero-intersection with D. The induced morphisms $\phi'_\pm : S' \to \mathbb{P}(E_\pm)$ are finite.

Let us now assume that the characteristic is $p \neq 2$. Then, $G \cong \mathbb{Z}/2\mathbb{Z} \cong \boldsymbol{\mu}_2$ and we denote by $\tau \in G$ the non-trivial element, which is the Enriques involution on X. As seen above, the τ-action decomposes E into the direct sum $E_+ \oplus E_-$ of $\pm$id-eigenspaces. The inclusions $E_\pm \subset E$ give rise to embeddings $\Lambda_\pm := \mathbb{P}(E_\pm) \subset \mathbb{P}(E)$ and to rational maps $p_\pm : \mathbb{P}(E) \dashrightarrow \mathbb{P}(E_\pm)$. Using the notations of Theorem 3.4.1, we thus obtain a factorization

$$\begin{tikzcd}
X \arrow[r, "\widetilde{\phi}_{|D|}"] \arrow[d, "\pi"'] & X' \arrow[r, "\subset"] \arrow[d, "\pi'"'] & \mathbb{P}^5 \arrow[d, dashed, "p_\pm"'] \\
S \arrow[r, "\psi"] & S' \arrow[r, "\phi'_\pm"] & \mathbb{P}(E_\pm)
\end{tikzcd}$$

with $\phi_\pm = \phi'_\pm \circ \psi$. As seen in Theorem 3.4.1, the map $\pi' : X' \to S'$ is a G-torsor, that is, the fixed-point free Enriques involution τ induces a fixed-point free involution τ' on X'.

The equations cutting out X are very special. More precisely, we have the following result.

Theorem 3.4.5 *Let S be an Enriques surface in characteristic $p \neq 2$, let D be a nef divisor with $D^2 = 4$ and $\Phi(D) = 2$, let $\pi : X \to S$ be the K3-cover, and let $\tau : X \to X$ be the covering involution. Let $E = E_+ \oplus E_-$ and $\widetilde{\phi}_{|D|} : X \to X' \subset \mathbb{P}(E)$ be as above. Then, there exist 3-dimensional linear subspaces $N_\pm$ of $S^2E_\pm$ and an isomorphism $\gamma : N_+ \to N_-$ such that*

$$X' = \bigcap_{q \in N_+} V\left(q + \gamma(q)\right) \subset \mathbb{P}(E). \tag{3.4.3}$$

Proof The covering involution $\tau : X \to X$ decomposes $E = H^0(X, \pi^*\mathcal{O}_S(D))$ into $\pm$id-eigenspaces $E = E_+ \oplus E_-$ of the induced τ-action. In particular, we obtain a τ-action on $\mathbb{P}(E)$ and the induced τ-action on $\Lambda_\pm = \mathbb{P}(E_\pm)$ is trivial.

Let N be the kernel of the restriction map

$$S^2E = H^0\left(\mathbb{P}(E), \mathcal{O}_{\mathbb{P}(E)}(2)\right) \to H^0\left(X', \mathcal{O}_{X'}(2)\right),$$

which defines the base locus of the net of quadrics in $\mathbb{P}(E)$ vanishing on $X' = \widetilde{\phi}_{|D|}(X)$, see also (3.4.2). The τ-action on E induces an action on S^2E, which decomposes into two eigensubspaces of dimensions 12 and 9 with eigenvalues 1 and -1, respectively. More precisely, we can decompose it as

$$S^2E \;=\; (S^2E)_1 \;\oplus\; (S^2E)_{-1} \;=\; \left(S^2E_+ \oplus S^2E_-\right) \;\oplus\; (E_+ \otimes E_-)\,.$$

Next, we choose coordinates x_0, x_1, x_2 in E_+ and coordinates y_0, y_1, y_2 in E_-. Then, the quadrics $x_i^2, x_ix_j,\; y_i^2, y_iy_i$ are a basis of $(S^2E)_+$ and the x_iy_j are a basis of $(S^2E_-)_{-1}$. We know that N is contained in either $(S^2E)_1$ or $(S^2E)_{-1}$, see the discussion around (3.4.2). If it is contained in $(S^2E)_{-1}$, the base-locus contains the planes $x_0 = x_1 = x_2 = 0$ and $y_0 = y_1 = y_2 = 0$, and hence, it is reducible. So we obtain that $N \subset (S^2E)_1$. Thus, we can find an injective map $N \to S^2E_+ \oplus S^2E_-$ such that its composition of with the projections onto each factor is injective (otherwise, $|N|$ contains a quadric with singular locus of dimension ≥ 2, in which case the base scheme X has a singular point on it). Let $N_\pm$ be the images of the compositions $\gamma_\pm : N \to N_\pm$. We define γ to be $\gamma_- \circ \gamma_+^{-1}$ and the assertion follows. □

For the future use, let us note the following.

Proposition 3.4.6 *Suppose that $\widetilde{\phi}_D : X \to X'$ is an isomorphism, or, equivalently, that the surface X' given by the equations* (3.4.3) *is smooth. Then, the two nets of conics $|N_\pm|$ have no base points.*

Proof Seeking a contradiction, assume that one of the nets $|N_\pm|$ has a base point, say, $x = [v]$ with $v \in E_\pm^\vee$. Without loss of generality, we may assume that $v \in E_+$. Thus, there exists a conic $V(q)$ in the net that has a singular point at x. Then, the point $[(v, 0)] \in \mathbb{P}(E) = \mathbb{P}(E_+ \oplus E_-)$ is a singular point of all quadrics $V(q + \gamma(q))$. Thus, X' contains a base point of the net $|N|$ of quadrics with base locus X'. Since $\mathbb{P}(E_\pm)$ consists of fixed points of the involution τ' and τ' acts freely on X', we obtain a contradiction. □

Consider the Veronese maps

$$\mathsf{v}_\pm \;:\; E_\pm \;\to\; S^2E_\pm \;\subset\; H^0(S, \mathcal{O}_S(2)),\; s \mapsto s^2 \;:=\; s \otimes s.$$

Both spaces $S^2E_\pm$ are 6-dimensional linear subspaces of $H^0(S, \mathcal{O}_S(2))$, which is 9-dimensional. Therefore, their intersection is of dimension ≥ 3 and we can choose a 3-dimensional subspace L of this intersection. Considering it as a subspace of $S^2E_\pm$, it defines a 3-dimensional linear space of quadratic forms on $E_\pm^\vee$ and thus, a net $N'_\pm$ of conics in $\mathbb{P}(E_\pm)$. Moreover, L defines a canonical bijection $\gamma' : N'_+ \to N'_-$. The so-constructed triple (N'_+, N'_-, γ') is similar to the triple (N_+, N_-, γ) from Theorem 4.7.2. In particular, it allows us to introduce a 3-dimensional linear space N' of quadratic forms in $E^\vee$ and a K3-like complete intersection of three quadrics

$$Y \;:=\; \bigcap_{q \in N'_+} V\left(q + \gamma'(q)\right). \tag{3.4.4}$$

Of course, one expects that $(N_+, N_-, \gamma) = (N'_+, N'_-, \gamma')$ and thus, $Y = X'$. This is indeed true and we will see this in Remark 3.4.9 below.

The maps $\phi_\pm : S \to \mathbb{P}(E_\pm)$ coincide with $\phi_{|D|}$ and $\phi_{|D+K_S|}$, respectively, and thus, the pre-images of lines are divisors in $|D|$ and $|D + K_S|$, respectively. Thus, the pre-images of double lines are divisors of type $2C, 2C' \in |2D|$, where $C \in |D|$, or $C' \in |D + K_S|$. Using the definition of the Veronese map, we identify $|L|$ with a non-complete linear subsystem of $|2D|$.

Proposition 3.4.7 *Consider the maps $\alpha_\pm : \mathbb{P}(E_\pm) \to \mathbb{P}(L)$ given by the linear system of conics $|N'_\pm|$ identified with $|L|$. Then, the map $\phi_+ \times \phi_- : S \to \mathbb{P}(E_+) \times \mathbb{P}(E_-)$ is a morphism that is birational map on its image. This image coincides with the fiber product of the maps α_+ and α_-. In other words, the commutative diagram*

$$\begin{tikzcd} S \arrow[r, "\phi_+"] \arrow[d, "\phi_-"] & \mathbb{P}(E_+) \arrow[d, "\alpha_+"] \\ \mathbb{P}(E_-) \arrow[r, "\alpha_-"] & \mathbb{P}(L) \end{tikzcd}$$

is a Cartesian square.

Proof The composition of the map $\phi_+ \times \phi_-$ and the Segre map $\mathbb{P}(E_+) \times \mathbb{P}(E_-) \to \mathbb{P}(E_+ \otimes E_-)$ is defined by the linear system $|2D + K_S|$. Since $\Phi(D) = 2$, we find $\Phi(2D + K_S) = 4$ and thus, by Theorem 2.4.16, the composition is a morphism that is birational onto its image. Let $\phi : S \to |L^\vee| = \mathbb{P}(L)$ be the map given by a non-complete linear system $|L| \subset |2D|$. It follows from the definitions above that $\phi = \alpha_+ \circ \phi_+ = \alpha_- \circ \phi_-$. Thus, the diagram in the assertion is commutative. The verification that the square is Cartesian is straightforward and we leave it to the reader. □

Next, consider the variety of secant lines $\ell_s \subset \mathbb{P}(E)$ with $s \in S$, joining pairs of points $(x', \tau'(x'))$ on X'. Obviously, it is birationally equivalent to S. Since each line is invariant with respect to the involution τ', there are two fixed points of τ' on ℓ_s. Since X' does not intersect $\mathbb{P}(E_\pm)$, one fixed point must be in $\mathbb{P}(E_+)$ and another one in $\mathbb{P}(E_-)$. This gives rise to a map

$$\iota : S \to \mathbb{P}(E_+) \times \mathbb{P}(E_-),\ s \mapsto (\ell_s \cap \mathbb{P}(E_+),\ \ell_s \cap \mathbb{P}(E_-)) \tag{3.4.5}$$

that factors over $S \to S'$. We have the following relation of ι relation to the maps in Proposition 3.4.7, the K3-cover $X \to S$ and the linear projections $p_\pm : \mathbb{P}(E) \dashrightarrow \mathbb{P}(E_\pm)$.

Proposition 3.4.8 *The morphism ι coincides with $\phi_+ \times \phi_-$. Moreover, the composition*

$$X \xrightarrow{\widetilde{\phi}_{|D|}} \mathbb{P}(E) \overset{p_+ \times p_-}{\dashrightarrow} \mathbb{P}(E_+) \times \mathbb{P}(E_-)$$

factors through the projection $\pi : X \to S$ and the induced map $S \to \mathbb{P}(E_+) \times \mathbb{P}(E_-)$ also coincides with ι and $\phi_+ \times \phi_-$.

Proof If $a \in \mathbb{P}(E_-)$, then $(p_+ \circ \widetilde{\phi}_{|D|})^{-1}(a)$ is equal to the intersection of the 3-dimensional subspace P_a spanned by $\mathbb{P}(E_-)$ and a with X. Let L_a be the pencil of quadrics in $|N|$ vanishing in the point a. For every point $x \in p_+^{-1}(a)$ every quadric in L_a vanishes in x and a. Since E_+ and E_- are orthogonal with respect to all polar symmetric bilinear forms associated with quadratic forms on N, we conclude that all quadrics in L_a contain the line $\overline{x,a}$. In fact, it follows from the equations of X that the restriction of $Q \in L_a$ to P_a is the quadratic cone with vertex at a and its intersection with the plane $\mathbb{P}(E_-)$ is a conic from the net N_-. Thus, we can identify L_a with a pencil of conics in N_-. Similarly, we can project x to a point $b \in \mathbb{P}(E_-)$ from $\mathbb{P}(E_+)$ and conclude that the line $\overline{x,b}$ is contained in L_b. The map $p_+ \times p_- : X \to \mathbb{P}(E_+) \times \mathbb{P}(E_-), x \mapsto (a,b)$ is obviously invariant with respect to τ and thus, its fiber over (a,b) contains the two points $\pi^{-1}(\pi(x))$. Hence, the line joining these points must coincide with the lines $\overline{x,a}$ and $\overline{x,b}$. This shows that the map ι coincides with the map (p_+, p_-). Also, our identification of L_a (resp. L_b) with the pencil of conics in N_+ (resp. (N_-)) with base point a (resp. b) shows that ι coincides with the map $\phi_+ \times \phi_-$. □

Remark 3.4.9 It follows from the proof and Proposition 3.4.7 that the triples (N_+, N_-, γ) and (N'_+, N'_-, γ') coincide and thus, the surface Y from (3.4.4) is equal to X'.

We now come back to the nets $N_\pm$ of conics in $\mathbb{P}(E_\pm)$. Quite generally, the discriminant curve Δ of a base-point-free net of conics is a plane cubic curve. The classification of such nets and their discriminant curves can be found in Example 7.2.10 in Volume II: up to projective equivalence, there are four families. The curve Δ is nonsingular if and only if the net does not contain double lines, that is, the net of conics is regular. This follows from this classification or from the description of the tangent space of the discriminant variety of quadrics in $\mathbb{P}^n$, see, for example [177, Example 1.2.3]. Quite generally, if C is a nonsingular (resp. one-nodal) plane cubic curve, then it follows from Plücker's formulas (see, for example, [177, Section 1.2.3]) that the dual curve C^* is a curve of degree 6 (resp. 4) with 9 (resp. 3) ordinary cusps. Also, if C is two-nodal, then C^* is nonsingular cubic curve and if C is three-nodal, then C^* is a set of three points. In any case, we define C^{**} to be the union of C^* with the double lines corresponding to the pencils of lines through the singular points. Then, C^{**} will be a plane curve of degree 6, which is non-reduced if C is singular.

Theorem 3.4.10 *Let S be an Enriques surface in characteristic $p \neq 2$ and let D be a nef divisor with $D^2 = 4$ and $\Phi(D) = 2$. Then, the maps*

$$\phi_{|D|} : S \to \mathbb{P}(E_+) \quad \textit{and} \quad \phi_{|D+K_S|} : S \to \mathbb{P}(E_-)$$

are morphisms that factor through the birational map $S \to S'$ that blows down the nodal cycle Z_D (the curves of S that have intersection number zero with D) and a finite morphisms of degree 4

$$\phi_\pm : S' \to \mathbb{P}(E_\pm).$$

The branch locus of $\phi_\pm$ is equal to the pre-image of the curve $\Delta_\pm^{**}$, where $\Delta_\pm$ is the discriminant curve of the net of conics $|N_\pm|$ under the map $|\mathbb{P}(E_\pm)| \to |N_\pm|$ given by the net. If this cubic curve is nonsingular, it is a curve of degree 12*. In general, it has* 36 *cusps lying on a plane sextic.*

Proof It follows from Proposition 3.4.7 that the branch curve of the map $\phi_\pm : S \to \mathbb{P}(E_\pm)$ is the pre-image of the branch curve of $\alpha_\mp$ on $\mathbb{P}(E_\pm)$. The discriminant curve $\Delta_\pm$ of the net of conics $|N_\pm|$ parametrizes singular conics of the net. A conic in the net is the pre-image of a line in $\mathbb{P}(E_\pm)$. It is singular if and only if it is tangent to $\Delta_\pm$ or passes through a singular point. Thus, the branch curve of $\alpha_\pm$ coincides with $\Delta_\pm^{**}$. Assume that $\Delta_\pm$ is nonsingular. Then, $\Delta_\pm^{**} = \Delta_\pm^{*}$ is a curve of degree 6 with 9 cusps. Since the pre-image of a line in $\mathbb{P}(L)$ is a conic in $\mathbb{P}(E_\pm)$, the degree of the branch curve is equal to 12. The pre-image of a cusp of $\Delta_\pm^{*}$ consists of 4 cusps, unless they are on the branch locus of $\alpha_\mp$. The 9 cusps of a nonsingular cubic lie on a cubic curve. Its pre-image under $\alpha_\mp$ is a curve of degree 6. ◻

Remark 3.4.11 An intersection point of the plane cubics Δ_+ and Δ_- corresponds to a quadric Q that is the join of a singular conic in $\mathbb{P}(E_+)$ and $\mathbb{P}(E_-)$, hence it has rank ≤ 4. Since the base locus of the net is irreducible, this easily implies that Δ_+ and Δ_- have no common irreducible components and hence, intersect in 9 points, the base points of the pencil of cubic curves spanned by Δ_+ and Δ_- (some of them could be infinitely near base points). We also see that all base points are simple base points since otherwise, we find a reducible quadric in the net.

The quadric Q contains two pencils of 3-dimensional subspaces and the restriction of the net of quadrics $\mathcal{N}$ to such a 3-dimension subspace is a quartic curve of genus one. This defines two genus one pencils $|F_1|$ and $|F_2|$ on X, which intersect in 4 points. Moreover, $F_1 + F_2$ is the divisor class of a hyperplane section. Both pencils are invariant under the involution τ and give rise to genus one pencils $|2G_1|$ and $|2G_2|$ on S. These satisfy $G_1 \cdot G_2 = 2$ and $|F_i|$ is the pre-image of $|2G_i|$. This shows that a choice of an intersection point q of Δ_+ and Δ_- defines a representative of the divisor class D as a sum of two genus one curves intersecting with multiplicity 2, compare also Proposition 2.6.6, case (3) with $k = 1$.

Remark 3.4.12 Let Y' be the double cover of the plane $|N|$ branched along the discriminant curve $\Delta = \Delta_+ \cup \Delta_-$ and let $Y \to Y'$ be the minimal resolution of singularities. By the previous remark, all singular points of Δ are simple singular points, hence a minimal resolution Y of Y' is a K3 surface that is isomorphic to the moduli space of rank 2 simple sheaves on X with Mukai vector $(2, c_1(\mathcal{O}_X(1)), 2)$, see [526, Example 0.9]. This is an example of a *Fourier–Mukai transform* between X and Y. Moreover, if X contains a smooth and rational curve, then X is isomorphic to Y, see [177, Lemma 10.3.1].

Finally, we assume that the characteristic of the ground field is $p = 2$ and that S is a $\boldsymbol{\mu}_2$-surface. In particular, the K3-cover $\pi : X \to S$ is étale, X is a K3 surface, and there exists an involution τ on X with quotient S. We keep the previous notations, but since $\omega_S \cong \mathcal{O}_S$, the maps ϕ_+ and ϕ_- coincide and we denote them simply by ϕ. We set $E_+ := \pi^*(H^0(S, \mathcal{O}_S(D))) \subset E = H^0(X, \pi^*\mathcal{O}_X(D))$ and note that the target of the map ϕ is the space $\mathbb{P}(E_+)$. The map $\widetilde{\phi} : X \to X' \subset \mathbb{P}(E)$ is birational onto its image and the composition $\phi \circ \pi : X \to \mathbb{P}(E_+)$ is the projection from $|E_+^\perp|$.

The involution τ on X extends uniquely to a linear involution $\tilde{\tau}$ of E, which has E_+ as its invariant part. Thus, the fixed locus of τ in the space $\mathbb{P}(E)$ where X' lies is equal to one plane $\mathbb{P}(E_+^\perp)$. Since τ extends to a fixed-point free involution τ' on X', we conclude $\mathbb{P}(E_+^\perp) \cap X' = \emptyset$. The Jordan normal form of the linear involution $\tilde{\tau}$ consists of three Jordan blocks of size 2×2. We can thus choose coordinates $(x_0, x_1, x_2, y_0, y_1, y_2)$ in E such that τ is given as follows:

$$\tau \;:\; (x_0, x_1, x_2, y_0, y_1, y_2) \;\mapsto\; (x_0, x_1, x_2, x_0 + y_0, x_1 + y_1, x_2 + y_2).$$

The space of τ-invariant quadratic forms is spanned by

$$x_i^2,\; x_i x_j,\; y_i^2 + y_i x_i,\; x_i y_j + x_j y_i,$$

see also Lemma 1.6.7. Thus, the net N of quadrics vanishing on X is generated by three quadrics $V(q_k)$, $k = 0, 1, 2$, which are of the form

$$q_k \;=\; q_k'(x_0, x_1, x_2) \;+\; \sum_{i=0}^{2} \alpha_i^{(k)} (y_i^2 + y_i x_i) \;+\; \sum_{0 \le i < j \le 2} \beta_{ij}^{(k)} (x_i y_j + x_j y_i) \;=\; 0\,.$$

Its restriction to $|E_+^\perp|$ is the net of conics generated by the three conics $V(q_k')$, $k = 0, 1, 2$.

Let $a \in \mathbb{P}(E_+)$ and let $P(a)$ be the pre-image of a under the projection $\mathbb{P}(E) \to \mathbb{P}(E_+)$. Then, the fiber of the map $X \to \mathbb{P}(E_+)$ is equal to the intersection $X \cap P(a)$. The restriction of $|N|$ to $P(a)$ can be written in the form

$$Q(t; a) \;=\; A(t; a) z^2 \;+\; \sum_{i=0}^{2} \Big(B_i(t) y_i^2 \;+\; C_i(t; a) y_i z \Big) \;=\; 0,$$

where $A(t; a)$ is a bihomogeneous form of degree 1 in t and degree 2 in a, where the $B_i(t)$ are linear forms in t, and where the C_i are bilinear forms in t, a. The base locus of $Q(t; a)$ is not reduced if and only if there exists a singular quadric with singular point inside the base locus. The quadric $Q(t; a)$ is singular if and only if the partial derivatives

$$\frac{\partial Q}{\partial y_i} \;=\; C_i(t; a) z \quad \text{and} \quad \frac{\partial Q}{\partial z} \;=\; C_i(t; a) y_i$$

have a common zero at a point satisfying $Q(t; a) = 0$. Since $X' \cap \mathbb{P}(E_+) = \emptyset$, we are interested only in singular points with $z \neq 0$. Such a point must satisfy $C_i(t; a) = 0$. Its singular locus is the plane $\sqrt{A(t; a)} z + \sum_{i=0}^{2} \sqrt{B_i} y_i = 0$. The quadrics of the net $|N|$ restrict to a pencil of conics in this plane and hence, they always have a common point. This shows that the branch locus of ϕ consists of the points $a \in \mathbb{P}(V_+)$ such that the three lines $C_i(t; a) = 0$ in the plane $|N|$ are concurrent. Since the $C_i(t; a)$ depend linearly on t, the branch locus is either a plane cubic curve or the whole plane. The latter happens if and only if the map ϕ is inseparable.

3.5 Birational Maps

Let D be a nef and big divisor with $D^2 = 2n$ on an Enriques surface S and let $\phi_{|D|} : S \dashrightarrow \mathbb{P}^n$ be the associated rational map. We classified the possibilities for the degree of $\phi_{|D|}$ (if finite) and the image $S' = \phi_{|D|}(S)$ in Proposition 3.1.1. In the previous sections, we studied the case where $\phi_{|D|}$ is generically finite of degree greater than 1. In this section, we will assume that $\deg \phi_{|D|} = 1$, that is, $\phi_{|D|}$ defines a birational map from S onto its image S'. Of course, this is the case for almost all polarizations. We will, therefore, only discuss three cases in some detail:

1. Enriques' sextic model $S' \subset \mathbb{P}^3$, which is non-normal,
2. the Fano model $S' \subset \mathbb{P}^5$, and
3. the Mukai model $S' \subset \mathbb{P}^9$.

Quite generally if $\phi_{|D|} : S \dashrightarrow S'$ is a birational (possibly rational) map, then we have $\Phi(D) \geq 2$ by Corollary 2.6.5 and then, by Theorem 2.4.14, the linear system $|D|$ has no base points, that is, $\phi_{|D|}$ is a morphism.

Let us recall from Theorem 2.4.16 and Proposition 3.1.1 that if D is a big and nef divisor with $\Phi(D) \geq 3$, then $D^2 = 2n \geq 10$ and $\phi_{|D|}$ defines a birational morphism $S \to S' \subset \mathbb{P}^n$. More precisely, S' is a normal surface with at worst rational double point singularities and $S \to S'$ is equal to the contraction morphism of all (-2)-curves on S that have zero-intersection with D.

If D is a big and nef divisor with $\Phi(D) = 2$, then $D = 2n \geq 4$ and if $\phi_{|D|}$ is birational onto its image in $\mathbb{P}^n$, then $n \geq 3$, that is, we have $D^2 \geq 6$. Since the image $S' = \phi_{|D|}(S)$ may not be a normal surface, we let $\bar{S}$ be the normalization of S' and then, $S \to S'$ factors through $\bar{S}$. If $D^2 \leq 8$, then it follows from the description of linear systems in Section 2.6 that the only cases where $\phi_{|D|}$ could be birational onto its image are the cases where $|D| = |D' + K_S|$, where $|D'|$ is a bielliptic linear system and $K_S \neq 0$. Then, $\phi_{|D|}$ is birational onto a non-normal surface of degree 6 (resp. 8) in $\mathbb{P}^3$ (resp. $\mathbb{P}^4$) if $D^2 = 6$ (resp. $D^2 = 8$).

Let us begin with the case $D^2 = 6$. This leads to Enriques's original construction of an Enriques surface as a non-normal sextic surface in $\mathbb{P}^3$ passing doubly through the edges of the coordinate tetrahedron, see also Example 1.6.2. We note that in Section 1.6, we also encountered a birationally equivalent model: using a suitable Cremona transformation, Enriques' examples can be transformed into non-normal quintic surfaces in $\mathbb{P}^3$, see Example 1.6.4. To describe the non-normal sextic models in detail, we will use the notation from Proposition 2.6.7, where we described nef divisors D with $D^2 = 6$. We start with the generic case.

Theorem 3.5.1 *Let S be an Enriques surface with $K_S \neq 0$ and let $D = F_1 + F_2 + F_3$ be a nef divisor with $\Phi(D) = 2$ as in case 1 of Proposition 2.6.7. Suppose that neither $|D|$ nor $|D + K_S|$ is a bielliptic linear system.*

Then, $S' = \phi_{|D|}(S) \subset \mathbb{P}^3$ is a surface of degree 6. Its singular locus consists of rational double points and 6 lines, which are the double locus of the union of four linearly independent planes in $\mathbb{P}^3$.

Moreover, the surface S' is projectively equivalent to a surface with equation

$$t_0t_1t_2t_3Q(t_0,t_1,t_2,t_3) + t_1^2t_2^2t_3^2 + t_0^2t_2^2t_3^2 + t_0^2t_1^2t_3^2 + t_0^2t_1^2t_2^2 = 0,$$

where Q is a homogeneous quadratic form. The surface S' is singular along the six edges $t_i = t_j = 0$ of the coordinate tetrahedron in $\mathbb{P}^3$.

Proof Since $D \cdot F_i = 2$, the restriction of $|D|$ to any the genus one curves F_i (resp. F_i') defines a map of degree 2 onto a line ℓ_i (resp. ℓ_i') in $\mathbb{P}^3$. Since $F_i' \sim F_i + K_S$, we have

$$F_1 + F_2 + F_3 \sim F_1' + F_2' + F_3 \sim F_1' + F_2 + F_3' \sim F_1 + F_2' + F_3'.$$

This shows that among the planes in $\mathbb{P}^3$ corresponding to divisors from the linear system $|D|$ there are four planes $H_1, \ldots, H_4$ that cut out the triples of lines

$$\ell_1 + \ell_2 + \ell_3, \quad \ell_1' + \ell_2' + \ell_3, \quad \ell_1' + \ell_2 + \ell_3', \quad \text{and} \quad \ell_1 + \ell_2' + \ell_3'. \tag{3.5.1}$$

Let us show that these four planes are linearly independent or, equivalently, their intersection is empty. Suppose they are linearly dependent. Two of the planes contain the line ℓ_1 and the other two contain the line ℓ_1'. This implies that the lines ℓ_1 and ℓ_1' intersect, hence they span a plane. It follows that $|D-F_1-F_1'| = |F_2+F_3-F_1+K_S| \neq \emptyset$. Applying Proposition 3.3.1, we conclude that the linear system $|D+K_S|$ is bielliptic, which contradicts our assumptions.

It remains for us to find the equation $\Phi_6(t_0,t_1,t_2,t_3) = 0$ of the sextic surface S'. After choosing coordinates in $\mathbb{P}^3$, we may assume that the planes are the coordinate hyperplanes $t_i = 0$. Since the line $t_i = t_j = 0$ is a double line of the surface, each monomial entering in Φ_6 is divisible by one of the monomials t_it_j, t_i^2, t_j^2. A linear combination of monomials divisible by $t_0t_1t_2t_3$ is equal to $t_0t_1t_2t_3Q(t_0,t_1,t_2,t_3)$, where Q is a homogeneous quadratic form. A monomial that does not contain, say t_3, must be of the form $x_0^2x_1^2x_2^2$. Each such monomial should enter with nonzero coefficient, otherwise the polynomial is reducible. After rescaling the variables, we may assume that the coefficients at monomials of type $x_0^2x_1^2x_2^2$ are equal to 1. This shows that Φ_6 is as claimed. ◻

Next, we consider degenerate cases. They can only appear if the surface S is nodal, that is, if it contains smooth rational curves. As we will see later, a general surface (in the sense of moduli) is unnodal. First, we continue to assume that $|D| = |F_1+F_2+F_3|$ is not bielliptic, but allow $|D+K_S|$ to be bielliptic. The difference here is that all the planes H_i intersects at one point and that all the lines ℓ_i pass through this point. The 6 lines are the intersection lines $H_i \cap H_j$. Arguments similar to the ones in the previous proof shows that the equation of $S' = \phi_{|D|}(S)$ is projectively equivalent to

$$t_0t_1t_2(t_0+t_1+t_2)Q(t_0,t_1,t_2,t_3) + a_1t_1^2t_2^2(t_0+a_2t_1+t_2)^2 + a_2t_0^2t_2^2(t_0+t_1+t_2)^2$$
$$+ a_3t_0^2t_1^2(t_0+t_1+t_2)^2 + a_4t_0^2t_1^2t_2^2 = 0.$$

Here, we can only scale the coordinates t_0, t_1, t_2 simultaneously. Also, we can use transformations $t_3 \mapsto \alpha t_3 + \beta t_2 + \gamma t_1 + \delta t_0$. The coefficient at t_3^2 in Q is not equal to zero for otherwise the surface has a point $[0,0,0,1]$ of multiplicity 5, in which case the surface would be rational. If the characteristic satisfies $p \neq 2$, then we can use the change of t_3 to transform Q to the form $Q_1(t_0, t_1, t_2) + t_3^2$. In this case, the equation of the sextic acquires the form

$$t_0t_1t_2(t_0 + t_1 + t_2)(Q_1(t_0, t_1, t_2) + t_3^2) + a_1t_1^2t_2^2(t_0 + a_2t_1 + t_2)^2 + a_2t_0^2t_2^2(t_0 + t_1 + t_2)^2 + a_3t_0^2t_1^2(t_0 + t_1 + t_2)^2 + t_0^2t_1^2t_2^2 = 0. \tag{3.5.2}$$

From this, we see that these surfaces depend on 9 parameters, whereas there are 10 parameters in the general case from Theorem 3.5.1. This is consistent with the moduli dimensions.

Second, we assume that the linear system is of the form $|F_1 + 2F_2 + R|$ as in case 2 from Proposition 2.6.7. The curves F_1, F_1', F_2, F_2' are mapped to the double lines $\ell_1, \ell_1', \ell_2, \ell_2'$ of the sextic surface S'. The curve R_1 is blown down to a point $P \in S'$. Since F_2, F_2' intersect R, the lines ℓ_2, ℓ_2' intersect at P. Also, since the restriction of $|D|$ to F_2 contains the divisor $F_1 \cap F_2$ and $R \cap F_2$, hence the intersection point $F_1 \cap F_2$ is mapped to P. Similarly, we find that the intersection point $F_1 \cap F_2'$ is mapped to P. Thus, the four lines ℓ_i, ℓ_i' intersect at P. They are not coplanar because the linear system $|D - F_1 - F_1' - F_2 - F_2'| = |R - F_1|$ is empty. However, $|D - F_2 - F_2' - F_1'| = |R|$ is not empty and thus, the lines ℓ_2, ℓ_2', ℓ_1' are coplanar. The plane spanned by ℓ_1, ℓ_2 (resp. ℓ_1', ℓ_2') cuts out the line ℓ_2 (resp. ℓ_2') with multiplicity 4. The plane spanned by ℓ_1, ℓ_1' cuts out additionally a conic equal to the image of a curve with the divisor class $D - F_1 - F_1' = 2F_2 - F_1 + R + K_S$. The normalization of the surface is always singular at the pre-image of the point P. In this case, the equation of the sextic surface is of the form

$$(t_0^2 + at_1^2)t_2^4 + bt_0t_1\left(t_2^2Q(t_0, t_1, t_3) + t_0t_1(t_0 + t_1)^2\right) = 0, \tag{3.5.3}$$

where Q is a homogeneous quadratic form. If $p \neq 0$, we can additionally reduce Q to the form $q(t_0, t_1) + t_3^2$. In this case, the equations of the lines are

$$\begin{aligned} \ell_1 &: t_0 = t_1 &&= 0, \\ \ell_1' &: t_2 = t_0 + t_1 &&= 0, \\ \ell_2 &: t_2 = t_0 &&= 0, \\ \ell_2' &: t_2 = t_1 &&= 0. \end{aligned}$$

Here, the double locus of the sextic surface consists of 4 lines and two lines that are infinitely near to ℓ_2 and ℓ_2'.

Finally, we assume that the linear system is of the form $|D| = |3F + 2R_1 + R_2|$, see Case 4 of Proposition 2.6.7. In this case, we have two coplanar lines ℓ, ℓ', which are the images of the curves F, F'. Their intersection point is the image of the curve $R_1 + R_2$. Let $t_0 = 0$ be the hyperplane corresponding to the divisor $3F + 2R_2 + R_1$ and $t_1 = 0$ be the hyperplane corresponding to the divisor $2F' + F + 2R_2 + R_1$. They intersect along the line ℓ, which we may assume to be given by $t_0 = t_1 = 0$. We may also assume that the equation of ℓ' is $t_1 = t_2 = 0$. Then, the equation of the sextic surface S' can be reduced to the form

$$x_1^6 + t_0\left(t_0t_1^2Q(t_0,t_1,t_2,t_3) + t_0^3t_2\right) = 0.$$

This finishes our discussion of non-normal sextic models of Enriques surfaces.

Now, we turn to big and nef divisors D with $\Phi(D) = 3$. As seen above, this implies $D^2 \geq 10$. A nef divisor class D with $D^2 = 10$ and $\Phi(D) = 3$ is called a *Fano polarization*. Its numerical class $h_{10} = [D] \in \mathrm{Num}(S)$ is called a *Fano numerical polarization*

The image $S' = \phi_{|D|}(S)$ in $\mathbb{P}^5$ is called a *Fano model* of the Enriques surface S.

By Corollary 1.5.4, there are two $W(\mathrm{Num}(S))$-orbits of vectors of norm 10 and only the orbit that contains ω_0 satisfies $\Phi = 3$. Moreover, it also follows from Corollary 1.5.4 that the class $3h_{10}$ is the sum over an isotropic 10-sequence $(f_1,\dots,f_{10})$ (see Section 1.5. We will prove in Section 6.1 of Volume II that the isotropic sequence must be a *canonical isotropic sequence* in the sense that it contains a certain number c of nef classes $f_{i_1},\dots,f_{i_c}$ and that every other f_j is obtained from some f_{i_k} by adding a chain of smooth rational curves. Thus, we obtain the following.

Lemma 3.5.2 *Let $h_{10} = [D]$ be a Fano numerical polarization on an Enriques surface S. Then,*

$$3h_{10} = f_1 + \cdots + f_{10}, \tag{3.5.4}$$

for some canonical isotropic 10*-sequence* $(f_1,\dots,f_{10})$*. Moreover, h_{10} lies in the* $W(\mathrm{Num}(S))$*-orbit of ω_0 in the notation Section 1.5.*

We note that the isotropic 10-sequence $(f_1,...,f_{10})$ consists of nef classes (we say that *non-degenerate* in this case) if and only if D is ample. In the case where the f_i's are nef numerical divisor classes, they are uniquely defined by h_{10} since they are characterized among nef classes by the property that $h_{10}\cdot f_i = 3$. In other words, the restriction of the genus one pencil $|2F_i|$ with $[F_i] = f_i$ to any smooth curve D with $[D] = h_{10}$ is the *gonality pencil*, that is, a linear system of type g_d^1: a pencil on a curve of minimal possible degree d.

Let $|D|$ be a Fano polarization of an Enriques surface such that $\phi_{|D|}(S)$ is not contained in a quadric. By Theorem 3.1.8, the embedding $\phi_{|D|}(S) = S' \subset \mathbb{P}^5$ is projectively normal. Therefore, S' is a scheme-theoretical intersection of $10 = h^0(\mathcal{O}_{\mathbb{P}^5}(3)) - h^0(\mathcal{O}_S(3D))$ cubic hypersurfaces. Assume moreover that D is ample so that the isotropic sequence $(f_1,\dots,f_{10})$ is non-degenerate. Note that the image of any curve representing one of the vectors f_i in (3.5.4) is a curve of degree 3, which spans a plane inside $\mathbb{P}^5 = |D|^*$. Thus, a choice $F_1,\dots,F_{10}$ of representatives of $(f_1,\dots,f_{10})$ defines an ordered sequence of 10 planes $(\Lambda_1,\dots,\Lambda_{10})$. If $D-F_i-F_j$ is nef, then $h^0(D-F_i-F_j) = 1$ and we conclude that $\Lambda_i\cap\Lambda_j = F_i\cap F_j$ is a point. Assume $K_S \neq 0$ and let $|F_i+K_S| = \{F_i'\}$. Then, $(F_1',\dots,F_{10}')$ defines another ordered sequence of 10 planes $(\Lambda_1',\dots,\Lambda_{10}')$. Since $h^0(D-F_i-F_j') = 1$, we conclude that $\Lambda_i\cap\Lambda_j'$ is a point if $i\neq j$. We will prove later in Proposition 7.10.3 in Volume II that $\Lambda_i\cap\Lambda_i' = \emptyset$ unless $\phi_{|D|}(S)$ lies on a quadric. We call the two 10-uples $(\Lambda_1,\dots,\Lambda_{10})$ and $(\Lambda_1',\dots,\Lambda_{10}')$ the *double-ten* of planes associated with a Fano polarization.

Another interesting polarization is the following: a nef divisor class D with $D^2 = 18$ and $\Phi(D) = 4$ on an Enriques surface S is called a *Mukai polarization*. We will call the image $S' = \phi_{|D|}(S)$ in $\mathbb{P}^9$ a *Mukai model* of the Enriques surface S.

Lemma 3.5.3 *Let* $\mathbf{v} \in (\mathsf{E}_{10})_{18}$ *with* $\Phi(\mathbf{v}) = 4$. *Then* $\mathbf{v}$ *belongs to the* $W(\mathsf{E}_{10})$*-orbir of the vector*

$$\mathbf{v}_1 = \omega_2 = 2\boldsymbol{\Delta} - \mathbf{f}_1 - \mathbf{f}_2$$

in the notation from Section 1.5.

Proof It follows from the intersection matrix of the fundamental weights given in Proposition 1.5.3 that the only positive integral linear combination of them with norm 18 is equal to ω_2 or $\omega_7 + \omega_{10}$. It follows that there are two $W(\mathrm{Num}(S))$-orbits of vectors of norm 18. These are represented by the two vectors

$$\mathbf{v}_1 = \omega_2 = 2\boldsymbol{\Delta} - \mathbf{f}_1 - \mathbf{f}_2 \quad \text{and} \quad \mathbf{v}_2 = \omega_7 + \omega_9 = \mathbf{f}_1 + \mathbf{f}_2 + \mathbf{f}_3 + 2\mathbf{f}_{10}.$$

It is easy to see that $\Phi(\mathbf{v}_2) = 3$ and we claim that $\Phi(\mathbf{v}_1) = 4$. Set

$$\mathbf{g}_i = \boldsymbol{\Delta} - \mathbf{f}_i - \mathbf{f}_{10}, \quad i = 1, \dots, 9.$$

Then, we find $\mathbf{g}_i \cdot \mathbf{g}_j = 1 - \delta_{ij}$, and $\sum \mathbf{g}_i = 9\boldsymbol{\Delta} - (\mathbf{f}_1 + \cdots + \mathbf{f}_9) - 9\boldsymbol{\Delta} = 6\boldsymbol{\Delta} - 8\mathbf{f}_{10}$. Let

$$\mathbf{v} := \frac{1}{2}(\mathbf{g}_1 + \cdots + \mathbf{g}_9) = 3\boldsymbol{\Delta} - 4\mathbf{f}_{10}.$$

It is clear that $\mathbf{v}^2 = 18$ and $\Phi(\mathbf{g}_1 + \cdots + \mathbf{g}_9) = 8$, hence $\Phi(\mathbf{v}) = 4$. Thus, $\mathbf{v}$ lies in a $W(\mathsf{E}_{10})$-orbit different from the $W(\mathsf{E}_{10})$-orbit of $\mathbf{v}_2$. This implies that $\Phi(\mathbf{v}_1) = \Phi(\mathbf{v}) = 4$. □

Note that replacing one $\mathbf{g}_i$ with $\mathbf{g}_{-i} = \mathbf{h} - \mathbf{g}_i$ gives an isotropic 9-sequence $(\mathbf{g}_1, \dots, \mathbf{g}_{-i}, \dots, \mathbf{g}_{10})$ with

$$\mathbf{g}_1 + \cdots + \mathbf{g}_{-i} + \cdots + \mathbf{g}_{10} = 2\mathbf{v} - \mathbf{g}_i + (\mathbf{h} - \mathbf{g}_i) = 2\mathbf{v} - 2\mathbf{g}_i + \mathbf{h}$$

which is not divisible by 2 in E_{10}. It can be extended to an isotropic 10-sequence.

It follows from Proposition 6.1.1 in Volume II that there are two $W(\mathsf{E}_{10})$-orbits of isotropic 9-sequences. One of them represents isotropic sequences that can be extended to an isotropic 10-sequence. The sum of the vectors in this sequence is not divisible by 2. The isotropic sequence $(\mathbf{g}_1, \dots, \mathbf{g}_9)$ in the proof of the previous lemma represents the other orbit.

Thus, if D is a Mukai polarization and $h_{18} = [D]$, we can write

$$2h_{18} = g_1 + \cdots + g_9, \tag{3.5.5}$$

where $(g_1, \dots, g_9)$ is a canonical isotropic 9-sequence that cannot be extended to a canonical isotropic 10-sequence. If D is ample, then all classes g_i are nef.

Since $\Phi(D) = 4$, the linear system $|D|$ defines a birational morphism $\phi_D : S \to S' = \phi_{|D|}(S) \subset \mathbb{P}^9$, where S' is a normal surface with at most rational double points as singularities, see Proposition 3.1.1. Note that the image of a divisor F_i representing g_i is a curve of degree 4 spanning a 3-dimensional subspace Λ_i. Similar to the case of a Fano model, we obtain a sequence $(\Lambda_1, \ldots, \Lambda_9)$ of 3-dimensional subspaces in $\mathbb{P}^9$. Since $(D - F_i - F_j)^2 = 4$, we conclude that any pair $\Lambda_i, \Lambda_j, i \neq j$, spans a codimension 3 subspace in $\mathbb{P}^9$ instead of a codimension 2 subspace, which would be the expected codimension. Since $F_i \cap F_j = 1$, we obtain that $\Lambda \cap \Lambda_j$ is a point. If $D - 2F_i$ is nef and $K_S \neq 0$, then $h^0(D - F_i - F_i') = 2$, and we obtain a *double-nine* of 3-dimensional subspaces $(\Lambda_1, \ldots, \Lambda_9)$ and $(\Lambda_1', \ldots, \Lambda_9')$ such that $\Lambda_i \cap \Lambda_j'$ is a point if $i \neq j$ and empty if $i = j$.

By Theorem 3.1.8, a Mukai model of an Enriques surface S in characteristic zero is a scheme-theoretical intersection of $18 = h^0(\mathcal{O}_{\mathbb{P}^9}(2)) - h^0(\mathcal{O}_S(2D))$ quadrics.

We end this section by explaining a close connection between Mukai polarizations and Cossec–Verra polarizations discussed in Section 3.4. Keeping the notations from the above and Section 1.5, we set

$$\mathbf{h} = \boldsymbol{\Delta} - \mathbf{f}_{10},$$

and let M be the sublattice of the Enriques lattice E_{10} generated by $\mathbf{h}, \mathbf{g}_1, \ldots, \mathbf{g}_9$.

Lemma 3.5.4 *The lattice M is isomorphic to $\mathsf{U} \oplus \mathsf{D}_8$.*

Proof We have $\mathbf{h} = \mathbf{g}_i + \mathbf{f}_i, i = 1, \ldots, 9$, so that M is generated by $\Delta - \mathbf{f}_{10} = -2\Delta + \mathbf{f}_1 + \cdots + \mathbf{f}_9, \mathbf{f}_2, \ldots, \mathbf{f}_9$. We note that $(\Delta, \mathbf{f}_1, \ldots, \mathbf{f}_9)$ is a basis of E_{10} and thus, the index of M in E_{10} is equal to 2. Next, the Gram matrix of M with respect to the basis $(\mathbf{h}, \mathbf{g}_1, \ldots, \mathbf{g}_9)$ is equal to

$$\begin{pmatrix} 4 & 2 & 2 & 2 & \ldots & 2 & 2 \\ 2 & 0 & 1 & 1 & \ldots & 1 & 1 \\ 2 & 1 & 0 & 1 & \ldots & 1 & 1 \\ \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots \\ 2 & 1 & 1 & 1 & \ldots & 0 & 1 \\ 2 & 1 & 1 & 1 & \ldots & 1 & 0 \end{pmatrix}$$

Let d_k be the greatest common divisor of all $k \times k$ minors. It is easy to see that $d_k = 1, k \leq 8$. Moreover, every 9×9-minor except the one defined by the last 9 rows and columns contains a row of even numbers. Thus, it is an even number. The remaining 9×9 minor is equal to 2^8. This implies that d_9 is even. The determinant of the Gram matrix is equal to the square of the index of M in E_{10}, which implies $d_{10} = 4$. Thus, the sequence of the elementary divisors $(d_1, \ldots, d_{10})$ is equal to $(1, \ldots, 1, 2, 2)$. This shows that the discriminant group of M is isomorphic to $(\mathbb{Z}/2\mathbb{Z})^2$. Thus, M and $\mathsf{U} \oplus \mathsf{D}_8$ have the same discriminant groups. We now conclude by applying Nikulin's results. Alternatively and more explicitly, we note that the vectors $\mathbf{g}_8, \mathbf{g}_9$ span a lattice isomorphic to U and that the vectors $\mathbf{g}_1 - \mathbf{g}_2, \ldots, \mathbf{g}_6 - \mathbf{g}_7, \mathbf{g}_7 + \mathbf{g}_8 + \mathbf{g}_9 - \mathbf{h}, \mathbf{h} - \mathbf{g}_1 - \mathbf{g}_2$ span a lattice isomorphic to D_8. Thus, M contains a sublattice isomorphic to $\mathsf{U} \oplus \mathsf{D}_8$ and thus, must be equal to it. □

Now, let D_{CV} be a Cossec–Verra polarization on an Enriques surface S, that is, D_{CV} is a nef divisor class with $D_{\mathrm{CV}}^2 = 4$ and $\Phi(D_{\mathrm{CV}}) = 2$. Assume additionally that D_{CV} is ample. We will freely use the notations and results from Section 3.4, where we studied these polarizations. We also assume that $p \neq 2$.

The pull-back $\pi^* D_{\mathrm{CV}}$ to the K3-cover $\pi : X \to S$ is very ample and defines an embedding $\widetilde{\phi}_{D_{\mathrm{CV}}} : X \to \mathbb{P}(E) \cong \mathbb{P}^5$, whose image is the complete intersection of three quadrics. Here, we have $E := H^0(X, \pi^* \mathcal{O}_S(D_{\mathrm{CV}}))$, and we let $|N|$ be the net generated by three such quadrics. We know that the discriminant curve Δ of $|N|$ is the union of two cubics $\Delta_+ \cup \Delta_-$. We assume that the cubics intersect transversally (this is the generic case) at 9 points $p_1, \ldots, p_9$. Each point p_i defines a quadric Q_i in $|N|$, whose singular locus is equal to a line ℓ_i. Moreover, the quadric Q_i contains two pencils of 3-planes containing ℓ_i. Fix one pencil. Then, the restriction of $|N|$ to each 3-plane of this pencil is equal to a complete intersection of two quadrics, hence it is a curve of arithmetic genus one. This defines an elliptic pencil $|F_i|$ on X. Replacing the pencil of planes with the other one, we obtain another pencil of elliptic curves which we denote by $|F_{-i}|$. Since the union of the 3-spaces, one from each family, spans a hyperplane, we obtain that, for each i, the divisor class of $\pi^* D_{\mathrm{CV}}$ in $\mathrm{Num}(X)$ is equal to $[F_i] + [F_{-i}]$. Note that two 3-planes from different pencils intersect along the singular line ℓ_i of Q_i. Thus, $F_i \cap F_{-i}$ consists of two points on this line taken with multiplicity 2. This agrees with the equality $8 = (\pi^*(D_{\mathrm{CV}}))^2 = 2F_i \cdot F_{-i}$. It follows from formula (3.4.3) that the pencils $|F_{\pm i}|$ are invariant under the involution τ of $\pi : X \to S$ and, hence are equal to the pre-images of genus one pencils $|2G_{\pm i}|$ on S. In $\mathrm{Num}(S)$, we have the equalities

$$[D_{\mathrm{CV}}] = g_i + g_{-i}, \quad i = 1, \ldots, 9,$$

where $g_{\pm i} = [G_{\pm i}]$. Since $D_{\mathrm{CV}}^2 = 4$, we find $g_i \cdot g_{-i} = 2$. We also have

$$2 = [D_{\mathrm{CV}}] \cdot g_j = (g_i + g_{-i}) \cdot g_j = g_i \cdot g_j + g_{-i} \cdot g_j.$$

Since $g_i \cdot g_j \geq 1$, we must have $g_i \cdot g_j = g_{-i} \cdot g_j = 1$. This defines an isotropic 9-sequence $(g_1, \ldots, g_9)$ in $\mathrm{Num}(S)$. We allow to replace here each g_i with g_{-i}. The Gram matrix of $(h, g_1, \ldots, g_9)$ coincides with the Gram matrix of $(\mathbf{h}, \mathbf{g}_1, \ldots, \mathbf{g}_9)$ from above. However, we doThus, $h, g_1, \ldots, g_9$ generate a sublattice $L \cong \mathsf{U} \oplus \mathsf{D}_8$ of E_{10} of index 2. The product of $v = \frac{1}{2}(g_1 + \cdots + g_9)$ with each vector in L is an integer, hence v belongs to $\mathrm{Num}(S)$. We have $v^2 = 18$, so it defines a Mukai polarization on S.

Bibliographical notes

A systematic study of linear systems $|D|$ on Enriques surfaces with $D^2 \leq 10$ was initiated by Cossec in [135] and [133] as a natural extension of the corresponding study of linear systems on a K3 surface due to Saint-Donat [630]. In his work, Cossec assumed that the characteristic of the ground field is different from two. This assumption was removed in [138].

The construction of a degree 2 map from an Enriques surface onto an anti-canonical del Pezzo surface defined by a bielliptic map (called a *superelliptic map* in [138]) is a birational version of the original *double plane construction* of Enriques [219]. He also considered the case of a degenerate bielliptic map defined by a degenerate U-pair of genus one pencils. The double plane constructions of Enriques were discussed in modern terminology by Artin [18] and Averbuch [36], [37]. Another birational version of a bielliptic map was given by Horikawa [309], but he did not compare it with Enriques' original construction. The double plane construction in positive characteristic $p \neq 2$ was first studied by Cossec in [133] and later in all characteristics in [138].

The fact that every Enriques surface over an algebraically closed field of characteristic $p \neq 2$ admits a degree 4 map onto the plane given by a complete linear system was proved independently by Cossec [133] and Verra [714]. This is equivalent to the fact that any Enriques surface can be obtained as the quotient of a K3 surface of degree 8 in $\mathbb{P}^5$ by a fixed-point-free involution. This latter construction for a general Enriques surface was first given by Enriques [220], although he did not prove that every Enriques surface can be obtained in this way. The construction of an Enriques surface as a 4-fold plane in characteristic 2 seems to be new. The corresponding construction as a quotient of a Gorenstein K3-like surface of degree 8 in $\mathbb{P}^5$ was first given by Liedtke [460].

The sextic model goes back to Enriques himself [219]. Its modern treatment was given in Artin's thesis [18]. For any Enriques surface S with $K_S \neq 0$, the construction of both degenerate and non-degenerate sextic models was obtained by W. Lang [435]. The existence of possibly degenerate sextic models for non extra-special surfaces of type $\tilde{E}_8$ was shown by Lang [435].

The fact that a general Enriques surface can be embedded by a complete linear system of degree 10 in $\mathbb{P}^5$ was first shown by Fano [227]. The Fano model was studied also in his later paper [228]. Fano also related it to the existence of 10 genus one pencils $|2F_i|$ with $F_i \cdot F_j = 1$. The Mukai polarization of degree 18 was introduced and studied in unpublished work of Mukai.

Chapter 4
Genus One Fibrations

In this chapter, we study genus one pencils and genus one fibrations on algebraic surfaces. We emphasize the situation in positive characteristic and genus one fibrations whose generic fiber is not smooth. We study the geometry of their fibers, multiple fibers, and their invariants, and we determine the basic invariants of surfaces with a genus one fibration. We study fibrations admitting a section, that is, jacobian genus one fibrations, their invariants, Mordell–Weil lattices, and Weierstrass models. We then turn to genus one fibrations without section and the relation to their associated jacobian fibrations, which is controlled by the Weil–Châtelet group. We finally apply these results to the study of genus one fibrations on rational surfaces and Enriques surfaces.

4.1 Elliptic and Quasi-Elliptic Pencils: Generalities

Let $\Bbbk$ be an algebraically closed field of arbitrary characteristic $p \geq 0$. In this section, we first collect some generalities about fibrations of relative dimension one. If $p = 0$, then the generic fiber of such a fibration is always smooth by Bertini's theorem, but if $p > 0$, then this need no longer be true. We will then turn to genus one fibrations, especially from surfaces onto curves: we will classify their local geometry, such as the degenerate fibers, as well the global geometry, such as Euler numbers and Betti numbers. For other textbooks treating the theory of (genus one) fibrations from surfaces onto curves, we refer to [38], [43], [513], [645], [646], and [683].

Let $f : X \to Y$ be a proper morphism between two varieties over $\Bbbk$. We set $Y' := \operatorname{Spec} f_*\mathcal{O}_X$, the affine spectrum of the $\mathcal{O}_Y$-Algebra $f_*\mathcal{O}_X$. We thus obtain a factorization $f = f'' \circ f'$ with $f' : X \to Y'$ and $f'' : Y' \to Y$, the *Stein factorization* of f. Since f is proper, $f_*\mathcal{O}_X$ is a coherent sheaf of $\mathcal{O}_Y$-modules, which implies that the natural morphism f'' is finite. We have $f'_*\mathcal{O}_X = \mathcal{O}_{Y'}$ by construction and thus, it follows from Zariski's Main Theorem that f' has connected fibers. We refer to [294, Chapter III.11] for details and proofs. We define a *fibration* between two varieties X and Y over $\Bbbk$ to be a proper and surjective morphism $f : X \to Y$ such that

$f_*\mathcal{O}_X = \mathcal{O}_Y$. As just mentioned, this implies that the fibers of f are connected. The dimension of the generic fiber of f is called the *relative dimension* of the fibration.

Theorem 4.1.1 *Let $f : X \to Y$ be a fibration of relative dimension one between normal varieties X and Y over an algebraically closed field $\Bbbk$ of characteristic $p \geq 0$.*

1. *If $p = 0$, then the generic fiber of f is a smooth curve over the function field $\Bbbk(Y)$. In particular, the generic fiber is geometrically integral and geometrically regular.*
2. *If $p > 0$ and $\dim(Y) = 1$, then the generic fiber of f is a regular curve over $\Bbbk(Y)$ that is geometrically integral.*
3. *If $\dim(Y) = 1$, then f is a flat morphism.*

Proof Let F_η be the generic fiber of f, which is a scheme of dimension one over $\Bbbk(Y)$ by assumption. Since all points in F_η are (non-closed) points in X, it follows that F_η is normal. Being of dimension one, F_η is regular.

If $p = 0$, then regularity implies geometric regularity and thus, F_η is smooth over $\Bbbk(Y)$, see [497, Section 28]. In particular, F_η is geometrically regular, geometrically integral, and geometrically reduced.

The assertion that F_η is geometrically integral if $\dim(Y) = 1$ and $p > 0$ is highly non-trivial and we refer to [38, Theorem 7.1] or [639, Corollary 2.5] for a proof. We also refer to [639] for a more general result, which implies the geometric integrality of F_η if $\dim(Y) = 1$.

If $\dim(Y) = 1$, then X is a normal surface, whence Cohen–Macaulay and Y is a normal curve, whence regular. This implies that f is flat, see Proposition 0.2.4. $\square$

Remark 4.1.2 The condition $\dim(Y) = 1$ in assertion (2) is really needed: for example, if $p = 2$, then there do exist examples of fibrations $f : X \to S$, such that X is a smooth threefold, S is a smooth surface, and such that every geometric fiber of f is non-reduced of multiplicity p and with reduction isomorphic to $\mathbb{P}^1$, see [522] or [406, Exercise IV.1.13.5]. In fact, such *wild conic bundles* play an important role in the classification of Fano threefolds in positive characteristic. We refer to [639] for some bounds on the embedding dimension of the geometric generic fiber of a fibration in characteristic $p > 0$.

Now, let $f : X \to Y$ be a fibration of relative dimension one from a normal surface X to a normal curve Y over an algebraically closed field $\Bbbk$ of characteristic $p > 0$. Let F_η be the generic fiber of f, which is a geometrically integral curve over $\Bbbk(Y)$ by the previous theorem. Let $F_{\overline{\eta}}$ be the base-change of F_η to some algebraic closure $\overline{\Bbbk(Y)}$ of $\Bbbk(Y)$, that is, $F_{\overline{\eta}}$ is the geometric generic fiber of f. Then, we consider the normalization morphism

$$\nu : \widetilde{F_{\overline{\eta}}} \to F_{\overline{\eta}}$$

and thus, $\widetilde{F_{\overline{\eta}}}$ is a normal curve, whence smooth over $\overline{\Bbbk(Y)}$. Moreover, $\mathcal{F} := \nu_*(\mathcal{O}_{\widetilde{F_{\overline{\eta}}}})/\mathcal{O}_{F_{\overline{\eta}}}$ is a torsion sheaf on $F_{\overline{\eta}}$, whose support is equal to the singular locus of $F_{\overline{\eta}}$. More precisely, if $x \in F_{\overline{\eta}}$ is a closed point, then $\delta(x) := \dim_{\Bbbk} \mathcal{F}_x$ is zero if and only if x is a smooth point. If $\delta(x) > 0$, then this is the *arithmetic genus* of the singular point x. The reason for this terminology is the equality

$$p_a(F_{\overline{\eta}}) = p_a(\widetilde{F_{\overline{\eta}}}) + \sum_{x \in F_{\overline{\eta}}} \delta(x),$$

where $p_a(-)$ denotes the arithmetic genus of an integral curve. The following theorem is a consequence of a slight generalization of Tate's theorem of genus change in inseparable field extensions [699].

Theorem 4.1.3 *We keep the assumptions and notations and assume moreover that $p \geq 3$. Then, every singular point $x \in F_{\overline{\eta}}$ satisfies*

$$\frac{p-1}{2} \mid \delta(x).$$

In particular, if $p > 2p_a(F_\eta) + 1$, then F_η is geometrically normal, that is, smooth over $\Bbbk(Y)$.

Proof The fact that $\frac{p-1}{2}$ divides $\sum_x \delta(x)$ is the classical theorem of Tate [699] and we refer to [638] and [669] for a modern treatment. The fact that each $\delta(x)$ divides $\frac{p-1}{2}$ was established in [303, Remark 2.18], also see [340, Theorem 5.7]. □

After these preparations, we now turn to fibrations of relative dimension one, whose generic fiber is a curve of arithmetic genus one. By a *genus one curve* over a field L we mean a projective and one-dimensional scheme E over L that satisfies

$$\dim_L H^0(E, \mathcal{O}_E) = \dim_L H^1(E, \mathcal{O}_E) = 1. \tag{4.1.1}$$

The first condition implies that E has no embedded components and that E is geometrically connected, that is, stays connected after any field extension. Since E has no embedded components, it is a Cohen–Macaulay scheme and thus, admits a dualizing sheaf ω_E. Serre duality then gives $\dim_L H^0(E, \omega_E) = \dim_L H^1(E, \mathcal{O}_E) = 1$, and thus, there exists a nonzero section $s : \mathcal{O}_E \to \omega_E$. If we also assume that E is reduced, then we obtain a short exact sequence

$$0 \to \mathcal{O}_E \to \omega_E \to \mathcal{F} \to 0,$$

where $\mathcal{F}$ is a torsion sheaf on E. Taking cohomology and using (4.1.1), we conclude that s is an isomorphism. In particular, a reduced genus one curve is a Gorenstein curve with trivial canonical sheaf. We will be mostly concerned with genus one curves lying on a nonsingular projective surface over an algebraically closed field $\Bbbk$.

A genus one curve is called an *elliptic curve* if it is smooth. Otherwise, we call it a *quasi-elliptic curve*.

A *genus one fibration* is a fibration $f : X \to Y$ between normal varieties over a field $\Bbbk$, such that f is flat and such that the generic fiber X_η is a geometrically integral and regular genus one curve. If $\Bbbk$ is algebraically closed and Y is of dimension one, then flatness of f and geometric integrality of the generic fiber are automatic, see Theorem 4.1.1. A genus one fibration is called *elliptic* if X_η is smooth and *quasi-elliptic* otherwise.

It follows from Theorem 4.1.3 that quasi-elliptic fibrations exist only if $p = 2, 3$. In Remark 4.4.3 below, we will give an independent proof of this fact.

By definition of a fibration, the canonical homomorphism $\mathcal{O}_C \to f_*\mathcal{O}_X$ is bijective. By general properties of morphisms of schemes, all geometric fibers are geometrically connected and there exists an open non-empty subset U of C such that an elliptic (resp. quasi-elliptic) f is smooth (resp. geometrically integral) over U, see [269, Part III, §9].

We will be mostly concerned with the case when the base is a regular integral noetherian scheme of dimension 1. To distinguish this case from the general one we will re-denote the base B by C. When C is a smooth projective curve over $\Bbbk$, the surface X, together with an elliptic (resp. quasi-elliptic) fibration $f : X \to C$ is called an *elliptic surface* (resp. *quasi-elliptic surface*).

As we saw in Section 2.2, an Enriques surface S always admits a base point free elliptic or quasi-elliptic pencil, which defines an elliptic or a quasi-elliptic fibration on S. In this chapter, we will develop a general theory of genus one fibrations $f : X \to C$, which, by far, exceeds our needs, but which will hopefully serve its purpose as a convenient reference to the theory.

Unless, stated otherwise, X will be assumed regular, and the base scheme C will be one of the following:

- **Global Case**: C is a smooth algebraic curve over an algebraically closed field $\Bbbk$. In this case, X is a smooth and irreducible algebraic surface.
- **Local Case** $C = \operatorname{Spec} R$, where R is a local ring of a smooth algebraic curve at its closed point or its completion or its henselization. In the latter cases, we say that C is *strictly local*.

In both cases, we will often denote by K the residue field $\Bbbk(\eta)$ of the generic point η of C.

For any morphism $\phi : Y \to C$ and a closed point $t \in C$, we denote by $\phi_t : Y(t) \to \operatorname{Spec} \mathcal{O}_{C,t}$ the base change of ϕ under the canonical morphism $\operatorname{Spec} \mathcal{O}_{C,t} \to C$. We call it the *localization* of ϕ at the point t. Similarly, we define the *strict localization* $\phi_t^h : Y(t)^h \to \operatorname{Spec} \mathcal{O}_{C,t}^h$ of ϕ. A genus one fibration $f : X \to C$ is called *relatively minimal* if the relative canonical sheaf $\omega_{X/C}$ is nef. Each fibration admits a birational morphism over C onto a relatively minimal fibration, which blows down (-1)-curves in fibers over closed points. From now on, we will assume that genus one fibrations are relatively minimal.

Let $f : X \to C$ be a genus one fibration and Σ be the set of closed points $t \in C$ such that the scheme-theoretical fiber X_t is not smooth if the fibration is elliptic or not irreducible if the fibration is quasi-elliptic. The fibers $X_t, t \in \Sigma$, are called *degenerate fibers* or *singular fibers*.

We consider a fiber X_t of f over a closed point t as an effective Cartier divisor, whose sheaf of ideals is equal to the pull-back of the ideal sheaf of the point t.

Since X is regular, we can identify X_t with the corresponding Weil divisor and write the fiber

$$X_t = \sum_{i \in I} n_i R_i$$

as a linear combination of its irreducible components. The number n_i is called the *multiplicity* of the component R_i. The greatest common divisor m_t of the n_i's is called the *multiplicity* of X_t.

We denote by $\mathrm{NS}_t(X)$ the free abelian group generated by the irreducible components of X_t inside $\mathrm{NS}(X)$. For effective divisors E and C on X, such that C is supported in a closed fiber, one defines the intersection product

$$E \cdot C := \deg_C(\mathcal{O}_X(E) \otimes \mathcal{O}_C),$$

see [151]. In particular, it equips $\mathrm{NS}_t(X)$ with the structure of a quadratic lattice.

The next result, originally due to Kodaira and Néron, describes the structure of possible degenerate fibers.

Theorem 4.1.4 *Let $X_t = \sum_{i \in I} n_i R_i$ be a degenerate fiber of multiplicity m. Then, $X_t = m\bar{X}_t$, where $\bar{X}_s$ is:*

1. *either an irreducible curve with an ordinary double point or a cuspidal point,*
2. *or a reducible divisor, whose irreducible components define a canonical root basis of affine type in the lattice $\mathrm{NS}_t(X)$.*

The type *of X_t is the type of the root basis, which is one of $\tilde{A}_n$, $\tilde{D}_n$, $\tilde{E}_6$, $\tilde{E}_7$, of $\tilde{E}_8$. The type determines the geometry of the fiber X_t uniquely except in the following cases*

1. *A fiber of type $\tilde{A}_1$ could be either the union of two components intersecting transversally at two points (notation: type $\tilde{A}_1$) or two components tangent at one point (notation: type $\tilde{A}_1^*$).*
2. *A fiber of type $\tilde{A}_2$ could be either the union of three components forming a divisor with normal crossings or the union of three components intersecting each other transversally at the same point (notation: type $\tilde{A}_2^*$).*

The multiplicities n_i/m are equal to the coefficient of the generator of the radical of NS_t with respect to the root basis. In the case $\tilde{A}_n$ all the coordinates are equal to 1. In other cases, they are indicated by the numbers above the vertices of the Dynkin diagrams.

The notation for the types of the fiber agrees with the notation for irreducible divisors of canonical types from Proposition 2.2.5. From now on we choose this notation and refer to Remark 2.2.6 for comparison with other notations of Kodaira or Néron.

Remark 4.1.5 We will see in Corollary 4.3.22 that this list is much smaller for quasi-elliptic fibrations. If the ground field is of characteristic $p \in \{2, 3\}$ and imperfect, then the classification is different, see [473] and [697].

The types $\tilde{A}_n$ are said to be *of multiplicative type* or *semi-stable*. They are the only singular fibers for which the first Betti number $b_1(X_t)$ is not equal to zero but rather equal to 1. All other degenerate fibers are said to be *of additive type* or *unstable*.

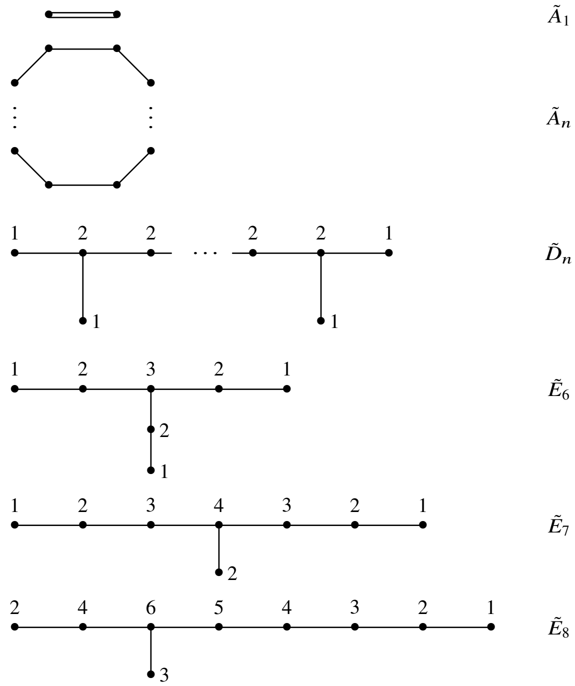


Fig. 4.1 Reducible fibers of genus one fibration

The reason for this terminology multiplicative and additive will become clear when discussing Néron models in Section 4.2. Moreover, base changes of semi-stable fibers remain semi-stable, whereas for every unstable fiber admits a finite base change that is semi-stable. This explains this part of the terminology.

It follows from the above classification that every fiber X_t is equal to $m_t \bar{X}_t$, where m_t is the multiplicity and where $\bar{X}_t$ has at least one reduced and irreducible component.

Another useful observation is that the number of components of $\bar{X}_t$ of multiplicity 1 is equal to the order of the discriminant group of the finite root system of the affine root system associated to $\bar{X}_t$. In particular, this number is equal to the determinant of the Cartan matrix of the finite root system.

Theorem 4.1.6 *Let $f : X \to C$ be a genus one fibration. Then*

$$\omega_{X/C} \;\cong\; f^* \mathcal{L}^{\otimes -1} \otimes O_X \left(\sum_{t \in C} a_t \bar{X}_t \right), \tag{4.1.2}$$

where $\mathcal{L}$ is an invertible sheaf on C that is defined by

$$R^1 f_* O_X \;\cong\; \mathcal{L} \oplus \mathcal{T} \quad \text{with} \quad \mathcal{T} \;=\; \mathrm{Tors}\ R^1 f_* O_X .$$

Moreover:

1. $\deg \mathcal{L} = -\chi(\mathcal{O}_X) - h^0(\mathcal{T})$.
2. $0 \le a_t < m_t$.
3. *The order* ν_t *of* $\mathcal{O}_{\bar{X}_t}(\bar{X}_t)$ *in* $\mathrm{Pic}(\bar{X}_t)$ *divides* m_t *and* $a_t + 1$.
4. $m_t = \nu_t\, p^{r_t}$, *where* $p = \mathrm{char}(\Bbbk)$.
5. $a_t = m_t - 1$ *if and only if* $\mathcal{T}_t = \{0\}$.

Proof We use the *relative duality theorem*. The complex $\omega_{X/C}[1]$ with $\omega_{X/C}$ placed at degree -1 is a dualizing complex in the sense that there is an isomorphism of functors

$$D_C \circ Rf_* \cong Rf_* \circ D_{X/C}, \tag{4.1.3}$$

where $D_C = \mathrm{RHom}(-, \mathcal{O}_C)$ and $D_{X/C} = \mathrm{RHom}(-, \omega_{X/C}[1])$ denote the Hom functors in the derived categories of coherent sheaves on C and X, respectively. In particular, for any locally free sheaf $\mathcal{E}$, we have an isomorphism

$$f_*(\omega_{X/C} \otimes \mathcal{E}^\vee) \cong (R^1 f_*\mathcal{E})^\vee = \mathcal{H}om_{\mathcal{O}_C}(R^1 f_*\mathcal{E}, \mathcal{O}_C). \tag{4.1.4}$$

Taking $\mathcal{E} = \mathcal{O}_X$, we obtain an isomorphism

$$f_*\omega_{X/C} \cong (R^1 f_*\mathcal{O}_X)^\vee = \mathcal{L}^{\otimes -1}. \tag{4.1.5}$$

Since f is of relative dimension one, we have $R^2 f_*\mathcal{O}_X = 0$, and hence, by the base change theorem [294, Chapter 3, §7]), the fiber $R^1 f_*\mathcal{O}_X(t)$ of $R^1 f_*\mathcal{O}_X$ at a point t is isomorphic to $H^1(X_t, \mathcal{O}_{X_t})$. For any fiber X_t with $h^1(\mathcal{O}_{X_t}) = 1$ we have $R^1 f_*\mathcal{O}_X(t) \cong \Bbbk(t)$, and hence, the generic rank of $R^1 f_*\mathcal{O}_X$ is equal to 1. Moreover, $R^1 f_*\mathcal{O}_X$ is invertible around a point t with $h^1(\mathcal{O}_{X_t}) = 1$. By Proposition 2.2.3, we have $h^1(\mathcal{O}_{X_t}) = 1$ for any non-multiple fiber. It follows that $R^1 f_*\mathcal{O}_X$ is isomorphic (non-canonically) to $\mathcal{L} \oplus \mathcal{T}$, where $\mathcal{L}$ is an invertible sheaf and $\mathcal{T}$ is a torsion sheaf. The length of $\mathcal{T}$ at a point t is equal to $h^1(\mathcal{O}_{X_t}) - 1$.

We have a canonical homomorphism of invertible sheaves on X

$$f^*\mathcal{L}^{\otimes -1} = f^* f_*\omega_{X/C} \to \omega_{X/C},$$

which is an isomorphism outside the set of multiple fibers. Tensoring with $\omega_{X/C}^{\otimes -1}$, we obtain the sheaf $f^*\mathcal{L}^{\otimes -1} \otimes \omega_{X/C}^{\otimes -1}$ as an ideal sheaf of $\mathcal{O}_X$, which defines an effective Cartier divisor D. Since both $f^*\mathcal{L}$ and $\omega_{X/C}$ restrict to $\mathcal{O}_{X_t}$ at each non-multiple fiber, we see that D is supported in multiple fibers. Since $\deg \omega_{X/C} \otimes \mathcal{O}_{X_t} = K_X \cdot X_t$ is constant and equal to zero for a non-multiple fiber, we obtain that it is equal to zero for all t. Also, each proper irreducible component R of X_t, is a (-2)-curve and hence, $\deg \omega_{X/C} \otimes \mathcal{O}_R = K_X \cdot R = 0$. It follows that the divisor class $[D_t]$ in each fiber D_t of D belongs to the radical of the sublattice of $\mathrm{NS}(X)$ generated by irreducible components of the fiber, and hence $D_t = [a_t \bar{X}_t]$ for each fiber X_t. This gives us formula (4.1.2) from the assertion of the theorem.

To prove Assertion 1, we use the Grothendieck–Leray spectral sequence for the morphism f and the Riemann–Roch theorem on C. They show that

$$\chi(R^1 f_* \mathcal{O}_X) = h^0(\mathcal{T}) + \deg \mathcal{L} + \chi(\mathcal{O}_X) = \chi(f_* \mathcal{O}_X) - \chi(\mathcal{O}_X) = \chi(\mathcal{O}_C) - \chi(\mathcal{O}_X),$$

hence

$$\deg \mathcal{L} = -\chi(\mathcal{O}_C) - h^0(\mathcal{T}).$$

Let us prove Assertion 2. Let $D = \sum_t D_t$, where $D_t = m_t \bar{X}_t$. Using (0.2.8) and applying f_* to $\omega_{X/C}$, we obtain by the projection formula

$$f_* \omega_{X/C} \cong \mathcal{L}^{\otimes -1} \cong \mathcal{L}^{\otimes -1} \otimes f_* \mathcal{O}_X(D).$$

Cancelling $\mathcal{L}^{\otimes -1}$, we get $f_* \mathcal{O}_X(D) \cong \mathcal{O}_C$. If $a_t > m_t$ for some t, then $f^* \mathcal{O}_X(t) \subset \mathcal{O}_X(D)$ and hence, $\mathcal{O}_X(t) \subset f_* \mathcal{O}_X(D) = \mathcal{O}_C$, a contradiction.

Since

$$\mathcal{O}_{\bar{X}_t}(m_t \bar{X}_t) \cong \mathcal{O}_{\bar{X}_t} \otimes \mathcal{O}_{X_t} \cong \mathcal{O}_{\bar{X}_t}.$$

we find $\nu_t | m_t$. By the adjunction formula, we have

$$\mathcal{O}_{\bar{X}_t} \cong \omega_{X/C}(\bar{X}_t) \otimes \mathcal{O}_{\bar{X}_t} \cong \mathcal{O}_{\bar{X}_t}((1 + a_t)\bar{X}_t).$$

We conclude $\nu_t | 1 + a_t$, which proves Assertion 3.

We will prove the last two assertions in the next section (Lemma 4.2.10 and Corollary 4.2.3). □

Corollary 4.1.7 *Let X be a surface that admits a relatively minimal genus one fibration. Then,*

$$K_X^2 = 0.$$

Definition 4.1.8 A fiber X_t is called *wild* if $\mathcal{T}_t \neq \{0\}$ and *tame* otherwise.

We have already noticed in the proof of the theorem that X_t is wild if and only if $h^1(\mathcal{O}_{X_t}) > h^1(\bar{X}_t) = 1$ and thus, by duality, if and only if $h^0(\mathcal{O}_{X_t}) > h^0(\bar{X}_t) = 1$. Also, the difference is equal to the length $l(\mathrm{Tors}(R^1 f_* \mathcal{O}_X)_t)$. We recall from [268, (7.8.1)] that a proper flat morphism of finite type $f : Y \to T$ is said to be *cohomologically flat* if taking f_* commutes with all base changes. Since $\mathcal{O}_T \to f_*(\mathcal{O}_Y)$ is an isomorphism in our case, it follows that a fiber is tame if and only if it is cohomologically flat, see also [606, Théorème 7.2.1].

In his unpublished manuscript [607], Raynaud gives a more precise computation of the coefficients a_t in the formula for $\omega_{X/C}$. We will reproduce it in the next section.

Remark 4.1.9 The fact that $a_t = m_t - 1$ if $\Bbbk = \mathbb{C}$ was proven by Kodaira and we refer to [43, Chapter V, Theorem 12.1] for a proof. The proof that the normal bundle $\mathcal{O}_{\bar{X}_t}(\bar{X}_t)$ is always of order equal to m_t is [43, Chapter III, Lemma 8.3]. We will prove in the next section that if $p > 0$, then this happens if and only if X_t is a tame fiber.

Remark 4.1.10 Suppose that $H^1(X, \mathcal{O}_X) = 0$. For example, X could be a rational surface or a classical Enriques surface. Then, the Grothendieck–Leray spectral sequence gives an exact sequence

$$0 \to H^1(C, O_C) \to H^1(X, O_X) \to H^0(C, R^1 f_* O_X) \to H^2(C, O_C) = 0.$$

This implies that $H^0(C, R^1 f_* O_X) = 0$ and hence, the torsion subsheaf $\mathcal{T}$ is zero. Thus, all fibers are tame and we have $a_t = m_t - 1$ for all $t \in C$.

Let

$$e(Z) = \sum_{i \geq 0} (-1)^i \dim_{\mathbb{Q}_\ell} H^i_{\text{ét}}(Z, \mathbb{Q}_\ell)$$

be the ℓ-adic Euler–Poincaré characteristic of a scheme Z of finite type over a field $\Bbbk$, see Section 0.10. Unless stated otherwise, we will assume that ℓ is a prime different from $p = \text{char}(\Bbbk)$. Straightforward computations give

$$e(X_t) = \begin{cases} 0 & \text{if } t \notin \Sigma \text{ and } f \text{ is an elliptic fibration,} \\ 2 & \text{if } t \notin \Sigma \text{ and } f \text{ is a quasi-elliptic fibration,} \\ 1 + \#\operatorname{Irr}(X_t) & \text{if } X_t \text{ is not of type } \tilde{A}_n, \text{ and} \\ \#\operatorname{Irr}(X_t) & \text{if } X_t \text{ is of type } \tilde{A}_n. \end{cases}$$

and

$$b_1(X_t) = \begin{cases} 0 \text{ if } X_t \text{ is of additive type,} \\ 1 \text{ if } X_t \text{ is of multiplicative type, and} \\ 2 \text{ if } X_t \text{ is smooth.} \end{cases}$$

Here, $\operatorname{Irr}(X_t)$ denotes the set of irreducible components of X_t. Next, we introduce the wild ramification invariant of a fiber: Let $\nu : R \to \mathbb{Z}$ be a discrete valuation ring with an algebraically closed residue field $\Bbbk$ of characteristic $p > 0$, let K be its fraction field, and let π be a generator of the maximal ideal of R. Let L/K be a finite Galois extension of K with group G. Let R_L be the integral closure of R in L, which is again a discrete valuation ring, and let π_L be a generator of the maximal ideal or R_L We define a function $a_G : G \to \mathbb{Z}$ via

$$a_G(g) := \begin{cases} -\nu({}^g\pi_L - \pi_L) & \text{if } g \neq 1, \\ -\sum_{g \neq 1} a_G(g) & \text{if } g = 1. \end{cases}$$

By a theorem of E. Artin, the function a_G is a character of some complex representation [658, Chapter 6], the *Artin representation*. If the ramification index of L/K is prime to p, that is, if L/K is tamely ramified, then this representation it coincides with the regular representation and a_G is the character of the regular representation, which we will denote by i_G. If p divides the ramification index of L/K, that is, if L/K is wildly ramified, then the Artin representation strictly contains the regular representation and $a_G - i_G$ takes zero value on elements of G of order prime to p. Artin used this to show that the Artin representation is rational over any ℓ-adic field $\mathbb{Q}_\ell$, where $\ell \neq p$. Next, we define a sequence of subgroups of G by setting $G_0 := G$ and

$$G_i := \{g \in G : a_G(g) \geq i + 1\}, \quad i \geq 1,$$

and denote their orders by $e_i := \#G_i$ and $e := e_0 := \#G$. Then, each G_{i+1} is a normal subgroup of G_i, that G/G_1 is a subgroup of $\Bbbk^*$, and that G_i/G_{i+1} is a subgroup of

$\Bbbk^+$ if $i > 0$. For any G-module M that is a finite abelian ℓ-group, we define

$$\delta(K, M) := \sum_{i=0} \frac{e_i}{e} \dim_{\mathbb{F}_\ell}(M/M^{G_i}).$$

One can show that the representation with character $a_G - i_G$ is defined by some projective $\mathbb{Z}_\ell[G]$-module P_G and that $\delta(K, M) = \dim_{\mathbb{F}_\ell} \mathrm{Hom}_G(P_G, M)$.

Let $f : X \to C$ be a fibration of relative dimension n over a global C as above. Let $R^i f_* \boldsymbol{\mu}_{\ell,X}$ be the étale sheaves on C killed by ℓ. We chose the prime ℓ large enough so that $\dim_{\mathbb{F}_\ell}(R^i f_* \boldsymbol{\mu}_\ell)_t = \dim_{\mathbb{Q}_\ell} H^i(X_t, \mathbb{Q}_\ell)$ for every geometric point t of C. For any generic point η_t of the localization of C at a closed point t, the fiber $(R^i f_* \boldsymbol{\mu}_{\ell,X})_{\eta_t}$ is a module over G_{K_t}, which will be trivialized over some Galois extension L_t/K_t, so that we can take it as the module M from above,

$$\delta_t(f;\ell)_i := \delta(K_t, (R^i f_* \boldsymbol{\mu}_{\ell,X})_{\eta_t}), \tag{4.1.6}$$

as the definition of the *invariant of the wild ramification*.

Theorem 4.1.11 *Let $f : X \to C$ be a fibration of relative dimension n over a global C as above. Then*

$$e(X) = e(X_{\bar{\eta}})\, e(C) + \sum_{t \in \bar{C}} \left(e(X_t) - e(X_{\bar{\eta}}) + \sum_{i=0}^{n} (-1)^i \delta_t(f;\ell)_i \right), \tag{4.1.7}$$

where $\bar{C}$ denotes the set of closed points of C.

Proof First, assume $n = 0$. In this case, $X \to C$ is a finite cover of complete smooth algebraic curves over $\Bbbk$ of some degree N. We have $e(X_{\bar{\eta}}) = N$ and $e(X_t) = \# f^{-1}(t)$ and we can rewrite the formula as

$$e(X) = N \cdot e(C) - \sum_{x \in \bar{X}} (e_x - 1 + \alpha_x), \tag{4.1.8}$$

where e_x is the ramification index at a closed point $x \in X$ and where $\sum_{x \in f^{-1}(t)} \alpha_x = \delta_t(f;\ell)_0$. Thus, the formula becomes the adjustment of the usual Hurwitz formula to characteristic p. The proof of the formula in this case can be found in [658, Chapter VI, Proposition 7].

The case $n > 0$ can be reduced to the case $n = 0$: we first use the Grothendieck–Leray spectral sequence for the sheaf $\boldsymbol{\mu}_{\ell,X}$, which gives

$$e(X) = \chi(X, \boldsymbol{\mu}_\ell) = \sum_{i=0}^{n} \chi(R^i f_* \boldsymbol{\mu}_{\ell,X}).$$

Then, one shows that the constructive sheaves $R^i f_* \boldsymbol{\mu}_{\ell,X}$ are trivialized over some finite separable extension $\phi : X \to C$ and coincide with the sheaves $\phi_*(\boldsymbol{\mu}_{\ell,X})$. We refer for the details to [604]. □

Now, we specialize this formula to the case of genus one fibrations. First of all, the invariants $\delta_t(f;\ell)_0$ and $\delta_t(f;\ell)_2$ are both equal to zero. The invariant

$$\delta_t = \delta_t(f,\ell)_1$$

is independent of ℓ and it is zero if X_t is smooth or of multiplicative type [570]. Since $(R^1 f_* \boldsymbol{\mu}_{l,X})_{\eta_t} \cong H^1(X_{\eta_t}, \boldsymbol{\mu}_\ell) \cong {}_\ell \operatorname{Pic}(X_{\eta_t})$, it depends only on the fiber of the corresponding jacobian fibration, see Section 4.3. Since the invariant of wild ramification $\delta(K,M)$ does not change after a separable extension and a jacobian fibration admits a semi-stable reduction after an extension of some degree dividing 24, it is equal to zero if $p \neq 2,3$. If $p = 2$ (resp. $p = 3$), then it is always non-zero unless the fiber is of type $\tilde{A}_2^*$ or $\tilde{E}_6$ (resp. $\tilde{A}_1^*, \tilde{E}_7, \tilde{D}_n$), see [477, Theorem 4.1]. It is also equal to zero if f is a quasi-elliptic fibration, because in this case, the sheaf $R^1 f_* \boldsymbol{\mu}_{\ell,X}$ is zero.

Corollary 4.1.12 *Assume that C is global and that $f : X \to C$ is a genus one fibration. Then,*

$$e(X) = \begin{cases} \sum_{t\in C} (e(X_t) + \delta_t) & \text{if } f \text{ is elliptic}, \\ 2e(C) + \sum_{t\in C} (b_2(X_t) - 1) & \text{if } f \text{ is quasi-elliptic}. \end{cases}$$

Remark 4.1.13 In the case when f is smooth outside of a finite set of points, formula (4.1.7) was proven by Deligne [152, Proposition 2.1]. There, he also shows that $\delta(f;\ell)_i = 0$ if $i \neq n$ and that $\delta(f;\ell)_n$ does not depend on ℓ. The term $e(X_t) - e(X_{\bar{\eta}}) + \delta_t(f)_n$ coincides with $(-1)^{n-1}\mu_{X/C}(x)$, where $\mu_{X/C}(x)$ is equal to the length of the module $\mathcal{E}xt^1(\Omega^1_{X/C}, \mathcal{O}_X)_x$. For example, if X is a singular fiber of multiplicative type of an elliptic fibration, then we have $\mu_{X/C}(x) = 1$ at each singular point of X_t and hence, the term is equal to $n = e(X_t)$. In particular, we see that $\delta_t = 0$ in this case.

Let $f : X \to C$ be a quasi-elliptic fibration. Its generic fiber X_η contains a unique non-smooth point, which becomes an ordinary cusp after passing to the algebraic closure of $\Bbbk(C)$. The closure of this point in X is an irreducible reduced subscheme $\mathfrak{C}$ of X, which we call the *curve of cusps*. For any $t \notin \Sigma$, the intersection of the curve of cusps with X_t is equal to the unique non-smooth point of X_t, the unique cusp.

Proposition 4.1.14 *The curve of cusps $\mathfrak{C}$ is a smooth curve on X with $\mathfrak{C} \cdot X_t = p = \operatorname{char}(\kappa(t))$. The restriction of f to $\mathfrak{C}$ is a purely inseparable finite cover of degree p.*

Proof It is proven in [77, Proposition 1] that $\mathfrak{C}$ intersects any irreducible fiber at its cusp with multiplicity equal to $p \in \{2,3\}$. Since $\mathfrak{C}_\eta$ is equal to the closure in X of the unique non-smooth point on the generic fiber X_η, it does not intersect any closed fiber at its smooth point. Since the intersection number $\mathfrak{C} \cdot F_t$ is constant, we see that $\mathfrak{C}$ must be smooth at each point (otherwise the intersection number would be larger than or equal to 4). It is clear that the restriction of f to $\mathfrak{C}$ is a homeomorphism and hence, it is an inseparable map of degree p. □

This proposition has the following interesting applications.

Corollary 4.1.15 *The multiplicity of a multiple fiber of a quasi-elliptic fibration is equal to the characteristic* $p \in \{2, 3\}$.

Corollary 4.1.16 *If* X *is a quasi-elliptic surface, then there exists a surface* Y *that is ruled over* $\mathfrak{C}$ *and a dominant morphism* $Y \to X$ *that is generically purely inseparable and finite of degree* p. *In particular,* X *is a uniruled surface and supersingular in the sense of Shioda.*

Proof Let S be a desingularization of the base-change $X \times_C \mathfrak{C} \to \mathfrak{C}$. The generic fiber of $S \to \mathfrak{C}$ is a smooth rational curve, that is, S is a ruled surface over $\mathfrak{C}$. The induced map $S \to X$ is dominant, generically finite of degree p and generically purely inseparable. Thus, X is a uniruled and thus, supersingular in the sense of Shioda. □

Proposition 4.1.17 *Let* $f : X \to C$ *be a genus one fibration and assume that there exists a multiple fiber* $X_t = m_t \bar{X}_t$ *with* $p \nmid m_t$. *Then,* f *is an elliptic fibration. Moreover,* X_t *is a smooth elliptic curve or a singular fiber of multiplicative type* $\tilde{A}_n$ *for some* n.

Proof Since the $m_t \neq p$, Corollary 4.1.15 implies that f is an elliptic fibration.

It follows from Theorem 4.1.6 that the normal sheaf $\mathcal{O}_{\bar{X}_t}(\bar{X}_t)$ is of order equal to m_t and thus, it is prime to p. Using the Kummer sequence and taking cohomology, we obtain an isomorphism

$$ {}_m\operatorname{Pic}(\bar{X}_t) \cong H^1_{\text{ét}}(\bar{X}_t, \boldsymbol{\mu}_m). $$

An elementary computation shows that the group on the right is non-trivial only if $\bar{X}_t$ is smooth or multiplicative of type $\tilde{A}_n$ for some n. (In these cases, this group is isomorphic to $(\mathbb{Z}/m\mathbb{Z})^2$ in the former case and isomorphic to $\mathbb{Z}/m\mathbb{Z}$ in the latter.) □

4.2 The Picard Group

Let $f : X \to C$ be a genus one fibration (global, local or strictly local) and let $\underline{\mathrm{Pic}}_{X/C}$ be the relative Picard functor. In this section, we study this relative Picard functor, Néron models, and associated invariants. In particular, we will consider degree homomorphisms and discriminant groups. Finally, we study non-reduced and wild fibers in genus one fibrations and give an estimate of the torsion of $R^1 f_* \mathcal{O}_X$.

We keep our assumption that C is a smooth curve over an algebraically closed field or a local ring of such curve or the henselization of such a local ring. In this case $\mathrm{Br}(C) = H^2_{\text{ét}}(C, \mathbb{G}_m) = \{0\}$, see [508, Chapter 4, 4.1]. It follows from Proposition 0.9.2 that the value of the functor $\underline{\mathrm{Pic}}_{X/C}$ on C is equal to the group $\mathrm{Pic}(X)/f^* \mathrm{Pic}(C)$. For any complete curve Z over a field with the set $\mathrm{Irr}(Z)$ of irreducible components, we let

$$\deg : \operatorname{Pic}(Z) \to \mathbb{Z}^{\operatorname{Irr}(Z)} \tag{4.2.1}$$

be the degree homomorphism that assigns to $\mathcal{L} \in \operatorname{Pic}(Z)$ the function, whose value on an irreducible component Z_i is equal to the degree of the restriction of $\mathcal{L}$ to Z_i. We set

$$\operatorname{Pic}^0(Z) := \operatorname{Ker}(\deg).$$

Proposition 4.2.1 *Let $f : X \to C$ be a genus one fibration with C a strictly local scheme with closed point t. Then, the restriction homomorphism*

$$r_t : \operatorname{Pic}(X) \to \operatorname{Pic}(X_t)$$

is surjective and its kernel is uniquely divisible by any integer prime to the characteristic.

Proof Since C is strictly local, we have

$$\operatorname{Pic}(X) \cong H^1_{\text{ét}}(X, \mathbb{G}_m) \cong H^0(C, R^1 f_* \mathbb{G}_m).$$

Next, the proper base change theorem asserts that $(R^1 f_* \mathbb{G}_m)(t) \cong H^1_{\text{ét}}(X_t, \mathbb{G}_m)$, see [508, Chapter 6, Corollary 2.3]. This implies the surjectivity of r_t. Moreover, by loc.cit., Corollary 2.7, the canonical homomorphism

$$H^i_{\text{ét}}(X, \boldsymbol{\mu}_n) \to H^i_{\text{ét}}(X_t, \boldsymbol{\mu}_n), \quad i \geq 0,$$

is bijective for any n that is prime to the characteristic. It follows from Theorem 4.3.11 that

$$H^2(X, \mathbb{G}_m) \cong H^2(X_t, \mathbb{G}_m) = 0.$$

The Kummer exact sequence (0.1.6) implies that the canonical homomorphisms

$${}_n\operatorname{Pic}(X) \to {}_n\operatorname{Pic}(X_t), \quad \text{and} \quad \operatorname{Pic}(X)/n\operatorname{Pic}(X) \to \operatorname{Pic}(X_t)/n\operatorname{Pic}(X_t)$$

are bijective. This proves the assertion. □

Let us assume that $C = \operatorname{Spec} R$ is strictly local and let $\mathfrak{m}$ be the maximal ideal of R. For all $i > 0$, we denote by R_i the artinian ring $R/\mathfrak{m}^i$.

For brevity of notation, we set $D := \bar{X}_t$ so that $X_t = mD$, where $m = d_0 \cdot p^r$ is the multiplicity of the closed fiber. Let X_n be the closed subscheme defined by the Cartier divisor nD with the ideal sheaf $\mathcal{I}_{X_n} = \mathcal{O}_X(-nD)$. By Oort's dévissage [582], the closed embeddings $X_n \hookrightarrow X_{n+1}$ define surjective homomorphism of groups

$$r_n : \operatorname{Pic}^\circ(X_n) \to \operatorname{Pic}^\circ(X_{n-1}) \tag{4.2.2}$$

with kernel isomorphic (as an abelian sheaf) to

$$\operatorname{Ker}\left(H^1(X_n, \mathcal{O}_{X_n}) \to H^1(X_{n-1}, \mathcal{O}_{X_{n-1}})\right) = \operatorname{Coker}\left(H^0(X_{n-1}, \omega_{X_{n-1}}) \to H^0(X_n, \omega_{X_n})\right)^\vee,$$

where $\omega_{X_n} = \omega_{X/C}(nD) \otimes \mathcal{O}_{X_n}$ is the canonical sheaf of X_n.

Proposition 4.2.2 *The restriction homomorphism r_n from* (4.2.2) *is surjective and its kernel is either trivial or isomorphic to the additive group of* $\Bbbk$. *The latter happens if and only if ω_{X_n} is isomorphic to $\mathcal{O}_{X_n}$.*

Proof Since X_1 is an indecomposable divisor of canonical type by Proposition 2.2.3, we have $H^0(X_1, \mathcal{O}_{X_1}) \cong \Bbbk$. Since ω_{X_n} is of degree zero on each component of X_1, we have $H^0(X_1, \omega_{X_n} \otimes \mathcal{O}_{X_1})$ is either zero or $\omega_{X_n} \otimes \mathcal{O}_{X_1} \cong \mathcal{O}_{X_1}$. In the first case, r_n is an isomorphism. We have $\omega_{X_n}(X_n) \cong \mathcal{O}_{X_n}(X_n)$. This gives a short exact sequence

$$0 \to \omega_{X_{n-1}} \to \omega_{X_n} \to \omega_{X_n} \otimes \mathcal{O}_{X_1} \to 0.$$

Since $\omega_{X_n} \in \mathrm{Pic}^0(X_n)$, we have $\omega_{X_n} \otimes \mathcal{O}_{X_1} \cong \mathcal{O}_{X_1}$ if and only if $H^0(X_1, \omega_{X_n} \otimes \mathcal{O}_{X_1}) \cong \Bbbk$. In this case, we use a commutative diagram

$$\begin{tikzcd} \mathcal{O}_{X_n} \arrow[r] \arrow[d] & \omega_{X_n} \otimes \mathcal{O}_{X_n} \arrow[d] \\ \mathcal{O}_{X_1} \arrow[r] & \omega_{X_1} \otimes \mathcal{O}_{X_1} \end{tikzcd}$$

to conclude that a non-zero section $\mathcal{O}_{X_1} \to \omega_{X_n} \otimes \mathcal{O}_{X_1}$ lifts to a non-zero section of $\omega_{X_n} \otimes \mathcal{O}_{X_n}$, hence $\omega_{X_n} \otimes \mathcal{O}_{X_{n+1}} \cong \mathcal{O}_{X_n}$. $\square$

Note that the projective system $(X_{nm})_n$ coincides with the projective system (X_n). By Artin's algebraization theorem [21, Theorem 3.5], we have an injective homomorphism

$$\mathrm{Pic}^\circ(X) \to \varprojlim{}_n \mathrm{Pic}^\circ(X_n)$$

with dense image. It follows from above that the kernel of each map $\mathrm{Pic}^\circ(X_n) \to \mathrm{Pic}^\circ(X_{n-1})$ is surjective with the kernel isomorphic to the additive group of a linear space over $\Bbbk$. This gives another way to see Proposition 4.2.1 and it also shows that the kernel on p-torsion points can be very large.

The following corollary is assertion 4 from Theorem 4.1.6.

Corollary 4.2.3 *Let d_0 be the order of $\mathcal{O}_{X_1}(X_1)$ in* $\mathrm{Pic}(X_1)$ *and m be the multiplicity of the closed fiber. Then, $m = d_0 \cdot p^r$ for some $r \geq 0$ and $m = d_0$ if $p = 0$.*

Proof The invertible sheaf $\mathcal{N} = \mathcal{O}_X(X_1) \in \mathrm{Pic}(X)$ is of order m and its image in $\mathrm{Pic}(X_1)$ is of order d_0. If $p = 0$, then we know that the kernels of the restriction maps r_n are vector spaces over a field of characteristic zero and hence r_n is an isomorphism on torsion subgroups. This proves that $m = d_0$. On the other hand, if $p > 0$, then the kernel of r_n is killed by p, so the set of orders d_n of $\mathcal{M} \otimes \mathcal{O}_{X_n}$ consists of numbers $d_0 \cdot p^{s(n)}$ with non-decreasing function $s(n)$ bounded by the multiplicity m. Since the image of $\mathrm{Pic}(X)$ in $\varprojlim{}_n \mathrm{Pic}(X_n)$ is dense, we obtain $m = d_0 \cdot p^r$ for some r. $\square$

At the end of this section, we will return to the study of the truncation maps $\mathrm{Pic}(X_{n+1}) \to \mathrm{Pic}(X_n)$ to give an application to Theorem 4.1.6 and to the computation of the length of the torsion sheaf $\mathcal{T}$ of $R^1 f_* \mathcal{O}_X$.

Assume that C is strictly local with closed point t and generic point η. Let $\mathrm{Div}(X)_{\mathrm{fib}}$ be the free abelian group generated by irreducible components R_i of X_t. Then, the kernel $\mathrm{Div}(X)^0_{\mathrm{fib}}$ of the natural homomorphism $\mathrm{Div}(X)_{\mathrm{fib}} \to \mathrm{Pic}(X)$ is the cyclic group $\mathbb{Z}[X_t]$ generated by the class of the fiber $X_t = m_t(\sum_i n_i R_i)$, where m_t denotes the multiplicity of the fiber. The image of $\mathrm{Div}_{\mathrm{fib}}(X)$ inside $\mathrm{Pic}(X)$ is equal to the kernel $\mathrm{Pic}_0(X)$ of the restriction homomorphism $r_\eta : \mathrm{Pic}(X) \to \mathrm{Pic}(X_\eta)$.

Let $\mathrm{Div}_{\mathrm{fib}}(X)^\vee = \mathrm{Hom}(\mathrm{Div}_{\mathrm{fib}}(X), \mathbb{Z})$ be the dual abelian group and let

$$\deg_t : \mathrm{Pic}(X) \to \mathrm{Div}_{\mathrm{fib}}(X)^\vee \tag{4.2.3}$$

be the homomorphism that assigns to a divisor class D the linear function $R_i \mapsto D \cdot R_i$. For any irreducible component R_i of X_t of multiplicity n_i, there exists a finite and flat S-scheme T of degree n_i and a regular C-embedding $u : T \to X$, such that $T \cap R_i = T \times_X R_i$ is a point, see [606, Corollary 7.1.2]. This shows that there exists an invertible sheaf $\mathcal{L} \in \mathrm{Pic}(X)$, such that $\deg_t(\mathcal{L})$ is the delta-function of the component R_i. In particular, the homomorphism $\deg_t$ is surjective.

Next, let

$$\beta_t : \mathrm{Div}_{\mathrm{fib}}(X)^\vee \to \mathbb{Z}, \quad l \mapsto l(X_t)$$

be the evaluation of X_t. Then, the composition $\beta_t \circ \deg_t$ assigns to $D \in \mathrm{Pic}(X)$ its *total degree*, that is, the intersection number $D \cdot X_t$.

Next, we consider the composition

$$\alpha_t : \mathrm{Div}_{\mathrm{fib}}(X) \to \mathrm{Pic}(X) \to \mathrm{Div}_{\mathrm{fib}}(X)^\vee, \tag{4.2.4}$$

where the first map is the natural homomorphism followed by $\deg_t$. Using the basis $\{R_i\}$ of $\mathrm{Div}_{\mathrm{fib}}(X)$ and the induced basis of $\mathrm{Div}_{\mathrm{fib}}^\vee$, the map α_t is given by the matrix $(R_i \cdot R_j)$. In the case where X_t is reducible, this is the Cartan matrix of an affine root system of type equal to the type of X_t. It equips $\mathrm{Div}_{\mathrm{fib}}(X)$ with the structure of a quadratic lattice. The kernel of α_t is the cyclic group generated by $\bar{X}_t$, which is also equal to the radical of the lattice. In particular,

$$\overline{\mathrm{Div}}_{\mathrm{fib}}(X) := \mathrm{Div}_{\mathrm{fib}}(X)/\mathrm{Ker}(\alpha_t) \cong \mathrm{Im}(\alpha_t)$$

is isomorphic to the root lattice of finite type corresponding to the affine root system associated to $\bar{X}_t$. In other words, it is of type A_n, D_n, E_n if X_t is of type $\tilde{A}_n, \tilde{D}_n, \tilde{R}_n$, respectively. Moreover, it follows that

$$\mathrm{Discr}_t(X) := \mathrm{Ker}(\beta_t)/\mathrm{Im}(\alpha_t)$$

is the discriminant group of the root lattice $\bar{\mathrm{Div}}_{\mathrm{fib}}$.

In the following, we review some results of Raynaud from [606] about the relative Picard functor $\underline{\mathrm{Pic}}_{X/C}$, which we will identify with its sheafication $\mathcal{P}_{X/C}$. First, we note that this functor is never representable by a separated group scheme unless the fiber X_t is an integral scheme. The reason is simple: assume that X_t is reducible or multiple. Then, the subgroup $\mathrm{Pic}_0(X)$ of $\mathrm{Pic}(X)$ is isomorphic to the non-trivial group $\mathrm{Div}_{\mathrm{fib}}/\mathrm{Div}_{\mathrm{fib}}(X)^0$. The restriction of $\mathcal{P}_{X/C}(C) = \mathrm{Pic}(X)$ to $\mathcal{P}_{X/C}(X_\eta)$ has a non-trivial kernel, from which it follows that $\mathcal{P}_{X/C}$ cannot be represented by a separated scheme.

To remedy this situation, Raynaud considers the scheme-theoretical closure $\mathcal{E}$ of the zero section inside $\mathcal{P}_{X/C}$. First, we recall that for any abelian sheaf $\mathcal{F}$ in the flat topology on some scheme S and every subsheaf G of $\mathcal{F}_\eta$, the scheme-theoretical closure $\bar{G}$ of G in $\mathcal{F}$ is an abelian subsheaf of $\mathcal{F}$, which is generated by morphisms $u : Z \to \mathcal{F}$, where Z is a flat S-scheme, such that $u_\eta : Z_\eta \to \mathcal{F}_\eta$ factors through G. In the case where the sheaf $\mathcal{F}$ is representable by a scheme F over S, then this coincides with the scheme-theoretical closure of the subscheme G in F, which is the unique flat subscheme of F, whose general fiber equal to G, see [269, (2.4.5)].

Let $\mathcal{P}'_{X/C}$ (resp. $\mathcal{P}^0_{X/C}$) denote the subsheaf of $\mathcal{P}_{X/C}$ whose values on any $T \to C$ are elements of $\mathcal{P}_{X/C}(T)$ such that their restriction to any fiber of $X_T \to T$ are the isomorphism classes of invertible sheaves of degree 0 (resp. degree zero on each irreducible component of the fiber).

Proposition 4.2.4 *Let $X \to C$ be a genus one fibration over a strictly local C with closed fiber X_t of multiplicity m_t. Let $\mathcal{E}$ be the scheme-theoretical closure of the zero section in $\mathcal{P}_{X/C}$.*

1. *The closed fiber $\mathcal{E}_t$ is representable by an affine group scheme* $\mathbf{e}$*, which is of finite type over $\Bbbk$ of dimension $h^0(\mathcal{O}_{X_0}) - 1$, the reduced scheme $\mathbf{e}^\circ_{\mathrm{red}}$ is a connected unipotent algebraic group of the same dimension.*
2. $\mathcal{E}(C) \cong \mathrm{Div}_{\mathrm{fib}}(X)/\mathrm{Ker}(\alpha_t)$.
3. $\mathcal{E}(C) \cap \mathcal{P}^0_{X/C}(C) \cong \mathrm{Div}_{\mathrm{fib}}(X)/\mathrm{Div}_{\mathrm{fib}}(X)^0 \cong \mathbb{Z}/m_t\mathbb{Z}$.
4. $\mathcal{P}'(C)/(E(C) + \mathcal{P}^0_{X/C}(C)) \cong \mathrm{Discr}_t(X)$.
5. $\mathbf{e}_t$ *is reduced if and only if X_t is a tame fiber or, equivalently, if and only if f is cohomologically flat.*

We note that $(\mathbf{e}_t)^\circ_{\mathrm{red}}$ is the vector space equal to the kernel of the canonical surjective map $H^1(X_t, \mathcal{O}^\times_{X_t}) \to H^1(\bar{X}_t, \mathcal{O}^\times_{\bar{X}_t})$ defined by the surjection $\mathcal{O}_{X_t} \to \mathcal{O}_{\bar{X}_t}$. The dimension of this kernel is equal to the dimension of the kernel of the map $H^1(X_t, \mathcal{O}_{X_t}) \to H^1(\bar{X}_t, \mathcal{O}_{\bar{X}_t})$, see [20].

In characteristic zero, every group scheme is reduced by Cartier's theorem, see Theorem 0.1.12 and Remark 0.1.13. Thus, in characteristic zero, Assertion (5) implies the following.

Corollary 4.2.5 *If $p = 0$, then all fibers of a genus one fibration $f : X \to C$ are tame and f is cohomologically flat. In particular, $\mathcal{E}_t$ is a constant group scheme associated to the abelian group*

$$\mathrm{Div}_{\mathrm{fib}}(X_t)/\mathbb{Z}[\bar{X}_t] \cong \overline{\mathsf{D}}_t \oplus \mathbb{Z}/m_t\mathbb{Z}.$$

The main result of Raynaud is the following theorem, see [606, Theorem (4.1.1)].

Theorem 4.2.6 *Let $X \to C$ be a genus one fibration, let $\mathcal{P}_{X/C} = \underline{\mathrm{Pic}}_{X/C}$ be the sheafication of the relative Picard functor, and let $\mathcal{E}$ be the scheme scheme-theoretical closure of the zero section in $\mathcal{P}_{X/C}$.*

1. *The sheaf $\mathcal{P}_{X/C}$ is representable by a separated group scheme over C (resp. algebraic space over C), which is locally finite type if and only if X_t is integral (resp. f is cohomologically flat).*
2. *The quotient sheaf $Q_{X/C} = \mathcal{P}_{X/C}/\mathcal{E}$ in the flat topology is representable by a separated smooth group C-scheme $\mathbf{Q}_{X/C}$, which is of locally finite type.*
3. *The quotient sheaf $Q'_{X/C} = \mathcal{P}'_{X/C}/\mathcal{E}$ in the flat topology is representable by a separated smooth group C-scheme $\mathbf{Q}'_{X/C}$ of finite type.*
4. *The quotient sheaf $Q^\circ_{X/C} = \mathcal{P}^\circ_{X/C}/\mathcal{E}$ is the identity component of $\mathbf{Q}'_{X/C}$.*
5. *Any C-homomorphism of $\mathcal{P}$ to a separated algebraic group space over C factors through Q.*

Remark 4.2.7 The results of Raynaud are stated in the case where C is the spectrum of a discrete valuation ring. However, they can be globalized to the case where C is a regular one-dimensional scheme. We define the maximal representable factor Q of $\underline{\mathrm{Pic}}_{X/C}$, whose strict localizations at any closed point coincides with Q from the above. Theorem 4.2.6 generalizes several previously known results concerning the representability of the functor $\mathcal{P}_{X/C}$, which we discussed earlier in Section 0.9.

From now we do not make the assumption that the base C of the fibration is strictly local. We recall from [86, 1.2] the definition of a Néron model.

Definition 4.2.8 Let S be a Dedekind scheme with residue field K at the generic point η. Let X_K be a smooth and separated K-scheme of finite type. A *Néron model* is a smooth and separated scheme $\mathbf{X}$ of finite type over S that satisfies the following universal property:

For each smooth S-scheme Y and each K-morphism $u_K : Y_K \to X_K$ there exists a unique S-morphism $u : Y \to \mathbf{X}$ extending u_K.

The universal property of this definition, which is called the *Néron mapping property*, is somewhat reminiscent of the valuative criterion for properness. Note that a Néron model (if it exists) is usually not proper over S, because if it is, then the valuative criterion of properness would allow one to extend any morphism u_K to a morphism $Y \to X$, where Y is the spectrum of a valuation ring. We refer for the many properties of Néron models to [86, Chapter 7].

It follows from the definition, taking $Y = S$, that there is a canonical bijection

$$X_K(K) \to \mathbf{X}(S).$$

If S is strictly local, then this bijectivity property even characterizes Néronian group schemes, see [86, Chapter 7, Theorem 1.1]. Moreover, G is a Néron model of G_η if and only if the strict localization at every closed point t is the Néron model of G_{η_t} (see [603, (2.3)]). Finally, let $\iota : \eta \to S$ be the inclusion of the generic point.

We will study Néron models only in the cases when $S = C$, global or local, and X_K is a group scheme G_K over K, in which case the Néron model $\mathbf{G}$, if it exists, is a group scheme over C that extends the group scheme G_K, see [86, Chapter 1, Proposition 6]. It is known that the Néron scheme exists if G_K is an abelian variety over K or if G_K is a non-K-unirational *wound unipotent algebraic group* over K that admits a regular compactification, see Section 4.8.

Here is our main example of a Néron model. Let $f : J \to C$ be a genus one fibration together with a fixed section O. Let $J^\sharp$ be the open subset of points $x \in J$ such that f is smooth at x. Assume C is strictly local. The closure of any rational point of J_K is a section of J that intersects each fiber at its smooth point. Thus, it defines a section $s : C \to J^\sharp$. Conversely, any section of $J^\sharp$ restricts to a rational point of J_K. This implies that $J^\sharp$ is a Néron model of $J^\sharp_\eta$ in the strictly local case and hence, it is a Néron model in a global case. Its identity component is obtained from $J^\sharp$ by throwing away all irreducible components of the fiber that do not intersect O. Moreover, it follows from the theory of relative minimal models of regular two-dimensional schemes (see the details in the next section) that J is the unique, up to isomorphism over C, relatively minimal genus one fibration over C that contains $J^\sharp$.

Theorem 4.2.9 *We keep the assumptions and notations of Theorem 4.2.6.*

1. *The group scheme* $\mathbf{Q}'_{X/C}$ *is the Néron model of its generic fiber* $(\mathbf{Q}_{X/C})'_\eta \cong \boldsymbol{Pic}^\circ_{X_\eta/\eta} \cong \mathrm{Jac}(X_\eta)$.
2. *The group scheme* $\mathbf{Q}^\circ_{X/C}$ *is the identity component of the Néron model of* $\mathrm{Jac}(X_\eta)$.
3. *For every closed point $t \in C$, there is a natural isomorphism*

$$\mathrm{Discr}_t(X) \to (\mathbf{Q}_{X/C})'_t/\mathbf{Q}^\circ_{X/C}.$$

The last statement is an improvement of a result of Raynaud [606, Proposition 8.1.2], whose proof can be found in [86, Theorem 9.6.1].

Let X_K be the generic fiber of $f : X \to C$ and $\mathrm{Jac}(X_K) = \mathbf{Pic}^\circ_{X_\eta}$ be its jacobian variety over K. It is an elliptic curve if f is an elliptic fibration and a one-dimensional unipotent algebraic group over K otherwise. Let $j : J \to C$ be a relatively minimal regular projective completion of the Néron model $\mathbf{J}$ of $\mathrm{Jac}(X_K)$. This is a genus one fibration over C with $J^\sharp \cong \mathbf{J}$. We call it the *jacobian fibration* associated to $f : X \to C$..

Recall that the Lie algebra of the relative Picard functor $\mathcal{P}_{X/C}$ coincides with the Lie algebra of the functor $\mathcal{P}^\circ_{X/C}$ and is isomorphic, as a O_C-Module, to $R^1 f_* O_X$. On the other hand, the Lie algebra of the Néron model $j : \mathbf{J} \to C$ is isomorphic to $R^1 j_* O_J$. The morphism

$$q : \mathcal{P}^\circ_{X/C} \to Q'_{X/C}$$

defines a homomorphism of O_C-Modules

$$\mathrm{Lie}(q) : R^1 f_* O_X \to \omega := R^1 j_* O_J . \tag{4.2.5}$$

We know from Theorem 4.1.6 that the quotient of the sheaf $R^1 f_* O_X$ by its torsion subsheaf $\mathcal{T}$ is an invertible sheaf $\mathcal{L}$ of degree $-\chi(O_J) - h^0(\mathcal{T})$. It follows from [473, Theorem 3.1] that the cokernel of $\mathrm{Lie}(q)$ is isomorphic to $\mathcal{T}$. In particular, this compares $\mathcal{T}$ of f with $\mathcal{T}$ of its Néron model.

For the remainder of this section, we study finer invariants associated to multiple fibers, which lead to the Raynaud polygon and give estimates for the length of the torsion subsheaf of $R^1 f_* \mathcal{O}_X$. Let

$$\phi(n) := h^1(X_n, \mathcal{O}_{X_n})$$

and let ν_n be the order of the invertible sheaf $\mathcal{O}_{X_n}(D)$ in $\mathrm{Pic}(X_n)$.

It follows from Proposition 4.2.2 that $\nu_{n+1} = \nu_n$, unless $\mathcal{O}_{X_{n+1}}(D)$ belongs to the kernel of r_n. Since the kernel is killed by p, we have $\nu_{n+1} = p\nu_n$ in this case. Moreover, we have $\nu_1 = d_0$. Let m_1 be the first n such that $\nu(n)$ jumps and becomes equal to $d_0 p$. Let m_2 be the first n that it jumps again and becomes equal to $d_0 p^2$. In this way, we obtain a sequence of numbers $(m_0 = 1, m_1, \ldots, m_r)$. It follows from the definition of m_i that $\mathcal{O}_{X_{m_i}}(D)$ belongs to $\mathrm{Ker}(r_{m_i})$. Hence, by Proposition 4.2.2, $\omega_{X_{m_i}}(m_i D) = \mathcal{O}_{X_{m_i}}(m_i D)$ is trivial.

Lemma 4.2.10 *Set* $m_0 := 1$. *Then:*

1. $m = d_0 \cdot p^r$.
2. $m_{i+1} = m_i + k_i d_0 p^i$ *for some positive integer* k_i.
3. $m_r = mh - a$ *for some positive integer* h *and* $0 \le a < m$.

Proof (1) We know that $\mathcal{O}_X(D)$ is of order m in $\mathrm{Pic}(X)$ and that $\mathcal{O}_{X_1}(D)$ is of order d_0 in $\mathrm{Pic}(X_1)$. The image of $\mathcal{O}_X(D)$ under the homomorphism $\mathrm{Pic}(X) \to \mathrm{Pic}(X_n)$ is of order $m/p^{n(i)}d$ and its image in $\mathrm{Pic}(X_1)$ is d_0. From this, the assertion follows.

(2) Let $j : J \to C$ be the jacobian fibration associated to f and let $\omega = (R^1 j_* \mathcal{O}_J)^\vee$. Let λ be defined by the equality

$$\omega_{X/C} \cong f^*\omega \otimes \mathcal{O}_X(\lambda D) \tag{4.2.6}$$

and set

$$\mathcal{M}_i = \omega_{X/C}(m_i D) = \mathcal{O}_X((\lambda + m_i)D).$$

Proposition 4.2.2 implies that $\omega_{X_{m_i}} = \mathcal{M}_i \otimes \mathcal{O}_{X_{m_i}}$ is trivial. Since $\mathcal{M}$ is a tensor power of $\mathcal{O}_X(D)$, its restriction to X_n is trivial for $n < m_{i+1}$ and also $\omega_{X_{m_i+1}} = \mathcal{M}_i \otimes \mathcal{O}_{X_{m_i+1}}$ is trivial. Thus, $\mathcal{O}_X((\lambda + m_i)D) \otimes \mathcal{O}_{X_{m_i}} = \mathcal{O}_{X_{m_i}}(D)^{\otimes \lambda + m_i}$ is trivial for $n < m_{i+1}$ and it is of order 1 or p for $n = m_{i+1}$. From this, it follows that for $n = m_i + h$ with $m_{i+1} - m_i > h > 0$, the sheaf $\omega_{X_n} = \mathcal{O}_X(hD) \otimes \mathcal{O}_{X_n}$ is trivial if and only if $d_0 p^i$ divides h.

Now, for $n = m_{i+1}$, $\omega_{X_n} = \mathcal{M} \otimes \mathcal{O}_{X_{m_i+1}}((m_{i+1} - m_i)D)$ is trivial and that $\mathcal{M} \otimes \mathcal{O}_{X_{m_i+1}}$ of order 1 or p. This implies that $\mathcal{O}_X(p(m_{i+1} - m_i)D) \otimes \mathcal{O}_{X_{m_i+1}}$ is trivial, hence $d_0 p^{i+1}$ divides $p(m_{i+1} - m_i)$. Therefore, there exists a positive integer k_i, such that $m_{i+1} = m_i + k_i d_0 p^i$.

(3) The sheaf $\mathcal{O}_{X_m}(mD)$ is the normal sheaf of a fiber, hence it is trivial. Thus, $\mathcal{O}_{X_m}(D)$ is of some order $d_0 p^i$. Hence, $m_r > m$ and can be written in the form $m_r = mh - a$ as stated in the assertion. □

Thus, with k_i as in the previous lemma, we have

$$\phi(m_0) = 1 \quad \text{and} \quad \phi(m_i) = 1 + k_1 + \cdots + k_{i-1}, \ 1 \le i \ge r.$$

Let $\mathrm{Rnd}_{X/R}(t) : [0, m_r] \to \mathbb{R}$ be the continuous piecewise affine convex function such that $\mathrm{Rnd}_{X/R}(m_i) = 1 + k_1 + \cdots + k_i, i = 1, \ldots, r$. We call its graph the *Raynaud polygon*. It has slope $\alpha_i := \frac{1}{d_0 p^i}$ in the interval $[m_i, m_{i+1}]$.

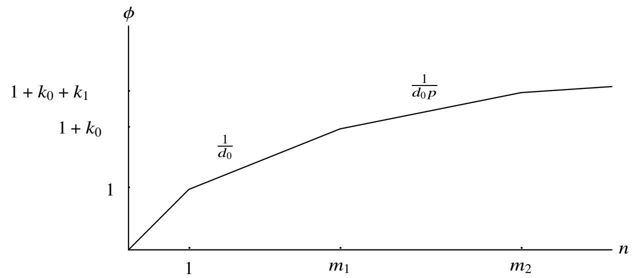


Fig. 4.2 The Raynaud polygon $\mathrm{Rnd}_{X/C}$

It follows that the Raynaud polygon of a tame fibration is just the linear function with slope 1 in the segment $[0, 1]$.

Let $\psi = \mathrm{Rnd}^{-1}_{X/C}$ be the inverse function. It is a piecewise linear strictly increasing function with

$$\psi(1) = 1 \quad \text{and} \quad \psi(1 + k_0 + \cdots + k_i) = m_i.$$

For $1 + k_0 + \cdots + k_{i-1} \leq n < 1 + k_0 + \cdots + k_i$ we have

$$\psi(n+1) = \psi(n) + \frac{m_{i+1} - m_i}{k_i} = d_0 p^i = \nu(n).$$

Let $q : \mathcal{P}^{\circ}_{X/C} \to Q^0_{X/C}$ be the quotient map from Theorem 4.2.9. Then, we have the following result, see [58, §4].

Proposition 4.2.11 *Let* $r_n : \mathcal{P}^0_{X/C}(C) = \mathrm{Pic}^0(X) \to \mathrm{Pic}^0(X_n)$ *and* $r'_n : Q^0_{X/C}(C) \to Q^0_{X/C}(R_n)$ *be the restriction homomorphisms. There exist homomorphisms* q_n : $\mathrm{Pic}^0(X_{\psi(n)}) \to Q^0_{X/C}(R_n)$ *that make the following diagram commutative:*

$$\begin{tikzcd} \mathrm{Pic}^0(X) \arrow[r, "q"] \arrow[d, "r_{\psi(n)}"] & Q^0_{X/C}(C) \arrow[d, "r'_n"] \\ \mathrm{Pic}^0(X_{\psi(n)}) \arrow[r, "q_n"] & Q^0_{X/C}(R_n). \end{tikzcd}$$

Moreover, each homomorphism q_n *is surjective and its kernel is the subgroup of* $\mathrm{Pic}^0(X_{\psi(n)})$ *generated by* $\mathcal{O}_{X_{\psi(n)}}(-D)$.

Finally, following [606], we give an application of the function $\phi(n)$ to the computation of the length of the torsion sheaf $\mathcal{T}$ from Theorem 4.1.6.

Theorem 4.2.12 *Let $R^1 f_* \mathcal{O}_X = \mathcal{L} \oplus \mathcal{T}$, where $\mathcal{L}$ is invertible and $\mathcal{T}$ is a torsion sheaf. Then,*

$$l(\mathcal{T}) = \left[\frac{\ell}{m}\right] = \left[(1 - \frac{1}{m}) + k_0(1 - \frac{1}{p^r}) + \cdots + k_{r-1}(1 - \frac{1}{p})\right].$$

Proof Taking the transpose of the homomorphism of Lie algebras (4.2.5), we obtain a homomorphism

$$\omega \to (R^1 f_* \mathcal{O}_X)^\vee = \mathcal{L}^{\otimes -1},$$

whose cokernel is isomorphic to the torsion sheaf $\mathcal{T}$. By definition of l in (4.2.6), we have $f_* \omega_{X/C} = \omega \otimes \mathfrak{m}^{-[\frac{\ell}{m}]}$. Comparing it with the equality $R^1 f_* \mathcal{O}_X = \mathcal{L} \oplus \mathcal{T}$, we get $\mathcal{T} \cong R/\mathfrak{m}^{[\frac{\ell}{m}]}$ and $l(\mathcal{T}) = \left[\frac{\ell}{m}\right]$.

It remains to compute $\frac{\ell}{m}$. Applying Lemma 4.2.10, we obtain

$$\phi(m_r) = 1 + k_0 + \cdots + k_{r-1} = \phi(m_r + a) = \phi(mh).$$

In particular, for $n \geq m_r$, we have $\phi(n) = \phi(n-1) + 1$ if and only if $n = m_r + km$. Since $m_r = mh - a$ by the previous lemma, we get

$$\phi(m_r) = \phi(m_r + a) = \phi(mh). \tag{4.2.7}$$

The coherent sheaves $R^1 f_* \mathcal{O}_X$, $\mathcal{L}$, and $\mathcal{T}$ correspond to R-modules M, L, and T, respectively. We have $f^* \mathfrak{m} = \mathcal{O}_X(-mD)$ and $R^1 f_* \mathcal{O}_{X_{mn}} = M/\mathfrak{m}^n M$. The short exact sequence

$$0 \to \mathfrak{m}^n/\mathfrak{m}^{n+1} \to M/\mathfrak{m}^{n+1} M \to M/\mathfrak{m}^n M \to 0$$

shows that $l(M/\mathfrak{m}^{n+1} M) = l(M/\mathfrak{m}^n M) + 1$. Hence, we find $l(R^1 f_* \mathcal{O}_{X_{mh}}) = l(T) + l(L/\mathfrak{m}^h L) = l(T) + h$. By the Base Change Theorem, we have $l(R^1 f_* \mathcal{O}_{X_{mn}}) = h^1(\mathcal{O}_{X_{mn}})$ for $n > h$, hence

$$\phi(hm) = l(T) + h. \tag{4.2.8}$$

Since $\omega_{X_{m_r}} = \mathcal{O}_X((l + m_r)D) \otimes \mathcal{O}_{X_{m_r}}$ is trivial and the order of $\mathcal{O}_X(D) \otimes \mathcal{O}_{X_{m_r}}$ is equal to m, we have

$$\ell + m_r = \alpha m$$

for some positive integer α. By Lemma 4.2.10, we have $\ell = (\alpha - h)m + a$ and taking into account (4.2.7) and (4.2.8), we obtain

$$l(T) = \left[\frac{\ell}{m}\right] = \alpha - h = \phi(m_r) - h.$$

Thus, $\alpha = \phi(m_r)$ and we get

$$\ell = \phi(m_r)m - m_r = m(1 + k_0 + \cdots + k_{r-1}) - (1 + k_0 d_0 + \cdots + k_{r-1} d_0 p^{r-1}).$$

This gives

$$\frac{\ell}{m} \ = \ (1 - \frac{1}{m}) + k_0(1 - \frac{1}{p^r}) + \cdots + k_{r-1}(1 - \frac{1}{p})$$

as asserted. □

Corollary 4.2.13 *Let a_t and $m_t = d_0 \cdot p^{r_t}$ be as in Theorem 4.1.6. Then*

$$\frac{a_t}{m_t} \ = \ \frac{\ell_t}{m_t} - \left[\frac{\ell_t}{m_t}\right],$$

where

$$\ell_t \ = \ m_t\big((1 - \frac{1}{m_t}) + k_0(1 - \frac{1}{p^r}) + \cdots + k_{r_t-1}(1 - \frac{1}{p})\big)$$

for some positive integers $k_0, \ldots, k_{r_t-1}$ if X_t is wild. In particular, we have $a_t = m_t - 1$ if and only if X_t is tame.

Proof We localize at $t \in C$ and may assume that C is strictly local. We let $l_t = l$, $m_t = m$, $a - t = a$ be as in the local computations from above. It follows from Theorem 4.1.6 that $\omega_{X/C} = f^*\mathcal{L}^{-1} \otimes \mathcal{O}_X(aD)$. On the other hand, it follows from the definition of l that $\omega_{X/C} = f^*\omega \otimes \mathcal{O}_X(lD)$. It remains to use that $f^*\mathcal{L}^{-1} = f^*\omega \otimes \mathcal{O}_X(l(T)mD)$, which implies that $\ell = a + l(T)m_t$. Then, we divide by m and apply Theorem 4.2.12□

Corollary 4.2.14 *Let $m = d_0 \cdot p^r$. Then*

$$l(\mathcal{T}) \ \geq \ r.$$

Proof Since $(1 - 1/p^r) \geq (1 - 1/2^r)$, we obtain

$$\frac{\ell}{m} \ \geq \ (1 - 2^{-r}) + r - (2^{-r} + \cdots + 2^{-1}) \ = \ (1 - 2^{-r}) + r - 1 + 2^{-r} \ = \ r,$$

hence $\left[\frac{\ell}{m}\right] \geq r - 1$. □

Example 4.2.15 Assume $l(T) = 1$. Then we have $r = 1$ by the above. Since $k_0 < p$, we compute

$$\frac{\ell}{m} \ = \ (1 - \frac{1}{d_0 p}) + k_0(1 - \frac{1}{p}) \ = \ (1 + k_0) - \frac{1 + d_0 k_0}{d_0 p}.$$

Since $\left[\frac{\ell}{m}\right] = 1$ and $\frac{1+d_0k_0}{d_0p} \leq 1$, we find $k_0 = 1, m_1 = 1 + d_0 p$, and $m = p$. Conversely, if $k_0 = 1$ then $\left[\frac{\ell}{m}\right] = 1$. If the fiber is wild with $d_0 = 1$, then $d_0 = 1$ and hence

$$\mathrm{Rnd}_{X/R}(t) = \begin{cases} 1 & \text{if } 0 \leq t \leq 1, \\ (x + p - 1)/p & \text{if } 1 \leq t \leq 1 + p. \end{cases}$$

We have $\frac{a_t}{d_0 p} = 1 - \frac{1+d_0k_0}{d_0p}$ and hence, using our assumption $l(T) = 1$,

$$a_t = m_t - 1 - d_0.$$

This result is shown by other means in the corollary to [78, Proposition 4].

4.3 Jacobian Fibrations

A genus one fibration $f : X \to C$ is called a *jacobian* fibration if it admits a section, that is, if $X(C) \neq \emptyset$. In the previous section, we showed how to associated to a genus one fibration $f : X \to C$ its jacobian fibration $j : J \to C$, but see also Proposition 4.3.1. In this section, we will study how invariants like the Brauer group, Betti numbers, the Euler–Poincaré characteristic, and the geometry of the fibers change when passing from f to j. On our way, we will introduce the *Mordell–Weil group* and prove the *Shioda–Tate formula*.

The generic fiber X_η of a jacobian fibration $f : X \to C$ is a regular and geometrically irreducible curve of genus one with a K-rational point. Here, K denotes the function field $\mathbb{k}(C)$ or, equivalently, the residue field of C at its generic point η, that is, $\eta = \operatorname{Spec} K$. If X_η is smooth, then the choice of such a K-rational point turns X_η into a one-dimensional abelian variety over K, that is, an *elliptic curve* over K. On the other hand, if X_η is not smooth, then there is a unique non-smooth point on X_η, the *cusp*, and the choice of a K-rational point turns the smooth locus $X_\eta^\sharp = X_\eta \setminus \{\text{cusp}\}$ of X_η into a one-dimensional unipotent algebraic group over K. In fact, the base change of X_η to an algebraic closure $\overline{K}$ of K is isomorphic to the additive group $\mathbb{G}_{a,\overline{K}}$ and thus, X_η is a *twisted form* of $\mathbb{G}_a$ over $\eta = \operatorname{Spec} K$ (see also Section 0.1).

For any morphism $g : X \to T$ of regular schemes, we set

$$X^\sharp := \{x \in X \ : \ g \text{ is smooth at } x\}.$$

The following proposition summarizes what we have found in the previous section.

Proposition 4.3.1 *Let $f : X \to C$ be a genus one fibration. Then, there exists a jacobian genus one fibration $j : J \to C$, unique up to a C-isomorphism, that satisfies the following properties:*

1. $J_\eta^\sharp \cong \operatorname{Jac}(X_\eta)$,
2. *the natural map of sections $J(C) \to J_\eta(\eta)$ is a bijection and defines the structure of an abelian group on $J(C)$,*
3. *the image of any section $C \to J$ lies in $J^\sharp$, and*
4. *there exists a natural group scheme structure over C on $f : J^\sharp \to C$, which is isomorphic to the Néron model of $J_\eta^\sharp$.*

It follows from Theorem 4.2.9 that $J_t^\sharp$ is a one-dimensional and commutative algebraic group, which is not necessarily proper, reduced, or connected. Its connected component of identity $(J_t^\sharp)^\circ$ is an elliptic curve if J_t is smooth, it is the multiplicative group $\mathbb{G}_{m,\Bbbk}$ if the fiber is singular of multiplicative type, and it is the additive group $\mathbb{G}_{a,\Bbbk}$ otherwise. The group of connected components $J_t^\sharp/(J_t^\sharp)^\circ$ is trivial in the first case and it is isomorphic to the discriminant group Discr_t otherwise. We recall that the structure of the discriminant groups is given by (0.8.5) and Table (0.2) from Section 0.8. We have

$$J_t^\sharp/(J_t^\sharp)^\circ \;\cong\; \begin{cases} \{1\} & \text{if } J_t \text{ is smooth, or of types } \tilde{A}_0^*, \tilde{A}_0^{**}, \\ \mathbb{Z}/(n+1)\mathbb{Z} & \text{if } J_t \text{ is of type } \tilde{A}_n, \tilde{A}_n^*, \\ (\mathbb{Z}/2\mathbb{Z})^{\oplus 2} & \text{if } J_t \text{ is of type } \tilde{D}_{2k}, \\ \mathbb{Z}/4\mathbb{Z} & \text{if } J_t \text{ is of type } \tilde{D}_{2k+1}, \\ \mathbb{Z}/3\mathbb{Z} & \text{if } J_t \text{ is of type } \tilde{E}_6, \\ \mathbb{Z}/2\mathbb{Z} & \text{if } J_t \text{ is of type } \tilde{E}_7, \\ \{1\} & \text{if } J_t \text{ is of type } \tilde{E}_8. \end{cases}$$

For a point $t \in C$ (not necessarily closed) we let

$$r_t \;:\; \mathrm{Pic}(X) \;\to\; \mathrm{Pic}(X_t)$$

be the restriction homomorphism. We set

$$\begin{aligned} \mathrm{Pic}_{\mathrm{fib}}(X) &:= \mathrm{Ker}(r_\eta), \\ \mathrm{Pic}_{\mathrm{fib}}(X/C) &:= \mathrm{Pic}_{\mathrm{fib}}(X)/f^*\,\mathrm{Pic}(C), \\ \mathrm{Pic}_0(X) &:= \mathrm{Ker}(\deg \circ r_\eta), \end{aligned}$$

where η denotes the generic point of C. Then, the following proposition follows from the definitions and the local information about $\mathrm{Pic}(X(t))$ that we established above.

Proposition 4.3.2 *There are the following isomorphisms of quadratic lattices:*

1. $\mathrm{Pic}_{\mathrm{fib}}(X/C) \;\cong\; \oplus_{t\in C}\mathsf{D}_t/\mathsf{D}_t^0$.
2. $\mathrm{Pic}_{\mathrm{fib}}(X/C) \cap \mathrm{Pic}_0(X) \;\cong\; \oplus_{t\in C}\,\mathrm{Ker}(\alpha_t)/\bar{\mathsf{D}}_t^0 \;\cong\; \oplus_{t\in C}\mathbb{Z}/m_t\mathbb{Z}$.
3. $\mathrm{Pic}_0(X)/\mathrm{Pic}_{\mathrm{fib}}(X) \;\cong\; \mathrm{Jac}(X_\eta)(\eta)$.

We will say that a genus one fibration $f : X \to C$ is *trivial* if it is isomorphic to a product, that is, isomorphic to $\mathrm{pr}_2 : F \times C \to C$ over C. Since we assumed X to be regular, this implies that F is smooth, that is, an elliptic curve. The following result is known as the *Mordell–Weil Theorem*.

Theorem 4.3.3 *Assume that C is global and let $f : J \to C$ be a non-trivial jacobian genus one fibration.*

1. The abelian group $J(K) \cong J(C)$ is a finitely generated abelian group.
2. If f is quasi-elliptic, then $J^\sharp(K)$ is an elementary abelian p-group.

Proof First, assume that f is an elliptic fibration. The pull-back morphism f^* : $\mathbf{Pic}_{C/\Bbbk} \to \mathbf{Pic}_{J/\Bbbk}$ induces a homomorphism of abelian varieties $\mathrm{Jac}(C) = \mathbf{Pic}^\circ_{C/\Bbbk} \to$

$\mathbf{Pic}^{\circ}_{J/\Bbbk}$. By the Poincaré Reducibility Theorem (see [541], Chapter 4, §19), there exists an abelian variety E_0 over $\Bbbk$ and an isogeny

$$E_0 \times \mathbf{Pic}^{\circ}_{C/\Bbbk} \to \mathbf{Pic}^{\circ}_{X/\Bbbk}.$$

Under the restriction morphism $r_\eta : \mathbf{Pic}^{\circ}_{X/\Bbbk} \to \mathbf{Pic}^{\circ}_{J_K/K}$, the image of E_0 is the $K/\Bbbk$-trace of J_K. We now assume $E_0 = 0$. Then,

$$\begin{aligned}
\mathrm{Pic}(J)/f^*(\mathrm{Pic}(C)) &\cong \mathbf{Pic}_{J/\Bbbk}(\Bbbk)/f^*\mathbf{Pic}_{C/\Bbbk}(\Bbbk),\\
\mathbf{Pic}_{J/\Bbbk}(\Bbbk)/\mathbf{Pic}^{\circ}_{J/\Bbbk}(\Bbbk) &\cong \mathrm{NS}(J),\\
\mathbf{Pic}_{C/\Bbbk}(\Bbbk)/\mathbf{Pic}^{\circ}_{C/\Bbbk}(\Bbbk) &\cong \mathbb{Z}.
\end{aligned}$$

This implies that $\mathrm{Pic}(J)/f^*\,\mathrm{Pic}(C)$ is a finitely generated abelian group of rank equal to $\rho(J) - 1$. Thus, $\mathrm{Pic}_0(J)/f^*\,\mathrm{Pic}(C)$ is finitely generated of rank $\rho(J) - 2$. Finally, $J(K)$ is finitely generated and it follows from Proposition 4.3.2 that

$$\mathrm{rank}\ J(K) = \rho(X) - 2 - \sum_{t\in C} (\#\,\mathrm{Irr}(J_t) - 1)\,. \tag{4.3.1}$$

Let us show that under our assumptions E_0 is always trivial. If E_0 was not trivial, then $\dim E_0 = 1$, and $J_K \cong E_0 \otimes_{\Bbbk} K$. Thus, the trivial minimal elliptic fibration $E_0 \times C \to C$ would have the same generic fiber as our fibration. By the uniqueness of relative minimal models, it would follow that $J \cong E \times C$ over C, that is, $f : J \to C$ is trivial, a contradiction to our assumptions.

Finally, assume that $f : J \to C$ is a quasi-elliptic fibration. Since its generic geometric fiber is a cuspidal cubic over the separable closure of $\kappa(\eta)$, it cannot contain an abelian variety. The previous argument implies that the group $J^\sharp(K)$ is a finitely generated subgroup of the additive group of $\overline{K}$, hence it is a finite elementary abelian p-torsion group. ◻

Definition 4.3.4 The group $J(K)$ is called the *Mordell–Weil group* of the jacobian genus one fibration $f : J \to C$ and it is denoted by $\mathrm{MW}(J/C)$ or $\mathrm{MW}(J_K)$. If $f : J \to C$ is the jacobian fibration associated to a genus one fibration $X \to C$, then $\mathrm{Jac}(X_K) \cong J_K$ and we extend the notion of the Mordell–Weil group to $X \to C$ by setting $\mathrm{MW}(X/C) := \mathrm{MW}(J/C)$.

In Section (4.5), we will study lattice structures on the Mordell–Weil group and its dual and relate these to the intersection pairing on $\mathrm{NS}(J)$. Formula (4.3.1) is called the *Shioda–Tate formula*.

We note that the Shioda–Tate formula for a quasi-elliptic fibration $f : X \to C$ becomes

$$\rho(J) = 2 + \sum_{t\in C} (\#\,\mathrm{Irr}(J_t) - 1)\,. \tag{4.3.2}$$

Next, we note that the proof of the Mordell–Weil theorem shows that in the case when the associated jacobian fibration is not trivial, then there is an isomorphism

$$\mathrm{MW}(J/C) \cong \mathrm{Num}(J)_0/\mathrm{Num}_{\mathrm{fib}}(J), \tag{4.3.3}$$

where $\mathrm{Num}(J)_0$ (resp. $\mathrm{Num}_{\mathrm{fib}}(J)$) is the image of $\mathrm{Pic}_0(J)$ (resp. $\mathrm{Pic}_{\mathrm{fib}}(J)$) in $\mathrm{Num}(J)$.

Proposition 4.3.5 *Let $f : J \to C$ be a global genus one jacobian fibration.*

1. *If f is non-trivial, then $b_1(J) = b_1(C)$.*
2. *If f is trivial, then $b_1(J) = b_1(C) + 2$.*
3. *If $\chi(\mathcal{O}_J) > 0$, then the Néron–Severi group $\mathrm{NS}(J)$ has no torsion and coincides with $\mathrm{Num}(J)$.*

Proof Assertion 1 follows essentially from the proof of Theorem 4.3.3. First, assume that the $\Bbbk$-trace E_0 of J_η is zero if f is an elliptic fibration. Then, we have an isogeny of abelian varieties $\mathrm{Jac}(C) \to \mathbf{Pic}^\circ(J)$. The usual Kummer exact sequence in the étale topology implies that $b_1(C) = 2\dim \mathrm{Jac}(C)$ and $b_1(J) = 2\dim \mathbf{Pic}^\circ(J)$. This proves the assertion in this case.

If E_0 is non-zero, then $E_0 \times \mathrm{Jac}(C)$ is isogenous to $\mathbf{Pic}^\circ(J)$, and Assertion 2 follows easily from that.

Let us prove the last assertion. We follow the proof of [646, Theorem 6.4]. Let $D \neq 0$ be a torsion divisor class. Then $h^0(D) + h^0(K_J - D) \geq \chi(\mathcal{O}_J) > 0$. Since a non-trivial torsion class cannot be effective, we have $K_J - D \sim D'$, where D' is an effective divisor. The restriction of K_J and D to each irreducible component of a fiber is of degree 0, hence D' is a linear combination of fibers and hence $D' = f^*(d')$ for some effective divisor class on C. By Theorem 4.1.6, $K_J = f^*(k)$ for some divisor class on C, we obtain that $D = f^*(k - d)$ where $k - d$ is a torsion divisor class on C. Hence, $D \in f^* \mathrm{Jac}(C)$ and thus, it is algebraically equivalent to zero. □

Corollary 4.3.6 *If $f : J \to C$ is a jacobian genus one fibration, then $\mathrm{NS}(J)$ is torsion free unless f is a smooth non-trivial elliptic fibration or a quasi-elliptic fibration $f : J \to C$, where C is an elliptic curve and all fibers are irreducible.*

Proof Suppose that f is not a smooth elliptic fibration. We will prove later in Proposition 4.4.9 that $\deg R^1 f_* \mathcal{O}_J > 0$, then Theorem 4.1.6 implies that $\chi(\mathcal{O}_J) > 0$. If f is a trivial fibration, then obviously $\mathrm{NS}(X)$ is torsion-free.

Next, suppose that f is a quasi-elliptic jacobian fibration. Then the inequality (4.4.35) from Section 4.4 shows that $\chi(\mathcal{O}_J) > 0$ unless C is an elliptic curve. Then we have $b_1(C) = b_1(J) = 2$, and, since J is obviously minimal, the classification Proposition 4.4.12 shows that J is a bielliptic surface (a surface from the last two rows in the table of Proposition 1.1.5). These surfaces are classified in [77]. □

We will see later in Proposition 4.3.14 that $b_1(X) = b_1(J)$ if $J \to C$ is the jacobian fibration of $X \to C$, hence the first two assertions are true for any elliptic fibration. Also, in Section 4.7 we will show that $\mathrm{NS}(J)$ may have torsion if f is a smooth fibration.

Let $f : J \to C$ be a jacobian genus one fibration. By definition, there exists a section and we will now fix a section $\mathtt{e} : C \to J$. Its image $\mathsf{E} := \mathtt{e}(C) \subset J$ is a one-dimensional subscheme. We therefore obtain the structure of a commutative group scheme on $J^\sharp$ over C, which is isomorphic to the Néron model of $J^\sharp_\eta$. Its zero section is equal to $\mathtt{e}$.

For every closed point $t \in C$, the intersection number $J_t \cdot \mathsf{E}$ is equal to 1, which implies that a jacobian fibration has no multiple fibers. In particular, the canonical bundle formula, that is, Theorem 4.1.6, simplifies in this case as follows.

Proposition 4.3.7 *Let $f : J \to C$ be a global genus one jacobian fibration. Then,*

$$\omega_J \;\cong\; f^*(\mathcal{L}^{-1} \otimes \omega_C),$$

where $\mathcal{L} = R^1 f_* \mathcal{O}_J$ is an invertible sheaf of degree $-\chi(\mathcal{O}_J)$ on C.

Corollary 4.3.8 *If $\mathsf{s} : C \to \mathsf{S} \subset J$ is a section of a jacobian fibration, then*

$$\mathsf{S}^2 \;=\; -\chi(\mathcal{O}_J),$$

where S^2 denotes the self-intersection number.

Proof By the adjunction formula, we have

$$\begin{aligned}\omega_C &\cong \mathsf{s}^*\omega_{\mathsf{S}} \;\cong\; \mathsf{s}^*(\mathcal{O}_{\mathsf{S}}(\mathsf{S}) \otimes \omega_J)\\ &\cong \mathsf{s}^*\mathcal{O}_{\mathsf{S}}(\mathsf{S}) \otimes \mathsf{s}^*(f^*(\omega_{\mathsf{S}} \otimes \mathcal{L}^{-1})) \;\cong\; \mathcal{O}_{\mathsf{S}}(\mathsf{S}) \otimes \omega_C \otimes \mathcal{L}^{-1}.\end{aligned}$$

This implies $\mathcal{O}_{\mathsf{S}}(\mathsf{S}) \cong \mathcal{L}$, hence $\mathsf{S}^2 = \deg \mathcal{O}_{\mathsf{S}}(\mathsf{S}) = \deg \mathcal{L} = -\chi(\mathcal{O}_J)$. □

Theorem 4.3.9 *Let $f : J \to C$ be a non-trivial jacobian genus one fibration and let* $\mathrm{MW}(J/C)$ *be its Mordell–Weil group.*

1. *For any closed point $t \in C$ and a prime number $\ell \neq p$, the natural restriction homomorphism*
$$_{\ell^\infty}\,\mathrm{MW}(J/C) \;\to\; {}_{\ell^\infty}J_t^\sharp$$
is injective.
2. *Any non-trivial torsion section is not contained in $(J^\sharp)^\circ$ unless f is a smooth fibration.*
3. *Assume that* $\mathrm{MW}(J/C)$ *is finite. Then,*
$$\#\,\mathrm{MW}(J/C)^2 \,\cdot\, \#\,\mathrm{Discr}(\mathrm{Num}(J)) \;=\; \prod_{t\in C} \#\,\mathrm{Discr}_t(J). \tag{4.3.4}$$

Proof (1) Since $J(C) \subset J(K_t)$, where K_t is the fraction field of a strict henselization $\mathcal{O}^h_{C,t}$, we may assume that C is strictly local. We know that $J^\sharp \to C$ represents the sheaf $\underline{\mathrm{Pic}}_{J/C}/\mathcal{E}$. Since C is strictly local, this implies that $J(C) = \mathrm{Pic}(J)/\mathcal{E}(C)$. By Proposition 4.2.1, the kernel of the restriction homomorphism $r_t : \mathrm{Pic}(J) \to \mathrm{Pic}(J_t)$ is a group that is uniquely divisible by any prime $\ell \neq p$. By Proposition 4.2.4, we have $_{\ell^\infty}\mathcal{E}(C) = \{0\}$. Now, we consider the following commutative diagram, whose vertical arrows are restriction homomorphisms.

$$\begin{tikzcd}
0 \arrow[r] & {}_{\ell^k}\operatorname{Pic}(J) \arrow[r] \arrow[d] & {}_{\ell^k}J(C) \arrow[r] \arrow[d] & E(C)^{(\ell^k)} \arrow[d] \\
0 \arrow[r] & \operatorname{Pic}(J_t) \arrow[r] & J_t^\sharp \arrow[r] & E(t)^{(\ell^k)}.
\end{tikzcd}$$

Since $\mathcal{E}(C) \to \mathcal{E}(t)$ is a bijection, we obtain that ${}_{\ell^\infty}\operatorname{Ker}(J(C) \to {}_{\ell^\infty}J_t^\sharp)$ is equal to the kernel of ${}_{\ell^\infty}\operatorname{Pic}(J) \to {}_{\ell^\infty}\operatorname{Pic}(J_t)$. Since it is divisible by ℓ, it has no non-trivial ℓ-torsion elements.

(2) Let $E_1, E_2 \in (J^\sharp)^\circ(C)$ be two different torsion sections. Then, there exists an integer n such that $n(E_1 - E_2) \in \operatorname{Pic}_{\mathrm{fib}}(J)$. Since E_1 and E_2 intersect the same irreducible components in each fiber, we have $E_1 - E_2 \in \operatorname{Pic}_{\mathrm{fib}}(J)^\perp$. This easily that

$$n(E_1 - E_2) \equiv mF,$$

where F is any closed fiber. Applying Corollary 4.3.8, we get

$$0 = (E_1 - E_2)^2 = E_1^2 + E_2^2 - 2E_1 \cdot E_2 = -2\chi(\mathcal{O}_J) - 2E_1 \cdot E_2.$$

Taking E_2 to be the zero section O, we get $0 \le E \cdot \mathsf{O} = -\chi(\mathcal{O}_J)$, hence

$$E \cdot \mathsf{O} = 0, \quad \chi(\mathcal{O}_J) = 0.$$

By Proposition 4.4.9 in the next section this happens if and only if $f : J \to C$ is smooth.

(3) It follows from the proof of the Mordell–Weil Theorem that the restriction homomorphism $\operatorname{Pic}_0(J) \to J(C)$ is surjective and that its kernel is generated by $\operatorname{Pic}_{\mathrm{fib}}(J)$. This homomorphism factors through a surjection $\operatorname{Num}_0(J)/\operatorname{Num}_{\mathrm{fib}}(J) \to J(C)$, where $\operatorname{Num}_0(J)$ is the group of numerical divisor classes, whose restriction to J_η is of degree 0, and $\operatorname{Num}_{\mathrm{fib}}(J)$ is generated by the numerical classes of irreducible components of fibers. Let L_1 be the sublattice of $\operatorname{Num}(J)$ generated by the class $[J_t]$ and the class of the zero section E. This is a sublattice of $\operatorname{Num}(J)$ isomorphic to the hyperbolic plane. The orthogonal complement L_2 of L_1 in $\operatorname{Num}_{\mathrm{fib}}(J)$ is contained in $\operatorname{Num}_0(J)$ and is generated by the components that do not intersect E. It is isomorphic to the direct sum of the root lattices of finite type $\mathsf{D}_t/\mathsf{D}_t^0$. We have

$$(\operatorname{Num}(J)/L_1) \perp L_2 \cong \operatorname{Num}_0(J)/\operatorname{Num}_{\mathrm{fib}}(J) \cong J(C).$$

Now, the assertion follows from the relationship between the discriminant of a lattice and its sublattice of finite index, see (0.8.2). □

We will say more about smooth genus one fibrations in Section 4.5 when discussing Mordell–Weil lattices.

Remark 4.3.10 The formula in Assertion 1 is a special case of the *determinant formula* in Section 4.5.

We will now study the relationship between the geometry of a genus one fibration and that of its associated jacobian fibration. We start with the Brauer group and to do so, we will use the following quite general vanishing theorem of M. Artin, see [271, II, Corollaire (3.2)].

Theorem 4.3.11 *Let $f : X \to Y$ be a proper and flat morphism of relative dimension one between locally noetherian regular schemes. Assume that all local rings of Y are Japanese. Then*

$$R^i f_*\mathbb{G}_m = 0 \quad \textit{for all} \quad i \geq 2.$$

Recall that a *Japanese ring* is an integral domain R, such that its normalization in any finite extension of its fraction field is finitely generated as an R-module, see [204, 4.2]. Our global or local bases C satisfy these conditions.

Corollary 4.3.12 *If $f : X \to C$ is a genus one fibration, then there exist canonical isomorphisms*

$$\begin{aligned}\mathrm{Br}(X) = H^2(X, \mathbb{G}_m) &\cong H^1_{\acute{e}t}(C, R^1 f_*\mathbb{G}_{m,X}),\\ H^3(X, \mathbb{G}_m) &\cong H^2_{\acute{e}t}(C, R^1 f_*\mathbb{G}_{m,X}),\\ H^n(X, \mathbb{G}_m) &= 0 \quad \textit{for all} \quad n > 3.\end{aligned}$$

Proof We know that $H^2(C, \mathbb{G}_m) = \mathrm{Br}(C) = 0$ by the Tsen's theorem. Applying Artin's vanishing theorem to the morphism $\pi : C \to \mathrm{Spec}\,\Bbbk$ and using that $\Bbbk$ is algebraically closed, we obtain that $R^i\pi_*\mathbb{G}_{m,C} = H^i(C, \mathbb{G}_{m,C}) = 0$ for $i \geq 2$. Thus, the Grothendieck–Leray spectral sequence

$$E_2^{p,q} = H^q_{\text{ét}}(C, R^p f_*\mathbb{G}_{m,X}) \Rightarrow \mathbf{H}^{p+q} = H^{p+q}_{\text{ét}}(X, \mathbb{G}_{m,X})$$

degenerates already on the E_2-page. Using that $H^i_{\text{ét}}(C, \mathbb{G}_m) = 0$ for $i > 1$, we obtain the stated isomorphism

$$\mathbf{H}^{k+1} = H^n_{\text{ét}}(X, \mathbb{G}_{m,X}) \to E_2^{k,1} = H^1_{\text{ét}}(C, R^1 f_*\mathbb{G}_{m,X}), \quad k \geq 1$$

and the result follows. □

We define the *index* $\mathrm{ind}(f)$ of a genus one fibration $f : X \to C$ to be the smallest degree of a multisection of f. By Corollary 4.6.6, this is the same as the index $\mathrm{ind}(X_\eta)$of the generic fiber. This latter is the smallest degree of a non-trivial and effective divisor on X_η or, equivalently, the gcd of all degrees of divisors on X_η, see Lemma 4.6.3. We will discuss the notion of index in detail in Section 4.6 when studying Weil–Châtelet groups.

Theorem 4.3.13 *Let $f : X \to C$ be a genus one fibration and let $j : J \to C$ be the associated jacobian fibration. Let $\mathrm{ind}(f)$ be the index of X_η and $\mathrm{ind}(f)' = \mathrm{ind}(f)/l$, where $l = l.c.m(\{m_t, t \in C\})$. Then, there exists an exact sequence of abelian groups*

$$0 \to \mathbb{Z}/\mathrm{ind}(f)'\mathbb{Z} \to \mathrm{Br}(J) \to \mathrm{Br}(X) \to \frac{\oplus_{t\in C}\,\mathbb{Z}/m_t\mathbb{Z}}{\mathbb{Z}/l\mathbb{Z}} \to A \to 0,$$

where the group $\mathbb{Z}/l\mathbb{Z}$ embeds diagonally and the group A is trivial if j is not a trivial fibration.

Proof From the previous section, we know that $R^1 f_*\mathbb{G}_{m,X}$, considered as a sheaf in the flat topology, coincides with the relative Picard sheaf $\mathcal{P}_{X/C}$ and its quotient by the subsheaf $\mathcal{E}_f$ is a Néronian sheaf Q_f. Since the sheaf Q_f is represented by a smooth group scheme that is of locally of finite type over C, we have $H^*_{\mathrm{fl}}(S,Q) \cong H^*_{\text{ét}}(C,Q)$, see [508], Chapter 3, Theorem 3.9. Applying Lemma 4.3.12, we obtain an exact sequence

$$H^1_{\text{ét}}(C,\mathcal{E}_f) \to \mathrm{Br}(X) \to H^1_{\text{ét}}(C,Q_f) \to H^2(C,\mathcal{E}_f).$$

Since $\mathcal{E}_f$ is supported in finitely many closed points, the cohomology groups H^1 and H^2 of $\mathcal{E}$ are zero. This gives us an isomorphism

$$\mathrm{Br}(X) \cong H^1_{\text{ét}}(C,Q_f).$$

Replacing f with j in the previous discussion, we obtain an isomorphism

$$\mathrm{Br}(J) \cong H^1_{\text{ét}}(C,Q_j).$$

Now, let Q'_f be the scheme-theoretic closure of $\mathrm{Pic}^0(X_\eta)$ in Q_f. We know that Q'_f coincides with Q'_j and that it is represented by the Néronian scheme $J^\sharp$. We denote it by $\mathcal{A}$ and have an exact sequence

$$0 \to \mathcal{A} \to Q_f \xrightarrow{\deg} \mathbb{Z}_S. \tag{4.3.5}$$

If $f = j$, then the degree map is surjective on sheaves and surjective on global sections. It is known that

$$H^1(C,\mathbb{Z}_C) = 0$$

(see [508, Chapter 3, §3]), and hence, we obtain an isomorphism

$$H^1(C,\mathcal{A}) \cong H^1(C,Q_j) \cong \mathrm{Br}(J).$$

We denote by $\mathbb{Z}'_C$ the subsheaf of the constant sheaf $\mathbb{Z}_C$ that is equal to the image of the homomorphism deg. Then, we have an exact sequence

$$0 \to \mathbb{Z}'_C \to \mathbb{Z}_C \to \oplus_{t\in C}(i_t)_*(\mathbb{Z}/m_t\mathbb{Z})_t \to 0, \tag{4.3.6}$$

where $i_t : t \hookrightarrow C$ denotes the inclusion of the closed point $t \in S$. Passing to cohomology in exact sequence

$$0 \to \mathcal{A} \to Q_f \to \mathcal{Z}'_S \to 0$$

gives an exact sequence

$$0 \to \mathrm{Coker}(H^0(C, \mathcal{Q}_f) \stackrel{\deg}{\to} H^0(C, \mathcal{Z}'_C)) \to \mathrm{Br}(J) \to \mathrm{Br}(X) \to \mathrm{Ker}(H^1(C, \mathbb{Z}'_C) \to H^2(C, \mathcal{A})) \to 0.$$

By the Chinese Remainder Theorem we obtain

$$H^0(C, \mathcal{Z}'_C)) \cong \mathrm{Ker}(\mathbb{Z} \to \oplus_{t\in C}\mathbb{Z}/m_t\mathbb{Z}) = l\mathbb{Z}$$

and

$$H^1(C, \mathcal{Z}'_C)) \cong \mathrm{Coker}(\mathbb{Z}/l\mathbb{Z} \to \oplus_{t\in C}\mathbb{Z}/m_t\mathbb{Z}).$$

This gives

$$\mathrm{Coker}\left(H^0(C, \mathcal{Q}_f) \stackrel{\deg}{\to} H^0(C, \mathcal{Z}'_C)\right) \cong \mathbb{Z}/\mathrm{ind}(f)'\mathbb{Z}.$$

By Theorem 4.7.9, the group $H^2(C, \mathcal{A})$ is trivial if j is not trivial □

Proposition 4.3.14 *Let $f : X \to C$ be a genus one fibration, let $j : J \to C$ be the associated jacobian fibration, and assume that C is global. Then,*

$$\chi(\mathcal{O}_X) = \chi(\mathcal{O}_J) \quad \text{and} \quad e(X) = e(J),$$

where χ and e denote the coherent and the topological (ℓ-adic) Euler–Poincaré characteristic, respectively.

Proof We already cited a result from [473] that the kernel and the cokernel of the map (4.2.5)

$$\mathrm{Lie}(q) : R^1 f_*\mathcal{O}_X \to R^1 j_*\mathcal{O}_J$$

is isomorphic to the torsion subsheaf $\mathcal{T}$ of $R^1 f_*\mathcal{O}_X$. This implies that $\chi(R^1 f_*\mathcal{O}_X) = \chi(R^1 j_*\mathcal{O}_J)$ and the equality $\chi(\mathcal{O}_X) = \chi(\mathcal{O}_J)$ follows now from the Grothendieck–Leray spectral sequence. By Corollary 4.1.7, $K_X^2 = K_J^2 = 0$. The second equality now follows from Noether's formula. □

We already noted that jacobian fibrations do not have multiple fibers. In particular, when passing from a genus one fibration to its jacobian fibration, there are no longer multiple fibers. Another way to get rid of a multiple fiber is via a suitable base change, which we may even assume not to affect the smooth fibers.

Definition 4.3.15 Let $f : X \to C$ be a genus one fibration and let Σ' be a set of points in C such that the fibers X_t with $t \in \Sigma'$ are multiple. We say that a finite and separable cover $\phi : C' \to C$ *eliminates multiple fibers* from Σ' if there exists a genus one fibration $f' : X' \to C'$ and a rational map $\Phi : X' \dashrightarrow X$ of finite degree making the following diagram commutative:

$$\begin{tikzcd} X' \arrow[r, dashed, "\Phi"] \arrow[d, "f'"] & X \arrow[d, "f"] \\ C' \arrow[r, "\phi"] & C \end{tikzcd}$$

and such that f' has no multiple fibers over points in $\phi^{-1}(\Sigma')$.

The fact that it is always possible to eliminate multiple fibers by a base change was first proved by Kodaira in the case $\Bbbk = \mathbb{C}$. The proof uses the fact that given a finite set of points $t_1, \ldots, t_k$ on a compact Riemann surface C, then one can find a finite Galois cover $\phi : C' \to C$ that is ramified over these points with prescribed ramification indices m_i. If we take m_i to be the multiplicity of the fiber X_{t_i}, then locally the base change is given by $u = v^{m_i}$, where u is a local parameter at t_i and v is a local parameter at a point $t_i' \in \phi^{-1}(t_i)$. Over $\mathbb{C}$, a multiple fiber is always of type $\tilde{A}_{n-1}$ and one shows that after the base change the fiber over $t' \in \phi^{-1}(t_i)$ becomes non-multiple of type $\tilde{A}_{m_i n-1}$, see [402, Section 4] or [43, Section V.7].

The situation in positive characteristic is more complicated: for example, multiple fibers can be of additive additive type [374] and thanks to wild ramification, it is more complicated to construct Galois covers of curves with prescribed ramification indices and prescribed inertia groups. To show that it is possible to eliminate multiple fibers in the case of positive characteristic, we use the following result, see [281, Theorem 3.4] and [377].

Theorem 4.3.16 *Let $\Bbbk$ be an algebraically closed field of characteristic $p > 0$. Let C be a smooth curve over $\Bbbk$ with closed points $t_i, i = 1, \ldots, t_k$. Let G be a finite group with subgroups $H_i, i = 1, \ldots, k$, such that each H_i is isomorphic to the product of a p-group with a cyclic group of order prime to p. Then, there exists a Galois G-cover $C' \to C$ of smooth curves over $\Bbbk$, whose branch divisor contains $\{t_1, \ldots, t_k\}$ and such that the inertia groups at t_i is isomorphic to H_i for all i.*

Now, in the situation of Definition 4.3.15, choose $\{t_1, \ldots, t_k\}$ to be the set Σ' and let m_i be the multiplicity of the fiber X_{t_i}. For any t_i, let $\mathcal{O}_{C,t_i}^h$ be the henselization of $\mathcal{O}_{C,t_i}$ and let $K_i = K_{t_i}^h$ be its field of fractions. Let $i_{t_i} : C(t_i) := \operatorname{Spec} \mathcal{O}_{C,t_i}^h \to C$ be the canonical morphism and let $f(t_i) : X(t_i) = X \times_C C(t_i) \to C(t_i)$ be the base change. The closed fiber of $f(t_i)$ is isomorphic to X_{t_i} and the generic fiber $X_{\eta_i} := X \times_C K_{t_i}^h$ is a genus one curve over the field $K_{t_i}^h$. We use the analogous notation for the jacobian fibration. The curve $X(t_i)_{\eta_i}^\sharp$ is a torsor under the curve $J \times_C K_i$ of order m_i. Let s_i be a closed point of degree m_i on X_{η_i}. Let L_i be the residue field of this point, which is an extension of K_i of degree equal to $m_t' p^n$, where m' is coprime to p. Its Galois group is the product of a p-group and a cyclic group of order prime to p. Now we apply Theorem 4.3.16. We find a finite group G that contains subgroups isomorphic to the groups H_i (for example, the direct product of these groups). Then we find a G-cover $C' \to C$, whose intertia subgroups over the points $t_1, \ldots, t_k$ are isomorphic to $H_1, \ldots, H_k$. We make the base change $X' = X \times_C C' \to C'$ and take a relative minimal model of the generic fiber $X'_{\eta'}$. Then, for any t_i' over t_i, $X(t_i)'_{\eta_i'}$ has a rational point and hence, the fiber $X'_{t_i'}$ is not multiple. Since the process of going to the relative minimal model gives us a rational map $\phi : X' \dashrightarrow X$, we have eliminated the multiple fibers over the points $t_i \in \Sigma'$.

For more effective results about elimination of multiple fibers of elliptic fibrations that are not of additive type, we refer to the work of Katsura and Ueno [374], [375].

Using the elimination process, we will now prove the following.

Proposition 4.3.17 *Let $f : X \to C$ be a genus one fibration, let $j : J \to C$ be the associated jacobian fibration, and let $t \in C$ be a closed point. Then,*

$$b_i(X_t) \;=\; b_i(J_t), \quad i \geq 0,$$

that is, the Betti numbers of the fibers coincide.

Proof This is clear for $i = 0$ and for $i \geq 3$. For $i = 1$, we argue as follows: for any integer n coprime to the characteristic p, the Kummer exact sequence gives an isomorphism of sheaves in étale topology

$$R^1 f_* \boldsymbol{\mu}_n \;=\; {}_n R^1 f_* \mathbb{G}_{m,X} \;:=\; \operatorname{Ker}\left(R^1 f_* \mathbb{G}_{m,X} \stackrel{[n]}{\to} R^1 f_* \mathbb{G}_{m,X} \right)$$

and we have similar isomorphism for $R^1 j_* \boldsymbol{\mu}_n$. Using the comparison of the sheaves $R^1 f_* \mathbb{G}_{m,X}$ and $R^1 j_* \mathbb{G}_{m,J}$ from the proof of Proposition 4.3.13, we obtain an isomorphism $R^1 f_* \boldsymbol{\mu}_n \cong R^1 j_* \boldsymbol{\mu}_n$. Passing to fibers, we get $b_1(X_t) = b_1(J_t)$.

It remains to deal with the case $i = 2$. Here, we will use the elimination of multiple fibers process. Fix one multiple fiber X_{t_1} and let $X_{t_2}, \ldots, X_{t_N}$ be the remaining multiple fibers. We apply the elimination process to the set $\Sigma' = \{t_2, \ldots, t_N\}$, but taking the base change $\phi : C' \to C$ to be unramified over t_1 (that is, we take $H_1 = \{1\}$ in Theorem 4.3.16). Let $X' \to C'$ be the relative minimal model of the base change $X \times_C C' \to C'$. We do the same for the base change of the jacobian fibration $j : J \to C$. For any $t_1' \in C'$ over t_1, the fibers of $X'_{t_1'}$ and $J'_{t_1'}$ are isomorphic. Both fibrations have no multiple fibers outside points over t_1. Thus, the fibrations are strictly locally isomorphic and hence, the fibers $X'_{t'}$ and $J'_{t'}$ are isomorphic for all points $t' \notin \phi^{-1}(t_1)$.

Now we invoke the formula for the Euler–Poincaré characteristic of a genus one fibration from Proposition 4.1.12:

$$e(X') = e(X_{\bar{\eta}'})e(C') \,+ \sum_{t' \notin \phi^{-1}(t_1)} \left(e(X'_{t'}) - e(X'_{\bar{\eta}'}) + \delta_{t'} \right) \,+\, N \left(e(X'_{t_1'}) - e(X'_{\bar{\eta}'}) + \delta_{t_1'} \right)$$

$$e(J') = e(J'_{\bar{\eta}'})e(C') \,+ \sum_{t' \notin \phi^{-1}(t_1)} \left(e(J'_{t'}) - e(J'_{\bar{\eta}'}) + \delta_{t'} \right) \,+\, N \left(e(J'_{t_1'}) - e(J'_{\bar{\eta}}) + \delta_{t_1'} \right),$$

where $N = \#\phi^{-1}(t_1)$. We already know that $e(X'_{t'}) = e(J'_{t'})$ for $t' \notin \phi^{-1}(t_1)$ and we also know that $e(X') = e(J')$. Moreover, the invariants of wild ramification $\delta_{t'}$ at $t' \in C'$ for X' and J' coincide: this is because their definition depends on the ramification of the sheaves $R^1 f_* \boldsymbol{\mu}_n$ and $R^1 j_* \boldsymbol{\mu}_n$ for $n = \ell^k$, where ℓ is a prime different from p, and by above, they are isomorphic if $(n, m_t) = 1$. Thus, we obtain that $e(X_t') = e(X_t') = e(J_t') = e(J_t)$. Since we already know that $b_1(X_t) = b_1(J_t)$, we obtain $b_2(X_t) = b_2(J_t)$. □

Corollary 4.3.18 *Let $f : X \to C$ be a genus one fibration, let $j : J \to C$ be the associated jacobian fibration, and assume that C is global. Then:*

$$\rho(X) = \rho(J) \quad \textit{and} \quad b_i(X) = b_i(J), \quad i \geq 0,$$

where ρ denotes the Picard number.

Proof The equalities for $b_i(X) = b_i(J)$ for $i = 0, 4$ are obvious. We have $b_1(X) = b_1(J)$ by Corollary 4.3.5 and then, we obtain $b_3(X) = b_3(J)$ by Poincaré duality. Moreover, we have $e(X) = e(J)$ by Proposition 4.3.14. This implies that $b_2(X) = b_2(J)$.

Finally, we know from (0.10.29) that $b_2(X) = \rho(X) + t_\ell(X)$ and $b_2(J) = \rho(J) + t_\ell(J)$, where $t_\ell(-)$ denotes the rank of the ℓ-adic Tate module $T_\ell(\mathrm{Br}(-))$ of the Brauer group. The equality $\rho(X) = \rho(J)$ now follows from Theorem 4.3.13. □

Remark 4.3.19 In fact, one can say a little bit more than just the equality $\rho(X) = \rho(J)$. Namely, it follows from Proposition 4.3.2.(3) that there is an isomorphism of groups

$$\mathrm{Pic}_0(X)/\mathrm{Pic}_{\mathrm{fib}}(X) \cong \mathrm{Pic}_0(J)/\mathrm{Pic}_{\mathrm{fib}}(J). \tag{4.3.7}$$

By Proposition 4.3.2.(1), the rank of the subgroup $\mathrm{Pic}_{\mathrm{fib}}(X)$ (resp. $\mathrm{Pic}_{\mathrm{fib}}(J)$) is equal to $\sum_t (\#\,\mathrm{Irr}(X_t) - 1)$ (resp. $\sum_t (\#\,\mathrm{Irr}(X_t) - 1)$). By Proposition 4.3.17, these numbers are equal.

Theorem 4.3.20 *Let $f : X \to C$ be a genus one fibration and let $j : J \to C$ be the associated jacobian fibration. Let $t \in C$ be a closed point. Then*

1. *X_t and J_t are of the same type.*
2. *Moreover, if X_t is not a multiple fiber, then $X_t \cong J_t$.*

Proof Passing to the strict localization, we may assume that C is strictly local. If X_t is not multiple, then, by Hensel's lemma, there exists a section of $X \to C$ that intersects X_t in a smooth point. Thus X_K is a trivial torsor and hence, $X \to C$ is isomorphic to $J \to C$. This proves Assertion 2.

To prove Assertion 1, we first use Artin's Approximation Theorem [22] to assume that C is global. We have $b_i(X_t) = b_i(J_t)$ by Proposition 4.3.17. Then, the classification of degenerate fibers shows that X_t and J_t must be of the same type unless X_t is of type $\tilde{E}_n$ (resp. $\tilde{D}_n$) and J_t is of type $\tilde{D}_n$ (resp. $\tilde{E}_n$). However, Theorem 4.2.9 implies that $\mathrm{Discr}_t(X)$ and $\mathrm{Discr}_t(J)$ are isomorphic. This is enough to see that also in these remaining cases the types of X_t and J_t are the same, see also Table 0.2 for the discriminant groups of root lattices of finite type. □

Remark 4.3.21 The crucial fact used in the proof of this fundamental result is Theorem 4.2.9. A more highbrow proof of the previous theorem can be found in [473, Theorem 6.6], where the authors define the discriminant of a genus one fibration in the strictly local situation and prove that it coincides with the discriminant of its associated jacobian fibration.

We end this section by showing that the list of possible degenerate fibers of a quasi-elliptic fibration is much smaller than the corresponding list of an elliptic fibration.

Corollary 4.3.22 *Let $f : X \to C$ be a quasi-elliptic fibration and let $t \in C$ be a closed point.*

1. *If $p = 2$, then X_t can be of type $\tilde{A}_0^{**}$, $\tilde{A}_1^{*}$, $\tilde{D}_{2k}$, $\tilde{E}_7$, or $\tilde{E}_8$.*
2. *If $p = 3$, then X_t can be of type $\tilde{A}_0^{**}$, $\tilde{A}_2^{*}$, $\tilde{E}_6$, or $\tilde{E}_8$.*
3. *If f is a jacobian fibration, then the discriminant group of the lattice* $\mathrm{Num}(X)$ *is an elementary p-group.*

Proof By the previous theorem, we may assume that f is a jacobian fibration and that C is strictly local. We also know that the Mordell–Weil group $\mathrm{MW}(X/C)$ is a finite p-group by Theorem 4.3.3. By Proposition 4.3.1, the Néron model $\mathbf{A}$ of X_K is a unipotent group, from which we conclude that $\mathbf{A}_t/\mathbf{A}_t^0$ is an elementary p-group. By Theorem 4.2.9, this group is isomorphic to $\mathrm{Discr}_t(X)$, hence $\mathrm{Discr}_t(X)$ is a p-group. Since $\mathbf{A}^0$ is a smooth group scheme with unipotent generic fiber, the closed fiber of $\mathbf{A}^\circ$ is isomorphic to $\mathbb{G}_a$. This implies that all singular fibers must be of additive type. Consulting again Table (0.2) and using Kodaira's classification of degenerate fibers of a genus one fibration, we obtain the lists asserted in (1) and (2). Applying the Shioda–Tate formula, we get (3). □

4.4 Weierstrass Models

In this section, we study genus one fibrations $f : X \to B$ together with a section $\mathfrak{e} : B \to X$. Contracting the fiber components that do not meet $\mathsf{E} := \mathfrak{e}(B)$, we arrive at a fibration $W \to B$ with a usually slightly singular total space W, the *Weierstrass model* of the fibration f. This W can be embedded into the projectivization of a locally free sheaf $\mathcal{E}_3$ of rank 3 on B, that is, a $\mathbb{P}^2$-bundle over B. More precisely, $W \subset \mathbb{P}(\mathcal{E}_3) \to B$ is locally given by an equation of degree 3, that is, $W \to B$ is a relative cubic, which makes the Weierstrass model a useful tool for explicit computations. For example, it allows us to analyze the singular fibers, to compute automorphisms, and to compute local and global invariants.

Let $f : X \to B$ be a genus one fibration, where B is any integral scheme over an algebraically closed field $\Bbbk$. We will also make the following assumptions:

1. f admits a section $\mathfrak{e} : B \to X$ such that f is smooth at each point of $\mathsf{E} = \mathfrak{e}(B)$,
2. f is cohomologically flat, and
3. for every point $t \in B$, the canonical sheaf ω_{X_t} is isomorphic to $\mathcal{O}_{X_t}$.

If X is normal and if B is a regular one-dimensional scheme, then the second condition follows from the first one, see [606, Théorème 7.2.1]. If X is even smooth over $\Bbbk$, then we are in the situation studied in the previous sections and we remind the reader of the discussion after Theorem 4.1.6, where we treated wild fibers and the relation to cohomological flatness. Since wild fibers are multiple, the existence of a section implies that there are no multiple fibers, in particular no wild fibers, and thus, f is cohomologically flat. For other criteria of cohomological flatness, we refer to [166, Proposition 2.7].

Coming back to the general setup above, we note that the third condition is satisfied, for example, if all fibers X_t are reduced or we are in the situation of Proposition 4.3.7.

It follows from the discussion in the beginning of this chapter that every fiber X_t of f is a genus one curve, whose canonical sheaf ω_{X_t} is isomorphic to $\mathcal{O}_{X_t}$.

In the case where $B = \operatorname{Spec} \mathbb{k}$, a genus one fibration is simply a genus one curve and the section E gives a distinguished $\mathbb{k}$-rational point. This point defines a Cartier divisor of degree 1 and three times this divisor is a very ample divisor on X that embeds it as a curve of degree 3 inside $\mathbb{P}^2$. The equation of this latter curve can be chosen in such way that the image of E is the point $[0, 1, 0]$ at infinity and linear coordinate changes yield the *Weierstrass normal form* of the cubic, see, for example, [682, Chapter 3, §1]. Coming back to the general setup $f : X \to B$ from above, we now want to construct a *relative Weierstrass model* $g : W \to B$. That is, we would like to derive a birational model $g : W \to B$ of $f : X \to B$, whose generic fiber is isomorphic to a plane cubic curve in $\mathbb{P}^2_\eta$ given by its Weierstrass normal form.

Lemma 4.4.1 *Under the above assumptions:*

1. *If $i > 0$ and $n > 0$, then $R^i f_* \mathcal{O}_X(n\mathsf{E}) = 0$.*
2. *If $n \geq 0$, then the sheaf $f_* \mathcal{O}_X(n\mathsf{E})$ is locally free of rank equal to n.*
3. *The natural homomorphism of sheaves $f_* \mathcal{O}_X \to f_* \mathcal{O}_X(\mathsf{E})$ is an isomorphism.*
4. *There is an isomorphism of invertible sheaves $(R^1 f_* \mathcal{O}_X)^{\otimes -1} \cong \mathsf{e}^* \Omega^1_{X/B}$, and*
5. *If $n > 0$, then the cokernel of the natural homomorphism $f_* \mathcal{O}_X(n\mathsf{E}) \to f_* \mathcal{O}_X((n+1)\mathsf{E})$ is isomorphic to $\mathcal{L}^{\otimes n}$, where $\mathcal{L} = R^1 f_* \mathcal{O}_X$.*

Proof We use some standard properties of cohomology of a projective morphism, see [294, Chapter III]. Since f is of relative dimension one, we have, for any coherent sheaf $\mathcal{F}$ on X that $R^i f_* \mathcal{F} = 0$ if $i > 1$. The base change theorem allows us to compute the fiber of $R^1 f_* \mathcal{F}$ at a point $t \in B$. We have

$$(R^1 f_* \mathcal{F})_t \;\cong\; H^1(X_t, \mathcal{F} \otimes \mathcal{O}_{X_t}).$$

Next, let us show that

$$H^1(X_t, \mathcal{O}_{X_t}(n\mathsf{E})) \;=\; 0, \tag{4.4.1}$$

which implies that

$$R^1 f_* \mathcal{O}_X(n\mathsf{E}) = 0 \quad \text{for all } n > 0. \tag{4.4.2}$$

Of course, if X_t is an integral scheme, then this follows from the vanishing of cohomology of any invertible sheaf with positive degree on a genus one curve. By Serre duality on X_t, we have $\dim H^1(X_t, \mathcal{O}_{X_t}(n\mathsf{E})) = \dim H^0(X_t, \mathcal{O}_{X_t}(-n\mathsf{E}))$. We will prove (4.4.1) by induction on n. First, we treat the case $n = 1$. Consider the usual short exact sequence

$$0 \to \mathcal{O}_X(-\mathsf{E}) \to \mathcal{O}_X \to \mathcal{O}_\mathsf{E} \to 0.$$

After tensoring with $\mathcal{O}_{X_t}$, we obtain the exact sequence

$$0 \to \mathcal{O}_{X_t}(-\mathsf{E}) \to \mathcal{O}_{X_t} \to \mathcal{O}_{\mathfrak{o}} \to 0,$$

where $\mathfrak{o} = \mathsf{e}(t) \in \mathsf{E}$. Here, we have used that $\mathrm{Tor}_1^{\mathcal{O}_X}(\mathcal{O}_{\mathsf{E}}, \mathcal{O}_{X_t}) = 0$, since X_t and E intersect transversally at $\mathfrak{o}$ and since X_t is smooth at $\mathfrak{o}_0$. By assumption, f is cohomologically flat and we find $H^0(X_t, \mathcal{O}_{X_t}) = \Bbbk$. Since the homomorphism $H^0(X_t, \mathcal{O}_{X_t}) \to \mathcal{O}_X(\mathsf{E}) \to \mathcal{O}_{\mathfrak{o}}$ is nonzero, we obtain that $H^0(X_t, \mathcal{O}_{X_t}(-\mathsf{E})) = 0$ and hence, $H^1(X_t, \mathcal{O}_{X_t}(\mathsf{E})) = 0$.

To prove the induction step, we tensor the exact sequence

$$0 \to \mathcal{O}_X(n\mathsf{E}) \to \mathcal{O}_X((n+1)\mathsf{E}) \to \mathcal{O}_{\mathsf{E}}((n+1)\mathsf{E}) \to 0 \tag{4.4.3}$$

with $\mathcal{O}_{X_t}$ and obtain an exact sequence

$$0 \to \mathcal{O}_{X_t}(n\mathsf{E}) \to \mathcal{O}_{X_t}((n+1)\mathsf{E}) \to \mathcal{O}_{\mathfrak{o}} \to 0.$$

Here, we have used that the sheaf $\mathcal{N} = \mathcal{O}_{\mathsf{E}}(\mathsf{E})$ is the normal sheaf of a regularly embedded hypersurface E, hence it is an invertible sheaf. Taking cohomology and using the induction hypothesis, we obtain (4.4.1). This proves (1).

It follows from (4.4.1) that $R^1 f_* \mathcal{O}_X(\mathsf{E}) = 0$. By the Base Change Theorem, this implies that $f_*\mathcal{O}_X(\mathsf{E})$ is a locally free sheaf. Its rank is equal to $\dim H^0(X_t, \mathcal{O}_{X_t}(\mathsf{E})) = \chi(\mathcal{O}_{X_t}(\mathsf{E}))$. Since f is flat, this number is the same for every t, which is why we may assume that X_t is an integral genus one curve. But then, we obtain by Riemann–Roch that this number is equal to $\deg \mathcal{O}_{X_t}(\mathsf{E}) = 1$. This proves (2) and (3).

Let us prove the remaining two assertions. We use exact sequence (4.4.3). The sheaf $\mathcal{O}_{\mathsf{E}}((n+1)\mathsf{E})$ is isomorphic to the $(n+1)$-fold tensor power of the normal sheaf $\mathcal{N} = \mathcal{O}_{\mathsf{E}}(\mathsf{E})$ of the section E. Let $n \geq 0$. Applying the functor f_* and using (2) and (4.4.2), we obtain an exact sequence

$$0 \to f_*\mathcal{O}_X(n\mathsf{E}) \to f_*\mathcal{O}_X((n+1)\mathsf{E}) \to f_*(\mathcal{N}^{\otimes(n+1)}) \to 0. \tag{4.4.4}$$

Let us take $n = 0$. Since f is cohomologically flat and of relative dimension one, we find $\dim(R^1 f_*\mathcal{O}_X)(t) = \dim H^1(X_t, \mathcal{O}_{X_t}) = 1$. Thus,

$$\mathcal{L} := R^1 f_* \mathcal{O}_X \tag{4.4.5}$$

is an invertible sheaf. In particular, the exact sequence (4.4.4) is an exact sequence of invertible sheaves. Since a surjective homomorphism of invertible sheaves is an isomorphism, we obtain that the arrow

$$f_*\mathcal{N} \cong \mathsf{e}^*\mathcal{N} \to \mathcal{L}$$

is an isomorphism and $f_*\mathcal{O}_X \to f_*\mathcal{O}_X(\mathsf{E})$ is bijective. This proves (3).

The exact sequence

$$0 \to \mathcal{N}^{-1} \to \Omega^1_{X/B} \to \Omega_{\mathsf{E}/B} \to 0$$

gives an isomorphism

$$\mathsf{e}^*\mathcal{N}^{-1} \;\cong\; \mathsf{e}^*\Omega^1_{X/B}$$

and establishes assertion (4). Since

$$f_*(\mathcal{N}^{\otimes n}) \;\cong\; \mathsf{e}^*\mathcal{N}^{\otimes n} \;\cong\; (\mathsf{e}^*\mathcal{N})^{\otimes n} \;\cong\; \mathcal{L}^{\otimes n},$$

we get (5). □

We now come to the construction of the Weierstrass model. We set

$$\mathcal{E}_n \;:=\; f_*\mathcal{O}_X(n\mathsf{E}). \tag{4.4.6}$$

For for every $n \geq 0$, we obtain a filtration

$$0 \;\subset\; \mathcal{E}_0 \;\subset\; \mathcal{E}_1 \;\subset\; \ldots \;\subset\; \mathcal{E}_n,$$

such that

$$\mathrm{gr}(\mathcal{E}_n) \;\cong\; \bigoplus_{k=0}^{n} \mathcal{L}^{\otimes k}.$$

Next, we recall that to give a B-morphism from X to a projective bundle $\mathbb{P}(\mathcal{E}) \to B$ is equivalent to giving an invertible sheaf $\mathcal{M}$ on X and a surjective homomorphism $\alpha : f^*\mathcal{E} \to \mathcal{M}$ of $\mathcal{O}_X$-modules. We take $\mathcal{M} = \mathcal{O}_X(3\mathsf{E})$ and $\mathcal{E} = \mathcal{E}_3$, so that we can choose α to be given by an isomorphism $\mathcal{E}_3 = f_*f^*(\mathcal{E}_3) \to f_*\mathcal{O}(3\mathsf{E})$. This isomorphism gives thus rise to a morphism

$$\phi \;:\; X \;\to\; \mathbb{P}(\mathcal{E}_3) \tag{4.4.7}$$

of schemes over B.

If $t \in B$ is a closed point and we restrict ϕ to the X_t, then we obtain a morphism $\phi_t : X_t \to \mathbb{P}^2$. This map ϕ_t corresponds to the complete linear system $|\mathcal{O}_{X_t}(3\mathfrak{o})|$. If X_t is irreducible, then ϕ_t is an isomorphism. If X_t is reducible, then ϕ_t blows down all reducible components that do not intersect the zero section. In particular, the morphism $\phi : X \to \mathbb{P}(\mathcal{E}_3)$ depends in general on the choice of the zero section E.

Definition 4.4.2 The image of ϕ is a closed subscheme $W \subset \mathbb{P}(\mathcal{E}_3)$, which we will call the *Weierstrass model* of the genus one fibration $f : X \to B$ with respect to the section E.

Next, we let x_U (resp. y_U) be a section of $\mathcal{E}_2$ over some open affine set U. We assume that U is small enough so that both projections $\mathcal{E}_2 \to \mathcal{L}^{\otimes 2}$ and $\mathcal{E}_3 \to \mathcal{L}^{\otimes 3}$ are split over U. Locally, this section generates a subsheaf of $\mathcal{E}_2$ (resp. $\mathcal{E}_3$), which is isomorphic to $\mathcal{L}^{\otimes 2}$ (resp. $\mathcal{L}^{\otimes 3}$).

We can consider x_U as a section of $\mathcal{O}_{X_U}(2\mathsf{E})$ and y_U as a section of $\mathcal{O}_{X_U}(3\mathsf{E})$. Thus, x_U^3 and y_U^2 are sections of $\mathcal{O}_{X_U}(6\mathsf{E})$ or $\mathcal{E}_6 = f_*\mathcal{O}_{X_U}(6\mathsf{E})$, whose images generate $\mathcal{L}^{\otimes 6}$. After multiplying by a unit, we may assume that $y_U^2 - x_U^3$ is a section of $\mathcal{E}_5$. Thus, after replacing x_U with $-x_U$, we can write

$$y_U^2 \,+\, a_1^U \cdot y_U x_U \,+\, a_3^U \cdot y_U \,+\, x_U^3 \,+\, a_2^U \cdot x_U^2 \,+\, a_4^U \cdot x_U \,+\, a_6^U \,=\, 0,$$

where a_k^U are certain sections over U. Our choice of indices will become clear below. This is a *local Weierstrass equation*.

Next, let $\mathfrak{U} = (U_i)$ be an open affine cover of B. We now study how x_U, y_U and a_k^U change when passing from some U_i to some U_j. We denote by $x_i, y_i, a_k^{(i)}$ the sections over U_i introduced above. Choose a local generator u_i of $\mathcal{L}$ in U_i. Let (c_{ij}) be the transition functions for the invertible sheaf $\mathcal{L}^{\otimes -1}$, that is, $u_j = c_{ij} u_i$ in $U_i \cap U_j$. Then, we can write

$$x_i \,=\, c_{ij}^2 x_j + \gamma_{ij} \quad \text{ and } \quad y_i \,=\, c_{ij}^3 y_j + c_{ij}^2 \alpha_{ij} x_j + \beta_{ij}, \tag{4.4.8}$$

where $\alpha_{ij}, \beta_{ij}, \gamma_{ij} \in \mathcal{O}_B(U_i \cap U_j)$. It follows that the transition functions for $\mathcal{L}$ are (c_{ij}^{-1}). We can rewrite (4.4.8) in the form

$$\begin{pmatrix} 1 \\ x_j \\ y_j \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ -\gamma_{ij} c_{ij}^{-2} & c_{ij}^{-2} & 0 \\ (\gamma_{ij}\alpha_{ij} - \beta_{ij}) c_{ij}^{-3} & -c_{ij}^{-3}\alpha_{ij} & c_{ij}^{-3} \end{pmatrix} \begin{pmatrix} 1 \\ x_i \\ y_i \end{pmatrix}. \tag{4.4.9}$$

This also gives the transition functions for $\mathcal{E}_3$

$$g_{ij} = {}^t\begin{pmatrix} 1 & 0 & 0 \\ -\gamma_{ij} c_{ij}^{-2} & c_{ij}^{-2} & 0 \\ (\gamma_{ij}\alpha_{ij} - \beta_{ij}) c_{ij}^{-3} & -c_{ij}^{-3}\alpha_{ij} & c_{ij}^{-3} \end{pmatrix} = \begin{pmatrix} 1 & -\gamma_{ij} c_{ij}^{-2} & (\gamma_{ij}\alpha_{ij} - \beta_{ij}) c_{ij}^{-3} \\ 0 & c_{ij}^{-2} & -c_{ij}^{-3}\alpha_{ij} \\ 0 & 0 & c_{ij}^{-3} \end{pmatrix}.$$

The transition matrices for $\mathcal{E}_2$ are

$$\begin{pmatrix} 1 & -\gamma_{ij} c_{ij}^{-2} \\ 0 & c_{ij}^{-2} \end{pmatrix}.$$

In particular, this shows that in general, $\mathcal{E}_2$ and $\mathcal{E}_3$ may not split into direct sums of invertible sheaves.

Using the coordinate change (4.4.8), we obtain that in order that the local equations glue together, the transition matrix for the coefficients is the following:

$$\begin{pmatrix} 1 \\ a_1^{(i)} \\ a_2^{(i)} \\ a_3^{(i)} \\ a_4^{(i)} \\ a_6^{(i)} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & c_{ij} & 0 & 0 & 0 & 0 \\ 0 & 0 & c_{ij}^2 & 0 & 0 & 0 \\ 0 & 0 & 0 & c_{ij}^3 & 0 & 0 \\ 0 & 0 & 0 & 0 & c_{ij}^4 & 0 \\ 0 & 0 & 0 & 0 & 0 & c_{ij}^6 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 2\alpha_{ij} & 1 & 0 & 0 & 0 & 0 \\ 3\gamma_{ij} + \alpha_j^2 & \alpha_{ij} & 1 & 0 & 0 & 0 \\ 2\beta_{ij} & \gamma_{ij} & 0 & 1 & 0 & 0 \\ 2\alpha_{ij}\beta_{ij} + 3\gamma_{ij}^2 & \alpha_{ij}\gamma_{ij} + \beta_{ij} & 2\gamma_{ij} & \alpha_{ij} & 1 & 0 \\ \beta_{ij}^2 + \gamma_{ij}^3 & \gamma_{ij}\beta_{ij} & \gamma_{ij}^2 & \beta_{ij} & \gamma_{ij} & 1 \end{pmatrix} \begin{pmatrix} 1 \\ a_1^{(j)} \\ a_2^{(j)} \\ a_3^{(j)} \\ a_4^{(j)} \\ a_6^{(j)} \end{pmatrix}. \tag{4.4.10}$$

This shows that the local equations

$$y_i^2 \,+\, a_1^{(i)} y_i x_i \,+\, a_3^{(i)} y_i \,+\, x_i^3 \,+\, a_2^{(i)} x_i^2 \,+\, a_4^{(i)} x_i \,+\, a_6^{(i)} \,=\, 0$$

glue together to define a closed subscheme of $\mathbb{P} := \mathbb{P}(\mathcal{E}_3 \otimes \mathcal{L}^{-2})$

$$y^2 + a_1 yx + a_3 y + x^3 + a_2 x^2 + a_4 x + a_6 = 0 \tag{4.4.11}$$

defined by an nonzero section in $H^0(\mathbb{P}, \mathcal{O}_{\mathbb{P}}(3)) \cong S^3(\mathcal{E}_3) \otimes \mathcal{L}^{\otimes -6}$. We call it a *global Weierstrass equation*.

To say more, we consider the following five possible cases in detail:

1. $p \neq 2, 3$,
2. $p = 2$ and $f : X \to B$ is an elliptic fibration,
3. $p = 3$ and $f : X \to B$ is an elliptic fibration,
4. $p = 2$ and $f : X \to B$ is a quasi-elliptic fibration,
5. $p = 3$ and $f : X \to B$ is a quasi-elliptic fibration.

We start with the most common case of an elliptic fibration and $p = \operatorname{char} \Bbbk \neq 2, 3$, that is:

- Case 1: characteristic $p \neq 2, 3$.

Replacing y_i with $y_i + \frac{1}{2}(a_1^{(i)} x_i + a_3^{(i)})$ and then replacing x_i with $x_i + \frac{1}{3} a_2$, we may assume that $a_1^{(i)} = a_2^{(i)} = a_3^{(i)} = 0$. In order for this form to be preserved under the change from U_i to U_j, we have to take $\alpha_{ij} = \beta_{ij} = \gamma_{ij} = 0$. In other words, for this to be fulfilled, the sheaves $\mathcal{E}_2$ and $\mathcal{E}_3$ must split as follows:

$$\begin{aligned} \mathcal{E}_2 \cong \operatorname{gr}(\mathcal{E}_2) &\cong \mathcal{O}_B \oplus \mathcal{L}^{\otimes 2}, \\ \mathcal{E}_3 \cong \operatorname{gr}(\mathcal{E}_3) &\cong \mathcal{O}_B \oplus \mathcal{L}^{\otimes 2} \oplus \mathcal{L}^{\otimes 3}. \end{aligned}$$

For example, this is always possible if $C = \mathbb{P}^1$ because in this case $\deg \mathcal{L} = -\chi(X, \mathcal{O}_X) \leq 0$, hence $\operatorname{Ext}^1(\mathcal{L}^{\otimes 2}, \mathcal{O}_C) \cong H^1(C, \mathcal{L}^{\otimes -2}) = 0$ and $\operatorname{Ext}^1(\mathcal{L}^{\otimes 2} \oplus \mathcal{L}^{\otimes 3}, \mathcal{O}_C) \cong H^1(C, \mathcal{L}^{\otimes -2} \oplus \mathcal{L}^{\otimes -2}) = 0$.

If $\mathcal{E}_2$ and $\mathcal{E}_3$ split into invertible sheaves as above, then the coefficients a_k become sections of $\mathcal{L}^{\otimes k}$, which also explains our choice of numbering the indices. Moreover, in this case we obtain a global equation of the form

$$y^2 + x^3 + a_4 x + a_6 = 0. \tag{4.4.12}$$

We cannot use this if the sheaves $\mathcal{E}_2$ and $\mathcal{E}_3$ do not split.

Localizing at the generic point, we get a Weierstrass equation of the elliptic curve X_η (whence also the name Weierstrass model). The condition that the generic fiber is indeed smooth is that

$$\Delta := 4a_4^3 + 27a_6^2 \in \Gamma(B, \mathcal{L}^{\otimes -12}) \tag{4.4.13}$$

is not zero. More precisely, it may vanish at some closed points of B, but it cannot be identically zero.

Remark 4.4.3 The Weierstrass form can be used to show that a regular curve of genus one over a field K is always smooth if $p = \operatorname{char}(K) \neq 2, 3$, which we already deduced from the much more general Theorem 4.1.3. Here is the argument: first, we pass to the Jacobian and thus, we may assume that $X(K) \neq \emptyset$. We may thus assume that X is given by a Weierstrass equation of the form $y^2 + x^3 + a_4 x + a_6 = 0$. (Here, we use $p \neq 2, 3$.) Computing the partial derivatives of this equation, we find that the curve is not smooth if and only if $\Delta = 4a_4^3 + 27a_6^2 = 0$ and in this case the non-smooth point has coordinates $(x, y) = (-\frac{3a_6}{2a_4}, 0)$ if $a_4 \neq 0$ and $(0, 0)$ if $a_4 = 0$. (Here, we use $p \neq 2, 3$ again.) We see from this explicit description that if there is a non-smooth point, then it is K-rational. In particular, if X is not smooth over K, then there is a non-smooth and K-rational point, and thus, X cannot be regular.

- Case 2: $f : X \to B$ is an elliptic fibration and $p = 2$.

First, we determine the condition for the generic fiber X_η to be smooth. Taking partial derivatives, a non-smooth point satisfies

$$a_1 x + a_3 \;=\; a_1 y + x^2 + a_4 \;=\; 0,$$

where we omit the upper indices. First, assume that $a_1 = 0$. We find that X_η is smooth if and only if $a_3 \neq 0$. If $a_1 \neq 0$, we get $x = a_3/a_1$, $y = (x^2 + a_4)/a_1 = (a_3^2 + a_4 a_1^2)/a_1^3$ and plugging this into the Weierstrass equation, we find

$$\begin{aligned} &\frac{(a_3^2 + a_4 a_1^2)^2}{a_1^6} + \frac{a_3(a_3^2 + a_4 a_1^2)}{a_1^3} + \frac{a_3(a_3^2 + a_4 a_1^2)}{a_1^3} + \frac{a_3^3}{a_1^3} + \frac{a_2 a_3^2}{a_1^2} + \frac{a_4 a_3}{a_1} + a_6 \\ &\quad = \; a_3^4 + a_1^3(a_3^3 + a_1 a_4^2 + a_1^2 a_3 a_4 + a_1 a_2 a_3^2 + a_1^3 a_6) \;=\; 0. \end{aligned}$$

Also, we see that if we replace a_k here with $a_k^{(i)}$ using the local coordinates y_i, x_i, then we obtain that they can be glued together to define a section

$$\Delta \;:=\; a_3^4 + a_1^3 a_3^3 + a_1^4(a_4^2 + a_1^2 a_3 a_4 + a_2 a_3^2 + a_1^2 a_6) \;\in\; \Gamma(B, \mathcal{L}^{\otimes -12}). \qquad (4.4.14)$$

This section is called the *discriminant of a jacobian elliptic fibration*. The fibration is elliptic if and only if $\Delta \neq 0$.

In particular, if $\Delta \neq 0$, then we see that at least one of the coefficients a_1 and a_3 must be non-zero. In this case, we may not be able to split $\mathcal{E}_2$ or $\mathcal{E}_3$. In general, the coefficients $(a_1, a_2, a_3, a_4, a_6)$ define a section of an affine bundle of rank 6 with transition functions

$$\begin{pmatrix} 1 \\ a_1^{(i)} \\ a_2^{(i)} \\ a_3^{(i)} \\ a_4^{(i)} \\ a_6^{(i)} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & c_{ij} & 0 & 0 & 0 & 0 \\ 0 & 0 & c_{ij}^2 & 0 & 0 & 0 \\ 0 & 0 & 0 & c_{ij}^3 & 0 & 0 \\ 0 & 0 & 0 & 0 & c_{ij}^4 & 0 \\ 0 & 0 & 0 & 0 & 0 & c_{ij}^6 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ \gamma_{ij} + \alpha_{ij}^2 & \alpha_{ij} & 1 & 0 & 0 & 0 \\ 0 & \gamma_{ij} & 0 & 1 & 0 & 0 \\ \gamma_{ij}^2 & \alpha_{ij}\gamma_{ij} + \beta_{ij} & 0 & \alpha_{ij} & 1 & 0 \\ \beta_{ij}^2 + \gamma_{ij}^3 & \gamma_{ij}\beta_{ij} & \gamma_{ij}^2 & \beta_{ij} & \gamma_{ij} & 1 \end{pmatrix} \begin{pmatrix} 1 \\ a_1^{(j)} \\ a_2^{(j)} \\ a_3^{(j)} \\ a_4^{(j)} \\ a_6^{(j)} \end{pmatrix}. \qquad (4.4.15)$$

We may locally kill the coefficient $a_2^{(i)}$ by replacing x_i with $x_i + a_2^{(i)}$. In order to eliminate a_2 in the global equations, we must have $\alpha_{ij} = 0$ if $a_1^{(i)} \neq 0$ and $\alpha_{ij}^2 + \gamma_{ij} = 0$ if $a_6^{(i)} \neq 0$. We can achieve this if $\mathcal{E}_2$ and $\mathcal{E}_3$ split.

- Case 3: $f : X \to B$ is an elliptic fibration and $p = 3$

Replacing y_i with $y_i + \frac{1}{2}(a_1^{(i)}x_i + a_3^{(i)})$, we may assume that $a_1^{(i)} = a_3^{(i)} = 0$. In order to preserve this form in the global equation, we must have $\gamma_{ij} = \beta_{ij} = 0$. Then, $\mathcal{E}_3$ must split into a direct sum $\mathcal{E}_2 \oplus \mathcal{L}^{\otimes 3}$, but $\mathcal{E}_2$ may not split. We obtain the global Weierstrass equation

$$y^2 + x^3 + a_2x^2 + a_4x + a_6 = 0. \tag{4.4.16}$$

Here, a_2 is a section of $\mathcal{L}^{\otimes -2}$. On the other hand, (a_2, a_4, a_6) is a section of a rank 3 affine bundle with transition functions

$$\begin{pmatrix} 1 \\ a_2^{(i)} \\ a_4^{(i)} \\ a_6^{(i)} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & c_{ij}^2 & 0 & 0 \\ 0 & 0 & c_{ij}^4 & 0 \\ 0 & 0 & 0 & c_{ij}^6 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ \alpha_{ij}^2 & 1 & 0 & 0 \\ -\alpha_{ij}\beta_{ij} & -\gamma_{ij} & 1 & 0 \\ \beta_{ij}^2 + \gamma_{ij}^3 & \gamma_{ij}^2 & \gamma_{ij} & 1 \end{pmatrix} \begin{pmatrix} 1 \\ a_2^{(j)} \\ a_4^{(j)} \\ a_6^{(j)} \end{pmatrix}. \tag{4.4.17}$$

Computing partial derivatives, we find that X_η is smooth if and only if

$$a_4^3 + 2a_2^2a_4^2 + a_2^3a_6 \neq 0,$$

where we consider the coefficients as rational functions on B. The expressions $a_4^{(i)3} + 2a_2^{(i)2}a_4^{(i)2} + a_2^{(i)3}a_6^{(i)}$ glue together to a section

$$\Delta := a_4^3 + 2a_2^2a_4^2 + a_2^3a_6 \in \Gamma(B, \mathcal{L}^{\otimes -12}). \tag{4.4.18}$$

The generic fiber X_η is an elliptic curve if and only if $\Delta \neq 0$.

- Case 4: $f : X \to B$ is a quasi-elliptic fibration and $p = 2$.

In this case, the expression Δ from (4.4.13) must be equal to zero after passing to an inseparable cover of $\Bbbk(B)$, but it cannot be equal to zero over $\Bbbk(B)$. We use (4.4.11), where we assume that the coefficients are rational functions on B.

If $a_1 \neq 0$, then we replace $a_1x + a_3$ with x' to assume $a_3 = 0, a_1 = 1$. Then, we replace y with $y + a_4$ and may assume that $a_4 = 0$. The discriminant Δ becomes equal to a_6^2. Thus, the curve is not smooth if and only if $a_6 = 0$. However, in this case it is also a non-regular point contradicting the assumption. So, $a_1 = 0$ and then, formula (4.4.13) shows that $a_3 = 0$. Replacing x_i with $x_i + \frac{a_2}{3}$, we may assume that the local equation becomes

$$y_i^2 + x_i^3 + a_4^{(i)}x + a_6^{(i)} = 0.$$

Taking the partial derivative with respect to the x_i's, we see that $x_i^2+a_4^{(i)}$ and $y_i^2+a_6^{(i)}$ both must vanish at a singular point of W.

Putting these observations together, we have thus shown that the fibration is quasi-elliptic if and only if $a_4^{(i)}$ or $a_6^{(i)}$ is not a square in $\Bbbk(B)$.

Since $a_2^{(i)} = a_3^{(i)} = 0$, the transition functions (4.4.15) show that $\gamma_{ij} = \alpha_{ij}^2$. This allows us to define a *global Weierstrass equation*

$$y^2 + x^3 + a_4 x + a_6 \;=\; 0. \tag{4.4.19}$$

The coefficients (a_4, a_6) are sections of an affine bundle $\mathcal{A}_2$ of rank 2 with transition functions defined by matrices

$$\begin{pmatrix} 1 \\ a_4^{(i)} \\ a_6^{(i)} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & c_{ij}^4 & 0 \\ 0 & 0 & c_{ij}^6 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ \gamma_{ij}^2 & 1 & 0 \\ \beta_{ij}^2 + \gamma_{ij}^3 & \gamma_{ij} & 1 \end{pmatrix} \begin{pmatrix} 1 \\ a_4^{(j)} \\ a_6^{(j)} \end{pmatrix}. \tag{4.4.20}$$

If $\mathcal{E}_2$ and $\mathcal{E}_3$ split, then we may choose $\gamma_{ij} = \beta_{ij} = 0$, and obtain that $a_4 \in \Gamma(C, \mathcal{L}^{\otimes -4})$ and $a_6 \in \Gamma(C, \mathcal{L}^{\otimes -6})$. If $a_4^{(i)} = b_i^2$ is a square in K, then the ideal of the cusp is given by $x_i + b_i = 0$ and it must be preserved under the transition functions. We have

$$\begin{aligned} x_i + b_i \;&=\; c_{ij}^2 x_j + \gamma_{ij} + c_{ij}^2 (b_j + \gamma_{ij}) \\ &=\; c_{ij}^2 (x_j + b_j) + (c_{ij}^2 + 1)\gamma_{ij}. \end{aligned}$$

Assume that $\deg \mathcal{L} = -\chi(B) < 0$. Then, $c_{ij}^i \neq 1, i > 0$, and hence $\gamma_{ij} = 0$.

Remark 4.4.4 Suppose $[K(a_4^{1/2}, a_6^{1/2}) : K] = 2$ (for example, if $K = \Bbbk((t))$ or $\Bbbk(t)$). If a_4 is not square in K, then we can write $a_6 = s^2 a_4 + r^2$, and replacing y with $y+r$, we may assume that $r = 0$ and $a_6 = s^2 a_4$. Now, replacing (x, y) with $(x+s^2, y+sx+s^3)$, we get a Weierstrass equation $y^2 + x^3 + (a + s^4)x = 0$ with $a_6 = 0$. The problem with this simple and local equation is that we cannot globalize it: because in order to do so, we would have to be able to choose $\gamma = \alpha = \beta = 0$, which may not always be possible.

- Case 5: $\quad f : X \to B$ is a quasi-elliptic fibration and $p = 3$.

We argue as in the previous case. First, we show that $a_1 = a_3 = 0$. If $a_2 \neq 0$, then we can eliminate a_4 by replacing x with $x - \frac{1}{2}a_2^{-1}a_4$. Then, the formula for the discriminant implies that $a_6 = 0$ and hence, $(x, y) = (0, 0) \in X_\eta$ is a singular point and X_η is not regular. So, we may assume that $a_2 = 0$ and the formula for the discriminant implies $a_4 = 0$. This gives us local equations of the form

$$y_i^2 + x_i^3 + a_6^{(i)} \;=\; 0.$$

The fibration is quasi-elliptic if and only if $a_6^{(i)}$ is not a cube in $\mathcal{O}_B(U_i)$. The local equations can be glued together to a global Weierstrass equation of the form

$$y^2 + x^3 + a_6 \;=\; 0 \tag{4.4.21}$$

only if $\alpha_{ij} = \beta_{ij} = 0$, that is, $\mathcal{E}_2$ may not split, but $\mathcal{E}_3$ has to split as $\mathcal{E}_3 \cong \mathcal{E}_2 \oplus \mathcal{L}^{\otimes 3}$. We have

$$a_6^{(i)} \;=\; c_{ij}^6 (a_6^{(j)} + \gamma_{ij}^3). \tag{4.4.22}$$

This finishes our case-by-case analysis.

Of course, we can use equation (4.4.11) in all Cases 1, 2, and 3 from above, that is, in the cases where the fibration is generically smooth, that is, is elliptic. Following [702], we can unify the formulas for the discriminant in these cases by introducing the following expressions:

$$\begin{array}{l} b_2 := a_1^2 - 4a_2 \\ b_4 := a_1a_3 - 2a_4 \\ b_6 := a_3^2 - 4a_6 \\ b_8 := a_1a_3a_4 - a_4^2 - a_1^2a_6 - a_2a_3^2 + 4a_2a_6. \end{array} \tag{4.4.23}$$

In terms of these, we have the following

Proposition 4.4.5 *The expression*

$$\Delta \;=\; -b_2^2b_8 + 9b_2b_4b_6 - 8b_4^3 - 27b_6^2$$

is equal to the discriminant of an elliptic fibration (up to multiplying by an integer scalar).

Proof Assume $p \neq 2, 3$. Then $a_1 = a_3 = a_2 = 0$ and we obtain

$$b_2 = 0, \quad b_4 = 2a_4, \quad b_6 = -4a_6, \quad b_8 = -a_4^2.$$

This gives

$$\Delta \;=\; -8b_4^3 - 27b_6^2 \;=\; -2^6a_4^3 - 3^32^4a_6^2 \;=\; -2^4(4a_4^3 + 27a_6^2),$$

which agrees with formula (4.4.13).

Next, assume $p = 2$. Then

$$b_2 = a_1^2, \quad b_4 = a_1a_3, \quad b_6 = a_3^2, \quad b_8 = a_1a_3a_4 + a_4^2 + a_1^2a_6 + a_2a_3^2.$$

This gives

$$\Delta \;=\; b_2^2b_8 + b_2b_4b_6 + b_6^2 \;=\; a_3^4 + a_1^3a_3^3 + a_1^4(a_4^2 + a_1^2a_3a_4 + a_2a_3^2 + a_1^2a_6),$$

which agrees with formula (4.4.14).

Finally, assume $p = 3$. Then $a_1 = a_3 = 0$ and

$$b_2 = -a_2,\; b_4 = -a_4,\; b_6 = -a_6,\; b_8 = -a_4^2 + a_2a_6.$$

This gives

$$\Delta = -b_2^2 b_8 + b_4^3 = -a_2^2(a_4^2 + a_2 a_6) - a_4^3,$$

which agrees with formula (4.4.18). □

We can simplify the formula for Δ further by introducing

$$\begin{aligned} c_4 &= b_2^2 - 24 b_4, \\ c_6 &= -b_2^3 + 36 b_4 b_2 - 216 b_6. \end{aligned}$$

Then, we have

$$1728\Delta = c_4^3 - c_6^2.$$

(If $p = 2, 3$, then this formula is also true, but there, it becomes $0 = 0$, which has not much content.) From now on we use this expression for the discriminant of an elliptic fibration.

The rational function on B

$$j := c_4^3/\Delta \tag{4.4.24}$$

is called the *absolute invariant* or the *j-invariant* of the elliptic fibration.

We have

$$j = \begin{cases} 1728 \frac{4a_4^3}{4a_4^3 + 27a_6^2} & \text{if } p \neq 2, 3, \\ \frac{a_1^{12}}{\Delta} & \text{if } p = 2, \\ \frac{a_2^6}{\Delta} & \text{if } p = 3, \end{cases} \tag{4.4.25}$$

and we refer to [682, Appendix A] for further details. The significance of the absolute invariant lies in the following proposition, which shows that it classifies elliptic fibrations geometrically, that is, up to finite covers of the base.

Proposition 4.4.6 *Let C be a global or local. Then, two relatively minimal jacobian fibrations $f : X \to C$ and $f' : X' \to C$ admit isomorphic relative minimal models after some separable finite cover $C' \to C$ if and only if their absolute invariants are equal.*

Proof A finite separable cover $C' \to C$ is determined, via normalization, by the field extension K'/K of their rational function fields. Applying the theory of relative minimal models, it is enough to assume that f and f' are elliptic curves over the field K. Let $e \in X(K)$ and $e' \in X'(K)$. Suppose the X and X' become isomorphic over some separable finite extension K'/K. After composing with a translation automorphism of X, we may assume that we have an isomorphism $\phi : X \to X'$ of elltiptic curves that sends e to e'.

Then, the isomorphism ϕ gives rise to an isomorphism $\phi : W \to W'$ of the Weierstrass models defined over K'. Let $\mathcal{E}_n$, $\mathcal{L}$, $\mathcal{E}'_n$, and $\mathcal{L}'$ be the $\mathcal{O}_{C'}$-modules defined in the previous discussion, which we may regard as linear spaces over K. An isomorphism ϕ defines isomorphisms $\omega \to \omega'$ and $\mathcal{E}_n \to \mathcal{E}'_n$. These are given by invertible matrices with entries in K'. It follows that an isomorphism $W \to W'$ is given by a projective automorphism of the form $(z, x, y) = (z, c^2x' + \gamma z', c^3y' + c^2\alpha x' + \beta z')$ that preserves the Weierstrass equations.

First, assume that we are in Case 1, that is, $p \neq 2, 3$. Then, we have $\alpha = \beta = \gamma = 0$. Thus, the automorphism is given by $(x, y, z) = (c^3 y, c^2 x, z)$. The projective curves $V(y^2 z + x^3 + a_4 xz^2 + a_6 z^3)$ and $V(y^2 z + x^3 + a_4' xz'^2 + a_6' z'^3)$ are isomorphic if and only if $a_2 = a_2' c^4$ and $a_6 = a_6' c^6$. This gives the equality of their absolute invariants. Conversely, assume $j = j'$ and $a_4, a_6 \neq 0$. We get $a_6^2/a_6'^2 = a_4^3/a_4'^3$. Let K'/K be a separable extension such that these ratio is of the form c^6 for some $c \in K'$. Then the projective transformation $(x, y, z) = (c^3 y, c^2 x, z)$ defines an isomorphism $X_{K'} \cong X'_{K'}$. If $j = j'$ and $a_4 = a_4' = 0$, then we have $a_6, a_6' \neq 0$. Then we find K'/K containing $c = (a_6/a_6')^{1/6}$ and we can define an isomorphism $X_{K'} \to X'_{K'}$. If $j = j'$ and $a_6 = a_6' = 0$, then $a_4, a_4' \neq 0$ and we find K'/K containing $c = (a_4/a_4')^{1/4}$ and define an isomorphism $X_{K'} \to X'_{K'}$.

Next, assume we are in Case 2, that is, $p = 2$ and the curves X and X' are smooth over K. We use Weierstrass equation (4.4.11). Suppose X and X' become isomorphic over some separable extension K'/K. Arguing as in the previous case, we obtain a projective transformation

$$(z, x, y) = (z, c^2 x + \gamma z, c^3 y + c^2 \alpha x + \beta z) \tag{4.4.26}$$

with coefficients $c \in K'^{\times}$ and $\alpha, \beta, \gamma \in K'$. The transition formulae for the coefficients of the Weierstrass equation easily give that $j = j'$. Conversely, assume that $j = j'$. We know that (a_1, a_3) and (a_1', a_3') are not equal to $(0, 0)$. Assume $a_1 = 0$. Then $\Delta = a_3^4 \neq 0$ and $j = 0$. Thus $j' = 0$ and we get $a_1' = 0, a_3' \neq 0$. Now, we can get rid of the coefficients a_2 and a_2'. We can solve the equations

$$a_3 = c^3 a_3',\ a_4 = c^4 a_4' + c^3 \alpha a_3' + \alpha^4,$$
$$a_6 = c^6 a_6' + c^4 \alpha^2 a_4' + c^3(\alpha^3 + \beta) a_3' + \beta^2$$

over a separable extension K'/K. The projective transformation (4.4.8) will define an isomorphism $X \to X'$ over K'.

Assume $a_1 \neq 0$. Then $j = j'$ implies that $a_1' \neq 0$. Replacing x with $a_1 x + a_3$ and x' with $a_1' x' + a_3'$, we get $a_3 = a_3' = 0$ and $a_1 = a_1' = 1$. Then $j = j'$ implies $\Delta = a_4^2 + a_6 = \Delta' = a_4'^2 + a_6'$. Now, replace y with $y + a_4$ and y' with $y' + a_4'$. We get the Weierstrass equations $y^2 + xy + x^3 + a_2 x^2 + \Delta = 0$ and $y^2 + xy + x^3 + a_2' x^2 + \Delta = 0$. Let $K' = K(\alpha)$, where α is a root of the separable equation $t^2 + t + a_2 + a_2' = 0$. Then, the transformation (4.4.8) with $c = 1$ and $\gamma = \beta = 0$ yields an isomorphism of the Weierstrass models.

Finally, let us consider Case 3, that is, $p = 3$ and X, X' are smooth over K. We check immediately that j is an invariant under an isomorphism. Assume $j = j' = a_2 = 0$. Let $c = (a_4/a_4')^{1/4}$ so that $c^4 = a_4/a_4'$. Let γ be a solution of the separable equation $t^3 + a_4' t - c^2 a_6' + a_6 = 0$. Then, the transformation (4.4.8) with $\alpha = \beta = 0$ defines an isomorphism of the Weierstrass models. □

The proof of the previous proposition allows us to determine the automorphism group scheme $\mathbf{Aut}_{\mathrm{gr}}(E)$ of an elliptic curve E over a field K, that is, the automorphism group scheme of E considered as an abelian variety over K.

Let E be an elliptic curve over a field K with Weierstrass equation

$$y^2 + a_1xy + a_3y + x^3 + a_2x^2 + a_4x + a_6 = 0.$$

We have seen in the proof of the previous proposition that any automorphism of E as an elliptic curve is a projective automorphism that preserves the Weierstrass equation. We now determine the group scheme of automorphisms $\mathbf{Aut}_{\mathrm{gr}}(E)$ of the elliptic curve E.

- Case 1, that is, $p \neq 2, 3$.

Here, $a_1 = a_2 = a_3 = 0$ and the only possible automorphisms are the projective automorphisms of the form $(z, x, y, z) = (z, c^2x, c^3y)$. Moreover, we have $c^4 = 1$ if $a_4 \neq 0$ and $c^6 = 1$ if $a_6 \neq 0$. This gives

$$\mathbf{Aut}_{\mathrm{gr}}(E/K) \cong \begin{cases} \boldsymbol{\mu}_{2,K} & \text{if } a_4, a_6 \neq 0, \\ \boldsymbol{\mu}_{6,K} & \text{if } a_4 = 0, \text{or, equivalently, } j = 0, \\ \boldsymbol{\mu}_{4,K} & \text{if } a_6 = 0, \text{or, equivalently, } j = 1728. \end{cases}$$

- Case 2, that is, $p = 2$.

As we observed in the proof of the previous proposition, we may assume that either $a_1 = 1, a_3 = a_4 = 0$ if $j \neq 0$, or $a_1 = a_2 = 0, a_3 \neq 0$ if $j = 0$.

If $j \neq 0$, then the only possible automorphism is given by the projective automorphism $g : (z, x, y, z) = (z, x, y + x)$. If $j = 0$, then an automorphism is given by (4.4.8), where $\gamma = \alpha^2$, and

$$c^3 = 1,\ (1+c)a_4+a_3\alpha+\alpha^4 = 0,\ c\alpha^2a_4+(\alpha^3+\beta)a_3+\alpha^6 = \beta^2+\beta a_3+\alpha^2a_4+\alpha^6 = 0.$$

The group scheme of automorphisms is étale of order 24. The kernel of the homomorphism $\mathrm{Aut}(E) \to \boldsymbol{\mu}_{3,K}, g \mapsto c$ is isomorphic to a separable form of the quaternion group Q_8. Its center is isomorphic to $\boldsymbol{\mu}_{2,K}$ and is generated by the transformation g_1 with $c = 1, \alpha = 0, \beta = a_3$. The quotient is a separable form of the group $(\mathbb{Z}/2\mathbb{Z})_K^2$. In fact, it is not too difficult to see that this group scheme is a separable form of the constant group scheme that is isomorphic to the binary tetrahedral group of order 24, which is isomorphic to the group $\mathrm{SL}(2, \mathbb{F}_3)$.

- Case 3, that is, $p = 3$.

If $j \neq 0$, then the only possible automorphism is given by $y \to -y, x \to x$.

Assume $j = 0$. Then, an automorphism is given by (4.4.8), where we drop the indices, with $c^4 = 1, \alpha = \beta = 0$ and γ equal to a root of the separable equation $t^3+a_4t+(1-c^2)a_6 = 0$. It is a group scheme of order 12. It admits a homomorphism onto $\boldsymbol{\mu}_{4,K}$, whose kernel a separable form of the group $(\mathbb{Z}/3\mathbb{Z})_K$.

Summarizing our calculations, we have the following.

Proposition 4.4.7 *Let E be an abelian curve over a field K. Then,* $\mathrm{Aut}_{\mathrm{gr}}(E/K)$ *is a separable form of a constant group scheme G over K, where G is as follows:*

$$G \cong \begin{cases} \mathbb{Z}/2\mathbb{Z} & \textit{if } j \neq 0, 1728, \\ \mathbb{Z}/4\mathbb{Z} & \textit{if } p \neq 2, 3,\ j = 1728, \\ \mathbb{Z}/6\mathbb{Z} & \textit{if } p \neq 2, 3,\ j = 0, \\ \mathrm{SL}(2, \mathbb{F}_3) & \textit{if } p = 2,\ j = 0 = 1728, \\ \mathbb{Z}/6\mathbb{Z} \rtimes \mathbb{Z}/2\mathbb{Z} & \textit{if } p = 3,\ j = 0 = 1728. \end{cases}$$

Example 4.4.8 Let $y^2 + x^3 + a_4 x + a_6 = 0$ be the Weierstrass equation of an elliptic curve E over a field K of characteristic $p \neq 2, 3$. Let $\alpha \in K^\times \setminus K^{\times 2}$. The curve $E' = V(\alpha y^2 + x^3 + a_4 x + a_6)$ becomes isomorphic to E over the quadratic field extension $K(\alpha)$. In fact, E' is an elliptic curve with Weierstrass equation $y^2 + x^3 + \alpha a_4 x + \alpha^2 a_6 = 0$. One says that E' is the *quadratic twist* of E with respect to α, see also [682, Chapter X]. By Proposition 4.4.6, the absolute invariants of E and E' must be equal, but in this explicit case, this also follows from a straightforward computation with their Weierstrass equations.

Quite generally, an elliptic curve E' over K that becomes isomorphic to E over some finite (and separable) field extension K'/K is called a *(separable) form* or a *(separable) twist* of E over K. By Proposition 4.4.6, an elliptic curve E' is a separable form of E if and only if $j(E) = j(E')$. Quite generally, separable twists of E are classified by the cohomology set $H^1(\mathrm{Gal}(\bar{K}/K), \mathrm{Aut}(E)(\bar{K}))$, which is an abelian group if $\mathrm{Aut}(E)(\bar{K})$ is an abelian group and which is merely a pointed set in the general case. Since $\mathrm{Aut}_{\mathrm{gr}}(E/K)$ is always a finite and étale group scheme over K, it follows that any twist (a priori not necessarily separable or finite) of E is in fact a finite and separable twist of E.

Moreover, Proposition 4.4.7 tells us that there are three possibilites for $\mathrm{Aut}(E)(\bar{K})$ (note that we assume $p \neq 2, 3$ in this example). If $\mathrm{Aut}(E) \cong \boldsymbol{\mu}_{2,K}$, then it follows from Kummer theory that $H^1(\mathrm{Gal}(\bar{K}/K), \mathrm{Aut}(E)(\bar{K}))$ is an abelian group that is isomorphic to $K^\times/K^{\times 2}$. In this case, every separable form is obtained as in the example just discussed and we refer to [682, Proposition X.5.4] for the details. However, if $\mathrm{Aut}(E)(\bar{K})$ is a cyclic group of order 4 or 6, then we have more separable forms and we refer again to [682, Proposition X.5.4] for details and equations.

Proposition 4.4.9 *Let $f : X \to C$ be a minimal jacobian elliptic fibration over a one-dimensional base C. Let* $\mathrm{div}(\Delta)$ *be the zero-dimensional subscheme of C defined by the discriminant $\Delta \in \Gamma(C, \mathcal{L}^{\otimes 12})$. Then:*

1. $t \in \mathrm{div}(\Delta)$ *if and only if X_t is not smooth.*
2. *If C is global, then* $\deg \mathrm{div}(\Delta) = c_2(X) = e(X) \geq 0$ *and the following assertions are equivalent:*
   a. $\deg \mathrm{div}(\Delta) = 0$,
   b. $\mathcal{L}^{\otimes 12} \cong \mathcal{O}_C$,
   c. *f is a smooth morphism,*
   d. *there exists a finite étale cover $C' \to C$ such that $X \times_C C' \cong C' \times E$, where E is an elliptic curve over $\Bbbk$.*

In these equivalent cases, all fibers of f are isomorphic, $e(X) = 0$, and $\chi(\mathcal{O}_X) = 0$. If f is not smooth, then $e(X) = 12\chi(\mathcal{O}_X) > 0$.

Proof (1) It follows from the construction of the Weierstrass model that the map $\phi : X \to W$ to the Weierstrass model is a birational morphism, which contracts all irreducible components of closed fibers X_t except the components X_t^o that contain the intersection point $X_t \cap \mathsf{E}$. It follows from the classification of singular fibers of genus one fibrations that X_t is either irreducible and then $u(t) : X_t \to W_t$ is an isomorphism, or else that X_t is reducible and that the component X_t^o is isomorphic to $\mathbb{P}^1$. Hence, W_t is a rational curve and thus, not smooth. This shows that W_t is singular if and only if X_t is singular.

(2) We know from Corollary 4.3.8 that $\deg \mathcal{L} = -\chi(\mathcal{O}_X)$. Applying Noether's formula, we obtain $\deg \operatorname{div}(\Delta) = \deg \mathcal{L}^{\otimes 12} = 12\chi(\mathcal{O}_X) = c_2(X) = e(X)$. Also, since an invertible sheaf of degree 0 is isomorphic to $\mathcal{O}_C$ if and only if it has a nonzero section, we see that (a) and (b) are equivalent. It follows from Proposition 4.1.12 that $e(X) \geq 0$ and that $e(X) = 0$ if and only if f is smooth. In particular, $\chi(\mathcal{O}_X) \geq 0$ and $\chi(\mathcal{O}_X) = 0$ if and only if f is smooth. This shows that (a) is equivalent to (c).

The implication (d)⇒(c) is easy and if (d) holds, then all fibers of f are isomorphic. Thus, it remains to show the implication (c)⇒(d).

Assume now that f is a smooth morphism, suppose $p \neq 2, 3$, and let $\phi : C' \to C$ be an étale cyclic cover of degree 12 that trivializes $\mathcal{L}$, that is, $\phi^*\mathcal{L} \cong \mathcal{O}_{C'}$. The elliptic fibration $f' : X' = X \times_C C' \to C'$ has a trivial sheaf $R^1 f'_*\mathcal{O}_{X'}$. It follows from the construction of the Weierstrass model W' that all its coefficients are sections of $\mathcal{O}_{C'}$, hence they are constants. This shows that W' is isomorphic to the product $C' \times E$. Since $u : X' \to W$ is an isomorphism in this case, we obtain that X' is a product, too. This proves the implication (c)⇒(d) in the case $p \neq 2, 3$.

The implication (c)⇒(d) if $p = 2, 3$ is more involved. We now give a rather abstract and technical, but also conceptual, argument that works for all characteristics. Here, we use the construction of a fine moduli space of elliptic curves together with a level structure, see, for example, [158]. Since f is smooth, the morphism $f : X \to C$ defines on X a structure of an abelian scheme $\mathbf{A}$ over C (a relative elliptic curve). Let $\ell > 3$ be a prime number that is different from p. The ℓ-torsion ${}_\ell\mathbf{A}$ is a relative étale group scheme of degree ℓ^2 over C and we set $C' :=_\ell \mathbf{A}$. Then, we consider the pull-back $X' = X \times_C C' \to C'$, which is an abelian scheme over C'. We have that ${}_\ell\mathbf{A}'$ gives rise a nonzero section of $X' \to C'$, hence defines a smooth family of elliptic curves equipped with a nontrivial ℓ-torsion point. Since $\ell > 3$, there exists a fine moduli scheme $\mathcal{M}_0(\ell)$ for families of elliptic curves with a non-trivial ℓ-torsion section. Thus, our family $X' \to C'$ gives rise to a classifying morphism $C' \to \mathcal{M}_0(\ell)$. It is known that $\mathcal{M}_0(\ell)$ is affine. Since C is global, it is proper and thus, the classifying morphism $C' \to \mathcal{M}_0(\ell)$ must be constant. Since $\mathcal{M}_0(\ell)$ is a fine moduli space, the family $X' \to C'$ is isomorphic to the pull-back of the universal family over $\mathcal{M}_0(\ell)$ via the classifying morphism. Since the classifying morphism is constant, it follows that $X' \to C'$ is isomorphic to the trivial family, which establishes (c)⇒(d). □

Let us assume now that $C = \mathbb{P}^1$, that is, we study (quasi-)elliptic fibrations over $\mathbb{P}^1$. Here, some of the previous discussions simplify - for example, it is easier to determine whether the sheaves $\mathcal{E}_2$ or $\mathcal{E}_3$ are split, as is the computation with singular fibers. This case will be of particular interest for the study of (quasi-)elliptic fibrations on Enriques surfaces and rational surfaces. Let $n = \chi(\mathcal{O}_X)$. It follows from Theorem 4.1.6 that

$$\mathcal{L} = R^1 f_* \mathcal{O}_X \cong \mathcal{O}_{\mathbb{P}^1}(-n).$$

In particular, $H^1(\mathbb{P}^1, \mathcal{L}^{\otimes k}) = 0$ if $k > 0$, and hence,

$$\begin{aligned}\mathcal{E}_2 &\cong \mathrm{gr}(\mathcal{E}_2) \cong \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-2n),\\ \mathcal{E}_3 &\cong \mathrm{gr}(\mathcal{E}_3) \cong \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-2n) \oplus \mathcal{O}_{\mathbb{P}^1}(-3n).\end{aligned}$$

Then, we can unify the five cases by writing down a general global Weierstrass equation

$$y^2 + a_1xy + a_3y + x^3 + a_2x^2 + a_4x + a_6 = 0, \tag{4.4.27}$$

where

$$a_k \in H^0(\mathbb{P}^1, \mathcal{L}^{\otimes -k}) = H^0(\mathbb{P}^1, \mathcal{O}(kn))$$

and where we realize the Weierstrass model as

$$W \subset \mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-2n) \oplus \mathcal{O}_{\mathbb{P}^1}(-3n)).$$

Locally, y_i (resp. x_i) generates the subsheaf $\mathcal{L}^{\otimes 3}$ (resp. $\mathcal{L}^{\otimes 2}$) of $\mathcal{E}_3$. It defines a section of $\mathcal{E}_3 \otimes p^*\mathcal{L}^{\otimes 3}$ (resp. $\mathcal{E}_3 \otimes p^*\mathcal{L}^{\otimes 2}$). Hence, the right-hand side of the global Weierstrass equation can be considered as a section of $\mathcal{E}_3 \otimes f^*\mathcal{L}^{\otimes 6}$ or as a section of $\mathcal{O}_X(3\mathsf{E}) \otimes p^*\mathcal{L}^{\otimes 6} = \mathcal{O}_X(3\mathsf{E}) \otimes f^*\mathcal{O}_{\mathbb{P}^1}(6n)$.

The inclusion of the symmetric algebras $S^\bullet(\mathcal{E}_2) \subset S^\bullet(\mathcal{E}_3)$ defines a rational map $\mathbb{P}(\mathcal{E}_3) \dashrightarrow \mathbb{P}(\mathcal{E}_2)$. It is the projection map from the section $\phi(\mathsf{E})$. The restriction of this projection to W is a cyclic double cover

$$\pi : W \to \mathbb{P}(\mathcal{E}_2) \cong \mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-2n)). \tag{4.4.28}$$

The surface $\mathbb{P}(\mathcal{E}_2)$ is a minimal ruled surface $\mathbf{F}_{2n}$. The image of E is the section of $\mathbb{P}(\mathcal{E}_2)$ defined by the surjection $\mathcal{E}_2 \to \mathcal{L}^{\otimes 2}$. It is the exceptional section $\mathfrak{e}$ with self-intersection $-2n$. The blow-down of this section defines a birational morphism $\mathbf{F}_{2n} \to \mathbb{P}(1,1,2n)$ onto the weighted projective plane $\mathbb{P}(1,1,2n)$, which is isomorphic to the cone over the Veronese curve $\nu_{2n}(\mathbb{P}^1)$ of degree $2n$ in $\mathbb{P}^{2n}$.

Let $\bar{W}$ be obtained from W by blowing down the section E (recall that its self-intersection is equal to $-2n$). Then, we obtain an embedding

$$\bar{W} \hookrightarrow \mathbb{P}(1,1,2n,3n)$$

and it is given by equation (4.4.11) of degree $6n$, where the weights of y, x are equal to $3n, 2n$ and the coordinates t_0, t_1 on $\mathbb{P}^1$ are of weight 1. The image of E under the composition $X \to W \to \bar{W}$ is the point with coordinates $[0,0,1,1]$. In the case

$n = 1$, this is an equation for the anti-canonical model of a del Pezzo surface of degree 1.

If f is an elliptic fibration, then the map $\pi : \bar{W} \to \mathbf{F}_{2n}$ is a split cyclic cover of degree 2 given by the data $(\mathcal{O}_X((n+1)\mathfrak{f} + 2\mathfrak{e}), s)$, where the zero scheme of the section s is equal to the union of the exceptional section $\mathfrak{e}$ and a divisor from the linear system $|3n\mathfrak{f} + 2\mathfrak{e}|$. The canonical class formula tells us that

$$\omega_{\bar{W}} \cong \pi^*\mathcal{O}_{\mathbb{P}^1}(-2(n+1)\mathfrak{f} - 2\mathfrak{e}) \otimes \mathcal{O}_{\mathbb{P}^1}(3n\mathfrak{f} + 2\mathfrak{e}) \cong \pi^*\mathcal{O}_{\mathbb{P}^1}(n-2). \tag{4.4.29}$$

Since the morphism $X \to W$ blows down nodal cycles of type A_k, D_k, E_k, all singular points of W are rational double points. This gives an isomorphism

$$\omega_X \cong f^*\mathcal{O}_{\mathbb{P}^1}(n-2). \tag{4.4.30}$$

Applying Riemann–Roch, this confirms in this case the formula $n = -\chi(\mathcal{O}_X)$ that we already established in Corollary 4.3.8. For example, if $n = 1$, then we conclude that X is a rational elliptic surface and if $n = 2$, then X is a K3 surface.

If $p = 2$ and if the fibration is quasi-elliptic, then the cover $\pi : \bar{W} \to \mathbf{F}_{2n}$ is inseparable. More precisely, it is a $\boldsymbol{\mu}_2$-cover defined by the invertible sheaf $\mathcal{O}_X(3n\mathfrak{f} + 2\mathfrak{e})$. The formula for the canonical class of X is the same.

We now return to the general study of jacobian (quasi-)elliptic fibrations. Our next task is to determine the types of fibers of a jacobian fibration $f : J \to C$ in terms of the coefficients of the Weierstrass model W of f. We know that the morphism $\phi : J \to W$ blows down those components of reducible fibers J_t that do not meet the zero section E. The type of a reducible fiber of $f : J \to C$ can be read off from the type of the corresponding rational double point on W.

To determine the type of a singular fiber, we may assume that C is local. In fact, we may even assume that C is strictly local, that is, $C = \operatorname{Spec} \Bbbk[[t]]$. Then, the sheaves ω and $\mathcal{E}_n$ can be identified with modules over $\Bbbk[[t]]$. Since the fibration $f : X \to C$ is relatively minimal, any automorphism of the generic fiber extends to an isomorphism over C. Assume $(a_1, a_2, a_3, a_4, a_6) = (ta_1', t^2a_2', t^3a_3, t^4a_4', t^6a_6')$. Then the generic fibers of the Weierstrass models with coefficients $(a_1, a_2, a_3, a_4, a_6)$ and $(a_1', a_2', a_3', a_4', a_6')$ are isomorphic via $x = t^2x'$, $y = t^3x'$. However, the isomorphism on generic fibers does not extend to an isomorphism of the Weierstrass models since t is not invertible on C. By successively performing isomorphisms of the generic fiber of this form, we may eventually assume that the order of vanishing satisfies

$$\nu(a_k) < k \tag{4.4.31}$$

for at least one nonzero coefficient a_k.

We now briefly discuss *Tate's algorithm* that determines the type of a singular fiber given a Weierstrass equation W of some elliptic fibration $f : X \to C = \operatorname{Spec} \Bbbk[[t]]$. For further details and for the more involved computations if $p = 2, 3$ we refer to Tate's original article [703] or to [683, Chapter IV.9]. Since C is strictly local, the sheaves $\mathcal{E}_2$ and $\mathcal{E}_3$ split into invertible sheaves. We assume $p \neq 2, 3$ and thus, we may assume that we have a Weierstrass equation of the form (4.4.12), that is,

$$W \;:\; y^2 + x^3 + a_4x + a_6 \;=\; 0$$

for some $a_4, a_6 \in \Bbbk[[t]]$. By (4.4.31), we may assume that $\nu(a_4) < 4$ or $\nu(a_6) < 6$.

First, concerning the regularity of W: taking the partial derivatives with respect to x, t, y we find that the scheme $W \subset \mathbb{A}^2_R$ is not regular if and only if

$$3x^2 + a_4, \quad -2x^3 + a_6, \quad \frac{da_4}{dt}x + \frac{da_6}{dt}$$

vanish at some point $(t, x, y) = (0, x_0, 0)$. The first two conditions imply that $\Delta = 4a_4^3 + 27a_6^2$ vanishes at $t = 0$, which is equivalent to saying that the special fiber of $f : X \to C$ is singular. The remaining condition implies that either $\frac{da_4}{dt}$ and $\frac{da_6}{dt}$ both vanish at $t = 0$ or else that $\frac{da_4}{dt}$ does not vanish and then $a_4(\frac{da_4}{dt})^2 = 3(\frac{da_6}{dt})^2$.

Next, assume that W is regular, that is, the morphism $\phi : X \to W$ over C is an isomorphism. Then, the closed fiber of W is irreducible with Weierstrass equation $y^2 + x^3 + a_4(0)x + a_6(0) = 0$. If $\Delta = 4a_4^3 + 27a_6$ does not vanish at $t = 0$, then this fiber is smooth. Otherwise, it is a singular rational curve that has either one node or one cusp as singular locus.

In particular, we obtain that, if W is regular, then the fiber is cuspidal if and only if $\nu(a_4) \geq 1$ and $\nu(a_6) = 1$.

Case (a): $a_4 = 0$.

After possibly scaling y, x by an invertible element of $\Bbbk[[t]]$, we may assume that $a_6 = t^n$ with $1 \leq n \leq 5$. If $n = 1$, then W is regular and the closed fiber is of type A_0^{**}, a cuspidal cubic. Otherwise, W has a rational double point of type A_1, D_4, E_6, E_8 if $n = 2, 3, 4, 5$, respectively. In this case, the closed fiber is of type $\tilde{A}_1^*, \tilde{D}_4, \tilde{E}_6, \tilde{E}_8$ if $n = 2, 3, 4, 5$, respectively.

Suppose now that $a_4 \neq 0$. After scaling, we may assume that $a_4 = t^m$ and $a_6 = \epsilon t^n$, where ϵ is a unit.

Case (b): $a_4 \neq 0$ and $m < 4$.

If $m = 0$, then $\Delta = 4a_4^3 + 27a_6^2$ does not vanish at $t = 0$ if $n > 0$. So, in this case W is regular and its closed fiber is a nonsingular elliptic curve with equation $y^2 + x^3 + x = 0$.

If $n = 0$, then W is not regular if $\Delta(0) = 4 + 27\epsilon(0)^2$, and $\frac{da_6}{dt}(0) = 0$. We write $\epsilon = \epsilon(0) + t^k\eta$ for some unit η. If $k = 1$, then the scheme W is regular. If $k > 1$, then we write $x^3 + x + \epsilon(0) = (x - \alpha)^2(x + \frac{\epsilon(0)}{\alpha^2})$, where $-2\alpha^3 + \epsilon(0) = 0$. We find a singular point $(\alpha, 0, 0)$ and the singularity is formally isomorphic to the singularity $V(xy + t^k)$, that is, a rational double point of type A_{k-1}, and the fiber is of type $\tilde{A}_{k-1}$. We also have $\nu(\Delta) = \nu(4 + 27(\epsilon(0) + t^k\eta)^2) = \nu(54t^k\eta) = k$.

If $m = 1$, then $\Delta = 4t^3 + 27t^{2n}\epsilon^2$. If $n = 0$, then $\Delta(0) \neq 0$ and W is regular and the fiber is isomorphic to $V(y^2 + x^3 + \epsilon(0))$, which is an elliptic curve with $j = 0$. If $n > 0$, then the equation is of the form $y^2 + tx + t^n\epsilon + \cdots$. It is regular if $n = 1$ and an ordinary double point if $n > 1$. In the former case, the fiber is of type $\tilde{A}_0^{**}$, a cuspidal cubic. In the latter case, we obtain that the fiber is of type $\tilde{A}_1^*$, a nodal cubic. We also have $\nu(\Delta) = 2$ in the first case and $\nu(\Delta) = 3$ in the second case.

If $m = 2$, then $\Delta = 4t^6 + 27t^{2n}\epsilon^2$. If $n = 0$, then the fiber is nonsingular, and if $n = 1$, then W is regular, and the fiber is a cuspidal cubic. If $n = 2$, then f is a semi-quasi-homogeneous polynomial with quasi-homogeneous part $f_1 = y^2 + x^3 + \epsilon(0)t^2$ of type A_2 (see Remark 0.4.14). Thus, the singular point is of type A_2 and the fiber is of type $\tilde{A}_2^*$. We find also that $\nu(\Delta) = 4$.

If $m = 2$ and $n > 3$, then f is a semi-quasi-homogeneous polynomial with quasi-homogeneous part $f_1 = y^2 + x^3 + t^2x$ of type D_4. Thus, the singular fiber is of type $\tilde{D}_4$. Applying Remark 0.4.14, we obtain that the singularity is of type D_4. Thus, the singular fiber is of type $\tilde{D}_4$. We also find that $\nu(\Delta) = 6$. Finally, if $n = 3$, then $\Delta = (4 + 27\epsilon(0)^2)t^6$ with $\nu(\Delta) = 6 + \nu((4 + 27\epsilon(0)^2) = 6 + k$. We can rewrite the equation in the form $y^2 + (x + \alpha t)^2(x + \beta t) + t^{k+3}\eta = 0$, where $3\alpha^2 = -1$, $\alpha^2\beta = \epsilon(0)$, and η is a unit. Replacing $x + \alpha$ with x, we find a semi-quasi-homogeneous equation with quasi-homogeneous part $f_1 = y^2 + x^2t + t^{k+3}$ of type D_{k+4}. This gives us a singular fiber of type $\tilde{D}_{k+4}$ and $\nu(\Delta) = 6 + k$.

If $m = 3$, then $\Delta = 4t^9 + 27t^{2n}\epsilon^2$. If $\nu(\Delta) = 2n \leq 6$, then we obtain the previous cases with the same value of $\nu(\Delta)$. If $n = 4$, then $\nu(\Delta) = 8$ and f is a semi-quasi-homogeneous polynomial with the quasi-homogeneous part $f_1 = y^2 + x^3 + t^3x$ of type $\tilde{E}_7$. This gives us a singular fiber of type $\tilde{E}_7$.

Case (c): $a_4 \neq 0$ and $n < 6$.

After possibly re-scaling, we may assume that $a_4 = t^m\epsilon$ with $m > 3$ and $a_6 = t^n$.

If $n = 0$ (resp. $n = 1$), then W is regular, the closed fiber is nonsingular (resp. a cuspidal cubic) and $\nu(\Delta) = 0$ (resp. 2).

If $n = 2, 3, 4, 5$, then f is a semi-quasi-homogeneous polynomial with the quasi-homogeneous part $f_1 = y^2 + x^3 + t^nx$ of type A_2, D_4, E_6, E_8, respectively. The closed fiber is of type $\tilde{A}_2^*, \tilde{D}_4, \tilde{E}_6, \tilde{E}_8$, respectively and we have $\nu(\Delta) = 4, 6, 8, 10$, respectively.

We now summarize our computations in Table 4.1.

Type	$\nu(a_4)$	$\nu(a_6)$	$\nu(\Delta)$	$\nu(j)$
$\tilde{A}_0$	0	≥ 0	0	≥ 0
$\tilde{A}_0$	≥ 0	0	0	≥ 0
$\tilde{A}_0^*$	0	0	1	-1
$\tilde{A}_0^{**}$	≥ 1	1	2	≥ 1
$\tilde{A}_1^*$	1	≥ 2	3	0
$\tilde{A}_2^*$	$m \geq 2$	2	4	≥ 1
$\tilde{A}_n$	0	0	$n - 1$	$1 - n$
$\tilde{D}_4$	2	≥ 3	6	≥ 0
$\tilde{D}_4$	3	2	6	≥ 0
$\tilde{D}_{4+k}, k > 0$	2	3	$6 + k$	$-6 - k$
$\tilde{E}_6$	$m \geq 3$	4	8	$3m - 8$
$\tilde{E}_7$	3	$n \geq 5$	9	0
$\tilde{E}_8$	$m \geq 4$	5	10	$3m - 2$

Table 4.1 Types of fibers of an elliptic fibration if $p \neq 2, 3$

Comparing $\nu(\Delta)$ and the ℓ-adic Euler characteristic $e(X_0)$ of the closed fiber, we obtain that $\nu(\Delta) = e(X_0)$. If we also allow $p = 2, 3$, then it follows from [570] that we have more generally

$$\nu(\Delta) \ = \ e(X_0) + \delta, \tag{4.4.32}$$

where δ is an invariant of the wild ramification that already occured in Proposition 4.1.12 and which can be non-zero only if $p = 2, 3$. The formula from this proposition gives the following.

Corollary 4.4.10 *Let $f : J \to C$ be a jacobian elliptic fibration over a global base C. Then,*

$$e(J) \ = \ \deg \Delta \ = \ -12 \ \deg \mathcal{L}.$$

As already mentioned above, there is also a Tate algorithm in small characteristics, that is, if $p = 2, 3$. Then, the algorithm becomes a little bit more involved and rather than reproducing it here, we refer to [703] or [683, Chapter IV.9]. As an application of Tate's algorithm we obtain Table 4.2 below.

	$\tilde{A}_0^{**}$	$\tilde{A}_1^{*}$	$\tilde{A}_2^{*}$	$\tilde{D}_4$	$\tilde{D}_n, n > 1$	$\tilde{E}_6$	$\tilde{E}_7$	$\tilde{E}_8$
min $\frac{\nu(a_i)}{i}$	$\frac{1}{6}$	$\frac{1}{4}$	$\frac{1}{3}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{2}{3}$	$\frac{3}{4}$	$\frac{5}{6}$
extra condition	–	–	$\nu(b_6) = 2$	$\nu(d) = 6$	$\nu(d) > 6, \nu(a_2^2 - 3a_4) = 2$	$\nu(b_6) = 4$	–	–

Table 4.2 Types of fibers of additive types

Here, we have

$$\begin{aligned} b_6 &= a_3^2 - 4a_6, \\ d &= \mathrm{Disc}(x^3 + a_2x^2 + a_4x + a_6) \ = \ -4a_2^3a_6 + a_2^2a_4^2 + 18a_2a_4a_6 - 4a_4^3 - 27a_6^2. \end{aligned}$$

We note that a simpler approach than a case-by-case analysis using Tate's algorithm has been given in [164].

For example, a fiber of additive type $\tilde{D}_{n+4}$ occurs if and only if:

- $\nu(a_2) = 1$,
- $\nu(a_i) \geq \frac{i}{2} + \left[\frac{i-1}{2}\right]\frac{n}{2}$,
- $\nu(d) = n + 6$ and $\nu(b_6) \geq n + 3$ if $2|n$ and $= n + 3$ otherwise.

Note that can combine Ogg's formula (4.4.32) together with Table 4.2 to compute the value of the invariant of wild ramification δ. We illustrate this with an example.

Example 4.4.11 Consider the Weierstrass equations

$$\begin{aligned} y^2 + t^3y + x^3 + tx^2 + tx + t = 0 \text{ if } p = 2, \\ y^2 + x^3 + t^4x + t = 0 \text{ if } p = 3. \end{aligned}$$

These define elliptic fibrations over $\Bbbk[[t]]$. Then, the Weierstrass model W is regular, we find $\Delta = t^{12}$, and we see that the type over $t = 0$ is a cuspidal cubic. Therefore, we conclude $\delta = 12 - 2 = 10$, which illustrates that the wild ramification invariant can become quite large.

Next, we study jacobian quasi-elliptic fibrations and their associated Weierstrass equations. We define a discriminant and a j-invariant, establish formulae similar to the ones for elliptic fibrations, and show how to read off the type of the fibers from the coefficients of a Weierstrass equation. We remind the reader that such fibrations can exist only in characteristic $p = 2, 3$.

First, we define the *discriminant* of a quasi-elliptic fibration. To point out the obvious, no fiber of a quasi-elliptic fibration $f : X \to C$ is smooth and the generic fiber has a cusp over the algebraic closure of $\Bbbk(C)$. Therefore, the zero locus of the discriminant corresponds to those fibers of f that are "more" singular than the cusp inherited from the generic fiber, see also Proposition 4.4.12 below.

In the following discussion, the base $B = C$ may be global, local, or strictly local. After covering C by sufficiently small open affines, we have local Weierstrass equations $y_i^2 + x_i^3 + a_4^{(i)} x_i + a_6^{(i)} = 0$, see Cases 4 and 5 in the above discussion. The singular points of these Weierstrass models satisfy $x_i^2 + a_4^{(i)} = 0$ and $x_i da_4^{(i)} + da_6^{(i)} = 0$. The transition functions for $a_4^{(i)}$ and $a_6^{(i)}$ show that the maps $a_4^{(i)} \to da_4^{(i)}$ and $a_6^{(i)} \mapsto da_6^{(i)}$ can be glued together to define global sections $da_4 \in H^0(\mathcal{L}^{-4} \otimes \omega_C)$ and $da_6 \in H^0(\mathcal{L}^{-6} \otimes \omega_C)$. After squaring, we define $\Delta_i := a_4^{(i)} (da_4^{(i)})^{\otimes 2} + (da_6^{(i)})^{\otimes 2}$. One can immediately check that the Δ_i's glued together to a global section

$$\Delta \ := \ a_4 da_4^{\otimes 2} + da_6^{\otimes 2} \ \in \ \Gamma(C, \mathcal{L}^{\otimes -12} \otimes \omega_C^{\otimes 2}). \tag{4.4.33}$$

We call Δ the *discriminant of a quasi-elliptic fibration*. In characteristic 3, we may assume that the local Weierstrass models satisfy $a_4^{(i)} = 0$ (see the discussion of Case 5 above) and in particular, we can arrange Δ to be of the form $da_6^{\otimes 2}$. In both characteristics 2 and 3, it is easy to see that the singularities of W lie over the zeros of the discriminant Δ in C.

Now, if C is global, comparing with formulas (4.1.7) and Theorem 4.1.6, we obtain

$$\deg \Delta \ = \ 12\chi(\mathcal{O}_J) - 2\chi(\mathcal{O}_C) \ = \ e(X) - e(C)e(J_\eta) \ = \ \sum_{s \in C} (e(J_s) - e(J_\eta)). \tag{4.4.34}$$

This is similar to the formulae for an elliptic fibration, where $\Delta \in H^0(C, \mathcal{L}^{\otimes -12})$.

For any closed point $s \in C$, we have the local ring $\mathcal{O}_{C,s}$, which comes with a discrete valuation $\nu_s : \mathcal{O}_{C,s} \to \mathbb{Z}$. We define $\nu_s(\Delta)$ to be $\nu_s(\Delta(t))$, where t is a local parameter of $\mathcal{O}_{C,s}$ and where we write locally $\Delta = \Delta(t)dt$ in $\mathcal{O}_{S,s}$. It is easy to check that this is well-defined, that is, does not depend on the choice of the local parameter t. Below, we will see that if $f : J \to C$ is a jacobian genus one fibration, then

$$\nu_s(\Delta) \ = \ e(J_s) - e(J_\eta) + \delta_s,$$

which is a local version of (4.4.34) and which generalizes (4.4.32) from elliptic fibrations to arbitrary jacobian genus one fibrations.

Proposition 4.4.12 *Let Δ be the discriminant of a jacobian quasi-elliptic fibration $f : J \to C$ and let $W \to C$ be its Weierstrass model. Then:*

1. *$t \notin \Delta$ if and only if J_t is irreducible if and only if W is regular over t.*
2. *If C is global, then* $\deg \Delta = c_2(J) - 4\chi(\mathcal{O}_C)$.

Proof (1) By construction and our discussion above, $t \in \Delta$ if and only if W is not regular over t. On the other hand, W is not regular over t if and only if the contraction morphism $J \to C$ is not an isomorphism, which is the case if and only if J_t is reducible.

(2) By the proof of Corollary 4.3.8, we have $\deg \mathcal{L} = \deg R^1 f_* \mathcal{O}_J = -\chi(\mathcal{O}_J)$. We have $\deg \Delta = -12 \deg \mathcal{L} + 2 \deg \omega_C$ by (4.4.33), and thus, applying Noether's formula, we obtain

$$\deg \Delta \;=\; 12\chi(\mathcal{O}_J) + 2\deg \omega_C \;=\; c_2(J) - 4\chi(\mathcal{O}_C).$$

By Corollary 4.3.22, a fiber of a jacobian quasi-elliptic fibration is irreducible if and only if it is of type $\widetilde{A}_0^{**}$, that is, an irreducible rational curve with a cusp. This is the generic case and thus, the discriminant detects precisely those fibers of a quasi-elliptic fibration that are "more" singular than the generic fiber or, equivalently, the reducible fibers of a jacobian quasi-elliptic fibration.

It follows from (2) that

$$\chi(\mathcal{O}_J) \;\geq\; \chi(\mathcal{O}_J) - \frac{1}{12}\deg \Delta \;=\; \frac{1}{3}\chi(\mathcal{O}_C). \tag{4.4.35}$$

In particular, if 3 does not divide $\chi(\mathcal{O}_C)$ (for example, if $C = \mathbb{P}^1$), then the inequality is strict, which implies $\deg \Delta > 0$ and then, we must have at least one reducible fiber.

Remark 4.4.13 Following [77, p. 203], there is a "ghost" of the classical *absolute invariant* or the *j-invariant* for quasi-elliptic fibrations: let U be an open subset of C where Δ does not vanish, i.e. the fibers over points in U are irreducible. Then,

$$\frac{da_4^{\otimes 3}}{\Delta} \;=\; \frac{da_4^{\otimes 3}}{a_4 da_4^{\otimes 2} + da_6^{\otimes 2}} \;\in\; H^0(U, \omega_U), \tag{4.4.36}$$

which should be compared to (4.4.24) and (4.4.25).

Finally, we study Tate's algorithm for jacobian quasi-elltiptic fibrations. Let

$$y^2 + x^3 + a_4 x + a_6 \;=\; 0,$$

be the Weierstrass model of a quasi-elliptic fibration over the strictly local base $\Bbbk[[t]]$. We consider the left-hand-side polynomial as a formal singularity in $\Bbbk[[x, y, t]]$. Since they arise from the Weierstrass models of a jacobian quasi-elliptic fibrations, they are formally isomorphic to rational double points. Moreover, since the equation is of the form $y^2 + f(x, t)$ if $p = 2$ and can be chosen to be of the form $x^3 + f(y, t)$ if $p = 3$, these singularities arise as purely inseparable covers of $\Bbbk[[x, t]]$ and

$\Bbbk[[y,t]]$, respectively, that is, they are *Zariski singularities*, see Section 10.2 of Volume II. In our case, it follows from the classification of reducible fibers of quasi-elliptic fibrations (see Corollary 4.3.22) that the rational double points are of types $A_1, D_{2k}^{(0)}, E_7^{(0)}, E_8^{(0)}$ if $p = 2$ and of types $A_2, E_6^{(0)}, E_8^{(0)}$ if $p = 3$. Here, the upper indices follow from the fact that the singularities are Zariski singularities. These singularities correspond to reducible fibers of respective types $\tilde{A}_1^*, \tilde{D}_{2k}, \tilde{E}_7, \tilde{E}_8$ if $p = 2$ and $\tilde{A}_2^*, \tilde{E}_6, \tilde{E}_8$ if $p = 3$. We will now analyze the Zariski double points on the Weierstrass models, which is sufficient to determine the singular fibers of the original quasi-elliptic fibration.

- Characteristic $p = 2$.

After a suitable change of variables from (x, y) to $(t^{2e}x, t^{3e}y)$, we may assume that

$$\nu(a_4) < 4 \quad \text{or that} \quad \nu(a_6) < 6, \tag{4.4.37}$$

see also the discussion around (4.4.31). If $\nu(a_4) = 0$ or $\nu(a_4) = 1$, then the ring $\Bbbk[[y,x,t]]/(f)$ is regular, so the fiber is irreducible. If $\nu(a_4) = 1$, then $f = y^2 + xt + \cdots$ and the singularity is an ordinary double point of type A_1. So, in the following we may exclude these cases.

By a change of coordinates $y \mapsto y + \alpha x + \beta, x \mapsto x + \alpha^2$, we may add α^4 to a_4 and add a square $(b + \alpha)^2$ to a_6. Thus, we may assume that a_4 (resp. a_6) does not contain monomials of degree $\equiv 0 \mod 4$ (resp. even degree). We can now write

$$a_4 \ = \ t^{4k_1+1}\alpha_1^4 + t^{4k_2+2}\alpha_2^4 + t^{4k_3+3}\alpha_3^4 \quad \text{and} \quad a_6 \ = \ t^{2k+1}\alpha_4^2,$$

where the α_i's are units or zero. This can be rewritten in the form

$$a_4 \ = \ t^{4s+2}\alpha^4 + t^{2m+1}\beta^2 \quad \text{and} \quad a_6 = t^{2k+1}\gamma^2, \tag{4.4.38}$$

where α, β, γ are units or zeros.

<u>Case 1</u>: $\gamma = 0$.

Since a_4 is not a square, we have $\beta \neq 0$.

If $s \neq 0$, then $\nu(a_4) = 2m + 1 < 4$ and hence, $m = 1$. We can write $f = f_1 + f_2$, where $f_1 = y^2 + x^3 + t^3x$ is a weighted homogeneous polynomial of degree 18 with weights $(9, 3, 2)$ and f_2 does not contain monomials of degree ≤ 18. Applying Remark 0.4.14 and Theorem 0.4.13, we see that the singularity is isomorphic to the rational double point of type $E_7^{(0)}$, which corresponds to a fiber of type $\tilde{E}_7$. On the other hand, if $s = 0$, then we get a singular point of type A_1, which corresponds to a fiber of type $\tilde{A}_1^*$.

<u>Case 2</u>: $\beta = 0$. Since a_6 is not a square, we have $\gamma \neq 0$.

In view of (4.4.37) and the excluded cases, it remains to consider the following cases:

1. $\alpha = 0$ or $s > 0$ and $k = 1, 2$.
2. $\alpha \neq 0$ and $s = 0$.

First, assume that we are in subcase (1). Then, we can write $f = f_1 + f_2$, where $f_1 = y^2 + x^3 + t^{2k+1}$ is a weighted-homogeneous polynomial of degree 6 with weights $(3, 2, 2)$ if $k = 1$ (resp. degree 30 with weights $(15, 10, 6)$ if $k = 2$) and all monomials in $f_2 = c_1 t^4 + \cdots$ are of degree > 6. Applying Remark 0.4.14, we see that the singularity is isomorphic to the rational double point of type $D_4^{(0)}$ (resp. of type $E_8^{(0)}$), which corresponds to a fiber of type $\tilde{D}_4$ (resp. $\tilde{E}_8$).

In subcase (2), we replace $x+\alpha^2 t$ by u and t by $(u+x)\alpha^{-2}$ and rewrite the equation in the form

$$f = y^2 + xu^2 + (u+x)^{2k+1}\epsilon^2 = y^2 + xu^2 + xu^{2k} + f_2,$$

where $f_1 = y^2 + xu^2 + ux^{2k}$ is a weighted homogeneous polynomial of degree $8k-2$ with weights $(4k-1, 4k-2, 2)$ and no monomial entering in f_2 has degree less than or equal to $8k-2$. Applying Remark 0.4.14 and Theorem 0.4.13, we obtain that the singularity is a rational double point of type $D_{4k}^{(0)}$.

<u>Case 3</u>: $\beta \neq 0$.

In view of (4.4.37), we have to consider the following cases:

1. $\alpha = 0$ or $s > 0$.
2. $\alpha \neq 0$ and $s = 0$.
3. $\alpha \neq 0$ and $s > 0$.

In subcase (1), we have $f = y^2 + x^3 + t^{2m+1}\beta^2 x + t^{2k+1}\gamma^2 = 0$. First, assume that $m < k$. Then, $m = 1$ and f is a semi-quasi-homogeneous polynomial with quasi-homogeneous part $f_1 = y^2 + x^3 + t^3 x$ of type E_7 (see Remark 0.4.14). Thus, f defines a rational double point of type $\tilde{E}_7^{(0)}$. Second, if $k \leq m$, then $k = 1, 2$ and we can write $f = y^2 + x^3 + t^{2k+1} + f_2$, and similar arguments show that the singularity is formally isomorphic to a double rational point of type $D_4^{(0)}$ if $k = 1$ and of type $E_8^{(0)}$ if $k = 2$.

In subcase (2) and if $k = 1$, we can write $f = y^2 + x^3 + t^2 x + t^3 + f_2$ and obtain a rational double point of type $D_4^{(0)}$. If $k > 1$ and $m < k$, then we write

$$f = y^2 + x(x + t\alpha^2)^2 + t^{2m+1}(x + t\alpha^2) + t^{2k+1}\gamma^2.$$

We replace $x + t\alpha^2$ by u and t by $(x+u)\alpha^2$ as in Case 2.(ii) and we write f as a semi-quasi-homogeneous polynomial with quasi-homogeneous part $f_1 = y^2 + xu^2 + ux^{2m+1}$ of type D_{4m+2}. This gives us a rational double point of type $D_{4m+2}^{(0)}$. If $m \leq k$, then we write $f = y^2 + x(x + t\alpha^2)^2 + t^{2k+1} + xt^{2m+1}$ and making a similar change of variables we obtain a rational double point of type $D_{4k}^{(0)}$.

Assume that we are in subcase (3). If $\nu(a_4) < 4$, then $f = y^2 + x^3 + t^3 x + f_2$ is a semi-quasi-homogeneous polynomial with quasi-homogeneous part of type E_7 and we obtain a rational double point of type $E_7^{(0)}$. If $\nu(a_4) \geq 4$, then $\nu(a_6) = 2k+1 < 6$, and we can write $f = y^2 + x^3 + t^{2k+1} + f_2$. From this, we obtain a rational double point of type $D_4^{(0)}$ if $k = 1$ and of type $E_8^{(0)}$ if $k = 3$.

- Characteristic $p = 3$.

After a suitably rescaling (x, y), we may assume that $\nu(a_6) < 6$. After adding a cube to a_6, we may write

$$a_6 \;=\; t^{3k+1}\alpha^3 + t^{3m+2}\beta^3$$

for some $k < m$ and where α, β are units or zero.

First, assume $3k + 1 < 3m + 2$. Then, $k = 0, 1$ and we obtain a singular point that is formally isomorphic to $y^2 + x^3 + t^{3k+1} = 0$. If $k = 0$, then it is nonsingular. If $k = 1$, then this is a rational double point of type $E_6^{(0)}$.

Second, assume $3k + 1 > 3m + 2$. Then we have $m = 0, 1$. If $m = 0$, then we get a singular point of type A_2. If $m = 1$, then we get a singular point of type $E_8^{(0)}$.

The analysis of these singular points allows us to find the type of a singular fiber. We use the normal form (4.4.38).

Char	$\nu(a_4)$	$\nu(a_6)$	$\nu(\Delta)$	Type
$p = 2$	any	1	0	$\tilde{A}_0^{**}$
	1	any	1	$\tilde{A}_1^{*}$
	> 3	3	4	$\tilde{D}_4^{(0)}$
	3	> 3	7	$\tilde{E}_7^{(0)}$
	> 4	5	8	$\tilde{E}_8^{(0)}$
	2	$2k + 1 \geq 5$	$4k$	$\tilde{D}_{4k}^{(0)}$
	$2, m > 1$	$> 2m + 1$	$4m + 2$	$\tilde{D}_{4m+2}^{(0)}$
$p = 3$	–	1	0	$\tilde{A}_0^{**}$
	–	2	2	$\tilde{A}_2^{*}$
	–	4	6	$\tilde{E}_6$
	–	5	8	$\tilde{E}_8$

Table 4.3 Types of degenerate fibers of a quasi-elliptic fibration

From this table, we conclude

$$\nu(\Delta) \;=\; e(X_0) - 2. \tag{4.4.39}$$

4.5 Mordell–Weil Lattices

Let $f : J \to C$ be a global jacobian genus one fibration. In Section 4.3, we introduced the Mordell–Weil group $\mathrm{MW}(J/C)$, the abelian group of sections $J_\eta(\eta)$. This maps to $\mathrm{Num}(J)$ and has a trivial intersection with $\mathrm{Num}_{\mathrm{fib}}(J)$, the subgroup of $\mathrm{Num}(J)$ generated by components of fibers of f. It follows from the Shioda–Tate formula (4.3.1) that the subgroup of $\mathrm{Num}(J)$ generated by $\mathrm{Num}_{\mathrm{fib}}(J)$, the zero section O, and $\mathrm{MW}(J/C)$ is of finite index. In this section, we study this index in detail. Moreover, we study the intersection forms on $\mathrm{Num}(J)$ and $\mathrm{Num}_{\mathrm{fib}}(J)$ and the relation to the intersection form on $\mathrm{MW}(J/C)$ coming from the canonical height pairing on J_η.

For further details, we refer the interested reader to [645], as well as to the book [646], which is entirely dedicated to the theory of Mordell–Weil lattices.

Let $f : J \to C$ be a global and jacobian genus one fibration. If f is elliptic, then we will assume that f is non-smooth. If f is quasi-elliptic, then we will assume that not all fibers of f are irreducible. Then, $\chi(\mathcal{O}_J) > 0$ by Proposition 4.4.9 if f is elliptic and the proof of Corollary 4.3.6 if f is quasi-elliptic In particular, we have $\mathrm{NS}(J) = \mathrm{Num}(J)$ and these groups have no torsion by Proposition 4.3.5, see also Corollary 4.3.6.

The generic fiber $J_\eta \to \eta$ is an elliptic curve (resp. cuspidal rational curve) if f is elliptic (resp. quasi-elliptic). Thus, the set of sections $J_\eta(\eta)$ is an abelian group, the *Mordell–Weil group* of J_η or f, which we denoted by $\mathrm{MW}(J/C)$ in Section 4.3, see also Proposition 4.3.1. By the Mordell–Weil theorem (Theorem 4.3.3), this abelian group is finitely generated and the rank is called the *Mordell–Weil rank*. Moreover, the sections $J_\eta(\eta)$ extend to divisors on J, which gives rise to a morphism of abelian groups $J_\eta(\eta) \to \mathrm{Pic}(J)$.

Let us recall from Section 4.3 that we defined $\mathrm{Pic}_{\mathrm{fib}}(J)$ to be the kernel of the restriction map $\mathrm{Pic}(J) \to \mathrm{Pic}(J_\eta)$. Then, we defined $\mathrm{Pic}_0(J)$ to be the kernel of $\mathrm{Pic}(J) \to \mathrm{Pic}(J_\eta) \to \mathbb{Z}$, where the last map is the degree map. We also defined $\mathrm{Pic}_{\mathrm{fib}}(J/C) := \mathrm{Pic}_{\mathrm{fib}}(J)/f^* \mathrm{Pic}(C)$. Let Num_0 and $\mathrm{Num}_{\mathrm{fib}}$ denote the images of Pic_0 and $\mathrm{Pic}_{\mathrm{fib}}$ inside $\mathrm{Num}(J)$. In Proposition 4.3.2 and (4.3.3), we showed the isomorphisms

$$\mathrm{MW}(J/C) = J_\eta(\eta) \cong \mathrm{Pic}_0(J)/\mathrm{Pic}_{\mathrm{fib}}(J) \cong \mathrm{Num}(J)_0/\mathrm{Num}_{\mathrm{fib}}(J)$$

Next, the Shioda–Tate formula (4.3.1) expresses the Picard rank $\rho(J)$ in terms of the Mordell–Weil rank and the number of components of all reducible fibers. Thus, one should be able to describe $\mathrm{Num}(J)$ in terms of $\mathrm{MW}(J/C)$ and $\mathrm{Num}_{\mathrm{fib}}(J/C)$. However, what makes the situation a little bit subtle is the following: first, the Shioda–Tate formula is about the ranks, which implies that the sublattice generated by $J_\eta(\eta)$ and $\mathrm{Num}_{\mathrm{fib}}(J)$ inside $\mathrm{Num}(J)$ is of finite index, but this index could be large. Second, $\mathrm{Num}(J)$ comes with an intersection form, that is, it is a lattice, which should somehow be explained by lattice structures on $\mathrm{Num}_{\mathrm{fib}}(J/C)$ and $\mathrm{MW}(J/C)$. In this section, we study these two observations in detail.

First, we introduce several lattices. Let O be the zero section of f, considered as a divisor class of J. Let T be the sublattice of $\mathrm{Num}(J)$ generated by $\mathrm{Num}_{\mathrm{fib}}(J/C)$ and O. Following [646], we will call T the *trivial lattice* of the jacobian genus one fibration. We remind the reader that we determined the lattice structure of $\mathrm{Num}_{\mathrm{fib}}(J/C)$ in Proposition 4.3.2. The trivial lattice has its name from the fact that a basis and the intersection numbers can be determined quite easily from the geometry of the reducible fibers and how they meet the zero section.

For a divisor class D we define $\deg_f(D) := D \cdot F$, where F is a fiber of f. Clearly, this number depends only of the class $[F] \in \mathrm{Num}(J)$. Recall that we have the class O of the zero section. Then, we have a surjective homomorphism

$$\mathrm{Num}(J) \to \mathrm{MW}(J/C) \qquad D \mapsto D - (\deg_f D)\mathsf{O} \mod \mathrm{Num}(S)_{\mathrm{fib}}.$$

Its kernel is generated by $\mathrm{Num}(S)_{\mathrm{fib}}$ and O and thus, equal to the trivial lattice T. Thus, we obtain an isomorphism

$$\mathrm{tr} \,:\, \mathrm{Num}(J)/T \;\cong\; \mathrm{MW}(J/C) \tag{4.5.1}$$

and we will identify these two groups via this isomorphism. We denote by

$$M \;:=\; T^{\perp} \;\subset\; \mathrm{Num}(J)$$

the orthogonal complement of T inside $\mathrm{Num}(J)$ and call it the *essential lattice*. It follows from the Hodge Index Theorem that M is negative definite.

The essential lattice leads to the Mordell–Weil lattice, the main topic of this section. Let

$$\mathrm{MW}(J/C)_{\mathrm{fr}} \;:=\; \mathrm{MW}(J/C)/\mathrm{MW}(J/C)_{\mathrm{tors}}$$

be the maximal free quotient of the Mordell–Weil group. We want to equip the dual group $\mathrm{MW}(J/C)_{\mathrm{fr}}^{\vee}$ with positive definite quadratic form with values in $\mathbb{Q}$. Above, we introduced the essential lattice $M \subset \mathrm{Num}(J)$. The intersection product induces a homorphism $M \to \mathrm{Num}(J)^{\vee}$ by sending $m \in M$ to the linear function $x \mapsto \langle x, m\rangle$. The image of M is equal to $(\mathrm{Num}(J)/T)^{\vee} = \mathrm{MW}(J/C)^{\vee}$. Passing to duals, we obtain an isomorphism

$$M \;\cong\; \mathrm{MW}(J/C)^{\vee} \;:=\; \mathrm{Hom}_{\mathbb{Z}}\left(\mathrm{MW}(J/C), \mathbb{Z}\right) \tag{4.5.2}$$

and hence an isomorphism

$$\phi \,:\, \mathrm{MW}(J/C)_{\mathrm{fr}} \;\to\; M^{\vee}. \tag{4.5.3}$$

We have a symmetric bilinear $\mathbb{Q}$-valued form on $M^{\vee}(-1)$ and use ϕ to obtain such a form on $\mathrm{MW}(J/C)$. By construction, we have

$$\langle \mathsf{S}, \mathsf{S}'\rangle \;=\; -\phi(\mathsf{S})\cdot\phi(\mathsf{S}') \;\in\; \mathbb{Q} \tag{4.5.4}$$

and this is called the *height pairing*. Thus, $\mathrm{MW}(J/C)_{\mathrm{fr}}$ carries the structure of a quadratic lattice over $\mathbb{Q}$, which is dual to the negative definite lattice M.

Definition 4.5.1 The positive definite quadratic $\mathbb{Q}$-lattice

$$\mathrm{MWL}(J/C) \;:=\; \mathrm{MW}(J/C)_{\mathrm{fr}}(-1) \;=\; M^{\vee}(-1)$$

is called the *Mordell–Weil lattice* of J/C.

Remark 4.5.2 The name "height pairing" needs some explanation: quite generally, a *height function* is a function that measures the "arithmetic complexity" of numbers in a local or global field or of closed points of a variety over some local or global field. For example, the *naive height* of a rational number $x = a/b \in \mathbb{Q}$ with $a, b \in \mathbb{Z}$ and coprime is $H(x) := \max\{|a|, |b|\}$ (multiplicative naive height) or its logarithm $h(x) := \log H(x)$ (logarithmic naive height). Given a height function h on a global field K, one can define the height of a point $[x_0, ..., x_N] \in \mathbb{P}^N_K$ with $x_i \in \mathcal{O}_K$ (ring of integers of K) with the $\{x_i\}$ assumed to be coprime, as $\max\{h(x_i)\}_i$. This defines a height function $h : \mathbb{P}^N_K(K) \to \mathbb{Q}$, which can be extended to all closed points of $\mathbb{P}^N_K$ and to subvarieties of $\mathbb{P}^N_K$. These functions are called the *Weil heights*. Of course, the Weil heights on a projective variety X over K are not unique and depend on many choices, such as an ample invertible sheaf on X that defines the embedding into $\mathbb{P}^N_K$. For an abelian variety A over a global field K, e.g., for an elliptic curve over K, there are special height functions, the *Néron–Tate heights*: one starts with an arbitrary Weil height $h : A(K) \to \mathbb{Q}$ and then, it turns out that the limit $\hat{h}(x) := \lim_{N\to\infty} h(Nx)/N^2$ defines a quadratic form on $A(K)$. For elliptic curves, the Néron–Tate heights are even unique. These heights are important in proving the Mordell–Weil theorem (Theorem 4.3.3) and they play a crucial role in the arithmetic of elliptic curves, see, for example, [682, Chapter VIII]. For a jacobian genus one fibration $f : J \to C$, one still has the Néron–Tate height on its generic fiber $J_\eta \to \operatorname{Spec} \Bbbk(C)$, although $\Bbbk(C)$ is not a global field if $\Bbbk$ is not a finite field, and it is given by (4.5.4), see [683, Chapter III]. We will say more about computing heights via local contributions in (4.5.9) and we will give some general computations of heights at the end of this section.

Let J_t be the fiber of $f : J \to C$ over a closed point $t \in C$. Let $R_{t,0}, R_{t,1}, \ldots, R_{t,n_t}$ be the irreducible components of a reduced fiber $(J_t)_{\mathrm{red}}$ and assume that $R_{t,0}$ is the unique component that meets the zero section. Let $A_t = (R_{t,i} \cdot R_{t,j})$ with $1 \le i, j \le n_t$ be the intersection matrix of those components that do not meet the zero section. The $R_{t,i}$ with $r > 0$ form a root basis of finite type, see Proposition 0.8.15 for the associated Dynkin diagram and the associated negative definite lattice. We also refer to Proposition 2.2.5 for the connection to genus one fibrations. Associated to the invertible matrix A_t, we have an even and negative definite lattice and let Discr_t be the associated discriminant group, which we computed in Table (0.2) in Section 0.8. It follows from the classification of fibers and the computation of discriminant groups that the order $|\,\mathrm{Discr}_t\,|$ is equal to the number r_t of the reduced components of J_t (components of J_t of multiplicity 1). If S is a section of f, then we have $\mathsf{S} \cdot J_t = 1$, which implies that S passes through exactly one of the reduced components of J_t. If S meets the component $R_{t,j}$, we let $(m_{t,1}, \ldots, m_{t,n_t})$ be the j.th column of the inverse matrix A_t^{-1}. The following proposition immediately follows from the definition of isomorphism (4.5.3).

Proposition 4.5.3 *For* $\mathsf{S} \in \mathrm{MWL}(J/C)$*, we have*

$$\phi(\mathsf{S}) \equiv \mathsf{S} - \mathsf{O} - [(\mathsf{S} - \mathsf{O}) \cdot \mathsf{O}]F - \sum_{t\in C} \sum_{i=1}^{n_t} m_{t,i} R_{t,i}.$$

Corollary 4.5.4 *For* $\mathsf{S} \in \mathrm{MW}(J/C)_{\mathrm{tors}}$*, we have*

$$\mathsf{S} \equiv \mathsf{O} + [(\mathsf{S} - \mathsf{O}) \cdot \mathsf{O}]F + \sum_{t\in C} \sum_{i=1}^{n_t} m_{t,i} R_{t,i}.$$

This corollary can also be proved directly by using that $n(\mathsf{S} - \mathsf{O}) \in \mathrm{Num}_{\mathrm{fib}}(J)$.

Let $\mathrm{MW}(J/C)^0$ be the subgroup of $\mathrm{MW}(J/C)$ that is generated by those sections that intersect each fiber at the irreducible component that meets the zero section. Since $f : J \to C$ is not smooth by assumption, Proposition 4.3.9 implies that $\mathrm{MW}(J/C)^0$ has no torsion and hence, can be identified with a subgroup of $\mathrm{MW}(J/C)_{\mathrm{fr}}$.

Proposition 4.5.5 *The image of* $\mathrm{MW}(J/C)^0$ *under isomorphism* (4.5.3) *is equal to the sublattice M of $M^\vee$. In particular, as a lattice,* $\mathrm{MWL}(J/C)^0$ *is isomorphic to* $M(-1)$.

Proof If $E \in \mathrm{MW}(J/C)^0$, then $E - \mathsf{O} \in T^\perp$ and thus, by definition of the map ϕ, the image of $\mathrm{MW}(J/C)^0$ is contained in M. Conversely, let $D \in M = T^\perp$. Then, the intersection of D with any component $R_{t,1}, \ldots, R_{t,n_t}$ is equal to zero, hence all $m_{t,i}$ are equal to zero and $\phi^{-1}(D - (\deg_f D)\mathsf{O}) \in \mathrm{MWL}(J/C)^0$. Thus, $\phi : \mathrm{MWL}(J/C)^0 \to M$ is bijective. □

Definition 4.5.6 The subgroup $\mathrm{MW}(J)^0$ of $\mathrm{MW}(J/C)_{\mathrm{fr}}$ equipped with the height pairing is called the *narrow Mordell–Weil lattice* and it is denoted by $\mathrm{MWL}(J/C)^0$.

The Shioda–Tate formula (4.3.1) implies that the Mordell–Weil lattice $\mathrm{MWL}(J/C)$ and the trivial lattice T generate a sublattice of $\mathrm{Num}(J)$ of finite index. We now address this index and relate the three lattice structures. In general, the sublattice $T \subset \mathrm{Num}(J)$ may not be primitive, that is, the quotient group $\mathrm{Num}(J)/T$ may not be free. Let $T' = (T^\perp)^\perp$ the double-dual of $\mathrm{Num}(J)$, which is the primitive closure of T in $\mathrm{Num}(J)$. Under the trace isomorphism (4.5.1), the image of T'/T is the torsion subgroup of $\mathrm{MW}(J/C)$

$$\mathrm{MW}(J/C)_{\mathrm{tors}} \cong T'/T. \tag{4.5.5}$$

It follows from the definition of T that we have $|\mathrm{discr}(T)| = \prod_{t\in C} |\,\mathrm{Discr}_t\,|$ and hence,

$$|\mathrm{discr}(T')| = \frac{\mathrm{discr}(T)}{|\,\mathrm{MW}(J/C)_{\mathrm{tors}}|^2} = \frac{\prod_{t\in C} |\,\mathrm{Discr}_t(J)|}{|\,\mathrm{MW}(J/C)_{\mathrm{tors}}|^2}. \tag{4.5.6}$$

Note formula (4.5.6) implies that if $\prod_{t\in C} |\,\mathrm{Discr}_t(J)|$ is square-free, then the Mordell–Weil group is torsion-free. Note that this formula agrees with formula (4.3.4) in the case where the Mordell–Weil group is finite. Next, we prove the following *discriminant formula*.

Proposition 4.5.7 *Let r be the rank of* $\mathrm{MW}(J/C)$. *Then,*

$$\begin{aligned} \mathrm{discr}\,(\mathrm{MWL}(J/C)) &= \frac{|\,\mathrm{discr}(\mathrm{Num}(J))| \cdot |\,\mathrm{MW}(J/C)_{\mathrm{tors}}|^2}{\mathrm{discr}(T)} \\ &= (-1)^r \cdot \mathrm{discr}\,(\mathrm{Num}(J)) \cdot \mathrm{discr}(T'). \end{aligned} \tag{4.5.7}$$

Proof Let m be the index of $M \oplus T'$ inside $\mathrm{Num}(S)$. It is equal to the index of M in $\mathrm{Num}(S)/T' = \mathrm{MW}(J/C)_{\mathrm{fr}}$, which is equal to $|M^\vee/M| = |\,\mathrm{discr}(M)|$. We compute

$$\begin{aligned} |\operatorname{discr}(M \oplus T')| &= \operatorname{discr}(M)\operatorname{discr}(T') = m\operatorname{discr}(T') \\ &= \frac{m|\operatorname{discr}(\operatorname{Num}(S)|}{m^2} = \frac{|\operatorname{discr}(\operatorname{Num}(S))|}{\operatorname{discr}(M)}. \end{aligned} \tag{4.5.8}$$

This implies

$$\begin{aligned} |\operatorname{discr}(\operatorname{MWL}(J/C))| &= |\operatorname{discr}(M^\vee)| = \frac{1}{|\operatorname{discr}(M)|} \\ &= \frac{|\operatorname{discr}(T')|\cdot|\operatorname{discr}(M)|}{m^2\cdot|\operatorname{discr}(T')|} = \frac{|\operatorname{discr}(\operatorname{Num}(J)|}{|\operatorname{discr}(T')|}. \end{aligned}$$

Taking into account its rank and using formula (4.5.6), we obtain the asserted formula. □

Next, we turn to the computation of the height pairing. For this, we need an explicit formula for computing the vectors $(m_{t,1},\dots,m_{t,n_t})$, where we remind the reader that these are the columns of the matrix A_t^{-1}. Using Proposition 4.5.3, we find

$$\langle \mathsf{S},\mathsf{S}'\rangle = -\mathsf{O}^2 + \mathsf{S}\cdot\mathsf{O} + \mathsf{S}'\cdot\mathsf{O} - \mathsf{S}\cdot\mathsf{S}' - \sum_{t\in C}\operatorname{contr}_t(\mathsf{S},\mathsf{S}'), \tag{4.5.9}$$

where $\operatorname{contr}_t(\mathsf{S},\mathsf{S}')$ is the *local contribution* expressed in terms of the vectors $(m_{t,1},\dots,m_{t,n_t})$ and $(m'_{t,1},\dots,m'_{t,n_t})$. Let us recall that we have $\mathsf{O}^2 = -\chi(O_J)$ by Corollary 4.3.8.

It remains to understand and to compute the local contributions $\operatorname{contr}_t(\mathsf{S},\mathsf{S}')$, which depend on the type of the reducible fiber J_t and on which components of J_t the sections S and S'. First, we observe that a section intersects only reduced components of a fiber. Thus, we have to compute the intersection of a section only with these components. To this end, we number the components as follows:

1. If J_t is of type $\tilde{A}_n$, then we index the components cyclically by $(0,1,\dots,n-1)$, where 0 corresponds to the component that intersects the zero section O and neighboring numbers correspond to components that intersect each other.
2. If J_t is of type $\tilde{D}_n$, then we index the components by $(0,1,2,3)$, where 0 corresponds to the component that intersects the zero section O and where 1 corresponds to the reduced component closest to the zero component in the Dynkin diagram.
3. If J_t is one of the remaining types, which is additive type, then we number O by 0 and all other reduced components are indexed in an arbitrary way.

Now, assume that S intersects J_t in the component $R_{t,i}$ and that S' intersect J_t in the component $R_{t,j}$. Then, we have

$$\operatorname{contr}_t(\mathsf{S},\mathsf{S}') = -(A_t^{-1})_{(i,j)},$$

the (i,j)-entry of the matrix A_t^{-1}. Here, i,j are assumed to be non-zero or otherwise the contribution is 0. The entries of the matrix A_t^{-1} can be computed easily for each type. The following table gives the computations of local contributions.

	$\tilde{A}_{n-1},\ \tilde{A}^*_{n-1}$	$\tilde{E}_6$	$\tilde{E}_7$	$\tilde{D}_n$	
$i=j$	$i(n-i)/n$	$\frac{4}{3}$	$\frac{3}{2}$	1	$i=1$
				$\frac{1}{2}+\frac{n-4}{4}$	$i=2,3$
$i<j$	$i(n-j)/n$	$\frac{2}{3}$	-	$\frac{1}{2}$	$i=1$
				$\frac{1}{2}+\frac{n-4}{4}$	$i=2$

Table 4.4 Local contributions to the height pairing

Example 4.5.8 Consider a general pencil of cubic curves in $\mathbb{P}^2$, let $p_0, p_1, \ldots, p_8$ be its base points, and let $\pi : J \to \mathbb{P}^2$ be the blow-up at these base points. Then the pull-back of this pencil of cubics to J gives rise to a genus one fibration $f : J \to \mathbb{P}^1$ and we obtain a rational elliptic surface. Since every exceptional divisor of π is a section of f, this fibration is jacobian. We let O be the exceptional divisor $\pi^{-1}(p_0)$. Moreover, using π, we see that $\mathrm{Num}(J)$ is a unimodular and odd lattice isomorphic to $\mathsf{I}^{1,9}$. Every fiber of f is irreducible and thus, the trivial lattice T is of rank 2 generated by the classes of the fixed section O and a fiber F. In particular, T is a primitive sublattice of $\mathrm{Num}(J)$, the intersection matrix is $\left(\begin{smallmatrix}-1 & 1\\ 1 & 0\end{smallmatrix}\right)$ and thus, T is unimodular. This implies that T is equal to its primitive closure, that is, $T' = T$. We also find $\mathrm{MWL}(J/\mathbb{P}^1) = \mathrm{MWL}(J/\mathbb{P}^1) \cong T^\perp$. Since the canonical divisor class satisfies $K_J = -[F]$, the essential lattice $M = T^\perp$ is contained in $K_J^\perp$ and hence, an even lattice. Thus, $\mathrm{MWL}(J/\mathbb{P}^1)$ is an even and positive definite unimodular lattice of rank 8, which is thus isomorphic to the lattice $\mathsf{E}_8(-1)$. In fact, it is naturally isomorphic to the lattice $\mathrm{Num}(J')(-1)$, where $J' \to \mathbb{P}^2$ is the blow-up of the points $p_1, \ldots, p_8$.

We have $\mathsf{O} = \pi^{-1}(p_0)$ and let S_i be the eight sections of f defined by the exceptional curves $\pi^{-1}(p_i), i \neq 0$. Since $S_i \cdot \mathsf{O} = S_i \cdot S_j = 0, i \neq 1$, the definition of the height shows that $h(S_i) = 2$ and $\langle S_i, S_j\rangle = 1$ for $i, j > 1$. The Gram matrix $(\langle S_i, S_j\rangle)$ has 1 as an eigenvalue of multiplicity 7 and is of trace equal to 16. Thus, the second eigenvalue is equal to 9, hence the determinant is equal to 9. This shows that the sublattice of the Mordell–Weil lattice spanned by the disjoint sections S_i has discriminant 9 and hence, its index is equal to 3.

Let S be a section of f. We consider its type, which is the vector $(d, a_0, \ldots, a_8)$ of coefficients in the geometric basis $(e, e_0, e_1, \ldots, e_8)$ of the blow-up $\mathrm{Bl}_{p_1,\ldots,p_8}(\mathbb{P}^2)$. Since S is a section, it is an exceptional (-1)-curve, which implies $\mathsf{S}^2 = -1$ and $\mathsf{S} \cdot K_J = -1$, from which we conclude $a^2 - \sum_{i=0} a_i^2 = -1$ and $3d - \sum_{i=0} a_i = 1$.

Next, assume that S lies in the subgroup of $\mathrm{MW}(J/C)$ generated by the disjoint sections $\mathsf{S}_1, \ldots, \mathsf{S}_8$. Thus, $\mathsf{S} = m_1 S_1 \oplus \cdots \oplus m_8 \mathsf{S}$ with respect to the group law of sections. According to [485, Theorem 6], its type is equal to $(3d, d - s - 1, d + m_1, \ldots, d + m_8)$, where

$$d = \sum_{i=1}^{8}(m_i^2 + m_i) + \sum_{1\le i<j\le 8} m_i m_j \quad \text{and} \quad s = \sum_{i=1}^{8} m_i.$$

In particular, we see that the section equal to the proper transform of a line through two points p_i, p_j does not belong to this subgroup. Also, there is a unique section S_0 with the property $3S_0 = \mathsf{S}_1 \oplus \cdots \oplus \mathsf{S}_8$. Its type is equal to $(4, 3, 1, 1, 1, 1, 1, 1, 1, 1)$, that is, it is equal to the proper transform of a plane curve of degree 4 with triple point at p_0 and simple points at the remaining points p_i. Summing up, one can choose the sections $\mathsf{S}_0, \mathsf{S}_1, \ldots, \mathsf{S}_7$ as a basis of the Mordell–Weil group.

It follows from the definition of the height that we have $h(\mathsf{S}) = 2 + 2a_0$ if S is of type $(d, a_0, a_1, \ldots, a_8)$. Replacing S_0 with $\mathsf{S}_0 - \mathsf{S}_1$, we compute its height to be equal to 2. Thus, the Mordell–Weil group has a basis formed by elements of minimal height 2.

Finally, we compare our definition of the height with the definition given in [683, III, §4], see also Remark 4.5.2. Assume that we have a global Weierstrass model

$$W \,:\, y^2 + a_1xy + a_3y + x^3 + a_2x^2 + a_4x + a_6 \;=\; 0 \tag{4.5.10}$$

of $f : J \to C$. Let us also assume for simplicity that $C = \mathbb{P}^1$, which will be enough for our applications. The coefficients a_i are binary forms in t_0, t_1 of degrees ik, where $k = \chi(\mathcal{O}_J) = -\mathsf{O}^2$. Next, we view the Weierstrass equation W as a hypersurface of degree $6k$ in the weighted projective space $\mathbb{P}(1, 1, 2k, 3k)$. A section S defines a morphism

$$s \,:\, \mathbb{P}^1 \to \mathbb{P}(1, 1, 2k, 3k),$$

which is given by

$$(u, v) \mapsto (t_0, t_1, x, y) = (A_0(u, v), A_1(u, v), A_2(u, v), A_3(u, v))\,,$$

where the A_0, A_1, A_2, A_3 are binary forms of degrees $d, d, 2kd, 3kd$, respectively. Since s is a section, the composition with the rational map given by the projection to $\mathbb{P}^1$ must be the identity. This implies that the first two coordinates can be written in the form uL, vL, where L is a binary form of degree $d - 1$. Its zeros correspond to the intersection points of S and O. In particular, we see that $d - 1 = \mathsf{S} \cdot \mathsf{O}$. The image $s(V(L))$ is equal to the point $[0, 0, 1, 1] \in X$, the image of the zero section O under the contraction map $\phi : J \to W$. Consider the image of the projection of $s(\mathbb{P}^1)$ to $\mathbb{P}(1, 1, 2k) \cong \overline{\mathbf{F}}_{2k}$: Its pre-image in $\mathbf{F}_{2k}$ is a curve from the divisor class $(2k+d-1)f + e$, where f is the class of a fiber on the minimal ruled surface $\mathbf{F}_{2k}$ and where e is the class of the exceptional section with self-intersection $-2k$. It intersects the exceptional section, whose pre-image in J is equal to O, with multiplicity $d - 1$.

Summing up, a section of f is defined by a solution (x, y) of the equation (4.5.10) given by binary forms A_2 and A_3 of degrees $2kd$ and $3kd$, respectively, such that the binary form of degree $6kd$

$$A_3^2 \,+\, a_1LA_2A_3 \,+\, a_3L^3A_3 \,+\, A_2^3 \,+\, L^2a_2A_2^2 \,+\, L^6a_6$$

is equal to zero.

If we assume that all fibers of $f : J \to \mathbb{P}^1$ are irreducible, then the height $h(\mathsf{S})$ of the section S is equal to $-(2\mathsf{O}^2 + 2\mathsf{O} \cdot \mathsf{S}) = 2k + 2d - 2$. In other words, the degree of the coordinate x as a binary form in u, v determines the height. If we view (x, y) as a rational point of the general fiber J_η, then x has to be considered as a rational function $\tilde{x} = x/v^{2dk} = A_2(t)$, where $t = u/v$ of degree $2dk$.

On the other hand, if some of the fibers of $f : J \to \mathbb{P}^1$ are reducible, then the components that do not meet the zero section get blown down to singular points of the Weierstrass model W. A section from the narrow Mordell–Weil lattice $\mathrm{MWL}(J/\mathbb{P}^1)^0$ defines a section of W that does not pass through any of these singular points. On the other hand, sections from $\mathrm{MWL}(J/\mathbb{P}^1) \setminus \mathrm{MWL}(J/\mathbb{P}^1)^0$ do pass through the singular points of W and the local correction terms in $h(\mathsf{S})$ reflect the behavior of the section at the singular points.

4.6 The Weil–Châtelet Group: the Local Case

In the previous sections, we studied genus one fibrations that admit a section, that is, jacobian genus one fibrations. In Proposition 4.3.1, we saw that every genus one fibration has an associated jacobian fibration. In the next three sections, we want to understand the set of all genus one fibrations whose associated jacobian fibrations are isomorphic to a fixed one. This set carries a group structure, the so-called Weil–Châtelet group. In this section, we will study this group over a local base, in the next section over a global base, and in Section 4.8, we will study quasi-elliptic fibrations.

We start with some generalities concerning torsors before turning to torsors under abelian varieties and elliptic curves. Let G be a commutative group scheme over a field F and let $\mathrm{PHS}(G/F)$ be the group of G-torsors over F as defined in Section 0.1. If G is a smooth quasi-projective group scheme, then Theorem 0.1.3 tells us that the group $\mathrm{PHS}(G/F)$ coincides with the cohomology groups

$$H^1_{\text{ét}}(F, G) \;:=\; H^1_{\text{ét}}(\operatorname{Spec} F, G) \;\cong\; H^1\left(\mathrm{Gal}(F^{\mathrm{sep}}/F), G(F^{\mathrm{sep}})\right),$$

where F^{sep} denotes a separable algebraic closure of F, where we identify a smooth commutative group scheme over a base scheme B with its abelian sheaf in the étale topology of B, and where the left hand side denotes étale cohomology and where the right hand side denotes Galois cohomology.

Let X be a G-torsor over $\operatorname{Spec} F$ and let $[X] \in \mathrm{PHS}(G/F)$ be its class. Using the isomorphisms $\mathrm{PHS}(G/F) \cong H^1_{\text{ét}}(F, G) \cong \mathrm{Ext}^1_F(\mathbb{Z}_F, G)$, the class $[X]$ defines an extension of group schemes

$$0 \to G \to G' \to \mathbb{Z}_F \to 0,$$

such that the image of 1 under the boundary homomorphism in the cohomology sequence is equal to $[X]$. The fiber of $G' \to \mathbb{Z}$ over 1 is isomorphic to G, see [277, VII, §1.4]. The pre-image of $n \in \mathbb{Z}$ is a G-torsor over $\operatorname{Spec} F$, whose isomorphism class in $\mathrm{PHS}(G/F)$ is equal to $n[X]$. Since the Galois cohomology is a torsion group, $[X]$ is an element of finite order in $\mathrm{PHS}(G/F)$. The order is called the *period* of the torsor X and denoted by $\mathrm{per}(X)$. A torsor of period n can be reconstructed from G

by means of the extension

$$0 \to G \to \bar{G}' \to (\mathbb{Z}/n\mathbb{Z})_F \to 0$$

as the pre-image of 1 in $\bar{G}'$.

Let Y be a regular, proper, and geometrically irreducible algebraic curve over a field F. For brevity of notation, we denote by $\mathcal{P}_{Y/F}$ the abelian sheaf in the flat topology associated to the relative Picard functor $\underline{\mathrm{Pic}}_{Y/F}$, which is representable by the Picard scheme $\mathbf{Pic}_{Y/F}$. Its connected component $\mathbf{Pic}^{\circ}_{Y/F}$ is a (generalized) Jacobian variety $\mathrm{Jac}(Y/F)$ of Y, which is a connected, commutative, and smooth algebraic group scheme over F. If Y is smooth over F, then $\mathrm{Jac}(Y/F)$ is an abelian variety over F. In any case, the degree homomorphism defines an exact sequence of group schemes over F

$$0 \to \mathrm{Jac}(Y/F) \to \mathbf{Pic}_{Y/F} \to \mathbb{Z}_F \to 0.$$

Taking cohomology, let δ be the boundary homomorphism $\mathbb{Z} = H^0_{\text{ét}}(F, \mathbb{Z}_F) \to H^1_{\text{ét}}(F, \mathrm{Jac}(Y/K))$. The image $\delta(1) \in H^1_{\text{ét}}(F, \mathrm{Jac}(Y/K))$ is the isomorphism class of the $\mathrm{Jac}(Y/F)$-torsor $Y \to \operatorname{Spec} F$. It is equal to the pre-image of 1 in the exact sequence and coincides with $\mathbf{Pic}^1_{Y/F}$, considered as a natural $\mathbf{Pic}^{\circ}_{Y/F}$-torsor.

Definition 4.6.1 A connected, commutative, and algebraic group scheme over a field F is called a *genus g algebraic group* if it is isomorphic to the Jacobian variety of a regular and geometrically irreducible projective algebraic curve Y over F.

The case that will be relevant to us is where G is either an elliptic curve E or a one-dimensional and inseparable form U of $\mathbb{G}_{a,K}$ over the field $F = \mathbb{k}(\eta)$, where η is a generic point of a local or global base C. Applying the theory of minimal models, we can always realize G as algebraic group $J^\sharp$, where $J \to C$ is a jacobian fibration over C. Then, we obtain that the group $H^1_{\text{ét}}(F, G)$ classifies isomorphism classes of genus one fibrations over C, whose jacobian fibrations are isomorphic to $f : J \to C$. Put differently, every genus one algebraic group over K can be realized as $J^\sharp$, where $f : J \to C$ is a genus one fibration.

Definition 4.6.2 Let A be an abelian variety over a field F. Then the group

$$\mathrm{WC}(A, F) := \mathrm{WC}(A/F) := H^1_{\text{ét}}(\operatorname{Spec} F,\ A)$$

is called the *Weil–Châtelet group* of A.

This group has been extensively studied, especially in the arithmetic situation where F is a number field or a local field.

Let $X \to \operatorname{Spec} F$ be an A-torsor, where A is an abelian variety A over F. There is a canonical isomorphism between the Picard variety $\mathbf{Pic}^{\circ}_{X/F}$ and the dual abelian A' variety of A, see [605, VII, 1.1]. In the one-dimensional case, that is, if E is an elliptic curve, then there is an isomorphism $E \to E' = \mathbf{Pic}^{\circ}_{E/F}$ and such an isomorphism depends on the choice of an F-rational point $E(F)$. More precisely, let $P \in E(F)$

(usually, one takes the neutral element of $E(F)$ of the group law) and then, a closed point $x \in E$ is sent to $\mathcal{O}_E(x - \deg(x)P)$. Similarly, there exists an isomorphism

$$X \cong \mathbf{Pic}^1_{X/F}$$

that assigns to an F-rational point $x \in X$ the isomorphism class of the invertible sheaf $\mathcal{O}_X(x)$. The degree of $\mathcal{O}_X(x)$ in $\mathrm{Pic}(X \otimes_F F(x))$ is equal to 1. Under an isomorphism $E \to \mathbf{Pic}^\circ_{E/F}$, the action of E on X corresponds to the natural action of $\mathbf{Pic}^\circ_{X/F}$ on $\mathbf{Pic}^1_{X/F}$, see [58, Lemma 3.2]. Coming back to the case of a general A-torsor $X \to \operatorname{Spec} F$, it follows from this discussion that the period of X is equal to smallest integer n, such that the $\mathbf{Pic}^\circ_{X/F}$-torsor $\mathbf{Pic}^n_{X/F}$ is a trivial torsor, that is, $\mathbf{Pic}^n_{X/F}(F) \neq \emptyset$. By abuse of notation, we will use the same notation for a torsor and its isomorphism class.

One also defines the *index* (resp. *separable index*) $\mathrm{ind}(X)$ (resp. $\mathrm{ind}_s(X)$) of X as the greatest common divisor of the degrees of closed points on it (resp. closed points whose residue fields are separable extensions of F). The following lemma shows that in our situation, this gcd is attained at one closed point.

Lemma 4.6.3 *Let Y be a regular genus one curve over a field F and $X = Y^\sharp$. Then, the index of the* $\mathrm{Jac}(Y)$*-torsor X is equal to the smallest degree of a closed point on X.*

Proof Since Y is regular, all Weil divisors are automatically Cartier divisors. By Riemann–Roch on a regular curve of arithmetic genus one, any divisor of positive degree is effective. Let $x \in X$ be a closed point of the smallest degree d. Suppose there exists a closed point $y \in X$ of degree coprime to d, say of degree $n = kd + r$ with $0 < r < d$. Then, the divisor $ny - kx$ is effective and of degree $r < d$, a contradiction. $\square$

Remark 4.6.4 One has to be a little bit careful with the notion of index because there are different notions in use that are related but usually not equal: For the purposes of this remark (this is not standard terminology), let us define the *min-index* of a variety X over a field F to be the smallest degree $[F(x) : F]$ among all closed points $x \in X$. Let us also define the *gcd-index* of X to be the gcd of the degrees $[F(x) : F]$ of all closed points $x \in X$. Clearly, the gcd-index is less than or equal to the min-index of X, but they need not be equal. For example, X has gcd-index 1 if and only if there exists a zero-cycle of degree 1. This latter does not necessarily imply that X has min-index 1, which is equivalent to X having an F-rational point. The previous lemma shows that the two indices coincide for regular curves of genus one. We refer to [472, Exercise 1.9 of Section 9.1] for the index of a curve that arises as generic fiber of a fibration.

Note that residue field of a closed point $x \in X$ of degree $\mathrm{ind}(X)$ splits X, which implies that $\mathrm{per}(X)$ divides $\mathrm{ind}(X)$. In particular, we have the following divisibility relations:

$$\mathrm{per}(X) \mid \mathrm{ind}(X) \mid \mathrm{ind}_s(X)\,. \tag{4.6.1}$$

Moreover, all three numbers have the same prime factors, but this is much harder to show, see, for example, [430, Proposition 5].

Proposition 4.6.5 *Let Y be a regular genus one curve over a field F. Set $X := Y^\sharp$. Assume that F is infinite and that* $\mathrm{Br}(F) = \{0\}$. *Then we have* $\mathrm{per}(X) = \mathrm{ind}(X) = \mathrm{ind}_s(X)$ *for the* $\mathrm{Jac}(Y)$*-torsor X.*

Proof We know from Section 0.9 that the étale sheaf $\mathcal{P}_{Y/F}$ is representable by a scheme $\mathbf{Pic}_{Y/F}$. By Proposition 0.9.2, we have $\mathbf{Pic}_{Y/F}(F) = \mathrm{Pic}(Y)$ if $Y(F) \neq \emptyset$ or $\mathrm{Br}(F) = 0$. Since the period is equal to the smallest n such that $\mathrm{Pic}^n(Y) \neq \emptyset$, the vanishing of $\mathrm{Br}(F)$ implies that there exists a F-rational divisor of degree n on Y. By Riemann–Roch, we may assume that it is effective and thus, coincides with a closed point of degree n. If $X = Y$ is an elliptic curve, we get $\mathrm{ind}(X) \mid \mathrm{per}(X)$ and hence $\mathrm{per}(X) = \mathrm{ind}(X)$. Since the residue field of the cusp on a quasi-elliptic curve Y is a purely inseparable extension of K of degree p, it remains to prove that $\mathrm{ind}(X) = \mathrm{ind}_s(X)$.

Let x be a point of degree $n = \mathrm{ind}(X)$. It suffices to find a point of the same degree with separable extension $F(x)/F$. If $F(x)/F$ is inseparable, then $n = pe$ for some e. By Riemann–Roch, the dimension of the linear system $|x|$ is equal to $n - 1$. We will show that one of the divisors in $|X|$ contains a separable point in its support. An effective Cartier divisor of degree d is an F-rational point of the symmetric product $Y^{(d)}$, where the latter denotes the Hilbert scheme $\mathrm{Hilb}^d(Y)$ of d points on Y. We consider the map of symmetric products $Y^{(e)} \to Y^{(n)}$, which is defined by multiplying an effective divisor of degree e by p. Since $e < n - 1$, unless $p = n = 2$, the image of this map does not contain the subvariety $|x|$ of $Y^{(n)}$ of dimension $n - 1$. Since F is an infinite field, we can find a divisor $D = \sum n_i x_i \in |x|$ of degree n, such that one of the points in its support has degree not divisible by p. This point is separable, and hence $\mathrm{ind}_s(X) \leq \mathrm{ind}(X)$, Together with $\mathrm{ind}(X) | \mathrm{ind}_s(X)$, this implies $\mathrm{ind}_s(X) = \mathrm{ind}(X)$.

It remains to consider the case $n = p = 2$. The image of Y in $Y^{(2)}$ is contained in the diagonal of $\bar{Y}^{(2)}$, where $\bar{Y} = Y \otimes_F \bar{F}$. Suppose that Y has no separable points of degree 2. Then, X is the union of linear systems $|y| \cong \mathbb{P}^1(F)$ and hence, the diagonal contains the union of $\mathbb{P}^1_{\bar{F}}$'s, which is a contradiction. □

By Tsen's theorem, $\mathrm{Br}(K) = 0$ if K is the field of rational functions on a algebraic curve over an algebraically closed field or its localization, see [660, Chapter X, §6]. We recall that that we defined the index $\mathrm{ind}(f)$ of a genus one fibration $f : X \to C$ to be the smallest degree of a multisection of f. Using this result, we obtain the following.

Corollary 4.6.6 *Let $f : X \to C$ be a genus one fibration. Then,*

$$\mathrm{per}(X_\eta) = \mathrm{ind}(X_\eta) = \mathrm{ind}_s(X_\eta) = \mathrm{ind}(f).$$

The multiplicity of any fiber divides this index. If C is strictly local, then it coincides with the multiplicity of the closed fiber.

Proof The generic fiber X_η is a genus one curve over the field K of rational functions on an algebraic curve over an algebraically closed field. By Tsen's theorem, we have $\mathrm{Br}(K) = 0$, see, for example, [660, Chapter X, §6]. The equality $\mathrm{per}(X_\eta) = \mathrm{ind}(X_\eta) = \mathrm{ind}_s(X_\eta)$ then follows from the previous proposition. Moreover, any closed point on X_η extends to a multi-section of f and every multi-section of f induces a closed point on X_η. This shows that $\mathrm{ind}(X_\eta) = \mathrm{ind}(f)$.

Similarly, any closed point on X_η extends to a multi-section of f, whose intersection number with every closed fiber is divisible by the multiplicity of the fiber.

Now, let C be strictly local. Let X_t be the fiber over the closed point $t \in C$ and let m_t be the multiplicity of X_t. In Section 4.2, we have already used the fact that there exists a regular closed embedding $T \hookrightarrow X$ such that $T \cdot X_t = m_t$, see also [606, Corollary 7.2.1]. The restriction of T to the generic fiber X_η is a point of degree m_t on X_η and thus, $\mathrm{ind}(f) \mid m_t$. The closure of a point x on the generic fiber is a finite cover of C of degree $\deg(x)$. It intersects X_t with multiplicity divisible by m_t. Thus, $\mathrm{ind}(f) = m_t$. □

From now on and until the end of this section, C is a strictly local base and $f : X \to C$ is an elliptic fibration. We will consider the case when f is a quasi-elliptic fibration in Section 4.8.

For any abelian group A, we denote by $A(\neq p)$ the direct sum of the ℓ-primary components of A with $\ell \neq p$, that is, the *prime-to-p part* of the abelian group A. The next result computes the prime-to-p part of the Weil–Châtelet group $\mathrm{WC}(J_K)$.

Theorem 4.6.7 *Let $f : J \to C$ be an elliptic fibration over a strictly local base C. Then,*

$$\mathrm{WC}(J_\eta)(\neq p) \;\cong\; (\mathbb{Q}/\mathbb{Z})(\neq p)^{b_1(J_s)}.$$

Proof We know that the order of any element $[X/K]$ of $\mathrm{WC}(J_K)$ is equal to the multiplicity of the closed fiber X_t of f. Let $E := J_K$, which we identify with the étale sheaf on $\eta = \mathrm{Spec}\, K$ represented by A. We have $\mathrm{WC}(J_\eta) \cong H^1(\eta, J_\eta)$ and, for every n coprime to p, the exact sequence

$$0 \to {}_nE \to E \xrightarrow{[n]} E \to 0$$

of group schemes over K gives an exact sequence of cohomology groups

$$0 \to E(K)^{(n)} \to H^1(\eta, {}_nE) \to {}_nWC(E,K) = {}_nH^1(K,E) \to 0. \tag{4.6.2}$$

Consider the Weil pairing

$${}_nE \times {}_nE \to \boldsymbol{\mu}_n,$$

see [682, Chapter III, §8]. The induced cup-product in cohomology produces a perfect pairing (see [508, Chapter V, §1, §2])

$$H^1(K, {}_nE) \times {}_nE(K) \to H^1(K, \boldsymbol{\mu}_n) \cong K^{*(n)} \cong \mathbb{Z}/n\mathbb{Z}. \tag{4.6.3}$$

Let $\mathbf{E} \to C$ be the Néron model of E. By the Néronian property, we have $E(K) \cong \mathbf{E}(R)$. Let $\mathbf{E}^\circ$ be the identity component of $\mathbf{E}$. The quotient group $\mathbf{E}/\mathbf{E}^\circ$ is isomorphic to the group Discr_t of reduced components of J_t. We have an exact sequence

$$0 \to E(K)^\circ \to E(K) \to \mathrm{Discr}_t \to 0,$$

where $E(K)^\circ = \mathbf{E}^\circ(R)$. Since $(n,p) = 1$ and $\mathbf{E}_t^\circ$ is a connected commutative algebraic group over $\Bbbk$, Proposition 4.2.1 implies that $(E(K)^\circ)^{(n)} \cong (\mathbf{E}_t^\circ)^{(n)} = \{0\}$. Thus, we obtain an exact sequence

$$_nE(K)^\circ \to {}_nE(K) \to {}_n\,\mathrm{Discr}_t \to 0. \tag{4.6.4}$$

The duality $H^1(K, {}_nE) \cong \mathrm{Hom}({}_nE(K), \mathbb{Z}/n\mathbb{Z})$ identifies $E(K)^{(n)} \cong (\mathrm{Discr}_t)^{(n)}$ in the exact sequence (4.6.2) with $\mathrm{Hom}({}_n\,\mathrm{Discr}_t, \mathbb{Z}/n\mathbb{Z})$, which is a finite group of the same order. It is easy to see that this defines an isomorphism

$${}_n\,\mathrm{WC}(E,K) \cong \mathrm{Hom}({}_nE(K)^\circ, \mathbb{Z}/n\mathbb{Z}) \cong \mathrm{Hom}({}_n\mathbf{E}_t^\circ, \mathbb{Z}/n\mathbb{Z}) \cong (\mathbb{Z}/n\mathbb{Z})^{b_1(J_t)} \tag{4.6.5}$$

and thus, finishes the proof. □

The structure of ${}_{p^\infty}\,\mathrm{WC}(E,K)$ is much more complicated. Understanding it will take up most of the remainder of this section.

Recall that at the end of Section 0.1 we associated to a group scheme G over a complete discrete valuation ring R a perfect pro-algebraic group $\mathcal{G} = (\mathcal{G}(G)_n)$, the perfect Greenberg realization of G. We define the fundamental group of G by

$$\pi_1(G) := \varprojlim_n (\mathcal{G}_n).$$

For our applications, we take G to be the Néron model $\mathbf{E}$. Since the fundamental group of a finite group scheme is trivial, we have

$$\pi_1(\mathbf{E}) = \pi_1(\mathbf{E}^\circ).$$

Let $\mathfrak{m}$ be the maximal ideal of R and set $R_n := R/\mathfrak{m}^n$. The reduction homomorphisms

$$r_n : \mathbf{E}^\circ(R_n) = \mathcal{G}(\mathsf{E})_n \to \mathbf{E}^\circ(\Bbbk)$$

define surjective homomorphisms $\pi_1(\mathcal{G}(\mathsf{E})_n) \to \pi_1(\mathbf{E}_t^\circ)$, and passing to the limit, we obtain a surjective homomorphism

$$\tilde{r} : \pi_1(\mathbf{E}^\circ) \to \pi_1(\mathbf{E}_t^\circ).$$

As we will see later, we have ${}_n\,\mathrm{Ker}(\tilde{r}) = \{0\}$ for any n prime to p and the computations from Section 0.10 show that ${}_n\pi_1(\mathbf{E}_t^\circ) \cong {}_n\mathbf{E}_t^\circ$. Thus, the duality isomorphism (4.6.5) can be re-stated as saying that the pairing

$$\mathrm{WC}(E,K)(\neq p) \times \pi_1(\mathbf{E})(\neq p) \to \mathbb{Q}/\mathbb{Z}(\neq p)$$

is a perfect duality. Shafarevich conjectured that there must be an extension of this duality to a perfect duality

$$\mathrm{WC}(A, K) \times \pi_1(\mathbf{A}^\vee) \to \mathbb{Q}/\mathbb{Z},$$

where A is an abelian variety over $\Bbbk$ and $\mathbf{A}^\vee$ is the Néron model of the dual abelian variety $A^\vee$. This conjecture has been solved by Bester [63] and Bertapelle [56, Theorem 3]. In the case of elliptic curves that satisfy some mild assumptions this was already proved much earlier in a series of papers by Vvedenskii.

Theorem 4.6.8 *Let C be stricly local and let K be its function field. Let A be an abelian variety over K, let $A^\vee$ be its dual abelian variety, and let $\mathbf{A}^\vee$ be the Néron model of A over C. Then,*

$$\mathrm{WC}(A/K) \cong \mathrm{Hom}\left(\pi_1(\mathbf{A}^\vee), \mathbb{Q}/\mathbb{Z}\right).$$

To compute the group $\mathrm{WC}(E, K)$ explicitly, we need to know the structure of $\pi_1(\mathbf{E}^\circ)$.

Let $\mathfrak{L}(E)$ be the kernel of the reduction homomorphism $r : \mathbf{E}(K) = \mathbf{E}(R) \to \mathbf{E}_t(\Bbbk)$. It is called the *Lutz group* of E. It coincides with the maximal ideal $\mathfrak{m}$ of R equipped with the group law defined by the formal group associated to the elliptic curve E over K, see [682, Chapter IV].

Let us recall the definition of the formal group $\widehat{E}$ associated to an elliptic curve E over an arbitrary field F, see, for example, [682, Chapter IV] for details. Let

$$w - (a_1 wz + a_3 w^2 + z^3 + a_2 wz^2 + a_4 w^2 z + a_6 w^3) = w - f(z, w) = 0$$

be the affine equation of E obtained from the Weierstrass equation

$$y^2 + a_1 xy + a_3 y + x^3 + a_2 x^2 + a_4 x + a_6 = 0$$

by the change the variables $z = -x/y$ and $w = -1/y$, so that the point at infinity in the Weierstrass equation becomes the point $(0, 0)$ in the new equation and z becomes a local uniformizer at $(0, 0)$. By recursively substituting $w = f(z, w)$ in the equation, we obtain a unique solution $w = w(z)$ as a formal power series of the equation $Y - f(X, Y) = 0$. This solution has the form

$$w(z) = z^3 \sum_{i=0}^{\infty} A_i z^i \in \mathbb{Z}[a_1, \ldots, a_6][[z]],$$

where the A_i's are quasi-homogeneous or weighted homogeneous polynomials in the coefficients a_i of the Weierstrass equation of E and $A_0 = 1$. Now, if z is any element of the maximal ideal $\mathfrak{m} = tF[[t]]$, then we plug in the above formula $z = z(t)$ and find a power series $w(z) = w(z(t))$, such that $(z, w(z))$ is a solution of the equation $Y - f(X, Y) = 0$ in formal power series in a variable t. Note that the substitution of a formal power series in a power series is not defined unless the power series has no constant term. Since our power series belongs to $\mathfrak{m}$, we have no problem with the substitution.

Considering $[1, z, w(z)]$ as a point in $\mathbb{P}^2(F)$, we can use the group law on $E(F)$ to obtain that $(z_1, w(z_1)) \oplus_E (z_2, w(z_2)) = (z_3, w(z_3))$, where

$$z_3 = -z_1 - z_2 + \frac{a_1\lambda + a_3\lambda^2 - a_2\nu - 2a_4\lambda\nu - 3a_6\lambda^2\nu}{1 + a_2\lambda + a_4\lambda^2 + a_5\lambda^3} = G(z_1, z_2) \in \mathbb{Z}[a_1, \ldots, a_6][z_1, z_2], \tag{4.6.6}$$

and

$$\lambda = \frac{w_2 - w_1}{z_2 - z_1} = \sum_{n=3}^{\infty} A_{n-3} \frac{z_2^n - z_1^n}{z_2 - z_1} \in \mathbb{Z}[a_1, \ldots, a_6][[z_1, z_2]]$$

is the slope of the line joining the two points, and where $\nu = w_1 - \lambda z_1$. The inverse in the group law on $E(F)$ is given by the formula

$$\iota(z) = \frac{z^{-2} - a_1 z^{-1} - \cdots}{-z^3 + 2a_1 z^{-2} + \cdots} \in \mathbb{Z}[a_1, a_2, a_3, a_4, a_6][[z]].$$

The formal power series $\Phi(X, Y) \in \mathbb{Z}[a_1, a_2, a_3, a_4, a_6][[X, Y]]$ defined by

$$\Phi(z_1, z_2) := \iota\,(G(z_1, z_2)),$$

for $z_1, z_2 \in \mathfrak{m} = tF[[t]]$, defines the structure of a formal group on the formal completion of the local ring of E at the origin isomorphic to the ring $F[[t]]$. If we equip $\mathfrak{m}$ with the structure of an abelian group with composition

$$x(t) \oplus y(t) := \Phi\,(x(t), y(t))$$

and inverse $\ominus x(t) = x(\iota(t))$, then we obtain a homomorphism of abelian groups

$$\mathfrak{m} \to E\,(F[[t]]), \quad z \mapsto (x(z), y(z)). \tag{4.6.7}$$

Suppose now that our field F is the quotient field K of the ring $R = \Bbbk[[t]]$, where $\Bbbk$ is an algebraically closed field. Let E be an elliptic curve over K, let $\mathbf{E}$ be its Néron model over R, and let $\mathfrak{L}(E)$ be its Lutz group. The formula for the map shows that its image lies in $\mathfrak{L}(E)$. It is proven in [682, Chapter VII, Proposition 2.2] that the map (4.6.7) is an isomorphism onto the Lutz group.

This allows us to equip the Lutz group $\mathfrak{L}(E)$ with the pro-algebraic structure defined by the filtration on $\mathfrak{m}$ by its powers $\mathfrak{m}^i$. It coincides with the pro-algebraic structure of the Greenberg realization of $\mathfrak{L}(E)$.

Let $[n] : \mathfrak{m} \to \mathfrak{m}$ be the multiplication-by-n homomorphism in the formal group $\widehat{E}$. It follows from the formula for the formal law $F(X, Y)$ in (4.6.6) that $[n](z) = nz+$ higher order terms. In particular, we see that $[n]$ is bijective if n is invertible in R, hence $\mathrm{Ker}([n])$ is uniquely divisible by any integer coprime to the characteristic. This agrees with Proposition 4.2.1.

If $n = p$ is the characteristic of $\Bbbk$ and K, then multiplication $[n]$ is more complicated and the height comes into play: recall from Section 0.1 that the height of the formal group $F(X, Y)$ on $R = F[[t]]$ of characteristic p is the largest power p^r of p such that $[p](t) = g(p^r)$ for some power series $g(T) \in R$. For formal groups arising from elliptic curves as just explained, we have the following result, see, for example, [682, Corollary 7.5].

Proposition 4.6.9 *The height of the formal group of an elliptic curve E over a field of characteristic $p > 0$ is equal to 1 if E is ordinary and it is equal to 2 if E is supersingular.*

Let $\mathfrak{m}_i = \mathfrak{m}^i$. It follows from the formula of the formal group law on $\mathfrak{m}$ that the filtration

$$\mathfrak{m} = \mathfrak{m}_1 \supset \mathfrak{m}_2 \supset \dots$$

has graded parts $\mathfrak{m}_i/\mathfrak{m}_{i+1}$ isomorphic to $\mathbb{G}_{a,\Bbbk}$.

We know that the height h_0 of the formal group associated to the reduction $\mathbf{E}_t^\circ$ of $\mathbf{E}^\circ$ is equal to 1 if $\mathbf{E}_t^\circ$ is $\mathbb{G}_{m,\Bbbk}$ or an ordinary elliptic curve, the height is equal to 2 if it is a supersingular elliptic curve, and the height is equal to ∞ if the reduction is additive. Let h be the height of $\mathfrak{L}(E)$.

It follows that the multiplication by p in the formal group $\mathfrak{L}(E)$ can be written in the form

$$[p](t) = f(t^p) = pt + c_0 t^p + c_1 t^{p^2} + \cdots, \tag{4.6.8}$$

where the following conditions on the values of the valuations at c_0, c_1 determine the heights of $\widehat{\mathbf{E}}_t^\circ$ and $\mathfrak{L}(E)$:

$\nu(c_0)$	$\nu(c_1)$	h_0	h_1
∞	$r_2 > 0$	∞	2
$r_1 > 0$	$r_2 > 0$	∞	1
0	any	1	1
$r_1 > 0$	0	2	1
∞	0	2	2

Proposition 4.6.10 *Let C be strictly local, let K be its function field K, and let E be an elliptic curve over K. Let $H = \mathfrak{L}(E)$ be the Lutz group of E considered as a perfect pro-algebraic group $G(\mathfrak{L}(E))^{\mathrm{pf}}$ over $\Bbbk$ with filtration*

$$H_1 := \mathfrak{m} \supset H_2 := \mathfrak{m}^2 \supset \cdots$$

There is a strictly increasing function $\lambda : \mathbb{Z}_+ \to \mathbb{Z}_+$, such that the morphism $[p] : H \to H, x \mapsto px$ maps H_n to $H_{\lambda(n)}$. Then:

1. *If $h_0 = 1$ and $h_1 = 1$, then $\lambda(n) = pn$.*
2. *If $h_0 = 2$ and $h_1 = 2$, then $\lambda(n) = p^2 n$.*
3. *If $h_0 = 2$ and $h_1 = 1$, then*

    a. *$\lambda(n) = p^2 n$ if $n \leq \nu(c_2)/p(p-1)$ and* *b.* $\lambda(n) = pn + \nu(c_2)$ *if* $n \geq \nu(c_2)/p(p-1)$.

The induced homomorphism $u_n : H_n/H_{n+1} \to H_{\lambda(n)}/H_{\lambda(n)+1}$ *is an isomorphism for all n.*

In Case (1), the formal group $\mathfrak{L}(E)$ is isomorphic to the formal group associated to the group U_K of units of K, and the proof in this case can be found in [658, 1.7]. The other cases were studied in [726] and [727].

Let W be the additive group of the ring of Witt vectors $W = W(\Bbbk)$. We consider W as a pro-algebraic group of the additive groups arising from the rings $W_n(\Bbbk)$ of Witt vectors of length n. The proof of the next lemma can be found in [658, 1.8, Proposition 7].

Lemma 4.6.11 *For each n, the isomorphism* $f_n : \mathbb{G}_{a,\Bbbk} \to H_n/H_{n+1}$ *and the natural projection* $\rho : W \to \mathbb{G}_a$ *can be lifted to fit into a commutative diagram*

$$\begin{tikzcd} W \arrow[r, "\phi_n"] \arrow[d, "\rho"] & H_n \arrow[d] \\ \mathbb{G}_a \arrow[r, "f_n"] & H_n/H_{n+1}. \end{tikzcd}$$

These diagrams define an isomorphism of pro-algebraic groups

$$W^I \cong H,$$

where $I := \mathbb{Z}_+ \setminus \lambda(\mathbb{Z}_+)$.

Corollary 4.6.12 *Let* E_K *be an elliptic curve over* $K = \Bbbk((t))$, *let* $\mathbf{E}$ *be its Néron model over* $R = \Bbbk[[t]]$, *and let* $\mathbf{E}_t$ *be the closed fiber of* $\mathbf{E}$. *Suppose that* $\mathbf{E}_t$ *is not of additive type. Then*

$${}_{p^\infty}\mathrm{WC}(E_K) \cong \mathrm{Hom}\left(\pi_1(\mathbf{E}^\circ), \mathbb{Q}_p/\mathbb{Z}_p\right) \cong \Bbbk^I \times (\mathbb{Q}_p/\mathbb{Z}_p)^\epsilon,$$

where $\epsilon = 1$ *if* $\mathbf{E}_t$ *is an ordinary elliptic curve and* $\epsilon = 0$ *otherwise.*

Proof We use Theorem 4.6.8 and Proposition 4.6.11. First, we have $\pi_1(\mathcal{L}(E)) = \pi_1(W(\Bbbk))^e$. It follows from [658, 8.5, Remarque] that

$$\mathrm{Hom}\,(\pi_1(W(\Bbbk)), \mathbb{Z}/p\mathbb{Z}) \cong \mathrm{Hom}\left(\pi_1(W(\Bbbk))^{(p)}, \mathbb{Z}/p\mathbb{Z}\right) \cong \mathrm{Hom}\,(\pi_1(\mathbb{G}_a), \mathbb{Z}/p\mathbb{Z}).$$

By [658, 8.3], the latter group is isomorphic to $\mathbb{G}_{a,k}$. Now

$$\pi_1(\mathbf{E}^\circ) \cong \pi_1(\mathcal{L}(E_K)) \times \pi_1(\mathbf{E}_t^\circ).$$

It remains to use the known structure of the group $\pi_1(\mathbf{E}_t^\circ)$, see Section 0.1. □

The treatment in the case of the additive reduction is more complicated and it is achieved by passing to a *semi-stable reduction*. By this, we mean the following.

Proposition 4.6.13 *Let C be strictly local with function field K and let E be an elliptic curve of K with additive reduction. Then, there exists a finite separable totally ramified extension L/K such that E_L has good or multiplicative reduction.*

Proof Choose a prime ℓ different from p and consider the action of the absolute Galois group Gal_K of K on ${}_\ell E(\bar{K}) \cong (\mathbb{Z}/\ell\mathbb{Z})^{\oplus 2}$. This becomes trivial after a finite and separable extension L/K, which we may assume to be totally ramified. Then, the base-change $E' = E_L$ has all its ℓ-torsion points defined over L. Let R' be the normalization of R in L and let t' be the pre-image of the closed point t. Let $\mathbf{E}'$ be the Néron model of E' over R'. We know from Proposition 4.2.1 that the kernel of the reduction homomorphism $E'(L) = \mathbf{E}'(R') \to \mathbf{E}'_{t'}$ has no torsion points of order prime to p. Thus, ${}_\ell\mathbf{E}'_{t'} \cong (\mathbb{Z}/\ell\mathbb{Z})^{\oplus 2}$ and hence either $\mathbf{E}'_{t'}$ is smooth or $\mathbf{E}'_{t'}$ contains a subgroup isomorphic to $\mathbb{G}_m \oplus (\mathbb{Z}/\ell\mathbb{Z})$. □

The previous proposition is a special case of Grothendieck's theorem on semi-stable reduction of abelian varieties over the field of fractions of any noetherian regular irreducible scheme of dimension 1, see [277, IX, Théorème 3.6]. In this general context, semi-stable reduction means that the closed fiber of the Néron model does not contain a non-trivial unipotent connected subgroup. A more elementary proof in the case of Jacobians was given in [32].

Example 4.6.14 If we avoid some small characteristics p, then the proof of the previous proposition can be made very explicit.

For example, suppose that the special fiber is additive of type $\tilde{D}_n$ and that $p \neq 2$. Let $J_0 := R_1 + R_2 + 2R_3 + \cdots + 2R_{n-1} + R_n + R_{n+1}$ as in Proposition 2.2.5. We have $J_0 = R + 2\sum_{i=3}^{n-1} R_i$. Thus, $\mathcal{O}_X(R)$ is divisible by 2 in $\mathrm{Pic}(J)$. Since $p \neq 2$, there exists a separable double cover $X \to J$ ramified over R. The proper transform of the components of R on X are disjoint (-1)-curves. The proper transforms of R_3 and R_{n-1} are smooth rational curves with self-intersection -4. The pre-images of other curves R_i split into the disjoint unions of two (-2)-curves. Blowing down the proper transform of R, we obtain a minimal relative model of $X \to C$. Its closed fiber is smooth if $n = 4$ or of type $\tilde{A}_{2n-9}$ if $n > 4$.

A similar procedure leads to a smooth fiber if the special fiber is additive of type E_6 or $\tilde{A}_2^*$ (assuming $p \neq 3$), of type $\tilde{E}_7$ (assuming $p \neq 2$), or of type $\tilde{E}_8$.

If the invariant of wild ramification δ is equal to zero (for example, if $p \neq 2, 3$), then a semi-stable reduction can always be attained after a tamely ramified finite extension K'/K. The smallest degree of a tame extension achieving this is equal to $2, 3, 4$ or 6 as in the following table shows. This table also gives the type of semi-stable reduction, see [646, Table 5.2].

	$\tilde{A}_0^{**}$	$\tilde{A}_1^*$	$\tilde{A}_2^*$	$\tilde{D}_4$	$\tilde{D}_n, n > 4$	$\tilde{E}_6$	$\tilde{E}_7$	$\tilde{E}_8$
$[K' : K]$	6	4	3	2	2	3	4	6
Type	$\tilde{A}_0$	$\tilde{A}_0$	$\tilde{A}_0$	$\tilde{A}_0$	$\tilde{A}_{2n-9}$	$\tilde{A}_0$	$\tilde{A}_0$	$\tilde{A}_0$

Table 4.5 Semi-stable reduction: the tame case

If $p \neq 2, 3$, then it follows from this table that the semi-stable reduction is of multiplicative type if and only if the singular fiber is of additive type $\tilde{D}_n$ with $n > 4$. If $p = 2$, then there are examples of elliptic curves over K, whose reduction is additive of type $\tilde{D}_n$ with $n > 4$ that have a good semi-stable reduction, see [478, Theorem 2.8].

Remark 4.6.15 One can reconstruct an elliptic fibration from a semi-stable reduction as the quotient of the minimal relative model of the semi-stable reduction by a finite group, which is isomorphic to the Galois group of the field extension. If the order of the group is prime to p or if the extension is tame, then it is easy to find the singular points of the quotient and resolve them. For example, if $p \neq 2$, $[K' : K] = 2$, and the semi-stable reduction has good reduction, then the quotient has 4 ordinary double points and resolving them we obtain a singular fiber of type $\tilde{D}_4$.

Let E be an elliptic curve over K and let E_0 be the special fiber of the Néron model of E. Let K'/K be a finite field extension such that $E_{K'}$ has semi-stable reduction. Let E_0' be the special fiber of the Néron model of $E_{K'}$. We use the notation from Proposition 4.6.10. We would like to see how the heights h_0, h_1 change under a semi-stable reduction.

First we use the following result, see [682, Appendix A, Corollary 1.4].

Lemma 4.6.16 *Let E be an elliptic curve over a local field K with ring of integers R. Then $E_{K'}$ has good semi-stable reduction if and only if $j(E) \in R$.*

Concerning supersingular elliptic curves, we have the following results of Deuring, see [429, Chapter 13, Theorem 6] and [682, Chapter V, Theorem 4.1] for details and proofs.

Theorem 4.6.17 *If E is a supersingular elliptic curve over a field of characteristic $p > 0$, then $j(E) \in \mathbb{F}_{p^2}$. Over an algebraically closed field of characteristic $p > 0$, there exist only a finite number of supersingular elliptic curves (in fact, roughly $p/12$) and they all can be defined over $\mathbb{F}_{p^2}$.*

Being supersingular does not depend on the ground field, that is, E is supersingular if and only if E_L is supersingular for any finite extension L/K. Moreover, if E has good reduction, say with special fiber E_0 of the Néron model, then the special fiber of the Néron model of E_L is isomorphic to E_0. In particular, having good and ordinary (resp. good and supersingular) reduction does not change after passing to finite extensions of K.

Remark 4.6.18 We thank Yuri Zarhin for the following remark: Let E be an ordinary elliptic curve over K with additive reduction E_0 and such that $j \in R$ and $j \mod \mathfrak{m}_R$ is the j-invariant of some supersingular elliptic curve E_0'. Then, by Lemma 4.6.16, there exists some finite extension K'/K such that $E' := E_{K'}$ has good reduction. In this case, the special fiber of the Néron model of E' is isomorphic to E_0'. In particular, E' is an ordinary elliptic curve, but E_0' is supersingular.

On the other hand, it is possible that E_0 is an ordinary elliptic curve but its semi-stable reduction E' has good supersingular reduction. If $p \geq 5$ and if $y^2 +$

$x^3 + a_4 x + a_6 = 0$ is the Weierstrass equation of E over K, then we may consider the *quadratic twist* E_d of E, that is, separable form E defined by an equation $y^2 + x^3 + d^2 a_4 x + d^3 a_6 = 0$, where $d \in \mathfrak{m}$ is not a square in K. The reduction of E_d is additive but its semi-stable reduction must have a supersingular elliptic curve. Since the two elliptic curves are isomorphic over $K(\sqrt{d})$, we obtain that E has a semi-stable reduction with good supersingular reduction.

We will use the Weil restriction functor, which we discussed in Section 0.1. We apply it to the following situation: let K be a local field and K' be a finite Galois extension of K (totally ramified in practice), say with Galois group Γ. Let E be an elliptic curve over K and $E' := E_{K'}$ and let $\mathbf{E}$ and $\mathbf{E}'$ be their Néron models over R and R', respectively. The Γ-action on K' induces a Γ-action on E' and $\mathbf{E}'$. We have the Weil restrictions $\mathfrak{R}_{K'/K}(E')$ and $\mathfrak{R}_{R'/R}(\mathbf{E}')$. We note that there are canonical homomorphisms of group schemes $E \to \mathfrak{R}_{K'/K}(E')$ and $\mathbf{E} \to \mathfrak{R}_{R'/R}(\mathbf{E}')$.

The following proposition [56, Lemma 15] plays an important role in the proof of the duality theorem 4.6.8. It allows one to reduce the proof to the case where the abelian variety in question has a semi-stable reduction.

Proposition 4.6.19 *Let A be an abelian variety over K and let K'/K be a finite and totally ramified Galois extension, such that $A' := A_{K'}$ has semistable reduction. Let $\mathbf{A}'$ be the Néron model of A' over R'. Let X and $\mathbf{X}$ be the Weil restrictions $\mathfrak{R}_{K'/K}(A')$ and $\mathfrak{R}_{R'/R}(\mathbf{A}')$, respectively. Then,*

$$\pi_1(\mathbf{X}) \cong \pi_1(\mathbf{A}'), \quad \mathrm{WC}(X/K) \cong \mathrm{WC}(A'/K).$$

Proof We have a canonical homomorphism of group schemes $\mathbf{A} \to \mathbf{X}$ that induces a homomorphism of Greenberg realizations $\mathcal{G}(\mathbf{A}) \to \mathcal{G}(\mathbf{X})$. So, it suffices to prove that $\mathcal{G}(\mathbf{X}) \cong \mathcal{G}(\mathbf{A}')$. Let $\mathfrak{m}'$ be the maximal ideal of the ring R' of integers of A_L. We have

$$R' \otimes_R R_i = R' \otimes_R R/\mathfrak{m}^i = R'/\mathfrak{m}^i R' = R'/\mathfrak{m}'^{ei} R' = R'_{ei},$$

where $e = [K' : K]$. By assumption, e coincides with the ramification index of K'/K. It is known that the Weil restriction commutes with the base changes. This gives

$$\mathfrak{R}_{R_i/\mathbb{k}}(\mathbf{X} \otimes_R R_i) = \mathfrak{R}_{R_i/\mathbb{k}}(\mathfrak{R}_{R'_{ei}/R_i}(\mathbf{A}' \otimes_R R'_{ei})) = \mathfrak{R}_{R'_{ei}/\mathbb{k}}(\mathbf{A}' \otimes_{R'} R'_{ei}).$$

Since the set $e\mathbb{Z}_+$ is a cofinal subset of $\mathbb{Z}_+$, we can pass to the projective limit and obtain that $\mathcal{G}(\mathbf{A}')$ is also the projective limit of $\mathfrak{R}_{R'_{ei}/\mathbb{k}}(\mathbf{A}' \otimes_{R'} R_{ei})$. ◻

This result has the following interesting corollary and we refer to Lemma 16 and Theorem 3 of [56] for details.

Corollary 4.6.20 *The canonical homomorphism of group schemes $\mathbf{A} \to \mathfrak{R}_{R'/R}(\mathbf{A}')$ induces a monomorphism*

$$\pi_1(A_K) \to \pi_1(A_{K'}).$$

Under the Shafarevich pairing from Theorem 4.6.8, its transpose is the corestriction homomorphism of the Weil–Châtelet groups $\mathrm{WC}(A'/K') \to \mathrm{WC}(A/K)$.

We can now complement Proposition 4.6.10 by treating the case when $h_0 = \infty$, that is, where the reduction of the elliptic curve E is of additive type. There are two possible cases to consider, namely (1) $h_1 = h'_0 = h'_1 = 1$ and (2) $h_1 = h'_0 = h'_1 = 2$.

Let $\mathcal{G} = \mathcal{G}(\mathfrak{L}(E))^{\mathrm{pf}}$ (resp. $\mathcal{G}' = \mathcal{G}(\mathfrak{L}(E'))^{\mathrm{pf}}$ be the perfect Greenberg realization of the Lutz group of E (resp. E'). It follows from the proof of Proposition 4.6.19 that we have a commutative diagram for every $n > 0$,

$$\begin{tikzcd} \mathcal{G}_n \arrow[r, "\phi_n"] \arrow[d, "{[p]}"] & \mathcal{G}'_{en} \arrow[d, "{[p]}"] \\ \mathcal{G}_{\lambda(n)} \arrow[r, "\phi_{\lambda(n)}"] & \mathcal{G}'_{e\lambda(n)} \end{tikzcd}$$

Applying Proposition 4.6.10, we obtain that the right vertical arrows are isomorphisms and that the function λ coincides with $\lambda'(en)/e$. The function λ' is given in Proposition 4.6.10 and it shows that $\lambda'(en)/e = \lambda'(n)$ in all cases. This allows us to compute λ. Applying Lemma 4.6.11, we obtain

$$\mathcal{G}(\mathbf{E})^{\mathrm{pf}} \;\cong\; W(\Bbbk)^I \times \mathbb{G}_{a,\Bbbk},$$

where $I = \mathbb{Z}_+ \setminus \lambda'(\mathbb{Z}_+)$.

Corollary 4.6.21 *Let E be an elliptic curve over K with additive reduction. Then*

$$_{p^\infty}\,\mathrm{WC}(E/K) \;\cong\; \mathrm{Hom}\left(\pi_1(\mathbf{E}^\circ), \mathbb{Q}_p/\mathbb{Z}_p\right) \;\cong\; \pi_1(W(\Bbbk))^I \times \Bbbk.$$

The next corollary was proved in [473, Corollary 6.3].

Corollary 4.6.22 *Let E be an elliptic curve over K. Then, for any $m > 0$,*

$$_{p^m}\,\mathrm{WC}(E/K) \;\neq\; \{1\}.$$

Proof Since K is a field of cohomological dimension 1, it follows that the multiplication-by-p^m map $H^1(K, E_K) \to H^1(K, E_K)$ is surjective. Thus, the group $\mathrm{WC}(E/K)$ if divisible by any power of p. Thus, it suffices to prove that $_p\,\mathrm{WC}(E/K) \neq \{1\}$. To see this, we note that this group is isomorphic to $\mathrm{Hom}(\pi_1(W(\Bbbk)^I), \mathbb{Z}/p\mathbb{Z})$. The surjection $W(\Bbbk) \to \mathbb{G}_{a,\Bbbk}$ induces a surjection $\pi_1(W(\Bbbk)) \to \pi_1(\mathbb{G}_{a,\Bbbk})$. It only remains to note that there is a natural isomorphism $\mathbb{G}_a \to \mathrm{Hom}(\mathbb{G}_{a,\Bbbk}, \mathbb{Z}/p\mathbb{Z})$, see [657, 8.3, Proposition 3]. □

As an application, we now can prove the following result, which was proven by other methods in [606], §9.

Theorem 4.6.23 *Let $\Bbbk$ be an algebraically closed field of characteristic $p > 0$, let $R := \Bbbk[[t]]$, and let K be the field of fractions of R. Let E be an elliptic curve K, let $\mathbf{E}$ be its Néron model over R, and let $\mathbf{E}_t$ be its closed fiber. Then,* *1. If $\mathbf{E}_t$ is an ordinary elliptic curve, then ${}_{p^\infty}\mathrm{WC}(E/K)$ is given by an extension*

$$0 \to \mathbb{Q}_p/\mathbb{Z}_p \to {}_{p^\infty}\mathrm{WC}(E/K) \to \mathrm{Hom}\left(\mathrm{Gal}(K^{\mathrm{ab}}/K), \mathbb{Q}_p/\mathbb{Z}_p\right) \to 0. \tag{4.6.9}$$

2. If $\mathbf{E}_t^\circ \cong \mathbb{G}_{m,\Bbbk}$, then

$${}_{p^\infty}\mathrm{WC}(E/K) \cong \mathrm{Hom}\left(\mathrm{Gal}(K^{\mathrm{ab}}/K), \mathbb{Q}_p/\mathbb{Z}_p\right).$$

Proof We apply Theorem 4.6.8. Applying the exact sequence of homotopy groups to the reduction homomorphism $r : E(K) = \mathbf{E}(R) \to \mathbf{E}(\Bbbk) = \mathbf{E}_t(\Bbbk)$, we obtain an exact sequence

$$0 \to \pi_1(\mathfrak{L}(E)) \to \pi_1(\mathbf{E}) \to \pi_1(\mathbf{E}_t) \to 0. \tag{4.6.10}$$

It follows from Examples 0.1.22 and 0.1.23 that

$$\mathrm{Hom}\left(\pi_1(\mathfrak{L}(E)), \mathbb{Q}_p/\mathbb{Z}_p\right) = \mathrm{Hom}\left(\mathrm{Gal}(K^{\mathrm{ab}}/K), \mathbb{Q}_p/\mathbb{Z}_p\right)$$

and

$$\mathrm{Hom}\left(\pi_1(\mathbf{E}_t), \mathbb{Q}_p/\mathbb{Z}_p\right) \cong \mathrm{Hom}\left(\mathbb{Z}_p, \mathbb{Q}_p/\mathbb{Z}_p\right) = \mathbb{Q}_p/\mathbb{Z}_p$$

if $\mathbf{E}_t$ is an ordinary elliptic curve. If $\mathbf{E}_t^0$ is of multiplicative type, we have

$$\mathrm{Hom}\left(\pi_1(\mathbf{E}_t^0), \mathbb{Q}_p/\mathbb{Z}_p\right) = 0.$$

Applying the functor $\mathrm{Hom}(-, \mathbb{Q}_p/\mathbb{Z}_p)$ to exact sequence 4.6.10, the assertions follow from Theorem 4.6.8. □

Remark 4.6.24 As already mentioned, this result was also shown by Raynaud in [606, §9]. He also showed that in Case (1), a non-zero element from ${}_{p^\infty}\mathrm{WC}(E/K)$ coming from an element from the subgroup $\mathbb{Q}_p/\mathbb{Z}_p$ has a tame closed fiber. In Case (2), all non-zero elements from ${}_{p^\infty}\mathrm{WC}(E/K)$ define torsors with wild closed fiber.

Let $\tilde{r} : \pi_1(\mathbf{E}^\circ) \to \pi_1(\mathbf{E}_t^\circ)$ be the reduction homomorphism. It gives an inclusion

$$\mathrm{Hom}\left(\pi_1(\mathbf{E}_t^\circ), \mathbb{Q}_p/\mathbb{Z}_p\right) \subseteq \mathrm{Hom}\left(\pi_1(\mathbf{E}^\circ), \mathbb{Q}_p/\mathbb{Z}_p\right) \cong {}_{p^\infty}\mathrm{WC}(E/K).$$

We denote by $\mathrm{WC}(E/K)^{\mathrm{tame}}$ the subgroup of $\mathrm{WC}(E/K)$ that is defined to be the image of $\mathrm{Hom}(\pi_1(\mathbf{E}_t^\circ), \mathbb{Q}_p/\mathbb{Z}_p)$ under the duality isomorphism from Theorem 4.6.8. The name and the notation are justified by the following.

Proposition 4.6.25 *Let $X \to C$ be a regular relatively minimal model of an E-torsor $X_K \to \mathrm{Spec}\, K$. Then its closed multiple fiber is tame if and only if the isomorphism class of X_K belongs to $\mathrm{WC}(E/K)^{tame}$.*

If $\mathbf{E}_t$ is additive or an ordinary elliptic curve, then the group $\mathrm{Hom}(\pi_1(\mathbf{E}_t^\circ), \mathbb{Q}_p/\mathbb{Z}_p)$ is non-trivial. These are also the only possible cases when a torsor may have a tame multiple fiber. We conjecture that

$$\mathrm{Hom}\left(\pi_1(\mathbf{E}_t^\circ), \mathbb{Q}_p/\mathbb{Z}_p\right) = \mathrm{WC}(E/K)^{\mathrm{tame}},$$

where $\mathrm{WC}(E,K)^{\mathrm{tame}}$ is the subgroup of isomorphism classes of torsors with tame multiple fiber. Raynaud's results imply that this is true if $\mathbf{E}_t$ is an ordinary elliptic curve.

It is an interesting open problem to find the subset of $\mathrm{WC}(E/K)$ of isomorphism classes of torsors with given type of a wild multiple fiber described by the function $\phi(n)$ discussed in Section 4.2.

4.7 The Weil–Châtelet Group: the Global Case

Having studied the Weil–Châtelet group in the local case, we now study it in the global case. Thus, we assume that C is global and that we have a jacobian elliptic fibration $f : J \to C$. Let $E := J_\eta$ be the generic fiber of f, which is an elliptic curve over the generic point $\eta = \mathrm{Spec}\, K$, where K is the function field of C. In this section, we show how to compute the group $\mathrm{WC}(E/K)$.

For every closed point $t \in C$, we have the generic point K_t^h of the strict localization of C at t. For the base change of $f : J \to C$ to K_t^h, the results of the previous section give us some control over the Weil–Châtelet group $\mathrm{WC}(E_{K_t^h}/K_t^h)$. An element of $\mathrm{WC}(E/K)$ defines for every $t \in C$ an element in $\mathrm{WC}(E_{K_t^h}/K_t^h)$ and thus, a homomorphism from the global Weil–Châtelet group $\mathrm{WC}(E/K)$ into the product (we will see that it lies in the direct sum) of local Weil–Châtelet groups

$$\mathrm{WC}(E/K) \to \prod_{t \in C} \mathrm{WC}(E_{K_t^h}/K_t^h).$$

In order to understand and compute the former, we have to understand the latter, as well as the kernel and cokernel of this homomorphism.

Quite generally, let $\mathcal{F}$ be an abelian sheaf in the étale topology on $\eta = \mathrm{Spec}\, K$. The Grothendieck–Leray spectral sequence for the inclusion morphism $i : \eta \hookrightarrow C$ gives a long exact sequence

$$0 \to H^1_{\text{ét}}(C, i_*\mathcal{F}) \to H^1_{\text{ét}}(\eta, \mathcal{F}_\eta) \to H^0_{\text{ét}}(C, R^1 i_*\mathcal{F}) \to H^2_{\text{ét}}(C, i_*\mathcal{F}) \to H^2_{\text{ét}}(\eta, \mathcal{F}_\eta). \tag{4.7.1}$$

Next, assume that $\mathcal{F}$ is the sheaf associated to a smooth connected group scheme G over η that admits a Néron model $\mathbf{G}$ over C. Then, the Néronian mapping property implies that $i_*G_K \cong \mathbf{G}$, where we identify a group scheme with its associated sheaf in the étale topology. The known computation of the fiber of direct images of étale sheaves [508, Chapter III, Theorem 1.15] shows that the fiber of the sheaf $(R^1 i_*G)_t$ at a closed point $t \in C$ is isomorphic to $H^1(\tilde{\eta}_t, \tilde{i}_t^*\mathbf{G})$, where $\tilde{i}_t : \tilde{\eta}_t^h \to \mathrm{Spec}(O_{C,t}^h) \to C$ and where $\tilde{\eta}_t = \mathrm{Spec}\, K_t^h$ is the generic point of the strict localization of C at a closed point t. By functoriality of Néron models, we obtain an isomorphism

$$H^1_{\text{ét}}(\tilde{\eta}_t, \tilde{i}_t^* \mathbf{G}) \cong H^1_{\text{ét}}\left(K_t^h, G \otimes_K K_t^h\right).$$

An element of $H^1_{\text{ét}}(K, G)$ is the isomorphism class of a G-torsor $X \to \operatorname{Spec} K$. It is trivialized over some finite separable finite extension L/K. Let C' be the normalization of C in L and $p : C' \to C$ be the corresponding finite map. For every $t \in C$, such that C' is not ramified over t, the torsor $X \otimes_K K_t^h$ is trivial and hence, its image in $(R^1 i_* G)_t$ is equal to zero. This shows that the image of each element is zero in almost all fibers of $R^1 i_* G_K$ and hence, we can replace $H^0(C, R^1 i_* \mathcal{F})$ in the exact sequence (4.7.1) by the direct sum $\oplus_{t \in C} H^1_{\text{ét}}(K_t^h, G \otimes_K K_t^h)$. The exact sequence

$$0 \to H^1(C, \mathbf{G}) \to H^1(\eta, G) \to \bigoplus_{t \in C} H^1(K_t^h, G \otimes_K K_t^h) \to H^2(C, \mathbf{G}) \to H^2(\eta, G) \tag{4.7.2}$$

is the *global-to-local* tool for the computation of the group of isomorphism classes of torsors under a commutative algebraic group G over K.

In this section, we apply this exact sequence to the case when G is an abelian variety A over K with Néron model $\mathbf{A}$.

Definition 4.7.1 The group $H^1(C, \mathbf{A})$ is called the *Tate–Shafarevich group* and it is denoted by $Ш(A, K)$ or by $Ш(A/K)$.

Note the order in the names is reversed, probably because in the Cyrillic alphabet the letter Ш goes after the letter T.

Theorem 4.7.2 *Let $f : J \to C$ be a jacobian elliptic fibration over a global base C. Let E be the generic fiber of f, which is an elliptic curve over the function field K of C.*

1. *If J is not a supersingular surface, or if $p_g(J) = 0$, then for any prime ℓ (possibly equal to p),*

$$_{\ell^\infty}Ш(E/K) \cong (\mathbb{Q}_\ell/\mathbb{Z}_\ell)^{\mathrm{t}_\ell(J)} \bigoplus {}_\ell \operatorname{NS}(J).$$

The group $_\ell \operatorname{NS}(J)$ is trivial if f is not smooth or f is a trivial fibration.

2. *If $p > 0$ and if the surface J is supersingular, then*

$$_{p^\infty}Ш(E/K) = \mathrm{U}(\Bbbk) \oplus (\mathbb{Q}_p/\mathbb{Z}_p)^{\mathrm{t}_p(J)} \bigoplus {}_p \operatorname{NS}(X),$$

for some unipotent algebraic group U. *(Conjecturally, we have $\mathrm{t}_p(J) = 0$.) The group $_p \operatorname{NS}(J)$ is trivial if f is not smooth or f is a trivial fibration.*

Proof Let $\mathbf{E}$ be the Néron model of E over C. We have proved in Corollary 4.3.12 that $\operatorname{Br}(J) = H^1_{\text{ét}}(C, R^1 f_* \mathbb{G}_m)$. The exact sequence

$$0 \to \mathbf{E} \to R^1 f_* \mathbb{G}_m \to \mathbb{Z}_C \to 0 \tag{4.7.3}$$

together with the vanishing of $H^1_{\text{ét}}(C, \mathbb{Z}_C)$ implies that

$$\text{Ш}(E/K) = H^1_{\text{ét}}(C, \mathbf{E}) \cong \text{Br}(J).$$

Now almost all the assertions follow from Theorem 0.10.2 and Theorem 0.10.29, which compute the Brauer group of a surface. The only remaining assertion is that $\text{NS}(J)$ has no torsion if f is not smooth or not trivial. However, this follows from Corollary 4.3.5 and the assertion is obvious if f is trivial. □

Let us give more information about the Tate–Shafarevich group in the case of a smooth elliptic jacobian fibration $f : J \to C$. In this case, the j-invariant of the generic fiber $E := J_\eta$ is a constant, that is, lies in $\mathbb{k}$, and all fibers are isomorphic to one and the same elliptic curve E_0 over $\mathbb{k}$. Since the j-invariant $E := J_\eta$ is equal to the j-invariant of E_0, it follows from Proposition 4.5.5 that E and E_0 become isomorphic over some finite extension L/K. In fact, this extension can be chosen to be separable of degree 2 if $j(E) \neq 0, 1728$ and of degree dividing 24 otherwise, see [429, Theorem 2, Appendix 1].

Moreover, we can choose L/K to trivialize the Tate module $T_\ell(E)$ for some suitable prime ℓ and thus, by [86, 7.4, Theorem 5], the extension L/K may assumed to be unramified. Passing to the Galois closure of L/K, we may assume the extension to be Galois, say with group G and it will still be unramified. Let $C' \to C$ be the normalization of C in L and thus, $C' \to C$ is a finite étale morphism of curves. Moreover, G acts on C' and the morphism $C' \to C$ is Galois with group G. In particular, we have $C \cong C'/G$. By construction, the base change $X := J \times_C C' \to C'$ is a trivial fibration, that is, isomorphic to $C' \times E_0 \to C'$. We note that X coincides with the Néron model of E_L over C'. In particular, we obtain an isomorphism

$$J \cong (E_0 \times C')/G,$$

where G acts freely on the product $E_0 \times C'$ by $g : (x, y) = (g(x), \rho(g)(y))$ for some homomorphism $\rho : G \to \text{Aut}(E_0)$. The projection $E_0 \times C' \to C'$ onto the second factor, which is a trivial elliptic fibration, induces an elliptic fibration $J \to C'/G \cong C$. In this case, we will say that the fibration is *étale isotrivial*.

Remark 4.7.3 The previous observations can also be viewed from the point of view of moduli stacks and actually, it lies at the very heart of what a stack is as opposed to a scheme: first, there exists a moduli stack $\mathbf{M}_{1,1,\mathbb{k}}$ of elliptic curves over $\mathbb{k}$, see also Example 5.1.1. Assigning to an elliptic curve its j-invariant yields a morphism $j : \mathbf{M}_{1,1,\mathbb{k}} \to \mathbb{A}^1_{\mathbb{k}}$, which is the closest approximation of $\mathbf{M}_{1,1,\mathbb{k}}$ to a scheme, namely its so-called coarse moduli space.

If $f : J \to C$ is a smooth elliptic fibration as above, then we have a classifying morphism $\gamma : C \to \mathbf{M}_{1,1}$. There is a universal elliptic curve $\mathcal{E} \to \mathbf{M}_{1,1}$ and $f : J \to C$ is isomorphic to the pull-back $\mathcal{E} \times_{\mathbf{M}_{1,1}} C \to C$ via γ. The composition $j \circ \gamma : C \to \mathbb{A}^1$ must be constant since C is proper and $\mathbb{A}^1$ is affine. In particular, every fiber of f has the same j-invariant, which implies that every fiber of f is isomorphic to the same elliptic curve E_0 over $\mathbb{k}$.

If j was an isomorphism, that is, if $\mathbf{M}_{1,1}$ was a scheme, then the pull-back $\mathcal{E} \times_{\mathbf{M}_{1,1}} C \to C$ would be a trivial product family, that is, of the form $E_0 \times C \to C$.

However, in general, we have that $J \cong (E_0 \times C')/G$ with the notation as above. This type of phenomenon is possible since $\mathbf{M}_{1,1}$ is a stack and in fact, a Deligne–Mumford stack: we have $f : J \to C$ and the trivial family $E_0 \times C \to C$, both of which give rise to classifying morphisms $\gamma, \gamma' : C \to \mathbf{M}_{1,1}$. Since $j \circ \gamma = j \circ \gamma'$, this implies that there exists a finite and étale cover $C' \to C$, that can be assumed, without loss of generality, to be a Galois cover with respect to some group G, such that $J \times_C C' \to C'$ and $E_0 \times C' \to C'$ are isomorphic. From there, one can argue as above to conclude the existence of an isomorphism $J \cong (E_0 \times C')/G \to C'/G = C$.

Proposition 4.7.4 *Let $f : J \to C$ be a non-trivial but étale isotrivial jacobian elliptic fibration. Then, there exists an isomorphism of abelian groups*

$$\mathrm{MW}(J/C) \cong \mathrm{Tors}\,(\mathrm{NS}(J))\,.$$

Proof Let $C' \to C$ be a Galois cover trivializing the fibration as above and let G be its Galois group. Let K and L be the function fields of C and C' as above. Let $E := J_\eta$ be the generic fiber of f, which is an elliptic curve over K. We have $E(K) = E(L)^G = E_0(\Bbbk)^G$. Let G_0 be the image of G in $\mathrm{Aut}_{\mathrm{gr}}(E_0)$. The group G_0 acts on E_0 with a finite set of fixed points, which form a subgroup isomorphic to $\mathrm{MW}(E)$. For example, if G is of order 2 and $p \neq 2$, then $\mathrm{MW}(E) \cong {}_2E_0 \cong (\mathbb{Z}/2\mathbb{Z})^{\oplus 2}$.

The Grothendieck–Leray spectral sequence for the morphism f and the sheaf $\boldsymbol{\mu}_n$ gives a long exact sequence

$$0 \to H^1_{\text{ét}}(C, \boldsymbol{\mu}_n) \to H^1_{\text{ét}}(J, \boldsymbol{\mu}_n) \to H^0_{\text{ét}}(C, R^1 f_* \boldsymbol{\mu}_n) \to H^2_{\text{ét}}(C, \boldsymbol{\mu}_n) \to H^2_{\text{ét}}(J, \boldsymbol{\mu}_n).$$

Next, we use the isomorphism $H^1_{\text{ét}}(Z, \boldsymbol{\mu}_n) \cong {}_n\mathrm{Pic}(Z)$ for $Z = C$ or J, as well as the fact that $R^1 f_* \boldsymbol{\mu}_n \cong {}_n R^1 f_* \mathbb{G}_m = {}_n\mathcal{P}_{J/C}$. Also we use that the map $H^2_{\text{ét}}(C, \boldsymbol{\mu}_n) \to H^2_{\text{ét}}(J, \boldsymbol{\mu}_n)$ is injective. Thus, we can rewrite the above long exact sequence as

$$0 \to {}_n\mathrm{Pic}(C) \to {}_n\mathrm{Pic}(J) \to {}_n\mathrm{Pic}(J/C) \to 0.$$

The group $\mathrm{Pic}^\circ(C)$ is n-divisible and the maximal n-divisible subgroup of $\mathrm{Pic}^\tau(J)$ is $\mathrm{Pic}^\circ(J)$. The homomorphism $f^* : \mathrm{Pic}^\circ(C) \to \mathrm{Pic}^\circ(J)$ is a bijection since f is non-trivial. This gives

$${}_n\mathrm{Tors}(\mathrm{NS}(J)) = {}_n\left(\mathrm{Pic}^\tau(J)/\mathrm{Pic}^\circ(J)\right) \cong {}_n\mathrm{MW}(J/C)$$

and finishes the proof. □

The group G acts on E_0 via $\rho : G \to \mathrm{Aut}(E_0)$ as above and we let G_0 be the subgroup of G that consists of those automorphisms of E_0 that fix the group structure. It follows from the description of the automorphism group of an elliptic curve in Proposition 4.4.7 that the torsion of $\mathrm{NS}(J)$ is restricted.

Corollary 4.7.5 *Let $f : J \to C$ be as before. Then,*

$$\mathrm{Tors}\,(\mathrm{NS}(J)) \in \left\{ \{0\},\ \mathbb{Z}/2\mathbb{Z},\ \mathbb{Z}/3\mathbb{Z},\ \mathbb{Z}/4\mathbb{Z},\ (\mathbb{Z}/2\mathbb{Z})^{\oplus 2} \right\}.$$

Example 4.7.6 Let us briefly digress on hyperelliptic or bielliptic surfaces: namely, we specialize to the case where $C := E$ is also an elliptic curve. Then, $E_1 := C' \to E$ is an étale G-cover and thus, a separable isogeny of elliptic curves with kernel G. Let K be the subgroup of G that acts on E_0 by translations. Then, we can replace E_0 with its quotient by K to assume that K is trivial. In this case, projection onto the first factor induces another elliptic fibration

$$f' : J \to \mathbb{P}^1 = E_0/G.$$

This fibration has multiple fibers and its jacobian is the trivial fibration. Since $\mathcal{L}^{\otimes 12} \cong \mathcal{O}_C$, the canonical class formula gives $\omega_J^{\otimes 12} \cong \mathcal{O}_J$. Since $b_1(J) = 2$, the surface J is a hyperelliptic surface of Kodaira dimension zero. We mentioned these surfaces already at the end of Section 1.1. We refer to [47] for classification of such surfaces over the complex numbers and to [78] or the survey [458] for the classification in all characteristics. Since $\dim H^1(J, \mathcal{O}_J) = 1$, we know from Example 4.2.15 that $l(\mathrm{Tors}(R^1 f'_* \mathcal{O}_J)) \leq 1$ and that there can be at most one wild multiple fiber. We refer to [78] for the classification of possible configuration of multiple fibers.

In Section 0.10, we discussed ordinary varieties. It follows from Example 0.10.24 that J is ordinary in degree 1 if and only if the base C of the fibration f is an ordinary curve of genus g and $\mathbf{Pic}_{J/\Bbbk}$ is reduced (here, we use that $\mathrm{Jac}(C)$ is isogenous to the Albanese variety of J). Being ordinary in degree 1, we see that $\mathbf{Pic}(J)$ is reduced, that $h^1(\mathcal{O}_J) = h^1(\mathcal{O}_C) = g$, and that $h^2(\mathcal{O}_J) = g - 1$.

Lemma 4.7.7 *Assume $p \geq 5$. Then, the following properties are equivalent:*

1. *E_0 and C are supersingular.*
2. *J is a supersingular surface.*

Proof The trivializing cover $\pi : J' := E_0 \times C' \to J$ is étale and, since we assumed $p \neq 2, 3$, its degree is prime to p. A trace map argument shows that the homomorphism $\pi^* : H^2(J, \mathcal{O}_J) \to H^2(J', \mathcal{O}_{J'})$ is injective. The map is the map of the Lie algebras of the homomorphism of formal Brauer groups $\pi^* : \widehat{\mathrm{Br}}(J) \to \widehat{\mathrm{Br}}(J')$. Since there are no non-trivial maps between formal groups of different heights, we see that J' is supersingular if and only if J is supersingular.

The Künneth formula in crystalline cohomology [61, Chapter 5, 4.1] yields a decomposition

$$H^2(J'/W) \cong \Big(H^1(E_0/W) \otimes H^1(C'/W)\Big) \oplus H^2(E_0/W) \oplus H^2(C'/W),$$

which is compatible with the action of Frobenius. We know that J' is supersingular if and only if $H^2(J'/W) \otimes_W K = (H^2(J'/W) \otimes_W K)_1$, where the subscript 1 indicates the slope 1 sub-isocrystal. This implies that J' is supersingular if and only if $H^1(E_0/W) \otimes_W K = (H^1(E_0/W) \otimes_W K)_1$ and $H^1(C'/W) \otimes_W K = (H^1(C'/W) \otimes_W K)_1$. The latter happens if and only if C' and E_0 are supersingular curves. Since C' is an étale cover of C, we have that C' is supersingular if and only if C is supersingular. □

We refer to [345] for more information about the ordinarity of isotrivial elliptic fibrations.

Next, we study the group $H^2(C, \mathbf{E})$. By (4.7.2), this group is crucial for the global-to-local tool: given a system of *local* classes, that is, elements in $H^1(K_t^h, G \otimes_K K_t^h)$ for every closed point $t \in C$, the obstructions to realizing them in a *global* fibration, lies in $H^2(C, \mathbf{E})$.

Lemma 4.7.8 *Let A be an abelian variety over K. Then, $H^2(K, A) = 0$.*

Proof Since $H^2(K, A)$ is a torsion group, it is enough to show that ${}_\ell H^2(K, A) = 0$ for every prime ℓ including the characteristic $p = \mathrm{char}(K)$. Using the long exact sequence in the flat cohomology associated to the short exact sequence

$$0 \to {}_\ell A \to A \xrightarrow{[\ell]} A \to 0,$$

we see that it is enough to prove that $H^2(K, {}_\ell A) = 0$. The sheaf ${}_\ell A$ is represented by a finite group scheme of height one over K. It is known that, for any scheme X and every finite group X-scheme G of height one, we have $H^i(X, G) = 0$ for $i > c(X) + 1$, where $c(X)$ is the cohomological dimension of X in the category of quasi-coherent sheaves on X, see [30, Corollary (1.3)]. By taking $X = \operatorname{Spec} K$, we obtain $H^i(K, {}_\ell A) = 0$ for $i > 1$, and we are done. □

Applying this lemma to the exact sequence (4.7.2), we find the exact sequence

$$0 \to Ш(A/K) \to \mathrm{WC}(K/A) \to \bigoplus_{t \in C} \mathrm{WC}(A_{K_t^h}/K_t^h) \to \mathrm{Tors}\left(H^2(C, \mathbf{A})\right) \to 0. \tag{4.7.4}$$

For non-trivial elliptic fibrations, we have the following fundamental result.

Theorem 4.7.9 *If $f : J \to C$ is a non-trivial elliptic jacobian fibration over a global base C, then*

$$\mathrm{Tors}\left(H^2(C, \mathbf{E})\right) = 0.$$

On the other hand, if f is trivial, say, $J \cong E_0 \times C$ for some elliptic curve E_0 defined over $\Bbbk$, then

$${}_n H^2(C, \mathbf{E}) \cong {}_n E_0$$

for every n.

Proof Assume that f is non-trivial. It suffices to prove that ${}_\ell H^2(C, \mathbf{E}) = 0$ for any prime ℓ including the characteristic $p = \mathrm{char}(\Bbbk)$.

First, assume that f is not smooth. By Corollary 4.3.12, we have $H^2(C, R^1 f_* \mathbb{G}_m) \cong H^3(J, \mathbb{G}_m)$. Next, we use the exact sequence (4.7.3) and the exact sequence of constant sheaves

$$0 \to \mathbb{Z}_C \to \mathbb{Q}_C \to (\mathbb{Q}/\mathbb{Z})_C \to 0$$

to conclude that

$$H^1(C, \mathbb{Z}_C) = 0, \quad \text{and} \quad H^2(C, \mathbb{Z}_C) = H^1(C, (\mathbb{Q}/\mathbb{Z})_C).$$

This gives an isomorphism

$$ {}_{\ell}H^2(C,\mathbf{E}) \cong \operatorname{Ker}\left({}_{\ell}H^3(J,\mathbb{G}_m) \to {}_{\ell}H^1(C,(\mathbb{Q}/\mathbb{Z})_C)\right). $$

Multiplication by ℓ in $(\mathbb{Q}/\mathbb{Z})_C$ shows that we have an isomorphism

$$ {}_{\ell}H^1\,(C,(\mathbb{Q}/\mathbb{Z})_C) \cong H^1\,(C,(\mathbb{Z}/\ell\mathbb{Z})_C)\,. $$

This gives

$$ {}_{\ell}H^2(C,\mathbf{E}) \cong \operatorname{Ker}\left({}_{\ell}H^3(J,\mathbb{G}_m) \to {}_{\ell}H^1(C,(\mathbb{Z}/\ell\mathbb{Z})_C)\right). \tag{4.7.5} $$

Next, the Kummer exact sequence on J gives an exact sequence

$$ 0 \to \operatorname{Br}(J)^{(\ell)} \to H^3(J,\boldsymbol{\mu}_\ell) \to {}_{\ell}H^3(J,\mathbb{G}_m) \to 0. $$

Since f is not smooth, Theorem 4.7.2 implies that $\operatorname{Br}(J)$ is a divisible group. Thus, $H^3(J,\boldsymbol{\mu}_\ell) \cong {}_{\ell}H^3(J,\mathbb{G}_m)$ and (4.7.7) gives that

$$ {}_{\ell}H^2(C,\mathbf{E}) \cong \operatorname{Ker}\left(H^3(J,\boldsymbol{\mu}_\ell) \to H^1(C,(\mathbb{Z}/\ell\mathbb{Z})_C)\right). \tag{4.7.6} $$

Assume $\ell \neq p$. Then, by Poincaré duality (0.10.18),

$$ \operatorname{Hom}\left({}_{\ell}H^2(C,\mathbf{E}),\mathbb{Z}/\ell\mathbb{Z}\right) \cong \operatorname{Coker}\left(f^*: H^1(C,\boldsymbol{\mu}_\ell) \to H^1(J,\boldsymbol{\mu}_\ell)\right). \tag{4.7.7} $$

By Corollary 4.3.5, we have $b_1(J) = b_1(C)$ and the map f^* is an isomorphism and hence, we find ${}_{\ell}H^2(C,\mathbf{E}) = 0$.

If $\ell = p$, then we use duality in $\boldsymbol{\mu}_p$-cohomology for curves from [30, Corollary 4.9] that gives

$$ \operatorname{Hom}\left(H^1(C,(\mathbb{Z}/p\mathbb{Z})_C),\mathbb{Q}/\mathbb{Z}\right) \cong H^1(C,\boldsymbol{\mu}_p) $$

and duality for flat cohomology of surfaces from [30, §5], which we discussed in Section 0.10, gives a short exact sequence

$$ 0 \to \mathsf{U}^2(J,\boldsymbol{\mu}_p)^\vee \to H^3(J,\boldsymbol{\mu}_p) \to \mathsf{D}^1(J,\boldsymbol{\mu}_p)^\vee \to 0. $$

Since $\mathsf{U}^1(C,\boldsymbol{\mu}_p) = 0$ and $\mathsf{D}^1(J,\boldsymbol{\mu}_p) \cong {}_p\operatorname{Jac}(C)$, we obtain an isomorphism of quasi-algebraic groups over $\Bbbk$

$$ {}_pH^2(C,\mathbf{E}) \cong \mathsf{U}^2(J,\boldsymbol{\mu}_p)^\vee \cong \operatorname{Coker}\left({}_p\operatorname{Jac}(C) \to {}_p\operatorname{Pic}(J)\right). $$

Since $b_1(C) = b_1(J)$, we get $\operatorname{Coker}({}_p\operatorname{Jac}(C) \to {}_p\operatorname{Pic}(J)) \cong {}_p\operatorname{NS}(J)$ and this concludes the proof in the case when the elliptic fibration is non-trivial.

Assume $J \cong E_0 \times C$. We consider J as a constant abelian scheme over C identify it with its Néron model. In this case

$$Ш(J) \cong \mathrm{Br}(J) \cong \mathrm{Br}(C) \times \mathrm{Br}(E_0) = \{0\}.$$

The Kummmer type exact sequence

$$0 \to {}_nJ \to J \xrightarrow{[n]} J \to 0 \to 0$$

shows that

$${}_nH^2(C, J) \cong H^2(C, {}_nJ).$$

Duality in étale cohomology gives an isomorphism

$$H^2(C, {}_nJ) \cong H^0(C, {}_nJ) \cong {}_nE_0,$$

finishing the proof. □

Remark 4.7.10 We have an exact sequence of sheaves of abelian groups on C

$$0 \to \mathbf{E}^\circ \to \mathbf{E} \to \mathcal{F} \to 0,$$

where $\mathcal{F}$ is a constant sky-scraper sheaf, whose fibers are the groups of connected components of fibers of $\mathbf{E}$. Taking flat or étale cohomology and using the Néronian property of $\mathbf{E}$, we obtain an exact sequence

$$0 \to \mathbf{E}^\circ(C) \to \mathbf{E}(C) \to \mathcal{F} \to Ш(E/K)' \to Ш(E/K) \to 0, \tag{4.7.8}$$

where $Ш(E/K)' := H^1(C, \mathbf{E}^\circ)$. Let F be the cokernel of the map $E(K) = \mathbf{E}(C) \to H^0(C, \mathcal{F})$, which is a finite group. We have an exact sequence

$$0 \to F \to Ш(E/K)' \to Ш(E/K) \to 0. \tag{4.7.9}$$

The group $Ш(E/K)'$ has a geometric interpretation as the group of locally trivial torsors of E of together with a choice of a component of multiplicity 1 in each reducible fiber of its relatively minimal model $X \to C$, see see [433, p. 486].

Remark 4.7.11 Let A be an abelian variety over a local or global field K and let $\mathbf{A}$ be its Néron model over C. The original Ogg–Shafarevich theory computes the Weil–Châtelet group ${}_n\,\mathrm{WC}(A/K)$ of A-torsors over K, where $(n, p) = 1$. If $\dim A \geq 2$, we cannot use the computation of the Brauer group of a minimal model of A. Thus, the theory of torsors of an abelian variety A over a global field K and, in particular, the computation of the Tate–Shafarevich group, is different. Raynaud [604], following Grothendieck, deduces the following formula for the Euler–Poincaré characteristic of any constructive sheaf $\mathcal{F}$ of finite modules over a commutative ring R in the étale topology of C or some open subset U of C. The formula, which is called the *Ogg–Shafarevich formula*, is as follows:

$$\chi_R(\mathcal{F}) = \sum_{i=0}^{2} (-1)^i \ \mathrm{cl}_R\left(H^i(C, \mathcal{F})\right) = (2 - 2g(C))\, \mathrm{cl}_R(\mathcal{F}_{\bar{\eta}}) - \sum_{x \in C^{(1)}} \epsilon_x^R(\mathcal{F}).$$

Here, $\mathrm{cl}_R(M)$ denotes the class of a finite R-module in the Grothendieck group of the abelian category of finite R-modules. The local invariant $\epsilon_x^R(\mathcal{F})$ is given by

$$\epsilon_x^R(\mathcal{F}) = \alpha_x^R(\mathcal{F}) + \mathrm{cl}_R(\mathcal{F}_{\bar{\eta}}) - \sum_{i=0}(-1)^i \,\mathrm{cl}_R\left(H_x^i(\mathcal{F})\right),$$

where $H_x^i(\mathcal{F})$ is the étale cohomology with support at x and the $\alpha_x^R(\mathcal{F})$ are the invariants of wild ramification $\delta(\hat{K}_x, M)$ for the $\mathrm{Gal}(\hat{K}_x^s/\hat{K}_x)$-module M defining the sheaf $\hat{i}_x^*\mathcal{F}$. We defined these invariants in Section 4.1. Note that both Ogg and Shafarevich assume that the sheaf ${}_n\mathcal{A}$ is moderately ramified, that is, that all the invariants $\alpha_x^R({}_n\mathcal{A})$ are zero.

We note that one can also deduce formula (4.1.7) for the Euler–Poincaré characteristic of an elliptic fibration from the Ogg–Shafarevich formula by taking $R = \mathbb{Z}/n\mathbb{Z}$ and $\mathcal{F} = R^1 f_*\boldsymbol{\mu}_n$.

4.8 The Weil–Châtelet group: Quasi-Elliptic Fibrations

In this section, we will study Weil–Châtelet groups for quasi-elliptic fibrations. We will use many ideas and results from the previous two sections and focus on the differences that arise from having a non-smooth generic fiber. We recall that quasi-elliptic fibrations can and do exist in characteristic $p \in \{2, 3\}$ only.

The setup is as follows: let $f : J \to C$ be a jacobian quasi-elliptic fibration over a global basis C. We let J_η be the generic fiber over $\eta = \operatorname{Spec} K$ and let $\mathsf{U}_K = J_\eta^\sharp$ be the smooth locus of J_η. Then, U_K is a wound unipotent group of dimension one and its Néron model $\mathbf{U}$ (which does exist despite that fact that U_K is not proper) is isomorphic to $J^\sharp \to C$. We can still apply exact sequence (4.7.2) by taking $G = \mathsf{U}_K$ and $\mathbf{G} = \mathbf{U}$. We denote $H^1(C, \mathbf{A})$ by $\text{Ш}(\mathsf{U}_K/K)$ and continue to call it the Tate–Shafarevich group of U_K. The sequence (4.7.2) thus becomes

$$0 \to \text{Ш}(\mathsf{U}_K/K) \to H^1(\eta, \mathsf{U}_K) \to \bigoplus_{t\in C} H^1(K_t^h, \mathsf{U}_{K_t^h}) \to H^2(C, \mathbf{U}) \to H^2(\eta, \mathsf{U}). \tag{4.8.1}$$

Since multiplication by p kills $\mathbb{G}_{a,K}$, it also kills U_K and $\mathbf{U}$. Thus, all groups in this exact sequence, are p-torsion groups.

To compute $H^1(\eta, \mathsf{U}_K)$, we need to understand and compute $\text{Ш}(\mathsf{U}_K/K)$, the groups of local invariants $H^1(K_t^h, U_{K_t^h})$, as well as the group of obstructions $\mathrm{Ker}(H^2(C, \mathbf{U}) \to H^2(K, \mathsf{U}_K))$.

We start with the Tate–Shafarevich group $\text{Ш}(\mathsf{U}_K/K) = H^1(C, \mathbf{U})$: as in the elliptic case, this is isomorphic to the Brauer group of the jacobian surface J. Since the Mordell–Weil group $\mathrm{MW}(J^\sharp/C) = \mathsf{U}_K(K)$ is a finite p-group, the Shioda–Tate formula gives $\rho = b_2$ (the surface J is supersingular in the sense of Shioda) and hence, $\mathrm{t}_p(J) = 0$. Thus, Theorem 4.7.2 shows that

$$Ш(\mathsf{U}_K/K) \;\cong\; \Bbbk^{p_g(X)} \,\oplus\, {}_{p^\infty}\,(\mathrm{NS}(J))\,. \tag{4.8.2}$$

We know from Corollary 4.3.5 that $\mathrm{Tors}(\mathrm{NS}(J)) = \{0\}$ unless $\chi(J) = 0$. Quasi-elliptic jacobian surfaces with $\chi(J) = 0$ are precisely the quasi-hyperelliptic surfaces, which we briefly discussed in Corollary 4.3.6. As in the case of hyperelliptic surfaces, we have a presentation of the form $J \cong (E_0 \times E_1)/G$, but now, E_0 is a cuspidal cubic curve over $\Bbbk$ (rather than an elliptic curve), $E_1 \to E$ is an isogeny of elliptic curves, and G is a finite group scheme (possibly not étale). It follows from the analysis of all possible G in [77, p. 214] that

$$\mathrm{Tors}\,(\mathrm{NS}(J)) \;=\; (E_0(\Bbbk) \setminus \{\mathrm{cusp}\})^{G_0} = \{0\},$$

where G_0 denotes the subgroup scheme of G that fixes 0. Let us record this discussion in the following.

Theorem 4.8.1 *Let $f : J \to \mathbb{P}^1$ be a jacobian quasi-elliptic surface over a global base C and let $\mathsf{U}_K = J_\eta^\sharp$. Then, there is an isomorphism of abelian groups*

$$Ш(\mathsf{U}_K/K) \;\cong\; \Bbbk^{p_g(J)}.$$

Example 4.8.2 Let X be a K3 surface that admits a jacobian quasi-elliptic fibration $f : X \to \mathbb{P}^1$. The theorem implies that the Brauer group is p-torsion and in fact isomorphic to the additive group of $\Bbbk$. In fact, X is a supersingular and unirational K3 surface, see also Corollary 4.1.16. The Tate–Shafarevich group is a p-torsion group also for any elliptic fibration $g : X \to \mathbb{P}^1$. Thus, any elliptic $K3$-surface whose jacobian fibration is isomorphic to $X : J \to \mathbb{P}^1$ has a multi-section of degree p and all other multi-sections are of degree divisible by p.

The proof of Theorem 4.7.9 extends word-by-word to quasi-elliptic fibrations. Since we do not have trivial quasi-elliptic fibrations (the total space would not be normal and in particular, not a smooth surface), we obtain the following.

Theorem 4.8.3 *Let $f : J \to C$ be a quasi-elliptic fibration over a global base C. Then,*

$$H^2(C, \mathbf{U}) \;=\; 0.$$

In particular, there are no obstructions in (4.8.1) to realize any given set of local conditions in a global quasi-elliptic fibration.

In order to finish our discussion on Ogg–Shafarevich theory for quasi-elliptic fibrations, it remains to compute the Weil–Châtelet group of a quasi-elliptic fibration over a strictly local base $C = \operatorname{Spec} R$. We start with explicit equations.

Proposition 4.8.4 *Let X be a quasi-elliptic curve over K that has a K-rational point. If $p = 2$, assume that the residue field of the cusp $\mathfrak{c}$ is a quadratic extension of K. Then, the unipotent group $\mathsf{U}_{\mathbf{K}} = \mathbf{X}^\sharp = \mathbf{X} \setminus \{\mathfrak{c}\}$ is isomorphic to a closed subgroup of the group scheme*

$$\mathbb{G}_{a,K}^2 \;\cong\; \operatorname{Spec} K[u, u^{-1}, v, v^{-1}]$$

given by the following equations:

1. $p = 2$

a. $u^2 + v + a_2 v^2 + a_6 v^4 = 0$, *where* $y^2 + x^3 + a_2^2 x + a_6 = 0$ *is the Weierstrass equation of* X*. The map given by the linear system* $|\mathfrak{c}|$ *is inseparable.*

b. $u^4 + v + a_4 v^2 = 0$, *where* a_4 *is not a square in* K *and one can choose a Weierstrass equation with* $a_6 = 0$*. The map given by the linear system* $|\mathfrak{c}|$ *is separable.*

2. $p = 3$

- $u^3 + v + a_6 v^3 = 0$, *where* $y^2 + x^3 + a_6 = 0$ *is the Weierstrass equation of* X*. The map given by the linear system* $|\mathfrak{c}|$ *can be separable or inseparable.*

Proof First, assume that $p = 2$. Let $f : X \to \mathbb{P}^1_K$ be a degree 2 map given by the linear system $|\mathfrak{c}|$.

Suppose f is inseparable. It follows from Example 0.2.22 and the formula for the canonical sheaf of a split cyclic cover that the equation of the curve in the weighted projective plane $\mathbb{P}(1, 1, 2)$ can be chosen to be

$$t_2^2 + b_4(t_0, t_1) = 0,$$

where $b_4 = \sum_{i=0}^4 c_i t_0^i t_1^{4-i}$. The non-smooth locus is the pre-image of the zero subscheme of da_4, which is of the form $V(c_1 t_0^2 + c_3 t_1^2)$. Since it coincides with the image of the cuspidal point $\mathfrak{c}$, which is of degree 2, the polynomial $c_1 t_0^2 + c_3 t_1^2$ must be the square of a linear polynomial. After a linear change of the coordinates (t_0, t_1), we may assume that $c_1 = 0, c_3 = 1$. Since X has a K-rational point, we may assume that its image is the point with coordinates $[0, 1, 0]$. This allows us to assume that $c_0 = 0$ and to write the equation of X in affine coordinates $x = t_1/t_0, y = t_2/t_0^2$ in the following form:

$$y^2 + x^3 + a_2 x^2 + a_6 = 0. \tag{4.8.3}$$

Using the change of variables $u = y/x^2, v = 1/x$, we arrive at equation (4.8.3).

$$u^2 + v + a_2 v^2 + a_6 v^4 = 0. \tag{4.8.4}$$

Note the Weierstrass equation $y^2 + x^3 + a_4 x + a_6 = 0$ can be reduced to (4.8.3) if $a_4 = a_2^2$ is a square.

Assume now that f is a separable map of degree 2. Then, the equation of X has the form

$$t_2^2 + b_2(t_0, t_1)t_2 + \sum_{i=0}^{4} c_i t_0^i t_1^{4-i} = 0,$$

where b_2 is a binary form of degree n. We may assume that $[0, 1, 0]$ is a K-rational point of X. We may thus assume $c_0 = 0$. Also, we may assume that the cusp is the point $[1, 0, c_0^{1/2}]$ and that $b_2 = t_1^2$. Taking partial derivatives, we see that $c_3 = 0$ and we obtain the equation

$$t_2^2 + t_1^2 t_2 + c_0 t_0^4 + c_1 t_0 t_1^3 + c_2 t_0^2 t_1^2 = 0.$$

Replacing t_2 by $t_2+\alpha t_0^2+\beta t_0t_1$ for suitable α and β, we may assume $c_1=c_2=0$. In affine coordinates $u=t_0/t_1, v=t_2/t_1^2$, we thus obtain an equation $v^2+v+c_0u^4=0$. Replacing v by c_0v and cancelling by c_0, we get equation

$$u^4+v+c_0v^2 = 0. \tag{4.8.5}$$

Let us see how to derive this equation from the Weierstrass equation of X. By the above, the coefficient a_4 in the Weierstrass form $y^2+x^3+a_4x+a_6=0$ is not a square. As we noted in Remark 4.4.4, we may find another Weierstrass equation of the same curve with $a_6=0$. Following [600], we set $v=\frac{a_4}{x^2+a_4}$ and $u=\frac{x}{y}$ and we check that $v^2+v+a_4u^4=0$. Replacing v by a_4v and cancelling by a_4, we get equation (4.8.5) with $a=a_4$.

This gives the relationship between the coefficients in (4.8.5) and the coefficients a_4, a_6 of the Weierstrass form.

Next, assume that $p=3$.

Suppose f is a separable map. Then, X has an equation of the form

$$t_2^3+b_2(t_0,t_1)t_2+c_0t_1^3+c_1t_1^2t_0+c_2t_1t_0^2+c_3t_0^3 = 0\,.$$

We may assume X has a K-rational point with coordinates $[0,1,\alpha]$, where $\alpha^3+b_2(0,1)\alpha+c_0=0$. After a linear change $t_2\mapsto t_2-\alpha$, we may assume that $c_0=0$ and $\alpha=0$. Taking partial derivatives, we find that b_2 must be square of a linear form and after a linear change of variables t_0,t_1, we may assume that $b_2=t_1^2, c_2=0$ and that the cusp has coordinates in $\bar{K}$ equal to $[1,0,-c_3^{1/3}]$. After the linear change $t_2\mapsto t_2+c_1t_0$, we may assume that $c_1=0$. In affine coordinates, $u=t_2/t_1, v=t_0/t_1$, the equation is

$$u^3+u+cv^3 = 0. \tag{4.8.6}$$

The same equation is obtained from the Weierstrass equation $y^2+x^3+a_6$ by a substitution $u=x/y, v=1/y$, which shows that $c=a_6$.

Finally, assume that f is inseparable and then, X is given by

$$t_2^3+c_0t_1^3+c_1t_1^2t_0+c_2t_1t_0^2+c_3t_0^3 = 0.$$

As above, we may assume that the K-rational point on X has coordinates $[0,1,0]$, which forces c_0 to be 0. Also, we may assume that the cusp has coordinates in $\bar{K}$ equal to $[1,0,-c_3^{1/3}]$. Taking partial derivatives, we find that the image of the cusp is the point $V(c_1t_1+c_2t_0)$. As above, we may assume that $c_0=0$ and hence, the pre-image of this point in $X_{\bar{K}}$ is $[-c_2,c_1,c_3^{1/3}]$. Taking partial derivatives, we obtain $c_2=0$. Now, after suitably scaling t_0 and working in the affine coordinates $u=t_2/t_1, v=t_0/t_1$, the equation becomes (4.8.6). □

Let us put these results into a broader perspective: Russell [629] studied purely inseparable twisted forms G of $\mathbb{G}_a$ over fields F of characteristic $p>0$, that is, G is a group scheme over F such that there exists purely inseparable extension L/F such that G_L is isomorphic $\mathbb{G}_a$. More precisely, he showed that a non-trivial inseparable form G of $\mathbb{G}_a$ is isomorphic to a closed subgroup of $\mathbb{G}_{a,F}^2$ given by an equation

$$u^{p^n} + v + a_1 v^p + \cdots + a_r v^{p^r} \; = \; 0. \tag{4.8.7}$$

where the coefficients a_i satisfy $a_i \notin F^{p^i}$. Thus, our groups $\mathsf{U} = J_\eta^\sharp$ correspond to the cases $p = 2$ and $n = 1, r = 2$ or $n = 2, r = 1$ (equation (4.8.5)) or $p = 3, n = 1$ (equation (4.8.6)). It is known that the number n is equal to the *height*, which is the smallest degree of a purely inseparable extension such that the base change to it is isomorphic to $\mathbb{G}_a$.

The group G admits a G-equivariant compactification, which is isomorphic to a plane curve X of degree $p^{\max\{m,n\}}$ in the weighted homogeneous plane $\mathbb{P}(1, 1, p^{\max\{m,n\}-\min\{m,n\}})$, which is given by an equation

$$t_2^{p^n} + t_0^{p^m-1} t_1 + \cdots + a_{m-1} t_0^p t_1^{p^{m-1}} + a_m t_1^{p^m} \; = \; 0 \tag{4.8.8}$$

if $n \leq m$ and which is given by an equation

$$t_2^{p^n} + t_1 t_0^{p^m} + \cdots + a_{m-1} t_0^p t_1^{p^{m-1}} + a_m t_1^{p^m} \; = \; 0 \tag{4.8.9}$$

if $m \leq n$. With respect to these equations, U is given by the complement of the hyperplane at infinity, that is, $V(t_0)$.

Using formula (4.1.6), we obtain

$$\omega_X \; \cong \; \mathcal{O}_X \left(-2 - p^{\max\{m,n\}-\min\{m,n\}} + p^{\max\{m,n\}} \right)$$

and

$$p_a(X) \; = \; \frac{1}{2} \left(p^{\min\{m,n\}} - 1 \right) \left(p^{\max\{m,n\}} - 2 \right) . \tag{4.8.10}$$

This curve is smooth except at the point at infinity $t_0 = 0$, where it has a singular unibranch point. More precisely, we make the following local computations: First, suppose $n \geq m$. In the open subset $t_1 \neq 0$, the affine equation is

$$1 + x^{p^m - p^{n-m}} y + \cdots + a_{m-1} x^p y^{p^m - p} + a_m y^{p^m} \; = \; 0 \, .$$

We can write it as $1 + a_m y^{p^m} + x^p \epsilon = 0$, where ϵ in the local ring at the non-smooth point. If $a_m \notin F^{p^s}$ for any $0 \leq s \leq m$, then x generates the maximal ideal $\mathfrak{m}$ and then, the curve is regular. However, if $a_m = c^{p^s}$, then $1 + c y^{p^{n-s}} \in \mathfrak{m}$ but does not belong to (x), so the curve is not normal and we have to take its normalization defined over F. Second, if $n \leq m$, then we obtain an affine equation

$$y^{p^n} + x^{p^m - 1} + \cdots + a_{m-1} x^{p^m - p^{m-1}} + a_m \; = \; 0$$

and come to a similar conclusion: the curve is regular if and only if a_m is not a p.th power.

Definition 4.8.5 A wound one-dimensional unipotent group is called a *unipotent group of genus g* if it admits a regular compactification of arithmetic genus g. A unipotent group of genus one is called a *quasi-elliptic group*.

We note that wound unipotent groups of genus 0 can exist only in characteristic $p = 2$ and that they have a regular compactification given by an equation of the form

$$x_2^2 + x_0x_1 + ax_1^2 = 0,$$

where a is not a square.

It follows from the genus formula that the weighted homogeneous compactification from above is a quasi-elliptic group with $(p; n, m) \in \{(2; 1, 2), (2; 2, 1), (3; 1, 1)\}$, which is in agreement with Proposition 4.8.4. In the case $(p; n, m) = (2; 2, 2)$ and the equation $u^4+v+av^2+c^2v^4 = 0$, the genus formula gives $p_a = 3$ and then, the weighted homogeneous compactification $t_2^4 + t_0^3t_1^2 + at_0^2t_1^2 + c^2t_1^4 = 0$ is not regular. The affine part $y^4+x^3+ax^2+b^2 = 0$ is not normal since $t = \frac{y^2+b}{x}$ satisfies $t^2 = x+a$ and has to be added to the coordinate ring in order to obtain the normalization, which is then given by equation $y^2 + w^3 + aw + b = 0$, see [445, Example 3.14]. This normalization is a regular curve of genus one. It follows from [600] that any quasi-elliptic curve is isomorphic to one of the four cases $(p : m, n) \in \{(2; 1, 2), (2; 2, 1), (3 : 1, 1), (2; 2, 2)\}$.

Proposition 4.8.6 *Let U be a unipotent group over K of genus $g > 0$. Then*

$$\mathrm{WC}(\mathsf{U}/K) \;\cong\; K/\Phi(K^{\oplus 2}),$$

where $\Phi : K^{\oplus 2} \to K$ is the homomorphism of additive groups given by

$$\Phi(u, v) \;=\; u^{p^n} + v + a_1v^p + \cdots + a_rv^{p^r},$$

where the a_i are the coefficients of U_K with respect to Russell's equation (4.8.7)*. In particular,*

$$\mathrm{WC}(\mathsf{U}/K) \;=\; {}_p\,\mathrm{WC}(\mathsf{U}/K),$$

that is, the Weil–Châtelet group of U is a p-torsion group.

Proof Given an equation (4.8.7) for U and defining Φ as in the proposition, we have a short exact sequence of group schemes in the flat topology

$$0 \;\to\; \mathsf{U} \;\to\; \mathbb{G}_{a,K}^2 \;\xrightarrow{\Phi}\; \mathbb{G}_{a,K} \;\to\; 0.$$

Taking cohomology and using (0.1.2), we obtain an isomorphism $\mathrm{WC}(\mathsf{U}/K) = H^1(\eta, J_\eta) \cong K/\Phi(K^{\oplus 2})$. □

Now, we are ready to compute the group $\mathrm{WC}(J_\eta/\eta)$ for a quasi-elliptic fibration $f : J \to C$ over a strictly local base C with function field K. As a first step, we make the equations from Proposition 4.8.4 more explicit.

Proposition 4.8.7 *A quasi-elliptic group over $K := \Bbbk((t))$ is isomorphic to a subgroup scheme of $\mathbb{G}_{a,K}^2 = \operatorname{Spec} K[u, u^{-1}, v, v^{-1}]$ given by one of the following equations:*

1. $p = 2$

a. $u^2 + v + t^{2k+1}v^4 = 0,\ k = 0, 1, 2.$
b. $u^2 + v + t^{2s+1}\epsilon^2 v^2 + t^{2k+1}v^4 = 0,\ k = 0, 1, 2.$
c. $u^4 + v + t^{2k+1}v^2 = 0,\ k = 0, 1, 2.$
d. $u^4 + v + (\epsilon^4 + t\epsilon_2^4 + t^2\epsilon_3^4 + t^3\epsilon_3^4)v^2 = 0.$
e. $u^4 + v + t^2(\epsilon^4 + t\epsilon_2^4 + t^2\epsilon_3^4 + t^3\epsilon_3^4)v^2 = 0,$

where ϵ is a unit and the ϵ_i's are units or zeroes.

2. $p = 3$

$$u^3 + v + t^s v^3 = 0, \quad s = 1, 2, 4, 5.$$

Proof Let $\nu : K^\times \to \mathbb{Z}$ be the discrete valuation with respect to the uniformizer t. We use Proposition 4.8.4.

Assume $p = 2$. case (1a). We know that the quasi-elliptic group arises from a quasi-elliptic curve of the form $y^2+x^3+a_2^2x+a_6 = 0$. Replacing y by $y+\alpha x+\beta+a_2\alpha$ and x by $x + \alpha^2$ changes (a_2, a_6) to $(a_2 + \alpha^2, a_6 + \beta^2)$. We also can change (x, y) to (c^3x, c^2y) in order to assume that $\nu(a_6) < 6$.

Since $a_6 \neq 0$ in this case, we may write $a_6 = t^{2k+1}\epsilon^2$ for some unit ϵ and $k = 0, 1, 2$. This gives us an equation

$$u^2 + v + t^{2s+1}\epsilon^2 v^2 + t^{2k+1}\eta v^4 = 0, \quad k = 0, 1, 2, \tag{4.8.11}$$

where ϵ is a unit or zero and η is a unit. Applying Hensel's lemma, we may change the local parameter to $t\eta^{1/2k+1}$ in order to assume that $\eta = 1$.

Assume that we are in case 1(b). Then, as already noted earlier, we may assume that $a_6 = 0$ and thus, deal with an equation

$$u^4 + v + t^s\epsilon v^2 = 0, \quad k = 0, 1, 2, 3, \tag{4.8.12}$$

where ϵ is a unit. If s is odd, then we may rescale t to assume that $\epsilon = 1$. If s is even, then we may assume that ϵ is not a square. Since we may also add a fourth power of an element from K to a_6, we may obtain the following equations

$$\begin{aligned} u^4 + v + \epsilon v^2 &= 0,\\ u^4 + v + tv^2 &= 0,\\ u^4 + v + t^2\epsilon &= 0,\\ u^4 + v + t^3 &= 0, \end{aligned}$$

where $\epsilon = \epsilon^4 + t\epsilon_1^4 + t^2\epsilon_2^4 + t^3\epsilon_3^4$ for some unit ϵ and some units or zeros ϵ_i.

Assume $p = 3$. Then the equation may be chosen to be $u^3+v+at^3 = 0$. Replacing u by $t^s u$ and v by $t^{3s}v$ for a suitably large s, we may assume that $a \in R$. Replacing u

by $u + t^k \epsilon v$, we may assume that $k = 1$. Using Hensel's lemma, we may assume that $\epsilon = 1$ and obtain an equation

$$u^3 + v + t^m v^3 = 0, \quad m = 1, 2, 4, 5, \tag{4.8.13}$$

which ends the proof. □

Using this proposition, we can now give explicit equations for representatives of the Weil–Châtelet group $\mathrm{WC}(\mathsf{U}/K)$ of a quasi-elliptic group U over K. By Proposition 4.8.6, we have that $\mathrm{WC}(\mathsf{U}/K)$ is isomorphic to $K/\Phi(K^{\oplus 2})$. In characteristic $p = 3$, the following realizations of elements of these groups were given by Lang in [431, Theorem 2.1].

Proposition 4.8.8 *Assume $p = 3$. Let $X \to \operatorname{Spec} K$ be a non-trivial torsor under a quasi-elliptic unipotent group U over K. Then, X is isomorphic to an affine curve over K given by an equation of the form*

$$u^3 + v + t^k v^3 + t^{-k} q_n(t^{-3}) = 0,$$

where $k = 1, 2, 4, 5$ and q_n is a polynomial of degree n.

Proof The equation of U can be chosen to be $\Phi := u^3 + v + t^k v^3 = 0$. We set $h(v) := \Phi(0, v) = v + t^k v^3$. Let $f(t) \in K$ be a representative of $K/\Phi(K^{\oplus 2})$. Using the fact that $\Phi(u, 0) = u^3$, we may assume that $f(t)$ does not contain cubes of monomials. Using $h(v) = v + t^k v^3$ and Hensel's lemma, we can find any given power series of $\Bbbk[[t]]$ in the image of Φ. Thus, we may assume that $f(t)$ is a Laurent polynomial in negative powers of t.

We we write $f \sim g$ if $f - g \in \operatorname{Im}(\Phi)$. For any constant $c \in \Bbbk$, we have

$$h(ct^{-i}) = \Phi(0, ct^{-i}) = ct^{-i} + c^3 t^{-3i+k}.$$

First, assume $k = 1$. We see that all monomials $t^{-i}, i \equiv 1 \mod 3$ enter only in one of these equations and that $t^{-2} \sim t^{-1}$, $t^{-5} \sim t^{-2} \sim t^{-1}$, and $t^{-8} \sim t^{-3} \sim 0$. Continuing in this way, we see that each monomial whose degree is not divisible by 3 is equivalent to a monomial of the form t^{-i} with $i \equiv 3 \mod 3$. Thus, we can choose a unique representative of $K/\Phi(K^{\oplus 2})$ of the form $f(t) = t^{-1} q_n(t^{-3})$ as claimed.

Next, if $k = 2$, then the same arguments as before show that monomials $t^{-i}, i \equiv 2 \mod 3$ form a basis of the cokernel of Φ. Thus, we can choose a unique representative of $K/\Phi(K^{\oplus 2})$ of the form $f(t) = t^{-2} q_n(t^{-3})$.

Finally, assume that $k = 4$ or $k = 5$. Then we see that t^{-1} and t^{-2} both lie in $\operatorname{Im}(\Phi)$. We can thus find representatives of the form $f(t) = t^{-5} q_n(t^{-3})$, if $k = 4$ and of the form $f(t) = t^{-4} q_n(t^{-3})$, if $k = 5$. □

Corollary 4.8.9 *Keeping the assumptions of the proposition, a non-trivial U-torsor admits an integral affine model over $R = \Bbbk[[t]]$ given by one of the following equations:*

1. $u^3 + t^{2n+2} v + t v^3 + t^2 p_n(t^3) = 0$.

2. $u^3 + t^{2n+2}v + t^2v^3 + tp_n(t^3) = 0$.
3. $u^3 + t^{2n+3}v + tv^3 + t^2p_n(t^3) = 0$.
4. $u^3 + t^{2n+3}v + t^2v^3 + tp_n(t^3) = 0$,

where p_n is a polynomial of degree $\leq n$ that does not vanish in 0.

Proof Assume $k = 1$. Multiplying the equation by t^{3n+3} and replacing u by $t^{n+1}u$, as well as v by $t^{n+1}v$, we obtain the equation $u^3 + t^{2n+2}v + tv^3 + t^2t^{3k}q_n(t^{-3}) = 0$. It remains to write $t^{3n}q_n(t^{-3}) = p_n(t^3)$. If $k = 3$, we multiply both sides by t^{3n+6} and replace (u, v) by $(t^{n+2}u, t^{n+1}v)$. The other two cases are treated similarly. □

The case $p = 2$ is more complicated.

Proposition 4.8.10 *Assume $p = 2$. Let $X \to \operatorname{Spec} K$ be a non-trivial torsor under a quasi-elliptic unipotent group U over K. Then, X is isomorphic to an affine curve over K given by one of the following equations:*

1. $u^2 + v + tv^4 + t^{-1}q_n(t^{-4}) = 0$,
2. $u^2 + v + t^3v^4 + t^{-3}q_n(t^{-4}) = 0$,
3. $u^2 + v + t^5v^4 + t^{-5}q_n(t^{-4}) = 0$,
4. $u^2 + v + t^{2s+1}\epsilon^2v^2 + tv^4 + t^{-1}q_n(t^{-4}) = 0$,
5. $u^2 + v + t^{2s+1}\epsilon^2v^2 + t^{2k+1}v^4 + t^{-5}q_n(t^{-4}) = 0, k = 1, 2$,
6. $u^4 + v + (\epsilon^4 + t\epsilon_2^4 + t^2\epsilon_3^4 + t^3\epsilon_3^4)v^2 + t^{-1}q_n(t^{-4}) = 0$,
7. $u^4 + v + tv^2 + t^{-2}q_n(t^{-4}) = 0$,
8. $u^4 + v + t^2(\epsilon^4 + t\epsilon_2^4 + t^2\epsilon_3^4 + t^3\epsilon_3^4)v^2 + t^{-3}q_n(t^{-2})$,
9. $u^4 + v + t^3v^2 + t^{-6}q_n(t^{-4}) = 0$,

where q_n is a polynomial of degree n.

Proof As in the characteristic 3 case before, we define $f \sim g$ if $f(t) - g(t) \in \operatorname{Im}(\Phi)$.

We start with equation (4.8.11). Assume first that $\epsilon = 0$. Replacing t with $t\eta^{\frac{1}{2k+1}}$, we may assume that $\eta = 1$, so that $h(v) = \Phi(0, v) = v + t^{2k+1}v^4$. As in the proof of Proposition 4.8.8, we may represent $f(t) \in K/\Phi(K^{\oplus 2})$ by a negative Laurent polynomial that is not a square. We have

$$h(ct^{-i}) = ct^{-i} + c^4t^{2k+1-4i}.$$

If $k = 0$, then all monomials t^{3-4i} enter only in one of the relations from above. All monomials of the form t^{1-4i} are equivalent to one such monomials. Thus, we can choose a unique representative of the form $f(t) = t^{-1}q_n(t^{-4})$.

If $k = 1$, then we have

$$h(ct^{-i}) = ct^{-i} + c^4t^{3-4i}.$$

Arguing as in the previous case, we find a representative of the form $t^{-3}q_n(t^{-4})$.

If $k = 2$, everything works as in the case $k = 0$, except that we can eliminate $t^{-1} \sim 0$ and $t^-3 \sim t^{-2}$. Thus, we find representatives of the form $f(t) = t^{-5}q_n(t^{-4})$.

Assume $\epsilon \neq 0$. If $k = 0$, then $c^4t^{-3} \sim ct^{-1} + c^2t^{2s-1}$ and hence, $t^{-3} \sim t^{-1}$. By induction, we see that any monomial t^{1-4i} is congruent to a monomial of the form

t^{3-4i}. Thus, we can find representatives of the form $t^{-1}q_n(t^{-4})$. Unfortunately, these representatives may be not unique.

If $k = 1, 2$, then we find $ct^{-1} \sim 0$ and thus, we find representatives of the form $t^{-5}q_n(t^{-4})$.

Next, let us consider equation (4.8.12). We write ϵ in the form $\epsilon^4+t\epsilon_1+t^2\epsilon_2^4+t^3\epsilon_3^4$. First, we eliminate all fourth powers.

Assume $k = 0$. Then, we have

$$h(ct^{-2i-1}) = ct^{-2i-1} + c^2t^{-4i-2}(\epsilon^4 + t\epsilon_1 + t^2\epsilon_2^4 + t^3\epsilon_3^4)$$

and we obtain that t^{-4i-2} is equivalent to a linear combination of monomials of odd degree and monomials of even degree larger than $-4i - 2$. By induction, we see that all monomials of even degree can be expressed in terms of monomials of odd degree. Thus, we find representatives of the form $t^{-1}q_n(t^{-2})$. Note however, that we do not claim that they are linearly independent modulo the image of h.

If $k = 1$, then we have $ct^{-i} \sim -c^2t^{-2i+1}$. We see that $(c + c^2)t^{-1} \sim 0$ and hence, $t^{-1} \sim 0$. Also $t^{-3} \sim t^{-2} \sim 0$. By induction, we see that all odd degree monomials are equivalent to monomials of even degree. Thus, we find representatives of the form $t^{-2}q(t^{-4})$.

If $k = 2$, then we get

$$t^{-2i-1} \sim t^{-4i}(\epsilon^4 + t\epsilon_1^4 + t^2\epsilon_2^4 + t^3\epsilon 3^4).$$

By induction, we see that any monomial of the form t^{-4i+2} can be expressed in terms of monomials of odd degree. This is similar to the case $k = 0$, but in this case we have $t^{-1} \sim 0$. Thus, we can find representatives of the form $t^{-3}q(t^{-2})$.

The case $k = 3$ can be treated in the same way as the case $k = 1$, except that we get $t^{-2} \sim 0$. Here, we can find representatives of the form $t^{-6}q(t^{-4})$. □

Multiplying the equations by a suitable power of the form t^{4m}, we obtain the following.

Corollary 4.8.11 *Keeping the assumptions of the proposition, a non-trivial* U*-torsor admits an integral affine model over* $R = \Bbbk[[t]]$ *given by one of the following equations:*

1. $u^2 + t^{3n+3}v + tv^4 + t^3p_n(t^4) = 0.$
2. $u^2 + t^{3n+3}v + t^3v^4 + tp_n(t^4) = 0.$
3. $u^2 + t^{3n+5}v + tv^4 + t^3p_n(t^4) = 0.$
4. $u^2 + t^{3n+3}v + t^{2s+2n+3}\epsilon^2v^2 + tv^4 + t^3p_n(t^4) = 0.$
5. $u^2 + t^{3n+6}v + t^{2s+2n+5}\epsilon^2v^2 + t^3v^4 + t^3p_n(t^2) = 0, k = 0, 1, 2.$
6. $u^2 + t^{3n+5}v + t^{2s+2n+7}\epsilon^2v^2 + tv^4 + t^3p_n(t^2) = 0, k = 0, 1, 2.$
7. $u^4 + t^{2n+2}v + (\epsilon^4 + t\epsilon_1^4 + t^2\epsilon_1^4 + t^3\epsilon_3^4)v^2 + t^3p_n(t^4) = 0.$
8. $u^4 + t^{2n+2}v + tv^2 + t^2p_n(t^4) = 0.$
9. $u^4 + t^{2n+3}v + (\epsilon^4 + t\epsilon_1^4 + t^2\epsilon_2^4 + t^3\epsilon_3^4)v^2 + tp_n(t^4) = 0.$
10. $u^4 + t^{2n+5}v + tv^2 + t^2p_n(t^4) = 0,$

where p_n is a polynomial of degree $\leq n$ that does not vanish at 0*. In cases (1),(2),(3), (8), and (10), the polynomial p_n is uniquely determined by the equation of* U*.*

Remark 4.8.12 The degree n of polynomials p_n must have something to do with the length $l(\mathcal{T})$ of the torsion sheaf $\mathrm{Tors}(\mathcal{T})$ from the canonical class formula Theorem 4.1.6.

In our case, the multiplicity of the closed fiber is equal to p. If X_t is wild, then $d_0 = 1$ and we have only $m_0 = 1$ and $m_1 = 1 + k_0$ in the formula for $l(\mathcal{T})$ from Section 4.2. It follows from this formula that

$$l(\mathcal{T}) = \begin{cases} \left[\frac{m_1}{2}\right] & \text{if } p = 2, \\ \left[\frac{2m_1}{3}\right] & \text{if } p = 3. \end{cases}$$

Lang conjectures in [433] that $l(\mathcal{T}) = \left[\frac{2n}{3}\right]$ if $p = 3$, or, in other words, that $m_1 = n$. We conjecture that $m_1 = n$ if $p = 2$ and that p_n is uniquely determined.

Let $X = mD$ be the multiple fiber. The restriction homomorphisms $\mathrm{Pic}(nD) \to \mathrm{Pic}((n-1)D)$ have kernels isomorphic to $\Bbbk$, so $\mathcal{O}_{mD}(D)$ depends on m parameters. Let k be as in Corollary 4.8.9 and Corollary 4.8.11. The number of parameters for multiple fibers with fixed n is equal to the number of possible non-zero coefficients in the polynomial p_n, that is, equal to $n + 1$. Then Lang conjectured in [431] that $m = n + 1$ if $p = 3$ and we conjecture that $m = n + 1$ if $p = 2$. In particular, the torsor is tame only if $n = 0$ in cases (1)–(6) if $p = 2$, and $n = 0$ in all cases if $p = 3$. In this case, the polynomial p_n is a non-zero constant from $\Bbbk$ that defines a p-torsion divisor class of the normal bundle of D.

Example 4.8.13 Assume that $p = 3$.

Consider Case (1) with $n = 0$. After scaling t, we obtain an equation

$$u^3 + t^2 v + t v^3 + a t^2 = 0$$

with $a \neq 0$. By homogenizing t, u, and v, we get an equation

$$t_0^2 x^3 + t_1^2 y z^2 + t_0 t_1 y^3 + a t_1^2 z^3 = 0.$$

This is a hypersurface in $\mathbb{P}^1 \times \mathbb{P}^2$ of bidegree $(2, 3)$ with trivial canonical sheaf. One checks that the surface is singular along the curve $t_0 x^3 + y^3 = z = 0$. Taking a minimal resolution of singularities of its normalization, we obtain a rational quasi-elliptic surface $f : V \to \mathbb{P}^1$. It follows from Remark 4.1.10 that this non-trivial torsor has only one multiple fiber and that it is tame. This verifies Lang's conjecture in this case.

In Case (2) and $n = 0$, we obtain a surface in $\mathbb{P}^1 \times \mathbb{P}^2$ of bidegree $(2, 3)$ with trivial canonical sheaf given by an equation

$$t_0^2 x^3 + t_1^2 y z^2 + t_1^2 y^3 + a t_1 t_0 z^3 = 0.$$

We find that this surface has two isolated singular points $([1, 0], [0, 1, 0])$ and $([0, 1], [1, 0, 0])$. The first one is a simple elliptic singularity and the second one is a rational double point. The minimal resolution of the surface is a rational surface. This verifies Lang's conjecture in this case.

In Case (3) and $n = 0$, we obtain a surface of bidegree $(3,3)$ with canonical divisor of bidegree $(1,0)$. The equation of the surface is

$$t_0^3x^3 + t_1^3yz^2 + t_1t_0^2y^3 + at_0t_1^2z^3 = 0.$$

It has a double curve $z = t_0 = 0$ and an isolated elliptic singularity $([1,0],[0,0,1])$. A resolution of singularities of its normalization is a rational surface.

Finally, in Case (4) and $n = 0$, we obtain a surface of bidegree $(3,3)$ with equation

$$t_0^3x^3 + t_1^3yz^2 + t_0t_1^2y^3 + at_0^2t_1z^3 = 0.$$

It has two elliptic singularities $([1,0],[0,1,0])$ and $([0,1],[1,0,0])$. A resolution of singularities of its normalization is a rational surface.

Let $f : J \to C$ be a non-trivial jacobian quasi-elliptic fibration over a global base C. Theorem 4.7.2 applies to this case too. We conclude that $\text{Ш}(\mathsf{U}/K) = H^1_{\text{ét}}(C,\mathbf{U})$ is a p-group and that $\mathrm{Tors}(H^2_{\text{ét}}(C,\mathbf{U})) = 0$. However, a more precise knowledge of the structure of the Néron model $\mathbf{U}$ allows us to give an independent proof and makes this assertion more explicit.

The following is an explicit equation of the identity component $\mathbf{U}^\circ$, see [734, Theorem 2.8].

Theorem 4.8.14 *Any smooth affine scheme over a discrete valuation ring R of characteristic $p > 0$ with connected closed fiber and wound unipotent generic fiber is isomorphic to a subgroup scheme of $\mathbb{G}_{a,R}^2$ defined by an equation of the form*

$$u^{p^n} + v + a_1v^p + \cdots + a_rv^{p^r} = 0,$$

where $a_i \in R$.

In Proposition 4.8.4, we gave three explicit models for quasi-elliptic unipotent algebraic groups over K – a Russell equation $\Phi = \Phi(u,v)$. From there, we can find such an equation as in the previous theorem by using a Russell equation of the generic fiber over K. Let t be a local parameter. We replace u by $t^{-s}u$ and v by $t^{-sp^n}v$ for a suitably large s in order to assume that $a_i \in R$. The point of the proposition is that any smooth connect group model of the generic fiber can be written in this way. In particular, we may assume that the identity component of the Néron model is given by such an equation.

Before proceeding, we recall that the Lie algebra of a relative group scheme $G \to S$ is the vector group scheme $\mathrm{Lie}(G)$, whose values on any affine scheme $f : \mathrm{Spec}\, A \to S$ is equal to the kernel of $G(A[\epsilon]) \to G(A)$, where $\epsilon^2 = 0$ and $A[\epsilon] \to A$, $a + b\epsilon \to a$. We know that the Lie algebra of the relative Picard functor $\mathcal{P}_{J/S}$ is given by $\mathbb{V}(\mathcal{L}^{\oplus -1}) \to S$.

We will now consider a general Russell equation over the function field K of a global base C

$$u^{p^n} + v + a_1 v^p + \cdots + a_m v^{p^m} = 0.$$

Of course, for our applications to quasi-elliptic fibrations, we will only need the cases $(n, m) = (1, 2), (2, 1), (2, 2)$ if $p = 2$ or $(n, m) = (1, 1)$ if $p = 3$. We will assume that the genus of the unipotent group U is not zero. In this case, U admits a Néron model $\mathbf{U}$ over C.

Applying Theorem 4.8.14, we find an open affine cover $(V_i)_{i\in I}$ of C such that the restriction $\mathbf{U}_i^\circ$ of $\mathbf{U}^\circ$ to each V_i is given by a Russell equation with coefficients in $\mathcal{O}_C(V_i)$

$$\Phi_i(u_i, v_i) = u_i^{p^n} + v_i + a_1^{(i)} v_i^p + \cdots + a_m^{(i)} v_i^{p^m} = 0.$$

Since any derivation of $\mathcal{O}(V_i)[u, v]/(\Phi_i)$ is a derivation of $\mathcal{O}(V_i)[u, v]$ that vanishes on v_i, the $\mathcal{O}(V_i)$-module $\mathrm{Lie}(\mathbf{U}_i) = \mathrm{Lie}(\mathbf{U}_i^\circ)$ is generated by $\frac{\partial}{\partial u_i}$.

For brevity of notation, we let $\mathcal{L}$ be the invertible sheaf on C equal to $\mathrm{Lie}(\mathbf{U}_C)$. Let (c_{ij}) be the transition functions of $\mathcal{L}$, so that $u_i = c_{ij}^{-1} u_j$.

The transition functions for $\mathbf{U}$ from (v_i, u_i) to (v_j, u_j) must be p-polynomials in $\mathcal{O}_C(V_i \cap V_j)$. Suppose $n \le m$. Then

$$\begin{aligned} v_i &= c_{ij}^{-p^n} v_j, \\ u_i &= c_{ij}^{-1} u_j + c_{ij}^{-p^n} \alpha_{ij}^{(1)} v_j + \cdots + c_{ij}^{-p^m} \alpha_{ij}^{(m)} v_j^{p^{m-n}}, \end{aligned}$$

and

$$a_k^{(i)} = c_{ij}^{p^{n+k} - p^n} a_k^{(j)} - (\alpha_{ij}^{(k)})^{-p^n}, \tag{4.8.14}$$

where $\alpha_{ij}^{(k)} = 0$ for $k < n$. If $n > m$, then the $\alpha_{ij}^{(k)}$ are all zero.

We can view $(1, a_1^{(i)}, \ldots, a_m^{(i)})$ as a section of a vector bundle $\mathcal{A}$ of rank $m + 1$ that sits in an extension

$$0 \to \mathcal{O}_C \to \mathcal{A} \to \mathcal{L}^{\otimes p^n - p^{n+1}} \oplus \cdots \oplus \mathcal{L}^{\otimes p^n - p^{m+n}} \to 0 \tag{4.8.15}$$

with transition functions inverse to the transition functions (4.8.14).

Next, let $\mathbf{V}$ be the group scheme over C that is locally isomorphic to $\mathbb{G}_a^2$ and whose transition functions are defined as above. The group scheme $\mathbf{V}$ fits into an extension of commutative group schemes over C

$$0 \to \mathbb{V}(\mathcal{L}^{\otimes -p^n}) \to \mathbf{V} \to \mathbb{V}(\mathcal{L}^{\otimes -1}) \to 0 \tag{4.8.16}$$

given by the projection $(u, v) \to u$. Taking cohomology, we obtain a long exact sequence

$$\begin{aligned} 0 &\to H^0(C, \mathcal{L}^{\otimes p^n}) \to \mathbf{V}(C) \to H^0(C, \mathcal{L}) \\ &\to H^1(C, \mathcal{L}^{\otimes p^n}) \to H^1(C, \mathbf{V}) \to H^1(C, \mathcal{L}) \to 0. \end{aligned}$$

If $n > m$, exact sequence (4.8.16) splits and in this case we get

$$H^i(C, \mathbf{V}) \cong H^i(C, \mathcal{L}) \oplus H^i(C, \mathcal{L}^{\otimes p^n}).$$

In any case, the local embeddings of the $\mathbf{U}_i^\circ$'s into the $\mathbb{G}_{a,V_i}^2$'s glue together, from which we obtain a short exact sequence

$$0 \to \mathbf{U}^\circ \to \mathbf{V} \xrightarrow{\mu} \mathbb{V}(\mathcal{L}^{\otimes - p^n}) \to 0. \tag{4.8.17}$$

of group schemes over C.

Remark 4.8.15 Note that $\mathbf{V}$ is a vector group if and only if $m = n$. In this case, the transition matrices are given by

$$\begin{pmatrix} c_{ij}^{p^n} & -\alpha_{ij}c_{ij}^p \\ 0 & c_{ij} \end{pmatrix}.$$

The vector group scheme $\mathbf{V}$ is equal to $\mathbb{V}(\mathcal{E}^\vee)$, where $\mathcal{E}$ sits in an extension

$$0 \to \mathcal{L}^{\otimes p^n} \to \mathcal{E} \to \mathcal{L} \to 0.$$

If $\mathrm{Ext}^1(\mathcal{L}, \mathcal{L}^{\otimes p^n}) \cong H^1(C, \mathcal{L}^{\otimes 1-p^n})$ is zero, then this extension splits, that is, we may assume that $\alpha_{ij} = 0$, if $H^1(C, \mathcal{L}^{\otimes 1-p^n}) = 0$ For example, this happens if $(1-p^n)\deg(\mathcal{L}) > 2g(C) - 2$. If $\mathcal{E}$ splits, then we may assume $\alpha_{ij}^{(k)} = 0$ and hence, $\mathcal{A}$ also splits.

We recall that quasi-elliptic surfaces is uniruled and supersingular in the sense of Shioda, see Corollary 4.1.16. Thus, the computation for the Brauer group gives

$$Ш(\mathsf{U}/K) = H^1_{\text{ét}}(C, \mathbf{U}) \cong \Bbbk^{p_g(J)} \oplus {}_p\mathrm{NS}(J), \tag{4.8.18}$$

where ${}_p\mathrm{NS}(J)$ can be non-zero only if $g(C) = 1$ and if all fibers are irreducible. We briefly discussed such surfaces in the previous sections.

Since there are no trivial quasi-elliptic fibrations (the total space of a trivial quasi-elliptic fibration would be non-normal), we get

$$H^2_{\text{ét}}(C, \mathbf{U}) = 0. \tag{4.8.19}$$

Thus, there are no obstructions for constructing a non-jacobian quasi-elliptic surface from a collection of local torsors.

Let us confirm (4.8.18) and (4.8.19) using a global equation of $\mathbf{U}^\circ$. For simplicity, let us assume that the Picard scheme of J is reduced. The Grothendieck–Leray spectral sequence for $f : J \to C$ gives a short exact sequence

$$0 \to H^1(C, \mathcal{O}_C) \to H^1(J, \mathcal{O}_J) \to H^0(C, R^1 f_* \mathcal{O}_X) \to 0.$$

The map $H^1(C, \mathcal{O}_C) \to H^1(J, \mathcal{O}_J)$ is the map $\mathrm{Lie}(f^*)$ of Lie algebras. We define $\mathcal{L} = R^1 f_* \mathcal{O}_J = \mathrm{Lie}(\mathbf{U})$, which is an invertible sheaf on C and thus, our assumption is equivalent to

$$H^0(C, \mathcal{L}) = 0.$$

Let us even assume the stronger condition that $\deg \mathcal{L} < 0$. Then, $H^0(C, \mathcal{L}) = H^0(C, \mathcal{L}^{\otimes p^n}) = \{0\}$ and hence, $H^0(C, \mathbf{V}) = \{0\}$. It follows that

$$H^0(C, \mathbf{U}^\circ) = \{0\}.$$

Taking cohomology in (4.8.17), we obtain an exact sequence

$$0 \to H^1_{\text{ét}}(C, \mathbf{U}^\circ) \to H^1_{\text{ét}}(C, \mathbf{V}) \to H^1_{\text{ét}}(C, \mathcal{L}^{\otimes p^n}) \to H^2_{\text{ét}}(C, \mathbf{U}^\circ) \to 0. \tag{4.8.20}$$

Here, we use that the étale cohomology of a vector group scheme $\mathbb{V}(\mathcal{E})$ is isomorphic to the Zariski cohomology of $\mathcal{E}^\vee$, see, for example, [508, Chapter III, Proposition 3.7].

Let $\mu : \mathbf{V} \to \mathbb{V}(\mathcal{L}^{\otimes -p^n})$ be the map from exact sequence (4.8.17) and let α be the restriction of μ to the subgroup $\mathbb{V}(\mathcal{L}^{\otimes -p^n})$. This is a surjective homomorphism in the étale topology and we denote its kernel by G. This is a finite group scheme over C, but not necessary flat over C. We have

$$H^1_{\text{ét}}(C, G) = \operatorname{Ker}\left(H^1(\alpha) : H^1(C, \mathcal{L}^{\otimes -p^n}) \to H^1(C, \mathcal{L}^{\otimes -p^n})\right),$$

and

$$H^2_{\text{ét}}(C, G) = \operatorname{Coker}\left(H^1(\alpha) : H^1(C, \mathcal{L}^{\otimes -p^n}) \to H^1(C, \mathcal{L}^{\otimes -p^n})\right).$$

It follows that $H^1_{\text{ét}}(C, \mathbf{U}^\circ)$ fits in an extension

$$0 \to H^1_{\text{ét}}(C, G) \to H^1_{\text{ét}}(C, \mathbf{U}^\circ) \to H^1(C, \mathcal{L}) \to 0. \tag{4.8.21}$$

We also have an isomorphism

$$H^2_{\text{ét}}(C, \mathbf{U}^\circ) \cong H^2_{\text{ét}}(C, G).$$

Unfortunately, we do not know how to compute the cohomology groups $H^i_{\text{ét}}(C, G)$ for a general unipotent group U of genus $g > 0$. In the particular case of quasi-elliptic groups in the cases (1b) or 2 of Proposition 4.8.4, we can apply the next proposition, which immediately follows from Lemma 0.10.23.

Proposition 4.8.16 *Let $u^{p^n} + v + a_1 v^p + \cdots + a_m v^{p^m} = 0$ be the equation of the Néron model $\mathbf{U}$ of a unipotent group U of genus $g > 0$ over K. Suppose that $a_1 = \cdots = a_{m-1} = 0$ and let*

$$\alpha : \mathcal{L}^{\otimes p^n} \to \mathcal{L}^{\otimes p^n}$$

be the map given by $v \mapsto v + a v^{p^m}$ and consider

$$H^1(\alpha) : H^1(C, \mathcal{L}^{\otimes p^n}) \to H^1(C, \mathcal{L}^{\otimes p^n})$$

be the induced map on cohomology. Then, $H^1(\alpha)$ is surjective and its kernel is a vector space over $\mathbb{F}_{p^m}$ of dimension equal to the stable rank r of the p^m-linear map $H^1(\alpha) - \mathrm{id}$. In particular, we have

$$H^1_{ét}(C,G) \cong (\mathbb{Z}/p^m\mathbb{Z})^r, \quad H^2_{ét}(C,\mathbf{U}) = H^2_{ét}(C,G) = 0.$$

Let

$$\pi_0(\mathbf{U}) := \mathbf{U}/\mathbf{U}^\circ,$$

which is a sky-scraper group scheme over C. It follows from the short exact sequence

$$0 \to \mathbf{U}^\circ \to \mathbf{U} \to \pi_0(\mathbf{U}) \to 0$$

that

$$H^2_{\text{ét}}(C,\mathbf{U}) \cong H^2_{\text{ét}}(C,\mathbf{U}^\circ) = H^2_{\text{ét}}(C,G) = 0.$$

We also have a short exact sequence

$$0 \to \pi_0(\mathbf{U})/\mathbf{U}(C) \to H^1_{\text{ét}}(C,\mathbf{U}^\circ) \to H^1_{\text{ét}}(C,\mathbf{U}) \to 0. \tag{4.8.22}$$

Assume $\mathrm{Tors}(\mathrm{NS}(X)) = \{0\}$. The images of the divisor classes of a fiber of $f : X \to C$ and of a section O generate a sublattice of $\mathrm{NS}(X)$ isomorphic to the integral hyperbolic plane U. It splits off as an orthogonal summand of $\mathrm{NS}(X)$. Let $\mathrm{NS}^0(X)$ be the orthogonal complement. The image of the restriction homomorphism $\mathrm{NS}^0(X) \to \mathrm{Pic}(X)$ is isomorphic to the group of sections $\mathbf{U}(C)$ and its kernel is the sublattice $\mathrm{NS}^0_{\mathrm{fib}}(X)$ that is generated by components of fibers not intersection O. Let us consider the chain of lattices and the corresponding dual lattices

$$\mathrm{NS}^0_{\mathrm{fib}}(X) \subset \mathrm{NS}(X)^0 \subset \mathrm{NS}^0(X)^\vee \subset \mathrm{NS}^0_{\mathrm{fib}}(X)^\vee.$$

The discriminant group $\mathrm{NS}^0_{\mathrm{fib}}(X)^\vee/\mathrm{NS}^0_{\mathrm{fib}}(X)$ of the lattice $\mathrm{NS}^0_{\mathrm{fib}}(X)$ is isomorphic to the group $\pi_0(\mathbf{U})$ (see [606, 8.1.2]) and the discriminant group $\mathrm{NS}^0(X)^\vee/\mathrm{NS}^0(X)$ of the lattice $\mathrm{NS}^0(X)$ is isomorphic to the discriminant group $D(\mathrm{NS}(X))$ of $\mathrm{NS}(X)$. This gives a chain of finite abelian groups

$$\mathbf{U}(C) \subset \mathbf{U}(C)' \subset \pi_0(\mathbf{U})$$

with quotients $\mathbf{U}(C)'/\mathbf{U} \cong D(\mathrm{NS}(X))$ and $\pi_0(\mathbf{U})/\mathbf{U}(C)' \cong \mathbf{U}(C)$. Comparing it with exact sequences (4.8.21) and (4.8.22), we make the following:

Conjecture 4.1 The intersection $H^1_{\text{ét}}(C,G)_0$ of the subgroups $H^1_{\text{ét}}(C,G)$ and $\pi_0(\mathbf{U})/\mathbf{U}(C)$ inside $H^1_{\text{ét}}(C,\mathbf{U}^\circ)$ splits the exact sequence

$$0 \to D(\mathrm{NS}(X)) \to \pi_0(\mathbf{U})/\mathbf{U}(C) \to \mathbf{U}(C) \to 0.$$

The group $H^1_{\text{ét}}(C,\mathbf{U})$ is isomorphic to $H^1(C,\mathcal{L})$ and fits in an extension

$$0 \to H^1_{\text{ét}}(C,G)/H^1_{\text{ét}}(G,G)_0 \to H^1_{\text{ét}}(C,\mathbf{U}) \to H^1(C,\mathcal{L}) \to 0.$$

We end this section with some explicit computations of a global quasi-elliptic fibrations $f : J \to C$.

Example 4.8.17 Assume that $g = 1$, $C = \mathbb{P}^1$, and that $\mathcal{L} \cong \mathcal{O}_{\mathbb{P}^1}(-k)$ for some $k > 0$. Then we have $H^0(C, \mathcal{L}^{\otimes n}) = 0$ for all $n \geq 0$ and hence,

$$H^0_{\text{ét}}(C, \mathbf{U}^\circ) = H^2_{\text{ét}}(C, \mathbf{U}) = 0, \quad H^1(C, \mathbf{U}) \cong H^1(C, \mathcal{L}).$$

The $\Bbbk$-vector space $H^1(C, \mathcal{L}^{\otimes p}) \cong H^1(C, \mathcal{O}_{\mathbb{P}^1}(-pk))$ has a basis given by the negative Laurent monomials $e_i = t_0^{-i} t_1^{pk-i}$ for $i = 1, \ldots, pk-1$, see also [294, III,§5].

Now, assume $p = 3$ and let $u^3 + v + a_{6k} v^3 = 0$ be the equation of $\mathbf{U}$, where $a_{6k} \in H^0(C, \mathcal{L}^{\otimes 6}) = H^0(C, \mathcal{O}_{\mathbb{P}^1}(6k))$ is a binary form of degree $6k$. Write $a_{6k} = \sum_{i=0}^{6k} c_i t_0^i t_1^{6k-i}$ and let $A = (c_{ij})$ be a matrix with entries defined by

$$\left(\frac{a_{6k}(t_0, t_1)}{t_0^{3i} t_1^{9k-3i}}\right)' = \sum_{j=1}^{3k-1} c_{ij} e_j, \quad i = 1, \ldots, 3k-1.$$

Here, $(-)'$ means that we eliminate all monomials $t_0^i t_1^j$ with non-negative i, or j from the Laurent polynomial. We compute the entries c_{ij} of A and obtain that

$$A = \begin{pmatrix} c_{3-1} & c_{6-1} & \cdots & c_{3d-1} \\ c_{3-2} & c_{6-2} & \cdots & c_{3d-2} \\ \vdots & \vdots & \vdots & \vdots \\ c_{3-d} & c_{6-d} & \cdots & c_{3d-d} \end{pmatrix},$$

where $c_j = 0$ if $j < 0$ and $d = 3k - 1$. The group $H^1(C, G)$ is isomorphic to $(\mathbb{Z}/3\mathbb{Z})^{\oplus r}$, where r is the stable rank of A, see (0.10.50).

The matrix A coincides with the *Hasse–Witt matrix* that computes the p-rank of the hyperelliptic curve H of genus d that is given by equation $t_2^2 + a_{6k}(t_0, t_1) = 0$, that is, the maximal r such that $(\mathbb{Z}/p\mathbb{Z})^r$ embeds in its Jacobian. In our case it may of course happen that the polynomial $a_{6k}(t_0, t_1)$ may degenerate and then, it does not define an hyperelliptic curve. Note that the projection $\pi : H \to \mathbb{P}^1$ is a double cover that is ramified over $V(a_6)$ that gives and exact sequence

$$0 \to \mathcal{O}_{\mathbb{P}^1} \to \pi_* \mathcal{O}_H \to \mathcal{O}_{\mathbb{P}^1}(-3k) \to 0$$

and an isomorphism $H^1(H, \mathcal{O}_H) = H^1(\mathbb{P}^1, \pi_* \mathcal{O}_H) \cong H^1(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(-3k))$. The matrix A describes the action of the Frobenius on the basis $(e_1, \ldots, e_{3k-1})$ of $H^1(H, \mathcal{O}_H)$.

Let us now discuss the cases $k = 1$ and $k = 2$ in detail.

1. Assume $k = 1$. The surface X is a rational quasi-elliptic surface with a section. The classification of such surfaces is known. In particular, the group $\mathbf{U}(C)$ is known in each case, see Section 4.9. The conjecture is checked in characteristic $p = 2, 3$ by explicit computations of the groups $H^1(C, G)$. Note that in this

case $D(\mathrm{NS}(X)) = \{0\}$ and $H^1(C, \mathcal{L}) = 0$, so that the Tate–Shafarevich group $Ш(\mathsf{U}/K) = H^1_{\text{ét}}(C, \mathbf{U})$ is trivial, which is in agreement with (4.7.2).

2. Assume $k = 2$. Thus, $\mathcal{L} = O_{\mathbb{P}^1}(-2)$ and we obtain a quasi-elliptic K3 surface $f : X \to \mathbb{P}^1$. In this case, we get

$$H^1_{\text{ét}}(C, \mathbf{U}) \cong H^1(C, \mathcal{L}) \cong \mathbb{G}_a(\Bbbk).$$

Example 4.8.18 The following two examples were communicated to us by T. Katsura.

1. It is known that the Fermat quartic surface $x^4 + y^4 + z^4 + w^4 = 0$ in characteristic 3 is a supersingular K3 surface with Artin invariant $\sigma = 1$. It admits a quasi-elliptic fibration with Weierstrass equation $y^2 + x^3 + t_0^2 t_1^2 (t_0^8 + t_1^8) = 0$, see [366]. A Russell equation of $\mathbf{U}^\circ$ is given by

$$u^3 + v + t_0^2 t_1^2 (t_0^8 + t_1^8) v^3 = 0.$$

The quasi-elliptic fibration has 10 reducible fibers of Kodaira type IV with $\pi_0(\mathbf{U}) \cong (\mathbb{Z}/3\mathbb{Z})^{\oplus 10}$ and the Mordell–Weil group $\mathbf{U}(C)$ is isomorphic to $(\mathbb{Z}/3\mathbb{Z})^{\oplus 4}$. The discriminant group $D(\mathrm{NS}(J))$ is isomorphic to $(\mathbb{Z}/3\mathbb{Z})^{\oplus 2}$. We compute the Hasse–Witt matrix A and find that $H^1(C, G) \cong (\mathbb{Z}/3\mathbb{Z})^{\oplus 4} \cong \mathbf{U}(C)$.

2. If we take the K3 surface in characteristic 3 given by the Weierstrass equation

$$y^2 + x^3 + t_0^2 t_1^{10} + t_0^5 t_1^7 + t_0^8 t_1^4 + t_0^{10} t_1^2 = 0,$$

then we obtain that its Mordell–Weil group $\mathbf{U}(C)$ is an elementary 3-group of rank 2. The quasi-elliptic fibration contains 10 reducible fibers of type IV. Thus, its Artin invariant is equal to $\sigma = 3$, so that $D(\mathrm{NS}(J))$ is an elementary 3-group of rank 6 and $\pi_0(\mathbf{U})$ is an elementary 3-group of rank 10. Computing the Hasse–Witt matrix A we find that its stable rank is equal to 4. Thus, $\mathbf{U}(C) \cong (\mathbb{Z}/3\mathbb{Z})^{\oplus 2}$ is isomorphic to a proper subgroup of $H^1_{\text{ét}}(C, G) \cong (\mathbb{Z}/3\mathbb{Z})^{\oplus 4}$. Katsura found an explicit isomorphism from a certain subgroup $H^1_{\text{ét}}(C, G)'$ of $H^1_{\text{ét}}(C, G)$ to $\mathbf{U}(C)$.

4.9 Genus One Fibrations on Rational Surfaces

In the remaining two sections of this chapter we apply the results from the previous sections to rational surfaces and Enriques surfaces. We will start with genus one fibrations on rational surfaces. In the next section, we will study genus one fibrations on Enriques surfaces. We will see in the next section that the jacobian fibration of a genus one fibration on an Enriques surface is a genus one fibration on a rational surface. A large part of this section will be concerned with the explicit classification of jacobian and minimal genus one fibrations on rational surfaces that are extremal, that is, where the classes of components of reducible fibers span a sublattice in the Picard lattice of rank 9, which is the maximum possible.

Let $f : J \to C = \mathbb{P}^1$ be a jacobian genus one fibration on a rational surface J. Since J is a rational surface, f is non-trivial and $\mathrm{Br}(J) = 0$. Using Theorem 4.7.2, Theorem 4.7.9, and applying it to the exact sequence (4.7.1), we obtain the following result concerning Weil–Châtelet groups of genus one fibrations on rational surfaces.

Theorem 4.9.1 *Let $f : J \to \mathbb{P}^1$ be a genus one fibration on a rational surface J. Let J_η be the generic fiber over the function field K of C and let $J_\eta^\sharp$ be the smooth locus. Then,*

$$\mathrm{loc} \,:\, \mathrm{WC}(J_\eta^\sharp/K) \;\to\; \bigoplus_{t\in C} \mathrm{WC}(J_\eta^\sharp \times_K K_t^h/K_t^h) \tag{4.9.1}$$

is an isomorphism.

Corollary 4.9.2 *Let $f : X \to \mathbb{P}^1$ be a genus one fibration, whose jacobian fibration $J \to \mathbb{P}^1$ is a rational surface. Let $(m_1, \ldots, m_r)$ be the multiplicities of the multiple fibers of f. Then, the order of $[f]$ in $\mathrm{WC}(J/\mathbb{P}^1)$ is equal to the least common multiple of $m_1, \ldots, m_r$.*

Proposition 4.9.3 *Let $f : X \to C$ be a relatively minimal genus one fibration on a rational surface and let $j : J \to C$ be the associated jacobian fibration. Then:*

1. *J is a rational surface, $C \cong \mathbb{P}^1$, and $K_J = -j^*F$, where F is a fiber of j.*
2. *If f has multiple fibers, then there exists precisely one multiple fiber m_0X_0. Moreover, this unique multiple fiber is tame and we have $K_X = -X_0$.*
3. *Any (-1)-curve E on X is an m-section of f for some $m \geq 1$, that is, $f|_E : E \to \mathbb{P}^1$ is a finite morphism of degree equal to m.*
4. *X is a* basic rational surface*, that is, it admits a birational morphism $\pi : X \to \mathbb{P}^2$.*

Proof (1) Applying Proposition 4.3.14 and Corollary 4.3.18 we obtain that $b_1(X) = b_1(J) = 0$ and $p_g(X) = p_g(J) = 0$. In particular, we find $C \cong \mathbb{P}^1$. By Corollary 4.3.8, we have $\mathsf{S}^2 = -1$ for any section S on J and hence, $K_J \cdot \mathsf{S} = -1$. By Proposition 4.3.7, we have $\omega_J \cong j^*(\mathcal{O}_{\mathbb{P}^1}(-1))$. This implies that J is a rational surface.

(2) By Theorem 4.1.6, we have $\mathrm{Tors}(R^1 f_*\mathcal{O}_X) = 0$ and $\deg \mathcal{L} = -1$. Thus, f cannot have wild fibers and we have $a_t = m_t - 1$ for all multiple fibers. By Theorem 4.1.6, we have $K_X = -F + \sum_{t\in\mathbb{P}^1}(m_t - 1)\bar{X}_t$, where F is some fiber. Since K_X is not nef, there exists some irreducible curve D with $K_X \cdot D < 0$. This curve D cannot be contained in a fiber and thus, we find $D \cdot F = D(m_t X_t) = a$ for some $a > 0$ and every $t \in C$. From this, we find $-1 + \sum_t((m_t - 1)/m_t) < 0$. This is only possible if we have at most one multiple fiber and if we have a multiple fiber, then we find $F \sim m_{t_0}X_{t_0}$ and $K_X = -X_{t_0}$.

(3) By the adjunction formula, we have $-1 = E \cdot K_X = -\frac{1}{m}E \cdot X_t$ and thus, $E \cdot X_t = m$.

(4) Since X is a relatively minimal rational genus one surface, it does not contain smooth rational curves with self-intersection < -2. The assertion now follows from Lemma 9.1.3 from Section 9.1 in Volume II. □

Let X be a smooth projective rational surface and let

$$X \;=\; X_N \xrightarrow{\pi_9} X_{N-1} \xrightarrow{\pi_8} \cdots \xrightarrow{\pi_2} X_1 \xrightarrow{\pi_1} X_0 \;=\; \mathbb{P}^2 \tag{4.9.2}$$

be the factorization of a birational morphism of $\pi : X \to \mathbb{P}^2$ into a composition of blow-ups $\pi_i : X_i \to X_{i-1}$ in closed points $x_i \in X_{i-1}$. Let $\mathcal{E}_1, \ldots, \mathcal{E}_N$ be the *exceptional configurations* and let $|dh - m_1x_1 - \cdots - m_Nx_N|$ be the linear system of plane curves of degree d with points x_i of multiplicity $\geq m_i$, see [177, 7.3.1]. The rational map $f : X \dashrightarrow \mathbb{P}^n$ defined by the linear system $|\pi^*(dh) - m_1\mathcal{E}_1 - \cdots - m_N\mathcal{E}_N|$ is obtained from the rational map $g : \mathbb{P}^2 \dashrightarrow \mathbb{P}^n$ defined by the linear system $|dh - m_1x_1 - \cdots - m_Nx_N|$ by (minimally) resolving its base points. We have $f = g \circ \pi$ as composition of rational maps.

We now apply this to the situation where X is a rational surface that admits a relatively minimal genus one fibration $f : X \to \mathbb{P}^1$. We have

$$K_X = \pi^* K_{\mathbb{P}^2} + \mathcal{E}_1 + \cdots + \mathcal{E}_N.$$

Since $K_X^2 = 0$, we have $N = 9$. Thus,

$$|-mK_X| = |X_t| = |-\pi^*(K_{\mathbb{P}^2}) - m\mathcal{E}_1 - \cdots - m\mathcal{E}_9| = |3mh - m(x_1 + \cdots + x_9)|.$$

Thus, the linear system $|3mh - m(x_1 + \cdots + x_9)|$ is a pencil and the genus one fibration $f : X \to \mathbb{P}^1$ is obtained from the rational map $\mathbb{P}^2 \dashrightarrow \mathbb{P}^1$ defined by resolving the base points of this pencil.

The pencil $|3mh - m(x_1 + \cdots + x_9)|$ is called a *Halphen pencil* of index m. Its general member is a curve of degree $3m$ and geometric genus 1 if the fibration is elliptic and geometric 0 if the fibration is quasi-elliptic. We have multiple points of multiplicity m at $x_1, \ldots, x_9$. Among the members of the pencil is the curve $m\pi(\bar{X}_{t_0})$, where $m\bar{X}_{t_0}$ is a multiple fiber of f of multiplicity m. The curve $\pi(\bar{X}_{t_0})$ is a plane cubic (unique if $m \geq 2$), which passes through the points $x_1, \ldots, x_9$. A rational surface obtained by minimally resolving the base points of a Halphen pencil of index m is called a *Halphen surface of index m*.

Remark 4.9.4 A set of points $x_1, \ldots, x_9 \in \mathbb{P}^2$, possibly including infinitely near points, is called a *Halphen set* if the linear system $|3mh - m(x_1 + \cdots + x_9)|$ is a pencil. Its mobile part of its full pre-image on the surface X obtained as a sequence of blow-ups (4.9.2) of $x_1, \ldots, x_9$ and defines a relatively minimal genus one fibration on X. Thus, we have bijections between:

1. The set of Halphen pencils in $\mathbb{P}^2$ (up to Cremona equivalence).
2. Halphen sets of points (up to Cremona equivalence).
3. Rational surfaces with genus one fibrations.

We refer for more details about the geometry of surfaces obtained by blowing up Halphen sets of points to [104].

Remark 4.9.5 Let $\lambda F + \mu G = 0$ be a pencil of plane curves, whose general member is birationally equivalent to a regular irreducible curve of arithmetic genus one. Let $\pi' : X' \to \mathbb{P}^2$ be a resolution of the base points of this pencil. It comes with a genus one fibration $f' : X' \to \mathbb{P}^1$, which is not necessary relatively minimal. Let $\phi : X' \to X$ be a birational morphism to a relatively minimal model. Then, f' factors through a relatively minimal genus one fibration $f : X \to \mathbb{P}^1$. The formula for the canonical class gives that $K_X = -F$, where mF is a fiber of f. Let $\pi : X \to X_N \to \cdots \to X_1 \to Y$ be a birational morphism onto a minimal rational surface. We can write it as a composition $\pi : X \to X_N \to \cdots \to X_1 \to Y$ of blow-ups in points. Suppose that Y is not isomorphic to $\mathbb{P}^2$. If $Y \cong \mathbf{F}_0$, then we find a birational morphism $X_1 \to \mathbb{P}^2$ that blows down proper transforms of two intersecting lines on $\mathbf{F}_0$. Thus, we get a birational morphism $\pi' : X \to \mathbb{P}^2$. Next, assume that $Y = \mathbf{F}_n$ for some $n \geq 2$ and let E_0 be the exceptional section on $Y \to \mathbb{P}^1$. If $X_1 \to Y$ is the blow-up of a point $x \notin E_0$, then we obtain a birational morphism $X_1 \to \mathbf{F}_{n-1}$ that blows down the proper transform of the fiber of $\mathbf{F}_n \to \mathbb{P}^1$ passing through x. By induction, we may assume that $x \in E_0$. Then, the proper inverse transform $\pi^{-1}(E_0)$ is a curve on X with self-intersection ≤ -3. The formula for K_X shows that there are no such curves on X.

The upshot of this discussion is that we may assume that $Y = \mathbb{P}^2$. The image of the genus one fibration in the plane is a Halphen pencil of index m. This proves a classical theorem of Bertini that states that any pencil of plane curves of geometric genus one can be reduced by a Cremona transformation T to a Halphen pencil. The transformation T here makes the following diagram of rational maps commutative:

$$\begin{tikzcd}
 & X' \arrow[r] \arrow[dl, "f'"'] \arrow[d, "\pi'"] & X \arrow[d, "\pi"] \arrow[dr, "f"] & \\
\mathbb{P}^1 & \mathbb{P}^2 \arrow[l, dashed] \arrow[r, dashed, "T"] & \mathbb{P}^2 \arrow[r, dashed] & \mathbb{P}^1
\end{tikzcd}$$

Let $V(G_3)$ be the plane cubic passing through a Halphen set of points $x_1, \ldots, x_9$ and let $V(F_{3m})$ be a member of this Halphen pencil different from $V(G_3)$. The Halphen pencil consists of curves $V(\lambda F_{3m} + \mu G_3^m)$. If the cubic $E = V(G_3)$ is a nonsingular elliptic curve and if there are no infinitely near points among $x_1, \ldots, x_9$, then the divisor $m(x_1 + \cdots + x_9)$ on $V(G_3)$ is cut out by a curve $V(F_{3m})$ of degree $3m$. This shows that $\mathcal{O}_E(m(x_1 + \cdots + x_9)) \cong \mathcal{O}_E(3m)$. By considering the group law on E with respect to an inflection point as its origin, this can be interpreted as that the sum $x_1 \oplus \cdots \oplus x_9$ in the group law is an m-torsion point. In fact, the order n of this sum must be equal to m, since reversing the argument we find a curve $V(F_{3m})$ that intersects E at the points $x_1, \ldots, x_m$ with multiplicities m. One can choose F_{3n} such that $V(F_{3n})$ has multiple points of multiplicities n at $x_1, \ldots, x_m$, see [174, Lemma 4.4]. This shows that $V(\lambda F_{3n} + \mu G_3^n)$ is a Halphen pencil of index n. If E is any reduced cubic curve and $x_1, \ldots, x_n$ are nonsingular points, then the same is true if one uses the group law on the set of nonsingular points on a reduced plane cubic, see [161, Appendix].

Without any assumption on E or on $x_1, \ldots, x_9$, let $D = \mathcal{E}_1 + \cdots + \mathcal{E}_9$ and $\mathcal{L} = \mathcal{O}_X(D)$. We have $\mathcal{L}^{\otimes m} \cong \mathcal{O}_X(mD) \cong \mathcal{O}_X(-mK_X) \cong \mathcal{O}_X(X_t)$. Let $X_{t_0} = m\bar{X}_{t_0}$ be a multiple fiber of multiplicity m of f and $\iota : \bar{X}_{t_0} \hookrightarrow X$ be the closed embedding. The invertible sheaf $\iota^*(\mathcal{L}) \cong \mathcal{O}_{\bar{X}_{t_0}}(\bar{X}_{t_0})$ satisfies $\iota(\mathcal{L})^{\otimes m} \cong \mathcal{O}_{\bar{X}_0}$, hence its isomorphism class in $\mathrm{Pic}(\bar{X}_{t_0})$ belongs to ${}_m\mathrm{Pic}(\bar{X}_{t_0})$. If the order is equal to n, then the exact sequence

$$0 \to \mathcal{O}_X \to \mathcal{O}_X(n\bar{X}_{t_0}) \to \mathcal{O}_X(\iota^*(\mathcal{L})^{\otimes n}) \to 0$$

shows that $h^0(\mathcal{O}_X(n\bar{X}_{t_0}) = 2$. Thus, $n\bar{X}_{t_0}$ moves in a pencil. This implies that $n = m$. This is the analog of the condition $m(x_1 \oplus \cdots \oplus x_9) = 0$ in the case of infinitely near points.

Let $f : J \to \mathbb{P}^1$ be a jacobian genus one fibration on a rational surface J. Since $\omega_X \cong f^*\mathcal{O}_{\mathbb{P}^1}(-1)$, we obtain that the sheaf $\omega = (R^1 f_*\mathcal{O}_X)^\vee$ is isomorphic to $\mathcal{O}_{\mathbb{P}^1}(1)$. Thus, the Weierstrass model W of j is a closed subscheme of $\mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-2) \oplus \mathcal{O}_{\mathbb{P}^1}(-3))$. The projection $X \to W \to \mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-2)) = \mathbf{F}_2$ is a composition of a birational morphism that blows down irreducible components of fibers not intersecting a fixed section E_0 and a finite morphism of degree 2 as described in Section 4.4.

By Theorem 4.3.20, the types of degenerate fibers of j are the same as the types of fibers on any genus one fibration $f : X \to \mathbb{P}^1$, where $[f] \in \mathrm{WC}(J/\mathbb{P}^1)$. Since $\rho_{\mathrm{fib}} \le 9$, all the reducible fibers can have at most 8 irreducible components.

All possible types of degenerate fibers on rational elliptic surfaces were classified by Persson [590] in characteristic zero, by Lang [438] in characteristic 2, and by Jarvis, Lang, et al. in characteristic 3 [350]. All quasi-elliptic rational surfaces were classified by Ito [336], [335].

In the sequel, we will give our own method for the classification of *extremal genus one fibrations* on jacobian rational surfaces. By definition, this means that ρ_{fib} takes the maximal possible value, which is equal to 9 on a rational surface. By formula (4.3.2), this is equivalent to the property that the Mordell–Weil group of the fibration is finite. All quasi-elliptic surfaces are automatically extremal. We will need the classification later for classifying Enriques surfaces with finite automorphism groups. The classification was done by Naruki [551] and by Miranda and Persson [515] in characteristic $p \neq 2, 3$. The classification in characteristic 2 and 3 is due to Lang [438], [439]. Their classification is based on the analysis of the Weierstrass equation of a jacobian rational elliptic surface. In the following we will do it in all characteristics by another, more geometric method.

Let $f : X \to \mathbb{P}^1$ be a relatively minimal genus one fibration on a rational surface. Then, $\mathrm{Pic}(X) \cong \mathrm{NS}(X) \cong \mathrm{Num}(X)$ is isomorphic to the unique odd unimodular lattice $\mathsf{I}^{1,9}$ of signature $(1, 9)$. Its orthonormal basis can be chosen to be $(e_0, e_1, \ldots, e_9)$, where $\pi^*(\mathrm{Pic}(\mathbb{P}^2)) = \mathbb{Z}e_0$ and e_i are the classes of the exceptional configurations $\mathcal{E}_i$. We have

$$K_X = -3e_0 + e_1 + \cdots + e_9.$$

If X is a Halphen surface of index m, then the class of a fiber of $f : X \to \mathbb{P}^1$ is equal to $-mK_X$. We have isomorphisms of quadratic lattices $K_X^\perp \cong \tilde{\mathsf{E}}_8$ and $K_X^\perp/\mathbb{Z}K_X \cong \mathsf{E}_8$.

It follows that $\mathrm{Pic}_{\mathrm{fib}}(X)$ is a primitive sublattice of $\tilde{E}_8$ and that the quotient by the class K_X is isomorphic to a primitive sublattice of E_8. This gives a useful information about possible structure of reducible fibers.

We have the following lemma, which we will use below for a detailed analysis of extremal and jacobian genus one fibrations on rational surfaces.

Lemma 4.9.6 *Let $f : J \to \mathbb{P}^1$ be an extremal jacobian genus one fibration on a rational surface. Then the types of reducible fibers belong to the following list:*

$\tilde{E}_8$, $\tilde{D}_8$, $\tilde{A}_8$, $\tilde{E}_7+\tilde{A}_1$, $\tilde{A}_7+\tilde{A}_1$, $\tilde{E}_6+\tilde{A}_2$, $\tilde{D}_5+\tilde{A}_3$, $\tilde{D}_4+\tilde{D}_4$, $\tilde{A}_4+\tilde{A}_4$, $\tilde{D}_6+\tilde{A}_1+\tilde{A}_1$, $\tilde{A}_5+\tilde{A}_2+\tilde{A}_1$, $\tilde{A}_3+\tilde{A}_3+\tilde{A}_1+\tilde{A}_1$, $\tilde{A}_2+\tilde{A}_2+\tilde{A}_2+\tilde{A}_2$.

Here, type $\tilde{A}_k$ for $k = 1,2$ means type $\tilde{A}_k$ or $\tilde{A}_k^*$.

Proof Let D_t be the subgroup of $\mathrm{Pic}(J)_t$ generated by irreducible components of a reducible fiber J_t and $\bar{D}_t = D_t/\mathbb{Z}[J_t]$ be the quotient by its radical. This is an irreducible root lattice. The orthogonal sum of lattices $\bar{D}_t$ is a root sublattice of E_8. Its reflection group is a subgroup of $W(\mathsf{E}_8)$. All such subgroups and hence, the types of root sublattices of $\mathbf{E}_8$, can be found using the *Borel–de Siebenthal–Dynkin algorithm*, see Section 6.4. In particular, we can find all root sublattices of maximal possible rank 8. Among them are sublattices whose Dynkin diagram is obtained by deleting one vertex from the Dynkin diagram of $\tilde{\mathsf{E}}_8$. These are the lattices

$$E_8,\ A_8,\ D_8,\ A_1+A_7,\ A_2+A_6,\ A_1+A_2+A_5,\ A_4+A_4,\ A_3+D_5,\ A_2+D_6,\ A_1+D_7.$$

Not all root sublattices are realized in extremal fibrations. Indeed, it follows from the Shioda–Tate formula that the discriminant of the lattice $\oplus_t \bar{D}_t$ must be a square of the order of the Mordell–Weil group. For example, the discriminants of the sublattices of types $A_2+A_6, A_2+D_6, A_1+D_7$ are not squares and hence, they cannot be realized. Applying the Borel–de Siebenthal–Dynkin algorithm and checking the square condition, we arrive at the asserted list. □

Remark 4.9.7 As we will see later, all types can indeed be realized as the types of reducible fibers on rational surfaces with a genus one fibration. However, some of them cannot be realized in characteristic zero.

We will need also the following result [645, Corollary 8.6].

Lemma 4.9.8 *Let $J \to \mathbb{P}^1$ be a jacobian elliptic fibration on a rational surface. Then, any two different torsion sections are disjoint.*

Proof In fact, we may assume that one section is the zero section O and that the other section S is a non-trivial torsion section. We use the theory of Mordell–Weil lattices from Section 4.5. Since J is a rational surface, we have $\mathsf{S}^2 = -1$. Suppose that S and O are not disjoint, that is, $\mathsf{S}\cdot\mathsf{O} \ge 1$. Since S is a torsion section, its height $\langle \mathsf{S},\mathsf{S}\rangle$ is equal to 0. Applying (4.5.9), we obtain that

$$0 = \langle \mathsf{S},\mathsf{S}\rangle \ge 4 - \sum_{t\in C} \mathrm{contr}_t(\mathsf{S},\mathsf{S}).$$

The sum of the local contributions is less or equal than the sum of local contributions on an extremal rational surface. Using Lemma 4.9.6 and Table 4.4, we check that it is always less than 4. This contradiction proves the assertion. □

Remark 4.9.9 Note that this assertion is false in general for quasi-elliptic rational surfaces. Given such a surface in characteristic $p = 2$ with 8 reducible fibers of type $\tilde{A}_1^*$, then its Mordell–Weil group is isomorphic to $(\mathbb{Z}/2\mathbb{Z})^{\oplus 4}$ and then, some 2-torsion sections intersect.

In fact, Table 4.4 gives us useful information on how a torsion section intersects the irreducible components of a reducible fiber. Note that on an extremal surface all sections are torsion sections (the Mordell–Weil rank is zero). First, we use the equality

$$\sum_{t \in C} \mathrm{contr}_t(\mathsf{S}, \mathsf{S}) = 2. \tag{4.9.3}$$

Let

$$y^2z + a_1xyz + a_3yz^2 + x^3 + a_2x^2z + a_4xz^2 + a_6z^3 = 0 \tag{4.9.4}$$

be the Weierstrass equation of the jacobian genus one fibration $j : J \to \mathbb{P}^1$. Since the base curve is $\mathbb{P}^1$ and since J is a rational surface, the $a_k \in \Bbbk[u, v]$ are binary forms of degree k. We now analyze the possible types of Lemma 4.9.6 case by case and deal with small characteristics p separately. These are the elliptic cases, but we will treat the quasi-elliptic cases of these types at the same time. We note that there are a couple of types that can show up on quasi-elliptic surfaces but not on rational extremal elliptic surfaces and we will deal with them afterwards.

- Type $\tilde{E}_8$

It follows from the determinant formula (4.3.4) that

$$\mathrm{MW}(j) = \{1\}.$$

Thus, we have the following diagram:

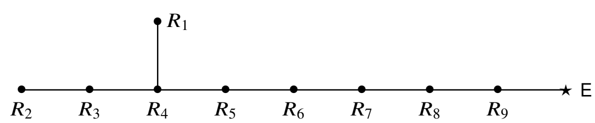


Fig. 4.3 Extremal rational elliptic surface: type $\tilde{E}_8$

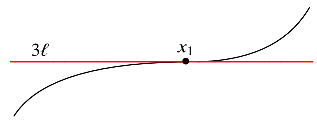
Starting from E, we blow down the 9 curves $R_9, \ldots, R_2$. The image of R_1 is a line ℓ taken with multiplicity 3. It is the tangent line of the image of a general fiber at its inflection point. Let $L = 0$ be the equation of this line and let $V(F_3)$ be the image of a nonsingular member if the fibration is elliptic and of an irreducible member otherwise. Our fibration is obtained from resolving the base points $x_9 \succ \cdots \succ x_1$ of the pencil of cubic curves $\lambda F_3 + \mu L^3 = 0$.

Choose projective coordinates, such that the equation of the pencil is given as

$$\lambda F_3 + \mu G_3 = \lambda(y^2z + c_1xyz + c_3yz^2 + x^3 + c_2x^2z + c_4xz^2) + \mu z^3 = 0. \tag{4.9.5}$$

Multiplying the equation by λ^5, replacing x by $\lambda^2 x$, and y by $\lambda^3 y$, we obtain an equation

$$y^2 + \lambda c_1xyz + \lambda^3c_3yz^2 + x^3 + \lambda^2c_2x^2z + \lambda^4c_4xz^2 + \lambda^5\mu z^3 = 0.$$



This equation is the global Weierstrass equation over $\mathbb{P}^1$ with coefficients

$$(a_1, a_2, a_3, a_4, a_6) = (\lambda c_1, \lambda^2 c_2, \lambda^3 c_3, \lambda^4 c_4, \lambda^5 \mu).$$

In characteristic $p \neq 2, 3$, we may assume that $c_1 = c_2 = c_3 = 0$. If $c_4 \neq 0$, then after scaling λ, we may assume that $c_4 = 1$. The coefficients in the Weierstrass equation become

$$(a_1, a_2, a_3, a_4, a_6) = (0, 0, 0, e\lambda^4, \lambda^5 \mu). \tag{4.9.6}$$

where $e = 0, 1$. The formula (4.4.14) and (4.4.24) gives

$$\Delta = -16\lambda^{10}(4e\lambda^2 + 27\mu^2), \quad j = 1728\frac{4e\lambda^2}{4e\lambda^2 + 27\mu^2}.$$

This shows that fibration has two irreducible singular fibers of type $\tilde{A}_0^*$ if $e = 1$ and one irreducible fiber of type $\tilde{A}_0^{**}$ if $e = 0$. In the latter case, we have $j = 0$.

Assume $p = 2$. If $c_1 \neq 0$, then replacing $c_1 x + c_3 \lambda^2 z^2$ by x, we may assume that $c_1 = 1$ and $c_3 = 0$. Replacing y by $y + \alpha\lambda x + \beta\lambda^2 z$, we may assume that $c_2 = c_4 = 0$.

The coefficients in the Weierstrass equation then become

$$(a_1, a_2, a_3, a_4, a_6) = (\lambda, 0, 0, 0, \lambda^5 \mu). \tag{4.9.7}$$

Computing the discriminants and the j-invariant, we find

$$\Delta = \lambda^{11}\mu, \quad j = \frac{\lambda}{\mu}.$$

Thus, we have two singular fibers, one of type $\tilde{E}_8$ and with wild ramification invariant $\delta = 1$ and the other of type $\tilde{A}_0^*$.

If $c_1 = 0$ and $c_3 \neq 0$, then we can arrange $c_3 = 1$ and $c_2 = c_4 = c_6 = 0$ and obtain that

$$(a_1, a_2, a_3, a_4, a_6) = (0, 0, \lambda^3, 0, 0, \lambda^5 \mu), \tag{4.9.8}$$

Thus, we have only one singular fibers of type $\tilde{E}_8$ with wild ramification invariant $\delta = 2$. The discriminant now is equal to λ^{12} and $j = 0$.

If $c_1 = c_3 = 0$, then the pencil is quasi-elliptic with Weierstrass equation

$$y^2 + x^3 + \lambda^4 x + \lambda^5 \mu = 0.$$

We have one reducible fiber of type $\tilde{E}_8$.

If $p = 3$, then we may assume that $c_1 = c_3 = 0$. If $c_2 \neq 0$ then after a linear transformation $x \mapsto x + \alpha\lambda^2$, we may assume that $c_4 = 0$. Thus,

$$(a_1, a_2, a_3, a_4, a_6) = (0, \lambda, 0, 0, \lambda^5\mu).$$

Computing the discriminant Δ and the j-invariant, we find that

$$\Delta = -\lambda^{11}\mu, \quad j = -\lambda/\mu.$$

In this case, we have two reducible fibers of types $\tilde{E}_8$ and $\tilde{A}_0^*$.

If $c_2 = 0$ and $c_4 \neq 0$, then after scaling we get

$$(a_1, a_2, a_3, a_4, a_6) = (0, 0, 0, \lambda^4, \lambda^5\mu).$$

and then

$$\Delta = \lambda^{12}, \quad j = 0.$$

This time, we have only one singular fiber of type $\tilde{E}_8$.

If $c_4 = 0$, then the pencil is quasi-elliptic with Weierstrass equation

$$y^2 + x^3 + \lambda^5\mu = 0.$$

We have one reducible fiber, which is of type $\tilde{E}_8$.

If $p = 2$ or $p = 3$ and if the fibration is quasi-elliptic, then we have $c_1 = c_3 = 0$ and Proposition 4.4.12 and Corollary 4.4.10 tell us that the fibration has only one reducible fiber, which is of type $\tilde{E}_8$.

- Type $\tilde{D}_8$

Applying the determinant formuls (4.3.4), we get

$$\mathrm{MW}(f) \cong \mathbb{Z}/2\mathbb{Z}.$$

Thus, we have the following diagram, where the curves E_1 and E_2 are sections.

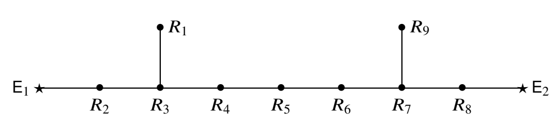
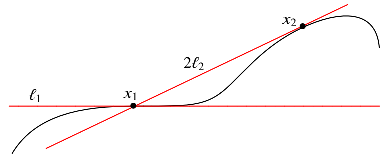


Fig. 4.4 Extremal rational elliptic surface: type $\tilde{D}_8$

First, we blow down E_1, R_2, R_3, R_4, R_5 and then, we blow down E_2, R_8, R_7, R_9 to points x_1 and x_2 in the plane. We obtain a birational morphism $\pi : J \to \mathbb{P}^2$. The images of R_1 and R_6 are a line ℓ_1 and a line ℓ_2, taken with multiplicity 2. The line ℓ_1 is the inflection tangent of the image of a general fiber at the point x_1, the line ℓ_2 is tangent to the image of a general fiber at the point x_2.



After a linear change of the parameters, we can write the equation of the pencil in the form

$$\lambda F_3 + \mu G_3 = \lambda(y^2z + c_1xyz + c_3yz^2 + x^3 + c_4xz^2 + c_6z^3) + \mu x^2 z = 0. \quad (4.9.9)$$

Since the line $x = 0$ is tangent to the curve $V(F_3)$, we must have

$$c_3^2 - 4c_6 = 0. \quad (4.9.10)$$

If $p \neq 2$, then we may assume that $c_1 = c_3 = 0$. Equation (4.9.10) gives $c_6 = 0$. After scaling the coordinates and λ, μ, we may assume that $c_4 = 1$. Arguing as in the previous case, we get a Weierstrass equation with

$$(a_1, a_2, a_3, a_4, a_6) = (0, \lambda\mu, 0, \lambda^4, 0). \quad (4.9.11)$$

The formula (4.4.14) gives

$$\Delta = 16\lambda^{10}(-4\lambda^2 + \mu^2), \quad j = 2^8 \frac{(-3\lambda^2 + \mu^2)^3}{\lambda^4(-4\lambda^2 + \mu^2)}. \quad (4.9.12)$$

In this case, there are two irreducible singular fibers of type A_0^*.

If $p \neq 2, 3$, then we can reduce the equation to a Weierstrass equation with $a_1 = a_2 = a_3 = 0$ and

$$(a_4, a_6) = (3u^2(3u^2 - v^2), u^3v(-9u^2 + 2v^2))$$

with $\Delta = 3^6u^{10}(4u^2 - v^2)$.

If $p = 2$, then it follows from (4.9.10) that $c_3 = 0$. If the pencil is elliptic, then $c_1 \neq 0$ and after scaling λ, we may assume that $c_1 = 1$. Replacing y by $y + c_4\lambda z$, we may assume that $c_4 = 0$. Thus, we obtain a Weierstrass equation with

$$(a_1, a_2, a_3, a_4, a_6) = (\lambda, \lambda\mu, 0, 0, c_6\lambda^6). \quad (4.9.13)$$

Here, $c_6 \neq 0$ since otherwise the curve is not regular. This gives

$$\Delta = \lambda^{12}c_6\lambda^{12}, \quad j = 1/c_6.$$

Thus, there is only one singular fiber and it has wild ramification index $\delta 2$. Note that the family depends on one parameter.

If $c_1 = 0$, then we may arrange $c_6 = 0$ and after a linear change $x \mapsto x + \lambda\mu z$, we get a quasi-elliptic fibration with Weierstrass equation

$$y^2 + x^3 + \lambda^2\mu^2 x + \lambda^5\mu = 0,$$

which has one reducible fiber of type $\tilde{D}_8$.

- Type $\tilde{A}_8$

Applying the determinant formula (4.3.4), we get

$$\mathrm{MW}(j) \cong \mathbb{Z}/3\mathbb{Z}.$$

We have the following diagram:

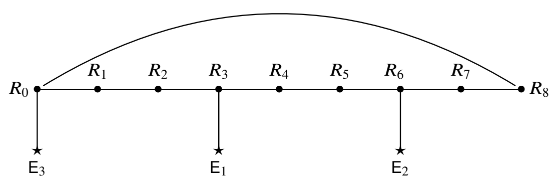
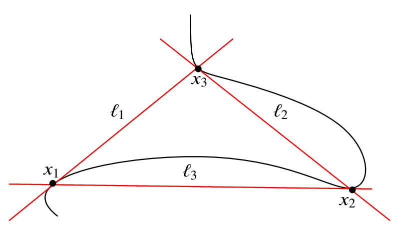


Fig. 4.5 Extremal rational elliptic surface: type $\tilde{A}_8$

We first blow down E_1, R_3, R_4, then E_2, R_6, R_7, and then E_3, R_0, R_8 to obtain a birational morphism $\pi : J \longrightarrow \mathbb{P}^2$. This is the blow up of 9 points $x_5 \succ x_4 \succ x_1, x_7 \succ x_6 \succ x_2, x_9 \succ x_8 \succ x_3$. The image of $R_2 + R_5 + R_8$ is a triangle of lines with vertices at x_1, x_2, x_3. The image of a general fiber is tangent to the sides of the triangle at its vertices. We can choose projective coordinates to write the equation of the pencil of cubics in the form

$$\lambda(y^2x + x^2z + z^2y) + \mu xyz = 0. \tag{4.9.14}$$



We conclude that the pencil is unique. In affine coordinates $u = x/z$, $v = y/z$, the equation is $\lambda uv^2 + u^2 + v + \mu uv = 0$. Multiplying by u and setting $w = uv$, we get the equation $\lambda w^2 + u^3 + w + \mu uw = 0$. Homogenizing again, we obtain that the generic fiber of the fibration is isomorphic to the Weierstrass curve

$$y^2 + \mu yx + \lambda^3 y + x^3 = 0. \tag{4.9.15}$$

We have

$$\Delta = -\lambda^9(27\lambda^3 + \mu^3), \quad j = -\mu^3(24\lambda^3 + \mu^3)^3/\lambda^9(27\lambda^3 + \mu^3).$$

If $p \neq 3$, then we have three additional singular fibers of type $\tilde{A}_0^*$.

If $p = 3$, then

$$\Delta = -\lambda^9\mu^3, \quad j = -\mu^9/\lambda^9.$$

In this case, there is only one additional fiber of type $\tilde{A}_0^{**}$ and it has wild ramification index $\delta = 1$.

If $p \neq 2, 3$, then we can reduce the equation to a standard Weierstrass form with

$$(a_4, a_6) = (-3\mu(\lambda^3 + 24\mu^3), -2(216\lambda^3 + 36\lambda^3\mu^3 + \mu^6).$$

Conversely, resolving the base points of the pencil (4.9.14), we obtain a genus one fibration on a rational surface with fibers of type $\tilde{A}_8$.

- Type $\tilde{E}_7 + \tilde{A}_1$

  Applying the determinant formula (4.3.4), we get

$$\mathrm{MW}(j) \cong \mathbb{Z}/2\mathbb{Z}.$$

We have the following diagram:

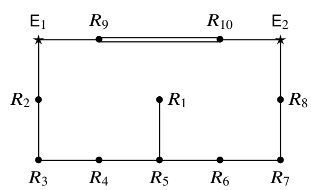


Fig. 4.6 Extremal rational elliptic surface: type $\tilde{E}_7 + \tilde{A}_1$

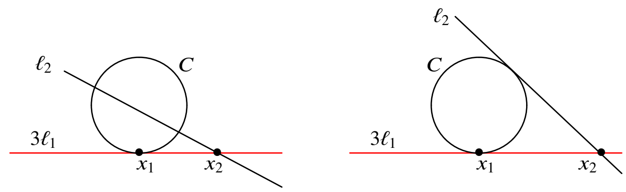
We blow down the curves $\mathsf{E}_1, R_2, R_3, R_4, R_5, R_1$ to a point x_1 and then, we blow down the curves E_2, R_8, R_7 to a point x_2. The image of the fiber of type $\tilde{E}_7$ is equal to the image of R_1. It is a line $\ell_1 = V(L_1)$ with multiplicity 3. The image of the fiber of type A_1 is the union of a nonsingular conic $C = V(Q)$ that is equal to the image of R_9 and a line $\ell_2 = V(L_2)$ that passes through $x_2 \in \ell_1$. The line ℓ_1 is tangent to C at the point x_1. This way, we obtain a pencil of cubic curves $V(\lambda Q L_2 + \mu L_1^3)$. We have to consider the following two possible cases: first, the fiber $R_9 + R_{10}$ is of type $\tilde{A}_1$ or second, it is of type $\tilde{A}_1^*$. In the first case, ℓ_2 and Q intersect transversally. In the second case they are tangent to each other. Fixing the equation of C, we are left with a 3-dimensional group of projective transformations that fixes the equations of the tangent line ℓ_1 and the line ℓ_2. In appropriate coordinates, we have

$$\lambda z((y+z)^2+xy)+\mu x^3 \;=\; \lambda(y^2z+2yz^2+yxz+z^3)+\mu x^3 \;=\; 0 \tag{4.9.16}$$

if ℓ_2 is not tangent to C and

$$\lambda z((y+z)^2+xz)+\mu x^3 \;=\; \lambda(y^2z+2yz^2+xz^2+z^3)+\mu x^3 \;=\; 0 \tag{4.9.17}$$

otherwise.



After replacing $y+\lambda^2\mu$ by y, the Weierstrass equation becomes

$$y^2+\lambda xy+x^3+\lambda^3\mu x \;=\; 0. \tag{4.9.18}$$

if ℓ_2 is not tangent to C and

$$y^2+x^3+\lambda^3\mu x \;=\; 0$$

otherwise. Moreover, we obtain

$$\Delta \;=\; \lambda^9\mu^2(\lambda-64\mu), \quad j \;=\; \frac{(\lambda-48\mu)^3}{\mu^2(\lambda-64\mu)},$$

in the former case and

$$\Delta \;=\; -64\lambda^9\mu^3, \quad j \;=\; 1728,$$

in the latter case. Assume that $p\neq 2$. Then, the pencil is elliptic with singular fibers of types $\tilde{E}_7, \tilde{A}_1, A_0^*$ or of types $\tilde{E}_7, \tilde{A}_1$. In the latter case, all nonsingular fibers are isomorphic. The standard Weierstrass equation is

$$y^2+x^3-3\lambda^3(\lambda-2\mu)x-2\lambda^5(\lambda-3\mu) \;=\; 0.$$

If $p=2$, then the first case is an elliptic pencil with

$$\Delta \;=\; \lambda^{10}\mu^2, \quad j \;=\; \lambda^2/\mu^2.$$

It has singular fibers of types $\tilde{E}_7$ (with $\delta=1$) and $\tilde{A}_1^*$. If $p=2$, then the second case is a quasi-elliptic pencil with Weierstrass equation

$$y^2+x^3+\lambda^3\mu x \;=\; 0. \tag{4.9.19}$$

It has two reducible fibers, which are of types $\tilde{E}_7$ and $\tilde{A}_1^*$.

- Type $\tilde{A}_7 + \tilde{A}_1$

Applying the discriminant formula (4.3.4), we get $\#\mathrm{MW}(j) = 4$. To choose one of the two possible groups of order 4, we use Lemma 4.9.8 that implies that $\mathrm{MW}(j)$ embeds in the discriminant group of each fiber (otherwise an element of the kernel intersects the zero section). This gives

$$\mathrm{MW}(j) \cong \mathbb{Z}/4\mathbb{Z}.$$

We have the following diagram:

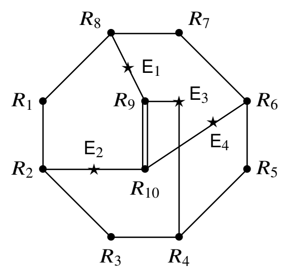
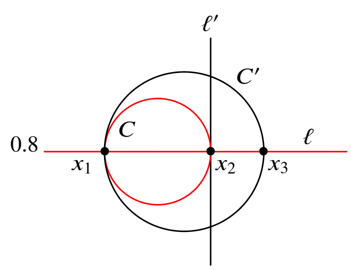


Fig. 4.7 Extremal rational elliptic surface: type $\tilde{A}_7 + \tilde{A}_1$

We blow down the exceptional configurations $\mathcal{E}_1 = E_1 + R_8 + R_7 + R_6 + R_5$, $\mathcal{E}_2 = E_2 + R_2 + R_3$, and $\mathcal{E}_3 = E_3$ to points x_1, x_2, x_3 in the plane. The image of R_1 in the plane is a conic C that passes through the points x_1, x_2. The image of R_4 is a line ℓ that passes through x_2, x_3. The image of R_9 is a conic C' that passes through x_1 and x_3. The image of R_{10} is a line ℓ' that passes through x_2. The line ℓ' is tangent to C at the point x_2. The conics C and C' are tangent at x_1 with multiplicity 4. We have the following picture:



We fix the equation of the conic C to be $yz + x^2 = 0$ and the equation of the line ℓ to be $x = 0$. The points x_1, x_2 have coordinates $[0, 0, 1]$ and $[0, 1, 0]$. The line ℓ' must now be equal to $V(z)$. The equation of a conic C' has to be of the form

$yz + x^2 + \alpha y^2 = 0$ with $\alpha \neq 0$. Using the transformation $(y, z) \mapsto (ty, t^{-1}z)$, we may assume that $\alpha = 1$. Note that ℓ' intersects C' at two distinct points if $p \neq 2$ and at one point if $p = 2$. Thus, we see that the fiber $R_9 + R_{10}$ is of type $\tilde{A}_1$ if $p \neq 2$ and of type $\tilde{A}_1^*$ if $p = 2$. Now, the pencil is uniquely determined up to a projective isomorphism, and its equation is

$$\lambda z(yz + x^2 + y^2) + \mu x(yz + x^2) = 0. \tag{4.9.20}$$

The Weierstrass equation is

$$y^2 + \mu xy + \lambda^2 \mu y + x^3 + \lambda^2 x^2 = 0. \tag{4.9.21}$$

Replacing $x + \lambda^2$ by x, we arrive at the equation

$$y^2 + \mu xy + x^3 - 2\lambda^2 x^2 + \lambda^4 x = 0$$

The discriminant and the absolute invariant are

$$\Delta = \lambda^8 \mu^2 (16\lambda^2 + \mu^2), \quad j = \frac{(16\lambda^4 + 16\lambda^2\mu^2 + \mu^4)^3}{\lambda^8\mu^2(16\lambda^2 + \mu^2)}.$$

The pencil is elliptic in all characteristics. If $p \neq 2$, then its singular fibers are of types $\tilde{A}_7, \tilde{A}_1, \tilde{A}_0^*, \tilde{A}_0^*$.

If $p = 2$, then we have

$$\Delta = \lambda^8 \mu^4, \quad j = \mu^8/\lambda^8,$$

and then, the pencil has two singular fibers of types $\tilde{A}_7$ and $\tilde{A}_1^*$.

If $p \neq 2, 3$, then we can transform the Weierstrass equation to an equation of the form

$$y^2 + x^3 + 3(\lambda^4 + 4u^2\mu^2 + v^4)x + (4\lambda^2 + \mu^2)(\lambda^4 - 8\lambda^2\mu^2 - 2\mu^4) = 0.$$

Conversely, resolving the base points of the pencil (4.9.20), we obtain a rational elliptic surface with reducible fibers of type $\tilde{A}_7, \tilde{A}_1$ or $\tilde{A}_7, \tilde{A}_1^*$.

- Type $\tilde{E}_6 + \tilde{A}_2$

  Applying the determinantal formula (4.3.4), we obtain

$$\mathrm{MW}(j) \cong \mathbb{Z}/3\mathbb{Z}.$$

We have the following diagram 4.8:

We blow down the curves E_0, R_1, R_2, then E_1, R_6, R_3, and then E_2, R_5, R_4 to the points x_1, x_2, x_3 in the plane. The image of R_0 is the image of the fiber of type $\tilde{E}_6$, which is a line ℓ in the plane of multiplicity 3. The image of the second reducible fiber is the triangle of lines, the images of R_7, R_8, R_9. The line ℓ does not pass through the vertices of the triangle. Thus, by choosing appropriate projective coordinates, we find that the equation of the pencil is

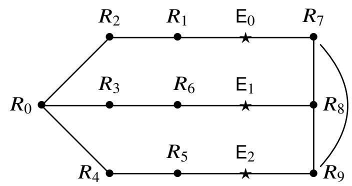
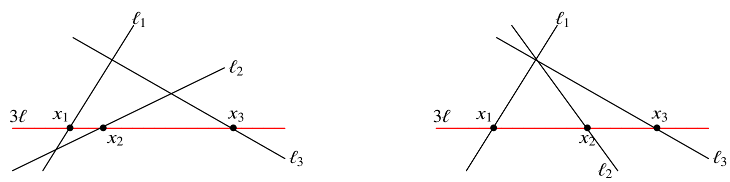


Fig. 4.8 Extremal rational elliptic surface: type $\tilde{E}_6 + \tilde{A}_2$

$$\lambda yz(y + z + \epsilon x) + \mu x^3 = \lambda(y^2 z + yz^2 + \epsilon xyz) + \mu x^3 = 0, \tag{4.9.22}$$

where $\epsilon = 1$ if the triangle consists of non-concurrent lines and 0 otherwise. The latter happens if the fiber $R_8 + R_9 + R_{10}$ is of type $\tilde{A}_2^*$.



Multiplying equation (4.9.22) by $\lambda^3\mu^2$ and changing $\lambda\mu x$ to x and $\lambda^2\mu y$ to y, we get the Weierstrass equation

$$y^2 + \lambda^2\mu y + \epsilon\lambda xy + x^3 = 0. \tag{4.9.23}$$

We have

$$\Delta = -\lambda^8\mu^3(\epsilon\lambda + 27\mu), \quad j = -\frac{\epsilon\lambda(\lambda + 24\mu)^3}{\mu^3(\epsilon\lambda + 27\mu)}.$$

Thus, if $e = 1$, we have three singular fibers of types $\tilde{E}_6, \tilde{A}_2$ and $\tilde{A}_0^*$. If $e = 0$, then we have two singular fibers of types $\tilde{E}_6$ and $\tilde{A}_3$.

If $p \neq 3$, then the fiber $F = R_8 + R_9 + R_{10}$ is of type $\tilde{A}_2$ or $\tilde{A}_2^*$. The latter happens if $\epsilon = 0$. Thus the singular fibers are of types $\tilde{E}_6, \tilde{A}_2, \tilde{A}_0^*$ or $\tilde{E}_6, \tilde{A}_2^*$.

If $p = 3$ and $\epsilon = 1$, we have

$$\Delta = \lambda^9\mu^3, \quad j = \lambda^3/\mu^3.$$

If $p = 3$ and $\epsilon = 0$, then the pencil is quasi-elliptic with Weierstrass equation

$$y^2 + x^3 + \lambda^4\mu^2 = 0.$$

It has two reducible fibers of types $\tilde{E}_6$ and $\tilde{A}_2^*$.

If $p \neq 2, 3$, we can reduce the Weierstrass equation to the form

$$y^2 + x^3 - 3\lambda^3(\lambda + 2\mu)x - 2u^4(2\lambda^2 + 3\lambda\mu + 18\mu^2) = 0.$$

Conversely, resolving the base points of the pencil (4.9.22), we obtain a rational surface with a genus one fibration with fibers of types described above.

- Type $\tilde{D}_5 + \tilde{A}_3$

Applying the determinantal formula (4.3.4), we obtain that $\#\mathrm{MW}(j) = 4$. As we explained earlier, Lemma 4.9.8 shows that

$$\mathrm{MW}(j) \cong \mathbb{Z}/4\mathbb{Z}.$$

We have the following diagram 4.9:

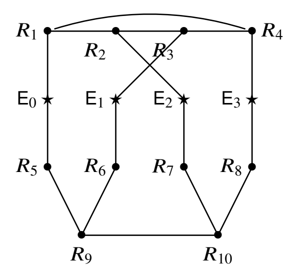
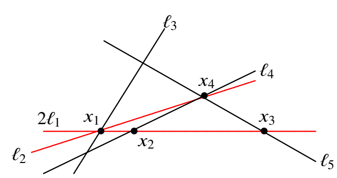


Fig. 4.9 Extremal rational elliptic surface: type $\tilde{D}_5 + \tilde{A}_3$

We blow down the curves $\mathsf{E}_1 + R_5 + R_9$, then E_2, R_3, then E_3, R_7, and, finally, E_4, R_8 to the points x_1, x_2, x_3, x_4 in the plane. The remaining curves in the diagram are mapped to lines. The curves R_6, R_{10} are mapped to a member of the pencil equal to the union of two lines $2\ell_1 + \ell_2$. The curves R_1, R_2, R_4 are mapped to a member of the pencil equal to the union of three non-concurrent lines $\ell_3 + \ell_4 + \ell_5$.



The equation of the pencil is

$$\lambda y(x+y)z + \mu x(x+z)^2 = \lambda xyz + y^2 z + \mu(x^3 + 2x^2 z + xz^2) = 0.$$

The Weierstrass equation is

$$y^2 + \lambda xy + x^3 + 2\lambda\mu x^2 + \lambda^2\mu^2 x = 0. \tag{4.9.24}$$

We have

$$\Delta = \lambda^7\mu^4(\lambda - 16\mu), \quad j = \frac{(\lambda^2 - 16\lambda\mu + 16\mu^2)^3}{\lambda\mu^4(\lambda - 4\mu)}.$$

In any case, the fibration is elliptic. If $p \neq 2$, then we have three singular fibers of types $\tilde{D}_5, \tilde{A}_3, A_0^*$.

If $p = 2$, we have a Weierstrass equation of the form

$$y^2 + \lambda xy + x^3 + \lambda^2\mu^2 x = 0$$

with $\Delta = \lambda^8\mu^4$ and $j = \frac{\lambda^4}{\mu^4}$. The fibration has two singular fibers of types $\tilde{D}_5$ with wild ramification invariant $\delta = 1$ and $\tilde{A}_3$.

If $p \neq 2, 3$, then we can reduce the Weierstrass equation to the form

$$y^2 + x^3 - 3u^2(u^2 - 4uv + v^2)x - 2u^3(u - 2v)(\lambda^2 - 4uv - 2v^2) = 0$$

with discriminant $\Delta = -\frac{1}{2^{12}}\lambda^7\mu^4(\lambda - 4\mu)$.

- Type $\tilde{D}_4 + \tilde{D}_4$.

Applying the determinantal formula 4.3.4, we obtain

$$\mathrm{MW}(j) \cong (\mathbb{Z}/2\mathbb{Z})^2.$$

We have the diagram (4.10) below.

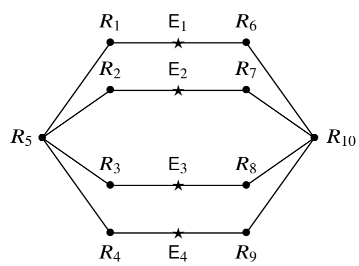


Fig. 4.10 Extremal rational elliptic surface: type $\tilde{D}_4 + \tilde{D}_4$

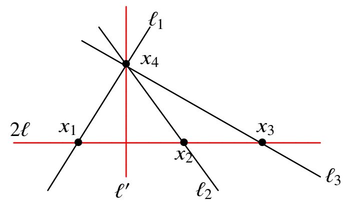
We blow down the curves E_1, R_1, R_5, then E_2, R_7, then E_3, R_8, and finally E_4, R_9. The image of the fiber $2R_5 + R_1 + R_2 + R_3 + R_4$ is the union of three concurrent lines ℓ_1, ℓ_2, ℓ_3, the image of the second reducible fiber is the union of two lines ℓ, ℓ', where one has to be taken with multiplicity two. The line with multiplicity two does not pass through the concurrency point. The equation of the pencil is

$$\lambda y^2 z + \mu x(x + z)(x + az) = \lambda y^2 z + \mu(x^3 + (a + 1)x^2 z + axz^2) = 0,$$

where $a \neq 0, 1$. The cross-ratio of the four lines $V(y), V(z), V(y + z), V(y + az)$ in the pencil of lines is an invariant of the surface.

The Weierstrass equation is

$$y^2 + x^3 + (a + 1)\lambda\mu x^2 + a\lambda^2\mu^2 x = 0 \tag{4.9.25}$$



and we have

$$\Delta = 16a^2(a-1)^2\lambda^6\mu^6, \quad j = \frac{2^8(a^2-a+1)^3}{a^2(a-1)^2}.$$

Note that we must have $a \neq 0, 1$, since otherwise the general member is not regular. If $p \neq 2$, then the fibration is elliptic with two singular fibers of type $\tilde{D}_4$.

If $p = 2$, then the fibration is quasi-elliptic with Weierstrass equation

$$y^2 + x^3 + (a^2 + a + 1)\lambda^2\mu^2 x + a(a+1)\lambda^3\mu^3 = 0.$$

There are no additional singular fibers.

If $p \neq 2, 3$, then the Weierstrass equation can be transformed to the form

$$y^2 + x^3 - 3(a^2 - a + 1)\lambda^2\mu^2 x + (a+1)(2a-1)(a-2)\lambda^3\mu^3 = 0$$

with discriminant Δ equal to $-3^6a^2(a-1)^2\lambda^6\mu^6$.

If $p = 2$, then the fibration is quasi-elliptic with two reducible fibers of type $\tilde{D}_4$. In this case, its Weierstrass equation is

$$y^2 + x^3 + \lambda^2\mu^2(a^2 + a + 1) + a(a+1)\lambda^3\mu^3 = 0.$$

- Type $\tilde{A}_4 + \tilde{A}_4$

Applying the discriminant formula 4.3.4, we obtain

$$\mathrm{MW}(j) \cong \mathbb{Z}/5\mathbb{Z}.$$

We have diagram 4.11:

Let $\sigma : X \to Y$ be the blow-down morphism of the curves $E_0, \ldots, E_4$. The images of the curves R_i, R_i' are (-1)-curves on Y, whose intersection graph is the Petersen graph.

Recall that the *Petersen graph* is the incidence graph of lines on a del Pezzo surface of degree 5, see [177, 8.5]. The surface is obtained by blowing up four points x_1, x_2, x_3, x_4 in the plane no three of which are collinear. The ten lines are the inverse transforms of the lines $\ell_{ij} = \overline{x_i, x_j}$. Our surface Y is isomorphic to a del Pezzo surface

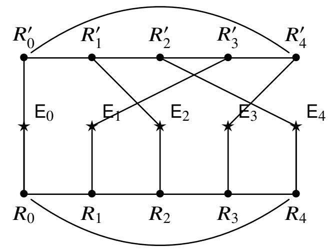


Fig. 4.11 Extremal rational elliptic surface: type $\tilde{A}_4 + \tilde{A}_4$

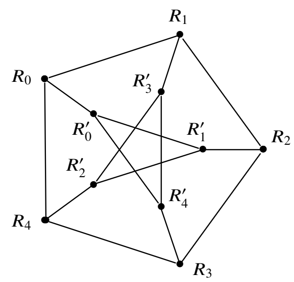


Fig. 4.12 Petersen graph

of degree 5 and we can blow down the curves R_0, R_2, R_4', R_2' on Y to four points in the plane.

We can also go to the plane in another way: we blow down $\mathsf{E}_0, R_0, R_1, R_2, R_3$ and $\mathsf{E}_4, R_2', R_3', R_4'$ to points x_1, x_2 in the plane. The image of the fiber $R_0 + \cdots + R_4$ is equal to the image of the curve R_4. It is a nodal cubic curve C with the node at x_1. The image of the fiber $R_0' + \cdots + R_4'$ is equal to the image of R_0' and R_1'. The image of R_0' is an irreducible conic Q that intersects C at x_1 with multiplicity 5 and passes through x_2. The image of R_1' is the inflection tangent line ℓ of C and the point x_2.

Choose the equation of C to be $x^3 + y^3 + xyz = 0$, such that $x_1 = [0, 0, 1]$ and $x_2 = [1, -1, 0]$ with the inflection tangent line $\ell = V(3x+3y-z)$. Then, the equation of the conic Q must be (up to switching x with y) $x^2 - y^2 + yz = 0$. This gives us the equation of the pencil:

$$\lambda(x^3 + y^3 + xyz) + \mu(x^2 - y^2 + yz)(3x + 3y - z) = 0. \tag{4.9.26}$$

After a linear change of variables $(x', y', z') = (x + y, y, 3x + 3y - z)$, the equation of the pencil becomes

$$\lambda(x^3 + y^2 z - xyz) + \mu(x^2 z + xyz - yz^2) = 0.$$

The Weierstrass equation is

$$y^2 + (-\lambda + \mu)xy - \lambda^2 \mu y + x^3 + \lambda\mu x^2 = 0, \tag{4.9.27}$$

which is always an elliptic pencil. We have

$$\Delta = \lambda^5\mu^5(-\lambda^2 - 11\lambda\mu + \mu^2), \quad j = \frac{(\lambda^4 + 12\lambda^3\mu + 14\lambda^2\mu^2 - 12\lambda\mu^3 + \mu^4)^3}{\Delta}.$$

The discriminant of $\lambda^2 + 11\lambda\mu - \mu^2$ is equal to 125. Thus if $p \neq 5$, then we have four singular fibers of types $\tilde{A}_4, \tilde{A}_4, \tilde{A}_0^*, \tilde{A}_0^*$. If $p = 5$, then we get three singular fibers of types $\tilde{A}_4, \tilde{A}_4, \tilde{A}_0^{**}$.

- Type $\tilde{D}_6 + \tilde{A}_1 + \tilde{A}_1$

  Applying the discriminant formula 4.3.4, we obtain

$$\mathrm{MW}(j) \cong (\mathbb{Z}/2\mathbb{Z})^2.$$

We have the following diagram 4.13:

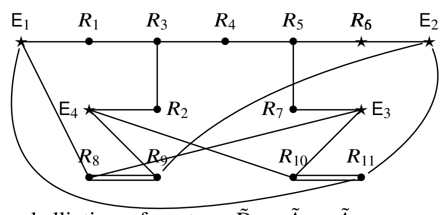


Fig. 4.13 Extremal rational elliptic surface: type $\tilde{D}_6 + \tilde{A}_1 + \tilde{A}_1$

We blow down the curves $\mathsf{E}_1, R_1, R_2, R_3, R_4$, then E_2, R_9, and then E_3, E_{10} to points x_1, x_2, x_3 in the plane. The image of the fiber of type $\tilde{D}_6$ is the union of three lines ℓ_1, ℓ_2, ℓ_3, which are equal to the images of R_2, R_6 and R_7, respectively. They intersect at the point x_1. The image of R_8 and R_{10} are singular irreducible cubics Q_1 and Q_2 with singular points at x_2, x_3, respectively. Each cubic passes simply through the node of the other and they are tangent with multiplicity 5 at x_1. The line ℓ_1 is the inflection tangent of Q_1 at the point x_1. The lines ℓ_2, ℓ_3 join the points x_1 with x_2 and x_3, respectively.

If $p \neq 2$, then in an appropriate coordinate system, the equation of the pencil can be written in the form

$$\lambda(y^2z + x^3 - x^2z) + \mu(x^2z - xz^2) = 0. \tag{4.9.28}$$

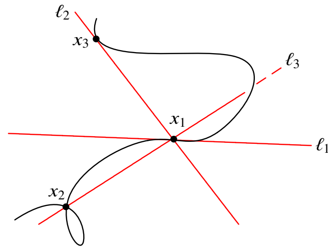
The Weierstrass equation is

$$y^2 + x^3 + (\mu\lambda - \lambda^2)x^2 - \mu\lambda^3 x = y^2 + x(x + \lambda\mu)(x - \lambda^2) = 0. \tag{4.9.29}$$

We get

$$\Delta = 16\lambda^8\mu^2(\lambda + \mu)^2, \quad j = 256\frac{(\lambda^2 + \lambda\mu + \mu^2)^3}{\lambda^2\mu^2(\lambda + \mu)^2}.$$

There are three singular fibers, which are of types $\tilde{D}_6, \tilde{A}_1, \tilde{A}_1$.



If $p = 2$, then the pencil is quasi-elliptic. The singular cubics Q_1 and Q_2 are cuspidal cubics. The equation of the pencil can be reduced to the form

$$\lambda(y^2z + x^3) + \mu(x^2z + xz^2) = 0.$$

The third singular fiber of the fibration corresponds to $\lambda = \mu = 1$. The Weierstrass equation can be transformed to the equation

$$y^2 + x^3 + \lambda\mu(\lambda + \mu)^2x + \lambda^4\mu^2 = 0.$$

Applying formula (4.4.33), we find that the pencil has three reducible fibers, which are of types $\tilde{D}_6$, $\tilde{A}_2^*$, and $\tilde{A}_2^*$.

- Type $\tilde{A}_5 + \tilde{A}_2 + \tilde{A}_1$

  Applying the discriminant formula 4.3.4, we obtain

$$\mathrm{MW}(j) \cong \mathbb{Z}/6\mathbb{Z}.$$

We have the following diagram (4.14).

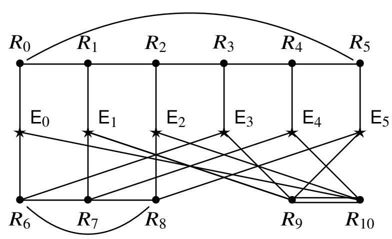


Fig. 4.14 Extremal rational elliptic surface: $\tilde{A}_5 + \tilde{A}_2 + \tilde{A}_1$

We blow down $\mathsf{E}_1 + R_1, E_3 + R_3, E_5 + R_5$ and E_0, E_2, E_4. The image of the fiber $R_0 + \cdots + R_5$ of type $\tilde{A}_5$ is a triangle of lines in the plane. The image of the fiber $R_6 + R_7 + R_8$ of type $\tilde{A}_2$ is another triangle of lines, whose sides pass through the vertices of the former triangle. The image of the fiber $R_9 + R_{10}$ of type $\tilde{A}_1$ is the union of a line that passes through the vertices of the second triangle and a conic which pass through the vertices of the first triangle and tangent direction equal to the sides of the second triangle. In appropriate coordinates, the equation of the pencil becomes

$$\lambda(x+y)(x+z)(y+z)+\mu xyz = 0. \tag{4.9.30}$$

Assume $p \neq 2$. We set $t := 2\lambda/\mu$ in order to avoid fractions in formulas. We apply a linear change of variables $(x', y', z') = (x-y, y+x, z)$, then multiply the equation by $(xt+z)$ and then replace y by $y(xt+z)$. We reduce the equation to the form

$$y^2 - t^2x^4 + 2t(t+1)x^3 + (t+1)^2x^2 + tx = 0.$$

Multiplying the equation by t^2/x^4, replacing ty/x^2 by y, $-t/x$ with x, we arrive at the Weierstrass equation

$$y^2 + x^3 - (1+t)^2x^2 + 2t^2(1+t)x - t^4 = y^2 + x^3 - ((1+t)x - t^2)^2 = 0.$$

Replacing y with $iy + (1+t)x + t^2$ and x with $-x$, we obtain

$$y^2 - 2(t+1)xy + 2t^2y + x^3 = 0.$$

We compute

$$\Delta = 16u^6v^3(4u+v)(u-2v)^2, \quad j = \frac{256(u+v)^3(u^3-3u^2v+3uv^2+v^3)^3}{u^6(4u+v)(u-2v)}.$$

If $p \neq 3$, then the fibration has one additional irreducible singular fiber of type $\tilde{A}_0^*$.

If $p = 3$, then we have three singular fibers of types $\tilde{A}_5, \tilde{A}_2, \tilde{A}_1^*$ and

$$\Delta = u^6v^3(u+v)^3, \quad j = \frac{(u+v)^9}{u^6v^3}.$$

If $p = 2$, then we take $t = \mu/\lambda$ and replace (x, y, z) by $(y, x, z+tx+y)$ and set $z = 1$, to obtain an equation

$$y^2 + y + txy + t(t+1)x^3 + x^2 + x = 0.$$

Multiplying by $t^2(t+1)^2$ and replacing $t(t+1)y$ by y and $t(t+1)x$ with x, we get a Weierstrass equation

$$y^2 + t(t+1)y + x^3 + x^2 + t(t+1)x = 0.$$

We have

$$\Delta = u^4v^6(u+v)^2, \quad j = \frac{u^8}{v^6(u+v)^2}.$$

We have three singular fibers, which are of types $\tilde{A}_1$, $\tilde{A}_2^*$, and $\tilde{A}_5$.

- Type $\tilde{A}_3 + \tilde{A}_3 + \tilde{A}_1 + \tilde{A}_1$

  Applying the discriminant formula 4.3.4, we obtain

$$\mathrm{MW}(j) \cong (\mathbb{Z}/2\mathbb{Z}) \oplus (\mathbb{Z}/4\mathbb{Z}).$$

Note that we have ${}_2\,\mathrm{MW}(J/C) \cong (\mathbb{Z}/2\mathbb{Z})^{\oplus 2}$ and that an elliptic curve in characteristic 2 does not contain such a 2-torsion group, we see an elliptic fibration of this type can exist only if $p \neq 2$. Moreover, a quasi-elliptic unipotent group does not contain a subgroup isomorphic to $\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/4\mathbb{Z}$. Thus, there exists no quasi-elliptic fibration of this type.

Being an extremal fibration, all sections are torsion sections and we have just seen that such fibrations are elliptic. In particular, all sections of j are disjoint by Lemma 4.9.8. Moreover, it follows from the formula for the Euler–Poincaré characteristic that there are no irreducible singular fibers.

We have the diagram (4.15) below.

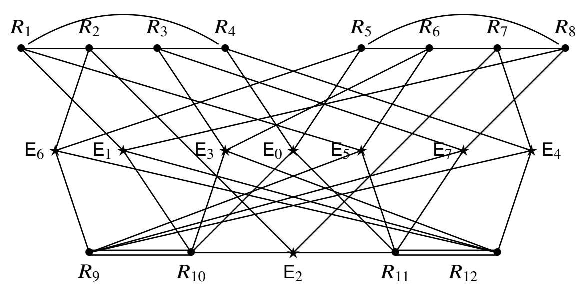


Fig. 4.15 Extremal rational elliptic surface: type $\tilde{A}_3 + \tilde{A}_3 + \tilde{A}_1 + \tilde{A}_1$

We blow down the curves E_0, R_4, then E_2, R_7, then $\mathsf{E}_1, \mathsf{E}_3, \mathsf{E}_5, \mathsf{E}_6, \mathsf{E}_7$ to points $x_1, \ldots, x_7$. The images of the fibers F_1 are two triangles of lines. One side of one triangle passes through the vertex of the another one. In appropriate coordinates, we can write the equation of the pencil in the form

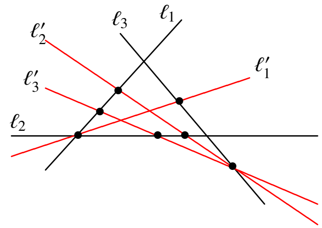
$$\lambda xyz + \mu(x + ay)(x + y + z)(x + y + bz) = 0,$$

where $b \neq 1$. We also have to use the condition that the point x_6 is collinear with two other points not on the same side of triangles. An easy check gives $a = b = -1$. Thus, the pencil is unique and has the equation

$$\lambda xyz + \mu(x - y)(x + y + z)(x + y - z) = 0. \tag{4.9.31}$$

One checks that the members of the pencil with $\lambda/\mu = \pm 4$ are singular. They are unions of a line and a conic given by equations $((x + y)^2 + z(y - z))(x - y + z) = 0$ and $((x + y)^2 + z(z - y))(x - y - z) = 0$. The conics are tangent at the base points of multiplicity 2.

To compute the Weierstrass form, we first make the variable change $x' = x - y$, $y' = x + y$ and $\mu' = 4\mu$, to transform the equation to the form



$$\lambda(y^2-x^2)z+\mu x(y^2-z^2) \;=\; y^2(\mu x+\lambda z)-xz(\lambda x+\mu z) \;=\; 0.$$

Multiplying by $\mu x+\lambda z$, replacing y by $y'=y(\mu x+\lambda z)$, and dividing by z^4, we get

$$y^2-x(\lambda x+\mu z)(\mu x+\lambda z) \;=\; y^2-\lambda\mu x^3-(\lambda^2+\mu^2)x^2-\lambda\mu x \;=\; 0.$$

Multiplying by $-\lambda^2\mu^2$, replacing y by $\lambda\mu y$, and replacing x by $-\lambda\mu$, we get a Weierstrass equation

$$y^2+x^3+(\lambda^2+\mu^2)x^2+\lambda^2\mu^2 x \;=\; 0. \tag{4.9.32}$$

We compute the discriminant and the absolute invariant to obtain

$$\Delta \;=\; 16\lambda^4\mu^4(\lambda-\mu)^2(\lambda+\mu)^2, \quad j \;=\; \frac{2^8(\lambda^4-\lambda^2\mu^2+\mu^4)^3}{\lambda^4\mu^4(\lambda^2-\mu^2)^2}.$$

If $p=2$, then the general fiber is not a regular curve. There are no irreducible singular fibers.

If $p\neq 2,3$, then we can transform the Weierstrass equation to the form

$$y^2-3(\lambda^4-\lambda^2\mu^2+\mu^4)x-3\lambda^4\mu^2+3(\lambda^2+\mu^2)(2\lambda^2-\mu^2)(\lambda^2-2\mu^2) \;=\; 0$$

with discriminant $\Delta=-\lambda^4\mu^4(\lambda^2-\mu^2)^2$.

- Type $\tilde{A}_2+\tilde{A}_2+\tilde{A}_2+\tilde{A}_2$

  Applying the discriminant formula 4.3.4, we obtain

$$\mathrm{MW}(j) \;\cong\; (\mathbb{Z}/3\mathbb{Z})^2.$$

Before entering the detailed analysis of this case, we note that if $p\neq 3$, then this type can be realized by the famous *Hesse pencil*, see [177, 3.1] or [16]:

$$\lambda(x^3+y^3+z^3)+\mu xyz \;=\; 0, \tag{4.9.33}$$

which we will encounter again several times in Volume II. The blow up of its nine base points is the universal elliptic curve with a fixed basis of 3-torsion points. The picture of the surface is given by the following Levi graph of the configuration of lines and points in the affine plane $\mathbb{A}^2(\mathbb{F}_3)$ over the field $\mathbb{F}_3$ of 3 elements.

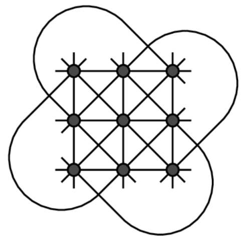


Fig. 4.16 Extremal rational elliptic surface: type $\tilde{A}_3 + \tilde{A}_3 + \tilde{A}_3 + \tilde{A}_3$

We now analyze this type in detail. First, let us find the Weierstrass form. Replacing λ/μ by s, then making the variable change $(x, y, z) \mapsto (x + y, -y, -z + 3sx)$, and finally dehomogenizing with respect to z, we get the equation

$$y^2 + xy + s(x^3 + (-1 + 3sx)^3) = 0.$$

Multiplying both sides by $s^2(1 + 27s^3)^2$ and replacing x by $s(1 + 27s^3)x$ and y by $s(1 + 27s^3)y)$, we get the equation

$$y^2 + yx + x^3 - 27s^3x^2 + 9s^3(1 + 27s^3)x - s^3(1 + 27s^3)^2 = 0.$$

Next, after a linear change $(x, y) = (x' + 9s^3, y' - x' - 6s^3)$, we obtain the Weierstrass equation

$$y^2 + vxy + 3u^3y + x^3 + 6u^3vx + 9u^6 - u^3v^3 = 0 \tag{4.9.34}$$

Using formula (4.4.14), we find

$$\Delta = u^3(v^3 - 27u^3)^3, \quad j = \frac{v^3(v^3 - 216u^3)^3}{u^3(v^3 - 27u^3)^3}.$$

This means that the singular fibers are over the points $v/u = \infty, -3, -3\omega, -3\omega^2$, where $\omega^3 = 1$ and $\omega \neq 1$. It is known that the singular members of the Hesse pencil (4.9.33) correspond to the same values of the parameter t.

Assume $p \neq 2, 3$. Then, after homogenizing the parameter and change (u, v) to $(u/6, v)$, we can rewrite the Weierstrass equation in terms of $\lambda = t/6$, and then, it acquires the following form:

$$y^2 + x^3 + 12v(u^3 - v^3)x + 2(u^6 - 20u^3v^3 - 8v^6) = 0 \tag{4.9.35}$$

with

$$\Delta = -2^6 3^3 u^3(u^3 + 8v^3)^3, \quad j = 2^{12}3^3 \frac{v^3(u^3 - v^3)}{u^3(u^3 + 8v^3)^3}.$$

This is the classical way to write the Weierstrass equation of the Hesse pencil

$$x^3 + y^3 + z^3 + 6txyz = 0,$$

where $t = v/u$, see [16].

Assume $p = 2$. Then, after substituting $(x, y) \mapsto (x + s^3, y + 1)$, we can rewrite the equation in the form

$$y^2 + vxy + u^3y + x^3 + u^3(u^3 + v^3) = 0. \tag{4.9.36}$$

We have

$$\Delta = u^3(u^3 + v^3)^3, \quad j = \frac{v^{12}}{u^3(u^3 + v^3)^3}.$$

Suppose $p = 3$. Note that the reduction of the Hesse pencil modulo 3 is the pencil

$$(x + y + z)^3 + txyz = 0.$$

Its fiber over $t = 0$ is of type $\tilde{E}_6$ and its fiber over $t = \infty$ is of type $\tilde{A}_2$. Thus, it is an extremal elliptic fibration of type $\tilde{E}_6 + \tilde{A}_2$, which have been already studied above. In particular, type under consideration, that is, $\tilde{A}_2 + \tilde{A}_2 + \tilde{A}_2 + \tilde{A}_2$, does not arise from the Hesse pencil. Instead, we have to look at a quasi-elliptic pencil in this case. It must have four reducible fibers of type $\tilde{A}_2^*$. It follows from (4.3) that the Weierstrass equation

$$y^2 + x^3 + a_6(u, v) = 0,$$

where da_6 has four simple zeros. The polynomial $a_6(u, v) = \sum_{i=0}^{6} a_{6-i}v^i$ is considered up to addition of a cube of a binary quadratic form. First, for a general form a_6, one can find a linear transformation of the variables to assume that $a_1 = a_5 = 0$, see [217, p. 287]. Then, after possibly adding a cube, we assume that $a_0 = a_3 = a_6 = 0$. After scaling, we reduce the polynomial to the form $u^2v^2(u^2+v^2)$. In the affine subset with coordinate $t = u/v$, we have $da_6 = d(t^4 + t^2) = (t^3 - t)dt$, so the differential has 3 simple roots at $t = 0, 1, -1$. In the open subset with affine coordinate $t = v/u$, it has additional simple root at $t = 0$. Thus, it has 4 simple roots and the order of the discriminant is equal to 2 at each root. By (4.1), we see that the fibration has four fibers of type $\tilde{A}_2^*$ with Weierstrass equation

$$y^2 + x^3 + u^2v^2(u^2 + v^2) = 0.$$

This suggests that there must be a unique quasi-elliptic fibration with such fibers. To confirm this, we use the conic bundle argument that the nine sections must be disjoint. Thus, the fibration comes from a pencil of cubic curves with nine distinct base points. It has four reducible members, each of which is equal to the union of three concurrent lines. It is easy to check that such a pencil is unique up to a linear transformation, and that it coincides with the pencil

$$\lambda y(x^2 - y^2) + \mu z(x^2 - z^2) = 0. \tag{4.9.37}$$

It singular members are four triples of concurrent lines

$$\begin{gathered} V(z(x-z)(x+z)), \quad V(y(x-y)(x+y)), \\ V((y-z)(x+y-z)(x-y+z)), V((y+z)(-x+y+z)(x+y+z)). \end{gathered} \tag{4.9.38}$$

It has nine base points

$$[1,0,0],\ [1,1,0],\ [1,-1,0],\ [1,0,1],\ [1,1,1],\ [1,-1,1],\ [1,0,-1],\ [1,1,-1],\ [1,-1,-1].$$

The 9 points and the 12 lines corresponding to the irreducible components of reducible fibers form a Hesse configuration $(12_3, 9_4)$. We can view the points as points in the projective plane $\mathbb{P}^2(\mathbb{F}_3)$ with coordinates (t_0, t_1, t_2) lying outside the line $t_0 = 0$ and take all lines in the plane except this one. In particular, the Hesse configuration can be realized over the finite field $\mathbb{F}_3$.

Note that the singular points of the reducible members of the pencil are $[0,1,0], [0,0,1], [0,1,-1]$. They are points at the line $t_0 = 0$. The line is the image of the curve of cusps on the rational quasi-elliptic surface.

We summarize our investigation with the following table. Even in characteristic $p \neq 2, 3$, for typographical reasons, we sometimes give a Weierstrass form with nonzero coefficients $a_1, a_2, a_3 \neq 0$. One can use formula (4.4.14) to recompute the coefficients satisfying $a_1 = a_2 = a_3 = 0$. Also, for typographical reason, we dehomogenize the coordinates (u, v) on the base $\mathbb{P}^1$.

Singular fibers	MW	Weierstrass equation	Δ	j
$\tilde{E}_8, 2\tilde{A}_0^*$	$\{0\}$	$y^2 + x^3 + x + t$	$-16(4 + 3^3t^2)$	$-\frac{2^{10}3^3}{4+3^3t^2)}$
$\tilde{E}_8, \tilde{A}_0^{**}$	$\{0\}$	$y^2 + x^3 + t$	t	0
$\tilde{D}_8, 2\tilde{A}_0^*$	$\mathbb{Z}/2\mathbb{Z}$	$y^2 + x^3 + tx^2 + x$	$16(t^2 - 4)$	$\frac{2^8(t^2-3)^3}{(t^2-4)}$
$\tilde{A}_8, 3\tilde{A}_0^*$	$\mathbb{Z}/3\mathbb{Z}$	$y^2 + txy + y + x^3$	$-(27 + t^3)$	$\frac{t^3(24+t^3)^3}{27+t^3}$
$\tilde{E}_7, \tilde{A}_1, \tilde{A}_0^*$	$\mathbb{Z}/2\mathbb{Z}$	$y^2 + xy + x^3 + x$	$t^2(1 - 64t)$	$\frac{(1-48t)^3}{t^2(1-64t)}$
$\tilde{E}_7, \tilde{A}_1^*$	$\mathbb{Z}/2\mathbb{Z}$	$y^2 + x^3 + tx$	$-64t^3$	1728
$\tilde{A}_7, \tilde{A}_1, 2\tilde{A}_0^*$	$\mathbb{Z}/4\mathbb{Z}$	$y^2 + txy + x(x-1)^2$	$t^2(16 + t^2)$	$\frac{(16+16t^2+t^4)^3}{\Delta}$
$\tilde{E}_6, \tilde{A}_2, \tilde{A}_0^*$	$\mathbb{Z}/3\mathbb{Z}$	$y^2 + xy + x^3 + tx$	$-t^3(1 + 27t)$	$-\frac{(1+24t)^3}{t^3(1+27t)}$
$\tilde{E}_6, \tilde{A}_2^*$	$\mathbb{Z}/3\mathbb{Z}$	$y^2 + ty + x^3$	$-27t^4$	0
$\tilde{D}_5, \tilde{A}_3, \tilde{A}_0^*$	$\mathbb{Z}/4\mathbb{Z}$	$y^2 + txy + x(x+t)^2$	$t^4(1 - 16t)$	$\frac{(1-16t+16t^2)^3}{t^4(1-16t)}$
$\tilde{D}_4, \tilde{D}_4$	$(\mathbb{Z}/2\mathbb{Z})^{\oplus 2}$	$y^2 + x^3 + (a+1)tx^2 + at^2x$	$16a^2(a-1)^2t^6, a \neq 0, 1$	$\frac{256(a^2-a+1)^3}{a^2(a-1)^2}$
$2\tilde{A}_4, 2\tilde{A}_0^*$	$\mathbb{Z}/5\mathbb{Z}$	$y^2 + (t-1)xy - ty + x^3 + tx^2$	$t^5(t^2 - 11t - 1)$	$\frac{(1+12t+14t^2-12t^3+t^4)^3}{\Delta}$
$2\tilde{A}_4, \tilde{A}_0^{**}, (p = 5)$	$\mathbb{Z}/5\mathbb{Z}$	$y^2 + (t-1)xy - ty + x^3 + tx^2$	$-t^5(t-3)^2$	$\frac{t-3}{t^5}$
$\tilde{D}_6, 2\tilde{A}_1$	$(\mathbb{Z}/2\mathbb{Z})^2$	$y^2 + x(x+1)(x-t)$	$16t^2(1+t)^2$	$\frac{(1+t+t^2)^3}{t^2(1+t)^2}$
$\tilde{A}_5, \tilde{A}_2, A_1, A_0^*$	$\mathbb{Z}/6\mathbb{Z}$	$y^2 - 2(1+t)xy + t^2y + x^3$	$16t^6(t-2)^2(1+4t)$	$\frac{2^{12}(1+t)^3(3+3t-3t^2-t^3)^3}{\Delta}$
$2\tilde{A}_3, 2\tilde{A}_1$	$\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/4\mathbb{Z}$	$y^2 + x^3 + (1+t^2)x^2 + t^2x$	$16t^4(1-t^2)^2$	$\frac{256(1-t^2+t^4)^3}{\Delta}$
$4\tilde{A}_2$	$(\mathbb{Z}/3\mathbb{Z})^{\oplus 2}$	$y^2 + x^3 + 12t(1-t^3)x + 2(1 - 20t^3 - 8t^6)$	$-2^63^3(1+8t^3)^3$	$\frac{2^{12}3^3t^3(1-t^3)^3}{(1+8t^3)^3}$

Table 4.6 Extremal elliptic rational surfaces ($p \neq 2, 3$)

Singular fibers	MW	Weierstrass equation	Δ	j
$\tilde{E}_8, \tilde{A}_0^*$	$\{0\}$	$y^2 + xy + x^3 + t$	t	$\frac{1}{t}$
$\tilde{E}_8$	$\{0\}$	$y^2 + ty + x^3 + t^5$	t^{12}	0
$\tilde{D}_8$	$\mathbb{Z}/2\mathbb{Z}$	$y^2 + txy + x^3 + tx^2 + at^4, a \neq 0$	at^{12}	a^{-1}
$\tilde{A}_8, 3\tilde{A}_0^*$	$\mathbb{Z}/3\mathbb{Z}$	$y^2 + txy + y + x^3$	$1 + t^3$	$\frac{t^{12}}{1+t^3}$
$\tilde{E}_7, \tilde{A}_1$	$\mathbb{Z}/2\mathbb{Z}$	$y^2 + xy + x^3 + tx$	t^2	$\frac{1}{t^2}$
$\tilde{A}_7, \tilde{A}_1^*$	$\mathbb{Z}/4\mathbb{Z}$	$y^2 + txy + x^3 + x$	t^4	t^8
$\tilde{E}_6, \tilde{A}_2, \tilde{A}_0^*$	$\mathbb{Z}/3\mathbb{Z}$	$y^2 + xy + ty + x^3$	$t^6(1+t)^4$	$\frac{1}{t^3(1+t)}$
$\tilde{E}_6, \tilde{A}_2^*$	$\mathbb{Z}/3\mathbb{Z}$	$y^2 + ty + x^3$	t^4	0
$\tilde{D}_5, \tilde{A}_3$	$\mathbb{Z}/4\mathbb{Z}$	$y^2 + xy + x^3 + t^2x$	t^4	$\frac{1}{t^4}$
$2\tilde{A}_4, 2\tilde{A}_0^*$	$\mathbb{Z}/5\mathbb{Z}$	$y^2 + x^3 + (t+1)xy + ty + tx^2)$	$t^5(1+t+t^2)$	$\frac{(1+t^4)^3}{\Delta}$
$\tilde{A}_5, \tilde{A}_2^*, A_1$	$\mathbb{Z}/6\mathbb{Z}$	$y^2 + x^3 + (t+1)(xy + ty + tx^2)$	$t^6(1+t)^4$	$\frac{t^8}{t^6(t+1)^2}$
$4\tilde{A}_2$	$(\mathbb{Z}/3\mathbb{Z})^{\oplus 2}$	$y^2 + txy + y + x^3 + 1 + t^3$	$(1+t^3)^3$	$\frac{t^{12}}{1+t^3)^3}$

Table 4.7 Extremal elliptic rational surfaces ($p = 2$)

Singular fibers	MW	Weierstrass equation	Δ	j
$\tilde{E}_8, \tilde{A}_0^*$	$\{0\}$	$y^2 + x^3 + x^2 + t$	$-t$	$0\frac{1}{t}$
$\tilde{E}_8$	$\{0\}$	$y^2 + x^3 + x + t$	1	0
$\tilde{D}_8, 2\tilde{A}_0^*$	$\mathbb{Z}/2\mathbb{Z}$	$y^2 + x^3 + tx^2 + x$	$t^2 - 1$	$\frac{t^6}{t^2-1}$
$\tilde{A}_8, \tilde{A}_0^*$	$\mathbb{Z}/3\mathbb{Z}$	$y^2 + txy + y + x^3$	$-t^3$	$-t^9$
$\tilde{E}_7, \tilde{A}_1, \tilde{A}_0^*$	$\mathbb{Z}/2\mathbb{Z}$	$y^2 + xy + x^3 + tx$	$t^2(1-t)$	$\frac{1}{t^2(1-t)}$
$\tilde{E}_7, \tilde{A}_1^*$	$\mathbb{Z}/2\mathbb{Z}$	$y^2 + x^3 + tx$	$-t^3$	0
$\tilde{A}_7, \tilde{A}_1, 2\tilde{A}_0^*$	$\mathbb{Z}/4\mathbb{Z}$	$y^2 + txy + x(x-1)^2$	$t^2(t^2+1)$	$\frac{(1-t^2)^6}{\Delta}$
$\tilde{E}_6, \tilde{A}_2$	$\mathbb{Z}/3\mathbb{Z}$	$y^2 + x^3 + xy + y + x^3$	t^3	$\frac{1}{t^3}$
$\tilde{D}_5, \tilde{A}_3, \tilde{A}_0^*$	$\mathbb{Z}/4\mathbb{Z}$	$y^2 + xy + x(x+t)^2$	$t^4(1-t)$	$\frac{(1-t)^6}{\Delta}$
$\tilde{D}_4, \tilde{D}_4$	$(\mathbb{Z}/2\mathbb{Z})^{\oplus 2}$	$y^2 + x^3 + (a+1)tx^2 + at^2x, a \neq 0, 1$	$a^2(a-1)^2t^6$	$\frac{(a+1)^6}{a^2(a-1)^2}$
$2\tilde{A}_4, 2\tilde{A}_0^*$	$\mathbb{Z}/5\mathbb{Z}$	$y^2 + (t-1)xy - ty + x^3 + tx^2$	$t^5(t^2+t-1)$	$\frac{((1-t)^6}{\Delta}$
$\tilde{D}_6, 2\tilde{A}_1$	$(\mathbb{Z}/2\mathbb{Z})^2$	$y^2 + x(x-1)(x-t)^3$	$t^2(1+t)^2$	$\frac{(1+t+t^2)^3}{t^2(1+t)^2}$
$\tilde{A}_5, \tilde{A}_2, A_1^*$	$\mathbb{Z}/6\mathbb{Z}$	$y^2 + (1+t)xy + (1-t)ty + x^3 + (1-t)x^2$	$t^3(t-1)^3$	$\frac{t^9}{t^3(t-1)^3}$
$2\tilde{A}_3, 2\tilde{A}_1$	$\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/4\mathbb{Z}$	$y^2 + x^3 + (1+t^2)x^2 + t^2x$	$t^4(1-t^2)^2$	$\frac{(1+t^2)^6}{\Delta}$

Table 4.8 Extremal elliptic rational surfaces ($p = 3$)

As a result of our explicit classification, we now draw a couple of conclusions. For example, by inspection of the table we find the following.

Corollary 4.9.10 *Let $f : J \to \mathbb{P}^1$ be an extremal jacobian elliptic fibration on a rational surface. Assume that the j-invariant is constant. Then one of the following cases occurs:*

1. $p \neq 2, 3$

- $j = 0$ *and f has two singular fibers of types $\tilde{E}_8$ and $\tilde{A}_0^{**}$.*
- $j = 1728$ *and f has two singular fibers of types $\tilde{E}_7$ and $\tilde{A}_1^*$.*
- $j = 0$ *and f has two singular fibers of types $\tilde{E}_6$ and $\tilde{A}_2^*$.*

- *j is constant depending on the parameter t of the elliptic fibration with two singular fibers of types $\tilde{D}_4$.*

2. $p = 2$

- $j = 0$ *and f has one singular fiber of types $\tilde{E}_8$.*
- $j = c \neq 0$ *and f has one singular fiber of types $\tilde{D}_8$.*
- $j = 0$ *and f has two singular fibers of types $\tilde{E}_6$ and $\tilde{A}_2^*$.*

3. $p = 3$

- $j = 0$ *and f has one singular fiber of types $\tilde{E}_8$.*
- $j = 0$ *and f has two singular fibers of types $\tilde{E}_7$ and $\tilde{A}_1^*$.*
- *j is a constant depending on the parameter t of the elliptic fibration with two singular fibers of types $\tilde{D}_4$.*

We can also give the table for quasi-elliptic fibrations. It follows from Proposition 4.4.12 that the Euler–Poincaré characteristics of reducible fibers add up to $8 + 2k$, where k is the number of reducible fibers.

Suppose $p = 3$. Then a possible reducible fiber must be of type $\tilde{E}_8, \tilde{E}_6$ or $\tilde{A}_2^*$. This allows us to list all possible configurations of reducible fibers

$$\tilde{E}_8, \quad \tilde{E}_6 + \tilde{A}_2^*, \quad 4\tilde{A}_2^*.$$

All such configurations occur on extremal rational elliptic surfaces and they occur as specialization of such surfaces in characteristic 3. We discussed these specializations in our classification of extremal rational elliptic surfaces.

Suppose that $p = 2$. Then a possible reducible fiber must be of type $\tilde{E}_8, \tilde{E}_7, \tilde{D}_{2k}, \tilde{A}_1^*$. We also know that the Mordell–Weil group is finite, that is, a quasi-elliptic fibration is automatically extremal, hence $\sum_{t \in C}(\#\,\mathrm{Irr}(J_t) - 1) = 8$. This allows us to list all possible configurations of reducible fibers. They are

$$\tilde{E}_8, \quad \tilde{D}_8, \quad \tilde{E}_7 + \tilde{A}_1^*, \quad 2\tilde{D}_4, \quad \tilde{D}_6 + 2\tilde{A}_1^*, \quad \tilde{D}_4 + 4\tilde{A}_1^*, \quad 8\tilde{A}_1^*.$$

Note that the first five cases have been already discussed in our analysis of extremal rational elliptic surfaces. The last two cases are new, and are not realized as elliptic fibrations on rational surfaces. Let us realize these two new cases.

- Type $\tilde{D}_4 + 4\tilde{A}_1^*$ (quasi-elliptic and $p = 2$)

For this type, the Mordell–Weil group $\mathrm{MW}(j)$ is isomorphic to $(\mathbb{Z}/2\mathbb{Z})^3$.

Let $F_1 = R_0 + R_1 + R_2 + R_3 + 2R_4$ be the fiber of type $\tilde{D}_4$ and $F_2 = R_5 + R_6, F_3 = R_7 + R_8, F_4 = R_9 + R_{10}, F_5 = R_{11} + R_{12}$ be other reducible fibers. Each $R_0, \ldots, R_3$ intersects two sections and each $R_i, i = 5, \ldots, 10$ intersects four sections. We assume that the zero section E_0 intersects R_0. Let E_1 intersect R_0, let E_2, E_3 intersect R_1, and the rest intersect R_2, R_3. We may assume that E_0 intersects R_5, R_7, R_9, R_{11} and we will call them the zero component. It follows from the computations of local contributions for the height of sections in the Mordell–Weil group that E_1 intersects

R_6, R_8, R_{10} and R_{12}. Also, E_2, E_3 intersect two zero components of fibers of type $\tilde{A}_1^*$ and other sections intersect three zero components of these fibers.

We blow down E_0, R_0, R_4 to a point x_0 and then blow down $E_2, \ldots, E_8$ to six points $x_1, \ldots, x_6$. The image of F_1 is the union of three lines passing through x_0. The images of the zero components of $F_2, \ldots, F_5$ are conics $C_1, \ldots, C_4$ that intersect each other with multiplicity 3 at the point x_0. The images of the other components of these fibers are lines $L_1, \ldots, L_4$, each passing through two points $x_1, \ldots, x_6$ and intersecting only at one these points. Each line L_i is tangent to the conic C_i.

Let us see that these properties define a one-parameter family of quasi-elliptic pencils that give rise to our surface.

First, we observe that the six points $x_1, \ldots, x_6$ must be the vertices of a complete quadrilateral. We choose projective coordinates, such that the sides are $V(x), V(y), V(z), V(x+y+z)$ and the points $x_1, \ldots, x_6$ have coordinates $[0,0,1], [0,1,0], [1,0,0], [1,1,0], [0,1,1], [1,0,1]$. The image of F_1 must be the union of the diagonals, that is, it must be equal to $V((x-y)(x-z)(y-z)$. Each conic C_i belongs to the pencil of conics through three non-collinear points x_i. The line component passes through the remaining three points. An easy computation shows that the four conic+line members belong to the following pencils:

$$a^2y(y+x) + (a^2+1)z(z+x) = 0, \quad x(x+z) + a^2y(y+z) = 0,$$

$$x(x+y) + (a^2+1)z(z+y) = 0 \quad y(x+z) + a^2x(y+z) = 0.$$

Each conic from each pencil passes through the point $q_1 = (1,1,1)$. It remains to choose one irreducible conic K_i from each pencil, such that the four conics are tangent at q_1 with multiplicity 3. Computing the equations of tangent lines of the conics at the point $(1,1,1)$, we easily find that the four conics must be

$$a^2y(y+x) + (a^2+1)z(z+x) = 0, \quad x(x+z) + a^2y(y+z) = 0,$$

$$x(x+y) + (a^2+1)z(z+y) = 0 \quad y(x+z) + a^2x(y+z) = 0.$$

The common tangent is $x + a^2y + (a^2+1)z = 0$. The singular points of the corresponding reducible cubics from the pencil are the points $[0, a+1, a], [a, 1, 0], [a+1, 0, 1], [1, a, a+1]$. They lie on the line $x + ay + (1+a)z = 0$. The inverse transform of this line on the blow-up surface is the curve of cusps $\mathfrak{C}$. The equation of our pencil is

$$\lambda(x+y)(x+z)(y+z) + \mu z(x^2 + xz + a^2y^2 + a^2yz) = 0,$$

where $a \neq 0, 1$. The pencil has four reducible fibers corresponding to the parameters $[\lambda, \mu] = [0,1], [1,0], [1,1]$ and $[1, a^{-1}]$. The Weierstrass equation

$$y^2 + x^3 + uv(u^2 + avu + (a+1)v^2)x = 0.$$

Replacing the parameter a with $\sqrt{a+1}$, we get the same equation as in [336, Table 1 (f)].

- Type $8\tilde{A}_1^*$ (quasi-elliptic and $p = 2$)

For this type, the Mordell–Weil group $\mathrm{MW}(j)$ is isomorphic to $(\mathbb{Z}/2\mathbb{Z})^4$.

Let $\mathfrak{C}$ be the curve of cusps of j, which is a smooth rational curve and a bisection (a 2-section). We have $\mathfrak{C}\cdot K_J = -2$, hence $\mathfrak{C}^2 = 0$.

Consider the map $\phi : J \to W \to \mathbf{F}_2$, where the first map is the birational morphism onto the Weierstrass model that blows the irreducible components that do not intersect a fixed section E. The morphism $W \to \mathbf{F}_2$ is purely inseparable of degree 2 given by the invertible sheaf $\mathcal{L} = \mathcal{O}_{\mathbf{F}_2}(3\mathfrak{f}+2\mathfrak{e})$, see (4.4.29). Applying Proposition 0.2.10, we expect that W has $c_2(\Omega_{\mathbf{F}_2}\otimes\mathcal{L}^{\otimes 2}) = 8$ ordinary double points. This can happen only if the quasi-elliptic fibration has 8 reducible fibers of type $\tilde{A}_1^*$. Taking a general curve $B \in |6\mathfrak{f}+3\mathfrak{e}|$, we see that we get a quasi-elliptic fibration of this type. It follows from the Weierstrass equation that such fibrations depend on 5 parameters, namely the coefficients of $(a_4(t_0,1), a_6(t_0,t_1))$ modulo projective transformations and modulo adding to a_6 the square of a binary form of degree 3.

Obviously, the 16 sections cannot be disjoint in this case. It follows from Section 4.5 that each non-zero section intersects the zero section with multiplicity 1 and does not intersect the zero component for any reducible fiber F_i (the sum of local contributions must be equal to 4). If two sections are disjoint, then they intersect exactly four common irreducible components of reducible fibers. Let $\mathsf{O}, P_1, \ldots, P_7$ be the eight sections that intersect the zero component of a reducible fiber F_1 and let $\mathsf{O}', Q_1, \ldots, Q_7$ be the sections intersecting the other component. We may assume that O' intersects O and that all other Q_i do not intersect O, but intersect exactly one P_i.

The theory of heights tells us that each section from the first group intersects one section from another group, say $\mathsf{O}\cdot\mathsf{O}', P_i\cdot Q_i = 1$. Following [336], we blow down the components $\mathsf{O}, P_1, \ldots, P_6, Q_7$ and the non-zero component of F_1. The image of F_1 is an irreducible cuspidal cubic. All other reducible fibers are mapped to the union of a conic and its tangent line. We have eight distinct base points $q_1, q_2, \ldots, q_8, q_8'$ and one base point $q_8' > q_8$ infinitely near to q_8. The point q_8 is the cusp of the image $\bar{F}_1$ of F_1. The image of seven sections $\mathsf{O}', Q_1, \ldots, Q_6$ are lines $\ell_1, \ldots, \ell_7$ passing through the point q_8 and one other point q_i, which is the image of P_i. The image of P_7 is the cuspidal tangent of the curve $\bar{F}_1$. The image of the section Q_7 is the line ℓ_9 passing through q_8, q_8' and other point q_i. We see that the set of points $q_1, \ldots, q_7$ and the lines $\ell_1, \ldots, \ell_7$ form the configuration of points and lines in the Fano plane $\mathbb{P}^2(\mathbb{F}_2)$. The net of cubics through $q_1, \ldots, q_8$ has equation

$$F(a,b,c) = a_1xy(x+y) + a_2xz(x+z) + a_3yz(y+z) = 0. \qquad (4.9.39)$$

The point $[\sqrt{b}, \sqrt{c}, \sqrt{a}]$ is the singular point of $F(a,b,c) = 0$. Let $\pi : Y \to \mathbb{P}^2$ be the blow-up of the base points $q_1, \ldots, q_7$. The net defines a quasi-elliptic fibration over Y. Its reducible fibers over the pre-images of the lines in the Fano plane with coordinates (a_1, a_2, a_3) are of type $\tilde{A}_1^*$. Thus, any general pencil in the net defines a quasi-elliptic surface with seven singular fibers of type $\tilde{A}_1^*$. Their line components are the lines in the Fano plane with coordinates (x, y, z). The points in the Fano plane are among its base points. Any other base point q_8 comes automatically with an infinitely near base point, and the member of the pencil with the cusp at q_8 gives the eighth reducible fiber of type $\tilde{A}_1^*$. Let the equation of the line defining the pencil be $a_1 = u, a_2 = v, a_3 = au + bv$. The equation of the pencil is

$$u(xy(x+y) + ayz(y+z)) + v(xz(x+z) + byz(y+z)) = 0.$$

We find that the base points are the seven points of the Fano plane, the eighth point

$$q_8 = [x_0, y_0, z_0] := [\sqrt{ab(a+b)}, \sqrt{a(a+1)}, \sqrt{b(b+1)}],$$

and the infinitely near point q_8' with tangent direction $z_0x + bz_0y + (x_0 + by_0)z = 0$. Since q_8 does not lie on any line in the Fano plane, we get

$$a, b \neq 0, \quad a \neq 1, \quad b \neq 1, \quad \text{and} \quad a \neq b.$$

(The restriction on the parameters is stronger than in [336], where the parameters a, b satisfy only $a \neq 0$.) The reduced fibers correspond to the points

$$[u, v] = [1,0], [0,1], [1,1], [b,a], [b,a{+}1], [a,b{+}1], [b{+}1,a{+}1], [b(b{+}1), a(a{+}1)].$$

Let us find the Weierstrass equation. Recall that the transformations $y \mapsto y + \alpha_1 x + \alpha_3$ changes the coefficients a_4, a_6 to

$$a_4' = a_4 + \alpha_1^4, \quad a_6' = a_6 + \alpha_1^2 a_4 + \beta_3.$$

Let $V(k)$ denote the linear space of binary forms of degree k. Counting parameters, we see that quasi-elliptic surfaces depend on $\dim \mathbb{P}(V(4) + V(6)) - \dim(V(1) + V(3)) - \dim \mathrm{PGL}(2) = 5 + 7 - (2 + 4 - 1 - 3) = 2$ parameters, as expected from the previous analysis. Write

$$a_4 = \sum_{i=0}^{4} r_i u^{4-i} v^i, \quad a_6 = \sum_{i=0}^{6} s_i u^{6-i} v^i.$$

We know that $\Delta = a_4 da_4^2 + da_6^2$ has eight simple distinct zeros. This implies that the coefficients r_2 and r_4 are non-zero. Taking appropriate β_3, we may assume that $s_0 = s_2 = s_4 = s_6 = 0$. Taking $\alpha_1 = Au + Bv$, we can make the coefficients s_1, s_5 equal to zero. Then again adding some square to a_6, and scaling the unknowns, we may assume that $a_6 = u^3v^3$. After a linear change of the coordinates (u, v), we may assume that a_4 has zeros $[0,1]$ and $[1,0]$. Thus, the Weierstrass equation acquires the form

$$y^2 + uv(u^2 + \alpha uv + \beta v^2)x + u^3v^3 = 0.$$

This agrees with the formula in [336, Table 1(g)].

The following table contains the classification of rational quasi-elliptic surfaces in characteristic 2 and 3. We refer to [336, Table 1] for explicit formula for sections in terms of the Weierstrass equations.

p	Types	$\mathrm{MW}(f)$	Weierstrass equation
2	$\tilde{E}_8$	$\{1\}$	y^2+x^3+t
2	$\tilde{D}_8$	$\mathbb{Z}/2\mathbb{Z}$	$y^2+x^3+t^2x+t$
2	$\tilde{E}_7, \tilde{A}_1^*$	$\mathbb{Z}/2\mathbb{Z}$	y^2+x^3+tx
2	$\tilde{D}_4, \tilde{D}_4$	$(\mathbb{Z}/2\mathbb{Z})^{\oplus 2}$	$y^2+x^3+(a^2+a+1)t^2x+a(a+1)t^3$
2	$\tilde{D}_6, 2\tilde{A}_1^*$	$(\mathbb{Z}/2\mathbb{Z})^2$	$y^2+x^3+t(1+t^2)x+t^2$
2	$\tilde{D}_4, 4\tilde{A}_1^*$	$(\mathbb{Z}/2\mathbb{Z})^3$	$y^2+x^3+(1+a+a^2)t^2x+a(a+1)t^3, a\neq 0,1$
2	$8\tilde{A}_1^*$	$(\mathbb{Z}/2\mathbb{Z})^4$	$y^2+x^3+t(1+at+bt^2)x+t^3$
3	$\tilde{E}_8$	$\{1\}$	y^2+x^3+t
3	$\tilde{E}_6, \tilde{A}_2^*$	$\mathbb{Z}/3\mathbb{Z}$	$y^2+x^3+t^2$
3	$4\tilde{A}_2^*$	$(\mathbb{Z}/3\mathbb{Z})^{\oplus 2}$	$y^2+x^3+t^2(1+t^2)$

Table 4.9 Quasi-elliptic rational surfaces

Remark 4.9.11 Suppose $p = 2$ and let $f : X \to \mathbb{P}^1$ be a jacobian quasi-elliptic fibration and $\mathfrak{C}$ be its curve of cusps. Then it intersects each fiber with multiplicity 2 at its singular point. Since $\mathfrak{C}$ is invariant with respect to a translation automorphism by a non-zero torsion point of the Mordell–Weil group, it intersects a fiber of type $\tilde{D}_{2k}$ at its central component of multiplicity 2. It follows from Figure 4.6 that it intersects the component R_1 of a fiber of type $\tilde{E}_7$. The description of the blow-down morphism $X \to \mathbb{P}^2$ shows that it intersects the component R_2 of a fiber of type $\tilde{E}_8$. In all other cases, it is clear which components the curve $\mathfrak{C}$ must intersect.

Remark 4.9.12 Assume $\Bbbk = \mathbb{C}$. Some of the extremal rational elliptic surfaces are isomorphic to *modular elliptic surfaces*. Let Γ be a subgroup of finite index of $\mathrm{SL}(2,\mathbb{Z})$ that does not contain -1. Given such a Γ, one defines the modular elliptic surface $S(\Gamma)$ to be the relatively minimal elliptic surface birationally isomorphic to the quotient

$$S(\Gamma) := (\Gamma \rtimes \mathbb{Z}^2)\backslash(\mathbb{H}\times\mathbb{C}) \to X(\Gamma) := \Gamma\backslash\mathbb{H},$$

where $\mathbb{H} = \{z = a + bi \in \mathbb{C} : b > 0\}$ is the upper half-plane and $\Gamma \rtimes \mathbb{Z}^2$ is the semi-direct product is taken with respect to the natural action of Γ on $\mathbb{Z}^2$, and $\Gamma \rtimes \mathbb{Z}^2$ acts on $\mathbb{H}\times\mathbb{C}$ by the formula

$$((m,n),g)\cdot(\tau,z) := \left(\frac{a\tau+b}{c\tau+d}, \frac{z+m\tau+n}{c\tau+d}\right).$$

The structure of an elliptic surface is given by the projection onto the first factor, that is, to $X(\Gamma) = \Gamma\backslash\mathbb{H}$. The unique smooth projective comptactification $\bar{X}(\Gamma)$ of $X(\Gamma)$ is called the *modular curve*. The name has to do with the fact that these curves often have descriptions as moduli spaces of elliptic curves with some extra structure. The complement $\bar{X}(\Gamma)\setminus X(\Gamma)$ consists of some finite number t of points and by definition, these points are called *cusps*. Cusps come in two kinds. The fibers of $S(\Gamma) \to \bar{X}(\Gamma)$ over cusps of the first kind (resp. second lind) are of type $\tilde{A}_n$ or $\tilde{A}_0^*$ (resp. $\tilde{D}_n, n > 4$). There could be also some singular fibers over points in $X(\Gamma)$. They correspond to elements of finite order 2 or 3 in Γ. (Note that we assumed if $-1 \notin \Gamma$. If we allow $-1 \in \Gamma$, then there may also be elements of order 4, 6.) The corresponding fibers are of type $\tilde{A}_2^*$ and $\tilde{E}_6$ in the former case and of type $\tilde{E}_7$ and $\tilde{E}_8$ in the latter case.

Let g be the genus of the modular curve $\bar{X}(\Gamma)$ and let p_g be the geometric genus of $S(\Gamma)$. We have (see [674])

$$g = 1 + \frac{\mu}{12} - \frac{r_2}{4} - \frac{r_3}{3} - \frac{t}{2},$$
$$p_g = 2g - 2 + t - \frac{t_1}{2} - \frac{r_2}{3},$$

where

$$\mu = \frac{1}{2}[SL(2,\mathbb{Z}) : \Gamma],$$

where t_1 denotes the number of cusps of the first kind, and where r_2 (resp. r_3) is the number of points in $X(\Gamma)$ corresponding to orbits with stabilizer subgroup of order 2 (resp. 3).

The connection between the modular elliptic surfaces and the surfaces discussed in this section is as follows: an modular elliptic surface has finite Mordell–Weil group and non-constant j-invariant [674]. Conversely, a jacobian elliptic surface with $p_g = 0$ over $\mathbb{C}$ with finite Mordell–Weil group, non-constant j-invariant and $p_g = 0$, and no singular fibers of type $\tilde{E}_8$ and $\tilde{E}_7$ is isomorphic to an modular elliptic surface [562].

Comparing this result with our list of extremal rational elliptic surfaces, we find that all of them are modular elliptic surfaces, except those of type $\tilde{E}_8, \tilde{E}_7+\tilde{A}_1, \tilde{D}_4+\tilde{D}_4$. In fact, we can say more. Recall that Γ is called a *congruence subgroup* if it contains a subgroup of the form

$$\Gamma(n) := \left\{ M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}) : M \equiv I_2 \mod n \right\}.$$

, for some $n \geq 2$. For $n \geq 3$, the group $\Gamma(n)$ does not contain -1 and we have

$$\mu = \frac{1}{2} n^3 \prod_{p|n} (1 - p^{-2}), \quad t = t_1 = \mu/n, \quad r_2 = r_3 = 0.$$

An example of a congruence subgroup is the group

$$\Gamma_m(n) := \left\{ M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}) : M \equiv \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix} \mod n, b \equiv 0 \mod m \right\},$$

where m, n are positive integers with $m|n$. Obviously, we have $\Gamma(n) = \Gamma_n(n)$. Moreover, we have (see [140]):

$$\mu = \begin{cases} 3 & \text{if } (n,m) = (2,1), \\ 6 & \text{if } (n,m) = (2,2), \\ \frac{1}{2} m n^2 \prod_{p|n}(1-p^{-2}) & \text{otherwise.} \end{cases}$$

$$t = \begin{cases} 2 & \text{if } (n,m) = (2,1), \\ 3 & \text{if } (n,m) = (2,2),(4,1), \\ \frac{1}{2} \prod_{p|n}(p-1)p^{\nu_p(mn)-2}(p+1+(p-1)\nu_p(n/m)) & \text{otherwise.} \end{cases}$$

We also recall the group

$$\Gamma_0(n) := \left\{ M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}) : c \equiv 0 \mod n \right\}.$$

Let $\bar{S}_m(n) \to \bar{X}_m(n)$ be the elliptic surface associated to the modular surface elliptic $S_m(n) := S(\Gamma_m(n))$ together with its elliptic fibration over the modular curve $X_m(n) = X(\Gamma_m(n))$. Its Mordell–Weil group is known to be isomorphic to $(\mathbb{Z}/n\mathbb{Z}) \oplus (\mathbb{Z}/m\mathbb{Z})$, see [140]. In particular, if $\bar{S}_m(n)$ is a rational surface, then it is an extremal rational elliptic surface. It is thus natural to identify the pairs (m,n), for which the surface is rational.

Next, it is known that over $\mathbb{C}$, a semi-stable jacobian elliptic surface over $\mathbb{P}^1$ has at least four singular fibers, see [48]. The surfaces with exactly four singular fibers are known as *Beauville surfaces*, they were classified in [49], and they coincide with extremal rational semi-stable elliptic surfaces. The corresponding modular groups are torsion-free congruence subgroups of the modular group with $g = 0$ and $\mu = 12$, which can be found in the list of torsion free congruence subgroups with $g = 0$, see [648]. All finite index subgroups of $\mathrm{SL}(2,\mathbb{Z})$ of genus $g \leq 24$ were found in [144].

In Table 4.10, we use their notation for some of the groups. Here, the *level* N of Γ is defined to be largest N, such that $\Gamma(N)$ is contained in Γ.

Singular fibers	Level	μ	Γ
$\tilde{A}_8, \tilde{A}_0^*, \tilde{A}_0^*, \tilde{A}_0^*$	9	12	$\Gamma_0(9) \cap \Gamma_1(3)$
$\tilde{A}_7, \tilde{A}_1, \tilde{A}_0^*, \tilde{A}_0^*$	8	12	$\Gamma_0(8) \cap \Gamma_1(4)$
$\tilde{A}_4, \tilde{A}_4, \tilde{A}_0^*, \tilde{A}_0^*$	5	12	$\Gamma_1(5)$
$\tilde{A}_5, \tilde{A}_2, \tilde{A}_1, \tilde{A}_0^*$	8	12	$\Gamma_1(6)$
$\tilde{A}_3, \tilde{A}_3, \tilde{A}_1, \tilde{A}_1$	4	12	$\Gamma_1(4) \cap \Gamma(2)$
$\tilde{A}_2, \tilde{A}_2, \tilde{A}_2, \tilde{A}_2$	3	12	$\Gamma(3)$
$\tilde{D}_5, \tilde{A}_3, \tilde{A}_0^*$	4	6	$\Gamma_1(4)$
$\tilde{D}_6, \tilde{A}_1, \tilde{A}_1$	4	6	$3C^0$
$\tilde{D}_8, \tilde{A}_0^*, \tilde{A}_0^*$	2	6	$2C^0$
$\tilde{E}_6, \tilde{A}_2, \tilde{A}_0^*$	3	4	$\Gamma_1(3)$
$\tilde{E}_7, \tilde{A}_1, \tilde{A}_0^*$	2	3	$2B^0$
$\tilde{E}_8, \tilde{A}_0^*, \tilde{A}_0^*$	2	2	$2A^0$

Table 4.10 Rational modular elliptic surfaces

Remark 4.9.13 Assume $\Bbbk = \mathbb{C}$ and consider extremal rational elliptic surfaces, all of whose singular fibers are of additive type or, equivalently, whose j-invariants are constant. There are four of them, and their fibers are the following types:

$$\tilde{D}_4 + \tilde{D}_4, \quad \tilde{E}_7 + \tilde{A}_1^*, \quad \tilde{E}_6 + \tilde{A}_0^{**}, \quad \tilde{E}_8 + \tilde{A}_0^{**}.$$

After an appropriate base change ramified over the two points corresponding to the two singular fibers, the surface is birationally equivalent to a constant elliptic fibration $(E \times C) \to C$. We may assume $C \to \mathbb{P}^1$ to be Galois, say with group G, and ramified over two points $\{x_1, x_2\}$. In particular, G is a quotient of $\pi_1(\mathbb{P}^1 - \{x_1, x_2\}) \cong \mathbb{Z}$, which implies that G is cyclic. Moreover, the Riemann–Hurwitz formula implies that $C \cong \mathbb{P}^1$. Thus, the original surface is isomorphic to the quotient $(E \times \mathbb{P}^1)/G \to \mathbb{P}^1/G$ by a cyclic group G of order $2, 4, 3, 6$, respectively to the order in the list.

4.10 Genus One Fibrations on Enriques Surfaces

Every Enriques surface S admits a genus one fibration, see Corollary 2.3.4. Moreover, there is a bijection between the set of genus one fibrations S and nef primitive isotropic vectors in $\mathrm{Num}(S)$. We also know from Corollary 2.2.9 that a genus one fibration on S cannot be jacobian and that it has one or two multiple fibers of multiplicity 2. The first case occurs if and only if S is a non-classical Enriques surface in characteristic $p = 2$.

Proposition 4.10.1 *Let $f : S \to C$ be a genus one fibration on an Enriques surface and let $j : J \to C$ be the associated jacobian fibration. Then, J is a rational surface.*

Conversely, let $j : J \to \mathbb{P}^1$ be a jacobian genus one fibration on a rational surface. Then, any torsor of j with two tame fibers of multiplicity 2 or with one wild fiber of multiplicity 2 and $h^0(\mathrm{Tors}(R^1 f_* \mathcal{O}_X)) = 1$ is a genus one fibration on an Enriques surface.

Proof The first assertion follows from Proposition 4.3.14, Corollary 4.3.18, and the formula (4.3.7) for the canonical class.

Concerning the second assertion, let $f : X \to \mathbb{P}^1$ be a torsor of j with at most two double fibers. Then, the same comparison assertions show that X is a surface with $b_2(X) = 10$ and $b_1(X) = 0$. If we have two tame double fibers, then the formula for the canonical class shows that K_X is the difference of two half-fibers. Thus, $K_X \neq 0$ but $2K_X = 0$, so that X is a classical Enriques surface. If we have one wild fiber X_t, then $p = 2$ and it follows from Theorem 4.1.6 that $a_t = 0$ and the formula for the canonical class shows that $K_X = 0$ and $\dim H^1(X, \mathcal{O}_X) = 1$. Using the classification of algebraic surfaces, we obtain that X is a non-classical Enriques surface. □

If $p \neq 2$, then it follows from Ogg–Shafarevich theory that an elliptic fibration on an Enriques surface is uniquely determined by its local invariants of order 2 at the pair of points where the associated jacobian fibration has a smooth or a multiplicative type singular fiber. Moreover, any pair of local invariants of order 2 is realized. If $p = 2$, then the situation is more complicated. It follows from Proposition 4.6.23 that a torsor with two tame double fibers exists if the local invariants are of order

2 and at arbitrary two points where the fibers of the jacobian fibration are ordinary elliptic curves. We can also construct a torsor with one wild double fiber, but we do not know how to control the torsion of $R^1 f_* \mathcal{O}_X$. We do not know how to compute the length of $\mathcal{T}$ for a torsor over a quasi-elliptic fibration.

By Theorem 4.3.20, the types of singular fibers of a genus one fibration on an Enriques surface are the same as the types of singular fibers on the associated jacobian fibrationa. For example, one can define an *extremal genus one fibration* on an Enriques surface to be a fibration such that the rank of $\mathrm{Pic}_{\mathrm{fib}}$ is maximal possible, that is, equal to 9. The types of singular fibers of such a fibration will be the same as for the extremal jacobian fibration. We classified them in the previous section.

Lemma 4.10.2 *Let S be a non-classical Enriques surface in characteristic $p = 2$.*

1. *If S is a $\boldsymbol{\mu}_2$-surface, then every genus one fibration is elliptic.*
2. *If S is a $\boldsymbol{\mu}_2$-surface (resp. $\boldsymbol{\alpha}_2$-surface), then the unique multiple fiber of an elliptic fibration on S is either an ordinary (resp. supersingular) elliptic curve or it is singular of multiplicative type (resp. additive type).*

Proof Let E be the unique half-fiber of a genus one fibration on S. The short exact sequence

$$0 \to \mathcal{O}_S(-E) \to \mathcal{O}_S \to \mathcal{O}_E \to 0$$

induces an isomorphism $H^1(S, \mathcal{O}_S) \to H^1(E, \mathcal{O}_E)$. One can check that it is compatible with the action of the Frobenius morphism $\mathbf{F}$. In particular, $\mathbf{F}$ is bijective (resp. zero) on $H^1(S, \mathcal{O}_S)$ if and only if it is bijective (resp. zero) on $H^1(E, \mathcal{O}_E)$. If E is nonsingular, then E is an ordinary (resp. supersingular) elliptic curve if $\mathbf{F}$ is bijective (resp. zero) on $H^1(E, \mathcal{O}_E)$. If E is singular, then $H^1(E, \mathcal{O}_E)$ is isomorphic to the Lie algebra of $\mathrm{Pic}^0_{E/\Bbbk}$, hence the latter is $\mathbb{G}_m$ (resp. $\mathbb{G}_a$) if $\mathbf{F}$ is bijective (resp. zero) on $H^1(E, \mathcal{O}_E)$. Finally, S is a $\boldsymbol{\mu}_2$-surface (resp. an $\boldsymbol{\alpha}_2$-surface) if $\mathbf{F}$ is bijective (resp. zero) on $H^1(S, \mathcal{O}_S)$. From this, all statements follow.

In particular, if S is a $\boldsymbol{\mu}_2$-surface, then the half-fiber is an ordinary elliptic curve or singular of multiplicative type, which implies that the generic fiber cannot be quasi-elliptic. Here is another argument: we know that S is not unirational by Theorem 1.3.11. By Corollary 4.1.16 and the fact that the base of a genus one fibration on an Enriques surface is $\mathbb{P}^1$, it follows that quasi-elliptic Enriques surfaces are unirational. Thus, S does not admit quasi-elliptic fibrations. □

The following theorem summarizes what we know about singular fibers of a genus one fibration on an Enriques surface.

Theorem 4.10.3 *Let $f : S \to \mathbb{P}^1$ be a genus one fibration on an Enriques surface.*

1. *If $p \neq 2$, then $K_S \neq 0$ and f is an elliptic fibration with two half-fibers, each of which is either nonsingular or singular of multiplicative type. In particular, there exist no quasi-elliptic fibrations on Enriques surfaces in characteristic $p = 3$.*
2. *If $p = 2$, then there are three cases:*

a. *If $K_S \neq 0$, that is S is classical, then f is either a quasi-elliptic or an elliptic fibration. It has two half-fibers, each of which is either an ordinary elliptic curve or singular of additive type.*
b. *If S is a non-classical $\boldsymbol{\mu}_2$-surface, then f is an elliptic fibration with one half-fiber that is a nonsingular ordinary elliptic curve or a singular curve of multiplicative type.*
c. *If S is a non-classical $\boldsymbol{\alpha}_2$-surface, then f is either an elliptic or a quasi-elliptic fibration with one half-fiber that is either a supersingular elliptic curve or a singular curve of additive type.*

Fibers and half-fibers on any S are of the same type as the corresponding fibers of the jacobian fibration on a rational surface.

Proof We have seen f has at least one and at most two multiple fibers in Corollary 2.2.9. Moreover, there is one multiple fiber if and only if $p = 2$ and S is non-classical. Multiple fibers are of multiplicity 2.

It follows from Corollary 4.1.15 that there are no quasi-elliptic fibrations on S if $p = 3$. Moreover, if there are no quasi-elliptic fibrations on S if $p = 2$ and S is a $\boldsymbol{\mu}_2$-surface by Lemma 4.10.2.

If $p \neq 2$, then the assertions concerning the types of the multiple fibers follow from Proposition 4.1.17.

If $p = 2$ and S is non-classical, then the assertions concerning the types of the multiple fibers have been established in Lemma 4.10.2.

If $p = 2$ and S is classical, then it follows from Theorem 4.1.6 that the Picard group of a half-fiber contains a non-trivial 2-torsion element. This happens only if it is either an ordinary elliptic curve or a singular fiber of additive type.

The last assertion follows from Theorem 4.3.20. □

Remark 4.10.4 In characteristic $p = 2$, there exist elliptic as well as quasi-elliptic Enriques surfaces. Let us do some (naive) moduli counts and note that we properly treat moduli spaces of Enriques surfaces in the next chapter.

Let $f : S \to \mathbb{P}^1$ be an elliptic fibration on an Enriques surface S, let $j : J \to \mathbb{P}^1$ be its associated jacobian fibration, and assume $p = 2$. We know that the Weierstrass fibration of j is of the form

$$y^2 + (a_1x + a_3)y + x^3 + a_2x^2 + a_4x + a_6 = 0,$$

where the a_k are binary forms of degree k. It follows that the generic fiber of j is supersingular if $a_1 = 0$. Otherwise, it has one fiber that is a supersingular elliptic curve or a singular fiber of additive type.

1. Concerning classical Enriques surfaces: Pencils of plane cubic curves depend on 8 parameters and a general pencil defines a jacobian elliptic fibration with non-constant j-invariant. Thus, classical Enriques elliptic surfaces depend on 10 parameters defined by a choice of two smooth ordinary fibers of its jacobian fibration and a non-trivial 2-torsion point on each of them. Note that a classical Enriques surface may not admit any elliptic fibrations but only quasi-elliptic fibrations, We will see examples of such surfaces in Volume II.

2. Concerning non-classical Enriques surfaces: First, we recall from Example 4.2.15 that a wild fiber of multiplicity $m = p$, where the length of the torsion sheaf is equal to 1, is defined by a choice of an element in the kernel of the homomorphism $\mathrm{Pic}(X_2) \to \mathrm{Pic}(X_1)$, which is isomorphic to the additive group of $\Bbbk$. This suggests that a wild fiber of an elliptic fibration on an Enriques surface depends on one parameter.

   a. Thus, a $\boldsymbol{\mu}_2$-surface is defined by choosing an ordinary smooth fiber or a fiber of multiplicative type and a wild fiber over it, which gives $10 = 8 + 2$ parameters.
   b. On the other hand, an $\boldsymbol{\alpha}_2$-surface is defined by the choice of a supersingular smooth fiber or a fiber of additive type that vary in a finite set unless $j \equiv 0$, which gives one condition on the moduli of the jacobian surface. This gives us at most $8 + 1 = 9$ parameters.

We can also do a moduli count for Enriques surfaces that admit a quasi-elliptic fibration. We know that jacobian quasi-elliptic fibrations on rational surfaces are extremal. It follows from Table 4.9 that they depend on at most two parameters. Similar to the above, we conclude that:

1. Classical Enriques surfaces admitting a quasi-elliptic fibration depend on at most 6 parameters (two points on the base and one parameter for the identity component of the corresponding fiber).
2. Non-classical $\boldsymbol{\alpha}_2$-surfaces admitting a quasi-elliptic fibration depend on at most 4 parameters.

Remark 4.10.5 In a recent paper [372], one can find more precise count of parameters of quasi-elliptic fibration on Enriques surfaces. First, the authors find normal forms for equations of a quasi-elliptic surface.

1. If S is classical, then it can be given by the following affine equation:

$$y^2 + t^2 a_1 y + tx^4 + t^3 a_0 x^2 + t^3 a_2 x + t^3(1 + t^4) = 0.$$

2. If S is supersingular surface, then it can be given by the following affine equation:

$$y^2 + t^4 a_1 y + tx^4 + t^5 a_0 x^2 + t^6 a_2 x + t^3 = 0.$$

coefficients a_i are polynomials in t of degree $\leq i$.

From this the authors Here the deduce that classical (resp. supersingular) quasi-elliptic surfaces with a fixed jacobian fibration with only irreducible fibers depend on 4 (resp. 3). This confirms our count in the classical case and also confirms our conjecture from Remark 4.8.12.

We refer to [438] where one can find the classification of all possible collections of singular fibers on rational elliptic surfaces in characteristic 2. This gives the classification of possible collections of singular fibers of an elliptic fibration on an Enriques surface in characteristic 2. For example, it follows that the singular non-multiple fibers of a $\boldsymbol{\alpha}_2$-surface whose jacobian fibration has a j-invariant equal to zero are all of multiplicative type.

The following proposition often allows us to distinguish an elliptic fibration from a quasi-elliptic fibration. Recall our discussion of bielliptic maps in Section 3.3.

Proposition 4.10.6 *Let S be an Enriques surface in characteristic $p = 2$, let $|2F|, |2G|$ be a non-degenerate U-pair of genus one pencils, and let $\phi : S \to \mathsf{D}$ be the corresponding bielliptic map.*

1. *If ϕ is inseparable, then both genus one pencils are quasi-elliptic.*
2. *If ϕ is separable, then at most one of the two pencils can be quasi-elliptic. If one of them is quasi-elliptic, then its curve of cusps is a component of a non-multiple fiber of the second pencil, as well as a component of the ramification curve of ϕ. If S is classical, then a general member of the other pencil, which is necassirly elliptic, is an ordinary elliptic curve.*
3. *F and G have no common irreducible components.*

Proof First, if ϕ is inseparable, a general member of $|2F|$ is equal to the pre-image of a conic on D under an inseparable cover. Obviously, it cannot be an elliptic curve. The same argument applies to $|2G|$. In particular, both pencils are quasi-elliptic.

Second, suppose that ϕ is separable. In this case, we already saw that at most one of the pencils can be quasi-elliptic in Proposition 3.3.23. Since a $\boldsymbol{\mu}_2$-surface does not admit quasi-elliptic fibrations, we may assume that S is either a classical or an $\boldsymbol{\alpha}_2$-surface and $\mathsf{D} = \mathsf{D}_1$ or D_3, respectively.

Suppose S is classical. In the notation of Theorem 3.3.11, the cover is defined by a quartic curve $Z(a)_0 \in |2e_0 - e_2 - e_4|$ that passes through the singular points of D and an octic $Z(b)_0 \in |-2K_{\mathsf{D}}|$. Suppose that one of the genus two fibrations is quasi-elliptic. We may assume that it is given by the pencil of lines $|e_0 - e_1|$ in the double plane model of S. Let C be the conic in the plane representing the curve $Z(a)_0$. If a general line from this pencil intersects C, transversally at two points, then its pre-image must be an elliptic curve. So, we obtain that point p_1 must be the strange point of the conic C, that is, a general line through p_1 is touching C at one point. If additionally $Z(b)$ has $Z(a)_0$ as a component of multiplicity 2 (the other components will be represented by lines passing through p_1), then the pre-image of each point of C will be a cusp of a member of $|2F|$ and the proper transform of C will be the curve of cusps. In the usual coordinates used in Section 0.5, the equation of C must be $x_0^2 + ax_1x_2 = 0$. The second genus one pencil is given by the pencil of conics $|2e_0 - e_2 - e_3 - e_4 - e_5| \subset |2e_0 - e_2 - e_4|$. This shows that the branch curve of ϕ on D_1 contains two disjoint lines corresponding to double fibers of the genus one fibration $|2G|$ defined by the pencil $|2e_0 - e_2 - e_3 - e_4 - e_5|$. Thus, any member of the pencil $|2G|$ different from the one defined by the conic C intersects it at two distinct points and, hence, it is an ordinary elliptic curve. Also, we see that the inverse transform of C on S is a simple fiber of the elliptic fibration. The proper inverse transform of C is its irreducible component.

Next, suppose S is an $\boldsymbol{\alpha}_2$-surface. Then $\mathsf{D} = \mathsf{D}_3$ and $Z(a)_0$ is a rational quartic curve from $|2e_0 - e_2 - e_3|$. The curve $Z(b)$ is the union of the exceptional curve $A = A_1 + A_2 + A_3 + A_4$ on D and a curve $Z(b)_0$ from $|-2K_{\mathsf{D}}|$. The pencils $|2F|$ and $|2G|$ are the pre-images of the pencils of conics $|2e_0 - e_2 - e_3 - e_4 - e_5|$ and $|e_0 - e_1|$

(recall that a minimal resolution of D is the blow-up of $p_1, p_5 > p_4 > p_3 > p_2$). We may assume that, in the plane model, a general member of $|2F|$ corresponds to a general line ℓ through p_1 and that $Z(a)_0$ is represented by a conic through p_2, p_3. If we choose coordinates in the plane such that $p_1 = [1,0,0]$ and $p_2 = [0,1,0]$, then a conic from $|2e_0 - p_2 - p_3|$ for which p_1 is its strange point must have an equation $x_0^2 + ax_1x_2 = 0$. So, if we choose such a conic for C and additionally choose $Z(b)_0$ singular along $Z(a)_0$, then the pencil $|e_0 - e_1|$ defines a quasi-elliptic pencil whose curve of cusps is equal to the proper inverse transform of C. A general member of the pencil $|2e_0 - e_2 - e_3 - e_4 - e_5|$ may intersect C at two points if $C \notin |2e_0 - e_2 - e_3 - e_4|$. In this case, the second pencil is elliptic and its general member is an ordinary elliptic curve. If $C \in |2e_0 - e_2 - e_3 - e_4|$ (resp. $C \in |2e_0 - e_2 - e_3 = e_4 - e_5|$), then C contributes to the branch curve on D the line ℓ_1 with class e_5 and $|2F|$ is an elliptic pencil whose general fiber is a supersingular elliptic curve.

To prove (3), we use that F is numerically 2-connected, that is, if we write F as a sum of two proper effective divisors $F = D_1 + D_2$, then $D_1 \cdot D_2 \geq 2$, see Section 2.5. To see this, we use that $D_1^2 < 0$, $D_2^2 < 0$, and $F \cdot D_1 = F \cdot D_2 = 0$, hence $2D_1 \cdot D_2 = -D_1^2 - D_2^2 + 2D_1 \leq -4$. Now, if D_1 is the maximal effective divisor with $D_1 \leq F$ and $D_1 \leq G$ and if we let $F = D_1 + D_2$ and $G = D_1 + D_2'$ be decompositions into effective divisors, then we have $D_2.D_2' \geq 0$. Therefore $1 = F \cdot G = (D_1 + D_2) \cdot G = (D_2 \cdot D_1 + D_2 \cdot D_2') \geq D_2 \cdot D_1$ and hence, $D_1 = 0$. □

In the following example, we construct a couple of genus one fibrations on an Enriques surface in characteristic $p = 2$ and describe their fibers.

Example 4.10.7 Suppose $S \to \mathsf{D}_1$ is a separable double cover of the anti-canonical del Pezzo surface D_1, that is, the quartic symmetroid surface in $\mathbb{P}^4$ given in Corollary 0.6.14, see also Theorem 3.3.4. We know that it is a split Artin–Schreier cover defined by minimally resolving the surface given by equation $V(F)$,

$$F = z^2 + t_1t_2A_2(t_0,t_1,t_2)z + t_1t_2B_6(t_0,t_1) = 0, \tag{4.10.1}$$

where $C = V(A_2)$ is a conic from the linear system $|2e_0 - e_2 - e_4|$, and where $W' = V(B_6)$ is a sextic from the linear system $|6e_0 - 2(e_1 + \cdots + e_5)|$. The polynomial B_6 is given in (3.3.9). Suppose the conic $A_2 = 0$ is nonsingular and that $\frac{\partial A_2}{\partial t_0} \neq 0$. The latter condition means that the conic has only two tangent lines containing the point $p_1 = [1,0,0]$. We also assume that none of the lines $t_1 = 0, t_2 = 0$ is the tangent line. Let $\ell : t_2 + \lambda t_1 = 0$ be a line passing through p_1 that intersects C at two distinct points. The restriction of the cover over the line is the Artin–Schreier split cover

$$u^2 + a(t)z + b(t) = 0,$$

where $t = t_1/t_0$ is the affine parameter on ℓ, $u = z/t^2$, $a(t) = 0$ (resp. $b(t) = 0$) give the intersection points of ℓ with C (resp. with W' outside p_1). We see that, for a general line ℓ, the cover is a nonsingular elliptic curve. It is a fiber of the elliptic fibration $|D_1|$ on S defined by the pencil $|e_0 - e_1|$. The fiber is singular if and only if $b'(c)^2 + b(c)a'(c)^2 = 0$, where c is one of the two roots of $a(t)$. For example, this happens if the conic C and the sextic W' intersect at some point q such that line $\ell = \overline{p_1, q}$ is tangent to W' at q. Now, let us specialize and see what happens.

1. First, suppose ℓ is tangent to C at some point $t = c \neq 0$. Replacing t by $t + c$, the equation becomes
$$u^2 + t^2 u + b(t + c) = 0. \tag{4.10.2}$$
   a. If $b'(c) \neq 0$, that is, if the coefficient of $b(t + c)$ at t is not zero, then the cover is a smooth supersingular elliptic curve. Using the equation of W' from (3.3.9), we find that the cover is singular if and only if
$$A_2(1, \lambda) + \lambda c^2 A_4(1, \lambda) = 0, \tag{4.10.3}$$
   where A_2 and A_3 are linear forms from (3.3.9). Thus, we may have 0, 1 or 2 supersingular elliptic fibers.
   b. If $b'_t(c) = 0$, then the pre-image of the line ℓ in the cover is a singular curve with an ordinary cusp. After minimally resolving the singular point of the cover, we obtain that the corresponding fiber is of additive type.

2. Next, we assume that a general line in $|e_0 - e_1|$ intersects the conic C with multiplicity 2. For example, C could be the double line $\ell = \overline{p_2, p_4}$. If we assume that equation (4.10.3) is not identically zero, then a general fiber of the elliptic fibration is a supersingular elliptic curve given by equation (4.10.2). The absolute invariant of the associated jacobian fibration is identically zero and all singular fibers must be of additive type.
   The equation of the conic C must be of the form $G = t_0^2 + at_1t_2 = 0$. A line $t_2 + \lambda t_1 = 0$ intersects the conic at the point $[1, 1, \sqrt{\lambda^{-1}a}]$. Thus, by (4.10.3), we must have $c^2 = \lambda^{-1}a$ and thus, the equation transforms to $aA_2(1, \lambda) + A_4(1, \lambda) = 0$. In this case, the equation of the curve W' takes the form
$$t_0^4 A_1(t_1, t_2) + t_0^2 t_1 t_2 A_3(t_1, t_2) + t_0 t_1 t_2 G(t_0, t_1, t_2) A_4(t_1, t_2) + t_1^2 t_2^2 A_5(t_1, t_2) = 0.$$
   Observe that if we take the partial derivatives of the equation of the surface, we get $\frac{\partial F}{\partial t_0} = \frac{\partial F}{\partial z} = 0$ and $\frac{\partial F}{\partial t_1} = \frac{\partial F}{\partial t_2}$ when restricted to the conic C. This shows that the surface has some singular points over the curve C. The proper transform of C must be the curve of cusps on S. Also, note that the exceptional curves e_3 and e_5 on D enter in the branch locus because the conic C belongs to the pencil $|2e_0 - e_2 - e_3 - e_4 - e_5|$. This shows that the genus one pencil $|D_2|$ on S defined by this pencil is an elliptic fibration. So, we have constructed an example of a pair of genus one fibrations $|D_1|$ and $|D_2|$, one is elliptic and the other is quasi-elliptic such that the linear system $|D_1 + D_2|$ is bielliptic and maps S to a 4-nodal quartic del Pezzo surface D_1. Any conic from the pencil $|2e_0 - e_2 - e_3 - e_4 - e_5|$ is tangent to a line from the pencil $|e_0 - e_1|$ at some point. We see that a general fiber of the genus one pencil $|D_2|$ is tangent to the general fiber of the quasi-elliptic pencil $|D_1|$ on S defined by the pencil $|e_0 - e_1|$ at two points.

Let $p = 2$ and S be a classical Enriques surface or an $\boldsymbol{\alpha}_2$-surface. In these two cases, there exists a non-zero global 1-form on S and more precisely, we have $\dim H^0(S, \Omega^1_{S/\Bbbk}) = 1$, see Section 1.4. The following proposition is taken from [364].

Proposition 4.10.8 *Assume $p = 2$ and let $f : S \to \mathbb{P}^1$ be a genus one fibration on a classical or an $\boldsymbol{\alpha}_2$-Enriques surface S as above. Let ω be a generator of $H^0(S, \Omega^1_{S/\Bbbk})$. Then there exists a rational 1-form η on $\mathbb{P}^1$ such that $\omega = f^*(\eta)$. More precisely, there exists an affine coordinate t on $\mathbb{P}^1$, such that $\omega = f^*(dt/t)$ (resp. $\omega = f^*(dt)$) if S is classical (resp. an $\boldsymbol{\alpha}_2$-surface).*

Proof First, assume that S is a classical Enriques surface and let f be as in the proposition. Choose an affine coordinate t on $\mathbb{P}^1$, such that the multiple fibers of f are over 0 and ∞. Set $\eta := dt/t$. Let R be an irreducible component of the fiber F_0 over 0 and let m be its multiplicity. Then, f is given locally at a general point of R by $t = \epsilon\phi^{2m}$, where $\phi = 0$ is a local equation of R and where η is a unit. Next, $f^*(\eta) = \phi^{2m} d\epsilon$ is regular at this point. A similar argument shows that $f^*(\eta)$ is regular at a general point of any irreducible component of the other multiple fiber. It is obviously regular over the complement of the two multiple fibers. Thus, $f^*(\eta)$ is regular outside ofa finite set of points, and hence regular everywhere. Since $H^0(S, \Omega^1_{S/\Bbbk})$ is one-dimensional, we obtain $\omega = f^*(c\eta)$ for a suitable constant c.

Second, assume that S is an $\boldsymbol{\alpha}_2$-surface. Let $F = 2E$ be the unique double fiber of a genus one fibration f. Since this fiber is wild, we have $\dim H^0(F, \mathcal{O}_F) \geq 2$. The exact sequence

$$0 \to \mathcal{O}_S(-F) \to \mathcal{O}_S \to \mathcal{O}_F \to 0 \tag{4.10.4}$$

defines a non-trivial coboundary homomorphism $\delta : H^0(F, \mathcal{O}_F) \to H^1(S, \mathcal{O}_S(-F))$. Let F' be any other fiber. As above, we chose an affine coordinate t to assume that F is a fiber over ∞ and F' is the fiber over 0. Then $\mathcal{O}_S(-F') \cong \mathcal{O}_S(-F)$ and the isomorphism is defined explicitly by multiplication by a rational function $f^*(t)$, which we identify with t. Replacing F by F' in exact sequence (4.10.4), we obtain an isomorphism $H^1(S, \mathcal{O}_S(-F')) \to H^1(S, \mathcal{O}_S)$. Let $\alpha \in H^0(F, \mathcal{O}_F)$ be such that $\delta(\alpha) \neq 0$. Choose an affine open cover (U_i) such that α is represented by regular functions g_i on U_i and $\delta(\alpha)$ is represented by a cocycle (f_{ij}) in this cover. The composition $H^0(F, \mathcal{O}_F) \to H^0(S, \mathcal{O}_S(-F)) \to H^0(S, \mathcal{O}_S(-F')) \to H^1(S, \mathcal{O}_S)$ allows us to write $g_i - g_j = f_{ij}t$. Since the Frobenius endomorphism acts trivially on $H^1(S, \mathcal{O}_S)$, we obtain that $f_{ij}^2 = f_i - f_j$ for some regular functions f_i on U_i. This gives

$$(g_i/t_i)^2 - f_i = (g_j/t_j)^2 - f_j \quad \text{on} \quad U_i \cap U_j.$$

We know that a nonzero regular 1-form ω on S can be defined locally by df_i. The previous equality shows that $\omega = d\phi$, where ϕ is a rational function on S defined by $\phi_i = (g_i/t_i)^2 - f_i$ on U_i. This rational function is regular outside F and hence, it is equal to $f^*(h(t))$ for some rational function $h(t)$ on the base. Thus, $\omega = f^*(dh(t))$. The function $h(t)$ must have pole of order 2 at ∞ and it must be regular outside ∞. Thus, $h(t) = at^2 + bt + c$ and hence, $dh(t) = bdt$. This shows that ω can be written in the form $f^*(dt)$. □

Next, we study Enriques surfaces, genus one fibrations, and the induced fibrations on their K3-covers. Let $\pi : X \to S$ be the K3-cover of S and let $f : S \to \mathbb{P}^1$ be a genus one fibration on S. Consider the composition $g = f \circ \pi : X \to S \to \mathbb{P}^1$. Since $g_*\mathcal{O}_X = f_*(\pi_*\mathcal{O}_X)$ is a locally free sheaf of algebras of rank 2, the Stein factorization gives us a degree 2 morphism $a : C = \mathrm{Spec}(g_*\mathcal{O}_X) \to \mathbb{P}^1$ such that the following diagram is commutative:

$$\begin{tikzcd} X \arrow[r, "\pi"] \arrow[d, "\tilde{f}"] & S \arrow[d, "f"] \\ C \arrow[r, "a"] & \mathbb{P}^1. \end{tikzcd} \tag{4.10.5}$$

Let $t \in \mathbb{P}^1$ be a point such that the fiber E_t of f is not multiple.

1. If π is a $\boldsymbol{\mu}_2$-cover corresponding to the canonical sheaf ω_S, then the restriction of ω_S to E is isomorphic to the canonical sheaf ω_E, which has has a nonzero section. This implies that the pre-image of E in X splits and hence, $a : C \to \mathbb{P}^1$ is étale over t. Thus, a is a separable cover of degree 2, which is ramified over two points corresponding to the multiple fibers.
2. Next, suppose π is not a $\boldsymbol{\mu}_2$-cover. The exact sequence

$$0 \to \mathcal{O}_S(K_S - E) \to \omega_S \to \omega_E \to 0$$

together with the fact that $h^1(\mathcal{O}_S(-E)) = h^0(\mathcal{O}_E) = 1$, shows that the homomorphism $H^1(S, \mathcal{O}_S) \to H^1(E, \mathcal{O}_E)$ is zero. Thus, the restriction of π defined by some $\alpha \in H^1(S, \mathcal{O}_S)$ to E is a trivial principal cover. It follows from $C \cong \mathbb{P}^1$ that the cover $a : C \to \mathbb{P}^1$ is an inseparable $\boldsymbol{\mu}_2$-cover defined by the sheaf $\mathcal{L} \cong \mathcal{O}_{\mathbb{P}^1}(-2)$ and a section s of this sheaf vanishing at the point corresponding to the unique half-fiber.

In particular, we see that a general fiber of $\tilde{f}$ is a genus one curve and that $C \cong \mathbb{P}^1$.

Let us now study the genus one fibration $\tilde{f} : X \to C$ and its singular fibers in detail.

First, assume that X is smooth, that is, we have $p \neq 2$ or that S is a $\boldsymbol{\mu}_2$-surface. The morphism $\tilde{f}$ defines a genus one fibration on the K3 surface X. We now discuss its singular fibers. Obviously, each non-multiple fiber of f defines two isomorphic fibers of $\tilde{f}$. If a half-fiber F is smooth, then its pre-image is a smooth fiber $\tilde{F}$ of $\tilde{f}$ and the morphism $\tilde{F} \to F$ is an étale double cover corresponding to the sheaf $\mathcal{O}_F(K_S)$. If F is of type $\tilde{A}_n$, then a straightforward computation shows that $\tilde{F}$ is of type $\tilde{A}_{2n+1}$.

Let us study a more interesting case when $p = 2$ and X is not smooth. Assume first that X is normal and that is has only rational double points (this is the generic case if X is not smooth). Let $\sigma : X' \to X$ be a minimal resolution of singularities of X, which is a K3 surface. The composition $\tilde{f} \circ \sigma : X' \to X$ is a relatively minimal genus one fibration. Let us study the singular fibers of this fibration and the singularities of X.

1. First, assume that E is a non-multiple fiber.

a. If it is of type $\tilde{A}_n$, then the morphism f is locally given at a singular point of the fiber as $t = uv$ and the pre-image of t on C is equal to $t^{1/2}$. Thus, the pre-image of the singular point of E is an ordinary double point of X. After resolving it, we obtain a fiber of type $\tilde{A}_{2n+1}$, as in the case of $p \neq 2$.
b. Next, assume that E is of type $\tilde{A}_0^{**}$. Then, we have $t = \epsilon(u^2 + v^3)$, where ϵ is a unit at the singular point of the fiber. The singular point of X over this point is locally given by $z^2 = \epsilon(u^2 + v^3)$. Replacing ϵ by $\epsilon^{1/3}$, we may assume that the equation is $z^2 = \epsilon u^2 + v^3$. Write $\epsilon = \epsilon_0^2 + \eta$ and replacing z by $z + \epsilon(0)u$, we may assume that the equation is $z^2 + a(u, v)u^2 + v^3 = 0$, where $a(u, v)$ vanishes at the singular point. It follows from the classification of rational double points that the singular point could be of type $\tilde{D}_4^{(0)}$, $\tilde{E}_7^{(0)}$, or $\tilde{E}_8$.
c. Next, suppose E is of type $\tilde{A}_1^*$. Similarly to the previous case, the equation of the singular point is $z^2 + a(u, v)u^2 + uv^2 = 0$. We get the singular fiber over this point is of type $\tilde{D}_n^{(0)}$ for some $n \geq 6$.
d. Finally, if E is of type $\tilde{A}_2^*$, then we get the equation $z^2 + a(x, y)x^3 + y^3 = 0$. The possible type of the singular fiber is $\tilde{D}_4^{(0)}$ or $\tilde{E}_7^{(0)}$.

2. Next, assume that F is a half-fiber of f, that is, $E = 2F$. Following the proof of Theorem 4.10.3, we see that the restriction of $X \to S$ over F is a principal cover of degree 2. Using Example 0.3.8, we conclude that if F is an ordinary (resp. supersingular) elliptic curve, then its pre-image is an ordinary (resp. supersingular) elliptic curve. If F is of type $\tilde{A}_0^{**}$, then the pre-image of F is a singular fiber of additive type. Unfortunately, nothing more can be said.

The following result describes the pre-image of rational 2-sections of an elliptic fibration $f : S \to \mathbb{P}^1$ on an Enriques on its K3 cover in characteristic $p \neq 2$. In particular, this shows that the induced elliptic fibration on the K3 cover is jacobian in this case.

Lemma 4.10.9 *Assume $p \neq 2$. Let $f : S \to \mathbb{P}^1$ be an elliptic fibration on an Enriques surface and let $\pi : X \to S$ be the K3 cover of S. Let R be an irreducible rational bisection of f with m ordinary nodes or smooth. Then, its pre-image under π splits into two smooth rational curves $R_+ + R_-$, which intersect transversally at m pairs of points and each pair is over one of the nodes.*

Proof First, if R is smooth, then the assertion is obvious.

Next, if $m = 1$, then R is of arithmetic genus one and since it is a bisection, it must be a half-fiber of some elliptic fibration on S. In this case, we know that $H^1(\mathcal{O}_S(-R + K_S)) = H^1(\mathcal{O}_S(R)) = 0$. The short exact sequence

$$0 \to \mathcal{O}_S(-R + K_S) \to \mathcal{O}_S(K_S) \to \mathcal{O}_R(K_S) \to 0$$

shows that $H^0(\mathcal{O}_R(K_S)) = 0$, and hence, the restriction of the canonical cover over R is a non-trivial étale cover.

Finally, if $m \geq 2$, then $R^2 > 0$ and $H^1(\mathcal{O}_S(R)) = 0$ by the Vanishing Theorem. Let $p : \tilde{R} \to R$ be the normalization map. The base change $X \times_S \tilde{R} \to \tilde{R}$ splits into components, whose images under the first projection are the curves R_+ and R_- as asserted. $\square$

The following extends a result of Kondō [409, Lemma (2.6)], see also [315]. It connects a genus one fibration $f : S \to \mathbb{P}^1$ on an Enriques surface with its jacobian fibration $j : J \to \mathbb{P}^1$, as well as with the induced fibration $\tilde{f} : X \to \mathbb{P}^1$ on the K3 cover $\pi : X \to S$. It also gives a recipe to construct an Enriques surface from a jacobian elliptic fibration on a rational surface and the choice of two points on the base of the fibration, which we will discuss after this result. We will refer to it as a *quadratic twist construction*.

Proposition 4.10.10 *Assume $p \neq 2$ and let $f : S \to \mathbb{P}^1$ be an elliptic fibration on an Enriques surface S with two half-fibers F_{t_1} and F_{t_2}. Let $\pi : X \to S$ be the K3 cover of S. Suppose that f admits a rational bisection R with m ordinary nodes.*

1. *The Stein factorization of $f \circ \pi : X \to \mathbb{P}^1$ is equal to the composition of a jacobian elliptic fibration $f' : X \to \widetilde{\mathbb{P}}^1$ followed by the double cover $\phi : \widetilde{\mathbb{P}}^1 \cong \mathbb{P}^1 \to \mathbb{P}^1$ that is ramified over t_1, t_2.*
   The surface X is a minimal resolution of the base change $j : J \to \mathbb{P}^1$ along the double cover $\widetilde{\mathbb{P}}^1 \to \mathbb{P}^1$ that is ramified over t_1, t_2. It coincides with J if the fibers J_{t_1}, J_{t_2} are smooth.
2. *There exists an involution $\sigma : X \to X$, such that the quotient $J' = X/(\sigma)$ is smooth and admits a birational morphism $\beta : J' \to J$, where $j : J \to \mathbb{P}^1$ is the jacobian fibration of $f : S \to \mathbb{P}^1$.*

If C denotes the image of $R_+ + \sigma(R_+)$ on J, then C is a rational bisection of j and we have the following additional properties:

1. *The bisection R splits into two sections R_+ and R_- of the induced fibration $\tilde{f}$ on X. The curve R_- is σ-invariant and its image on J is a section O of $j : J \to \mathbb{P}^1$.*
2. *The bisection C is tangent to the fibers J_{t_1}, J_{t_2} if they are smooth.*
3. *If J_{t_i} is singular of type $\tilde{A}_{2k-1}$ (resp. $\tilde{A}_{2k}$), then C is tangent to an irreducible component (resp. passes through the singular point) opposite to the component intersected by O.*
4. *The bisection C intersects the zero section O at m ordinary double points.*
5. *If none of the fibers F_{t_i} is singular of type $\tilde{A}_{2k+1}$, then the section C is invariant with respect to the negation involution of $J \to \mathbb{P}^1$ with respect to the zero section O.*

$$\begin{tikzcd}
S \times_{\mathbb{P}^1} \widetilde{\mathbb{P}}^1 \arrow[r, equal] & X \arrow[r, "/\sigma"] \arrow[dl, "\pi"'] \arrow[d, "f'"] & J' \arrow[d, "\beta"] \\
S \arrow[d, "f"'] & \widetilde{\mathbb{P}}^1 \times_{\mathbb{P}^1} J \arrow[r, "p_2"] \arrow[d, "p_1"] & J \arrow[d, "j"] \\
\mathbb{P}^1 & \tilde{\mathbb{P}}^1 \arrow[l] \arrow[r] & \mathbb{P}^1
\end{tikzcd}$$

Proof By Lemma 4.10.9, the bisection R splits on the canonical cover X into the union of smooth rational curves $R_+ + R_-$. They intersect at $2m$ points, where m is the number of double points of R. As explained earlier in (4.10.5), the canonical cover $\pi : X \to S$ is obtained from the base change $a : \tilde{\mathbb{P}}^1 \to \mathbb{P}^1$ ramified at the points t_1, t_2. One can see also that the base change cover coincides with the composition $\tilde{R} \to R \xrightarrow{\pi} \mathbb{P}^1$, where $\tilde{R} \to R$ is the normalization map.

We use R_+ to be the zero section of the elliptic fibration $f' : X \to \tilde{\mathbb{P}}^1$ and we let $t_{R_-} : X \to X$ be the translation automorphism that sends R_+ to R_-. Let

$$\sigma := t_{R_-} \circ \tau,$$

where τ is the deck involution of the K3-cover. Set $R'_- := t_{R_-}(R_-)$, where t_{R_-} is the translation automorphism with respect to the zero section R_+. Then

$$\sigma(R_+) = t_{R_-}(R_-) = R'_-, \quad \sigma(R_-) = t_{R_-}(R_+) = R_-.$$

The automorphism σ preserves the elliptic fibration $\tilde{f}$, but it acts as an involution on the base of the fibration. Thus, σ^2 acts is identity on the fibration and fixes its section R_-. It must act on the general fiber X_η as an automorphism of the elliptic curve X_η with the zero point R_-. Since τ (resp. t_{R_-}) acts as -1 (resp. identically) on $H^0(X, \omega_X) \cong H^0(X_\eta, \omega_{X_\eta})$, we see that σ acts as -1 on $H^0(X, \omega_X)$. Hence, σ^2 acts identity on this vector space. Since a non-trivial automorphism of an elliptic curve that acts non-identically on the regular 1-form, we get that σ^2 must be the identity.

Let $J' = X/(\sigma)$ be the quotient by the cyclic group (σ). The involution σ switches the non-multiple fibers and its fixed points have to lie on the half-fibers. The curve $R_\pm$ intersects $\tilde{F}_i$ at the point $p_+^{(i)}$ and $p_-^{(i)}$. If F_i is singular of type $\tilde{A}_{k_i-1}$ (type $\tilde{A}_0*$ if $k_i = 1$), then $\tilde{F}_i$ is singular of type $\tilde{A}_{2k_i-1}$. We index them by $E_s^{(i)}$, where $s \in \mathbb{Z}/2k_i\mathbb{Z}$, $p_+ \in E_0^{(i)}$, and $p_- \in E_{k_i}^{(i)}$. We call the components $E_s^{(i)}$ *even* (resp. *odd*) if $s \equiv 0 \mod 2$ (resp. $s \equiv 1 \mod 2$).

The restriction of t_{R_-} and τ to $\tilde{F}_i$ sends $p_+^{(i)}$ to $p_-^{(i)}$. If F is smooth, then they are both translations by the same point, so that σ acts as the identity on $\tilde{F}_i$. Suppose that F_i is singular. Then τ and t_{R_-} send $E_s^{(i)}$ to $E_{-s}^{(i)}$ and hence, σ leaves all components invariant. Since $E_{k_i}^{(i)}$ contains 3 fixed points of σ, it is pointwise fixed. Suppose that one of the intersection points $E_{k_i-1}^{(i)} \cap E_{k_i-2}^{(i)}$ or $E_{k_i-1}^{(i)} \cap E_{k_i-2}^{(i)}$ is an isolated fixed point of σ, say the first case occurs. Then the extension of σ to the blow-up of this point has no isolated fixed points on the proper transform of $E_{k-1}^{(i)}$, whose self-intersection is equal to -3. Taking the quotient by σ, we find a contradiction using how intersection numbers behave under finite maps. Continuing in this way, we obtain that the components $E_s^{(i)}$ with $s \equiv k_i \mod 2$ are pointwise fixed. In particular, if k is even, then E_0^i is pointwise fixed. Otherwise, σ is an involution on E_0^i that fixes two points $E_0^{(i)} \cap E_{\pm 1}^{(i)}$ and sends $p_+^{(i)}$ to $q_+^{(i)} = E_0^{(i)} \cap R'_-$. The point $q_+^{(i)}$ is a 2-torsion point of $\tilde{F}_i^\sharp$ with respect to the group law, where $p_+^{(i)}$ is the zero point.

Thus, we infer from the above discussion that the fixed locus X^σ of σ consists of disjoint union of $k_1 + k_2$ isolated curves contained in the fibers X_{t_1} and X_{t_2}. Two (resp. one) of them are elliptic curves if $k_1 + k_2 = 0$ (resp. $k_1 + k_2 = 1$), the rest of them are (-2)-curves. If $k_i > 0$, then the image of $\tilde{F}_i$ in $J' = X/(\sigma)$ is a $(2k_i)$-gon of smooth rational curves. If k_i is odd (even), then the odd (even) components are in X^σ. Their images on J' are (-2)-curves. The images of the remaining components are (-1)-curves on J'. Blowing down these (-1)-curves, we obtain a rational elliptic surface $j : J \to \mathbb{P}^1$ with a section O that is equal to the image of R_- and a 2-section C, the image of $R_+ + R'_-$. The involutions τ and σ of X differ by a translation automorphism and act on the base $\widetilde{\mathbb{P}}^1$ of the fibration $f' : X \to \widetilde{\mathbb{P}}^1$ in the same way. This shows that the generic fiber of f' is obtained from the generic fiber of f or j by the same base change $\widetilde{\eta} \to \eta$, where $\widetilde{\eta}$ is the generic point of $\widetilde{\mathbb{P}}^1$. This proves that $j : J \to \mathbb{P}^1$ is the jacobian fibration of $f : S \to \mathbb{P}^1$. We also see that the base change $\widetilde{\mathbb{P}}^1 \times_{\mathbb{P}^1} J$ is singular at the pre-images of the singular points of $J_{t_1} \cup J_{t_2}$ and hence, the morphism $X \to \widetilde{\mathbb{P}}^1 \times_{\mathbb{P}^1} J$ is a minimal resolution of singularities. If k_i is odd, the exceptional curves over singular points of J_{t_i} are even (odd) components of X_{t_i}.

Suppose F_{t_i} is smooth. Then, R_+ and R'_- intersect it at the same point $p_+^{(i)}$. Since F_i lies in the ramification locus of $X \to J'$, their image is a bisection C which is tangent to J'_{t_i} at the image of $p_+^{(i)}$. If F_{t_i} is singular and k_i is odd, then R_+ and R'_- intersect at different points in an even component of F_{t_i}. Their images belong to a (-1)-curve of $J_{t'_i}$, which is blown down to a point on J_{t_i}. Thus, the images of $R_+ + R'_-$ is a bisection on J that passes through the singular point of J_{t_i}. It is opposite of the component that intersects O.

To see the last property we use that

$$R_+ \cdot R_- = \sigma(R_+) \cdot \sigma(R_-) = R'_- \cdot R_- = 2m.$$

This shows that the image C of $R_+ + R_-$ intersects the image O of R_- at m double points, which are the images of the intersection points of R_+ and R_-. $\square$

We now explain a converse result, which gives an explicit construction of Enriques surfaces. Let $J \to W$ be the birational morphism from J to its Weierstrass model. Recall that W is a double cover of the rational minimal ruled surface $\mathbf{F}_2$ branched along the union of the special section E and a curve $B \in |6\mathfrak{f} + 3\mathfrak{e}|$, where $\mathfrak{f}$ is the divisor class of a fiber of $p : \mathbf{F}_2 \to \mathbb{P}^1$ and $\mathfrak{e}$ is the divisor class of E. The linear system $|2\mathsf{O} - 2K_J|$ defines a map equal to the composition of the map $J \to \mathbf{F}_2$ and the contraction of the special section.

1. Suppose that J_{t_i} is smooth. Since R_+ and R'_- intersect X_{t_i} at a 2-torsion point with respect to the zero section R_-, the bisection C is tangent to F_{t_i} at a 2-torsion point with respect to the zero section O. The image of J_{t_i} on $\mathbf{F}_2$ is the fiber $p^{-1}(t_i)$, which intersects B transversally at 3 points. The image $\bar{C}$ of C in $\mathbf{F}_2$ passes through one of them and intersects the fiber transversally at this point. This shows that $\bar{C}$ is a section of p and hence, belongs to the linear system $|a\mathfrak{f} + \mathfrak{e}|$. Since $C \cdot \mathsf{O} = 2m$, the curve $\bar{C}$ is tangent to E at m points and hence, belongs to the linear system $|(2m+2)\mathfrak{f} + \mathfrak{e}|$. We have $\bar{C} \cdot B = ((2m+2)\mathfrak{f} + \mathfrak{e}) \cdot (6\mathfrak{f} + 3\mathfrak{e}) = 6(m+1)$.

If $\bar{C}$ intersects B transversally at some point not lying over t_1 and t_2, then the composition of the normalization map $R_+ \to C$ and the double cover $C \to \bar{C}$ is ramified over more than 2 points and hence is not rational. This shows that $\bar{C}$ must be tangent at all $\frac{1}{2}(6(m+1)-2) = 3m+2$ intersection points with B not over t_1, t_2. Since $\dim((2m+2)\mathfrak{f}+\mathfrak{e}) = 3+2m$, we see that one does not expect that such curve C exist. This agrees with the fact that a general Enriques surface does not contain a rational bisection R.

2. Suppose that J_{t_i} is of type $\tilde{A}_{k_i-1}$. Then B has a simple singular point b_i of type a_{k_i-1} on $p^{-1}(t_i)$. The fiber intersects B with multiplicity 2 at this point. If k_i is odd, then the curve $\bar{C}$ is nonsingular at b_i and intersects B with multiplicity $2k_i$ at this point. It belongs to the linear system $|(2m+2)\mathfrak{f}+\mathfrak{e})|$ and the rest of the analysis is similar to the previous case. If k_i is even, then b_i is a simple singular point of $\bar{C}$ of type a_{k_i+1}. It intersects B with multiplicity $4k_i$ at this point. The curve $\bar{C}$ is a bisection of p and splits under the cover $J \to \mathbf{F}_2$ into the sum of C and $C' = \iota(C)$, where C is the negation involution. It belongs to the linear system $|(4m+4)\mathfrak{f}+2\mathfrak{e}|$. The singularities of $\bar{C}$ on E are tacnodes locally isomorphic to $y^2+x^4 = 0$, where $y = 0$ is the local equation of E. We have $\bar{C}\cdot B = 6(m+2)$ and we have to impose some further conditions to guarantee that C is a rational curve.

Example 4.10.11 Let S be an unnodal Enriques surface (that is, S does not contain smooth rational curves) that admits an elliptic fibration with an irreducible nodal half-fiber. By Lemma 4.10.9, this half-fiber splits in the K3-cover $\pi : X \to S$. To construct the surface S from its jacobian fibration, we have to find a curve $\bar{C}$ in the linear system $|4\mathfrak{f}+\mathfrak{e}|$ on $\mathbf{F}_2$ that satisfies the following condition: it is tangent to the exceptional section at one point, it passes through a point $b_1 \in B$ where it is tangent to the fiber of $p : \mathbf{F}_2 \to \mathbb{P}^1$, and it passes through another point $b_2 \in B$, where the fiber intersects B transversally and tangent to other four points on B. Since $\dim|4\mathfrak{f}+\mathfrak{e}| = 7$ and each tangency condition imposes one constraint, counting constants, we can always find an irreducible member of $|4\mathfrak{f}+\mathfrak{e}|$ satisfying our conditions. The moduli space of curves B is of dimension 8 and the curves $\bar{C}$ depend on one parameter corresponding to a choice of one smooth half-fiber on S. From this, we see that the moduli space of unnodal Enriques surfaces that admit an elliptic fibration with an irreducible nodal half-fiber depends on 9 parameters. Over $\mathbb{C}$, this corresponds to the fact that the Picard number of the K3-cover is equal to 11, see Section 5.3.

Example 4.10.12 Assume that S is a nodal surface, that is, S contains at least one smooth and rational curve. We will prove in Theorem 6.3.3in Volume II that S admits an elliptic fibration with a special bisection R. Suppose for simplicity, that its half-fibers are smooth. Then, $m = 0$ and we have to look for a curve $\bar{C} \in |2\mathfrak{f}+\mathfrak{e}|$ that passes through two fixed points on B and also tangent to B at two other points. The set of pairs $(\bar{C}, B) \in |2\mathfrak{f}+\mathfrak{e}| \times |6\mathfrak{f}+3\mathfrak{e}|$ satisfying these conditions is an irreducible subvariety of $|2\mathfrak{f}+\mathfrak{e}| \times |6\mathfrak{f}+3\mathfrak{e}|$ that is fibered over $|6\mathfrak{f}+3\mathfrak{e}|$ with general fiber isomorphic to a hypersurface in $|2\mathfrak{f}+\mathfrak{e}|$. It is easy to see that the variety of such pairs modulo $\mathrm{Aut}(\mathbf{F}_2)$ is of dimension 9. This confirms our expectation that the moduli space of nodal Enriques surface is of dimension 9: all Enriques surfaces form a 10-dimensional moduli space and containing at least one smooth rational curve should impose one condition.

Example 4.10.13 Assume $p \neq 2, 3$ and let

$$F(t) := x^3 + y^3 + z^3 + txyz = 0$$

be the Hesse pencil of plane cubics (4.9.33). It has nine base points

$$p_0 = (0,1,-1),\ p_4 = (0,1,-\epsilon),\ p_7 = (0,1,-\epsilon^2),$$

$$p_2 = (1,0,-1),\ p_5 = (1,0,-\epsilon^2),\ p_8 = (1,0,-\epsilon),$$

$$p_3 = (1,-1,0),\ p_6 = (1,-\epsilon,0),\ p_9 = (1,-\epsilon^2,0),$$

where ϵ denotes a primitive third root of unity. Let $J \to \mathbb{P}^1$ be the corresponding rational elliptic surface. The base points lead to sections of this elliptic fibrations and we fix one of the base base points, say $p_1 = (0,1,-1)$ to define a group law on $J \to \mathbb{P}^1$ by declaring the corresponding section to be the zero section.

The polar conic of $V(F_t)$ splits into the union of the tangent line $V(3y + 3z - tx)$ at the point $(0,1,-1)$ and the line $V(y+z)$ that does not depend on t. It is called the *harmonic polar line*. It intersects any member at the set of 2-torsion points of the connected component of identity of $(J_t^\sharp)^0$. Its image in $\mathbf{F}_2$ is our curve B. It has four cusps, the images of the components of singular fibers different from the one that intersects O. Fix two cusps c_i, c_j on B and consider the pencil $K(i,j)$ in $|2\mathfrak{f} + \mathfrak{e}|$ of conics (in the embedding of $\mathbf{F}_2$ into $\mathbb{P}^3$) passing through c_i, c_j. A general member $\bar{C}$ of the pencil intersects B with multiplicity 2 outside c_i, c_j. It is tangent to B at two points b_1, b_2.

Now we can make our construction of a torsor $f : S \to \mathbb{P}^1$ with two smooth fibers over the projections of b_1, b_2 under the projection $p : B \to \mathbb{P}^1$. This way, we construct a pencil of Enriques surfaces with an elliptic fibration of Hesse type (that is, with four singular fibers of type $\tilde{A}_2$) and a special bisection.

We can do this even more explicitly: it follows from [16, Remark 6.2] that the map $\phi' : \mathbb{P}^2 \dashrightarrow J \xrightarrow{\phi} \mathbf{F}_2$ is given by the linear system

$$a(x^3 + y^3 + z^3)^2 + bx^2y^2z^2 + cxyz(x^3 + y^3 + z^3) + d\Phi_6(x, y, z)$$

of plane curves of degree 6 with double points at the base points $p_2, \ldots, p_9$. Here, Φ_6 is a certain invariant of degree 6 with respect to the Hesse group G_{216} of projective automorphisms leaving invariant the Hesse pencil. The deck transformation of the cover $\phi : J \to \mathbf{F}_2$ corresponds to the projective involution

$$g_0 : [x, y, z] \mapsto [x, z, y].$$

The pre-image of a curve from $|2\mathfrak{f} + \mathfrak{e}|$ passing through three of the cusps is a pair of disjoint sections on J that add up to O in the Mordell–Weil group. Its pre-image under

ϕ' is the union of three pairs of sides of the triangles of lines corresponding to the reducible members of the pencil lines that intersect the sections. For example, assume that the sections correspond to the base points $p_2 = [1, -1, 0]$ and $p_3 = [1, 0, -1]$. They intersect the six components $V(y)$, $V(z)$, $V(x + y + \epsilon z)$, and $V(x + \epsilon y + z)$ of reducible members of the Hesse pencil that do not contain p_1. The pencil $K(1, 2)$ is generated by conics from $|2\mathfrak{f} + \mathfrak{e}|$ that pass through the cusps c_1, c_2, c_3 and c_1, c_2, c_4 of B. For example, we may assume the first curve splits into a pair of sections corresponding to the base points p_2, p_3 and the pair of base points $p_5 = [1, -\epsilon, 0]$ and $p_6 = [1, 0, -\omega]$. This shows that the pre-image of a conic from the pencil $K(1, 2)$ belongs to the pencil generated by the curves $V(yz(x + y + \epsilon z)(x + y + \omega^2 z)(x + \epsilon y + z)(x + \epsilon^2 y + z))$ and $V(yz(x + \epsilon y + \omega^2 z), (x + \epsilon^2 y + \epsilon z)(x + y + \epsilon z)(x + y + \epsilon^2 z))$. Getting rid of the common irreducible components, we obtain a pencil of conics

$$x^2 - xy + y^2 - yz + z^2 + sxz = 0.$$

It has four base points, the remaining base points p_4, p_7, p_8, p_9 of the Hesse pencil. The two reducible fibers correspond to the parameters $s = -1, 2$.

The intersection of a general member $H(s)$ of this pencil with a general member F_t of the Hesse pencil consists of six points, four of which are base points of both pencils. This shows that the residual set of two intersection points is given by a quadratic polynomial $P(T, s, t)$ in a rational parameter T of $H(s)$. We parameterize a general member of the pencil of conics

$$[x, y, z] = [-\epsilon T^2 - \epsilon^2 sT^2 + (1 - \epsilon^2)T, -\epsilon(T^2 + Ts + 1), \epsilon^2 T^2 - \epsilon T + 1].$$

Then we find the equation for T that determines the intersection points of $H(s)$ with F_t:

$$P(T, s, t) = T^2(-\epsilon^2 s^2 - 2\epsilon s + \epsilon t - 1) + (-s^2\epsilon + (-2\epsilon^2 + \epsilon t)s + 2)T + (\epsilon^2 + 2)s - \epsilon^2 + \epsilon t + \epsilon.$$

If $s \neq -1, 2$ corresponding to $t = -3, -3\epsilon^2$, then this is a quadratic equation in T and its discriminant is equal to

$$R(s, t) := (s - 2)\left(t^2(s + 2) - 2t(s^2 + 2s - 2) + s^3 + 6s^2 + 4\right).$$

This shows that fixing a pair of cusps of B, the pre-image $H(s_0)$ of a member of $|2\mathfrak{f} + \mathfrak{e}|$ that passes through these cusps is tangent to two members F_{t_1}, F_{t_2} of the Hesse pencil, where t_1, t_2 are solutions of the quadratic equation $(s - 2)^{-1} R(s_0, t) = 0$. Note that the special values of the parameter $s = 2$ and $s = -1$ (in this case we get a double root $t = -3$) correspond to the reducible members $E + f_i, E + f_j$ of the pencil $|2\mathfrak{f} + \mathfrak{e}|$, where f_i, f_j are the fibers of $\mathbf{F}_2 \to \mathbb{P}^1$ that pass through the two fixed cusps of B.

We see an explicit relationship between two multiple fibers of an elliptic fibration of Hesse type that admit a special bisection. The pairs of fibers, considered as a point in $(\mathbb{P}^1)^{(2)} \cong \mathbb{P}^2$ (Hilbert scheme of two points of $\mathbb{P}^1$) is a cubic curve given in parametric form as

$$[x, y, z] = [(u + 2v)v^2, -2v(u^2 + 2uv - 2v^2), u^3 + 6u^2 v + 4v^3)]$$

or in an explicit form as

$$432x^3 - 216x^2 y + 72xy^2 - 5y^3 - 144x^2 z + 12xyz + y^2 z - 4xz^2 = 0.$$

This is an irreducible plane cubic with a node at $[1, 24, 90]$ (It corresponds to the parameters $s = -7 \pm 3\sqrt{-3}$, whose geometric meaning we do not know).

Note that the Hesse group G_{216} of automorphisms of J acts transitively on pairs of reducible fibers, so we may also fix the choice of the pair of cusps c_i, c_j. This shows that each irreducible component of the moduli space of Enriques surfaces together with an elliptic fibration of Hesse type that admits a special bisection is a rational curve.

Remark 4.10.14 Let $f : S \to \mathbb{P}^1$ be an extremal elliptic fibration on an Enriques surface (that is, the associated jacobian fibration is an extremal rational surface) with a special bisection R. Suppose we can choose one irreducible component in each reducible fiber, such that the remaining components of reducible fibers together with R generate a negative root lattice, which is then necessarily of rank 9. Blowing down the nine (-2)-curves, we obtain a surface with rational double points and Picard number equal to 1, a $\mathbb{Q}$-*homology projective plane*. In [322] and [643], it is shown that there are 31 different isomorphism classes of negative definite lattices of rank 9 that can be realized in this way (in characteristic $p = 0$). The latter paper describes the moduli spaces of Enriques surfaces supporting such lattices. In our example, the special bisection that we constructed intersects two reducible fibers at one component and the other two reducible fibers at two components. This gives a lattice isomorphic to $\mathsf{A}_2 \oplus \mathsf{A}_2 \oplus \mathsf{A}_5$. The other possibility is when R intersects one component in one fiber and two components in other three fibers. This gives the lattice isomorphic to $\mathsf{A}_2 \oplus \mathsf{A}_2 \oplus \mathsf{A}_2 \oplus \mathsf{A}_3$. The parametric equation of the corresponding curve parameterizing the locus of Enriques surfaces of Hesse type admitting such bisections is given in [643, Table 5]. It is a nodal cubic curve.

Bibliographical notes

The study of elliptic pencils on algebraic surfaces over the complex numbers was initiated by F. Enriques in his book [221]. A modern treatment of Enriques's work was given by I. Shafarevich in [5, Chapter 7] and K. Kodaira in [401]. Since then, many expositions of this theory in the case where the ground field is $\mathbb{C}$ have been included in classical textbooks such as [47], [43], [240], [513]. Many general facts about elliptic fibrations which we treat in this chapter have been proven by Kodaira in the complex analytic setting.

The first systematic study of genus one fibrations (including quasi-elliptic fibrations) on algebraic surfaces over algebraically closed fields of positive characteristic was given by E. Bombieri and D. Mumford [78] and [77]. We have also borrowed a lot from unpublished notes of M. Raynaud [607]. In particular, we supplied the proofs of Raynaud's improvement of the formula for the canonical sheaf of a non-jacobian genus one fibration, see also [58, Proposition 3.17].

The material in Section 2 is mostly due to M. Raynaud and can be found in [603] and [606]. The theory of Néron models is discussed in the monograph [86] by S. Bosch, W. Lütkebohmert, and M. Raynaud.

The study of elliptic fibrations as torsors under their jacobian fibrations goes back to F. Enriques [221] and the theory of torsors applied to elliptic fibrations is discussed by I. Shafarevich in [5, Chapter VII]. The transcendental construction of torsors based on logarithmic transformations is due to K. Kodaira [401]. In his work Kodaira also introduced the complex-analytic analog of the Tate–Shafarevich group and showed that its elements define a locally trivial complex analytic torsors and that its torsion elements define locally trivial algebraic torsors. An exposition of this theory can be found in [43] and [240], we are not discussing it here. The formula for the rank of the Mordell–Weil group is due to [674], see also [701]. It is often referred to as the Shioda–Tate formula.. The description of all possible finite Mordell–Weil groups given by formula (4.3.4) appeared first in [674] and [141]. The relationships between Brauer groups and other invariants of an elliptic surface and its jacobian surface are discussed in [25] and [607]. The result that the types of reducible fibers of torsors and their jacobian fibrations coincide is proven in [473] and a partial result can be found in [138].

The notion of a Mordell–Weil lattice of an elliptic surface was introduced by T. Shioda [678], [679] and, independently, by N. Elkies (unpublished). However, many aspects of this theory already appeared earlier in the work of Yu. Manin [485] and D. Cox and S. Zucker [141]. For more history, we refer to the Historical Notes in chapter 6 of M. Schütt's and T. Shioda's book [646] that contains the by now most complete exposition of the theory of Mordell–Weil lattices and their various applications.

The Weil–Châtelet group was first introduced by A. Weil [736]. Its cohomological interpretation was first given by I. Shafarevich [664] and independently, by S. Lang and J. Tate [430]. The extensive study of this group was undertaken by I. Shafarevich [665] and, independently, by A. Ogg [569]. A complex analytic version of the theory of elliptic fibrations was developed at the same time by K. Kodaira [401]. Almost all the results that we discussed in this chapter have analogs in this situation. The analog of the twist construction of a torsor from a 1-cocycle is Kodaira's *logarithmic transformation*. A beautiful exposition of Kodaira's theory can be found in [240].

Ogg and Shafarevich compute the prime-to-p part of the Weil–Châtelet group for the field of algebraic functions in one variable with algebraically closed field of constants. An exposition of their work based on Grothendieck's theory of cohomology of constructive sheaves on algebraic curves was the subject of a Bourbaki talk by M. Raynaud [604]. The relationship between the Tate–Shafarevich group of a generic fiber of a jacobian elliptic fibration on a surface and the Brauer group of the surface is due to A. Grothendieck [271].

The work on the p-part of the Weil–Châtelet groups was initiated in a series of articles by O. Vvedenskii [726], [727], [728]. He proved the duality theorem for elliptic curves with all possible types of reduction, but omitted the case of additive reduction in characteristic $p = 2, 3$. The general duality theorem for abelian varieties was proved by M. Bester [53] in the case of a good reduction and by A. Bertapelle [56] with no restriction on the reduction.

The theory of torsors for quasi-elliptic fibrations is based on the work of P. Russell [629]. The first geometric application was given by W. Lang in the case where $p = 3$ [431]. We have extended some of his results to the case where $p = 2$. The results discussed at the end of the section are taken from [183].

The classification of singular fibers of extremal rational elliptic surfaces was given over the complex numbers by R. Miranda and U. Persson [515] in terms of Weierstrass equations and by I. Naruki [551] in terms of pencils of cubic curves. In special cases $p = 2, 3, 5$, this requires some special arguments, and the classification in terms of the Weierstrass equations was first given by W. Lang [436], [437] . Some misprints from the former article have been corrected by A. Schweizer [647]. The classification of rational quasi-elliptic surfaces can be found in H. Ito's work [336] and [335]. The equations of the one-dimensional unipotent group that arise from a jacobian quasi-elliptic fibration were given first in [600], [601]. The computation of the Weil–Châtelet group of a wound unipotent group over global or local base C was initiated by W. Lang [431]. It was extended to characteristic two in [183].

Chapter 5
Moduli Spaces

In this chapter, we study moduli spaces of Enriques surfaces. Over the complex numbers, this can be done via lattice-polarized K3 surfaces and their moduli spaces, which leads to constructions of moduli spaces of marked, unmarked, polarized, and nodal Enriques surfaces. We discuss maps between these moduli spaces and the birational geometry of these moduli spaces, that is, their dimensions, Kodaira dimensions, and (uni-)rationality questions. We study some classical compactifications of some of these moduli spaces and we address the question whether the boundary itself has a modular interpretation. This leads to the study of Coble surfaces of K3 type and Kulikov degenerations of Enriques surfaces. Finally, we study moduli spaces in positive and mixed characteristic.

5.1 Moduli Problems and Moduli Spaces

In this section, we discuss various moduli spaces of Enriques surfaces. The idea and the terminology of a moduli space goes back to Riemann, who showed that the set of isomorphism classes of compact Riemann surfaces of genus $g \geq 2$ depends on $3g-3$ parameters or moduli, that is, this set forms a $(3g-3)$-dimensional space. See, for example, [259, Chapter 2.3] for Riemann's original heuristics and considerations that led to these insights.

To make this idea and some of these heuristics more precise, one has to consider not individual objects, but families of the objects one wants to parametrize from the very beginning. Next, one has to introduce appropriate equivalence relations on these families, and then, one seeks for a universal family, from which all families arise via base change. This leads to the very conceptual approach to moduli problems using the language of functors. However, one is then led to the problem of showing that such a functor is representable by a suitable space (a scheme, an algebraic space, or an algebraic stack), which is the sought moduli space, and which automatically comes with a universal family of the objects one wants to parameterize. A necessary condition for the representability of the moduli functor is that it satisfies a sheaf axiom (with respect to some fixed Grothendieck topology) and that one can find some family that contains all the objects one wants to parametrize. The idea of Deligne and Mumford was to turn this approach into a definition: by definition, a groupoid–valued functor that satisfies a sheaf axiom is a stack, and if it can be covered by some algebraic families then it is algebraic. In this section, we will sketch the ideas and notions of this approach, but we claim by no means a thorough treatment of moduli theory, of algebraic stacks, etc. Working out this program in detail requires quite an amount of theoretical foundations and is rather complicated and lengthy. Here, we will only sketch some of the main ideas so that we can work with the objects and refer the reader to [581] for a thorough treatment.

In this chapter, our objects of concern will be K3 surfaces or Enriques surfaces. We fix an algebraically closed field $\Bbbk$. An *algebraic family* of such surfaces is a smooth morphism $f : \mathcal{X} \to T$ in the category of schemes (or algebraic spaces) over $\Bbbk$, such that the fiber $\iota_{\bar{t}} : \mathcal{X}_{\bar{t}} \to \mathcal{X}$ over every geometric point $\bar{t} : \operatorname{Spec} K \to T$ is a K3 surface (resp. Enriques surface) over K. Often, we will put additional structures on these families. For example, the families may come together with an invertible sheaf $\mathcal{L}$ on $\mathcal{X}$, such that $\iota_{\bar{t}}^*(\mathcal{L})$ is ample or nef on $\mathcal{X}_{\bar{t}}$. We will give details later, when discussing moduli of polarized surfaces.

Next, we define a contravariant functor

$$\mathcal{M} \;:\; (\text{Schemes}/\Bbbk) \;\to\; (\text{Sets})$$

that associates to every scheme T over $\Bbbk$ the set of all families over T modulo some appropriate notion of isomorphism. If $T' \to T$ is a morphism of schemes, then $\mathcal{M}(T)$ is a family $\mathcal{X} \to T$ and then, the fiber product $\mathcal{X} \times_T T' \to T'$ is a family over T', that is, an element of $\mathcal{M}(T')$. This defines a map $\mathcal{M}(T) \to \mathcal{M}(T')$ and turns $\mathcal{M}$ into a contravariant functor. The idea behind this approach is as follows: we hope for the existence of a scheme $\mathbf{M}$ over $\Bbbk$ together with a family $\mathcal{U} \to \mathbf{M}$, such that for every family $f : \mathcal{X} \to T$ as above there exists a unique morphism $T \to \mathbf{M}$, the *classifying morphism*, such that the family $\mathcal{X} \to T$ is isomorphic to the pull-back $\mathcal{U} \times_{\mathbf{M}} T \to T$ along the classifying morphism. In particular, if K is a field extension of $\Bbbk$, then the set of all objects we are interested over K is in bijection with the set of K-valued points of $\mathbf{M}$, that is, $\mathbf{M}(K)$. In this case, the space $\mathbf{M}$ is called a *fine moduli space* and the family $\mathcal{U} \to \mathbf{M}$ is called the *universal family*. The relation to the functor $\mathcal{M}$ is as follows: if such an $\mathbf{M}$ exists, then the functor $\mathcal{M}$ is isomorphic to the Yoneda functor $h_{\mathbf{M}} : T \to \operatorname{Mor}_{\Bbbk}(T, \mathbf{M})$. Moreover, the universal family $\mathcal{U}$ corresponds to the identity morphism in $\operatorname{Mor}_{\Bbbk}(\mathbf{M}, \mathbf{M})$.

In view of this discussion, we say that our moduli problem admits a *fine moduli space* if the functor $\mathcal{M}$ is representable by a scheme, that is, if there exists a scheme $\mathbf{M}$ over $\Bbbk$, such that the functor $\mathcal{M}$ is isomorphic to the Yoneda functor $h_{\mathbf{M}} : T \to \operatorname{Mor}_{\Bbbk}(T, \mathbf{M})$. It follows from the Yoneda lemma that a fine moduli space, if it exists, is unique up to isomorphism. As explained above, it comes with a *universal family* $\mathcal{U} \to \mathbf{M}$, namely, the family over $\mathbf{M}$ corresponding to the identity in $h_{\mathbf{M}}(\mathbf{M})$. It is easy to see that this universal family has the property that every family over T is isomorphic to the base change of the universal family along the classifying morphism $T \to \mathbf{M}$.

Unfortunately, fine moduli spaces exist rather rarely, unless we put some additional structure on the families. One of the reasons is the existence of automorphisms of objects that we want to parametrize. Here is a classical example.

Example 5.1.1 Let $\mathcal{M}_{1,1}$ be the moduli problem of pairs (E, O) of elliptic curves together with a fixed point O (the neutral element of the group law) over $\Bbbk$. Thus, a family $f : \mathcal{E} \to \mathcal{T}$ is a smooth genus one fibration together with a section $s : T \to \mathcal{E}$. Suppose that a fine moduli space $\mathbf{M}_{1,1}$ exists and let $\mathcal{U}_{1,1} \to \mathbf{M}_{1,1}$ be the universal family. Then, for every family $\mathcal{X} \to T$, we have a Cartesian diagram

$$\begin{tikzcd} \mathcal{X} \arrow[r, "\tilde{\phi}"] \arrow[d] & \mathcal{U}_{1,1} \arrow[d] \\ T \arrow[r, "\phi"] & \mathbf{M}_{1,1} \end{tikzcd}$$

Now, consider the negation involution $\iota : x \mapsto -x$ with respect to the group law of the relative elliptic curve $\mathcal{X} \to T$. Then, the composition $\mathcal{X} \xrightarrow{\iota} \mathcal{X} \to T$ yields an isomorphic family, hence defines, by uniqueness, the same classifying morphism $T \to \mathbf{M}_{1,1}$. If $\mathbf{M}_{1,1}$ was a fine moduli space, then $T \to \mathbf{M}_{1,1}$ would correspond to a unique family of elliptic curves rather than two distinct ones that are isomorphic. However, even when identifying isomorphism classes of families, then this does not help: suppose that there exists a non-trivial $(\mathbb{Z}/2\mathbb{Z})$-torsor $S \to T$. Then, one can form the twisted family $\mathcal{X}' := (\mathcal{X} \times_T S)/(\iota) \to T$, where the quotient is with respect to the diagonal action given by ι on the first factor and by the $(\mathbb{Z}/2\mathbb{Z})$-action on the second factor. Both families over T have isomorphic geometric fibers and thus, define the same classifying morphism $\mathcal{T} \to \mathbf{M}_{1,1}$. In general, these two families over T are not isomorphic, but since they have the same classifying morphism, at most one can be the pull-back of the universal family over $\mathbf{M}_{1,1}$. This contradiction shows that there does not exist a universal family and thus, the functor $\mathcal{M}_{1,1}$ is not representable. To give an explicit example, we let $T = \mathbb{A}^1_{\Bbbk} - \{0\} = \operatorname{Spec} k[t]_{(t)}$ and consider the two families of elliptic curves over T

$$\mathcal{X} \;:\; y^2 = x^3 - 1 \quad \text{and} \quad \mathcal{X}' \;:\; y^2 = x^3 - t.$$

In both cases, all geometric fibers are isomorphic to the elliptic curve $y^2 = x^3 - 1$, that is, in both cases, the classifying morphism $T \to \mathbf{M}_{1,1}$ would be constant. In particular, the pull-back from the universal family $\mathcal{U} \to \mathbf{M}_{1,1}$ along this classifying morphism would yield a trivial product family. This is indeed the case for $\mathcal{X}$, but not for $\mathcal{X}'$, and we refer to [581, Preface] for details and further discussion of this example.

Similar arguments show that the moduli problem for hyperelliptic curves of genus $g \geq 2$ is not representable. We refer to [290, Section 2.A] for details and further discussion.

However, the following approach can remedy this problem: let $n \geq 3$ be an integer that is coprime to the characteristic p of $\Bbbk$. Now, consider the functor that associates to a scheme T over $\Bbbk$ the set of families $\mathcal{E} \to T$ of smooth genus one fibrations together with a choice of section $s : T \to \mathcal{E}$ (the zero section that turns this family into a family of elliptic curves) and another section that is fiberwise an n-torsion point. Then, there are no automorphisms of such families and in fact, this time, a fine moduli space does exist, namely the modular curve $X_0(n)$.

Since fine moduli spaces do not exist in many cases of interest, one often asks for less, namely for a *coarse moduli space*. Instead of requiring the representability of the functor $\mathcal{M}$, we merely ask for the existence of a scheme (or an algebraic space and we note that we discuss algebraic spaces below) $\mathbf{M}$ over $\Bbbk$ and a morphism of functors $\pi : \mathcal{M} \to h_{\mathbf{M}}$, such that $\mathcal{M}(\mathrm{Spec}\ K) \to h_{\mathbf{M}}(\mathrm{Spec}\ K)$ is bijective for every algebraically closed field K and such that for every scheme (or algebraic space) N over $\Bbbk$ and every morphism $g : \mathcal{M} \to h_N$, there exists a unique morphism $f_* : h_{\mathbf{M}} \to h_N$ (or, equivalently, a unique morphism $f : \mathbf{M} \to N$), such that $g = f_* \circ \pi$. A coarse moduli space, if it exists, is unique up to isomorphism. In the sense made precise by its definition, it is the closest approximation of $\mathcal{M}$ to a scheme (or algebraic space). We refer to [581, Chapter 11] for details.

For example, a coarse moduli space of the moduli functor $\mathcal{M}_{1,1}$ from Example 5.1.1 exists and coincides with the affine line $\mathbb{A}^1_{\Bbbk}$. More precisely, for every family $\mathcal{E} \to T$ in $\mathcal{M}_{1,1}(T)$, the j-invariant $t \to j(\mathcal{E}_t)$ defines a map $T \to \mathbb{A}^1_{\Bbbk}$. This defines a morphism of functors $\mathcal{M}_{1,1}(T) \to h_{\mathbb{A}^1_{\Bbbk}}$ that satisfies the properties of a coarse moduli space. The j-invariant is the classifying morphism in this case.

Unfortunately, even this weaker notion of representing a moduli functor may not exist, as the following example shows.

Example 5.1.2 Let G be an affine group scheme over $\Bbbk$, let X be a variety over $\Bbbk$, and assume that X comes with a G-action. Let $\mathcal{M}(T)$ be the set of diagrams

$$\begin{array}{ccc} P & \xrightarrow{f} & X \\ \downarrow{\scriptstyle p} & & \\ T & & \end{array}$$

where f is a G-equivariant morphism and where $p : P \to T$ is a G-torsor over T, that is, p is a G-equivariant morphism $f : X \to T$, where G acts trivially on T and such that locally for some chosen Grothendieck topology, X is isomorphic to $T \times G$. A morphism $\phi : T' \to T$ defines a map from $\mathcal{M}(T) \to \mathcal{M}(T')$ that sends a pair (p, f) as above to $(p' : P \times_T T' \to T', f \circ \mathrm{pr}_2)$.

Now, suppose that X is affine. We claim that if the functor $\mathcal{M}$ admits a coarse moduli space, then it is isomorphic to $\mathbb{M} := \mathrm{Spec}\ \mathcal{O}_X(X)^G$. In fact, it follows from the definition of a torsor that $\mathcal{O}_T = (p_*\mathcal{O}_P)^G$ and hence, $\mathcal{O}_T(T) = \mathcal{O}_P(P)^G$. The G-equivariant morphism $f : P \to X$ defines a homomorphism of coordinate rings $\mathcal{O}_X(X)^G \to \mathcal{O}_P(P)^G = \mathcal{O}_T(T)$. This defines a morphism $T \to \mathbb{M}$. Using this, it is easy to check that a coarse moduli space, if it exists, coincides with $\mathbb{M}$. Taking

$T = \operatorname{Spec}\ \Bbbk$, we see that $\mathcal{M}(T)$ consists of G-orbits in X. On the other hand, it must be equal to $\operatorname{Hom}_{\Bbbk}(\mathcal{O}_X(X)^G, \Bbbk)$. However, this cannot be always true. For example, one could take $G = \mathbb{C}^\times$, $X = \mathbb{C}$, and the G-action on X defined by $z \mapsto \lambda z$. Then, we have $\mathcal{O}_X(X) = \mathbb{C}$, but we have the two orbits $\{0\}$ and $\mathbb{C} - \{0\}$.

To overcome these difficulties, Deligne and Mumford [157] introduced a novel idea that a moduli problem should not be considered as a contravariant functor $(\text{Schemes}/\Bbbk) \to (\text{Sets})$, but rather as a contravariant functor from $(\text{Schemes}/\Bbbk)$ to the category to *groupoids* rather than sets. (A groupoid is a category, in which every morphism is an isomorphism. A set becomes a groupoid, whose objects are the elements of the sets and whose morphisms are only the identity maps.) In particular, when considering a family $\mathcal{X} \to T$ as above, then we do not identify isomorphic families, that is, $\mathcal{M}(T)$ is the groupoid of all families over T. All such families must form a category $\mathcal{M}$ that comes with a functor $p : \mathcal{M} \to (\text{Schemes}/\Bbbk)$. More precisely, the category $\mathcal{M}(T)$ is a fiber of this functor, that is, it consists of objects f in $\mathcal{M}$, such that $p(f) = T$. One also says that $\mathcal{M}$ is a category fibred in groupoids. We will write objects of $\mathcal{M}(T)$ as arrows $\phi : X \to T$. The morphisms $(f : X \to T) \to (f' : X' \to T')$ consist of morphisms $\phi : T' \to T$ in $(\text{Schemes}/\Bbbk)$ and a morphism $\tilde{\phi} : X' \to X$, such that the diagram

$$\begin{tikzcd} X \arrow[d, "f"] & X' \arrow[l, "\tilde{\phi}"'] \arrow[d, "f'"] \\ T & T' \arrow[l, "\phi"'] \end{tikzcd}$$

is Cartesian, that is, a pull-back diagram.

This definition is too general to work with. Without being very precise (we refer to [581] for details and precise definitions), we say that $\mathcal{M}$ is a *stack* if the association $T \mapsto \mathcal{M}(T)$ is a presheaf with values in groupoids, that is, it satisfies the axiom of a sheaf with respect to coverings $\{T_i \to T\}_{i \in I}$ with respect to some chosen Grothendieck topology (usually flat or étale) and it also is required that for any $f, g \in \mathcal{M}(T)$ the presheaf $\operatorname{Isom}(f, g)$ on the category $(\text{Schemes}/T)$ is a sheaf. Roughly speaking, the first condition says that to give a family $\mathcal{X} \to T$ is equivalent to giving a covering $\{T_i \to T\}_{i \in I}$ and families $\mathcal{X}_i \to T_i$, such that the pull-backs of $\mathcal{X}_i$ and $\mathcal{X}_j$ to $T_i \times_T T_j$ are isomorphic and fulfill a certain cocycle condition on triple fiber products over T. Similarly, the second condition says that to give a morphism $f : \mathcal{X} \to \mathcal{Y}$ of families over T is equivalent to giving a covering $\{T_i \to T\}_{i \in I}$ and morphisms $f_i : \mathcal{X} \times_T T_i \to \mathcal{Y} \times_T T_i$, such that the pull-backs of f_i and f_j to $\mathcal{X} \times_T (T_i \times T_j)$ coincide. Even more roughly, these conditions ensure that one can constructs families and morphisms between them locally if they glue on overlaps. An example of a stack is the category $(\text{Schemes}/X)$ for any $X \in (\text{Schemes}/\Bbbk)$. We denote it by $\underline{X}$. More precisely, the groupoid $\underline{X}(T)$ is the set of all morphisms of schemes to $T \to X$, that is, $h_X(T)$.

A *morphism* of stacks $\mathcal{M}' \to \mathcal{M}$ is a functor $F : \mathcal{M}' \to \mathcal{M}$ that commutes with the projection functors $p' : \mathcal{M}' \to (\mathrm{Schemes}/\Bbbk)$ and $p : \mathcal{M} \to (\mathrm{Schemes}/\Bbbk)$ and the pull-back functors.

For example, a morphism $\underline{X} \to \mathcal{M}$ assigns to any morphisms of schemes $f : T \to X$ a "family" $f \in \mathcal{M}(T)$. We say that X is a *fine moduli space* for the stack $\mathcal{M}$ if there exists an equivalence of categories $\underline{X} \to \mathcal{M}$. One can also define a *coarse moduli space* of a stack. If it exists, then it is an object $\mathbf{M}$ of $(\mathrm{Schemes}/\Bbbk)$ (or $(\mathrm{AlgebraicSpaces}/\Bbbk)$), together with a morphism of stacks $\pi : \mathcal{M} \to \underline{\mathbf{M}}$, such that $\mathcal{M}(\mathrm{Spec}\, K) \to X(K)$ is an equivalence of categories for every algebraically closed field extension K of $\Bbbk$ and such that for every scheme (or algebraic space) N and every morphism $g : \mathcal{M} \to \underline{N}$, there exists a unique morphism $f : \mathbf{M} \to N$ such that $g = f \circ \pi$. Clearly, a coarse moduli space, if it exists, is unique up to isomorphism.

Next, one may try to cover a stack by a scheme X, that is, to find a morphism of stacks $\underline{X} \to \mathcal{M}$, such the functor $X(T) \to \mathcal{M}(T)$ is surjective on objects. Such stacks are said to be *algebraic*.

An example of such stacks, which generalize the notion of a scheme, is an *algebraic space*, where $\mathcal{M}$ is defined to be the quotient of some scheme X by an equivalence relation $R \to X \times_{\Bbbk} X$, where the two projections $R \to X$ are étale morphisms. Then, the sheaf in the étale topology associated to the pre-sheaf of quotients $X(T)/R(T)$ defines a stack and since it can be covered by X, it is algebraic. A morphism of algebraic spaces $\alpha : \mathcal{F} \to \mathcal{G}$ is a morphism of étale sheaves. We say that α is representable by a scheme if for every $T \in (\mathrm{Schemes})/\Bbbk$ and every morphism of sheaves $h_T \to \mathcal{G}$ the fiber product

$$\mathcal{F} \times_{\mathcal{G}} T \,:\, U \;\mapsto\; \{(a \in \mathcal{F}(U), f : U \to T) \,|\, \alpha(a) = G(f)\}$$

is representable by a scheme. This means that the restriction of the morphism $\mathcal{F} \to \mathcal{G}$ to the subcategory $(\mathrm{Schemes}/T)$ of $(\mathrm{Schemes}/\Bbbk)$ coincides with a morphism of a scheme. We say that a representable morphism $\mathcal{F} \to \mathcal{G}$ has property P (for example, affine, étale, a closed immersion, an open immersion, proper, or smooth) if, for every scheme T, the projection morphism of schemes $\mathcal{F} \times_{\mathcal{G}} T \to T$ has property P.

An equivalent definition of an algebraic space $\mathcal{F}$ is that $\mathcal{F}$ is a sheaf in the étale topology, that the diagonal morphism $\Delta : \mathcal{F} \to \mathcal{F} \times_{\Bbbk} \mathcal{F}$ is representable by a scheme, and that there exists a surjective and étale morphism $U \to \mathcal{F}$. The property of the diagonal map being representable guarantees that for every scheme T any morphism $h_T \to \mathcal{F}$ is representable by a scheme. Conversely, the latter property implies that $\mathcal{F}$ is a quotient of a scheme by an étale equivalence relation.

It is known that, for any algebraic space $\mathcal{F}$, there exists a morphism from a $\Bbbk$-scheme $f : X \to \mathcal{F}$ that is representable by a birational morphism of algebraic varieties. In particular, there exists an open and dense subalgebraic space of $\mathcal{F}$ that is isomorphic to a scheme.

Example 5.1.3 The following two examples are typical sources of algebraic spaces that are not schemes:

1. Let $\Bbbk$ be an algebraically closed field of characteristic zero. Let $C_0 \subset \mathbb{P}^2_{\Bbbk}$ a smooth cubic curve, let $P_1, ..., P_{10}$ be ten distinct points on C_0, let X' be the blow-up

of $\mathbb{P}^2$ in these ten points, and let C be the strict transform of C_0 on X'. Thus, C is isomorphic to C_0 and satisfies $C^2 = -1$. Then, there exists a contraction $X' \to X$ of C in the category of algebraic spaces, that is the morphism is proper, birational, equal to its own Stein factorization, it contracts C to a point, and induces an isomorphism on $X' - C$. If the $P_1, \ldots, P_{10}$ are chosen to be sufficiently general, then X is not isomorphic to a scheme. This classical example is due to Mumford and Nagata and we refer to [23, Section 4] for details.

2. Second, if a finite group G acts on a quasi-projective variety X, then the quotient X/G exists in the category of quasi-projective schemes. In fact, quasi-projectivity ensures that there exists an open and affine cover $\{U_i = \operatorname{Spec} R_i\}_{i\in I}$ of X such that each U_i is G-stable. Given such a cover, the individual quotients U_i/G exist, namely, $\operatorname{Spec} R_i^G$, where R_i^G denotes the ring of G-invariants. These quotients $\{U_i/G\}_{i\in I}$ then glue to X/G. On the other hand, if X is not quasi-projective, then there exist examples, where the quotient X/G exists as an algebraic space, but not as a scheme. We refer to [399, Chapter 4] for details.

Moreover, a theorem of Keel and Mori asserts that the quotient of an algebraic variety by a proper action of a group scheme acting with finite stabilizers always exists as a separated algebraic space [381]. This collection of results should illustrate that the category of algebraic spaces does have some advantages over the category of schemes. We refer to [23, 399, 581] for more about algebraic spaces.

Remark 5.1.4 Let X be a compact analytic manifold over $\mathbb{C}$. Then, its field of meromorphic functions $\mathbb{C}(X)$ is of transcendence degree at most equal $\dim(X)$. If it is equal to $\dim(X)$, then X is said to be a *Moishezon manifold*. Now, given a smooth and proper algebraic space Y over $\mathbb{C}$, there is an associated complex analytic manifold Y^{an} and since the function field of Y is of transcendence degree equal to the dimension of Y over $\mathbb{C}$, it follows that Y^{an} is a Moishezon manifold. Conversely, every Moishezon manifold arises as analytification of a smooth and proper algebraic space over $\mathbb{C}$. We refer to [294, Appendix B] for details and references.

Finally, we come to the notion of an algebraic stack and a Deligne–Mumford stack. We say that a stack is an *algebraic stack* or an *Artin stack* if the diagonal morphism $\Delta : \mathcal{M} \to \mathcal{M} \times_{\Bbbk} \mathcal{M}$ is representable and if there exists a surjective morphism $\underline{X} \to \mathcal{M}$ for some scheme X. If additionally this surjective morphism can be chosen to be étale, we say that $\mathcal{M}$ is a *Deligne–Mumford stack*. We note that an algebraic stack $\mathcal{M}$ is a Deligne–Mumford stack if for every algebraically closed field K the automorphism group scheme of any object in $\mathcal{M}(K)$ is a finite reduced group scheme, see [581], Theorem 8.3.3 and Remark 8.3.4.

Similar to the above notion for algebraic spaces a morphism $F : \mathcal{M} \to \mathcal{M}'$ is said to be representable if for any $\Bbbk$-scheme T the fiber product $\mathcal{M} \times_{\mathcal{M}'} T$ is representable by an algebraic space. The condition that the diagonal morphism is representable is equivalent to the following: for any $\Bbbk$-scheme T and two objects $f_1, f_2 \in \mathcal{M}(T)$, the sheaf $\underline{\mathrm{Isom}}(u_1, u_2)$ on $(\mathrm{Schemes}/T)$ is an algebraic space.

The following crucial results show that this approach via stacks does indeed work in the sense that some very important and central moduli problems are representable by Deligne–Mumford stacks.

Example 5.1.5 For non-negative integers g, n, one defines the moduli functor

$$\mathcal{M}_{g,n} \; : \; (\text{Schemes}/\Bbbk) \; \to \; (\text{Groupoids}),$$

that associates to every $\Bbbk$-scheme T the groupoid of families $\mathcal{X} \to T$, whose geometric fibers are smooth and projective curves of genus g, together with disjoint sections $s_1, ..., s_n : T \to \mathcal{X}$. That is, $\mathcal{M}_{g,n}$ is the moduli functor for *n-marked curves of genus g*. If $2g-2+n > 0$, then $\mathcal{M}_{g,n}$ is a smooth Deligne–Mumford stack over $\Bbbk$. In particular, the set-valued moduli functor from Example 5.1.1 does not admit a fine moduli space, but the just constructed groupoid-valued functor can be represented by a Deligne–Mumford stack. The key point is that, if $2g-2+n > 0$, then the automorphism group scheme of an n-marked curve of genus g over a field is finite and étale. We refer to [581, Chapter 13] for details, proof, and further discussion.

Example 5.1.6 This is the follow-up of Example 5.1.2: first, we convert the moduli problem $\mathcal{M}$ into a stack that will be denoted by $[X/G]$. We set $[X/G]$ to be the category whose objects are $(P \to T, f : P \to X)$ as before and whose morphisms from $(P' \to T', f' : P' \to X)$ to $(P \to T, f : P \to X)$ are isomorphisms of torsors $\Phi : (P' \to P \times_T T' \to T') \to (P' \to T')$ such that $f \circ \mathrm{pr}_1 \circ \Phi = f'$. This is a fibered category $[X/G] \to (\text{Schemes}/\Bbbk)$ whose fibers $[X/G](T)$ are the category, whose objects are pairs $(P \to T, f : P \to X)$ as above and whose morphisms $(P \to T, f : P \to X) \to (P' \to T, f' : P \to X)$ are isomorphisms of torsors $\Phi_T : P' \to P$ with $f \circ \Phi = f$. Note the stack makes sense even if G acts trivially on X. One can check that $[X/G]$ is an algebraic stack, and that if G is an algebraic group that acts with finite and reduced stabilizer subgroup schemes, then it is a Deligne–Mumford stack [157], Theorem 4.21. For example, if $X = \operatorname{Spec} \Bbbk$, then we obtain the so-called *classifying stack* for the group G.

Finally, let us mention the following fundamental result of Keel and Mori [381, Corollary 1.3]: a separated algebraic stack of finite type over $\Bbbk$ and with finite inertia stack always admits a coarse moduli space in the category of algebraic spaces (but not necessarily in the category of schemes). In Example 5.1.6, assuming that G acts with finite stabilizers, the coarse moduli space of $[X/G]$ is the geometric quotient of the subset $X^{\mathrm{s}}(\mathrm{Pre})$ of pre-stable points, that is, points admitting an open affine neightborhood that is G-invariant.

This suggests the following strategy when constructing moduli spaces: one first sets up a functor $\mathcal{M}$ from $(\text{Schemes}/\Bbbk)$ to (Groupoids) that associates to T the groupoid of families over T one is interested in. If $\mathcal{M}$ is a sheaf with respect to the étale topology, say, and if the automorphism group schemes are finite and étale, then there is a good chance to represent $\mathcal{M}$ by a Deligne–Mumford stack, which is indeed true in many cases of interest. Moreover, since the stabilizers are finite, a coarse moduli space exists at least in the category of algebraic spaces. Moreover, whenever a "parameter space" was classically constructed using geometric invariant theory, then the discussion of the previous paragraph suggests that this might actually be the coarse space for $\mathcal{M}$, which is true in many cases.

5.2 Lattice Polarized K3 Surfaces

In this section, we consider moduli spaces for pairs (X, j), where X is a K3 surface and $j : M \to \mathrm{Pic}(X)$ is a primitive embedding of lattices for some fixed lattice M. These moduli spaces will be used in the next section to construct moduli spaces for Enriques surfaces. For more details on moduli spaces of lattice polarized K3 surfaces, we refer to [2, 52, 173, 620].

Let X be an algebraic K3 surface. By Definition 1.1.10, this means that X is a smooth and projective surface with $K_X = 0$ and $h^1(\mathcal{O}_X) = 0$. Note that there exist compact complex but not algebraic surfaces satisfying these conditions. They admit a structure of a Kähler manifold [52, Exp. XII]. However, we will be mainly concerned with K3 surfaces that are K3 covers of Enriques surfaces and these are always algebraic.

First, assume that X is a complex K3 surface. Let us briefly run through the computations of some important invariants and we refer to Section 0.10 for background and further discussion of the results we are using. Recall that $K_X = 0$ and Wu's formula (0.10.12) imply that $H^2(X, \mathbb{Z})$ is an even lattice with respect to the cup-product, which is a symmetric bilinear form. By Poincaré duality, it is unimodular. Since $h^1(X, \mathcal{O}_X) = 0$, we have $b_1(X) = b_3(X) = 0$ and thus, by Noether's formula (0.10.22), we find $b_2(X) = 22$. By Hirzebruch's signature theorem (0.10.8), the signature of the cup-product on $H^2(X, \mathbb{R})$ is $(3, 19)$. Moreover, by Theorem 1.4.4, the Hodge numbers of X are given by $h^{2,0}(X) = h^{0,2}(X) = 1$ and $h^{1,1}(X) = 20$. Thus, by Proposition 0.8.8, there is an isomorphism of lattices

$$H^2(X, \mathbb{Z}) \cong \mathsf{U}^{\oplus 3} \oplus \mathsf{E}_8^{\oplus 2} \cong \mathsf{U} \oplus \mathsf{E}_{10}^2,$$

where E_{10} denotes the Enriques lattice studied in Section 1.5. We denote the lattice on the right-hand side by $\mathbf{L}$ and call it the *K3 lattice*. It follows from Proposition 1.1.9 that the Picard scheme of X is reduced and its connected component of identity is trivial. There is an isomorphism of abelian groups

$$\mathrm{Pic}(X) \cong \mathrm{NS}(X) \cong \mathrm{Num}(X) \cong \mathbb{Z}^\rho,$$

where ρ is the Picard number of X. The Chern class homomorphism

$$c_1 : \mathrm{Pic}(X) \to H^2(X, \mathbb{Z})$$

is injective and its image lies in $H^2(X, \mathbb{Z}) \cap H^{1,1}(X)$. This implies that the Picard number satisfies

$$1 \le \rho(X) \le 20.$$

The intersection form on $\mathrm{Pic}(X)$ defines a structure of a quadratic lattice on $\mathrm{Pic}(X)$. The Chern class homomorphism c_1, which respects the intersection forms on both sides, and thus, identifies $\mathrm{Pic}(X)$ with a sublattice of $H^2(X, \mathbb{Z})$. Moreover, it follows from Lefschetz's theorem on $(1, 1)$-classes that this embedding of lattices is primitive.

Now, we use the terminology from Section 0.8. Assume $\rho > 1$, so that the lattice $\mathrm{Pic}(X)$ is hyperbolic. We let $W_X := \mathrm{Ref}_2(\mathrm{Pic}(X))$ to be the Weyl group of the lattice $\mathrm{Pic}(X)$, which is generated by reflections in the divisor classes δ with $\delta^2 = -2$. We have already seen in Section 2.2 that this group coincides with the nodal Weyl group W_X^{nod} defined by the root basis $\mathcal{R}(X)$ that consists of (-2)-curves on X. By Proposition 2.2.1, the fundamental chamber of this root basis coincides with the nef cone $\mathrm{Nef}(X)$ of X. The interior of $\mathrm{Nef}(X)$ is the ample cone $\mathrm{Amp}(X)$ of X. The boundary $\partial\mathrm{Nef}(X) := \mathrm{Nef}(X) \setminus \mathrm{Amp}(X)$ consists of those numerical classes $[D]$ in $\mathrm{Nef}(X)$ with $D^2 = 0$ or $D^2 > 0$, such that $D \cdot R = 0$ for some $R \in \mathcal{R}(X)$. We set

$$\begin{aligned} \mathrm{Pic}(X)^{\mathrm{pa}} &= \{x \in \partial\mathrm{Nef}(X) : x^2 > 0\} \cap \mathrm{Pic}(X), \\ \mathrm{Pic}(X)^{\mathrm{a}} &= \mathrm{Amp}(X) \cap \mathrm{Pic}(X). \end{aligned}$$

which are the line bundles on X that are big and nef (also called "pseudo-ample") and ample, respectively.

Let us now generalize and formalize this setup: let M be an even and non-degenerate lattice of signature $(1, t)$. Let

$$V(M) \ := \ \left\{M_{\mathbb{R}} \,|\, x^2 > 0\right\} \ \subset \ M_{\mathbb{R}},$$

We choose one of the connected components of $V(M)$ and denote it by $V(M)^+$. Then, we define

$$C(M) \ := \ \text{fundamental chamber of } W(M) \ := \ \mathrm{Ref}_2(M) \text{ in } V(M)^+.$$

With respect to this setup, we now make the following definition and refer to [173] for background and further information.

Definition 5.2.1 An *M-polarized K3 surface* is a pair (X, j), where X is a K3 surface, and

$$j \ : \ M \ \to \ \mathrm{Pic}(X)$$

is a primitive lattice embedding and $j(C(M))$ contains a big and nef divisor class D. Moreover, we say that the M-polarization (X, j) is *ample* if $j(C(M))$ contains an ample class.

Remark 5.2.2 Let $C(M)^\circ$ be the interior of $C(M)$. It follows from this definition that a connected component of $j(C(M)^\circ)$ intersects the interior of the nef (resp. ample) cone. Since both sets are convex, there exists a convex cone $C(M)_+$ in $C(M)$, such that $j(C(M)_+)$ is contained in the nef (resp. ample) cone of X.

If one is interested only in ample lattice polarizations (as it seems to be in many applications of the theory of lattice polarized K3 surfaces), then the definition should only require that the image $j(h)$ of some $h \in M$ is ample. This will fix $(V(M)^+, C(M))$ by requiring that $h \in C(M)^\circ$.

Remark 5.2.3 It is important to understand that a lattice polarization comes with a choice of the data $(V(M)^+, C(M))$. Composition with an isometry of the lattice M

changes the polarization but also may change the choice of $(V(M)^+, C(M))$. It sends an ample polarization to an ample polarization.

Example 5.2.4 Let $M = \mathbb{Z} \cdot e$ with $e^2 = 2d$ and $d > 0$. Choose $V(M)^+$ to be one of the rays of $M_{\mathbb{R}} \backslash \{0\}$. In this particular case, an M-polarized K3 surface (X, j) is called a *degree* $2d$ *polarized K3 surface*. A polarization is ample if and only if $j(e)$ is an ample divisor class. (A polarization that is big and nef, but not ample, is often called a *quasi-polarization*.) There are three possibilities depending on the behavior of the complete linear system $|j(e)|$ and the associated rational map f.

1. (Unigonal case) The linear system $|j(e)| = |C + (d+1)E|$, where C is a smooth rational curve and E is a genus one curve with $C \cdot E = 1$. It has C as fixed component and the map associated to $|(d+1)E|$ is a morphism $X \to \mathbb{P}^{d+1}$, whose image is a rational normal curve of degree $d + 1$.
2. (Hyperelliptic case) The linear system $|j(e)|$ has no base points and f is a morphism of degree 2 onto a normal surface of degree d in $\mathbb{P}^{d+1}$, whose singular points are at worst rational double points.
3. (Birational case) The linear system $|j(e)|$ has no base points and f is a morphism of degree 1 onto a normal surface of degree $2d$ in $\mathbb{P}^{d+1}$, whose singular points are at worst rational double points.

We refer to [52, Exposé IV] for details.

Example 5.2.5 If S is an Enriques surface, then there exists an isometry $\mathrm{Num}(S) \cong \mathsf{E}_{10}$, where E_{10} is the Enriques lattice, see Section 1.5. If $\pi : X \to S$ denotes the canonical K3 cover, then $\pi^*(\mathrm{Pic}(Y)) \to \mathrm{Pic}(X)$ yields a sublattice isometric to $\mathsf{E}_{10}(2) \cong \mathsf{U}(2) \oplus \mathsf{E}_8(2)$, and we obtain a lattice $\mathsf{E}_{10}(2)$ polarized K3 surface

$$j : \mathsf{E}_{10}(2) \to \mathrm{Pic}(X).$$

In Proposition 5.2.12 below we will see that we obtain in this way a bijection of sets of marked Enriques surfaces and ample $\mathsf{E}_{10}(2)$-polarized K3 surfaces.

Next, we want to construct moduli spaces of lattice polarized K3 surfaces. First, we set up the moduli functor as discussed in Section 5.3. Then, we first describe their automorphisms and infinitesimal deformations, which gives a local description of the moduli spaces we look for and allow us to conclude that they exist as smooth Deligne–Mumford stacks. Let us slightly enhance the setup and work over arbitrary fields or even arbitrary base schemes, where we loosely follow [2, 173, 576, 620] and refer to these articles for more details and proofs.

First, we define *families of (ample) M-polarized of K3 surfaces* and associated moduli functors in the following two settings.

1. (Algebraic) In this case, we fix a base scheme B, for example, $B = \mathrm{Spec}\, \Bbbk$, where $\Bbbk$ is a field or $B = \mathrm{Spec}\, \mathbb{Z}$, which is the universal case since every scheme is a scheme over $\mathbb{Z}$. Given a lattice M, we define two functors

$$\mathcal{K}3_M \quad (\text{resp. } \mathcal{K}3_M^{\mathrm{a}}) : (\mathrm{Schemes}/B) \to (\mathrm{Groupoids})$$

that assign to every scheme T over B the groupoid of pairs $(\mathcal{X} \to T, j)$, where $\mathcal{X} \to T$ is a smooth projective family of K3 surfaces and where $j : M_T \hookrightarrow \underline{\mathrm{Pic}}_{\mathcal{X}/T}$ is a map of sheaves in the étale topology such that $j(C(M))$ contains a nef and big invertible sheaf (resp. $j(V(M)^+)$ contains an ample invertible sheaf). Clearly, $\mathcal{K}3^{\mathrm{a}}_M$ is an open subfunctor of $\mathcal{K}3_M$, that is, for any T, the morphism $\mathcal{K}3^a_M \times_{\mathcal{K}3_M} h_T \to h_T$ is represented by an open embedding of schemes.
2. (Analytic) Then, a family of M-polarized K3 surfaces is a morphism of complex spaces $\mathcal{X} \to T$, each of whose fibers is a complex K3 surface, and a morphism $M_S \to \underline{\mathrm{Pic}}_{\mathcal{X}/T}$ as before.
3. (Analytic and marked) We fix a lattice M and an embedding $M \to \mathbf{L}$ of M into the K3 lattice and then, we define a *marked* (ample) M-polarized K3 surface to be a pair (X, ϕ), where X is a K3 surface and $\phi : H^2(X, \mathbb{Z}) \to \mathbf{L}$ is an isomorphism of lattices, such that $\phi^{-1}|_M : M \to \mathrm{Pic}(X)$ is an (ample) M-polarization. Similarly, families of (ample) marked M-polarized K3 surfaces are defined, which leads to the functors

$$\mathcal{K}_M \quad \text{and} \quad \mathcal{K}^{\mathrm{a}}_M \; : \; (\text{ComplexSpaces}) \; \to \; (\text{Groupoids}).$$

We refer to [173] for details.

The following result gives the local structure and local charts for the moduli spaces of M-polarized K3 surfaces that we want to construct.

We refer to Section 5.11 for some basic definitions and results in the theory of local deformations.

Proposition 5.2.6 *Let M be an even non-degenerate lattice of signature $(1, t)$.*

1. *Let (X, j) be an M-polarized K3 surface over an algebraically closed field $\Bbbk$ of characteristic $p \geq 0$. Then, there exists a formal deformation space S_M of M-polarized K3 surfaces that is a formal scheme over $\Bbbk$. Assume that one of the following holds:*

   a. *$p = 0$,*
   b. *$p > 0$ and p does not divide the discriminant $\mathrm{disc}(M)$,*
   c. *$p > 0$ and X is not a supersingular K3 surface.*

   Then, S_M is formally smooth and of dimension $(19 - t)$ over $\Bbbk$. If $p > 0$, there even exists a formal deformation space S_M over the ring of Witt vectors $W(\Bbbk)$ that is smooth of relative dimension $(19 - t)$.
2. *If, additionally to assumptions (a), (b) and (c), we assume that (X, j) is an ample M-polarization, then the formal deformation space can be algebraized, that is, there exists a smooth proper morphism $f : \mathcal{X} \to \mathrm{Spec}\, R$, where $R \cong \Bbbk[[t_1, \ldots, t_{19-t}]]$ whose fiber over the closed point is isomorphic to X and every ample $\mathcal{L}$ from $j(M)$ lifts to an ample invertible sheaf on $\mathcal{X}$. If $p > 0$ one may choose R to be $W(\Bbbk)[[t_1, \ldots, t_{19-t}]]$.*
3. *Let (X, j) be a marked or ample M-polarized K3 surface over $\mathbb{C}$. Then, there exists a local moduli space of M-polarized K3 surfaces that is a formal complex space that is smooth of dimension $(19 - t)$.*

Proof There exists a formal deformation space $\mathrm{Def}(X)$ of X that is a smooth formal scheme of dimension 20 over $\mathrm{Spf}(\mathbb{k})$ (or even $\mathrm{Spf}(W(\mathbb{k}))$ if $p > 0$). Next, Deligne [155, Théorème 1.6] showed that if $\mathcal{L}$ is an invertible sheaf on X, then the formal deformation space of the pair $(X, \mathcal{L})$ is a formal Cartier divisor $\mathrm{Def}(X, \mathcal{L})$ inside $\mathrm{Def}(X)$, which is flat over $W(\mathbb{k})$ when $p > 0$ and allows deformations in mixed characteristic. Thus, after choosing a basis $m_0, \ldots, m_t$ of M, the desired local moduli space S_M is the intersection of the $(t + 1)$ formal Cartier divisors $\mathrm{Def}(X, j(m_i))$ inside $\mathrm{Def}(X)$, and it is easy to see that it is flat over $\mathbb{k}$ (resp. $W(\mathbb{k})$) of relative dimension at least $(19 - t)$. Next, $d \log$ induces a map $\delta : M \otimes_{\mathbb{Z}} k \to H^1(X, \Omega^1_X)$, which is compatible with intersection forms on both sides. The Zariski tangent space of S_M is the orthogonal complement of $\delta(M \otimes k)$ in $H^1(X, \Omega^1_X)$. In the case where $p = 0$ or where p does not divide $\mathrm{disc}(M)$, then the intersection form on $M \otimes_{\mathbb{Z}} k$ is non-degenerate, and thus, δ is injective. Thus, in these cases, the Zariski tangent space of S_M has dimension $(19 - t)$, and S_M is at least of dimension $(19 - t)$ over $\mathbb{k}$ (resp. $W(\mathbb{k})$). Being flat over $\mathbb{k}$ (resp. $W(\mathbb{k})$), it follows that S_M is formally smooth of relative dimension $(19 - t)$ over $\mathbb{k}$ (resp. $W(\mathbb{k})$). See also [2, Proposition 3.8]. If $p > 0$ and X is ordinary, then the claimed dimension and formal smoothness have been shown in [2, Proposition 3.3 and Section 3.5]. In fact, this argument works more generally if $p > 0$ and if X is not supersingular, which is slightly implicit in [454], see also [117, Proposition 2.11]. The algebraization assertion follows from [155, Corollary 1.8].

We leave the proof in the complex analytic setting to the reader. In the marked case, we refer to [173, Proposition (2.1)] for details. □

This result is the key to proving the following result, which shows that the moduli functors introduced above give rise to reasonable moduli spaces.

Proposition 5.2.7 *Let M be an even non-degenerate lattice of signature $(1, t)$.*

1. *The functors $\mathcal{K}3_M$ and $\mathcal{K}3^{\mathrm{a}}_M$ can be represented by Deligne–Mumford stacks. Moreover, $\mathcal{K}3^{\mathrm{a}}_M$ is separated. When non-empty, both stacks are smooth of relative dimension $(19 - t)$ over $\mathrm{Spec}\,\mathbb{Z}[\frac{1}{d}]$, where $d = \mathrm{disc}(M)$.*
2. *For complex K3 surfaces, there exists a fine moduli space $\mathcal{K}_M$ of marked M-polarized K3 surfaces as a smooth and non-separated complex space.*

Proof In the first case, the assertion about $\mathcal{K}3^a_M$ is proven essentially in [50, Proposition 2.6] (over $\mathbb{C}$) with details in the general case given in [2, Proposition 3.3]. The latter proof relies on results of [620] on the moduli functor for polarized K3 surfaces with ample primitive polarization. This latter has been extended to the case of quasi-polarized K3 surfaces in [500] and [483]. Let us briefly sketch the arguments: it is easy to see that the functors $\mathcal{K}3_M$ and $\mathcal{K}3^{\mathrm{a}}_M$ are sheaves in the étale topology and thus, give rise to stacks. Next, the formal deformation spaces that we established in Proposition 5.2.6 can be algebraized because the families considered come with (pseudo-)ample polarizations. From these algebraizations, we obtain morphisms to the functors $\mathcal{K}3_M$ and $\mathcal{K}3^{\mathrm{a}}_M$. In fact, these morphisms can be combined to give an algebraic family (usually not connected when constructed this way) that maps surjectively to $\mathcal{K}3_M$ and $\mathcal{K}3^{\mathrm{a}}_M$. This shows that both stacks are algebraic stacks or Artin stacks. Moreover, the automorphism group scheme of a (quasi-)polarized family is finite and since K3 surfaces have no global vector fields, these group schemes are finite and étale. This implies that the stacks $\mathcal{K}3_M$ and $\mathcal{K}3^{\mathrm{a}}_M$ are Deligne–Mumford stacks. Finally, the statement about smoothness and dimensions of these spaces follow again from the local description of these stacks provided by Proposition 5.2.6.

For the second statement, we refer to [173, Section 3], [52, Exposé XIII], and [555]. It uses the theory of periods of K3 surfaces, which we will discuss later in this section. □

Remark 5.2.8 We end this discussion with two remarks.

1. The non-separatedness of these moduli spaces when working with non-ample polarizations has to do with *flops* or *elementary modifications*. More precisely, there may exist the spectrum T of a discrete valuation ring, say with algebraically closed residue field, and smooth families $\mathcal{X} \to T$ of K3 surfaces together with an invertible sheaf $\mathcal{L}$ such that $\mathcal{L}$ is ample on the generic fiber $\mathcal{X}_\eta$, that is big and nef, but not ample on the special fiber $\mathcal{X}_0$. In this case, there exists at least one (-2)-curve $R \subset \mathcal{X}_0$ such that $\mathcal{L}$ has zero intersection with R and a rational and birational map
$$\phi : \mathcal{X} \dashrightarrow \mathcal{X}^+$$
of smooth and proper algebraic spaces over T that is defined outside R and such that ϕ restricted to $\mathcal{X} - R$ is an isomorphism onto its image. Although ϕ does not extend to a morphism, it induces an isomorphism of the special fibers $\mathcal{X}_0$ and $\mathcal{X}_0^+$. These two families $\mathcal{X}$ and $\mathcal{X}^+$ over T are not isomorphic, but have isomorphic geometric fibers. In particular, they violate the uniqueness requirement for the valuative criterion of separatedness of the moduli stack $\mathcal{K}3_M$. This type of phenomenon cannot happen for ample polarized families.
2. The moduli space $\mathcal{M}_{1,1}$ of elliptic curves exists as a Deligne–Mumford stack, but not as an algebraic space or scheme, and we mentioned in Example 5.1.1 the modular curves $X_0(n) \to \mathcal{M}_{1,1}$, which are representable by algebraic spaces (in this case even schemes). Now, if one wants to have moduli spaces that exist as algebraic spaces rather than Deligne–Mumford stacks in the case of M-polarized K3 surfaces, then one can add level structures as follows: one considers families of M-polarized K3 surface $f : \mathcal{X} \to S$ together with an isomorphism $Rf_*(\mathbb{Z}/n\mathbb{Z}) \cong (\mathbf{L}/n\mathbf{L})$ of the relative étale cohomology. If n is sufficiently large, then such families do not admit non-trivial automorphisms and then, the corresponding moduli functor can be represented by Deligne–Mumford stacks with trivial stabilizers, which are representable by algebraic spaces. We refer to [620] and [483] for further information, details, and results.

The previous results show the existence of moduli spaces for lattice polarized K3 surfaces, give smoothness and dimension (if non-empty). However, to understand the geometry of these spaces, such as non-emptiness, number of connected components, or their birational geometry, such as uniruledness or unirationality of these spaces, one needs another approach to these moduli spaces. Over the complex numbers, such an approach is provided by period maps, period spaces, and yields coarse moduli spaces.

For the remainder of this section, we work with complex algebraic K3 surfaces. Let M be an even non-degenerate lattice of signature $(1, t)$ together with an embedding $\iota_M : M \to \mathbf{L}$ into the K3 lattice. Then, we define

$$N := M^{\perp}$$

to be the orthogonal complement of M in $\mathbf{L}$ and note that it is a lattice of signature $(2, 19 - t)$. Now, given a marked M-polarized K3 surface (X, ϕ), the Hodge decomposition of $H^2(X, \mathbb{C})$ defines a point $[\omega] := \phi(H^{2,0}(X))$ in $|\mathbf{L}_{\mathbb{C}})| = \mathbb{P}(\mathbf{L}_{\mathbb{C}}^{\vee})$. More precisely, $H^{2,0}(X)$ is orthogonal to $H^{1,1}(X)$ with respect to the cup-product, and thus, $[\omega]$ is orthogonal to $M \subset \mathbf{L}$, which implies that $[\omega]$ is a point in $|N_{\mathbb{C}}| \subset |M_{\mathbb{C}}|$. Since $\omega^2 := \omega \cdot \omega$ lies in $H^{4,0}(X)$, which is zero, we conclude that $[\omega]$ lies in the quadric

$$Q_N := \left\{ [\omega] \in |N_{\mathbb{C}}| \mid \omega^2 = 0 \right\} \subset |N_{\mathbb{C}}|.$$

Next, we observe that $\omega \cdot \overline{\omega}$ lies in $H^{2,2}(X) \cap H^4(X, \mathbb{R}) \cong \mathbb{R}$ and that it is positive. Putting these observations together, we find that

$$\phi(H^{2,0}(X)) \in \mathcal{D}_N := \left\{ \mathbb{C} \cdot \omega \in |N_{\mathbb{C}}| \, \omega^2 = 0, \, \omega \cdot \overline{\omega} > 0 \right\} \subset |N_{\mathbb{C}}|.$$

We call the open subset $\mathcal{D}_N$ of Q_N the *period space* or *period domain*. Note that $\mathcal{D}_N$ is not connected and consists of two connected components, each of which is analytically isomorphic to a Hermitian symmetric domain (of type IV or of orthogonal type). We refer to Section 5.9 and especially Example 5.9.2 for details.

To see these two components, we choose a basis in $N_{\mathbb{C}}$ with coordinates $t_1, t_2, \ldots, t_k$, where $n := \operatorname{rank}(N) = 19 - t$, such that $\mathcal{D}_N$ consists of points in $\mathbb{P}(N_{\mathbb{C}})$ with projective coordinates $[z_1, \ldots, z_n]$ satisfying

$$\begin{aligned} &z_1^2 + z_2^2 - z_3^2 - \cdots - z_n^2 = 0, \\ &|z_1|^2 + |z_2|^2 - |z_3|^2 - \cdots - |z_n|^2 > 0, \end{aligned}$$

This set consists of two connected components that are distinguished by the sign of $\operatorname{Im}(t_1/t_2)$. Another way to see this is to consider a real plane $P(z) \subset N_{\mathbb{R}}$ spanned by the imaginary and real part of a vector $z = x + iy \in N_{\mathbb{C}}$ that represents a point $[z] \in Q_N$. Then, $0 = z^2 = (x + iy)^2$ implies $x^2 - y^2 = x \cdot y = 0$ and $z \cdot \bar{z} = (x + iy) \cdot (x - iy) > 0$ implies $x^2 + y^2 > 0$. Thus, $x^2 = y^2 > 0$ and $x \cdot y = 0$ implies that $P(z)$ is a positive definite plane in $N_{\mathbb{R}}$. This defines a map from Q_N to the Grassmannian $G(2, N_{\mathbb{R}})^+$ of positive definite planes in $N_{\mathbb{R}}$. This consists of two connected components defined by a choice of an orientation of the plane.

Via the natural action of $\mathrm{O}(N_{\mathbb{R}}) \cong \mathrm{O}(2, 19 - t)$ on $G(2, N_{\mathbb{R}})$, the Grassmannian inherits the structure of a homogeneous space isomorphic to $\mathrm{O}(2, 19 - t)/\mathrm{SO}(2) \times \mathrm{O}(19 - t)$. The connected component containing the image of neutral element is isomorphic to $\mathrm{SO}(2, 19 - t)/\mathrm{SO}(2) \times \mathrm{SO}(19 - t)$ and it is a Hermitian symmetric domain. The complex involution switches the two components and the action on the periods correspond to switching the complex structure to the conjugate one.

Let (X, ϕ) be a marked M-lattice polarized K3 surface and let $\mathcal{X} \to S_M$ be the local moduli space together with its universal family around (X, ϕ) established in Proposition 5.2.6. Shrinking if necessary, we may assume that S_M is contractible, and thus, we may assume that the marking ϕ extends to a marking $\phi : H^2(\mathcal{X}_s, \mathbb{C}) \to \mathbf{L}$ for all fibers $\mathcal{X}_s$ with $s \in S_M$. Thus, we obtain a period point $\phi(H^{2,0}(\mathcal{X}_s)) \in \mathcal{D}_N$ for all $s \in S_M$ with respect to the extended marking. By the local Torelli theorem for K3 surfaces [52, Exposé V], the resulting local period map $p : S_M \to \mathcal{D}_N$ is holomorphic and locally an isomorphism in a neighborhood around the point corresponding to (X, ϕ). Thus, if $\mathcal{K}_M$ is the fine moduli space of marked M-polarized K3 surfaces from Proposition 5.2.7, then these local period maps glue to a holomorphic map

$$p : \mathcal{K}_M \to \mathcal{D}_N .$$

Being locally an isomorphism, p is étale, but not necessarily injective. In the following proposition we describe the fibers. Let $z \in Q_N$ and let π be a positive definite oriented plane in $N_{\mathbb{R}}$ associated to z. Let $\pi^{\perp}$ be its orthogonal complement in $L_{\mathbb{R}}$. It contains $M_{\mathbb{R}}$. Let W_π be the 2-reflection group of $\pi^{\perp} \cap L$. It acts on M and its subgroup leaving $C(M)$ invariant is contained in the reflection group $W(\pi^{\perp} \cap N)$. We denote it by $W_\pi(N)$. Applying the Global Torelli Theorem for Kähler K3 surfaces from [100], we obtain the following theorem (see [173, Theorem (3.1)]).

Theorem 5.2.9 *The restriction*

$$p' := p|_{\mathcal{K}_M} : \mathcal{K}_M \to \mathcal{D}_N$$

of the period map p to the subset $\mathcal{K}_M$ of marked M-polarized K3 surfaces is surjective. For all $\pi \in \mathcal{D}_N$, there is a natural bijection between the fiber $p'^{-1}(\pi)$ and the subgroup $W_\pi(N)$.

To determine the image of $\mathcal{K}_M^{\mathrm{a}}$ under the period map, we recall and define

$$\begin{array}{ll} N_{-2} & = \left\{\delta \in N \mid \delta^2 = -2\right\} \\ H_{N,\delta} & := \{x \in N_{\mathbb{C}} \mid x \cdot \delta = 0\}\,, \quad \delta \in N_{-2}, \\ \mathcal{H}_N(-n) & := \bigcup_{\delta \in N_{-n}} H_\delta \cap \mathcal{D}_N\,, \\ \mathcal{D}_N^{\circ} & := \mathcal{D}_N \setminus \mathcal{H}_N(-2) \end{array}$$

and note that $\mathcal{D}_N^{\circ}$ is an open and dense subset of $\mathcal{D}_N$.

The divisor $\mathcal{H}_N(-n)$ is called the *Heegner divisor*. If $n = 2$, then we also call it the *discriminant of the period domain*. Now, if (X, ϕ) is an *ample* marked M-polarized K3 surface, then the ampleness assumption implies $c_1(j(M))^{\perp} \cap H^{1,1}(X)$ does not contain vectors δ with $\delta^2 = -2$. Thus, the period point $\pi := \phi(H^{2,0}(X))$ satisfies $\pi^{\perp} \cap N_{-2} = \emptyset$, and we obtain the following corollary, see [173, Corollary (3.2)].

Corollary 5.2.10 *The restriction*

$$p|_{\mathcal{K}_M^{\mathrm{a}}} : \mathcal{K}_M^{\mathrm{a}} \to \mathcal{D}_N^{\circ}$$

of the period map p to the subset $\mathcal{K}_M^{\mathrm{a}}$ of marked ample M-polarized K3 surfaces is bijective. In particular, the points in $\mathcal{H}_N(-2)$ are the period points of marked pseudo-ample but not ample M-polarized K3 surfaces.

Note that during the proof of Theorem 5.2.9 and its corollary, we used the *Global Torelli Theorem* for K3 surfaces given below. In fact, this result is the key to the injectivity of the period map p. We refer to [599] for the original proof; other proofs can be found in [43] or [52].

Theorem 5.2.11 *Let X, X' be complex algebraic K3 surfaces and let $\phi : H^2(X,\mathbb{Z}) \to H^2(X',\mathbb{Z})$ be an isometry such that the induced linear isomorphism $\phi_{\mathbb{C}} : H^2(X,\mathbb{C}) \to H^2(X',\mathbb{C})$ sends $H^{2,0}(X)$ to $H^{2,0}(X')$ and that it also sends the nef cone* $\mathrm{Nef}(X)$ *to the nef cone* $\mathrm{Nef}(X')$. *Then, there exists a unique isomorphism $f : X' \to X$, such that $\phi = f^*$.*

Note that the condition $\phi(\mathrm{Nef}(X)) = \mathrm{Nef}(X')$ is equivalent to one of the following conditions:

- $f(\mathrm{Eff}(X)) = \mathrm{Eff}(X')$.
- $f(\mathrm{Amp}(X)) \cap \mathrm{Amp}(X') \neq \emptyset$.
- $f(\mathrm{Amp}(X)) = \mathrm{Amp}(X')$.
- $f(\mathcal{R}(X)) = \mathcal{R}(X')$.

In Proposition 5.2.7, we constructed algebraic moduli spaces for (ample) M-polarized K3 surfaces. To link these moduli spaces to the analytic moduli spaces of marked and M-polarized K3 surfaces, we have to get rid of the markings. Let M be a primitive sublattice of finite index of a non-degenerate lattice L. Suppose an isometry σ of M acts as identity on the discriminant group $D(M)$. Using the chain of sublattices (see Section (0.8)), we see that σ acts identically on L/M, transforming any $x \in L$ to $x+m \in L, m \in M$. In particular, it defines an isometry of L that restricts to the isometry σ. Applying this to our situation, let

$$\mathrm{O}(N)^\sharp := \mathrm{Ker}\,(\mathrm{O}(N) \to \mathrm{O}(D(N))\,. \tag{5.2.1}$$

We introduced this notation in Section 0.8 for an arbitrary even non-degenerate lattice N. As we explained in this section, for any $\sigma \in \mathrm{O}(N)^\sharp$, the isometry $\mathrm{id}_M \oplus \sigma$ extends to an isometry of L. Conversely, if $\tilde{\sigma}$ is an isometry of L that leaves M invariant and acts as identity on it, then its restriction to $M^\perp = N$ belongs to $\mathrm{O}(N)^\sharp$.

An element $\sigma \in \mathrm{O}(N)^\sharp$ acts as identity on $\mathrm{O}(D(M \oplus N))$ and hence extends to an isometry $\tilde{\sigma} \in \mathrm{O}(L)$ that leaves both M and N invariant and acts as identity on M. Therefore, the group $\mathrm{O}(N)^\sharp$ acts on $\mathcal{K}_M$ by replacing the marking of a K3 surface, but not changing the lattice polarization. The group $\mathrm{O}(N)$ contains a subgroup of index ≤ 2 that is an arithmetic group of automorphisms of a Hermitian symmetric domain. This implies that $\mathrm{O}(N)$ and hence $\mathrm{O}(N)^\sharp$, acts discretely on $\mathcal{D}_N$ and that the quotient has a uniquely defined structure of a quasi-projective algebraic variety, see [44]. We come back to the structure of this quasi-projective variety in Section 5.9 below.

Since any reflection in an element $\delta \in N_{-2}$ belongs to the subgroup $\mathrm{O}(N)^\sharp$, each fiber of the map $p : \mathcal{K}_M \to \mathcal{D}_N$ is mapped to the same $\mathrm{O}(N)^\sharp$-orbit $\mathcal{K}_M$. Thus, by Theorem 5.2.9, we obtain a bijection

$$\mathcal{M}_{K3,M} := \mathrm{O}(N)^\sharp \backslash \mathcal{K}_M \cong \mathrm{O}(N)^\sharp \backslash \mathcal{D}_N. \tag{5.2.2}$$

The points of the quotient on the left are precisely the isomorphism classes of M-polarized K3 surfaces, where we fixed the embedding of M into $\mathbf{L}$. Changing the embedding may change N and hence the group $\mathrm{O}(N)^\sharp$ and the corresponding quotient.

The quotient on the right carries the structure of a quasi-projective variety. However, we should warn that the moduli functor $\mathcal{K}3_M$ is not a separated Deligne–Mumford stack and that $\mathcal{M}_{K3,M}$ is not its coarse moduli space. We only have a bijection of points between the isomorphism classes of complex M-polarized K3 surfaces and points of $\mathcal{M}_{K3,M}$.

Finally, we assume that M has the property that every two primitive embeddings of M into $\mathbf{L}$ differ by an isometry of $\mathbf{L}$. For example, it is true if rank $M \le 9$, hence $l(M) \le 9$. Then rank $N \ge 13 > 2 + l(N) = l(M)$. Theorem 0.8.6 implies that all primitive embedding of N, and hence of M are equivalent. Now, if (X, j) is an M-polarized K3 surface, we obtain a primitive embeddings $M \to \mathrm{Pic}(X) \to H^2(X, \mathbb{Z}) \cong \mathbf{L}$ into the K3 lattice. By assumption, we can change this previous embedding so that it coincides with the fixed one. Thus, the pair (X, j) occurs as a point of $\mathrm{O}(N)^\sharp \backslash \mathcal{D}_N$ and we may view this latter quotient as the moduli space of M-polarized K3 surfaces. In fact, following [52, Exposé XIII], one can show that $\mathrm{O}(N)^\sharp \backslash \mathcal{D}_N$ is a coarse moduli space of M-polarized K3 surfaces. In Section 5.11 we will discuss compactifications of the quotients $\mathrm{O}(N)^\sharp \backslash \mathcal{D}_N$, which will be projective algebraic varieties. The existence of such compactifications implies that the just–considered quotients are quasi-projective varieties. Moreover, it follows from [173, Proposition 5.6] that these varieties are irreducible if N contains a direct summand isomorphic to $\mathsf{U}(m)$.

Proposition 5.2.12 *Let M be an even non-degenerate sublattice of $\mathbf{L}$ of signature $(1, t)$ with $t \le 9$ and set $N = M^\perp$.*

1. *The quotient $\mathrm{O}(N)^\sharp \backslash \mathcal{D}_N$ has the structure of a quasi-projective variety and its points are in a natural bijection with isomorphism classes of lattice M polarized K3 surfaces. It is irreducible if N contains a direct summand isomorphic to $\mathsf{U}(m)$ (this condition is satisfied, for example, if* rank $N \ge l(N) + 3$ *[556, Proposition 1.13.5]).*
2. *The quotient $\mathrm{O}(N)^\sharp \backslash \mathcal{D}_N^\circ$ is an open subvariety of $\mathrm{O}(N)^\sharp \backslash \mathcal{D}_N$, and it is the coarse moduli space for the moduli space $\mathcal{K}3_M^{\mathrm{a}}$ of ample M-polarized K3 surfaces.*
3. *The quotient $\mathrm{O}(N)^\sharp \backslash \mathcal{D}_N^\circ$ is a Zariski open subset of $\mathrm{O}(N)^\sharp \backslash \mathcal{D}_N$, whose complement is a union of finitely many hypersurfaces.*

Example 5.2.13 Assume $M = \langle 2d \rangle$. We have $\mathrm{O}(M) = \{\pm 1\}$ and $\mathrm{O}(D(M)) = \mathrm{O}(\langle \frac{1}{2d} \rangle) \cong (\mathbb{Z}/2\mathbb{Z})^{p(d)}$, where $p(d)$ is the number of prime divisors of d, see [636, Lemma 3.6.1]. Although the homomorphism $\rho_M : \mathrm{O}(M) \to \mathrm{O}(D(M))$ is not surjective, the homomorphism $\rho_N : \mathrm{O}(N) \to \mathrm{O}(D(N))$ is surjective. This follows from Theorem 0.8.6. The same theorem implies that all primitive embeddings of $N = \mathsf{E}_{10}^{\oplus 2} \oplus \langle -2d \rangle$ into $\mathbf{L}$ are equivalent. Thus, there is no ambiguity and we can denote the moduli space of lattice M polarized K3 surfaces by $\mathcal{M}_{K3,2d}$. It is an irreducible space of dimension 19 and it is isomorphic to an arithmetic quotient of $\Gamma_{2d}^{\sharp}\backslash\mathcal{D}_N$, where $\Gamma_{2d} = \mathrm{O}(N)$.

Let

$$\check{M} := \mathsf{E}_{10} \oplus \mathsf{E}_8 \oplus \langle -2d \rangle$$

be the orthogonal direct sum decomposition of $\mathsf{U}^{\perp}$ in N. The moduli space $\mathcal{M}_{K3,\check{M}}$ is one-dimensional and it is the *mirror moduli space* in the sense of [173]. We have $\check{M}^{\perp} \cong \mathsf{U} \oplus \langle 2d \rangle$. The period space $\mathcal{D}_{\check{M}^{\perp}}$ is the union of two copies of the upper half plane $\mathcal{H} = \{z = a + bi \in \mathbb{C} : b > 0\}$. The subgroup of $\mathrm{O}(\check{M}^{\perp})^{\sharp}$ that fixes one of the copies is isomorphic to the modular group $\Gamma_0(d)^+$. It is a subgroup of $\mathrm{PSL}(2,\mathbb{R})$ generated by the modular group $\Gamma_0(d) \subset \mathrm{PSL}(2,\mathbb{Z})$ and the *Fricke involution* defined by the matrix

$$\begin{pmatrix} 0 & -\frac{1}{\sqrt{n}} \\ \frac{1}{\sqrt{n}} & 0 \end{pmatrix}.$$

We thus obtain an isomorphism

$$\mathcal{M}_{K3,\check{M}} \cong \Gamma_0(d)^+\backslash\mathcal{H}.$$

To obtain $\mathcal{M}^a_{K3,\check{M}}$ we have to throw away N points, namely the orbits of the points $\frac{c}{b}+\frac{i}{b\sqrt{d}}$, $c \in \mathbb{Z}$ and $b|c^2d+1$. They correspond to isomorphism classes of K3 surfaces with Picard number 20 that contain a (-2)-class in their Picard group orthogonal to $\check{M}$. In other words, their transcendental lattices $T(X)$ are contained in the orthogonal complement of a (-2)-vector in the lattice $\langle 2d \rangle \oplus \mathsf{U}$. One can compute the number N and finds

$$N = \begin{cases} 1 & \text{if } d \le 4, \\ 2h(-4d) & \text{if } d \equiv 7 \mod 8, \\ 4h(-4d)/3 & \text{if } d \equiv 3 \mod 8, d \ge 4, \\ h(-4n) & \text{otherwise}, \end{cases}$$

where $h(k)$ denotes the class number of integral primitive positive definite binary forms with discriminant k, see [173, Theorem 7.3].

Example 5.2.14 Let X be a K3 surface with $M \cong \mathrm{Pic}(X)$. This corresponds to a general point of $\mathcal{M}_{K3,M}$. We fix one lattice polarization $j_0 : M \to \mathrm{Pic}(X)$ and choose $C(M)$ to be the pre–image of the nef cone $\mathrm{Nef}(X)$. Obviously, any polarization is ample and differs by an isometry of $\mathrm{Pic}(X)$ (or M) that preserves the nef cone $\mathrm{Nef}(X)$. We denote the group of such isometries by $A(X)$. The elements of $A(X)^{\sharp}$ lift to isometries of $H^2(X,\mathbb{Z})$ that preserve the nef cone and act identically on $T(X)$. Thus, they are realized by automorphisms of X.

Let $A(C(M))$ be the subgroup of $\mathrm{O}(M)$ that leaves invariant $C(M)$. It follows from above that $A(C(M))_0 = A(C(M)) \cap \mathrm{O}(M)^\sharp$ acts as identity on $\mathcal{M}_{K3,M}$. Assume that $\rho_M : \mathrm{O}(M) \to \mathrm{O}(D(M))$ and $\rho_N : \mathrm{O}(M) \to \mathrm{O}(D(M))$ are surjective. Choose an isomorphism $\gamma : D(M) \to D(N)$ such that $q_{D(N)} \circ \gamma = -q_{D(M)}$. This gives rise to an isomorphism of groups $\mathrm{O}(D(M)) \to \mathrm{O}(D(N))$ and in this way $\mathrm{O}(D(N)) = \mathrm{O}(N)/\mathrm{O}(N)^\sharp$ acts on $\mathcal{M}_{K3,M}$ by composing lattice polarizations $j : M \to \mathrm{Pic}(X)$ with elements of $A(C(M))$. In the case where $A(C(M)) = \mathrm{O}(M)'$, we get a geometric realization of the quotient space

$$\mathcal{M}_{K3,M}/\mathrm{O}(D(M))' = \mathrm{O}(N)\backslash\mathcal{D}_N.$$

There is a dense subset that parameterizes K3 surfaces with Picard number $\rho = \mathrm{rank}\ M$ that admit a lattice M polarization.

Let $\alpha : M' \hookrightarrow M$ be a primitive sublattice of M of signature $(1, t')$. We choose the data $(V(M')^+, C(M')^+)$ such that we have $\alpha((V(M')^+, C(M')^\circ)) \subset (V(M)^+, C(M)^\circ)$. Note that the latter condition is equivalent to the condition that the orthogonal complement K of M' in M does not contain vectors of norm square -2.

The forgetful functor defines a morphism of Deligne–Mumford stacks

$$F(\alpha) : \mathcal{M}_{K3,M} \to \mathcal{M}_{K3,M,M'}, \quad F(\alpha)^a : \mathcal{M}^a_{K3,M} \to \mathcal{M}^a_{K3,M,M'}, \tag{5.2.3}$$

where the target denotes the stack of lattice M' polarized K3 surfaces where we choose the embedding $M' \hookrightarrow \mathbf{L}$ equal to the composition $M' \overset{\alpha}{\hookrightarrow} M \hookrightarrow \mathbf{L}$. Given a T-point of $\mathcal{M}_{K3,M,M'}$, the fiber

$$\mathcal{M}_{K3,M} \times_{\mathcal{M}_{K3,M,M'}} T$$

over this point is either empty or a torsor under the subgroup $\mathrm{O}(M)_\alpha$ of $\mathrm{O}(M)$ that restricts to $\mathrm{id}_{M'}$ on M'. Since the orthogonal complement K of M' in M is negative definite, the group $\mathrm{O}(M)_\alpha$ is finite. Also, it follows from the assumption on α that any $\sigma \in \mathrm{O}(M)_\alpha$ leaves invariant $(V(M)^+, C(M)^+)$.

Example 5.2.15 Choose $M' = \mathbb{Z}v \cong \langle v^2 \rangle$, where $v \in C(M)^\circ$. This defines a primitive embedding $\alpha : M' \hookrightarrow M$. Then $\mathcal{M}_{K3,M,v}$ is an open substack of the stack $\mathcal{P}_{K3}$ of polarized K3 surfaces of degree $2d = v^2$. The fibers of $F(\alpha)$ are either empty or torsors under the group $\mathrm{O}(M)_\alpha = \mathrm{O}(M)_v$.

Over the complex numbers, α gives rise to a morphism of quasi-projective varieties

$$F(\alpha) : \mathrm{O}(N)^\sharp\backslash\mathcal{D}_N \to \mathrm{O}(N')^\sharp\backslash\mathcal{D}_{N'},$$

where N (resp. N') is the orthogonal complement of M (resp. M') in $\mathbf{L}$. Let $\mathrm{O}(N', N)^\sharp$ be the subgroup of $\mathrm{O}(N')^\sharp$ of elements that leave $N \subset N'$ invariant. The image of the natural homomorphism $r_N : \mathrm{O}(N', N)^\sharp \to \mathrm{O}(N)$ is the subgroup of $\mathrm{O}(N)$ of isometries that can be lifted to isometries of $\mathbf{L}$ that leave invariant $M = N^\perp$ and acts as identity on M'. Let $\tilde{G}(\alpha)$ be the subgroup of the image of r_N that in its action on

M leaves invariant $(V(M)^+, C(M))$. It acts on $\mathcal{D}_N$ with kernel of the action equal to $\mathrm{O}(N)^\sharp$. Let $G(\alpha)$ be the quotient group $\tilde{G}(\alpha)/\mathrm{O}(N)^\sharp$. It is isomorphic to the group $\mathrm{O}(M)_\alpha$ from above. In this way, we see that

$$\mathcal{M}_{K3,M,M'} = \mathrm{O}(N')^\sharp\backslash\mathcal{D}_{N'} \cong \tilde{G}\backslash\mathcal{D}_N \cong \mathcal{M}_{K3,M}/G(\alpha). \tag{5.2.4}$$

5.3 Marked and Unmarked Enriques Surfaces

In this section, we construct and discuss coarse moduli spaces of marked and unmarked Enriques surfaces over the complex numbers using moduli spaces of ample $\mathsf{E}_{10}(2)$-lattice polarized K3 surfaces. On our way, we show that every two Enriques surfaces are diffeomorphic, we discuss their automorphism groups, we establish the Global Torelli Theorem, and we introduce Coble surfaces.

Let us start with a more general situation, where we only assume that the characteristic satisfies $p \neq 2$ so that the canonical cover of an Enriques surface is a K3 surface. In the previous section, we defined the Deligne–Mumford stacks $\mathcal{K}3_M$ and $\mathcal{K}3^{\mathrm{a}}_M$. In the sequel, we will consider the case where M is equal to or contains $\mathsf{E}_{10}(2)$. Following Achter [2], we modify these functors and define the functor in groupoids $\mathcal{F}_{M,\rho}$ (resp. $\mathcal{F}^{\mathrm{a}}_{M,\rho}$) of triples $(\mathcal{X} \xrightarrow{f} T, \alpha, \rho)$, where $(\mathcal{X} \xrightarrow{f} T, \alpha) \in \mathcal{K}3_M$ (resp. in $\mathcal{K}3^{\mathrm{a}}_M$) and $\rho : \boldsymbol{\mu}_{2,T} \to \mathrm{Aut}(\mathcal{X}/T)$ is a non-trivial homomorphism such that $(\underline{\mathrm{Pic}}_{\mathcal{X}/T})^{\boldsymbol{\mu}_{2,T}} = \alpha(M)$ and $\boldsymbol{\mu}_{2,T}$ acts as minus the identity on $\mathcal{H}^2(\mathcal{X}, \mathcal{O}_\mathcal{X}) = R^2 f_* \mathcal{O}_\mathcal{X}$. The following assertion is proven in [2, Proposition 3.6].

Proposition 5.3.1 *The functors $\mathcal{F}_{\mathsf{E}_{10}(2),\rho}$ and $\mathcal{F}^{\mathrm{a}}_{\mathsf{E}_{10}(2),\rho}$ are represented by smooth Deligne–Mumford stacks over $\mathbb{Z}$ of relative dimension* 10*. The stack $\mathcal{F}^{\mathrm{a}}_{\mathsf{E}_{10}(2),\rho}$ is separated.*

Applying the main theorem of Keel and Mori from [381] we obtain the following.

Corollary 5.3.2 *The stacks $\mathcal{F}_{\mathsf{E}_{10}(2),\rho}$ and $\mathcal{F}^{\mathrm{a}}_{\mathsf{E}_{10}(2),\rho}$ admit coarse moduli spaces in the category of algebraic spaces. The coarse moduli space of $\mathcal{F}^{\mathrm{a}}_{\mathsf{E}_{10}(2),\rho}$ is separated.*

We will now work over the complex numbers. We will show that the coarse moduli space of $\mathcal{F}^{a}_{\mathsf{E}_{10}(2),\rho}$ is isomorphic to the coarse moduli space of $\mathcal{K}3^{a}_{\mathsf{E}_{10}(2)}$ and, as such, it is isomorphic to an open subset of the arithmetic quotient $\mathrm{O}(N)^\sharp\backslash\mathcal{D}_N$, where N is the orthogonal complement of $\mathsf{E}_{10}(2)$ in the K3-lattice $\mathbf{L}$. Since such a description as an arithmetic quotient is not available in positive characteristic p, it is not clear whether a coarse space of $\mathcal{F}^{a}_{\mathsf{E}_{10}(2),\rho}$, if it exists, is irreducible if $p > 0$.

First, we fix a primitive embedding of the lattice $\mathsf{E}_{10}(2)$ in the K3-lattice $\mathbf{L}$. Since $D(\mathsf{E}_{10}(2)) \cong \mathsf{u}_2^{\oplus 5}$, we can apply Theorem 0.8.6 to obtain that all primitive embeddings of $\mathsf{E}_{10}(2)$ are equivalent with respect to $\mathrm{O}(\mathbf{L})$. We can fix one such an embedding as follows. We write $\mathsf{E}_{10}(2)$ as the orthogonal sum $\mathsf{U}(2) \oplus \mathsf{E}_8(2)$ and write $\mathbf{L}$ as the orthogonal sum $\mathsf{U}^{\oplus 3} \oplus \mathsf{E}_8^{\oplus 2}$. Then, we embed diagonally $\mathsf{U}(2)$ into $\mathsf{U} \oplus \mathsf{U}$ and $\mathsf{E}_8(2)$ into $\mathsf{E}_8 \oplus \mathsf{E}_8$.

Lemma 5.3.3 *Let X be a complex algebraic K3 surface.*

1. *Assume that there exists an involution g that acts as* $-\mathrm{id}$ *on the transcendental lattice* $T(X)$. *Then, the sublattice* $\mathrm{Pic}(X)^{g^*}$ *of* $\mathrm{Pic}(X)$ *of divisor classes fixed by* g^* *is a* 2*-elementary lattice that contains an ample divisor class.*
2. *Conversely, let* $\mathrm{Pic}(X)'$ *be a* 2*-elementary primitive sublattice of* $\mathrm{Pic}(X)$ *that contains an ample divisor class. Then, there exists a unique involution g of X that acts as the identity on* $\mathrm{Pic}(X)'$ *and as the minus the identity on the transcendental lattice* T_X.

Proof Let g be an involution of X such that g^* acts as $-\mathrm{id}_{T(X)}$ on the transcendental lattice $T(X)$. Thus, $H^2(X,\mathbb{Z})^{g^*}$ is contained in $\mathrm{Pic}(X)$, which actually shows that $\mathrm{Pic}(X)^{g^*}$ coincides with $H^2(X,\mathbb{Z})^{g^*}$. Since g is of finite order, we can always find a g-invariant ample divisor class. Let $M := \mathrm{Pic}(X)^{g^*}$ and let N be the orthogonal complement of M in $H^2(X,\mathbb{Z}) \cong \mathbf{L}$. The restriction of g^* to N acts as $-\mathrm{id}_N$. Since it extends to an involution of $\mathbf{L}$ that acts trivially on the orthogonal complement of N, it acts as identity on $D(N)$. This is possible only if $D(N)$ (and hence $D(M)$) is a 2-elementary abelian group. Thus, N is an 2-elementary lattice and so is its orthogonal complement M.

To prove the converse, we use the Global Torelli Theorem for K3 surfaces 5.2.11. Let $M = \mathrm{Pic}(X)$ and $N = M^\perp$ inside $H^2(X,\mathbb{Z})$. We define an involution σ of $H^2(X,\mathbb{Z})$ that extends the involution $\sigma_0 = \mathrm{id}_M \oplus -\mathrm{id}_N$ to an involution on $H^2(X,\mathbb{Z})$. Since M is 2-elementary, N is also 2-elementary and thus, σ_0 acts trivially on its discriminant lattice, which implies that it extends to $H^2(X,\mathbb{Z})$. Now, σ acts as $-\mathrm{id}_{T(X)}$ on $T(X)$ and hence, preserves $H^{2,0}(X)$ and leaves invariant an ample divisor on X. By the Global Torelli Theorem, there exists a unique involution g of X such that $\sigma = g^*$. It acts as id_M on M and as $-\mathrm{id}_{T(X)}$ on $T(X)$. □

Let M be a 2-*elementary lattice* of rank r and signature $(1, r-1)$, that is, the discriminant group of M is a 2-elementary abelian group $(\mathbb{Z}/2\mathbb{Z})^l$. All such lattices that admit a primitive embedding into the K3 lattice $\mathbf{L}$ were classified by Nikulin [557]. It follows that the isomorphism class of M is uniquely determined by a triple (r, l, δ), where δ is equal to 0 or 1 depending on whether the quadratic form on $D(M)$ is of even or odd type. All possible invariants (r, l, δ) can be found in Nikulin's triangle diagram in [**?**]Section 6]Nikulin2. In particular, we find that $l \le 11$ and $r \le 11$. Of course, the lattice $\mathsf{E}_{10}(2)$ is one of them and it corresponds to the triple $(r, l, \delta) = (10, 10, 0)$.

Let g be an involution on a complex algebraic K3 surface X that acts as minus the identity on $T(X)$. By the previous result $M := \mathrm{Pic}(X)^{g^*}$ is a 2-elementary lattice and we let (r, l, δ) be the triple associated to M as defined above. Then the locus of fixed points is described in [557, Theorem 4.2.2]. We have

$$X^g = \begin{cases} \emptyset & \text{if } (r,l,\delta) = (10,10,0), \\ C_1^{(1)} + C_2^{(1)} & \text{if } (r,l,\delta) = (10,8,0), \\ C^{(g)} + \sum_{i=1}^k R_i & \text{otherwise,} \end{cases} \tag{5.3.1}$$

where $C^{(g)}$ denotes a curve of genus $g \geq 0$, R_i are disjoint (-2)-curves, and

$$g = \frac{1}{2}(22 - r - l), \qquad k = \frac{1}{2}(r - l).$$

Proposition 5.3.4 *Let M be a 2-elementary lattice. Then, the two functors $\mathcal{F}^a_{M,\rho}$ and $\mathcal{K}3^a_M$ coincide over the complex numbers.*

Proof Given a scheme $T \to \operatorname{Spec}\mathbb{C}$, a family $(\mathcal{X} \to T, j_T, \rho) \in \mathcal{F}^a_{M,\rho}(T)$ defines a family in $\mathcal{F}^a_M(T)$ by forgetting the action ρ.

Conversely, let $(f : \mathcal{X} \to T, j_T, \rho) \in \mathcal{F}^a_M(T)$. Replacing T by a contractible set, we introduce a marking of the family $\phi_T : \mathbf{L}_T \to R^2 f_*\mathbb{Z}_\mathcal{X}$ with $\phi = j_T$ on M and, using Lemma 5.3.3, define an involution σ on $H^2(\mathcal{X}, \mathbb{Z})$ that induces an involution g_t on each fiber $\mathcal{X}_t$ with $g_t^* = \sigma$. The union of the graphs of these involutions on $\mathcal{X} \times \mathcal{X}$ is the graph of an involution $\tilde{g}$ of $\mathcal{X}/T$ that restricts to the involutions g_t on fibers. The involution $\tilde{g}$ defines an action $\rho : \boldsymbol{\mu}_{2,T} \to \operatorname{Aut}(\mathcal{X}/T)$ that turns the pair (f, j_T) into a family from $\mathcal{F}^a_{M,\rho}(T)$. Since the functor $\mathcal{F}^a_{M,\rho}$ is a stack, the local families glue together to a global family over T. □

Definition 5.3.5 A *marked Enriques surface* is a pair (S, ϕ) of an Enriques surface S and an isomorphism $\phi : \operatorname{Num}(S) \to \mathsf{E}_{10}$ (called a *marking*). Two marked Enriques surfaces (S_1, ϕ_1) and (S_2, ϕ_2) are said to be *isomorphic* if there exists an isomorphism of surfaces $f : S_1 \to S_2$ such that $\phi_1 \circ f^* = \pm\phi_2$. A *family of marked Enriques surfaces* is a pair $(f : \mathcal{S} \to T, j_T)$ that consists of a family of Enriques surfaces $f : \mathcal{S} \to T$ and an isomorphism of abelian sheaves $j_T : (\mathsf{E}_{10})_T \to \underline{\operatorname{Pic}}_{\mathcal{S}/T}/\underline{\operatorname{Pic}}^\tau_{\mathcal{S}/T}$ that is compatible with the quadratic form on E_{10} and the intersection form on $\underline{\operatorname{Pic}}_{\mathcal{S}/T}/\underline{\operatorname{Pic}}^\tau_{\mathcal{S}/T}$.

We define an isomorphism of marked families $(\mathcal{S} \to T, \phi) \to (\mathcal{S}' \to T, \phi')$ to be an isomorphism $f : \mathcal{S}/\mathcal{T} \to \mathcal{S}'/T$, such that $\phi = f^* \circ \phi'$. It follows that for any $t \in T$, an automorphism of a marked family acts as identity on the group $\operatorname{Num}(\mathcal{S}_t)$. It follows from Proposition 8.2.1 in Volume II that the group of such automorphisms is finite. This allows us to show that the functor $\mathcal{E}^{\mathrm{m}}$ of families of marked Enriques surfaces is a Deligne–Mumford stack. We denote by $\mathcal{M}^{\mathrm{m}}_{\mathrm{Enr}}$ its coarse moduli space.

Corollary 5.3.6 *Over an algebraically closed field of characteristic $p \neq 2$, the functor $\mathcal{F}^a_{\mathsf{E}_{10}(2),\rho}$ coincides with the functor $\mathcal{E}^m$ of marked Enriques surfaces.*

Proof Let $M := \mathsf{E}_{10}(2)$ and let g be an involution on a lattice M-polarized K3 surface X with $\operatorname{Pic}(X)^{g^*} = j(M)$. Since $p \neq 2$, we can apply the Lefschetz fixed–point formula

$$\operatorname{Lef}(g) = \operatorname{Tr}\left(g^* : H^*_{\text{ét}}(X, \mathbb{Q}_\ell) \to H^*_{\text{ét}}(X, \mathbb{Q}_\ell)\right) = e(X^g)$$

to conclude $e(X^g) = 0$. We set $Y = X/(g)$ and we note that the map $X \to Y$ is generically étale of degree 2. The Hurwitz–type formula $e(X) = 2e(Y) - e(X^g)$ gives that $e(Y) = 12$. Since g acts non-trivially on $H^0(X, \omega_X)$ and the map $X \to Y$ is generically étale, the classification of algebraic surfaces shows that Y must be either an Enriques surface or a rational surface. A rational surface with $e(Y) = 12$ is not minimal, hence contains a (-1)-curve. Its pre-image on X is a g-invariant (-2)-curve. Obviously, its class does not belong to $j(M)$. This contradiction shows that Y must be an Enriques surface.

Let $(\mathcal{X} \to T, j_T, \rho) \in \mathcal{F}^a_{M,\rho}$ and let g be the involution on $\mathcal{X}$ defined by ρ. The restriction of g to each fiber has no fixed points, hence g has no fixed points on $\mathcal{X}$ and the quotient family is a smooth family of Enriques surfaces. The lattice polarization j_T descends to a marking of the family. Conversely, let $(\mathcal{S} \to T, \phi)$ be a smooth family of marked Enriques surfaces. Then, $\omega_{\mathcal{S}/T} \in \mathrm{Pic}^\tau_{\mathcal{S}/T}(T)$ is a non-zero 2-torsion element. It defines an étale degree 2 cover $\pi : \mathcal{X} \to \mathcal{S}$, whose fibers $\mathcal{X}_t$ are the K3-covers of the fibers $\mathcal{S}_t$. Thus, it is a smooth family $\mathcal{X} \to T$ of K3 surfaces. The marking $\phi : (\mathsf{E}_{10})_T \to \underline{\mathrm{Pic}}_{\mathcal{S}/T}/\underline{\mathrm{Pic}}^\tau_{\mathcal{S}/T}$ defines a lattice $\mathsf{E}_{10}(2)$-polarization $\phi^*(J_T) : \mathsf{E}_{10}(2) \to \underline{\mathrm{Pic}}_{\mathcal{X}/T}$. Since $\underline{\mathrm{Pic}}_{\mathcal{S}/T}(T)$ contains a relatively ample invertible sheaf, the lattice polarization is ample. ◻

Corollary 5.3.7 *Over an algebraically closed field of characteristic $p \neq 2$, the coarse moduli space of the functor $\mathcal{F}^a_{\mathsf{E}_{10}(2)}$ is isomorphic to $\mathcal{M}^m_{\mathrm{Enr}}$.*

We have just established that any ample lattice $\mathsf{E}_{10}(2)$-polarized complex K3 surface (X, j) admits a fixed-point involution τ. If we choose a marking $\phi : H^2(X, \mathbb{Z}) \to \mathbf{L}$ compatible with j, then $T(X) = \mathrm{Pic}(X)^\perp$ admits a primitive embedding in $N = \mathsf{E}_{10}(2)^\perp = \mathsf{U} \oplus \mathsf{E}_{10}(2)$. Since the polarization j is ample, the orthogonal complement of $j(E_{10})$ in $\mathrm{Pic}(X)$ does not contain divisor classes with self-intersection -2. Since its image K under ϕ coincides with the orthogonal complement of $\phi(T(X))$ in N, we have $K_{-2} = \emptyset$. We now prove the converse (see [383, Theorem 1]).

Theorem 5.3.8 *A complex K3 surface X admits a fixed-point-free involution if and only if $T(X)$ admits a primitive embedding into the lattice $N = \mathsf{U} \oplus \mathsf{E}_{10}(2)$ such that the orthogonal complement of the image does not contain vectors of norm square -2.*

Proof Let $\iota : T(X) \hookrightarrow N$ be a primitive embedding satisfying the assumption from the theorem. The composition with the inclusion $N \subset \mathbf{L}$ gives a primitive embedding of $T(X)$ into $\mathbf{L}$. If we choose a marking $\phi : H^2(X, \mathbb{Z}) \to \mathbf{L}$, then its restriction to $T(X)$ defines another primitive embedding of $T(X)$ into $\mathbf{L}$. Suppose we prove that all primitive embeddings of $T(X)$ into $\mathbf{L}$ are equivalent. Then, after composing with an isometry of $\mathbf{L}$, we may assume that $\phi(T(X)) \subset N$. Then $M = \mathsf{E}_{10}(2) \subset \phi(\mathrm{Pic}(X))$ and hence, $\phi|\,\mathrm{Pic}(X)$ defines a lattice M polarization $j : M \hookrightarrow \mathrm{Pic}(X)$ of X. After choosing $V(M)^+$ appropriately, we may assume that $j(V(M)^+) = j(C(M)) \subset V(\mathrm{Pic}(X))^+$. By assumption, $\mathrm{Pic}(X)$ has no (-2)-divisor classes in the orthogonal complement of $j(M)$, hence the image of $V(M)^+$ lies in the interior of $\mathrm{Nef}(X)$ and hence contains an ample divisor class. Thus, the lattice M polarization is ample and hence, X admits a fixed-point-free involution.

It remains to prove that all primitive embedding of $T(X)$ into $\mathbf{L}$ are equivalent. We follow an argument due to Ohashi. Let $t(X) = 22 - \rho(X) \le 12$ be the rank of $T(X)$. If $t(X) = 12$, then $M \cong \mathrm{Pic}(X)$ and we know that all primitive embeddings of the Enriques lattice into the K3 lattice are equivalent. If $t(X) \le 10$, then $\rho(X) \ge 12$ and $l(\mathrm{Pic}(X)) \le 10$. Applying Theorem 0.8.6, we obtain that all primitive embeddings of $\mathrm{Pic}(X)$ into $\mathbf{L}$ are equivalent, hence all primitive embeddings of $T(X)$ into $\mathbf{L}$ are equivalent.

It remains to consider the case $t(X) = \rho(X) = 11$. In this case rank $K = 1$ and hence $K \cong \langle -2n\rangle$. Since, by assumption, $K_{-2} = \emptyset$, we have $n \ge 2$. The lattice $\mathrm{Pic}(X) = T(X)^\perp$ contains $\mathsf{E}_{10}(2) \oplus \langle -2n\rangle$ as a sublattice of finite index and hence, the embedding $\mathsf{E}_{10}(2) \oplus \langle -2n\rangle \hookrightarrow \mathrm{Pic}(X)$ corresponds to an isomorphism $\gamma : H \to H' \subset D(\langle -2n\rangle$ of a subgroup H of $\mathsf{u}_2^{\oplus 5}$ to a subgroup H' of $D(\langle -2n\rangle)$. Since H is an elementary 2-group and $D(\langle -2n\rangle)$ is cyclic of order $2n$, we have two possibilities:

1. $H = H' = \{1\}$.
2. H and H are of order 2.

Since $\mathsf{u}_2^{\oplus 5}$ is isomorphic to the even non-degenerate quadratic space $\mathbb{F}_2^{10}$ and the group of its symmetries acts transitively on its non-zero elements, we can fix H and hence γ. In case (1) we get $M = \mathsf{E}_{10}(2) \oplus \langle -2n\rangle$ and in case (2) we get $M \cong \mathsf{U} \oplus \mathsf{E}_8(2) \oplus \langle -4k\rangle$. It remains to apply Theorem 0.8.6 to conclude the uniqueness of a primitive embedding of $\mathrm{Pic}(X)$ into $\mathbf{L}$. $\square$

All possible isomorphism classes of transcendental lattices of K3 surfaces admitting a fixed-point-free involution have been classified in [93]. There are only 11 with rank$T(X) \ge 7$.

We will further discuss the functor $\mathcal{E}^m$ in characteristic $p = 2$ in Section 5.11.

We remind the reader that we introduced period spaces for K3 surfaces, orthogonal groups acting on them, and Heegner divisors in Section 5.2. To simplify the notation, we set

$$\begin{aligned}
\mathcal{D}_{\mathrm{Enr}} &:= \mathcal{D}_{\mathsf{E}_{10}(2)^\perp},\\
H_\delta &:= H_{\mathsf{E}_{10}(2)^\perp,\delta},\ \delta \in \mathsf{E}_{10}(2)^\perp,\\
\mathcal{H}(-2n) &:= \cup_{\delta\in(\mathsf{E}_{10}(2)^\perp)_{-2n}} H_{\mathsf{E}_{10}(2)^\perp,\delta},\\
\mathcal{D}^\circ_{\mathrm{Enr}} &:= \mathcal{D}^\circ_{\mathsf{E}_{10}(2)^\perp}.
\end{aligned}$$

We also set

$$\Gamma_{\mathrm{Enr}} = \mathrm{O}(\mathsf{E}_{10}(2)^\perp)^\sharp, \quad \Gamma_{\mathrm{Enr}}(2)^\sharp := \mathrm{O}(\mathsf{E}_{10}^\perp)^\sharp$$

Taking into account Proposition 5.2.12, Proposition 5.3.4, Corollary 5.3.6, and Corollary 5.2.10, we obtain the following.

Theorem 5.3.9 *Over the complex numbers, the period map $\mathcal{K}^a_{\mathsf{E}_{10}(2)} \to \mathcal{D}_{\mathrm{Enr}}$ defines an isomorphism*

$$\mathcal{M}^{m}_{\mathrm{Enr}} := \Gamma^{\sharp}_{\mathrm{Enr}} \backslash \mathcal{D}^{\circ}_{\mathrm{Enr}}.$$

In particular, $\mathcal{M}^{m}_{\mathrm{Enr}}$ *carries the structure of a* 10*-dimensional irreducible quasi-projective variety over* $\mathbb{C}$*.*

A projective model of a compactification of the moduli space $\mathcal{M}^{\mathrm{m}}_{\mathrm{Enr}}$ is described in the appendix.

Since this space is irreducible, it is connected, which has the following important application to the differential topology of complex Enriques surfaces.

Corollary 5.3.10 *Any two complex Enriques surfaces are homeomorphic and diffeomorphic as four-manifolds.*

To get rid of a marking, we consider the group $\mathsf{O}(\mathsf{E}_{10}(2))' \cong \mathsf{O}(\mathsf{E}_{10})' = W(\mathsf{E}_{10})$ and note that we have to restrict to isometries of the lattice M that preserve $V(M)^+$. We have a natural isomorphism

$$\mathsf{E}_{10}(\tfrac{1}{2})/2\mathsf{E}_{10}(\tfrac{1}{2}) \to \mathsf{E}_{10}(2)^{\vee}/\mathsf{E}(2) = D(\mathsf{E}_{10}(2)). \tag{5.3.2}$$

Using this, we can identify the quadratic form $q_{\mathsf{E}_{10}(2)}$ with an even quadratic form on the vector space $\mathsf{E}_{10}/2\mathsf{E}_{10} \cong \mathbb{F}_2^{10}$ and obtain an isomorphism

$$\mathsf{O}\left(D(\mathsf{E}_{10}(2), q_{\mathsf{E}_{10}(2)})\right) \cong \mathsf{O}^+(10, \mathbb{F}_2),$$

see Corollary 6.4.7 in Section 6.4 of Volume II. Also, the natural homomorphisms give rise to a commutative diagram

$$\begin{tikzcd}
\mathsf{O}(E_{10}) \arrow[r, "\phi"] \arrow[d] & \mathsf{O}(\mathsf{E}_{10}(2)) \arrow[d] \\
\mathsf{O}(\mathsf{E}_{10}/2\mathsf{E}_{10}) \arrow[r, "\bar{\phi}"] & \mathsf{O}(D(\mathsf{E}_{10}(2)))
\end{tikzcd}$$

where the horizontal arrows are isomorphisms. Applying Theorem 0.8.6, we see that the right vertical arrow is surjective, so the left vertical arrow is also surjective. By the same theorem, an isometry of E_{10}, identified with an isometry of $\mathsf{E}_{10}(2)$, lifts to an isometry of $\mathsf{E}_{10}(2)^{\perp}$ and hence, acts on $\mathcal{M}^{m}_{\mathrm{Enr}}$ by changing the markings. The kernel of this action is the 2-congruence subgroup $W(\mathsf{E}_{10})(2)$ identified with $\mathsf{O}(\mathsf{E}_{10}(2))^{\sharp}$. The quotient by the action of

$$\overline{W(\mathsf{E}_{10})} := W(\mathsf{E}_{10})/W(\mathsf{E}_{10})(2) \cong \mathsf{O}^+(10, \mathbb{F}_2)$$

is identified with the action of the quotient of $\Gamma_{\mathrm{Enr}}/\Gamma^{\sharp}_{\mathrm{Enr}}$ on $\mathcal{D}_{\mathrm{Enr}}$.

We set

$$\mathcal{M}_{\mathrm{Enr}} := \Gamma_{\mathrm{Enr}} \backslash \mathcal{M}^{\mathrm{m}}_{\mathrm{Enr}} = \mathcal{M}^{\mathrm{m}}_{K3}/\overline{W(\mathsf{E}_{10})}$$

and call it the *moduli space of (unmarked) complex Enriques surfaces*.

The group $\overline{W(\mathsf{E}_{10})}$ does not act freely on $\mathcal{M}^{\mathrm{m}}_{\mathrm{Enr}}$. The stabilizer subgroup of (X, ϕ) is equal to the image $\overline{\mathrm{Aut}(S)}^{*}_{\phi}$ of the group

$$\phi^*(\mathrm{Aut}(S)^*) := \phi^{-1} \circ \mathrm{Aut}(S) \circ \phi.$$

in $\overline{W(\mathsf{E}_{10})}$. In other words, we have

$$\overline{\mathrm{Aut}(S)}^*_\phi = \phi^*(\mathrm{Aut}(S)^*)/\phi^*(\mathrm{Aut}(S)^*) \cap W(\mathsf{E}_{10})(2). \tag{5.3.3}$$

Remark 5.3.11 As we will see in Chapter 8 of Volume II, the group of automorphisms of an unnodal Enriques surface is an infinite discrete group. Thus, although we can define the stack of Enriques surfaces, it is not an algebraic stack and it is unclear whether it admits a coarse moduli space. In particular, our moduli space $\mathcal{M}_{\mathrm{Enr}}$ is not a coarse moduli space in the usual sense because the automorphism group of a general Enriques surface is infinite. What we can really say is that its points are in a bijective correspondence with the set of isomorphism classes of complex Enriques surface and that this correspondence is compatible with the bijection between the set of isomorphism classes of marked Enriques surfaces and the points of its coarse moduli space.

Remark 5.3.12 Note that via the isomorphism (5.3.2), a marking $i : \mathrm{Num}(S) \to \mathsf{E}_{10}$ of an Enriques surface S defines an isomorphism

$$(\mathsf{E}_{10}/2\mathsf{E}_{10}, q_{\mathsf{E}_{10}} \mod 2) \cong (\mathbb{F}_2^{10}, q) \to (\mathrm{Num}(S)/2\,\mathrm{Num}(S), q_{\mathrm{Num}(S)} \mod 2)$$

that can be thought as a 2-*level structure* on $\mathrm{Num}(S)$. This is in analogy with the notion of a 2-level structure on a principally polarized abelian variety of dimension g, which defines an isomorphism between the group of 2-torsion points in its Picard variety equipped with the Weil pairing and the space $\mathbb{F}_2^{2g}$ equipped with the standard non-degenerate symplectic form $\sum_{i=1}^g (x_i y_{i+g} + x_{i+g} y_i)$, see [541, Chapter IV, 20].

We have just established that the canonical cover X of a general Enriques surface S admits a unique lattice $\mathsf{E}_{10}(2)$ polarization, and hence, its transcendental lattice is isomorphic to $\mathsf{U} \oplus \mathsf{E}_{10}(2)$.

Let (X, j) be a K3 surface with an ample lattice polarization $j : M \to \mathrm{Pic}(X)$ such that j is bijective. We discussed such lattice polarizations in Example 5.2.14. We have $N = M^\perp \cong T(X)$, where $T(X)$ is the transcendental lattice of X.

It is a natural question whether a given K3 surface X admits a fixed-point-free involution and how many of them there are, up to conjugacy by automorphisms of X. In the case $\Bbbk = \mathbb{C}$ we gave a necessary and sufficient condition in Theorem 5.3.8 for the existence of such an involution in terms of the transcendental lattice $T(X)$ of X.

If τ is a fixed-point-free involution on X, then $\mathrm{Pic}(X)^\tau$ is isomorphic to $\mathrm{Num}(X/(\tau))(2) \cong \mathsf{E}_{10}(2)$. Thus, any fixed-point-free involution of X defines a primitive sublattice of $\mathrm{Pic}(X)$ isomorphic to $\mathsf{E}_{10}(2)$ that does not contain a (-2)-vector in its orthogonal complement. In turn, it defines an ample lattice $\mathsf{E}_{10}(2)$ polarization on X. However, we identify two such polarizations if they differ by an isometry of $\mathsf{E}_{10}(2)$. Let $\mathfrak{M}$ be the set of such sublattices of $\mathrm{Pic}(X)$ and let $\{M_1, \ldots, M_k\}$ be representatives of the orbits of $\mathrm{O}(\mathrm{Pic}(X))$ on $\mathfrak{M}$. For any M_i let

$\bar{G}_i$ be the image of the stabilizer subgroup G_i of M_i under the homomorphism $\rho_{\mathrm{Pic}(X)} : \mathrm{O}(\mathrm{Pic}(X)) \to \mathrm{O}(D(\mathrm{Pic}(X)))$. We set

$$B := \sum_{i=1}^{k} [\mathrm{O}(D(\mathrm{Pic}(X))) : \bar{G}_i]. \tag{5.3.4}$$

and then, we have the following theorem of Ohashi [577].

Theorem 5.3.13 *The number of conjugacy classes of fixed-point-free involutions on a complex algebraic K3 surface is less than or equal to the number B from* (5.3.4). *It is equal to this number if the homomorphism* $\mathrm{O}(\mathrm{Pic}(X)) \to \mathrm{O}(D(\mathrm{Pic}(X)))$ *is surjective and* $\mathrm{Aut}_{\mathrm{Hdg}}(T(X)) = \{\pm 1\}$.

Proof Let N_i be the number of fixed-point-free involutions τ of X such that $\mathrm{Pic}(X)^\tau$ belongs to the orbit of M_i. Let $\sigma \in \mathrm{O}(\mathrm{Pic}(X))'$ and after composing it with some reflection $w \in W_X$, we may assume that $M_i \to \mathrm{Pic}(X)$ is an ample lattice polarization $j_i : \mathsf{E}_{10}(2) \to \mathrm{Pic}(X)$. Then we know that it defines a fixed-point-free involution τ_i of X. The stabilizer subgroup of j_i in $\mathrm{O}(\mathrm{Pic}(X))$ is equal to the subgroup $A(X)$ that preserves the ample cone of X. Its subgroup $A(X)^\sharp$ extends to a group of automorphisms of X and all involutions from the orbit of $A(X)^\sharp$ of j_i define conjugate involutions. Thus the number of conjugacy classes of involutions defined by the orbit of M_j is at most $[A(X) : A(X)^\sharp] = [\mathrm{O}(D(\mathrm{Pic}(X))) : \bar{G}_i]$. This proves the asserted bound.

Suppose $\rho = \rho_{\mathrm{Pic}(X)}$ is surjective. Let K_i be the orthogonal complement of M_i. Since M is 2-elementary, the isometry $(-\mathrm{id}_M, \mathrm{id}_{K_i})$ of $M \oplus K_i$ extends to an isometry of $\mathrm{Pic}(X)$ that does not preserve $V(M)^+$ and hence, does not preserve the ample cone of X. Since its image in $D(M \oplus K_i)$ is trivial, its lift to $\mathrm{O}(\mathrm{Pic}(X))$ belongs to $\mathrm{O}(\mathrm{Pic}(X))^\sharp$. Thus, the restriction of ρ to the index 2 subgroup $\mathrm{O}(\mathrm{Pic}(X))'$ of elements preserving $V(\mathrm{Pic}(X))^+$ is surjective. Since $W(\mathrm{Num}(X))$ is contained in $\mathrm{O}(\mathrm{Pic}(X))^\sharp$ and $\mathrm{O}(\mathrm{Pic}(X))' = W(\mathrm{Num}(X)) \rtimes A(X)$, we see that ρ defines a surjective homomorphism $A(X) \to \mathrm{O}(D(\mathrm{Pic}(X)))$. Thus, the subgroup $\bar{G}_i$ of $\mathrm{O}(D(\mathrm{Pic}(X)))$ is equal to the image of some subgroup of $A(X)$ and we may assume that G_i is such a subgroup. The bound is sharp if no non-zero element σ from $\bar{G}_i$ is equal to the image of g^* for some $g \in \mathrm{Aut}(X)$. Here, the second assumption is used. If σ is such an element, then it extends to an isometry of $H^2(X, \mathbb{Z})$ that leaves $T(X)$ invariant and defines a non-trivial element of $\mathrm{O}(D(T(X)) \cong \mathrm{O}(D(\mathrm{Pic}(X)))$. But then, it defines a Hodge isometry of $T(X)$ different from $\pm\,\mathrm{id}_{T(X)}$, contradicting our assumption. □

Example 5.3.14 The following example is taken from [577]. It shows that the number of possible fixed-point-free involutions for complex K3 surfaces can be arbitrarily large.

Let X be a K3 surface and assume that X admits an ample $\mathsf{E}_{10}(2)$-polarization and $\mathrm{rank}(T(X)) = 11$. It follows from the proof of Theorem 5.3.8 that $\mathrm{Pic}(X) \cong M$, where $M = \mathsf{E}_{10}(2) \oplus \langle -2d \rangle$ with $d \geq 2$, or $M \cong \mathsf{U} \oplus \mathsf{E}_8(2) \oplus \langle -4d \rangle$. We have $\mathrm{O}(D(\mathrm{Pic}(X))) \cong \mathrm{O}(10, \mathbb{F}_2)^+ \times (\mathbb{Z}/2\mathbb{Z})^{p(d)}$ in the first case and that it is isomorphic to $\mathsf{O}(8,\mathbb{F}_2)^+ \times (\mathbb{Z}/2\mathbb{Z})^{\oplus p(2d)}$ in the latter case. It follows from Theorem 0.8.6 that the homomorphism $\rho : \mathsf{O}(\mathrm{Pic}(X)) \to \mathsf{O}(D(\mathrm{Pic}(X)))$ is surjective. Since the rank of $T(X)$ is odd, we have $\mathrm{Aut}_{\mathrm{Hdg}}(T(X) = \{\pm 1\}$ [321, Corollary 3.5].

Assume that M is as in the first case. Since $\mathrm{Pic}(X)(\frac{1}{2})$ is an integral lattice and any primitive embedding of $\mathsf{E}_{10}(2)$ in M is defined by a primitive embedding of the unimodular lattice E_{10} in $\mathrm{Pic}(X)(\frac{1}{2})$, it follows from Theorem 0.8.6 that all such embedding are equivalent. Let K be the orthogonal complement of $\mathsf{E}_{10}(2)$ in M. It is isomorphic to $\langle -2d\rangle$. The stabilizer of the embedding $\mathsf{E}_{10}(2) \to M$ is equal to $\mathsf{O}(\mathsf{E}_{10}(2)) \times \mathsf{O}(\langle -2d\rangle)$. Since the homomorphism $\rho_{\mathsf{E}_{10}(2)}$ is surjective, we obtain that the image of K in $\mathsf{O}(D(M))$ is isomorphic to $\mathsf{O}(D(\mathsf{E}_{10}(2)) \times \{\pm 1\}$. Its index B in $\mathsf{O}(D(M))$ is at least $\frac{1}{2}\#O(\langle -\frac{1}{2d}\rangle)$. Using [636], we see that $B \geq 2^{p(d)-1}$ – in fact, it follows from [577, Lemma 3.1] that $B = 2^{10+p(d)}$.

Example 5.3.15 Let A be an abelian surface and let $X = \widetilde{\mathrm{Kum}}(A)$ be the minimal resolution of the Kummer surface $\mathrm{Kum}(A) = A/(\iota)$, where ι is the sign involution. In case there is no danger of confusion, we continue to call it a Kummer surface. Let $\tilde{A} \to A$ be the blow-up of 16 fixed points of ι and let $\pi : \tilde{A} \to X$ be the quotient map for the action of the lift of ι to $\tilde{A}$. The homomorphism $\pi^* : H^2(X,\mathbb{Z}) \to H^2(\tilde{A},\mathbb{Z})$ defines an isomorphism of lattices $T(X)(2) \to T(A)$[321, Chapter 3,2.5]. Since the sum of the classes of the exceptional curves of $X \to \mathrm{Kum}(A)$ is divisible by 2, the sublattice of $\mathrm{Pic}(X)$ generated by the classes of the exceptional curves is not primitive. In fact, its primitive closure (called the *Kummer lattice*) is defined by an isotropic subgroup of $D(\langle -2\rangle^{\oplus 16})$ isomorphic to $(\mathbb{Z}/2\mathbb{Z})^{\oplus 5}$, see [321, Chapter 14.3.3].

Since $H^2(A,\mathbb{Z})$ is a unimodular even lattice of signature $(3,3)$, it is isomorphic to $\mathsf{U}^{\oplus 3}$. The Néron–Severi lattice $\mathrm{NS}(A)$ is equal to the image of $\mathrm{Pic}(A)$ in $H^2(A,\mathbb{Z})$ under the Chern class map c_1. It is isomorphic to the numerical lattice $\mathrm{Num}(A)$. If A is moduli general, then $\mathrm{Num}(A)$ is isomorphic to $\langle 2d\rangle$ for some positive integer d. Thus, $\mathrm{rank}\, T(X) = 5$ and $\rho(X) = 17$. We have $T(X) = T(A)(2) \cong \mathsf{U}(2)^{\oplus 2} \oplus \langle -4d\rangle$. If $\rho(X) = 18$ (resp. 19 or 20), then $T(X) \cong T(2) \oplus \mathsf{U}(2)$ (resp. $T(2)$) for some even hyperbolic lattice T. For example, if $A = E_1 \times E_2$, where E_1, E_2 are two non-isogenous elliptic curves, then we have $\rho(X) = 18$. If $A = E_1 \times E_2$ are two isogenous elliptic curves then $\rho(X) \geq 19$. If $A = E \times E$ and $\#\,\mathrm{Aut}(E) > 2$, then $\rho(X) = 20$.

Since $T(X)$ is always contained in $\mathsf{U}(2)^{\oplus 2} \oplus \langle -4d\rangle$, we can primitively embed it in $N = \mathsf{U} \oplus \mathsf{U}(2) \oplus \mathsf{E}_8(2)$ by embedding $\mathsf{U}(2) \oplus \langle -4d\rangle$ into $\mathsf{U} \oplus \mathsf{E}_8(2)$. A less obvious fact is that we can find an embedding such that its orthogonal complement does not contain (-2)-vectors, see [383, Theorem 2]. This shows that any Kummer surface admits a fixed-point-free involution.

To compute the number of these involutions, we apply Theorem 5.3.13. For example, assume that $A \cong \mathrm{Jac}(C)$ for some general curve C of genus two. In this case, $\mathrm{Pic}(A) \cong \langle 2\rangle$ and $T(X) \cong \mathsf{U}(2)^{\oplus 2} \oplus \langle -4\rangle$. One can choose a representative Θ of a generator of $\mathrm{NS}(A)$ such that the linear system $|2\Theta|$ defines a map $A \to \mathbb{P}^3$ that factors through a closed embedding of $\mathrm{Kum}(A)$. Its image is a quartic surface with 16 ordinary nodes, a *Kummer quartic surface* . The sign involution of A acts on $H^0(A,\mathcal{O}_A(4\Theta))$ and the linear subsystem of $|H^0(A,\mathcal{O}_A(4\Theta))^-|$ of anti-invariant sections is of dimension 5. The 2-torsion points of A are its base points. It defines a map of $\tilde{A}$ that factors through a closed embedding of X into $\mathbb{P}^5$ as a surface of degree 8. In suitable projective coordinates it can be given by equations

$$\sum_{i=0}^{5} x_i^2 = \sum_{i=0}^{5} a_i x_i^2 = \sum_{i=0}^{5} a_i^2 x_i^2 = 0,$$

where $(a_0, a_1, \dots, a_5)$ are the branch points of the cover $C \to \mathbb{P}^1$ defined by the hyperelliptic involution of C, see [177, 10.3]. The group of automorphisms of X generated by changing the signs of the coordinates is isomorphic to $(\mathbb{Z}/2\mathbb{Z})^{\oplus 5}$. An involution obtained by changing the signs of odd number of coordinates is called a *switch*) (for reasons that we do not discuss here). A switch changing the signs of three coordinates is fixed-point-free. Note that the subring A of invariant element of the projective coordinate ring of X in $\mathbb{P}^5$ is generated by x_3, x_4, x_5 and $y_{ij} = x_i x_j, i, j \le 2$. Thus, the quotient Enriques surface S is isomorphic to Proj A and admits two maps to $\mathbb{P}^2$ defined by the homomorphism $\Bbbk[x_3, x_4, x_5] \to A$ and $\Bbbk[x_0, x_1, x_2]^{(2)} \to A$. This agrees with our models of Enriques surfaces defined by a polarization of degree 4 discussed in Section 3.4. We can express x_0^2, x_1^2, x_3^2 as quadratic forms in x_3, x_4, x_5 and obtain the equations of S in $\mathbb{P}(1,1,1,2,2,2)$ given by these quadratic forms and the quadratic forms in x_0, x_1, x_2 corresponding to the equations of the Veronese surface $\nu_2(\mathbb{P}^2)$ in $\mathbb{P}^5$.

There are also other examples of fixed-point-free involutions.

It is known that a minimal nonsingular model of the Hesse quartic surface of a Sylvester non-degenerate cubic surface is a K3 surface that admits a natural fixed-point-free involution, see Example 6.4.20 in Volume II. It was shown by Hutchinson that the Kummer surface associated to a the Jacobian variety of a curve of genus two admits a birational model isomorphic to the Hessian surface of a cubic surface (it must be a special cubic surface), see [187] and the references there. Each such model is defined by a choice of a Weber hexad of nodes on Kum(A). Another type of a fixed-point-free involution is also due to Hutchinson. It is defined by a cubic Cremona involution with a set of four fundamental points forming a Göpel tetrad of nodes. It is shown in [578] that the set of conjugacy classes of fixed-point-free involutions on X can be represented by 10 switches, 6 Hutchinson-Weber involutions and 15 Hutchinson-Göpel involutions, see Theorem 10.7.5 in Volume II. We will give later in Volume II many examples of fixed-point-free involutions on Kummer surfaces of other types.

5.4 Moduli of Coble Surfaces of K3 Type

In this section, we will show that the Heegner divisor

$$\Gamma_{\mathrm{Enr}}^{\sharp} \backslash \mathcal{H}(-2) = \Gamma_{\mathrm{Enr}}^{\sharp} \backslash (\mathcal{D}_{\mathrm{Enr}} - \mathcal{D}_{\mathrm{Enr}}^{\circ})$$

is irreducible and that it contains a dense and locally closed subset that parameterizes isomorphism classes of marked Coble surfaces, which we will discuss in this section. We denote this devisor by $\mathcal{M}^{\mathrm{m}}_{\mathrm{Coble}}$ and its image in $\Gamma_{\mathrm{Enr}}\backslash\mathcal{D}_{\mathrm{Enr}}$ by $\mathcal{M}_{\mathrm{Coble}}$.

We will start with the following nice observation of Allcock [6, Lemma 1].

Lemma 5.4.1 *Let M be an even unimodular lattice and set $N := M(2) \oplus \mathsf{U}$. Then, there exists an odd lattice N' isomorphic to $M \oplus \mathsf{I}^{1,1}$ with $N'_{\mathbb{R}} = N_{\mathbb{R}}$, such that every isometry of N extended to $N_{\mathbb{R}}$ preserves N' and vice-versa. In particular, $\mathsf{O}(M(2) \oplus \mathsf{U}) \cong \mathsf{O}(M \oplus \mathsf{I}^{1,1})$.*

Proof Note that the lattice $\frac{1}{\sqrt{2}}M(2) \subset M_{\mathbb{R}}$ is isomorphic to M and that $\frac{1}{\sqrt{2}}\mathsf{U} \cong \mathsf{U}(\frac{1}{2})$. Thus, there exists an isometry of lattices $(\frac{1}{\sqrt{2}}N)^\vee \cong M \oplus \mathsf{U}(2)$. Their discriminant quadratic forms are isomorphic to the discriminant quadratic form of $\mathsf{U}(2)$ given by the Gram matrix $\frac{1}{2}\cdot\left(\begin{smallmatrix}0&1\\1&0\end{smallmatrix}\right)$. Adding a suitable non-zero non-isotropic vector, gives an unimodular odd overlattice of $\mathsf{U}(2)$, which must be isomorphic to $\mathsf{I}^{1,1}$. Thus, $(\frac{1}{\sqrt{2}}N)^\vee$ lies in $N' = M \oplus \mathsf{I}^{1,1}$. We can recover N from N' as $(\sqrt{2}N'^{\mathrm{ev}})^\vee$, where N'^{ev} is the maximal even sublattice of N'. It is clear that any isometry of N or N' preserves the other. □

Corollary 5.4.2 *The boundary*

$$\mathcal{M}^{m}_{Coble} := \Gamma^{\sharp}_{\mathrm{Enr}}\backslash\mathcal{H}(-2)$$

of $\mathcal{M}^{m}_{\mathrm{Enr}}$ in $\Gamma^{\sharp}_{\mathrm{Enr}}\backslash\mathcal{D}_{\mathrm{Enr}}$ is an irreducible divisor.

Proof This follows from the fact that all (-2)-vectors in $\mathsf{E}_{10}(2) \oplus \mathsf{U}$ form one orbit with respect to the $\Gamma^{\sharp}_{\mathrm{Enr}}$. First, we see that they form one orbit with respect to $\Gamma_{\mathrm{Enr}} = \mathsf{O}(\mathsf{E}_{10})'$. We apply Lemma 5.4.1 to $M = \mathsf{E}_{10}$. Then, the set of vectors of norm -2 in $N := \mathsf{E}_{10}(2)^\perp = \mathsf{E}_{10}(2) \oplus \mathsf{U}$ corresponds bijectively to the set of vectors of norm -1 in $N' := \mathsf{E}_{10} \oplus \mathsf{I}^{1,1}$. Next, we use that $\mathsf{O}(\mathsf{E}_{10} \oplus \mathsf{I}^{1,1})$ acts transitively on vectors of norm -1. In fact, for any such vector v, the orthogonal complement $\langle v\rangle^\perp$ is an odd unimodular lattice of signature $(1, 10)$ and hence, by Proposition 0.8.8, it must be isomorphic to $\mathsf{I}^{1,9}$, which gives a decomposition

$$\mathsf{E}_{10} \oplus \mathsf{I}^{1,1} = \langle v\rangle \oplus \langle v\rangle^\perp = \langle -1\rangle \oplus \mathsf{I}^{1,10}.$$

Given two vectors of norm -1, it is immediate that there exists an isometry sending one to another.

To see that there is only one orbit under the $\Gamma^{\sharp}_{\mathrm{Enr}}$-action, we use an argument from [410, Lemma 2.1]. Since $\Gamma^{\sharp}_{\mathrm{Enr}}$ is a normal subgroup of Γ_{Enr} with quotient $D(\mathsf{E}_{20}(2))$, it follows that the number of orbits is equal to

$$c := \frac{[\Gamma_{\mathrm{Enr}} : \Gamma^{\sharp}_{\mathrm{Enr}}]}{[(\Gamma_{\mathrm{Enr}})_r : (\Gamma^{\sharp}_{\mathrm{Enr}})_r]},$$

where $r \in N_{-2}$. However, the reflection s_r defined by r belongs to $\Gamma_{\mathrm{Enr}}^{\sharp}$, so that the natural map from $(\Gamma_{\mathrm{Enr}})_r/(\Gamma_{\mathrm{Enr}}^{\sharp})_r$ to $D(\mathsf{E}_{10}(2))$ is an isomorphism. This implies that c is equal to 1. □

We recall the definition of a Coble surface from [195].

Definition 5.4.3 A *Coble surface* is a smooth projective rational surface V with $|-K_{\mathsf{V}}| = \emptyset$ and $|-2K_{\mathsf{V}}| \neq \emptyset$.

We will be interested only in *terminal Coble surfaces of K3 type*, see [195, Section 6.1]. These are Coble surfaces with $|-2K_{\mathsf{V}}| = \{C_1 + \cdots + C_n\}$, where $C_1, \ldots, C_n$ are disjoint smooth rational surfaces with self-intersection -4, which we will call (-4)-*curves* for short. Following Mukai, we call these (-4)-curves the *boundary components* and the union of them the *boundary* of V.

In characteristic $p \neq 2$, the double cover $\pi : X \to \mathsf{V}$ branched along $C_1 + \cdots + C_n$ is a K3 surface X. The pre-images $\bar{C}_i$ of the curves C_i are disjoint (-2)-curves on X. Generically, the ramification divisor of the cover is described by (5.3.1), where $g = 0$ and $k = n - 1$. Thus, $\mathrm{Pic}(X)$ is a 2-elementary lattice of rank $r = 10 + n$ and the rank l of the 2-elementary discriminant group is equal to $12 - n$.

It is known that V is a *basic* rational surface, that is, it admits a birational morphism $f : \mathsf{V} \to \mathbb{P}^2$ that decomposes as the blow-up of N points, which are allowed to be infinitely near (see Proposition 9.1.3 in Volume II). The condition $|-K_{\mathsf{V}}| = \emptyset$ implies that $N \geq 10$, so that $\mathrm{rank}\,\mathrm{Pic}(\mathsf{V}) \geq 11$. We have

$$K_{\mathsf{V}}^2 = 9 - N = -n. \tag{5.4.1}$$

Now, let $(e_0, e_1, \ldots, e_N)$ be a geometric basis in $\mathrm{Pic}(V)$ as introduced in (0.5.5) in Section 0.3. A *marking* of V is a choice of such a geometric basis. It defines an isomorphism of lattices $\mathrm{Pic}(\mathsf{V}) \to \mathsf{I}^{1,N}$. The sublattice $\mathrm{Pic}(\mathsf{V})'$ of $\mathrm{Pic}(\mathsf{V})$ spanned by $e_0, e_1, \ldots, e_{10}$ is a sublattice of $\mathsf{I}^{1,N}$, which is isomorphic to $\mathsf{I}^{1,10}$. It contains the lattice isomorphic to E_{10} as the orthogonal complement to the vector $k_{10} := 3e_0 - e_1 - \cdots - e_{10}$.

Let $M = \mathrm{Pic}(X)'$ be the sublattice of $\mathrm{Pic}(X)$ generated by $\pi^*(\mathrm{Pic}(\mathsf{V}))$ and the classes of the curves $\bar{C}_i$. It is a 2-elementary sublattice of $\mathrm{Pic}(X)$ of index 2^{k-1} in $\pi^*(\mathrm{Pic}(\mathsf{V})) \cong \mathsf{I}^{1,N}(2)$. Taking into account formula (5.4.1), we obtain

$$l(D(M)) = l(\mathsf{I}^{1,N}(2)) - 2(n-1) = 21 - N.$$

This agrees with formula (5.3.1), where $g = 0$ and $k = n - 1$. Thus, generically, $\mathrm{Pic}(X)$ is a 2-elementary lattice of rank $r = 10 + n$ and the rank l of the 2-elementary discriminant group is equal to $12 - n$. The discriminant quadratic form could be of even or of odd type. Table 5.1 below is based on Nikulin's classification of 2-elementary lattices, see [557, 4.3] or Theorem 10.1.6 in Volume II.

k	N	K_V^2	(r, l, δ)	2-elementary lattice M	$N = M^\perp$
1	10	-1	$(11, 11, 1)$	$\mathsf{E}_{10}(2) \oplus \mathsf{A}_1$	$\mathsf{I}^{2,9}(2)$
2	11	-2	$(12, 10, 1)$	$\mathsf{E}_{10}(2) \oplus \mathsf{A}^{\oplus 2}$	$\mathsf{I}^{2,8}(2)$
3	12	-3	$(13, 9, 1)$	$\mathsf{D}_4^{\oplus 2} \oplus \mathsf{A}_1^{\oplus 3} \oplus \mathsf{U}(2)$	$\mathsf{I}^{2,7}(2)$
4	13	-4	$(14, 8, 1)$	$\mathsf{E}_7 \oplus \mathsf{A}_1^{\oplus 5} \oplus \mathsf{U}(2)$	$\mathsf{I}^{2,6}(2)$
5	14	-5	$(15, 7, 1)$	$\mathsf{E}_8 \oplus \mathsf{A}_1^{\oplus 5} \oplus \mathsf{U}(2)$	$\mathsf{I}^{2,5}(2)$
6	15	-6	$(16, 6, 1)$	$\mathsf{E}_{10} \oplus \mathsf{A}_1^{\oplus 6}$	$\mathsf{I}^{2,4}(2)$
7	16	-7	$(17, 5, 1)$	$\mathsf{E}_8 \oplus \mathsf{D}_6 \oplus \mathsf{A}_1 \oplus \mathsf{U}(2)$	$\mathsf{I}^{2,3}(2)$
8	17	-8	$(18, 4, 0)$	$\mathsf{E}_8 \oplus \mathsf{D}_8 \oplus \mathsf{U}(2)$	$\mathsf{U}(2)^{\oplus 2}$
8	17	-8	$(18, 4, 1)$	$\mathsf{E}_{10} \oplus \mathsf{D}_6 \oplus \mathsf{A}_1^{\oplus 2}$	$\mathsf{I}^{2,2}(2)$
9	18	-9	$(19, 3, 1)$	$\mathsf{E}_{10} \oplus \mathsf{D}_8 \oplus \mathsf{A}_1$	$\mathsf{I}^{2,1}(2)$
10	20	-10	$(20, 2, 1)$	$\mathsf{E}_{10} \oplus \mathsf{D}_{10}$	$\langle 2\rangle^{\oplus 2}$

Table 5.1 Types of Coble surfaces

Note all such lattices are marked on the right–hand side of Nilulin's triangle table. We now give some explicit constructions of Coble surfaces. Let $\mathcal{D}_5$ be a del Pezzo surface of degree 5, see also Section 0.5.

Example 5.4.4 We recall that a $\mathcal{D}_5$ is obtained by blowing up four points p_1, p_2, p_3, p_4 in the projective plane $\mathbb{P}^2$, which are in general position, that is, no three of them are collinear. A general member of the linear system $|-2K_{\mathcal{D}_5}|$ is of proper transform of a plane curve of degree 6 with double points $p_1, \ldots, p_4$. Let B be a member from $|-2K_{\mathcal{D}_5}|$ that is either a rational irreducible curve with double points (including infinitely near) or a reducible curve which consists of $h > 1$ irreducible rational components $B_1, \ldots, B_h$ and such that all singular points of B are double points. Let V be the blow-up of $\mathcal{D}_5$ with centers at the singular points of B. It is a Coble surface with $N = 4 + \delta$, where δ is the number of singular points of B. Since $p_a(B) = 6$, the formula $p_a = \sum_{i=1}^h g_i + \delta - h + 1$ implies that $\delta = 5 + h$. Moreover, in the anti-canonical embedding $\mathcal{D}_5 \to \mathbb{P}^5$, the curve B is of degree 10, so we obtain $h \leq 10$. We have the following extreme cases:

1. If B is irreducible, then $h = 1, \delta = 6, N = 10$, and V is the Coble surface originally considered by A. Coble, namely, the blow-up of the 10 nodes of a plane rational sextic curve. This realizes the top row of Table 5.1.
2. If the Coble surface V is obtained by blowing up the 15 intersection points of ten lines on $\mathcal{D}_5$, then we have $h = 10$, $\delta = 15$, and $N = 19$. This realizes the bottom row of Table 5.1. Moreover, the double cover of V branched over B is isomorphic to one of the two *most algebraic K3 surfaces* studied by E. Vinberg in [721].

Example 5.4.5 It is known that a $\mathcal{D}_5$ contains five pencils of conics, see also the end of Section 0.5. They are the proper transforms of pencils of lines through each point p_i and the pencil of conics through the four points p_i. Two conics from different families intersect transversally at one point. We choose one conic K_i from each pencil. They mutually intersect in $10 = \binom{5}{2}$ points $q_{ij} = K_i \cap K_j$. Suppose $K_1, \ldots, K_s$ with

$0 \le s \le 5$ are reducible conics and let $q_1, \dots, q_s$ be their singular points if $s > 0$. Let V be the blow-up of $\mathcal{D}_5$ with centers at the $10+s$ points $q_{ij}, q_1, \dots, q_s$.

This is a Coble surface with the boundary $C_1 + \cdots + C_{5+s} \in |-2K_V|$. A general surface obtained in this way depends on $5-s$ parameters and represents a general Coble surface with $n = 5, \dots, 10$ or, equivalently, $N = 14, \dots, 19$, whose K3 double cover has transcendental lattice $\mathsf{I}^{2,5-s}(2)$.

To construct the other case with $n = 8$, where $M^\perp \cong \mathsf{U}(2)^{\oplus 2}$ of Table 5.1, we consider the double cover of $\mathbb{P}^1 \times \mathbb{P}^1$ branched along four curves of bidegree $(0,1)$ and four curves of type $(1,0)$. The corresponding K3 double cover is a nonsingular model of the Kummer surface associated with the product of two elliptic curves. Of course, it can also obtained by blowing up points on $\mathcal{D}_5$ or $\mathbb{P}^2$.

The proof of the next proposition is completely analogous to Theorem 5.3.9 and we leave it to the reader.

Proposition 5.4.6 *Let $M_1 = \mathsf{E}_{10}(2) \oplus \mathsf{A}_1$, that is, the first lattice from Table 5.1. Then, $\mathcal{K}3^a_{M_1}$ admits an open embedding into $\mathcal{K}3_{\mathsf{E}_{10}(2)}$, such that its image under the period map is an open subset of $\mathcal{M}_{Coble}$. The coarse moduli space of $\mathcal{K}3^a_{M_1}$ is the coarse moduli space of marked Coble surfaces with $n = 1$.*

Similarly, one can prove that. for each lattice M of rank $10+n$ from Table 5.1, the functor $\mathcal{K}3^a_M$ admits an embedding into $\mathcal{K}3_{\mathsf{E}_{10}(2)}$ as an open subset of an irreducible and closed subset of codimension $n-1$ of $\Gamma^\sharp_{\mathrm{Enr}} \backslash \mathcal{H}(-2)$. It can be interpreted as a coarse moduli space of Coble surfaces with n boundary components.

5.5 Automorphisms of Complex Enriques Surfaces

In this section, we study automorphisms of complex Enriques surfaces. The case where the ground field is of arbitrary characteristic will be discussed in Chapter 8 of Volume II. We will see that the group of automorphisms of any Enriques surface is discrete and finitely generated and will determine the automorphism group in the very general case.

We recall from Section 2.3 that a nodal curve on an Enriques surface S is a (-2)-curve, that is, a smooth and rational curve, and we denote by $\mathcal{R}(S)$ the set of all nodal curves. The nodal Weyl group W_S^{nod} is the Weyl group associated to the root basis formed by the classes of $\mathcal{R}(S)$ inside $\mathrm{Num}(S)$. Let $\widetilde{W}_S^{\mathrm{nod}}$ be the image of W_S^{nod} in $\mathrm{O}(\mathsf{E}_{10}(2))$ under the homomorphism $\pi^* : \mathrm{Num}(S) \to \mathrm{Num}(X) = \mathrm{Pic}(X)$. For a smooth rational curve $R \in \mathcal{R}(S)$ with class $\delta = [R] \in \mathrm{Num}(S)$, we denote by $r_\delta \in W_S^{\mathrm{nod}}$ the associated reflection. On the K3-cover $\pi : X \to S$, the curve R splits into the disjoint union of two (-2)-curves R_1' and R_2'. Thus, $\pi^*(\delta) = \delta_1' + \delta_2'$, where $\delta_i' = [R_i']$. For any $x \in \mathrm{Num}(S)$, we compute

$$\begin{aligned}\pi^*\,(r_\delta(x)) &= \pi^*(x + (x\cdot\delta)\delta) \;=\; \pi^*(x) + (x\cdot\delta)(\delta_1' + \delta_2')\\ &= \pi^*(x) + \frac{1}{2}(\pi^*(x)\cdot(\delta_1'+\delta_2'))(\delta_1'+\delta_2')\\ &= \pi^*(x) + (\pi^*(x)\cdot\delta_1')\delta_1' + (\pi^*(x)\cdot\delta_2')\delta_2'\\ &= r_{\delta_1'}\circ r_{\delta_2'}(\pi^*(x)).\end{aligned}$$

This shows that $\widetilde{W}_S^{\mathrm{nod}}$ can be extended to a subgroup of the Weyl group $W_X^{\mathrm{nod}} := W(\mathrm{Pic}(X))$ of $\mathrm{Pic}(X)$ by setting $\tilde{r}_\delta =: r_{\delta_1'}\circ r_{\delta_2'}$. It is easy to see that this subgroup is contained in $W_X^{\mathrm{nod}}\cap \mathrm{Cent}(\tau^*)$, where $\mathrm{Cent}(\tau^*)$ is the centralizer of the covering involution τ^* of $X \to S$. In general, it does not coincide with the centralizer of τ^* in W_X^{nod}. In fact, if S admits an elliptic fibration with an irreducible nodal double fiber $2F_0$, then the pre-image of F_0 in X splits into the sum of two (-2)-curves R_1 and R_2 with $R_1\cdot R_2 = 2$ and the product of the reflections $r_{[R_1]}\circ r_{[R_2]}$ belongs to the centralizer, but does not come from W_S^{nod}.

Theorem 5.5.1 *Let* $\mathrm{Aut}(S)^*$ *be the image of the automorphism group of* S *in* $\mathrm{O}(\mathrm{Num}(S))' = W(\mathrm{Num}(S))$. *Then,*

$$W(\mathrm{Num}(S))(2) \;\subset\; W_S^{\mathrm{nod}} \rtimes \mathrm{Aut}(S)^*.$$

Proof We fix a marking $\mathrm{Num}(S)\to \mathsf{E}_{10}$ and its lift to a marking $\phi_0 : H^2(X.\mathbb{Z}) \to \mathbf{L}$ such that we can identify $\mathrm{Num}(S)$ with E_{10} and its image in $\mathrm{Pic}(X)$ with $\mathsf{E}_{10}(2)$. The elements of the group $W(\mathrm{Num}(S))(2)$ are lifted to isometries of $\mathbf{L}$ that act identically on $\mathsf{E}_{10}(2)^\perp$. Let h be an ample divisor class of S. We know that its pre-image $\tilde{h} = \pi^*(h)$ is an ample class in $\mathrm{Pic}(X)$. Let $\tilde{\sigma} \in \mathrm{O}(\mathsf{E}_{10}(2))^\sharp$ be a lift of $\sigma \in W(E_{10})(2)$. Composing $\tilde{\sigma}$ with the lift $\tilde{w}$ of some $w\in W_S^{\mathrm{nod}}$, we may assume that $\tilde{w}\circ\tilde{\sigma}(\tilde{h})$ belongs to the ample cone $\mathrm{Amp}(X)$. By the Global Torelli Theorem for K3 surfaces 5.2.11, we obtain that $\tilde{w}\circ\tilde{\sigma} = g'^*$ for some automorphism g' of X. Since both $\tilde{w}$ and $\tilde{\sigma}$ commute with τ, we obtain that g' descends to an automorphism g of S. Thus, any element of $W(\mathsf{E}_{10})(2)$ can be written as the product of an element of W_S^{nod} and an element of $\mathrm{Aut}(S)^*$. So we obtain that $W(\mathrm{Num}(S))(2)$ is contained in the subgroup $G = W_S^{\mathrm{nod}}\cdot \mathrm{Aut}(S)^*$ generated by W_S^{nod} and $\mathrm{Aut}(S)^*$. Clearly, $\mathrm{Aut}(S)^*$ normalizes W_S^{nod} and hence W_S^{nod} is a normal subgroup of G. Since no element of W_S^{nod} leaves the nef cone invariant, we see that $W_S^{\mathrm{nod}}\cap\mathrm{Aut}(S)^* = \{1\}$. This proves the assertion of the theorem. □

Corollary 5.5.2 $\mathrm{Aut}(S)$ *is a finitely generated group.*

Proof We use that $W(\mathrm{Num}(S))$ is isomorphic to the Coxeter group $W_{2,3,7}$, which is finitely generated by its Coxeter generators. We have just seen that the subgroup $W_S^{\mathrm{nod}}\rtimes\mathrm{Aut}(S)^*$ of $W(\mathrm{Num}(S))$ contains a finitely generated subgroup of finite index, hence it is finitely generated. Since $\mathrm{Aut}(S)^*$ is its quotient, it is also finitely generated. Finally, we use that the kernel of $\mathrm{Aut}(S)\to\mathrm{Aut}(S)^*$ is a finite group, see Section 8.2 in Volume II. □

Corollary 5.5.3 *The action of* $\mathrm{Aut}(S)$ *in* $\mathrm{Nef}(S)_{\mathbb{R}}$ *has a rational polyhedral cone as a fundamental domain. In particular,* $\mathrm{Aut}(S)$ *has finitely many orbits on the set of divisor classes of irreducible curves with fixed arithmetic genus.*

Proof We use that the Coxeter group $W_{2,3,7}$ has a natural linear representation in $\mathbb{R}^B$, where B is the set of Coxeter generators. This linear action preserves the quadratic form and the lattice $\mathsf{E}_{2,3,7} = \mathsf{E}_{10}$. Passing to the associated hyperbolic space $\mathbb{H}^9$, it becomes isomorphic to a discrete group of its motions with a fundamental domain equal to a convex rational simplicial polyhedron, see 0.8. We have seen that $W_S^{\mathrm{nod}} \rtimes \mathrm{Aut}(S)^*$ is a subgroup of finite index inside $W(\mathrm{Num}(S))$. Thus, its fundamental domain Π in $\mathbb{H}^9$ is of finite covolume. Fix an interior point $h \in \Pi$ defined by an ample class $H \in \mathrm{Num}(S)$. Let $\rho(x, y)$ be the hyperbolic distance defined by formula (0.8.9). It is known that Π is a *Dirichlet domain* for Γ, that is,

$$\Pi = \{x \in \mathbb{H}^9 : \rho(x, h) \le \rho(\gamma(x), h), \gamma \in \Gamma\} = \{x \in \mathbb{H}^9 : (x, h) \le (\gamma(x), h)\},$$

see [724], 1.4. A Dirichlet domain is bounded by hyperplanes H_γ in $\mathbb{H}^9$ passing through the middle point of the geodesic connecting h with $\gamma(h)$. Thus, it is a rational convex polyhedron in our case. Taking $\gamma = s_r \in W_S^{\mathrm{nod}}$, we obtain that, for any $x \in \Pi$, we have $(x, h) \le (s_r(x), h) = (x + (r, x)r, h) = (x, h) + (r, x)(x, h)$, hence $(r, x) \ge 0$. This shows that Π is a subset of the image $\overline{\mathrm{Nef}}(S)$ of the nef cone in $\mathbb{H}^9$. Thus, Π is a fundamental domain for $\mathrm{Aut}(S)^*$ in $\mathbb{H}^9$. Taking the pre-images of Π, $\overline{\mathrm{Nef}}(S)$ in $\mathrm{Num}(S)_{\mathbb{R}}$, we conclude that the fundamental domain of $\mathrm{Aut}(S)^*$ in $\mathrm{Num}(S)_{\mathbb{R}}$ is a rational polyhedral cone inside the nef cone.

Since Π is of finite volume, its closure in $\bar{\mathbb{H}}^9$ has only finitely many points on the boundary, the cusps. The rational cusps correspond to primitive isotropic vectors in the lattice representing elliptic curves. This proves the assertion for curves of arithmetic genus one. An irreducible curve of positive arithmetic genus corresponds to a rational point in the interior. There is also only finitely many of them. It remains to prove the assertion for smooth rational curves. These correspond to the faces $H_r = \{x : x \cdot r = 0\}$ of the nef cone. The group $\mathrm{Aut}(S)^*$ acts on this set and has only finitely many orbits represented by H_r inside of Π. □

Corollary 5.5.4 *The following properties are equivalent:*

1. $\mathrm{Aut}(S)$ *is a finite group.*
2. W_S^{nod} *is of finite index in* $W(\mathrm{Num}(S))$.
3. *The set of smooth rational curves on S forms a crystallographic basis in* $\mathrm{Num}(S)$.

Remark 5.5.5 Properties 1 and 2 can be stated also for a K3 surface and can be proven along the same lines using the Global Torelli Theorem for K3 surfaces, see [321, Chapter 4]. The *Kawamata–Morrison conjecture* states that the automorphism group of a Calabi–Yau manifold has a rational polyhedral cone as a fundamental domain with respect to its action on the nef cone. We refer to Section 8.1 in Volume II, where we extend the previous two corollaries to the case of arbitrary characteristic.

Corollary 5.5.6 *Assume that S is unnodal. Then,* $\mathrm{Aut}(S)^*$ *contains* $\mathrm{O}(\mathrm{Num}(S))'(2) \cong W_{2,3,7}(2)$.

In Section 8.2 from Volume II, we will prove the same result for Enriques surfaces over fields of arbitrary characteristic. We also prove that the homomorphism

$$\rho : \operatorname{Aut}(S) \to \operatorname{Aut}(S)^*$$

is an isomorphism if S is an unnodal surface or a general nodal surface.

Now, suppose that S is unnodal and let $g \in \operatorname{Aut}(S)$ be an automorphism such that g^* does not belong to $\mathrm{O}(\mathrm{Num}(S))(2)$. Its lift $\tilde{g}$ to the K3-cover X fixes the period point and hence, has $H^{2,0}(X)$ as an eigensubspace with some eigenvalue λ. Since $\tilde{g}^*$ preserves $H^2(X,\mathbb{Q})$, the eigenvalues of $\tilde{g}^*$ are algebraic integers and the degree of the minimal polynomial divides $\dim T(X)_{\mathbb{Q}} = 12$. This gives the following possibilities for possible orders of g:

$$\operatorname{ord}(g) \in \{1,\ldots,16,18,20,21,22,24,26,28,30,36,42\}. \tag{5.5.1}$$

This also shows that the period point of a surface admitting such an automorphism belongs to a countable set of eigensubspaces of elements $\sigma \in \Gamma_{\mathrm{Enr}} \setminus \{\pm\}$ in $(\mathsf{E}_{10}(2)^{\perp})_{\mathbb{C}}$. Let $\mathcal{F}$ be the union of the images of these eigensubspaces in $\mathcal{M}_{\mathrm{Enr}}^{\mathrm{m}}$.

Definition 5.5.7 An unnodal Enriques surface with $\operatorname{Aut}(S) \neq W(\mathrm{Num}(S))(2)$ is called an *unnodal Enriques surface with extra automorphisms*.

We will discuss unnodal surfaces with extra automorphisms in Section 8.3 in Volume II and refer to [490] for more information about such surfaces.

Let $\mathcal{M}_{\mathrm{Enr}}^{\mathrm{un,m}}$ be the open and dense subset of $\mathcal{M}_{\mathrm{Enr}}^{\mathrm{m}}$ of isomorphism classes of unnodal Enriques surfaces. We will describe the nodal locus in detail in the next section. The action of the discriminant group $D(\mathsf{E}_{10}(2))$ on $\mathcal{M}_{\mathrm{Enr}}^{\mathrm{m}}$ is free outside $\mathcal{F}$. Its locus of fixed points corresponds to the isomorphism classes of unnodal Enriques surfaces with extra automorphisms. Since these surfaces belong to $\mathcal{F}$, a very general Enriques surface in $\mathcal{M}_{\mathrm{Enr}}^{\mathrm{m}}$ will be unnodal and it will not have extra automorphisms.

We remark that, in general, a fixed point of $D(\mathsf{E}_{10})$ on $\mathcal{M}_{\mathrm{Enr}}^{\mathrm{m}}$ is the isomorphism class of an Enriques surface with non-trivial image of $\operatorname{Aut}(S)^*$ in $D(\mathrm{Num}(S))$.

Let us consider the natural projection map

$$p : \mathcal{M}_{\mathrm{Enr}}^{\mathrm{m}} \cong \Gamma_{\mathrm{Enr}}^{\sharp}\backslash\mathcal{D}_{\mathsf{E}_{10}(2)^{\perp}}^{\circ} \longrightarrow \mathcal{M}_{\mathrm{Enr}} \cong \Gamma_{\mathrm{Enr}}\backslash\mathcal{D}_{\mathsf{E}_{10}(2)^{\perp}}^{\circ}.$$

Over the open and dense subset $\mathcal{M}_{\mathrm{Enr}}^{\mathrm{un}}$ of unnodal Enriques surfaces (we will describe the nodal locus in detail in the next section), it is a finite map of degree

$$\#(\Gamma_{\mathsf{E}_{10}}/\Gamma_{\mathsf{E}_{10}}^{\sharp}) = \#W(\mathsf{E}_{10})/W(\mathsf{E}_{10})(2) = \#\mathrm{O}(10,\mathbb{F}_2)^+ = 2^{21}\cdot 3^5\cdot 5^2\cdot 7\cdot 17\cdot 31.$$

A point in $p^{-1}(\mathcal{M}_{\mathrm{Enr}}^{\mathrm{un}})$ represents the isomorphism classes of a marked unnodal Enriques surface. The map p is unramified over the open subset of $\mathcal{M}_{\mathrm{Enr}}^{\mathrm{un}}$ that consists of isomorphism classes of unnodal Enriques surfaces without extra automorphisms. The map is ramified over the locus $\mathcal{M}_{\mathrm{Enr}}^{\mathrm{nod}}$ of nodal surfaces.

Now, let $z \in \mathcal{M}_{\mathrm{Enr}}$ be a point corresponding to the isomorphism class of an Enriques surface S and let W_S^{nod} be its nodal Weyl group. Then the fiber of p over z corresponds to a *weak isomorphism class* of marked Enriques surfaces:

Definition 5.5.8 Let S be an Enriques surfaces. Two marked Enriques surfaces $(S, \imath)$ and $(S', \imath')$ are called *weakly isomorphic* if there exists an isomorphism $f : S \to S'$, an element $w \in W_S^{\mathrm{nod}}$ in the nodal Weyl group of S, and an isometry $\sigma \in \mathrm{O}(\mathsf{E}_{10})$ such that

$$\pm w \circ f^* \circ \imath = \imath' \circ \sigma.$$

The surfaces are called *isomorphic* if we can choose $\sigma = \mathrm{id}$.

In fact, this is a special case of the following result, the Global Torelli Theorem for Enriques surfaces:

Theorem 5.5.9 *Let S_1 and S_2 be two complex Enriques surfaces, let X_1 and X_2 be their K3-covers, and let $f : \mathrm{Num}(S_1) \to \mathrm{Num}(S_2)$ be an isometry of lattices such that:*

1. *f extends to an isometry $\widetilde{f} : H^2(X_2, \mathbb{Z}) \to H^2(X_1, \mathbb{Z})$ that preserves the period, that is, we have $\widetilde{f}(H^{2,0}(X_2)) = H^{2,0}(X_1)$, and*
2. *f maps each effective class of $H^2(S_2, \mathbb{Z})$ to an effective class of $H^2(S_1, \mathbb{Z})$.*

Then, f is induced by an isomorphism $a : S_1 \to S_2$.

Proof From the assumptions, we obtain $f(C(S_2)^+) = C(S_1)^+$, which implies that $\widetilde{f}(C(X_2)^+) \cap C(X_1)^+ \neq \emptyset$, which implies that $\widetilde{f}(C(X_2)^+) = C(X_1)^+$. Thus, by the Global Torelli Theorem for K3 surfaces, $\widetilde{f}$ is induced by an isomorphism $\widetilde{a} : X_1 \to X_2$ that commutes with the covering involutions. Therefore $\widetilde{a}$ descends to an isomorphism $a : S_1 \to S_2$ that induces f. □

Remark 5.5.10 Assuming condition 1 of the theorem holds, the following assertions are equivalent

1. Condition 2 of the theorem.
2. $f_{\mathbb{R}}(V(S_2)^+) \subseteq V(S_1)^+$ and $f(\Delta(S_2)^+) \subseteq \Delta(S_2)^+$.
3. $f(C(S_2)^+) = C(S_1)^+$.
4. $f(C(S_2)^+) \cap C(S_1)^+ \neq \emptyset$.

In the case of K3 surfaces, the Global Torelli Theorem even gives uniqueness of the isomorphism a. However, for an Enriques surface X, the homomorphism $\rho : \mathrm{Aut}(S) \to \mathrm{O}(\mathrm{Num}(S))$ may fail to be injective: there are two 2-dimensional families of Enriques surfaces with involutions, and a 1-dimensional family of Enriques surfaces with an automorphism of order 4, such that ρ is not injective, see Section 8.2 in Volume II. Thus, the isomorphism a, whose existence is assured in Theorem 5.2.11 is even unique except for the three families just explained, see also [549, Remark 5.4].

The next corollary is Theorem (3.3) from [42].

Corollary 5.5.11 *Let $A(S)$ be the subgroup of* $\mathrm{O}(\mathrm{Num}(S))$ *that preserves the nef cone* $\mathrm{Nef}(S)$ *of a complex Enriques surface S. Then,*

$$G := A(S) \cap W(\mathrm{Num}(S))(2) \subset \mathrm{Aut}(S)^*.$$

Proof Let $\pi : X \to S$ be the K3-cover. We know that $W(\mathrm{Num}(S))(2)$ lifts to a group of isometries of $\mathrm{Pic}(X)$ that acts as the identity on the orthogonal complement of $\pi^*(\mathrm{Num}(S))$ in $H^2(X, \mathbb{Z})$. In particular, it leaves the period of X invariant. Since any element of G leaves invariant the nef cone, it leaves invariant the cone of effective divisors. The previous theorem finishes the proof. □

5.6 Moduli of Nodal Enriques Surfaces

In this section, we discuss and construct coarse moduli spaces for complex (marked and unmarked) nodal Enriques surfaces, that is Enriques surfaces with (-2)-curves. These are constructed via ample $\mathsf{U} \oplus \mathsf{E}_8(2) \oplus \langle -4\rangle$-polarized K3 surfaces. Finally, we construct moduli spaces of nodal Enriques surfaces with fixed Nikulin R-invariant.

Let S be a nodal Enriques surface and R be a (-2)-curve on it. Then, its pre-image under the K3-cover $\pi : X \to S$ splits into a disjoint sum of two (-2)-curves $R_1' + R_2'$. Fix a $\mathsf{E}_{10}(2)$-lattice polarization of X, fix a marking $\phi : H^2(X, \mathbb{Z}) \to \mathbf{L}$, and let ι be the involution of $\mathbf{L}$ corresponding to the covering involution. Obviously, $[R_1 - R_2] \in \pi^*(\mathrm{Pic}(S))^\perp$ and $(R_1 - R_2)^2 = -4$, hence $\delta := \phi([R_1' - R_2']) \in (\mathsf{E}_{10}(2)^\perp)_{-4}$.

The following lemma is due to Namikawa [549, Theorem 2.15].

Lemma 5.6.1 *Let $N = \mathsf{E}_{10}(2) \oplus \mathsf{U}$ and let $v \in N_{-4}$. Then its orthogonal complement $v^\perp$ in N is isomorphic to either $\mathsf{E}_8(2) \oplus \mathsf{U} \oplus \langle 4\rangle$ or to $\mathsf{E}_8(2) \oplus \mathsf{U}(2) \oplus \langle 4\rangle$. In particular, there are two orbits of such vectors.*

Proof The discriminant group of N is equal to the discriminant group of $\mathsf{E}_{10}(2)$. By (5.3.2), it is isomorphic to the even type quadratic space $\mathbb{F}_2^{10}$ and hence, it is also isomorphic to $\mathsf{u}_1^{\oplus 5}$. Here, we use the notation for the discriminant quadratic forms from Section 0.8. The discriminant group of $\mathbb{Z}v$ is $\mathbb{Z}/4\mathbb{Z}$ with quadratic form $\langle -\frac{1}{4}\rangle$. As explained in Section 0.8, the discriminant group of the overlattice N of $\langle -4\rangle \oplus v^\perp$ is isomorphic to $H^\perp/H$, where H is an isotropic subgroup of $\langle -\frac{1}{4}\rangle \oplus D(v^\perp)$. This gives only two possibilities, namely $D(v^\perp) \cong \mathsf{u}_1^{\oplus 4} \oplus \langle \frac{1}{4}\rangle$ or $\mathsf{u}_1^{\oplus 5} \oplus \langle \frac{1}{4}\rangle$. Applying Theorem 0.8.6, we conclude that an even hyperbolic lattice P of rank 11 with such a discriminant quadratic form is unique up to isometry and the canonical homomorphism $\rho_{v^\perp}$ is surjective. We may take $P = \mathsf{E}_8(2) \oplus \mathsf{U} \oplus \langle 4\rangle$ in the first case and $P = \mathsf{E}_8(2) \oplus \mathsf{U}(2) \oplus \langle 4\rangle$ in the second case. Applying the theorem again, we find that there are two orbits of v: one is represented by a vector $v = f_1 + g_1$ in the $\mathsf{U}(2)$-summand of N and the other one is represented by a vector $v = 2f_2 + g_2$ in the U-summand. Here, (f_i, g_i) is the standard basis of $\mathsf{U}(2)$ or U. □

We call a vector $v \in N_{-4}$ of *even type* if its orthogonal complement is isomorphic to $\mathsf{E}_8(2) \oplus \mathsf{U} \oplus \langle 4\rangle$ and of *odd type* otherwise. It follows from the proof of the previous lemma that v is of even type if and only if $\frac{1}{2}v$ belongs to $N^\vee$. Also, it follows from the proof of Lemma 5.4.1 that an even vector corresponds to a vector of square norm -2 in the unimodular lattice $\mathsf{E} \oplus \mathsf{I}^{1,1}$ and an odd vector corresponds to a vector of norm -8 in this lattice.

Theorem 5.6.2 *Let*

$$\mathcal{H}(-4)_{ev} := \bigcup_{even\ v\ \in\ N_{-4}} \{z \in \mathcal{D}_{\mathrm{Enr}} : z \cdot v = 0\}$$

and let $\mathcal{H}(-4)_{ev}^\circ$ *be the complement of the discriminant* $\mathcal{H}(-2)$*, which is open. Then,*

$$\mathcal{M}_{\mathrm{Enr}}^{\mathrm{nod},m} \cong \Gamma_{\mathrm{Enr}}^\sharp \backslash \mathcal{H}(-4)_{ev}{}^\circ,$$
$$\mathcal{M}_{\mathrm{Enr}}^{\mathrm{nod}} \cong \Gamma_{\mathrm{Enr}} \backslash \mathcal{H}(-4)_{ev}{}^\circ.$$

Both varieties are irreducible and quasi-projective of dimension 9*.*

Proof Let $\pi : X \to S$ be the canonical cover of an Enriques surface. We fix an isomorphism $\mathrm{Pic}(X) \to \mathbf{L}$ and identify $\pi^*(\mathrm{Num}(S))^\perp$ with the lattice $N = \mathsf{E}_{10}(2)^\perp$. Suppose that S contains a smooth rational curve R. In the K3-cover, it splits as $\pi^*(R) = R_1 + R_2$ and we set $\delta_+ = [R_1 + R_2]$ and $\delta_- = [R_1 - R_2]$. Obviously, $\delta_- \in N$. Since $\delta_+ \in \mathsf{E}_{10}(2)$, $\frac{1}{2}\delta_- = [R_1] - \frac{1}{2}\delta_+ \in N^\vee$. Thus δ_- is an even vector in N.

Since $\delta_\pm \in \mathrm{Pic}(X)$, the period of X must belong to $\mathcal{H}(-4)_{\mathrm{ev}}$. Since $\pi^*(\mathrm{Pic}(S))$ contains an ample divisor, it does not belong to $\mathcal{H}(-2)$. Thus, the isomorphism class of a marked (resp. unmarked) nodal Enriques surface belongs to $\mathrm{O}(N)^\sharp \backslash \mathcal{H}_N(-4)^\circ$ (resp. $\mathrm{O}(N)\backslash \mathcal{H}_N(-4)^\circ$).

Suppose that the period point of a marked ample $\mathsf{E}_{10}(2)$-polarized K3-surface X belongs to a hyperplane $H_v = \{[z] : z\cdot v = 0\}$, where $v^2 = -4$. First, assume that v is of odd type. By the characterization given above, we may assume that $v = f_2 - 2g_2 \in \mathsf{U}$ with f_2, g_2 as above. Applying an element w from $W_X^{\mathrm{nod}} = W(\mathrm{Pic}(X))$, we may assume that g' is a nef isotropic vector. Then $(f_2 - g_2)^2 = -2$ and thus, $r = f_2 - g_2$ is effective or $-r$ is effective by Riemann–Roch. Since $r \cdot g_2 = 1$, it follows that r must be effective. Hence, if the period of (X, ϕ) belongs to H_v, then it is also belongs to $\mathcal{H}_N(-2)$. However, since the lattice polarization is ample, we find a contradiction. Thus, we may assume that v is of even type. In this case, the period point lies in the orthogonal complement of v, which is isomorphic to $\mathbb{P}(N_{\mathbb{C}})$, where $N = \mathsf{U} \oplus \mathsf{E}_8(2) \oplus \langle 4 \rangle$. Also, the stabilizer of $\mathcal{H}_N(-4)$ is isomorphic to $\mathrm{O}(N)^\sharp$.

Thus, suppose that the period point lies in H_{δ_-} for some vector δ_- with $\delta_-^2 = -4$ and of even type. Note that $\delta_- \in \phi(\mathrm{Pic}(X))$ and $\iota(\delta_-) = -\delta_-$. By taking a particular nodal Enriques surface, we may assume that one of these vectors comes from the previous scenario. In this case, there exists a vector $\delta_+ \in \mathsf{E}_{10}(2)_{-4}$, such that $\frac{1}{2}(\delta_+ + \delta_-) = r$ for some $r \in \mathrm{Pic}(X)$. Since all vectors of even type form one orbit with respect to the orthogonal group, we may assume that δ_- satisfies this property. Thus, returning to our marked $\mathsf{E}_{10}(2)^\perp$-lattice polarized K3 cover of S, we obtain that there exists $r_+ \in \mathrm{Pic}(S)_{-2}$ such that $r = \frac{1}{2}(\pi^*(r_+) + r_-) \in \mathrm{Pic}(X) \subset H^2(X, \mathbb{Z})$, where $r_+^2 = r_-^2 = -4$ and $\tau^*(r_-) = -r_-$. We have $r^2 = -2$, hence r or $-r$ is effective.

Since $\pi_*(r) = \pi_*(\pi^*(r_+) + \pi^*(r_-)) = 2r_+$, and we may assume that r is effective, we obtain that r_+ is effective. This shows that S is a nodal Enriques surface. □

Definition 5.6.3 A *Cayley lattice* is a lattice isomorphic to the lattice

$$\mathsf{Ca} := \mathsf{U} \oplus \mathsf{E}_8(2) \oplus \langle -4 \rangle.$$

The reason for this name is that, by Proposition 7.7.3, the Picard lattice of a minimal resolution X of a general Cayley quartic symmetroid is isomorphic to such a lattice, see also Section 8.4 in Volume II. By Corollary 7.9.9, the Picard lattice of the K3-cover of a general nodal Enriques surface is isomorphic to the lattice Ca.

We define a primitive embedding $\mathsf{E}_{10}(2) \hookrightarrow \mathsf{Ca}$ by taking the identity on the summand $\mathsf{E}_8(2)$ and take the embedding $\mathsf{U}(2) \hookrightarrow \mathsf{U} \oplus \langle -4 \rangle$ that sends the basis (f_1, g_1) of $\mathsf{U}(2)$ to $(2f_2+g_2+e, g_2)$, where e is a basis of $\langle -4 \rangle$. Using this embedding, we see that the coarse moduli space $\mathcal{M}^{\mathrm{m}}_{\mathsf{Ca}}$ of marked Cayley lattice polarized K3 surfaces is included naturally in $\mathcal{M}^{\mathrm{m}}_{\mathrm{Enr}}$ and that it coincides with $\mathcal{M}^{\mathrm{nod,m}}_{\mathrm{Enr}}$.

We end this section by defining subloci in $\mathcal{M}^{\mathrm{nod,m}}_{\mathrm{Enr}}$ that correspond to nodal Enriques surface with special configurations of nodal curves. These are described using *Nikulin R-invariants* and we refer to Section 6.4 for details. A Nikulin R-invariant on an Enriques surface S consists of a pair of abelian groups (K, H), where K is a root lattice of finite type and where H is a finite abelian group. This data is associated to a nodal Enriques surface as follows: Let K' be the sublattice of $\mathrm{Pic}(X) \cap \pi^*(\mathrm{Pic}(S))^\perp$ that is generated by the classes of $[R^+ - R^-]$, where $R^+ + R^- = \pi^*(R)$ for some (-2)-curve $R \subset S$. We set $K := K'(\frac{1}{2})$. The finite abelian group H is defined to be the kernel of the homomorphism $\gamma : D(K) \to \mathrm{Num}(S)/2\,\mathrm{Num}(S)$ that sends $[R^+ - R^-]$ to the class of $[R]$. The group H is a 2-torsion subgroup of the discriminant group of K and it defines an odd overlattice $\tilde{K}$ of K with $\tilde{K}/K \cong H$. Equivalently, it defines an even overlattice $\tilde{K}'$ of K' contained in $\mathrm{Pic}(X)$ with $\tilde{K}'/K' \cong H$.

Theorem 5.6.4 *Let (K, H) be a Nikukin R-invariant and let r be the rank of K. Let $K' := K(2) \hookrightarrow \mathsf{E}_{10}(2)^\perp$ be a primitive embedding and let $\tilde{K}'$ be an overlattice with $\tilde{K}'/K' \cong H$. Let $\mathcal{M}^{(K,H)}_{\mathrm{Enr}}$ be the closure of the locus of Enriques surfaces in $\mathcal{M}_{\mathrm{Enr}}$ with Nikulin R-invariant (K, H). Then,*

$$\mathcal{M}^{(K,H)}_{\mathrm{Enr}} \cong \Gamma_{\mathrm{Enr},K} \backslash \mathcal{D}^\circ_{K^\perp},$$

where $K^\perp$ is the orthogonal complement of K in $\mathsf{E}_{10}(2)^\perp$ and where $\Gamma_{\mathrm{Enr},K}$ is the stabilizer subgroup of K in Γ_{Enr}. In particular,

$$\dim \mathcal{M}^{(K,H)}_{\mathrm{Enr}} = 10 - r.$$

Proof Choose a marking of S and a marking ϕ of the corresponding $\mathsf{E}_{10}(2)$-lattice polarized K3 cover (X, j). Then, $\phi(K)$ is a root sublattice of $\mathsf{E}_{10}(2)^\perp$ and since

$K \subset \mathrm{Pic}(X)$, we see that the period of (X, ϕ) lies in $\mathbb{P}((K^\perp)_\mathbb{C})$, where the orthogonal complement $K^\perp$ is taken in the lattice $\mathsf{E}_{10}(2)^\perp$. This shows that the isomorphism class of (X, j) belongs to $\Gamma_K \backslash \mathcal{D}^\circ_{K^\perp}$, where Γ_K is the stabilizer subgroup of K in $\Gamma^\sharp_{\mathrm{Enr}}$. More precisely, if $(k_1, \ldots, k_r)$ is a root basis of K and $\delta_i = \phi(k_i)$, then the period of (X, j) belongs to the intersection of the hyperplanes H_{δ_i}, which is contained in the Heegner divisor $\mathcal{H}(-4)$.

Conversely, suppose the period of (X, ϕ) belongs to the intersection of r hypersurfaces H_{δ_i}, which are contained in $\mathcal{H}(-4)$. Assume also that $\delta_1, \ldots, \delta_r$ are linearly independent over $\mathbb{Z}$. Then, we define K' to be the lattice $\phi^{-1}(\mathbb{Z}\delta_1 + \cdots + \mathbb{Z}\delta_r)$. It is a negative definite sublattice generated by vectors of norm -4. Moreover, the lattice $K = K'(\frac{1}{2})$ is a root lattice of rank r. Using the proof of Theorem 5.6.2, we can show that the vectors $k_i = \phi^{-1}(\delta_i)$ are equal to $R_i^+ - R_i^-$, where $R_i^\pm$ are smooth rational curves that are interchanged by the covering involution. Thus, K together with the group H defined above form the Nikulin R-invariant of S. $\square$

Example 5.6.5 If $K = \mathsf{A}_1 = \langle -2 \rangle$ and $H = \{0\}$, then an Enriques surface with Nikulin R-invariant (K, H) is a *general nodal Enriques surface* as defined in Section 6.5 in Volume II. In this case, $\mathcal{M}^{(K,H)}_{\mathrm{Enr}}$ is equal to $\mathcal{M}^{\mathrm{nod}}_{\mathrm{Enr}}$, the locus of all nodal Enriques surfaces in $\mathcal{M}_{\mathrm{Enr}}$. We will give more examples and computations of Nikulin R-invariants in Section 6.4 of Volume II.

Remark 5.6.6 We note that the notation $\mathcal{D}^\circ_{K^\perp}$ is somewhat misleading since there could be different primitive embeddings of $K(2)$ in $\mathsf{E}_{10}(2)^\perp$ that are not equivalent with respect to the orthogonal group of $\mathsf{E}_{10}(2)^\perp$. Thus, the moduli space of Enriques surfaces with a fixed abstract Nikulin R-invariant could be reducible. However, if we fix a sublattice $K(2)$, then it consists of at most two irreducible components. For example, it is irreducible if the orthogonal complement of $K(2)$ in $\mathsf{E}_{10}(2)^\perp$ contains a lattice isomorphic to U or $\mathsf{U}(2)$.

5.7 Moduli of Polarized Enriques Surfaces

Having constructed the moduli spaces $\mathcal{M}_{\mathrm{Enr}}$ and $\mathcal{M}^{\mathrm{m}}_{\mathrm{Enr}}$ of (marked) Enriques surfaces, we construct and study moduli spaces of polarized Enriques surfaces in this section.

Slightly changing the terminology, we call a choice of isomorphism class of a big and nef (=pseudo-ample) invertible sheaf $\mathcal{L}$ on a smooth projective algebraic variety X a *polarization*. We say that the polarization is an *ample polarization* if $\mathcal{L}$ is ample.

A *numerical (ample) polarization* is a choice of a numerical equivalence class of a big and nef (ample) of an invertible sheaf on X In [542] an ample numerical polarization is called an *inhomogeneous polarization*. A homogeneous polarization is a choice of the set of rational multiples of a numerical polarization.

A smooth projective variety equipped with a (numerical) polarization are called (numerically) polarized varieties. Two polarized varieties $(X, [\mathcal{L}])$ and $(X', [\mathcal{L}'])$

are called isomorphic if there exists an isomorphism $f : X \to Y$ such that $f^*([\mathcal{L}']) = [\mathcal{L}]$ in $\mathrm{Pic}(X)$ (resp. $\mathrm{Num}(X)$).

There is a natural notion of a family of (numerically) polarized varieties $(f : \mathcal{X} \to T, \mathcal{L})$. It consists of a smooth projective morphism $f : \mathcal{X} \to T$ and a relatively pseudo-ample (ample) invertible sheaf $\mathcal{L}$ on $\mathcal{X}$. Two families $(f : \mathcal{X} \to T, \mathcal{L})$ and $(f' : \mathcal{X}' \to T, \mathcal{L}')$ are isomorphic if there exists an isomorphism $\phi : \mathcal{X}/T \to \mathcal{X}'/T$ such that $\phi^*(\mathcal{L}') \cong \mathcal{L}$ (resp. their isomorphism classes are numerically equivalent). In particular, taking $T = \mathrm{Spec}\ \Bbbk$, we see that an isomorphism class of a polarized variety (resp. numerically polarized) defines the isomorphism class of a pseudo-ample (ample) invertible sheaf (resp. the numerical class of such a sheaf).

Example 5.7.1 Let A be an abelian variety over a field K. Then, a choice of an invertible sheaf defines a map $\lambda_{\mathcal{L}} : A \to \hat{A} := \mathbf{Pic}^0_{A/K}$. If $\mathcal{L}$ is algebraically equivalent to zero, then this map is zero. This allows one to associate a unique map λ to any numerical equivalence class of an invertible sheaf. If the numerical class of $\mathcal{L}$ is not zero, then $\mathcal{L}$ is ample and the map $\lambda_{\mathcal{L}}$ is an isogeny. A polarized abelian variety is an abelian variety equipped with an isogeny $\lambda : A \to \hat{A}$, see [542, Chapter 6,§2]. Using the universal Poincaré sheaf $\mathcal{P}$ on $A \times \hat{A}$, one can show that each isogeny λ is equal to $\lambda_{\mathcal{L}}$ for some $\mathcal{L}$. Thus, the notion of a numerical polarization of an abelian variety coincides with the notion of a numerically polarization from above.

More precisely, there are two types of polarizations, namely numerical and Picard polarizations, giving rise to different types of moduli spaces that are related by an étale double cover. We also refer to [262] for details.

We have a functor $\mathcal{P}$ ($\mathcal{P}^{\mathrm{num}}$) (resp. $\mathcal{P}^a$ ($\mathcal{P}^{\mathrm{num},a}$) from the category Schemes$/\Bbbk$ to the category of groupoids that assigns to T the groupoid of families of polarized (numerically polarized) (resp. ample polarized (resp. numerically polarized varieties)). It is known that it defines an algebraic stack.

We could also consider $\mathcal{P}^{\mathrm{num}}$ as the quotient of $\mathcal{P}$ by the equivalence relation $\mathcal{R} \to \mathcal{P}$ such that

$$\mathcal{R}(T) = \{(X \to T, \mathcal{L}), (X' \to T, \mathcal{L}') : \exists\, \phi : X/T \xrightarrow{\cong} X'/T \text{ such that } \phi^*(\mathcal{L}') \equiv \mathcal{L}\}.$$

The quotient by this equivalence relation is the algebraic stack $\mathcal{P}^{\mathrm{num}}$.

For any $Q \in \mathbb{Q}[t]$ we can consider the subfunctor of families $(\mathcal{X} \to T, \mathcal{L})$ such that for all points $t \in T$, we have $\chi(\mathcal{L}^{\otimes m} \otimes \mathcal{O}_{\mathcal{X}_t}) = Q(m)$. It is an open substack of $\mathcal{P}$ which we denote by $\mathcal{P}^Q$. We have similar notations $\mathcal{P}^{Q,a}, \mathcal{P}^{Q,\mathrm{num}}, \mathcal{P}^{Q,\mathrm{num},a}$.

Let $\mathrm{Hilb}^Q_{\mathbb{P}^n_{\Bbbk}}$ be the Hilbert schemes parametrizing closed subschemes $Z \subset \mathbb{P}^N_{\Bbbk}$ with $\chi(Z, \mathcal{O}_Z(n)) = Q(n)$. We have a natural morphism of algebraic stacks

$$\mathrm{Hilb}^Q_{\mathbb{P}^n_{\Bbbk}} \to \mathcal{P}^{Q,a}.$$

Its image parameterizes families with very ample polarizations. Its image is contained in $\mathcal{P}^{Q,a}$ and denoted by $\mathcal{P}^{Q,\mathrm{va}}$.

Now, we specialize and consider families of polarized (numerically polarized) Enriques surfaces with a polarization $\mathcal{L}$ satisfying $(\mathcal{L}, \mathcal{L}) = 2d$ (called the *degree* *of the polarization*) restricted to any fiber of a family. It is an open subfunctor of $\mathcal{P}^Q$ (resp. $\mathcal{P}^{Q,\mathrm{num}}$), where $Q = 2dt^2 + 1$. We denote it by $\tilde{\mathcal{E}}_{\mathrm{Enr},2d}$ (resp. $\mathcal{E}_{\mathrm{Enr},2d}$). They are open substacks of $\mathcal{P}$ (resp. $\mathcal{P}^{\mathrm{num}}$).

Proposition 5.7.2 *Assume $p \neq 2$. The algebraic stacks $\tilde{\mathcal{E}}_{\mathrm{Enr},2d}$ and $\mathcal{E}^{\mathrm{num}}_{\mathrm{Enr},2d}$ are Deligne–Mumford stacks.*

Proof It follows from [581, Theorem 8.3.3] that to prove it we have to verify that the subscheme G of the scheme of automorphisms $\mathbf{Aut}_{S/\Bbbk}$ of an Enriques surface S that leaves invariant the numerical class of $\mathcal{L}$ is a finite étale group scheme. We will prove in Theorem 8.1.1 in Volume II that, if $p \neq 2$, then the identity component $\mathbf{Aut}^{\circ}_{S/\Bbbk}$ is trivial, hence $\mathbf{Aut}_{S/\Bbbk}$ is reduced and coincides with the trivial group scheme $\mathrm{Aut}(S)$ over $\Bbbk$. We will also prove in Proposition 8.2.1 that the kernel of its natural action on $\mathrm{Num}(S)$ is a finite group. This proves the assertion. □

Let $(S, \phi, \mathcal{L})$ be a marked polarized Enriques surface, say, $\phi(\mathcal{L}) = v \in \mathsf{E}_{10}$ with $v^2 = 2n$. We assume that v is a primitive vector. Taking the pre-image under the canonical K3-cover $\pi : X \longrightarrow S$, we get a lattice $\mathsf{E}_{10}(2)$ polarization of X and a sublattice $\mathbb{Z}\tilde{v} \cong \langle 4n \rangle \subset \mathsf{E}_{10}(2)$, where $\tilde{v}$ is the image of v in $\mathsf{E}_{10}(2)$. It defines a primitive embedding $\alpha : \langle 4d \rangle \hookrightarrow \mathsf{E}_{10}(2)$ with the image of a generator equal to $\tilde{v}$ and the corresponding morphism

$$F(v) : \mathcal{M}^m_{\mathrm{Enr}} = \mathcal{M}^a_{K3,\mathsf{E}_{10}} \to \mathcal{M}^{\mathrm{m}}_{\mathrm{Enr},v} := \mathcal{M}^a_{K3,\mathsf{E}_{10}(2),\tilde{v}},$$

which we introduced at the end of Section 5.2. It follows from (5.2.4) that

$$\mathcal{M}_{\mathrm{Enr},v} \cong \mathcal{M}^{\mathrm{m}}_{\mathrm{Enr}}/G(v),$$

where

$$G(v) = \mathrm{O}(\mathsf{E}_{10}(2))_{\tilde{v}}/\mathrm{O}(\mathsf{E}_{10}(2))^{\sharp}_{v} \cong W(\mathsf{E}_{10})_v/W(\mathsf{E}_{10})(2)_v.$$

The group $W(\mathsf{E}_{10})_v/W(\mathsf{E}_{10})(2)_v$ is equal to the image of $W(\mathsf{E}_{10})_v$ in $\mathrm{O}(\mathsf{E}_{10}/2\mathsf{E}_{10}) \cong \mathrm{O}^+(10, \mathbb{F}_2)$. We denote it by $\overline{W(\mathsf{E}_{10})}_v$.

Note that the moduli space $\mathcal{M}^{\mathrm{m}}_{\mathrm{Enr}}$ admits an explicit quasi-projective model described in the appendix. This gives quasi-projective models of $\mathcal{M}_{\mathrm{Enr},v}$ as finite quotients of this model.

We know that the group $\overline{W(\mathsf{E}_{10})}$ acts on $\mathcal{M}^{\mathrm{m}}_{\mathrm{Enr}}$ with the stabilizer subgroup at (S, ϕ) over the isomorphism class $[S]$ of a surface S isomorphic to the group

$$\overline{\mathrm{Aut}(S)^*_{\phi}} := \phi^{-1} \circ \big(\mathrm{Aut}(S)^*/\mathrm{Aut}(S)^* \cap W(\mathrm{Num}(S))(2)\big) \circ \phi.$$

Since the forgetful map $\mathcal{M}^{\mathrm{m}}_{\mathrm{Enr}} \to \mathcal{M}^{\mathrm{m}}_{\mathrm{Enr},v}$ is a torsor under the group $G(v)$, we obtain the formula for the number $P(S, v)$ isomorphism classes of polarizations on X that contains $\phi(v)$:

$$P(S, v) = [\overline{W(\mathsf{E}_{10})} : \langle G(v), \overline{\mathrm{Aut}(S)^*_{\phi_0}} \rangle], \tag{5.7.1}$$

where we fix one marking $\phi_0 : \mathsf{E}_{10} \to \mathrm{Num}(S)$ with $\phi_0(v) = h$.

Note that applying to v an element from $W(\mathsf{E}_{10})(2)$ changes $\phi(v) = h$ to a divisor class, that may be not nef and hence, does not define a polarization. Thus, we may change v only by isometries from the group $A(S)_{\phi_0} = \phi_0^{-1} \circ A(S) \circ \phi_0$. So, in the case when $\mathrm{Nef}(S) \neq V(\mathrm{Num}(S))^+$, the number of isomorphism classes of polarizations h with $h^2 = 2n$ is larger then the number of $\mathrm{O}(\mathsf{E}_{10})$-orbits on the set $(\mathsf{E}_{10})_{2n}$.

Recall that Corollary 5.5.11 gives that $A(S) \cap W(\mathrm{Num}(S))(2) \subset \mathrm{Aut}(S)^*$, so this part disappears in $\overline{\mathrm{Aut}(S)}^*_{\phi}$. In the case where S is an unnodal surface without extra automorphisms, we know that $\mathrm{Aut}(S)^* = W(\mathrm{Num}(S))(2)$, so the contribution of this group in the counting formula disappears altogether. We will see in Section 8.4 in Volume II that $\mathrm{Aut}(S)^*$ is contained in $W(\mathrm{Num}(S))(2)$ in the case of general nodal surfaces. So again, we can ignore its contribution to $P(S, v)$.

Before we present some examples of computations of the number $P(S, v)$, let us list the orbits of $W(\mathsf{E}_{10})$ of vectors of small square norm ≤ 10. The following table can be deduced using the information that can be found in Section 1.5.

$2n$	#{orbits}	representatives of $W_{2,3,7}$-orbits
0	1	$\omega_9 = \mathbf{f}_{10}$
2	1	$\omega_8 = \mathbf{f}_9 + \mathbf{f}_{10}$
4	2	$\omega_1 = \mathbf{\Delta} - \mathbf{f}_1, \quad \omega_8 + \omega_9 = \mathbf{f}_9 + 2\mathbf{f}_{10}$
6	2	$\omega_7 = \mathbf{f}_8 + \mathbf{f}_9 + \mathbf{f}_{10}, \quad \omega_8 + 2\omega_9 = \mathbf{f}_9 + 3\mathbf{f}_{10}$
8	3	$2\omega_8 = 2(\mathbf{f}_9 + \mathbf{f}_{10}), \quad \omega_8 + 3\omega_9 = \mathbf{f}_9 + 4\mathbf{f}_{10}, \quad \omega_1 + \omega_9 = \mathbf{\Delta} - \mathbf{f}_1 + \mathbf{f}_{10}$
10	3	$\omega_0 = \mathbf{\Delta}, \quad \omega_7 + 3\omega_9 = \mathbf{f}_8 + \mathbf{f}_9 + 4\mathbf{f}_{10}, \quad \omega_8 + 4\omega_9 = \mathbf{f}_9 + 5\mathbf{f}_{10}$

Example 5.7.3 Let S be an unnodal Enriques surface without extra automorphisms. We know that $\mathrm{Aut}(S)^* = W(\mathrm{Num}(S))(2)$, hence $\overline{\mathrm{Aut}(S)}^*_{\phi}$ is trivial. This gives

$$P(S; v) \; = \; [\mathrm{O}^+(10, \mathbb{F}_2) : \overline{W(\mathsf{E}_{10})}_v] = \frac{2^{21} \cdot 3^5 \cdot 5^2 \cdot 7 \cdot 17 \cdot 31}{\#\overline{W(\mathsf{E}_{10})}_v}. \tag{5.7.2}$$

We have the following values of $P(S, v)$ for $n \leq 5$ and primitive v satisfying $\Phi(v) \geq 2$ that define linear systems without fixed points:

$2n$	orbit	$\Phi(v)$	$P(S, v)$
0	ω_9	–	$17 \cdot 31$
2	ω_8	1	$2^7 \cdot 17 \cdot 31$
4	ω_1	2	$2^6 \cdot 3 \cdot 5 \cdot 17 \cdot 31$
6	ω_7	2	$2^{10} \cdot 5 \cdot 17 \cdot 31$
8	$\omega_1 + \omega_9$	2	$2^9 \cdot 17 \cdot 31$
10	ω_0	3	$2^{13} \cdot 3 \cdot 17 \cdot 31.$

(5.7.3)

Let us explain the computations. We will skip the case $n \geq 3$ and $\Phi(v) = 1$, which define polarizations $\phi(v)$ such that the linear system $|\phi(v)|$ has base points.

Although we have assumed so far that the polarization is defined by a vector of positive square norm, we can do similar computations for any primitive isotropic vector. The fibers of the map $\mathcal{M}_{\mathrm{Enr},v} \to \mathcal{M}_{\mathrm{Enr}}$ can be interpreted as the number of isomorphism classes of elliptic pencils on S. We can still use formula 5.7.2 to find $(\bar{W}_{2,3,7})_v$, we can still complete ω to a hyperbolic plane U, which we assume to be the usual direct summand U of $\mathsf{E}_{10} = \mathsf{E}_8 \oplus \mathsf{U}$. Using this realization, we find that $W(\mathsf{E}_{10})_{\omega_8} = \mathsf{E}_8 \rtimes W(\mathsf{E}_8) \cong \mathrm{O}(\mathsf{E}_9)$. Its image in $\mathrm{O}(\bar{\mathsf{E}}_{10}) \cong \mathrm{O}^+(10, \mathbb{F}_2)$ is equal to the stabilizer subgroup of an isotropic vector. We know that these are all equivalent under the orthogonal group and that their number is equal to $527 = 2^4(2^5+1) - 1$. Thus, the index of the stabilizer subgroup is equal to $527 = 17 \cdot 31$. This gives the first row of the table.

If $n = 1$, then we embed $\langle v \rangle$ in U as the sum of two canonical generators of U to obtain that $v^\perp = \langle \mathbf{f}_9 - \mathbf{f}_{10} \rangle \oplus \mathsf{E}_8$. Each isometry from $(W(\mathsf{E}_{10})_v$ of this lattice leaves invariant the first summand, and hence $W(\mathsf{E}_{10})_v = \{\pm 1\} \times W(\mathsf{E}_8)$. Its image modulo $W(\mathsf{E}_{10})(2)$ is a group of order equal to $\# W(\mathsf{E}_8) = 2^{14} \cdot 3^3 \cdot 5^2 \cdot 7$. This gives the second row of the table.

If $n = 2$ and $v = \omega_8$, then we check that $\langle v \rangle^\perp$ is spanned by the vectors $\boldsymbol{\alpha}_i, i \neq 1$, where $(\boldsymbol{\alpha}_0, \ldots, \boldsymbol{\alpha}_9)$ is a standard root basis in E_{10}. They generate a sublattice isomorphic to D_9 with $W(\mathsf{D}_9) \cong (\mathbb{Z}/2\mathbb{Z})^{\oplus 8} \rtimes \mathsf{S}_9$. The group $W(\mathsf{E}_{10})_v$ is a subgroup $\cong \mathrm{O}(\mathsf{D}_9)^\sharp$ of index 2 in $\mathrm{O}(\mathsf{D}_9)$. It is mapped isomorphically to a subgroupof $\mathrm{O}^+(10, \mathbb{F}_2)$ of order $2^{14} \cdot 3^4 \cdot 5 \cdot 7$. This gives row 3 of the table.

If $n = 2$ and $v = \boldsymbol{\omega}_8 + \boldsymbol{\omega}_9$, then we check that $\langle v \rangle^\perp$ is spanned by the vectors $\boldsymbol{\alpha}_0, \ldots, \alpha_7, \alpha_9 - \boldsymbol{\omega}_9$ and it is isomorphic to $L = \mathsf{E}_8 \oplus \langle -4 \rangle$. We have $\mathrm{O}(L)^\sharp \cong W(\mathsf{E}_8)$. It is mapped isomorphically to a subgroup of $\mathrm{O}^+(10, \mathbb{F}_2)$ isomorphic to the group $\mathrm{O}^+(8, \mathbb{F}_2)$ of order $2^{13} \cdot 3^3 \cdot 5^2 \cdot 7$. This gives row 4 of the table.

Assume $n = 3$. In this case $\langle v \rangle^\perp$ is generated by $\boldsymbol{\alpha}_0, \ldots, \boldsymbol{\alpha}_5$ and $\boldsymbol{\alpha}_8, \boldsymbol{\alpha}_9$ and it is isomorphic to $L = \mathsf{E}_7 \oplus \mathsf{A}_2$. The group $W(\mathsf{E}_{10})_v$ is isomorphic to $\mathrm{O}(L)^\sharp = \mathrm{Ker}(\rho_L)$. Since the homomorphism ρ_L is surjective by Theorem 0.8.6 and $\mathrm{O}(D(L)) = \mathrm{O}(\langle \frac{1}{2} \rangle \oplus \langle \frac{1}{3} \rangle \cong \mathrm{O}(\langle \frac{5}{6} \rangle) = \{1\}$, we get $\mathrm{O}(L)^\sharp = \mathrm{O}(L)$. We have $W(\mathsf{E}_{10})(2) \cap \mathrm{O}(L) = \langle -\mathrm{id}_{\mathsf{E}_7} \oplus \mathrm{id}_{\mathsf{A}_2} \rangle$. Thus $W(\mathsf{E}_{10})_v$ is mapped isomorphically to a subgroup of $\mathrm{O}^+(10, \mathbb{F}_2)$ of order $2^{10} \cdot 3^5 \cdot 5 \cdot 7$. This gives row 5 of our table.

Assume $n = 4$ and $v = \boldsymbol{\omega}_1 + \boldsymbol{\omega}_9$. In this case, the linear system $|\phi(v)|$ defines a birational map onto a non-normal octic surface in $\mathbb{P}^4$. We can write $v = (\boldsymbol{\Delta} - \mathbf{f}_1 - \mathbf{f}_9) + 2g$, where $f = \boldsymbol{\Delta} - \mathbf{f}_1 - \mathbf{f}_9, g = \mathbf{f}_9$ are two primitive isotropic vectors with $f \cdot g = 2$. Their sum $f + g$ defines a polarization of degree 4 considered before. As in the case $n = 2, v' = \boldsymbol{\omega}_8 + \boldsymbol{\omega}_9$, we obtain that there is a degree 2 map $\mathcal{M}_{\mathrm{Enr},v} \to \mathcal{M}_{\mathrm{Enr},v'}$. This gives row 6 of the table.

Finally assume $n = 5$. In this case $\langle v \rangle^\perp$ is generated by $\boldsymbol{\alpha}_1, \ldots, \boldsymbol{\alpha}_9$ and isomorphic to A_9. We have $W(\mathrm{Num}(S))_2 \cong \mathrm{O}(\mathsf{A}_9)^\sharp \cong \mathfrak{S}_{10}$ (for this, we use that $\mathrm{O}(D(\langle \frac{1}{10} \rangle) = \{1\})$. It is mapped isomorphically onto a subgroup of $\mathrm{O}^+(10, \mathbb{F}_2)$ of order $2^8 \cdot 3^4 \cdot 5 \cdot 7$. This gives row 7 of our table.

We will prove in Section 8.4 in Volume II that $\mathrm{Aut}(S)^*$ is contained in $W(\mathrm{Num}(S))(2)$ in the case when S is a general nodal surface. So, we can still use formula (5.7.2). However, in this case $A(S) \neq W(\mathrm{Num}(S))$ and there are more $A(S)$-orbits of vectors of fixed norm square. For example, in case $n = 1$, we have four orbits. The computation of the number $P(S, v)$ for $v^2 \leq 10$ in this case can be found in Table 5.7.3.

Remark 5.7.4 Observe that $P(S, \omega_8 + \omega_9) = 2P(S, \omega_8)$. For an ample marking ϕ, an ample representative of $\phi(\omega_8 + \omega_9)$ defines a linear system $|F_1 + 2F_2|$ with two base points that defines a double plane model of S. An ample representative of $2\phi(\omega_8)$ defines a bielliptic linear system $|2F_1 + 2F_2|$ and therefore a bielliptic map onto a 4-nodal quartic surface D_1. We see that two linear systems $|F_1 + 2F_2|$ and $|2F_1 + F_2|$ correspond to the same bielliptic linear system. They define two rational maps $S \dashrightarrow \mathbb{P}^2$ that differ by a Cremona involution $[x, y, z] \mapsto [z^2, xy, xz]$ that switches the pencil of lines $|e_0 - e_1|$ with the pencil of conics $|2e_0 - e_1 - \cdots - e_5|$.

By analogy with the spaces $\mathcal{M}_{K3,M,M'}$ introduced in Section 5.2 we can introduce the spaces $\mathcal{M}_{\mathrm{Enr},M}$, where M is a primitive sublattice of E_{10}. Over $\mathbb{C}$, we can define them to coincide with $\mathcal{M}^a_{\mathsf{E}_{10}(2),M(2)}$.

Example 5.7.5 Let us take $M = \mathsf{U}_{[k]}$ as defined in Section 0.8. It is generated by isotropic vectors $\mathbf{f}_1, \ldots, \mathbf{f}_k$ forming an isotropic k-sequence. It contains the vector $\mathbf{v}_k = \mathbf{f}_1 + \ldots + \mathbf{f}_k$ and the forgetful map

$$F(\alpha) : \mathcal{M}^a_{\mathsf{E}_{10}(2),M(2)} = \mathcal{M}^{\mathrm{m}}_{\mathrm{Enr}} \to \mathcal{M}_{\mathrm{Enr},M},$$

defined by the primitive embedding $M(2) \hookrightarrow \mathsf{E}_{10}(2)$, is a torsor over the group $\mathrm{O}(M^\perp)^\sharp$. The discriminant group $D(\mathsf{U}_{[k]})$ is a cyclic group of order k generated by $\frac{1}{k-1}\mathbf{v}_k$ and it is isomorphic to $\langle \frac{k-2}{k-1} \rangle$. We know that $\mathsf{U}_{[10]}$ is a sublattice of index 3 in E_{10}. It follows from Proposition 6.1.1 in Section 6.1 in Volume II that all primitive embeddings of $\mathsf{U}_{[k]}, k \leq 9$ are equivalent. There is a non-primitive embedding of $\mathsf{U}_{[9]}$ with $\frac{1}{2}\mathbf{v}_9 \in \mathsf{E}_{10}$. Its image under an ample marking of an Enriques surface is the Mukai polarization of degree 18, see Section 3.5.

Assume $k \leq 7$. Then, we find

$$\mathsf{U}^\perp_{[k]} \in \{\mathsf{E}_8, \mathsf{E}_7, \mathsf{E}_6, \mathsf{D}_5, \mathsf{A}_4, \mathsf{A}_1 \oplus \mathsf{A}_2\}.$$

We have $\mathrm{O}(\langle \frac{k-2}{k-1} \rangle) = \mathbb{Z}/2\mathbb{Z}$ if $k \neq 1, 2$ and trivial otherwise. From this we deduce that

$$\mathrm{O}(\mathsf{U}^\perp_{[k]})^\sharp \cong W(\mathsf{U}^\perp_k), \quad k = 1, \ldots, 7.$$

If $k = 8$, then the lattice $\mathsf{U}^\perp_{[8]}$ has a basis α, β with the Gram matrix $\begin{pmatrix} -2 & 7 \\ 7 & -28 \end{pmatrix}$. The group $\mathrm{O}(\mathsf{U}^\perp_{[8]})$ is generated by the reflection s_α. If $k = 9$, then $\mathsf{U}_{[9]}$ is generated by $\mathbf{v}_9 - 8\mathbf{f}_{10}$ and again $\mathrm{O}(\mathsf{U}^\perp_{[k]})^\sharp$ is of order 2. This computation allows one to compute the degree of the forgetful map

$$\mathcal{M}_{\mathrm{Enr},\mathsf{U}_{[k]}} \to \mathcal{M}_{\mathrm{Enr}}.$$

It is equal to $\frac{\#\mathrm{O}^+(10,\mathbb{F}_2)}{\#\mathrm{O}(\mathsf{U}^\perp_{[k]})^\sharp}$. For example, if $k = 2$, the degree is equal to $2^8 \cdot 17 \cdot 31$.

As a corollary we see that all moduli spaces $\mathcal{M}_{\mathsf{U}_{[k]}}$ are irreducible.

The orthogonal complement $\mathbf{v}_k^\perp$ in $\mathsf{U}_{[k]}$ is isomorphic to the root lattice A_{k-1}. Since $\mathrm{O}(D(\mathsf{A}_{k-1}))$ is trivial, we find that the forgetful map

$$\mathcal{M}_{\mathrm{Enr},\mathsf{U}_{[k]}} \to \mathcal{M}_{\mathrm{Enr},\mathsf{v}_k}$$

is a Galois cover whose Galois group isomorphic to the symmetric group S_k. For example, we obtain that the forgetful map from $\mathcal{M}_{\mathrm{Enr},\mathbf{v}_{10}}$ to $\mathcal{M}_{\mathrm{Enr}}$ is a Galois cover with the group $\mathfrak{S}_{10}$, as expected. The moduli space $\mathcal{M}_{\mathrm{Enr},\frac{1}{3}\mathbf{v}_{10}}$ coincides with the moduli space $\mathcal{M}_{\mathrm{Enr},\mathbf{w}_0}$. We re-denote it by $\mathcal{M}_{\mathrm{Enr,Fano}}$. It is the moduli space of Enriques surfaces with a numerical Fano polarization.

Our final remark is that we can consider the forgetful maps

$$f_k : \mathcal{M}_{\mathrm{Enr},\mathsf{U}[k+1]} \to \mathcal{M}_{\mathrm{Enr},\mathsf{U}[k]}, \quad k = 1, \dots, 7$$

by considering a primitive embedding $\mathsf{U}_k \hookrightarrow \mathsf{U}_{k+1}$ defined by forgetting $\mathbf{f}_{k+1}$. This corresponds to the natural embedding of $W(\mathsf{U}_{[k]})$ into $W(\mathsf{U}_{[k+1]}$ as the stabilizer of the fundamental weight $\mathbf{w}_{k+1}$. Thus, we obtain that the degree of the map f_k is equal to $[W(\mathsf{U}_{[k]}) : W(\mathsf{U}_{[k+1]})]$. This coincides with the number of lines on a smooth anti-canonical del Pezzo surface of degree $k-1$. For example, the degree of f_4 is equal to 27. So the tower

$$\mathcal{M}_{\mathrm{Enr},\mathsf{U}_{[7]}} = \mathcal{M}_{\mathrm{Enr}}^{\mathrm{m}}/W(\mathsf{A}_2 \oplus \mathsf{A}_1) \to \mathcal{M}_{\mathrm{Enr},\mathsf{U}_{[6]}} \to \cdots \to \mathcal{M}_{\mathrm{Enr},\mathsf{U}_{[2]}} = \mathcal{M}_{\mathrm{Enr}}/W(\mathsf{E}_8)$$

is similar to the tower of the moduli spaces of marked del Pezzo surfaces together with a choice of line

$$\mathcal{M}_{dP,1}^{\mathrm{m}} \to \mathcal{M}_{dP,2}^{\mathrm{m}} \to \cdots \to \mathcal{M}_{dP,6}^{\mathrm{m}},$$

where the morphisms are defined by blowing down the last member in an exceptional sequence of $(e_1, \dots, e_{9-k})$ whose image under a marking is an ordered set of skew lines that defines the marking, see [177, Remark 9.4.19].

Example 5.7.6 Let S be an Enriques surface over an algebraically closed field of characteristic $p \neq 2$. An ample numerical polarization of degree two on S is defined by the image of v_2 of vector $\mathbf{v}_2 = \omega_8$ in $\mathrm{Num}(S)$ such that v belongs to the ample cone. Since $\Phi(v_2) = 1$, one can find a unique pair $\{f, g\}$ of isotropic divisor classes such that $v_2 = f + g$. We also may assume that they are nef divisors such that $|2f + 2g|$ defines a bielliptic map $\phi : S \to \mathsf{D}_1$ onto a non-degenerate quartic symmetroid del Pezzo surface D_1. Since v_2 is ample, the branch curve $W \in |\mathcal{O}_{\mathsf{D}_1}(2)|$ is smooth. Ordering the pair $\{f, g\}$ defines an ample U-marking of S. Conversely, such a lattice marking defines a unique ample degree two polarization.

So far we dealt with numerical polarizations. We know that the moduli space of polarized Enriques surfaces $\mathcal{P}_{\mathrm{Enr}}$ is a Deligne–Mumford stack that comes with an étale morphism of degree 2 onto the stack $\mathcal{P}_{\mathrm{Enr}}^{\mathrm{num}}$ of numerically polarized Enriques surfaces. We denote its component over $\mathcal{M}_{\mathrm{Enr},v}$ by $\widetilde{\mathcal{M}}_{\mathrm{Enr},v}$.

Using Hilbert schemes, we can construct the substack $\widetilde{\mathcal{M}}_{\mathrm{Enr},v}^{a}$ with $\Phi(v) \geq 3$ as follows:

Let S be an Enriques surface embedded into $\mathbb{P}^n$ by a complete linear system $|D|$ with $D^2 = 2n$. Then, the Hilbert polynomial of $S \subset \mathbb{P}^n$ is given by $P_S(t) =$

$\chi(S, \mathcal{O}_S(t)) = 2nt^2 + 1$. We let $\mathrm{Hilb}_{\mathrm{Enr},2n}$ be the Hilbert scheme of subschemes of $\mathbb{P}^n$ with Hilbert polynomial $P(t) = 2nt^2 + 1$. Next, we compute the tangent space of $\mathrm{Hilb}_{\mathrm{Enr},2n}$ at the point $[S]$. Let $\mathcal{N}_S$ be the normal bundle of $S \subset \mathbb{P}^n$. We have a natural short exact sequence

$$0 \to \Theta_S \to \Theta_{\mathbb{P}^n} \otimes \mathcal{O}_S \to \mathcal{N}_S \to 0, \tag{5.7.4}$$

where Θ_S and $\Theta_{\mathbb{P}^n}$ denote the tangent sheaves of S and $\mathbb{P}^n$, respectively. We know from Section 1.4 that

$$h^i(S, \Theta_S) = h^i(S, \Omega_S^1) - h^i(S, \Omega_S^1) = \begin{cases} 10 & \text{if } i = 1\ , \\ 0 & \text{otherwise.} \end{cases}$$

Applying the exact sequence (5.7.4) and the exact sequence

$$0 \to \mathcal{O}_S \to \mathcal{O}_S(1)^{\oplus(n+1)} \to \Theta_{\mathbb{P}^n} \otimes \mathcal{O}_S \to 0$$

obtained from the known resolution of the tangent sheaf of projective space (the Euler sequence), we obtain

$$h^i(S, \mathcal{N}_S) = \dim H^i(S, \mathcal{N}_S) = \begin{cases} 10 + n^2 + 2n & \text{if } i = 1\ , \\ 0 & \text{otherwise.} \end{cases} \tag{5.7.5}$$

By the deformation theory of Hilbert schemes, the tangent space of the Hilbert scheme at the point corresponding to S is isomorphic to $H^0(S, \mathcal{N}_S)$ and that it is smooth at this point if $H^1(S, \mathcal{N}_S) = 0$, see [265, §5] or [651, Theorem 4.3.5 and Proposition 4.3.6]. Thus, we see that $\mathrm{Hilb}_{\mathrm{Enr},2n}$ is smooth at the point $[S]$ and that it is of dimension $n^2+2n+10$ at $[S]$. Since this dimension is equal to $\dim \mathrm{PGL}(n+1)+10$, it follows that there is an open neighborhood of $[S]$ in $\mathrm{Hilb}_{\mathrm{Enr},2n}$ that parametrizes only Enriques surfaces that are embedded via a complete and ample linear system $|D|$ with $D^2 = 2n$.

Proposition 5.7.7 *The Hilbert scheme* $\mathrm{Hilb}_{\mathrm{Enr},2n}$ *of Enriques surfaces embedded into* $\mathbb{P}^n$ *by a complete linear system is a smooth variety. The dimension of each of its connected (=irreducible) component is equal to* $n^2 + 2n + 10 = \dim \mathrm{PGL}(n + 1) + 10$.

The group $\mathrm{PGL}(n + 1)$ acts on $\mathrm{Hilb}_{\mathrm{Enr},2n}$ via its action on $\mathbb{P}^n$. Since the algebraic group $\mathrm{PGL}(n + 1)$ is reductive, this action is proper, that is, orbits are closed, see [542, 0.8]. It also has finite stabilizer groups: indeed, by Proposition 8.2.1 in Volume II, the kernel of the homomorphism $\mathrm{Aut}(S) \to \mathrm{Aut}(S)^* \subset \mathrm{O}(\mathrm{Num}(S))$ is finite. Since $\mathrm{Aut}(S)^*$ leaves the numerical class $[h]$ of a big and nef divisor h invariant, as well as its negative definite orthogonal complement $[h]^\perp$, the group $\mathrm{Aut}(S)^*_{[h]}$ is also finite. Now, we can apply [381, Theorem 1.1] to conclude that the geometric quotient $\mathrm{Hilb}_{\mathrm{Enr},2n}/\mathrm{PGL}(n + 1)$ exists as a separated algebraic space.

Proposition 5.7.8 *The quotient* $\mathcal{M}^a_{\mathrm{Enr},2n} := \mathrm{Hilb}_{\mathrm{Enr},2n}/\mathrm{PGL}(n+1)$ *is a coarse moduli space for the functor* $\mathcal{P}_{\mathrm{Enr},2n}$.

This result is analogous to the existence of coarse moduli spaces for the moduli functor of polarized K3 surfaces, see [320, Chapter 5, Theorem 2.4] and its proof.

We can similarly treat polarizations (S, D) with $\Phi(D) = 2$. For example, if $D^2 = 2$, then we know that $|2D|$ defines a bielliptic map $S \to \mathsf{D}$ to a quartic symmetroid del Pezzo surface in $\mathbb{P}^4$. In characteristic $p \neq 2$ it is defined by a section of $\mathcal{O}_{\mathsf{D}}(2)$ and we construct the moduli space $\mathcal{M}_{\mathrm{Enr},v}$ as the quotient of the Hilbert scheme of curves in $|\mathcal{O}_{\mathsf{D}}(2)|$ by $\mathrm{Aut}(\mathsf{D})$. We will discuss this quotient in the next section.

Example 5.7.9 Let us consider Fano polarizations as discussed in Section 3.5. It follows from Example 5.7.5 that the moduli space space $\mathcal{M}_{\mathrm{Enr,Fano}}$ of Enriques surfaces with a numerical Fano polarization is irreducible. Let us consider its double cover $\widetilde{\mathcal{M}}_{\mathrm{Enr,Fano}}$, that is, the moduli space of Enriques surfaces with a Fano polarization.

It follows from Theorem 3.5.1 that $\widetilde{\mathcal{M}}_{\mathrm{Enr,v_3}}$ is rationally dominated by a 10-dimensional affine space $\mathbb{A}^{10}$ of quadratic polynomials in four variables. In particular, $\widetilde{\mathcal{M}}_{\mathrm{Enr,v_3}}$ is an irreducible and unirational variety. The pre-image of each edge $\ell_{ij} = V(x_i, x_j)$ of the coordinate tetrahedron $V(x_1x_2x_3x_4)$ is a half-fiber F_{ij} of an elliptic pencil on S. In fact, a choice of such an equation defines a family of degree 6 polarized Enriques surfaces over an open and dense subset of $\mathbb{A}^{10}$ that dominates the coarse moduli space.

In [716] Verra uses the family $\mathcal{X} \to T$ from the previous example to prove the irreducibility of the moduli space $\widetilde{\mathcal{M}}_{\mathrm{Enr,Fano}}$. Since he does not state this result explicitly, we give a brief sketch of his argument and refer to [121] for details. Verra chooses the edges ℓ_{12} and ℓ_{34} and considers the family $\mathcal{F}$ of quintic elliptic curves in $\mathbb{P}^3$ that do not pass through the vertices of the coordinate tetrahedron, that intersect exactly in one point the edges ℓ_{12}, ℓ_{34}, and that intersect exactly at two points of the remaining edges. If we choose a Fano polarization H, then such curve can be taken from the linear system $|\Delta - F_{13} - F_{14}|$. This suggests to look for Δ as the divisor class of $F_{13} + F_{14} + E$, where E is the pre-image of a quintic elliptic curve from the family Q the pre-image is a Fano polarization curve from the family Q, see [716, Proposition 3.1]. In [716, Proposition 1.1], he shows that $\mathcal{F}$ is an irreducible rational variety of dimension 10 that dominates $\widetilde{\mathcal{M}}_{\mathrm{Enr,Fano}}$.

Let us now briefly discuss supermarked Enriques surfaces.

Definition 5.7.10 A *supermarking* of an Enriques surface S is an isomorphism $\tilde{\phi} : \mathsf{E}_{10} \to \mathrm{Pic}(S)$ such that its composition with the natural map $\mathrm{Pic}(S) \to \mathrm{Num}(S)$ is a marking $\phi : \mathsf{E}_{10} \to \mathrm{Num}(S)$. A supermarking $\tilde{\phi}$ is called *ample* if ϕ is an ample marking.

Two supermarkings $\tilde{\phi}$ and $\tilde{\phi}'$ are called isomorphic if there exists an automorphism g of S such that $\tilde{\phi}' = \pm\tilde{\phi} \circ g^*$.

It is clear that $W(\mathsf{E}_{10})$ acts on supermarkings by compositions on the right. Also any $l \in \overline{\mathsf{E}}_{10} = \mathsf{E}_{10}/2\mathsf{E}_{10} \cong \mathbb{F}_2^{10}$ acts on supermarkings by replacing $\tilde{\phi}$ with $\tilde{\phi} + l$ defined by $x \mapsto \tilde{\phi}(x) + (l \cdot x)K_S$. In this way, the group

$$\widetilde{W}(\mathsf{E}_{10}) := \overline{\mathsf{E}}_{10} \rtimes W(\mathsf{E}_{10})$$

acts on the set of isomorphism classes of supermarkings and the kernel of the action is the subgroup $\widetilde{\mathrm{Aut}(S)^*} := \overline{\mathrm{Aut}}(S)^* \rtimes \mathrm{Aut}(S)^*$, where $\overline{\mathrm{Aut}}(S)^*$ is the image of $\mathrm{Aut}(S)^*$ in its action on $\overline{\mathsf{E}}_{10}$. In particular, if S is a general unnodal surface without extra automorphisms, then the normal subgroup $\overline{\mathsf{E}}_{10}$ of $\widetilde{W}(\mathsf{E}_{10})$ acts freely on the set of isomorphism classes of supermarkings.

We can define a family of supermarked surfaces as a family $(\mathcal{X} \to T, \phi : (\mathsf{E}_{10})_T \to \underline{\mathrm{Pic}}_{\mathcal{X}/T})$ such that the composition of ϕ with the map $\underline{\mathrm{Pic}}_{\mathcal{X}/T} \to \underline{\mathrm{Pic}}_{\mathcal{X}/T}/\underline{\mathrm{Pic}}^{\tau}_{\mathcal{X}/T}$ defines a family of marked surfaces. This gives us a functor with values in groupoids on the category schemes over $\Bbbk$ and a stack $\mathcal{E}^{\mathrm{sm}}_{\mathrm{Enr}}$ on which the group $\mathbb{F}_2^{10}$ acts with quotient isomorphic to the Deligne–Mumford stack $\mathcal{E}^{\mathrm{m}}_{\mathrm{Enr}}$ of marked Enriques surfaces. One can prove that the stack $\mathcal{E}^{\mathrm{sm},a}_{\mathrm{Enr}}$ of ample supermarked Enriques surfaces is a separated Deligne–Mumford stack.

The proof of the next theorem is rather involved and we skip it.

Theorem 5.7.11 *Assume $\Bbbk = \mathbb{C}$. Then $\mathcal{E}^{\mathrm{sm},a}_{\mathrm{Enr}}$ is an irreducible Deligne–Mumford stack.*

Remark 5.7.12 It is natural to expect that the complex analytic space $\mathcal{M}^{\mathrm{sm},a}_{\mathrm{Enr}}$ is isomorphic to the quotient of an open subset of the period domain for a normal subgroup of the monodromy group $\Gamma^{\sharp}_{\mathrm{Enr}}$ with quotient group isomorphic to $\overline{\mathsf{E}}_{10}$. According to R. Borcherds, Γ_{Enr} indeed contains such a subgroup. Let us explain his construction:

Let $N = \mathsf{E}_{10}(2)^{\perp}$. We have

$$N/2N^{\vee} \cong (\mathsf{E}_{10}(2) \oplus \mathsf{U})/(2\mathsf{E}_{10}(2)^{\vee} \oplus 2\mathsf{U}(2)^{\vee} \oplus 2\mathsf{U}^{\vee}) \cong \mathsf{U}/2\mathsf{U} \cong \mathbb{F}_2^2.$$

Let f, g be the standard isotropic generators of U and let $\bar{f}, \bar{g}$ be their cosets in $\mathsf{U}/2\mathsf{U}$. The subgroup $\Gamma^{\sharp}_{\mathrm{Enr}}$ of $\mathrm{O}(N)$ in its natural action on $(N/2N)^{\vee}$ leaves the vector $\eta = \bar{f} + \bar{g}$ invariant (since it is the only vector of square 2 mod 4 in $\mathbb{F}_2^2$ with quadratic form inherited from U). Let A be the quotient $\mathbb{F}_2^2/\mathbb{F}_2\eta \cong \mathbb{F}_2$. Define a map

$$\alpha : \Gamma^{\sharp}_{\mathrm{Enr}} \to \mathrm{Hom}(N^{\vee}/N, A) \cong \mathbb{F}_2^{10}$$

as follows: The image of $g \in \Gamma_{\mathrm{Enr}}$ is equal to the linear function $l(w+N) = g(w) - w \mod \mathbb{F}_2\eta$. One can show that the images of reflections in vectors of square 2 are nonzero and generate $\mathrm{Hom}(N^{\vee}/N, A) \cong \mathbb{F}_2^{10}$. One can hope that the quotient space of an open subset of $\mathcal{D}_{\mathrm{Enr}}$ by the group $\mathrm{Ker}(\alpha)$ is the coarse moduli space of the stack $\mathcal{E}^{\mathrm{sm}}_{\mathrm{Enr}}$.

We know that over $\mathbb{C}$ the stack $\mathcal{E}^{\mathrm{m},a}_{\mathrm{Enr}}$ admits the coarse moduli space $\mathcal{M}^{a}_{\mathrm{Enr}}$, which is isomorphic to $\Gamma^{\circ}_{\mathrm{Enr}}\backslash\mathcal{D}_{\mathrm{Enr}}$. The map of stacks $\mathcal{E}^{\mathrm{sm},a}_{\mathrm{Enr}} \to \mathcal{E}^{\mathrm{m},a}_{\mathrm{Enr}}$ defines a Galois cover $\mathcal{M}^{\mathrm{sm}}_{\mathrm{Enr}}$ of $\mathcal{M}^{\mathrm{m}}_{\mathrm{Enr}}$ with Galois group $\overline{\mathbf{E}}_{10}$.

Using the irreducibility of $\mathcal{M}^{\mathrm{sm}}_{\mathrm{Enr}}$, we can prove the following theorem that was proved by other methods in [396].

Theorem 5.7.13 *The moduli space $\widetilde{\mathcal{M}}_{\mathrm{Enr},v}$ is irreducible for all primitive ample polarizations $v \in \mathsf{E}_{10}$.*

Proof Let $\bar{v}$ be the image of v in $\overline{\mathsf{E}}_{10}$ and let $\langle\bar{v}\rangle^{\perp}$ be its orthogonal complement in the corresponding quadratic space. We know that $\mathcal{M}_{\mathrm{Enr}}^{\mathrm{sm},a}/\overline{\mathsf{E}}_{10} \cong \mathcal{M}_{\mathrm{Enr}}^{\mathrm{m},a}$, therefore $\widetilde{\mathcal{M}}_{\mathrm{Enr}}^{\mathrm{m},a} := \mathcal{M}_{\mathrm{Enr}}^{\mathrm{sm},a}/\langle\bar{v}\rangle^{\perp}$ is a double cover of $\mathcal{M}_{\mathrm{Enr}}^{\mathrm{m},a}$. We also know that $\mathcal{M}_{\mathrm{Enr},v}^{a} = \mathcal{M}_{\mathrm{Enr}}^{\mathrm{m},a}/G(v)$, where $G(v)$ is a subgroup of $W(\mathsf{E}_{10})$ that fixes $\bar{v}$. It follows that the pre-image of $G(v)$ in $\overline{\mathsf{E}}_{10} \rtimes W(\mathsf{E}_{10}$ contains the subgroup $\langle\bar{v}\rangle^{\perp} \rtimes G(v)$ and that the quotient of $\mathcal{M}_{\mathrm{Enr}}^{\mathrm{sm},a}$ by this subgroup is a double cover of $\mathcal{M}_{\mathrm{Enr},v}^{a}$.

An element $l \in \overline{\mathsf{E}}_{10}$ acts on supermarkings $\tilde{\phi} : \mathsf{E}_{10} \to \mathrm{Pic}(D)$ by replacing $\tilde{\phi}$ with $\tilde{\phi}+l$, where $l(D) = D + l([D] + 2\,\mathrm{Num}(S))K_S$, where $[D]$ is the numerical class of D. Under this action, the subgroup $\langle\bar{v}\rangle$ fixes the divisor class D such $[D] = \phi(v)$. It shows that, forgetting the supermarking, we get an isomorphism class of S together with a choice of a divisor class D with $[D] = \phi(v)$. This is the definition of the moduli space $\widetilde{\mathcal{M}}_{\mathrm{Enr},v}^{a}$ of ample polarized Enriques surface with polarization defined by a vector v. We get

$$\widetilde{\mathcal{M}}_{\mathrm{Enr},v}^{a} \cong \mathcal{M}_{\mathrm{Enr}}^{\mathrm{sm},a}/\langle\bar{v}\rangle^{\perp} \rtimes G(v) \cong \widetilde{\mathcal{M}}_{\mathrm{Enr}}^{\mathrm{m},a}/G(v).$$

As a finite quotient of an irreducible space $\mathcal{M}_{\mathrm{Enr}}^{\mathrm{sm}}$, the moduli space $\widetilde{\mathcal{M}}_{\mathrm{Enr},v}^{a}$ is irreducible. □

Note that for non-primitive polarizations (which we have ignored) the moduli space could be reducible if v is divisible by 2. For example, take $v = 2w$ with $v^2 = 2$. We used this numerical polarization to define bielliptic linear systems $|2F_1 + 2F_2|$. However, we know from Section 3.3 that if we take the adjoint polarization $|2F_1 + 2F_2 + K_S|$, t then he map is not bielliptic but defines a birational map onto a surface of degree 8 in $\mathbb{P}^4$. It is proven in [397] that the reduciblity of the moduli space occurs only in the case if the polarization is divisible by 2.

We remark also that restricting $\mathcal{M}_{\mathrm{Enr},v}$ to the hypersurface corresponding to isomorphism classes of nodal Enriques surfaces the space usually becomes reducible. We will compute the number of components with $v^2 \leq 10$ in Section 8.4 in Volume II. As we mentioned earlier, there are four different irreducible components if $v^2 = 2$. It is an interesting problem to answer in general when the restriction of $\widetilde{\mathcal{M}}_{\mathrm{Enr},v}$ to these components becomes irreducible.

5.8 Birational Geometry of Moduli Spaces

In this section, we study birational geometry of coarse moduli spaces of – marked, unmarked, polarized, and nodal – Enriques surfaces over the complex numbers. Since these moduli spaces are of dimension 9 or 10, this means that we ask for the Kodaira dimension κ of these spaces, see Section 1.1 If $\kappa = -\infty$, one can ask whether these spaces are (uni-)ruled or (uni-)rational. We refer to [121] and [262] for further results.

We start with the following result of S. Kondō [410]:

Theorem 5.8.1 *The moduli space $\mathcal{M}_{\mathrm{Enr}}$ is a rational variety.*

Proof The proof is based on a birational isomorphism (of non-geometric nature) between the moduli space $\mathcal{M}_{\mathrm{Enr}}$ and the moduli space $\mathcal{M}_{5,\mathrm{cusp}}$ of $K3$ surfaces that are isomorphic to the double cover of $\mathbb{P}^2$ branched along the union of a cuspidal plane quintic curve and the cuspidal tangent line.

Let $N = \mathsf{E}_{10}^{\perp} \cong \mathsf{U} \oplus \mathsf{U}(2) \oplus \mathsf{E}_8(2)$ and recall that $\mathcal{M}_{\mathrm{Enr}} \cong \mathrm{O}(N)' \backslash \mathsf{D}_N$. We have $N^{\vee} \cong \mathsf{U} \oplus \mathsf{U}(1/2) \oplus \mathsf{E}_8(1/2)$, from which we obtain that

$$N^{\vee}(2) \cong \mathsf{U}(2) \oplus \mathsf{U} \oplus \mathsf{E}_8 \cong \mathsf{U} \oplus \mathsf{U} \oplus \mathsf{D}_8.$$

Here, we use that the lattices $\mathsf{U}(2) \oplus \mathsf{E}_8$ and $\mathsf{U} \oplus \mathsf{D}_8$ have isomorphic discriminant groups together with their quadratic forms, hence, applying Nikulin's theorem 0.8.6, they must be isomorphic. Since $\mathrm{O}(N)' \cong \mathrm{O}(N^{\vee}(2)$, we obtain that

$$\mathcal{M}_{\mathrm{Enr}} \cong \mathrm{O}(N^{\vee}(2))' \backslash \mathsf{D}_N.$$

Let X be a K3 surface that is birationally equivalent to the double cover of $\mathbb{P}^2$ branched along a curve of degree 6 that is equal to the union of a plane quintic C with a cusp $c_0 \in C$ and the cuspidal tangent line ℓ, that is, the line that intersects C at c_0 with multiplicity 3. Thus, the local equation of C at the point c_0 is given by $(y^2 - x^3)y = 0$. It follows from Proposition 0.4.13 that the double cover acquires a rational double point over c_0 that is of type E_7. This cover has two more ordinary double points over the remaining intersection points of ℓ with C. Consider the pencil of lines through the point c_0. Its pre-image on X is an elliptic pencil with two reducible fibers of type $\tilde{A}_1$ and one reducible fiber of type $\tilde{D}_8$. The pre-image of the line ℓ is a section of this fibration. If τ denotes the covering involution of X as a double cover $\mathbb{P}^2$, then $\mathrm{Pic}(X)^{\tau}$ contains a sublattice M isomorphic to $\mathsf{U} \oplus \mathsf{D}_8$. Let $\mathcal{M}_{5,\mathrm{cusp}}$ be the moduli space of lattice M polarized K3 surfaces. Using the isomorphism

$$M^{\perp} \cong \mathsf{U} \oplus \mathsf{U} \oplus \mathsf{D}_8 \cong N^{\vee}(2),$$

we conclude that $\mathcal{M}_{\mathrm{Enr}}$ and $\mathcal{M}_{5,\mathrm{cusp}}$ both are isomorphic to the quotient of a 10-dimensional period space by the same discrete group.

Now, $\mathcal{M}_{5,\mathrm{cusp}}$ is birationally equivalent to the quotient of the space of cuspidal curves of degree 5 by the group pf projective automorphisms. Fixing the line ℓ and a point c_0 on it, the quotient becomes isomorphic to the quotient of a linear space by a subgroup of PGL(3) that fixes the flag (c_0, ℓ), which is a solvable linear group. A well-known result of Vinberg [720] and Miyata [519] asserts that this quotient is rational. □

We refer to [190] for a geometric explanation of the birational equivalence between $\mathcal{M}_{\mathrm{Enr}}$ and $\mathcal{M}_{5,\mathrm{cusp}}$. In the same article, one can find a proof of the following:

Theorem 5.8.2 *The moduli space of nodal Enriques surfaces $\mathcal{M}_{\mathrm{Enr}}^{\mathrm{nod}}$ and the moduli space of Coble surfaces of K3 type $\mathcal{M}_{\mathrm{Cob}}$ are rational varieties.*

On the other extreme, according to a yet unpublished result of V. Gritsenko, the moduli space $\mathcal{M}_{\mathrm{Enr}}^{\mathrm{m}}$ of marked Enriques surfaces, which is a Galois cover of $\mathcal{M}_{\mathrm{Enr}}$ with Galois group $\mathrm{O}(10, \mathbb{F}_2)^+$, is of general type.

For a vector $v \in \mathsf{E}_{10}$, we constructed the corresponding coarse moduli space $\mathcal{M}_{\mathrm{Enr},v}$ of numerically polarized Enriques surfaces in Section 5.5. We have also seen that there exist dominant maps

$$\mathcal{M}_{\mathrm{Enr}}^{\mathrm{m}} \to \mathcal{M}_{\mathrm{Enr},v} \to \mathcal{M}_{\mathrm{Enr}}, \tag{5.8.1}$$

where the first map is the quotient map by a finite subgroup $G(v)$ of $\overline{W}(\mathsf{E}_{10}) \cong \mathrm{O}^+(10, \mathbb{F}_2)$. We also have the irreducible moduli space $\widetilde{\mathcal{M}}_{\mathrm{Enr},v}$ of primitively polarized Enriques surfaces of type v that is a double cover of $\mathcal{M}_{\mathrm{Enr},v}$ and is a quotient of a double cover $\widetilde{\mathcal{M}}_{\mathrm{Enr}}^{\mathrm{m}}$ of $\mathcal{M}_{\mathrm{Enr}}^{\mathrm{m}}$ by the group $G(v)$.

The left–hand side being of general type, the right–hand side being rational (and thus of negative Kodaira dimension), this begs the question for the Kodaira dimension and (uni-)rationality of the moduli spaces $\mathcal{M}_{\mathrm{Enr},v}$ and $\widetilde{\mathcal{M}}_{\mathrm{Enr},v}$. Among other things, Gritsenko and Hulek [262] prove that the extremal cases do occur.

A standard approach to birational geometry of arithmetic quotients of period spaces $\mathcal{D}_N$ is via the theory of automorphic forms on it. We refer to the Appendix for the definition of a holomorphic automorphic form of weight $k \geq 0$ and character χ on $\mathcal{D}_N$. If the signature of N is equal to $(2, 1)$, this is the usual definition of an automorphic form in one variable. We assume that the signature of N is equal to $(2, n)$, where $n \geq 3$. In fact, for our applications to Enriques surfaces, we may take $N = \mathsf{U} \oplus \mathsf{E}_{10}(2)$ and $n = 10$. Let Γ be a subgroup of finite index of $\mathrm{O}(N)$. We will consider the algebra of modular forms $M_k(\Gamma, \chi)$ of weight k, character χ and the group Γ.

Let $\mathcal{M}_\Gamma := \Gamma\backslash\mathcal{D}_N$. When we refer to a birational property of this quasi-projective algebraic variety we mean the birational property of a smooth projective model. We will discuss in Section 5.9 various compactifications of $\mathcal{M}_\Gamma$ and in particular, a torodal compactification $\overline{\mathcal{M}_\Gamma}^{\mathrm{trd}}$. It is known that there exists a toroidal compactification with only canonical singularities [261, Theorem 1]. To investigate the Kodaira dimension of $\overline{\mathcal{M}_\Gamma}^{\mathrm{trd}}$ we have to study the linear spaces of holomorphic differentials $H^0(\overline{\mathcal{M}_\Gamma}^{\mathrm{trd}}, \omega_{\overline{\mathcal{M}_\Gamma}^{\mathrm{trd}}}^{\otimes k})$. Let $\pi_\gamma : \mathcal{D}_N \to \mathcal{M}_\Gamma$ be the projection map to the quotient. If it was unramified, then it would follow from the definition of a modular form that the space $H^0(\mathcal{M}_\Gamma, \omega_{\mathcal{M}_\Gamma}^{\otimes k})$ coincides with the vector space $M_{nk}(\Gamma, 1)$. However, the projection map π_Γ is ramified along the Heegner divisors

$$\mathcal{H}_N(\delta) = \{x \in \mathcal{D}_N : x \cdot \delta = 0\},$$

where $\delta \in N$ such that, for any $v \in N$, $\frac{2\delta \cdot v}{\delta^2} \in \mathbb{Z}$. In other words δ defines a reflection $s_\delta : v \mapsto v - \frac{2\delta \cdot v}{\delta^2}\delta$ on N. see [261, Corollary 2.13]. Let

$$\mathcal{H}_N(-n) = \cup_{\delta \in N_{-n}} \mathcal{H}_N(\delta).$$

In our case $N = \mathsf{U} \oplus \mathsf{E}_{10}(2)$, so there are only two types of Heegner divisors, namely $\mathcal{H}_N(-2)$ and $\mathcal{H}_N(-4)$, which we denoted in Section 5.3 by $\mathcal{H}(-2)$ and $\mathcal{H}(-4)$. The first one is the discriminant which we have to throw away to consider the moduli space of Enriques surfaces. The quotient by $\Gamma_{\mathrm{Enr}} = \mathsf{O}(N)^\sharp$ is the moduli space $\mathcal{M}_{\mathrm{Coble}}$ of Coble surfaces. The quotient of the second one by the same group is the moduli space $\mathcal{M}_{\mathrm{Enr}}^{\mathrm{nod}}$ of nodal Enriques surfaces. To see the relations between differential forms and modular forms we have to restrict ourselves with modular forms from $M_{10k}(\Gamma, 1)$ that vanish on the ramification locus of order k and extend to a holomorphic form on the compactification. According to the Koecher principle any modular form extends to the boundary of the Baily–Borel compactification of $\mathcal{M}_\Gamma$ (because we assumed that $n \geq 3$). But it is not clear whether it extends to the toroidal compactification $\mathcal{M}_\Gamma^{\mathrm{trd}}$. To construct modular form that extend to a holomorphic form on the compactification one uses Borcherds forms, whose construction we review in the Appendix.

We restrict ourselves to the case $N = \mathsf{U} \oplus \mathsf{E}_{10}(2)$. In this case, we have two Borcherds automorphic forms with zeros on Heegner divisors. These are the forms $\Phi_4(z)$ of weight 4 that vanishes with order 1 on the Heegner divisor $\mathcal{H}(-2)$ and $\Phi_{124}(z)$ of weight 124 that vanishes or order 1 on the Heegner divisor $\mathcal{H}(-4)$, see see Corollary A.2.9 and Corollary A.2.11 in the Appendix. The vectors $\delta \in N(-4)$ that define the Heegner divisors $\mathcal{H}_\delta$ are of even type, that is, $\frac{1}{2}\delta \in N^\vee$.

Suppose $F_{10k}(Z) \in M_{10k}(\Gamma, 1)$ that vanishes of order k on $\mathcal{H}(-2)$. Then, applying the Koecher principle, we obtain that

$$F_{6k}(z) = \frac{F_{10k}(z)}{\Phi_4^k(z)} \in M_{6k}(\Gamma, \chi),$$

where χ is a character that depends on Γ. We put

$$F(z) = \prod_{\gamma \in \Gamma_{\mathrm{Enr}}} F_{6k}(g \cdot z) \in M_{6k[\Gamma_{\mathrm{Enr}}:\Gamma]}(\Gamma_{\mathrm{Enr}}, \chi').$$

Here, the product is finite since each factor depends only on the coset of γ modulo Γ_{Enr}. The factor ${}^\gamma F(z) = F_{6k}(\gamma \cdot z)$ is an automorphic form with respect to the group $\gamma^{-1} \cdot \Gamma \cdot \gamma$. If F_{6k} vanishes on the Heegner divisor $\mathcal{H}_\delta$, then ${}^\gamma F(z)$ vanishes on the Heegner divisor $\mathcal{H}_{\gamma^{-1}(\delta)}$. By Corollary 5.6.2, all even vectors $\delta \in N_{-4}$ form one orbit with respect to Γ_{Enr}. We know that the orbits of $\Gamma_{\mathrm{Enr}}^\sharp$ on the set of even vectors $\delta \in N_{-4}$ are in bijective correspondence with 496 non-isotropic vectors in $\overline{\mathsf{E}}_{10} \cong \mathbb{F}_2^{10}$. Let $R = R(\Gamma)$ be the number of orbits of such vectors with respect to Γ. If $\Gamma = \Gamma_{\mathrm{Enr}}$, then this number is equal to 1 and in another extreme case, if $\Gamma = \Gamma_{\mathrm{Enr}}^\sharp$, then R is equal to 496. It follows that $F(z)$ vanishes on the image of a Heegner divisor $\mathcal{H}_\delta$ in $\mathcal{M}_{\mathrm{Enr}}$ with multiplicity

$$m = \frac{kR[\Gamma_{\mathrm{Enr}} : \Gamma]}{496}.$$

This implies that F is divisible by the automorphic form Φ_{124}^m and that

$$6k[\Gamma_{\mathrm{Enr}} : \Gamma] \geq 124m = \frac{kR[\Gamma_{\mathrm{Enr}} : \Gamma]}{4}.$$

Thus, $R \leq 24$ is the necessary condition in order that an automorphic form from $M_{10k}(\Gamma, 1)$ extends to a non-zero holomorphic form on $\mathcal{M}_{\Gamma}^{\mathrm{trd}}$. We have thus proved the following theorem, which is [262, Theorem 5.1].

Theorem 5.8.3 *Assume that the image $\overline{\Gamma}$ of Γ in $\mathsf{O}(\overline{\mathsf{E}}_{10}) \cong \mathsf{O}^+(10, \mathbb{F}_2)$ contains at least 25 reflections with respect to non-isotropic vectors. Then the Kodaira dimension of $\mathcal{M}_{\Gamma}$ is negative.*

Corollary 5.8.4 *Let $\mathcal{M}_{\mathrm{Enr},M}$ be the moduli space of Enriques surfaces with lattice M polarization. Assume that the image of $\mathsf{O}(M)^{\sharp}$ in $\mathsf{O}(\overline{\mathsf{E}}_{10})$ contains at least 25 reflections in non-isotropic vectors. Then $\mathcal{M}_{\mathrm{Enr},M}$ has negative Kodaira dimension.*

Example 5.8.5 Here are two examples of moduli spaces of Enriques surfaces that are of negative Kodaira dimension.

1. Let $M = \mathsf{U}_{[k]}$. We know from Example 5.7.5 that $\mathsf{O}(\mathsf{U}_{[k]})^{\sharp} \cong W(\mathsf{E}_{10-k})$ for $k = 2, 3, 4$. Its image in $W(\overline{E}_{10})$ contains 120, 56, 36 non-isotropic vectors. Thus, the moduli space $\mathcal{M}_{\mathrm{Enr},\mathsf{U}_{[k]}}$ is of negative Kodaira dimension if $k = 2, 3, 4$.
2. Let $v \in (\mathsf{U}_{[k]})_{2n}$ and assume $k \leq 4$. Then the projection $\mathcal{M}_{\mathrm{Enr},M} \to \mathcal{M}_{\mathrm{Enr},v}$ shows that $\mathcal{M}_{\mathrm{Enr},v}$ is of negative Kodaira dimension. In fact, computations from [262] show that $\mathcal{M}_{\mathrm{Enr},v}$ is of negative Kodaira dimension for all v with $v^2 \leq 32$.

On the other extreme, we have the following:

Theorem 5.8.6 *There exists a $v \in \mathsf{E}_{10}$ such that $\mathcal{M}_{\mathrm{Enr}}^m = \mathcal{M}_{\mathrm{Enr},v}$ is of general type.*

Let C be the fundamental chamber in E_{10} of the Weyl group defined by the root basis $\boldsymbol{\alpha}_0, \ldots, \boldsymbol{\alpha}_9$ and fundamental weights $\boldsymbol{\omega}_0, \ldots, \boldsymbol{\omega}_9$. We know from Section 1.5 that we can write $v \in C$ as in (1.5.7)

$$v = m\boldsymbol{\Delta} - \sum_{i=1}^{10} m_i \boldsymbol{\omega}_i, \tag{5.8.2}$$

where $m_i \geq 0, 3m = m_1 + \cdots + m_{10}$ and $m \geq m_1 + m_2 + m_3, m_1 \geq \ldots \geq m_{10}$. We call the vector $(m, m_1, \ldots, m_{10})$ the *numerical type* of v. For any vector v we choose $w \in W(\mathbf{E}_{10})$ such that $w(v) \in C$ and say that the numerical type of v is the numerical type of $w(v)$. If $m_1 \geq \ldots \geq m_k > m_{k+1} = \cdots = m_{10}$, then we say that k is the *threshold* of the numerical type (it is equal to zero if $m_1 = \cdots = m_{10}$).

We have the following:

Lemma 5.8.7 *If $C^2 > 0$, then $\Phi(v) = m_{10}$.*

Proof Since $v \in C$ and f_{10} is the unique primitive isotropic vector in C, we have for any other primitive isotropic vector f $f = \mathbf{f}_{10} + \sum a_i \boldsymbol{\alpha}_i, a_i \geq 0$. This gives $v \cdot f \geq v \cdot f_{10}$. It follows from Lemma 1.5.8 that $v \cdot \mathbf{f}_{10} = m_{10}$. □

Lemma 5.8.8 *Let $v \in C$ be of numerical type $(m_1, \dots, m_{10})$ and threshold k. Then:*

1. *$v^2 = 0$ if and only if $m_{k+1} = 0$ and in this case $v = \mathbf{f}_{10}$.*
2. *If $k = 0$, then $v = a\boldsymbol{\omega}_0$.*
3. *If $k = 1$, then $v = a\boldsymbol{\omega}_1 + b\boldsymbol{\omega}_0$.*
4. *If $k = 2$, then $v = a\boldsymbol{\omega}_2 + b\boldsymbol{\omega}_1 + c\boldsymbol{\omega}_0$.*
5. *If $k \ge 3$, then $v = a_4\mathbf{f}_4 + \cdots + a_{10}\mathbf{f}_{10} + v'$, where $a_4 \le \ldots \le a_{10}$ and $v' \in C$ is of threshold $k' \le 2$.*

Proof (1) By the previous lemma, $m_{10} = 0$. Then $v \cdot \mathbf{f}_{10} = 3m - \sum_{i=1}^{10} m_i = 0$. Since the lattice E_{10} has signature $(1, 9)$ we obtain that $v = a\mathbf{f}_{10}$.

(2) In this case $3m = 10m_1$, hence $v = m_1(\frac{10}{3}\boldsymbol{\Delta} - \mathbf{f}_1 - \cdots - \mathbf{f}_{10}) = \frac{m_1}{3}\boldsymbol{\Delta}$. So, $m_1 = 3a$ and $v = a\boldsymbol{\Delta}$. Since $\mathbf{f}_{1,2} = \boldsymbol{\Delta} - \mathbf{f}_1$, we get $\boldsymbol{\Delta} = \mathbf{f}_{1,2} + \mathbf{f}_1$.

(3) We have

$$\boldsymbol{\omega}_1 = \boldsymbol{\Delta} - \mathbf{f}_1 = \mathbf{f}_{12} + \mathbf{f}_2 = 7\boldsymbol{\Delta} - 3\mathbf{f}_1 - 2(\mathbf{f}_2 + \cdots + \mathbf{f}_{10}).$$

So, its numerical type has threshold equal to 1. Suppose $v \in C$ has threshold equal to 1. We can write

$$v = m\boldsymbol{\Delta} - m_1\mathbf{f}_1 - m_2(f_2 + \cdots + f_{10}),$$

where $3m = m_1 + 9m_2$ and $m \ge m_1 + 2m_2, m_1 \ge m_2$. This gives $3m = m_1 + 9m_2 \ge 3m_1 + 6m_2$, hence $3m_2 \le 2m_1 > 2m_2$. This implies that $m_2 \ge 2, m_3 \ge 3$ and $m \ge 7$. Since $v \cdot \boldsymbol{\alpha}_1 = m_1 - m_2 \ge 1$, subtracting $\boldsymbol{\omega}_1$, we obtain that $v - \boldsymbol{\alpha}_1$ is still in C and that it is equal to $v - \boldsymbol{\alpha}_1 = (m-7)\boldsymbol{\Delta} - (m_1 - 3)\mathbf{f}_1 - (m_2 - 2)(\mathbf{f}_2 + \cdots + \mathbf{f}_{10}$. Continuing in this way, we will either get either a vector with threshold 0 or a vector with threshold 1 and $m'_{10} = 2$. It is equal to $\boldsymbol{\omega}_1$. Thus we obtain $v = a\boldsymbol{\omega}_1 + b\boldsymbol{\Delta}$.

(4) We have

$$\boldsymbol{\omega}_2 = \boldsymbol{\Delta} - \mathbf{f}_1 - \mathbf{f}_2 = 4\Delta - 2\mathbf{f}_1 - 2\mathbf{f}_2 - (\mathbf{f}_3 + \cdots + \mathbf{f}_{10})$$

has threshold equal to 2. As in the previous case, we can subtract subtract $a\boldsymbol{\omega}_2$ until we either get zero or a vector with threshold ≤ 1.

(5) We have

$$v' = v - (\mathbf{f}_{k+1} + \cdots + \mathbf{f}_{10}) = (m+3k-30)\boldsymbol{\Delta} - \sum_{i=1}^{k}(m_i + k - 10)\mathbf{f}_i - (m_{k+1} + k - 9)(\mathbf{f}_{k+1} + \cdots + \mathbf{f}_{10})$$

satisfies $v' \cdot \boldsymbol{\alpha}_0 = v \cdot \boldsymbol{\alpha}_0 \ge 0$ and $v' \cdots \boldsymbol{\alpha}_i \ge 0, i \ne 0$, thus $v' \in C$. Continuing subtracting this vector, we get a vector with $m'_k = m'_{k+1} = \cdots m'_{10}$ with threshold $k' < k$. Then we subtract $(\mathbf{f}_{k'+1} + \cdots + \mathbf{f}_{10})$ and decrease the threshold again. In this way, we arrive at a vector with threshold ≤ 2. □

The next corollary is [397, Proposition 2.3] that improves [397, Lemma 2.12].

Corollary 5.8.9 *Every effective divisor class D with $D^2 \ge 0$ on an Enriques surface can be written as a positive integer linear combination*

$$D = a_0F_0 + a_1F_1 + \cdots + a_7F_7 + a_8F_8 + a_9F_9 + a_{10}F_{10} + \epsilon K_S,$$

where $(F_1, \ldots, F_{10})$ *is an effective lift on an isotropic* 10*-sequence and* F_0 *is an effective lift of an isotropic vector* f_0 *with* $f_0 \cdot f_9 = f_0 \cdot f_{10} = 2$ *and* $f_0 \cdot f_i = 1, i \neq 9, 10$. *Moreover:*

1. $a_8 = 0$.
2. $a_1 \leq \cdots \leq a_7$.
3. $a_9 + a_{10} \geq a_0 \geq a_9 \geq a_{10}$.

Proof We fix a marking $\phi : \mathrm{Num}(S) \to \mathsf{E}_{10}$ and let $v = \phi([D])$. Applying $w \in W(\mathsf{E}_{10})$, we may assume that v belongs to the fundamental chamber C. Let $(m, m_1, \ldots, m_{10})$ be the numerical type of v. If $D^2 = 0$, then we get $v = a\mathbf{f}_{10}$ and the assertion is true. Let k be the threshold of v. If $k = 0$, then $v = a\omega_0 = a\mathbf{\Delta} = a(\mathbf{f}_{1,2} + \mathbf{f}_1 + \mathbf{f}_2)$, so we take F_0 with $[F_0] = \mathbf{f}_{1,2}$ and F_i with $[F_i] = \mathbf{f}_i$ and get $(a_0, a_1, \ldots, a_{10}) = (a, a, a, 0, \ldots, 0)$. After permuting $\mathbf{f}_i$, we get $(a_0, a_1, \ldots, a_{10}) = (a, 0, \ldots, a, a)$ and the assertion follows.

If $k = 1$, then we get

$$v = a\omega_1 + b\omega_0 = a(\mathbf{f}_{1,2} + \mathbf{f}_2) + b(\mathbf{f}_{1,2} + \mathbf{f}_1 + \mathbf{f}_2),$$

hence $(a_0, \ldots, a_{10}) = (a+b, b, a+b, 0, \ldots, 0)$. After permuting $\mathbf{f}_i$, we get $(a_0, \ldots, a_{10}) = (a + b, 0, \ldots, 0, a + b, b)$, and the assertion follows.

If $k = 2$, then we get

$$v = a\omega_2 + b\omega_1 + c\omega_0 = a(2\mathbf{f}_{12} + \mathbf{f}_1 + \mathbf{f}_2) + b(\mathbf{f}_{1,2} + \mathbf{f}_2) + c(\mathbf{f}_{1,2} + \mathbf{f}_1 + bff_2)$$

$$= (2a + b + c)\mathbf{f}_{1,2} + (a + c)\mathbf{f}_1 + (a + b + c)\mathbf{f}_2.$$

Thus, after permuting the $\mathbf{f}_i$'s, we find $(a_0, a_1, \ldots, a_{10}) = (2a + b + c, 0, \ldots, 0, a + b + c, a + c)$ and again the assertion follows.

Finally, if $k \geq 3$, then we can write v as a sum of a vector with $k \leq 2$ and a vector $a_4\mathbf{f}_4 + \cdots + a_{10}\mathbf{f}_{10}$ with $a_4 \leq \cdots \leq a_{10}$. After permuting $\mathbf{f}_i$'s, we get $(a_0, a_1, \ldots, a_{10}) = (a_0, a_{10}, \ldots, a_4, 0, a_9, a_{10})$ and the assertion follows again. □

Note that it follows from the uniqueness of the numerical type of a numerical class $[D]$ that it can be written uniquely in the form $a_0f_0 + \cdots + a_{10}f_{10}$ as above.

To state the next theorem, we define, following [121], a *simple isotropic decomposition* of a numerical class h with $h^2 > 0$ to be a choice of primitive isotropic vectors $g_1, \ldots, g_n$ such that

$$h = k_1g_1 + \cdots + k_ng_n, \quad k_1, \ldots, k_n \in \mathbb{Z}_+,$$

where one of the following conditions is satisfied:

1. $n \neq 9$, and $(g_1, \ldots, g_n)$ is an isotropic n-sequence.
2. $n \neq 10$, and $g_1 \cdot g_2 = 2$ and $g_i \cdot g_j = 1$ for all other $i \neq j$.
3. $g_1 \cdot g_2 = g_1 \cdot g_3$ and $g_i \cdot g_j = 1$ for all other $i \neq j$.

It follows from Corollary 5.8.9 that a simple isotropic decomposition always exists although it is not unique (we take $g_1 = [F_0]$ if $a_0 \neq 0$ and $g_1 = [F_9]$ if $a_9 \neq 0, a_{10} = 0$ and $g_1 = [F_9], g_2 = [F_{10}]$ if $a_{10} \neq 0$).

The next theorem was proven in [121] using a construction similar to one we used in Example 5.7.9 proving the rationality of $\mathcal{M}_{\mathrm{Enr,Fano}}$.

Theorem 5.8.10 *$\mathcal{M}^a_{\mathrm{Enr},v}$ is unirational in the following cases:*

1. *There exists a marking $\phi : \mathsf{E}_{10} \to \mathrm{Num}(S)$ such that $\phi(v)$ admits a simple isotropic decomposition with $n \leq 4$.*
2. *There exists a marking $\phi : \mathsf{E}_{10} \to \mathrm{Num}(S)$ such that $\phi(v)$ admits a simple isotropic decomposition with $n = 5$ satisfying (1) and it is uniruled otherwise.*

Note that in Example 5.8.5 we have proved that $\mathcal{M}_{\mathrm{Enr},v}$ being of negative Kodaira dimension if $v \in \mathsf{U}_{[4]}$ that implies that $n \leq 4$, so the case in the theorem improves on this result.

We now turn to the moduli spaces $\mathcal{M}_{\mathrm{Enr},v}$ and we show that if v^2 is small, then not only the Kodaira dimension is negative, but that in some cases, these spaces are actually rational. We start with polarizations of degree 2. We recall from Corollary 1.5.4 that every vector $v \in \mathsf{E}_{10}$ with v^2 lies in the $\mathsf{O}(\mathsf{E}_{10})$-orbit of ω_8.

Theorem 5.8.11 *The moduli space $\mathcal{M}_{\mathrm{Enr},2} = \mathcal{M}_{\mathrm{Enr},\omega_8}$ of degree 2 numericallly polarized Enriques surfaces is connected and rational.*

Proof If $\mathcal{L}$ is a polarization of degree 2 on a general Enriques surface S, then the complete linear system $|\mathcal{L}^{\otimes 2}|$ defines a bielliptic map of degree 2 from S onto a non-degenerate 4-nodal quartic del Pezzo surface $\mathsf{D}_1 \subset \mathbb{P}^4$, see Section 3.3. The branch curve is cut out by a quadric. Thus, the moduli space $\mathcal{M}_{\mathrm{Enr},\omega_8}$ is birationally equivalent to the quotient of the projective space $|\mathcal{O}_{\mathsf{D}_1}(2)|$ by the automorphism group of D_1, which is isomorphic to the semi-direct product $G = \mathbb{G}_m^2 \rtimes D_4$. We will now use the equations and computations of Section 0.6.

First, we choose projective coordinates to write D_1 in the equations

$$x_0^2 + x_1x_2 = x_0^2 + x_3x_4 = 0.$$

Then, the automorphism group G is generated by the transformations

$$\begin{aligned} g_{\lambda,\mu} &: [x_0, x_1, x_2, x_3, x_4] \mapsto [x_0, \lambda x_1, \lambda^{-1}x_2, \mu x_3, \mu^{-1}x_4] \\ g_1 &: [x_0, x_1, x_2, x_3, x_4] \to [x_0, x_2, x_1, x_3, x_4], \\ g_2 &: [x_0, x_1, x_2, x_3, x_4] \to [x_0, x_1, x_2, x_4, x_3], \\ g_3 &: [x_0, x_1, x_2, x_3, x_4] \to [x_0, x_3, x_4, x_1, x_2]. \end{aligned}$$

We set $V(15) := H^0(\mathbb{P}^4, \mathcal{O}_{\mathbb{P}^4}(2))$ and let $V(13) \subset V(15)$ be the subspace of *harmonic quadratic forms* with respect to the linear space of quadratic forms spanned by $q_1 = x_0^2 + x_1x_2$ and $q_2 = x_0^2 + x_3x_4$. This means that $V(13)$ is equal to those quadratic forms that are annihilated by the differential operators

$$\frac{\partial^2}{\partial x_0^2} + 2\frac{\partial^2}{\partial x_1 \partial x_2} \quad \text{and} \quad \frac{\partial^2}{\partial x_0^2} + 2\frac{\partial^2}{\partial x_3 \partial x_4}.$$

If $V(2) \subset V(15)$ denotes the subspace spanned by the quadratic forms q_1, q_2, then we obtain a direct sum decomposition of linear G-representations $V(15) = V(13) \oplus V(2)$. This gives an isomorphism $V(15)/V(2) \cong V(13)$ of linear G-representations. The space $V(13)$ is spanned by the following thirteen polynomials:

$$x_1^2,\ x_2^2,\ x_3^2,\ x_4^2,\ x_0x_1,\ x_0x_2,\ x_0x_3,\ x_0x_4,\ x_1x_3,\ x_1x_4,\ x_2x_3,\ x_2x_4,\ 2x_0^2 - x_1x_2 - x_3x_4,$$

which we denote by m_i', $i = 1, \dots, 13$. Passing to the projective space $\mathbb{P}(V(13))$, we introduce the 12 invariant rational functions $m_i := m_i'/{m_{13}'}^{d_i}$, where d_i is the degree of the monomial m_i.

We have to show that the field of invariants $K = \mathbb{C}(m_1, \dots, m_{12})^G$ is a purely transcendental extension of $\mathbb{C}$. The torus $\mathbb{G}_m^2$ acts on $\mathbb{P}(V(13))$ with a basis $m_1, \dots, m_{12}$ via the diagonal matrix by the characters

$$\mathrm{diag}(\lambda^2, \lambda^{-2}, \mu^2, \mu^{-2}, \lambda, \lambda^{-1}, \mu, \mu^{-1}, \lambda\mu, \lambda\mu^{-1}, \lambda^{-1}\mu, \lambda^{-1}\mu^{-1}, 1).$$

A monomial $m_1^{a_1} \cdots m_{12}^{a_{12}}$ in the basis is invariant if and only if

$$\begin{aligned} &2a_1 - 2a_2 + a_5 - a_6 + a_9 + a_{10} - a_{11} - a_{12} = 0, \\ &2a_3 - 2a_4 + a_7 - a_8 + a_9 - a_{10} + a_{11} - a_{12} = 0. \end{aligned}$$

Solving these equations, we conclude that the algebra of $\mathbb{G}_m^2$-invariant polynomials in m_i is freely generated by the following ten Laurent monomials:

$$\begin{aligned} (n_1, \dots, n_{10}) = (&m_1m_2,\ m_3m_4,\ m_5m_6,\ m_7m_8,\ m_1m_2^{-1}m_5^{-2}m_6^2, \\ &m_3m_4^{-1}m_7^{-2}m_8,\ m_9m_{10}m_{11}m_{12},\ m_9m_{12}m_{10}m_{11}^{-1}, \\ &m_5^{-4}m_6^{-4}m_9m_{10}m_{11}^{-1}m_{12}^{-1},\ m_7^4m_8^4m_9m_{10}m_{11}^{-1}m_{12}^{-1}). \end{aligned}$$

Now, let us see how the finite subgroup $G_0 \cong D_8$ of G acts on this basis. We have

$$\begin{aligned} g_1 &: (n_1, \dots, n_{10}) \mapsto (n_1, n_2, n_3, n_4, n_5^{-1}, n_6, n_7, n_8^{-1}, n_9^{-1}, n_{10}), \\ g_2 &: (n_1, \dots, n_{10}) \mapsto (n_1, n_2, n_3, n_4, n_5, n_6^{-1}, n_7, n_8^{-1}, n_9, n_{10}^{-1}), \\ g_3 &: (n_1, \dots, n_{10}) \mapsto (n_2, n_1, n_4, n_3, n_6, n_5, n_7, n_8^{-1}, n_{10}^{-1}, n_9^{-1}). \end{aligned}$$

This shows that the field K is generated by ten rational functions

$$n_1 + n_2,\ n_1n_2,\ n_3 + n_4,\ n_3n_4,\ n_5 + n_5^{-1} + n_6 + n_6^{-1},\ n_5n_6 + n_5^{-1}n_6 + n_5^{-1}n_6^{-1},$$

$$n_7,\ n_8 + n_8^{-1},\ n_9 + n_{10} + n_9^{-1} + n_{10}^{-1},\ n_9n_{10} + n_9^{-1}n_{10}^{-1} + n_9n_{10}^{-1} + n_9^{-1}n_{10}.$$

This proves the assertion. □

Next, we turn to polarizations of degree 4. If $v \in \mathsf{E}_{10}$ satisfies $v^2 = 4$, then it lies in the $\mathrm{O}(E_{10})$-orbit of ω_1 or of $\omega_8 + \omega_9$. Since $\Phi(\omega_8 + \omega_9) = 1$, polarizations of this type correspond to hyperelliptic linear systems. On the other hand, $\Phi(\omega_1) = 2$ and polarizations of this type correspond to *Cossec–Verra polarizations*, which we studied in Section 3.4. The following result was proven by G. Casnati [107], but the proof is too involved to be reproduced here. It is based on the construction of degree 4 covers of the projective plane using sections of vector bundles [108].

Theorem 5.8.12 *The moduli space $\mathcal{M}_{\mathrm{Enr,CV}} := \mathcal{M}_{\mathrm{Enr},\omega_1}$ of Cossec–Verra polarized Enriques surfaces is connected and rational.*

Remark 5.8.13 We know from Section 3.4 that a general Cossec–Verra polarized Enriques surface is defined by the choice of two nonsingular cubic curves $\Delta_\pm = V(F_\pm)$ in $\mathbb{P}^2$ with nontrivial 2-torsion points $\eta_\pm$ on both. It is well-known that the Hessian curve $V(\mathrm{Hess}(F))$ of a cubic curve $V(F)$, which is defined by the Hessian determinant $\mathrm{Hess}(F)$, comes with a unique nontrivial 2-torsion point and assigning the Hessian curve to $V(F)$ establishes a birational equivalence between the projective space of plane cubics and its triple cover corresponding to pairs (C, η) consisting of a plane cubic and a nontrivial 2-torsion point on it, see also [177], Section 3.2. We conclude that $\mathcal{M}_{\mathrm{Enr,CV}}$ is isomorphic to the moduli space $\mathcal{M}$ of unordered pairs (C_1, C_2) of plane cubic curves modulo projective equivalence. Assigning to (C, C') the pencil of cubic curves spanned by C and C', we obtain that $\mathcal{M}$ is birationally equivalent to the product $\mathbb{P}^2$ (more precisely: the symmetric product of the pencil) and the 8-dimensional moduli space of pencils of plane cubics. We do not how to prove the rationality of the latter moduli space, but it follows from Casnati's result that this space is stably rational.

Next we consider polarizations of degree 6. By Corollary 1.5.4, there are two $\mathrm{O}(\mathsf{E}_{10})$-orbits of vectors $v \in \mathsf{E}_{10}$ with $v^2 = 6$. The orbit that does not correspond to hyperelliptic linear systems is the orbit containing ω_7. By Proposition 3.1.1, there are two polarizations $\mathcal{L}$ on an Enriques surface S with $\mathcal{L}^2 = 6$ and $\Phi(\mathcal{L}) = 2$, namely a bielliptic one and one that gives rise to a birational morphism. For the former, we refer to Section 3.3 and for the latter to Section 3.5. Moreover, by Theorem 3.5.1, if $\mathcal{L}$ is a polarization of the latter type, then $|\mathcal{L}|$ defines a morphism to $\mathbb{P}^3$, whose image is a sextic surface that is singular along the lines of a tetrahedron, see also Example 1.6.2. In fact, using the explicit description of these surfaces from Theorem 3.5.1, we find that the moduli space of such surfaces is isomorphic to the quotient space of the space of quadratic polynomials in four variables modulo the action of the symmetric group $\mathfrak{S}_4$ that permutes the variables. This is the key to the following result.

Theorem 5.8.14 *The moduli space $\widetilde{\mathcal{M}}_{\mathrm{Enr},\omega_7}$ is connected and rational. An open and dense subset parametrizes pairs $(S, \mathcal{L})$ of an Enriques surface S together with a degree 6-polarization $\mathcal{L}$ with $\Phi(\mathcal{L}) = 2$, such that the image of $|\mathcal{L}|$ is an Enriques sextic in $\mathbb{P}^3$.*

Proof This is an easy exercise. First, the representation of $\mathfrak{S}_4$ in the space V_{10} of quadratic polynomials in four variables $\{x_1, ..., x_4\}$ decomposes as the direct sum of representations $V_4 \oplus V_3 \oplus V_3'$, where

$$\begin{aligned}V_4 &= \langle x_1^2, x_2^2, x_3^2, x_4^2\rangle,\\ V_3 &= \langle x_1x_2 + x_3x_4, x_1x_3 + x_2x_4, x_1x_4 + x_2x_3\rangle,\\ V_3' &= \langle x_1x_2 - x_3x_4, x_1x_3 - x_2x_4, x_1x_4 - x_2x_3\rangle.\end{aligned}$$

This implies that there is an isomorphism

$$\mathbb{C}[V_{10}]^{\mathfrak{S}_4} \cong \mathbb{C}[V_4]^{\mathfrak{S}_4} \otimes \mathbb{C}[V_3]^{\mathfrak{S}_4} \otimes \mathbb{C}[V_3']^{\mathfrak{S}_4}.$$

The first space $\mathbb{C}[V_4]^{\mathfrak{S}_4}$ is isomorphic to the polynomial algebra in elementary symmetric polynomials in the x_i^2. The varieties $V_3/\mathfrak{S}_4 \cong \mathrm{Spec}\ \mathbb{C}[V_3]^{\mathfrak{S}_4}$ and $V_3'/\mathfrak{S}_4 = \mathrm{Spec}\ \mathbb{C}[V_3]^{\mathfrak{S}_4}$ are both birationally equivalent to affine cones over the rational surfaces $\mathbb{P}^2/\mathfrak{S}_4$. All of them are obviously rational varieties. □

Remark 5.8.15 A bielliptic polarization $\mathcal{L}$ of degree 6 has class $[\mathcal{L}] = \omega_7$. However, such polarizations only exist on nodal Enriques surfaces and thus, form locally closed subsets of $\mathcal{M}_{\mathrm{Enr},\omega_7}$ and $\mathcal{M}_{\mathrm{Enr},\omega_7}$ that are not dense. We refer to Table 5.7.3 at the end of Section 8.4 in Volume II for details.

Finally, there are three $\mathsf{O}(\mathsf{E}_{10})$-orbits of vectors $v \in \mathsf{E}_{10}$ with $v^2 = 10$. One of them is the orbit of the class ω_0, which corresponds to the class of a *Fano polarization*, see Section 3.5. We denote by $\mathcal{M}_{\mathrm{Enr,Fano}}$ be the component of the $\mathcal{M}_{\mathrm{Enr},\omega_0}$ corresponding to Fano polarizations. Finally, we sketch the proof of the following result due to Verra [716]:

Theorem 5.8.16 *The moduli space $\widetilde{\mathcal{M}}_{\mathrm{Enr,Fano}}$ of Fano-polarized Enriques surfaces is irreducible and unirational.*

Proof Let U be the 10-dimensional linear space of Enriques sextics given by the equations from Theorem 3.5.1. Let $T = V(t_0t_1t_2t_3)$ be the coordinate tetrahedron. Let $\ell_{ij} : t_i = t_j = 0$ be its edges and let P_i be the vertex with $t_i \neq 0$. Let h be the ample divisor class on a general Enriques surface from U that defines the sextic model. We may assume that its numerical class is equal to $f_1 + f_2 + f_3$, where (f_1, f_2, f_3) is a non-degenerate isotropic 3-sequence. Let $\mathcal{F}$ be the variety of elliptic curves of degree 5 in $\mathbb{P}^3$ that intersect the edges ℓ_{23} and ℓ_{01} with multiplicity 1 and that intersect the other edges with multiplicity 2. If $C \in \mathcal{F}$ lies on X, then its pre-image under the normalization map $S \to X$ is an elliptic curve C', such that $C' \cdot f_1 = 1, C' \cdot f_2 = C' \cdot f_3 = 2$. Then, the divisor class $\Delta = C' + f_2 + f_3$ satisfies $\Delta^2 = 10$ and $\Phi(\Delta) \geq 3$. This is a Fano polarization on S.

Claim 1: For a general $C \in \mathcal{F}$ there exists a unique sextic $X \in U$ that contains C.

To see this, we consider the 10-dimensional linear system of sextics $X_q, q \in U$. Each such surface cuts out on C a divisor of the form $2d_0 + d_q$, where d_0 is a divisor of degree 10 that is supported on the edges. Since C is an elliptic curve, the dimension of the linear series $|d_q|$ is equal to 9, hence there exists a surface X that contains C. To show that X is unique, suppose that C is contained in two such surfaces X, X'. Subtracting the normalized equations of X, X', we see that C is contained in the base locus of the pencil $\lambda Q(t_0, t_1, t_2, t_3) + \mu Q(t_0, t_1, t_2, t_3)' = 0$ of quadrics. Since a general C does not lie on a quadric, we find $X = X'$.

Claim 2: There exists a dominant and rational map : $\mathcal{F} \dashrightarrow \mathcal{M}_{\mathrm{Enr,Fano}}$.

It follows from the previous step that we have a rational map $f : \mathcal{F} \dashrightarrow U$. As we saw above, each $C \in \mathcal{F}$ determines a Fano polarization on $X_{f(C)}$. Thus, we obtain a map $\mathcal{F} \dashrightarrow \mathcal{M}_{\mathrm{Enr,Fano}}$. Since $U \dashrightarrow \mathcal{M}_{\mathrm{Enr}}$ and $\mathcal{M}_{\mathrm{Enr,Fano}} \to \mathcal{M}_{\mathrm{Enr}}$ are finite rational maps, it follows that f is dominant. Thus, the map is dominant.

Claim 3: $\mathcal{F}$ is an irreducible rational variety of dimension 10.

Let $C \in \mathcal{F}$ and let $p : \mathbb{P}^3 \dashrightarrow V(t_3) \cong \mathbb{P}^2$ be the projection map from the point P_3 to the plane $t_3 = 0$. The image of C is a plane quintic. It passes doubly through the points $q_0 = p(\ell_{03})$ and $q_1 := p(\ell_{13})$ and it passes simply through the point $q_2 = p(\ell_{23})$. It has also 3 other double points A, B, C.

Counting the dimension of variety $\mathcal{F}'$ of such plane elliptic quintics, we find that it is equal to 10. However, two curves C and C' may project to the same plane. This happens only if they differ by a projective automorphism that preserves the coordinate tetrahedron, that fixes the vertex P_3, and that acts as identity on the coordinate plane $t_3 = 0$. The set of such automorphisms is a one-parameter group of homotheties. This seems to imply that $\dim \mathcal{F} = 11$. An explanation for this paradox is that not every plane quintic from the above is the projection of a curve from $\mathcal{F}$. Verra proves in [716], Lemma 1.1 that a necessary and sufficient condition for a quintic $C' \in \mathcal{F}'$ to be a projection is that there exists a plane cubic that passes through the singular points q_0, q_1, q_2, A, B, C of C' and also through the 5 residual points of the intersection of C' with the edges ℓ_{03}, ℓ_{13}, ℓ_{23} lying in the plane $t_3 = 0$. This make $\mathcal{F}$ to be birationally equivalent to a line bundle over a 9-dimensional variety $\mathcal{F}_0'$.

Finally, it remains to prove that $\mathcal{F}_0'$ is a rational variety. First, we use that there is a natural rational map from $\mathcal{F}_0'$ to the symmetric product $\mathrm{Sym}^3(\mathbb{P}^2)$ of the plane that assigns to $\mathcal{F}'$ the set of singular points $\{A, B, C\}$. The fiber of this map $\{A, B, C\}$ is isomorphic to the 3-dimensional linear projective space of plane cubics passing through the points q_0, q_1, q_2, A, B, C. Since $\mathrm{Sym}^3(\mathbb{P}^2)$ is a rational variety, we are done. □

Remark 5.8.17 Verra uses this beautiful construction in order to give a proof of the unirationality of the moduli space $\mathcal{A}_5$ of principally polarized abelian varieties and its cover $\mathcal{R}_6$, the moduli space of genus 6 curves together with a non-trivial 2-torsion divisor class. The cover $\mathcal{R}_6 \to \mathcal{A}_5$ is given by the Prym variety construction. In order to prove this, he introduces a $\mathbb{P}^5$-bundle over $\mathcal{M}_{\mathrm{Enr,Fano}}$, whose fiber over the isomorphism class (S, Δ) is the linear system $|\Delta|$. A curve D from $|\Delta|$ is a genus six curve that comes with a non-trivial 2-torsion divisor class defined by $\omega_S \otimes \mathcal{O}_D$.

We refer to [121] for more results on unirationality of the moduli spaces $\mathcal{M}_{\mathrm{Enr},v}$. For example, if $0 < \omega^2 \leq 58$, then all known components of these spaces are unirational.

5.9 Compactifications of Moduli Spaces

In Section 5.3 and Section 5.5, we constructed the coarse moduli spaces $\mathcal{M}_{\mathrm{Enr}}$ and $\mathcal{M}_{\mathrm{Enr},v}$ of unmarked and of numerically polarized Enriques surfaces, respectively, as quotients of the form $\Gamma_{\mathrm{Enr}}\backslash\mathcal{D}^{\circ}_{\mathrm{Enr}}$ and $\Gamma_{\mathrm{Enr}}\backslash\mathcal{D}^{\circ}_{\mathrm{Enr},v}$. These moduli spaces are quasi-projective varieties, but not proper over $\mathbb{C}$. In Section 5.4, we have already found a partial compactification $\mathcal{M}^{m}_{\mathrm{Enr}}$ and $\mathcal{M}_{\mathrm{Enr}}$ and interpreted its boundary as a moduli space of Coble surfaces.

The goal of this section is to discuss compactifications of these moduli spaces in the following way: using the Baily–Borel–Satake compactification $\mathcal{D}_{\mathrm{Enr}} \subset \overline{\mathcal{D}}_{\mathrm{Enr}}$ of the period domain, we obtain a compactification

$$\Gamma_{\mathrm{Enr}}\backslash\mathcal{D}^{\circ}_{\mathrm{Enr}} \subset \Gamma_{\mathrm{Enr}}\backslash\mathcal{D}_{\mathrm{Enr}} \subset \Gamma_{\mathrm{Enr}}\backslash\overline{\mathcal{D}}_{\mathrm{Enr}},$$

which is a normal complex projective variety, such that the complement of $\Gamma_{\mathrm{Enr}}\backslash\mathcal{D}_{\mathrm{Enr}}$ in $\Gamma_{\mathrm{Enr}}\backslash\overline{\mathcal{D}}_{\mathrm{Enr}}$ has only components of dimension 0 and 1.

We refer to [173] for more background and to [34] for a thorough treatment of compactifications of symmetric domains.

Before doing so, we first briefly review some general properties of the *Baily–Borel–Satake* compactifications of bounded Hermitian symmetric domains.

We recall, for example from [301], that a *symmetric Hermitian space* is a connected complex manifold M together with a Hermitian Riemannian metric, such that every point is an isolated fixed point for some involution in the group $A(M)$ of holomorphic automorphisms of M. Both, the group $A(M)$ and its connected component of the identity $A(M)^{\circ}$, admit structure of real algebraic Lie groups. The group $A(M)^{\circ}$ acts transitively on M and the stabilizer subgroup of every point is a maximal compact subgroup K of $A(M)^{\circ}$. If a symmetric Hermitian space is not the product of two such spaces, it is said to be *irreducible*. Each such space is either a Euclidean space or a homogeneous space G/K, where G is either a simply connected non-compact (resp. compact) real Lie group and K is a maximal (resp. maximal proper) subgroup. In the former case, it is of non-compact type and in the latter case it is of compact type. Each irreducible symmetric Hermitian space G/K of non-compact type admits a natural open embedding into an irreducible symmetric Hermitian space G^c/K of compact type, where G^c is a compact real form of the complex Lie group $G_{\mathbb{C}}$. It is called the *dual compact form* of G/K. The dual compact form admits the structure of a complex homogeneous space $G_{\mathbb{C}}/P$, where P is a parabolic subgroup of $G_{\mathbb{C}}$, such that $P \cap G = K$. An irreducible symmetric Hermitian space of non-compact type also admits an open embedding into $\mathbb{C}^n$, whose image is a bounded domain $\mathcal{D}$.

It can be realized as an open subset of $\check{\mathcal{D}} = G^c/K^c$. A bounded domain obtained in this way is called a *bounded Hermitian symmetric domain*.

Example 5.9.1 The simplest example is the *upper half-plane* $\mathcal{H} = \{z = a + bi \in \mathbb{C} : b > 0\}$. In this case, it is a homogeneous space with respect to $G = \mathrm{SL}(2, \mathbb{R})$ and $K \cong \mathrm{SO}(2)$. Here, K is embedded into G as the subgroup of matrices of the form $\left(\begin{smallmatrix} a & b \\ -b & a \end{smallmatrix}\right)$. The group G acts on $\mathcal{H}$ by the Möbius transformations $z \mapsto \frac{az+b}{cz+d}$. Although $\mathcal{H}$ is not a bounded domain, it is holomorphically isomorphic to the unit disk

$$\Delta := \{z \in \mathbb{C} : |z| < 1\} \subset \mathbb{C}.$$

via the map $z \mapsto \frac{z-i}{z+i}$. In this latter case, we obtain a description of Δ as a homogeneous space with respect to $G \cong \mathrm{SU}(1, 1)$, the group of complex unimodular matrices preserving the Hermitian form $|z_1|^2 - |z_2|^2$. The subgroup K is the group of diagonal matrices in G. Since $G^{\mathbb{C}} = \mathrm{SL}(2, \mathbb{C})$ and $K^{\mathbb{C}} = \mathrm{SO}(2, \mathbb{C})$, we obtain an embedding

$$\mathcal{D} \subset \check{\mathcal{D}} = \mathrm{SL}(2, \mathbb{C})/\mathrm{SO}(2, \mathbb{C}) \cong \mathbb{P}^1_{\mathbb{C}}.$$

The upper-half plane $\mathcal{H}$ is the one-dimensional case of the *Siegel upper-half space* $\mathcal{H}_g$, which is defined as the set of complex symmetric $g \times g$ matrices with positive definite imaginary part. It is an irreducible symmetric Hermitian space of non-compact type with $(G, K) = (\mathrm{Sp}(2g, \mathbb{R}), \mathrm{U}(g))$. It can be realized as a bounded domain in $\mathbb{C}^{\frac{1}{2}g(g+1)}$ of complex symmetric $g \times g$-matrices Z with $I_g - \bar{Z} \cdot Z > 0$. The bounded domain $\mathcal{H}_g$ serves as the period space for abelian varieties of dimension g. It is of *Type III* in Cartan's classification. Its compact form is the Grassmannian of maximal isotropic subspaces in a complex symplectic space of dimension $2g$.

Example 5.9.2 This example will be of the main interest for us. Let (V, q) be a real quadratic space with signature $(2, t)$ and let $V_{\mathbb{C}}$ be its complexification. The complex manifold

$$\mathcal{D} := \{\mathbb{C}z \in \mathbb{P}(V_{\mathbb{C}}) : (z, z) = 0, (z, \bar{z}) > 0\}$$

consists of two connected components, each isomorphic to an irreducible Hermitian symmetric space G/K, where $G = \mathrm{SO}(V, q) \cong \mathrm{SO}(2, t)$ and $K = \mathrm{SO}(2) \times \mathrm{SO}(t)$. This is an example of an irreducible symmetric Hermitian spaces of *orthogonal type* or of *Type IV* in Cartan's classification of such spaces. The corresponding compact symmetric Hermitian space is

$$\check{\mathcal{D}} = \mathrm{SO}(21 - t, \mathbb{C})/\mathrm{SO}(2, \mathbb{C}) \times \mathrm{SO}(19 - t, \mathbb{C}),$$

which is isomorphic to a quadric hypersurface $V(q) \subset \mathbb{P}(V_{\mathbb{C}})$. Another model of $\mathcal{D}$ is the subset $G^+(2, V)$ of the real Grassmannian $G(2, V)$ that parameterizes positive definite subspaces of V. An isomorphism is given by assigning to $z \in \mathcal{D}$ the real plane spanned by the real and the imaginary part of z. To choose a connected component we put an orientation in this real plane.

To see that a connected component of $\mathcal{D}$ is realized as a Hermitian bounded homogeneous domain, we use coordinates $(x_1, \ldots, x_t, x_{t+1}, x_{t+2})$ in V, such that the quadric hypersurface is given by $q = x_{t+1}^2 + x_{t+2}^2 - \sum_{i=3}^{t} x_i^2$. Let $z_i = x_i + iy_i$ be the corresponding coordinates in $V_{\mathbb{C}}$. Choose a connected component of $\mathcal{D}$ by requiring that $\operatorname{Im}\frac{z_{t+1}}{z_{t+2}} > 0$. Then, the map is given by the formula

$$w_1 = \frac{z_1 + iz_2}{z_{t+1} + iz_{t+2}}, \quad w_2 = \frac{z_1 - iz_2}{z_{t+1} + iz_{t+2}}, \quad \text{and} \quad w_k = \frac{z_k}{z_{t+1} + iz_{t+2}},\ k = 3, \ldots, t,$$

see [598, Chapter 2, §8], and [634, Appendix, §6]. The image is a domain in $\mathbb{C}^t$ given by inequalities

$$|w_1|^2 + |w_2|^2 + 2\sum_{k=3}^{t} |w_k|^2 \ < \ 1 + |w_1 w_2 + w_3^2 + \cdots + w_t^2| \ < \ 2.$$

As we have seen in Section 5.3, the period spaces for K3 surfaces and Enriques surfaces are examples of such spaces $\mathcal{D}$ with $t \leq 19$. We keep calling $\mathcal{D}$ a *period domain*. If $t = 1$, then $\mathcal{D}$ coincides with the upper half-plane $\mathcal{H}$ from the previous example.

Being isomorphic to $G_{\mathbb{C}}/P$, the compact form $\check{\mathcal{D}}$ of $\mathcal{D} = G/K$ is a projective algebraic variety. Thus, the open embedding $\mathcal{D} \subset \check{\mathcal{D}}$ could be viewed as a compactification of $\mathcal{D}$. However, this compactification has no geometric meaning, which is why we proceed to define a smaller partial compactification of $\mathcal{D}$, which leads to a compactification of an arithmetic quotient of the period domain.

A subset F of the boundary $\partial\mathcal{D} := \check{\mathcal{D}}\backslash\mathcal{D}$ is called a *boundary component* if it satisfies the following properties:

1. F is an analytic subset of $\partial\mathcal{D}$ in an open neighborhood of each of its points,
2. any holomorphic curve in $\partial\mathcal{D}$ that intersects F is entirely contained in F, and
3. F is minimal with respect to the previous properties.

An equivalent definition is that F is a minimal analytic subset of the boundary, such that any two points lie in the image of a holomorphic map from the unit disk to F.

The group G acts transitively on the set of boundary components and the stabilizer subgroup of a boundary component F is a maximal parabolic subgroup G_F of G (this means that the Zariski closure in $G(\mathbb{C})$ is a parabolic subgroup). The assignment

$$\begin{array}{ccc} \left\{\text{boundary components of } \mathcal{D} \subset \check{\mathcal{D}}\right\} & \to & \{\text{maximal parabolic subgroups of } G\} \\ F & \to & G_F \ := \ \{g \in G \mid g(F) = F\} \end{array}$$

is a bijection of sets.

A choice of boundary component of a non-compact Hermitian symmetric domain $\mathcal{B}$ gives a realization of $\mathcal{B}$ as a certain *Siegel domain* in a complex affine space. It is given by the data $(U, \mathcal{E}, W, C^+, B, H)$, where U is a real linear space, where $\pi : E \to B$ is a complex vector bundle over a bounded domain B in a complex linear space W, where C^+ is an open convex cone in U, and where H is a semi-hermitian form on $\mathcal{E}$ with values in the trivial vector bundle $(U_{\mathbb{C}})_B$ (a semi-hermitian form is the sum of a hermitian form and a symmetric bilinear form). A Siegel domain associated to this data is the set

$$\mathcal{S} := \{(x+iy, v) \in U_{\mathbb{C}} \times \mathcal{E} : y - \mathrm{Re}(H_{\pi(v)}(v,v)) \in C^+\}. \tag{5.9.1}$$

There are three kinds of Siegel domains:

1. In the case where $W = \mathcal{E} = \{0\}$, we obtain

$$\mathcal{S} = \{x+iy : y \in C^+\},$$

which is called a *Siegel domain of the first kind*. An example of such a domain is the Siegel half-space $\mathcal{H}_g$, where C^+ is the cone of positive definite symmetric matrices.
2. In the case where $W = \mathcal{E} = E \times \{0\}$ and H is a Hermitian form, we obtain

$$\mathcal{S} = \{(x+iy, v) \in U_{\mathbb{C}} \times E : y - H(v,v) \in C^+\},$$

which is called a *Siegel domain of the second kind*. One can show that such a domain is holomorphically isomorphic to the unit ball in $\mathbb{C}^n$, where $n = \dim U + \dim E$.
3. In the case where $\dim B > 0$, a Siegel domain is called a *Siegel domain of the third kind*.

We are now interested in the case of a period space $\mathcal{D}$ (more precisely, their connected components) associated with a real quadratic space (V,q) of signature $(2,t)$, that is, we consider

$$\mathcal{D} = \{\mathbb{C}z \in \mathbb{P}(V_{\mathbb{C}}) : (z,z) = 0, (z,\bar{z}) > 0\}$$

as introduced in Example 5.9.2.

A boundary component corresponds to a parabolic subgroup of $\mathrm{SO}(V)$. Such groups are stabilizer subgroups of flags of isotropic subspaces in V. Since the signature of V is $(2,t)$, there are three types of such flags: lines, planes, and lines contained in planes. The first two types define maximal parabolic subgroups. More precisely, we can describe the boundary components as follows: the boundary of $\mathcal{D}$ consists of points $\mathbb{C}z \in \check{\mathcal{D}} \subset \mathbb{P}(V_{\mathbb{C}})$ with $(z,z) = (z,\bar{z}) = 0$. If we write $z = x+iy$, then this means that $x^2 = y^2 = (x,y) = 0$. Thus x, y span an isotropic subspace of dimension 1 or 2.

1. In the first case, we have $\mathbb{C}z = \mathbb{C}f$, where f is an isotropic vector representing a real point on the quadric $\check{\mathcal{D}}$. This is a 0-dimensional boundary component.
2. In the second case, x, y span an isotropic plane J in V and the closure of the boundary component F is equal to $|J_{\mathbb{C}}| \cong \mathbb{P}^1$. The conjugation involution $z \mapsto \bar{z}$ switches the two connected components of $\mathcal{D}$, so the intersection of $|J_{\mathbb{C}}|$ with one of the components, say $\mathcal{D}^+$, is the upper half-plane $\{x+iy, u > 0\}$, and the intersection with the other component $\mathcal{D}^-$ is the lower half-planne $\{x+iy, y < 0\}$. This gives rise to a 1-dimensional boundary component.

Next, we realize $\mathcal{D}^{\pm}$ as Siegel domains. Below, we will use this to describe neighborhoods of an arithmetic quotient of $\mathcal{D}^{\pm}$ near the boundary components.

First, let $I = \mathbb{R}f$ be an isotropic line corresponding to a 0-dimensional boundary component F. The tangent hyperplane of the quadric $Q = \check{\mathcal{D}}$ at the point $[f]$ is given by the linear function $v \to (v, f)$ that vanishes on $|(I^{\perp})_{\mathbb{C}}|$. Since any point $z \in \mathcal{D}$ corresponds to a real positive definite subspace (spanned by the real and the imaginary part of $z \in V_{\mathbb{C}}$), it does not lie in the tangent hyperplane. Projecting Q from the point $[f]$, we obtain an open embedding

$$\pi_I : \mathcal{D} \hookrightarrow A_I = |(V/I)_{\mathbb{C}}| - |(I^{\perp}/I)_{\mathbb{C}}| = \{z = x + iy \in (V/I)_{\mathbb{C}} | (x, f) = 1, (y, f) = 0\}.$$

A point $z \in \mathcal{D}$ can be represented by a vector $\lambda f + x + iy$ with x, y as above. From $(z, z) = 0$, we find $(x, x) - (y, y) + 2\mathrm{Re}(\lambda) = 0$ and from $(z, \bar{z}) > 0$, we find $(x, x) + (y, y) + \mathrm{Re}(\lambda) > 0$. This gives $(y, y) > 0$ and conversely, if $(y, y) > 0$, then we get $(z, z) = 0$ and $(z, \bar{z}) > 0$. Thus, we obtain that a choice of a 0-dimensional boundary component defines an isomorphism

$$\pi_I : \mathcal{D}^+ \to \mathcal{T}_I := \{x + iy \in V/I + i(I^{\perp}/I) : (x, f) = 1, y \in C^+\},$$

where C^+ is connected component of the cone $C = \{y \in (I^{\perp}/I)_{\mathbb{R}} : (y, y) > 0\}$. The real part $\{x \in V/I : (x, f) = 1\}$ is a real affine space associated to the vector space $(I^{\perp}/I)_{\mathbb{R}}$.

If $t > 0$, that is, if the quadratic space (V, q) is not positive definite (otherwise $\mathcal{D}^+$ is a singleton), then we can choose an isotropic vector g with $(f, g) = 1$, we can identify both the real and the imaginary parts with the linear subspace $U = \langle f, g\rangle^{\perp}$ of V, so that $\mathcal{T}_I$ becomes a Siegel domain of the first kind. This is also a special case of a *tube domain*, that is, a subset of $\mathbb{C}^n$ of the form $\{x + iy : y \in C\}$, where C is a convex open subset in $\mathbb{R}^n$.

Now let us look at 1-dimensional rational boundary component $F \subset |J_{\mathbb{C}}|$, where $J \subset V$ is an isotropic plane. Since J is negative definite, we find $\mathcal{D}^+ \cap |J_{\mathbb{C}}| = \emptyset$. Thus, we can project $\mathcal{D}^+$ to $|(V/J)_{\mathbb{C}}| \cong \mathbb{P}^{t-1}$. The fibers of the projection of Q from the line $|J_{\mathbb{C}}|$ are lines, namely the residual lines of intersections of Q with planes containing the line. The fibers of $\pi_I : \mathcal{D}^+ \to |(V/J)_{\mathbb{C}}|$ are isomorphic to upper half-planes. Next we project $|(V/J)_{\mathbb{C}}|$ to $|(V/J^{\perp})_{\mathbb{C}}| \cong |J_{\mathbb{C}}^{\vee}| \cong \mathbb{P}^1$ from the subspace $|(J^{\perp}/J)_{\mathbb{C}}|$. The image of $\mathcal{D}^+$ under the composition of the projections will be our boundary component F. The fibers of the second projection $\pi_{J^{\perp}} : \pi_J(\mathcal{D}^+) \to F$ are vector spaces of dimension $t - 2$ isomorphic to $(J^{\perp}/J)_{\mathbb{C}}$. As a result, we find an isomorphism from $\mathcal{D}^+$ to an upper half-plane bundle over a complex vector bundle of rank $t - 2$ over F. One can show that this gives a realization of $\mathcal{D}^+$ as a Siegel domain of the third kind.

The Siegel domain realizations allow us to describe the neighborhoods of an arithmetic quotient of $\mathcal{D}^+$ near the images of the boundary components. To do this, we first have to describe the stabilizers of the boundary components.

Lemma 5.9.3 *Let F be a 0-dimensional boundary component corresponding to a one-dimensional isotropic subspace $I = \mathbb{R}f$. Then, there is a homomorphism*

$G_F \to \mathrm{O}(I^\perp/I)$, whose kernel Z_F is isomorphic to $I^\perp/I$, where each $w \in I^\perp/I$ is identified with a transformation of V

$$E_{w,f} \ : \ v \ \mapsto \ v - (w,v)f + (v,f)w - \frac{1}{2}(w,w)(v,f)f$$

a so-called Siegel–Eichler transformation.

Proof One checks immediately that each $E_{w,f}$ is an orthogonal transformation of V that leaves I invariant and that induces transformations $v \mapsto v - (v,f)f$ on $I^\perp$. Thus, $I^\perp/I \subset Z_F$. If $\sigma \in Z_F$, then for any $v \in I^\perp$, we have $\sigma(v) = v - \alpha(v)f$, where $\alpha \in (I^\perp)^\vee$. Since $\sigma \in \mathrm{SO}(V)$, it acts as identity on I. Hence, $\alpha \in (I^\perp/I)^\vee$ and it can be written as $\alpha(v) = (w,v)$. This shows that σ coincides with $E_{w,f}$ on $I^\perp$. Choosing some splitting (not orthogonal) $V = I \oplus I^\perp$, we see that the only possible orthogonal extension of σ to the whole V is equal to $E_{w,f}$ for some $w \in I^\perp/I$. □

Lemma 5.9.4 *Let F be a 1-dimensional boundary component corresponding to a two-dimensional isotropic subspace J of V. Then, there is a homomorphism $G_F \to$* $\mathrm{GL}(J) \times \mathrm{O}(J^\perp/J)$*, whose kernel Z_F in its action on J preserves the flag* $0 \subset J \subset J^\perp$*, and whose center $Z_F^0 \cong \Lambda^2 J \cong \mathbb{R}$ is the subgroup that acts trivially on the quotients of the flag.*

Proof For any $\sigma \in Z_F$, the restriction of $\sigma - \mathrm{id}$ to $J^\perp$ is the identity on J and it defines a homomorphism $J^\perp/J \to J$. Thus, we have a homomorphism $Z_F \to \mathrm{Hom}(J^\perp/J, J) \cong (J^\perp/J) \otimes J \cong \mathbb{R}^{2t}$. Using some splitting $V = J \oplus J^\perp$, one can show that the homomorphism is surjective. Its kernel Z_F^0 consists of transformations that induce the identity on $J^\perp/J$. Each such transformation can be written in the form $v \mapsto\mapsto v + \alpha(v,f')f + \beta(v,f)f'$ in some basis (f,f') of J. It depends only on the image of $f \wedge f' \in \Lambda^2 J$ and thus, it is isomorphic to this group. □

Remark 5.9.5 It follows from this lemma that we have a non-trivial and central extension of groups

$$1 \to Z_F^0 \to Z_F \to (J^\perp/J) \otimes J \to 1, \tag{5.9.2}$$

so that Z_F is a non-abelian nilpotent group, a *Heisenberg group*.

In order to define an action on an arithmetic group Γ on $\mathcal{D}$, one has to put a $\mathbb{Q}$-structure on the group G. We do it by fixing a sublattice N in V of signature $(2,t)$, so that V can be identified with $N_\mathbb{R}$. An *arithmetic subgroup* of G is a subgroup Γ of finite index of $\mathrm{O}(N)$. Recall that this means that Γ is a subgroup of $G(\mathbb{Q})$, which is commensurable with a subgroup of $G(\mathbb{Z})$. An arithmetic subgroup of G is a discrete subgroup, but the converse is in general not true.

We will call a boundary component F *rational* if G_F is defined over $\mathbb{Q}$. This is equivalent to the isotropic subspace corresponding to F being of the form $J_\mathbb{R}$, where J is a primitive isotropic sublattice of N. Thus, we have a bijection between rational boundary components of dimension 0 (resp. 1) and primitive isotropic sublattices of rank 1 (resp. 2). To distinguish the two different types of a rational boundary domain, we will use I for a rank one primitive isotropic sublattice and we will use J otherwise.

For an arithmetic subgroup Γ, we set

$$\Gamma_F \ := \ G_F(\mathbb{Q}) \ \cap \ \Gamma,$$

we denote by $\partial_{\mathrm{rat}}\mathcal{D}$ the union of all rational boundary components, and we set

$$\mathcal{D}^{\mathrm{c}} \ := \ \mathcal{D} \ \cup \ \partial_{\mathrm{rat}}\mathcal{D}.$$

Thus, we have

$$\Gamma\backslash\mathcal{D}^{\mathrm{c}} \ = \ \Gamma\backslash\mathcal{D} \ \cup \ \{\Gamma_F\backslash F \ : \ F \text{ is a rational boundary component}\}\,.$$

By a theorem of Cartan, the space $\mathcal{D}^{\mathrm{c}}$ can be given a topology, such that $\Gamma\backslash\mathcal{D}^{\mathrm{c}}$ is a Hausdorff and compact topological space and such that it is the topological space underlying a normal analytic space containing $\Gamma\backslash\mathcal{D}$ as an open and dense analytic subset.

Next, let $\mathbb{L}$ be the line bundle on $Q = \check{\mathcal{D}} \subset |N_{\mathbb{C}}|$ that arises as pull-back of $\mathcal{O}(1)$ of the ambient projective space. There is an $\mathrm{O}(N_{\mathbb{C}})$-action on $|N_{\mathbb{C}}|$, which gives rise to an action on Q and therefore, all line bundles $\mathbb{L}^{\otimes k}$ are equivariant with respect to the group $\mathrm{O}(N_{\mathbb{C}})$. The canonical bundle of Q satisfies $\omega_Q \cong \mathbb{L}^{\otimes t}$ and hence, sections of $\mathbb{L}^{\otimes k}$ restricted to $\mathcal{D} \subset Q$ can be viewed as *automorphic forms* of weight $\frac{k}{t}$. We refer to $\mathbb{L}$ as the *automorphic line bundle*.

By a theorem of Baily–Borel [44], the automorphic line bundle descends to the quotient $\Gamma\backslash\mathcal{D}$ and it can be extended to an ample line bundle on $\Gamma\backslash\mathcal{D}^{\mathrm{c}}$. It particular, this shows that it is a projective algebraic variety. Moreover, it even carries the structure of a normal projective algebraic variety, which is isomorphic to the projective spectrum of a certain graded ring of automorphic forms. This is the *Baily–Borel compactification* of $\Gamma\backslash\mathcal{D}$ and we will be denote it by $(\Gamma\backslash\mathcal{D})^{\mathrm{BB}}$.

Example 5.9.6 We return to the upper half-plane $\mathcal{H}$, which we already studied in Example 5.9.1. It is easy to see that a boundary component of $\mathcal{H} \subset \mathbb{C}$ is defined by points $z = a + bi$ with $b = 0$ and the point $\infty \in \mathbb{P}^1(\mathbb{C})$. The group $G = \mathrm{SL}(2, \mathbb{R})$ acts transitively on the set of boundary components. The stabilizer subgroup G_∞ of ∞ is the parabolic subgroup of matrices $\left(\begin{smallmatrix} 1 & b \\ 0 & 1 \end{smallmatrix}\right)$. A rational boundary component is simply a point with rational coordinates on the real line and the point ∞. If $\Gamma = \mathrm{SL}(2, \mathbb{Z})$, then Γ_F consists of matrices as above where b is an integer.

The group $\mathrm{SL}(2, \mathbb{Z})$ acts transitively on the set of rational boundary components and thus, all Γ_F's, where F runs through the rational boundary components, are conjugate by elements from $\mathrm{SL}(2, \mathbb{Z})$. Now, let Γ be a subgroup of $\mathrm{SL}(2, \mathbb{Z})$ of finite index, which is an arithmetic subgroup. Then, $X := \Gamma\backslash\mathbb{H}$ is an affine curve over $\mathbb{C}$, and the Baily–Borel compactification $\overline{X} := \overline{\mathbb{H}}/\Gamma$ is a smooth projective curve over $\mathbb{C}$, the *modular curve* of level Γ.

The boundary $\partial\overline{X} := \overline{X}\backslash X$ is a finite set of points, the so-called *cusps*, whose number is equal to the index of $\Gamma_F \cap \Gamma$ inside $\Gamma \cap \mathrm{SL}(2, \mathbb{Z})$. For example, if Γ is one of the congruence subgroups

$$\Gamma_0(n) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \ c \equiv 0 \mod n \right\} \subset \mathrm{SL}(2, \mathbb{Z}),$$

then $X_0(n) := \overline{X} = \overline{\mathbb{H}}/\Gamma_0(n)$ is called the *modular curve of level n*. We refer to [673, Chapter 1] for the genus of $X_0(N)$ and the number c_n of cusps, as well as proofs and details.

We now return to the period space $\mathcal{D}$ associated to a real quadratic space (V, q) of signature $(2, t)$. We fix a lattice $N \subset V$, which equips the orthogonal group $\mathrm{O}(V)$ with a $\mathbb{Q}$-structure. We let Γ be a subgroup of finite index in $\mathrm{O}(N)'$. Then, the two previous Lemmas 5.9.3 and 5.9.4 give the following structure of the groups Γ_I, Γ^I, Γ_J, and Γ^J.

$$\begin{aligned} \Gamma^I &= I^\perp/I \quad \cong \quad \mathbb{Z}^t, \\ \Gamma_I &= \Gamma^I \rtimes \overline{\Gamma}_I, \end{aligned}$$

where $\overline{\Gamma}_I$ is a subgroup of finite index in $\mathrm{O}(I^\perp/I)$.

$$\begin{aligned} Z(\Gamma^J) &= \Lambda^2 J \quad \cong \quad \mathbb{Z}, \\ \Gamma^J/Z(\Gamma^J) &= J^\perp/J \quad \cong \quad \mathbb{Z}^{t-2}, \\ \Gamma_J &= \Gamma^J \rtimes \overline{\Gamma}^J, \end{aligned}$$

where $\overline{\Gamma}^J$ is a finite subgroup of $\mathrm{O}(J^\perp/J)$. We consider the quotient map

$$\mathcal{D}_N \to \Gamma\backslash\mathcal{D}_N,$$

which extends to a continuous map $\mathcal{D}_N^c \to (\Gamma\backslash\mathcal{D}_N)^{\mathrm{BB}}$.

First, assume that $F = |I|$ is a 0-dimensional rational boundary component. The group Γ_I (or its subgroup of index 2) preserves the tube domain realization $\mathcal{T}_I = \pi_I(\mathcal{D}_N^+)$. The subgroup Γ^I acts trivially on the subspace $(I^\perp/I)_\mathbb{R}$, but acts on the affine space A_I by real translations. If we choose coordinates in V, such that the quadratic form in $(I^\perp/I)_\mathbb{R}$ is given by $y_1^2 - y_2^2 - \cdots - y_t^2$, then we see that C^+ is given by inequalities $y_i > 0$. Using the vector exponential map $\exp(2\pi i z)$, the quotient $\Gamma^I\backslash\mathcal{T}_I$ becomes isomorphic to the product $(\Delta^*)^t$, where Δ^* is the punctured unit disk. It is an open subset U_I of the complex algebraic torus $(I^\perp/I)_\mathbb{C}/\Gamma^I \cong (\mathbb{C}^*)^t$. The group Γ_I acts on U_I via its action on the image $e^{2\pi C^+}$ of the cone C^+ in the purely imaginary part of $\mathcal{T}_I$.

Next, assume that $F \subset |J|$ is a 1-dimensional boundary component. The center of Γ^J is isomorphic to $\mathbb{Z}$ and it acts on the upper half-plane bundle with quotient isomorphic to a punctured unit disk bundle. The lattice $\Gamma^J/Z(\Gamma^J)$ acts on the vector bundle with quotient a compact complex torus bundle. One can show that the punctured unit disk bundle embeds into a principal $\mathbb{C}^*$-bundle, whose associate line bundle $\mathcal{L}$ is anti-ample. The line bundle $\mathcal{L}^{-1}$ and the bracket skew-symmetric bilinear form $\Gamma^J \times \Gamma^J \to Z(\Gamma^J)$ defined by the extension class of the exact sequence (5.9.2) is the first Chern class of $\mathcal{L}^{-1}$. This gives a structure of a smooth family of abelian varieties of dimension $(t-2)$ for the map

$$(\Gamma^J/Z(\Gamma^J))\backslash\pi_J(\mathcal{D}_N) \to \Gamma^J\backslash\pi_{J^\perp}(\mathcal{D}_N) \cong \Gamma^J\backslash F.$$

Finally, the finite group $\overline{\Gamma}_J$ acts on the family and we have to take the quotient by this group.

A *toroidal compactification* $(\Gamma\backslash\mathcal{D}_N)^{\mathrm{trd}}$ is a certain blow-up of the Baily–Borel compactification. First, in a neighborhood of a *cusp* (the image of a 0-dimensional rational boundary component), one partially compactifies the open subset of the torus T_I by using a toric compactification $X(\Sigma_J)$, where Σ_J is a certain (admissible) fan of rational polyhedral cones inside of C^+. Here, a choice of fan Σ_J has to be made. Over the images of 1-dimensional components, one adds to the punctured unit disk bundle the zero section of the line bundle $\mathcal{L}$. Then, the map

$$\sigma_J : (\Gamma^J\backslash\mathcal{D}_N)^{\mathrm{trd}} \to (\Gamma^J\backslash\mathcal{D}_N)^{\mathrm{BB}}$$

blows down the zero section to the modular curve $\bar{F} = \Gamma^J\backslash F$. Finally, we have to take a finite quotient by $\overline{\Gamma}_J$. All these local constructions can be glued together (here one uses the definition of the admissibility of the fans Σ_J). As a result, we obtain a birational morphism

$$\sigma : (\Gamma\backslash\mathcal{D}_N)^{\mathrm{trd}} \to (\Gamma\backslash\mathcal{D}_N)^{\mathrm{BB}}.$$

The fibers of the map σ over a general point of the image of a 1-dimensional boundary component are (finite quotients) of polarized abelian varieties of dimension $(t-2)$. The fibers over the cusps are their degenerations.

A *semi-toric compactification* as introduced by Looijenga is a certain version of the toroidal compactification. In it, the lattice $J^\perp/J$ is replaced with a sublattice M that contains J. This results in enlarging the boundary of the toroidal compactification. More precisely, an appropriate choice of fan $X(\Sigma_M)$ leads to the blow-up of a Weil divisor on the Baily–Borel compactification. In all known applications, one takes for M the primitive hull of $J + (J^\perp \cap \mathcal{A})$, where $\mathcal{A}$ is the set of vectors in N of negative norm forming a finite set of Γ-orbits. It defines an arrangement of hyperplanes in $\mathcal{D}_N$. We refer to [476] for the details.

We now apply this general machinery to our period space $\mathcal{D}$. To obtain a rational structure, we use (of course) the lattice $N = \mathsf{E}_{10}(2)^\perp = \mathsf{E}_8(2) \oplus \mathsf{U}(2) \oplus \mathsf{U}$ and $\Gamma = \mathrm{O}(\mathsf{E}_{10}(2)^\perp)^\sharp$ or $\Gamma = \mathrm{O}(\mathsf{E}_{10}(2))'$.

To understand the 0-dimensional rational boundary components, we have to understand primitive isotropic vectors. Let (f, g) (resp. (f', g')) be the standard basis of the summand U (resp. $\mathsf{U}(2)$) that consists of isotropic vectors. It is clear that the vectors f and f' belong to two different $\mathrm{O}(N)'$ orbits of primitive isotropic vectors in N. Thus, there are at least two rational 0-dimensional boundary components. The complete picture is given by the following result.

Proposition 5.9.7 *Every primitive isotropic vector of N is either $\mathrm{O}(N)'$-equivalent to f or to f'. In particular, there are two Γ_{Enr}-orbits of rational 0-dimensional boundary components of $\mathcal{D}_{\mathrm{Enr}}$.*

There are also two orbits of primitive isotropic planes and hence, two Γ_{Enr}-orbits of 1-dimensional rational boundary components of $\mathcal{D}_{\mathrm{Enr}}$.

Proof We apply Lemma 5.4.1 to replace the lattice K with the odd unimodular lattice $\mathsf{U} \oplus \mathsf{E}_8 \oplus \mathsf{I}^{1,1}$ and follow the proof of Corollary 3 from [6]. By construction of K, there is a bijection between the orbits of primitive vectors in N and in K. Any primitive isotropic vector v in an odd unimodular indefinite lattice $\mathsf{I}^{p,q}$ defines the lattice $M_v = \langle v\rangle^\perp/\langle v\rangle$, which is unimodular of signature $(p-1, q-1)$. The lattice M_v embeds into $\mathsf{I}^{p,q}$ with orthogonal complement a hyperbolic plane containing v. This defines a bijection between orbits of primitive isotropic vectors in $\mathsf{I}^{p,q}$ and isomorphism classes of unimodular lattices of signature $(p-1, q-1)$. Applying this to our case, we see that we have two orbits corresponding to lattices E_{10} and $\mathsf{I}^{1,9}$. This proves the first assertion.

Next, consider a primitive isotropic plane P in K and let P^* be the set of primitive isotropic vectors in P. We claim that P^* contains an odd vector. If this were not true, then the orthogonal complement of an even vector in P^* contains a sublattice $A \cong \mathsf{E}_{10}$. Since K is an odd lattice, the orthogonal complement of A is isomorphic to $\mathsf{I}^{1,1}$. The intersection $P \cap A$ contains a vector v with $\langle v\rangle^\perp$ containing a sublattice isomorphic to $\mathsf{I}^{1,1}$, so that $\langle v\rangle^\perp$ is odd and hence, $v \in P^*$ is odd. Since odd primitive isotropic vectors form one orbit, any primitive isotropic plane can be transformed to a plane P' containing v. Let $\langle w\rangle$ (resp. $\langle w'\rangle$) be the image of P (resp. P') in $M = \langle v\rangle^\perp/\langle v\rangle \cong \mathsf{I}^{1,9}$. There are two $\mathrm{O}(M)$-orbits of such sublattices corresponding to whether w or w' is odd or even. If w and w' are in the same $\mathrm{O}(M)$-orbit, then we can transform P' to P. This gives one orbit. The other orbit is represented by a plane P' that is spanned by an odd and an even vector. □

The previous proposition allows us to give an explicit description of the Baily–Borel compactification of $\mathcal{M}_{\mathrm{Enr}}$.

Theorem 5.9.8 *There exist immersions*

$$\mathcal{M}_{\mathrm{Enr}} = \Gamma_{\mathrm{Enr}}\backslash\mathcal{D}^\circ_{\mathrm{Enr}} \subset \Gamma_{\mathrm{Enr}}\backslash\mathcal{D}_{\mathrm{Enr}} \subset \overline{\mathcal{M}}_{\mathrm{Enr}} := \Gamma_{\mathrm{Enr}}\backslash\overline{\mathcal{D}}^\circ_{\mathrm{Enr}}$$

of complex quasi-projective varieties, such that:

1. *$\Gamma_{\mathrm{Enr}}\backslash\mathcal{D}_{\mathrm{Enr}} - \mathcal{M}_{\mathrm{Enr}}$ is an irreducible divisor, which is isomorphic to $\mathcal{M}_{\mathrm{Cob}}$.*
2. *The boundary $\overline{\mathcal{M}}_{\mathrm{Enr}} - (\mathcal{M}_{\mathrm{Enr}} \cup \mathcal{M}_{\mathrm{Cob}})$ is the union of two curves, one isomorphic to the modular curve $X_0(2) \cong \mathbb{P}^1$ and the another one isomorphic to the modular curve $X \cong \mathbb{P}^1$ corresponding to the group $\mathrm{SL}(2,\mathbb{Z})$. The two cusps of the $X_0(2)$ coincide with the two cusps of $\overline{\mathcal{M}}_{\mathrm{Enr}}$. The two curves intersect at one of the cusps.*

Proof By Theorem 5.3.9 and Corollary 5.4.6, it only remains to show claim 2. By Proposition 5.9.7, there are two 0-dimensional boundary components p_1 and p_2 and they correspond to the Γ_{Enr}-orbits of the isotropic vectors $f \in \mathsf{U}$ and $f' \in$

$\mathsf{U}(2)$, respectively. Moreover, by the same proposition, there are two 1-dimensional boundary components C_1 and C_2 and they correspond to Γ_{Enr}-orbits of two isotropic planes P_1 and P_2. One easily sees that they can be represented by the planes $\langle f, f'\rangle$ and $\langle f, f'+g'+\alpha\rangle$, where $\alpha \in (\mathsf{E}_8)_{-4}$.

To determine the structure of $\overline{C}_i$, we have to compute the groups $N_{\Gamma_{\mathrm{Enr}}}(P_i)/Z_{\Gamma_{\mathrm{Enr}}}(P_i)$.

Case 1: $P_1 = \langle f, f'\rangle$. Given $\sigma \in N_{\Gamma_{\mathrm{Enr}}}(P_1)$, there exist integers a, b, c, d such that

$$\sigma(f') = af' + cf \quad \text{and} \quad \sigma(f) = bf' + df.$$

Since $f \cdot N = \mathbb{Z}$ and $f' \cdot N = 2\mathbb{Z}$, we find $d \notin 2\mathbb{Z}$ and $c \in 2\mathbb{Z}$. Thus, we have a homomorphism

$$\begin{aligned} \varphi : N_{\Gamma_{\mathrm{Enr}}}(P_1) &\to \Gamma_0(2) \\ \sigma &\mapsto \begin{pmatrix} a & b \\ c & d \end{pmatrix}. \end{aligned}$$

Since $P^\perp/P \cong \mathsf{E}_8(2)$, we can write N as

$$N = \mathsf{U} \oplus \mathsf{U}(2) \oplus (P^\perp/P),$$

and thus, given a matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(2)$, we can define an isometry σ_A by setting it to be the identity on $P^\perp/P$ and by setting it to be $f' \mapsto af' + cf$, $f \mapsto bf' + df$, $g' \mapsto -\frac{c}{2}g$, and $g \mapsto ag - \frac{c}{2}g'$ on $\mathsf{U} \oplus \mathsf{U}(2)$. Then, $\varphi(\sigma_A) = A$, which shows that φ is surjective. By definition, the kernel of φ is $Z_{\Gamma_{\mathrm{Enr}}}(P_1)$. Thus, $\overline{C}_1$ is isomorphic to the modular curve $X_0(2) = \overline{\Gamma_0(2)\backslash\mathbb{H}}$. We recall that the modular curve $X_0(2)$ is a compactification of the moduli space of elliptic curves together with a choice of a non-trivial 2-torsion point. It is isomorphic to $\mathbb{P}^1$ and the boundary consists of two cusps, see, for example, [673].

Case 2: We may choose P to be $\langle f', 2f + 2g + \alpha\rangle$, where $\alpha \in \mathsf{E}_8(2)$ is of norm -8. Since this plane is generated by isotropic vectors v with $v \cdot N \subset 2\mathbb{Z}$, it does not belong to the orbit of the plane $\langle f, f'\rangle$. This time, we have a surjective homomorphism $N_{\Gamma_{\mathrm{Enr}}}(F) \to \mathrm{SL}(2,\mathbb{Z})$. In this case, the boundary component is the modular curve $X = \overline{\mathrm{SL}(2,\mathbb{Z})\backslash\mathcal{H}}$ with one cusp.

Since we have only two 0-dimensional boundary components, we see that the two boundary components intersect at one point. □

Corollary 5.9.9 *The number of zero–dimensional (resp. one–dimensional) boundary components in $\mathcal{M}^m_{\mathrm{Enr}}$ is equal to* $2 \cdot 17 \cdot 31$ *(resp.* $2^2 \cdot 3^3 \cdot 5 \cdot 17 \cdot 31$*).*

Proof The group $\Gamma_{\mathrm{Enr}}/\Gamma^\sharp_{\mathrm{Enr}} \cong D(\mathsf{E}_{10}(2)) \cong \mathrm{O}(10,\mathbb{F}_2)^+$ acts on the set of $\Gamma^\sharp_{\mathrm{Enr}}$-orbits of primitive isotropic vectors (resp. planes) with stabilizer subgroup isomorphic to the stabilizer subgroup of an isotropic vector (resp. isotropic line) in the even quadratic space $\mathbb{F}_2^{10}$. It is isomorphic to $2^8 \rtimes \mathrm{O}(8,\mathbb{F}_2)^+$ (resp. $2^{16} \rtimes \mathrm{O}(6,\mathbb{F}_2)^+$). This implies that the number orbits of 0-dimensional (resp. 1-dimensional) boundary components is equal to $2[\#\mathrm{O}(10,\mathbb{F}_2)^+ : 2^8\#\mathrm{O}(8,\mathbb{F}_2)^+] = 2 \cdot 17 \cdot 31$ (resp. $2[\#\mathrm{O}(10,\mathbb{F}_2)^+ : 2^{12}\#\mathrm{O}(6,\mathbb{F}_2)^+] = 2^2 \cdot 3^3 \cdot 5 \cdot 17 \cdot 31$). □

Corollary 5.9.10 *Let $\overline{\mathcal{M}}_{\mathrm{Cob}}$ be the closure of $\mathcal{M}_{\mathrm{Cob}}$ in $\overline{\mathcal{M}}_{\mathrm{Enr}}$. Then, the boundary has a unique zero–dimensional component and a unique one–dimensional component, which is isomorphic to the modular curve $X = \overline{\mathrm{SL}(2,\mathbb{Z})\backslash\mathcal{H}}$.*

Proof The orthogonal complement of any $\delta \in (\mathsf{E}_{10}(2)^{\perp})_{-2}$ contains primitive isotropic vectors only of even type (that is, the corresponding linear form takes values in $2\mathbb{Z}$). Thus, the closure of $\mathcal{M}_{\mathrm{Cob}}$ misses the boundary components isomorphic to $X_0(2)$. □

Theorem 5.9.11 *The moduli spaces $\mathcal{M}^m_{\mathrm{Enr}}$, $\mathcal{M}_{\mathrm{Enr}}$, and $\mathcal{M}_{\mathrm{Enr},v}$ for $\omega \in \mathsf{E}_{10}$ are ten–dimensional quasi-affine varieties over $\mathbb{C}$.*

Proof Since the maps from (5.8.1) are quasi-finite, it is enough to prove that $\mathcal{M}_{\mathrm{Enr},v}$ is quasi-affine for one $\omega \in \mathsf{E}_{10}$ or for the quotient $\mathcal{M}_{\mathrm{Enr},v} = \Gamma\backslash\mathcal{D}^\circ$ with respect to some arithmetic subgroup Γ of Γ_{Enr}. We follow the proof of Pappas [587], and in the Appendix we will give the original proof of this result due to Borcherds.

The proof consists of three steps. In Step 1, we fix an $\omega \in \mathsf{E}_{10}$ with $\omega^2 = 2$, then we construct a normal subgroup Γ of finite index in Γ_{Enr} such that $\mathcal{M} = \mathcal{M}_{\mathrm{Enr},v} = \Gamma\backslash\mathcal{D}_{\mathrm{Enr}}$ can be realized as the base of smooth family $f : \mathcal{S} \to \mathcal{M}$ of Enriques surfaces. In Step 2, we prove that $f_*(\omega_{\mathcal{S}/\mathcal{M}})$ coincides with the descent $\mathbb{L}_{\mathcal{M}}$ of the automorphic line bundle $\mathbb{L}$ to $\mathcal{M}$. In Step 3, we prove that $\omega_{\mathcal{S}/\mathcal{M}}$ is a torsion line bundle. The theorem now follows from this, because we know by Baily–Borel that some tensor power of $\mathbb{L}^{\otimes n}$ extends to a very ample sheaf on the compactification $\mathcal{M}^{\mathrm{BB}}$. Thus, it has a section, whose divisor of zeros is contained in the boundary of $\mathcal{M}^{\mathrm{BB}}$, which implies that $\mathcal{M}$ is an open subset of an affine variety, that is, quasi-affine.

Step 1: To choose Γ, we first consider a congruence subgroups of $\Gamma_{\mathrm{Enr},v}$ with $\omega^2 = 2$ defined by

$$\Gamma_n' := \{g \in \Gamma_{\mathrm{Enr}} : g(\omega) = \omega, g \equiv \mathrm{id} \mod n\mathsf{E}_{10}^{\perp}\}.$$

For sufficiently large n, the congruence subgroup in $\mathrm{SL}(N,\mathbb{Z})$ will be torsion-free, so we choose an n so that Γ_n' is torsion free, see [84, Proposition 17.4]. Since we need a normal subgroup of Γ_{Enr}, we define Γ to be the normal subgroup $\cap_{g\in\Gamma_{\mathrm{Enr}}} g \cdot \Gamma_n' \cdot g^{-1}$ contained in Γ_n'. In general, an arithmetic quotient of the period space for lattice polarized K3 surfaces $\mathcal{D}_{\mathrm{Enr}}$ is only a coarse moduli space, that is, it does not admit a universal family. However, it is a fine moduli space if the group acts without fixed points, see Remark 5.4.8 and the discussion in [321], Chapter 6. We denote by $f : \mathcal{S} \to \mathcal{M} := \Gamma\backslash\mathcal{D}_{\mathrm{Enr}}$ the universal family over $\mathcal{M}$.

Step 2: We identify the family $f : \mathcal{S} \to \mathcal{M}$ with the corresponding family of lattice polarized K3 surfaces and consider the Hodge line bundle $f_*(\omega_{\mathcal{S}/\mathcal{M}}) \subset R^2 f_*\mathbb{C}$, whose fibers are the subspaces $H^{2,0} \subset H^2(f^{-1}(m),\mathbb{C})$ spanned by a holomorphic 2−form on the fiber. It follows from the definition of the automorphic line bundle that $f_*(\omega_{\mathcal{S}/\mathcal{M}})$ coincides with $\mathbb{L}_{\mathcal{M}}$.

Step 3: Since the restriction of $\omega^{\otimes 2}_{\mathcal{S}/\mathcal{M}}$ to each fiber is the pull-back of the bicanonical bundle on an Enriques surface, it is trivial along the fibers of f. Hence, there exists a line bundle $\mathcal{L}$ on $\mathcal{M}$, such that $\omega^{\otimes 2}_{\mathcal{S}/\mathcal{M}} = f^*(\mathcal{L})$. Applying the projection formula and using that $f_*\mathcal{O}_{\mathcal{S}} = \mathcal{O}_{\mathcal{M}}$, we conclude that $\mathcal{L} \cong \mathbb{L}_{\mathcal{M}}^{\otimes 2}$. Thus, $\Omega_{\mathcal{S}/\mathcal{M}}^{\otimes 2} \cong f^*(\mathbb{L}_{\mathcal{M}}^{\otimes 2})$.

Now, we apply the Grothendieck–Riemann–Roch formula for the proper morphism f and the sheaf $\mathcal{O}_{\mathcal{S}}$, see, for example, [242]. Let λ be the first Chern class of $f_*(\omega_{\mathcal{S}/\mathcal{M}})$. By the above, we find $f^*(\lambda) = -2c_1(\omega_{\mathcal{S}/\mathcal{M}}^{-1}) = -2c_1(\Theta_{\mathcal{S}/\mathcal{M}})$, where $\Theta_{\mathcal{S}/\mathcal{M}}$ is the relative tangent bundle. To simplify the notation, we let c_1, c_2 be the Chern classes of this bundle. The Grothendieck–Riemann–Roch formula gives the equality

$$c_1(Rf_*\mathcal{O}_{\mathcal{S}}) = \frac{1}{24} f_*(c_1 \cdot c_2)$$

in $\mathrm{Pic}(\mathcal{M})_{\mathbb{Q}}$. Since $R^i f_*\mathcal{O}_{\mathcal{S}} = 0$ for $i \neq 0$, we obtain $f_*(c_1 \cdot c_2) = 0$. Hence

$$0 = 2f_*(c_1 \cdot c_2) = -f_*(f_*(c_1) \cdot c_2) = -\lambda \cdot f_*(c_2).$$

Noether's formula (0.10.22) implies that $f_*(c_2) = 12$ in the Chow group of $\mathcal{M}$ tensored with $\mathbb{Q}$. Thus, λ is trivial in $\mathrm{Pic}(\mathcal{M})_{\mathbb{Q}}$, hence it is a torsion class. This ends the proof. □

Remark 5.9.12 In fact, one can say more. By Theorem 5.9.8, $\mathcal{M}_{\mathrm{Enr}}^{\mathrm{BB}} - \mathcal{M}_{\mathrm{Enr}}$ consists of the closure H of the irreducible divisor $\mathcal{M}_{\mathrm{Cob}}$ and the union of two modular curves X and $X_0(2)$, both of which are isomorphic to $\mathbb{P}^1$. It follows from Corollary 5.9.10 that the closure of the Coble divisor contains the modular curve X. This shows that the complement of H in the Baily–Borel compactification is an affine open set that consists of $\mathcal{M}_{\mathrm{Enr}}$ and the affine line $X \setminus \{\mathrm{cusp}\}$.

The fact that moduli spaces $\mathcal{M}_{\mathrm{Enr}}$ and $\mathcal{M}_{\mathrm{Enr}}^{\mathrm{m}}$ are quasi-affine has the following interesting application to families of Enriques surfaces.

Corollary 5.9.13 *Let $\mathcal{Y} \to T$ be a smooth family of Enriques surfaces, where B is connected, proper and of dimension ≥ 1 over the complex numbers $\mathbb{C}$. Then, this family has no moduli, that is, for every two points $t_1, t_2 \in T$, the fibers $\mathcal{Y}_{t_1}$ and $\mathcal{Y}_{t_2}$ are isomorphic.*

Proof Given a flat family $\sigma : \mathcal{Y} \to T$ of Enriques surfaces, we let $\mathcal{P}_{\mathcal{Y}/T} = R^1\sigma_*\mathbb{G}_m$ be the relative Picard sheaf and we let $\mathcal{P}_{\mathcal{Y}/T}^{\tau}$ be the quotient sheaf modulo numerical equivalence. Then, a *marking* of the family is an isomorphism $\mathsf{E}_{10,T} \to \mathcal{P}_{\mathcal{Y}/T}^{\tau}$ of sheaves equipped with the structure of sheaves of quadratic lattices. Here, $\mathsf{E}_{10,T}$ denotes the constant sheaf associated with the quadratic lattice E_{10}.

Now, let $\mathcal{Y} \to T$ be as in the statement of the corollary. Passing to some finite cover of T, we may assume that the general fiber $(\mathcal{P}_{\mathcal{Y}/T}^{\tau})_\eta$ of $\mathcal{P}_{\mathcal{Y}/T}^{\tau}$ contains an isotropic 10-sequence $(f_1, \ldots, f_{10})$ that defines a marking of the general fiber $\mathcal{Y}_\eta$ of the family. Using the specialization homomorphism of the Picard groups, which is known to preserve the intersection form, we specialize $(f_1, \ldots, f_{10})$ to define a marking of the whole family, that is, we trivialize the sheaf $\mathcal{P}_{\mathcal{Y}/T}^{\tau}$ and obtain a family of marked Enriques surfaces. Since $\mathcal{M}_{\mathrm{Enr}}^{\mathrm{m}}$ is a coarse moduli space (in category of analytic spaces), we get a holomorphic map $T \to \mathcal{M}_{\mathrm{Enr}}^{\mathrm{m}}$ that must be constant since $\tilde{T}$ is proper and $\mathcal{M}_{\mathrm{Enr}}^{\mathrm{m}}$ is quasi-affine. □

Moreover, one can consider the compactifications and the boundaries of the moduli spaces $\mathcal{M}_{\mathrm{Enr},v}$ of polarized and numerically polarized Enriques surfaces. As far as we know, the only known case that has been worked out it is the case of degree 2 polarizations in work of Sterk [690], [691].

Let us briefly discuss Sterk's results, that is, we consider the compactification $\overline{\mathcal{M}}_{\mathrm{Enr},2}$ of the coarse moduli space $\mathcal{M}_{\mathrm{Enr},2} = \mathcal{M}_{\mathrm{Enr},\omega_8} = \Gamma_{\mathsf{E}_{10}(2),2}\backslash\mathcal{D}_{\mathrm{Enr}}$ of degree-2 numerically polarized Enriques surfaces. Moreover, there is a finite cover

$$\overline{\mathcal{M}}_{\mathrm{Enr},2} \to \overline{\mathcal{M}}_{\mathrm{Enr}},$$

which is of degree $2^7 \cdot 17 \cdot 31$, see the end of Section 5.7. In these cases, the geometry of the boundary is considerably more complicated than in the case of unpolarized surfaces.

Theorem 5.9.14 . *The boundary* $\Gamma_{\mathsf{E}_{10}(2),2}\backslash(\overline{\mathcal{D}}_{\mathrm{Enr}} - \mathcal{D}_{\mathrm{Enr}})$ *of* $\overline{\mathcal{M}}_{\mathrm{Enr},2}$ *consists of* 5 *zero–dimensional (marked by* $(1), (2), (3), (4), (5)$ *in the following diagram) and* 9 *one-dimensional components. The following graph describes the incidence relation between the components:*

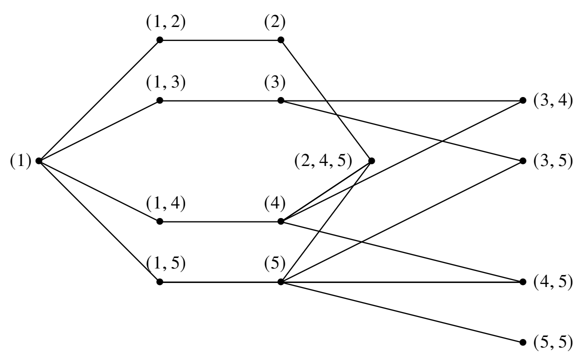


Fig. 5.1 Boundary components of $\overline{\mathcal{M}}_{\mathrm{Enr},2}$

The boundary components marked by $(3,5), (3,4), (4,5)$ *are the modular curves* $X^1(2)$. *The boundary component marked by* $(5,5)$ *is the modular curve X for the full modular group and the boundary component marked by* $(2,4,5)$ *is the modular curve* $X(2)$ *for the 2-level congruence subgroup of* $\mathrm{SL}(2,\mathbb{Z})$.

Let H be a non-special Picard polarization of degree 2 of an Enriques surface S. Then, the linear system $|2H|$ is bielliptic and defines a degree 2 map $f : S \to \mathsf{D}_1$, where D_1 is a non-degenerate 4-nodal quartic del Pezzo surface in $\mathbb{P}^4$. It ramifies over the four nodes and a curve W cut out by a quadric. Let $|\mathcal{O}_{\mathsf{D}_1}(2)|$ be the projective space of such curves. In Theorem 5.8.11, we studied the rational quotient of this space by the group $G = \mathrm{Aut}(\mathsf{D}_1)$, which is isomorphic to $(\mathbb{C}^*) \rtimes D_8$. Let $\mathcal{M} := |\mathcal{O}_{\mathsf{D}_1}(2)|//G$ be the GIT-quotient. Passing to K3-covers, it is isomorphic to the GIT-quotient $W//G$, where W is the projective space of curves of bi-degree $(4,4)$ on $\mathbb{P}^1 \times \mathbb{P}^1$ that are invariant with respect to the involution with four isolated fixed points. The analysis of stable and semi-stable points for the GIT-quotient was done by J. Shah [667]. It follows from this description that the points represented by hyperplane sections of D_1 passing through two opposite vertices of the quadrangle of lines and taken with multiplicity 2 form a minimal orbit of strictly semi-stable points. Let $\mathcal{M}' \to \mathcal{M}$ be the blow-up of this point (in fact, a certain weighted blow-up). Sterk shows that points in the exceptional divisor E can be viewed as minimal semi-stable orbits of curves in the linear system $|\mathcal{O}_{\mathsf{D}_1'}(2)|$, where D_1' is a degenerate 4-nodal quartic del Pezzo surface. Next, he defines the blow up of $\mathcal{M}'$ along the proper inverse transform on $\mathcal{M}'$ of one-dimensional strata of the union of two conics passing through the opposite vertices of the quadrangle of lines on D_1. Let $\mathcal{M}$ be the result of the two blow-ups. Let $[B]$ be a point in the GIT-quotient $\mathcal{M}$ representing a smooth curve B in $|\mathcal{O}_{\mathsf{D}_1}(2)|$. The double cover $S \to \mathsf{D}_1$ branched along B and the singular locus of D_1 is an Enriques surface S. The double cover is given by a non-degenerate bielliptic linear system $|2F_1 + 2F_2|$ and $h = [F_1] + [F_2]$ equips S with a numerical ample polarization of degree 2. This defines a rational map $\mathcal{M} \dashrightarrow \mathcal{M}_{\mathrm{Enr},2}$, which is not defined on the divisor of special polarizations. Let $\mathcal{M}_{\mathrm{Enr},2}^{\mathrm{strd}} \to \overline{\mathcal{M}}_{\mathrm{Enr},2}$ be the *semi-toric compactification* of Looijenga that blows up the Weil divisor of special polarizations. The proof of the following theorem can be found in [691].

Theorem 5.9.15 *The rational map $\mathcal{M} \dashrightarrow \overline{\mathcal{M}}_{\mathrm{Enr},2}$ extends to a birational morphism*

$$f : \mathcal{M} \dashrightarrow \mathcal{M}_{\mathrm{Enr},2}^{\mathrm{strd}}$$

to the semi-toric compactification of $\mathcal{M}_{\mathrm{Enr},2}$ obtained by blowing up the Weil divisor H that corresponds to the locus (E, h) of Enriques surfaces with special degree 2 polarizations. The strict transform of the exceptional divisor E_1 of the first blow-up $\mathcal{M}' \to \mathcal{M}$ is mapped to the strict inverse transform of the divisor H.

Sterk's paper also describes a match between different strata of semi-stable points in $|\mathcal{O}_{\mathsf{D}_1}(2)|$ and the pre-images of the boundary components in the semi-toric compactification. We come back to these two compactifications of $\mathcal{M}_{\mathrm{Enr},2}$ at the end of Section 5.10.

5.10 Degenerations of Enriques Surfaces

Having studied compactifications of moduli spaces of Enriques surfaces, it is natural to ask whether the boundary has a modular description, that is, whether it also parametrizes some interesting objects. In the previous section, we have interpreted an open set of the boundary divisor as moduli space of Coble surfaces of K3 type. We have also seen that the remaining parts of the boundary in the Baily–Borel compactification are unions of zero-dimensional and one-dimensional sets. To understand these, we are led to studying degenerations of K3 surfaces and Enriques surfaces. We refer to [421], [422], [523], [589], and [591] for details and further background.

More generally, let $\Delta := \{z \in \mathbb{C} | |z| < 1\}$ be the complex disk and $\Delta^* := \{z \in \mathbb{C} | 0 < |z| < 1\}$ the punctured disk.

Definition 5.10.1 A *degeneration of a family of surfaces* is proper and flat map $\mathcal{X} \to \Delta$ from a 3-dimensional complex manifold such that for all $t \in \Delta^*$, the fiber $\mathcal{X}_t$ is smooth. The degeneration is called *semi-stable* if the central fiber $\mathcal{X}_0$ is a simple normal crossing divisor.

Let us recall that $\mathcal{X}_0$ being a simple normal crossing divisor means that all its irreducible components are smooth, and that it is locally analytically at each point $x \in \mathcal{X}_0$ of the form $z_0 \cdot \ldots \cdot z_k = 0$, where the z_i are local parameters of x in $\mathcal{X}$. We will say that two degenerations $\mathcal{X} \to \Delta$ and $\mathcal{X}' \to \Delta$ are *bimeromorphically equivalent*, if there exists a bimeromorphic map $\mathcal{X} \dashrightarrow \mathcal{X}'$ over Δ that is an isomorphism outside the central fibers. By a theorem of Mumford [34], [382], given a degeneration $\mathcal{X} \to \Delta$, then, after some finite base change $\Delta' \to \Delta$, it is bimeromorphically equivalent to a semi-stable degeneration.

Now, let $\mathcal{X} \to \Delta$ be a semi-stable degeneration of a family of surfaces. Let $V_1, \ldots, V_k$ be the irreducible components of the special fiber $\mathcal{X}_0$. Being semi-stable, all V_i are smooth surfaces, and their intersections $C_{ij} := V_i \cap V_j$ are smooth curves. We shall call the irreducible components of the C_{ij} the *double curves* of $\mathcal{X}_0$. Being semi-stable, the intersection $V_i \cap V_j \cap V_k$ for pairwise distinct i, j, k is either empty or consists of *triple points* that are analytically locally of the form $z_1 z_2 z_3 = 0$. The *dual graph* Γ associated to $\mathcal{X}_0$ is the undirected graph with a vertex v_i for every component V_i, and an edge (v_i, v_j) for every component of C_{ij}, and a two-simplex (v_i, v_j, v_k) for every triple point of $V_i \cap V_j \cap V_k$.

For two components $V, W \subset \mathcal{X}_0$, and a double curve $C \subset V \cap W$, we have *Persson's triple point formula*

$$C_V^2 + C_W^2 = -\tau_C,$$

where C_V^2 (resp. C_W^2) denotes the self-intersection number of C as a curve on the surface V (resp. W), and where τ_C denotes the number of triple points of X on C, see [589, Corollary 2.4.2]. Next, we have the *normal bundle formula*

$$N_{V_i/\mathcal{X}} \cong \mathcal{O}_{V_i}(-\sum_j C_{ij}),$$

where the sum runs over all double curves C_{ij} contained in V_i.

In particular, if $\mathcal{X} \to \Delta$ is a semi-stable degeneration of a family of surfaces with numerically trivial canonical sheaves, then the normal bundle formula implies that the components of the special fiber are ruled surfaces or also have a numerically trivial canonical sheaf. For example, if the special fiber $\mathcal{X}_0$ is irreducible, then it is a smooth surface with numerically trivial canonical class of the same type (K3, Enriques, Abelian, Hyperelliptic) as the general fiber of that family. In the remaining cases, we distinguish whether there exist triple points in $\mathcal{X}_0$ or not. We start with the case where there are no triple points, and refer to [589], Proposition 3.3.1 for details and proofs.

Proposition 5.10.2 *Let $\mathcal{X} \to \Delta$ be a semi-stable degeneration of a family of surfaces, whose general fiber has a numerically trivial canonical sheaf. Assume that $\mathcal{X}_0$ has no triple points, and that the components of $\mathcal{X}_0$ are algebraic. Then, after having blown down exceptional components of the special fiber, we have the following possibilities:*

1. *$\mathcal{X}_0$ is a smooth surface with numerically trivial canonical class (trivial degeneration).*
2. *$\mathcal{X}_0$ is a cycle $V_1 \cup \ldots \cup V_k$ of elliptic ruled surfaces. The double curves are smooth elliptic curves, and, more precisely, disjoint sections of the V_i.*
3. *$\mathcal{X}_0$ is a chain of surfaces $V_1 \cup \ldots \cup V_k$. The "interior" components $V_2, \ldots, V_{k-1}$ are elliptic ruled surfaces. The "end" components V_1 and V_k are either elliptic ruled or rational surfaces. The double curves are smooth elliptic curves. The general fiber will be:*

K3	*in the case of two rational components*
Enriques	*in the case of only one rational component*
Hyperelliptic	*if all components are elliptic ruled.*

4. *$\mathcal{X}_0$ is a* flower pot*, and the general fiber is an Enriques surface. All components and the double curves are rational. Each flower P is a $\mathbb{P}^2$, its double curve is a conic. The stalk consists of rational minimal ruled surfaces $\mathbf{F}_4$, its double curves are disjoint sections. All the double curves of the pot B have self-intersection* -4.

Example 5.10.3 Let $\mathcal{X} \to \Delta$ be a smooth family of K3 surfaces that is equipped with an involution $\tau : \mathcal{X} \to \mathcal{X}$ over Δ that acts without fixed points on every fiber $\mathcal{X}_t$, $t \neq 0$, and whose fixed–point set on the special fiber $\mathcal{X}_0$ is equal to a (-2)-curve C. We consider the quotient $\pi : \mathcal{X} \to \mathcal{Y}' := \mathcal{X}/\tau$, which is singular along $D := \pi(C)$. Then, $\mathcal{Y}'^* \to \Delta^*$ is a smooth family of Enriques surfaces. In Example 5.4.5, we showed that $\mathcal{X}_0 \to Y := \mathcal{X}_0/\tau$ is a double cover of the Coble surface Y, which is branched over the smooth rational curve $D := \pi(C)$ with $D^2 = -4$. To obtain a semi-stable degeneration, we have to resolve the double curve $D \subset \mathcal{Y}'$. For this, let $\widetilde{\mathcal{X}} \to \mathcal{X}$ be the blow-up along C. Then, the exceptional divisor is isomorphic to a smooth quadric surface intersecting the proper transform of $\mathcal{X}_0$ along a conic (use the triple point formula and the normal bundle formula). Next, the involution τ extends to an involution $\widetilde{\tau}$ on $\widetilde{\mathcal{X}}$. The restriction of $\widetilde{\tau}$ to the exceptional divisor acts without fixed points outside the conic. Then,

$$\mathcal{Y} := \widetilde{\mathcal{X}}/\widetilde{\tau} \to \Delta$$

is a semi-stable degeneration of a family of Enriques surfaces. The central fiber $\mathcal{Y}_0$ is the union of the Coble surface Y and a $\mathbb{P}^2$. The double curve of the intersection is a smooth rational curve, which is a conic in $\mathbb{P}^2$ and a (-4)-curve on Y. In particular, $\mathcal{Y} \to \Delta$ is an example of a flower pot degeneration.

In the case where the special fiber $\mathcal{X}_0$ of a semi-stable degeneration has triple points, we have a detailed description for which we refer to [523].

Now, if we do not insist on finding a semi-stable degeneration over Δ, but also allow finite base-changes $\Delta' \to \Delta$, then we have the existence of *Kulikov models*, where the geometry of the special fiber becomes much simpler. Let us first introduce these models.

Theorem 5.10.4 *Let $\mathcal{X} \to \Delta$ be a semi-stable degeneration of a K3 surface or an Enriques surface. Then, after a finite base change $\Delta' \to \Delta$, it is bimeromorphically equivalent to a semi-stable fibration $\mathcal{X}' \to \Delta'$, such that $\omega_{\mathcal{X}'/\Delta'}$ is numerically trivial.*

Proof The original proof is due to Viktor Kulikov [421], but see also [591]. A more recent approach is via the semi-stable minimal model program, which, given $\mathcal{X} \to \Delta$, produces a bimeromorphic (weakly) semi-stable model $\mathcal{Y} \to \Delta$ with terminal singularities such that $\omega_{\mathcal{Y}/\Delta}$ is numerically trivial. Weakly in the sense that the special fiber $\mathcal{Y}_0$ is a normal crossing divisor but maybe not a simple normal crossing divisor (the components of $\mathcal{Y}_0$ need not be smooth). However, there exists a finite base-change $\Delta' \to \Delta$ after which the singularities of $\mathcal{Y}$ and those of the components of $\mathcal{Y}_0$ can be resolved. We refer to [500] for this point of view and [467] for an overview. □

Now, when dealing with the Kulikov models, the classification of the special fiber becomes much easier, see [421], [523], [589], and [591].

Theorem 5.10.5 *Let $\mathcal{X} \to \Delta$ be semi-stable degeneration of K3 surfaces or Enriques surfaces, and assume that $\omega_{\mathcal{X}/\Delta}$ is numerically trivial. Then, the special fiber is one of the following:*

1. *$\mathcal{X}_0$ is smooth or a flower pot degeneration.*
2. *2. $\mathcal{X}_0$ is not smooth, but contains no triple points. Then, $\mathcal{X}_0 = V_1 \cup \ldots \cup V_k$ is a chain of elliptic ruled surfaces, whose end components are either elliptic ruled or rational, meeting along elliptic smooth elliptic curves. There are two (resp. one) rational components in the case of degeneration of a K3 surface (resp. Enriques surface).*
3. *$\mathcal{X}_0$ contains triple points. Then, $\mathcal{X}_0$ is a union of rational surfaces V_i, the double curves on each V_i form a cycle of smooth rational curves, and the dual graph Γ associated to $\mathcal{X}_0$ is topologically homeomorphic to the 2-sphere $\mathbb{S}^2$ (degeneration of K3 surfaces), or the real projective 2-plane $\mathbb{RP}^2$ (degeneration of Enriques surfaces).*

According to the three cases of the theorem, one also speaks of a *Kulikov model of type I* (resp. *type II*, resp. *type III*).

Example 5.10.6 1. The degenerations of K3 surfaces and Enriques surfaces given in Proposition 5.10.2 give examples of Kulikov models for families of K3 surfaces and a family of Enriques surfaces, which are of of type III, that is, with triple points.

2. This example is taken from [208]. Let Δ be a simplicial complex on the set $E = \{e_0, \ldots, e_n\}$. It defines a closed subscheme $X(\Delta)$ of $\mathbb{P}^n_{\Bbbk}$ called the *face variety of* Δ defined by the monomial ideal generated by monomials $x_{i_0} \cdots x_{i_m}$

such that $\{i_0, \ldots, i_m\}$ is not a face of Δ. The projective coordinate ring is the *Stanley–Reisner ring* of Δ, see [687]. If the topological realization $|\Delta|$ of Δ is a manifold, then $X(\Delta)$ is a locally Gorenstein scheme with dualizing sheaf $\omega_{X(\Delta)}^{\otimes 2} \cong \mathcal{O}_{X(\Delta)}$. Moreover, $\omega_{X(\Delta)} \cong \mathcal{O}_{X(\Delta)}$ if and only if $|\Delta|$ is an orientable manifold. We have $H^i(X(\Delta), \mathcal{O}_{X(\Delta)}) \cong H^i(|\Delta|, \Bbbk)$.

Now, we take Δ to be a minimal simplicial triangulation of $\mathbb{P}^2(\mathbb{R})$ given by the following figure.

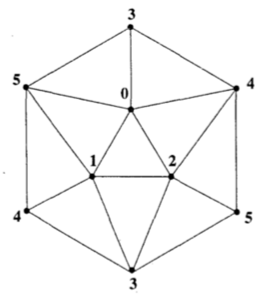


Fig. 5.2 A minimal simplicial triangulation of $\mathbb{P}^2(\mathbb{R})$

Depending on the characteristic p of $\Bbbk$, we have the following:

a. If $p \neq 2$, then $\omega_{X(\Delta)} \not\cong \mathcal{O}_{X(\Delta)}$ since $|\Delta|$ is not orientable. Moreover, $H^1(X(\Delta), \mathcal{O}_{X(\Delta)}) \cong H^2(X(\Delta), \mathcal{O}_{X(\Delta)}) = 0$. The surface $X(\Delta)$ is a degeneration of type III of an Enriques surface.
b. If $p = 2$, then $\omega_{X(\Delta)} \cong \mathcal{O}_{X(\Delta)}$ since $|\Delta|$ is orientable with respect to mod 2 coefficients. Moreover, $H^1(X(\Delta), \mathcal{O}_{X(\Delta)}) \cong H^2(X(\Delta), \mathcal{O}_{X(\Delta)}) \cong \Bbbk$ because these singular cohomology groups of $\mathbb{P}^2(\mathbb{R})$ with coefficients in $\mathbb{Z}/2\mathbb{Z}$ in degree 1 and 2 are non-zero. The surface $X(\Delta)$ is a degeneration of type III of a non-classical Enriques surface.

In both cases, $X(\Delta)$ is the union of ten planes in $\mathbb{P}^5$ and it is a projective degeneration of a Fano model of an Enriques surface.

3. Finally, the flower pot degeneration of Example 5.10.3 is not a Kulikov model.

An important feature of semi-stable degenerations and Kulikov models of K3 surfaces is that the type of the special fiber can be detected by monodromy. Let $\pi : \mathcal{X} \to \Delta$ be a semi-stable degeneration of a family of K3 surfaces, and let Λ be a commutative ring. Then, the restriction of $R^2\pi_*\Lambda$ to the punctured disk Δ^* is a locally constant sheaf with fibers $(R^2\pi_*\Lambda)_t \cong H^2(\mathcal{X}_t, \Lambda)$ for all $t \in \Delta^*$. Next, we identify the universal cover $U := \widetilde{\Delta^*} \to \Delta^*$ with the exponential map $\exp : \Delta \to \Delta^*$, we fix some $t_0 \in \Delta^*$, and then, we choose a trivialization of the pull-back of the local system $\exp^*(R^2\pi_*\Lambda) \cong H^2(\mathcal{X}_t, \Lambda) \times U$. This gives rise to the *monodromy representation*

$$\pi_1(\Delta^*, t_0) \ \to\ \mathrm{Aut}\left(H^2(\mathcal{X}_{t_0}, \Lambda)\right),$$

and the image of a generator of $\pi_1(\Delta^*, t_0) \cong \mathbb{Z}$ under this representation is called a *monodromy operator*. Since $\mathcal{X}$ is a degeneration of K3 surfaces, we can represent this operator by a matrix $T \in \mathrm{GL}(22, \Lambda)$, the *monodromy matrix*. By the *Grothendieck–Landman–Monodromy Theorem* [428] and [277, Theorem I.1.2], T is quasi-unipotent, that is, there exist positive integers r and n such that

$$(T^r - 1)^n \ =\ 0.$$

Since the degeneration is semi-stable, we have $r = 1$. Moreover, being an operator on H^2, we may choose $n = 2 + 1$. Thus, $N := T - 1$ is nilpotent and gives rise to a filtration

$$0 \ \subseteq\ W_0 \ \subseteq\ W_1 \ \subseteq\ W_2 \ \subseteq\ W_3 \ \subseteq\ W_4 \ :=\ H^2(\mathcal{X}_{t_0}, \Lambda),$$

by setting

$$\begin{aligned}
&W_0 := N^2(W_4)\\
&W_3 := \mathrm{Ker}(N^2)\\
&W_1 := \text{the inverse image of } \mathrm{Ker}(N : W_3/W_0 \to W_3/W_0) \text{ in } W_3\\
&W_2 := \text{image of } N : W_3/W_0 \to W_3.
\end{aligned}$$

This is called the *monodromy weight filtration*. The monodromy operator N satisfies $N(W_k) \subseteq W_{k-2}$ and N^k induces isomorphisms $W_{k+2}/W_{k+1} \to W_{2-k}/W_{1-k}$ for all k, see also the discussion on [589], page 66.

In the case $\Lambda = \mathbb{C}$, there is another filtration

$$0 \ \subseteq\ F^2_\infty \ \subseteq\ F^1_\infty \ \subseteq\ F^0_\infty \ :=\ H^2(\mathcal{X}_{t_0}, \mathbb{C}),$$

the *limit Hodge filtration*, that is defined as follows: for the family $\mathcal{X}^* \to \Delta^*$ of K3 surfaces, let $U \to \Delta^*$ be the universal cover, and let $p : U \to \mathcal{D}$ be the period map. By definition, $p(z)$ is the line $\ell(z) = H^{2,0}(\mathcal{X}_z)$, considered as a point of $\mathcal{D}$. As in Example 5.9.6, we identify U with the upper half plane $\mathbb{H}$, and use this to define

$$\begin{aligned}
&F^2_\infty := \lim\nolimits_{\mathrm{Im}(z)\to\infty} \exp(-zN)\ell(z),\\
&F^1_\infty := F^2_\infty \ +\ \left(F^2_\infty \cap \overline{F}^2_\infty\right).
\end{aligned}$$

where $\overline{F}^2_\infty$ denotes complex conjugation. This structure of two filtrations, an ascending weight filtration W_m and a descending Hodge filtration F^j_∞ such that the F^j_∞ induce Hodge structures on the subquotients W_m/W_{m-1} is called a *mixed Hodge structure*. For example, if $N = 0$, then the weight filtration is trivial, that is, $W_m = 0$ for $m = 0, 1$ and $W_m = H^2(\mathcal{X}_{t_0}, \Lambda)$ for $m = 2, 3$ in the example above, and we obtain a *pure Hodge structure*. We refer to [725] for details and precise definitions.

After these preparations, we have the following result, which shows that the monodromy can detect potential good reduction of K3 surfaces.

Theorem 5.10.7 *Let $\mathcal{X} \to \Delta$ be a semi-stable degeneration of K3 surfaces that is a Kulikov model. Let N be the monodromy operator on $H^2(\mathcal{X}_{t_0}, \mathbb{C})$ for some $t_0 \in \Delta^*$. Then:*

1. *$\mathcal{X}_0$ is smooth if and only if $N = 0$.*
2. *$\mathcal{X}_0$ is a chain of ruled surfaces if and only if $N \neq 0$ and $N^2 = 0$.*
3. *$\mathcal{X}_0$ is a union of rational surfaces, whose dual graph Γ is homeomorphic to $\mathbb{S}^2$, if and only if $N^2 \neq 0$.*

Proof See [422, Chapter V,§6] and also [467] for a discussion. □

Remark 5.10.8 In fact, the assumption that the degeneration is semi-stable and already a Kulikov model is not needed: given a smooth family $\mathcal{X}^* \to \Delta^*$ of K3 surfaces with trivial monodromy (resp. unipotent monodromy), that is, if N is trivial (resp. N is unipotent), then this family can be completed to a smooth family (resp. a Kulikov model) $\mathcal{X} \to \Delta$ of K3 surfaces. We refer to [298, Theorem 35], who attribute this result to R. Friedman, D. Morrison, and F. Scattone.

If $\mathcal{X}_0$ is smooth, one also says that the family $\mathcal{X}^* \to \Delta^*$ has *good reduction*. We note that if $\mathcal{X}_i \to \Delta$ are two smooth families of K3 surfaces that extend the given $\mathcal{X}^* \to \Delta^*$, then these families are related by a sequence of *flops* or *elementary modifications* in (-2)-curves of the special fiber $\mathcal{X}_0$. On the other hand, the special fibers of the two models are birationally equivalent by a theorem of Matsusaka and Mumford [499] and since they are K3 surfaces, they are isomorphic. Thus, the special fibers are unique, but the completed families $\mathcal{X}_i \to \Delta$ usually are not.

For semi-stable degenerations of Enriques surfaces, we have the following analog of the previous theorem:

Theorem 5.10.9 *Let $\mathcal{Y} \to \Delta$ be a semi-stable degeneration of Enriques surfaces that is a Kulikov model. Let N be the monodromy operator on $H^2(\widetilde{\mathcal{Y}}_{t_0}, \mathbb{C})$ for some $t_0 \in \Delta^*$, where $\widetilde{\mathcal{Y}}_{t_0} \to \mathcal{Y}_{t_0}$ denotes the K3 cover. Then,*

1. *$\mathcal{Y}_0$ is smooth or a flower pot degeneration if and only if $N = 0$.*
2. *$\mathcal{Y}_0$ is a chain of ruled surfaces if and only if $N \neq 0$ and $N^2 = 0$.*
3. *$\mathcal{Y}_0$ is a union of rational surfaces, whose dual graph Γ is homeomorphic to $\mathbb{RP}^2$, if and only if $N^2 \neq 0$.*

In particular, for Enriques surfaces, the monodromy operator cannot detect good reduction. The typical example of this phenomenon is given in Example 5.10.3: here, one has a family of Enriques surfaces $\mathcal{Y}^* \to \Delta^*$, such that the associated family $\mathcal{X}^* \to \Delta^*$ of K3 surfaces has trivial monodromy. Therefore, the latter family has good reduction, say to $\mathcal{X} \to \Delta$. The covering involution on $\mathcal{X}^* \to \mathcal{Y}^* \to \Delta^*$ extends to $\mathcal{X}$, however, it acquires a fixed locus on $\mathcal{X}_0$. Thus, although the monodromy operator does detect good reduction of $\mathcal{X}^* \to \Delta^*$, it cannot detect whether the involution acts without fixed points on the special fiber $\mathcal{X}_0$.

Let us also briefly discuss the algebraic case: The roles of Δ and Δ^* are played by $\operatorname{Spec} R$ and $\operatorname{Spec} K$, where R is some DVR, where K is its field of fractions, and where $\Bbbk$ is the residue field. As above, a degeneration of some surface X over

K is a proper flat map $\mathcal{X} \to \operatorname{Spec} R$ from a regular 3-dimensional algebraic space, whose generic fiber is isomorphic to X. When discussing the Kulikov models in the complex analytic setting, one has to work in the category of complex analytic Moishezon manifolds and thus, it is not surprising that in the purely algebraic situation one has to work with algebraic spaces. Again, we can define a notion of semi-stable families, by requiring the components of the special fiber to be smooth, and that the strict local ring at each point is étale locally of the form $z_0 \cdot \ldots \cdot z_k = 0$. Unfortunately, the semi-stable reduction for surfaces is not known in this setting (if the characteristic of $\Bbbk$ is zero, then it follows from [34] and [382]). Assuming this result, K3 surfaces and Enriques surfaces do admit Kulikov models after possibly replacing R by a finite extension, which is a theorem of Maulik [500], see also the discussion in [467]. The classification of the special fibers of Kulikov models in this setting is due to Nakajima [548]. For the computation of the monodromy operator and good reduction results, we refer to [118], [119], [467], and [495].

Finally, we come back to the compactifications of the moduli spaces of Enriques surfaces. By Theorem 5.9.8, we have immersions

$$\mathcal{M}_{\mathrm{Enr}} = \Gamma_{\mathrm{Enr}} \backslash \mathcal{D}^{\circ}_{\mathrm{Enr}} \subset \Gamma_{\mathrm{Enr}} \backslash \mathcal{D}_{\mathrm{Enr}} \subset \overline{\mathcal{M}}_{\mathrm{Enr}},$$

where the left space is the moduli space of Enriques surfaces. Moreover, the complement $\Gamma_{\mathrm{Enr}} \backslash \mathcal{D}_{\mathrm{Enr}} - \mathcal{M}_{\mathrm{Enr}}$ is an irreducible divisor, which we identified with the moduli space $\mathcal{M}_{\mathrm{Cob}}$ of Coble surfaces. Next, the complement $\overline{\mathcal{M}}_{\mathrm{Enr}} - \Gamma_{\mathrm{Enr}} \backslash \mathcal{D}_{\mathrm{Enr}}$ is the union of two curves X and $X_0(2)$, which intersect in one point. There are two cusps, which coincide with the cusps of X and $X_0(2)$. Moreover, we also discussed Sterk's compactification [690], [691]

$$\mathcal{M}_{\mathrm{Enr},2} = \Gamma_{\mathsf{E}_{10}(2),2} \backslash \mathcal{D}^{\circ}_{\mathrm{Enr}} \subset \Gamma_{\mathsf{E}_{10}(2),2} \backslash \mathcal{D}_{\mathrm{Enr}} \subset \overline{\mathcal{M}}_{\mathrm{Enr},2},$$

in Theorem 5.9.14, which has nine 1-dimensional boundary components and five cusps.

The general expectation is that the period space should correspond to surfaces with trivial monodromy operator N. Since the period space $\mathcal{D}_{\mathrm{Enr}}$ of Enriques surfaces is really that of their K3-covers, Theorem 5.10.9 suggests that this would correspond to $\Gamma_{\mathrm{Enr}} \backslash \mathcal{D}_{\mathrm{Enr}} = \mathcal{M}_{\mathrm{Enr}} \cup \mathcal{M}_{\mathrm{Cob}}$. On the other extreme, the rational cusps of $\mathcal{D}_{\mathrm{Enr}}$ should correspond to semi-stable degenerations with unipotent monodromy operator N, such that $N^2 \neq 0$, that is, *maximal unipotent monodromy*. Moreover, the one-dimensional boundary components outside the cusps should correspond to semi-stable degenerations with $N \neq 0$ and $N^2 = 0$. Thus, one would hope to identify the cusps (resp. the 1-dimensional boundary components) of $\overline{\mathcal{M}}_{\mathrm{Enr}}$ with Kulikov models of Enriques surfaces ot type III (resp. type II). The main technical difficulty here is that in general, the Baily–Borel compactification is "too small" to extend a universal family to it, which is why in practice complicated blow-ups of boundary components are needed. We illustrate this problem in the case of $\mathcal{M}_{\mathrm{Enr},2}$.

In Section 5.9, we discussed two compactifications of the moduli space $\mathcal{M}_{\mathrm{Enr},2}$ of numerically polarized Enriques surfaces of degree 2. The first one $\overline{\mathcal{M}}_{\mathrm{Enr},2}$ is defined to be $\Gamma_{\mathsf{E}_{10}(2),2}\backslash\overline{\mathcal{D}}_{\mathrm{Enr}}$. The other one is Looijenga's semi-toric compactification $\mathcal{M}^{\mathrm{strd}}_{\mathrm{Enr},2} \to \overline{\mathcal{M}}_{\mathrm{Enr},2}$, which is the blow-up of $\overline{\mathcal{M}}_{\mathrm{Enr},2}$ along the Weil divisor of special polarizations. In [667], Shah studied and classified projective degenerations of Enriques surfaces. For example, he found 9 types of type II degenerations, compare Theorem 5.9.14. Now, Shah's results can used to "explain" the boundary components of the compactifications of $\mathcal{M}_{\mathrm{Enr},2}$. It turns out that Shah's results can be used to interprete $\overline{\mathcal{M}}_{\mathrm{Enr},2}$ outside the cusps as the coarse moduli space of Enriques surfaces including type II degenerations. If one also wants to include Shah's degenerations of type III, it turns out that the cusps of $\overline{\mathcal{M}}_{\mathrm{Enr},2}$ are too small and that one has to use the blow-up $\mathcal{M}^{\mathrm{strd}}_{\mathrm{Enr},2}$ instead. This gives the following modular interpretation of the compactifications discussed at the end of Section 5.9.

Theorem 5.10.10 *If C denotes the set of cusps of $\overline{\mathcal{M}}_{\mathrm{Enr},2}$, then $\overline{\mathcal{M}}_{\mathrm{Enr},2} - C$ is the coarse moduli space of numerically degree 2 polarized Enriques surfaces together with Shah's degree 2 polarized degenerations of type II.*

Moreover, $\mathcal{M}^{\mathrm{strd}}_{\mathrm{Enr},2}$ is the coarse moduli space of numerically degree 2 polarized Enriques surfaces together with Shah's degree 2 polarized degenerations.

5.11 Deformation Theory and Arithmetic Moduli

In this section, we study the deformation theory and moduli spaces of Enriques surfaces in positive characteristic, especially in characteristic 2, and then, over $\operatorname{Spec}\mathbb{Z}$. We construct the moduli space of numerically Cossec–Verra polarized Enriques surface over $\operatorname{Spec}\mathbb{Z}$ and describe its geometry. We end by discussing crystalline period maps and crystalline period spaces for unipotent Enriques surfaces in characteristic 2.

As general references for the theory of deformations, we refer to [230, Part 3] or [651]. We start by adapting this theory to deformations of Enriques surfaces, which is easy in characteristic $p \neq 2$, and rather subtle if $p = 2$. First, we recall Schlessinger's setup [637]: let Λ be a complete, Noetherian, local ring, with maximal ideal μ and residue field $\Bbbk = \Lambda/\mu$. Associated, we consider the following two categories:

$\mathbf{C} = \mathbf{C}_\Lambda$ Artinian local Λ-algebras with residue field $\Bbbk$, and
$\widehat{\mathbf{C}} = \widehat{\mathbf{C}}_\Lambda$ Noetherian local Λ-algebras $(R, \mathfrak{m})$ such that $R/\mathfrak{m}^n \in \mathbf{C}_\Lambda$ for all $n \geq 1$.

The first category is a full subcategory of the second. A surjective morphism $B \to A$ in $\mathbf{C}$ is called a *small extension* if its kernel is a principal ideal (t) such that $\mathfrak{m}_B t = 0$, where $\mathfrak{m}_B$ is the maximal ideal of B. Its importance comes from the fact that any surjection in $\mathbf{C}$ can be factored into a sequence of small extensions.

We will be only interested in covariant functors $F : \mathbf{C} \to (\mathrm{Sets})$ such that $F(\Bbbk)$ contains just one element. If $F, G : \mathbf{C} \to (\mathrm{Sets})$ are two such functors as above, then a morphism of functors $F \to G$ is called *smooth*, if for any surjection $B \to A$ in $\mathbf{C}$, the canonical morphism $F(B) \to F(A) \times_{G(A)} G(B)$ is surjective. In fact, it suffices to check small extensions $B \to A$. Also, we extend a covariant functor $F : \mathbf{C} \to (\text{Sets})$ to a functor $\widehat{F} : \widehat{\mathbf{C}} \to (\text{Sets})$ by setting $\widehat{F}(R) := \varprojlim F(R/\mathfrak{m}^n)$ for every $R \in \widehat{\mathbf{C}}$, where $\mathfrak{m}$ denotes the maximal ideal of R. Next, $t_F := F(\mathbb{k}[\varepsilon]/(\varepsilon^2))$ is called the *tangent space* to F. In practice, t_F always carries naturally the structure of a $\mathbb{k}$-vector space, see [637, Lemma 2.10]. An important example of such functors arises as follows: for every $R \in \widehat{\mathbf{C}}_\Lambda$, we set

$$\begin{array}{rcl} h_R : \mathbf{C}_\Lambda & \to & (\text{Sets}) \\ A & \mapsto & \operatorname{Hom}(R, A), \end{array}$$

and then, $t_{h_R} \cong \operatorname{Hom}(R, \mathbb{k}[\varepsilon]/(\varepsilon^2))$, which is dual as $\mathbb{k}$-vector space to $\mathfrak{m}/(\mathfrak{m}^2 + \mu R) \cong \operatorname{Der}_\Lambda(R, \mathbb{k})$, where $\mathfrak{m} \subseteq R$, $\mu \subseteq \Lambda$ are the respective maximal ideals. In the case where a functor F as above is isomorphic to h_R for some $R \in \widehat{\mathbf{C}}$, it is called *pro-representable*. In this case, there exists $R \in \widehat{\mathbf{C}}$ and a smooth morphism $h_R \to F$, such that the induced map on tangent spaces $t_{h_R} \to t_F$ is an isomorphism of $\mathbb{k}$-vector spaces, R is called a *hull* (or *miniversal defomation*) for F.

Let us now turn to the deformation theory of a scheme X over a field $\mathbb{k}$, that is, we want to classify all flat schemes over $R \in \mathbf{C}$ with special fiber X. Now, if $\mathbb{k}$ is of characteristic zero, one usually sets $\Lambda = \mathbb{k}$. On the other hand, if $\mathbb{k}$ is of positive characteristic p, then there are two natural choices: the first one is $\Lambda = \mathbb{k}$, which means that one only studies deformations over rings in characteristic p, so called *equi-characteristic* deformations. Secondly, if $\mathbb{k}$ is assumed to be perfect, one can also consider the case $\Lambda = W(\mathbb{k})$, i.e., the Witt ring over $\mathbb{k}$. In this latter case, one also allows *liftings* to characteristic zero. This said, we make the following definition.

Definition 5.11.1 Let X be a scheme over $\mathbb{k}$.

1. An *infinitesimal deformation* of X is a scheme $\mathcal{X} \to \operatorname{Spec} R$ with $R \in \mathbf{C}$ that is flat over R, and with special fiber $\mathcal{X} \times_{\operatorname{Spec} R} \operatorname{Spec} \mathbb{k}$ isomorphic to X.
2. The functor $\operatorname{Def}_X : \mathbf{C} \to (\text{Sets})$ that associates to each $R \in \mathbf{C}$ the set of infinitesimal deformations of X over R modulo isomorphism is called the *deformation functor* of X.

Before proceeding, let us mention a technical point: when constructing moduli stacks below, it is crucial to look at deformations that are allowed to be algebraic spaces rather than only schemes, even if one is only interested in moduli spaces for schemes. Now, by [399, Corollary 3.6], an algebraic space over $R \in \mathbf{C}$, whose special fiber over $\mathbb{k}$ is a scheme, is automatically a scheme. So, to understand the deformation theory of a scheme X over $\mathbb{k}$, it suffices to study deformations as previously defined.

Now, we want to understand Def_X for a scheme X of finite type over a field $\mathbb{k}$ in detail. Quite generally, whenever X is proper over $\mathbb{k}$, or affine with isolated singularities, then Def_X possesses a hull, see [637, Proposition 3.10]. On the other hand, pro-representability of this functor is more subtle and not even true for all Enriques surfaces.

To understand pro-representability of Def_X, let us briefly digress on tangent-obstruction theories: first, let X be an affine scheme that is of finite type over $\mathbb{k}$, let

$f : A \to B$ be a small extension in $\mathbf{C}$, and let $\mathcal{X} \to \operatorname{Spec} B$ be a deformation of X. Consider the pre–image of $\mathcal{X} \in \operatorname{Def}_X(B)$ under the map $\operatorname{Def}_X(A) \to \operatorname{Def}_X(B)$, which is the set of deformations of X over A that become isomorphic to $\mathcal{X}$ when restricting to B. This set, which may be empty, is a torsor under $\operatorname{Hom}_X(\Omega_{X/\mathbb{k}}, \mathcal{O}_X) \otimes_{\mathbb{k}} \operatorname{Ker}(f)$. We note that if $\Lambda = \mathbb{k}$, then there exist distinguished deformations, namely trivial product families, which allow us to compare deformations with trivial ones, and thus, the torsor-structure is in fact a module structure. However, if Λ is not a $\mathbb{k}$-algebra – for example, if $\mathbb{k}$ is perfect of positive characteristic and $\Lambda = W(\mathbb{k})$ – then there is no such thing as a trivial product family.

Now, let X be a scheme that is smooth over $\mathbb{k}$, let $f : A \to B$ be a small extension in $\mathbf{C}$, and let $\mathcal{X} \to \operatorname{Spec} B$ be a deformation of X. Next, we choose an open affine cover $\mathcal{U}_\alpha$, $\alpha \in I$ of $\mathcal{X}$. Since X is smooth over k, also $\mathcal{X}$ is smooth over B by openness of smoothness, and then, for every $\mathcal{U}_\alpha \to \operatorname{Spec} B$, there exists a scheme $\mathcal{V}_\alpha \to \operatorname{Spec} A$ that is smooth over A and with $\mathcal{V}_\alpha \times_{\operatorname{Spec} A} \operatorname{Spec} B \cong \mathcal{U}_\alpha$. From this, a standard computation with cocycles shows that there exists a cohomology class in

$$\operatorname{Ext}^2_X(\Omega_{X/\mathbb{k}}, \mathcal{O}_X) \otimes_{\mathbb{k}} \operatorname{Ker}(f) \cong H^2(T_X) \otimes_{\mathbb{k}} \operatorname{Ker}(f),$$

whose vanishing is necessary and sufficient for the existence of a scheme $\mathcal{Y} \to \operatorname{Spec} A$ with $\mathcal{Y} \times_{\operatorname{Spec} A} \operatorname{Spec} B \cong \mathcal{X}$, i.e. for an extension of the deformation $\mathcal{X}$ from B to A. If this class is zero, then the set of all such extensions is non-empty and is a torsor under

$$\operatorname{Ext}^1_X(\Omega_{X/\mathbb{k}}, \mathcal{O}_X) \otimes_{\mathbb{k}} \operatorname{Ker}(f) \cong H^1(T_X) \otimes_{\mathbb{k}} \operatorname{Ker}(f).$$

And finally, the set of automorphisms of an extension $\mathcal{Y} \to \operatorname{Spec} A$ that are trivial over $\mathcal{X} \to \operatorname{Spec} B$ is isomorphic to $\operatorname{Ext}^1_X(\Omega_{X/\mathbb{k}}, \mathcal{O}_X) \otimes_{\mathbb{k}} \operatorname{Ker}(f) \cong H^1(T_X) \otimes_{\mathbb{k}} \operatorname{Ker}(f)$. If X is also assumed to be proper over $\mathbb{k}$, then all the $H^i(T_X) \otimes_{\mathbb{k}} \operatorname{Ker}(f)$ are finite-dimensional $\mathbb{k}$-vector spaces. These considerations show that if X is a scheme that is smooth and proper over $\mathbb{k}$, then the functor possesses a *tangent-obstruction theory* via $T^i_X := H^i(T_X)$, see also [230, Part 3, Definition 6.1.21] for definitions and details. Let us now apply these general techniques to Enriques surfaces.

Proposition 5.11.2 *Let S be an Enriques surface over a perfect field $\mathbb{k}$ of characteristic $p \geq 0$. Then,* Def_S *possesses a hull. If we assume moreover that $h^0(T_S) = 0$, which holds, for example, if $p \neq 2$ (see Table 1.2 for all cases), then* Def_S *is pro-representable by R, where*

$$R = \begin{cases} \mathbb{k}[[x_1, \ldots, x_{10}]] & \text{if } \Lambda = \mathbb{k}, \\ W(\mathbb{k})[[x_1, \ldots, x_{10}]] & \text{if } p \neq 0,\ \mathbb{k} \text{ is perfect, and } \Lambda = W(\mathbb{k}), \end{cases}$$

and where $W(\mathbb{k})$ denotes the ring of Witt vectors of $\mathbb{k}$.

Proof Being a proper variety over a field, Def_S possesses a hull by [637, Proposition 3.10]. Next, assume that $H^0(T_S) = 0$. Since this is the tangent space to the identity component of $\operatorname{Aut}_{S/\mathbb{k}}$, it follows that $\operatorname{Aut}_{S/\mathbb{k}}$ is reduced, and thus, smooth. By loc. cit., Def_S is pro-representable by some Noetherian, local and complete Λ-algebra R.

Moreover, by Table 1.2, we have $h^2(T_S) = 0$ and $h^1(T_S) = 10$. The former implies that R is formally smooth over Λ, and since the latter gives the dimension of the Zariski tangent space of R, the remaining assertions follow. □

Similar to Definition 5.11.1 above, one can also consider a scheme X over $\Bbbk$ together with an invertible sheaf $\mathcal{L} \in \mathrm{Pic}(X)$, and define the functor $\mathrm{Def}_{(X,\mathcal{L})}$: $\mathbf{C} \to$ (Sets) that associates to each $R \in \mathbf{C}$ the set of deformations of the pair $(X, \mathcal{L})$ over R modulo isomorphism. Assume for simplicity that X is smooth over $\Bbbk$ so that the tangent sheaf T_X is a locally free of rank equal to $\dim(X)$. The first Chern class, which can be purely algebraically defined via the $d\log$-map, $c_1(\mathcal{L}) \in H^1(\Omega^1_X) \cong \mathrm{Ext}^1(T_X, \mathcal{O}_X)$ defines an extension

$$0 \to \mathcal{O}_X \to \mathcal{E}_{\mathcal{L}} \to T_X \to 0,$$

the *Atiyah extension* of $\mathcal{L}$. Then, the deformation theory of the pair $(X, \mathcal{L})$ has a tangent-obstruction theory given by $H^i(\mathcal{E}_{\mathcal{L}})$ with $i = 0, 1, 2$. If S is an Enriques surface in characteristic $\neq 2$, and $\mathcal{L} \in \mathrm{Pic}(S)$ is arbitrary, then taking cohomology in the Atiyah extension sequence and using Table 1.2 we find $h^0(\mathcal{E}_{\mathcal{L}}) = 1$, $h^1(\mathcal{E}_{\mathcal{L}}) = 10$, and $h^2(\mathcal{E}_{\mathcal{L}}) = 0$. We refer to [651, Section 3.3] for details and background. We obtain the following result, whose proof is completely analogous to that of Proposition 5.11.2.

Proposition 5.11.3 *Let S be an Enriques surface over a perfect field $\Bbbk$ of characteristic $p \neq 2$, and $\mathcal{L} \in \mathrm{Pic}(S)$ be an invertible sheaf. Then, $\mathrm{Def}_{(S,\mathcal{L})}$ is pro-representable by a formally smooth Λ-algebra, where $\Lambda = \Bbbk$ or $\Lambda = W(\Bbbk)$ as above.*

The tangent space to (the identity component of) the automorphism group scheme $\mathrm{Aut}^0_{S/\Bbbk}$ of an Enriques surface S is isomorphic to $H^0(T_S)$. In particular, this group scheme is smooth if and only if it is reduced if and only if $h^0(T_S) = 0$. By [637, Proposition 3.10], this is also equivalent to Def_S being pro-representable. In particular, for Enriques surfaces with $h^0(T_S) \neq 0$, which exist in characteristic 2 by Table 1.2, the situation is much more complicated. The following highly-non-trivial result is due to Ekedahl, Hyland, and Shepherd-Barron [214].

Theorem 5.11.4 *Let S be an Enriques surface over an algebraically closed field $\Bbbk$ of characteristic $p = 2$, and let $\Lambda = W(\Bbbk)$.*

1. *If S is an $\boldsymbol{\alpha}_2$-surface, then the hull of Def_S is given by*

$$R = W(\Bbbk)[[x_1, \ldots, x_{12}]]/(fg - 2),$$

   where f, g lie in the ideal $(2, x_1, \ldots, x_{12})$.
2. *If S is a classical Enriques surface with $h^0(T_S) = h^2(T_S) = 1$, then the hull of Def_S is given by*

$$R = W(\Bbbk)[[x_1, \ldots, x_{12}]]/(h),$$

   where the power series $h = h(x_1, \ldots, x_{12})$ that satisfies $p \nmid h$.

In both cases, Def_S is not pro-representable.

We refer to [214, Section 4] for a more detailed analysis of the deformation functors of Enriques surfaces (polarized as well as without polarization) in characteristic 2.

An important application is the liftability of Enriques surfaces to characteristic zero. Let us recall the definitions: let X be a smooth and proper variety over some perfect field $\mathbb{k}$, and let R be a Noetherian, local, and complete DVR with residue field $\mathbb{k}$. Since $\mathbb{k}$ is perfect, R contains naturally the ring $W(\mathbb{k})$ of Witt vectors. Moreover, if R is a finite and integral extension of $W(\mathbb{k})$, then the valuation (normalized such that a uniformizer of R has valuation 1) of p with respect to the discrete valuation of R is called the *(absolute) ramification index*. An *algebraic (resp. formal) lift* of X over R is a scheme (resp. formal scheme) $\mathcal{X} \to \operatorname{Spec} R$ (resp. $\mathcal{X} \to \operatorname{Spf} R$), which is flat over R with special fiber X. By passing to the completion along its special fiber, every algebraic lift yields a formal lift. Moreover, since R is automatically a $W(\mathbb{k})$-algebra, the most desireable lifts are algebraic lifts over $W(\mathbb{k})$. We refer to [458, Section 11] for more about lifting of varieties and references. The following combines several results of Lang [433], Liedtke [460], as well as Ekedahl, Hyland, and Shepherd-Barron [214].

Theorem 5.11.5 *Let S be an Enriques surface over an algebraically closed field $\mathbb{k}$ of positive characteristic p.*

1. *If $h^0(T_S) = 0$, which holds, for example, if $p \neq 2$, then S admits an algebraic lift over $W(\mathbb{k})$.*
2. *In the remaining cases, there exists a an algebraic lift of S over a finite extension $R \supseteq W(\mathbb{k})$, whose ramification index divides $2^9 N$, where $N \in \{9, 49, 56, 60, 128, 192\}$.*
3. *If S is an $\boldsymbol{\alpha}_2$-surface, then it does not lift over $W_2(\mathbb{k})$. In particular, it does not lift over $W(\mathbb{k})$, not even formally.*

Proof In the first case, we have $h^0(T_S) = h^2(T_S) = 0$, and then, it follows from Proposition 5.11.2 that S admits a formal lift over $W(\mathbb{k})$. If S is classical, then $h^2(\mathcal{O}_S) = 0$, which implies that invertible sheaves can be lifted to every formal lift. In particular, one can lift an ample invertible sheaf to this formal lift, which is thus algebraizable by Grothendieck's existence theorem. If $p = 2$ and S is a $\boldsymbol{\mu}_2$-surface, then for every $\mathcal{L} \in \operatorname{Pic}(S)$, the invertible sheaf $\mathcal{L}^{\otimes 2}$ lifts to every formal lift by [214], Proposition 4.2, [460], Proposition 4.4, see also [433], Theorem 1.4. From this, algebraization follows as before.

In the remaining cases, $p = 2$ and S is an $\boldsymbol{\alpha}_2$-surface or a classical Enriques surface with $h^0(T_S) = h^2(T_S) = 1$. But then, there exists a formal lift over a possibly ramified extension $R \supseteq W(\mathbb{k})$ by Theorem 5.11.4. Algebraicity of the formal lift follows as in the case of $\boldsymbol{\mu}_2$-surfaces. Next, to bound the ramification, we use that by Corollary 5.11.10 (see also [460, Theorem 4.9]), there exists a birational morphism $S \to S'$ such that S' has at worst rational double point singularities, and such that S' lifts algebraically over $W(\mathbb{k})$ (if it not an $\boldsymbol{\alpha}_2$-surface), or to $W(\mathbb{k})[\sqrt{2}]$ (if S is an $\boldsymbol{\alpha}_2$-surface). By Artin's theorem on simultaneous resolutions of rational double point singularities [24], there exists an algebraic lift (via an algebraic space) to some

$R \supseteq W(\Bbbk)$. Extending an ample invertible sheaf from the special fiber to the total space of the lift, it follows that the algebraic space is projective over R, whence a scheme. A careful analysis of the possible singularities of S' then yields the estimate on the bound of ramification, and we refer to [214, Corollary 5.7] for details.

Finally, if S is an $\boldsymbol{\alpha}_2$-surface, then $\mathbf{Pic}^{\tau}_{S/\Bbbk} \cong \boldsymbol{\alpha}_2$. If S were to lift to $W_2(\Bbbk)$, then so would $\mathbf{Pic}^{\tau}_{S/\Bbbk}$. On the other hand, it follows from the description of all deformations of $\boldsymbol{\alpha}_2$, see, for example, Example 1.6.6 or [584] that $\boldsymbol{\alpha}_2$ does not admit a lift to $W_2(\Bbbk)$. □

We now turn to moduli stacks of Enriques surfaces over arbitrary base rings and remind the reader of the short introduction given in Section 5.3. We start with a rather technical observation: we saw in Section 5.5 that a general Enriques surface over $\mathbb{C}$ has an infinite and discrete automorphism group. Thus, when considering moduli of Enriques surfaces with no extra data, we obtain a stack, whose stabilizers are in general infinite discrete groups. Thus, the diagonal morphism of this stack is not quasi-compact, that is, this stack is not quasi-separated. However, quasi-separatedness is usually built in from the very beginning when discussing stacks for several technical reasons, see for example [444] or [581]. Therefore, we shall only consider moduli spaces of (numerically) polarized Enriques surfaces in the sequel, as this will force the automorphism groups of the objects to be finite, and then, the moduli spaces will exist as algebraic Artin stacks. In characteristic $\neq 2$, these will even be Deligne–Mumford stacks.

We fix a base scheme B or $\mathbb{Z}$. We fix a positive and even integer n and consider the functor of *degree n polarized Enriques surfaces*

$$\begin{array}{rcl} \widetilde{\mathcal{F}}_{\mathrm{Enr},n,B} : (\mathrm{Schemes}/B) & \to & (\mathrm{Groupoids}) \\ T & \mapsto & \left\{ \begin{array}{l} \text{morphisms of algebraic spaces } (\mathcal{S},\mathcal{L}) \to T, \\ \text{whose geometric fibers are Enriques surfaces,} \\ \text{and where } \mathcal{L} \in \mathrm{Pic}(\mathcal{S}/T) \text{ restricts to a big and nef} \\ \text{invertible sheaf with self-intersection number } n \\ \text{in every geometric fiber.} \end{array} \right\} \end{array}$$

We have the following related functors: $\widetilde{\mathcal{F}}^{\mathrm{a}}_{\mathrm{Enr},n,B}$ denotes the open subfunctor of $\widetilde{\mathcal{F}}_{\mathrm{Enr},n,B}$ that takes values in those families where $\mathcal{L}$ is relatively ample, that is, *degree n ample polarized Enriques surfaces*. Next, we denote by $\widetilde{\mathcal{F}}^{\mathrm{a}}_{\mathrm{EnrRDP},n,B}$ the functor that takes values in families whose geometric fibers are pairs of Enriques surfaces with at worst rational double point (RDP) singularities and ample invertible sheaves of self-intersection n. Given a pair $(Y,\mathcal{L})$, where Y is an Enriques surface and $\mathcal{L}$ is big and nef, we consider the map:

$$Y \to Y' := \mathrm{Proj} \bigoplus_{i\geq 0} H^0(Y, \mathcal{L}^{\otimes i}), \tag{5.11.1}$$

which is a birational morphism that contracts precisely those curves on S that have zero-intersection with $\mathcal{L}$ and nothing else. Since contracted cycles have negative intersection matrix, and since an integral curve on a surface with numerically trivial canonical sheaf must be a $\mathbb{P}^1$ with self-intersection number -2 by the adjunction formula, we conclude that S' has at worst RDP singularities. Moreover, the $\mathcal{O}_{S'}(1)$ is an ample invertible sheaf with self-intersection number equal to that of $\mathcal{L}$. Since (5.11.1) works in families, we obtain a contraction morphism Φ of functors, and we obtain the following:

$$\widetilde{\mathcal{F}}^{\mathrm{a}}_{\mathrm{Enr},n,B} \xrightarrow{\iota} \widetilde{\mathcal{F}}_{\mathrm{Enr},n,B} \xrightarrow{\Phi} \widetilde{\mathcal{F}}^{\mathrm{a}}_{\mathrm{EnrRDP},n,B} .$$

Here, ι is an open immersion, and $\Phi \circ \iota$ is still an open immersion. Moreover, since RDP singularities have unique minimal resolutions, it follows that Φ is a bijection on geometric points. However, in general Φ is not an isomorphism, which is related to functor of simultaneous resolution of deformations of RDP singularities.

We may also consider the functor of *degree n numerically polarized Enriques surfaces*, denoted $\mathcal{F}_{\mathrm{Enr},n,B}$, where we consider the class of $\mathcal{L} \in \mathrm{Pic}(\mathcal{S}/T) = \mathrm{Pic}_{\mathcal{S}/T}(T)$ inside $\mathrm{Num}(\mathcal{S}/T)$. Similarly, we can define functors $\mathcal{F}^{\mathrm{a}}_{\mathrm{Enr},n,B}$ and $\mathcal{F}^{\mathrm{a}}_{\mathrm{EnrRDP},n,B}$, Again, we have a natural forgetful morphism

$$\widetilde{\mathcal{F}}_{\mathrm{Enr},n,B} \to \mathcal{F}_{\mathrm{Enr},n,B}$$

and similarly for the other functors. This map is a torsor under a finite and flat group scheme of length 2. More precisely, if $\mathcal{Y} \to T$ is a family of numerically polarized Enriques surfaces, then the fiber over it is a torsor under $\mathrm{Pic}^{\tau}_{\mathcal{Y}/T}$ of In particular, if 2 is invertible in B, then this morphism of functors is a $\mathbb{Z}/2\mathbb{Z}$-torsor and thus, étale. This follows easily from Theorem 1.2.1 and we note that the $\mathbb{Z}/2\mathbb{Z}$-action is given on geometric points by $(Y, \mathcal{L}) \mapsto (Y, \mathcal{L} \otimes \omega_Y)$.

After these preparations, we have the following representability results, which are straight forward applications of Artin's work on deformation theory, algebraization, and stacks.

Theorem 5.11.6 *For all positive integers n and base schemes B, the functors*

$$\mathcal{F}^{\mathrm{a}}_{\mathrm{Enr},n,T}, \quad \mathcal{F}_{\mathrm{Enr},n,S}, \quad \mathcal{F}^{a}_{\mathrm{EnrRDP},nT}, \quad \mathcal{F}^{\mathrm{a}}_{\mathrm{Enr},n,T}, \quad \mathcal{F}^{\mathrm{a}}_{\mathrm{Enr},n,T}, \quad \text{and} \quad \mathcal{F}^{\mathrm{a}}_{\mathrm{EnrRDP},n,T}$$

are representable by quasi-separated Artin stacks of finite type over S. Moreover,

$$\mathcal{F}^{\mathrm{a}}_{\mathrm{Enr},n,B}, \quad \mathcal{F}^{\mathrm{a}}_{\mathrm{Enr},n,B}, \quad \mathcal{F}^{\mathrm{a}}_{\mathrm{EnrRDP},n,B}, \quad \text{and} \quad \mathcal{M}^{\mathrm{a}}_{\mathrm{EnrRDP},n,B}$$

are separated over B. If 2 *is invertible on B, then*

$$\mathcal{F}_{\mathrm{Enr},n,B}, \quad \mathcal{F}^{\mathrm{a}}_{\mathrm{Enr},n,B}, \quad \mathcal{F}_{\mathrm{Enr},n,B}, \quad \text{and} \quad \mathcal{F}^{\mathrm{a}}_{\mathrm{Enr},n,B}$$

are representable by Deligne–Mumford stacks that are smooth over B.

Proof The fact that all these functors are representable by quasi-separated Artin stacks of finite type over R follows from Artin's work, see, for example, [26, Example 5.5]. The separatedness assertion for functors of polarized families follows from the

Matsusaka–Mumford theorem [499], and we refer the reader to [620, Theorem 4.3.3] for details.

If 2 is invertible on B then every geometric point $(Y, \mathcal{L})$ of $\mathcal{F}_{\mathrm{Enr},n,B}$ or of $\mathcal{F}^{\mathrm{a}}_{\mathrm{Enr},n,B}$ corresponds to a polarized Enriques surfaces in characteristic $\neq 2$. By Table 1.2, we have $h^0(\Theta_Y) = 0$, which implies that the automorphism group scheme of Y is discrete and reduced. In particular, the automorphism group of $(Y, \mathcal{L})$ is discrete, reduced, and finite, whence étale. In particular, $\widetilde{\mathcal{F}}_{\mathrm{Enr},n,B}$ and $\widetilde{\mathcal{F}}^{\mathrm{a}}_{\mathrm{Enr},n,B}$ are representable by Deligne–Mumford stacks. Since deformations of polarized Enriques are unobstructed by Proposition 5.11.3, these stacks are smooth over B. Using the finite étale maps $\widetilde{\mathcal{F}}_{\mathrm{Enr},n,B} \to \mathcal{F}_{\mathrm{Enr},n,B}$ and $\widetilde{\mathcal{F}}^{\mathrm{a}}_{\mathrm{Enr},n,B} \to \mathcal{F}^{\mathrm{a}}_{\mathrm{Enr},n,B}$, we obtain the remaining assertions. We refer to [460] and [620] for more details. □

This settles existence of these moduli spaces, but it says nothing about their geometry, such as dimension, number of components, and birational geometry (such as unirationality). For a field $\Bbbk$ and a vector $\omega \in \mathsf{E}_{10}$, we denote by $\mathcal{F}_{\mathrm{Enr},\omega,\Bbbk}$ (resp. $\mathcal{F}_{\mathrm{Enr},\omega,\Bbbk}$) the component of $\mathcal{F}_{\mathrm{Enr},\omega^2,\Bbbk}$ (resp. $\mathcal{F}_{\mathrm{Enr},\omega^2,\Bbbk}$) corresponding to Enriques surfaces that are numerically (resp. Picard) ω-polarized. In Section 5.5, we constructed quasi-projective varieties $\widetilde{\mathcal{M}}_{\mathrm{Enr},v}$ and $\widetilde{\mathcal{M}}_{\mathrm{Enr},v}$ over $\mathbb{C}$. Using period spaces and complex analytic methods, we were able to describe their birational geometry in some cases. It seems plausible that these latter spaces are the coarse moduli spaces of the former stacks. This is indeed the case.

Theorem 5.11.7 *Let* $\omega \in \mathsf{E}_{10}$.

1. *The moduli stacks* $\mathcal{F}_{\mathrm{Enr},\omega,\mathbb{C}}$ *and* $\widetilde{\mathcal{F}}_{\mathrm{Enr},\omega,\mathbb{C}}$ *admit coarse moduli spaces* $\mathcal{M}_{\mathrm{Enr},\omega,\mathbb{C}}$ *and* $\widetilde{\mathcal{M}}_{\mathrm{Enr},\omega,\mathbb{C}}$. *Moreover, we have a commutative diagram of algebraic stacks*

$$\begin{array}{ccc} \widetilde{\mathcal{F}}_{\mathrm{Enr},\omega,\mathbb{C}} & \to & \widetilde{\mathcal{M}}_{\mathrm{Enr},\omega,\mathbb{C}} \\ \downarrow & & \downarrow \\ \mathcal{F}_{\mathrm{Enr},\omega,\mathbb{C}} & \to & \mathcal{M}_{\mathrm{Enr},\omega,\mathbb{C}}, \end{array}$$

   where the horizontal morphisms are the universal maps of coarse moduli spaces and where the vertical morphisms are étale of degree 2.
2. *The complex algebraic space* $\mathcal{M}_{\mathrm{Enr},\omega,\mathbb{C}}$ *(resp.* $\widetilde{\mathcal{M}}_{\mathrm{Enr},\omega,\mathbb{C}}$*) is isomorphic to the complex quasi-projective variety* $\mathcal{M}_{\mathrm{Enr},v}$ *(resp.* $\widetilde{\mathcal{M}}_{\mathrm{Enr},v}$*) constructed in Section 5.5.*

Proof Existence of the coarse moduli spaces $\mathcal{M}_{\mathrm{Enr},\omega,\mathbb{C}}$ and $\widetilde{\mathcal{M}}_{\mathrm{Enr},\omega,\mathbb{C}}$ follows quite generally from the theorem of Keel and Mori [381]. Moreover, we have already noted above that there is an étale morphism $\tilde{\mathcal{F}}_{\mathrm{Enr},\omega,\mathbb{C}} \to \mathcal{F}_{\mathrm{Enr},\omega,\mathbb{C}}$ of degree 2 of stacks. This induces a morphism $\widetilde{\mathcal{M}}_{\mathrm{Enr},\omega,\mathbb{C}} \to \mathcal{M}_{\mathrm{Enr},\omega,\mathbb{C}}$ of coarse moduli spaces and we obtain the stated commutative diagram. See also [262, Proposition 4.1].

Let T be a scheme over $\mathbb{C}$. We denote by M_ω the complex quasi-projective variety $\mathcal{M}_{\mathrm{Enr},v}$ constructed in Section 5.5. To give a morphism $T \to \mathcal{F}_{\mathrm{Enr},\omega,\mathbb{C}}$ is the same as to give a family $\mathcal{S} \to T$ of numerically ω-polarized Enriques surfaces. In Section 5.5, we saw how this gives rise to morphism $T \to M_\omega$. By the Yoneda lemma, this induces a morphism of stacks $\mathcal{F}_{\mathrm{Enr},\omega,\mathbb{C}} \to M_\omega$ and by the universal property of coarse moduli spaces, we obtain a morphism of algebraic spaces $\mathcal{M}_{\mathrm{Enr},\omega,\mathbb{C}} \to M_\omega$. It follows from the construction of M_ω in Section 5.5 that this latter map induces a bijection on the level of $\mathbb{C}$-valued points. Moreover, $\mathcal{M}_{\mathrm{Enr},\omega,\mathbb{C}}$ is a normal algebraic space and by construction, M_ω is a normal and quasi-projective variety. Thus, $\mathcal{M}_{\mathrm{Enr},\omega,\mathbb{C}} \to M_\omega$ is an isomorphism by Zariski's Main Theorem. We leave the case of Picard polarized Enriques surfaces to the reader. □

In the case $B = \operatorname{Spec} \Bbbk$, especially if $\Bbbk$ is a field of characteristic 2 or even $B = \operatorname{Spec} \mathbb{Z}$, currently only one moduli space has been studied in detail, namely the *moduli stack for numerically Cossec–Verra polarized Enriques surfaces*. Let us recall (for example, from Corollary 1.5.4 or the discussion in Section 5.5), that there are two $W(\mathsf{E}_{10})$-orbits of vectors of self-intersection number 4 in E_{10}, namely the orbit of ω_1 and the orbit of $\omega_8+\omega_9$. The vectors of the first orbit satisfy $\Phi = 2$ and the vectors of the second orbit satisfy $\Phi = 1$ with respect to the Φ-function introduced in (2.4.6). Clearly, the Φ-function is constant on every component of a (coarse) moduli space of (numerically) polarized Enriques surfaces. In particular, moduli spaces of (numerically) degree 4 polarized Enriques surfaces have component(s) corresponding to $\Phi = 1$-polarizations and component(s) corresponding to $\Phi = 2$-polarizations. We are now interested in the latter case. A (numerical) polarization $\mathcal{L}$, whose class belongs to the second case, that is, a big and nef (numerical) invertible sheaf with $\mathcal{L}^2 = 4$ and $\Phi(\mathcal{L}) = 2$ is called a *Cossec–Verra polarization* and we studied such polarizations in Section 3.4. In particular, we denote the union of components of the moduli stacks $\mathcal{F}^{\mathrm{a}}_{\mathrm{Enr},4,B}$, $\mathcal{F}_{\mathrm{Enr},4,B}$, and $\mathcal{F}^{\mathrm{a}}_{\mathrm{EnrRDP},4,B}$ by

$$\mathcal{F}^{\mathrm{a}}_{\mathrm{Enr,CV},B}, \quad \mathcal{F}_{\mathrm{Enr,CV},B}, \quad \text{and} \quad \mathcal{F}^{\mathrm{a}}_{\mathrm{EnrRDP,CV},B}$$

respectively, and similarly for moduli stacks of numerically polarized Enriques surfaces. The following result shows that the moduli stack $\mathcal{M}^{\mathrm{a}}_{\mathrm{EnrRDP,CV},\mathbb{Z}}$ has a very easy and beautiful geometry.

Theorem 5.11.8 *There exists a smooth morphism of relative dimension* 10

$$\mathcal{F}^{\mathrm{a}}_{\mathrm{EnrRDP,CV},\mathbb{Z}} \xrightarrow{\mathrm{Pic}^\tau} [\operatorname{Spec} \mathbb{Z}[a,b]/(ab-2)\,/\,\mathbb{G}_m].$$

More precisely,

1. *For every field* $\Bbbk$ *of characteristic* $p \neq 2$, $\mathcal{F}^{\mathrm{a}}_{\mathrm{EnrRDP,CV},\Bbbk}$ *is a smooth, geometrically irreducible,* 10*-dimensional, and unirational Artin stack over* $\operatorname{Spec} \Bbbk$.
   If $p \notin \{3,5,7\}$*, then this stack is even a Deligne–Mumford stack.*
2. *For every field* $\Bbbk$ *of characteristic* 2*,*
   a. $\mathcal{F}^{a}_{\mathrm{EnRDP,CV},\Bbbk}$ *consists of two components* $\mathcal{F}^{\mathbb{Z}/2\mathbb{Z}}$ *and* $\mathcal{F}^{\mu_2}$*, both of which are smooth, geometrically irreducible,* 10*-dimensional, and unirational Artin stacks over* $\operatorname{Spec} \Bbbk$.
   b. *They intersect transversally along a smooth, geometrically irreducible,* 9*-dimensional, and unirational substack* $\mathcal{F}^{\alpha_2}$ *over* $\operatorname{Spec} \Bbbk$.

c. *The geometric points of $\mathcal{F}^{\alpha_2}$ are $\boldsymbol{\alpha}_2$-Enriques surfaces, whereas for $G = \boldsymbol{\mu}_2$ and $G = \mathbb{Z}/2\mathbb{Z}$, the geometric points of $\mathcal{F}^G\backslash\mathcal{F}^{\alpha_2}$ are G-Enriques surfaces.*

Proof Let us sketch the idea of proof and refer to [460] for details: Let S be an Enriques surface with at worst RDP singularities, let $\pi : X \to S$ be its K3-cover, and let $\mathcal{L}$ be an ample numerical Cossec–Verra polarization. Then, $\pi^*\mathcal{L}$ is a globally generated invertible sheaf on X, and defines a morphism $f : X \to \mathbb{P}^5$ that is an isomorphism onto its image, which is the complete intersection of three quadrics, see Theorem 3.4.1. Moreover, the $(\mathrm{Pic}^{\tau}_{S/\Bbbk})^*$-action on X extends to a linear action on $\mathbb{P}^5$, and the three quadrics cutting out $f(X)$ can be chosen to be $(\mathrm{Pic}^{\tau}_{S/\Bbbk})^*$-invariant. Thus, we obtain a description of X and S as in the Bombieri–Mumford–Reid examples from Example 1.6.8. Using that complete intersections have unobstructed deformations, the only obstruction to deforming X comes from deforming $\mathrm{Pic}^{\tau}_{S/\Bbbk}$, from which it follows that the deformation functor of the pair $(S, \mathcal{L})$ is smooth over the deformation functor of $\mathrm{Pic}^{\tau}_{S/\Bbbk}$. Using the Oort-Tate classification [584] (see also Example 1.6.6), we thus obtain a smooth morphism of relative dimension 10 from $\mathcal{F}^{\mathrm{a}}_{\mathrm{EnrRDP,CV},\mathbb{Z}}$ to $[\mathrm{Spec}\,\mathbb{Z}[a,b]/(ab-2)/\mathbb{G}_m]$. Next, let G be a group scheme of length 2 over a field $\Bbbk$ and recall the G-invariant quadrics from Lemma 1.6.7. Then, complete intersections of three G-invariant quadrics in $\mathbb{P}^5$ are parametrized by some open dense set of some affine space, which is rational. From this, we obtain an over-parametrization of a component of $\mathcal{F}^{\mathrm{a}}_{\mathrm{EnrRDP,CV},\Bbbk}$, and the stated unirationality and irreducibility results follow. □

Remark 5.11.9 Since an Enriques surface S in characteristic $p \neq 2$ satisfies $h^0(S, T_S) = 0$, the automorphism group $\mathrm{Aut}(S)$ is reduced and thus, stacks of smooth and polarized Enriques surfaces are then Deligne–Mumford stacks. On the other hand, $\mathcal{F}^{a}_{\mathrm{EnRDP,CV},\Bbbk}$ parametrizes Enriques surfaces with RDP singularities. Now, it could happen that such a surface does have global vector fields, which would be related to the fact that the RDP singularities occuring on such a surface admit non-trivial global vector fields that do not lift to the resolution of singularities and refer to [305] for results concerning this phenomenon. At points of $\mathcal{F}^{a}_{\mathrm{EnRDP,CV},\Bbbk}$ corresponding to such a surface, the stack cannot be Deligne–Mumford. We do not know whether such RDP-Enriques surfaces with global vector fields exist in characteristic $p \geq 3$. However, if $p \geq 11$, then no RDP singularity that can possibly occur an Enriques surface admits such non-trivial and non-liftable vector field, that is, no RDP-Enriques surface admits global vector fields, which shows that the stacks $\mathcal{F}^{a}_{\mathrm{EnRDP,CV},\Bbbk}$ are Deligne–Mumford if $p \geq 11$.

In the proof of the theorem we have seen that the deformation functor of the pair $(S, \mathcal{L})$, where S is an Enriques surface with at worst RDP singularities and where $\mathcal{L}$ is an ample numerical Cossec–Verra polarization, is smooth over the deformation functor of $\mathrm{Pic}^{\tau}_{S/\Bbbk}$. It follows that deformations and liftings of S to characteristic zero are controlled by deformations and liftings of $\mathrm{Pic}^{\tau}_{S/\Bbbk}$. This has the following interesting consequence.

Corollary 5.11.10 *Let S be an Enriques surface over an algebraically closed field $\Bbbk$ of characteristic $p > 0$.*

1. If $p \neq 2$, then S lifts algebraically over $W(\Bbbk)$.

2. If $p = 2$, then let $\mathcal{L}$ be an ample Cossec–Verra polarization and let $S \to S'$ be the associated contraction morphism (5.11.1). *Then, S' is an Enriques surface with RDP singularities and S' lifts algebraically to $W(\Bbbk)$ (resp. $W(\Bbbk)[\sqrt{p}]$) if S' is not an $\boldsymbol{\alpha}_2$-surface (resp. an $\boldsymbol{\alpha}_2$-surface).*

Remark 5.11.11 The contraction morphism

$$\Phi : \mathcal{F}_{\mathrm{Enr,CV},\mathbb{Z}} \to \mathcal{F}^{\mathrm{a}}_{\mathrm{EnrRDP,CV},\mathbb{Z}}$$

induces a bijection on geometric points, but it is not an isomorphism of functors. The fibers of Φ are related to the functor of simultaneous resolutions of families of RDP singularities, and we refer to [24] for details. By Theorem 5.11.4, the deformation functor of classical Enriques surfaces in characteristic 2 may not be pro-representable by a smooth algebra. In particular, $\mathcal{F}_{\mathrm{Enr,CV},\mathbb{Z}}$ may not be smooth at points corresponding to classical Enriques surfaces in characteristic 2. By the previous theorem, these singularities can be explained from the fibers of Φ, whereas $\mathcal{F}^{\mathrm{a}}_{\mathrm{EnrRDP,CV},\mathbb{Z}}$ is smooth at points corresponding to classical Enriques surfaces. We refer to [460] for details.

We discussed coarse moduli spaces already in Section 5.3 and constructed coarse moduli spaces for (numerically) polarized Enriques surfaces in Section 5.5. In the case of $\mathcal{F}^{\mathrm{a}}_{\mathrm{EnrRDP,CV},\Bbbk}$ for a field $\Bbbk$, the previous theorem gives an explicit construction of the coarse space: let V be the space of $\mathcal{G}_{a,b}$-invariant quadrics from Lemma 1.6.7 and let $U \subset \mathbb{P}(V)^3$ be the open and dense subset that consists of triples of quadrics (Q_1, Q_2, Q_3) such that $Q_1 \cap Q_2 \cap Q_3 \subset \mathbb{P}^3$ is a complete intersection of three quadrics, such that the quotient by the $\mathcal{G}_{a,b}$-action is an Enriques surface with at worst RDP singularities. Then, the sought components of the coarse moduli spaces arise as quotients of U by a linear algebraic group. In particular, also the coarse moduli space is unirational. Moreover, the coarse moduli space of the double cover $\widetilde{\mathcal{F}}_{\mathrm{Enr,CV},\mathbb{C}}$ is even rational by a theorem of Casnati, see Theorem 5.8.12 or [107].

We end this section by discussing crystalline periods of unipotent Enriques surfaces in characteristic 2, as developed by Ekedahl, Hyland, and Shepherd-Barron [214]. Here, an Enriques surface S in characteristic 2 is called *unipotent* if Pic^{τ}_S is a unipotent group scheme, that is, if S is an $\boldsymbol{\alpha}_2$-, or an $\mathbb{Z}/2\mathbb{Z}$-Enriques surface. Note that the K3-cover $X \to S$ of a unipotent Enriques surface is finite and flat, but not étale. As before, a marking of S is a choice of isomorphism $\phi : \mathsf{E}_{10} \to \mathrm{Num}(S)$. Inside the positive cone of $\mathsf{E}_{10} \otimes \mathbb{R}$, we have the chamber $\mathcal{D}_0$ defined by the roots of E_{10}. More precisely, if $\boldsymbol{\omega}_0, ..., \boldsymbol{\omega}_9$ denote the fundamental dominant weights of E_{10} defined by the root basis, then $\mathcal{D}_0$ is the $\mathbb{R}_{\geq 0}$-span of all the $\boldsymbol{\omega}_i$. Then, we define the moduli functor

$$\begin{array}{rcl} \mathcal{E}_{\mathrm{uni}} : (\mathrm{Schemes}/\mathbb{F}_2) & \to & (\mathrm{Groupoids}) \\ B & \mapsto & \left\{ \begin{array}{l} \text{morphisms of algebraic spaces } (\mathcal{S}, \phi) \to B, \\ \text{whose geometric fibers are unipotent Enriques surfaces,} \\ \text{and where } \phi : \mathsf{E}_{10,B} \to \mathrm{Num}_{\mathcal{S}/B} \\ \text{is a marking such that } \phi(\mathcal{D}_0) \text{lies fiberwise in the ample cone.} \end{array} \right\} \end{array}$$

Via such a marking, the fundamental dominant weight ω_1 corresponds to a Cossec–Verra polarization, see [214, Lemma 5.3]. Thus, there exists a natural forgetful functor

$$\mathcal{E}_{\mathrm{uni}} \longrightarrow \mathcal{F}_{\mathrm{Enr,CV},\mathbb{F}_2}$$

that is surjective onto the $\mathcal{F}^{\mathbb{Z}/2\mathbb{Z}}$-component of the latter space, that is, the component parametrizing unipotent Enriques surfaces.

Next, let $\mathcal{E}_{\mathrm{uni},K3}$ be the open and dense substack of $\mathcal{E}_{\mathrm{uni}}$ of those Enriques surfaces, whose associated flat double cover $\pi : X \to S$ is a K3 surface with, at worst, RDP singularities. More precisely, for a generic surface S in $\mathcal{E}_{\mathrm{uni},K3}$, this double cover X has 12 RDP singularities of type A_1, and the tangent sheaf Θ_X (the dual of the sheaf of Kähler differentials) is trivial. Since π is a torsor under an infinitesimal group scheme, it corresponds to a p-closed foliation $\mathcal{F} \subset \Theta_X$, and since S is an Enriques surface, we find $\mathcal{F} \cong \mathcal{O}_X$. This leads us to considering the moduli stack of $\mathsf{E}_{10}(2)$-marked K3 surfaces with RDP singularities of RDP rank 12 and with trivial tangent sheaf:

$$\mathcal{K} : (\mathrm{Schemes}/\mathbb{F}_2) \to (\mathrm{Groupoids})$$
$$T \quad \mapsto \left\{ \begin{array}{l} \text{Families } (\mathcal{X},\phi) \to T \text{ of algebraic spaces, whose} \\ \text{geometric fibers are non-smooth K3 surfaces with} \\ \text{RDP-singularities of RDP rank 12, and such that} \\ f^*f_*\Theta_{\mathcal{X}/T} \to \Theta_{\mathcal{X}/T} \text{ is an isomorphism, and where} \\ \phi : \mathsf{E}_{10}(2)_T \to \mathrm{Pic}_{\mathcal{X}/T} \text{ is a marking such that} \\ \phi(\mathcal{D}_0(2)) \text{ lies fiberwise in the ample cone.} \end{array} \right\}$$

see also [214, Section 8]. Next, we consider the $\mathbb{P}^1$-bundle $\mathcal{P} := \mathbb{P}((f_*\mathcal{T}_{\mathcal{X}/\mathcal{K}})^\vee) \to \mathcal{K}$, where $\mathcal{T}_{\mathcal{X}/\mathcal{K}}$ is the relative tangent sheaf on the universal K3 surface $f : \mathcal{X} \to \mathcal{K}$. Thus, a geometric point of $\mathcal{P}$ is a marked K3 surface X from $\mathcal{K}$ together with a line $\Xi \subset H^0(X, \Theta_X)$. Since Ξ is automatically p-closed, we always have a purely inseparable quotient map $X \to X/\Xi$, and in the case where Ξ vanishes at the singularities of X, the quotient is in fact a unipotent Enriques surface. More precisely, if Ξ is a multiplicative vector field, which is the generic case, then X/Ξ is a $\mathbb{Z}/2\mathbb{Z}$-Enriques surface, and in the case where Ξ is an additive vector field, then X/Ξ is an $\boldsymbol{\alpha}_2$-Enriques surface.

Example 5.11.12 An explicit example of a one–dimensional family of Enriques surfaces that arises as quotient of a K3 surface by a family of p-closed vector fields was constructed in [368]: in characteristic 2, the equation

$$y^2 + y + x^3 + t^2 \cdot x \cdot (y^2 + y + 1) = 0$$

is the Weierstrass model $X_t' \to \mathbb{P}^1$ of an elliptic surface $X \to \mathbb{P}^1$, which has four singular fibers of type I_6. This surface is the unique supersingular K3 surface with Artin invariant $\sigma_0 = 1$. The model X' has twelve RDP singularities of type A_1, which correspond to singular points of the singular fibers. For parameters a, b with $a + b = ab$ and $a^3 \neq 1$ we obtain a family

$$D_{a,b} := \frac{1}{1+t}\left((t+1)(t+a)(t+b)\frac{\partial}{\partial t} + (1+t^2x)\frac{\partial}{\partial x}\right)$$

of p-closed vector fields on X' and we let

$$S_{a,b} := X'/D_{a,b}$$

be the corresponding quotient. Then, if moreover $a \notin \mathbb{F}_4$ (resp. $a = b = 0$), then $S_{a,b}$ is a classical Enriques surface (resp. α_2-surface).

If $\mathcal{U} \subset \mathcal{P}$ denotes the open substack, where Ξ vanishes at the singularities, then we obtain the following diagram of moduli stacks:

$$\begin{array}{ccc} \mathcal{P} \supseteq \mathcal{U} & \xrightarrow{\phi} & \mathcal{E}_{\mathrm{uni},K3} \\ \downarrow & & \\ \mathcal{K} & & \end{array}$$

As shown in [214, Theorem 8.2], ϕ is a smooth and surjective morphism, that is in fact a gerbe associated with the height one subgroup scheme of the stabilizer of $\mathcal{K}$ over Spec $\mathbb{F}_2$.

Finally, $\mathcal{K}$ can be described using an extension of Ogus' period map to moduli spaces of supersingular K3 crystals. More precisely, consider the lattice $\langle -2\rangle^{12}$, say with generators $e_1, ..., e_{12}$, such that $e_i \cdot e_j = -2\delta_{ij}$, and let R be the lattice generated by $e_1, ..., e_{12}$ and $\frac{1}{2}\sum_{i=1}^{12} e_i$.

Next, let $M := R \oplus \mathsf{E}_{10}(2)$. Then, extending Ogus' moduli space of supersingular K3 crystals from [575] to characteristic 2, there exists a moduli space $\mathcal{M}_M^{\mathrm{crys}}$. Its geometric points of $\mathcal{M}_M^{\mathrm{crys}}$ are supersingular K3 crystals $(H, \langle -, -\rangle, \Phi)$, where H is a free $W(\Bbbk)$-module of rank 22 together with an injective Frobenius-linear map Φ, $\langle -, -\rangle$ is a non-degenerate symmetric bilinear form, together with an isometric embedding $M \to T_H$ of M into the Tate-module. This data satisfies certain axioms see [214] or [575].

This space is called the *period space for supersingular M-polarized K3 surfaces*, and $\mathcal{M}_M^{\mathrm{crys}} \times \mathbb{F}_2$ has two components, both of which are smooth, 9-dimensional, and unirational.

Next, the locus $\mathcal{H} \subset \mathcal{M}_M^{\mathrm{crys}}$, whose points correspond to M-marked crystals $(H, \phi : M \to T_H)$, such that there exists a root in the saturation of $\mathsf{E}_{10}(2) \to M \xrightarrow{\iota} T_H \subset H$, is a divisor by [214, Lemma 6.11]. We denote its complement by $\mathcal{M}_M^{\mathrm{crys},\circ}$, and by [214, Proposition 8.6], there exist morphisms

$$\mathcal{M}_M^{\mathrm{crys},\circ} \to [\mathcal{M}_M^{\mathrm{crys},\circ}/\mathfrak{S}_{12}] \to \mathcal{K},$$

where $\mathfrak{S}_{12}$ denotes the symmetric group on 12 letters that acts on the $\{e_i\}$ in R by permutations. In loc. cit., it is shown that the induced maps on geometric quotients induces a bijection on geometric points.

Put a little bit sloppily, we thus obtain a description of an open substack of $\mathcal{E}_{\mathrm{uni}}$ as an open subset of a $\mathbb{P}^1$-bundle over an open subset of the period space of M-polarized supersingular K3 crystals. The latter space parametrizes M-polarized supersingular K3 surfaces and let X be such a surface. After contracting the (-2)-curves corresponding to M, one obtains a K3 surface X' with RDP singularities and trivial tangent sheaf $\Theta_{X'}$. There is an open and dense subset of $\mathbb{P}(H^0(X', \Theta_{X'})) \cong \mathbb{P}^1$, whose points correspond to p-closed vector fields D on X', such that the quotient X'/D is a unipotent Enriques surface.

Bibliographical notes

The theory of periods of algebraic polarized K3 surfaces was initiated in the fundamental work of I.J. Pyatetskii-Shapiro and I.R. Shafarevich [599]. The notion of a lattice polarized K3 surface was introduced (under a different name) by V. Nikulin in [555]. The theory of periods of lattice polarized K3 surfaces was used for introducing and explaining a certain duality of moduli spaces of lattice polarized K3 surfaces, which is a 2-dimensional analog of mirror symmetry for Calabi–Yau threefolds. It was discovered by the first-named author and V. Nikulin and independently, by H. Pinkham in 1977. The exposition for this theory can be found in [173] and we followed this in Section 5.2.

In Section 5.3 we discussed the moduli space of marked and unmarked Enriques surfaces via the periods of their canonical covers. This approach to the moduli problem was first initiated by E. Horikawa [309],[310] and independently, as the periods of lattice polarized K3 surfaces, by V. Nikulin [555]. Some of the problems left from Horikawa's work were solved later by Y. Namikawa [549]. At the end of the section we give an exposition of the work of J. Keum [383] and H. Ohashi [577], [578] on fixed-point-free involutions on complex K3 surfaces and, in particular, K3 surfaces birationally isomorphic to a Kummer surface associated to an abelian surface.

In Section 5.4 we discussed complex Coble surfaces whose isomorphism classes lie in the boundary of the moduli space of Enriques surfaces. We will return and give more details about rational Coble surfaces defined over fields of arbitrary characteristic in Chapter 10 of Volume II. The irreducibility of the moduli space of marked or unmarked Coble surfaces was first proven by Y. Namikawa. We gave another proof of this result based on a note of D. Allcock [6]. Yet another proof of this result is due to R. Borcherds [81]. Namikawa also gave several applications of the Global Torelli Theorem for Enriques surface. For example, he proved that the number of orbits of the automorphism group on the set of smooth rational curves and elliptic pencils is finite. The same result was proved independently by E. Looijenga and H. Sterk [689]. The fact that the automorphism group of an unnodal Enriques surface contains a subgroup isomorphic to the 2-level congruence subgroup $W_{2,3,7}(2)$ of $W_{2,3,7}$ was deduced from the Global Torelli Theorem by Barth and Peters in [42]. In a short note [559], Nikulin described a much more general result about automorphisms of Enriques surfaces, from which the same theorem follows as a special case. Around the same time, a geometric proof of this theorem based on a result of A. Coble about the normal generation of the subgroup $W_{2,3,7}(2)$ was given by the second author [172]. We will return to this in Volume II, Chapter 8.

The irreducibility of the moduli space $\mathcal{M}_{\mathrm{Enr}}^{\mathrm{nod}}$ of nodal Enriques surfaces was proven first by Y. Namikawa [549]. We give a simpler proof following D. Allcock [6]. The moduli spaces $\mathcal{M}_{\mathrm{Enr},v}$ of polarized Enriques surfaces for low degree $h^2 \leq 10$ were studied in earlier chapters. The degrees of the forgetful map $\mathcal{M}_{\mathrm{Enr},v} \to \mathcal{M}_{\mathrm{Enr}}$ were first computed by Barth and Peters [42]. In the case $h^2 = 2$, a systematic study of the GIT-quotient model of this space and its relationship with the moduli space $\mathcal{M}_{\mathrm{Enr},v}$, as well as various compactifications were studied by H. Sterk [691]. The rationality of the moduli spaces $\mathcal{M}_{\mathrm{Enr}}$ was first proven by S. Kondō [410]. Much later, the second author and S. Kondō proved the rationality of the moduli spaces of $\mathcal{M}_{\mathrm{Enr}}^{\mathrm{nod}}$ and $\mathcal{M}_{\mathrm{Cob}}$ [190]. The fact that there are only finitely many birational types of moduli spaces of polarized Enriques surfaces was first observed by V. Gritsenko and K. Hulek [262]. They also proved that some of these spaces are not unirational. A recent paper [121] gives many examples of unirational irreducible moduli spaces of polarized Enriques surfaces. Another recent paper [397] answers the question of when the moduli space is irreducible. We did not discuss the toroidal compactifications of the moduli spaces of Enriques surfaces.

The study of deformations of Enriques surfaces in positive characteristic was initiated by P. Blass [67] and W.E. Lang [433], [435]. The construction and description of moduli spaces for Enriques surfaces and especially in characteristic 2 is due to the third author [460] and the work of T. Ekedahl, J.M.E. Hyland, and N.I. Shepherd-Barron [213], [214].

Appendix A
Automorphic Forms and Moduli Spaces by S. Kondō

In this Appendix, we will discuss a theory of automorphic forms on bounded symmetric domains of type IV associated with an even lattice of signature $(2, n)$. In particular, we will give a proof of the existence of a discriminant form on the period space of Enriques surfaces mentioned in Volume I, Chap. 5, Theorem 5.9.11.

A.1 Multiplicative and Additive Liftings

We follow the notation of Chapter 5. Let L be an even lattice of signature $(2, n)$ and let

$$\mathcal{D}_L = \{[\omega] \in \mathbb{P}(L_\mathbb{C}) \mid \omega^2 = 0,\ \omega \cdot \bar{\omega} > 0\}.$$

Let Γ be a subgroup of $\mathrm{O}(L)$ of finite index. We recall the notion of automorphic forms on $\mathcal{D}_L$ with respect to Γ. Let

$$\tilde{\mathcal{D}}_L = \{\omega \in L_\mathbb{C} \mid \omega^2 = 0,\ \omega \cdot \bar{\omega} > 0\}.$$

The natural map $\tilde{\mathcal{D}}_L \to \mathcal{D}_L$ gives a structure of a principal $\mathbb{C}^*$-bundle. A holomorphic (resp. meromorphic) function $f : \tilde{\mathcal{D}}_L \to \mathbb{C}$ is called a holomorphic (resp. meromorphic) *automorphic form* on $\mathcal{D}_L$ of weight $k \in \mathbb{Z}_{\geq 0}$ if

$$\begin{cases} f(\gamma \cdot \omega) = \chi(\gamma) f(\omega) \ \ (\forall \gamma \in \Gamma), \\ f(\alpha \cdot \omega) = \alpha^{-k} f(\omega) \ \ (\alpha \in \mathbb{C}^*), \end{cases} \tag{A.1.1}$$

where $\chi : \Gamma \to \mathrm{U}(1)$ is a unitary character of Γ. There is also some technically stated condition requiring that a holomorphic form must be holomorphic at cusps. This condition is automatically satisfied when $n \geq 3$, the fact often referred to as the *Koecher principle* [405]. Since $\tilde{\mathcal{D}}_L \to \mathcal{D}_L$ can be identified with the principal $\mathbb{C}^*$-bundle associated to the nth root of the cotangent bundle of $\mathcal{D}_L$, an equivalent definition is that f is a holomorphic (meromorphic) function on $\mathcal{D}_L$ satisfying

$$f(\gamma \cdot z) = \chi(\gamma) \frac{1}{(\det d\gamma_z)^{\frac{k}{n}}} f(z) \quad (\forall \gamma \in \Gamma). \tag{A.1.2}$$

In this way, the notion of an automorphic form extends to the notion of an automorphic form on any complex domain and a discrete group Γ of its holomorphic automorphisms. The notion of a *modular form* is a special case of this definition when $n = 1$. In this case $\mathcal{D}_L$ is identified with two copies of the upper half-plane $\mathbb{H} = \{z = a + bi \in \mathbb{C} : b > 0\}$, $\gamma = \frac{az+b}{cz+d}$ is a fractional-linear transformation with $ad - bc = \pm 1$ and $d\gamma_z = \frac{1}{(cz+d)^2}$. Let $D(L) = L^\vee/L$ be the discriminant group of L, $q_L : D(L) \to \mathbb{Q}/2\mathbb{Z}$ the discriminant quadratic form and $b_L : D(L) \times D(L) \to \mathbb{Q}/\mathbb{Z}$ the discriminant bilinear form of $D(L)$.

Now, for simplicity, we assume that $n = 2m$ is an even integer because this is enough for applications to the case of Enriques surfaces. Let

$$T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \tag{A.1.3}$$

be the standard generators of $\mathrm{SL}(2, \mathbb{Z})$. Let $\mathbb{C}[D(L)]$ be the group ring of $D(L)$ and let $\{e_\alpha\}_{\alpha \in D(L)}$ be the standard generators of $\mathbb{C}[D(L)]$. There is a unitary representation ρ_L of $\mathrm{SL}(2, \mathbb{Z})$ on $\mathbb{C}[D(L)]$ defined by

$$\begin{cases} \rho_L(T) \cdot e_\alpha = e^{\pi\sqrt{-1} q_L(\alpha)} e_\alpha, \\ \rho_L(S) \cdot e_\alpha = \frac{\sqrt{-1}^{m-1}}{\sqrt{|D(L)|}} \sum_{\beta \in D(L)} e^{-2\pi\sqrt{-1} b_L(\alpha,\beta)} e_\beta. \end{cases} \tag{A.1.4}$$

This representation ρ_L is called the *Weil representation* associated with the discriminant quadratic form $(D(L), q_L)$. In the general case, that is, not assuming the evenness of n, one can consider a double cover of $\mathrm{SL}(2, \mathbb{Z})$ called the *metaplectic group*. We remark that the natural action of the orthogonal group $\mathrm{O}(D(L))$ on $\mathbb{C}[D(L)]$ commutes with the action of $\mathrm{SL}(2, \mathbb{Z})$ on $\mathbb{C}[D(L)]$. If we denote by N the smallest integer such that $Nb_L(\alpha, \beta) = Nq_L(\alpha)/2 = 0$ for all $\alpha, \beta \in D(L)$, then the representation ρ_L factors through the finite group $\mathrm{SL}(2, \mathbb{Z}/N\mathbb{Z})$.

One extends the notion of a holomorphic (resp. meromorphic) modular form of weight k with respect to $\Gamma = \mathrm{SL}(2, \mathbb{Z})$ by considering a holomorphic map $f : \mathbb{H} \to \mathbb{C}[D(L)]$ satisfying

$$f(\gamma \cdot \tau) = (c\tau + d)^k \rho_L(\gamma) \cdot f(\tau)$$

for any $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2, \mathbb{Z})$, and requiring that f is holomorphic (resp. meromorphic) at ∞. If L is not unimodular, this is a vector-valued function. As usual, f has a Fourier expansion

$$f(\tau) = \sum_{\lambda \in D(L)} e_\lambda \sum_{m \in \mathbb{Q}} c_\lambda(m) e^{2\pi\sqrt{-1} m\tau}.$$

The following theorem, due to Borcherds [82] (see also [99]), gives a criterion for the existence of an automorphic form whose zero or pole divisor is a given Heegner divisor.

Theorem A.1.1 *Suppose L is an even lattice of signature $(2, n)$ and let f be a modular form of weight $1 - \frac{n}{2}$ with respect to ρ_L whose Fourier coefficients $c_\lambda(\ell)$ are integers for $\ell \le 0$. Then, there exists a meromorphic function $\Psi(\omega)$ for $\omega \in \tilde{\mathcal{D}}_L$ with the following properties*:

1. *Ψ is an automorphic form of weight $c_0(0)/2$.*
2. *The only zeros or poles of Ψ lie on a Heegner divisor $\lambda^\perp$ for $\lambda \in L$ with $\lambda^2 < 0$ and their order is equal to*

$$\sum_{0<x\in\mathbb{R}, x\lambda\in L^\vee} c_{x\lambda}(x^2\lambda^2/2)$$

(or poles if this number is negative).

3. *Ψ is a holomorphic function if the orders of all zeros in item 2 above are nonnegative. If, in addition, $\operatorname{rank}(L) \ge 5$, or if $\operatorname{rank}(L) = 4$ and L contains no 2-dimensional isotropic sublattice, then Ψ is a holomorphic automorphic form.*

The function Ψ is called a *Borcherds product* or a *multiplicative lifting*.

On the other hand, Borcherds generalizes constructions of Saito-Kurokawa, Shimura, Maass, Gritsenko, Oda, and others for liftings of modular forms to automorphic forms. For our application, it is sufficient to consider the most simple case; that is, modular forms of weight 0 with respect to ρ_L.

Theorem A.1.2 *Suppose L is an even lattice of signature $(2, n)$ with an even integer $n \ge 4$. Then there exists an $\mathrm{O}(L)$-equivariant linear map*

$$\mathbb{C}[D(L)]^{\mathrm{SL}(2,\mathbb{Z})} \to [\mathrm{O}(L)^\sharp, n/2 - 1], \tag{A.1.5}$$

where $[\mathrm{O}(L)^\sharp, n/2 - 1]$ is the space of holomorphic automorphic forms of weight $n/2 - 1$ with respect to $\mathrm{O}(L)^\sharp$, and with trivial character. Here $\mathrm{O}(L)$ acts on the target of the map because $\mathrm{O}(L)^\#$ is a normal subgroup, and it acts on the source of the map via the action on $D(L)$.

The linear map from the theorem is called an *additive lifting*. The weight $n/2 - 1$ is called the singular weight which is the smallest weight of non-zero holomorphic automorphic forms. The Fourier series of the obtained automorphic form is given explicitly in terms of Fourier coefficients of the original modular form. Here we give only the constant term of the Fourier expansion.

Theorem A.1.3 *Suppose L is an even lattice of signature $(2, n)$ with an even integer $n \ge 6$. Let F be the additive lifting corresponding to $\sum c_\alpha e_\alpha \in \mathbb{C}[D(L)]^{\mathrm{SL}(2,\mathbb{Z})}$. Let z be a primitive isotropic vector of L. Then the constant term of the Fourier expansion of F around z is given by*

$$- \sum_{\delta \in \mathbb{Z}/N\mathbb{Z}} c_{\delta z/N} \sum_{0<\epsilon \leq N} N^{n/2-2} e^{2\pi\sqrt{-1}\delta\epsilon/N} B_{n/2-1}(\epsilon/N)/(n-2), \qquad \text{(A.1.6)}$$

where $B_{n/2-1}(x)$ is the Bernoulli polynomial of degree $n/2-1$.

A.2 Borcherds Φ-Function

In this section, we restrict ourselves to the case $q_L = \mathsf{u}_1^{\oplus l}$, where $\mathsf{u}_1 = q_{U(2)}$ is given by the matrix $\begin{pmatrix} 0 & 2^{-1} \\ 2^{-1} & 0 \end{pmatrix}$. Its discriminant group and quadratic form can be identified with the quadratic space $(\mathbb{F}_2^{2l}, q)$ of even type. We will show the existence of the Borcherds Φ-function found in the paper [81], that is, an automorphic form on the period space of Enriques surfaces vanishing exactly on the locus of the periods of Coble surfaces. Recall that in this case $L = \mathsf{E}_{10}(2) \oplus \mathsf{U}$ and the discriminant group $D(\mathsf{E}_{10}(2))$ can be identified with the quadratic space $(\mathsf{E}_{10}/2\mathsf{E}_{10}, q)$, where $q(x + 2\mathsf{E}_{10}) = \frac{1}{2}x^2 \mod 2$ (see (5.3.2)). The quadratic form is of even type, i.e. can be reduced to the form $\sum_{i=1}^{5} x_i x_{i+5}$. It is known that all non-zero vectors form two orbits with respect to the orthogonal group represented by a vector with $q(x) = 1$ and $q(x) = 0$. The number of vectors in each orbit is equal to $2^4(2^5 - 1) = 496$ (resp. $2^4(2^5 + 1) = 528$). This extends to any lattice L with discriminant quadratic form isomorphic to $\mathsf{u}_1^{\oplus l}$. An easy count of vectors with $q(x) = 0$ (by induction on l or using Witt's Theorem) gives the following.

Lemma A.2.1 *The discriminant group $D(L)$ consists of the following 2^{2l} vectors:*
Type (00) : $\alpha = 0$*;*
Type (0) : $\alpha \neq 0,\ q_L(\alpha) = 0,\ \#\{q^{-1}(0) - \{0\}\} = 2^{l-1}(2^l + 1) - 1$;
Type (1) : $q_L(\alpha) = 1,\ \#q_L^{-1}(1) = 2^{l-1}(2^l - 1)$.

Let $m_{ij}(k)$ be the number of nonzero vectors of type $(j) = (0)$ or (1) which have the inner product $k = 0$ or 1 with respect to $2b_L$ with a fixed vector of type $(i) = (0)$ or (1). A straightforward inductive computation shows that

$$\begin{aligned} m_{00}(0) &= 2^{l-1}(2^{l-1} + 1) - 1, & m_{01}(0) &= 2^{l-1}(2^{l-1} - 1), \\ m_{00}(1) &= 2^{2l-2}, & m_{01}(1) &= 2^{2l-2}, \\ m_{10}(0) &= 2^{2l-2} - 1, & m_{11}(0) &= 2^{2l-2}, \\ m_{10}(1) &= 2^{l-1}(2^{l-1} + 1), & m_{11}(1) &= 2^{l-1}(2^{l-1} - 1). \end{aligned}$$

By definition of the Weil representation A.1.4, we have $\rho(S)^2 = \rho(T)^2 = 1$ on $\mathbb{C}[D(L)]$. Hence the action of $\mathrm{SL}(2, \mathbb{Z})$ factorizes through that of $\mathrm{SL}(2, \mathbb{Z}/2\mathbb{Z})$. The conjugacy classes of $\mathrm{SL}(2, \mathbb{Z}/2\mathbb{Z}) \cong \mathfrak{S}_3$ consist of E, T, ST, where E is the identity matrix. Let χ_1, χ_2, and χ_3 be the trivial character, the alternating character, and the irreducible character of degree 2 of $\mathfrak{S}_3$, respectively. For the convenience of the reader we give the character table of $\mathfrak{S}_3$.

	E	T	ST
χ_1	1	1	1
χ_2	1	−1	1
χ_3	2	0	−1

Table A.1 Character table of $\mathfrak{S}_3$

Lemma A.2.2 *Let χ be the character of the Weil representation ρ_L of* $\mathrm{SL}(2, \mathbb{Z}/2\mathbb{Z})$ *on* $\mathbb{C}[D(L)]$. *Let*

$$\chi = \sum_{i=1}^{3} m_i \chi_i$$

be the decomposition of χ into irreducible characters. Then,

$$\chi = \frac{1}{6}(2^{2l} + 3 \cdot 2^l + 2)\chi_1 + \frac{1}{6}(2^{2l} - 3 \cdot 2^l + 2)\chi_2 + \frac{1}{3}(2^{2l} - 1)\chi_3.$$

Proof It follows from the definition of ρ_L and computation of the numbers $m_{ij}(k)$ that $\mathrm{trace}(\rho_L(E)) = 2^{2\ell}$, $\mathrm{trace}(\rho_L(T)) = 2^{\ell-1}(2^\ell + 1) - 2^{\ell-1}(2^\ell - 1) = 2^\ell$, $\mathrm{trace}(\rho_L(ST)) = 2^{-\ell}2^\ell = 1$. Thus, we have

$$m_1 + m_2 + 2m_3 = 2^{2\ell}, \;\; m_1 - m_2 = 2^\ell, \;\; m_1 + m_2 - m_3 = 1,$$

and hence we have proved the Lemma. □

Let

$$\begin{cases} f_{00}(\tau) = 8\eta(2\tau)^8/\eta(\tau)^{16}, \\ f_0(\tau) = -8\eta(2\tau)^8/\eta(\tau)^{16}, \\ f_1(\tau) = 8\eta(2\tau)^8/\eta(\tau)^{16} + \eta(\tau/2)^8/\eta(\tau)^{16}, \end{cases}$$

where $\eta(\tau)$ is the *Dedekind eta function*. Recall that

$$\eta(\tau)^{24} = q\prod_{n=1}^{\infty}(1 - q^n)^{24} \qquad (q = e^{2\pi\sqrt{-1}\tau}).$$

Then, we have

$$f_{00}(\tau + 1) = f_{00}(\tau), \;\; f_0(\tau + 1) = f_0(\tau), \;\; f_1(\tau + 1) = -f_1(\tau),$$

$$f_{00}(-1/\tau) = \tau^{-4}(-f_{00}(\tau) + f_1(\tau))/2,$$

$$f_0(-1/\tau) = \tau^{-4}(f_{00}(\tau) - f_1(\tau))/2,$$

$$f_1(-1/\tau) = \tau^{-4}(3f_{00}(\tau) + f_1(\tau))/2.$$

By using these, we have the following:

Lemma A.2.3 *Let*

$$h_{00} = (2^l - 1) f_{00}, \quad h_0 = f_0, \quad h_1 = f_1.$$

Then $\{h_\alpha\}_{\alpha \in D(L)}$ *is a modular form of weight* -4 *and of type* ρ_L*, and their q-expansion are given by*

$$\begin{cases} h_{00} = 8(2^l - 1) + 128(2^l - 1)q + \cdots \\ h_0 = -8 - 128q + \cdots \\ h_1 = q^{-1/2} + 36q^{1/2} + \cdots . \end{cases}$$

By applying Theorem A.1.1 to $\{h_\alpha\}_{\alpha \in D(L)}$ in Lemma A.2.3, we now have the following theorem.

Theorem A.2.4 *Let L be an even lattice of signature* $(2, 10)$ *with* $q_L = \mathsf{u}_1^{\oplus l}$. *Then there exists an automorphic form* Ψ *of weight* $4(2^l - 1)$ *on the bounded symmetric domain* $\mathcal{D}_L$ *of type IV associated to L such that the zero divisor* (Ψ) *is the Heegner divisor* $\mathcal{H}_{-1}$.

We keep the assumption that L is an even lattice of signature $(2, 10)$ with $q_L = \mathsf{u}_1^{\oplus l}$. Let $\alpha \in D(L)$ be a non-isotropic vector, i.e. $q_L(\alpha) = 1$. We define a map

$$t_\alpha : D(L) \to D(L), \quad x \to x + 2b_L(x, \alpha)\alpha,$$

which is called a transvection and is contained in $\mathrm{O}(q_L)$. Let r be a vector in L with $r^2 = -4$ such that $r/2 + L = \alpha$. Since $r/2 \in L^\vee$, the reflection

$$s_r : x \to x - \frac{2\langle x, r\rangle}{r^2} r$$

associated with r is contained in $\mathrm{O}(L)$. Note that s_r induces the transvection t_α.

Let V be a subspace of $D(L) = \mathbb{F}_2^{2l}$. Then V is called totally isotropic if $q_L|V \equiv 0$ and totally singular if it is generated by mutually orthogonal non-isotropic vectors. For each maximal totally singular subspace V in $D(L)$, let I be the maximal totally isotropic subspace in V. Then there exist exactly two maximal totally isotropic subspaces M_+ and M_- in $D(L)$ containing I because $I^\perp/I (\cong \mathbb{F}_2^2)$ is a hyperbolic plane. Let

$$\theta_V = \sum_{\alpha \in M_+} e_\alpha - \sum_{\alpha \in M_-} e_\alpha.$$

Lemma A.2.5 (1) $\theta_V \in \mathbb{C}[D(L)]^{\mathrm{SL}(2,\mathbb{Z})}$.

(2) $t_\alpha(\theta_V) = -\theta_V$ *for any* $\alpha \in V$ *with* $q_L(\alpha) = 1$.

Proof (1) It suffices to see that, for any maximal totally isotropic subspace M, $\sum_{\alpha \in M} e_\alpha \in \mathbb{C}[D(L)]^{\mathrm{SL}(2,\mathbb{Z})}$. By definition, $\rho_L(T)(\sum_{\alpha \in M} e_\alpha) = \sum_{\alpha \in M} e_\alpha$. On the other hand,

$$\rho(S)(\sum_{\alpha \in M} e_\alpha) = \frac{1}{2^\ell} \sum_{\beta \in M} \left(\sum_{\alpha \in M} (-1)^{2b_L(\alpha,\beta)} \right) e_\beta + \frac{1}{2^\ell} \sum_{\beta \notin M} \left(\sum_{\alpha \in M} (-1)^{2b_L(\alpha,\beta)} \right) e_\beta.$$

Obviously $\sum_{\alpha\in M}(-1)^{2b_L(\alpha,\beta)} = 2^\ell$ for any $\beta \in M$ and $\sum_{\alpha\in M}(-1)^{2b_L(\alpha,\beta)} = 0$ for any $\beta \notin M$.

(2) Since the projection of α in $I^\perp/I$ is non-isotropic, t_α exchanges M_+ and M_-, and hence the assertion follows. □

Recall that zero-dimensional rational boundary components of $\mathcal{D}_L$ bijectively correspond to primitive isotropic vectors in L. Since L is a 2-elementary indefinite lattice, by Theorem 0.8.6, the natural map

$$\mathrm{O}(L) \to \mathrm{O}(D(L))$$

is surjective and hence $\mathrm{O}(L)$ acts transitively on the set of nonzero isotropic vectors in $D(L)$. This implies that any nonzero isotropic vector in $D(L)$ represents a 0-dimensional boundary component.

Proposition A.2.6 *Let F_V be the additive lifting associated to θ_V. Then F_V is a non-zero automorphic form of weight* 4*.*

Proof Note that $c_0 = 0$ for θ_V. It follows from Theorem A.1.3 that there exists a primitive isotropic vector $z \in L^\vee$ satisfying the formula $F_V(z) = a \cdot c_\alpha$, where $a = a_\ell$ is a nonzero constant and $\alpha = z + L \neq 0 \in D(L)$. □

Since the additive lifting is $\mathrm{O}(D(L))$-equivariant and the transvections t_α is represented by (-4)-reflections in $\mathrm{O}(L)$, Lemma A.2.5, (2) implies that the zero divisor (F_V) contains the Heegner divisor

$$\mathcal{H}(V) := \sum_{\alpha\in V,\ q_L(\alpha)=1} \mathcal{H}_\alpha.$$

Let m be the number of all maximal totally singular subspaces V in $D(L)$. Put

$$k = m/(2^l - 1).$$

Consider the ratio

$$\prod_V F_V/\Psi^k,$$

where V varies in the set of all maximal totally singular subspaces in $D(L)$. Note that $\prod F_V$ is of weight $4m$ and vanishes along each Heegner divisor $\mathcal{H}_\alpha, \alpha \in D(L), q_L(\alpha) = 1$, with multiplicity $\geq 2^{l-1}m/2^{l-1}(2^l - 1) = m/(2^l - 1) = k$. On the other-hand, Ψ^k is of weight $4(2^l - 1)k = 4m$ and vanishes along $\mathcal{H}_\alpha$ with multiplicity k. Thus, the ratio is a holomorphic automorphic form of weight zero, and hence it is constant. This gives the following:

Theorem A.2.7 *Let L be an even lattice of signature* $(2, 10)$ *with $q_L = \mathsf{u}_1^{\oplus l}$. Then F_V is a holomorphic automorphic form of weight* 4 *with $(F_V) = \mathcal{H}(V)$.*

Now we apply these results to the case of complex Enriques surfaces. Recall that the moduli space of Enriques surfaces $\mathcal{M}_{\mathrm{Enr}}$ is given by an arithmetic quotient

$$\mathcal{M}_{\mathrm{Enr}} := \mathsf{O}(\mathsf{E}_{10}(2)^{\perp})\backslash\mathcal{D}^{\circ}_{\mathsf{E}_{10}(2)^{\perp}}.$$

For simplicity we denote $\mathsf{E}_{10}(2)^{\perp}$ by N, which is isomorphic to $\mathsf{U} \oplus \mathsf{U}(2) \oplus \mathsf{E}_8(2)$. Then $D(N) = (\mathbb{Z}/2\mathbb{Z})^{10}$ and $q_N = \mathsf{u}_1^{\oplus 5}$. Note that

$$N^{\vee}(2) \cong \mathsf{U}(2) \oplus \mathsf{U} \oplus \mathsf{E}_8, \quad \mathrm{O}(N) \cong \mathrm{O}(N^{\vee}(2)).$$

We denote by M the lattice $N^{\vee}(2)$. Obviously $D(M) = (\mathbb{Z}/2\mathbb{Z})^2$ and $q_M = \mathsf{u}_1$.

By applying Theorem A.2.4 to the lattice M we have the following:

Corollary A.2.8 *There exists an automorphic form* Φ *of weight* 4 *on* $\mathcal{D}_M$ *whose zero divisor is the Heegner divisor* $\mathcal{H}(-4)$.

Recall from the proof of Corollary 5.4.2 that (-2)-vectors in N form one orbit with respect to $\mathrm{O}(N)$. Therefore, any such vector is conjugate to a (-2)-vector in the factor U in N under the action of $\mathrm{O}(N)$. Therefore (-2)-vectors in N bijectively correspond to (-4)-vectors r in M with $r/2 \in M^{\vee}$, or, in other words, (-1)-vectors in $M^{\vee}$. Moreover, we identify $\mathcal{D}_N$ and $\mathcal{D}_{N^{\vee}(2)}$ under $\mathrm{O}(N) \cong \mathrm{O}(N^{\vee}(2))$-equivarinant isomorphism. Then Corollary A.2.8 implies the following:

Corollary A.2.9 *There exists an automorphic form* Φ *of weight* 4 *on* $\mathcal{D}_N$ *whose zero divisor is the Heegner divisor* $\mathcal{H}(-2)$.

It is surprising that the above corollary came from the denominator formula in some generalized super Kac–Moody algebra (Borcherds [81]). Later Borcherds [82] reproved the corollary as considered in this section.

Recall that $\mathcal{D}_N^{\circ} = \mathcal{D}_N \setminus \mathcal{H}_{-2}$ and Φ defines an ample line bundle on the Baily–Borel compactification of $\mathrm{O}(N)\backslash\mathcal{D}_N$. Therefore, Corollary A.2.9 implies:

Corollary A.2.10 *The moduli space of Enriques surfaces is quasi-affine.*

Recall that we have proved this result in Theorem 5.9.11 by a different method. On the other hand, by applying Theorem A.2.4 to N, we have the following:

Corollary A.2.11 *There exists an automorphic form* Φ' *of weight* 124 *on* $\mathcal{D}_N$ *whose zero divisor is the Heegner divisor* $\mathcal{H}(-4)$.

Since $\mathcal{H}(-4)$ is the locus of nodal Enriques surfaces, we have the following Corollary as in the case of Corollary A.2.10.

Corollary A.2.12 *The moduli space of unnodal Enriques surfaces is quasi-affine.*

We can also apply Theorem A.2.7 to the lattice M. Note that u_1 has a unique non-isotropic vector and hence $M^{\vee}/M$ has a unique 1-dimensional singular subspace. Thus, we have an automorphic form F of weight 4 whose zero divisor is the Heegner divisor $\mathcal{H}(-4)$. This automorphic form F is nonzero by calculating the Fourier expansion at a cusp. Since the ratio F/Φ is a holomorphic automorphic form of weight zero, F/Φ is constant. Thus, we have the following:

Corollary A.2.13 *The Borcherds product* Φ *is obtained as an additive lifting.*

A.3 A Projective Model of the Moduli Space of Marked Enriques Surfaces

By applying the results in the previous sections, we will show the following theorem. Let $\mathcal{M}^m_{\mathrm{Enr}}$ be the moduli space of marked Enriques surfaces and let $\overline{\mathcal{M}}^m_{\mathrm{Enr}}$ be its Baily–Borel compactification.

Theorem A.3.1 *There exists a holomorphic* $\mathrm{O}(D(N))$*-equivariant map*

$$\varphi : \overline{\mathcal{M}}^m_{\mathrm{Enr}} \to \mathbb{P}^{186},$$

defined by automorphic forms of weight 4 *obtained as additive liftings. The map is birational onto its image.*

We use the same notation as in the previous section and assume $\ell = 5$. Then the discriminant group $D(N)$ consists of 0, 527 nonzero isotropic vectors and 496 non-isotropic vectors. By applying Lemma A.2.2, we have the following:

Proposition A.3.2 *The character of the Weil representation* ρ_N *is given by*

$$\chi = 187\chi_1 + 155\chi_2 + 341\chi_3.$$

Denote by W the subspace of $\mathbb{C}[D(N)]$ of dimension 187. Since the action ρ_N of $\mathrm{SL}(2, \mathbb{Z})$ and that of $\mathrm{O}(D(N))$ on $\mathbb{C}[D(N)]$ commute, $\mathrm{O}(D(N))$ acts on W. The degrees of irreducible representations of $\mathrm{O}(D(N))$ are $1, 155, 186, \ldots$. On the other hand, the character of the action of $\mathrm{O}(D(N))$ on the space $\mathbb{C}^{4590}$ of maximal totally isotropic subspaces in $D(N)$ is given by ([131], p. 146)

$$2\chi_1 + 2\chi_{186} + 2\chi_{2108}.$$

Thus we can see that W decomposes into 1-dimensional and 186-dimensional subspaces W_1 and W_{186} on each of which $\mathrm{O}(D(N))$ acts irreducibly. Note that W_1 is fixed under the action of $\mathrm{SL}(2, \mathbb{Z}) \times \mathrm{O}(D(N))$. Proposition A.2.6 and Schur's lemma imply that the additive lifting

$$W_{186} \to [\mathrm{O}(N)^{\#}, 4]$$

is injective. On the other hand, W_1 is generated by

$$\theta = \sum_{M} \sum_{\alpha \in M} e_\alpha,$$

where M varies on the set of all maximal totally isotropic subspaces in $D(N)$. We can easily see that

$$\theta = 270 \left(17 e_0 + \sum_{\alpha \neq 0, q_L(\alpha) = 0} e_\alpha \right).$$

It follows from Theorem A.1.3 that the additive lifting F_0 of θ is a non-zero automorphic form of weight 4 with respect to the full orthogonal group $\mathrm{O}(N)$. Thus we have a rational map

$$\varphi : \overline{\mathcal{M}}^m_{\mathrm{Enr}} \to \mathbb{P}^{186}. \tag{A.3.1}$$

For the base-point freeness of the map φ, we use the following lemmas.

Lemma A.3.3

$$\bigcap_V \mathcal{H}(V) = \emptyset,$$

where V varies on the set of all maximal totally singular subspaces in $D(N)$.

For the proof of this lemma we need a case-by-case analysis and hence omit it here (see [412, Theorem 5.1]).

We denote by $\overline{\mathcal{H}}(V)$ the closure of $\mathcal{H}(V)$ in $Q_N = \{[\omega] \in \mathbb{P}(L \otimes \mathbb{C}) \mid \omega^2 = 0\}$. Recall that 0-dimensional rational boundary components of the Baily–Borel compactification $\mathrm{O}(N)^\# \backslash \mathcal{D}_N$ bijectively correspond to $\mathrm{O}(N)^\#$-orbits of isotropic vectors in $D(N)$. Denote by e a primitive isotropic vector in N such that $e/2$ is not in $N^\vee$. Let F_e denote the 0-dimensional boundary component corresponding to e. By the proof of Proposition A.2.6, we have the following (see [412, Remark 5.12]).

Corollary A.3.4

$$\bigcap_V \overline{\mathcal{H}}(V) = \mathrm{O}(N) \cdot F_e,$$

where V varies on the set of all maximal totally singular subspaces in $D(N)$.

On the other hand, by Theorem A.1.3, we see that the additive lifting F_0 of θ does not vanish at e. Thus, combining this with Theorem A.2.7, we have proved that the map φ is holomorphic. Since $\overline{\mathcal{M}}^m_{\mathrm{Enr}}$ is compact, φ is a proper map.

To prove that φ is birational, we consider a special point in $\mathcal{M}^m_{\mathrm{Enr}}$. Let $\omega_0 \in \mathcal{D}_N$ be the period of the marked Enriques surface of type VI in Section 8.9. It is known that

$$\omega_0^\perp \cap N = E_6(2) \oplus A_4(2)$$

(see Proposition 8.9.26).

Lemma A.3.5 *Let $\omega \in \overline{\mathcal{D}}_N$. Assume that ω is not equivalent to ω_0 under the action of* $\mathrm{O}(N)^\#$. *Then there exists a maximal totally singular subspace V satisfying $\omega \in \overline{\mathcal{H}}(V)$, $\omega_0 \notin \mathcal{H}(V)$ or $\omega \notin \overline{\mathcal{H}}(V)$, $\omega_0 \in \mathcal{H}(V)$.*

It follows that $\varphi^{-1}(\varphi([\omega_0])) = [\omega_0]$ and hence φ has degree 1. Thus φ is birational.

Remark A.3.6 Corollary 7.2 in [412] is wrong. Theorem A.3.1 is the correction of the corollary. This was pointed out by Freitag and Salvati Manni [239].

Remark A.3.7 In the case $\ell = 3$, we have an analogy of Theorem A.3.1, that is, we get an embedding of the moduli space of semi-stable ordered sets of eight points in the projective line into $\mathbb{P}^{13}$ which is $\mathrm{O}(D(L))(\cong \mathfrak{S}_8)$-equivariant ([413]). This embedding coincides with the one defined by the "cross ratios" of ordered sets of eight points in the projective line.

Remark A.3.8 It would be very interesting to find a geometric meaning of the map φ in Theorem A.3.1.

Bibliographical Notes

The construction of modular forms on Hermitian symmetric homogeneous domains of orthogonal type from modular forms in one variable is the special case of the theta correspondence that transforms automorphic forms on the Lie group $\mathrm{Sp}(V_1)$ to the automorphic form on the Lie group $\mathrm{SO}(V_2)$ [311]. The fundamental discovery of Borcherds was the fact that the denominator of the Kac–Weyl character formula for some hyperbolic generalized Kac–Moody infinite-dimensional Lie algebras is the infinite product of some such modular forms.

Later, based on the ideas of physicists J. Harvey and G. Moore who used Borcherds automorphic forms in string theory, Borcherds gave another less computational treatment of his theory [82]. In between, he was already able to give the first application to algebraic geometry by proving that the moduli space of Enriques surfaces is quasi-affine [81]. We follow his proof in this appendix. Bruinier's book [99] gives a good exposition of Borcherds's theory.

Allcock and Freitag [9] were the first to apply Borcherds additive and multiplicative liftings to moduli problems. They considered the moduli space of marked cubic surfaces and obtained an embedding of the moduli space into $\mathbb{P}^{14}$ which is $W(E_6)$-equivariant. This embedding coincides with the one due to Naruki [550], defined by Cayley's cross ratios of cubic surfaces. The systematic use of Borcherds automorphic forms for the study of different moduli spaces in algebraic geometry, in particular. the moduli spaces of K3 and Enriques surfaces can be found in the works of S. Kondō [412], [416], [415], [414]. A brief exposition of some of this work is given in this appendix.

References

1. Abdallah, N., Emsalem, J., Iarrobino, A.: Nets of conics and associated Artinian algebras of length 7. Eur. J. Math. **9** (2021)
2. Achter, J.: Arithmetic occult period maps. Algebr. Geom. **7**, 581–606 (2020)
3. Achter, J., Howe, E.: Hasse–Witt and Cartier–Manin matrices: a warning and a request. Arithmetic geometry: computation and applications, 1–18, Contemp. Math., **722**, Amer. Math. Soc., Providence, RI (2019)
4. Alekseevskij, D., Vinberg, E., Solodovnikov, A.: Geometry of spaces of constant curvature. Geometry, II, 1–138, Encyclopaedia Math. Sci., 29, Springer, Berlin (1993)
5. Algebraic surfaces. ed. by Shafarevich, I. R., Proc. Steklov Math. Inst., **75** (1964) [Engl.transl.: AMS, Providence.R.I. (1967)]
6. Allcock, D.: The period lattice for Enriques surfaces. Math. Ann. **317**, 483–488 (2000)
7. Allcock, D.: Congruence subgroups and Enriques surface automorphisms, J. Lond. Math. Soc. (2) **98**, 1–11 (2018)
8. Allcock, D., Dolgachev, I.: The tetrahedron and automorphisms of Enriques and Coble surfaces of Hessian type. Annales Henri Lebesgue **3**, 1133–1159 (2020)
9. Allcock, D., Freitag, E.: Cubic surfaces and Borcherds products. Comm. Math. Helv., **77**, 270–296 (2002)
10. Aramova, A.: Reductive derivations of local rings of characteristic p. J. Algebra **109**, no. 2, 394–414 (1987)
11. Aramova, A., Avramov, L.: Singularities of quotients by vector fields in characteristic p. Math. Ann. **273**, no. 4, 629–645 (1986)
12. Arbarello, E., Cornalba, M., Griffiths, P., Harris, J.: Geometry of algebraic curves. vol.I, Springer–Verlag (1984)
13. Arnold, V.: Critical points of smooth functions. Proc. Int. Congress of Math. Vancouver, vol. 1, 19–39 (1978)
14. E. Arrondo, M. Bertolini, C. Turrini, *Focal loci*, Asian J. Math. **9** 449–472 (2005)
15. Arrondo, E., Gross, M.: On smooth surfaces in Gr(1,$\mathbf{P}^3$) with a fundamental curve. Manuscripta Math. **79**, 283–298 (1993)
16. Artebani, M., Dolgachev, I : The Hesse pencil of plane cubic curves. Enseign. Math. (2) **55**, 235–273 (2009)
17. Artin, E.: Geometric algebra. Interscience Publ. New York (1957)
18. Artin, M.: On Enriques surfaces. Harvard thesis (1960)
19. Artin, M.: Some numerical criteria for contractability of curves on algebraic surfaces. Amer. J. Math. **84**, 485–496 (1962)
20. Artin, M.: On isolated rational singularities of surfaces. Amer. J. Math. **88**, 129–136 (1966)
21. Artin, M.: Algebrization of formal moduli : In "Global analysis", Princeton. Univ. Press, Princeton, 21–71 (1969)

22. Artin, M.: Algebraic approximation of structures over complete local rings. Publ. Math. IHES, **36**, 23–58 (1969)
23. Artin, M.: Algebraic spaces. A James K. Whittemore Lecture in Mathematics given at Yale University, 1969, Yale Mathematical Monographs, 3. Yale University Press, New Haven, Conn.-London (1971)
24. Artin, M.: Algebraic construction of Brieskorn's resolutions. J. Algebra **29**, 330–348 (1974)
25. Artin, M.: Supersingular K3 Surfaces. Ann. Scient. Éc. Norm. Sup. 4e Serie, **7**, 543–570 (1974)
26. Artin, M.: Versal deformations and algebraic stacks. Invent. Math. **27**, 165–189 (1974)
27. Artin, M.: Wildly ramified $\mathbf{Z}/2$ actions in dimension two. Proc. Amer. Math. Soc. **52**, 60–64 (1975)
28. Artin, M.: Coverings of the rational double points in characteristic p. In "Complex Analysis and Algebraic Geometry", Iwahami-Shoten, Tokyo, 11–22 (1977)
29. Artin, M., Mazur, B.: Formal groups arising from algebraic varieties. Ann. Sci. École Norm. Sup. (4) **10**, 87–131 (1977)
30. Artin, M., Milne, J.: Duality in the flat cohomology of curves. Invent. Math., **35**, 111–129 (1976)
31. Artin, M., Mumford, D.: Some elementary examples of unirational varieties which are not rational. Proc. London Math. Soc. (3) **25**, 75–95 (1972)
32. Artin, M., Winters, M.: Degenerate fibers and stable reduction of curves. Topology **10**, 373–383 (1971)
33. Artin, M., Verdier, J.-L.: Reflexive modules over rational double points. Math. Ann. **270**, no. 1, 79–82 (1985)
34. Ash, A., Mumford, D., Rapoport, Tai, Y.-S.: Smooth compactifications of locally symmetric varieties. Second edition. With the collaboration of Peter Scholze. Cambridge Mathematical Library. Cambridge University Press, Cambridge (2010)
35. Atiyah, M., Singer, I.:The index of elliptic operators.III. Ann. Math. **87**, 546–604 (1968)
36. Averbuh, B.: Enriques surfaces. Chapter X in Algebraic surfaces. Trudy Mat. Inst. Steklov. **75** (1965) [Engl. Transl.: Proc. Steklov Inst. Math. **75** (1965)]
37. Averbuh, B.: Kummer and Enriques surfaces of special type. Izv. Akad. Nauk SSSR, Ser. mat. **29**, 1095–1118 (1965)
38. Bădescu, L.: Algebraic surfaces. Translated from the 1981 Romanian original by Vladimir Maşek and revised by the author. Universitext. Springer–Verlag, New York (2001)
39. Baker, H.F.: Principles of geometry vol. II. Cambridge University Press (1922); vol. IV Cambridge University Press (1940)
40. Baldisserri, N.: Nets of conics over a field of characteristic 2. Rendiconti di Matematica, (7), **5** (1985), no.3–4, 355–365 (1988)
41. Barth, W.: Lectures on K3-and Enriques surfaces. In "Algebraic geometry. Sitges 1983", Lect. Notes in Math. vol. **1124**, Springer–Verlag, pp. 21–57 (1985)
42. Barth, W., Peters, C.: Automorphisms of Enriques surfaces. Invent. Math., **73**, 383–411.(1983),
43. Barth, W., Hulek, K., Peters, C., Van de Ven: A.: Compact complex surfaces. Second edition. Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. Springer–Verlag, Berlin (2004)
44. Baily, W., Borel, A.: Compactification of arithmetic quotients of bounded symmetric domains. Ann. of Math. (2) **84**, 442–528 (1966)
45. Bauer, T., Di Rocco, S., Harbourne, B., Kapustka, M., Knutsen, A., Syzdek, W., Szemberg, T.: A primer on Seshadri constants. Interactions of classical and numerical algebraic geometry, 33-70, Contemp. Math., **496**, Amer. Math. Soc., Providence, RI, (2009)
46. Beauville, A.: Variétes de Prym et jacobiennes intermediaires. Ann. Sci. Éc. Norm. Sup. **10**, 309–331 (1977)
47. Beauville, A.: Surfaces algébriques complexes. Astérisque. **54**, Soc. Mat. de France (1980) [Engl. Transl.:Complex algebraic surfaces, London Math. Soc. Lecture Notes. **68**, Cambridge Univ. Press. (1983)].
48. Beauville, A.: Le nombre minimum de fibres singuliéres d'une courbe stable sur $\mathbf{P}^1$. Seminar on Pencils of Curves of Genus at Least Two, Astérisque No. **86**, 97–108 (1981)

49. Beauville, A.: Les familles stables de courbes elliptiques sur P^1 admettant quatre fibres singulières. C. R. Acad. Sci. Paris Sér. I Math. **294**, no. 19, 657–660 (1982)
50. Beauville, A.: Fano threefolds and K3 surfaces. The Fano Conference, 175–184, Univ. Torino, Turin (2004)
51. Beauville, A.: On the Brauer group of Enriques surfaces. Math. Res. Lett. **16**, 927–934 (2009)
52. Beauville, A. et al.: Géometrie des surfaces K3: modules et périodes. (Palaiseau, 1981/1982). Astérisque No. **126** (1985)
53. Bégueri, L.: Dualité sur un corps local é corps résiduel algébriquement clos. Mémoire Soc. Math. France, Ser. 2, **4**, 121 pp. (1980)
54. Behrens, K.: On the number of Enriques quotients for supersingular K3 surfaces. arXiv:2003.02132v3 (2020)
55. Benoist, O.: Construction de courbes sur les surfaces K3 (d'après Bogomolov-Hassett-Tschinkel, Charles, Li-Liedtke, Madapusi Pera, Maulik...). Astérisque No. **367**-**368**, Exp. 1081, 219–253 (2015)
56. Bertapelle, A.: Local flat duality of abelian varieties. Manuscripta Math. **111**, 141–161 (2003)
57. Bertapelle, A.: González-Avilés, C., The Greenberg functor revisited. Eur. J. Math. **4**, 1340–1389 (2018)
58. Bertapelle, A., Tong, J.: On torsors under elliptic curves and Serre's pro-algebraic structures. Math. Z. **277**, 91–147 (2014)
59. Berthelot, P.: Sur le "théorème de Lefschetz faible" en cohomologie cristalline. C. R. Acad. Sci. Paris Sér. A-B **277**, A955–A958 (1973)
60. Berthelot, P., Cohomologie crystalline des schémas de caractéristique $p > 0$. Lecture Notes in Math., **407**, Springer–Verlag, Berlin-New York (1974)
61. Berthelot, P.: Le thèoreme de dualitè plate pour les surfaces (d'apr'ees J.S. Milne). In "Surfaces Algébriques", Lect. Notes in Math., **868**, Springer–Verlag, 203–237 (1981)
62. Berthelot, P., Ogus, A.: Notes on crystalline cohomology. Princeton University Press, Princeton, University of Tokyo Press, Tokyo (1978)
63. Bester, M.: Local flat duality of abelian varieties. Math. Ann. **235**, 149–174 (1978)
64. Bhatt, B., Scholze, P.: The pro-étale topology for schemes. Astérisque **369**, 99–201 (2015)
65. Bhosle, U.: Classification of pencils of quadrics in characteristic two. Proceedings of the Indo-French Conference on Geometry (Bombay, 1989), 13–27, Hindustan Book Agency, Delhi, (1993)
66. Blass, P.: Zariski surfaces. Thesis. Univ. of Michigan (1977)
67. Blass, P.: Unirationality of Enriques surfaces in characteristic two. Comp. Math., **45**, 393–398 (1982)
68. Blass, P., Lang, J.: Zariski surfaces and differential equations in characteristic $p > 0$. Monographs and Textbooks in Pure and Applied Mathematics, **106**, Marcel Dekker, Inc., New York (1987)
69. Bloch, S.: Algebraic K-theory and crystalline cohomology. Inst. Hautes Études Sci. Publ. Math. **47**, 187–268 (1977)
70. Bloch, S., Kato, K.: p-adic étale cohomology. Inst. Hautes Études Sci. Publ. Math. No. **63**, 107–152 (1986)
71. Bloch, S., Ogus, A.: Gersten's conjecture and the homology of schemes. Ann. Sci. École Norm. Sup. (4) **7** (1974); 181–201 (1975)
72. Bogomolov, F.: Holomorphic tensors and vector bundles on projective manifolds. Izv. Akad. Nauk SSSR Ser. Mat. **42**, no. 6, 1227–1287, 1439 (1978)
73. Bogomolov, F.: The theory of invariants and applications to some problems in algebraic geometry. In "Algebraic surfaces", Proc. CIME Summer School in Cortona, Liguore, Napoli, pp. 217–245 (1981)
74. Bogomolov, F., Zarhin, Y.: Ordinary reduction of K3 surfaces. Cent. Eur. J. Math. **7**, no. 2, 206–213 (2009)

75. Bolognese, B., Harris, C., Jelisiejew, J.: Equations and tropicalization of Enriques surfaces. Combinatorial algebraic geometry, 181–200, Fields Inst. Commun., 80, Fields Inst. Res. Math. Sci., Toronto, ON (2017)
76. Bombieri, E.: Canonical models of surfaces of general type. Publ. Math. IHES. **42**, 171–229 (1973)
77. Bombieri, E., Mumford, D.: Enriques classification in char. p, III. Invent. Math. **35**, 197–232 (1976)
78. Bombieri, E., Mumford, D.: Enriques classification in char. p, II. Complex Analysis and Alg. Geometry, Iwanami-Shoten, Tokyo, 23–42 (1977)
79. Borcherds, R.: Automorphism groups of Lorentzian lattices. J. Algebra **111**, 133–153 (1987)
80. Borcherds, R.: Automorphic forms on $O_{s+2,2}(\mathbf{R})$ and infinite products. Invent. Math. **120**, 161–213 (1995)
81. Borcherds, R.: The moduli space of Enriques surfaces and the fake monster Lie superalgebra. Topology **35**, 699–710 (1996)
82. Borcherds, R.: Automorphic forms with singularities on Grassmannians. Invent. Math. **132**, 491–562 (1998)
83. Borcherds, R.: Coxeter groups, Lorentzian lattices, and K3 surfaces. Internat. Math. Res. Notices **1998**, no. 19, 1011–1031 (1998)
84. Borel, A.: Introduction aux groupes arithmétiques. Publications de l'Institut de Mathématique de l'Université de Strasbourg, XV. Actualités Scientifiques et Industrielles, No. 1341 Hermann, Paris (1969)
85. Borel, A., De Siebenthal, J.: Les sous-groupes fermés de rang maximum des groupes de Lie clos. Comment. Math. Helv. **23**, 200–221 (1949)
86. Bosch, S., Lütkebohmert, W., Raynaud, M.: Néron models. Ergebnisse der Mathematik und ihrer Grenzgebiete (3) [Results in Mathematics and Related Areas (3)], 21. Springer–Verlag, Berlin (1990)
87. Bourbaki, N.: Algebra I. Chapters 1–3. Translated from the French. Reprint of the 1989 English translation . Elements of Mathematics (Berlin). Springer–Verlag, Berlin (1998)
88. Bourbaki, N.: Lie groups and Lie algebras. Chapters 4–6. Translated from the 1968 French original by Andrew Pressley. Elements of Mathematics (Berlin). Springer–Verlag, Berlin (2002)
89. Bourbaki, N.: Algebra. Chapter 8. Second revised edition of the 1958 edition. Springer, Berlin (2012)
90. Bragg, D., Lieblich, M.: Twistor spaces for supersingular K3 surfaces. arXiv:1804.07282 (2018)
91. Brandhorst, S., Shimada, I.: Borcherds' method for Enriques surfaces. Michigan Math. J. **71** (1), 3–18 (2022)
92. Brandhorst, S., Shimada, I.: Automorphism groups of certain Enriques surfaces. Found. Comput. Math. **22**, 1463–1512 (2022)
93. Brandhorst, S., Sonel, S., Veniani, D.: Idoneal genera and the classification of K3 surfaces covering an Enriques surface. arXiv:2003.08914v4 (2020)
94. Brieskorn, E.: Rationale Singularitäten Komplexer Flächen, Inv. Math. **4**, 336–358 (1968)
95. Brieskorn, E.: Die Milnorgitter der exzeptionellen unimodularen Singularitäten. Bonner Mathematische Schriften, 150. Universität Bonn, Mathematisches Institut, Bonn (1983)
96. Brion, M.: Some structure results for algebraic groups. https://arxiv.org/pdf/1509.03059.pdf (2015)
97. Brion, M., Kumar, S.: Frobenius Splitting Methods in Geometry and Representation Theory. Progress in Mathematics **231**, Birkhäuser (2005)
98. Broer, A., Reiner, V., Smith, L., Webb, P.: Extending the coinvariant theorems of Chevalley, Shephard-Todd, Mitchell, and Springer. Proc. Lond. Math. Soc. (3) **103**, 747–785 (2011)
99. Bruinier, J.: Borcherds products on O(2,l) and Chern classes of Heegner divisors. Lecture Notes in Mathematics, 1780. Springer–Verlag, Berlin (2002)
100. Burns, D., Rapoport, M.: On the Torelli problem for kählerian K3 surfaces. Ann. Sci. École Norm. Sup. (4) **8**, no. 2, 235–274 (1975)

101. Bruns, W., Herzog, J.: Cohen-Macaulay rings. Cambridge Studies in Advanced Mathematics **39**. Cambridge University Press, Cambridge (1993)
102. Burnside, W.: On the Hessian Configuration and its Connection with the Group of 360 Plane Collineations. Proc. London Math. Soc. **4**, 54–71 (1906)
103. Calabri, A., Ciliberto, C., Mendes Lopes, M.: Even sets of four nodes on rational surfaces. Math. Res. Lett. **11**, 799–808 (2004)
104. Cantat, S., Dolgachev, I.: Rational surfaces with a large group of automorphisms. J. Amer. Math. Soc. **25**, 863–905 (2012)
105. Carter, R.: Conjugacy classes in the Weyl group. Seminar on Algebraic Groups and Related Finite Groups, The Institute for Advanced Study, Princeton, N.J., pp. 297–318, Springer, Berlin (1968/69)
106. Cartier, P.: Questions de rationalité de diviseurs en géometrie algébrique. Bull. Soc. Math. France **86**, 117–251 (1958)
107. Casnati, G.: The moduli space of Enriques surfaces with a polarization of degree 4 is rational. Geom. Dedicata **106**, 185–194 (2004)
108. Casnati, G., Ekedahl, T.: Covers of algebraic varieties. I. A general structure theorem, covers of degree 3,4 and Enriques surfaces. J. Algebraic Geom. **5**, 439–460 (1996)
109. Castelnuovo, G.: Sulle superficie di genere zero. Mem. delle Soc. Ital. delle Scienze, ser. III, **10**, 103–123 (1894-96)
110. Catanese, F.: Babbage's conjecture, contact of surfaces, symmetric determinantal varieties and applications. Invent. Math. **63**, 433–465 (1981)
111. Catanese, F.: On the rationality of certain moduli spaces of curves of genus 4. Algebraic geometry, Lecture notes in Math. **1008**, Springer–Verlag, 30–50 (1982)
112. Catanese, F.: Pluricanonical-Gorenstein-curves. Enumerative geometry and classical algebraic geometry (Nice, 1981), 51–95, Progr. Math., **24**, Birkhäuser Boston, Boston, MA (1982)
113. Catanese, F.: Singular bidouble covers and the construction of interesting algebraic surfaces. Algebraic geometry: Hirzebruch 70 (Warsaw, 1998), 97–120, Contemp. Math. **241**, Amer. Math. Soc. (1999)
114. Catanese, F., Franciosi, M., Hulek, K., Reid, M., Embeddings of curves and surfaces. Nagoya Math. J. **154**, 185–220 (1999)
115. Cayley, A.: Memoir on quartic surfaces. Proceedings of the London Mathematical Society, **3**, 19–69 (1869/70)[Collected Papers, vol. VII, 133–181].
116. Chambert-Loir, A.: Cohomologie cristalline: un survol. Exposition. Math. **16**, 333–382 (1998)
117. Chen, X., Gounelas, F., Liedtke, C.: Curves on K3 Surfaces. Duke Math. J. **171**, 3283–3362 (2022)
118. Chiarellotto, B., Lazda, C.: Combinatorial degenerations of surfaces and Calabi-Yau threefolds. Algebra Number Theory **10**, no. 10, 2235–2266 (2016)
119. Chiarellotto, B., Lazda, C., Liedtke, C.: A Néron-Ogg-Shafarevich criterion for K3 surfaces. Proc. Lond. Math. Soc. (3) **119**, no. 2, 469–514 (2019)
120. Chinburg, T.: Minimal models for curves over Dedekind ring. Arithmetical Geometry, Springer–Verlag, pp. 309–325 (1986)
121. Ciliberto, C., Dedieu, T., Galati, C., Knutsen, A.: Irreducible unirational and uniruled components of moduli spaces of polarized Enriques surfaces. Math. Z. **303** no. 3, Paper No. 73 (2023).
122. Coble, A.: Theta modular groups determined by point sets. Amer.J. Math., **40**, 317–340 (1918)
123. Coble, A.: The ten nodes of the rational sextic and of the Cayley symmetroid. Amer. J. Math. **41**, 243–265 (1919)
124. Coble, A.: Associated sets of points. Trans. Amer. Math. Soc. **24**, 1–20 (1922)
125. Coble, A.: Algebraic geometry and theta functions. Amer. Math. Soc. Coll. Publ. vol. **10**, Providence, R.I. (1929; 4d ed., 1982)
126. Coble, A.: Cremona transformations with an invariant rational sextic. Bull. Amer. Math. Soc. **45**, no. 4, 285–288 (1939)

127. Conrad, B.: Chow's K/k-image and K/k-trace, and the Lang-Néron theorem. Enseign. Math. (2) **52**, 37–108 (2006)
128. Conte, A., Verra, A.: Reye constructions for nodal Enriques surfaces. Trans. Amer. Math. Soc. **336**, 79–100 (1993)
129. Conway, J.: Three lectures on exceptional groups. In: Finite simple groups (Proc. Instructional Conf., Oxford, 1969), 215–247. Academic Press, London (1971) (Chapter 10 in Conway, J., Sloane, N., Sphere packings, lattices and groups. Third edition. Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], **290**. Springer–Verlag, New York (1999)
130. Conway, J.: The automorphism group of the 26 dimensional even Lorentzian lattice. J. Algebra **80**, 159–163 (1983) (Chapter 27 in Conway, J., Sloane, N.: Sphere packings, lattices and groups. Third edition. Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], **290**. Springer–Verlag, New York (1999)
131. Conway, J., Curtis, R.T. Norton, S.P., Parker, R.A., Wilson, R.A.: Atlas of finite groups. Maximal subgroups and ordinary characters for simple groups. With computational assistance from J. G. Thackray. Oxford University Press, Eynsham (1985)
132. Corti, A., Kollár, J., Smith, K. E.: Rational and nearly rational varieties. Cambridge Studies in Advanced Mathematics **92**, Cambridge University Press (2004)
133. Cossec, F.: Projective models of Enriques surfaces. Math. Ann. **265**, 283–334 (1983)
134. Cossec, F.: Reye congruences. Trans. Amer. Math. Soc. **280**, 737–751 (1983)
135. Cossec, F.: On the Picard group of Enriques surfaces. Math. Ann. **271**, 577–600 (1985)
136. Cossec, F., Dolgachev, I.: Rational curves on Enriques surfaces. Math. Ann. **272**, 369–384 (1985)
137. Cossec, F., Dolgachev, I.: On automorphisms of nodal Enriques surfaces. Bull. Amer. Math. Soc. **12**, 247–249 (1985)
138. Cossec, F., Dolgachev, I.: Enriques surfaces. I. Progress in Mathematics, **76**. Birkhäuser Boston, Inc., Boston, MA (1989)
139. Cox, D., Katz, S.: Mirror symmetry and algebraic geometry. Mathematical Surveys and Monographs **68**, American Mathematical Society, Providence, RI (1999)
140. Cox, D., Parry, W.: Torsion in elliptic curves over $k(t)$. Compositio Math. **41**, 337–354 (1980)
141. Cox, D., Zucker, S.: Intersection numbers of sections of elliptic surfaces. Invent. Math. **53**, 1–44 (1979)
142. Coxeter, H.S.M.: Finite groups generated by reflections and their finite subgroups generated by reflections. Proc. Cambridge Phil. Soc. **30**, 466–482 (1934)
143. Crew, R.: Étale p-covers in characteristic p. Compos. Math. **52**, 31–45 (1984)
144. Cummins, C.J., Pauli, S.: Congruence subgroups of $\mathrm{PSL}(2, \mathbf{Z})$ of genus less than or equal to 24. Experiment. Math. **12**, 243–255 (2003)
145. Danilov, V.: Cohomology of algebraic varieties. Algebraic geometry, II, 1–125, 255–262, Encyclopaedia Math. Sci., **35**, Springer, Berlin (1996)
146. Darboux, G.: Sur systèmes linéaires de coniques et de surfaces du seconde ordre. Bull. Sci. Math. Astr., **1**, 348–358 (1870)
147. Dardanelli, E., van Geemen, B.: Hessians and the moduli space of cubic surfaces. Algebraic geometry, 17–36, Contemp. Math., **422**, Amer. Math. Soc., Providence, RI (2007)
148. Davis, M.: The geometry and topology of Coxeter groups. London Mathematical Society Monographs Series, 32. Princeton University Press, Princeton, NJ (2008)
149. Degtyarev, A.: Lines on smooth polarized K3-surfaces. Discrete Comput. Geom. **62**, no. 3, 601–648 (2019)
150. Deligne, P.: La formule de dualité globale. Théorie des topos et cohomologie étale des schémas. Tome 3. Séminaire de Géométrie Algébrique du Bois-Marie 1963–1964 (SGA 4). Dirigé par Artin, M., Grothendieck, A. et Verdier., J. L., Lecture Notes in Math., **305**, Springer–Verlag, Berlin-New York, Exposé XVIII, 481–587 (1973)
151. P. Deligne: Intersections sur les surfaces régulières. Groupes de Monodromie en Géométrie Algébrique, Séminaire de Géométrie Algébrique du Bois-Marie 1967–1969 (SGA 7 II). Dirigé par P. Deligne et N. Katz. Lecture Notes in Math., **340**, Springer–Verlag, Berlin-New York, pp. 1–38 (1973)

152. Deligne, P.: La formule de Milnor. Groupes de Monodromie en Géométrie Algébrique, Séminaire de Géométrie Algébrique du Bois-Marie 1967–1969 (SGA 7 II). Dirigé par P. Deligne et N. Katz. Lecture Notes in Math., **340**, Springer–Verlag, Berlin-New York, pp. 197–211 (1973)
153. Deligne, P.: Courbes elliptiques: Formulaire (d'aprés J. Tate). Modular functions in one variable, IV, Lect. Notes in Math., **476**, Springer–Verlag, pp. 53–74 (1975)
154. Deligne, P.: Cohomologie étale. Séminaire de Géométrie Algébrique du Bois-Marie SGA $4\frac{1}{2}$. Avec la collaboration de J. F. Boutot, A. Grothendieck, L. Illusie et J. L. Verdier, Lecture Notes in Math., **569**, Springer (1977)
155. Deligne, P.: Relèvement des surfaces K3 en caractéristique nulle. Prepared for publication by Luc Illusie. Lecture Notes in Math., **868**, Algebraic surfaces (Orsay, 1976–78), pp. 58–79, Springer, Berlin-New York (1981)
156. Deligne, P., Illusie, L.: Relévements modulo p et decomposition du complexe de de Rham. Invent. Math. **89**, 247–270 (1987)
157. Deligne, P., Mumford, D.: The irreducibility of the space of curves of given genus. Inst. Hautes Études Sci. Publ. Math. No.**36**, 75–109 (1969)
158. Deligne, P., Rapoport, M.: Les schémas de modules de courbes elliptiques. Modular functions in one variable II, Lecture Notes in Math., **349**, Springer, 143–316 (1973)
159. Demazure, M., Gabriel, P.: Groupes algébriques. Tome I: Géométrie algébrique, généralitesés, groupes commutatifs. Masson and Cie, Éditeur, Paris; North-Holland Publishing Co., Amsterdam (1970)
160. Demazure, M., et al.: Surfaces de del Pezzo I,II,III,IV,V. Séminaire sur les Singularités des Surfaces, Lecture Notes in Math., **777**, Springer, 23–69 (1980)
161. Diller, J.: Cremona transformations, surface automorphisms, and plane cubics. With an appendix by Igor Dolgachev. Michigan Math. J. **60**, 409–440 (2011)
162. Dimca, A., Singularities and Topology of Hypersurfaces. Universitext, Springer–Verlag, New York (1992)
163. Dixon, J., Mortimer, B.: Permutation groups. Graduate Texts in Mathematics, 163. Springer–Verlag, New York (1996)
164. Dokchitser, T., Dokchitser, V.: A remark on Tate's algorithm and Kodaira types. Acta Arith. **160**, 95–100 (2013)
165. Dolgachev, I.: Rational surfaces with a pencil of elliptic curves. Izv. Akad. Sci. SSSR, Ser. Math. **30**, 1073–1100 (1966)
166. Dolgachev, I.: On the purity of the degeneration loci of families of curves. Invent. Math. **8**, 34–54 (1969)
167. Dolgachev, I.: Euler characteristic of a family of algebraic varieties. Mat. Sbornik, **89**, 297–312 (1972)
168. Dolgachev, I.: Algebraic surfaces with $p_g = q = 0$. In: Algebraic surfaces, Proc. CIME Summer School in Cortona, Liguore, Napoli, pp. 97–216 (1981)
169. Dolgachev, I.: Integral quadratic forms: Application to algebraic geometry (after V. Nikulin). Séminaire Bourbaki 1982/83, no. 611, Astérisque, vol. **105/106**. Soc. Mat. de France, Paris, pp. 251–275.
170. Dolgachev, I.: Weyl groups and Cremona transformations. Singularities, Part 1 (Arcata, Calif., 1981), 283–294, Proc. Sympos. Pure Math., **40**, Amer. Math. Soc., Providence, RI (1983)
171. Dolgachev, I.: Automorphisms of Enriques surfaces. Invent. Math. **76**, 63–177 (1984)
172. Dolgachev, I.: Infinite Coxeter groups and automorphisms of algebraic surfaces. The Lefschetz centennial conference, Part I (Mexico City, 1984), 91–106, Contemp. Math., **58**, Amer. Math. Soc., Providence, RI (1986)
173. Dolgachev, I.: Mirror symmetry for lattice polarized K3 surfaces. Algebraic geometry, **4**, J. Math. Sci. **81**, 2599–2630 (1996)
174. Dolgachev, I.: Lectures on invariant theory. London Mathematical Society Lecture Note Series, 296. Cambridge University Press, Cambridge (2003)
175. Dolgachev, I.: Reflection groups in algebraic geometry. Bull. Amer. Math. Soc. (N.S.) **45**, 1–60 (2008)

176. Dolgachev, I.: Cremona special sets of points in products of projective spaces. Complex and differential geometry, 115–134, Springer Proc. Math., **8**, Springer, Heidelberg (2011)
177. Dolgachev, I.: Classical algebraic geometry:a modern view. Cambridge Univ. Press (2012)
178. Dolgachev, I.: Numerical automorphisms of Enriques surfaces in arbitrary characteristic. Arithmetic and geometry of K3 surfaces and Calabi-Yau threefold, ed. R. Lazu, M. Schütt, N. Yui, Fields Institute Communications, vol. **67**, Springer, pp.267–284 (2013)
179. I. Dolgachev, A brief introduction to Enriques surfaces. In: Development in Moduli theory, Kyoto-2013, Advanced Studies in Pure Math., vol. **69** (2016)
180. Dolgachev, I.: Quartic surfaces with icosahedral symmetry. Adv. Geom. **18**, 119–132 (2018)
181. Dolgachev, I.: Salem numbers and Enriques surfaces. Exp. Math. **27**, 287–301 (2018)
182. Dolgachev, I.: 15-nodal quartic surfaces, I. In: Recent Developments in Algebraic Geometry, ed. A. Hamid, G. Brown, A. Kasprzyk, S. Mori, London Mathematical Society Lecture Notes in Math. 2022 (to appear).
183. Dolgachev, I.: Integral models and torsors of inseparable forms of G_a. Mich. Math. J., **72**, 209–242 (2022)
184. Dolgachev, I., Duncan, A.: Pencils of quadrics in characteristic 2. J. Algebra Number Theory **12**, 99–130 (2018)
185. Dolgachev, I., Duncan, A.: Automorphisms of cubic surfaces in positive characteristic. Izv. Ross. Akad. Nauk Ser. Mat. **83**, no. 3, 15–92 (2019)
186. Dolgachev, I., Keum, J.: Wild p-cyclic actions on K3-surfaces. J. Algebraic Geom. **10**, 101–131 (2001)
187. Dolgachev, I., Keum, J.: Birational automorphisms of quartic Hessian surfaces. Trans. Amer. Math. Soc. **354**, no. 8, 3031–3057 (2002)
188. Dolgachev, I., Keum, J.: Finite groups of symplectic automorphisms of K3 surfaces in positive characteristic. Ann. of Math. (2) **169**, 269–313 (2009)
189. Dolgachev, I., Kondō, S.: A supersingular $K3$ surface in characteristic 2 and Leech lattice. IMRN **2003**, 1–23 (2003)
190. Dolgachev, I., Kondō, S.: Rationality of moduli spaces of Coble surfaces and general nodal Enriques surfaces. Izv. Russ. Akad. Nauk, Ser. Mat. **77**, 77–92 (2013)
191. Dolgachev, I., Martin, G.: Numerically trivial automorphisms of Enriques surfaces in characteristic 2. J. Math. Soc. Japan, **71**, 1181–1200 (2019)
192. Dolgachev, I., Martin, G.: Automorphism groups of rational elliptic and quasi-elliptic surfaces in all characteristics. Adv. Math. **400** (2022), Paper No. 108274, 46 pp.
193. Dolgachev, I., Ortland, D.: Point sets in projective spaces and theta functions. Astérisque, **165** (1988)
194. Dolgachev, I., Reider, I.: On rank 2 vector bundles with $c_1^2 = 10$ and $c_2 = 3$ on Enriques surfaces. Algebraic geometry (Chicago, IL, 1989), 39–49, Lecture Notes in Math., **1479**, Springer, Berlin (1991)
195. Dolgachev, I., Zhang, De-Qi: Coble rational surfaces. Amer. J. Math. **123**, 79–114 (2001)
196. Donovan, P.:The Lefschetz-Riemann-Roch formula Bull. Soc. Math. de France, **97**, 257–273 (1969)
197. Dornhoff, L.: Group representation theory, Part A: ordinary representation theory. Dekker (1971)
198. Douglass, J., Pfeiffer, G. Röhrle, G.: On reflection subgroups of finite Coxeter groups. Comm. Algebra **41**, 2574–2592 (2013)
199. Durfee, A. H.: Fifteen characterizations of rational double points and simple critical points. Enseign. Math. (2) **25**, 131–163 (1979)
200. Du Val, P.: On isolated singularities which do not affect the condition of adjunction, I,II,IV. Proc. Cambridge Phil.Soc. **30**, 453–491 (1934)
201. Du Val, P.: On the Kantor group of a set of points in a plane. Proc. London Math. Soc. (2) **42**, 18–51 (1936)
202. Dynkin, E.: Semi-simple subalgebras of semi-simple Lie algebras. Mat. Sbornik, **30**, 349–462 (1952)
203. Edixhoven, B.: Néron models and tame ramification. Compositio Math.**81**, 291–306 (1992)

204. Eisenbud, D.: Commutative algebra. With a view toward algebraic geometry. Graduate Texts in Mathematics **150**, Springer–Verlag, New York (1995)
205. Eisenbud, D., Harris, J.: On varieties of minimal degree (a centennial account). Algebraic geometry, Bowdoin, 1985 (Brunswick, Maine, 1985), 3–13, Proc. Sympos. Pure Math. **46**, Part 1, Amer. Math. Soc., Providence, RI (1987)
206. Eisenbud, D., Harris, J.: 3264 and all that:a second course in algebraic geometry. Cambridge University Press, Cambridge (2016)
207. Eisenbud, D., Hulek, K., Popescu, S.: A note on the intersection of Veronese surfaces. Commutative algebra, singularities and computer algebra (Sinaia, 2002), 127–139, NATO Sci. Ser. II Math. Phys. Chem., **115**, Kluwer Acad. Publ., Dordrecht (2003)
208. Eisenbud, D., Popescu, S., Walter, C.: Lagrangian subbundles and codimension 3 subcanonical subschemes. Duke Math. J. **107**, 427–467 (2001)
209. Eisenbud, D., Popescu, S., Walter, C.: Enriques surfaces and other non-Pfaffian subcanonical subschemes of codimension 3. Special issue in honor of Robin Hartshorne, Comm. Algebra **28**, 5629–5653 (2000)
210. Ekedahl, T.: Foliations and inseparable morphisms. Algebraic Geometry, Part. 2, Proc. Symp. Pure Math. **46**, AMS, Providence, 139–150 (1987)
211. Ekedahl, T.: Canonical models of surfaces of general type in positive characteristic. Inst. Hautes Études Sci. Publ. Math. **67**, 97–144 (1988)
212. Ekedahl, T.: Vector fields on classical Enriques surfaces. unpublished manuscript.
213. Ekedahl, T., Shepherd-Barron, N.I.: On exceptional Enriques surfaces. math/0405510 (2004)
214. Ekedahl, T., Hyland, J.M.E., Shepherd-Barron, N.I.: Moduli and periods of simply connected Enriques surfaces. arXiv:1210.0342 (2012)
215. Elkies, N., Schütt, M.: Genus 1 fibrations on the supersingular $K3$ surface in characteristic 2 with Artin invariant 1. Asian J. Math., **19**, 555–581 (2015)
216. Elkik, R.: Solutions d'équations à coefficients dans un anneau hensélien. Ann. Sci. École Norm. Sup. (4) **6**, 553–603 (1973)
217. Elliott, E.: An introduction to the algebra of quantics. Oxford Univ. Press (1895) [2nd edition reprinted by Chelsea Publ. Co. (1964)]
218. Enriques, F.: Introduzione alla geometria sopra le superficie algebriche. Mem Soc. Ital. delle Scienze, ser. 3a, **10**, 1–81 (1896) [Memorie Scelete di Geometria", t. 1, Nicolla Zanichelli. Bologna, pp. 211–312 (1956)]
219. Enriques, F.: Sopra le superficie algebriche di bigenere uno. Mem Soc. Ital. delle Scienze, ser. 3a, **14**, 39-366 (1906) [Memorie Scelte di Geometria", t. 2, Zanichelli, Bologna, pp. 241–272 (1959)]
220. Enriques, F.: Un' ossevazione relativa alle superficie di bigenere uno. Rend. Acad. Scienze Inst. Bologna **12**, 40–45 (1908) [Memorie Scelte di Geometria", t. 2, Zanichelli, Bologna, pp. 303–306 (1959)]
221. Enriques, F.: Le superficie algebriche. Zanichelli, Bologna (1949)
222. Enriques, F.: Riposto armonie. Letters di Federigi Enriques a Guido Castelnuovo. ed. Umberto Bottazini, Alberto Conti, Paolo Gario, Bollati Boringhieri (1996)
223. Esnault, H., Viehweg, E.: Lectures on vanishing theorems. DMV Seminar, 20. Birkhäuser Verlag (1992)
224. Esnault, H., Viehweg, E.: Surface singularities dominated by smooth varieties. J. Reine Angew. Math. **649**, 1–9 (2010)
225. Esselmann, F.: Ueber die maximale Dimension von Lorenz-Gittern mit coendlicher Spiegelungsgruppe. J. Number Theory **61**, 103–144 (1996)
226. Fano, G.: Nuovo ricerche sulle congruenze di retta del 3 ordine. Mem. Acad. Sci. Torino, **50**, 1–79 (1901)
227. Fano, G.: Superficie algebriche di genere zero bigenere uno e loro casi particulari. Rend. Circ. Mat. Palernmo, **29**, 98–118 (1910)
228. Fano, G.: Superficie regolari di genere zero e bigenere uno. Revista de matematica y fisica Teoretica Univ. de Tucuman Argentina, **4**, 69–79 (1944)
229. Faltings, G.: p-adic Hodge theory. J. Amer. Math. Soc. **1**, 255–299 (1988)

230. Fantechi,B., Göttsche, L., Illusie, L., Kleiman, S., Nitsure, N., Vistoli, A.: Fundamental algebraic geometry. Math. Surveys Monogr., **123**, American Mathematical Society, Providence, RI, 2005.
231. Fedder, R.: F-purity and rational singularity. Trans. Amer. Math. Soc. **278**, 461–480 (1983)
232. Fedder, R.: A Frobenius characterization of rational singularity in 2-dimensional graded rings. Trans. Amer. Math. Soc. **340**, 655–668 (1993)
233. Fedder, R., Watanabe, K.: A characterization of F-regularity in terms of F-purity. Commutative algebra (Berkeley, CA, 1987), 227–245, Math. Sci. Res. Inst. Publ. **15**, Springer, New York (1989)
234. Flenner, H.: Reine lokale Ringe der Dimension zwei Math. Ann. **216**, 253–263 (1975)
235. Fogarty, J.:Fixed point schemes. Amer. J. Math. **95** 35–51 (1973)
236. Fogarty, J.; Norman, P.: A fixed-point characterization of linearly reductive groups. Academic Press, New York, 1977, pp. 151–155.
237. Freitag, E., Kiehl, R.: Étale cohomology and the Weil conjecture. Ergebnisse der Mathematik und ihrer Grenzgebiete (3) 13. Springer–Verlag, Berlin (1988)
238. Freitag, E.: Some modular forms related to cubic surfaces. Kyungpook Math. J., **43**, 433–462 (2003)
239. Freitag, E., Salvati-Manni, R.: Modular forms for the even unimodular lattice of signature $(2, 10)$. J. Algebraic Geometry **16**, 753–791 (2007)
240. Friedman, R., Morgan, J.: Smooth four-manifolds and complex surfaces. Ergebnisse der Mathematik und ihrer Grenzgebiete (3) [Results in Mathematics and Related Areas (3)], 27. Springer–Verlag, Berlin (1994)
241. Fujita, T.: On polarized manifolds whose adjoint bundles are not semipositive. Algebraic geometry, Sendai, 1985, Adv. Stud. Pure Math., **10**, North-Holland, Amsterdam, pp. 167–178 (1987)
242. Fulton, W.: Intersection theory. Springer–Verlag (1984)
243. Fulton, W.: Introduction to toric varieties. Annals of Mathematics Studies **131**, The William H. Roever Lectures in Geometry, Princeton University Press, Princeton, NJ (1993)
244. Fulton, W., Harris, J.: Representation theory. A first course. Graduate Texts in Mathematics, 129. Readings in Mathematics. Springer–Verlag, New York (1991)
245. Gabriel, P.: Etude infinitesimal des schémas en groupes. Schémas en groupes (SGAA 3), t.1, Lect. Notes in Math. **151**, Springer–Verlag, pp. 474–560 (1970)
246. Galati, C., Knutsen, A.,:Rational curves and Seshadri constant on Enriques surfaces. arXiv22122.08191v1 [mathAG] 15 Dec 2022.
247. Garbagnati, A., Schütt, M.: Enriques surfaces: Brauer groups and Kummer structures. Michigan Math. J. **61**, 297–330 (2012)
248. van der Geer, G.: On the geometry of a Siegel modular threefold. Math. Ann. **260**, 317–350 (1982)
249. Gille, P., Szamuely, T.: Central simple algebras and Galois cohomology. Cambridge Studies in Advanced Mathematics **101**, Cambridge University Press, Cambridge (2006)
250. Giraldo, L., Lopez, A.F., Muñoz, R.: On the projective normality of Enriques surfaces. With an appendix by Angelo Felice Lopez and Alessandro Verra. Math. Ann. **324**, 135–158 (2002)
251. Gizatullin, M.: Rational G-surfaces. (Russian) Izv. Akad. Nauk SSSR Ser. Mat.**44**, no. 1, 110–144 (1980)
252. Godeaux, L.: Variétés algébriques généralisant la surface d'Enriques. Acad. Roy. Belg. Bull. Cl. Sci. (5) **54**, 1401–1409 (1968)
253. Godeaux, L.: Variétés algébriques généralisant la surface d'Enriques. Acad. Roy. Belg. Bull. Cl. Sci. (5) **55**, 1034–1039 (1969)
254. Goldstein, N.: The geometry of surfaces in the 4-quadric. Rend. Sem. Mat. Univ. Politec. Torino **43**, 467–499 (1985)
255. J. Gonzalez-Sprinberg, J., Verdier, J.-L.: Construction géometrique de la correspondence de McKay. Ann. Scient. Éc. Norm. Sup. 4e Ser., **16**, 409–449 (1983)
256. Greenberg, M.: Rational points in Henselian discrete valuation rings. Inst. Hautes Études Sci. Publ. Math. **31**, 59–64 (1966)

257. Greuel, F.-M., H. Kröning, H.: Simple singularities in positive characteristic. Math. Z. **203**, 339–354 (1990)
258. Griess, R.: Quotients of infinite reflection groups. Math. Ann **263**, 267–288 (1983)
259. Griffiths, P., Harris, J.: Principles of algebraic geometry. John Wiley and Sons, New York (1978)
260. Griffiths, P., Harris, J.: Residues and zero cycles on algebraic varieties. Ann. Math. **108**, 461–505 (1978)
261. Gritsenko, V., Hulek, K., Sankaran, G.: The Kodaira dimension of the moduli of K3 surfaces. Invent. Math. **169**, 519–567 (2007)
262. Gritsenko, V., Hulek, K.: Moduli of polarized Enriques surfaces. K3 surfaces and their moduli, 55–72, Progr. Math., **315**, Birkhäuser/Springer (2016)
263. Gross, M.: Surfaces of degree 10 in the Grassmannian of lines in 3-space. J. Reine Angew. Math. **436**, 87–127 (1993)
264. Grothendieck, A.: Le Théorie des classes de Chern. Bull. Soc. Math. France., **86** , 137–154 (1958)
265. Grothendieck, A.: Techniques de construction et théorèmes d'existence en géométrie algébrique. IV. Les schémas de Hilbert. Séminaire Bourbaki, Vol. 6, Exp. No. 221, 1960-61, Soc. Math. France, Paris (1995)
266. Grothendieck, A.: Fondéments de Géométrie Algébrique. Séminaire Bourbaki 1957-1962, Secr. Math. Paris (1962)
267. Grothendieck, A.: Éléments de géométrie algébrique. II. Étude globale élémentaire de quelques classes de morphismes. Inst. Hautes Études Sci. Publ. Math. **8** (1961)
268. Grothendieck, A.: Éléments de géométrie algébrique. III. Étude cohomologique des faisceaux cohérents. I. Inst. Hautes Études Sci. Publ. Math. **11**, 1–167 (1961)
269. Grothendieck, A.: Éléments de géométrie algébrique. IV, Parts I-IV, Étude locale des schémas et des morphismes de schémas. I. Inst. Hautes Études Sci. Publ. Math. **20**, 1–259 (1964); **24**, 1–231 (1965); **28**, 1–255 (1966); **32**, 1–361 (1967)
270. Grothendieck, A.: On the de Rham cohomology of algebraic varieties. Inst. Hautes Études Sci. Publ. Math. **29**, 95–103 (1966)
271. Grothendieck, A.: Brauer Group I,II, III. Dix Exposes sur cohomologie des schémas", North-Holland, Amsterdam, 46–188 (1968)
272. Grothendieck, A.: Revêtements étales et groupe fondamental (SGA1). Séminaire de géométrie algébrique du Bois Marie 1960–1961. Directed by Grothendieck, A.: With two papers by M. Raynaud. Updated and annotated reprint of the 1971 original, Documents Mathématiques (Paris) 3. Société Mathématique de France, Paris (2003)
273. Grothendieck, A.: Cohomologie locale des faisceaux cohérents et théorèmes de Lefschetz locaux et globaux (SGA 2). Augmenté d'un exposé par MichÌe Raynaud. Séminaire de Géométrie Algébrique du Bois-Marie, 1962. Advanced Studies in Pure Mathematics, Vol. 2. North-Holland Publishing Co., Paris (1968)
274. Grothendieck, A.: Spécialization en theorie des intersections. Theorie des intersections et théoreme de Riemann-Roch, (SGA6), Lect. Notes in Math. **225**, Springer–Verlag, 560–594 (1971)
275. Grothendieck, A.: Modéles de Néron et monodromie. Groupes de monodromie en géométrie algébrique, (SGA 7 I)", Lect. Notes in Math. **288**, Springer–Verlag, 313–521 (1972)
276. Séminaire de Géométrie Algébrique du Bois-Marie 1966/67 (SGA 6). Lecture Notes in Math. **225**, Springer, Berlin (1971)
277. Groupes de monodromie en gémétrie algébrique. I. Séminaire de Géométrie Alg'ebrique du Bois-Marie 1967/69 (SGA 7 I). Dirigé par A. Grothendieck. Avec la collaboration de M. Raynaud et D. S. Rim. Lecture Notes in Math. **288**, Springer–Verlag, Berlin-New York (1972)
278. Groupes de monodromie en géométrie algébrique. II. Séminaire de Géométrie Algébrique du Bois-Marie 1967/1969 (SGA 7 II). Dirigé par P. Deligne et N. Katz. Lecture Notes in Math. **340**, Springer–Verlag, Berlin-New York (1973)
279. Grove, L.: Classical groups and geometric algebra. Graduate Studies in Mathematics, **39**. American Mathematical Society, Providence, RI (2002)

280. Grünbaum, B.: Configurations of points and lines. Graduate Studies in Mathematics, **103**. American Mathematical Society, Providence, RI (2009)
281. Guralnick, R., Stevenson, K.: Prescribing ramification. Arithmetic fundamental groups and noncommutative algebra (Berkeley, CA, 1999), 387–406, Proc. Sympos. Pure Math., **70**, Amer. Math. Soc., Providence, RI (2002)
282. Halphen, M.: Sur les courbes planes du sixiéme degré a neuf points doubles, Bull. Soc. Math. France, **10**, 162–172 (1981)
283. Hara, N.: Classification of Two-Dimensional F-Regular and F-Pure Singularities. Adv. Math. **133**, 33–53 (1998)
284. Hara, N., Watanabe, K.: The injectivity of Frobenius acting on cohomology and local cohomology modules. Manuscripta Math. **90**, 301–315 (1996)
285. Harbourne, B.: Blowings-up of $\mathbf{P}^2$ and their blowings-down. Duke Math. J. **52**, 129–148 (1985)
286. Harbourne, B., Lang, W.: Multiple fibers on rational elliptic surfaces. Trans. Amer. Math. Soc. **307**, 205–223 (1988)
287. Harbourne, B., Miranda, R.: Exceptional curves on rational numerically elliptic surfaces. J. Algebra **128**, 405–433 (1990)
288. Harris, J.: Theta-characteristics on algebraic curves. Trans. Amer. Math. Soc. **271**, 611–638 (1982)
289. Harris, J.: Algebraic geometry. A first course. Graduate Texts in Mathematics **133**, Springer–Verlag, New York (1992)
290. Harris, J., Morrison, I.: Moduli of Curves. Graduate Texts in Mathematics **187**, Springer-Verla (1998)
291. Harris, J., Tu, L.: On symmetric and skew-symmetric determinantal varieties. Topology **23**, 71–84 (1984)
292. Hartshorne, R.: Residues and duality. Lecture Notes in Math. **20**, Springer–Verlag, Berlin-New York (1966)
293. Hartshorne, R.: On the De Rham cohomology of algebraic varieties. Publ. Math. IHES, **45**, 5–99 (1976)
294. Hartshorne, R.: Algebraic Geometry. Springer–Verlag (1977)
295. Hartshorne, R.: Stable reflexive sheaves. Math. Ann. **254**, 121–176 (1980)
296. Hashimoto, K.: Period map of a certain K3 family with an $\mathfrak{S}_5$ -action. J. Reine Angew. Math. **652**, 1–65 (2011).
297. Hashimoto, M.: Classification of the linearly reductive finite subgroup schemes of SL_2. Acta Math. Vietnam. **40**, no. 3, 527–534 (2015)
298. Hassett, B., Tschinkel, Y.: Rational points on K3 surfaces and derived equivalence. In: Brauer groups and obstruction problems, 87–113, Progr. Math., **320**, Birkhäuser/Springer, Cham (2017)
299. Hatcher, A.: Algebraic topology. Cambridge University Press, Cambridge (2002)
300. Hazewinkel, M.: Formal groups and applications. Corrected reprint of the 1978 original, AMS Chelsea Publishing (2012)
301. Helgason, S.: Differential geometry, Lie groups, and symmetric spaces. Corrected reprint of the 1978 original. Graduate Studies in Mathematics, **34**. American Mathematical Society, Providence, RI (2001)
302. Hernàndez, R., Sols, I.: On a family of rank 3 bundles on Gr(1,3). J. Reine Angew. Math. **360**, 124–135 (1985)
303. Hilario, M., Stöhr, K.-O.: On regular, but non-smooth integral curves, arxiv:2211.16962 (2022)
304. Hirokado, M.: Singularities of multiplicative p-closed vector fields and global 1-forms of Zariski surfaces. J. Math. Kyoto Univ. **39**, 455–468 (1999)
305. Hirokado, M.: Further evaluation of Wahl vanishing theorems for surface singularities in characteristic p. Michigan Math. J. **68**, no. 3, 621–636 (2019)
306. Hirzebruch, F.: Neue topologische Methoden in der algebraischen Geometrie. Zweite ergänzte Auflage. Ergebnisse der Mathematik und ihrer Grenzgebiete **9**, Springer (1962)

307. Hochster, M., Roberts, J.: Rings of invariants of reductive groups acting on regular rings are Cohen–Macaulay. Advances in Math. **13**, 115–175 (1974)
308. Homma, M.: Singular hyperelliptic curves. Manuscripta Math. **98**, no. 1, 21–36 (1999)
309. Horikawa, E.: On the periods of Enriques surfaces, I. Math. Ann. **234**, 73–108 (1978)
310. Horikawa, E.: On the periods of Enriques surfaces, II. Math. Ann. **235**, 217–246 (1978)
311. Howe, R.: θ-series and invariant theory. Automorphic forms, representations and L-functions (Proc. Sympos. Pure Math., Oregon State Univ., Corvallis, Ore., 1977), Part 1, pp. 275–285, Proc. Sympos. Pure Math., XXXIII, Amer. Math. Soc., Providence, R.I. (1979)
312. Hudson, H.: Kummer's quartic surface. Cambridge Univ. Press (1905)
313. Hudson, H.: Cremona transformations in plane and space. Cambridge Univ. Press (1927)
314. Hughes, D.R., Piper, F.C.: Design Theory. Cambridge Univ. Press (1985)
315. Hulek, K., Schütt, M.: Enriques surfaces and Jacobian elliptic K3 surfaces. Math. Z. **268**, 1025–1056 (2011)
316. Hulek, K., Schütt, M.: Arithmetic of singular Enriques surfaces. Algebra Number Theory **6**, 195–230 (2012)
317. Humphreys, J.: Reflection groups and Coxeter groups. Cambridge Studies in Advanced Mathematics **29**, Cambridge University Press, Cambridge (1990)
318. Hutchinson, J.I.: The Hessian of the cubic surface. II, Bull. Amer. Math. Soc., (2) **5**, 282–292 (1899); ibid. **6**, 328–337 (1899)
319. Hutchinson, J.I.: On some birational transformations of the Kummer surfaces into itself. Bull. Amer. Math. Soc., (2) **7**, 211–217 (1901)
320. Huybrechts, D.: Complex geometry. An introduction. Universitext. Springer (2005)
321. Huybrechts, D.: Lectures on K3 surfaces. Cambridge Studies in Advanced Mathematics, **158**. Cambridge University Press, Cambridge (2016)
322. Hwang, D., Keum, J., Ohashi, H.: Gorenstein **Q**-homology projective planes. Sci. China Math. **58**, 501–512 (2015)
323. Iarrobino, A., Kanev, V.: Power sums, Gorenstein algebras, and determinantal loci. Lect. Notes in Math., **1721**. Springer–Verlag, Berlin (1999)
324. Igusa, J.: A fundamental inequality in the theory of Picard varieties. Proc. Nat. Acad. Sci. U.S.A. **41**, 317–320 (1955)
325. Igusa, J.: On some problems in abstract algebraic geometry. Proc. Nat. Acad. Sci. U.S.A. **41**, 964–967 (1955)
326. Igusa, J.: Betti and Picard numbers of abstract algebraic surfaces. Proc. Nat. Acad. Sci. U.S.A. **46**, 724–726 (1960)
327. Illusie, L.: Report on crystalline cohomology. Algebraic geometry (Proc. Sympos. Pure Math., Vol. **29**, Humboldt State Univ., Arcata, Calif., 1974), pp. 459–478, Amer. Math. Soc., Providence, R.I. (1975)
328. Illusie, L.: Formule de Lefschetz, par A. Grothendieck. Cohomologie ℓ-adique et fonctions L. Séminaire de Géometrie Algébrique du Bois-Marie 1965–1966 (SGA 5). Edité par Luc Illusie. Lecture Notes in Math., **589**. Springer–Verlag, Berlin-New York, pp. 73–137 (1977)
329. Illusie, L.: Complexe de De Rham-Witt et cohomologie crystalline. Ann. Sci. Éc. Norm. Sup, 4e ser., **12**, 501–661 (1979)
330. Illusie, L.: Ordinarité des intersections complètes générales. The Grothendieck Festschrift, Vol. II, 376–405, Progr. Math., **87**, Birkhüser Boston, Boston, MA (1990)
331. Illusie, L.: Frobenius et dégénérescence de Hodge. Introduction à la théorie de Hodge, 113-168, Panor. Synthèses **3**, Soc. Math. France (1996)
332. Illusie, L., Raynaud, M.: Les suites spectrales associées au complexe de de Rham-Witt. Inst. Hautes Études Sci. Publ. Math. No. **57**, 73–212 (1983)
333. Ingalls, C., Kuznetsov, A.: On nodal Enriques surfaces and quartic double solids. Math. Ann. **361**, 107–133 (2015)
334. Isaacs, I.: Character theory of finite groups. Corrected reprint of the 1976 original [Academic Press, New York]; AMS Chelsea Publishing, Providence, RI (2006)
335. Ito, H.: The Mordell-Weil groups of unirational quasi-elliptic surfaces in characteristic 3. Math. Z. **211**, 1–39 (1992)

336. Ito, H.: The Mordell-Weil groups of unirational quasi-elliptic surfaces in characteristic 2. Tohoku Math. J. (2) **46**, 221–251 (1994)
337. Ito, H.: On extremal elliptic surfaces in characteristic 2 and 3. Hiroshima Math. J. **32**, no. 2, 179–188 (2002)
338. Ito, H., Miyanishi, M.: Algebraic surfaces in positive characteristics, purely inseparable phenomena in curves and surfaces. World Sci. Publ. Co. Pte. Ltd., Hackensack, NJ (2021)
339. Ito, H., Ohashi, H.: Classification of involutions on Enriques surfaces. Michigan Math. J. **63**, 159–188 (2014)
340. Ito, K., Ito, T., Liedtke, C.: Deformations of rational curves in positive characteristic. J. reine Angew. Math. **769**, 55–86 (2020)
341. Iversen, B.: A fixed point formula for action of tori on algebraic varieties. Invent. Math. **16**, 229–236 (1972)
342. Jacobson, N.: Lie algebras. Interscience Tracts in Pure and Applied Mathematics, No. 10 Interscience Publishers (a division of John Wiley and Sons), New York-London (1962) [republished by Dover Publ. (1979)]
343. Jacobson, N.: Lectures in abstract algebra. III. Theory of fields and Galois theory. Second corrected printing. Graduate Texts in Mathematics, No. 32. Springer–Verlag, New York-Heidelberg (1975)
344. James, D.: On Witt's theorem for unimodular quadratic forms. Pacific J. Math. **26**, 303–316 (1968)
345. Jang, J.: The ordinarity of an isotrivial elliptic fibration. Manuscripta Math. **134**, 343–358 (2011)
346. Jang, J.: An Enriques involution of a supersingular K3 surface over odd characteristic. Int. Math. Res. Not.**2014**, 3158–3175 (2014)
347. Jang, J.: Néron-Severi group preserving lifting of K3 surfaces and applications. Math. Res. Lett. **22**, 789–802 (2015)
348. Jang, J.: A lifting of an automorphism of a K3 surface over odd characteristic. Int. Math. Res. Not.**2017**, no. 6, 1787–1804 (2017)
349. Janusz, G., Rotman, J.: Outer automorphisms of S_6. Amer. Math. Monthly **89**, 407–410 (1982)
350. Jarvis, T., Lang, W., Rimmasch, G., Rogers, J., Summers, E., Petrosyan, N.: Classification of singular fibers on rational elliptic surfaces in characteristic three. Comm. Algebra **33**, 4533–4566 (2005)
351. Jensen, S.T.: Picard schemes of quotients by finite commutative group schemes. Math. Scand. **42**, 197–210 (1978)
352. Jessop, C.: A treatise of the line complex. Cambridge University Press (1903) [reprinted by Chelsea Publ. Co., New York (1969)]
353. Jordan, C.: Réduction d'um reseaux de formes quadratiques or bilinéaire. J. Math. pures et appl, **6**, Series 2, 403–438 (1906)
354. Joshi, K., Rajan, C.S.: Frobenius splitting and ordinarity. Int. Math. Res. Not. **2003**, no. 2, 109–121 (2003)
355. Jouanolou, J.P.: Théorèmes de Bertini et applications. Progress in Mathematics, **42**, Birkhäuser Boston, Inc., Boston, MA (1983)
356. Józefiak, T., Lascoux, A., Pragacz, P.: Classes of determinantal varieties associated with symmetric and skew-symmetric matrices. Izv. Akad. Nauk SSSR Ser. Mat. **45**, 662–673 (1981)
357. Józefiak, T., Pragacz, P.: Ideals generated by Pfaffians. J. Algebra **61**, 189–198 (1979)
358. Jung, H.: Algebraische Flächen. Hannover, Helwig (1925)
359. Kac, V.: Infinite-dimensional Lie Algebras. Birkhäuser (1983)
360. Kambayashi, T., Miyanishi, M., Takeuchi, M.: Unipotent algebraic groups. Lecture Notes in Math, **414**. Springer–Verlag, Berlin-New York (1974)
361. Kaplansky, I.: Infinite abelian groups. Ann Arbor, The University of Michigan Press (1971)
362. Kato, K., Saito, T.: Ramification theory for varieties over a perfect field. Ann. Math. **68**, 33–96 (2008)

363. Katsura, T.: Surfaces unirationnelles en caracteristique p. C.R. Acad. Sci. Paris, **288**, 45–47 (1979)
364. Katsura, T.: A note on Enriques surfaces in characteristic 2. Comp. Math. **47**, 207–216 (1982)
365. Katsura, T.:Multicanonical systems of elliptic surfaces in small characteristics. Special issue in honour of Frans Oort, Compositio Math. **97**, 119–134 (1995)
366. Katsura, T.: Lefschetz pencils on a certain hypersurface in positive characteristic. Higher dimensional algebraic geometry–in honour of Professor Yujiro Kawamata's sixtieth birthday, 265–278, Adv. Stud. Pure Math., **74**, Math. Soc. Japan, Tokyo (2017)
367. Katsura, T., Kondō, S.: Rational curves on the supersingular K3 surface with Artin invariant 1 in characteristic 3. J. Algebra **352**, 299–321 (2012)
368. Katsura, T., Kondō, S.: A 1-dimensional family of Enriques surfaces in characteristic 2 covered by the supersingular K3 surface with Artin invariant 1. Pure Appl. Math. Q. **11**, no. 4, 683–709 (2015)
369. Katsura, T., Kondō, S.: Enriques surfaces in characteristic 2 with a finite group of automorphisms. J. Alg. Geom. **27**, 173–202 (2018)
370. Katsura, T., Kondō, S.: Coble surfaces in characteristic two. J. Math. Soc. Japan, **75**, 1287–1337 (2023)
371. Katsura, T., Kondō, S., Martin, G.: Classification of Enriques surfaces with finite automorphism group in characteristic 2. Algebraic Geometry **7**, 390–459 (2020)
372. Katsura, T., Schütt M.: Normal forms for quasi-elliptic surfaces and applications. ArXive:2304.12599v1 [mathAG] 25 April 2023.
373. Katsura, T., Takeda, Y.: Quotients of abelian and hyperelliptic surfaces by rational vector fields. J. Algebra **124**, 472–492 (1989)
374. Katsura, T., Ueno, K.: Multiple singular fibers of type G_a of elliptic surfaces in characteristic p. Algebraic and topological theories, pp. 405–429 (1985)
375. Katsura, T., Ueno, K.: On elliptic surfaces in characteristic p. Math. Ann. **272**, 291–330 (1985)
376. Katz, N.: Nilpotent connections and the monodromy theorem. Publ. Math. I.H.E.S., **39**, 175–232 (1970)
377. Katz, N.: Local-to-global extensions of representations of fundamental groups. Ann. Inst. Fourier (Grenoble) **36**, 69–106 (1986)
378. Katz, N., Messing, W.: Some consequences of the Riemann hypothesis for varieties over finite fields. Invent. Math. **23**, 73–77 (1974)
379. Kawamata, Y.: A generalization of Kodaira-Ramanujam's vanishing theorem. Math. Ann. **261**, 43–46 (1982)
380. Kawamata, Y.: Finite generation of a canonical ring. Current developments in mathematics, 2007, 43–76, Int. Press, Somerville, MA (2009)
381. Keel, S., Mori, S.: Quotients by groupoids. Ann. of Math. (2) **145**, 193–213 (1997)
382. Kempf, G., Knudsen, K., Mumford, D., Saint-Donat, B.: Toroidal embeddings. I. Lecture Notes in Math. **339**, Springer–Verlag, Berlin-New York (1973)
383. Keum, J.: Every algebraic Kummer surface is the K3-cover of an Enriques surface. Nagoya Math. J. **118**, 99–110 (1990)
384. Keum, J.: Automorphisms of Jacobian Kummer surfaces. Compositio Math., **107**, 269–288 (1997)
385. Keum, J.: Wild p-cyclic actions on smooth projective surfaces with $p_g = q = 0$. J. Algebra **244**, 45–58 (2001)
386. Keum, J.: Orders of automorphisms of K3 surfaces. Adv. Math. **303**, 39–87 (2016)
387. Kim, H.: Stable vector bundles on Enriques surfaces. Ph.D. Thesis, Univ. of Michigan (1990)
388. Kim, H.: Stable vector bundles of rank two on Enriques surfaces. J. Korean Math. Soc. **43** (2006), 765–782.
389. Kim, Y.: Normal quintic Enriques surfaces. J. Korean Math. Soc. **36**, 545–566 (1999)
390. Kim, W., Madapusi Pera, K.: 2-adic integral canonical models. Forum Math. Sigma **4**, e28, 34 pp (2016)

391. Klein, F.: Vorlesungen über das Ikosaeder und die Auflösung der Gleichungen vom fünften Grade. Reprint of the 1884 original. Edited, with an introduction and commentary by Peter Slodowy, Birkhäuser Verlag (1993)
392. Kleiman, S.L.: The Picard scheme. Fundamental algebraic geometry, 235-321, Math. Surveys Monogr. **123**, Amer. Math. Soc., Providence, RI (2005)
393. Kleppe, H., Laksov, D.: The algebraic structure and deformation of Pfaffian schemes. J. Algebra **64**, no. 1, 167–189 (1980)
394. Kneser, M.: Quadratische Formen. Revised and edited in collaboration with Rudolf Scharlau, Springer–Verlag, Berlin (2002)
395. Knörrer, H.: Group representations and the resolution of rational double points. Finite groups coming of age (Montreal, Que., 1982), 175–222, Contemp. Math., **45**, Amer. Math. Soc., Providence, RI (1985)
396. Knutsen, A.: On kth-order embeddings of K3 surfaces and Enriques surfaces. Manuscripta Math. **104**, 211–237 (2001)
397. Knutsen, A.: On moduli spaces of polarized Enriques surfaces. J. Math. Pures Appl. **144**, 106–136 (2020)
398. Knutsen, A., Lopez, A.: Projective normality and the generation of the ideal of an Enriques surface. Adv. Geom. **15**, 339–348 (2015)
399. Knutson, D.: Algebraic Spaces. Lecture Notes in Math. **203**, Springer (1971)
400. Kodaira, K.: On a differential-geometric method in the theory of analytic stacks. Proc. Nat. Acad. Sci. U. S. A. **39**, 1268–1273 (1953)
401. Kodaira, K.: On compact complex analytic surfaces. I. Ann. Math. **71**, 111–152 (1960); II, ibid, **77**, 563–626 (1963); III, ibid, **78**, 1–40 (1963)
402. Kodaira, K.: On the structure of compact complex analytic surfaces. I. Amer. J. Math. **86**, 751–798 (1964)
403. Kodaira, K.: Pluricanonical systems on algebraic surfaces of general type. J. Math. Soc. Japan, **20**, 180–192 (1968)
404. Kodaira, K.: On homotopy K3-surfaces. Essays in topology and related topics, Springer–Verlag, pp. 56–69 (1970)
405. Koecher, M.: Zur Theorie der Modulformen n-ten Grades. I. Math. Z. **59**, 399–416 (1954)
406. Kollár, J.: Rational curves on algebraic varieties. Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge **32**, Springer, Berlin (1996)
407. Kollár, J.: Lectures on Resolutions of Singularities. Ann. of Math. Studies **166**, Princeton University Press (2007)
408. Kollár, J., Mori, S.: Birational geometry of algebraic varieties. With the collaboration of C. H. Clemens and A. Corti. Translated from the 1998 Japanese original. Cambridge Tracts in Mathematics **134**, Cambridge University Press, Cambridge (1998)
409. Kondō, S.: Enriques surfaces with finite automorphism group. Japan. J. Math. **12**, 192–282 (1986)
410. Kondō, S.: The rationality of the moduli space of Enriques surfaces. Comp. Math. **91**, 151–173 (1994)
411. Kondō, S.: The automorphism group of a generic Jacobian Kummer surface. J. Algebraic Geometry **7**, 589–609 (1998)
412. Kondō, S.: The moduli space of Enriques surfaces and Borcherds products. J. Algebraic Geometry **11**, 601–627 (2002)
413. Kondō, S.: The moduli space of 8 points on $\mathbf{P}^1$ and automorphic forms. Algebraic geometry, 89–106, Contemp. Math., **422**, Amer. Math. Soc., Providence, RI (2007)
414. Kondō, S.: Moduli of plane quartics, Göpel invariants and Borcherds products. Int. Math. Res. Not. IMRN **2011**, no. 12, 2825–2860 (2011)
415. Kondō, S.: The moduli space of Hessian quartic surfaces and automorphic forms. J. Pure Appl. Algebra **216**, 2233–2240 (2012)
416. Kondō, S.: The Segre cubic and Borcherds products. Arithmetic and geometry of K3 surfaces and Calabi-Yau threefolds, 549–565, Fields Inst. Commun., 67, Springer, New York (2013)
417. Kondō, S.: K3 surfaces. Tracts in Mathematics **32**, European Math. Soc. (2020) (Translation of the Japanese original).

418. Kondō, S.: Classification of Enriques surfaces covered by the supersingular $K3$ surface with Artin invariant 1 in characteristic 2. J. Math. Soc. Japan **73**, 301–328 (2021)
419. Kondō, S.: Coble surfaces with finite automorphism group. Rend. Circ. Mat. Palermo **71**, 829–864 (2022)
420. Kondō, S., Shimada, I.: On a certain duality of Néron-Severi lattices of supersingular $K3$ surfaces. Algebraic Geometry **3**, 311–333 (2014)
421. Kulikov, V.: Degenerations of K3 surfaces and Enriques surfaces. Izv. Akad. Nauk SSSR Ser. Mat. **41**, 1008–1042 (1977)
422. Kulikov, V., Kurchanov, P.:Complex algebraic varieties: periods of integrals and Hodge structures [MR1060327] Encyclopaedia Math. Sci., **36** Springer–Verlag, Berlin, 1998, 1–217, 263–270.
423. Kunz, E.: Characterizations of regular local rings for characteristic p. Amer. J. Math. **91**, 772–784 (1969)
424. Kurke, H.: Vorlesungen über algebraische Flächen. With English, French and Russian summaries. Teubner-Texte zur Mathematik [Teubner Texts in Mathematics], 43. BSB B. G. Teubner Verlagsgesellschaft, Leipzig (1982)
425. Kuznetsov, A.: Derived categories of quadric fibrations and intersections of quadrics. Adv. Math. **218**, 1340–1369 (2008)
426. Laface, A., Testa, D.: Nef and semiample divisors on rational surfaces. Torsors, étale homotopy and applications to rational points. London Math. Soc. Lecture Note Ser., **405**, 429–446, Cambridge Univ. Press, Cambridge (2013)
427. Lamotke, K.: Regular Solids and Isolated Singularities. Advanced Lectures in Mathematics. Vieweg und Sohn, Braunschweig (1986)
428. Landman, A.: On the Picard-Lefschetz transformation for algebraic manifolds acquiring general singularities. Trans. Amer. Math. Soc. **181**, 89–126 (1973)
429. Lang, S.: Elliptic functions. With an appendix by J. Tate. Second edition. Graduate Texts in Mathematics, 112. Springer, New York (1987)
430. Lang, S., Tate, J.: Principal homogeneous spaces over abelian varieties. Amer. J. Math. **80**, 659–684 (1958)
431. Lang, W.: Quasi-elliptic surfaces in characteristic three. Ann. Scient. Éc. Norm. Sup. 4e Ser., **12**, 473–500 (1979)
432. Lang, W.: Two theorems on the De Rham cohomology. Comp. Math. **40**, 417–423 (1980)
433. Lang, W.: On Enriques surfaces in char. p. I. Math. Ann. **265**, 45–65 (1983)
434. Lang, W.: An analogue of the logarithmic transform in characteristic p. Proceedings of. 1984 Vancouver Conf. in Alg. Geometry, CMS Conf. vol. 6, pp. 337–340 (1986)
435. Lang, W.: On Enriques surfaces in char. p. II. Math. Ann. **281**, 671–685 (1988)
436. Lang, W.: Extremal rational elliptic surfaces in characteristic p. I. Beauville surfaces. Math. Zeit. **207**, 429–437 (1991)
437. Lang, W.: Extremal rational elliptic surfaces in characteristic p. II. Surfaces with three or fewer singular fibres. Ark. Mat. **32**, 423–448 (1994)
438. Lang, W.: Configurations of singular fibers on rational elliptic surfaces in characteristic two. Special issue in honor of Robin Hartshorne. Comm. in Algebra **28**, 5813–5836 (2000)
439. Lang, W.: Classification of singular fibers on rational elliptic surfaces in characteristic three. Comm. Algebra **33**, 4533–4566 (2005)
440. Lang, W., Nygaard, N.: A short proof of the Rudakov-Shafarevich theorem. Math. Ann. **251**, no. 2, 171–173 (1980)
441. Langer, A.: A note on Bogomolov's instability and Higgs sheaves. Algebraic geometry, 237–256, de Gruyter, Berlin (2002)
442. Langer, A.: Bogomolov's inequality for Higgs sheaves in positive characteristic. Invent. Math. **199**, no. 3, 889–920 (2015)
443. Laufer, H.B.: Taut two-dimensional singularities. Math. Ann. **205**, 131–164 (1973)
444. Laumon, G., Moret-Bailly, L.: Champs algébriques. Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. **39**, Springer–Verlag, Berlin (2000)
445. Laurent, B.: Almost homogeneous curves over an arbitrary field. Transform. Groups **24** (2019), 845–886.

446. Lawson, H.B., Michelsohn, M,: Spin geometry. Princeton Mathematical Series **38**, Princeton University Press (1989)
447. Lazarsfeld, R.: Positivity in algebraic geometry. vols. I-II, Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge, **49**, Springer–Verlag, Berlin (2004)
448. Le Stum, B.: Rigid cohomology. Cambridge Tracts in Mathematics **172**, Cambridge University Press (2007)
449. Lee, Y., Nakayama, N.: Simply connected surfaces of general type in positive characteristic via deformation theory. Proc. Lond. Math. Soc. (3) **106**, 225–286 (2013)
450. Li, C.: The forms of the Witt group schemes. J. Algebra **186**, 182–206 (1996)
451. Lichtenbaum, S.: Curves over discrete valuation rings. Amer. J. Math. **85**, 380–405 (1968)
452. Lichtenbaum, S.: The period-index problem for elliptic curves. Amer. J. Math. **90**, 1209–1223 (1968)
453. Lieberman, D.: Compactness of the Chow scheme: applications to automorphisms and deformations of Kähler manifolds. Fonctions de plusieurs variables complexes, III (Sém. Francois Norguet, 1975-1977), pp. 140–186, Lecture Notes in Math., **670**, Springer, Berlin (1978)
454. Lieblich, M., Maulik, D.: A note on the cone conjecture for K3 surfaces in positive characteristic. Math. Res. Lett. **25**, no. 6, 1879–1891 (2018)
455. Liedtke, C.: Singular abelian covers of algebraic surfaces. Manuscripta Math. **112**, no. 3, 375–390 (2003)
456. Liedtke, C.: Uniruled surfaces of general type. Math. Z. **259**, 775–797 (2008)
457. Liedtke, C.: A note on non-reduced Picard schemes. J. Pure Appl. Algebra **213**, 737–741 (2009)
458. Liedtke, C.: Algebraic surfaces in positive characteristic. Birational geometry, rational curves, and arithmetic, 229–292, Springer, New York (2013)
459. Liedtke, C.: The canonical map and Horikawa surfaces in positive characteristic. Int. Math. Res. Not. IMRN **2013**, no. 2, 422–462 (2013)
460. Liedtke, C.: Arithmetic moduli and lifting of Enriques surfaces. J. Reine Angew. Math. **706**, 35–65 (2015)
461. Liedtke, C.: Supersingular K3 surfaces are unirational. Invent. Math. **200**, 979–1014 (2015)
462. Liedtke, C.: Lectures on Supersingular K3 Surfaces and the Crystalline Torelli Theorem. K3 Surfaces and Their Moduli. Progress in Mathematics **315**, Birkhäuser, 171–235 (2016)
463. Liedtke, C.: Morphisms to Brauer–Severi varieties with applications to del Pezzo surfaces. Geometry over nonclosed fields, 157–196, Simon Symp.,Springer, Cham (2017)
464. Liedtke, C.: The Picard Rank of an Enriques Surface. Math. Res. Lett. **24**, no. 6, 1729–1737 (2017)
465. Liedtke, C., Martin, G., Matsumoto, Y.: Linearly Reductive Quotient Singularities. arXiv:2102.01067 (2021), to appear in Astérisque
466. Liedtke, C., Martin, G., Matsumoto, Y.: Torsors over the rational double points in characteristic p, arXiv:2110.03650 (2021), to appear in Astérisque
467. Liedtke, C., Matsumoto, Y.: Good Reduction of K3 Surfaces. Compos. Math. **154**, 1–35 (2018)
468. Liedtke, C., Satriano, M.: On the birational nature of lifting. Adv. Math. **254**, 118–137 (2014)
469. Lipman, J.: Rational singularities, with applications to algebraic surfaces and unique factorization. Inst. Hautes Études Sci. Publ. Math. **36**, 195–279 (1969)
470. Lipman, J.: The Picard group of a scheme over an Artin ring. Inst. Hautes Études Sci. Publ. Math. **46**, 15–86 (1976)
471. Lipman, J.: Dualizing sheaves, differentials and residues on algebraic varieties. Astérisque, **117**, Soc. Math. de France (1984)
472. Liu, Q.: Algebraic geometry and arithmetic curves. Oxf. Grad. Texts Math., **6** Oxford Sci. Publ. Oxford University Press, Oxford, 2002
473. Liu, Q., Lorenzini, D., Raynaud, M.: Néron models, Lie algebras, and reduction of curves of genus one. Invent. Math. **157**, 455–518 (2004)
474. Looijenga, E.: Invariant theory for generalized root systems. Invent. Math. **61**, 1–32 (1980)

475. Looijenga, E.: Rational surfaces with an anticanonical cycle. Ann. Math. (2) **114**, 267–322 (1981)
476. Looijenga, E.: Compactifications defined by arrangements. II. Locally symmetric varieties of type IV. Duke Math. J. **119**, 527–588 (2003)
477. Lorenzini, D.: Groups of components of Néron models of Jacobians. Compositio Math. **73**, 145–160 (1990)
478. Lorenzini, D.: Wild models of curves. Algebra Number Theory **8**, 331–367 (2014)
479. Lorenzini, D.: Néron models. Eur. J. Math. **3**, 171–198 (2017)
480. Lorenzini, D., Schröer, S.: Moderately ramified actions in positive characteristic. Math. Z. **295**, 1095–1142 (2020).
481. Lorenzini, D., Schröer, S.: Discriminant groups of wild cyclic quotient singularities. Algebra and Number Theory **17**, 1017–1068 (2023)
482. Lutz, E.: Sur l'equation $y^2 = x^3 - Ax - B$ dans le corps p-adiques. J. für Math. **177**, 238–247 (1937)
483. Madapusi Pera, K.: The Tate conjecture for K3 surfaces in odd characteristic. Invent. Math. **201**, 625–668 (2015)
484. Manin, Y.: Theory of commutative formal groups over fields of finite characteristic. Uspehi Mat. Nauk **18**, 3–90 (1963)
485. Manin, J.: The Tate height of points on an Abelian variety, its variants and applications. (Russian) Izv. Akad. Nauk SSSR Ser. Mat. **28**, 1363–1390 (1964) [Engl. Transl.:Amer. Math. Soc. Translations, ser. 2, **59**, 82–110 (1966)]
486. Manin, Y.: Cubic forms: algebra, geometry, arithmetic. Nauka. Moscow (1972) [English translation:North-Holland, Amsterdam, 1974, 2nd edition (1986)]
487. Marrama, A.: A Purity Theorem for Torsors. Master Thesis, Bordeaux (2016)
488. Martin, G.: Enriques surfaces with finite automorphism group in positive characteristic. Algebraic Geometry **6**, 592–649 (2019)
489. Martin, G.: A note on Enriques surfaces with non-normal K3-covers. in preparation.
490. Martin, G.: Automorphisms of unnodal Enriques surfaces. arXiv:1908.000449v1 (2019)
491. Martin, G.: Infinitesimal automorphisms of algebraic varieties and vector fields on elliptic surfaces, Algebra Number Theory, **16**, 1655–1704 (2022)
492. Martin, G., Mezzedimi, G., Veniani, D.C.: On extra-special Enriques surfaces. Math. Ann. **387**, 133–143 (2023)
493. Martin, G., Mezzedimi, G., Veniani, D.C.: Nodal Enriques surfaces and Reye congruences. arXiv.20306.116611, [mathAG] June 2023.
494. Matsumoto, Y.: Canonical coverings of Enriques surfaces in characteristic 2. J. Math. Soc. Japan **74**, 849–872 (2022)
495. Matsumoto, Y.:Good reduction criterion for K3 surfaces. Math. Z. **279**, 241–266 (2015)
496. Matsumoto, Y., Ohashi, H., Rams, S.: On automorphisms of Enriques surfaces and their entropy. Math. Nachr. **291**, 2084–2098 (2018)
497. Matsumura, H.: Commutative ring theory. Cambridge studies in advanced mathematics 8, Cambridge University Press (1986)
498. Matsumura, H., Oort, F.: Representability of group functors, and automorphisms of algebraic schemes. Invent. Math. **4**, 1–25 (1967)
499. Matsusaka, T., Mumford, D.: Two fundamental theorems on deformations of polarized varieties. Amer. J. Math. **86**, 668–684 (1964)
500. Maulik, D.: Supersingular K3 surfaces for large primes. With an appendix by Andrew Snowden. Duke Math. J. **163**, 2357–2425 (2014)
501. Mazur, B.: Frobenius and the Hodge filtration. Bull. Amer. Math. Soc. **78**, 653–667 (1972)
502. Mazur, B.: Frobenius and the Hodge filtration (estimates). Ann. of Math. (2) **98** , 58–95 (1973)
503. McKay, J., Sebbar, A.: Arithmetic semistable elliptic surfaces. Proceedings on Moonshine and related topics (Montreal, QC, 1999), 119–130, CRM Proc. Lecture Notes, 30, Amer. Math. Soc., Providence, RI (2001)
504. Mclaughlin, J.: Some subgroups of $SL_n(\mathbf{F}_2)$. Ill. J. Math. **13**, 108–115 (1969)

505. Mehta, V.B., Srinivas, V.: Varieties in positive characteristic with trivial tangent bundle. With an appendix by Srinivas and M. V. Nori. Compositio Math. **64**, no. 2, 191–212 (1987)
506. Milne, J.: Flat homology. Bull. Amer. Math. Soc. **82**, 118–120 (1976)
507. Milne, J.: Duality in the flat cohomology of a surface. Ann. Scient. Éc. Norm. Sup. 4e ser., **9**, 171–202 (1976)
508. Milne, J.: Étale cohomology. Princeton Univ. Press (1980)
509. Milne, J.: Arithmetic duality theorems. Princeton Univ. Press (1986)
510. Milne, J.:Algebraic groups. The theory of group schemes of finite type over a field. Cambridge Stud. Adv. Math., **170** Cambridge University Press, Cambridge, 2017.
511. Milnor, J.: Singular points of complex hypersurfaces. Ann. Math. Studies **61**, Princeton Univ. Press (1968)
512. Milnor, J., Stasheff, J.: Characteristic classes. Ann. Math. Studies **75**, Princeton Univ. Press (1974)
513. Miranda, R.: The basic theory of elliptic surfaces. Dottorato di Ricerca in Matematica. [Doctorate in Mathematical Research] ETS Editrice, Pisa (1989)
514. Miranda, R., Morrison, D.: Integral quadratic forms. Seminar notes at the Institute for Advanced Study, 2000, (available online:web.math.ucsb.edu/ drm/manuscripts/eiqf.pdf).
515. Miranda, R., Persson, U.: On extremal rational elliptic surfaces. Math. Zeit. **193**, 537–558 (1986)
516. Miranda, R., Persson, U.: Mordell-Weil groups of extremal elliptic K3 surfaces. Problems in the theory of surfaces and their classification (Cortona, 1988), 167–192, Sympos. Math. XXXII, Academic Press, London (1991)
517. Miyanishi, M., Nomura, T.: Finite group scheme actions on the affine plane. J. Pure Appl. Algebra **71**, 249–64 (1991)
518. Miyanishi, M., Ito, H.: Algebraic surfaces in positive characteristics. Purely inseparable phenomena in curves and surfaces. World Scientific Publ. Co. (2021)
519. Miyata, T., Invariants of certain groups. I. Nagoya Math. J. **41**, 69–73 (1971)
520. Miyaoka, Y.: Geometry of Rational Curves on Varieties. In: Miyaoka, Y., Peternell, T.: Geometry of higher-dimensional algebraic varieties. DMV Seminar **26**, Birkhäuser (1997)
521. Moishezon, B.: Complex algebraic surfaces and connected sums of complex projective planes. Lect. Notes in Math. **603**, Springer–Verlag (1977)
522. Mori, S., Saito, N.: Fano threefolds with wild conic bundle structures. Proc. Japan Acad. Ser. A Math. Sci. **79**, no. 6, 111–114 (2003)
523. Morrison, D.: Semistable degenerations of Enriques' and hyperelliptic surfaces. Duke Math. J. **48**, 197–249 (1981)
524. Morrison, D., Saito, M.: Cremona transformations and degrees of period maps for K3 surfaces with ordinary double points. Algebraic geometry, Sendai, 1985, 477–513, Adv. Stud. Pure Math., **10**, North-Holland, Amsterdam (1987)
525. Moschetti, R., Rota, F., Schaffer L.:A computational view on the non-degeneracy invariant for Enriques surfaces. arXiv:22-2.01775v1 [math.AG] 3 Feb 2022.
526. Mukai, S.: Symplectic structure of the moduli space of sheaves on an abelian or K3 surface. Invent. Math. **77**, 101–116 (1984)
527. Mukai, S.: Finite groups of automorphisms of K3 surfaces and the Mathieu group. Invent. Math. **94**, 183–221 (1988)
528. Mukai, S.: Numerically trivial involutions of Kummer type of an Enriques surface. Kyoto J. Math. **50**, 889–902 (2010)
529. Mukai, S.: Kummer's quartics and numerically reflective involutions of Enriques surfaces. J. Math. Soc. Japan **64**, 231–246 (2012)
530. Mukai, S.: Lecture notes on K3 and Enriques surfaces. Contributions to algebraic geometry, 389–405, EMS Ser. Congr. Rep., Eur. Math. Soc., Zürich (2012)
531. Mukai, S.: Kummer surfaces, and Enriques surfaces with tree structure (in Japanese). Proceedings of Algebraic Geometry Symposium, Kinosaki 2018, 134–144 (2018)
532. Mukai, S., Namikawa, Y.: Automorphisms of Enriques surfaces which act trivially on the cohomology groups. Invent. Math. **77**, 383–397 (1984)

533. Mukai, S., Ohashi, H.: Enriques surfaces of Hutchinson-Göpel type and Mathieu automorphisms. Arithmetic and geometry of K3 surfaces and Calabi-Yau threefolds, 429–454, Fields Inst. Commun., 67, Springer, New York (2013)
534. Mukai, S., Ohashi, H.: Finite groups of automorphisms of Enriques surfaces and the Mathieu group M_{12}. arXiv:1410.7535v2. August 2015
535. Mukai, S., Ohashi, H.: The automorphism groups of Enriques surfaces covered by symmetric quartic surfaces. Recent advances in Algebraic geometry, ed. Ch. Hacon, M. Mustata and M. Popa, Cambridge Univ. Press (2015)
536. Mumford, D.: Pathologies of modular algebraic surfaces. Amer. J. Math., **83**, 339–342 (1961)
537. Mumford, D.: The topology of normal singularities of an algebraic surface and a criterion for simplicity. Inst. Hautes Études Sci. Publ. Math. **9**, 5–22 (1961)
538. Mumford, D.: Lectures on curves on an algebraic surface. Ann. Math. Studies, vol. 59, Princeton Univ. Press, Princeton (1966)
539. Mumford, D.: Enriques' classification of surfaces in char. p, I. Global analysis, pp. 325–339, Princeton. Univ. Press, Princeton (1969)
540. Mumford, D.: Varieties defined by quadratic equations. 1970 Questions on Algebraic Varieties (C.I.M.E., III Ciclo, Varenna, 1969), 29–100. Edizioni Cremonese, Rome (1970)
541. Mumford, D.: Abelian varieties. Oxford Univ. Press (1970)
542. Mumford, D., Fogarty, J., Kirwan, F.: Geometric invariant theory. Third edition. Ergebnisse der Mathematik und ihrer Grenzgebiete, **34**. Springer–Verlag, Berlin (1994)
543. Mumford, D., Suomininen, K.: Introduction to the theory of moduli. Algebraic geometry, pp.171–222, Oslo 1970, Wolters-Noordhoff Publ. (1972)
544. Munkres, J.: Elements of algebraic topology. Addison-Wesley Publishing Company, Menlo Park, CA (1984)
545. Murre, J.: On contravarinat functors from the category of schemes to the category of abelian groups (with applications to the Picard functor). Publ. Math. IHES, **23**, 581–619 (1964)
546. Nagata, M.: On rational surfaces I. Mem. Coll. Sci. Univ. Kyoto **37**, 271–293 (1960)
547. Nagata, M.: Complete reducibility of rational representations of a matric group. J. Math. Kyoto Univ. **1**, 87–99 (1961/1962)
548. Nakkajima, Y.: Liftings of simple normal crossing log K3 and log Enriques surfaces in mixed characteristics. J. Algebraic Geom. **9**, no. 2, 355–393 (2000)
549. Namikawa, Y.: Periods of Enriques surfaces. Math. Ann. **270**, 201–222 (1985)
550. Naruki, I.: Cross ration variety as moduli space of cubic surfacs. Proc. London Math. Soc., **45**, 1–30 (1982)
551. Naruki, I.: Configurations related to maximal rational elliptic surfaces. Complex analytic singularities 315–347, Adv. Stud. Pure Math. **8**, North-Holland, Amsterdam (1987)
552. Némethi, A.: Normal surface singularities. Ergeb. Math. Grenzgeb. (3), **74** Springer, Cham, 2022
553. Néron, A.: Modéles minimaux des variétetes abéliennes. Publ. Math. Inst. IHES. **21** (1964)
554. Nikulin, V.: Kummer surfaces. Izv. Akad. Nauk SSSR Ser. Mat. **39**, 278–293 (1975)
555. Nikulin, V.: Finite groups of automorphisms of Kähler K3 surfaces. Proc.Moscow Math. Society, **38**, 71–135 (1980)
556. Nikulin, V.: Integral quadratic forms and some of its geometric applications. Izv. Akad. nauk SSSR, Ser. Math. **43**, 111–177 (1979)
557. Nikulin, V.: Quotient-groups of groups of automorphisms of hyperbolic forms modulo subgroups generated by 2-reflections, Algebraic geometric applications. Current Problems of Mathematics, t. 18, VINITI, Moscow, 3–114 (1981) [English translation:J. Soviet Math. **22**, 1401–1476 (1983)]
558. Nikulin, V.: Surfaces of type K3 with finite automorphism group and Picard group of rank 3. Trudy. Steklov Inst. Math. **165**, 119–142 (1984) [English translation: Proc. Inst. Steklov, pp.131–156 (1985)].
559. Nikulin, V.: On a description of the automorphism groups of an Enriques surfaces. Dokl. Akad, Nauk SSSR **277**, 1324–1327 (1984) [English translation: Soviet Math. Doklady **30** (1984), 282–285].

560. Nishiyama, K.: The Jacobian fibrations on some $K3$ surfaces and their Mordell-Weil groups. Japan. J. Math. **22**, no. 2, 293–347 (1996)
561. Noohi, B.: Fundamental groups of algebraic stacks. J. Inst. Math. Jussieu **3**, no. 1, 69–103 (2004)
562. Nori, M.: On certain elliptic surfaces with maximal Picard number. Topology **24**, 175–186 (1985)
563. Nygaard, N.: A p-adic proof of the nonexistence of vector fields on K3 surfaces. Ann. of Math. (2) **110**, 515–528 (1979)
564. Nygaard, N.: Slopes of powers of Frobenius on crystalline cohomology. Ann. Sci. École Norm. Sup. (4) **14**, 369–401 (1981)
565. Nygaard, N.: The Tate conjecture for ordinary K3-surfaces over finite fields. Invent. Math. **74**, 213–237 (1983)
566. Nygaard, N., Ogus, A.: Tate's conjecture for K3 surfaces of finite height. Ann. of Math. (2) **122**, no. 3, 461–507 (1985)
567. Oda, T.: The first de Rham cohomology group and Diéudonné modules. Ann. Sci. Éc. Norm. Sup. , 4e ser., **2**, 63–135 (1969)
568. Oesterlé, J.: Dégénerescence de la suite spectrale de Hodge vers De Rham (d'aprés Deligne et Illusie). Séminaire Bourbaki, 1986/87, no. 673, Astérisque **152–153** (1987), 3, 67–83 (1988)
569. Ogg, A.: Cohomology of abelian varieties over function fields. Ann. Math. **76**, 185–212 (1962)
570. Ogg, A.: Elliptic curves and wild ramification. Amer. J. Math. **89**, 1–21 (1967)
571. O'Grady, K.: Irreducible symplectic 4-folds and Eisenbud-Popescu-Walter sextics. Duke Math. J. **134**, 99–137 (2006)
572. O'Grady, K.: Double covers of EPW-sextics. Michigan Math. J. **62**, 143–184 (2013)
573. O'Grady, K.: Periods of double EPW-sextics. Math. Z. **280**, 485–524 (2015)
574. Oguiso, K., Shioda, T.: The Mordell-Weil lattice of a rational elliptic surface. Comment. Math. Univ. St. Paul. **40**, 83–99 (1991)
575. Ogus, A.: Supersingular K3 crystals. Journées de Géométrie Algébrique de Rennes (Rennes, 1978), Vol. II, pp. 3–86, Astérisque, **64**, Soc. Math. France, Paris (1979)
576. Ogus, A.: A crystalline Torelli theorem for supersingular K3 surfaces. Arithmetic and geometry, Vol. II, 361–394, Progr. Math., **36**, Birkhäuser Boston, Boston, MA (1983)
577. Ohashi, H.: On the number of Enriques quotients of a K3 surface. Publ. Res. Inst. Math. Sci. **43**, 181–200 (2007)
578. Ohashi, H.: Enriques surfaces covered by Jacobian Kummer surfaces. Nagoya Math. J., **195**, 165–186 (2009)
579. Ohashi, H.: Bi-canonical representations of finite automorphisms acting on Enriques surfaces, arXiv:1504.00728 (2015)
580. Okonek, C.: Fake Enriques surfaces. Topology **27**, 415–427 (1988)
581. Olsson, M.: Algebraic spaces and stacks. American Mathematical Society Colloquium Publications, 62. American Mathematical Society, Providence, RI (2016)
582. Oort, F.: Sur le schéma de Picard. Bull. Soc. Math. France **90**, 1–14 (1962)
583. Oort, F.: Commutative group schemes. Lect. Notes in Math. **15**, Springer–Verlag (1966)
584. Oort, F., Tate, J.: Group schemes of prime order. Ann. Sci. École Norm. Sup. (4) **3**, 1–21 (1970)
585. Pan, I.: Les transformations de Cremona stellaires. Proc. Amer. Math. Soc. **129**, 1257–1262 (2001)
586. S. Papadakis, S., Reid, M.: Kustin-Miller unprojection without complexes. J. Algebraic Geom. **13**, 563–577 (2004)
587. Pappas, G.: Grothendieck-Riemann-Roch and the moduli of Enriques surfaces. Math. Res. Lett. **15**, 117–120 (2008)
588. Pardini, R.: Abelian covers of algebraic varieties. J. Reine Angew. Math. **417**, 191–213 (1991)
589. Persson, U.: On degenerations of algebraic surfaces. Mem. Amer. Math. Soc. **11**, no. 189 (1977)
590. Persson, U.: Configurations of Kodaira fibers on rational elliptic surfaces. Math. Z. **205**, 1–47 (1990)

591. Persson, U., Pinkham, H.: Degeneration of surfaces with trivial canonical bundle. Ann. of Math. (2) **113**, no. 1, 45–66 (1981)
592. Peskin, B.: Quotient-singularities and wild p-cyclic actions. J. Algebra **81**, 72–99 (1983)
593. Peters, C.: K3-surfaces and Enriques surfaces: highlights from two decades. Problems in the theory of surfaces and their classification (Cortona, 1988), 259–270, Sympos. Math., XXXII, Academic Press, London (1991)
594. Pho, D.T., Shimada, I.: Unirationality of certain supersingular K3 surfaces in characteristic 5. Manuscripta Math. **121**, 425–435 (2006)
595. Pinkham, H.: Singularit??s exceptionnelles, la dualit?? ètrange d'Arnold et les surfaces K???3. C. R. Acad. Sci. Paris Sér. A-B **284**, A615???-A618 (1977)
596. Pinkham, H.: Singularités de Klein. Singularités de surfaces. Lect. Notes in Math. **777**, pp. 1–20, Springer–Verlag (1979)
597. Pinkham, H.: Singularités rationnelles de surfaces. Singularités de surfaces. Lect. Notes in Math. **777**, pp. 147–178, Springer–Verlag (1979)
598. Pyateskii-Shapiro, I.: Automorphic functions and the geometry of classical domains. Translated from the Russian. Mathematics and Its Applications, Vol. 8, Gordon and Breach Science Publishers, New York-London-Paris (1969)
599. Pyatetskii-Shapiro, I., Shafarevich, I.: A Torelli theorem for algebraic surfaces of type K3. Izv. Akad. Nauk SSSR, ser. math., **35**, 503–572 (1971) [Collected Mathematical Papers, pp. 516–557. Springer–Verlag (1989)]
600. Queen, C.: Non-conservative function fields of genus one. I. Arch. Math. (Basel) **22**, 612–623 (1971)
601. Queen, C.: Non-conservative function fields of genus one. II. Arch. Math. (Basel) **23**, 30–37 (1972)
602. Ramanujam, C.: Remarks on Kodaira vanishing theorem. J. Indian Math. Soc., **36**, 41–51 (1972)
603. Raynaud, M.: Modeles de Néron. C.R. Acad. Sci. Paris, **262**, 413–414 (1966)
604. Raynaud, M.: Characteristique d'Euler-Poincaré d'un faisceau et cohomologie des varietés abeliennes. Séminaire Bourbaki, Vol. 9, Exp. No. 286, 129–147, Soc. Math. France, Paris (1995) [Dix Exposés sur cohomologie des schémas, North-Holland. Amsterdam, pp. 12–30 (1968)].
605. Raynaud, M.: Faisceaux amples sur les schémas en groupes et les espaces homogènes. Lecture Notes in Math. **119**, Springer–Verlag, Berlin-New York (1970)
606. Raynaud, M.: Spécialization du foncteur de Picard. Publ. Math. IHES, **38**, 27–76 (1970)
607. Raynaud, M.: Surfaces elliptique et quasi-elliptiques. unpublished notes (1976)
608. Raynaud, M.: Contre-exemple au "vanishing theorem" en characteristic $p > 0$. C. P. Ramanujam". Publ. Tata Insitute, pp. 273–278 (1978)
609. Recillas, S.: Jacobians of curves with g_1^4's are the Prym's of trigonal curves. Bol. de la Soc. Mat. Mexicana, **19**, 9–13 (1974)
610. Reid, M.: Bogomolov's theorem $c_1^2 \le 4c_2$. Proceedings of the International Symposium on Algebraic Geometry (Kyoto Univ., Kyoto, 1977), pp. 623–642, Kinokuniya Book Store, Tokyo (1978)
611. Reid, M.: Nonnormal del Pezzo surfaces. Publ. Res. Inst. Math. Sci. **30**, 695–727 (1994)
612. Reid, M.: Chapters on algebraic surfaces. Complex algebraic geometry IAS/Park City Math. Series, Vol. 3, Amer. Math. Soc., pp. 5–159. Providence, RI (1997)
613. Reider, I.: Vector bundles of rank 2 and linear systems on algebraic surfaces. Ann. Math. **127**, 309–316 (1988)
614. Reye, T.: Die Geometrie der Lage. 3 vols., Hannover, C. Rümpler (1877–1880)
615. Reye, T.: Über lineare Systeme und Gewebe von Flächen zweiten Grades. J. Reine und Angew. Math., **82**, 54–83 (1877)
616. Reznick, B.: Some concrete aspects of Hilbert's 17th Problem. Real algebraic geometry and ordered structures (Baton Rouge, LA, 1996), 251–272, Contemp. Math., **253**, Amer. Math. Soc., Providence, RI (2000)
617. Reznick, B.: On Hilbert's construction of positive polynomials. arXiv:0707.2156 (2007)

618. Richardson, R.: Conjugacy classes of involutions in Coxeter groups. Bull. Austral. Math. Soc. **26**, 1–15 (1982)
619. Riener, C.: On the degree and half-degree principle for symmetric polynomials. J. Pure Appl. Algebra **216**, no. 4, 850–856 (2012)
620. Rizov, J.: Moduli stacks of polarized K3 surfaces in mixed characteristic. Serdica Math. J. **32**, 131–178 (2006)
621. Roczen, M.: Recognition of simple singularities in positive characteristic. Math. Z. **210**, 641–653 (1992). An. Ştiinţ. Univ. Ovidius Constanţa Ser. Mat. **5**, no. 2, 99–104 (1997)
622. Romagny, M.: Group actions on stacks and applications. Michigan Math. J. **53**, 209–236 (2005)
623. Room, T.G.: Geometry of determinantal loci. Cambridge University Press, London/New York (1938)
624. Rosenberg, J.: Geometry of moduli space of cubic surfaces. Univ. Michigan, Ph.D. Thesis (1999)
625. Rosenlicht, M.: Toroidal algebraic groups. Proc. Amer. Math. Soc. **12**, 984–988 (1961)
626. Rudakov, A., Shafarevich, I.: Inseparable morphisms of algebraic surfaces. Izv. Akad. Nauk SSSR, Ser. Math., **40**, 1269–1307 (1976) [Igor R. Shafarevich, Collected Mathematical Papers, Springer–Verlag, pp. 577–609 (1989)]
627. Rudakov, A., Shafarevich, I.: Supersingular surfaces of type K3 in characteristic 2. Izv. Akad. Nauk SSSR, Ser. Math., **42**, 848–869 (1978) [Igor R. Shafarevich, Collected Mathematical Papers, Springer–Verlag, pp. 614–632 (1989)].
628. Rudakov, A., Shafarevich, I.: Surfaces of type K3 over fields of finite characteristic. Current Problems of Mathematics, t. 18, VINITI, Moscow, pp. 3–114 (1981) [Igor R. Shafarevich, Collected Mathematical Papers, Springer–Verlag, pp. 657–714 (1989)].
629. Russell, P.: Forms of the affine line and its additive group. Pacif. J. Math. **32**, 527–539 (1970)
630. Saint-Donat, B.: Projective models of K3-surfaces. Amer. J. Math.,**96**, 602–639 (1974)
631. Salmon, G.: On the degree of the surface reciprocal to a given one. Cambridge and Dublin Math. J. **2**, 65–73 (1847)
632. Salmon, G.: A treatise on the analytic geometry of three dimensions. Revised by R. A. P. Rogers. 7th ed. Vol. 1 Edited by C. H. Rowe. Chelsea Publ. Company, New York (1958)
633. Salomonsson, P.: Equations for some very special Enriques surfaces in characteristic two. math.AG.0309210 (2003)
634. Satake, I.:Algebraic structures of symmetric domains Satake. Iwanami Shoten, Tokyo; Princeton University Press, Princeton, N.J., 1980,
635. Satriano, M.: The Chevalley-Shephard-Todd theorem for finite linearly reductive group schemes. Algebra Number Theory **6**, 1–26 (2012)
636. Scattone, F.: On the compactification of moduli spaces for algebraic K3 surfaces. Mem. Amer. Math. Soc. **70**, no. 374 (1987)
637. Schlessinger, M.: Functors of Artin rings. Trans. Amer. Math. Soc. **130**, 208–222 (1968)
638. Schröer, S.: On genus change in algebraic curves over imperfect fields. Proc. Amer. Math. Soc. **137**, no. 4, 1239–1243 (2009)
639. Schröer, S.: On fibrations whose geometric fibers are nonreduced. Nagoya Math. J. **200**, 35–57 (2010)
640. Schröer, S.: Enriques surfaces with normal K3-like coverings. J. Math. Soc. Japan **73**, 433–496 (2021)
641. Schüller, F.: On taut singularities in arbitrary characteristics. Ph.D. thesis, Düsseldorf University (2012)
642. Schütt, M.: $\mathbf{Q}_l$-cohomology projective planes and Enriques surfaces in characteristic two. Épijournal Géom. Algébrique **3**, Art. 10, 24 pp. (2019)
643. Schütt, M.: Moduli of Gorenstein **Q**-homology projective planes. J. Math. Soc. Japan **75** , 329–366 (2023)
644. Schütt, M., Schweizer, A.: On the uniqueness of elliptic K3 surfaces with maximal singular fibre. Ann.Inst. Fourier **63**, 689–713 (2013)
645. Schütt, M., Shioda, T.: Elliptic surfaces. In: Algebraic geometry in East Asia–Seoul 2008, 51–160, Adv. Stud. Pure Math., **60**, Math. Soc. Japan, Tokyo (2010)

646. Schütt, M., Shioda, T.: Mordell-Weil lattices. Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics, 70. Springer, Singapore (2019)
647. Schweizer, A.: Extremal elliptic surfaces in characteristic 2 and 3. Manuscripta Math. **102**, 505–521 (2000)
648. Sebbar, A.: Classification of torsion-free genus zero congruence groups. Proc. Amer. Math. Soc. **129**, 2517–2527 (2001)
649. Segre, B.: On the quartic surface $x_1^4 + x_2^4 + x_3^4 + x_4^4 = 0$. Proc. Camb. Publ. Soc. **40**, 121–145 (1944)
650. Segre, C.: Surfaces du 4e ordre a conique double. Math. Ann. **24**, 313–344 (1884)
651. Sernesi, E.: Deformations of algebraic schemes. Grundlehren der Mathematischen Wissenschaften 334, Springer–Verlag, Berlin (2006)
652. Serre, J.P.: Géométrie algébrique et géométrie analytique. Ann. Inst. Fourier, Grenoble **6**, 1–42 (1955–1956)
653. Serre, J.P.: Sur la topologie des variétés algébriques en caractéristique p. 1958 International symposium on algebraic topology pp. 24–53, Universidad Nacional Autónoma de México and UNESCO, Mexico City.
654. Serre, J.P.: Morphismes universels et variété d'Albanese. Séminaire Claude Chevalley, tome 4, exp. no. 10, 1–22 (1958-1959)
655. Serre, J.P.: On the fundamental group of a unirational variety. J. London Math. Soc. **34**, 481–484 (1959)
656. Serre, J.P.: Groupes algébrique et corps de classes. Hermann. Paris (1959)
657. Serre, J.P.: Groupes proalgébriques. Publ. Math. IHES, **7** (1960)
658. Serre, J.P.: Sur les corps locaux é corps résiduel algébriquement clos. Bull. Soc. Math. France, **89**, 105–154 (1961)
659. Serre, J.P.: Cohomologie Galoisienne. Lect. Notes in Math. **5**, Springer–Verlag (1964)
660. Serre, J.P.: Corps locaux. Deuxième édition. Publications de l'Université de Nancago, No. VIII. Hermann, Paris (1968) [English Translation:Local fields. Graduate Texts in Mathematics, 67. Springer–Verlag, New York-Berlin (1979)].
661. Serre, J.P.: Cours de Arithmétique. Pres. Univ. de France, Paris (1970)
662. Serre, J.P.: Oeuvres, collected papers. vols. 1–4, Berlin, Springer–Verlag (1986)
663. Serre, J.P.: Le groupe de Cremona et ses sous-groupes finis. Séminaire Bourbaki. Volume 2008/2009. Astérisque No. 332, Exp. No. 1000, 75–100 (2010)
664. Shafarevich, I.: Birational equivalence of elliptical curves. (Russian) Dokl. Akad. Nauk SSSR (N.S.) **114**, 267–270 (1957) [Collected Mathematical Papers, Springer–Verlag, pp. 192–196 (1989)]
665. Shafarevich, I.: Principal homogeneous spaces over function fields. Proc. Steklov Inst. Math. **64**, 316–346 (1961) [Collected Mathematical Papers, Springer–Verlag, pp. 237–266 (1989)]
666. Shafarevich, I.: Lectures on minimal models and birational transformations of two-dimensional schemes. Tata Lect. Notes Math. Bombay (1966)
667. Shah, J.: Projective degenerations of Enriques surfaces. Math. Anal.**256**, 475–495 (1981)
668. Shepherd-Barron, N.: Unstable vector bundles and linear systems on surfaces in characteristic p. Invent. Math. **106**, 243–262 (1991)
669. Shepherd-Barron, N.: Geography for surfaces of general type in positive characteristic. Invent. Math. **106**, 263–274 (1991)
670. Shepherd-Barron, N.: Some foliations on surfaces in characteristic 2. J. Algebraic Geom. **5**, 521–535 (1996)
671. Shimada, I.: Automorphisms of supersingular K3 surfaces and Salem polynomials. Exp. Math. **25**, 389–398 (2016)
672. Shimada, I.: On an Enriques surface associated with a quartic Hessian surface. Canad. J. Math. **71**, 213–246 (2019) [Corrigendum:Canad. J. Math. **74**, no. 2, 603–605 (2022)]
673. Shimura, G.: Introduction to the arithmetic theory of automorphic functions. Reprint of the 1971 original. Publications of the Mathematical Society of Japan, 11. Kanô Memorial Lectures, 1. Princeton University Press, Princeton, NJ (1994)

674. Shioda, T.: On elliptic modular surfaces. J. Math. Soc. Japan, **24**, 20–59 (1972)
675. Shioda, T.: An example of unirational surfaces in characteristic p. Math. Ann. **211**, 233–236 (1974)
676. Shioda, T.: Some results on unirationality of algebraic surfaces. Math. Ann. **230**, 153–168 (1977)
677. Shioda, T.: Supersingular K3 surfaces. Algebraic geometry (Proc. Summer Meeting, Univ. Copenhagen, Copenhagen, 1978), pp. 564–591, Lecture Notes in Math., **732**, Springer, Berlin (1979)
678. Shioda, T.: Mordell-Weil lattices and Galois representation, I, II, III. Proc. Japan Acad. Ser. A Math. Sci. **65**, no. 7, 268–271; no. 8, 296-303 (1989)
679. Shioda, T.: On the Mordell-Weil lattices. Comment. Math. Univ. St. Paul. **39**, 211–240 (1990)
680. Shioda, T.: The abc-theorem, Davenport's inequality and elliptic surfaces. Proc. Japan Acad. Ser. A Math. Sci. **84**, 51–56 (2008)
681. Siegel, C.: Einheiten quadratischer Formen. Math. Sem. Hansischen Univ. **13**, 209–239 (1940) [Gesammelten Abhandlungen, B. II, Springer–Verlag, pp. 138–169 (1966)]
682. Silverman, J.: The arithmetic of elliptic curves. Second edition. Graduate Texts in Mathematics, 106. Springer, Dordrecht (2009)
683. Silverman, J.: Advanced topics in the arithmetic of elliptic curves. Graduate Texts in Mathematics, 151. Springer–Verlag, New York (1994)
684. Springer, T.: Poincaré series of binary polyhedral groups and McKay's correspondence. Math. Ann. **278**, 99–116 (1987)
685. Springer, T.: Some remarks on characters of binary polyhedral groups. J. Algebra **131**, 641–647 (1990)
686. Stagnaro, E.: Constructing Enriques surfaces from quintics in P^3_k. Algebraic Geometry-Open Problems, Lect. Notes in Math. **997**, pp. 400–403, Springer–Verlag, (1983)
687. Stanley, R.: Combinatorics and commutative algebra. Second edition. Progress in Mathematics, 41. Birkhäuser Boston, Inc., Boston, MA (1996)
688. Steinberg, R.: Finite reflection groups. Trans. Amer. Math. Soc. **91**, 493–504 (1959)
689. Sterk, H.: Finiteness results for algebraic K3 surfaces. Math. Z. **189**, 507–513 (1985)
690. Sterk, H.: Compactifications of the period space of Enriques surfaces. I. Math. Z. **207**, 1–36 (1991)
691. Sterk, H.: Compactifications of the period space of Enriques surfaces. II. Math. Z. **220**, 427–444 (1995)
692. Stöhr, K.O.: Hyperelliptic Gorenstein curves. J. Pure Appl. Algebra **135**, no. 1, 93–105 (1999)
693. Sturm, R.: Die Gebilde ersten und zweiten Grades der Liniengeometrie in synthetischer Behandlung. Thiel 1–3, Leipzig (1892)
694. Suzuki, M.: Group theory. I. Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], **247**; Springer–Verlag, Berlin-New York (1982)
695. Sylvester, J.: Elementary researches in the analysis of combinatorial aggregation. Phil. Mag. **24**, 285–296 (1844) [Collected Math. P: vol. 1, 91–102 (1904)]
696. Szemberg, T.: On positivity of line bundles on Enriques surfaces. Trans. Amer. Math. Soc. **353**, 496–4972 (2001)
697. Szydlo, M.:Elliptic fibers over non-perfect residue fields. J. Number Theory **104** (2004), 75–99 (2004).
698. Takeda, Y.: Artin-Schreier coverings of algebraic varieties. J. Math. Soc. Japan, **41**, 415–435 (1989)
699. Tate, J.: Genus change in purely inseparable extensions of function fields. Proc. Amer. Math. Soc. **3**, 400–406 (1952)
700. Tate, J.: Algebraic cycles and poles of zeta functions. Arithmetical Algebraic Geometry (Proc. Conf. Purdue Univ., 1963), Harper and Row, 93–110 (1965)
701. Tate, J.: On the conjecture of Birch and Swinnerton-Dyer and a geometric analog. Séminaire Bourbaki 1965/66, 306 [Dix exposés sur cohomologie des schémas, North-Holland. Amsterdam, pp. 189–214 (1968)]

702. Tate, J.: The arithmetic of elliptic curves. Invent. Math. **23**, 179–206 (1974)
703. Tate, J.: Algorithm for determining the type of a singular fiberin an elliptic pencil. Modular functions in one variable, IV, Lect. Notes in Math. **476**, Springer–Verlag, pp 33–52 (1975)
704. Tate, J.: Conjectures on algebraic cycles in ℓ-adic cohomology. Motives (Seattle, WA, 1991), Proc. Sympos. Pure Math. 55, Amer. Math. Soc., 71–83 (1994)
705. Timms, G.: The nodal cubic surfaces and the surfaces from which are derived by projections. Proc. Roy. Soc, Ser.A, **119**, 213–248 (1928)
706. Totaro, B.: Recent progress on the Tate conjecture. Bull. Amer. Math. Soc. (N.S.) 54, no. 4, 575–590 (2017)
707. Tyurin, A.: The intersection of quadrics. Uspekhi Mat. Nauk, **30** , no. 6 (186), 51–99 (1975)
708. Tyurina, G.: On a type of contractible curves. Doklady Akad. Nauk SSSR, **173**, 529–531 (1967) [English translation: Soviet Math. Doklady **8**, 441–443 (1967)]
709. Tyurina, G.: The rigidity of rationally contractible curves on a surface. Izv. Akad. Nauk SSSR Ser. Mat. **32**, 943–970 (1968)
710. Tziolas, N.: Quotients of schemes by α_p or μ_p actions in characteristic $p > 0$. Manuscripta Math. **152**, 247–279 (2017)
711. Umezu, Y.: Normal quintic surfaces which are birationally Enriques surfaces. Publ. Res. Inst. Math. Sci. **33**, 359–384 (1997)
712. Urabe, T.: On singularities on degenerate del Pezzo surfaces of degree 1,2. Singularities, Part 2, Proc. Symp. Pure Math., v.40, Part I, pp. 587–591 (1983)
713. Urzúa G.: The Coble-Mukai lattice from $\mathbb{Q}$-Gorenstein deformations. arXiv:2308.05625v1 [mathAG], 10 Aug 2023.
714. Verra, A.: On Enriques surfaces as a fourfold cover of P^2. Math. Anal. **266**, 241–250 (1983)
715. Verra, A.: The étale double covering of an Enriques surface. Rend. Sem. Mat. Univ. Politec. Torino **41**, 131–167 (1983)
716. Verra, A.: A short proof of the unirationality of $\mathcal{A}_5$. Nederl. Akad. Witensch. Indag. Math. **46**, 339–355 (1984)
717. Viehweg, E.: Vanishing theorems. J. Reine Angew. Math. **335**, 1–8 (1982)
718. Vinberg, E.: Discrete groups generated by reflections. Izv. Akad. Nauk SSSR, Ser. math. **51** (1971) [English translation: Math. USSR-Izvestija, **5**, 1083–1119 (1971)]
719. Vinberg, E.: Some arithmetical discrete groups in Lobachevskii spaces. Discrete subgroups of Lie groups, Oxford Univ. Press. pp. 323–348 (1973)
720. Vinberg, E.: Rationality of the field of invariants of a triangular group. (Russian. English summary) Vestnik Moskov. Univ. Ser. I Mat. Mekh. no. 2, 23–24 (1982) [English translation:Moscow Univ. Math. Bull. **37**, no. 2, 27–29 (1982)]
721. Vinberg, E.: The two most algebraic K3-surfaces. Math Ann. **26**, 1–21 (1983)
722. Vinberg, E.: Classification of 2-reflective hyperbolic lattices of rank 4. Tr. Mosk. Mat. Obs. **68**, 44–76 (2007); translation in Trans. Moscow Math. Soc., 39–66 (2007)
723. Vinberg, E., Kaplinskaja, I.M.: The groups $\mathrm{O}_{18,1}(\mathbf{Z})$, $\mathrm{O}_{19,1}(\mathbf{Z})$ (in Russian). Dokl. Akad. Nauk SSSR **238**, no. 6, 1273–1275 (1978)
724. Vinberg, E., Schvartsman, O.: Discrete groups of motions of spaces of constant curvature. Geometry, II, 139–248, Encyclopaedia Math. Sci., 29, Springer, Berlin (1993)
725. Voisin, C.: Théorie de Hodge et géométrie algébrique complexe. Cours Spécialisés 10, Société Mathématique de France (2002)
726. Vvedenskii, O.: Duality in elliptic curves over a local field, I, II. Izv. Akad. Nauk SSSR, Ser. Math., **28**, 1091–1112 (1964); **30**, 891–922 (1966)
727. Vvedenskii, O.: On the Galois cohomology of elliptic curves defined over a local field. Math. Sbornik, **83**, 474–484 (1970)
728. Vvedenskii, O.: Quasilocal "class fields" of elliptic curves. I. Izv. Akad. Nauk SSSR Ser. Mat. **40**, no. 5, 969–992, 1199 (1976)
729. Wagreich, P.: Elliptic singularities of surfaces. Amer. J. Math. **92**, 419–454 (1970)
730. Wall, C.T.C.: On the orthogonal groups of unimodular quadratic forms. Math. Ann. **147**, 328–338 (1962)
731. Wang, L.: On automorphisms and the cone conjecture for Enriques surfaces in odd characteristic. Math. Res. Lett. **28**, no. 4, 1263–1281 (2021)

732. Warner, F.W.: Foundations of Differentiable Manifolds and Lie Groups. GTM **94**, Springer (1983)
733. Waterhouse, W.C.: Introduction to Affine Group Schemes. GTM **66**, Springer (1979)
734. Waterhouse, C., Weisfeiler, B.: One-dimensional affine group schemes. J. Algebra **66**, 550–568 (1980)
735. Wedhorn, T.: De Rham cohomology of varieties over fields of positive characteristic. Higher-dimensional geometry over finite fields, 269–314, NATO Sci. Peace Secur. Ser. D Inf. Commun. Secur. **16**, IOS, Amsterdam (2008)
736. Weil, A.: On algebraic groups and homogeneous spaces. Amer. J. Math. **77**, 493–512 (1955)
737. Winger, R.: Self-Projective Rational Sextics. Amer. J. Math. **38**, 45–56 (1916)
738. Winger, R.: On the invariants of the ternary icosahedral group. Math. Ann. **93**, 210–216 (1925)
739. Zariski, O.: On the purity of the branch locus of algebraic functions. Proc. Nat. Acad. Sci. U.S.A. **44**, 791–796 (1958)
740. Zariski, O.: On Castelnuovo's criterion of rationality $p_a = P_2 = 0$ of an algebraic surface. Illinois J. Math. **2**, 303–315 (1958)
741. Zariski, O.: The theorem of Riemann-Roch for high multiples of an effective divisor on an algebraic surface. Ann. of Math. (2) **76**, 560–615 (1962)
742. Zariski, O.: Characterization of plane algebroid curves whose module of differentials has maximum torsion. Proc. Nat. Acad. Sci. U.S.A. **56**, 781–786 (1966)

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