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Enriques Surfaces II

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Preface

The book gives a contemporary account of the study of the class of projective algebraic surfaces known as Enriques surfaces. These surfaces were discovered more than 125 years ago in an attempt to extend the characterization of rational algebraic curves via the absence of regular (or holomorphic) differential 1-forms to the two-dimensional case.

The theory of differential forms on complex algebraic varieties of arbitrary dimension and their birational invariance was laid out in the works of Clebsch and Noether between 1870 and 1880. Further developments of these ideas and clarification of their geometric meaning were undertaken by the school of Italian algebraic geometers, who were probably the first to define one of the main goals of algebraic geometry, namely the classification of algebraic varieties up to birational equivalence. They also understood the significance of vector spaces of regular differential forms. One of the main achievements of their work was the classification of algebraic surfaces, mainly due to Castelnuovo and Enriques. Central results of this classification are achieved via the analysis of the canonical and pluri-canonical linear systems and the Albanese map. The main numerical invariants are q, p_g, and P_n, which are, by definition, the dimensions of the vector spaces of regular 1-forms, regular 2-forms, and regular n-pluri-canonical forms, respectively. A rational variety, that is, an algebraic variety birationally equivalent to projective space, has no nonzero regular forms, and the converse is true for algebraic curves. In 1894, Castelnuovo proved that the vanishing of q, p_g, and P_2 is sufficient for the rationality of an algebraic surface. In discussions with Enriques about whether the condition $P_2 = 0$ can be eliminated, each came up with an example that shows that it cannot be done. In the example of Enriques, one has $P_{2n} = 1$ and $P_{2n+1} = 0$ for all $n \geq 0$, and in the example of Castelnuovo, one has $P_n = [1 + \frac{n}{2}]$, that is, linear growth as n tends to infinity. Enriques mentions this example in a letter to Castelnuovo on July 22, 1894 [222, Letter 11], and he also mentions it in his 1896 paper [218, §39]. Castelnuovo's example is discussed in his 1896 paper [109]. In the later development of the classification of algebraic surfaces, these two examples occupy different places: Enriques' example is of Kodaira dimension 0 and shares this class with abelian surfaces, K3 surfaces, and hyperelliptic surfaces. On the other hand, Castelnuovo's example is a surface

of Kodaira dimension 1. The Enriques construction has a birational model that is a non-normal surface of degree 6 in $\mathbb{P}^3$ that passes through the edges of the coordinate tetrahedron with multiplicity 2. It was dubbed an Enriques sextic surface and the notion of an Enriques surface as a smooth projective surface with $q = 0$ and $P_2 = 1$ occurs in Artin's thesis from 1960 [18], in Shafarevich's seminar in 1961–1963 [5], as well as in Kodaira's 1963 paper [401, part 3, p. 719].

In 1906, Enriques proved that every (general) surface with invariants $p_g = q = 0$ and $P_2 = 1$ is birationally equivalent to an Enriques sextic. He also gave other birational models of his surfaces, for example, as double planes branched along a certain curve of degree 8, an Enriques octic. A special case of the double plane construction was known to Enriques already in 1896 [222, Letter 302].

Still over the complex numbers, a minimal, smooth, and projective surface satisfies $p_g = q = 0$, $P_2 = 1$ if and only if its fundamental group is of order two and its universal cover is isomorphic to a K3 surface, which is characterized by being a minimal, smooth, and projective projective surface with invariants $q = 0$ and $p_g = P_2 = 1$. Enriques already understood this and proved that the pre-image of his sextic surface under the double cover of $\mathbb{P}^3$ branched along the union of four coordinate planes is birationally equivalent to a K3 surface [220]. This result leads to the modern definition of an Enriques surface as the quotient of a K3 surface by a fixed-point-free involution. This point of view suggests that the theory of Enriques surface may be understood as a part of the theory of K3 surfaces, which is widely discussed and used in the modern literature, see, for example, [43], [321], or [417]. However, most usage of K3 surfaces in the study of Enriques surfaces consists of applying transcendental methods related to the theory of periods of K3 surfaces, which has little to do with the fascinating intrinsic geometry of Enriques surfaces.

The classification of algebraic surfaces was extended to algebraically closed fields of positive characteristic in the work of Bombieri and Mumford [539], [78] and [77]. In particular, they gave a characteristic-free definition of Enriques surfaces. It turns out that Enriques surfaces in characteristic two live in a completely different and beautiful world that has many features that have no analogs in characteristic $\neq 2$. For example, the canonical double cover still exists but is a torsor under one of the three finite group schemes $\boldsymbol{\mu}_2$, $\mathbb{Z}/2\mathbb{Z}$, $\boldsymbol{\alpha}_2$ of order 2. Accordingly, this divides Enriques surfaces in characteristic two into three different classes, which are called classical, $\boldsymbol{\mu}_2$-surfaces (or singular surfaces), and $\boldsymbol{\alpha}_2$-surfaces (or supersingular surfaces). In the case where the canonical cover is inseparable, it is never a smooth surface, and in some cases, it is a rational surface, so it is not even birationally equivalent to a K3 surface. There are many good modern expositions of the theory of algebraic surfaces, and, in particular, Enriques surfaces over the complex numbers (see, for example [43]). Our priority is to provide the first complete as possible treatment of Enriques surfaces over fields of arbitrary characteristic. The price that we have to pay for this goal is reflected in the size of our book and also in requiring many more technical tools that we use. We collect all these needed tools in Chapter 0 and, in fact, more than we need in the hope that this may serve as a helpful reference for the study of algebraic surfaces over fields of arbitrary characteristic.

The authors have to admit that the initial goal of providing a complete exposition of the theory of Enriques surfaces over fields of arbitrary characteristic turned out to be too ambitious. Among the important topics that had to be left out are vector bundles on Enriques surfaces, derived categories of coherent sheaves on Enriques surfaces, arithmetic properties such as the (non-)existence of rational points on Enriques surfaces over number fields, as well as the theory of special subvarieties of the moduli spaces of algebraic curves that represent curves lying on Enriques surfaces.

Each chapter ends with a bibliographical note, where we tried our best to give credit to the original research discussed in the chapter.

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Contents

Introduction to Volume 2

Chapter 6 'Nodal Enriques Surfaces' is an expanded revision of Chapter 3 from the first edition of Volume I [138]. In Section 6.1 we discuss canonical isotropic sequences and introduce the *non-degeneracy invariant* $\mathsf{nd}(S)$ of an Enriques surface. It is the maximal length of a sequence $(f_1, \ldots, f_k)$ of nef numerical classes of divisors with $f_i \cdot f_j = 1 - \delta_{ij}$. The main result of this section is Corollary 6.1.14, which asserts that we always have $\mathsf{nd}(S) \geq 3$ in characteristic $\neq 2$. We give a new conceptual proof of this result that differs from the case-by-case proof sketched by Cossec in [135]. The situation in characteristic 2 is much more complicated. There are extra-special surfaces of three different types discussed in Section 6.2 for which $\mathsf{nd}(S) \leq 2$. One expects that $\mathsf{nd}(S) \geq 3$ for all other surfaces. The first edition contains a proof of this result that consists of more than 30 pages of a case-by-case analysis of the possible dual graphs of sets of smooth rational curves arising from the assumption that $\mathsf{nd}(S) \leq 2$. A more conceptual proof of this result in all characteristics was recently provided by Martin, Mezzedimi and Veniani [492]. In Section 6.3, we discuss, among other things, results of Cossec and the first author [136] on the minimal degrees of smooth rational curves on polarized Enriques surfaces. In Section 6.4, we introduce different invariants of an Enriques surface that control the set of smooth rational curves. Among them are Nikulin root invariants, as well as nodal and Reye lattices. We also discuss an important class of special Enriques surfaces arising as the Hessian surfaces of cubic surfaces and compute their nodal invariants. Finally, in Section 6.5 we define the notion of a general nodal Enriques surface and give their different geometric characterizations.

Chapter 7 'Reye Congruences' discusses a classical construction of nodal Enriques surfaces as a smooth congruence of lines in the Grassmannian of lines in $\mathbb{P}^3$. The canonical cover of such surfaces is birationally isomorphic to the famous Cayley quartic symmetroid surfaces. Some of the modern expositions of this theory can be found in an article by Cossec [134] and the book of the first-named author [177]. The novel feature here is the analogous construction of Reye congruences and their relationship to Enriques surfaces in characteristic 2.

Chapter 8 'Automorphisms of Enriques Surfaces' plays the central role in Volume II. In Section 8.1, after recalling some general facts about automorphism group

schemes of algebraic varieties, we prove that the automorphism group of an Enriques surface is discrete, hence it is determined by its representation as subgroup of isometries of the lattice $\mathrm{Num}(S)$ of numerical divisor classes. This opens a way to relate these groups to discrete groups of isometries of hyperbolic space $\mathbb{H}^9$. In Section 8.2, we study finite groups of automorphisms of an Enriques surface that act trivially on $\mathrm{Num}(S)$ or $\mathrm{Pic}(S)$. The Enriques surfaces for which such a group can be non-trivial are very rare and have been completely classified over the field of complex numbers by Mukai and Namikawa [532], [528]. We supply the classification in all characteristics different from 2. The characteristic 2 case is subject of [191] by the first-named author and Martin. We give an exposition of the main results from this article. The group of automorphisms of a general Enriques surface over the complex numbers was determined in the early eighties by Barth and Peters [42] and, independently, by Nikulin [557]. In Section 8.3, we reprove their result by using purely geometric methods that do not make any assumptions on the characteristic of the ground field. Section 10 of Chapter 2 of the first edition of Volume I contains preparatory computer computations needed for this proof. The new proof eliminates this by using some nice and short lattice-theoretical arguments due to Allcock [7]. Section 8.4 extends the results of the previous section to the case of a general nodal Enriques surface. The structure of the group of automorphisms of such a surface over an algebraically closed field of arbitrary characteristic was announced by Cossec and the first-named author in a short note [137] and we give complete proof of these announced results. We replace some computer computations from [138] with a lattice-theoretical argument due to Allcock [7]. As an application, we discuss in Section 8.5 the automorphism group of a Cayley quartic symmetroid. Next, Sections 8.6 and 8.7 are devoted to expositions of results of Ito and Ohashi [339] on cyclic groups of automorphisms of complex Enriques surfaces. In Section 8.8, we also discuss the results of Mukai and Ohashi about automorphisms of Mathieu type of Enriques surfaces [533], [534] and [535]. Again, we use different methods not relying on the theory of periods of Enriques surfaces that allow us to extend these results to positive characteristic. The question of the existence of an Enriques surface whose group of automorphisms is finite was raised by Enriques. We refer to the history of this question to Bibliographical Notes in Chapter 8. In Section 8.9, we discuss a complete classification of Enriques surfaces with finite automorphism group over a field of complex numbers due to Nikulin [559] in terms of periods of their K3 covers and a purely geometric classification over fields of characteristic zero due to the second-named author [409]. In Section 8.10, we give a brief exposition of a recent classification of such surfaces over algebraically closed fields of characteristic $\neq 2$ due to Martin [488] and characteristic 2 due to Katsura, the second-named author and Martin [371].

Chapter 9 'Rational Coble Surfaces' is devoted to close cousins of Enriques surfaces, which are smooth rational surfaces that contain an isolated curve in their anti-bicanonical linear systems. They appear as nonsingular models of type II degenerations of Enriques surfaces and they share many common properties with Enriques surfaces. Over a field of characteristic different from 2 they appear as quotients of K3 surfaces by an involution that is not fixed-point-free, but whose fixed locus is

the disjoint union of smooth rational curves. They also arise as the blow-up of the projective plane at ≥ 10 double points that lie on a curve of degree 6. Such curves and their Cremona equivalence classes were intensively studied by Coble in the first half of the last century [123], [123]. In Section 9.1, we discuss the relationship between Coble surfaces, Enriques surfaces, and K3 surfaces. In Section 9.2, we introduce the Coble–Mukai lattice and prove that it is isomorphic to the Enriques lattice. In Sections 9.4 and 9.5, we discuss the work of Winger [737] and [738] on the classification of irreducible rational plane curves of degree 6 with non-trivial projective group of automorphisms. These give rise, to Coble surfaces that admit a finite group of automorphisms preserving a Fano polarization. In Section 9.6, following Coble's work [123], we find the automorphism group of a general Coble surface, which turns out to have the same structure as the automorphism group of a general Enriques surface. The group of automorphisms of a Coble surface, as well as of an Enriques surface, could also be a finite group. In the last section of this chapter, we classify Coble surfaces with finite automorphism group over fields of arbitrary characteristic referring for proofs to the work of Katsura and the second-named author [370], [419]. There are only three types of complex Coble surfaces with finite automorphism group discovered by Mukai. However, there are many more in positive characteristic. Some of them arise as either a reduction to positive characteristic of Enriques surfaces with finite automorphism group or as the limits in families of such surfaces in characteristic 0 or 2.

Chapter 10 'Enriques Surfaces and supersingular K3 Surfaces' deals with classical or α_2-Enriques surfaces in characteristic 2, whose canonical cover is birationally isomorphic to a K3 surface X. It turns out that X is always a supersingular K3 surface and conversely, a general such Enriques surface can be obtained in this way by considering its quotient by a rational vector field on X. We tried to be self-contained by giving an introduction to the theory of supersingular K3 surfaces over algebraically closed fields of arbitrary positive characteristic. In particular, we discuss the periods of such surfaces and Ogus's Global Torelli Theorem. Each supersingular K3 surface X comes with an Artin invariant σ_0 that is determined by the discriminant of the Néron–Severi lattice. The moduli space of supersingular K3 surfaces is of dimension $\sigma_0 - 1$ and the K3 surface with $\sigma_0 = 1$ is unique up to an isomorphism. In the last section, we discuss Enriques and Coble surfaces whose canonical cover is birationally isomorphic to a supersingular K3 surface of Artin invariant $\sigma_0 = 1$.

We did not discuss the dynamical properties of automorphisms of Enriques surfaces. A reader interested in this topic may consult [181] and [496], where one can also find some other references.

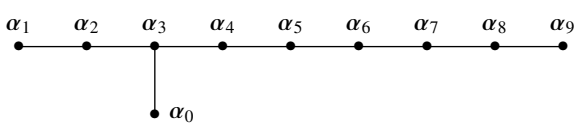
Chapter 6
Nodal Enriques Surfaces

A general Enriques surface does not contain smooth rational curves. In this chapter we will present some basic facts about Enriques surfaces that contain such curves. They are called nodal Enriques surfaces. In Volume I, it was shown that, over the complex numbers, the isomorphism classes of nodal Enriques surfaces form a codimension one subvariety of the moduli space of Enriques surfaces.

6.1 Canonical Isotropic Sequences

In this section, we will introduce one of the main tools for studying a nodal Enriques surface S, the notion of a canonical isotropic sequence of divisor classes realized as half-fibers of genus one fibrations on S. It will lead us to the definition of an important invariant $\mathsf{nd}(S)$, the non-degeneracy of S. We will show that in characteristic $p \neq 2$, the non-degeneracy invariant is greater than or equal to three. This implies that an Enriques surface always admits a degenerate $U_{[3]}$-marking.

Recall from Section 1.5 of Volume I that the Enriques lattice E_{10} has a root basis $(\alpha_0, \dots, \alpha_9)$ with the Dynkin diagram



(6.1.1)

It contains 10 isotropic vectors $\mathbf{f}_1, \dots, \mathbf{f}_{10}$ such that $\alpha_i = \mathbf{f}_i - \mathbf{f}_{i+1}, i \geq 1$, and $\alpha_0 = \mathbf{\Delta} - \mathbf{f}_1 - \mathbf{f}_2 - \mathbf{f}_3$ for some $\mathbf{\Delta} \in \mathsf{E}_{10}$. We have $\mathbf{f}_i \cdot \mathbf{f}_j = 1, i \neq j$, and

$$3\mathbf{\Delta} = \mathbf{f}_1 + \cdots + \mathbf{f}_{10}.$$

An ordered set $(f_1, \dots, f_k)$ of $k \geq 2$ isotropic vectors with $f_i \cdot f_j = 1, i \neq j$, will be called an *isotropic k-sequence*. If $k = 1$, we assume that f_1 is primitive. It follows

from above that an isotropic k-sequences exists for any $1 \le k \le 10$ and does not exist for $k > 10$. Consider the standard embedding of the lattice E_{10} in the standard hyperbolic lattice $\mathsf{I}^{1,10}$ as described in Section 1.5. Let $\mathbf{k}_{10} = 3\mathbf{e}_0 - \mathbf{e}_1 - \cdots - \mathbf{e}_{10}$ generate the orthogonal complement of E_{10} in $\mathsf{I}^{1,10}$. Let $(\mathbf{f}_1, \dots, \mathbf{f}_k)$ be an isotropic k-sequence. Then, the vectors $\mathbf{v}_i = \mathbf{f}_i - \mathbf{k}_{10}$ satisfy ${\mathbf{v}_i}^2 = -1, \mathbf{v}_i \cdot \mathbf{v}_j = 0, i \ne j$. A sequence $(v_1, \dots, v_k)$ of vectors in $\mathsf{I}^{1,10}$ with $v_i^2 = -v_i \cdot \mathbf{k}_{10} = -1, v_i \cdot v_j = 0$, is called an *exceptional k-sequence*.

We use the notation from Section 1.5.

Proposition 6.1.1 *The Weyl group $W(\mathsf{E}_{10})$ acts transitively on the set of isotropic k-sequences with $1 \le k \ne 9$ and has two orbits of isotropic 9-sequences represented by $(\mathbf{k}_{10} + \mathbf{e}_0 - \mathbf{e}_1 - \mathbf{e}_2, \mathbf{f}_3, \dots, \mathbf{f}_{10})$ and $(\mathbf{f}_2, \dots, \mathbf{f}_{10})$.*

Proof Suppose $k = 2$. An isotropic 2-sequence $(\mathbf{f}_1, \mathbf{f}_2)$ (resp. $(\mathbf{f}'_1, \mathbf{f}'_2)$) generates a unimodular hyperbolic sublattice U (resp. U') of E_{10}. Let $\mathsf{E}_{10} = U \oplus M = U' \oplus M'$. By Witt's theorem, there exists an element $\sigma \in \mathrm{O}(\mathsf{E}_{10})$ such that $\sigma(U) = U'$. This gives $(\sigma(\mathbf{f}_1), \sigma(\mathbf{f}_2)) = (\pm\mathbf{f}'_1, \pm\mathbf{f}'_2)$, or $(\pm\mathbf{f}'_2, \pm\mathbf{f}'_1)$. Composing σ with $-\mathrm{id}_{\mathsf{E}_{10}}$, we may assume that $\sigma \in W(\mathsf{E}_{10})$. Composing σ with an isometry of U, which acts identically on M, we may assume that $(\sigma(\mathbf{f}_1), \sigma(\mathbf{f}_2)) = (\mathbf{f}'_1, \mathbf{f}'_2)$. Thus, the assertion is true for $k = 2$. It also shows that $W(\mathsf{E}_{10})$ has one orbit on the set of exceptional 2-sequences. In particular, we may assume that the first two vectors in an exceptional k-sequence $(v_1, \dots, v_k)$ coincide with the vectors $\mathbf{e}_1, \mathbf{e}_2$ from the standard basis $(\mathbf{e}_0, \mathbf{e}_1, \dots, \mathbf{e}_{10})$ of $\mathsf{I}^{1,10}$. The orthogonal complement of the sublattice spanned by $(\mathbf{e}_1, \mathbf{e}_2)$ is the lattice $\mathsf{I}^{1,8}$. Since all roots in $\mathsf{I}^{1,8}$ belong to its sublattice $\mathsf{E}_8 = \mathbf{k}_8^\perp$, its Weyl group coincides with $W(\mathsf{E}_8)$ that embeds naturally in $W(\mathsf{E}_{10})$ by acting identically on the orthogonal complement. It is known that $W(\mathsf{E}_8)$ acts transitively on exceptional r-sequences with $r \ne 7$ and has two orbits on the set of exceptional 7-sequences (see [160, II:Proposition 4] or [486, Chapter 4, Corollary 4.8]). The two orbits are represented by the exceptional 7-sequences $(\mathbf{e}_0 - \mathbf{e}_1 - \mathbf{e}_2, \mathbf{e}_3, \dots, \mathbf{e}_7)$ and $(\mathbf{e}_2, \mathbf{e}_3, \dots, \mathbf{e}_7)$. This corresponds to the isotropic sequences $(\mathbf{f}, \mathbf{f}_3, \dots, \mathbf{f}_{10})$ and $(\mathbf{f}_2, \dots, \mathbf{f}_{10})$, where $\mathbf{f} = \mathbf{k}_{10} + \mathbf{e}_0 - \mathbf{e}_1 - \mathbf{e}_2$. This proves the assertion. □

Corollary 6.1.2 *If $k \ne 9$, any isotropic k-sequence $(g_1, \dots, g_k)$ in E_{10} can be extended to a canonical isotropic 10-sequence $(g_1, \dots, g_k, g_{k+1}, \dots, g_{10})$.*

Proof Consider an isotropic 10-sequence $(\mathbf{f}_1, \dots, \mathbf{f}_{10})$ and find $w \in W(\mathsf{E}_{10})$ such that $w(g_i) = \mathbf{f}_i, i = 1, \dots, k$. Then, set $g_{k+i} = w^{-1}(\mathbf{f}_{k+i}), i = 1, \dots, 10 - k$. □

Recall that we denoted by U an abstract quadratic lattice with a basis $(\mathbf{f}_1, \mathbf{f}_2)$ formed by two isotropic vectors with $\mathbf{f}_1 \cdot \mathbf{f}_2 = 1$. Its generalization is the lattice $\mathsf{U}_{[k]}$ with a basis formed by isotropic vectors $(\mathbf{f}_1, \dots, \mathbf{f}_k)$ such that $\mathbf{f}_i \cdot \mathbf{f}_j = 1, i \ne j$.

Proposition 6.1.3 *The lattice $\mathsf{U}_{[k]}$ is an even hyperbolic lattice with cyclic discriminant group of order $k - 1$. Every isotropic k-sequence contained in $\mathsf{U}_{[k]}$ is a basis. The vector $\mathfrak{s} = \mathbf{f}_1 + \cdots + \mathbf{f}_k$ is a unique vector in $\mathsf{U}_{[k]}$ such that $\mathfrak{s}^2 = k(k-1)$ and $|\mathfrak{s} \cdot \mathbf{f}| \ge k - 1$ for any isotropic vector $\mathbf{f} \in \mathsf{U}_{[k]}$.*

Proof Obviously, the Gram matrix of the basis $(\mathbf{f}_1, \dots, \mathbf{f}_k)$ has -1 and $k-1$ as its eigenvalues of multiplicities $k-1$ and 1, respectively. So, its discriminant is equal to $(-1)^k(k-1)$. This also shows that any isotropic k-sequences forms a basis of $\mathsf{U}_{[k]}$. The theory of elementary divisors implies that the discriminant group is cyclic. Writing any vector $v \in \mathsf{U}_{[k]}$ as a linear combination of the vectors $\mathbf{f}_1, \dots, \mathbf{f}_k$, and computing $v \cdot \mathbf{f}_i$ and $v \cdot v$, we obtain that v must coincide with $\mathfrak{s}$. □

We continue to use S to denote an Enriques surface and continue to denote the reflection group generated by reflections in classes of (-2)-curves on S by W_S^{nod}.

Definition 6.1.4 A *nef isotropic k-sequence* in $\mathrm{Num}(S)$ is an isotropic k-sequence that consists of nef vectors.

Proposition 6.1.5 *Let $(f_1, \dots, f_k)$ be an isotropic k-sequence in $\mathrm{Num}(S)$ of effective isotropic classes. There exists a unique $w \in W_S^{\mathrm{nod}}$ such that, after reindexing, the sequence $(f_1', \dots, f_k') := (w(f_1), \dots, w(f_k))$ contains $c \geq 1$ nef vectors forming an isotropic subsequence $f_{i_1}', f_{i_2}', \dots, f_{i_c}'$ with $1 = i_1 < i_2 < \dots < i_c \leq k$, such that, for any $i_s < i < i_{s+1}$,*

$$f_i' = f_{i_s}' + R_{i_s,1} + \dots + R_{i_s,i-i_s} \in W_S^{\mathrm{nod}} \cdot f_{i_s},$$

where $R_{i_s,1} + \dots + R_{i_s,i-i_s}$ is a nodal cycle of a rational double point of type A_{i-i_s}.

Proof Let $f = f_1 + \dots + f_k$. Let $w' \in W_S^{\mathrm{nod}}$ such that $h = w'(f) = f_1' + \dots + f_k'$ is nef. Since W_S^{nod} sends effective divisors with non-negative self-intersection to effective divisors, the classes f_i' are effective. Then, $h^2 = k(k-1)$ and $\Phi(h) \leq h_{10} \cdot f_i' = k-1$. Let $\Phi(h) = h_{10} \cdot f_0$ for some isotropic vector f_0, where we use the function Φ defined in Section 2.4. By Theorem 2.3.3, we can write $f_0 = g_0 + \sum R_i$, where g_0 is a nef isotropic class and R_i are (-2)-curves. Since h is nef, we get $h_{10} \cdot f_0 \geq h_{10} \cdot g_0$, hence we may assume that f_0 is nef. Since $f_0 \cdot f_i' \geq 0$, we obtain that f_0 is one of the f_i'. After reindexing, we may assume that there exists a sequence $1 = i_1 < \dots < i_c$ such that f_{i_s}' are nef, and $f_i', i_s < i < i_{s+1}$ belong to the W_S^{nod}-orbit of f_{i_s}'. If $c = 10$, there is nothing to prove. Assume $c < 10$, and $i_{s+1} - i_s > 1$. For any $i_s < i < i_{s+1}$, we can write

$$f_i' = w(f_{i_s}) = f_{i_s}' + \sum m_\alpha R_\alpha,$$

where R_α are different (-2)-curves and $m_\alpha > 0$. Intersecting with f_{i_s}', we find a unique R_α such that $f_{i_s}' \cdot R_\alpha = 1$ and $m_\alpha = 1$. The class $f_{i_s}' + R_\alpha$ is isotropic and

$$h_{10} \cdot (f_{i_s}' + R_\alpha) = h_{10} \cdot (f_{i_s+1}' - \sum_{\beta \neq \alpha} m_\beta R_\beta) \leq h_{10} \cdot f_{i_s+1}' \leq k-1.$$

Since $\Phi(h) = k-1$, the class $f_{i_s}' + R_\alpha$ must be one of the classes $f_i', i_s < i < i_{s+1}$. After reindexing, we may assume that $f_{i_s+1}' = f_{i_s}' + R_{i_s,1}$, where $R_\alpha = R_{i_s,1}$. Assume $i > i_s + 1$. Then,

$$1 = f_{i_s+1}' \cdot f_i' = (f_{i_s}' + R_{i_s,1}) \cdot (f_{i_s}' + R_{i_s+1} + \sum_{\beta \neq \alpha} m_\beta R_\beta) = R_{i_s,1} \cdot (\sum_{\beta \neq \alpha} m_\beta R_\beta).$$

This shows that there exists a unique β such that $R_{i_s,1} \cdot R_\beta = 1$ and $m_\beta = 1$. As mentioned above, we show that $f'_{i_s} + R_{i_s,1} + R_\beta$ is equal to one of $f'_i, i_s < i < i_{s+1}$. After reindexing, we may assume that $f_{i_s+2} = f_{i_s} + R_{i_s,1} + R_{i_s,2}$, where $R_{i_s,2} = R_\beta$. Continuing in this way, we show that, after reindexing $f'_i = f'_{i_s} + R_{i_s,1} + \cdots + R_{i_s,i-i_s-1}, i_s < i < i_{s+1}$, where $R_{i_s,1} + \cdots + R_{i_s,i-i_s}$ is a nodal cycle of type A_{i-i_s}. □

An isotropic k-sequence $(f_1, \ldots, f_k)$ which, after reindexing, is equal to the sequence $(f'_1, \ldots, f'_k)$ described in the previous lemma, is called *canonical*.

It also follows from the lemma that for any isotropic k-sequence there exists a unique $w \in W_S^{\mathrm{nod}}$ such that $(w(f_1), \ldots, w(f_k))$ is a canonical isotropic k-sequence.

The number c of nef members in a canonical isotropic k-sequence is called the *non-degeneracy* invariant of the sequence. We say that a canonical isotropic k-sequence is c-non-degenerate if its non-degeneracy invariant is equal to c. A canonical isotropic k-sequence with the non-degeneracy invariant equal to k is called *non-degenerate*.

Proposition 6.1.6 *A canonical isotropic sequence $(f_1, \ldots, f_{10})$ is non-degenerate if and only if $f = f_1 + \cdots + f_{10}$ is an ample numerical divisor class.*

Proof Assume that $(f_1, \ldots, f_{10})$ is non-degenerate. By definition, each f_i is nef. Then, f is nef and $f^2 = 90 > 0$. If f is not ample, then there exists some (-2)-curve R such that $f \cdot R = 0$. Then, $R \cdot f_i = 0$ for all i. Since $f_1, \ldots, f_{10}$ generate $\mathrm{Num}(S) \otimes \mathbb{Q}$, we get a contradiction. Conversely, suppose f is ample but $(f_1, \ldots, f_{10})$ is degenerate. Then, we can find some f_i that is equal to $f_{i-1} + R$, where f_{i-1} is nef and $R \cdot f_{i-1} = 1$. It follows from the definition of a canonical isotropic sequence that $(f_1 + \cdots + f_{10}) \cdot R = 0$, hence f is not ample. □

Proposition 6.1.7 *A canonical isotropic sequence $(f_1, \ldots, f_k)$ with $k \neq 9$ with degeneracy invariant c can be extended to a canonical isotropic 10-sequence with degeneracy invariant $c' \geq c$.*

Proof Applying Corollary 6.1.2, we can extend $(f_1, \ldots, f_k)$ to a maximal isotropic 10-sequence $(f_1, \ldots, f_k, f_{k+1}, \ldots, f_{10})$. Applying some $w \in W_S^{\mathrm{nod}}$, we obtain a canonical isotropic sequence $(f'_1, \ldots, f'_{10})$. Let $f'_1, \ldots, f'_{c'}$ be the nef vectors in this sequence. Then, each vectors f'_i belongs to the W_S^{nod}-orbits of one of these vectors. In particular, the nef vectors in $(f_1, \ldots, f_k)$ belong to the orbits of $f'_1, \ldots, f'_{c'}$. Since two different nef vectors cannot belong to the same orbit, we may assume that $f'_i = f_i, i = 1, \ldots, c$. The vectors $f_{c+1}, \ldots, f_k$ belong to the W_S^{nod}-orbits of $f_1, \ldots, f_c$. The vectors $f'_{c+1}, \ldots, f'_k$ belong to the W_S^{nod}-orbits of $f'_1, \ldots, f'_{c'}$. Since the orbits of different isotropic vectors are disjoint, we obtain that $f_{c+i} = f'_{c+i}, i = c+1, \ldots, k$. □

Definition 6.1.8 A primitive lattice embedding $j : \mathsf{U}_{[k]} \hookrightarrow \mathrm{Num}(S)$ is called a $\mathsf{U}_{[k]}$*-marking* of S. Two $\mathsf{U}_{[k]}$-markings j and j' are called equivalent if there exists an isometry σ of $\mathsf{U}_{[k]}$ and an element $w \in W_S^{\mathrm{nod}}$ such that $j' = w \circ j \circ \sigma$.

We say that a $\mathsf{U}_{[k]}$-marking is *canonical* (resp. *non-degenerate*) if the image $(f_1, \ldots, f_k)$ of its canonical basis $(\mathbf{f}_1, \ldots, \mathbf{f}_k)$ is a canonical (resp. non-degenerate)

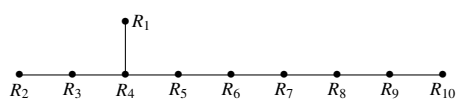
isotropic k-sequence. It follows from Proposition 6.1.5 that any $\mathsf{U}_{[k]}$-marking is equivalent to a canonical $\mathsf{U}_{[k]}$-marking.

When $k = 2$, the lattice $\mathsf{U}_{[2]}$ coincides with the standard hyperbolic plane lattice U. So, in this case we just say a U-marking. A canonical U-marking is called a *U-pair*. A non-degenerate U-pair is uniquely defined by an ordered pair of nef isotropic vectors f_1, f_2 with $f_1 \cdot f_2 = 1$. A canonical degenerate U-pair is uniquely defined by a choice of a nef isotropic vector f and the class r of a smooth rational curve such that $f \cdot r = 1$.

Definition 6.1.9 The *non-degeneracy invariant* of an Enriques surface is the maximal length $\mathsf{nd}(S)$ of a non-degenerate isotropic sequence.

Of course, if S has no (-2)-curves, then any isotropic k-sequence is non-degenerate and the non-degeneracy invariant is equal to 10.

In characteristic 2 it may happen that $\mathsf{nd}(S) = 1$ if S is *extra $\tilde{E}_8$-special* in the following sense. It contains a half-fiber F of type $\tilde{E}_8$ and a (-2)-curve intersecting F with multiplicity 1. The dual graph of the ten (-2)-curves is a $T_{2,3,7}$-diagram (6.1.1).



(6.1.2)

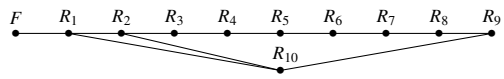
The classes of the curves R_i form a *crystallographic root basis* in $\mathrm{Num}(S)$. Recall that this means that the reflection subgroup G of W_S^{nod} generated by the reflections s_{R_i} is of finite index in $\mathrm{O}(\mathrm{Num}(S))$. It follows from Proposition 0.8.20 that G coincides with W_S^{nod}, hence the set of W_S^{nod}-orbits of isotropic vectors coincides with the set of primitive nef isotropic classes. It follows from the theory of reflection groups that the set of W_S^{nod}-orbits is equal to the number of parabolic subdiagrams of maximal rank. In our case, the rank is equal to 8. Looking at the diagram, we find that there is only one such subdiagram, hence there is only one genus one fibration on S.

Theorem 6.1.10 *Suppose $p \neq 2$ or $p = 2$ and S is not extra-$\tilde{E}_8$-special. Then, any isotropic nef class f can be extended to a non-degenerate canonical 2-sequence (f_1, f_2). In particular, the non-degeneracy invariant $\mathsf{nd}(S)$ of a not extra-$\tilde{E}_8$-special Enriques surface is greater than or equal to 2.*

Proof It is enough to show that one can find a canonical 10-sequence with the non-degeneracy invariant ≥ 2. Suppose it does not exist. Starting from any isotropic class $f = f_1$, we extend it to an isotropic 10-sequence and then apply an element of W_S^{nod} to transform the latter to a canonical 10-sequence $(f_1, f_2, \ldots, f_{10})$ with the non-degeneracy invariant c equal to 1. By Proposition 6.1.5, we may assume that $f_i = f_1 + R_1 + \cdots + R_{i-1}$, where $R_1 + \cdots + R_9$ is a nodal cycle of type A_9 with $(R_1 + \cdots + R_9) \cdot f_1 = R_1 \cdot f_1 = 1$. Let $f_1 = [F]$ for some genus one curve F and $|2F|$ be the corresponding genus one pencil. Since $R_i \cdot f_1 = 0$, the nodal cycle $R_2 + \cdots + R_9$ is contained in some member D of $|2F|$ and $D \cdot R_1 = 1$ or $D \cdot R_1 = 2$. The classification of genus one curves on S shows that D is of type $\tilde{A}_8$ or $\tilde{E}_8$.

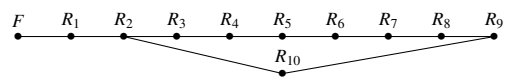
Case 1: D is of type $\tilde{A}_8$ and $D = D_{\mathrm{red}}$.

Let $D = R_2 + \cdots + R_9 + R_{10}$. Since $R_1 \cdot D = 2$ and R_1 intersects $R_2 + \cdots + R_9$ with multiplicity 1, we must have $R_1 \cdot R_{10} = 1$. This is pictured on the following diagram:



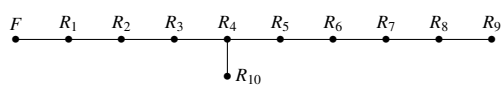
Consider the divisor $F' = R_1 + R_2 + R_{10}$. It is a genus one curve of type $\tilde{A}_2$ intersecting F with multiplicity 1. The pair $[F], [F']$ is a non-degenerate isotropic 2-sequence.

Case 2: D is of type $\tilde{A}_8$ and $D \in |2F|$. We have the following picture:



We see that the divisor $D = R_6 + 2R_7 + 3R_8 + 4R_9 + 5R_{10} + 6R_2 + 3R_1 + 4R_3 + 2R_4$ is of type $\tilde{E}_8$ and since $D \cdot R_5 = 3$, it must be a half-fiber. therefore, $p = 2$ and the surface is not a $\boldsymbol{\mu}_2$-surface. On the other hand, we see that the divisor $R_2 + \cdots + R_{10}$ is of type $\tilde{A}_8$ and $R_1 \cdot D = 1$. Thus, the surface admits an elliptic fibration with a double fiber of type $\tilde{A}_8$, and hence it is a $\boldsymbol{\mu}_2$-surface. This contradiction excludes this case.

Case 3: D is of type $\tilde{E}_8$. One of the two possible cases is when S is extra $\tilde{E}_8$-special. In other case, we have the following picture:



Since R_{10} comes with multiplicity 3, the curves R_1 and R_{10} do not intersect.

We see that $D' = R_1 + D - R_8 - R_9$ is the support of a divisor of type $\tilde{E}_7$ such that $F \cdot D' = 1$. Thus, (F, D') is a non-degenerate U-pair. □

Corollary 6.1.11 *Suppose S is not $\tilde{E}_8$-special. Then, there exists a degree 2 cover $f : S \to \mathsf{D}$, where D is a symmetroid quartic del Pezzo surface $\mathsf{D}_1, \mathsf{D}_2$ or D_3. In particular, if $p \neq 2$, D is a 4-nodal anti-canonical quartic del Pezzo surface.*

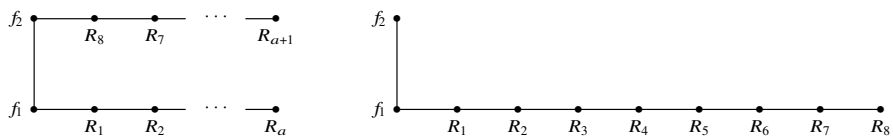
Proof Let $|2F|$ and $|2F'|$ be two genus one fibrations such that $([F], [F'])$ is a non-degenerate U-pair. Then, the linear system $|2F + 2F'|$ is a non-special bielliptic linear system and the assertion follows from Theorem 6.1.10. □

Theorem 6.1.12 *Assume that $p \neq 2$. Then, any non-degenerate canonical isotropic 2-sequence (f_1, f_2) can be extended to a non-degenerate canonical isotropic 3-sequence (f_1, f_2, f_3).*

Proof Suppose we cannot extend (f_1, f_2) to a non-degenerate canonical 3-sequence (f_1, f_2, f_3). We can always extend (f_1, f_2) to a canonical isotropic sequence of two types:

(a) $f_1, f_1 + R_1, \ldots, f_1 + R_1 + \cdots + R_a, f_2, f_2 + R_8, \ldots, f_2 + R_8 + \cdots + R_{a+1}$, or
(b) $f_1,\ f_2,\ f_1 + R_1, \ldots, f_1 + R_1 + \cdots + R_8$.

In case (a), each (-2)-curve from the sets $Z_1 = \{R_2, \ldots, R_a\}$ and $Z_2 = \{R_{a+1}, \ldots, R_7\}$ has zero intersection with F_1 and F_2. In case (b), the same is true for curves in the set $Z = \{R_2, \ldots, R_8\}$. In case (a), we let $Z = Z_1 + Z_2$.



(a) (b)

Let $|2F_1|$ and $|2F_2|$ be the elliptic pencils with $f_1 = [F_1]$ and $f_2 = [F_2]$. Then, each curve $R \in Z$ is contained in some fiber D_1 of the first pencil and some fiber D_2 of the second pencil. Let $\phi : S \to \mathsf{D}_1$ be a bielliptic map defined by the linear system $|2F_1 + 2F_2|$. Since $R \cdot F_1 = R \cdot F_2 = 0$, the map ϕ blows down R to a singular point P of the branch curve $W \in |O_{\mathsf{D}_1}(2)|$ of ϕ.

Assume that Z contains a curve R not contained in a half-fiber of $|2F_1|$ or $|2F_2|$.

This guarantees that the point $P = \phi(R)$ does not lie on any line in D_1.

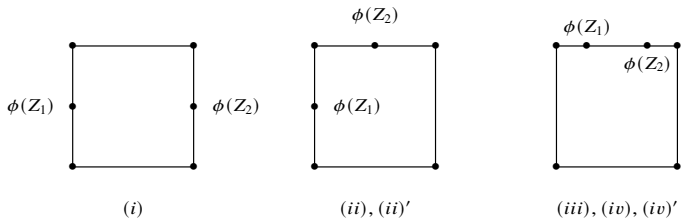
Recall from Section 3.3 that $|D| = |2F_1 + 2F_2 - R|$ is a bielliptic linear system with $D^2 = 6$ that defines a degree 2 map $\phi' : S \to \mathsf{C}_1$ to a 4-nodal cubic surface C_1. It is known that a 4-nodal cubic surface is isomorphic to the blow-up of the vertices of a complete quadrilateral of lines in the plane (see Example 8.7.17 in Section 8.7). It contains nine lines, the images of the six exceptional curves of the blow-up and the images of the proper transforms of the three diagonals. Each line defines a pencil of residual conics in the pencil of planes passing through the line. There are three distinguished pencils of conics, each containing two double lines. These are the pencils of conics residual to the proper transforms of the three diagonals. The genus one pencils $|2F_1|$ and $|2F_2|$ are the pre-images on S of two of such pencils. The pre-image of the third one gives a genus one pencil $|2F_3|$ with $F_1 \cdot F_3 = F_2 \cdot F_3$. The pencil $|2F_3|$ is the pre-image on S of the pencil of hyperplane sections of D_1 that are tangent to D_1 at the point P.

Assume now that any curve from Z is contained in a half-fiber of $|2F_1|$ or $|2F_2|$.

Since the lines may intersect only at singular points of D_1 and W does not contain these points, no half-fiber of $|2F_1|$ contains a common irreducible component with some half-fiber of $|2F_2|$.

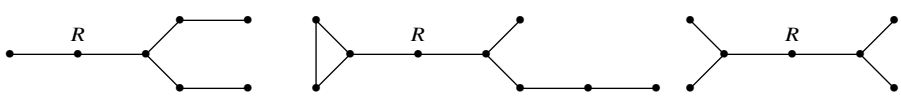
We have to consider the following possible cases:

(i) Z_1 is contained in F_1 and Z_2 contained in $F_1' \in |F_1 + K_S|$;
(ii) Z_1 is contained in F_1 and Z_2 is contained in F_2;
(ii)' Z_1 is contained in F_2 and Z_2 is contained in F_1;
(iii) Z is not connected and contained in F_1 or F_2;
(iv) Z is connected and contained in F_1;
(iv)' Z is connected and contained in F_2. □



We will use the following lemma whose proof we leave to the reader.

Lemma 6.1.13 *Suppose we have one of the following diagrams of* (-2)*-curves:*



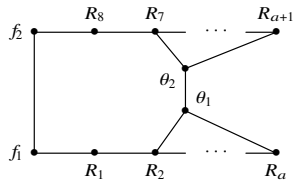
In case (1), (resp. (2), resp. (3)) there exists a divisor D of type $\tilde{E}_6$ (resp. $\tilde{E}_7$, resp. $\tilde{D}_6$) with irreducible components from the curves in the diagram that contains R with multiplicity 2 *in cases (1) and (3), and contains R with multiplicity* 3 *in cases (1) and (3), and multiplicity* 3 *in case (2).* □

Case (i). Since R_8 intersects Z_2 but does not intersect F_2, it must be contained in F_1'. Then, F_1 is of type $\tilde{A}_{k_1}$ with $k_1 \geq a - 1$ and F_1' is of type $\tilde{A}_{k_2}$ with $k_2 \geq 8 - a$. The curve R_1 is a special bisection of $|2F_1|$. We are in a situation of Lemma 6.1.13 with $R = R_1$. If $a = 2$, we get case (1), if $a = 3, 6$, we get case (2), and if $a = 4, 5$, we get case (3). The divisor D is a simple fiber of an elliptic fibration $|2F_3|$ with $F_1 \cdot F_3 = F_2 \cdot F_3 = 1$.

Case (ii). Here, we may also assume that $a \leq 4$. Let $F_2 = Z_2 + \Theta_2$, where Θ_2 is a chain of (-2) curves connecting R_7 with R_{a+1}. Similarly, let $F_1 = Z_1 + \Theta_1$. The half-fibers intersect at one of the points q_i that lies on an irreducible component θ_1 of Θ_1 and on an irreducible component θ_2 of Θ_2.

The following picture illustrates this case when $\Theta_1 = \theta_1$ and $\Theta_2 = \theta_2$:

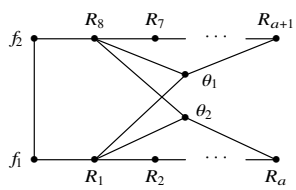
Assume that $a = 2, 3$, hence Z_2 contains $R_7, R_6, \ldots, R_{a+1} \neq R_5$. Choose a chain $\Theta_2' \subset \Theta_2$ that connects θ_2 with R_7 and let R be an irreducible component



of Θ_2 that intersects θ_2. It could be R_{a+1} or some component from Θ_2. Let $D = R_8 + 2R_7 + R_6 + 2\Theta' + R + \theta_1$. Then, D is a simple fiber of type $\tilde{D}_n$ in an elliptic pencil. It satisfies $D \cdot F_1 = D \cdot F_2 = 2$ that achieves our goal.

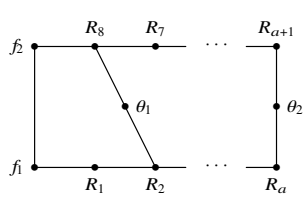
If $a = 4$, we find a divisor D of type $\tilde{D}_5$ that contains $2\theta_1 + 2\theta_2$. It satisfies $D \cdot F_1 = D \cdot F_2 = 2$.

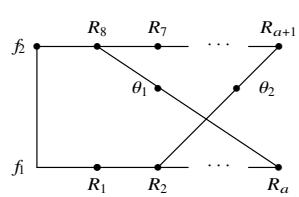
Case (ii'). In this case F_1 contains $R_8 + Z_2$ and F_2 contains $R_1 + Z_1$. The two chains are disjoint and the total number of (-2)-curves is equal to 8. Since all irreducible components, except one, of F_1 and F_2 are blown down under ϕ, $F_1 = R_8 + Z_2 + \theta_1$ and $F_2 = R_1 + Z_1 + \theta_2$ for some (-2)-curves θ_1 and θ_2. Since R_1 must intersect F_1, it intersects θ_1. Similarly R_8 intersects θ_2. Let $F_3 = R_8 + \theta_1 + R_1 + \theta_2$. It is a half-fiber of type $\tilde{A}_3$ of the elliptic fibration $|2F_3|$ satisfying $F_1 \cdot F_3 = F_2 \cdot F_3 = 1$.



If $a \neq 4$, we let $D = R_1 + 2\theta_2 + 3\theta_1 + 2R_8 + R_7 + 2R_{a+1} + R_{a+2}$ be a simple fiber of type $\tilde{E}_6$ of some elliptic fibration. Then, $f_1 \cdot D = 1$, and we get a contradiction. If $a = 4$, we take $D = R_8 + R_4 + 2\theta_1 + 2\theta_2 + R_7 + R_8$ of type $\tilde{D}_6$ with $D \cdot f_1 = D \cdot f_2 = 1$, and get a contradiction again.

Case (iii). Without loss of generality, we may assume that $Z = Z_1 + Z_2$ is contained in F_1. Then, R_8 is also contained in F_1 and hence F_1 has already two disconnected chain of seven irreducible components. This implies that F_1 is of type $\tilde{A}_8$ and there are two new components θ_1 and θ_2, one connects R_2 with R_8 or R_{a+1}, another connects R_a with R_{a+1} or R_8. This gives the following possible diagrams:



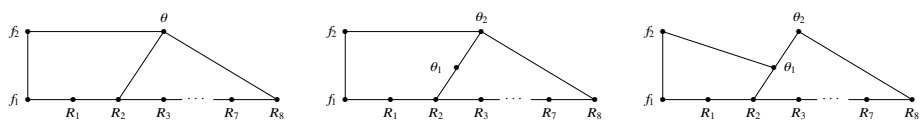


Here. we see a half-fiber F of type $\tilde{A}_8$ and its special bisection R_1. This is impossible for any Enriques surface in characteristic $p \neq 2$. In fact, in this case, there would be a divisor of type $\tilde{E}_8$ with irreducible components R_1 and other irreducible components from F that intersects F with multiplicity 3.

Case (iv). The fiber F_1 must be the union of Z and either one additional component connecting R_2 with R_8 or two intersecting components $\theta_1 + \theta_2$ with θ_1 intersecting R_2 and θ_2 intersecting R_8. The half-fiber F_2 intersects θ_1 or θ_2.

In the first case, $D = R_7 + 2R_8 + 3\theta + 4R_2 + 3R_3 + 2R_4 + R_5 + 2R_1$ is of type $\tilde{E}_7$ and intersects F_2 with multiplicity 3, a contradiction.

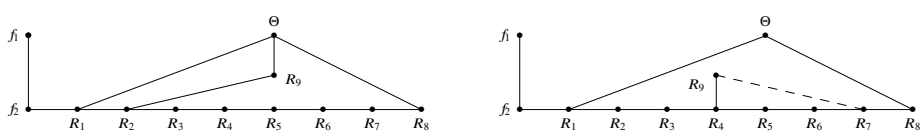
In the second case, the elliptic fibration $|2F_2|$ contains a fiber with nine components $R_1, \dots, R_8, \theta_1$ or $R_1, \dots, R_8, \theta_2$. They span a root lattice of type D_9 or A_9, no such fiber is possible.



Case (iv'). In this case, $R_1 + Z$ is a part of F_2. We have $F_2 = R_1 + Z + \Theta$, where Θ connects R_1 with R_8. It is of type $\tilde{A}_8$. Since $F_1 \cdot F_2 = 1$ and F_1 intersects R_1, we have $F_1 \cdot \Theta = 0$. Thus, $R_2 + \cdots + R_8 + \Theta$ is a part of a simple fiber D from $|2F_1|$. It can be extended to a fiber F of type $\tilde{A}_8$ or $\tilde{E}_8$ by adding one more curve R_9.

In the first case, we use that the Mordell–Weil group of the jacobian fibration of $|2F_1|$ is isomorphic to $\mathbb{Z}/3\mathbb{Z}$ and it acts on S by translations. The orbit of the rational bisection R_1 consists of three rational bisections. One of them R_0 intersects the irreducible components R_6 and R_8. The divisor $D = R_1 + R_9 + R_0 + R_7 + 2(\theta + R_8)$ is of type D_5 and intersects F_1 with multiplicity 3, a contradiction.

In the second case, we find that $R_1 \cdot F = 3$, a contradiction.



Corollary 6.1.14 *Let* $\mathsf{nd}(S)$ *be the non-degeneracy invariant of an Enriques surface over a field of characteristic* $p \neq 2$*. Then,*

$$\mathsf{nd}(S) \geq 3.$$

Remark 6.1.15 It is claimed in [138, Theorem 3.5.1] that the assertion of the theorem remains true in characteristic 2, except for explicitly classified extra-special Enriques surfaces with $\mathsf{nd}(S) \leq 2$ (see the next section for their classification). The proof is

similar to the proof of Theorem 6.1.10, but there are many more cases to consider, and the proof occupies 32 pages of the book. There may be some cases that the authors have missed, fortunately, a more conceptual and shorter proof has been recently given in [492].

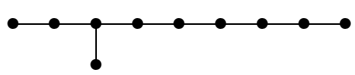
We refer to Sections 8.8 and 8.10, where we compute $\mathsf{nd}(S)$ for Enriques surfaces with finite automorphism group.

Remark 6.1.16 Suppose we have a non-degenerate isotropic 3-sequence that defines three genus one pencils $|2F_i|, i = 1, 2, 3$. We have $(2F_1 + 2F_2 - F_3)^2 = 0$, hence the pencil $|2F_1 + 2F_2|$ contains irreducible divisor $F_1 + F_1'$. Suppose F_1' is nef, i.e. a genus one curve, and has no common irreducible components with F_1. In this case $F_1 + F_1'$ is the pre-image of a hyperplane section of D_1 that splits under the cover $\phi : S \to \mathsf{D}_1$.

6.2 Extra-Special and Exceptional Enriques Surfaces

In this section, we will study Enriques surface with small non-degeneracy invariant $\mathsf{nd}(S)$ and compare them with special surfaces that admit a non-zero regular vector field.

We have already encountered a surface with $\mathsf{nd}(S) = 1$. It contains (-2)-curves forming the following intersection graph of type $T_{2,3,7}$:



It follows from the definition of a crystallographic root basis that the classes of the curves corresponding to the vertices of this diagram form a crystallographic root basis in $\mathrm{Num}(S)$.

Definition 6.2.1 An Enriques surface is called *nodal reflective* surface if it contains a crystallographic root basis formed by the divisor classes of (-2)-curves.

We will later give a classification of such surfaces in §8.9, 8.10. It coincides with the classification of Enriques surfaces with finite automorphism group. It follows from this classification that in the case when $\mathrm{char}(\Bbbk) \neq 2$, the cardinality c of such a root basis is larger than or equal to 12. However, in characteristic 2 we have several possibilities with $c < 12$ given in the following theorem.

First let us prove the following lemma, which has multiple uses.

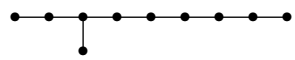
Lemma 6.2.2 *Let F_1, F_2 form a non-degenerate U-pair. Then, F_1 and F_2 have no common irreducible components.*

Proof By Proposition 2.5.2, a fiber F_1 is numerically 2-connected, i.e. if we write F_1 as a sum of two proper effective divisors $F_1 = D_1 + D_2$, then $D_1 \cdot D_2 \geq 2$. Now,

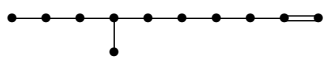
if D_1 is the maximal effective divisor with $D_1 \leq F_1$ and $D_1 \leq F_2$ and if we let $F_1 = D_1 + D_2$ and $F_2 = D_1 + D_2'$ be decompositions into effective divisors, we have $D_2.D_2' \geq 0$. therefore, $1 = F_1 \cdot F_2 = (D_1 + D_2) \cdot F_2 \geq D_2 \cdot D_1 + D_2 \cdot D_2' \geq D_2 \cdot D_1$, hence $D_1 = 0$. □

Theorem 6.2.3 *Let* B *be a crystallographic root basis in* $\mathrm{Num}(S)$ *formed by k classes of* (-2)*-curves such that, for any* $\alpha, \beta \in \mathsf{B}$, $\alpha \cdot \beta \leq 2$. *Assume that* $k \leq 11$. *Then, the intersection graph of the curves is one of the following:*

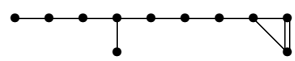
$\tilde{E}_8$



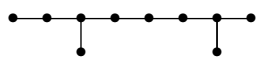
$\tilde{E}_7^{(1)}$



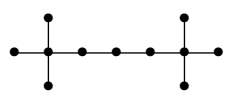
$\tilde{E}_7^{(2)}$



$\tilde{D}_8$



$\tilde{D}_4 + \tilde{D}_4$



Every such crystallographic basis of (-2)*-curves is realized in characteristic* 2.

Proof We will show later in Section 8.10 that such crystallographic bases are realized in characteristic 2. Now, let us show that the five diagrams are the only ones that can be realized under the assumption of the theorem.

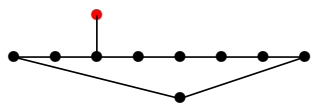
Let Γ be the Coxeter diagram of the reflection group W_B of the crystallographic basis formed by the curves R_i. Since the fundamental polytope $P(\mathsf{B})$ is of finite volume and is not compact (since $W(\mathsf{E}_{10})$ is not cocompact), Γ contains a maximal rank parabolic subdiagram P of rank 8. Let $P_1, \ldots, P_m$ be its connected components, and let n_i be the number of vertices in P_i. Then,

$$8 = \sum_{i=1}^{m} (n_i - 1) = -m + \sum_{i=1}^{m} n_i \leq k - 1 - m$$

gives $m \leq k - 9$.

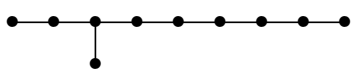
Assume $k = 10$. Then, $m = 1$. A connected parabolic diagram with 9 vertices must be of type $\tilde{A}_8$, $\tilde{E}_8$, or $\tilde{D}_8$. We must have an additional vertex v_{10}. If it intersects

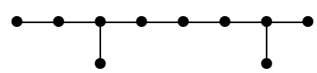
two vectors $v_i, i \neq 10$, then we find a parabolic subdiagram of type $\tilde{A}_k$ with $k < 8$. It is easy to see that it cannot be extended to a parabolic subdiagram of rank 8. Thus, v_{10} intersects only one other vector. By assumption, $\alpha \cdot \beta \leq 2$, hence the edge between (v_i, v_{10}) is simple or double. In the latter case, it defines a parabolic subdiagram that is not contained in a unique maximal rank parabolic subdiagram P. Thus, the edge (v_i, v_{10}) is simple. In the case $\tilde{A}_8$, we get the following diagram:



Here, the red vertex corresponds to v_{10}. It is clear that the graph contains a parabolic subdiagram of type $\tilde{E}_7$. It is not of maximal rank and it is not contained in a maximal rank parabolic subdiagram. This contradicts the assumption $k = 10$.

A similar argument shows that in the case $\tilde{E}_8$ and $\tilde{D}_8$, we get the following diagrams:





Let us assume that $k = 11$. We will apply Vinberg's Theorem 0.8.23. The assumption on B is equivalent to the property that the fundamental polytope $P(\mathsf{B})$ has no divergent faces.

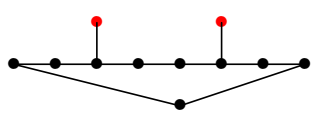
Suppose there is a non-connected maximal rank parabolic subdiagram P. Then, there exists a unique vertex $v \notin P$. Suppose we have another non-connected maximal rank parabolic subdiagram P'. One of its connected components must contain v. Another one must be contained in P, a contradiction. We conclude that any other maximal rank parabolic subdiagram must be connected. In particular, v is connected to only one vertex of each connected component of P unless the component is of type $\tilde{A}_1$ (or $\tilde{A}_1^*$, we will always assume this alternative). To sum up, there are two possibilities: either all maximal rank parabolic subdiagrams are connected, or there exists a unique non-connected maximal rank parabolic subdiagram.

Case 1: All maximal rank parabolic subdiagrams are connected.

In particular, the diagram has no multiple edges and any $v \notin P$ is connected to at most one other vertex.

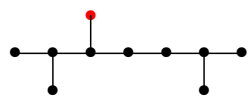
Case 1a: Γ contains a maximal rank parabolic subdiagram P of type $\tilde{A}_8$.

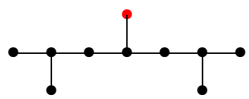
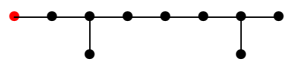
The diagrams are of the following pattern:



Let r be the smallest number of vertices between the vertices connected to the red vertices. Then, if $r = 0, 1, 2, 3$, we find a parabolic subdiagram $\tilde{D}_5$, $\tilde{D}_6$, $\tilde{D}_7$, $\tilde{E}_7$, respectively. This obviously contradicts the assumption that all maximal rank parabolic subdiagrams are connected.

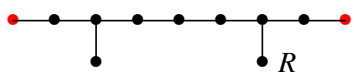
Case 1b: Γ contains a maximal rank parabolic subdiagram P of type $\tilde{D}_8$.
Adding only one new vertex, we get the following possible diagrams:

(a)  (b) 

(c)  (d) 

In cases (a), (b), and (c) we have a parabolic subdiagram of type $\tilde{D}_4, \tilde{D}_5, \tilde{D}_6$, For the same reason as in the previous case, it cannot be contained in a connected parabolic subdiagram of maximal rank.

In case (d), adding the second red vertex connected to any vertex except the one of the two extreme vertices on the right, we obtain a parabolic subdiagram of one of the types $\tilde{D}_5, \tilde{D}_4, \tilde{E}_6, \tilde{D}_6, \tilde{D}_7$, which is not contained in a maximal rank parabolic subdiagram. It remains to consider the case of the following possible diagram:

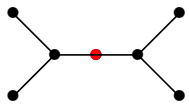


In this case, the diagram contains a root lattice of rank 10 isomorphic to $\mathsf{E}_6 \oplus \mathsf{A}_4$, which is impossible because $\mathrm{NS}(S)$ has signature $(1, 9)$.

Case 2: There exists a maximal rank parabolic subdiagrams P that consists of two connected components P_1, P_2.

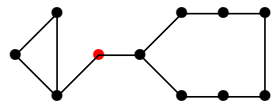
In this case there is only one additional vertex v. As we remarked before, there could be only one disconnected parabolic subdiagram of maximal rank. So, any parabolic subdiagram different from P_1 and P_2 must be a connected maximal rank parabolic subdiagram. In particular, there are no double edges unless it represents a parabolic subdiagram of type $\tilde{A}_1$. Also, v is connected to only one vertex of P_1 and P_2 unless P_1 or P_2 is of type $\tilde{A}_1$.

Case 2a: P_1 and P_2 are of types $\tilde{A}_s, \tilde{A}_t$ with $s + t = 8$.
If $s, t > 2$, the diagram contains a parabolic subdiagram of type $\tilde{D}_6$:



It cannot be a part of any connected maximal rank parabolic subdiagram.

If $s = 2, t = 6$, we have the following subdiagram:



In this case, we see a subdiagram of type $\tilde{E}_6$ not contained in any maximal rank parabolic subdiagram.

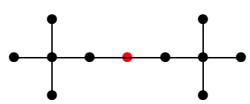
If $s = 1$, we have the following possible subdiagrams:

In both cases, we have a parabolic subdiagram of type $\tilde{E}_7$ that is not a part of a maximal rank parabolic subdiagram.

Case 2b: P_1 is of type $\tilde{D}_s$ and P_2 is of type $\tilde{A}_t$ with $s + t = 8, s \geq 4$.

If $s \neq 4$ and v is connected to one of the tri-valent vertices of the parabolic subdiagram P_1, we obtain a parabolic subdiagram of type $\tilde{D}_4$ that is not contained in a maximal rank parabolic subdiagram. If v is connected to one of the extreme vertices of P_1, then, adding some vertices from P_2, we obtain a parabolic subdiagram of type $\tilde{D}_7$ (if $s = 4, 5$), or $\tilde{E}_7$ (if $s = 6, 7$). None of them can be extended to a parabolic subdiagram of rank 8.

Case 2c: P_1 and P_2 are of type $\tilde{D}_4$. If the new vertex is connected to a 4-valent vertex of the subdiagrams of type $\tilde{D}_4$, we obtain a parabolic subdiagram of type $\tilde{D}_6$ or $\tilde{D}_7$ that is not contained in a maximal rank parabolic subdiagram. This gives the following diagram:



(6.2.1)

Since the fibers of type $\tilde{D}_4$ are double fibers, we see that $K_S \neq 0$ and $p = 2$. We know that the jacobian fibration with these types of reducible fibers must be a quasi-elliptic fibration.

Case 2d: P_1 is of type $\tilde{E}_6$ and P_2 is of type $\tilde{A}_2$.

If the new vertex joins a vertex of P_1 different from the extreme one, we find a parabolic subdiagram of type $\tilde{E}_6$ that is not contained in a maximal rank parabolic subdiagram. The only possibility is the following:

(6.2.2)

Observe that we have two parabolic subdiagrams of types $\tilde{E}_8$ that define two genus one fibrations $|2F_1|$ and $|2F_2|$ with $F_1 \cdot F_2 = 3$. The half-fibers share common irreducible components contradicting Lemma 6.2.2. This case is excluded.

Case 2d: P_1 is of type $\tilde{E}_7$ and P_2 is of type $\tilde{A}_1$.

The only possibilities here are given in cases $(\tilde{E}_7^1)$ and $(\tilde{E}_7^2)$ of the theorem. □

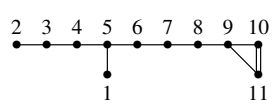
Definition 6.2.4 An Enriques surface is called *extra-special* if $\mathsf{nd}(S) \leq 2$.

It follows from Theorems 6.1.10 and 6.1.12 that there are no extra-special surfaces in characteristic different from two.

Proposition 6.2.5 *Let S be an Enriques surface that admits a crystallographic basis of one of the types $\tilde{E}_8, \tilde{E}_7^{(1)}, \tilde{D}_8$ from the assertion of Theorem 6.2.3. Then, S is extra-special. The surface of type $\tilde{E}_7^{(2)}$ has $\mathsf{nd}(S) = 3$ and the surface of type $\tilde{D}_4 + \tilde{D}_4$ has $\mathsf{nd}(S) = 4$.*

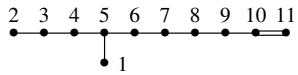
Proof Assume that S has a crystallographic basis of type $\tilde{E}_8$. It has only one genus one fibration, hence $\mathsf{nd}(S) = 1$.

Suppose S contains a crystallographic basis of (-2)-curves of type $\tilde{E}_7^{(2)}$. Then, S has one genus one fibration $|2F_1|$ with reducible fibers F_1 of type $\tilde{E}_7$ and $\tilde{A}_1$ and two genus one fibrations $|2F_2|, |2F_3|$ with reducible fibers of type $\tilde{E}_8$. We may assume that F_1 is of type $\tilde{E}_7$. Let D_2, D_3 be the divisors defined by the parabolic diagrams of type $\tilde{E}_8$. We have to decide whether $D_i \sim 2F_i$ or $D_i \sim F_i$. Let us index the vertices by the corresponding (-2)-curves as follows:



and denote the corresponding curves by R_i. Consider the genus one pencil $|D_1|$ with two fibers of types $\tilde{E}_7$ and $\tilde{A}_1$ (or $\tilde{A}_1^*$). The curve R_9 is its bisection and we see that the first fiber is double and the second is not. Thus, $[R_{10} + R_{11}]$ is divisible by 2 in $\mathrm{Num}(S)$ and hence $v = \frac{1}{2}[R_{10} - R_{11}] \in \mathrm{Num}(S)$. Let M be the sublattice of $\mathrm{Num}(S)$ generated by the curves R_i and v. It is easy to see that M is isomorphic to the lattice $\mathsf{E}_7 \oplus \mathsf{A}_1 \oplus \mathsf{U}$, where A_1 is generated by v. Its discriminant group is generated by $r_1 = \frac{1}{2}v \mod M$ and $r_2 = \frac{1}{2}[R_2 + R_4 + R_1] \mod M$ with $r_1^2 = -\frac{1}{2}, r_2^2 = -\frac{3}{2}, r_1 \cdot r_2 = 0$. We see that $r = r_1 + r_2$ is a unique non-zero isotropic vector in the discriminant group. Adding it to M, we obtain a unimodular lattice that must coincide with $\mathrm{Num}(S)$. Since $D_2 \cdot r$ and $D_3 \cdot r$ are even, we see that $D_2 \in |2F_2|, D_3 \in |2F_3|$ are simple fibers. Now, it is easy to see that $F_1 \cdot F_2 = F_1 \cdot F_3 = F_2 \cdot F_3 = 1$. This shows that $\mathsf{nd}(S) = 3$.

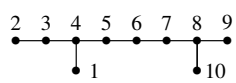
Suppose S contains a crystallographic basis of (-2)-curves of type $\tilde{E}_7^{(1)}$.



(6.2.3)

Then, S has two genus one pencils $|2F_1|$ and $|2F_2|$. The first one has two half-fibers of type $\tilde{E}_7$ and $\tilde{A}_1^*$ and the second one has a fiber (or half-fiber) D_2 of type $\tilde{E}_8$. We observe that $F_1 \cdot D_2 = 2$. Consider the sublattice M of $\mathrm{Num}(S)$ spanned by the classes of (-2)-curves represented by the vertices of the diagram. It is easy to see that it is isomorphic to the lattice $\mathsf{A}_1 \oplus \mathsf{E}_7 \oplus \mathsf{U}$. It is a sublattice of index 2 in $\mathrm{Num}(S)$. The vectors $v_1 = [R_1 + R_2 + R_4]$ and $v_2 = [R_1 + R_6 + R_8 + R_{10}]$ have even intersection with all curves in the diagram, hence, $r_1 = \frac{1}{2}v_1$ and $r_2 = \frac{1}{2}v_2$ belong to the dual lattice $M^\vee$. The residues of r_1, r_2 modulo M generate the discriminant group of M. We have $r_2^2 = 0$, hence $r_2 \mod M$ is the unique isotropic vector in $\mathrm{discr}(M)$ and thus joining r_2 to M, we obtain the unimodular lattice that must coincide with $\mathrm{Num}(S)$. Since $[D_2]$ intersects any vector in M evenly and $[D_2] \cdot v_2 = 0$, we obtain that D_2 intersects all classes in $\mathrm{Num}(S)$ evenly, and hence D_2 is a double fiber. Thus, $F_1 \cdot F_2 = 1$ and $\mathsf{nd}(S) = 2$.

Suppose S contains a crystallographic basis of (-2)-curves of type $\tilde{D}_8$.

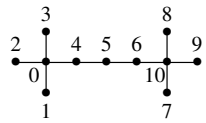


(6.2.4)

Then, S has one genus one fibration $|2F_1|$ with reducible half-fiber F_1 of type $\tilde{D}_8$ and two genus one fibrations $|2F_2|, |2F_3|$ with reducible fibers of type $\tilde{E}_8$. Let D_2, D_3 be the divisors defined by the parabolic diagrams of type $\tilde{E}_8$. We have to decide whether $D_i \sim 2F_i$ or $D_i \sim F_i$.

Consider the sublattice M of $\mathrm{Num}(S)$ generated by the (-2)-curves represented by the vertices of the diagram. It is easy to see that it is isomorphic to $\mathsf{D}_8 \oplus \mathsf{U}$, a sublattice of index 2 in E_{10}. The curve $R_1 + R_5 + R_7 + R_{10}$ has even intersection with all curves in the diagram. As in the previous cases, we show that the discriminant group of M is generated by vectors $r_1 = \frac{1}{2}[R_1+R_5+R_7+R_9] \mod M$ and $r_2 = \frac{1}{2}[R_1+R_5+R_7+R_{10}] \mod M$ with $r_1 + r_2 = \frac{1}{2}[R_9 + R_{10}] \mod M$. We have $r_1^2 \equiv r_2^2 \equiv 0, (r_1 + r_2)^2 \equiv 1$. Adding one of the vectors r_1, r_2 to M generates a sublattice isomorphic to E_{10}. Since both of r_1, r_2 cannot belong to $\mathrm{Num}(S)$, only one of them belongs to $\mathrm{Num}(S)$. This shows that one of the fibers of type $\tilde{E}_8$ is non-double, let it be $D_2 \sim 2F_2$ and let $D_3 = F_3$ be a half-fiber. Now, we find that $F_1 \cdot F_2 = 1, F_1 \cdot F_3 = 2, F_2 \cdot F_3 = 1$, and hence $([F_2], [F_3])$ and $([F_1], [F_2])$ are the only non-degenerate isotropic 2-sequences. This gives $\mathsf{nd}(S) = 2$.

Suppose S is of type $\tilde{D}_4 + \tilde{D}_4$, we index the vertices as follows:



(6.2.5)

and denote by R_i the corresponding (-2)-curves. We have one maximal parabolic subdiagram of type $\tilde{D}_4 + \tilde{D}_4$ and nine maximal parabolic subdiagrams of type $\tilde{D}_8$. Using the classification of extremal jacobian genus one fibrations on rational surfaces from Section 4.9 in Volume I, we find that the genus one fibration corresponding to the diagram of type $\tilde{D}_4 + \tilde{D}_4$ is quasi-elliptic. The curve R_5 is the curve of cusps and we have two half-fibers $F_1 = 2R_0 + R_1 + R_2 + R_3 + R_4$ and $F_2 = 2R_{10} + R_6 + R_7 + R_8 + R_9$ of type $\tilde{D}_4$.

Let $M \cong \mathsf{U} \oplus \mathsf{D}_4 \oplus \mathsf{D}_4$ be the sublattice $\mathrm{Num}(S)$ of index 4 spanned by the numerical divisor classes of the curves R_i. The numerical classes $r_1 = \frac{1}{2}[R_2 + R_3 + R_8 + R_9]$ and $r_2 = \frac{1}{2}[R_1 + R_2 + R_7 + R_9]$ generate a maximal isotropic subspace in the discriminant group of M, and hence, as above, we obtain that $\mathrm{Num}(S)$ is generated by M and the classes r_1, r_2. Let $F_{a,b}$, $a = 1, 2, 3$, $b = 7, 8, 9$, be the parabolic subdiagram spanned by the curves R_i except R_a and R_b. We check that $F_{1,7}, F_{3,8}$ and $F_{2,9}$ intersect both r_1 and r_2 with multiplicity 2. All other $F_{a,b}$ intersect one of the r_i with multiplicity 1. This implies that $F_{1,7}, F_{3,8}$ and $F_{2,9}$ are simple fibers of genus one fibrations, all other $F_{a,b}$ are half-fibers. We also check that $F_{1,7}, F_{3,8}$ and $F_{2,9}$ intersect each other with multiplicity 4 and intersect F_1 with multiplicity 2, hence the half-fibers of these fibrations together with F_1 form a non-degenerate canonical 4-sequence, and the other $F_{a,b}$ are not involved in non-degenerate canonical sequences. It is a maximal such sequence, hence $\mathsf{nd}(S) = 4$. □

From now on, referring to an extra-special surface of type $\tilde{E}_7^{(1)}$, we say that it is an *extra-special surface of type* $\tilde{E}_7$.

The following theorem was claimed in [138, Theorem 3.5.2]: however, its proof was based on lengthy case-by-case considerations and there is no guarantee that it is correct. A more conceptional proof was recently supplied in [492].

Theorem 6.2.6 *An extra-special surface must be one of the three types* $\tilde{E}_8, \tilde{E}_7, \tilde{D}_8$.

Proposition 6.2.7 *Let S be an extra-special Enriques surface.*

- *An extra-special surface of type $\tilde{E}_8$ has only one genus one fibration. It is quasi-elliptic and has a half-fiber of type $\tilde{E}_8$.*
- *An extra-special surface of type $\tilde{E}_7$ has two genus one fibrations $|2F_1|$ and $|2F_2|$. Both of them are quasi-elliptic. One of them has two half-fibers of types $\tilde{E}_7$ and $\tilde{A}_1^*$, hence S is a classical Enriques surface. Another one has a simple fiber of type $\tilde{E}_8$. We have $F_1 \cdot F_2 = 1$.*
- *An extra-special surface of type $\tilde{D}_8$ has three genus one fibrations $|2F_1|, |2F_2|, |2F_3|$. The first fibration has a half-fiber of type $\tilde{D}_8$. The fibration $|2F_2|$ has a simple fiber of type $\tilde{E}_8$ and it is elliptic and the fibration $|2F_3|$ has a half-fiber of type $\tilde{E}_8$. We have $F_1 \cdot F_2 = F_2 \cdot F_3 = 1, F_1 \cdot F_3 = 2$.*

Proof Suppose S is an extra-special surface of type $\tilde{E}_8$. It has only one parabolic subdiagram and hence only one genus one fibration. We have to prove only that it is a quasi-elliptic fibration.

Suppose it is an elliptic fibration. We know that the set of (-2)-curves on S generates $\mathrm{Num}(S)$ and only one of them is not contained in fibers. It is a special bisection R. It follows that any bisection of the fibration must be numerically equivalent to $R + A$, where A is contained in fibers. Thus, the generic fiber has a unique degree two point. However, by Riemann–Roch, an effective Cartier divisor of degree 2 on a genus one curve moves in a pencil. Thus, f is a quasi-elliptic fibration and R is its curve of cusps. Recall from Table 4.9 in Volume I that the jacobian fibration of f has a unique reducible fiber.

Suppose S is an extra-special surface of type $\tilde{E}_7$. It has one genus one fibration $|2F_1|$ of type $\tilde{E}_7+\tilde{A}_1$ and one of type $|2F_2|$ of type $\tilde{E}_8$. Since $|2F_1|$ has two half-fibers, we also see that $K_S \neq 0$. This implies that the fiber of type $\tilde{A}_1$ is of additive type. If $|2F_1|$ is an elliptic pencil, then it should be of multiplicative type by the classification of extremal rational surfaces in characteristic 2 given in Table 4.7 in Volume I. Thus, $|2F_1|$ is quasi-elliptic. Similarly, the second pencil $|2F_2|$ contains two half-fibers and they are neither of multiplicative type nor supersingular elliptic curves. Suppose $|2F_2|$ is an elliptic pencil. Then, the jacobian fibration either contains one half-fiber of type $\tilde{A}_0^*$ or has absolute invariant equal to zero. So, we get a contradiction and conclude that $|2F_2|$ is a quasi-elliptic fibration and R_{11} is its curve of cusps.

Suppose S is an extra-special surface of type $\tilde{D}_8$. It has one genus one pencil $|2F_1|$ with a half-fiber F_1 of type $\tilde{D}_8$ and two genus one pencils $|2F_2|, |2F_3|$ of type $\tilde{E}_8$. We have seen in the proof of Proposition 6.2.5 that only one of the latter fibrations has a half-fiber of type $\tilde{E}_8$, so we may assume that F_2 is of this type and F_3 is irreducible. We also have $F_1 \cdot F_2 = 1, F_1 \cdot F_3 = 2, F_2 \cdot F_3 = 1$. Since the pencil $|2F_2|$ has no special bisections, it cannot be quasi-elliptic.

Note that in cases of $\tilde{E}_7$, $\tilde{D}_8$, the set of (-2)-curves spans a sublattice of index 2 in E_{10}, so we cannot use the previous argument in case of $\tilde{E}_8$ to show that some of the fibrations must be quasi-elliptic. □

Remark 6.2.8 It follows from [371, Remark 12.4] that in the third case of Proposition 6.2.7, $|2F_1|$ is a quasi-elliptic fibration and the remaining two fibrations are elliptic.

We also have a partial converse statement.

Proposition 6.2.9 *Let S be an -Enriques surface in characteristic* 2. *Then:*

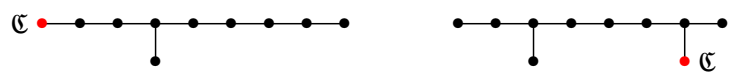
- *S is extra-special of type $\tilde{E}_8$ if and only if it is classical or $\boldsymbol{\alpha}_2$-surface and admits a quasi-elliptic fibration with a half-fiber of type $\tilde{E}_8$.*
- *S is extra-special of type $\tilde{E}_7$ if and only if it is a classical Enriques surface that admits a quasi-elliptic fibration with a simple fiber of type $\tilde{E}_8$ and a quasi-elliptic fibration with a half-fiber of type $\tilde{E}_7$.*

Proof We proved in the previous proposition that an extra-special of type $\tilde{E}_8$ is quasi-elliptic. Since it admits a quasi-elliptic fibration, it cannot be an $\boldsymbol{\mu}_2$-surface.

So, the first properties are necessary. In the second case, we proved that S must be classical and the stated properties of the fibrations are necessary.

Suppose S has a quasi-elliptic fibration with a half-fiber F of type $\tilde{E}_8$. Then, the curve of cusps $\mathfrak{C}$ is its special bisection. Together with irreducible components of F they form a crystallographic basis of type $\tilde{E}_{10}$. Thus, the surface is extra-special of type $\tilde{E}_8$.

Suppose S has a quasi-elliptic fibration with simple fiber F of type $\tilde{E}_8$. Let $\mathfrak{C}$ be the curve of cusps. Then, it intersects F at an irreducible component of multiplicity 2. Thus, we have two possible diagrams of (-2)-curves:



The first diagram shows that S admits a fibration with a half-fiber of type $\tilde{E}_7$. Since the curves in the second diagram form a crystallographic basis that contains an affine root basis of type $\tilde{D}_8$, there are no more (-2)-curves on S. Therefore, S does not contain a fiber of type $\tilde{E}_7$. In fact, it describes an extra-special surface of type $\tilde{D}_8$. □

Example 6.2.10 In this example, we construct a one-dimensional family of classical extra-special surfaces of type $\tilde{E}_8$.

The surface has only one bielliptic linear system. It corresponds to the degenerate isotropic sequence $(f, f+R)$, where f is the numerical class of a half-fiber of type $\tilde{E}_8$ and R is the special bisection. The linear system defines a degree 2 map $\phi : S \to \mathsf{D}_1'$, where D_1' is a symmetroid quartic del Pezzo surface with singularities $2A_1, A_3$. We know from Section 0.6 that S has a double plane model given by the equation $w^2 + xyf(x,y,z) = 0$, where $V(f(x,y,z))$ is a plane sextic passing through 5 points $p_3 \succ p_2 \succ p_1, p_4 \succ p_5$, where p_1, p_2, p_3 and p_1, p_4, p_5 are on a line. We choose projective coordinates such that $p_1 = [0,0,1]$ and $p_2 = [0,1,0]$. We take

$$f(x,y,z) = xy(y^2z^4 + x^2y^3z + x^6 + \lambda x^2y^4). \tag{6.2.6}$$

We use the affine open set $z = 1$ and blow-up at the point $p_1 = [0,0,1]$ using the substitution $w = xu, y = xv$. After the normalization, we get the equation of the surface in a neighborhood of the point p_2:

$$u^2 + v(v^2 + x^4 + x^3v^3 + \lambda x^4u^4) = 0.$$

After the blow-up p_2, using the substitution $u = zt, v = xt$ and the normalization, we get the equation in a neighborhood of the point p_3:

$$z^2 + t^3 + t + t^4t^5 + \lambda t^5x^8 = 0.$$

The coordinate t of the intersection point P of the proper transform of the sextic $V(f)$ with the exceptional curve (one of the two lines on D_1') over p_3 is equal to 0. However, after we replace t with $t+1$, and replace z with $z+t$, we get the following

equation:

$$z^2 + t^3 + x^5(1 + \gamma(t^5 + t^4 + t + \gamma)x^3) = 0.$$

This shows that the surface has a singular point of type $E_8^{(0)}$ at a point $Q \neq P$ on one of the lines on D_1'. After we resolve this singularity of the surface at Q, we obtain an extra-special surface of type $\tilde{E}_8$.

Example 6.2.11 In this example we construct an extra-special classical Enriques surface of type $\tilde{E}_7$. We use the unique non-degenerate U-pair of half-fibers (F_1, F_2) of types $\tilde{E}_7$ and $\tilde{A}_1^*$. It defines an inseparable degree 2 map $\phi : S \to \mathsf{D}_1$. The surface D_1 is the blow-up of 5 points $p_1, p_3 \succ p_2, p_5 \succ p_3$ where p_1, p_2, p_3 and p_1, p_4, p_5 are on a line. We choose the coordinates such that $p_1 = [0, 0, 1], p_2 = [0, 1, 0], p_3 = [1, 0, 0]$ and consider the double plane model of S given by equation $w^2 + xyf(x, y, z) = 0$, where the curve $W = V(f)$ is the union of a line ℓ and a quintic curve $V(g)$. We require that $V(g)$ has a simple point at p_1 and double points at $p_2, \dots, p_5$ and also that the line ℓ is tangent to $V(g)$ at p_1 with multiplicity 3. We choose $\ell = V(x + y)$ and

$$g(x, y, z) = (x + y)z^4 + xy(c_1x + c_2y)z^2 + x^2y^2(x + y).$$

We check that the double plane $w^2 + xy(x + y)g(x, y, z) = 0$ has a simple singular point of type A_2 over the point $q = [1, 1, 0]$ and a simple point of type E_7 over p_1. The pre-image of the line ℓ gives us a simple fiber of type $\tilde{E}_8$, the pre-image of the exceptional curve over p_1 gives a half-fiber of type $\tilde{E}_7$ and the pre-image of the line $V(z)$ gives a half-fiber of type $\tilde{A}_1^*$.

In Section 8.10, following [371], we will give a construction of all extra-special surfaces.

In the remaining part of this section, we will follow the work of T. Ekedahl and N. Shepherd-Barron [213] and provide more details for their proof of Theorem 1.4.10. Following the authors, we introduce the following definition.

Definition 6.2.12 A classical Enriques surface S in characteristic two is called *exceptional* if it admits a non-zero regular vector field.

As we shall see, not every extra-special surface is exceptional and vice versa, not every exceptional surface is extra-special.

Assume that the K3-cover X of S is not normal and let A be the conductrix of S. Recall from Section 1.3 that $2A$ is equal to the divisorial part of a 1-form ω spanning $H^0(S, \Omega^1_{S/\Bbbk})$. We have proved that $A^2 = -2, h^0(A) = 1$, each irreducible component of A is a (-2)-curve, and A is numericaly connected. Also, we have the short exact sequence

$$0 \to \mathcal{O}_S \to \tilde{\pi}_*\mathcal{O}_Y \to \omega_S(A) \to 0,$$

where $\tilde{\pi} : Y \to S$ is the composition of the K3-cover $\pi : X \to S$ and the normalization map $\sigma : Y \to X$. We also have the short exact sequence

$$0 \to \mathcal{O}_S(2A) \to \Omega^1_{S/\Bbbk} \to \mathcal{J}_Z(K_S - 2A) \to 0,$$

where Z is the 0-dimensional part of the scheme (ω) of zeros of ω with $h^0(\mathcal{O}_Z) = 4$. For any point $x \in S$, we denote the length of $\mathcal{O}_Z$ at x by $\langle\omega\rangle_x$.

Suppose R is an irreducible component of multiplicity m of a non-multiple fiber F of f. Let t be a local parameter at $f(F)$. If m is odd, then we can write $f^*(t) = \epsilon u^m$, where $u = 0$ is a local equation of R at its general (not generic) point x and ϵ is a unit at x. Passing to the formal completion of $\mathcal{O}_{S,x}$, and applying Hensel's Lemma, we can replace u with $\epsilon^{1/m}u$ and obtain $f^*(t) = u^m$. This gives $\omega = df^*(t) = mu^{m-1}du$, hence R enters in $2A$ with multiplicity $m-1$. On the other hand, if m is even, we obtain $\omega = u^m d\epsilon$, so that R enters in $2A$ with multiplicity $\geq m$. If f is quasi-elliptic, then at a general point of the curve of cusps $\mathfrak{C}$, we have $f^*(t) = y^2 + x^3$, so that $\omega = x^2dx$ vanishes on C with multiplicity 2. Thus, $\mathfrak{C}$ enters in A with multiplicity 1.

Suppose f is smooth at a closed point x. We can choose a local parameter u at x such that the equation of the fiber F through x is given by local equation $u = 0$. Then, $f^*(t) = \epsilon u$, where ϵ is a unit in $\mathcal{O}_{S,x}$. Replacing u with ϵu, we may assume that $\epsilon = 1$. Thus, $\omega = du$ and hence does not vanish at x. Suppose f is an elliptic fibration. Then, any curve C not contained in a fiber intersects some fiber at its smooth point. This implies that ω does not vanish at a general point of C.

To summarize, we obtain that A is a combination of irreducible components of fibers if f is elliptic and $2A = 2\mathfrak{C} + 2A'$, where A' is a combination of irreducible components of fibers, if f is quasi-elliptic.

Suppose x is an isolated singular point of a non-multiple fiber F of an elliptic fibration. We have

$$\langle\omega\rangle_x = \dim_{\Bbbk} \mathrm{Ext}^1(\Omega^1_{S/\mathbb{P}^1,x}, \mathcal{O}_{S,x})$$

(see [152]). It is easy to see, by using the exact sequence of sheaves of relative differentials, that the number $\langle\omega\rangle_x$ coincides with the *Milnor number* (see Section 0.4) of the fiber F at x.

If x is an ordinary double point, then, in some local coordinates u, v at x, we can write $f^*(t) = uv$. This implies that $\langle\omega\rangle_x = 1$. Thus, if the fiber is of type I_n, then

$$\sum_{x\in F} \langle\omega\rangle_x = \nu_x(\Delta), \tag{6.2.7}$$

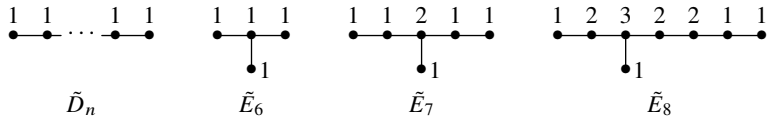
where Δ is the discriminant of the associated jacobian fibration.

If x is an ordinary cusp (resp. a triple point) of the fiber, then $f^*(t) = \epsilon(u^2 + v^3)$ (resp. $\epsilon(u^2v+v^2u)$), and $\langle\omega\rangle_x$ depends on the unit ϵ. It follows from [151], Théorème 2.6, that (6.2.7) still holds. This time $\langle\omega\rangle_x = e(F) + \alpha$, where α is the invariant of wild ramification. In our case $\langle\omega\rangle_x = 4$, hence $Z = \{x\}$ and the singularity of the K3-cover over x is a rational double point of type $D_4^{(0)}$. If F has a cusp at x, then the invariant of the wild ramification is equal to 2, otherwise it is equal to zero.

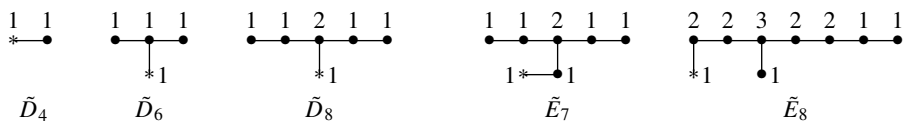
Let us derive some immediate corollaries of our previous discussion.

Proposition 6.2.13 *Let $p = 2$ and let S be a classical or α_2-surface. Let ω be a generator of $H^0(S, \Omega^1_{S/\Bbbk})$. Let $f : S \to \mathbb{P}^1$ be an elliptic fibration with a non-multiple fiber of type $\tilde{D}_n$ or $\tilde{E}_n$. Then, the divisorial part D of the scheme of zeros of*

ω is equal to $2A$, where A is defined by the following weighted graph with weights indicating the multiplicities of the irreducible components:



(in case $\tilde{D}_n$ the components are all multiple components of the fiber). If $f : S \to \mathbb{P}^1$ is a quasi-elliptic fibration, then D is described by the following diagrams, where the star indicates the curve of cusps:



Proof If R is an irreducible component of a non-multiple fiber of a genus one fibration of even multiplicity m with local equation $u = 0$ at its general point, then $d\pi^*(t) = u^m d\epsilon$. It is easy to see that $d\epsilon$ can vanish on R only with even multiplicity. Suppose f is an elliptic fibration, then A is equal to $A' + B$, where A' is as in the assertion of the proposition and B is a combination of components R_i entering in the fiber containing A. One easily computes $A^2 = (A' + B)^2$ and obtain that $A^2 < -2$ if $B \neq 0$ contradicting (1.3.7). If f is quasi-elliptic, then we know that the curve of cusps $\mathfrak{C}$ enters in A with multiplicity 1. We also know which component of F the curve $\mathfrak{C}$ intersects (see Remark 4.9.11). The rest of the argument is the same. □

Corollary 6.2.14 *Assume $p = 2$ and let $\pi : X \to S$ be the canonical cover of an Enriques surface. Suppose X is birationally isomorphic to a K3 surface. Then, S does not admit a quasi-elliptic fibration.*

Proof By Lemma 4.10.2 from Volume I, a $\boldsymbol{\mu}_2$-surface does not admit a quasi-elliptic fibration. In other cases the canonical cover is inseparable. Moreover, we know that $H^0(S, \Omega^1_{S/\mathbb{k}}) \neq \{0\}$ and, by Proposition 0.2.21 from Volume I, X is singular over zeros of a non-zero regular 1-form ω. However, we know that the curve of cusps $\mathfrak{C}$ enters in the scheme of zeros of ω. Thus, X is not normal, and hence not birationally isomorphic to a K3 surface. This contradicts the assumption of the corollary. □

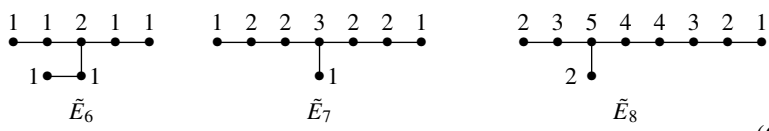
One can analyze the conductrix A for surfaces with $H^0(S, \Theta_{S/\mathbb{k}}) \neq 0$ not assuming that it is supported at non-multiple fibers. The classification of possible configurations of A is more delicate, and we refer for this to [213]. As we noted before in the proof of Theorem 1.4.10, this analysis allows the authors to classify exceptional Enriques surfaces.

Recall from the proof of Theorem 1.4.10 from Volume I that $H^0(S, \Theta_{S/\mathbb{k}}) \neq 0$ if and only if $h^0(2A + K_S) \neq \{0\}$. Suppose S is exceptional and the dual graph of the support of A is a part of a Dynkin diagram of finite type. Let $D \in |2A + K_S|$. We have

$4A \sim 2D$, hence $4A$ and $2D$ span a pencil. Since its moving part has non-negative self-intersection, we get a contradiction. Thus, $h^0(2A + K_S) = 0$ and S cannot be exceptional. So, we have proved the following:

Proposition 6.2.15 *Suppose S is an exceptional Enriques surface. Then, the dual graph of the support of the conductrix A is not a part of a Dynkin diagram of finite type.*

Suppose f is an elliptic fibration. By inspection of the graphs from Proposition 6.2.13, we see that the support of A is equal to the support of a half-fiber. The further classification of possible conductrices shows that the only possibilities for the dual graphs of A are the following:

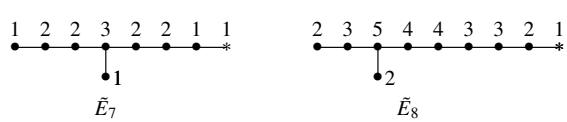


(6.2.8)

Assume $K_S \neq 0$ and let $\bar{F}$ be a half-fiber containing the support of A and let $\bar{F}'$ be another half-fiber. In all three cases $A = \bar{F} - B$, where $0 < 2B \leq \bar{F}$. Thus, $2A + K_S \sim 2\bar{F} - 2B + K_S \sim \bar{F}' + (\bar{F} - 2B) > 0$. Thus,in all these cases we have a non-zero regular vector field.

Assume now that f is a quasi-elliptic fibration. Let $A = A' + \mathfrak{C}$, where $\mathfrak{C}$ is the curve of cusps. Suppose the support of A' is contained in a non-multiple fiber F. By inspection of the list in Proposition 6.2.13, we find that the assertion of Proposition 6.2.15 is satisfied in the last two cases. In the case E_7, we observe that A is contained in a half-fiber F of type $\tilde{E}_6$ in some genus one fibration on S. So, this case has been already considered and we have concluded that in this case $2A+K_S$ is effective. In the case E_8, let A'' is obtained from A by deleting the component R, which is extreme on the right side of the diagram. Then, $A''+\mathfrak{C}$ is a part of half-fiber G of type $\tilde{E}_7$ of some genus one fibration with R being a special 2-section. Moreover, $A'' + \mathfrak{C} = G - B$, where $0 < 2B < G$. Thus, $2A + K_S \sim 2R + 2A'' + 2\mathfrak{C} \sim 2R + 2G - 2B + K_S \sim 2R + G' + (G - 2B) > 0$, where G' is another half-fiber of $|2G|$.

Next, we assume that f is a quasi-elliptic fibration, A is contained in a half-fiber F of f. It follows from [213] that there are two possible cases for A:



(6.2.9)

In both cases, $2A$ contains F, hence $2A + K_S$ is linearly equivalent to an effective divisor containing the second half-fiber. Note that in the second case S is an $\tilde{E}_8$-special Enriques surface.

The above discussion provides more details for the proof of Theorem 1.4.10. Let us summarize what we have found.

Theorem 6.2.16 *Let S be an exceptional Enriques surface and let ω be a basis of $H^0(S, \Theta_{S/\Bbbk})$. Then, its divisorial part D is equal to $2A$, where A is a curve with dual weighted graphs from* (6.2.8)*,* (6.2.9) *and the last diagram from Proposition 6.2.13. The surface contains a genus one fibration with a half-fiber of type $\tilde{E}_6$, or $\tilde{E}_7$, or $\tilde{E}_8$. If the fibration is quasi-elliptic with a half-fiber of type $\tilde{E}_7$ or $\tilde{E}_8$, then there exists a special 2-section. In particular, in the last case the surface is extra-special of type $\tilde{E}_8$.*

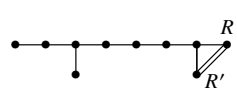
Thus, we infer that S must admit a quasi-elliptic fibration such that the conductrix A is supported on a half-fiber.

Writing K_S as the difference of two half-fibers $F_1 - F_2$, we obtain that $2A + K_S \sim F_2 - B$ is not effective. If f is quasi-elliptic, we get $2A = F - B + \mathfrak{C}$. It follows from the above proposition that $\mathfrak{C} \cdot A = 1$ and $\mathfrak{C} \cdot B = 0$. Then, $2A + K_S \sim F_2 - B + \mathfrak{C}$, and intersecting with $\mathfrak{C}$, we obtain $(2A + K_S) \cdot \mathfrak{C} = \mathfrak{C}^2 + (F_2 - B) \cdot \mathfrak{C}) = -1$. This implies that $\mathfrak{C}$ must be an irreducible component of an effective divisor in $|2A + K_S|$. Thus,$F_2 - B$ must be effective, and we get again a contradiction. It follows that the support of conductrix A is contained in the union of a half-fiber and the curve of cusps, if f is quasi-elliptic.

Since the divisorial part D of the scheme of zeros of ω does not depend on a choice of an elliptic fibration, we obtain that D enters in a fiber of any other elliptic fibration on S. Since we know that two non-reduced non-multiple fibers must be of types $\tilde{D}_4$ and they occur only for quasi-elliptic fibrations, we see that the type of a non-reduced non-multiple fiber of an elliptic fibration on a classical Enriques surface does not depend on the fibration.

Example 6.2.17 Let S be as above, and let f be an elliptic fibration on S with a non-multiple fiber F of type $\tilde{E}_8$. Note that it is possible to find a classical (resp. $\boldsymbol{\alpha}_2$-) surface with a non-multiple fiber of type $\tilde{E}_8$. We take an extremal rational elliptic surface with a fiber of type $\tilde{E}_8$ with the invariant of wild ramification δ equal to 1 (resp. 2). We choose two (resp. one) smooth fiber and apply the Ogg–Shafarevich theory from Section 4.7 in Volume I to create a torsor with two double fibers (resp. one wild double fiber). Note that in the case of a rational elliptic surface with $\delta = 2$, all smooth fibers are supersingular curves, so the torsor must have a wild fiber. This is our Enriques surface.

It follows from Theorem 6.1.10 from Section 6.1 that there always exists another genus one pencil $f' : S \to \mathbb{P}^1$ on S such that its general fiber intersects F with multiplicity 4. Since f is an elliptic fibration, the bielliptic map $S \to \mathsf{D}$ onto a quartic del Pezzo surface must be separable. This implies that f' is also an elliptic fibration. Since the conductrix does not depend on the choice of a fibration, we see that f' also contains a non-multiple fiber F' of type $\tilde{E}_8$. The fibers F and F' share all components except their reduced components R and R'. It is easy to see that we must have the following diagram of the components:



The divisor $R + R'$ (or $2R + 2R'$) defines a third genus one fibration f'' on S. Its other reducible fiber or a half-fiber must be of type $\tilde{E}_7$. It consists of irreducible components of A and an irreducible component R_0 from the following diagram:

(6.2.10)

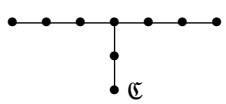
Note that $\mathfrak{C} \cdot R_0 = 0$, otherwise, $\mathfrak{C} \cdot R_0 = 1$ and hence there exists a genus one fibration with a double fiber of type $\tilde{A}_7$, which is a contradiction.

Since the curve of cusps $\mathfrak{C}$ intersects $R + R'$ with multiplicity 2, the fiber $R + R'$ is a simple fiber. The eleven components form a crystallographic root basis in $\mathrm{Num}(S)$, so there are no more (-2)-curves on S. The surface is $\tilde{E}_7^{(1)}$-special.

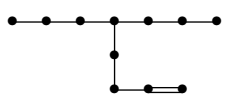
It remains to see the existence of an exceptional Enriques surface.

In the previous example, we constructed an extra-special $\tilde{E}_7^{(1)}$-surface starting from a quasi-elliptic fibration with a non-multiple fiber of type $\tilde{E}_8$. As we have shown this implies that the surface is exceptional.

Example 6.2.18 Take a quasi-elliptic surface with a non-multiple fiber F of type $\tilde{E}_7$ and a reducible non-wild half-fiber of type $\tilde{A}_1^*$. It obviously exists as a torsor of a rational quasi-elliptic surface. This surface admits another genus one fibration f' such that the conductrix A is supported in a half-fiber F' of type $\tilde{E}_6$. This fibration must be elliptic. The conductrix A is supported in F', so we get an exceptional Enriques surface. We obtain the following diagram:

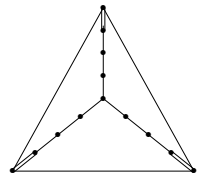


Here, $\mathfrak{C}$ is the curve of cusps of the quasi-elliptic pencil. Since it has a reducible half-fiber, we obtain the following diagram:



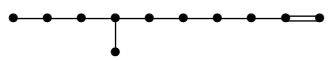
Now, we see that our diagram contains two more parabolic subdiagrams of type $\tilde{E}_7$ and each must contain another reducible half-fiber of type $\tilde{A}_1^*$. We also see that the second fibration with fiber of type $\tilde{E}_6$ has another reducible fiber, an irreducible component of the half-fiber of the first quasi-elliptic fibration. The only possible configuration is the following one.

Note that the Mordell–Weil group of the jacobian fibration of the elliptic fibration of type $\tilde{E}_6 + A_2^*$ is of order 3 and it acts on this diagram by a symmetry of order 3. The vertices of the diagram define a crystallographic root basis in $\mathrm{Num}(S)$ of cardinality



13. We will see in Theorem 8.10.20 (resp.Theorem 8.10.27) that the automorphism group of the surface S is isomorphic to the group $\mathfrak{S}_3 \times \mathbb{Z}/5\mathbb{Z}$ (resp. $\mathfrak{S}_3$) if S is a α_2-surface (resp. a classical surface).

Example 6.2.19 We take an Enriques surface with a quasi-elliptic fibration with a reducible half-fiber of type $\tilde{E}_7$ or $\tilde{E}_8$ defining the conductrix from diagram (6.2.9). In the former case, the quasi-elliptic fibration must have an additional reducible fiber of type $\tilde{A}_1^*$. If this fiber were not multiple, we get diagram (6.2.10). This defines an extra-special Enriques surface of type $\tilde{E}_7^{(1)}$. If the fiber is multiple, we get the following diagram:



(6.2.11)

In the latter case, when the conductrix is supported on a half-fiber of a quasi-elliptic fibration of type $\tilde{E}_8$, we get an extra $\tilde{E}_8$-special Enriques surface.

Recall that, by Theorem 1.4.10 an exceptional Enriques surfaces admits a crystallographic basis of (-2)-curves of one of the types $T_{2,3,7}, T_{2,4,5}$, or $T_{3,3,4}$. In the first case, the surface is extra-special of type $\tilde{E}_8$. In the second case, it could be an extra-special of type $\tilde{E}_7$, or contains a crystallographic basis of type $\tilde{E}_7^{(2)}$ from Theorem 6.2.3. In the latter case, we get a surface from Example 6.2.18. We say that an exceptional surface is of type $\tilde{E}_8$ (resp. $\tilde{E}_7^{(1)}$ or $\tilde{E}_7^{(2)}$, resp. $\tilde{E}_6$) if it contains a crystallographic basis of (-2)-curves of type $T_{2,3,7}$ (resp. $T_{2,4,5}$, resp. $T_{3,3,4}$).

6.3 Smooth Rational Curves on an Enriques Surface

A smooth rational curve on a nodal Enriques surface can appear as a component of a reducible fiber of a genus one fibration, or as its bisection, or as an exceptional curve of a bielliptic map of the surface. In this section, we will study all such possible appearances of a smooth rational curve. We will also study possible degrees of such curves on a polarized Enriques surface.

Let $|2F|$ be a genus one pencil on S. If it contains a reducible member, then its irreducible components are (-2)-curves. The next theorem shows that each (-2)-curve occurs in this way.

Recall the function $\Phi : \mathrm{Num}(S)^+ \to \mathbb{Z}_{\geq 0}$ defined in Chapter 2 (2.4.6). We use the same formula to define $\Phi(R)$, where R is a (-2)-curve. A (-2)-curve is an irreducible component of some genus one pencil if and only if $\Phi(R) = 0$.

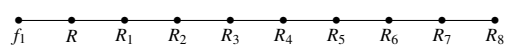
We have proved in Proposition 2.4.11 in Volume I that, for any $h \in \mathrm{Num}(S)$ with $h^2 > 0$,

$$\Phi(h) \leq \sqrt{h^2}.$$

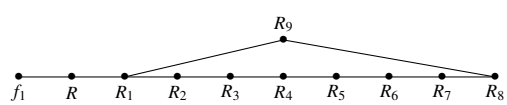
Theorem 6.3.1 *Suppose $p \neq 2$ or $p = 2$ and S is not extra-special of type $\tilde{E}_8$. For any (-2)-curve R, there exists a genus one pencil $|2F|$ such that R is an irreducible component of its member.*

Proof We have to show that $\Phi(R) = 0$. Suppose $\Phi(R) = R \cdot f_0 \geq 2$ and let F_0 be a genus one curve with $[F_0] = f_0$. Since f_0 is nef, the divisor class $C = F_0 + R$ is nef. Since $C^2 = 2(F_0 \cdot R - 1) \geq 2$, the above inequality implies that there exists a genus one pencil $|2F|$ such that $C \cdot F = F \cdot F_0 + F \cdot R \leq \frac{1}{2}C^2 = F_0 \cdot R - 1$. Since F_0 is nef, this gives that $F \cdot R \leq F_0 \cdot R - 1$, contradicting the choice of F_0.

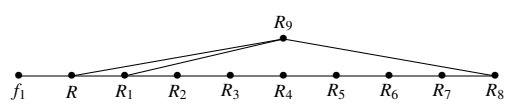
Suppose $\Phi(R) = R \cdot F_0 = 1$. In this case $C^2 = 0$. We can find a canonical isotropic sequence $(f_1, \ldots, f_{10})$, where $f_1 = [F_0]$, $f_2 = [F_0 + R] = [C]$. If there exists a nef numerical class $f_i, i > 2$, then $f_i \cdot f_2 = 1$ implies that $f_i \cdot R = 0$, and we are done. So, assume that the sequence contains only one nef isotropic divisor class f_1. Then, $f_3 = f_1 + R + R_1, \ldots, f_{10} = f_1 + R + R_1 + \cdots + R_8$, where R_i are (-2)-curves, and we have the following diagram:



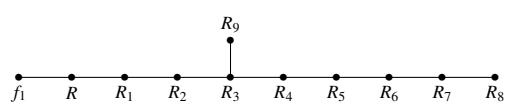
Let D be a member of $|2F_0|$ that contains the curves $R_1, \ldots, R_8$. Since the surface is not extra-special of type $\tilde{E}_8$, D cannot be of type $\tilde{E}_8$. Hence, there are three possibilities:



or



or



In the first case, the curves $R, R_1, R_2, R_3, R_4, R_9, R_8, R_7$ support a genus one curve F' of type $\tilde{E}_7$ that contains R as an irreducible component. Hence R is an irreducible component of a member of $|2F'|$. In the second case, R is an irreducible component of a curve $R + R_1 + R_9$ of type $\tilde{A}_2$. Finally, in the third case, R is an irreducible component of a curve of type $\tilde{E}_7$ formed by the curves $R, R_1, \ldots, R_6, R_9$. This proves the assertion. □

Definition 6.3.2 Let $|2F|$ be a genus one pencil on an Enriques surface. A smooth rational curve R with $F \cdot R = 1$ is called a *special bisection*.

Theorem 6.3.3 *Let S be a nodal Enriques surface. Then, there exists a genus one fibration on S with a special bisection.*

Proof If S is extra-special of type $\tilde{E}_8$, then, by its definition, it contains a bisection. By Proposition 6.2.7, it is the curve $\mathfrak{C}$ of cusps. Assume that S is not extra-special of type $\tilde{E}_8$. By the previous theorem, R is contained in a fiber D of some genus one fibration $|2F_1|$. By Theorem 6.1.10, there exists a genus one fibration $|2F_2|$ such that $F_1 \cdot F_2 = 1$. Since $D \cdot F_2 = 2$, either $R \cdot F_2 = 1$, and we are done, or there exists an irreducible component R' of D such that $R' \cdot F_2 = 0$. Let U be the sublattice of $\mathrm{Num}(S)$ generated by $[F_1], [F_2]$. Its orthogonal complement $U^\perp$ is isomorphic to E_8, which contains $r' = [R']$ and a root α with $\alpha \cdot r' = 1$. Then, $f_3 = [F_1] + [F_2] + \alpha$ is an isotropic vector in $\mathrm{Num}(S)$ with $f_3 \cdot r' = 1$. If f_3 is nef, we have a genus one fibration with a half-fiber F_3 with $[F_3] = f_3$ and its bisection R'.

Suppose f_3 is not nef. Let $f_3 = [F_3]$, where F_3 is an effective divisor. Then, there exists a (-2)-curve R'' such that $F_3 \cdot R'' < 0$. This implies that R'' is an irreducible component of F_3 and $(F_3 - R'')^2 = -2F_3 \cdot R'' - 2 \geq 0$. Since $D = F_3 - R''$ is effective, intersecting it with F_1 we obtain $0 \leq D \cdot F_1 = 1 - R'' \cdot F_1$ and hence $0 \leq R'' \cdot F_1 \leq 1$. If $F_1 \cdot R'' = 1$, we have found a special bisection of the pencil $|2F_1|$. So, we obtain $F_1 \cdot R'' = 0$ and similarly, we get $F_2 \cdot R'' = 0$. Thus, the divisor class of any (-2)-curve R'' with $R'' \cdot F_3 < 0$ belongs to the orthogonal complement $U^\perp$. Applying the reflections in the divisor classes of such curves, we may assume that F_3 is nef and r' is the sum $\mathcal{R}$ of the divisor classes of (-2)-curves in $U^\perp$. The (-2)-curves from the support of $\mathcal{R}$ intersect F_1 with multiplicity 0, and hence they are components of some fibers of $|2F_1|$. Since $F_1 \cdot F_3 = 1$, we see that $F_3 \cdot R' = 1$, hence R' is a bisection of a genus one pencil $|2F_3|$, and we are done. □

Remark 6.3.4 It is not true that any (-2)-curve is a special bisection of some genus one fibration. For example, in Figure 8.1 that gives the intersection graph of all (-2)-curves on a surface S with finite automorphism group, the curves R_2, R_4, R_6, R_8 are not bisections of any genus one fibration. We do not know examples of surfaces with infinitely many (-2)-curves such that neither of them is realized as a special bisection of a genus one fibration.

Corollary 6.3.5 *Let S be a nodal Enriques surface. Then:*

(i) S contains a genus one fibration with a reducible fiber,
(ii) S contains a genus one fibration with a special bisection,

(iii) S admits a special bielliptic linear system,
(iv) if S is not an extra-special of type $\tilde{E}_8$, there exists a non-special bielliptic map $S \to \mathsf{D}$ that blows down a (-2)-curve to a point.

Recall from Section 5.7 in Volume I that a choice of a nef and big divisor class H (resp. numerical class h) on a smooth projective algebraic surface X is called a *polarization* (resp. numerical polarization). The number h^2 is called the *degree* of the polarization. The pair (X, H) (resp. (X, h)) is called a polarized (resp. numericallly polarized) algebraic surface.

Let h be a numerical polarization and let $\mathcal{R}_S$ be the set of (-2)-curves on S. We set

$$\delta(h) = \min\{h \cdot R \ : \ R \in \mathcal{R}_S\}. \tag{6.3.1}$$

Clearly, $\delta(h) > 0$ if and only if h is ample.

We refer for the proof of the following lemma to [136, Theorem 1.5].

Lemma 6.3.6 *Let α be a root in the Enriques lattice E_{10} and let x be a vector in E_{10} with $x^2 > 0$. Then, there exists an isotropic vector f such that $f \cdot \alpha = 0$ and*

$$x \cdot f < x \cdot \alpha, \quad \text{or} \quad x \cdot \alpha \leq \frac{3x^2}{\Phi(x)},$$

where $\Phi(x) = \min\{|x \cdot g| \ : \ g^2 = 0\}$.

Note that, by Hodge Index Theorem, $\Phi(x) > 0$. Also, note that the assertion is true for x if and only if it is true for $-x$.

Theorem 6.3.7 *Let (S, h) be a numerically polarized nodal Enriques surface. Assume that $\Phi(h) \geq 3$. Then,*

$$0 \leq \delta(h) \leq 3h^2/\Phi(h) \leq h^2.$$

Proof Let R_0 be a (-2)-curve on S. Assume

$$R_0 \cdot h > 3h^2/\Phi(h). \tag{6.3.2}$$

Applying the previous lemma, we find an effective isotropic divisor class f such that $R_0 \cdot f = 0$ and $h \cdot f < R_0 \cdot h$. Obviously, we may assume that f is primitive and $f = [D]$, where $D > 0$. If R_0 is an irreducible component of D, then $h \cdot R_0 \leq f \cdot h$, and hence Lemma 6.3.6 gives the inequality that contradicts (6.3.2). Thus, R_0 is not an irreducible component of D. Suppose D is not nef. Let R be a (-2)-curve such that $R \cdot D < 0$. Obviously, R is an irreducible component of D. Since R_0 is not an irreducible component of D and $D \cdot R_0 = 0$, we have $R \cdot R_0 = 0$. Next, we apply reflections with respect to curves R. We have

$$s_R(f) \cdot h = (f + (R \cdot f)R) \cdot h = f \cdot h + (R \cdot f)(R \cdot h) \leq f \cdot h < R_0 \cdot h,$$

$$s_R(f) \cdot R_0 = (f + (R \cdot f)R) \cdot R_0 = f \cdot R_0.$$

Replacing f with $s_R(f)$, and continue in this way, we may assume that f is nef. Thus, $f = [F]$, where $|2F|$ is a genus one pencil. Since $R_0 \cdot D = R_0 \cdot R = 0$, we did not use R_0 in the process of applying reflections s_R. Thus, $R_0 \cdot F = 0$, and hence R_0 is an irreducible component of some fiber D' of $|2F|$. Let R' be another irreducible component of D. We have

$$h \cdot R' \leq h \cdot (D' - R_0) = h \cdot D' - h \cdot R_0 < 2h \cdot f - h \cdot R_0 < h \cdot f < R_0 \cdot h.$$

Continuing in this way, we obtain that $\delta(h) \leq \frac{3h^2}{\Phi(h)}$. The assertion has been proven.□

Let H_{10} be a Fano polarization on S and let $h_{10} = [H_{10}]$ be its numerical class, we can write

$$3h_{10} = f_1 + \cdots + f_{10}$$

for some canonical isotropic 10-sequence (see (3.5.4)).

The next result shows that the bound from the previous theorem is not the best one.

Theorem 6.3.8 *Let h_{10} be a numerical Fano polarization of a nodal Enriques surface. Then, $\delta(h_{10}) \leq 4$.*

Proof Assume that $R \cdot h_{10} > 4$ for any (-2)-curve R on S. In particular, h_{10} is ample and, by Proposition 6.1.6, each f_i is nef. Thus, $(f_1, \ldots, f_{10})$ is a non-degenerate canonical isotropic sequence.

Let us show that for every $w \in W(\mathrm{Num}(S))$, $w(h_{10})$ is ample and $\delta(w(h_{10})) > 4$. The proof is by induction on the length $\lg(w)$ of w as an element of the Coxeter group $(W(\mathrm{Num}(S)), B)$, where B is the set of reflections corresponding to the root basis formed by $\alpha_0 = h_{10} - f_1 - f_2 - f_3, \alpha_i = f_i - f_{i+1}$. If $\lg(w) = 1$, then $w = s_{\alpha_i}$. If $i > 0$, we have $\alpha_i \cdot h_{10} = 0$, hence $w(h_{10}) = h_{10}$. If $i = 0$, then

$$3w(h_{10}) = w(f_1) + \cdots + w(f_{10}) = f_{23} + f_{13} + f_{12} + f_4 + \cdots + f_{10},$$

where $f_{ij} = h_{10} - f_i - f_j$ are isotropic classes with $h_{10} \cdot f_{ij} = 4$. If f_{ij} is not nef, then $f_{ij} - R' \geq 0$ for some (-2)-curve R', hence $h_{10} \cdot R' \leq 4$, contradicting our assumption.

Thus, $(f'_1, \ldots, f'_{10}) = (w(f_1), \ldots, w(f_{10}))$ is a non-degenerate canonical isotropic sequence. If R is a (-2)-curve with $f'_i \cdot R = 0$, then R is a component of a fiber F of a genus one fibration $|2F'_i|$ with $[F'_i] = f'_i$, hence $2f'_i - R - R' \geq 0$ for some (-2)-curve R'. Since $h_{10} \cdot f'_i = w(h_{10}) \cdot f_i = (2h - f_1 - f_2 - f_3) \cdot f_i \leq 4$, this implies that $h_{10} \cdot R$ or $h_{10} \cdot R'$ is less than or equal to 4, contradicting our assumption. Therefore, $f'_i \cdot R \geq 1$ for all i, hence $w(h_{10}) \cdot R \geq 10/3$. Thus, $\delta(w(h_{10})) \geq 4$. If $\delta(w(h_{10})) > 4$, we have proved our claim. Suppose the equality holds. Then, $12 = R \cdot 3w(h_{10}) = R \cdot (f'_1 + \cdots + f'_{10})$ implies that there exists f'_i such that $f'_i \cdot R = 3$ and $f'_j \cdot R = 1$ for $j \neq i$, or $f'_i \cdot R = f'_j \cdot R = 2$ and $f'_k \cdot R = 1$ for $k \neq i, j$.

Suppose $f_i' \cdot R = 3$. Then, $(w(h_{10}) - 2w(f_i)) \cdot w(f_i) = R \cdot w(f_i) = 3$, and $(w(h_{10}) - 2w(f_i)) \cdot w(f_j) = R \cdot w(f_j) = 1$. Since $w(f_1), \dots, w(f_{10})$ generate $\mathrm{Num}(S) \otimes \mathbb{Q}$, we obtain that the numerical class $[R]$ of R is equal to $w(h_{10}) - 2w(f_i)$.

If $i > 3$, then $w(f_i) = f_i$ and

$$R \cdot f_{1i} = R \cdot (h_{10} - f_1 - f_i) = (w(h_{10}) - 2f_i) \cdot (h_{10} - f_1 - f_i)$$

$$= (2h_{10} - f_1 - f_2 - f_3 - 2f_i) \cdot (h_{10} - f_1 - f_i) = 0.$$

This gives $2h_{10} - 2f_1 - 2f_i - R = R' \geq 0$ and implies that $h_{10} \cdot R' \leq 8 - h_{10} \cdot R \leq 3$, contradicting $\delta(h_{10}) > 4$. If $i \leq 3$, say $i = 1$, then $w(f_1) = h_{10} - f_2 - f_3$ and we can repeat the argument by taking f_2 instead of f_{1i}.

Suppose $f_i' \cdot R = f_j' \cdot R = 2$. Again, by comparing the intersection numbers, we find that

$$R = w(h_{10}) - w(f_i) - w(f_j).$$

However, this implies that $R^2 = 0$, a contradiction. So, we have checked that $\delta(w(h_{10})) > 4$ if $\lg(w) = 1$. Applying the induction, we obtain that $w(h_{10}) \cdot R > 4$ for all $w \in W(\mathrm{Num}(S))$ and $R \in \mathcal{R}_S$. By taking $w = s_R$, we obtain $w(h_{10}) \cdot R = h_{10} \cdot w(R) = -h_{10} \cdot R < 0$, which is absurd. □

Recall that the linear space over $\mathbb{F}_2$

$$\mathsf{E}_{10}/2\mathsf{E}_{10} \cong \mathsf{U}/2\mathsf{U} \oplus \mathsf{E}_8/2\mathsf{E}_8 \cong \mathbb{F}_2^{10}$$

is equipped with the induced quadratic form which is a non-degenerate quadratic form of even type. It is known that it contains exactly $496 = 2^4(2^5 - 1)$ vectors of norm 1. These vectors are the cosets of vectors $v \in \mathsf{E}_{10}$ with $v^2 = -2$.

Corollary 6.3.9 *Let h_{10} be a numerical Fano polarization on a nodal Enriques surface S. Write $3h_{10} = f_1 + \cdots + f_{10}$ for some canonical isotropic sequence.*

Then, S contains a (-2)-curve R such that its class in $\mathrm{Num}(S)$ is equal to the one of the following classes representing 496 cosets in $\mathrm{Num}(S)/2\,\mathrm{Num}(S)$ of roots in $\mathrm{Num}(S)$:

$$\begin{aligned}
&R \cdot h_{10} = 0 : f_j - f_i,\ 1 \leq i < j \leq 10, \\
&R \cdot h_{10} = 1 : h_{10} - f_i - f_j - f_k,\ 1 \leq i < j < k \leq 10, \\
&R \cdot h_{10} = 2 : f_i + f_j + f_k + f_l - h_{10},\ 1 \leq i < j < k < l \leq 10, \\
&R \cdot h_{10} = 3 : f_i + f_j - f_k,\ 1 \leq i < j < k \leq 10, \\
&R \cdot h_{10} = 4 : h_{10} - 2f_1.
\end{aligned} \tag{6.3.3}$$

Proof By the previous theorem, we can find a (-2)-curve R such that $h_{10} \cdot R \leq 4$. Assume $R \cdot h_{10} = 0$. Then, $R \cdot f_i \neq 0$ for some f_i. It follows from the description of a canonical isotropic sequence that we may assume that $f_i = f_j + R_1 + \cdots + R_{i-j}$, where $R_{i-j} = R$ and f_j is nef. Thus, $f_{i-1} = f_j + R_1 + \cdots + R_{i-j-1}$ and $R \sim f_i - f_{i-1}$.

Let $G = (f_i \cdot f_j)$ be the Gram matrix of $(f_1, \dots, f_{10})$. Its inverse G^{-1} is equal to $\frac{1}{9}G$ except the diagonal elements are changed to $-8/9$.

Assume $d = h_{10} \cdot R > 0$. Let $m_i = f_i \cdot R$, where we may assume that $m_1 \geq m_2 \geq m_3 \geq \cdots \geq m_{10}$ with

$$3d = m_1 + \cdots + m_{10}.$$

Since $R \cdot f_i < 0$ implies that $3h_{10} \cdot R = (f_{i-1} + f_i) \cdot R = 1 - 1 = 0$, we may assume that all $m_i \geq 0$. Write $R = \sum_{i=1}^{10} x_i f_i$ with $x_i \in \mathbb{Q}$. Then, the vector ${}^t x = (x_1, \ldots, x_{10})$ is equal to the product of $G^{-1} \cdot v$, where ${}^t v = (m_1, \ldots, m_{10})$.

Assume $d = 1$. Then, $m_1 + m_2 + m_3 = 3$. If $m_1 = m_2 = m_3 = 1$,

$$R = -\frac{2}{3}(f_1 + f_2 + f_3) + \frac{1}{3}(f_4 + \cdots + f_{10}) = h_{10} - f_1 - f_2 - f_3,$$

which agrees with the assertion. If $(m_1, m_2, m_3) = (2, 1, 0)$, we get, in the same way, that $R = h_{10} - 2f_1 - f_2$ but $R^2 = -4$ in this case, so this case is not realized. If $(m_1, m_2, m_3) = (3, 0, 0)$, then $R = h_{10} - 3f_1$ and again $R^2 \neq -2$.

Assume $d = 2$. We have $m_1 + \cdots + m_6 = 6$. If $(m_1, \ldots, m_6) = (1, 1, 1, 1, 1, 1)$, we get $R = -\frac{1}{3}(f_1 + \cdots + f_6) + \frac{2}{3}(f_7 + f_8 + f_9 + f_{10}) = -h_{10} + f_7 + f_8 + f_9 + f_{10}$, again in agreement with the assertion. All other possibilities lead to the wrong value of R^2.

Assume $d = 3$. Then, we have $m_1 + \cdots + m_{10} = 9$. If we take $(m_1, \ldots, m_{10}) = (2, 1, 1, 1, 1, 1, 1, 1, 0, 0)$, we obtain $R = -f_1 + f_9 + f_{10}$ which agrees with the assertion. Other choices are discarded (for example, $(m_1, \ldots, m_{10}) = (1, 1, 1, 1, 1, 1, 1, 1, 1, 0)$ gives $R = f_{10}$).

Assume $d = 4$. Then, $m_{10} > 0$, since otherwise R is an irreducible component of an effective divisor D with $[D] = 2f_{10}$, and $h_{10} \cdot D = 2h_{10} \cdot f_{10} = 6$ implies that $R' = D - R$ is an effective divisor with $h_{10} \cdot R' = 2, R'^2 = -2$. Thus, R' is congruent mod $2\,\mathrm{Num}(S)$ to a sum of (-2)-curves that intersect h_{10} with multiplicities ≤ 2. The previous computations show that $R = -R' + 2f_{10}$ is congruent mod $2\,\mathrm{Num}(S)$ to one of the previous vectors, giving a contradiction. This leaves us with two possible cases: $(m_1, \ldots, m_{10}) = (3, 1, \ldots, 1)$ or $(2, 2, 1, \ldots, 1)$. In the first case, $x = (-\frac{5}{3}, \frac{1}{3}, \ldots, \frac{1}{3})$, hence $R = h_{10} - 2f_1$ which agrees with the assertion. The second case gives $R = -\frac{2}{3}(f_1 + f_2) + \frac{1}{3}(f_3 + \ldots + f_{10}) = h_{10} - f_1 - f_2$ with $R^2 = 0$, a contradiction. □

For any numerical Fano polarization h_{10}, we denote by $\Pi_{h_{10}}$ the set of cosets of $\mathrm{Num}(S)/2\,\mathrm{Num}(S)$ represented by effective classes R with $R^2 = -2, R \cdot h_{10} \leq 4$. The previous corollary gives a choice of possible representatives of $\Pi_{h_{10}}$.

Proposition 6.3.10 *Let $\Pi_{h_{10}}^{\mathrm{nod}}$ be the subset of $\Pi_{h_{10}}$ represented by smooth rational curves. Then, these curves form a root basis whose Dynkin diagram Γ has only simple or double edges.*

Proof Under the map $S \to \mathbb{P}^5$ defined by the Fano polarization h_{10}, each (-2) curve is either contracted, or is mapped to a rational curve R_i of degree ≤ 4. It follows from the description of the set $\Pi_{h_{10}}$ that $|x \cdot y| \leq 2$ for any two distinct $x, y \in \Pi_{h_{10}}$. This proves the assertion. □

Remark 6.3.11 In fact, one observes that, if $|x \cdot y| = |x \cdot y'| = 2$ then $y = y'$. Also, if $|x \cdot y| = 2$, then $|h_{10} \cdot x| + |h_{10} \cdot y| \leq 4$.

Now, we will give another (much simpler) proof of Theorem 6.3.8 and its Corollary that works if $p \neq 2$ or the canonical cover is birationally isomorphic to a K3-surface. It is based on the following lemma communicated to us by E. Looijenga.

Lemma 6.3.12 *Assume the canonical cover $X \to S$ of an Enriques surface S is birationally isomorphic to a K3 surface. Let R be a (-2)-curve on S, H an ample divisor, $\alpha \in \mathrm{Num}(S)$ with $\alpha^2 = -2, \alpha \cdot H > 0$ and $R - \alpha \in 2\,\mathrm{Num}(S)$. Then, α is the numerical class of an effective divisor.*

Proof Let $\alpha = R + 2x$ for some divisor class x. Since $\mathcal{O}_R(K_S) \cong \mathcal{O}_R$, we obtain that the canonical cover splits over R.

Assume first that $\pi : X \to S$ is separable; this means that $\pi^*(R) = R_1 + R_2$, where R_1, R_2 are disjoint (-2)-curves on X. We have

$$\pi^*(\alpha) = \pi^*(R) + 2\pi^*(x) = R_1 + R_2 + 2\pi^*(x) = (R_1 + \pi^*(x)) + (R_2 + \pi^*(x)).$$

Using that

$$-4 = \pi^*(\alpha)^2 = 2(R_1 + \pi^*(x))^2 + 2(R_1 + \pi^*(x)) \cdot (R_2 + \pi^*(x))$$

$$= 2(R_1 + \pi^*(x))^2 + 2(2R_1 \cdot \pi^*(x) + \pi^*(x)^2) = 4(R_1 + \pi^*(x))^2 + 4,$$

we obtain

$$(R_i + \pi^*(x))^2 = -2, \ (R_1 + \pi^*(x)) \cdot (R_2 + \pi^*(x)) = 0.$$

Since

$$(R_1 + \pi^*(x)) \cdot \pi^*(H) = \frac{1}{2}\pi^*(\alpha) \cdot \pi^*(H) = \alpha \cdot H > 0,$$

applying Riemann–Roch, we obtain that $R_i + \pi^*(x)$ is linearly equivalent to an effective divisor R_i', $i = 1, 2$. Thus, $\pi^*(\alpha)$ is represented by the sum of two effective divisors which are permuted by the cover involution. This immediately implies that α is linearly equivalent to an effective divisor D on S equal to the image of the curve R_i' on S.

Next, we assume that π is inseparable. Then, $\pi^*(R) = 2R'$ for some curve R' isomorphic to R. Let $\sigma : X' \to X$ be the minimal resolution of singularities. Let $x' = \pi^*(\alpha)$; we have $\pi^*(\alpha) = 2(R' + x')$ and $\sigma^*(\pi^*(\alpha)) = 2\bar{R}' + 2\sigma^*(x') + \mathcal{R}$, where $\mathcal{R}$ is the exceptional divisors over the points lying on R. We will see later in Lemma 10.2.9 that there are two such points and $\mathcal{R}^2 = -4$ and $\bar{R}' \cdot \mathcal{R} = 2$ (see Example 10.6.8). This gives $-4 = 4(\bar{R}' + \sigma^*(x'))^2 + 4$, hence $(\bar{R}' + \sigma^*(x'))^2 = -2$. As in the separable case, we check that $\bar{R}' + \sigma^*(x')$ is an effective divisor and its image on S is linearly equivalent to α. □

The proof of Theorem 6.3.8 and Corollary 6.3.9 is now easy. The coset $R + 2\,\mathrm{Num}(S)$ contains a class from the list (6.3.3). By the previous lemma, it is an effective class. Since $\alpha^2 = -2$, one of the irreducible components of an effective representative of α is (-2)-curve R'. Since $\alpha \cdot h_{10} \leq 4$, we obtain that $R' \cdot h_{10} \leq 4$.

Lemma 6.3.13 *Let $(|2F_1|, |2F_2|)$ be a non-degenerate U-pair on a nodal Enriques surface S. There exists a (-2)- curve R with $F_1 \cdot R \leq 1, F_2 \cdot R \leq 1$.*

Proof We extend $f_1 = [F_1]$, $f_2 = [F_2]$ to a canonical c-degenerate maximal isotropic sequence $(f_1, f_2, \dots, f_{10})$. If $c < 10$, there exists a (-2)-curve R satisfying $R \cdot f_1 = R \cdot f_2 = 0$, hence R satisfies the assertion. Thus, we may assume that the sequence is non-degenerate and $3h_{10} = f_1 + \cdots + f_{10}$ is a numerical Fano polarization. By Theorem 6.3.8, we can find a (-2)-curve R with $0 < d = h_{10} \cdot R \leq 4$ and we can apply Corollary 6.3.9.

If $h_{10} \cdot R = 1$, then we can choose R with $[R] = h_{10} - f_i - f_j - f_k$ and $R \cdot f_s \in \{0, 1\}$, so we are done.

If $h_{10} \cdot R = 2$, then we can choose R with $[R] = f_i + f_j + f_k + f_l - h_{10}$ and $R \cdot f_s \in \{0, 1\}$ again.

If $d = 3$, then $[R] = f_i + f_j - f_k$ and, for any s, $R \cdot f_s \in \{0, 1\}$ unless $s = k$ in which case $R \cdot f_k = 2$. We have $m = R \cdot (f_1 + f_2) \leq 3$. If $m \leq 1$, we are done. Suppose $m = 2$ and the assertions of the lemma are not satisfied. Without loss of generality, we may assume that $R \cdot f_1 = 2, R \cdot f_2 = 0$. Then, R is an irreducible component of some divisor D with $[D] = 2f_2$. It follows that $f_1 \cdot D = 2$ and any other irreducible component R' of D does not intersect f_1. Hence, $R \cdot f_1 = R \cdot f_2 = 0$, and we are done. If $m = 3$, we may assume that $[R] = f_3 + f_4 - f_1$ so that $R \cdot f_1 = 2$ and $R \cdot f_2 = 1$. It follows that $R \cdot f_3 = 0$, and we can choose an irreducible component R' of a divisor D with $[D] = 2f_3$ with $f_1 \cdot R' = 0, f_2 \cdot R_1 \leq 1$.

Finally, if $R \cdot h_{10} = 4$, we can choose R with $[R] = h_{10} - 2f_i$. If $i > 2$, then $R \cdot f_1 = R \cdot f_2 = 1$, and we are done. So, we may assume that $[R] = h_{10} - 2f_1$ in which case $R \cdot f_1 = 3,\ R \cdot f_j = 1,\ j \neq 1$.

Let $f_{1,3} = h_{10} - f_1 - f_3$. It satisfies $f_{1,3}^2 = f_{1,3} \cdot R = 0$. Write an effective divisor D' representing $f_{1,3}$ as a sum of a nef divisor F and some (-2)-curves. Since $2 = f_1 \cdot D' < f_1 \cdot R = 3$, we see that R is not an irreducible component of D'. Thus, $R \cdot F = 0$ and hence R is an irreducible component of an effective divisor $D'' \in |2F|$. Let R' be another irreducible component of D''. It is a (-2)-curve that satisfies $R' \cdot f_1 \leq (D'' - R) \cdot f_1 \leq 4 - 3 = 1$ and $R' \cdot f_2 \leq D'' \cdot f_2 - R \cdot f_2 \leq 2 - 1 = 1$. So, R' satisfies our assertion. □

Corollary 6.3.14 *Let (f_1, f_2, f_3) be a non-degenerate canonical sequence on a nodal Enriques surface. Then, there exists a (-2)-curve such that*

$$f_1 \cdot R \leq 2,\ f_2 \cdot R \leq 1,\ f_3 \cdot R \leq 1.$$

Proof Let $3h_{10} = f_1 + f_2 + f_3 + \cdots + f_{10}$ be as in the proof of the previous lemma and let R be a (-2)-curve with $d = h_{10} \cdot R \leq 4$ such that $R \cdot f_2 \leq 1, R \cdot f_3 \leq 1$. Assume that $f_1 \cdot R \geq 3$. As in the proof of the lemma, we may assume that $d > 0$ and $(f_1, \dots, f_{10})$ is a non-degenerate isotropic sequence.

Assume $d = 3$, then $9 \geq 3 + \sum_{i=2}^{10} R \cdot f_i$ implies that $R \cdot f_i = 0$ for some $i \geq 4$. Thus, R is an irreducible component of an effective divisor $D \in |2F_i|$, where $[F_i] = f_i$. Then, $3 \leq R \cdot f_1 \leq 2f_i \cdot f_1 = 2$ which is a contradiction.

Assume $d = 4$. By the above argument, we may assume that $R \cdot f_i \geq 1$ for any i. Since, $12 = (f_1 + \cdots + f_{10}) \cdot R$, we get $R \cdot f_1 = 3$. In the Fano model, the numerical class $f = h_{10} - f_1 - f_4$ represents a curve F of degree 4 and arithmetic genus one. For any nef primitive isotropic class $f' \neq f_i, i = 1, \dots, f_{10}$, we have $h_{10} \cdot f' \geq 4$,

showing that the numerical class f is nef. Since $R \cdot f = 4 - 3 - 1 = 0$, there exists a divisor D with $[D] = 2f$ such that R is its irreducible component. For any other irreducible component R' of D, we have $R' \cdot f_1 \leq 2f \cdot f_1 - R \cdot f_1 = 1, R' \cdot f_i \leq 2f \cdot f_i - R \cdot f_i = 1, i = 2, 3$. So, R' satisfies the assertion. □

Lemma 6.3.15 *Let $|2F_1|, |2F_2|$ be two genus one pencils on a nodal Enriques surface with $F_1 \cdot F_2 = 2$. Then, there exists a (-2)-curve R such that $F_1 \cdot R \leq 2,\ F_2 \cdot R \leq 1$.*

Proof Let $f_1 = [F_1], f_2 = [F_2]$. The divisor class $v = f_1 + f_2 \in \mathrm{Num}(S)$ satisfies $v^2 = 4, \Phi(v) = 2$. By Corollary 1.5.4, $v = h - f$ with $\Phi(h) = 3, h^2 = 10, f^2 = 0$ and $h \cdot f = 3$. If h is not effective, intersecting $h - f$ with $-h$, we get a contradiction. Similarly, intersecting $h - f$ with $-f$, we see that f is effective.

Suppose h and f are both nef devisors. By Theorem 6.3.8, we can find a (-2)-curve R with $R \cdot h \leq 4$. Then $(f_1 + f_2) \cdot R \leq h \cdot R - f \cdot R \leq 4$. If the inequality is strict, we are done unless we may assume that $f_1 \cdot R = 3,\ f_2 \cdot R = 0$. In this case R is an irreducible component of some $D \in |2F_2|$, hence there exists an irreducible component R' of D such that $f_1 \cdot R' \leq 2f_2 \cdot f_1 - R \cdot f_1 = 1$, and we are done.

Suppose that $(f_1 + f_2) \cdot R = 4$. Then, $f \cdot R = 0$ and R is an irreducible component of a divisor $D \in |2F|$, where $[F] = f$, and we can find another component R' that satisfies $h \cdot R' \leq 2h \cdot f - 4 = 6 - 4 = 2$, and we are done again.

Let h_0 (resp. f_0) be a nef divisor in the same W_S^{nod}-orbit of h (resp. f). We can write $h = h_0 + \mathcal{R}_1$ and $f = f_0 + \mathcal{R}_2$, where $\mathcal{R}_1, \mathcal{R}_2$ are the zero divisors or sums of (-2)-curves. This gives us $f_1 + f_2 = h_0 - f_0 + \mathcal{R}_1' - \mathcal{R}_2'$, where $\mathcal{R}_i' \subset \mathcal{R}_i$ have no common components. Suppose there is a (-2)-curve R such that $R \cdot h < 0$. Then, $R \in \mathcal{R}_1'$ and $R \cdot (f_0 + \mathcal{R}_2') \geq 0$, thus $R \cdot (f_1 + f_2) < 0$, a contradiction. Thus, h is a nef divisor and $\mathcal{R}_1' = 0$.

Intersecting $f_1 + f_2 = h - f$ with f, we obtain $3 = h \cdot f = (f_1 + f_2) \cdot f$. Since f_1, f_2 are nef and f is effective with $f^2 = 0$, we have $f_i \cdot f \geq 0$. If $f_i \cdot f = 0$, then $f = f_i$ and we have $h^2 = (2f_1 + f_2)^2 = 8$, a contradiction. Thus, we may assume that $f_1 \cdot f = 1, f_2 \cdot f = 2$. If f is not nef, an irreducible component R of $\mathcal{R}_2'$ satisfies $f_1 \cdot R = 0, f_2 \cdot R \leq 1$, and we are done. Thus, both h and f must be nef, and, by the above, we are done. □

Theorem 6.3.16 *Let (S, h) be a numerically polarized nodal Enriques surface. Then,*

$$\delta(h) \leq h^2.$$

Proof If $\Phi(h) \geq 3$, this follows from Theorem 6.3.7. Assume $\Phi(h) = 1$. By Proposition 2.6.1, h is one of the following forms: $h = nf_1 + f_2$ for some nef isotropic classes f_1, f_2 with $f_1 \cdot f_2 = 1$, or $h = (n+1)f + R$ for some nef isotropic class f and a (-2)-curve R with $f \cdot R = 1$.

Note that in both cases $h^2 = 2n$. In the first case we apply Lemma 6.3.13 to find a (-2)-curve R such that $f_i \cdot R \leq 1, i = 1, 2$. Then $h \cdot R \leq n + 1 \leq 2n = h^2$. In the second case, $h \cdot R = n - 1 = \frac{1}{2}h^2 - 1 \leq h^2$.

Assume $\Phi(h) = 2$ and $h^2 = 4k$. We apply Proposition 2.6.6. In case (1), we have $h = kf_1 + 2f_2$, where f_i are as above. We choose R as above, and get $h \cdot R \leq k + 2 =$

$\frac{1}{4}h^2 + 2 \leq h^2$. In case (2), we have $h = (k+2)f + 2R$, where f, R are as above. We get $h \cdot R = k = \frac{1}{4}h^2 \leq h^2$. In case (3), $h = kf_1 + f_2$, where f_1, f_2 are nef isotropic classes with $f_1 \cdot f_2 = 2$. We apply Lemma 6.3.15 and find a (-2)-curve R such that $h \cdot R \leq 2k + 1 = \frac{1}{2}h^2 + 1 \leq h^2$. In cases (4)–(6), we take the nodal cycle R in the notation of Proposition 2.6.6 to obtain $h \cdot R = k - 1 \leq \frac{1}{4}h^2 - 1 \leq h^2$.

Assume $\Phi(h) = 2$ and $h^2 = 4k + 2$. We apply Proposition 2.6.7. In case (i), we have $h = kf_1 + f_2 + f_3$, where f_i are as in Lemma 6.3.14. We take R from the assertion of the lemma to obtain $h \cdot R \leq 2 + 2k = \frac{1}{2}h^2 + 1 \leq h^2$. In case (ii), there exists a curve R such that $h \cdot R = 0$. In cases (iii) and (iv), $R_1 \cdot h = k - 1 = \frac{1}{4}h^2 - \frac{3}{2}$.□

It follows from the proof that the bound $\delta(h) \leq h^2$ can be improved for polarizations h with $\Phi(h) < 3$.

Corollary 6.3.17 *Assume* $\Phi(h) \leq 2$. *Then*

$$\delta(h) \leq 1 + \frac{1}{2}h^2.$$

Remark 6.3.18 A long and elaborate proof of the following inequality:

$$\delta(h) \leq \frac{h^2}{\Phi(h)} + \frac{\Phi(h)}{3}$$

can be found in unpublished manuscripts of F. Cossec.

We will see later that a general nodal surface has infinitely many smooth rational curves. However, in a special case, the set of such curves could be finite. By Theorem 8.1.12 this happens if and only if the automorphism group of S is finite. We will classify all such surfaces in Sections 8.9 and 8.10.

6.4 Nodal Invariants

In this section, we introduce other invariants of nodal Enriques surfaces expressed in terms of quadratic lattices. In particular, we extend the notion of the Nikulin R-invariant of a complex Enriques surface to the case when the ground field is of arbitrary characteristic.

Let $\mathcal{R}_S$ be the set of (-2)-curves on an Enriques surface S. We assume that it is not empty. It spans a non-zero sublattice N_S of $\mathrm{Num}(S)$, which we will call the *nodal sublattice*. We define the *r-invariant* $\mathsf{Nod}(S)$ of S as the image of $\mathcal{R}_S$ in the vector space

$$\overline{\mathrm{Num}}(S) := \mathrm{Num}(S)/2\,\mathrm{Num}(S) \cong \mathsf{E}_{10}/2\mathsf{E}_{10} \cong \mathbb{F}_2^{10}.$$

The homomorphism of multiplication by 2

$$\mathsf{E}_{10}(2)^{\vee}/\mathsf{E}_{10}(2) \to \mathsf{E}_{10}/2\mathsf{E}_{10}$$

is an isomorphism from the discriminant group of the lattice $\mathsf{E}_{10}(2)$ to the additive group of the vector space $\overline{\mathrm{Num}}(S)$. Under this identification, the discriminant quadratic form from Section 0.6 in Volume I equips $\overline{\mathrm{Num}}(S)$ with a non-degenerate symmetric bilinear form defined by

$$(x + 2\,\mathrm{Num}(S)) \cdot (y + 2\,\mathrm{Num}(S)) = x \cdot y \mod 2$$

and the quadratic form q defined by

$$(x + 2\,\mathrm{Num}(S))^2 = \frac{1}{2}x^2 \mod 2.$$

Let

$$\langle \mathsf{Nod}(S) \rangle \subset \overline{\mathrm{Num}}(S)$$

be the linear span of $\mathsf{Nod}(S)$.

It is clear that

$$\mathsf{N}_S/2\,\mathrm{Num}(S) \cap \mathsf{N}_S \cong \langle \mathsf{Nod}(S) \rangle.$$

Obviously,

$$\mathsf{Nod}(S) \subset q^{-1}(1).$$

Choose a numerical Fano polarization h_{10} and let $\Pi^{\mathrm{nod}}_{h_{10}}$ be the set of elements in $\Pi_{h_{10}}$ represented by the classes of (-2)-curves modulo $2\,\mathrm{Num}(S)$. Let $\mathsf{N}^{h_{10}}_S$ be the sublattice spanned by these classes. By Proposition 6.3.10, the set $\Pi^{\mathrm{nod}}_{h_{10}}$ forms a root basis in $\mathsf{N}^{h_{10}}_S$ whose Dynkin diagram has only simple or double edges.

Since any element in N_S is congruent modulo $2\,\mathrm{Num}(S)$ to an element from $\mathsf{N}^{h_{10}}_S$, the image of $\mathsf{N}^{h_{10}}_S$ in $\overline{\mathrm{Num}}(S)$ spans the same subspace $\langle \mathsf{Nod}(S) \rangle$. We also see that the kernel

$$\mathsf{N}_0 := \mathrm{Ker}(\mathsf{N}_S/2\mathsf{N}_S \to \overline{\mathrm{Num}}(S)) = \mathsf{N}_S \cap 2\,\mathrm{Num}(S)/2\mathsf{N}_S$$

coincides with the kernel

$$\mathsf{G}_{h_{10}} := \mathrm{Ker}(\mathsf{N}^{h_{10}}_S/2\mathsf{N}^{h_{10}}_S \to \overline{\mathrm{Num}}(S)) = \mathsf{N}^{h_{10}}_S \cap 2\,\mathrm{Num}(S)/2\mathsf{N}^{h_{10}}_S.$$

Since any $g \in \mathsf{G}_{h_{10}}$ is represented by an element $2x$, where $x \in \mathrm{Num}(S)$, we have $x = \frac{1}{2}g \in (\mathsf{N}^{h_{10}}_S)^\vee$ and $x^2 \in 2\mathbb{Z}$, we see that $\mathsf{G}_{h_{10}}$ defines an isotropic subgroup of the 2-torsion group of $D(\mathsf{N}^{h_{10}}_S)$.

Definition 6.4.1 The pair $(\mathsf{N}^{h_{10}}_S, \mathsf{G}_{h_{10}})$ is called the *Fano root invariant* of S.

We believe that the Fano root invariant $(\mathsf{N}^{h_{10}}_S, \mathsf{G}_{h_{10}})$ does not depend on a Fano polarization. Next, we give a partial confirmation of this under the assumption that $p \neq 2$ or S is a $\boldsymbol{\mu}_2$-surface.

Let $\pi : X \to S$ be the K3-cover. Assume that it is an étale map, i.e. $p \neq 2$ or S is a $\boldsymbol{\mu}_2$-surface. Let σ be the covering involution and let

$$\mathrm{Pic}(X)^{\pm} := \{L \in \mathrm{Pic}(X) : \sigma^*(L) = L^{\otimes \pm 1}\}.$$

We have $\mathrm{Pic}(X)^+ = \pi^*(\mathrm{Pic}(S)) = \pi^*(\mathrm{Num}(S))$. Let

$$\Delta_{\pm} = \{\delta_{\pm} \in \mathrm{Pic}(X)^{\pm} : \delta_{\pm}^2 = -4, \text{there exists } \delta_{\mp} \in \mathrm{Pic}(X)^{\mp} \text{ such that } \delta_{\pm} + \delta_{\mp} \in 2\,\mathrm{Pic}(X)\}.$$

Since $(\delta_+ + \delta_-)^2 = -8$, $\frac{1}{2}(\delta_+ + \delta_-)^2 = -2$. Replacing δ_+, δ_- with $-\delta_+, -\delta_-$, we may assume that $D_+ = \frac{1}{2}(\delta_+ + \delta_-)$ and $D_- = \frac{1}{2}(\delta_+ - \delta_-)$ are effective divisors on X with $D_{\pm}^2 = -2$ and

$$D_+ + D_- = \pi^*(C),$$

where C is an effective divisor on S with $C^2 = -2$. Write $C = \sum n_i C_i + \sum m_i R_i$ as a sum of irreducible curves C_i of positive arithmetic genus with primitive numerical class $[C_i]$ and smooth rational curves R_i. The exact sequence

$$0 \to \mathcal{O}_S(K_S - C_i) \to \mathcal{O}_S(K_S) \to \mathcal{O}_{C_i}(K_S) \to 0$$

together with the Vanishing Theorem 2.1.16 from Volume I, shows that the étale cover π does not split over C_i. It splits over each R_i. Thus, we can write

$$\pi^*(C) = \sum n_i \pi^*(C_i) + \sum m_i \pi^*(R_i) = \sum n_i \pi^*(C_i) + \sum m_i (R_i^+ + R_i^-) = D_+ + D_-.$$

The only way to split the sum into the sum of two positive divisors which are conjugate with respect to the covering involution σ is to take $D_+ = \frac{1}{2}\sum n_i \pi^*(C_i) + \sum m_i R_i^+, D_- = \frac{1}{2}\sum n_i \pi^*(C_i) + \sum m_i R_i^-$. Thus, we obtain that $\delta_- = D_+ - D_- = \sum m_i (R_i^+ - R_i^-)$.

Let K' be the sublattice of $\mathrm{Pic}(X)^-$ spanned by the classes $\delta_- = R^+ - R^-$, where $R^{\pm}$ is an effective divisor class on X with $(R^{\pm})^2 = -2$ and $\sigma^*(R^+) = R^-$. Since $\delta_-^2 = -4$, the lattice $K = K'(\frac{1}{2})$ is a negative definite (because $\mathrm{Pic}(X)^+$ contains an ample divisor class) and generated by the vectors of norm -2. Thus, K is a root lattice. It follows from above that $K(2)$ is generated by the classes

$$\delta_R = R^+ - R^-, \tag{6.4.1}$$

where $\pi^*(R) = R^+ + R^-$ for some (-2)-curve R.

Let H be an isotropic subgroup of $\frac{1}{2}K'/K'$ such that it defines a sublattice $\tilde{K}'$ of $\mathrm{Pic}(X)^-$ that contains K'. Since $\mathrm{Pic}(X)^+$ contains an ample divisor, $\mathrm{Pic}(X)^-$ does not contain divisor classes of smooth rational curves, i.e. the lattice $\mathrm{Pic}(X)^-$ does not contain vectors of norm -2, hence $\tilde{K}'$ does not contain vectors of norm -2.

Under the isomorphism $\frac{1}{2}K'/K' = \frac{1}{2}K(2)/K(2) \cong K/2K$, we can consider H as an isotoropic subgroup of $K/2K$ with respect to the quadratic forms $\frac{1}{2}x^2 (x \in K)$ satisfying the condition that the lattice K_H generated by K and $\{x/2 : x \mod 2K \in H\}$ contains no (-1)-vectors.

Definition 6.4.2 The pair (K, H) is called the *Nikulin R-invariant* of S and is denoted by $\mathrm{Nik}(S)$.

Remark 6.4.3 Define a homomorphism

$$\phi : K/2K \to \mathrm{Num}(S)/2\,\mathrm{Num}(S) \tag{6.4.2}$$

by assigning to δ_- modulo $2K$ the class of δ_+ modulo $2\,\mathrm{Num}(S)$. Here, we use the identification $\mathrm{Pic}(X)^+/2\,\mathrm{Pic}(X)^+ \cong \mathrm{Num}(S)/2\,\mathrm{Num}(S)$. One can show that the homomorphism ϕ is well-defined and $H \cong \mathrm{Ker}(\phi)$.

Theorem 6.4.4 *Suppose* $\mathsf{N}_S^{h_{10}}$ *is negative definite. Then, it is isomorphic to the lattice* K.

Proof If R' is a (-2)-curve with $[R'] = [R+2D]$ for some divisor D on S, we have $[\pi^*(R')] = [R'^+ + R'^-] = [\pi^*(R) + 2\pi^*(D)] = [(R^+ + \pi^*(D))] + [(R^- + \pi^*(D))]$. It follows from the proof of Lemma 6.3.12, that $(R^+ + \pi^*(D))^2 = (R^- + \pi^*(D))^2 = -2$ and both classes $[R^+ + \pi^*(D)]$ and $[R^- + \pi^*(D)]$ are effective. Since $h^0(R'^+ + R'^-) = 1$, we must have $R'^+ \sim R^+ + \pi^*(D)$ or $R'^+ \sim R^- + \pi^*(D)$. This implies that $[R'^+ - R'^-] = \pm[R^+ - R^-]$. Thus, we see that the lattice K' is generated by the classes of $R^+ - R^-$, where $R \in \Pi_{h_{10}}^{\mathrm{nod}}$.

Let $R, R' \in \Pi_{h_{10}}^{\mathrm{nod}}$ with $R \cdot R' = 0$. Then, R^+, R^-, R'^+, R'^- are disjoint (-2)-curves and hence $\delta_R \cdot \delta_{R'} = 0$. Assume $\mathsf{N}_S^{h_{10}}$ is negative definite. Choose a root basis of $\mathsf{N}_S^{h_{10}}$ represented by (-2)-curves $R_1, \ldots, R_l$. Its Dynkin diagram Γ has no loops and we may assume that it is connected. This implies that the canonical cover splits over $R = \sum R_i$. We may assume that $\pi^*(R) = R^+ + R^-$, where $R^+ = \sum R_i^+, R^- = \sum R_i^-$ are connected with the Dynkin graph isomorphic to Γ. This implies that the matrix $(\delta_{R_i} \cdot \delta_{R_j}) = 2C$, where C is the Cartan matrix of Γ. Thus, $\mathsf{N}_S^{h_{10}} \cong K'(\frac{1}{2}) = K$. □

Recall some terminology and facts about quadratic forms over a field of two elements. A quadratic form $q : V \to \mathbb{F}_2$ on a linear space V of dimension n over $\mathbb{F}_2$ defines an associated symmetric bilinear form $b_q(x, y) = q(x + y) + q(x) + q(y)$. It is always a symplectic form, i.e. satisfies $b_q(x, x) = 0$. The *kernel* of q is the subspace $\ker(q)$ of the radical of b_q on which q vanishes. A quadratic form is *regular* if its kernel is trivial. Geometrically this means that the quadric $V(q)$ is regular over $\mathbb{F}_2$. The *rank* of q is $\mathrm{rank}(q) := n - \dim\ker(q)$ and the *rank* of b_q is $\mathrm{rank}(b_q) := n - \dim\mathrm{rad}(b_q)$. Since $q|\,\mathrm{rad}(b_q)$ is a semi-linear map to $\mathbb{F}_2$, we have $\dim\mathrm{rad}(b_q) - \dim\ker(q) \le 1$. This difference is called the *defect* of q and the quadratic form is called *defective* if it is not equal to zero. Since b_q is alternating, q is regular if and only if its radical is trivial. In this case, b_q is a symplectic form. Otherwise, we have $\dim\mathrm{rad}(b_q) \ge 2$. However, when n is odd, $\mathrm{rad}(b_q)$ is always non-trivial, and q is regular if and only if it is defective and $\dim\mathrm{rad}(b_q) = 1$.

A linear subspace of V is called *isotropic* if the restriction of q to this subspace is identically zero. A vector is isotropic if it spans an isotropic line. The *Witt index* of q is the maximal dimension of an isotropic subspace. It is equal to $\frac{1}{2}(r-1)$ if $r = \mathrm{rank}(b_q)$ is odd and it is equal to $\frac{1}{2}r$ or $\frac{1}{2}(r-2)$ if r is even. A quadratic form q of even rank r and the Witt index equal to $\frac{1}{2}r$ (resp. $\frac{1}{2}(r-2)$) is said to be of *even type* (resp. *odd type*).

After an appropriate choice of a basis $(e_1, \dots, e_n)$ in V such that the radical of V is spanned by the last $n - 2m$ vectors, a quadratic form with defect δ can be written in one of the following forms:

$$1.\ q(\sum x_i e_i) = \sum_{i=1}^{m} x_i x_{m+i} + x_{2m+1}^2 \text{ if } \delta = 1; \tag{6.4.3}$$

$$2.\ q(\sum x_i e_i) = \sum_{i=1}^{m} x_i x_{m+i} \text{ if } \delta = 0;$$

$$3.\ q(\sum x_i e_i) = \sum_{i=1}^{m-1} x_i x_{m+i-1} + x_{2m-1}^2 + x_{2m-1}x_{2m} + x_{2m}^2 \text{ if } \delta = 0.$$

In the second (resp. third) case, q is even (resp. odd). We assign to it sign 1 (resp. -1) if it is even (resp. odd). Viewing the sign as an element ϵ of the group $\{+1, -1\} \cong \mathbb{Z}/2\mathbb{Z}$, the sign of the direct sum of quadratic forms (defined in a natural way) is equal to the product of the signs of the summands. A linear space equipped with a quadratic form is called a *quadratic space*.

We denote by $\mathrm{O}(V, q)$ or just $\mathrm{O}(V)$ the orthogonal group of a quadratic space (V, q). If q is regular and n is odd, $\mathrm{O}(V, q) \cong \mathrm{Sp}(V/\mathrm{rad}(V), b_q)$. If q is a regular quadratic form on $\mathbb{F}_2^n$ given by formula (6.4.3), we denote the orthogonal group by $\mathrm{Sp}(n-1, \mathbb{F}_2)$ if n is odd and by $\mathrm{O}^+(n, \mathbb{F}_2)$ (resp. $\mathrm{O}^-(n, \mathbb{F}_2)$) if n is even and q is even (resp. odd). Note that here, we differ from the terminology in [131], where this notation is reserved to the subgroup of index 2, the kernel of the spin norm. Our groups are denoted by $GO_n^{\pm}(q)$.

The type of a quadratic form of rank $r = 2m$ and defect δ on a quadratic space of dimension n can be recognized by the number $\#q^{-1}(1)$ of non-zero non-isotropic vectors (see [17, Chapter III, §6]).

$$\#q^{-1}(1) = \begin{cases} 0 & \text{if } q = 0, \\ 2^{n-1} & \text{if } \delta = 1, \\ 2^{n-m-1}(2^m - 1) & \text{if } q \text{ is even, } \delta = 0, \\ 2^{n-m-1}(2^m + 1) & \text{if } q \text{ is odd, } \delta = 0. \end{cases} \tag{6.4.4}$$

We will also use *Witt's Theorem* (see [279, Theorem 12.10]).

Theorem 6.4.5 *Let L_1 and L_2 be two isomorphic linear subspaces of a non-defective quadratic space V. Then, there exists an orthogonal transformation of V that maps L_1 onto L_2 and an orthogonal transformation that maps $L_1^{\perp}$ onto $L_2^{\perp}$.*

Let M be an even lattice of rank l and let $\bar{M} = M/2M$, considered as a vector space over $\mathbb{F}_2$. We equip $\bar{M}$ with the quadratic form defined by

$$q(x + 2M) = \frac{1}{2}x^2 \mod 2\mathbb{Z}.$$

The polar bilinear form b_q associated with q is defined by

$$b_q(x+2M, y+2M) = x \cdot y \mod 2\mathbb{Z}.$$

Lemma 6.4.6 *Let M be a root lattice and let (r, δ, ϵ) be the rank of b_q, defect and the sign of the quadratic form q on $\bar{M}$. Then:*

$$(r,\delta,\epsilon) = \begin{cases} (2m, \frac{1}{2}(1+(-1)^m), (-1)^{\frac{1}{2}m(m+1)}) & \text{if } M = \mathsf{A}_{2m+1}, \\ (2m, 0, (-1)^{\frac{1}{2}m(m+1)}) & \text{if } M = \mathsf{A}_{2m}, \\ (4m-2, 0, (-1)^m) & \text{if } M = \mathsf{D}_{4m}, \\ (4m, 1, (-1)^{(m+1)}) & \text{if } M = \mathsf{D}_{4m+2}, \\ (2m, 0, (-1)^{\frac{1}{2}m(m+1)}) & \text{if } M = \mathsf{D}_{2m+1}, \\ (6, 0, -1) & \text{if } M = \mathsf{E}_6, \\ (6, 1, 1) & \text{if } M = \mathsf{E}_7, \\ (8, 0, 1) & \text{if } M = \mathsf{E}_8. \end{cases}$$

Proof The rank of b_q is equal to $l - s$, where s is the order of the 2-torsion group of the discriminant group of M. The known discriminant groups given, for example, in Bourbaki's tables [88], check the assertion for the rank.

Let $M = \mathsf{A}_{2m+1}$ with the standard root basis $\alpha_1 = e_1 - e_2, \alpha_2 = e_2 - e_3, \ldots, \alpha_{2m+1} = e_{2m+1} - e_{2m+2}$, where $(e_1, \ldots, e_{2m+2})$ is the standard basis of $\mathsf{I}^{0,2m+2}$. The radical $\bar{M}^\perp$ of $\bar{M}$ is spanned by the vector $r = \alpha_1 + \alpha_3 + \cdots + \alpha_{2m+1} \mod 2M$. We have $q(r) = m + 1 \mod 2$. Thus, the defect δ is equal to $\frac{1}{2}(1 + (-1)^m)$. We check that $\bar{M}/\bar{M}^\perp$ decomposes into the direct sum of m subspaces $V_1, \ldots, V_m$, where V_i is spanned by the cosets of $\alpha_1 + \alpha_3 + \cdots + \alpha_{2i-1}$ and α_{2i}. The quadratic form on V_i is odd if i is odd and even otherwise. Thus, the sign of the quadratic form on $\bar{M}/\bar{M}^\perp$ is -1 if $m \equiv 1, 2 \mod 4$ and it is equal to 1 otherwise.

Suppose $M = \mathsf{A}_{2m}$. The bilinear form is non-degenerate and, as above, we find the decomposition $\bar{M} = V_1 \oplus \cdots \oplus V_m$ into the sum of quadratic forms with the signs indicated in the above paragraph.

If $M = \mathsf{D}_l$, we use our notation for the simple roots $\alpha_0, \alpha_1, \ldots, \alpha_{l-1}$, where α_2 is the tri-valent vertex in the Dynkin diagram and α_0, α_1 correspond to the short arms. The radical $\bar{M}^\perp$ of $\bar{M}$ is generated by the cosets of $\alpha_0 + \alpha_1$ if $l = 2m + 1$ and cosets of $\alpha_0 + \alpha_1, \alpha_1 + \alpha_3 + \cdots + \alpha_{l-1}$ if l is even. If l is odd, $\bar{M}$ is not defective. If $l = 2m$, then $\bar{M}$ is defective if and only if m is even. The quotient by the radical $\bar{M}/\bar{M}^\perp$ is spanned by the cosets of $\alpha_2, \ldots, \alpha_{l-1}$ if l is even and $\alpha_1, \ldots, \alpha_{l-1}$ if l is odd. If l is even (resp. odd) the quadratic space is isomorphic to $\bar{\mathsf{A}}_{l-2}$ (resp. $\bar{\mathsf{A}}_{l-1}$). Now, we can apply the previous case and find the sign of the reduced quadratic form.

Suppose $M = \mathsf{E}_6$. The quadratic form on $\bar{M}$ is regular of dimension 6. It contains 36 non-isotropic vectors, the cosets of positive roots. Comparing the known number of such vectors, we obtain that the form is odd.

If $M = \mathsf{E}_7$, the radical is spanned by the coset of the vector $r = \alpha_0 + \alpha_4 + \alpha_6$, where $\alpha_0, \ldots, \alpha_6$ are as in the diagram (6.1.1). Modulo the radical, $\bar{M} \cong \bar{\mathsf{E}}_6$. Since $q(r) = 1$, we obtain $(r, \delta, \epsilon) = (6, 1, 1)$. It is also confirmed by computing the number of positive roots which is equal to 63. Their cosets are non-isotropic vectors.

Finally, if $M = \mathsf{E}_8$, then the form is non-degenerate and contains 120 non-isotropic vectors, the cosets of positive roots. This shows that q is even. □

Using that $\mathsf{E}_{10} \cong \mathsf{E}_8 \oplus \mathsf{U}$, and $\mathsf{U}/2\mathsf{U}$ is an even quadratic space of rank 2, we obtain:

Corollary 6.4.7 *The quadratic form on* $\bar{\mathsf{E}}_{10} = \mathsf{E}_{10}/2\mathsf{E}_{10}$ *is a regular even form of rank 10.*

Let $(\mathsf{N}_S^{h_{10}}, \mathsf{G}_{h_{10}})$ be the Fano root invariant of an Enriques surface S. To determine the image of $\bar{\mathsf{N}}^{h_{10}} = \mathsf{N}_S^{h_{10}}/2\mathsf{N}_S^{h_{10}}$ in the quadratic space $\overline{\mathrm{Num}}(S)$, we write $\mathsf{N}_S^{h_{10}}$ as the direct sum of irreducible root lattices and use the previous lemma. If $\bar{\mathsf{N}}^{h_{10}}$ is a regular quadratic space, the group $\mathsf{G}_{h_{10}}$ is trivial. Otherwise, it is a subgroup of the radical of $\overline{\mathsf{N}}^{h_{10}}$. Its structure depends on the embedding of $\mathsf{N}_S^{h_{10}}$ in $\mathrm{Num}(S)$.

The following well-known fact from the theory of Coxeter groups (see [88, Chapter V, §4, Exercise 2(d)]) will be useful for us here and later.

Lemma 6.4.8 *Let H be a finite subgroup of a Coxeter group (W, S). Then, there exists a finite subset J of S such that the subgroup W_J generated by $s \in J$ is finite and H is conjugate to a subgroup of W_J.*

A subgroup W_J of a Coxeter group is called a *parabolic subgroup*. It is a Coxeter group (W_J, J) and its Coxeter graph is called the type of W_J. In our case (W, S) is the Weyl group $W(\mathsf{E}_{10})$ with the set of generators corresponding to the canonical root basis with the Coxeter–Dynkin diagram of type $T_{2,3,7}$. Finite parabolic subgroups of $W(\mathsf{E}_{10})$ with connected Coxeter–Dynkin diagrams are of types A_k, D_k, E_6, E_7, E_8.

Note that different subdiagrams of the same type in the Coxeter graph $T_{2,3,7}$ of $W_{2,3,7}$ may define non-conjugate subgroups. In general, there is an algorithm, due to V. Deodhar and R. Howlett (see [618]) that decides whether two subsets J, J' of the set S of Coxeter generators of a Coxeter group (W, S) are W-equivalent. In our case, it works as follows. For any $J \subset S$ generating a finite Coxeter group and $s \in S$, let us denote by $L = L(J, s)$ the connected component of $J \cup \{s\}$ that contains s. Let $A(J)$ be the set of elements $s \in S$ such that the Coxeter group W_L generated by $L(J, s)$ is of type $A_k, k > 1, D_{2k+1}$, or E_6. Let $s \in A(J)$ and $s' = w_0(s)$, where w_0 is the element of the maximal length (it acts as the symmetry of the diagram) in W_L. We set $K(s, J) = (J \cup \{s'\}) \setminus \{s\}$. We say that J and K are related by an *elementary operation*. Two subsets J, J' define W-equivalent subgroups $W_J, W_{J'}$ if and only if there is a sequence of elementary operations relating J and J'.

Example 6.4.9 In most cases, two parabolic subgroups W_J and $W_{J'}$ of the Weyl group W of an irreducible root lattice are conjugate in W if and only if the subdiagrams corresponding to the subsets J and J' are isomorphic. However, it is not always true.

For example, in notation of the Dynkin diagram from Figure 0.3 in Volume I, two non-equivalent subsets of simple roots in E_7 are $\{\alpha_1, \alpha_3, \alpha_5\}$ and $\{\alpha_0, \alpha_4, \alpha_6\}$. However, if we add one more root α_7 to get a root lattice of type E_8, the two subsets become equivalent.

On the other hand, if we consider the affine root lattice $\tilde{\mathsf{E}}_7$ with simple roots $\alpha_0, \dots, \alpha_7$, there will be still two non-equivalent subdiagrams of type $A_1 + A_1 + A_1$. They are defined by the subsets $\{\alpha_0, \alpha_5, \alpha_7\}$ and $\{\alpha_0, \alpha_1, \alpha_3\}$.

Any finite parabolic subgroup W_J of $W(\mathsf{E}_{10})$ is contained in a parabolic subgroup $W_{J'}$, where $\#J' = 9$. They are of the following types:

$$A_9,\ D_9,\ E_8{+}A_1,\ A_1{+}A_8,\ A_6{+}A_2{+}A_1,\ A_4{+}A_5,\ E_6{+}A_3,\ E_7{+}A_2,\ D_5{+}A_4. \quad (6.4.5)$$

A subgroup of a Coxeter group generated by reflections is a Coxeter group; however, it is not necessarily conjugate to a parabolic subgroup. For example, the lattice E_8 contains a sublattice isomorphic to $\mathsf{A}_1^{\oplus 8}$ but its Weyl group is not conjugate to a parabolic subgroup of $W(\mathsf{E}_8)$ because E_8 does not contain 8 orthogonal simple roots.

However, if W is the Weyl group of some root lattice, then any reflection subgroup of W is conjugate to the Weyl group of some root system of rank r (see, for example, [198]). The latter can be found using the following the *Borel–de Siebenthal–Dynkin algorithm* that can be derived from [85] or [202].

Let M be an irreducible root lattice with Dynkin diagram Γ. Extend it to the affine Dynkin diagram $\tilde{\Gamma}$ by adding the maximal root $\alpha_{\max}$. Then, throw away one vertex v different from the new one. The remaining diagram $\Gamma_v = \tilde{\Gamma} \setminus \{v\}$ is the Dynkin diagram of a root sublattice of M of the same rank, maybe equal to M. We repeat the process until no new sublattices are created in this way. For example, a subdiagrams of type A_n does not create new sublattices of the same rank. All root sublattices of the same rank are obtained in this way.

If M is reducible, we write it as a direct sum of irreducible root lattices M_i. Since the set of roots of a sublattice N of M is equal to the disjoint sum of the subsets of roots of each direct summand, we obtain that N is the direct sum of root sublattices of each M_i.

We can also create all root sublattices of smaller rank in this way. To do so, at each step we delete more than one vertex.

Example 6.4.10 Let $\alpha_0, \dots, \alpha_9$ is the canonical root basis of E_{10} from Figure 6.1.1. The subdiagrams of type A_8 in the Dynkin diagram $T_{2,3,7}$ of E_{10} defined by the sets of simple roots $J = \{\alpha_0, \alpha_3, \dots, \alpha_9\}$ and $\{\alpha_2, \dots, \alpha_9\}$ are related by one elementary operation and define conjugate parabolic subgroups. However, the subdiagrams $\alpha_1, \dots, \alpha_8$ and $\alpha_0, \alpha_3, \dots, \alpha_9$ are not W-equivalent. On the other hand, all subsets of type A_7 are W-equivalent.

For another example, let M be a root lattice of type $A_1^{\oplus 4}$ spanned by $\alpha_0, \alpha_2, \alpha_4, \alpha_6$ embedded in E_7 which we consider as a sublattice of E_{10} spanned by $\alpha_0, \dots, \alpha_6$. Using the Borel–de Siebenthal–Dynkin algorithm, one easily shows that all parabolic subgroups corresponding to a set of disjoint vertices of the Dynkin diagram are conjugate. However, if we take roots $(\beta_1, \beta_2, \beta_3, \beta_4) = (\alpha_{\max}^{\mathsf{E}_7}, \alpha_2, \alpha_4, \alpha_6)$, where $\alpha_{\max}^{\mathsf{E}_7} = 2\alpha_0 + 2\alpha_1 + 3\alpha_2 + 4\alpha_3 + 3\alpha_4 + 2\alpha_5 + \alpha_6$ is the maximal root of the lattice E_7, we find a non-conjugate root sublattice isomorphic to $\mathsf{A}_1^{\oplus 4}$. To see that we have obtained different embeddings of the same lattice, one checks that the sum

$\alpha_0 + \alpha_1 + \alpha_4 + \alpha_6 \notin 2\mathsf{E}_7$ in the parabolic embedding and $\beta_1 + \beta_2 + \beta_3 + \beta_4 \in 2\mathsf{E}_7$ in the non-parabolic embedding.

One can consider another $N = \mathsf{A}_1^{\oplus 4} \hookrightarrow \mathsf{E}_7$ defined by $(\beta_1, \beta_2, \beta_3, \beta_4) = (\alpha_{\max}^{\mathsf{D}_4}, \alpha_0, \alpha_2, \alpha_4)$, where $\alpha_{\max}^{\mathsf{D}_4} = \alpha_0 + \alpha_2 + \alpha_4 + 2\alpha_3$ is the maximal root of the lattice D_4. We leave it to the reader to check that it is conjugate to the previous one. Note that $\frac{1}{2}(\beta_1 + \beta_2 + \beta_3 + \beta_4)$ defines an isotropic vector in $D(N)$. In the non-parabolic case it corresponds to an embedding $\mathsf{A}_1^{\oplus 4}$ in D_4.

To decide whether the group $\mathsf{G}_{h_{10}}$ in the Fano root invariant $(\mathsf{N}_S^{h_{10}}, \mathsf{G}_{h_{10}})$ is non-trivial, one checks whether the discriminant group of $\mathsf{N}_S^{h_{10}}$ has a non-trivial 2-torsion subgroup. We refer to Table 0.2 from Volume I for the information about the discriminant forms of root lattices. We also use that the discriminant form of $\mathsf{U}(2)$ is equal to u_2.

We will also use the following lemma.

Lemma 6.4.11 *Let $\pi : X \to S$ be the canonical cover which we assume to be étale. Let $R_1, \dots, R_k$ be (-2)-curves on S. Let M' be the sublattice of* $\mathrm{Pic}(X)$ *spanned by the divisor classes δ_{R_i} from* (6.4.1) *and $M = M'(\frac{1}{2})$. Then:*

(i) M is a root lattice.
(ii) If $R_1, \dots, R_k$ span a negative definite lattice Q, then $M \cong Q$.
(iii) If $\sum_{i=1}^k R_i$ is the irreducible decomposition of a half-fiber of type $\tilde{A}_{k-1}$ on S, then $M \cong \mathsf{A}_1 \oplus \mathsf{A}_1$ if $k = 2$, A_3 if $k = 3$ and D_k if $k \geq 4$.
(iv) Suppose $R_1, \dots, R_k$ span a half-fiber of type $\tilde{A}_{k-1}$. Let R_0 be a special bisection. Then, $k \leq 8$ and $\delta_{R_0}, \cdots \delta_{R_k}$ span a sublattice N such that

$$N(\tfrac{1}{2}) \cong \begin{cases} \mathsf{A}_1 \oplus \mathsf{A}_2 & \text{if } k = 2, \\ \mathsf{A}_4 & \text{if } k = 3, \\ \mathsf{D}_5 & \text{if } k = 4, \\ \mathsf{E}_6 & \text{if } k = 5, \\ \mathsf{E}_7 & \text{if } k = 6, \\ \mathsf{E}_8 & \text{if } k \geq 7. \end{cases}$$

Proof (i) Choose an ample divisor D on S. Then, $\pi^*(D)^2 > 0$ and each δ_R is orthogonal to $\pi^*(D)$. Thus, the divisor classes δ_R span a negative definite lattice. Also, since $\delta_R^2 = -4$, the lattice M is spanned by vectors of norm square -2 and hence must be a root lattice.

(ii) Since R_i is simply connected, the cover splits over $\mathcal{R}$ into a disjoint sum of two nodal cycles whose components span a lattice isomorphic to Q. Choosing the notation $R_i^{\pm}$ for irreducible components of $\pi^*(R_i)$, we can write

$$\pi^*(\mathcal{R}) = \sum_{i=1}^k R_i^+ + \sum_{i=1}^k R_k^-, \tag{6.4.6}$$

where $R_i^+ \cdot R_j^- = 0$. We have

$$(R_i^+ - R_i^-) \cdot (R_j^+ - R_j^-) = R_i^+ \cdot R_j^+ + R_i^- \cdot R_j^- = (R_i^+ + R_i^-) \cdot (R_j^+ + R_i^-) = 2R_i \cdot R_j.$$

Thus, $\delta_{R_1}, \dots, \delta_{R_k}$ is a root basis of M isomorphic to the root basis of Q.

(iii) As above we can choose the notation to write $\pi^*(R)$ as in (6.4.6), where R_i^+ and R_i^- are opposite vertices in the dual graph of the components $R_i^\pm$ isomorphic to a $2k$-gon. If $k = 2$, we see that $M \cong \mathsf{A}_1 \oplus \mathsf{A}_1$. Assume $k \geq 3$. We see that $(R_1^+ - R_1^-) \cdot (R_k^+ - R_k^-) = -2$ and $(R_i^+ - R_i^-) \cdot (R_j^+ - R_j^-) = 2$ if $i = 1, \dots, k-1$, $j = i+1$, and $(R_i^+ - R_i^-) \cdot (R_j^+ - R_j^-) = 0$ if $i, j = 1, \dots, k-1$, $j \neq i+1$. Replacing the basis $(e_1, \dots, e_k) = (\delta_{R_1}, \dots, \delta_{R_k})$ of M' with $(e_{k-1}, e_{k-2}, \dots, e_1, -(e_2 + \cdots + e_k))$ we find a root basis in M of type A_3 if $k = 3$ and D_k if $k \geq 4$. This proves (iii).

(iv) Deleting R_0, we find a root basis $(e_{k-1}, e_{k-2}, \dots, e_2, -(e_2 + \cdots + e_k), e_1)$ as in (iii). We may assume that $\pi^*(R_0) = R_0^+ + R_0^-$, where R_0^+ intersects R_1^+ and R_0^- intersects R_1^-. Adding to the previous basis δ_{R_0} as the last vector, we obtain a root basis in M of type $\mathsf{A}_1 \oplus \mathsf{A}_2$ if $k = 2$, A_4 if $k = 3$, D_5 if $k = 4$ and E_{k+1} if $k = 5, 6, 7$. If $k = 8$ or 9, the Gram matrix becomes singular, and hence the vectors $\delta_{R_0}, \dots, \delta_{R_k}$ are linearly dependent. So, we obtain a root lattice of rank 8. By deleting some $R_i, i \neq 0$, and using (i) and (ii) we see that the lattice M contains sublattices isomorphic to $\mathsf{A}_8, \mathsf{D}_8, \mathsf{A}_1 \oplus \mathsf{A}_7, \mathsf{E}_8$. Since the rank of M is equal to 8, we see that the only possibility is that $M \cong \mathsf{E}_8$. □

All of this makes it feasible to classify all possible Fano root invariants and the corresponding quadratic subspaces $\langle \mathsf{Nod}(S) \rangle$ of $\overline{\mathrm{Num}}(S)$. We give the classification in Table 6.4, where we restrict ourselves only to the cases where the Fano root invariants are of rank $l \leq 4$. Here, $\tilde{K}$ is the primitive closure of K from the Nukulin R-invariant (K, H).

Remark 6.4.12 We see from the previous table that two different Fano root invariants may define the same quadratic space $\langle \mathsf{Nod}(S) \rangle$. Also, note that for a larger rank the Fano root invariant is not necessarily defined by a subdiagram of the Dynkin diagram of E_{10}. An example is the lattice $\mathsf{A}_1^{\oplus 8}$ that can be realized by a quasi-elliptic surface.

The next invariant of a nodal Enriques surface is the *Reye lattice* defined by

$$\mathrm{Rey}(S) := \{x \in \mathrm{Num}(S) : x \cdot R \in 2\mathbb{Z} \text{ for all } R \in \mathcal{R}_S\} = \{x \in \mathrm{Num}(S) : \frac{1}{2}x \in \mathsf{N}_S^\vee\}. \tag{6.4.7}$$

It is clear that

$$\mathrm{Rey}(S) = p^{-1}(\langle \mathsf{Nod}(S) \rangle^\perp),$$

where $p : \mathrm{Num}(S) \to \mathrm{Num}(S)/2\,\mathrm{Num}(S)$ is the projection to the factor-space. Applying Witt's Theorem 6.4.5 and the fact that the homomorphism $\mathrm{O}(\mathsf{E}_{10}) \to \mathrm{O}(\bar{\mathsf{E}}_{10})$ is surjective, we obtain:

Proposition 6.4.13 *The isomorphism class of* $\mathrm{Rey}(S)$ *depends only on the isomorphism class of the quadratic space* $\langle \mathsf{Nod}(S) \rangle$.

It follows that

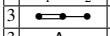
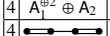
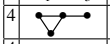
l	$\mathsf{N}_S^{h_{10}}$	K	$\tilde{K}$	$\dim\langle\mathsf{Nod}(S)\rangle$	(r,δ)	ϵ
1	$\tilde{\mathsf{A}}_1$	A_1	A_1	1	$(0,1)$	
2	$\mathsf{A}_1^{\oplus 2}$	$\mathsf{A}_1^{\oplus 2}$	$\mathsf{A}_1^{\oplus 2}$	2	$(0,1)$	
2	A_2	A_2	A_2	2	$(2,0)$	-1
2	$\tilde{\mathsf{A}}_1$	$\mathsf{A}_1^{\oplus 2}$	$\mathsf{A}_1^{\oplus 2}$	2	$(0,1)$	
3	$\mathsf{A}_1^{\oplus 3}$	$\mathsf{A}_1^{\oplus 3}$	$\mathsf{A}_1^{\oplus 3}$	3	$(0,1)$	
3	$\mathsf{A}_1 \oplus \tilde{\mathsf{A}}_1$	$\mathsf{A}_1^{\oplus 3}$	$\mathsf{A}_1^{\oplus 3}$	3	$(0,1)$	
3	$\mathsf{A}_1 \oplus \mathsf{A}_2$	$\mathsf{A}_1 \oplus \mathsf{A}_2$	$\mathsf{A}_1 \oplus \mathsf{A}_2$	3	$(2,1)$	
3		$\mathsf{A}_1 \oplus \mathsf{A}_2$	$\mathsf{A}_1 \oplus \mathsf{A}_2$	3	$(2,1)$	
3	A_3	A_3	A_3	3	$(2,0)$	-1
3	$\tilde{\mathsf{A}}_2$	A_2	A_2	2	$(2,0)$	-1
4	$\mathsf{A}_1^{\oplus 4}$	$\mathsf{A}_1^{\oplus 4}$	D_4	3	$(0,1)$	
4	$\mathsf{A}_1^{\oplus 4}$	$\mathsf{A}_1^{\oplus 4}$	$\mathsf{A}_1^{\oplus 4}$	4	$(0,1)$	
4	$\tilde{\mathsf{A}}_1 \oplus \tilde{\mathsf{A}}_1$	$\mathsf{A}_1^{\oplus 4}$	$\mathsf{A}_1^{\oplus 4}$	4	$(0,1)$	
4	$\mathsf{A}_1^{\oplus 2} \oplus \mathsf{A}_2$	$\mathsf{A}_1^{\oplus 2} \oplus \mathsf{A}_2$	$\mathsf{A}_1^{\oplus 2} \oplus \mathsf{A}_2$	4	$(2,1)$	
4		$\mathsf{A}_1^{\oplus 2} \oplus \mathsf{A}_2$	$\mathsf{A}_1^{\oplus 2} \oplus \mathsf{A}_2$	4	$(2,1)$	
4	$\tilde{\mathsf{A}}_1 \oplus \mathsf{A}_2$	$\mathsf{A}_1^{\oplus 2} \oplus \mathsf{A}_2$	$\mathsf{A}_1^{\oplus 2} \oplus \mathsf{A}_2$	4	$(2,1)$	
4	$\tilde{\mathsf{A}}_2 \oplus \mathsf{A}_1$	$\mathsf{A}_1 \oplus \mathsf{A}_2$	$\mathsf{A}_1 \oplus \mathsf{A}_2$	4	$(2,1)$	
4	A_4	A_4	A_4	4	$(4,0)$	-1
4	$\mathsf{A}_1 \oplus \mathsf{A}_3$	$\mathsf{A}_1 \oplus \mathsf{A}_3$	$\mathsf{A}_1 \oplus \mathsf{A}_3$	4	$(2,1)$	
4		A_4	A_4	4	$(4,0)$	-1
4		A_4	A_4	4	$(4,0)$	-1
4	D_4	D_4	D_4	4	$(2,0)$	-1

Table 6.1 Fano root invariants of rank $l \leq 4$

$$\mathrm{Num}(S)/\mathrm{Rey}(S) \cong \overline{\mathrm{Num}}(S)/\mathsf{Nod}(S)^{\perp} \cong \mathbb{F}_2^a, \tag{6.4.8}$$

$$\mathrm{Rey}(S)/2\,\mathrm{Num}(S) \cong \mathbb{F}_2^{10-a}. \tag{6.4.9}$$

We have

$$\mathrm{Rey}(S) \subset \mathrm{Num}(S) \subset \mathrm{Rey}(S)^{\vee}. \tag{6.4.10}$$

Thus, the discriminant group $D(\mathrm{Rey}(S))$ fits into an exact sequence

$$0 \to (\mathbb{Z}/2\mathbb{Z})^a \to D(\mathrm{Rey}(S)) \to (\mathbb{Z}/2\mathbb{Z})^a \to 0. \tag{6.4.11}$$

Since $\mathrm{Num}(S)$ is a unimodular lattice, we infer that the maximal isotropic subgroup of $D(\mathrm{Rey}(S))$ is of order $2^a = \frac{1}{2}\#D(\mathrm{Rey}(S))$ and

$$D(\mathrm{Rey}(S)) = (\mathbb{Z}/2\mathbb{Z})^{\oplus\alpha} \oplus (\mathbb{Z}/4\mathbb{Z})^{\oplus\beta}. \tag{6.4.12}$$

It follows from the classification of finite discriminant forms on such groups (see Theorem 0.8.1 from Volume I) that the discriminant quadratic form on the group $(\mathbb{Z}/2\mathbb{Z})^{\oplus\alpha}$ is the orthogonal sum of $\mathsf{w}_{2,1}^{\epsilon}, \mathsf{u}_1$ and v_1. Also, the discriminant quadratic form on the group $(\mathbb{Z}/4\mathbb{Z})^{\oplus\beta}$ is the orthogonal sum of $\mathsf{w}_{2,2}^{\epsilon}, \mathsf{w}_{2,2}^{5\epsilon}, \mathsf{u}_2, \mathsf{v}_2$. Since the maximal isotropic subgroup of $\mathsf{w}_{2,2}^{\epsilon}$ is trivial, we obtain that β must be even and the quadratic form on $(\mathbb{Z}/4\mathbb{Z})^{\oplus\beta}$ must be the direct sum of quadratic forms $\mathsf{u}_2, \mathsf{v}_2, \mathsf{w}_{2,2}^{1} \oplus \mathsf{w}_{2,2}^{-1}$ or $\mathsf{w}_{2,2}^{5} \oplus \mathsf{w}_{2,2}^{-5}$.

It follows from [556, Proposition 1.11.2] that the lattice with discriminant form isomorphic to v_2 must have a signature congruent to 0 modulo 8. This can happen

in our case since it is either a hyperbolic lattice of rank 10 (we will see later that it is isomorphic to $\mathsf{E}_{4,4,4}$) or a negative definite lattice of rank 8 (we will see that it is a direct summand of $\mathsf{E}_{4,4,4}$).

Note that the decomposition of the discriminant quadratic form into the orthogonal sum is not unique. For example,

$$\mathsf{v}_k \oplus \mathsf{v}_k \cong \mathsf{u}_k \oplus \mathsf{u}_k.$$

It follows from the definition that the image $\overline{\mathrm{Rey}}(S)$ in $\overline{\mathrm{Num}}(S)$ coincides with the orthogonal complement of the quadratic space $\langle \mathsf{Nod}(S)\rangle$

$$\overline{\mathrm{Rey}}(S) = \langle \mathsf{Nod}(S)^{\perp}\rangle. \tag{6.4.13}$$

Since $\dim \overline{\mathrm{Rey}}(S) = \dim \mathrm{Rey}(S)/2\,\mathrm{Num}(S)$, we obtain that

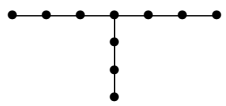
$$a = \dim\langle \mathsf{Nod}(S)\rangle.$$

Lemma 6.4.14 *Let L be a sublattice of E_{10} with $\mathsf{E}_{10}/L \cong (\mathbb{Z}/2\mathbb{Z})^a$. Let $\bar{L}$ be its reduction modulo $2\mathsf{E}_{10}$ in the quadratic space $\bar{\mathsf{E}}_{10} = \mathsf{E}_{10}/2\mathsf{E}_{10}$. Then, its orthogonal complement $\bar{L}^{\perp}$ is isomorphic to the quotient group $D(L)/D(L)_0$ of $D(L)$ by the maximal isotropic subspace $D(L)_0$ that corresponds to the embedding $L \hookrightarrow \mathsf{E}_{10}$. The quadratic form on $\bar{L}^{\perp}$ is defined by $q(x) = 2x^2 \mod 2\mathbb{Z}$, where x^2 is the quadratic form on $D(L)$.*

Proof Since $L^{\vee}/\mathsf{E}_{10}$ is a 2-elementary group, $2L^{\vee} \subset \mathsf{E}_{10}$, and hence we can define a natural homomorphism $L^{\vee} \to \bar{\mathsf{E}}_{10}, x \mapsto 2x + 2\mathsf{E}_{10}$. Obviously, $L \subset L^{\vee}$ is contained in the kernel, hence we get a homomorphism $\phi : D(L) = L^{\vee}/L \to \bar{\mathsf{E}}_{10}$. It is immediate to see that its kernel is equal to E_{10}/L, which can be identified with the maximal isotropic subgroup $D(L)_0$ of $D(L)$ that corresponds to the embedding $L \hookrightarrow \mathsf{E}_{10}$. The image of ϕ is equal to $\bar{L}^{\perp}$. The assertion about the quadratic form follows from the definition. □

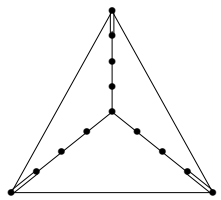
Example 6.4.15 Let $L = \mathsf{D}_4 \oplus \mathsf{D}_4 \oplus \mathsf{U}$ be the lattice generated by (-2)-curves $(\alpha_0, \dots, \alpha_{10})$ given in (6.2.5). Here, $\alpha_0, \alpha_1, \alpha_2, \alpha_3$ generate one copy of D_4, and $\alpha_7, \alpha_8, \alpha_9, \alpha_{10}$ generate another copy of D_4 and the summand U is generated by $f = \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 + 2\alpha_0, g = f + \alpha_5$. Note that $\alpha_6 + \alpha_7 + \alpha_8 + \alpha_8 + 2\alpha_{10}$ and f are half-fibers of the same genus one fibration. The lattice L is isomorphic to a sublattice of E_{10} of index 2^2. The discriminant group of L is generated by $v_1 = \frac{1}{2}(\alpha_1+\alpha_2), v_2 = \frac{1}{2}(\alpha_1+\alpha_3), v_3 = \frac{1}{2}(\alpha_7+\alpha_8), v_4 = \frac{1}{2}(\alpha_7+\alpha_9)$. The discriminant quadratic form is isomorphic to $\mathsf{v}_1 \oplus \mathsf{v}_1 \cong \mathsf{u}_1 \oplus \mathsf{u}_1$. The maximal isotropic subspace corresponding to this embedding is generated by $v_1 + v_3, v_2 + v_4$. The quotient is generated by the cosets of v_1, v_2. Since $v_1^2 = v_2^2 = 1, v_1 \cdot v_2 = \frac{1}{2}$ in $D(L)$, we see that the subspace $\bar{L}$ is isomorphic to a regular quadratic space with quadratic form $q = x_1x_2$.

Example 6.4.16 Consider the lattice $\mathsf{E}_{4,4,4}$ spanned by a root basis defined by the following diagram:



We already noticed that the discriminant form of this lattice is isomorphic to the finite discriminant form v_2 defined by the matrix $\begin{pmatrix} \frac{1}{2} & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{2} \end{pmatrix}$.

One can embed this lattice in E_{10} by using the following diagram:



As we will see later, it is realized as the dual graph of (-2)-curves on an exceptional classical Enriques surface of type $\tilde{E}_6$ (see Table 8.13). We use the notation of the vertices of any $T_{p,q,r}$-diagram from Figure 0.3. We find that $v_1 = \frac{1}{4}(\alpha_0 + 2\alpha_1 + 3\alpha_2 + \alpha_5 + 2\alpha_4 + 3\alpha_3), v_2 = \frac{1}{4}(\alpha_9 + 2\alpha_8 + 3\alpha_7 + \alpha_5 + 2\alpha_4 + 3\alpha_3)$ generate $D(L)$ and $2v_1, 2v_2$ generate $D(L)_0$. We have $v_1^2 = v_2^2 = \frac{1}{2}, v_1 \cdot v_2 = \frac{1}{4}$ as expected. We see that under an isomorphism $D(L)/D(L)_0 \to \bar{L}^\perp$, this corresponds to a basis of the quadratic space of vectors e_1, e_2 with $e_1^2 = e_2^2 = 1$ and $e_1 \cdot e_2 = 1$. This defines the odd quadratic form $x_1^2 + x_1x_2 + x_2^2$.

For future use, let us observe that the lattice $\mathsf{E}_{4,4,4} \cong \mathsf{U} \oplus \mathsf{E}'_{4,4,4}$, where $\mathsf{E}'_{4,4,4}$ is generated by the roots $\alpha_4, \alpha_5, \alpha_6, \alpha_7, \alpha_8, \alpha_2$ spanning a copy of E_6 and vectors $\alpha_3 - \alpha_0, \alpha_9 - \alpha_0$. The hyperbolic plane is spanned by isotropic vectors of type $\tilde{E}_6$ and half of an isotropic vector of type $\tilde{E}_7$.

Theorem 6.4.17 *The Reye lattice* $\mathrm{Rey}(S)$ *with* $a \leq 5$ *is isomorphic to one of the lattices from Table 6.2. Here, the lattices* L_1, L_2, L_3 *are given by the matrices* M_1, M_2, M_3 *below. The triple* (r, δ, ϵ) *denote the rank, the defect and the sign of the quadratic space* $\bar{L}^\perp \subset \bar{E}_{10}$.

Proof The embedding $\mathrm{Rey}(S) \subset \mathrm{Num}(S)$ corresponds to a 2-elementary isotropic subgroup of $D(\mathrm{Rey}(S))$ of order a. This gives a restriction for a possible discriminant quadratic form. This number is always less than or equal to 8 unless $(\alpha, \beta) = (10, 0), (8, 2), (6, 4), (4, 6), (2, 8), (0, 10)$. In the first case, $\overline{\mathrm{Rey}}(S)$ is a maximal isotropic subspace of $\bar{\mathsf{E}}_{10}$, hence its orthogonal complement does not contain a vector v with $q(v) = 1$. Hence, it cannot be realized as $\langle \mathrm{Nod}(S) \rangle$. In the last case, we obtain that $\mathrm{Rey}(S) = 2\,\mathrm{Num}(S)$, hence $\mathrm{Rey}(S) \cong \mathsf{E}_{10}(4)$. Besides this, we have $\alpha + \beta \leq 8$, so we may apply Theorem 0.8.6 to obtain that the lattice is uniquely determined by its discriminant form.

a	$\mathrm{Rey}(S)$	rank	defect	sign
1	$\mathsf{A}_1 \oplus \mathsf{E}_7 \oplus \mathsf{U} \cong \mathsf{E}_{2,4,6}$	0	0	
2	$\mathsf{E}_{4,4,4}$	2	0	-1
2	$\mathsf{A}_1 \oplus \mathsf{E}_7 \oplus \mathsf{U}(2)$	0	1	
3	$\mathsf{A}_1 \oplus \mathsf{E}_7 \oplus \mathsf{U}(4)$	2	1	
3	$\mathsf{E}'_{4,4,4} \oplus \mathsf{U}(2)$	2	0	-1
3	$\mathsf{D}_4 \oplus \mathsf{A}_1^{\oplus 4} \oplus \mathsf{U}$	0	1	
4	$\mathsf{D}_4 \oplus \mathsf{D}_4 \oplus \mathsf{U}(4)$	4	0	1
4	$\mathsf{E}'_{4,4,4} \oplus \mathsf{U}(4)$	4	0	-1
4	$\mathsf{D}_6 \oplus \mathsf{A}_1^{\oplus 2} \oplus \mathsf{U}(4)$	2	1	
4	$L_1 \oplus \mathsf{U}$	2	0	1
4	$\mathsf{A}_1^{\oplus 8} \oplus \mathsf{U}$	0	1	
5	$\mathsf{A}_1^{\oplus 4} \oplus \mathsf{D}_4 \oplus \mathsf{U}(4)$	2	1	
5	$\mathsf{A}_1^{\oplus 8} \oplus \mathsf{U}(2)$	0	1	
5	$L_2 \oplus \mathsf{U}$	4	1	
5	$L_3 \oplus \mathsf{U}$	4	0	1

Table 6.2 Reye lattices with $a \leq 5$

In the following we denote $\mathrm{Rey}(S)$ by L. Assume $a = 1$. There is only one isomorphism class of a non-trivial one-dimensional quadratic space. Applying Proposition 6.4.13 we obtain that there is only one possible Reye lattice. Since $D(L) = (\mathbb{Z}/2\mathbb{Z})^{\oplus 2}$, possible discriminant forms with non-trivial isotropic subgroup are u_1 and $\mathsf{w}_{2,1}^1 \oplus \mathsf{w}_{2,1}^{-1}$. In the first case, $D(L)/D(L)_0$ is generated by an isotropic vector, so this case is discarded since $\langle \mathrm{Nod}(S) \rangle = \bar{L}^{\perp}$ is generated by a non-isotropic vector. The remaining case is realized by a lattice $L \cong \mathsf{A}_1 \oplus \mathsf{E}_7 \oplus \mathsf{U}$.

Assume $a = 2$. In this case, there are two possible 2-dimensional quadratic spaces generated by non-isotropic vectors. They are defined by a non-defective odd quadratic form or a defective quadratic form of rank 0. To realize the first case we can take the lattice $L = \mathsf{A}_1 \oplus \mathsf{E}_7 \oplus \mathsf{U}(2)$ with discriminant form $D(L) \cong (\mathsf{w}_{2,1}^1 \oplus \mathsf{w}_{2,1}^{-1}) \oplus \mathsf{u}_1$. The second possible quadratic space corresponds to the case $(\alpha, \beta) = (0, 2)$. The only possible discriminant form satisfying our conditions is v_2. It is realized by the lattice $\mathsf{E}_{4,4,4}$.

Assume $a = 3$. There are three possible 3-dimensional quadratic spaces generated by non-isotropic vectors: a non-defective odd quadratic form of rank 2, a defective quadratic form of rank 2, and a quadratic form of rank 0. Using the previous case, we realize the first quadratic form by the lattice $L = \mathsf{E}'_{4,4,4} \oplus \mathsf{U}(2)$ with discriminant form $\mathsf{v}_2 \oplus \mathsf{u}_1$. Here, $\mathsf{E}'_{4,4,4}$ is the negative definite lattice of rank 8 with finite discriminant quadratic form v_2 introduced in Example 6.4.16. We have $(\alpha, \beta) = (6, 0), (2, 2)$. So, the lattice L corresponds to the case $(\alpha, \beta) = (2, 2)$. Another possibility in this case is the discriminant quadratic form $(\mathsf{w}_{2,1}^1 \oplus \mathsf{w}_{2,1}^{-1}) \oplus \mathsf{u}_2$. It is realized by the lattice $L = \mathsf{A}_1 \oplus \mathsf{E}_7 \oplus \mathsf{U}(4)$. The case $(\alpha, \beta) = (6, 0)$ corresponds to a lattice of rank 10 with a 2-elementary discriminant group. According to Nikulin [557], there are two isomorphism classes of such lattices $L = \mathsf{D}_4^{\oplus 2} \oplus \mathsf{U}(2)$ and $\mathsf{D}_4 \oplus \mathsf{A}_1^4 \oplus \mathsf{U} \cong \mathsf{D}_6 \oplus \mathsf{A}_1^{\oplus 3} \oplus \mathsf{A}_1(-1)$. We check that in the first case the quadratic space is an even

non-defective of rank 4. It has only two non-isotropic vectors, so this case does not occur in our classification. The remaining case realizes a quadratic defective space of rank 4.

Assume $a = 4$. We have $(\alpha, \beta) = (8, 0), (4, 2)$, or $(0, 4)$. In the first case, the discriminant group is 2-elementary. According to Nikulin, there are two isomorphism classes of such lattices of rank 10. They are represented by the lattices $\mathsf{A}_1^{\oplus 8} \oplus \mathsf{U}$ and $\mathsf{E}_8(2) \oplus \mathsf{U}$. The second lattice does not embed in E_{10} (a maximal isotropic subgroup corresponds to an embedding into an odd unimodular lattice $\mathsf{I}^{1,9}$). Since $D(L)$ in the first case is generated by orthogonal vectors with value of the quadratic form equal to $\frac{1}{2}$, we see that $\bar{L}$ has rank 0. So, L realizes the quadratic space of rank 0.

Assume now that $(\alpha, \beta) = (4, 2)$. Possible discriminant quadratic forms are $(\mathsf{w}_{2,1}^1)^{\oplus 2} \oplus (\mathsf{w}_{2,1}^{-1})^{\oplus 2} \oplus \mathsf{u}_2$ and $\mathsf{u}_1^{\oplus 2} \oplus \mathsf{u}_2$. The first one is realized by the lattice $L = \mathsf{D}_6 \oplus \mathsf{A}_1^{\oplus 2} \oplus \mathsf{U}(4)$. The quadratic space $\bar{L}^\perp$ has $r = 2$ and $\delta = 1$. The second quadratic form is realized by the lattice $\mathsf{D}_4 \oplus \mathsf{D}_4 \oplus \mathsf{U}(4)$. The quadratic space $\bar{L}^\perp$ is non-defective, even of rank 4.

In the case $(\alpha, \beta) = (0, 4)$, we must have $D(L) = \mathsf{u}_2 \oplus \mathsf{v}_2$ or $\mathsf{u}_2^{\oplus 2}$. The first case is realized by the lattice $L = \mathsf{E}'_{4,4,4} \oplus \mathsf{U}(4)$. The quadratic space is the direct sum of an odd rank 2 space and an even rank 2 space. So, $\bar{L}$ has $(r, \delta, \epsilon) = (4, 0, -1)$. Note that we do not see an R-invariant M of rank ≤ 5 with $\bar{M}$ isomorphic to such quadratic space. However, this space is realized by a Fano root invariant $M \cong \mathsf{D}_6$ of rank $l = 6$. By Proposition 1.11.2 from [556], the signature of L realizing $\mathsf{u}_2^{\oplus 2}$ must be equal to $0 \mod 8$. We have not realized the possibility that the quadratic space $\bar{L}^\perp$ is even of rank 2. So, this must be the case when L has the discriminant quadratic form $\mathsf{u}_2^{\oplus 2}$. We use that $L \mod 2$ is orthogonal to the Fano root invariant $M = \mathsf{D}_4$. By embedding D_4, in an obvious way, into E_{10} we find that L is generated by $\alpha_0 + \alpha_2 + \alpha_4, 2\alpha_i, i \neq 4, 6, 7, 8, 9$ and $\alpha_i, i = 6, 7, 8, 9$. Thus, $L \cong L_1 \oplus \mathsf{U}$, where L_1 is given by the following matrix:

$$M_1 = \begin{pmatrix} -6 & -4 & 2 & -4 & 0 & 2 & 0 & 0 \\ -4 & -8 & 4 & 0 & 0 & 0 & 0 & 0 \\ 2 & 4 & -8 & 4 & 0 & 0 & 0 & 0 \\ -4 & 0 & 4 & -8 & 4 & 0 & 0 & 0 \\ 0 & 0 & 0 & 4 & -8 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 & 0 & -8 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2 & -2 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & -2 \end{pmatrix}$$

The integral Smith normal form of M_1 gives that $(\alpha, \beta) = (2, 4)$.

Assume $a = 5$. We have $(\alpha, \beta) = (10, 0), (6, 2)$ or $(2, 4)$. According to [557], there are only two even hyperbolic lattices of rank 10 with 2-elementary discriminant group of rank 10. They are $\mathsf{E}_{10}(2)$ and $\mathsf{I}^{1,9}(2) \cong \mathsf{A}_1^8 \oplus \mathsf{U}(2)$. Only the second one embeds in E_{10} as $\mathsf{A}_1^8 \oplus \mathsf{U}(2) \hookrightarrow \mathsf{E}_8 \oplus \mathsf{U}$. Using the previous case, we find that $\bar{L}^\perp$ is an even quadratic form with $(r, \delta) = (0, 1)$. The lattice $L = \mathsf{D}_8(2) \oplus \mathsf{U}$ realizes the case $(6, 2)$. Its discriminant quadratic form is $\mathsf{u}_1^{\oplus 3} \oplus \mathsf{u}_2$. We see that $\bar{L}^\perp$ is a quadratic space with $(r, \delta) = (2, 0, 1)$.

We can also realize the case $(6,2)$ by the lattice $\mathsf{A}_1^{\oplus 4} \oplus \mathsf{D}_4 \oplus \mathsf{U}(4)$ with discriminant quadratic form $(\mathsf{w}_{2,1}^{-1})^{\oplus 4} \oplus \mathsf{v}_1 \oplus \mathsf{u}_2$. We can choose $D(\mathsf{A}_1^{\oplus 4} \oplus \mathsf{D}_4)_0$ generated by $v_1+v_2+w_1, v_1+v_3+w_2, v_1+v_2+v_3+v_4$, where (v_1,v_2,v_3,v_4) is an orthogonal basis of $\mathsf{A}_1^{\oplus 4}$ and w_1, w_2 is the standard basis of v_1. The quadratic space $\bar{L}^{\perp}$ is defective of rank 2.

Unfortunately, we do not see how to realize other three possible cases by using the orthogonal sum of root lattices and scaled hyperbolic planes. So we have to find them by straightforward computations.

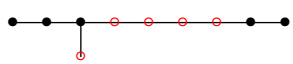
Assume that the quadratic space is defective of rank 4. We may take $M = \mathsf{A}_1 \oplus \mathsf{A}_4$ generated by simple roots $\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_6$.

The Reye lattice is spanned by $2\alpha_0, 2\alpha_1, 2\alpha_2, 2\alpha_3, \alpha_0+\alpha_4, \alpha_1+\alpha_3+\alpha_5+\alpha_7, \alpha_6, 2\alpha_7, \alpha_8, \alpha_9$. The lattice is the orthogonal sum of U and the lattice L_2 given by the matrix

$$M_2 = \begin{pmatrix} -8 & 0 & 0 & 4 & -4 & 2 & 0 & 0 \\ 0 & -8 & 4 & 0 & 0 & -4 & 0 & 0 \\ 0 & 4 & -8 & 4 & 0 & 4 & 0 & 0 \\ 4 & 0 & 4 & -8 & 4 & -4 & 0 & 0 \\ -4 & 0 & 0 & 4 & -4 & 3 & 0 & 0 \\ 2 & -4 & 4 & -4 & 3 & -8 & 2 & -4 \\ 0 & 0 & 0 & 0 & 0 & 2 & -2 & 2 \\ 0 & 0 & 0 & 0 & 0 & -4 & 2 & -8 \end{pmatrix}.$$

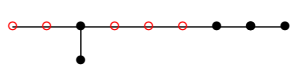
The integral Smith normal form of the matrix M_2 shows that the discriminant group has $(\alpha,\beta) = (2,4)$.

Note that there is another, non-isomorphic, embeddings of $\mathsf{A}_1 \oplus \mathsf{A}_4$ in E_{10}. It is represented by the diagrams



In both cases M defines the Fano root invariant $(\mathsf{N}_S^{h_{10}}, G_{h_{10}})$ with $G_{h_{10}} = \{0\}$ and $\bar{M}$ does not depend on the isomorphism class of the embedding.

Finally, to realize the quadratic space with $(r,\delta,\epsilon) = (4,0,1)$ we take $\mathsf{N}_S^{h_{10}} = \mathsf{A}_2 \oplus \mathsf{A}_3$ generated by $\alpha_1, \alpha_2, \alpha_4, \alpha_5, \alpha_6$.



The Reye lattice is spanned by $\alpha_0, 2\alpha_1, 2\alpha_2, 2\alpha_3, \alpha_4+\alpha_6, 2\alpha_4, 2\alpha_5, 2\alpha_7, \alpha_8, \alpha_9$. It is isomorphic to $\mathsf{U} \oplus L_3$, where L_4 is given by the matrix

$$M_3 = \begin{pmatrix} -2 & 0 & 0 & 2 & 0 & 0 & 0 & 1 \\ 0 & -8 & 4 & 0 & 0 & 0 & 0 & -4 \\ 0 & 4 & -8 & 4 & 0 & 0 & 0 & 4 \\ 2 & 0 & 4 & -8 & 2 & 4 & 0 & -4 \\ 0 & 0 & 0 & 2 & -4 & -4 & 4 & 4 \\ 0 & 0 & 0 & 4 & -4 & -8 & 4 & 4 \\ 0 & 0 & 0 & 0 & 4 & 4 & -8 & -4 \\ 1 & -4 & 4 & -4 & 4 & 4 & -4 & -8 \end{pmatrix}$$

The integral Smith normal form of M_3 shows that the discriminant group has $(\alpha, \beta) = (2, 4)$. □

Example 6.4.18 In this example, we will compute the Fano root invariants and the Reye lattices of extra-special Enriques surfaces.

Suppose that S is extra-special of type $\tilde{E}_8$ with crystallographic nodal basis equal to the canonical root basis of $\mathrm{Num}(S) \cong \mathsf{E}_{10}$. Obviously, $\mathsf{N}_S \cong \mathsf{E}_{10}$ and $\mathrm{Rey}(S) = 2\,\mathrm{Num}(S) \cong \mathsf{E}_{10}(4)$. There is only one numerical Fano polarization h_{10} defined by the canonical isotropic sequence

$$(f, f + R_{10}, f + R_{10} + R_9, \ldots, f + R_{10} + \cdots + R_2),$$

where we index R_i as in diagram (6.1.2). We have $h_{10} \cdot R_i = 0, i \neq 0$ and $h_{10} \cdot R_1 = 1$. Thus, the Fano model has one line and a rational double point of type A_9 lying on this line. Obviously, $\mathsf{N}_S^{h_{10}} = \mathsf{N}_S$.

Let S be an extra-special surface of type $\tilde{E}_7$. Then, the (-2)-curves on S are given by vertices of the diagram $\tilde{E}_7^1$ from Theorem 6.2.3. It is obvious that $\mathsf{N}_S \cong \mathsf{A}_1 \oplus \mathsf{E}_7 \oplus \mathsf{U}$.

There is only one numerical Fano polarization h_{10} defined by the canonical isotropic sequence

$$(f, f + R_9, f + R_9 + R_8, \ldots, f + R_9 + \cdots + R_3, g, g + R_{11}),$$

where f is the class of half-fiber of type $\tilde{E}_7$ and g is the class of a half-fiber of the genus one fibration with simple fiber of type $\tilde{E}_8$ (see diagram (6.2.3)). We find that $h_{10} \cdot R_i = 0$ for $i \neq 1, 2, 10$ and $h_{10} \cdot R_1 = h_{10} \cdot R_2 = 1, h_{10} \cdot R_{10} = 3$. Thus, $\mathsf{N}_S^{h_{10}} = \mathsf{N}_S$. We see that $G_{h_{10}}$ is generated by $\frac{1}{2}[R_1 + R_6 + R_8 + R_{10}]$, hence $G_{h_{10}} \cong \mathbb{Z}/2\mathbb{Z}$. Thus, the Fano root invariant of S is equal to $(\mathsf{A}_1 \oplus \mathsf{E}_7 + \mathsf{U}, \mathbb{Z}/2\mathbb{Z})$.

It follows now that the quadratic space $\bar{\mathsf{N}}^{\mathrm{nod}}$ is 9-dimensional, defective of rank 8. Its orthogonal complement is a defective quadratic space of rank 0 and dimension 1. The Reye lattice coincides with its pre-image under the map $p : \mathrm{Num}(S) \to \overline{\mathrm{Num}}(S)$, hence it must be isomorphic to $2\mathsf{E}_9 \oplus \mathsf{A}_1$.

Finally, let S be an extra-special surface of type $\tilde{D}_8$. In notation of diagram (6.2.4), we see that $D(\mathsf{N}_S)$ is generated by $\frac{1}{2}[R_1 + R_5 + R_7 + R_9]$ and $\frac{1}{2}[R_9 + R_{10}]$. Thus, again N_S has 2-elementary discriminant form. Computing the discriminant quadratic form, we find that $\mathsf{N}_S \cong \mathsf{E}_8 \oplus \mathsf{U}(2)$.

We can confirm the previous computations by computing $\mathsf{N}_S^{h_{10}}$. Note that $R_1 + R_5 + R_7 + R_9 \in 2\,\mathrm{Num}(S)$ if we assume that it is supported in a simple fiber of type

$\tilde{E}_8$. We take a Fano polarization corresponding to a canonical isotropic sequence $(f, f+R_2, f+R_2+R_3, \dots, f+R_2+\cdots+R_8+R_9, g)$, where f is the numerical class of the half-fiber of the genus one fibration of type $\tilde{D}_8$ and g is the numerical class of a half-fiber of the genus one fibration with a simple fiber of type $\tilde{E}_8$. We assume that the latter contains the component R_9. We check that $h_{10} \cdot R_i = 0$ for $i \neq 1, 10$ and $h_{10} \cdot R_2 = 2, h_{10} \cdot R_{10} = 1$. Thus, again, $\mathsf{N}_S^{h_{10}} = \mathsf{N}_S$. The group $G_{h_{10}}$ is generated by the cosets of $\frac{1}{2}[R_9 + R_{10}]$ and $\frac{1}{2}[R_1 + R_5 + R_7 + R_9]$. It is isomorphic to $(\mathbb{Z}/2\mathbb{Z})^{\oplus 2}$. Thus, the Fano root invariant of S is equal to $(\mathsf{E}_8 \oplus \mathsf{U}(2), (\mathbb{Z}/2\mathbb{Z})^{\oplus 2})$.

We see that $\bar{\mathsf{N}}^{\mathrm{nod}} = p(\mathsf{E}_8 \oplus \mathsf{U}(2))$ is a regular, non-defective even quadratic space of dimension 8. Its orthogonal complement is a regular non-defective even quadratic space of dimension 2. This easily implies that $\mathrm{Rey}(S) \cong 2\mathsf{E}_8 \oplus \mathsf{U}$.

The Reye lattice will play an important role when we will study automorphisms of Enriques surfaces.

Remark 6.4.19 In [42, p. 393], the authors define the *nodal type* of a complex Enriques surface. Let $\pi : X \to S$ be the canonical cover of S. Consider the following sublattices of $L = H^2(X, \mathbb{Z}) \cong L_{K3}$. Let $L^+ \oplus L^- \subset L$ be the orthogonal direct sum of the invariant and anti-invariant sublattices with respect to the deck transformation of X/S. First, let $L_1 = \pi^*(\mathsf{N}_S)' \subset L^+$, where $(\mathsf{N}_S)'$ is the primitive closure of N_S. Let L_2 be the smallest sublattice of L^- containing the classes $R^+ - R^-$, where $\pi^*(R) = R^+ + R^-$ for a (-2)-curve R on S. Let $L_3 = L_2^\perp \cap \mathrm{Pic}(X) \cap L^-$. The nodal type of S is defined to be the smallest primitive sublattice N of L containing $T_X = \mathrm{Pic}(X)^\perp$ and L_3. It is clear that N determines L_2 which is the lattice K in the Nikulin R-invariant $\mathrm{Nik}(S)$.

Assume that $\Bbbk = \mathbb{C}$. Following a communication from S. Mukai, one can give the following refinement of the Nikulin R-invariant. We identify the fundamental group $\pi_1(S)$ with $\langle\sigma\rangle$, where σ is the Enriques involution of X. Let $\mathbb{Z}_S^\omega$ be the local coefficient system on S defined by the unique non-trivial homomorphism $\mathbb{Z}/2\mathbb{Z} = \pi_1(S) \to \mathrm{GL}(\mathbb{Z}) = \{\pm 1\}$. We have $\pi^*(\mathbb{Z}_S^\omega) = \mathbb{Z}_X$ and, hence, a canonical homomorphism $\mathbb{Z}_S^\omega \to \pi_*(\mathbb{Z}_X)$. It identifies $\mathbb{Z}_S^\omega$ with a subsheaf of $\pi_*(\mathbb{Z}_X)$. The quotient sheaf is the sheaf $\mathbb{Z}_S$, so that we get an exact sequence of local coefficient systems on S

$$0 \to \mathbb{Z}_S^\omega \to \pi_*(\mathbb{Z}_X) \to \mathbb{Z}_S \to 0$$

The map $\pi_*(\mathbb{Z}_X) \to \mathbb{Z}_S$ is the local trace map. The cup-product defines a perfect pairing

$$\mathbb{Z}_S^\omega \times \mathbb{Z}_S^\omega \to \mathbb{Z}_S.$$

It allows one to extend the usual Poincaré duality and the universal coefficient theorem from Section 0.10 to the cohomology with coefficients in $\mathbb{Z}_S^\omega$. Since $\mathbb{Z}_S^\omega$ is not trivial, we have $H^0(S, \mathbb{Z}_S^\omega) = \{0\}$. The map $H^0(S, \pi_*(\mathbb{Z}_X)) = H^0(X, \mathbb{Z}) \to H^0(S, \mathbb{Z}_S)$ is the multiplication by 2 map $\mathbb{Z} \to \mathbb{Z}$. This gives $H^1(S, \mathbb{Z}_S^\omega) \cong \mathbb{Z}/2\mathbb{Z}$ and $H^3(S, \mathbb{Z}_S^\omega) = 0$. Since $H^1(S, \mathbb{Z}_S) = 0$, we obtain an exact sequence

$$0 \to H^2(S, \mathbb{Z}_S^\omega) \to H^2(X, \mathbb{Z}) \to H^2(S, \mathbb{Z}) \to 0. \tag{6.4.14}$$

This gives

$$H^2(S, \mathbb{Z}_S^\omega) \cong \mathbb{Z}^{12}.$$

The Poincaré duality defines a structure of a unimodular quadratic lattice on $H^2(S, \mathbb{Z}_S^\omega)$ of signature $(2, 10)$. The lattice $H^2(S, \mathbb{Z}_S^\omega)(2)$ is a sublattice of $H^2(X, \mathbb{Z})$. Since $H^2(S, \mathbb{Z})$ has torsion subgroup of order 2, it is not a primitive sublattice.

For any $\gamma \in H^2(X, \mathbb{Z})$, we can write $2\gamma = (\gamma + \sigma^*(\gamma)) + (\gamma - \sigma^*(\gamma))$. Thus, for any $\alpha \in H^2(S, \mathbb{Z})$, we have

$$(2\gamma, \pi^*(\alpha))_X = (\gamma + \sigma^*(\gamma), \pi^*(\alpha))_X = 2(\beta, \alpha)_S,$$

where $\pi^*(\beta) = \gamma + \sigma^*(\gamma)$. This shows that the homomorphism $H^2(X, \mathbb{Z}) \to H^2(S, \mathbb{Z})$ in (6.4.14) coincides with the *Gysin map* $\pi_* : H^2(X, \mathbb{Z}) \to H^2(S, \mathbb{Z})$. We saw before that it is a surjective map (see another proof in [51]). Thus,

$$H^2(S, \mathbb{Z}_S^\omega)(2) = \mathrm{Ker}(\pi_*).$$

The saturation of the sublattice $H^2(S, \mathbb{Z}_S^\omega)$ in $H^2(X, \mathbb{Z})$ coincides with the sublattice $\mathrm{Ker}(1+\sigma^*)$. As we explained in Section 5.3 in Volume I, one can find a decomposition $H^2(X, \mathbb{Z}) = \mathsf{E}_{10}^{\oplus 2} \oplus \mathsf{U}$ such that, for any $x, y \in \mathsf{E}_{10}, a \in \mathsf{U}$, we have $\sigma^*(x, y, a) = (y, x, -a)$. This shows that

$$\mathrm{Ker}(1 + \sigma^*) = \{(x, -x, a), x \in \mathsf{E}_{10}, a \in \mathsf{U}\} \cong \mathsf{E}_{10}(2) \oplus \mathsf{U}.$$

It is equal to the orthogonal complement of $\pi^*(H^2(S, \mathbb{Z})) \cong \mathsf{E}_{10}(2)$ in $H^2(X, \mathbb{Z})$. It is easy to see that the only unimodular lattice M of rank 12 of signature $(2, 10)$ such that $M(2)$ embeds in $\mathsf{E}_{10}(2) \oplus \mathsf{U}$ as a sublattice of index 2 is the odd lattice

$$\mathsf{I}^{2,10} \cong \mathsf{E}_{10} \oplus \langle 1 \rangle \oplus \langle -1 \rangle \cong \langle 1 \rangle^{\oplus 2} \oplus \langle -1 \rangle^{\oplus 10},$$

where $\mathsf{I}^{1,1}(2) = \langle 2 \rangle \oplus \langle -2 \rangle$ embeds in U as a sublattice generated by $f + g$ and $f - g$, where f, g is a canonical basis of U. This realization of the lattice $\mathsf{I}^{2,10}$ was first introduced in [6] who used it to simplify the theory of periods for Enriques surfaces. Thus, we obtain an isomorphism of quadratic lattices

$$H^2(S, \mathbb{Z}_S^\omega) \cong \mathsf{I}^{2,10}.$$

It follows from the previous discussion that

$$\mathrm{Ker}(1 + \sigma^*) = \langle H^2(S, \mathbb{Z}_S^\omega)(2), \alpha \rangle,$$

where $\pi_*(\alpha) = K_S$.

Suppose now that $\Bbbk$ is not necessarily the field of complex number but the canonical cover $\pi : X \to S$ is étale. Following Mukai, we set

$$\mathrm{Pic}^\omega(S) := \mathrm{Ker}(\mathrm{Nm}),$$

where $\mathrm{Nm} : \mathrm{Pic}(X) \to \mathrm{Pic}(S)$ is the norm map defined in Section 1.3. It follows from [242, Theorem 1.4] that it coincides with the map $\pi_* : \mathrm{Pic}(X) \to \mathrm{Pic}(S)$ defined on the divisor classes. If $\Bbbk = \mathbb{C}$, we identify $\mathrm{Pic}^\omega(S)(\frac{1}{2})$ with a sublattice of $H^2(S, \mathbb{Z}^\omega)$. It is clear now that

$$\mathrm{Im}(1 - \sigma^*) \subset \mathrm{Pic}^\omega(S) \subset \mathrm{Ker}(1 + \sigma^*),$$

and the first factor is non-trivial if and only if the homomorphism $\mathrm{Br}(S) \to \mathrm{Br}(X)$ is the zero map, and the second factor is non-zero if and only if $K_S \in \mathrm{Im}(\mathrm{Nm})$. Since $2x = (x + \sigma^*(x)) + (x - \sigma^*(x))$, we obtain that $\mathrm{Ker}(1 + \sigma^*)/\mathrm{Im}(1 - \sigma^*)$ is killed by 2.

It is easy to describe elements of the group $\mathrm{Im}(1 - \sigma^*)$. Note that, for any irreducible curve C on X, we have $C \cdot \sigma^*(C)$ is even (otherwise σ has a fixed point on C). This implies that $(C - \sigma^*(C))^2 \equiv 0 \mod 4$, and hence $\mathrm{Im}(1 - \sigma^*)$ is an even sublattice of $\mathrm{Pic}^\omega(S)(\frac{1}{2})$. Thus, if $\mathrm{Pic}^\omega(S)(\frac{1}{2})$ is an odd sublattice of $H^2(S, \mathbb{Z}_S^\omega)$, we have $\mathrm{Pic}^\omega(S) \neq \mathrm{Im}(1 - \sigma^*)$, hence the map $\pi^* : \mathrm{Br}(S) \to \mathrm{Br}(X)$ is the zero map. Conversely, if this map is the zero map, then, by Corollary 5.7 in [51], there exists a divisor class $D \in \mathrm{Ker}(\mathrm{Nm})$ such that $D^2 \equiv 2 \mod 4$. This shows that $\mathrm{Pic}^\omega(S)(\frac{1}{2})$ is an odd lattice in this case.

Note that the definition of $\mathrm{Pic}^\omega(S)$ makes sense in any characteristic $p \neq 2$ and in characteristic 2 if S is a $\boldsymbol{\mu}_2$-surface. It is a quadratic lattice such that $\mathrm{Pic}^\omega(S)(2)$ coincides with $\mathrm{Ker}(\mathrm{Nm})$. In particular, $\mathrm{Pic}^\omega(S)$ lies in the orthogonal complement of $\pi^*(\mathrm{Pic}(S))$ and hence must be a negative definite quadratic lattice. Also, by Riemann–Roch, any divisor of norm -2 on X must be effective or anti-effective, and, since $\pi^*(\mathrm{Pic}(S))$ contains an ample divisor class, we see that $\mathrm{Pic}^\omega(S)$ does not contain elements of norm -1.

For any smooth compact oriented 4-manifold M, the exact sequence

$$0 \to \mathbb{Z} \xrightarrow{[2]} \mathbb{Z} \to \mathbb{Z}/2\mathbb{Z} \to 0 \tag{6.4.15}$$

defines an exact sequence

$$0 \to H^2(M, \mathbb{Z}) \otimes \mathbb{Z}/2\mathbb{Z} \to H^2(M, \mathbb{Z}/2\mathbb{Z}) \to H^3(M, \mathbb{Z})[2] \to 0.$$

The Poincaré duality defines a non-degenerate symmetric form on $H^2(M, \mathbb{Z}/2\mathbb{Z})$ with values in $\mathbb{Z}/2\mathbb{Z}$. The cohomology $H^2(M, \mathbb{Z}/2\mathbb{Z})$ contains a special class, the Stiefel–Whitney class $w_2(M)$ such that, for any $x \in H^2(M, \mathbb{Z}/2\mathbb{Z})$, we have $(w_2, x) = x^2$. If M is a complex surface, then $w_2(M)$ is the image of $c_1(M) = -K_M$ in $H^2(M, \mathbb{Z}/2\mathbb{Z})$.

Applying this to $M = X$, we get

$$H^2(X, \mathbb{Z}) \otimes \mathbb{Z}/2\mathbb{Z} \cong H^2(X, \mathbb{Z}/2\mathbb{Z}).$$

Applying this to $M = S$, we get an exact sequence

$$0 \to H^2(S, \mathbb{Z}) \otimes \mathbb{Z}/2\mathbb{Z} \xrightarrow{p} H^2(S, \mathbb{Z}/2\mathbb{Z}) \xrightarrow{\phi} H^3(S, \mathbb{Z})[2] \to 0.$$

We have $H^2(S,\mathbb{Z})\otimes\mathbb{Z}/2\mathbb{Z}\cong\overline{\mathrm{Num}}(S)\oplus\mathbb{Z}K_S\cong\mathbb{F}_2^{11}$, and $H^3(S,\mathbb{Z})[2]\cong\mathrm{Br}(S)$. For any x in the image of p, we have $w_2(S)\cdot x=0$, hence $x^2=0$.

One has the analog of the exact sequence (6.4.15) for cohomology with coefficient in local systems. For our needs, we use the exact sequence

$$0\to\mathbb{Z}_S^\omega\xrightarrow{[2]}\mathbb{Z}_S^\omega\to(\mathbb{Z}/2\mathbb{Z})_S^\omega\to 0.$$

Since $\mathrm{GL}(\mathbb{Z}/2\mathbb{Z})=\{1\}$, the local coefficient system $(\mathbb{Z}/2\mathbb{Z})_S^\omega$ is trivial, hence isomorphic to $(\mathbb{Z}/2\mathbb{Z})_S$. The exact sequence of cohomology gives an exact sequence

$$0\to H^1(S,\mathbb{Z}/2\mathbb{Z})\to H^2(S,\mathbb{Z}_S^\omega)\otimes\mathbb{Z}/2\mathbb{Z}\to H^2(S,\mathbb{Z}/2\mathbb{Z})\to 0.$$

Let

$$c:\mathrm{Pic}^\omega(S)\to H^2(S,\mathbb{Z}_S^\omega)\to H^2(S,\mathbb{Z}/2\mathbb{Z})$$

be the composition of the inclusion of $\mathrm{Pic}^\omega(S)$ in $H^2(S,\mathbb{Z}_S^\omega)$ with the reduction mod 2 map $H^2(S,\mathbb{Z}_S^\omega)\to H^2(S,\mathbb{Z}_S^\omega)\otimes\mathbb{Z}/2\mathbb{Z}$ followed by the projection map $H^2(S,\mathbb{Z}_S^\omega)\otimes\mathbb{Z}/2\mathbb{Z}\to H^2(S,\mathbb{Z}/2\mathbb{Z})$. For any $x\in H^2(X,\mathbb{Z})$, we can write $2x=(x+\sigma^*(x))+(x-\sigma^*(x))$. This shows that the restriction of c to the subgroup $\mathrm{Im}(1-\sigma^*)$ is defined as follows. Let $x=\sigma^*(y)-y\in\mathrm{Pic}^\omega(S)$. Then, $\sigma^*(y)+y=\pi^*(z)$ for some $z\in\mathrm{Pic}(S)$. It is immediate to see that $z \mod 2\,\mathrm{Pic}(S)$ is independent of a choice of y and hence defines an element in $\overline{\mathrm{Num}}(S)\subset H^2(S,\mathbb{Z})\otimes\mathbb{Z}/2\mathbb{Z}$. Its image in $H^2(S,\mathbb{Z}/2\mathbb{Z})$ is equal to $c(x)$.

Let $w_2(S)$ be the image of K_S in $H^2(S,\mathbb{Z}/2\mathbb{Z})$. Suppose it belongs to $c(\mathrm{Pic}^\omega(S))$. Then, by Wu's formula (0.10.12) from Volume I, for any x in the orthogonal complement of $\mathrm{Pic}^\omega(S)(\frac{1}{2})$ in $H^2(S,\mathbb{Z}_S^\omega)$, we have $x^2\equiv 0 \mod 2$. This shows that $\mathrm{Pic}^\omega(S)(\frac{1}{2})^\perp$ is an even lattice. The converse is also true, if $\mathrm{Pic}^\omega(S)(\frac{1}{2})^\perp$ is an even sublattice of $H^2(S,\mathbb{Z}_S^\omega)$, then $w_2(S)\in c(\mathrm{Pic}^\omega(S))$. Also note that the latter happens if and only if there exists a divisor class D in $\mathrm{Pic}(X)$ such that $\mathrm{Nm}(D)=K_S$.

Let L be a negative definite quadratic lattice that does not contain elements of norm -1. We say that an Enriques surface is of *Mukai's nodal type* L if there is a primitive embedding of L in $\mathrm{Pic}^\omega(S)(\frac{1}{2})$. We say that the nodal type is *odd* (resp. *even*) if the orthogonal complement of L in $\mathrm{Pic}^\omega(S)(\frac{1}{2})$ is odd (resp. even). When we want to distinguish these cases, we will write (L,even) (resp. (L,odd)).

Let (K,H) be the Nikulin R-invariant of S. It is clear that K is a sublattice of $\mathrm{Pic}^\omega(S)(\frac{1}{2})$. It is the maximal root sublattice of $\mathrm{Pic}^\omega(S)(\frac{1}{2})$. Also, one can show that the maximal even sublattice of $\mathrm{Pic}^\omega(S)(\frac{1}{2})$ coincides with the overlattice of K corresponding to the subgroup H, i.e. $\mathrm{Pic}^\omega(S)(\frac{1}{2})$ is generated by K and elements $\frac{1}{2}h\in K^\vee$, such that $h \mod 2K\in H$. Note that, in general, the maximal even sublattice L of $\mathrm{Pic}^\omega(S)(\frac{1}{2})$ could be larger than K. For example, if S admits an elliptic fibration with a double fiber $F=2F_0$ of type $\tilde{A}_0^*$, we have $\pi^*(F_0)=R_1+R_2$, where R_1+R_2 is a fiber of type $\tilde{A}_1$ on X, and $R_1-R_2\in L(\frac{1}{2})$ but $R_1-R_2\notin K(\frac{1}{2})$.

Example 6.4.20 Let $C=V(F)$ be a nonsingular cubic surface in $\mathbb{P}^3$ over a field $\Bbbk$ of characteristic different from 3. It is called *Sylvester non-degenerate* if the

homogeneous cubic form F can be written as $l_1^3 + \cdots + l_5^3 = 0$, where l_i are linear forms spanning the 4-dimensional linear space of such forms, no two of which are proportional. It is known that a general cubic surface is Sylvester non-degenerate. The Sylvester Theorem asserts that the linear forms $(l_1, \ldots, l_5)$ are defined uniquely by C up to common scaling ([177, Theorem 9.4.1]). One can rewrite its equation in the form

$$t_1 + \cdots + t_5 = a_1 t_1^3 + \cdots + a_5 t_5^3 = 0, \ a_i \neq 0.$$

Assume additionally that char($\Bbbk$) is not equal to 2. The *Hessian surface* of C is a quartic surface $H(C)$ defined by the determinant of the matrix of second-order partial derivatives of the polynomial F. In variables $t_1, \ldots, t_5$, the equation of the Hessian surface can be written in the form

$$t_1 + \cdots + t_5 = \frac{1}{a_1 t_1} + \cdots + \frac{1}{a_5 t_5} = 0. \tag{6.4.16}$$

The last sum is shorthand for the quartic polynomial obtained by clearing denominators. It contains 10 ordinary double points P_{ab} satisfying $t_i = 0, i \neq a, b$, and 10 lines ℓ_{ab} satisfying $t_i = 0, i \in \{a, b\}$. For any 2-subsets α, β of $\{1, 2, 3, 4, 5\}$, we have $P_\alpha \in \ell_\beta$ if and only if $\alpha \cap \beta = \emptyset$. The union Π of the hyperplanes $t_i = 0$ in $\mathbb{P}^3$ identified with the hyperplane $t_1 + \cdots + t_5 = 0$ is called the *Sylvester pentahedron*. The points P_α are its vertices and the lines ℓ_α are its edges.

The Hessian surface may acquire other singular points. It is easy to see that they do not lie on the coordinate hyperplanes $t_i = 0$ and satisfy the equations $a_i t_i^2 - a_j t_j^2 = 0$. They coincide with singular points of the cubic surface C.

The birational involution of $\mathbb{P}^4$ defined by

$$T : (t_1, \ldots, t_5) \mapsto \left(\frac{1}{a_1 t_1}, \ldots, \frac{1}{a_5 t_5}\right)$$

leaves the surface $H(C)$ invariant. It extends to a biregular involution σ of a surface X' obtained from $H(C)$ by resolving the ten nodes P_α. Let E_α be the exceptional curve over the point P_α and let L_α be the proper transform of the line ℓ_α. The involution switches E_α with L_α. The fixed points of T coincide with singular points of $H(C)$ different from the nodes P_α.

In particular, if C is a smooth cubic surface, we see that X' is a K3 surface X and the involution σ has no fixed points. Thus, $\pi : X \to S = X/(\sigma)$ is the canonical cover of an Enriques surface. We call it an *Enriques surface of Hessian type*. The curve $E_\alpha + L_\alpha$ descends to a (-2)-curve U_α on S. The intersection graph of the curves U_α is the famous *Petersen graph* given in Figure 6.1.

One computes the determinant of the intersection matrix of the curves U_α and finds that it is equal to -256. Thus, the classes $[U_\alpha]$ span $\mathrm{Num}(S)_{\mathbb{Q}}$. In fact, any U_α is an irreducible component of a singular fiber of an elliptic pencil $|2F_\alpha|$ with a simple fiber $\sum_{\beta, \#\beta\cap\alpha=1} U_\beta$ of type $\tilde{A}_5$. The pre-image of this elliptic pencil on $H(C)$ is the pencil of residual cubics in hyperplane sections passing through the line ℓ_α. We have

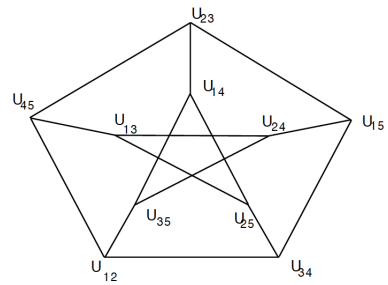


Fig. 6.1 Petersen graph

$$h = \sum_{\alpha} U_\alpha \tag{6.4.17}$$

is a Fano polarization on S and

$$3h \equiv \sum_{\alpha} F_\alpha. \tag{6.4.18}$$

The pre-image of h on the canonical cover is equal to $h' + \sigma^*(h')$, where h' is the class of a hyperplane section (see [181]).

The sublattice L of $\mathrm{Pic}(X)$ generated by the curves E_α and L_α is of rank 16 and discriminant 48. Its discriminant quadratic form is $\mathsf{u}_1 \oplus q_{\mathsf{A}_2(-2)}$. There exists a unique (up to isometry) hyperbolic lattice with the same rank and the discriminant form. This lattice primitively embeds into the K3-lattice and its orthogonal complement is isomorphic to $\mathsf{U} \oplus \mathsf{U}(2) \oplus \mathsf{A}_2(2)$. If C is a general cubic surface (this can be made precise but we are not going to elaborate on this), the sublattice L coincides with the Picard lattice $\mathrm{Pic}(X)$ (see [187]).

We assume now that C is a general cubic surface. It is clear that the divisor classes $r_\alpha = L_\alpha - E_\alpha$ generate the subgroup $\mathrm{Im}(1-\sigma^*)$ of $\mathrm{Pic}^\omega(S)$. They also generate the whole group $\mathrm{Ker}(1+\sigma^*)$. In fact, suppose $D = \sum n_\alpha L_\alpha + \sum m_\alpha E_\alpha$ belongs to this group. Then, $\sigma^*(D) = \sum n_\alpha E_\alpha + \sum m_\alpha L_\alpha$,

$$D + \sigma^*(D) = \sum (n_\alpha + m_\alpha)(L_\alpha + E_\alpha) = \sum (n_\alpha + m_\alpha)\pi^* U_\alpha = 0.$$

As mentioned above, the divisor classes U_α on S are linearly independent. This implies that $n_\alpha + m_\alpha = 0$, hence $D \in \mathrm{Im}(1-\sigma^*)$.

We have $r_\alpha^2 = -4$ and $r_\alpha \cdot r_{\alpha'} = -2U_\alpha \cdot U_{\alpha'}$. Consider a subgraph of the Petersen graph obtained by deleting the vertices $U_{23}, U_{24}, U_{25}, U_{14}$. It is the Coxeter–Dynkin diagram of the root system of type E_6. The vectors r_α corresponding to the vertices of this diagram span the lattice M isomorphic to $\mathsf{E}_6(2)$. It is shown in [187] that, for any $i = 1, 2, 3, 4, 5$,

$$r_{ia} + r_{ib} + r_{ic} + r_{id} = 2(h' - \sigma^*(h')), \tag{6.4.19}$$

where $\{a, b, c, d, i\} = \{1, 2, 3, 4, 5\}$ and h' is the class of a plane section of $H(C)$. Using relations (6.4.19), one can easily show that the remaining divisor classes $r_{14}, r_{23}, r_{24}, r_{25}$ are linear combination of the previous six divisor classes and $2(h' - \sigma^*(h'))$. This shows that the sublattice $\mathrm{Im}(1 - \sigma^*)$ of $\mathrm{Pic}^{\omega}(S)$ is isomorphic to E_6.

Let us now compute the lattice $\mathsf{N}_S^{h_{10}}$. It follows from (6.4.17) that each U_α becomes a line in the Fano model defined by $|h| = |h_{10}|$. Suppose R is a (-2)-curve with $h_{10} \cdot R \leq 4$, then $R \cdot \sum_\alpha U_\alpha \leq 4$ implies that R coincides with one of the curves U_α. It follows that $\mathsf{N}_S^{h_{10}}$ is generated by the ten classes $[U_\alpha]$ and the Dynkin graph of its root basis formed by $[U_\alpha]$ is equal to the Petersen graph. The elliptic fibration $|2F_\alpha|$ contains a simple fiber of type $\tilde{A}_5$ with components U_β, where $\alpha \cap \beta \neq \emptyset$. This shows that $\sum_{\beta \cap \alpha \neq \emptyset} [U_\beta] \in 2\,\mathrm{Num}(S)$. Computing the matrix $A = (U_\alpha \cdot U_\beta)$, where we order α's as $(12, 13, 14, 15, 23, 24, 25, 34, 35, 45)$, we find that the null-space of A over $\mathbb{F}_2$ is generated by four linearly independent vectors

$$v_1 = (1, 1, 1, 1, 0, 0, 0, 0, 0, 0),\ v_2 = (1, 0, 0, 0, 1, 1, 1, 0, 0, 0),$$
$$v_3 = (0, 1, 0, 0, 1, 0, 0, 1, 1, 0),\ v_4 = (0, 0, 1, 0, 0, 1, 0, 1, 0, 1).$$

This shows that the discriminant quadratic form has isotropic subgroup isomorphic to $(\mathbb{Z}/2\mathbb{Z})^4$. Thus, the Fano root invariant with respect to h_{10} is equal to $(\mathsf{N}_S^{h_{10}}, \mathsf{G}_{h_{10}})$, where $\mathsf{G}_{h_{10}} = (\mathbb{Z}/2\mathbb{Z})^4$. In particular, the quadratic space $\overline{\mathsf{N}}^{h_{10}}$ is of dimension 6. By taking the representatives of $\Pi_{h_{10}}^{\mathrm{nod}}/\mathsf{G}_{h_{10}}$, we find that it is a regular odd quadratic space that coincides with $\langle \mathsf{Nod}(S) \rangle$ defined by the Nikulin R-invariant $(\mathsf{E}_6, 0)$.

6.5 General Nodal Surfaces

In this section, we will study nodal Enriques surfaces with the smallest possible Nikulin R-invariant. Over the complex numbers, they are general in the sense of the moduli space of nodal Enriques surfaces.

A nodal Enriques surface S is said to be *general* if $\mathsf{Nod}(S)$ consists of one element (in other words, all (-2)-curves are congruent modulo $2\,\mathrm{Num}(S)$).

Lemma 6.5.1 *Assume S is general nodal. Then, for any two (-2)-curves R and R'*

$$R \cdot R' \equiv 2 \mod 4.$$

Proof We have $R' - R = 2x$ for some $x \in \mathrm{Num}(S)$. This yields $-4 - 2(R \cdot R') = 4x^2 \equiv 0 \mod 8$, hence $R \cdot R' \equiv 2 \mod 4$. □

Corollary 6.5.2 *Let S be a general nodal Enriques surface. Then, S has neither chains of (-2)-curves nor disjoint (-2)-curves.*

Corollary 6.5.3 *A general nodal surface does not admit quasi-elliptic fibrations.*

Proof It follows from Proposition 4.1.14 that the curve of cusps $\mathfrak{C}$ in a quasi-elliptic fibration is a smooth rational curve. By Corollary 4.1.12, a quasi-elliptic fibration on an Enriques surface contains a reducible fiber. Obviously, an irreducible component of a reducible fiber that intersects the curve of cusps is not congruent to $\mathfrak{C}$ modulo $2\,\mathrm{Num}(S)$. □

The following corollary follows from Theorems 6.3.3 and 6.3.1.

Corollary 6.5.4 *Every (-2)-curve on a general nodal Enriques surface is realized as an irreducible component of some genus one fibration or as a special bisection.*

Two (-2)-curves R and R' are called *f-equivalent* if there exists a sequence of genus one fibrations $|2F_1|, \dots, |2F_{k-1}|$ and a sequence of (-2)-curves $R_1 = R, \dots, R_k = R'$ such that

$$R_1 + R_2 \in |2F_1|, R_2 + R_3 \in |2F_2|, \dots, R_{k-1} + R_k \in |2F_{k-1}|.$$

Obviously, the f-equivalence is an equivalence relation on the set of (-2)-curves.

The following result gives some characterizations of general nodal Enriques surfaces.

Theorem 6.5.5 *The following properties are equivalent:*

(i) S is a general nodal Enriques surface.
(ii) Any genus one fibration on S contains at most one reducible fiber that consists of two irreducible components. A half-fiber is irreducible.
(ii)' Any genus one fibration on S contains at most one reducible fiber that consists of two irreducible components.
(iii) The degeneracy invariant c of any canonical isotropic 10-sequence is larger than or equal to 9.
(iv) Any two (-2)-curves are f-equivalent.
(v) For any Fano polarization, the set $\Pi^{\mathrm{nod}}_{h_{10}}$ consists of one element.
(vi) For any $d \le 4$, S admits a Fano polarization such that $\Pi^{\mathrm{nod}}_{h_{10}}$ consists of one vector represented by a (-2)-curve R with $R \cdot h_{10} = d$.
(vii) A genus one pencil that admits a special bisection does not contain reducible fibers.

Proof (i) $\Rightarrow$ (ii) The first assertion follows from Corollary 6.5.2. It remains to show that a half-fiber F is irreducible. If not, then, by the same corollary, it must consist of two components R_1, R_2 intersecting with multiplicity 2. Then, $R_1 + R_2 \equiv 2R_1 \mod 2\,\mathrm{Num}(S)$, hence $[F]$ is divisible by 2, a contradiction.

(ii) $\Rightarrow$ (ii)'. Obvious.

(ii)' $\Rightarrow$ (iii) Suppose we find a canonical isotropic sequence $(f_1, \dots, f_{10})$ with the non-degeneracy invariant ≤ 8. It follows from (ii)' that no nodal cycle of type $A_k, k > 1$, has zero intersection with a nef isotropic class f. Without loss of generality, we may assume that f_1, f_2 are nef and $f_3 = f_1 + R_1, f_4 = f_2 + R_2$ with notations from Proposition 6.1.5. We have $f_1 \cdot R_1 = 1, f_1 \cdot R_2 = 0$. Again, since there

are no chains of (-2)-curves of length larger than 1, we can find a nef class f_i, for some $i > 4$, such that $f_i \cdot R_1 = f_i \cdot R_2 = 0$. This gives us a genus one fibration that does not satisfy property (ii)'.

(iii) $\Rightarrow$ (ii) Suppose (ii) is not true. Then, there exist two (-2)-curves R_1 and R_2 with $R_1 - R_2 \notin 2\,\mathrm{Num}(S)$ and a nef isotropic vector f_1 such that $f_1 \cdot R_1 = f_1 \cdot R_2 = 0$. Let us extend f_1 to a canonical isotropic sequence $(f_1, \ldots, f_{10})$. By assumption, its degeneracy invariant ≥ 9. Consider the vectors $v_i = (f_1 \cdot R_i, \ldots, f_{10} \cdot R_i) \mod 2$. Since the images of f_i form a basis in $\overline{\mathrm{Num}}(S)$, these vectors are non-zero and $v_1 \neq v_2$. Moreover, since $R_1 \cdot R_2$ is equal to 0 or 1 (if they lie in the same fiber), the image of $R_1 + R_2$ in $\overline{\mathrm{Num}}(S)$ is of square norm 0. This implies that the vectors v_1, v_2 have at least two different coordinates. Without loss of generality, we may assume $f_2 \cdot R_1 = f_3 \cdot R_2 = 1$ and $f_3 \cdot R_1 = f_2 \cdot R_2 = 0$. By assumption, only one of the classes f_2, f_3 is not nef. Suppose that f_2 is nef. If R_1, R_2 are components of the same fiber with $R_1 \cdot R_2 = 1$, then $(f_1, f_2, f_2 + R_1, f_2 + R_1 + R_2)$ can be extended to a canonical isotropic sequence with degeneracy invariant ≤ 8. Thus, we have proved that no fiber contains more than two irreducible components, so we may assume that R_1 and R_2 are components of different fibers.

Suppose that f_2, f_3 are both nef. Then, $(f_1, f_2 + R_1, f_3 + R_2)$ extends to a canonical isotropic sequence with the non-degeneracy invariant ≤ 8. If f_2 is not nef and f_3 is nef, then one of $f_i, i > 3$ is equal to $f_2 + R$. Since $f_i \cdot f_1 = f_i \cdot f_2 = 1$, we obtain $R \cdot R_1 = f_i \cdot R = 0$. Thus, $(f_1, f_2, f_2 + R, f_3, f_3 + R_1)$ extends to a canonical isotropic sequence with the non-degeneracy invariant ≤ 8.

(iii) $\Rightarrow$ (iv) Let R and R' be two (-2)-curves with $R \cdot R' = n$. We use induction on n. If $n = 2$, by (iii), $R + R'$ must be a simple fiber of a genus one fibration, hence R and R' are f-equivalent. Assume $n > 2$. Since S has no chains of (-2)-curves of length larger than one, by Theorem 6.3.1, there exists a nef isotropic vector f_1 with $f_1 \cdot R = 0$. We use the argument and the notation from the previous paragraph to find an isotropic sequence $(f_1, \ldots, f_{10})$ with f_1 nef and $f_2 \cdot R = 1$. Now, we extend $(f_1, f_2, f_2 + R)$ to a canonical isotropic sequence $(g_1, g_2, g_3, g_4, \ldots, g_{10})$. By assumption (iii), all $g_i, i \neq 3$, are nef. Since $g_j \cdot R = 0$ for $j \geq 3$, the curve R is contained in a simple fiber of each pencil $|2G_j|, j \neq 3$, with $[G_j] = g_j$. Let R_j be another component of a fiber of $|2G_j|$. If $R_j \cdot R' < n$, we are done by induction, so we may assume that $R_j \cdot R' \geq n$ for $j \neq 3$. Let $h = \frac{1}{3}(g_1 + \ldots + g_{10})$. We know that $R' \cdot g_j \geq R' \cdot R_j \geq n$ for $j \neq 3$ and also $R' \cdot g_1 \geq R' \cdot R \geq n$. Since we may assume that R' is not a fiber component of $|2G_1|$, we have $R' \cdot g_3 = R' \cdot (R + R') \geq n$. Intersecting h_{10} with R', we get

$$h_{10} \cdot R' \geq \frac{1}{3}(10n) \geq 10 = \frac{3h^2}{\Phi(h)}.$$

Applying Theorem 6.3.7, we find a genus one pencil $|2F'|$ such that $F' \cdot R' = 0$, and $0 < h_{10} \cdot F' < h_{10} \cdot R'$. Let $R' + R''$ be a fiber of $|2F'|$. We have $h_{10} \cdot R'' = 2h_{10} \cdot F' - h_{10} \cdot R' < h_{10} \cdot R'$. The curve R'' is f-equivalent to R', if $R' \cdot R, R'' \cdot R > n$, we repeat the argument, replacing R' with R'' to decrease further $h_{10} \cdot R''$. Continuing in this way, we get a (-2)-curve equivalent to R' which intersects R or R' with

multiplicity $< n$. Note that if $h_{10} \cdot R'' = 0$, the curve R'' must be in one of the fibers of $|2G_j|$.

(iv) $\Rightarrow$ (i) We use that f-equivalence implies that two (-2)-curves are congruent modulo $2\,\mathrm{Num}(S)$.

(i) $\Leftrightarrow$ (v) Obvious.

(v) $\Rightarrow$ (vi) Obvious.

(v) $\Leftarrow$ (vi) We have to show that, for any $0 \le d \le 4$, there exists a Fano polarization h_{10} such that the unique nodal curve R in $\Pi_{h_{10}}$ satisfies $h_{10} \cdot R = d$. We know that S contains a nodal curve R with $d = h_{10} \cdot R \le 4$. Suppose $d < 4$. We use Corollary 6.3.9. If $d = 0$, we represent R by some $f_i - f_j$. Then, $\alpha = h_{10} - f_i - f_k - f_l$, $j \ne k, l$, satisfies $\alpha^2 = -2, \alpha \cdot h_{10} = 1$. If it were effective, it must be linearly equivalent to a (-2)-curve R'. However, α is not congruent to $R \mod 2\,\mathrm{Num}(S)$. Hence, it is not effective, and we can apply the reflection $x \mapsto x + (x \cdot \alpha)\alpha$ to transform h_{10} to a new polarization h'_{10} such that the class of R becomes equal to $h_{10} - f_j - f_k - f_l$. We have now $h' \cdot R = 1$. Now, take $\alpha = h_{10} - f_m - f_n - f_r$, where $\{j, k, l\} \cap \{m, n, r\} = \emptyset$. As above, applying the reflection in α, we obtain the class of a (-2)-curve R with $h_{10} \cdot R = 2$ representing $2h_{10} - f_j - f_k - f_l - f_m - f_n - f_r = -h + f_a + f_b + f_c + f_d$. Next, we use $\alpha = h_{10} - f_c - f_d - f_j$ to transform h_{10} to h'_{10} such that the (-2)-curve with the class $f_a + f_b - f_j$ satisfies $R \cdot h'_{10} = 3$. Finally, we use $\alpha = h_{10} - f_a - f_b - f_j$ to find a (-2)-curve with $R \cdot h'_{10} = 4$.

(i) $\Rightarrow$ (vii) Suppose $|2F|$ is a genus one fibration with a special bisection R and a reducible fiber D. Then, R intersects one of the components R' of D with multiplicity 1 or 0. This contradicts Lemma 6.5.1.

(vii) $\Rightarrow$ (iii) Suppose there is a canonical isotropic sequence $(f_1, \dots, f_{10})$ with the degeneracy invariant $c \le 8$. Without loss of generality, we may assume that $(f_1, f_2, f_3, f_4) = (f_1, f_1{+}R, f_3, f_3{+}R')$ or $(f_1, f_2, f_3, f_4) = (f_1, f_1{+}R, f_1{+}R{+}R', f_4)$. In the first case $R \cdot f_1 = 1, R' \cdot f_1 = 0$ contradicting condition (vii). In the second case, $R' \cdot f_1 = 0, R \cdot f_1 = 1$, again contradicting (vii). □

Remark 6.5.6 Some of the properties of elliptic fibrations on a general nodal Enriques surface follow also from the analysis of isomorphism classes of elliptic fibrations on such surfaces based on the known structure of the automorphism group (see Section 8.4). For example, this analysis shows that a general nodal surface does not admit two elliptic fibrations $|2F_1|$ and $|2F_2|$ with $F_1 \cdot F_2 = 2$ that share a common irreducible fiber component.

Remark 6.5.7 We will prove later in Corollary 7.9.9 that, assuming $p \ne 2$, the Picard lattice of the canonical cover of a general nodal Enriques surface S is isomorphic to $\mathsf{U} \oplus \mathsf{E}_8(2) \oplus \mathsf{A}_1(2)$.

The following corollary follows from the discussions in the previous section.

Corollary 6.5.8 *Let S be a general nodal Enriques surface. The lattice $\mathsf{N}_S^{h_{10}}$ is isomorphic to A_1 and the R-invariant is $(\mathsf{A}_1, \{0\})$. It coincides with the Nikulin R-invariant when the canonical cover is étale. The Reye lattice is isomorphic to $\mathsf{E}_{2,4,6}$ and $\langle \mathsf{Nod}(S) \rangle$ is a one-dimensional defective quadratic space. Conversely, any of these properties characterize a general nodal Enriques surface.*

Corollary 6.5.9 *Let R be a (-2)-curve on a general nodal Enriques surface S. Then, there exists a canonical isotropic 10-sequence $(f_1, \dots, f_{10})$ with the non-degeneracy invariant 9 such that $f_1, \dots, f_9$ are nef, and $f_{10} = f_9 + R$.*

Proof By Theorem 6.5.5 (vi), there exists a numerical Fano polarization h_{10} and a (-2)-curve R with $h_{10} \cdot R = 0$. Thus, if we write $3h_{10} = f_1 + \cdots + f_{10}$, as usual, the non-degeneracy invariant c of the isotropic sequence $(f_1, \dots, f_{10})$ will be ≤ 9. Thus, we may assume that $f_1, \dots, f_9$ are nef and $f_{10} = f_9 + r$, where r is the class of a (-2)-curve. Intersecting with $[R]$, we get $0 = f_1 \cdot [R] + \cdots + 2f_9 \cdot [R] + r \cdot [R]$. This obviously implies $r = [R]$. □

Recall that we proved in Theorem 6.3.1 that any (-2)-curve on a nodal, not extra-special, Enriques surface occurs as an irreducible component of a fiber of some genus one fibration. We also proved that in Theorem 6.3.3 that there exists a genus one fibration with a special bisection. The next corollary strengthens the latter result.

Corollary 6.5.10 *Any (-2)-curve on a general nodal Enriques surface is a special bisection of some elliptic fibration.*

Corollary 6.5.11 *Let S be a general nodal surface. Then, its canonical cover is a normal surface. Moreover, $H^0(S, \Theta_S) = \{0\}$ unless S is an $\boldsymbol{\alpha}_2$-surface.*

Proof It follows from Corollary 10.2.7 in Section 10.2 that the singular points of the canonical cover lie over singular points of fibers in any genus one fibration on S. By Corollary 6.5.3, all genus one fibrations are elliptic fibrations. Since S is a general nodal Enriques surface, there is at most one reducible fiber and its type is $\tilde{A}_1$. Thus, the canonical cover has only isolated singular points, hence it must be a normal surface birationally isomorphic to a K3 surface or a rational surface with one elliptic singularity. If $K_S \neq 0$, then the second assertion follows from Theorem 1.4.10. Also, by Corollary 1.4.5 from Volume I, a $\boldsymbol{\mu}_2$-surface has no non-zero regular global fields and an $\boldsymbol{\alpha}_2$-surface has them always. □

Remark 6.5.12 It is not true that the canonical cover of a unnodal Enriques surface must be birationally isomorphic to a K3 surface. There are examples due to S. Schröer [639] and Y. Matsumoto [494] of an unnodal Enriques surface S such that all its elliptic fibrations have only one singular irreducible fiber and its canonical cover is a normal rational surface. It is proven by Matsumoto that S must be an $\boldsymbol{\alpha}_2$-surface [494, Proposition 3.2].

Bibliographical Notes

The notion of an isotropic 10-sequences of the numerical classes of elliptic curves on an Enriques surface goes back to G. Fano [226]. He used it to study of a degree 10 birational model of Enriques surfaces, the Fano models we discussed in Volume I. Their first use in modern literature can be found in [42]. The first systematic study of such sequences, and in particular, introducing the notion of a canonical isotropic sequence and the proof of Proposition 6.1.5 were given by F. Cossec [135].

The notion of the non-degeneracy invariant $\mathsf{nd}(S)$ of an Enriques surface was introduced in [138]. A recent paper [525] gives an algorithm for its computation.

The proof of Theorem 6.1.10 that asserts that $\mathsf{nd}(S) \geq 2$ for any Enriques surface in characteristic different from 2 was first given by Cossec [135, Proposition 3.4]. It was extended to the case of Enriques surfaces in characteristic 2 in [138]. The proof of Theorem 6.1.12 in the case $p \neq 2$ was indicated by Cossec [135, Theorem 3.5], who left to complete the proof to the reader. The proof that works in all characteristics, based on consideration of many cases arising from possible extensions of a non-degenerate isotropic pair to a canonical maximal isotropic sequence, was undertaken in [138]. It required more than thirty pages and almost surely the analysis missed some cases. The proof of Theorem 6.1.12 in the case $p \neq 2$ is new and based on entirely different ideas. Recently, the theorem was proved in all characteristics in [492].

The notion of extra-special Enriques surfaces of types $\tilde{E}_8$, $\tilde{D}_8$ and $\tilde{A}_1 + \tilde{E}_7$ (that combines our types $\tilde{E}_7^{(1)}$ and $\tilde{E}_7^{(2)}$) was introduced in [138, Chapter III, §5]. It was wrongly asserted there that the non-degeneracy invariant of the surface of type $\tilde{E}_7^{(2)}$ is equal to 2. In Section 6.2 of this chapter, we defined an extra-special surface as a surface with $\mathsf{nd}(S) \leq 2$ and proved in Theorem 6.2.6 that the definitions are equivalent. An extra-special surface of type $\tilde{E}_8$ was considered earlier by W. Lang [435, Appendix A], where it was called a surface of hyperbolic type. The first construction of extra-special surfaces was given in an unpublished paper by Salomonsson [633].

The fact that any rational smooth curve is contained in a fiber of some genus one fibration was proven by Cossec [135, Theorem 4.1] in the case $p \neq 2$. He also proved that any nodal Enriques surface contains an elliptic fibration with a special bisection. The extension of this result to characteristic 2 is due to W. Lang [435, Theorem A3].

Theorems 6.3.8 and 6.3.16 from Section 6.3 were first proven in [136].

The notions of nodal and Nikulin R-invariant from Section 6.4 were first introduced in [559]. The Fano root invariant that can be defined in all characteristics seems to be new. The notion of the Reye lattice coincides with the notion of the Reye lattice introduced in [138] only in the case when the nodal invariant consists of one vector. We also discussed in this section a slightly different definition of R-invariant given by S. Mukai, and we followed A. Beauville [51] to relate it with the Brauer group of the K3 cover of an Enriques surface.

The systematic study and different characterization of general nodal Enriques surfaces are new. Over the field of complex numbers the fact that a general, in the sense of moduli, nodal surface can be characterized by the condition that the classes of all smooth rational curves are congruent modulo 2 follows from the work of Nikulin [559].

Chapter 7
Reye Congruences

In this chapter, we discuss a classical theory of congruences of lines in $\mathbb{P}^3$ and examples of nodal Enriques surfaces called Reye congruences. A general nodal Enriques surface, in the sense of moduli, is isomorphic to a Reye congruence. We also present an analog of a Reye congruence in characteristic 2.

7.1 Congruences of Lines

According to classical terminology, a *congruence of lines* in $\mathbb{P}^3$ is an irreducible surface S in the Grassmann variety $\mathbb{G} = G_1(\mathbb{P}^3)$ of lines in $\mathbb{P}^3$. The line ℓ_s in $\mathbb{P}^3$ corresponding to a point $s \in S$ is a *ray* of the congruence. In this section, we recall some basic attributes of a congruence of lines such as its bidegree, sectional genus, fundamental points and focal surface. We will see later that a general nodal Enriques surface is isomorphic to a smooth congruence of lines of bidegree $(7, 3)$, a Reye congruence.

For any point x, a line ℓ and a plane Π in $\mathbb{P}^3$, let

$$\sigma_x = \{\ell \in \mathbb{G} : x \in \ell\}, \tag{7.1.1}$$

$$\sigma_\Pi = \{\ell \in \mathbb{G} : \ell \subset \Pi\}, \tag{7.1.2}$$

$$\sigma_\ell = \{\ell' \in \mathbb{G} : \ell' \cap \ell \neq \emptyset\}, \tag{7.1.3}$$

$$\sigma_{x,\Pi} = \{\ell \in \mathbb{G} : x \in \ell \subset \Pi\}, \tag{7.1.4}$$

be the *Schubert subvarieties* of $G_1(\mathbb{P}^3)$ (see [242] or [177, Chapter 10]). The Chow ring $A^*(\mathbb{G})$ is generated by the classes of these varieties, the class of a point and the class of $\mathbb{G}$:

$$A^*(\mathbb{G}) = \bigoplus_{i=0}^{4} A^i = \mathbb{Z}[\mathbb{G}] \oplus \mathbb{Z}[\sigma_\ell] \oplus \big(\mathbb{Z}[\sigma_x] \oplus \mathbb{Z}[\sigma_\Pi]\big) \oplus \mathbb{Z}[\sigma_{x,\Pi}] \oplus \mathbb{Z}[\text{point}].$$

The multiplication in $A^*(\mathbb{G})$ is determined by formulas:

$$\begin{aligned}&[\sigma_\Pi]^2 = [\sigma_x]^2 = [\text{point}], \quad [\sigma_x]\cdot[\sigma_\Pi] = 0,\\ &[\sigma_x]\cdot[\sigma_\ell] = [\sigma_{x,\Pi}], \quad [\sigma_\Pi]\cdot[\sigma_\ell] = [\sigma_{x,\Pi}], \quad [\sigma_\ell]^2 = [\sigma_\Pi] + [\sigma_x].\end{aligned} \tag{7.1.5}$$

The algebraic cocycle class $[S]$ of a congruence is determined by the two numbers (m, n) called the *order* and the *class* of S:

$$[S] = m[\sigma_x] + n[\sigma_\Pi].$$

It follows from the previous formulas that

$$m = [S]\cdot[\sigma_x],\ n = [S]\cdot[\sigma_\Pi],\ m + n = [S]\cdot[\sigma_\ell]^2.$$

The Grassmann variety $\mathbb{G}$ is isomorphic to a nonsingular quadric in $\mathbb{P}^5$ embedded via the Plücker embedding. The class $[\sigma_\ell]$ is the class of a hyperplane section of $\mathbb{G}$. Thus, the number $m + n$ coincides with the degree of S in the Plücker embedding. The pair (m, n) will be called the *bidegree* of S.

In a coordinate-free way, we consider $\mathbb{P}^3$ as the variety $|E|$ of lines in a linear space E of dimension 4 over $\Bbbk$. In Grothendieck's notation, $|E| = \mathbb{P}(E^\vee)$, where $E^\vee$ is the dual linear space. The Plücker space becomes $|\bigwedge^2 E| = \mathbb{P}(\bigwedge^2 E^\vee)$. A line in $|E|$ is $|U|$, where U is a 2-dimensional subspace of E, and the Plücker embedding is $|U| \mapsto |\bigwedge^2 U| \subset |\bigwedge^2 E|$. The Grassmann variety comes equipped with a natural exact sequence of locally free sheaves

$$0 \to \mathcal{S}_\mathbb{G} \to E \otimes \mathcal{O}_\mathbb{G} \to \mathcal{Q}_\mathbb{G} \to 0$$

and the dual exact sequence

$$0 \to \mathcal{Q}_\mathbb{G}^\vee \to E^\vee \otimes \mathcal{O}_\mathbb{G} \to \mathcal{S}_\mathbb{G}^\vee \to 0. \tag{7.1.6}$$

The geometric vector bundle $\mathbb{V}(\mathcal{S}_\mathbb{G}^\vee)$ is called the *universal subbundle* or *tautological subbundle*. Its fiber over a point $\ell = [U] \in \mathbb{G}$ is the subspace U of the fiber of $\mathbb{V}(E \otimes \mathcal{O}_\mathbb{G}^\vee) = E$, where we consider E as the associated affine space over $\Bbbk$. The geometric vector bundle $\mathbb{V}(\mathcal{Q}_\mathbb{G}^\vee)$ is called the *universal quotient bundle* or *tautological quotient bundle*. Its fiber over a point $\ell = [U] \in \mathbb{G}$ is the quotient space E/U. The surjection $E^\vee \otimes \mathcal{O}_\mathbb{G} \to \mathcal{S}_\mathbb{G}^\vee$ defines a canonical closed embedding

$$Z_\mathbb{G} := \mathbb{P}(\mathcal{S}_\mathbb{G}^\vee) \hookrightarrow \mathbb{P}(E \otimes \mathcal{O}_\mathbb{G}) = |E| \times \mathbb{G}.$$

The Plücker embedding is given by the surjection $\bigwedge^2 E^\vee \otimes \mathcal{O}_\mathbb{G} \to \bigwedge^2 \mathcal{S}_\mathbb{G}^\vee$. In particular,

$$\bigwedge^2 \mathcal{S}_\mathbb{G}^\vee \cong \bigwedge^2 \mathcal{Q}_\mathbb{G} \cong \mathcal{O}_\mathbb{G}(1). \tag{7.1.7}$$

The usual properties of the Chern classes and equalities (7.1.5) give

$$c_1(\mathcal{Q}_{\mathbb{G}}) = [\sigma_\ell] = c_1(\mathcal{O}_{\mathbb{G}}(1)),$$
$$c_2(\mathcal{S}_{\mathbb{G}}) = [\sigma_\Pi],$$
$$c_2(\mathcal{Q}_{\mathbb{G}}) = [\sigma_x].$$

Let

$$\begin{tikzcd} & Z_{\mathbb{G}} \arrow[dl, "p"'] \arrow[dr, "q"] & \\ |E| & & \mathbb{G} \end{tikzcd}$$

be the projection maps. The variety $Z_{\mathbb{G}}$ coincides with the flag variety of points–lines in $|E|$:

$$Z_{\mathbb{G}} = \{(x, \ell) \in |E| \times \mathbb{G} : x \in \ell\}.$$

The projection $q : Z_{\mathbb{G}} \to \mathbb{G}$ is a projective bundle. It is the projective subbundle of the trivial bundle $E_{\mathbb{G}} := \mathbb{P}(E^\vee \otimes \mathcal{O}_{\mathbb{G}})$. We have $\mathcal{O}_{E_{\mathbb{G}}}(1) \cong p^*\mathcal{O}_{|E|}(1)$. The restriction $\mathcal{O}_{Z_{\mathbb{G}}}(1)$ of $\mathcal{O}_{E_{\mathbb{G}}}(1)$ to $Z_{\mathbb{G}}$ is the tautological invertible sheaf corresponding to the choice of an isomorphism $Z_{\mathbb{G}} \cong \mathbb{P}(\mathcal{S}_{\mathbb{G}}^\vee)$.

The projection map $p : Z_{\mathbb{G}} \to |E|$ is a projective bundle over $|E|$. The Euler exact sequence

$$0 \to \Omega^1_{|E|} \to E^\vee \otimes \mathcal{O}_{|E|}(-1) \to \mathcal{O}_{|E|} \to 0 \tag{7.1.8}$$

defines a canonical isomorphism of the projective bundles

$$Z_{\mathbb{G}} \cong \mathbb{P}(\Omega^1_{|E|}(1)). \tag{7.1.9}$$

Twisting (7.1.8) by $\mathcal{O}_{|E|}(2)$, we obtain a canonical isomorphism

$$H^0(|E|, \Omega^1_{|E|}(2)) \cong \mathrm{Ker}(E^\vee \otimes E^\vee \to S^2E^\vee) = \bigwedge^2 E^\vee. \tag{7.1.10}$$

Let $Z_S = q^{-1}(S) \subset Z_{\mathbb{G}}$ and let

$$\begin{tikzcd} & Z_S \arrow[dl, "p_S"'] \arrow[dr, "q_S"] & \\ |E| & & S \end{tikzcd}$$

be the restrictions of the projections p and q to Z_S. For any point $x \in |E|$, the projection q_S defines an isomorphism of the fiber $p_S^{-1}(x)$ with $S \cap \sigma_x$. In particular, if $m > 0$, $p_S : Z_S \to |E|$ is a morphism of degree m. It is known, and it is easy to prove, that a congruence S with $m = 0$ is equal to a Schubert variety σ_Π for some plane Π.

There is a natural duality isomorphism $G_1(|E|) \cong G_1(|E^\vee|)$ that assigns to a line ℓ the pencil of planes containing ℓ. Under this isomorphism a congruence of bidegree (m, n) is mapped to a congruence of bidegree (n, m). In particular, a congruence

with $n = 0$ has order $m = 1$ and coincides with a Schubert variety σ_x. The varieties $\sigma_x, x \in |E|$, (resp. $\sigma_\Pi, \Pi \in \mathbb{P}(E)$) are planes, classically called *α-planes* (resp. *β-planes*). They form two rulings of the quadric $\mathbb{G}$ by planes.

From now on, we assume that S is smooth and $m, n > 0$, if not stated otherwise. We also assume that the projection map p_S is separable.

Let $\mathsf{R}(S) \subset Z_S$ be the set of points $z = (x, \ell) \in Z_S$ such that the morphism p_S is not smooth at z. Since S is smooth, Z_S is also smooth and consists of points at which the relative differential sheaf $\Omega^1_{Z_S/\mathbb{P}^3}$ is not zero. Thus, the set $\mathsf{R}(S)$ is the support of the closed subscheme of Z_S given by the Fitting ideal defined by the map of locally free sheaves $p_S^*\Omega^1_{\mathbb{P}^3} \to \Omega^1_{Z_S}$. We equip $\mathsf{R}(S)$ with the structure of this subscheme. Recall that the Fitting ideal sheaf is the image of the canonical map

$$\bigwedge^3 p_S^*\Omega_{\mathbb{P}^3} \otimes (\bigwedge^3 \Omega^1_{Z_S})^{-1} \to \mathcal{O}_{Z_S}.$$

Locally, it is given by the determinant of a 3×3-matrix, in particular $\mathsf{R}(S)$ is an effective divisor on Z_S such that

$$K_{Z_S} \sim \mathsf{R}(S) + p_S^*(K_{\mathbb{P}^3}). \tag{7.1.11}$$

The following lemma computes the canonical sheaf of $Z_{\mathbb{G}}$ and Z_S (see [177, 10.1.1]).

Lemma 7.1.1

$$\omega_{Z_{\mathbb{G}}} \cong p_S^*\mathcal{O}_{\mathbb{P}^3}(-2) \otimes q_S^*\mathcal{O}_{\mathbb{G}}(-3),$$
$$\omega_{Z_S} \cong p_S^*\mathcal{O}_{\mathbb{P}^3}(-2) \otimes q_S^*\omega_S(1).$$

Applying (7.1.11), we obtain

$$\mathcal{O}_{Z_S}(\mathsf{R}(S)) \cong q_S^*\omega_S(1) \otimes p_S^*\mathcal{O}_{\mathbb{P}^3}(2) \cong q_S^*\omega_S(1) \otimes \mathcal{O}_{Z_S}(2). \tag{7.1.12}$$

Let

$$p_S : Z_S \xrightarrow{\alpha_S} Z'_S \xrightarrow{p'_S} \mathbb{P}^3$$

be the Stein factorization. The map $\alpha_S : Z_S \to Z'_S$ is a birational morphism onto a normal variety Z'_S and the map $p'_S : Z'_S \to \mathbb{P}^3$ is a finite separable morphism of degree m.

The closed subset $p_S(\mathsf{R}(S))$ of $\mathbb{P}^3$ is called the *focal locus* of S. It contains the image $\mathrm{Foc}(S)$ of the ramification divisor of p'_S (if $m > 1$), called the *focal surface*. The image of the exceptional locus of α_S is called the *fundamental locus* and denoted by $\mathrm{Fund}(S)$. Its points are *fundamental points* of the congruence. If $m > 1$, its isolated points belong to $\mathrm{Foc}(S)$. The one-dimensional part of $\mathrm{Fund}(S)$ is called the *fundamental curve* of S..

In general, the divisor $\mathsf{R}(S)$ could be very complicated. For our needs, we assume that $\mathsf{R}(S)$ is reduced. Let $\mathsf{R}(S) = \mathsf{R}(S)_1 + \mathsf{R}(S)_2$, where $\mathsf{R}(S)_1$ is the union of irreducible components of $\mathsf{R}(S)$ that are mapped dominantly onto the focal surface

$\mathrm{Foc}(S)$ and $\mathsf{R}(S)_2$ is the union of irreducible components that are mapped onto the fundamental curve of S. Applying (7.1.12), and using the projection formula, we find that the restriction

$$q'_S : \mathsf{R}(S) \to S$$

of q_S to $\mathsf{R}(S)$ is of degree 2. Let d_1 (resp. d_2) be the degree of the restriction of q'_S to $\mathsf{R}(S)_1$ (resp. $\mathsf{R}(S)_2$).

Assume $m > 1$ so that $\mathsf{R}(S)_1 \neq 0$. We have the following possible cases for (d_1, d_2):

(i) $(d_1, d_2) = (2, 0)$: $\mathsf{R}(S)_1$ (and hence $\mathrm{Foc}(S)$) either consists of two irreducible components which are rational sections of $q_S : Z_S \to S$, or $\mathsf{R}(S)_1$ is irreducible (hence $\mathrm{Foc}(S)$ is irreducible) and it is a rational bisection of q_S.
(ii) $(d_1, d_2) = (1, 1)$: the fundamental curve of S is irreducible, and both $\mathsf{R}(S)_1$ and $\mathsf{R}(S)_2$ are rational sections of q_S.
(iii) $(d_1, d_2) = (0, 2)$: $\mathrm{Foc}(S)$ is the union of *focal rays* of S, i.e. rays contained in the focal surface). The fundamental curve is either irreducible and defines a rational bisection of q_S, or it consists of two irreducible components which are rational sections of q_S.

Assume that we are in case (i).

Let

$$\mathsf{R}(S) \to \mathsf{R}(S)' \to S$$

be the Stein factorization of the map $q'_S : \mathsf{R}(S)_1 \to S$. The fibers of the first map are the pre-images of focal rays under the projection q_S. The images of these fibers in $\mathbb{P}^3$ are focal rays.

Assume that $\mathrm{Foc}(S)$ does not contain focal rays, we apply Lemma 7.1.1 and the adjunction formula to obtain:

$$\omega_{\mathsf{R}(S)'} \cong q_S^* \omega_S(1)^{\otimes 2} \otimes \mathcal{O}_{\mathsf{R}(S)}. \tag{7.1.13}$$

Comparing it with the formula for the canonical sheaf of a cyclic cover of degree 2 from (0.2.8) in Volume I, we find that the double cover $q'_S : \mathsf{R}(S)_1 \to S$ is defined by the invertible sheaf $\mathcal{L} = \omega_S(1)$. If the double cover is separable, its branch curve B satisfies

$$\mathcal{O}_S(\mathsf{B}) \cong \mathcal{L}^{\otimes 2} \cong \omega_S^{\otimes 2} \otimes \mathcal{O}_S(4). \tag{7.1.14}$$

In particular,

$$[\mathsf{B}] = 2K_S + 4h, \tag{7.1.15}$$

where $h = c_1(\mathcal{O}_S(1))$.

It follows that the fibers of q_S intersect $\mathsf{R}(S)$ (in Z_S) with multiplicity 2 and are not contained in $\mathsf{R}(S)$. This implies that their images in $\mathbb{P}^3$ which are the rays of S intersect the focal surface $\mathrm{Foc}(S)$ at two points with multiplicity 2. These points coincide if the ray is the image of a fiber over a point in B. In other words,

$$S \subset \mathrm{Bit}(\mathrm{Foc}(S)), \tag{7.1.16}$$

where, for any reduced surface X in $\mathbb{P}^3$, the *bitangent surface* $\mathrm{Bit}(X)$ of X is defined to be the closure in $\mathbb{G}$ of the set of lines in $\mathbb{P}^3$ that are tangent to X at two points (maybe equal). Note that $\mathrm{Bit}(\mathrm{Foc}(S))$ could be a reducible surface, so S is one of its irreducible components (see Remark 7.4.9).

Let g be the genus of a general hyperplane section of S. It is called the *sectional genus* of S.

Proposition 7.1.2 *Let S be a smooth congruence of bidegree (m, n). Assume that $\mathsf{R}(S)$ is reduced. Then,*

$$\deg \mathrm{Foc}(S) = 2g - 2 + 2m.$$

Proof We use the intersection theory in the Chow ring $A^*(Z_S)$ (see [242]). Let ℓ be a general line in $\mathbb{P}^3$ and $[\ell] \in A^2(\mathbb{P}^3)$ be its class in the Chow ring of $\mathbb{P}^3$. Its pull-back $p_S^*([\ell])$ in $A^*(Z_S)$ is equal to H^2, where $H = c_1(p_S^* \mathcal{O}_{\mathbb{P}^3}(1))$. Applying (7.1.12) and the projection formula, we obtain:

$$\begin{aligned}
&\deg(\mathrm{Foc}(S)) = H^2 \cdot \mathsf{R}(S) = H^2 \cdot \mathsf{R}(S)_1 = H^2 \cdot (2H + q_S^*(h + K_S)) \\
&= 2H^3 + H^2 \cdot q_S^*(h + K_S) = 2m + (q_S)_*(H^2) \cdot (h + K_S) \\
&= 2m + 2 + h \cdot (h + K_S) = 2m + 2g - 2.
\end{aligned} \tag{7.1.17}$$

The following formula can be found in [302]. For the sake of completeness, we supply the proof.

Proposition 7.1.3 *Let S be a smooth congruence of bidegree (m, n) and let g be its sectional genus. Then,*

$$m^2 + n^2 = 3(m + n) + 8(g - 1) + 2K_S^2 - 12\chi(\mathcal{O}_S). \tag{7.1.18}$$

Proof Using the intersection theory on $\mathbb{G}$, we obtain $[S]^2 = m^2 + n^2$. On the other hand, this number is equal to the second Chern class of the normal sheaf $\mathcal{N}_{S/\mathbb{G}} = (\mathcal{I}_S/\mathcal{I}_S^2)^\vee$ of S in $\mathbb{G}$. The standard exact sequence

$$0 \to \mathcal{I}_S/\mathcal{I}_S^2 \to \Omega^1_{\mathbb{G}} \otimes \mathcal{O}_S \to \Omega^1_S \to 0,$$

after passing to the dual sequence and taking the Chern classes, gives

$$\begin{aligned}
&c_1(\mathcal{N}_{S/\mathbb{G}}) = -K_{\mathbb{G}} \cdot S + K_S = -4 \deg c_1(\mathcal{O}_S(1)) + K_S = 4c_1(\mathcal{O}_S(1)) + K_S, \\
&c_2(\mathcal{N}_{S/\mathbb{G}}) = c_2(\mathbb{G}) \cdot S - c_2(S) + K_S \cdot c_1(\mathcal{N}_{S/\mathbb{G}}).
\end{aligned}$$

The second Chern class of the quadric $\mathbb{G}$ in $\mathbb{P}^5$ is computed using the short exact sequence:

$$0 \to \Theta_{\mathbb{G}} \to \Theta_{\mathbb{P}^5} \otimes \mathcal{O}_{\mathbb{G}} \to \mathcal{O}_{\mathbb{G}}(2) \to 0.$$

It easily gives $c_2(\mathbb{G}) = 7c_1(\mathcal{O}_{\mathbb{G}}(1))^2$. Now, we apply Noether's formula $c_2(S) + K_S^2 = 12\chi(\mathcal{O}_S)$ and obtain

$$c_2(\mathcal{N}_{S/\mathbb{G}}) = 7(m + n) - (12\chi(\mathcal{O}_S) - K_S^2) + 4c_1(\mathcal{O}_S(1)) \cdot K_S + K_S^2$$

$$= 7(m+n) - 12\chi(\mathcal{O}_S) + 2K_S^2 + 4c_1(\mathcal{O}_S(1)) \cdot (K_S + c_1(\mathcal{O}_S(1))) - 4c_1(\mathcal{O}_S(1))^2$$

$$= 7(m+n) - 12\chi(\mathcal{O}_S) + 2K_S^2 + 4(2g-2) - 4(m+n) = 3(m+n) - 12\chi(\mathcal{O}_S) + 2K_S^2 + 8(g-1).$$

Corollary 7.1.4 *Let S be a smooth congruence of lines of bidegree $(7,3)$ and sectional genus six. Then, S is the Fano model of an Enriques surface. If S is classical, then it is a nodal Enriques surface.*

Proof For the proof of the first assertion we follow the argument from [263, §3]. It follows from the proposition that $K_S^2 = 6(\chi(\mathcal{O}_S) - 1)$.

Let H be a general hyperplane section. Then, $H^2 = 10$, and the adjunction formula, $2g - 2 = H^2 + H \cdot K_S$ gives

$$K_S \cdot H = -K_S \cdot H = 0. \tag{7.1.19}$$

This implies that $|mK_S|$ and $|-mK_S|$ are empty for any $m > 0$.

Suppose $\chi(\mathcal{O}_S) \geq 1$. Then, $K_S^2 \geq 0$, and, by Riemann-Roch, $h^0(mK_S) \geq \frac{1}{2}(m^2 - 1)K_S^2 + \chi(\mathcal{O}_S) - 1 \geq 0$. This implies that $K_S^2 = 0$. Applying Riemann-Roch and Serre duality to $-K_S$, we obtain $h^0(-K_S) \geq K_S^2 + \chi(\mathcal{O}_S) - h^0(2K_S) \geq 1$. If $h^0(2K_S) = 0$, we get $|-K_S| \neq \emptyset$ contradicting (7.1.19). So, $h^0(2K_S) = 1$, hence $2K_S = 0$, $\chi(\mathcal{O}_S) = 1$, and the surface is an Enriques surface.

Suppose $\chi(\mathcal{O}_S) = 0$. Since $p_g = 0$, we get $q = 1$. Also, we get $K_S^2 = -6$, and $(K_S + H)^2 = 4$. This implies that $K_S \cdot (H + K_S) = -6$ and hence $|H + K_S|$ is a linear system with a general member of genus zero. Thus S is rational contradicting the equality $q = 1$.

Let us prove the second assertion. Since H is an ample divisor with $H^2 = 10$, the Plücker embedding $S \hookrightarrow \mathbb{P}^5$ is a Fano model of S. Suppose S is unnodal, then $3H \sim F_1 + \cdots + F_{10}$, where $([F_1], \ldots, [F_{10}])$ is a non-degenerate canonical sequence of isotropic numerical divisor classes. Since $H \cdot F_i = 3$, the curves F_i are irreducible plane cubics, each spans a plane contained in the quadric $\mathbb{G}$. A smooth quadric in $\mathbb{P}^5$ has two families of planes– α-planes and β-planes. Since $F_1 \cdot F_2 = F_1' \cdot F_2 = 1$, the planes $\langle F_1 \rangle, \langle F_2 \rangle, \langle F_1' \rangle$ spanned by the cubics F_2, F_1 and $F_1' \in |K_S + F_1|$ belong to the same family. In particular, $\langle F_1 \rangle$ and $\langle F_1' \rangle$ intersect. This implies that there exists a hyperplane section H containing F_1 and F_1'. But then $H - F_1 - F_1'$ is effective $(H - F_1 - F_1')^2 = -2$. Hence, one of the irreducible components of an effective divisor in $|H - F_1 - F_1'|$ is a (-2)-curve. This contradiction proves the assertion. □

Remark 7.1.5 It follows from [263, Proposition 1.8] that the Hilbert scheme of congruences of bidegree $(7,3)$ (or $(3,7)$) in the Grassmannian $G_1(\mathbb{P}^3)$ is of dimension 26. Subtracting 15 equal to the dimension of the group $\mathrm{Aut}(G_1(\mathbb{P}^3))$, we obtain that the moduli space of congruences of bidegree $(7,3)$ is of dimension 9 that agrees with the number of moduli of nodal Enriques surfaces. We will see later in Corollary 7.9.8 that any general nodal Enriques surface is isomorphic to a Reye congruence of lines of bidegree $(7,3)$ and sectional genus six.

Unfortunately, we do not know whether an Enriques surface isomorphic to any congruence of lines of bidegree $(7,3)$ and sectional genus equal to six is a nodal Enriques surface.

We finish this section with some examples.

Example 7.1.6 Let C be a smooth connected curve of degree d and genus g in $\mathbb{P}^3$. Consider the congruence $S = \mathrm{Bis}(C)$ of bisecants of C. Projecting from a general point $x \in \mathbb{P}^3$, we obtain a plane curve of degree d and genus $g = \frac{1}{2}(d-1)(d-2) - m$, where m is the order of S. Thus

$$m = \frac{1}{2}(d-1)(d-2) - g.$$

A general plane intersects C at d points. The class n of S is equal to the number of joints of two points in this set, i.e.

$$n = \frac{1}{2}d(d-1).$$

In similar way, we obtain the bidegree of the congruence $\mathrm{Join}(C_1, C_2)$ of joints of two disjoint smooth projective curves of degrees d_1, d_2 and genus g_1, g_2. A joint is a line $\langle x_1, x_2\rangle$, where $x_i \in C_i$. As before, the projection of C_i is a plane curve of genus $g_i = \frac{1}{2}(d_i - 1)(d_i - 2) - m_i$, where m_i is the order of $\mathrm{Bis}(C_i)$. The number of common bisecants is expected to be equal to $\#\mathrm{Bis}(C_1) \cap \mathrm{Bis}(C_2) = m_1m_2 + n_1n_2$. We take the center of the projection not lying on any common bisecant. Taking x even more general, we may assume that the projections intersect transversally at d_1d_2 points. Thus, $m = d_1d_2$. The class is equal to the number of joints of d_1 on C_1 and d_2 points on C_2, i.e.

$$n = m = d_1d_2.$$

It is easy to see that $\mathrm{Join}(C_1, C_2)$ is a complete intersection of two hypersurfaces of degrees d_1 and d_2 in $G_1(\mathbb{P}^3)$. The divisors are the Chow forms of the curves.

We will need two special cases. In the first one, we take $d = 4$ and $g = 1$. The curve C is the intersection of two quadrics. We get a congruence S of bidegree $(2, 6)$. It is isomorphic to the symmetric product $C^{(2)}$ of C. The natural map $C^{(2)} \to \mathrm{Pic}^2(C) \cong C$ is a $\mathbb{P}^1$-bundle. Its fiber over $c \in \mathrm{Pic}^2(C)$ is equal to the linear system $|c|$ of degree 2. The surface is a minimal elliptic ruled surface. We have $K_S^2 = 0, q = 1, p_g = 0$. By formula (7.1.18), its sectional genus is equal to 3. Note that $\mathrm{Bis}(C)$ contains four special lines. They parameterize rays through a point x_i such that the projection from x_i is a double cover of a plane conic. The points x_i are the singular points of four quadric cones containing C. They are also the fundamental points of the congruence. The focal surface is the union of the four singular quadrics containing C. Its degree is 8 which agrees with the formula from Proposition 7.1.2. The ramification divisor $\mathsf{R}(S) \subset Z_S$ is the union of the pre-images of the four special fibers of the ruled surface S.

Another special case is when $S = \mathrm{Join}(C_1, C_2)$, where $d_1 = d_2 = 1$. The congruence is a nonsingular quadric in $G_1(\mathbb{P}^3)$ cut out by a linear space of codimension 2. The lines C_1, C_2 are the fundamental curves of S. The map $\pi_1 : Z_S \to \mathbb{P}^3$ is the blow-up of C_1 and C_2. The focal surface is not defined.

7.2 Hyperwebs of Quadrics

In this section, we study linear systems of quadrics in a projective space and their many attributes. One of them is the Reye variety which, in the case of a general web (= three-dimensional linear system) of quadrics, is isomorphic to an Enriques surface. One can also find here the classification of base-point-free nets of conics.

Let

$$W = |L| \subset |\mathcal{O}_{\mathbb{P}^n}(2)|$$

be an n-dimensional linear system of quadrics in $\mathbb{P}^n = |E|$, a *hyperweb of quadrics*. In this section, we assume that $\mathrm{char}(\Bbbk) \neq 2$. We refer to [177, Chapter 1], for the following varieties associated with W:

- The *discriminant variety* $\mathcal{D}(W) \subset W$ parameterizing singular quadrics in W. It is either the whole $\mathbb{P}^n$ or a hypersurface of degree $n+1$ given by the intersection of W with the discriminant variety of singular quadrics in $\mathbb{P}^n$. If $(\lambda_0, \dots, \lambda_n)$ are projective coordinates in W corresponding to a basis formed by quadrics associated to symmetric matrices $A_0, \dots, A_n$, then

$$\mathcal{D}(W) : \det(\lambda_0 A_0 + \cdots + \lambda_n A_n) = 0. \tag{7.2.1}$$

- The *Steinerian variety* $\mathrm{St}(W) \subset \mathbb{P}^n$, the union of $\mathrm{Sing}(Q)$, $Q \in W$. It is either the whole $\mathbb{P}^n$ or a hypersurface in $\mathbb{P}^n$ of degree $n+1$. If $(t_0, \dots, t_n)$ are projective coordinates in $\mathbb{P}^n$, then

$$\mathrm{St}(W) : \det\left(\left[A_0 \begin{pmatrix} t_0 \\ \vdots \\ t_n \end{pmatrix}, \dots, A_n \begin{pmatrix} t_0 \\ \vdots \\ t_n \end{pmatrix}\right]\right) = 0$$

  (here the matrix is written as the collection of its columns).
- The *base scheme* $\mathrm{Bs}(W)$ of W, the scheme-theoretical intersection of quadrics in W.
- The *polar base scheme* $\mathrm{PB}(W) \subset \mathbb{P}^n \times \mathbb{P}^n$, the base scheme of the linear system $B_W = |\tilde{L}|$ of divisors of bidegree $(1,1)$ on $\mathbb{P}(E) \times \mathbb{P}(E)$ defined by the image $\tilde{L}$ of the polarization map $\mathfrak{p} : L \to S^2 E^\vee$. In the notation from the above, it is given by $n+1$ bilinear equations

$$\mathrm{PB}(W) : (t_0, \dots, t_n) \cdot A_i \cdot \begin{pmatrix} t'_0 \\ \vdots \\ t'_n \end{pmatrix} = 0, \ i = 0, \dots, n. \tag{7.2.2}$$

- The *Reye variety* $\mathrm{Rey}(W) \subset G_1(|E|)$ parameterizing lines in $|E|$ contained in the base scheme of linear subspace of W of codimension ≤ 2. They are called *Reye lines*.

Suppose that $\mathcal{D}(W)$ is a reduced hypersurface. Then, it contains an open subset $\mathcal{D}(W)^0$ of quadrics of corank 1. The map that assigns to such a quadric its unique

singular point defines a rational map, the *Steinerian map*

$$\mathrm{st} : \mathcal{D}(W) \dashrightarrow \mathrm{St}(W).$$

If the Steinerian is a hypersurface, then this map is a birational map. Note that the open subset $\mathcal{D}(W)^{\mathrm{sm}}$ of nonsingular points in $\mathcal{D}(W)$ is contained in $\mathcal{D}(W)^0$. The image $\mathrm{st}(\mathcal{D}(W)^{\mathrm{sm}})$ is the open subset $\mathrm{St}(W)^0$ of points x such that there exists a unique quadric $Q \in W$ with $x \in \mathrm{Sing}(Q)$.

Let

$$\widetilde{\mathcal{D}}(W) = \{(x, Q) \in \mathbb{P}^n \times \mathcal{D}(W) : x \in \mathrm{Sing}(Q)\}.$$

In coordinates, $(\lambda_0, \dots, \lambda_n)$ in W and $(t_0, \dots, t_n)$ in $|E|$, it is given by equations

$$(\sum_{k=0}^{n} \lambda_k A_k) \cdot \begin{pmatrix} t_0 \\ \vdots \\ t_n \end{pmatrix} = 0.$$

We assume that the Steinerian map is defined. The variety $\widetilde{\mathcal{D}}(W)$ comes with the projections to $\mathrm{pr}_W : \widetilde{\mathcal{D}}(W) \to W$ and $\mathrm{pr}_{|E|} : \widetilde{\mathcal{D}}(W) \to \mathrm{St}(W)$ that make the following diagram commutative:

$$\begin{tikzcd} & \widetilde{\mathcal{D}}(W) \arrow[dl, "\mathrm{pr}_W"'] \arrow[dr, "\mathrm{pr}_{|E|}"] & \\ \mathcal{D}(W) \arrow[rr, dashed, "\mathrm{st}"] & & \mathrm{St}(W). \end{tikzcd}$$

For any $Q \in \mathcal{D}(W)$, the fiber $\mathrm{pr}_W^{-1}(Q)$ is isomorphic to $\mathrm{Sing}(Q)$. For any $x \in \mathrm{St}(W)$, the fiber $\mathrm{pr}_{|E|}^{-1}(x)$ is isomorphic to the linear subspace of $\mathcal{D}(W)$ of quadrics Q with $x \in \mathrm{Sing}(Q)$.

Recall that a quadratic form $q \in S^2E^\vee$ defines a symmetric bilinear form $b_q \in (S^2E)^\vee$. We will identify it with a linear map $b_q : E \to E^\vee$ that coincides with its transpose. In the language of projective geometry, the corresponding map $|E| \to |E^\vee|$ is given by $x \mapsto P_x(Q)$, where $P_x(Q)$ is the first polar hypersurface of the quadric hypersurface $Q = V(q)$. A quadric $Q = V(q)$ is singular at $x = [v] \in |E|$ if $v \in \mathrm{Ker}(b_q)$, or, equivalently, $P_x(Q) = |E|$. For any point $x = [v] \in |E|$, the intersection $\cap_{q \in L} b_q(v) \subset E^\vee$ is equal to $\{0\}$ unless there exists $q \in L$ such that $b_q(v) = 0$. In the language of projective geometry this means that $\cap_{Q \in W} P_x(Q) \neq \emptyset$ if and only if there exists $y \in \mathbb{P}^n$ such that $y \in P_x(Q)$ for all $Q \in W$. This is also equivalent to that there exists $Q \in \mathcal{D}(W)$ such that $x \in \mathrm{Sing}(Q)$. This implies the following:

Proposition 7.2.1 *Let $p_1, p_2 : \mathrm{PB}(W) \to \mathbb{P}^n$ be the projection maps induced by the projections $\mathbb{P}^n \times \mathbb{P}^n \to \mathbb{P}^n$. Then, the image of p_i is equal to* $\mathrm{St}(W)$ *and the projection p_1 defines an isomorphism*

$$p_2^{-1}(x) \cong \cap_{Q \in W} P_x(Q).$$

Remark 7.2.2 The images of the varieties $\widetilde{\mathcal{D}}(W)$ and $\mathrm{PB}(W)$ under the projection to $\mathbb{P}^3$ coincide with $\mathrm{St}(W)$, and the fibers of the projection maps are isomorphic. However, this does not mean that the varieties $\widetilde{\mathcal{D}}(W)$ and $\mathrm{PB}(W)$ are isomorphic. In fact, they are the *Auslander transposes* to each other in the following sense (see [204, Exercise A3.22]). Let us consider the natural polarization map $L \to E^\vee \otimes E^\vee, q \mapsto b_q$. We view this map as a homomorphism of locally free sheaves on $|E|$:

$$\mathfrak{p} : \underline{L}(-1) \to \underline{E}^\vee, \tag{7.2.3}$$

where, for any linear space V, we denote by $\underline{V}$ the sheaf of sections of the trivial vector bundle associated to V. The corresponding map of the fibers over a point $x = [v] \in |E|$ is given by

$$\mathfrak{p}([v]) : L \to E^\vee, q \mapsto b_q(v, \cdot).$$

We assume that a general quadric in $|L|$ is nonsingular. Then, the map $\mathfrak{p}$ is injective at a generic point of W. Since a locally free sheaf does not contain torsion subsheaves, the homomorphism $\mathfrak{p}$ is injective. Let $C = \mathrm{Coker}(\mathfrak{p})$. Its support is equal to $\mathcal{D}(W)$. Passing to fibers, we find that $C([v]) = \mathrm{Coker}(\mathfrak{p}([v]))$ and $C([v])^\vee = \{w \in E : b_q(v, w) = 0, \text{for all } q \in L\}$. It is easy to see that $C([v])$ can be identified with the fiber of the projection $\widetilde{\mathcal{D}}(W) \to \mathrm{St}(W)$ over $[v]$ and $C([v])^\vee$ with the fiber of $\mathrm{PB}(W) \to \mathrm{St}(W)$ over the same point. Although the fibers are isomorphic, the isomorphisms can be glued to an isomorphism of the total spaces of the fibrations.

It follows from the previous remark that the variety $\mathrm{St}(W)$ coincides with the degeneracy scheme of the homomorphism of vector bundles $\mathfrak{p}$ in the sense of [242, Chapter 14]. Let

$$\mathrm{St}(W)_r = \{x \in \mathrm{St}(W) : \dim \widetilde{\mathcal{D}}(W)_x \geq r\}.$$

It is known that the expected codimension (i.e. for general W) of $\mathrm{St}(W)_r$ in $\mathrm{St}(W)$ is equal to r^2 and

$$\deg \mathrm{St}(W)_r = \prod_{i=0}^{r} \frac{(n+1+i)!i!}{(n-r+i)!(r+1+i)!}. \tag{7.2.4}$$

The proof of the following proposition can be found in [177, Proposition 1.1.28 and Proposition 1.1.30].

Proposition 7.2.3 *A point $(x, y) \in \mathrm{PB}(W)$ is singular if and only if $x = y$, or there exists a quadric $Q \in W$ such that $x, y \in \mathrm{Sing}(Q)$. In this case, the point $(x, Q) \in \mathrm{Sing}(\widetilde{\mathcal{D}}(W))$. Conversely, (x, Q) is a singular point of $\widetilde{\mathcal{D}}(W)$ if and only if $x \in \mathrm{Bs}(W)$ or there exists a point $y \neq x$ such that $\langle x, y\rangle \subset \mathrm{Sing}(Q)$.*

Assume that $\mathrm{Bs}(W) = \emptyset$. Then, $\mathrm{PB}(W)$ does not intersect the diagonal Δ of $\mathbb{P}^n \times \mathbb{P}^n$. Consider a map

$$f : \mathrm{PB}(W) \to G_1(|E|)$$

that assigns to $(x, y) \in \mathrm{PB}(W)$ the line $\ell = \langle x, y\rangle$ spanned by x and y.

Proposition 7.2.4 *The map f is of degree 2 onto* $\mathrm{Rey}(W)$.

Proof A quadric $Q = V(q)$ contains a line ℓ if and only if q vanishes at three distinct points on ℓ. Let Q contain $x = [v]$ and $y = [w]$, then $q(v + w) = b_q(v, w) + q(v) + q(w) = 0$. This shows that Q contains ℓ. Thus, ℓ imposes only two conditions on quadrics to contain it. This implies that $\ell \in \mathrm{Rey}(W)$. It is clear that f factors through the involution $(x, y) \mapsto (y, x)$ of $\mathbb{P}^n \times \mathbb{P}^n$. For any $\ell \in \mathrm{Rey}(W)$, the restriction of W to ℓ is of dimension ≤ 1. Since W is base-point-free, the dimension is equal to 1. Let $|L - \ell|$ be the linear system of quadrics in W that contain ℓ. A base-point free pencil of divisors of degree 2 on a line contains two ramification points. If $\ell = \langle x, y\rangle$ for some $(x, y) \in \mathrm{PB}(W)$, then there exists a quadric $Q \in W$ such that $x \in \mathrm{Sing}(Q)$. This easily shows that all quadrics in W intersect ℓ at x with multiplicity 2. Thus, x and y are the two ramification points. Conversely, if x, y are the ramification points of the pencil, then, for any quadric $Q \in W$, the line ℓ is contained in the tangent plane $P_x(Q)$ and $P_y(Q)$. Thus, $y \in P_x(Q)$ and $x \in P_y(Q)$, i.e. $(x, y) \in \mathrm{PB}(W)$. □

We see that any Reye line contains two points x, y such that $(x, y) \in \mathrm{PB}(W)$.

Proposition 7.2.5 *Assume* $\mathrm{Bs}(W) = \emptyset$. *Let $\ell = \langle x, y\rangle$ be a Reye line, where* $(x, y) \in \mathrm{PB}(W)$. *Then $x, y \in \mathrm{St}(W)$. Let $x \in \mathrm{Sing}(Q)$ for some quadric $Q \in W$. Then, Q contains ℓ if and only if there exists a pencil of quadrics in W with singular point at x.*

Proof The restriction of W to ℓ is a base-point-free pencil $\mathcal{P}$ of quadrics. It defines a degree 2 map $f : \ell \to \mathcal{P}^*$ with two ramification points x', y'. Let $Q \in W$ be a quadric that intersects ℓ at x' with multiplicity 2 but does not contain ℓ. Together with the $(n - 2)$-dimensional linear system $|L - \ell| \subset W$ of quadrics containing ℓ they span a hyperplane in W of quadrics touching ℓ at the point x'. By dimension count, one of the quadrics in the linear system must have the tangent space equal to the whole $\mathbb{P}^n$, i.e. it is singular at x'. Thus, $x' \in \mathrm{St}(W)$. Let $Q = V(q_0)$ with $x' = [v] \in \mathrm{Sing}(Q)$. Then, $b_{q_0}(v, w) = 0$ for all $w \in E$, hence the image of the map $L \to E^\vee, q \mapsto b_q(v, \cdot)$, is contained in a hyperplane of $E^\vee$. This shows that we can find $w \in E$ such that $b_q(v, w) = 0$ for all $q \in L$. Therefore, $(x', [w]) \in \mathrm{PB}(B)$ and x' is equal to x or y.

Choose projective coordinates such that $\ell = \{t_2 = \ldots = t_n = 0\}$ and $x = [1, 0, \ldots, 0], y = [0, 1, 0, \ldots, 0]$. Then, we can find a basis in L such that W consists of quadrics of the form

$$q(\lambda) = \lambda_0 t_0^2 + \lambda_1 t_1^2 + \sum_{i=2}^{n} \lambda_i L_i(t_0, \ldots, t_n) = 0,$$

where the coefficients in L_i are linear forms in $t_2, \ldots, t_n$. Computing the partials at the point x, we obtain that the conditions for x to belong to $Q(\lambda)$ are:

$$L_j(1, 0, \ldots, 0) = 0,\ j = 2, \ldots, n,\ \lambda_0 = 0.$$

Let $A_0^{(j)}$ be the coefficients in L_j at t_0. The linear forms $A_0^{(j)}$ are linearly dependent if and only if there exists a quadric containing ℓ with singular point at x. In this case,

we can find $a_2, \dots, a_n$ not all zeros such that the quadrics $Q(a) = V(\sum a_i q_i)$ and $Q_1 = V(q_1)$ are linearly dependent and have x as their singular point. □

Definition 7.2.6 A hyperweb of quadrics W is called *regular* if $\mathrm{PB}(W)$ (or, equivalently, $\widetilde{\mathcal{D}}(W)$) is smooth.

Note that, by the adjunction formula, $\mathrm{PB}(W)$ has trivial canonical class, so, when it is nonsingular, it is a *Calabi–Yau* variety.

Theorem 7.2.7 *Let W be a regular hyperweb of quadrics in $\mathbb{P}^n$. The following properties hold:*

(i) $\mathrm{Bs}(W) = \emptyset$.
(ii) The map $f : \mathrm{PB}(W) \to \mathrm{Rey}(W)$ is an étale finite map of degree 2.
(iii) $\mathrm{Rey}(W)$ *is smooth,*
(iv) $\mathrm{Sing}(\mathcal{D}(W))$ *consists of quadrics $Q \in W$ of corank > 1.*
(v) $\widetilde{\mathcal{D}}(W) \to \mathcal{D}(W)$ is a resolution of singularities.
(vi) The projections $\mathrm{PB}(W) \to \mathrm{St}(W)$ *and* $\widetilde{\mathcal{D}}(W) \to \mathrm{St}(W)$ *are weak resolutions of singularities.*[1]

Proof Assertion (i) follows from Proposition 7.2.3.

Let us prove (ii). By Proposition 7.2.4, a fiber of the map f consists of two distinct points. We have to show that the differential at each point is bijective. Let $(v_0, w_0) \in E \times E$ represent such a point. The fiber of the map $F : |E| \times |E| \to G_1(|E|), (x, y) \mapsto \langle x, y\rangle$ over a line ℓ is equal to $\ell \times \ell \subset |E| \times |E|$. If $\ell = f(x, y)$, $F^{-1}(\ell)$ consists of pairs $([\lambda v_0 + \mu w_0], [\lambda' v_0 + \mu' w_0])$, so the fiber has the natural coordinates $[\lambda\mu, \lambda\mu', \lambda'\mu, \lambda'\mu']$ in the Segre embedding of $\ell \times \ell$. The intersection of $F^{-1}(\ell) \cap f^{-1}(\ell)$ is given by the equations

$$b_q(\lambda v_0 + \mu w_0, \lambda' v_0 + \mu' w_0) = (\lambda\mu' + \lambda'\mu) b_q(v_0, w_0) + \lambda\lambda' q(v_0) + \mu\mu' q(w_0)$$

$$= \lambda\lambda' q(v_0) + \mu\mu' q(w_0) = 0,$$

where $V(q) \in W$. These are bilinear equations on $\ell \times \ell$. By (i), not all coefficients are zeros. Passing to the symmetric product $\ell^{(2)}$, we see that these bilinear equations arise from linear equations in the plane. Since we know that there is only one solution of these equations, we obtain that a fiber of the differential of the map f consists of one point, hence f is étale.

(iii) Follows from (ii).

(iv) Let $\mathsf{D}(n)$ be the discriminant variety of singular quadrics in $\mathbb{P}^n$ and $\mathsf{D}(n)_k$ be the closed subvariety of quadrics of corank $\geq k$. It is known that $\mathsf{D}(n)_{k+1} = \mathrm{Sing}(\mathsf{D}(n)_k)$ (see Theorem 7.2.11). In particular, the singular locus of $\mathsf{D}(n)$ consists of quadrics of corank > 1. The discriminant variety $\mathcal{D}(W)$ is equal to the intersection $W \cap \mathsf{D}(n)_1$. The tangent space of $\mathsf{D}(n)$ at a nonsingular point Q can be canonically identified with the space of quadrics passing through the unique singular point of Q

[1] A weak resolution of singularities requires only to be a birational morphism, not necessarily an isomorphism over the set of nonsingular points.

(see [177, p. 32]). Thus, a quadric of corank 1 in W is a singular point in $\mathcal{D}(W)$ if and only if the singular point of Q is a base point of W. Thus, a quadric of corank 1 in a regular web W is always a nonsingular point of $\mathcal{D}(W)$. On the other hand, a quadric Q of corank ≥ 2 is a singular point of the discriminant hypersurface $\mathsf{D}(n)_1$, hence it is a singular point of $\mathcal{D}(W)$ if we assume that $\mathcal{D}(W) \neq \mathbb{P}^n$.

(v) By Proposition 7.2.3, $\widetilde{\mathcal{D}}(W)$ is smooth. It is known that a nonsingular point Q of $\mathcal{D}(W)$ is of corank one. Thus, the fiber $\mathrm{pr}_W^{-1}(Q)$ consists of one point, hence pr_W is an isomorphism over the subset of nonsingular points. Note that a quadric $Q \in \mathcal{D}(W)$ of corank 1 could be a singular point of $\mathcal{D}(W)$ but this implies that $\widetilde{\mathcal{D}}(W)$ is singular over Q.

(vi) We know that $\mathrm{pr}_{|E|} : \widetilde{D}(W) \to \mathrm{St}(W)$ and $p_2 : \mathrm{PB}(W) \to \mathrm{St}(W)$ are isomorphic as schemes over $\mathrm{St}(W)$. It remains only to prove that p_2 is a birational morphism. By (iv), the set of quadrics in W of corank 1 (i.e. with isolated singular point) is the open subset of smooth points on $\mathcal{D}(W)$. The Steinerian map st : $\widetilde{\mathcal{D}}(W)^{\mathrm{sm}} \to \mathrm{St}(W)$ has linear spaces as fibers. Since both the source and the target are hypersurfaces in an n-dimensional projective space, the map is a birational morphism. Thus, $\mathrm{St}(W)$ contains an open subset $\mathrm{St}(W)^0$ that consists of points x such that there exists a unique quadric Q in W with $x \in \mathrm{Sing}(Q)$. Moreover, this quadric is of corank 1. This implies that $\dim \cap_{Q \in W} P_x(Q) = 0$, hence, by Proposition 7.2.1, the projection $\mathrm{PB}(W) \to \mathrm{St}(W)$ is an isomorphism over x.

Definition 7.2.8 A hyperweb of quadrics in $\mathbb{P}^n$ is called *excellent* if it is regular and the projection map $\mathrm{PB}(W) \to \mathrm{St}(W)$ is an isomorphism.

Proposition 7.2.9 *A regular hyperweb W is excellent if and only if $\mathcal{D}(W)$ does not contain lines.*

Proof Suppose W is excellent. Then, for any point $x \in \mathrm{St}(W)$, the intersection of polar hyperplanes $P_x(Q), Q \in W$, consists of a single point. Thus, there exists a unique quadric Q with $x \in \mathrm{Sing}(Q)$. Suppose $\mathcal{D}(W)$ contains a pencil. By Bertini Theorem, there exists a point $x \in \mathbb{P}^n$ such that each quadric in this pencil is singular at x. This gives a contradiction. This argument also proves the converse. □

Example 7.2.10 Assume $n = 2$, so we are dealing with a net W of conics in $\mathbb{P}^2$. The classification of nets of conics up to projective equivalence over $\mathbb{C}$ is due to C. Jordan [353] (see a modern survey in [1]). It consists of 15 isomorphism classes. For our future application, we restrict ourselves to the case when the net is base-point-free. There are the following four non-projectively equivalent nets:

(i)

$$\lambda(2x_0x_1) + \mu(2x_0x_2 + x_1^2) + \gamma((-a - \frac{3}{4}\alpha^2)x_0^2 - \alpha x_1^2 + x_2^2 + \alpha x_0x_2) = 0,$$

where α is one of the three distinct roots of the equation $\alpha^3 + a\alpha + b = 0$. The discriminant curve is a nonsingular plane cubic

$$\mathcal{D}(W) : \lambda^2\gamma + \mu^3 + a\mu\gamma^2 + b\gamma^3 = 0,\ 4a^3 + 27b^2 \neq 0.$$

The Steinerian curve is a nonsingular plane cubic curve:

$$\mathrm{St}(W) : x_1^2x_2 + \frac{3}{2}\alpha x_0x_1^2 - x_0x_2^2 - (a - \frac{3}{4}\alpha^2)x_0^3 = 0.$$

The curve $\mathrm{PB}(W)$ and $\widetilde{\mathcal{D}}(W)$ are isomorphic to $\mathcal{D}(W)$. The Reye curve lies in the dual plane and parameterizes line components of singular conics in W. It coincides with the *Cayleyan curve* of $\mathrm{St}(W)$ [177, 3.2]. The map $\mathrm{PB}(W) \to \mathrm{Rey}(W)$ is an étale double cover.

(ii)

$$\lambda x_0^2 + 2\mu x_1x_2 + \gamma(x_1^2 + x_2^2 + 2x_0x_1) = 0.$$

The discriminant curve is a nodal irreducible cubic:

$$\mathcal{D}(W) : \gamma^3 + \lambda\mu^2 - \lambda\gamma^2 = 0.$$

The Steinerian curve is the union of a line and a conic intersecting transversally:

$$\mathrm{St}(W) : x_0(x_2^2 - x_1^2 - x_0x_1) = 0.$$

The curve $\mathrm{PB}(W)$ and $\widetilde{\mathcal{D}}(W)$ are isomorphic to $\mathrm{St}(W)$. The Reye curve is a nodal cubic. The map $\mathrm{PB}(W) \to \mathrm{Rey}(W)$ is an étale double cover.

(iii)

$$\lambda x_0^2 + \mu x_1^2 + \gamma(2x_0x_1 + x_2^2) = 0.$$

The discriminant curve is the union of a line and a conic intersecting at two points:

$$\mathcal{D}(W) : \gamma(\lambda\mu - \gamma^2) = 0.$$

The Steinerian curve is the union of three non-concurrent lines:

$$\mathrm{St}(W) : x_0x_1x_2 = 0.$$

The curve $\mathrm{PB}(W)$ consists of four irreducible components. In the Segre embedding of $\mathbb{P}^2 \times \mathbb{P}^2$, two components are lines and other two are conics. The map $\mathrm{PB}(W) \to \mathrm{Rey}(W)$ is a double cover with the deck transformation switching two conics and two lines.

(iv)

$$\lambda x_0^2 + \mu x_1^2 + \gamma x_2^2 = 0.$$

The discriminant curve is the union of three non-concurrent lines:

$$\lambda\mu\gamma = 0.$$

The Steinerian curve is the union of three non-concurrent lines:

$$\mathrm{St}(W) : x_0x_1x_2 = 0.$$

The curve $\mathrm{PB}(W)$ consists of six irreducible components. In the Segre embedding of $\mathbb{P}^2 \times \mathbb{P}^2$, they are lines. The curve $\widetilde{\mathcal{D}}(W)$ is isomorphic to $\mathrm{PB}(W)$. The map $\mathrm{PB}(W) \to \mathrm{Rey}(W)$ is a double cover with the deck transformation switching two skew lines.

A net of conics is regular if and only if the discriminant curve $\widetilde{\mathcal{D}}(W)$ is nonsingular. A regular net of conics is an excellent net.

The proof of the following theorem can be found in [291] or [356].

Theorem 7.2.11 *Let $\mathsf{D}(n)_k$ be the variety of quadrics of corank k in $\mathbb{P}^n$.*

- *$\mathsf{D}(n)_k$ is an irreducible Cohen–Macaulay subvariety of codimension $\frac{1}{2}k(k+1)$;*
- $\mathrm{Sing}(\mathsf{D}(n)_k) = \mathsf{D}(n)_{k+1}$;
- $\deg \mathsf{D}(n)_k = \prod_{0 \le i \le k-1} \frac{\binom{n+1+i}{k-i}}{\binom{2i+1}{i}}$.

A general W intersects $\mathsf{D}(n)_k$ transversally and $\widetilde{\mathcal{D}}(W)$ inherits all the properties from Theorem 7.2.11. We do not know if a regular hyperweb of quadrics satisfies these properties.

Finally, we introduce one more variety associated to a hyperweb of quadrics $|L|$. Since we assumed that $p \neq 2$,

$$E^\vee \otimes E^\vee = \bigwedge^2 E^\vee \oplus S^2 E^\vee. \tag{7.2.5}$$

We have $E^\vee \otimes E^\vee = H^0(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3 \times \mathbb{P}^3}(1,1))$, and the linear subsystem $|\bigwedge^2 E^\vee|$ of $|E^\vee \otimes E^\vee|$ maps $|E| \times |E|$ to $|\bigwedge^2 E|$. The linear subsystem $|S^2 E^\vee|$ maps $|E| \times |E|$ to $|S^2 E|$.

Consider the restriction of these maps to $\mathrm{PB}(W)$. The restriction of the first map is the *Reye map*

$$\mathfrak{r} : \mathrm{PB}(W) \to \mathrm{Rey}(W) \subset |\bigwedge^2 E|. \tag{7.2.6}$$

The restriction of the second map is the *Cayley map*

$$\mathfrak{c} : \mathrm{PB}(W) \to \mathrm{Cay}(W) \subset |S^2 E|, \tag{7.2.7}$$

where $\mathrm{Cay}(W)$ is a subvariety of quadrics of rank 2 in $|L^\perp| \subset |S^2 E|$. It is called the *Cayley variety* of W. Note that, by Theorem 7.2.11, one expects that $\mathrm{Cay}(W)$ is of codimension $n(n-1)/2$ in $|L^\perp| \cong \mathbb{P}^{\frac{1}{2}n(n+1)-1}$, i.e., it is of dimension $n-1 = \dim \mathrm{PB}(W)$. Its degree is equal to $\frac{1}{2}\binom{2n}{n}$. The complete linear system $|E^\vee \otimes E^\vee|$ defines the Segre map $|E| \times |E| \to |E \otimes E|$. The Reye variety (resp. the Cayley variety) is the projection of the Segre variety from the subspace $|L^\perp|$ (resp. $|S^2 E|$).

Proposition 7.2.12 *The Cayley variety of a regular hyperweb of quadrics is a smooth subvariety of $\mathbb{P}^{\frac{1}{2}n(n+1)-1}$ of degree $\frac{1}{2}\binom{2n}{n}$. It is isomorphic to $\mathrm{Rey}(W)$ and to the quotient of $\mathrm{PB}(W)$ by a fixed-point free involution $(x, y) \mapsto (y, x)$.*

Proof The Cayley variety is equal to the image of the restriction of the map $|E| \times |E| \to |S^2E| = \mathbb{P}(S^2E^\vee)$ given by $(x, y) \to H_x \cup H_y$, where H_x and H_y are hyperplanes in $|S^2E^\vee|$ corresponding to the points $x, y \in |E|$. Since W is regular, the base scheme of W is empty. This implies that $\mathrm{PB}(W)$ does not intersect the diagonal. Hence, $\mathrm{Cay}(W) \cong \mathrm{PB}(W)/(\tau)$, where τ is the involution $(x, y) \mapsto (y, x)$. □

7.3 Hyperwebs of Quadrics in Characteristic 2

In this section, we extend the discussion from the previous section to hyperwebs W of quadrics in characteristic 2. We will introduce two substitutes of $\mathrm{PB}(W)$ in this case. One uses a general web B of symmetric bilinear forms, its base locus is a separable canonical cover of a Reye congruence. Another one still uses a general web of quadrics W, and the Reye map is an inseparable $\boldsymbol{\mu}_2$-cover of a Reye congruence. Here, we will also introduce the notion of the half-discriminant of a quadratic form in odd number of variables.

The crucial difference between the cases $p = 2$ and $p \neq 2$ is that a quadratic form q is not determined uniquely by its polar bilinear form b_q. In coordinates, a quadratic form $q \in S^2E^\vee$ can be written in the form

$$q = \sum_{i=0}^{n} a_{ii}t_i^2 + \sum_{0 \le i < j \le n} a_{ij}t_it_j, \tag{7.3.1}$$

and

$$b_q = \sum_{0 \le i < j \le n} a_{ij}(t_it'_j + t'_it_j),$$

so it could be zero without q being zero. We have

$$\mathrm{Sing}(V(q)) \subset V(\mathrm{Ker}(b_q)), \tag{7.3.2}$$

where $\mathrm{Ker}(b_q) = \{v \in E : b_q(v, w) = 0 \text{ for all } w \in E\}$. The projective space $|\mathrm{Ker}(b_q)|$ is called the *nullspace* of Q. A quadric is called *defective* if the inclusion (7.3.2) is strict. In this case, the set of singular points of Q is the subspace of codimension 1 in the nullspace. In geometric terms, a point x with $P_x(Q) = \mathbb{P}^n$ is not necessarily a singular point of Q as it is in the case $p \neq 2$. Recall that any quadratic form $q(t_0, \dots, t_n)$ over an algebraically closed field $\Bbbk$ of characteristic 2 can be reduced to the form $t_0t_1 + \cdots + t_{2k-2}t_{2k-1}$ or $t_0t_1 + \cdots + t_{2k-2}t_{2k-1} + t_{2k}^2$, where $2k$ is the rank of the bilinear form b_q. Such a quadric $Q = V(q)$ is nonsingular if and only if $k = [\frac{n}{2}]$. In the first case, the discriminant of the polar bilinear form is zero, and, in the second case, it is not zero.

If $n = 2k + 1$ is odd, the quadric $V(q)$ is nonsingular if and only if the polar bilinear form is non-degenerate. In coordinates, this means that the determinant of its matrix is non-zero. Since the associated bilinear form is alternate and the determinant of a general alternate form is the square of the *pfaffian* determinant, we

see that the *discriminant hypersurface* $\mathsf{D}(n)$ is of degree $k+1$ and is given by the pfaffian determinant of the matrix (a_{ij}).

If $n = 2k$ is even, the analog of the discriminant of a quadric is the *half-discriminant* [394]. In order to define it, one considers the universal quadratic form

$$\mathbf{q} = \sum \sum_{0 \le i \le j \le n} A_{ij} T_i T_j \in \mathbb{Z}[A_{ij}, T_i].$$

Let $\mathrm{discr}(\mathbf{q})$ be the determinant of the symmetric matrix defining the associated symmetric bilinear form $\sum_{i=0}^{n} 2A_{ii}T_iT_i' + \sum_{0 \le i < j \le n} A_{ij}(T_iT_j' + T_jT_i')$. Reducing modulo 2, we get a polynomial in $\mathbb{F}_2[A_{ij}]$. If n is even, evaluating it on any quadratic form, we get zero. This implies that all coefficients of $\mathrm{discr}(\mathbf{q})$ are even integers. We define the *universal discriminant* $\mathrm{discr}'(\mathbf{q})$ by setting $\mathrm{discr}'(\mathbf{q}) = \frac{1}{2}\,\mathrm{discr}(\mathbf{q})$.

We define the *half-discriminant* $\mathrm{discr}'(q)$ of any quadratic form $q(t_0, \dots, t_n)$ to be the value of $\mathrm{discr}'(\mathbf{q})$ on q if n is even. If n is odd, we do not correct the definition of the discriminant. As is for the usual discriminant, the half-discriminant does not depend on a choice of a basis, up to a multiplicative factor which is a square in $\Bbbk^\times$.

One can give an explicit formula for the half-discriminant as follows. Let $q \in S^2E^\vee$ and $b_q \in \bigwedge^2 E^\vee$ the associated polar bilinear form. The symbolic power $b_q^{(k)} \in \bigwedge^{2k} E^\vee$ can be identified, via a choice of a basis in $\bigwedge^{2k+1} E^\vee$, with a vector $\mathrm{pf}(q)$ from E. Its coordinates are the pfaffians of the principal minors of the alternate matrix of b_q (see [204, Exercise A2.11]). We have

$$\mathrm{discr}'(q) = q(\mathrm{pf}(q)).$$

This allows one to write an explicit formula for the half-discriminant hypersurface $\mathsf{D}(n)$ in the space of quadrics $S^2E^\vee$. If we use the coefficients a_{ij} in (7.3.1), we get

$$\mathsf{D}(2) : a_{00}a_{12}^2 + a_{11}a_{02}^2 + a_{22}a_{01}^2 + a_{01}a_{12}a_{02} = 0, \tag{7.3.3}$$

$$\begin{aligned}\mathsf{D}(4) : {}& (a_{00}a_{12}^2a_{34}^2 + \cdots) + (a_{01}^2a_{23}a_{34}a_{24} + \cdots) \\ & + (a_{01}a_{12}a_{23}a_{34}a_{04} + \cdots) = 0.\end{aligned} \tag{7.3.4}$$

We leave it to the reader to guess and prove the general formula.

We denote the discriminant variety by $\mathcal{D}(W) = W \cap \mathsf{D}(n)$ by using the pfaffian equation if n is odd and the half-discriminant if n is even.

Recall from the previous section that a linear subspace L of $S^2E^\vee$ defines a linear subspace $\tilde{L}$ of the linear space $(S^2E)^\vee$ of symmetric bilinear forms. It is equal to the image of L under the polarization map

$$\mathfrak{p} : S^2E^\vee \to (S^2E)^\vee.$$

Since $p = 2$, the image is contained in the subspace $\bigwedge^2 E^\vee$ of alternating forms. The kernel of $\mathfrak{p}$ consists of quadratic forms l^2, where $l \in E^\vee$. We assume that L does not contain such quadratic forms, and hence can be identified with a linear subspace of $\bigwedge^2 E^\vee$.

We have the canonical exact sequence

$$0 \to \bigwedge^2 E^\vee \to E^\vee \otimes E^\vee \to S^2 E^\vee \to 0 \tag{7.3.5}$$

that comes from the definition of the symmetric square of a linear space. In characteristic $\neq 2$, the polarization map splits this exact sequence.

The base scheme $\mathrm{Bs}(B_W)$ of the linear system $B_W = |\tilde{L}|$ on $|E| \times |E|$ contains the diagonal Δ. We denote by $\mathrm{Bs}(B_W)^0$ the residual component of dimension 2 of B_W.

Let $(S^2E)^\vee$ be the subspace of $E^\vee \otimes E^\vee$ of symmetric bilinear forms. As in the previous section, we view $|(S^2E)^\vee|$ as the space of symmetric divisors of type $(1,1)$ on $|E|\times|E|$. Restricting to $\mathrm{Bs}(\tilde{W}) \subset |E|\times|E|$, we obtain a linear system $|(S^2E)^\vee/\tilde{L}|$. It defines a rational map

$$\mathrm{Bs}(B_W)^0 \dashrightarrow |\tilde{L}^\perp| \subset |S^2E| = |(S^2(E)^\vee)^\vee|.$$

This is the analog of the Cayley map (7.2.7) in characteristic $\neq 2$. It assigns to a point $(x, y) \in \mathrm{Bs}(B_W)^0$ the reducible quadric $H_x + H_y$ in $|E^\vee|$, where H_x, H_y are hyperplanes associated to the points $x, y \in |E|$.

Note that the polarization map $L \hookrightarrow S^2(E^\vee) \to \mathrm{Sym}^2(E)^\vee$ defines a linear map

$$\mathfrak{p} : L \to \bigwedge^2 E^\vee.$$

We will assume that this map is injective. This happens if and only if L does not contain quadrics of rank 1. Let

$$\mathcal{D}(W)' = \{Q = V(q) \in W : \mathrm{Ker}(b_q) \neq \{0\}\}. \tag{7.3.6}$$

If n is even, $\mathcal{D}(W)'$ is equal to the whole $|L|$, if n is odd, it is a hypersurface of degree $\frac{1}{2}(n+1)$ given by the pfaffian of the matrix b_q, where $q = \sum_{i=0}^n \lambda_i q_i$ for some basis (q_i) of L.

Proposition 7.3.1 *Let $\mathbb{A}^{\binom{n+1}{2}}$ be the variety of alternating matrices of size $n+1$. Let $\mathrm{Pf}_n(r)$ be the (reduced) closed subvariety of matrices of rank $\leq r$.*

(i) $\mathrm{Pf}_n(2c) = \mathrm{Pf}_n(2c+1)$, $2c \leq n$. *It is defined by pfaffians of principal* $(2c \times 2c)$-*submatrices of size* $2c$,
(ii) $\mathrm{Pf}_n(2c)$ *is a Cohen–Macaulay variety of codimension* $\frac{1}{2}(m-2c+2)(m-2c+1)$,
(iii) $\mathrm{Pf}_n(2c-2) = \mathrm{Sing}(\mathrm{Pf}_n(2c))$ *if* $2c < n+1$,
(iv) $\deg \mathrm{Pf}_n(2c) = 2^{-n+2c} \prod_{i=0}^{n-2c-1} \frac{\binom{n+1+i}{n-2c-i}}{\binom{2i+1}{i}}$.

The proof of properties (i)–(iii) can be found in [393]. The formula for the degrees can be found in [356] or [291, Proposition 12].

Let $\mathsf{D}(n)_s$ be the set of quadrics $V(q)$ in $|E|$ with $\dim \mathrm{Sing}(V(q)) \geq s$. If $n = 2k+1$, then $\mathsf{D}(n)_s = \mathrm{Pf}_n(s)$, hence $\mathsf{D}(n)_{2s} = \mathsf{D}(n)_{2s+1}$, $s \geq 0$, and $\mathsf{D}(n)_0 = \mathsf{D}(n)$. We do not know the analog of Proposition 7.2.11 in the case when n is even.

We set

$$\widetilde{\mathcal{D}}(W)' = \{([v], Q) \in |E| \times W : v \in \mathrm{Ker}(b_q)\},$$
$$\widetilde{\mathcal{D}}(W) = \{(x, Q) \in |E| \times W : x \in \mathrm{Sing}(Q)\}.$$

Proposition 7.3.2 *The projection $\widetilde{\mathcal{D}}(W)' \to |E|$ is surjective. It is a birational morphism if a general quadric in W is of corank ≤ 2.*

Proof In coordinates, $\widetilde{\mathcal{D}}(W)'$ is given by (7.2.1), where the matrices A_i are the matrices of the polar bilinear form. Since the matrices A_i are alternating, for any point $x = [x_0, \ldots, x_n]$, we have ${}^t x \cdot A_i \cdot x = 0$. This shows that x belongs to the left kernel of the matrix $M(x) = [A_0 \cdot x, \ldots, A_n \cdot x]$. Thus, $\det M(x) = 0$, and there exists a quadric $Q = V(q)$ such that $x \in |\mathrm{Ker}(b_q)|$. Note that the right kernel of $M(x)$ can be identified with the linear space of $q \in L$ such that $x \in |\mathrm{Ker}(b_q)|$. The assumption on W implies that $\dim \widetilde{\mathcal{D}}(W)' = n$. This implies the assertion. □

Let $\mathrm{St}(W)$ be the image of the projection of $\widetilde{\mathcal{D}}(W)$ to $|E|$. We assume that $\dim \widetilde{\mathcal{D}}(W) = n - 1$, i.e. a general member of W is a nonsingular quadric, and a general point of $\widetilde{\mathcal{D}}(W)$ represents a quadric with isolated singular point. Then, $\mathrm{St}(W)$ is a proper subvariety of $\mathbb{P}^n$. We continue to call it the Steinerian hypersurface.

Proposition 7.3.3 *Assume that* $\dim \widetilde{\mathcal{D}}(W) = n - 1$. *Then,* $\mathrm{St}(W)$ *is a hypersurface of degree* $n + 1$.

Proof A point $x \in \mathbb{P}^n$ belongs to $\mathrm{St}(W)$ if and only if there exists a quadric $Q = V(q)$ such that $x \in |\mathrm{Ker}(b_q)|$ and $x \in Q$. Choose a basis $q_0, \ldots, q_n$ in L and coordinates $(t_0, \ldots, t_n)$ in $|E|$. Then, $Q(\lambda) = V(\sum \lambda_i q_i)$ contains $x = [x_0, \ldots, x_n]$ in its singular locus if and only if $M(x) \cdot \lambda = 0$ and $x \in Q(\lambda)$, where $M(x)$ is the matrix from the proof of the previous proposition. Assume that $\mathrm{St}(W)$ is not contained in the subset $\{x : \mathrm{corank}\, M(x) > 1\}$. Then, for a general point x, there exists a unique quadric $Q(\lambda)$ such that $x \in \mathrm{Sing}(Q(\lambda))$. We can take $[\lambda_0, \ldots, \lambda_n]$ to be a column of the adjugate matrix $\mathrm{adj}(M(x))$. Thus, each λ_i is a polynomial of degree n in coordinates of x. Since x belongs to the right kernel of the matrix $M(x)$, we see that each entry C_{ij} in a row $(C_{i0}, \ldots, C_{in})$ of $\mathrm{adj}(M(x))$ is divisible by x_i. This shows that each column $(C_{0j}, \ldots, C_{nj})$ of $\mathrm{adj}(M(x))$ is divisible by x_j. Thus, we can take λ_i to be polynomials of degree $n - 1$ in x. This gives the equation of $\mathrm{St}(W)$ of degree $n + 1$. □

To extend the construction of the double cover $f : \mathrm{PB}(W) \to \mathrm{Rey}(W)$ to characteristic 2, we have to change the definition of $\mathrm{PB}(W)$. In fact, in characteristic 2, the old definition of $\mathrm{PB}(W)$ shows that it contains the diagonal of $|E| \times |E|$. We will now define $\mathrm{PB}(W)$ as a certain closed subvariety of the projective completion

$$\mathbb{T}(|E|) = \mathbb{P}(\Omega^1_{\mathbb{P}^3} \oplus \mathcal{O}_{\mathbb{P}^3})$$

of the (geometric) tangent bundle

$$T(|E|) = \mathbb{V}(\Omega^1_{\mathbb{P}^3}).$$

Recall that for any quasi-coherent sheaf $\mathcal{F}$ on a scheme Y, one defines the following schemes:

$$\begin{aligned}
C &= \mathbb{V}(\mathcal{F}) = \mathrm{Spec}(\mathbf{S}(\mathcal{F})),\\
X &= \mathbb{P}(\mathcal{F}) = \mathrm{Proj}(\mathbf{S}(\mathcal{F})),\\
\hat{C} &= \overline{\mathbb{V}}(\mathcal{F}) := \mathbb{P}(\mathcal{F} \oplus \mathcal{O}_Y) \cong \mathrm{Proj}(\mathbf{S}(\mathcal{F})[z]),\\
C_0 &= \text{the closed subscheme of } C \text{ defined by } \mathbf{S}(\mathcal{F}) \to S^0(\mathcal{F}) := \mathcal{O}_Y,\\
C^* &= C \setminus C_0,\\
s_0 : Y = \mathbb{P}(\mathcal{O}_Y) &\to \hat{C} = \text{the closed embedding defined by } \mathcal{F} \oplus \mathcal{O}_Y \to \mathcal{O}_Y,\\
i : C \hookrightarrow \hat{C} &= \text{ the open embedding with the complement } s_0(Y),\\
\hat{C}^* &= \hat{C} \setminus s_0(Y),\\
s_\infty : X \to \hat{C} &= \text{ closed embedding corresponding to the surjection } \mathcal{F} \oplus \mathcal{O}_Y \to \mathcal{F},\\
p : \hat{C}^* \to X &= \text{ the morphism corresponding to the natural inclusion } \mathbf{S}(\mathcal{F}) \subset \mathbf{S}(\mathcal{F})[z],\\
\pi : C^* \to X &= \text{ the composition } p \circ i.
\end{aligned}$$

We specialize, taking $\mathcal{F}$ to be a locally free sheaf of rank $r+1$. Then, $C = \mathbb{V}(\mathcal{F})$ is the geometric vector bundle associated to $\mathcal{F}$. The sheaf $\mathcal{F}^\vee$ is the sheaf of local sections of $\mathbb{V}(\mathcal{F})$. The scheme $X = \mathbb{P}(\mathcal{F})$ is the projectivization of the vector bundle C. It is a projective r-bundle. Its fibers X_y are the projective spaces $\mathbb{P}(\mathcal{F}(y)) = |\mathcal{F}(y)^\vee|$ of dimension r. The scheme $\hat{C} = \bar{\mathbb{V}}(\mathcal{F})$ is the projective completion of the vector bundle $\mathbb{V}(\mathcal{F})$.

To specialize it further, we take $Y = \mathbb{P}^n$ and $\mathcal{F} = \Omega^1_{\mathbb{P}^n}$. Then, $\mathbb{V}(\Omega^1_{\mathbb{P}^n})$ is the tangent bundle $T(\mathbb{P}^n)$ of $\mathbb{P}^n$, $\mathbb{P}(\Omega^1_{\mathbb{P}^n}) = \mathbb{P}\,\mathrm{T}(\mathbb{P}^n)$ is its projectivization, $\overline{\mathbb{V}}(\Omega^1_{\mathbb{P}^n}) = \mathbb{T}(\mathbb{P}^n)$ is the completion of the tangent bundle. We denote by $\mathbb{T}(\mathbb{P}^n)^\infty$ (resp. $\mathbb{T}(\mathbb{P}^n)^0$) the image of the projectivized tangent bundle $\mathbb{P}\,\mathrm{T}(\mathbb{P}^n)$ (resp. of $\mathbb{P}^n$) under the morphism s_∞ (resp. s_0). Note that $\mathbb{T}(\mathbb{P}^n)^0$ is the analog of the diagonal of $\mathbb{P}^n \times \mathbb{P}^n$.

We use the canonical isomorphism from (7.1.10)

$$\bigwedge^2 E^\vee \to H^0(|E|, \Omega^1_{|E|}(2)),$$

which, in coordinates, is given by $t_i \wedge t_j \mapsto t_j^2 d\frac{t_i}{t_j}$. Thus, if we write

$$\mathbb{T}(|E|) = \mathbb{P}(\Omega^1_{\mathbb{P}^n} \oplus \mathcal{O}_{|E|}) = \mathbb{P}(\Omega^1_{\mathbb{P}^n}(2) \oplus \mathcal{O}_{|E|}(2)),$$

we will be able to identify the vector space $\bigwedge^2 E^\vee \oplus S^2 E^\vee$ with the space of sections of $\mathcal{O}_{\mathbb{T}(|E|)}(2)$. If $p \neq 2$, exact sequence (7.3.5) splits and we can identify $\bigwedge^2 E^\vee \oplus S^2 E^\vee$ with the linear space of bilinear forms $H^0(\mathbb{P}^1 \times \mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(1,1))$.

Let $\mathcal{O}_{\mathbb{T}(|E|)}(1)$ be the invertible sheaf on $\mathbb{T}(|E|)$ whose direct image in the base is equal to $\Omega^1_{|E|}(1) \oplus \mathcal{O}_{|E|}(2)$.

Recall that a tangent vector $\mathsf{t}_x \in T(|E|)_x$ at a point x is canonically identified with a linear map $\mathsf{t}_x : x \to E/x$, where x is considered as the line in E corresponding to the point $x \in |E|$. We identify the space $\mathbb{T}(|E|)_x$ with $|\operatorname{Hom}(x, E/x) \oplus \Bbbk|$. We denote vectors from $\operatorname{Hom}(x, E/x) \oplus \Bbbk$ by $\bar{\mathsf{t}}_x$, write them as (t_x, a), and denote the corresponding point in $\mathbb{T}(|E|)_x$ by $\tilde{x}$ or by $[\bar{\mathsf{t}}_x]$. The tangent space $T(|E|)_x = \operatorname{Hom}(x, E/x)$ embeds in $\mathbb{T}(|E|)_x$ as the set of points $[\mathsf{t}_x, 1]$. The projectivized tangent space $|T(|E|)_x|$ embeds as the hyperplane of points $[\mathsf{t}_x, 0]$. The point $[0, 1]$ belongs to $\mathbb{T}(|E|)^0$.

By definition, the *corrected* polar bilinear form b'_q of a quadratic form $q \in S^2E^\vee$ is defined to be a section (b_q, q) of $\bigwedge^2 E^\vee \oplus S^2E^\vee$. The scheme of zeros $Z(b'_q)$ of the section b'_q of $\mathcal{O}_{\mathbb{T}(|E|)}(2)$ is a divisor on $\mathbb{T}(|E|)$. We have $\bar{\mathsf{t}}_x = [\mathsf{t}_x, a] \in Z(b'_q)$ if and only if $b_q(v, \mathsf{t}_x(v)) + aq(v) = 0$, where $x = [v]$.

In characteristic two, we define the polar base variety $\mathrm{PB}(W)$ of a hyperweb W as the intersection scheme $\cap_{q \in W} Z(b'_q)$. Note that

$$\mathrm{PB}(W)^0 := \mathrm{PB}(W) \cap \mathbb{T}(|E|)^0 = s_0(\mathrm{Bs}(W)), \tag{7.3.7}$$

$$\mathrm{PB}(W)^\infty := \mathrm{PB}(W) \cap \mathbb{T}(|E|)^\infty = \cap_{q \in L} V(b_q) \subset \mathbb{P}\,\mathrm{T}(|E|). \tag{7.3.8}$$

Here, $b_q \in \bigwedge^2 E^\vee$ is considered as a section of the invertible sheaf $\mathcal{O}_{\mathbb{P}\,\mathrm{T}(|E|)}(1) \otimes p^*\mathcal{O}_{|E|}(2)$ on $\mathbb{P}\,\mathrm{T}(|E|) = \mathbb{P}(\Omega^1_{|E|})$. Recall from [177, 10.2.3] that a point in $|\bigwedge^2 E^\vee|$, viewed as a hyperplane in the Plücker space $|\bigwedge^2 E|$, is identified with a *linear complex of lines*, i.e., a hyperplane section of the Grassmann variety $G(2, E)$ (see [177, Chapter 10]). We assume that the polarization map $L \to \bigwedge^2 E^\vee$ is injective, that is, W does not contain quadrics of rank 1. Then, W defines a linear system of linear complexes in $\mathbb{G}$. In this terminology, the variety $\mathrm{PB}(W)^\infty$ becomes isomorphic to the pre-image in $Z_{\mathbb{G}}$ of the base scheme of the corresponding linear system of linear line complexes in $\mathbb{G}$. Its dimension is equal to $n - 2$. Its degree is equal to $\frac{(2n-2)!}{(n-1)!n!}$, the degree of the Grassmannian $\mathbb{G}$.

We keep the definitions of the Reye variety $\mathrm{Rey}(W)$ and the Reye line.

For any point $\tilde{x} = [\bar{\mathsf{t}}_x, a] \in \mathrm{PB}(W) \setminus \mathrm{PB}(W)_0$, let $\ell_{\tilde{x}}$ denote the line through x with the tangent direction t_x.

Lemma 7.3.4 *For any $\tilde{x} \in \mathrm{PB}(W) \setminus \mathrm{PB}(W)_0$, the line $\ell_{\tilde{x}}$ is a Reye line. All quadrics in W are tangent to $\ell_{\tilde{x}}$ at the point x. For any Reye line ℓ not passing through a base point of W, there exists a unique $\tilde{x}$ such that $\ell = \ell_{\tilde{x}}$.*

Proof Let $Q = V(q)$ be a quadric from W containing $\ell = \ell_{\tilde{x}}$. Let $x = [v]$ and $y = [w] \in \ell_{\tilde{x}},\ y \neq x$. Since $q(v) = q(w) = q(v + w) = 0$, we get $b_q(v, w) = 0$. Conversely, if $q(v) + b_q(v, w) = 0$ and $q(w) = 0$ for some $y = [w] \in \ell, y \neq x$, we obtain that $\ell \subset Q$. This shows that ℓ imposes two (instead of expected three) linear conditions on a quadric $Q \in W$ to contain ℓ. This implies that ℓ is a Reye line. Also observe that $q(v) = 0$ implies $b_q(v, w) = 0$, hence the line ℓ is tangent at the point x to all quadrics from W.

Conversely, let ℓ be a Reye line. Then, the restriction of W to ℓ is of dimension ≤ 1. If the dimension is equal to zero, all quadrics in W have a base point x on ℓ. Thus, we may assume that the dimension is equal to 1 and the pencil is base-point-free.

Let $\phi : \ell \to \mathbb{P}^1$ be the corresponding map of degree 2.

Assume ϕ is a separable map. Then, ϕ has a unique ramification point $x = [v] \in \ell$. Thus, there exists a quadric in the pencil that is tangent to ℓ at x. This implies that there exists a hyperplane H of quadrics in W such that all quadrics from H are tangent to ℓ at x. For any $\mathfrak{t}_x \in T(\ell)_x$, we have $b_q(v, \mathfrak{t}_x(v)) = q(v) = 0$ for all $V(q) \in H$. We can find a scalar a such that $b_q(v, \mathfrak{t}_x(v)) + aq(v) = 0$ for all $q \in L$. This implies that $\tilde{x} = [\bar{\mathfrak{t}}_x, a]$ belongs to $\mathrm{PB}(W)$ and $\ell = \ell_{\tilde{x}}$.

Assume ϕ is an inseparable map. Then, for each point $x \in \ell$, there exists a hyperplane $H(x) \subset W$ of quadrics tangent to ℓ at x. Take two different points $x_1 = [v_1], x_2 = [v_2]$ on ℓ, and let $Q_i = V(q_i)$ be quadrics in W that touch ℓ at the points x_i. Take a point $\tilde{x}_1 \in \mathbb{T}(|E|)^\infty_x$ defined by $[\mathfrak{t}_{x_1}] = [v_2 - v_1] \in \mathbb{P}(T(|E|))_x$. Since $H(x_1)$ and $H(x_2)$ generate W, we find that $b'_q(\tilde{x}_1) = b_q(v_1, \mathfrak{t}_{x_1}(v_1)) = 0$ for all $q \in L$. Thus, ℓ coincides with $\ell_{\tilde{x}_1}$.

Definition 7.3.5 We say that a Reye line that does not pass through a base point of W is *separable* (resp. *inseparable*) if the restriction of W to ℓ defines a separable (resp. inseparable) map $\ell \to \mathbb{P}^1$ of degree 2.

It follows from the proof of the previous lemma that a separable (resp. inseparable) Reye line is equal to a line $\ell_{\tilde{x}}$ for some point $\tilde{x} \in \mathrm{PB}(W) \setminus (\mathrm{PB}(W)_0 \cup \mathrm{PB}(W)^\infty)$ (resp. $\tilde{x} \in \mathrm{PB}(W)^\infty$). We know from the description of the subvariety $\mathrm{PB}(W)^\infty$ that the subvariety $\mathrm{Rey}(W)_{\mathrm{ins}} \subset \mathrm{Rey}(W)$ of inseparable Reye lines is of dimension $n-3$ and its degree in the Plücker embedding is equal to $\deg \mathbb{G} = \frac{(2n-2)!}{n!(n-1)!}$. The formula for the canonical class of $\mathbb{G} = G_1(\mathbb{P}^n)$ shows that

$$\omega_{\mathrm{Rey}(W)_{\mathrm{ins}}} \cong \mathcal{O}_{\mathrm{Rey}(W)_{\mathrm{ins}}}. \tag{7.3.9}$$

Assume that $\mathrm{Bs}(W) = \emptyset$. Consider the *Reye map*

$$\mathfrak{r} : \mathrm{PB}(W) \to \mathrm{Rey}(W), \quad \tilde{x} \mapsto \ell_{\tilde{x}}.$$

It is equal to the restriction to $\mathrm{PB}(W)$ of the composition of the projections

$$\mathbb{T}(|E|)^* \xrightarrow{\mathrm{pr}} \mathbb{P}\,\mathrm{T}(|E|) = Z_{\mathbb{G}} \xrightarrow{q} \mathbb{G}.$$

It is defined by the linear subsystem $|V|$ of $|i^*\mathcal{O}_{\mathbb{T}(|E|)}(1)|$, where $i : \mathrm{PB}(W) \hookrightarrow \mathbb{T}(|E|)$ is the closed embedding, and the tautological line bundle q corresponds to an isomorphism $\mathbb{T}(|E|) \cong \mathbb{P}(\Omega^1_{|E|}(2) \oplus \mathcal{O}_{|E|}(2))$. If $W = |L|$, where $L \subset H^0(\mathbb{T}(|E|), \mathcal{O}_{\mathbb{T}(|E|)}(1)) \cong \bigwedge^2 E^\vee \oplus S^2E^\vee$, then V is equal to the image $\bigwedge^2 E^\vee$ in $H^0(\mathbb{T}(|E|), \mathcal{O}_{\mathbb{T}(|E|)}(1))/L$. The image of $S^2E^\vee$ in $H^0(\mathbb{T}(|E|), \mathcal{O}_{\mathbb{T}(|E|)}(1))/L$ is a linear space of dimension 6 that defines the Cayley map

$$\mathfrak{c} : \mathrm{PB}(W) \to \mathrm{Cay}(W). \tag{7.3.10}$$

Proposition 7.3.6 *Let*

$$\mathfrak{r} : \mathrm{PB}(W) \xrightarrow{\sigma} \mathrm{PB}(W)' \xrightarrow{\mathfrak{r}'} \mathrm{Rey}(W)$$

be the Stein factorization of the Reye map $\mathfrak{r}$. *Then,* $\mathfrak{r}'$ *is an inseparable finite map of degree 2 and* σ *is the blowing down of* $\mathrm{PB}(W)^\infty$ *onto* $\mathfrak{r}'^{-1}(\mathrm{Rey}(W)_{\mathrm{ins}})$.

Proof Let ℓ be a Reye line, and let $\mathbb{T}(\mathbb{P}^n)_\ell$ be the restriction of the projection $\mathbb{T}(\mathbb{P}^n) \to \mathbb{P}^n$ over ℓ. Then, $\mathbb{T}(\mathbb{P}^n)_\ell \cap \mathrm{PB}(W)$ is isomorphic $\mathrm{PB}(W) \cap \mathbb{T}(\ell)$, where $\mathbb{T}(\ell) = \mathbb{P}(\Omega^1_\ell \oplus \mathcal{O}_\ell)$ is the rational minimal ruled surface $\mathbf{F}_2$ embedded in $\mathbb{T}(\mathbb{P}^n)|_\ell$ via the surjection $\Omega^1_{\mathbb{P}^n} \oplus \mathcal{O}_\ell \to \Omega^1_\ell \oplus \mathcal{O}_\ell$. Since ℓ is a Reye line, the restriction of the linear system $|b'_q|$ of corrected polar forms to $\mathbb{T}(\ell)$ is the scheme-theoretical intersection of two sections s_1 and s_2 of the ruled surface $\mathbf{F}_2$. If ℓ is a separable Reye line $\ell_{\tilde{x}}$, then s_1 and s_2 do not intersect the exceptional section $e = \mathbb{T}(\ell) \cap \mathrm{PB}(W)_0$. They intersect at one point $\tilde{x}$ with multiplicity 2. If ℓ is an inseparable line, then $s_1 = s_2 = e$. So, the fiber is the section taken with multiplicity 2. This proves the assertion. □

Remark 7.3.7 Note that the projection $\mathbb{T}(\mathbb{P}^n)^* \to \mathbb{P}\,\mathrm{T}(\mathbb{P}^n)$ is a line bundle, a torsor over the group $\mathbb{G}_m$. The subscheme $\mathrm{PB}(W)$ is invariant with respect to the subgroup $\boldsymbol{\mu}_2$ and the quotient $\mathrm{PB}(W)/\boldsymbol{\mu}_2$ is isomorphic to the blow-up of the locus of inseparable Reye lines in $\mathrm{Rey}(W)'$.

Note that, for any locally free sheaf $\mathcal{V}$ over a scheme S, the relative Euler exact sequence gives an isomorphism $\omega_{\mathbb{P}(\mathcal{V})/S} \cong \det(p^*\mathcal{E}(-1))$. This gives

$$\omega_{\mathbb{T}(|E|)} \cong p^*\omega_{|E|} \otimes \omega_{\mathbb{T}(|E|)/|E|} \cong p^*\omega_{|E|} \otimes p^*\omega_{|E|}(-n-1) = \mathcal{O}_{\mathbb{T}(|E|)}(-n-1).$$

Applying the adjunction formula, we obtain

$$\omega_{\mathrm{PB}(W)} \cong \mathcal{O}_{\mathrm{PB}(W)}.$$

However, as we will see later, the variety $\mathrm{PB}(W)$ is always singular.

Consider the canonical projection $p : \mathbb{T}(|E|) \to |E|$. The restriction of the projection to $\mathrm{PB}(W)$ defines a morphism $q : \mathrm{PB}(W) \to |E|$. The scheme-theoretical image is the Steinerian variety $\mathrm{St}(W)$. In fact, $x = [v] \in p(\mathbb{T}(|E|))$ if and only if $b_q(v, \mathfrak{t}_x(v)) + aq(v) = 0$ for some $[\bar{\mathfrak{t}}_x, a] \in \mathbb{T}(|E|)_x$ and all $q \in L$. Let q_0 be an element of the kernel of the linear map $L \to (E/x)^\vee, q \mapsto (\bar{\mathfrak{t}}_x \mapsto b_q(v, \bar{\mathfrak{t}}_x(v)))$. Then $b_{q_0}(v, w) = 0$ for all $w \in E$ and $q_0(v) = 0$, i.e. $[v] \in \mathrm{Sing}(V(q_0))$.

Proposition 7.3.8 *A point* $(x, Q) \in \widetilde{\mathcal{D}}(W)$ *(resp.* $\widetilde{\mathcal{D}}(W)'$*) is singular if and only if* $x \in \mathrm{Bs}(W)$*, or there exists a point* $\tilde{x} \in \mathrm{PB}(W)$ *(resp.* $\tilde{x} \in \mathrm{PB}(W)^\infty$*) such that the line* $\ell_{\tilde{x}} \subset |\operatorname{Ker}(b_q)|$.

Proof Suppose (x, Q) is a singular point of $\widetilde{\mathcal{D}}(W)$. Choose coordinates $(\lambda_0, \ldots, \lambda_n)$ in L and coordinates $(t_0, \ldots, t_n)$ in E. Let $q_k = \sum a_{ij}^{(k)} t_i t_j, k = 0, \ldots, n$ be the

corresponding basis in L, $A^{(k)} = (a_{ij}^{(k)})$, and let $B^{(k)} = (b_{ij}^{(k)})$ be the alternating matrices defining the polar bilinear form b_{q_k}. We have $b_{ii}^{(k)} = 0, b_{ij}^{(k)} = b_{ji}^{(k)} = a_{ij}^{(k)}$. We write $A(\lambda) = \sum \lambda_k A^{(k)}, B(\lambda) = \sum \lambda_k B^{(k)}$. The equations of $\widetilde{\mathcal{D}}(W)$ are

$$F_i = \sum_{0 \le j,k \le n} \lambda_k b_{ij}^{(k)} t_j = 0,\ i = 0, \dots, n,$$
$$G = \sum_{0 \le j,k \le n} \lambda_k a_{ij}^{(k)} t_i t_j = 0.$$

The Jacobian matrix at the point $\lambda_i = \alpha_i, t_i = x_i$ is equal to

$$\begin{pmatrix} \sum_{j=0}^n b_{0j}^{(0)} x_j & \dots & \sum_{j=0}^n b_{0j}^{(n)} x_j & \sum_{k=0}^n \alpha_k b_{00}^{(k)} & \dots & \sum_{k=0}^n \alpha_k b_{0n}^{(k)} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \\ \sum_{j=0}^n b_{nj}^{(0)} x_j & \dots & \sum_{j=0}^n b_{nj}^{(n)} x_j & \sum_{k=0}^n \alpha_k b_{n0}^{(k)} & \dots & \sum_{k=0}^n \alpha_k b_{nn}^{(k)} \\ \sum_{i,j=0}^n a_{ij}^{(0)} x_i x_j & \dots & \sum_{i,j=0}^n a_{ij}^{(n)} x_i x_j & \sum_{k,j=0}^n \alpha_k b_{0j}^{(k)} x_j & \dots & \sum_{k,j=0}^n \alpha_k b_{nj}^{(k)} x_j \end{pmatrix}$$

Note that the last $n+1$ entries in the last row are equal to zero because $B(\alpha) \cdot x = 0$. Since the point (α, x) is singular, there exist a non-zero vector $(\beta_0, \dots, \beta_{n+1})$ such that the linear combination of the rows with coefficients β_i is equal to zero. This gives

$$\sum_{0 \le i,j \le n} a_{ij}^{(0)} \beta_i x_j + \beta_{n+1} a_{ij}^{(0)} x_i x_j = \dots = \sum_{0 \le i,j \le n} b_{ij}^{(n)} \beta_i x_j + \beta_{n+1} a_{ij}^{(n)} x_i x_j = 0,$$

$$\sum_{0 \le i,k \le n} a_{i0}^{(k)} \beta_i \alpha_k = \dots = \sum_{0 \le i,k \le n} a_{in}^{(k)} \beta_i \alpha_k = 0.$$

If $\beta_i = 0$ for $i \ne n+1$, we obtain that $q_k(x) = 0$ for all k, i.e. $x \in \mathrm{Bs}(W)$. Assume $(\beta_0, \dots, \beta_n) \ne 0$. The first $n+1$ equations imply that $b_{q_k}(x, \beta) + \beta_{n+1} q_k(x) = 0, k = 0, \dots, n$. The last $n+1$ equations imply that $A(\alpha) \cdot \beta = 0$. Let $\bar{\mathfrak{t}}_x : x \to E = \Bbbk^{n+1}$ be defined by $(\beta_0, \dots, \beta_n)$ and let $\mathfrak{a} : x \to x$ be defined by β_{n+1}. This shows that the point $\tilde{x} = (\mathfrak{t}_x, \mathfrak{a})$ belongs to $\mathrm{PB}(W)$ and the line $\ell_{\tilde{x}}$ belongs to the nullspace of the matrix $B(\alpha)$. Conversely, if this happens, the point (x, Q) is a singular point.

If we follow the previous proof and take $G = 0$, we obtain the assertion about singularities of $\widetilde{\mathcal{D}}(W)'$. □

Assume n is odd. One can restate the previous proposition in terms of the projections $\pi : \widetilde{\mathcal{D}}(W) \to \mathcal{D}(W)$ and $\pi : \widetilde{\mathcal{D}}(W)' \to \mathcal{D}(W)$. A point $(x, Q) \in \widetilde{\mathcal{D}}(W)$ (resp. $(x, Q) \in \widetilde{\mathcal{D}}(W)'$) is singular if and only if $x \in \mathrm{Bs}(W)$, or the fiber $\pi'^{-1}(Q)$ does not contain a Reye line $\ell_{\tilde{x}}$ (resp. an inseparable Reye line $\ell_{\tilde{x}}$).

Proposition 7.3.9 *A point $\tilde{x} = [\mathfrak{t}_x, a] \in \mathrm{PB}(|E|)$ is a singular point if and only if $\tilde{x} \in \mathrm{PB}(W)_0$ (equivalently, $x \in \mathrm{Bs}(W)$), or there exists a quadric $Q = V(q) \in W$ such that $x \in \mathrm{Sing}(Q)$ and the line $\ell_{\tilde{x}} \subset |\mathrm{Ker}(b_q)|$.*

Proof We keep the notation from the proof of the previous proposition. Let $\tilde{x} \in \mathrm{Sing}(\mathrm{PB}(W))$. We may assume that $x = [e_0]$, where $(e_0, \ldots, e_n)$ is a basis in E corresponding to coordinates $(t_0, \ldots, t_n)$. We identify E/x with the span of the basis vectors $e_1, \ldots, e_n$. A point $\tilde{x}$ is defined by linear maps $\mathfrak{t}_x : x \to E/x$ and $a \in \Bbbk$. We identify these maps with a vector $\alpha = (\alpha_0, \ldots, \alpha_n)$ and a scalar $a = \alpha_{n+1}$, respectively. The point $\tilde{x}$ belongs to $\mathrm{PB}(W)_0$ if and only if $\mathfrak{t}_x = 0$, or, equivalently, $x \in \mathrm{Bs}(W)$.

The equations of $\mathrm{PB}(W)$ are

$$F_k = \sum_{i,j=0}^{n} b_{ij}^{(k)} t_j y_i + y_{n+1} a_{ij}^{(k)} t_i t_j = 0, \ k = 0, \ldots, n. \tag{7.3.11}$$

We have

$$\frac{\partial F_k}{\partial t_j}(x, \alpha) = \sum_{i=0}^{n} b_{ij}^{(k)} \alpha_i + \alpha_{n+1} \sum_{i=0}^{n} a_{ij}^{(k)} x_i, \ j = 0, \ldots, n, \tag{7.3.12}$$

$$\frac{\partial F_k}{\partial y_i}(x, \alpha) = \sum_{j=0}^{n} b_{ij}^{(k)} x_i, \ i = 0, \ldots, n, \tag{7.3.13}$$

$$\frac{\partial F_k}{\partial y_{n+1}}(x, \alpha) = \sum_{i,j=0}^{n} a_{ij}^{(k)} x_i x_j. \tag{7.3.14}$$

The point $\tilde{x}$ is singular if and only if there exists $[\beta_0, \ldots, \beta_n] \in \mathbb{P}^n$ such that

$$\sum_{k=0}^{n} \beta_k \frac{\partial F_k}{\partial t_j}(x, a) = \sum_{k=0}^{n} \beta_k \frac{\partial F_k}{\partial y_j}(x, a) = 0$$

for all $i = 0, \ldots, n$ and $j = 0, \ldots, n+1$. The last two equations imply that $x \cdot A(\beta) \cdot x = 0$ and $B(\beta) \cdot x = 0$, i.e. $(x, Q(\beta)) \in \widetilde{\mathcal{D}}(W)$. The first equation implies that the line $\langle x, [\alpha] \rangle$ belongs to $\mathrm{Ker}(b_{q(\beta)})$. Conversely, if this happens, the point $\tilde{x}$ is a singular point of $\mathrm{PB}(W)$. □

Assume n is odd. Let $\mathcal{D}(W)_2$ be the subset of quadrics $Q = V(q)$ such that $\dim \mathrm{Ker}(b_q) = 2$ and let $\widetilde{\mathcal{D}}(W)_2'$ be its pre-image in $\widetilde{\mathcal{D}}(W)'$. Then, the projection $\mathrm{pr}_W : \widetilde{\mathcal{D}}(W)_2' \to \mathcal{D}(W)_2$ is a $\mathbb{P}^1$-bundle. Consider the map $\mathfrak{a} : \mathcal{D}(W)_2 \to G_1(\mathbb{P}^n)$ that sends $Q = V(q)$ to $|\mathrm{Ker}(b_q)|$. The pre-image $\mathfrak{a}^{-1}(\ell)$ of a line $\ell \in G_1(\mathbb{P}^n)$ is the linear system of quadrics containing ℓ in its nullspace. If the fiber is of positive dimension, then all quadrics from $\mathfrak{a}^{-1}(\ell)$ are tangent to ℓ at some points. Let P be a pencil of quadrics contained in $\mathfrak{a}^{-1}(\ell)$. We may assume that ℓ is given by equations $t_2 = \ldots = t_n = 0$. Thus, the coefficients at $t_0 t_i, i = 1, \ldots, n$ in equation of any quadric in P are equal to zero. We can find a quadric $Q = V(q)$ in P such that the rank of b_q, considered as a bilinear form in $t_2, \ldots, t_n$, is less than $n-1$. Thus, b_q has corank > 2. The set of such quadrics is of expected codimension 15. Assume that $\mathcal{D}(W)$ does not consist of quadrics with nullspace of dimension > 1. In this case,

the map a is birational. So, the null-lines of quadrics in W are parameterized by a subvariety of dimension $n-1$ in $G_1(\mathbb{P}^n)$. Since the Reye variety is of dimension $n-1$, we expect that they intersect. By Proposition 7.3.8, the point $(Q,x) \in \widetilde{\mathcal{D}}(W)$ such that x lies on a Reye line contained in a nullspace of Q is a singular point. Thus, we expect that $\widetilde{\mathcal{D}}(W)$ is always singular. In the same way, we see that $\mathrm{PB}(W)$ is expected to be singular.

Applying Proposition 7.2.3, we also see that $\widetilde{\mathcal{D}}(W)$ is singular.

Proposition 7.3.10 *The image of the projection* $p : \mathrm{PB}(W) \to \mathbb{P}^n$ *is equal to* $\mathrm{St}(W)$.

Proof Let $\tilde{x} \in \mathrm{PB}(W)$ be represented by $\mathfrak{t}_x : x \to E/x$ and $a \in \Bbbk$. Taking coordinates such that $x = [1,0,\dots,0]$, we may assume that E/x has coordinates $y_1,\dots,y_n$ so $\bar{\mathfrak{t}}_x = (\mathfrak{t}_x, a)$. Choose a basis $(q_0,\dots,q_n)$ in L and let $q_k = \sum a_{ij}^{(k)} t_i t_j$ and let b_{q_k} be represented by an alternating matrix $A^{(k)}$. The equations of $\mathrm{PB}(W)$ imply that $(y_0, y_1, \dots, y_n)$ satisfy the equations

$$\sum_{j=0}^{n}\sum_{i=1}^{n} y_i x_j \lambda_k b_{ij}^{(k)} + y_0 \sum_{0\le i,j\le n} a_{ij}^{(k)} t_i t_j = 0,\ k = 0,\dots,n.$$

A point x belongs to the image of the projection if and only if

$$\det\begin{pmatrix} \sum_{j=0}^{n} b_{1j}^{(0)} x_j & \dots & \sum_{j=0}^{n} b_{nj}^{(0)} x_j & \sum_{0\le i,j\le n} a_{ij}^{(0)} t_i t_j \\ \vdots & \vdots & \vdots & \vdots \\ \sum_{j=0}^{n} b_{1j}^{(n)} x_j & \dots & \sum_{j=0}^{n} b_{nj}^{(n)} x_j & \sum_{0\le i,j\le n} a_{ij}^{(n)} t_i t_j \end{pmatrix} = 0.$$

Expanding the determinant along the last column of the matrix, we obtain that the equation of $p(\widetilde{\mathcal{D}}(W))$ coincides with the equation of $\mathrm{St}(W)$. □

Example 7.3.11 One can classify the projective equivalence classes of nets of conics in characteristic 2. There are 19 projective equivalence classes [40].

The base-point-free nets of conics are represented by the following families:

(i)

$$\lambda x_0 x_1 + \mu(x_0 x_2 + x_1^2) + \gamma(\frac{a_6}{a_1^2} x_0^2 + x_2^2 + a_1 x_1 x_2 + \sqrt{a_4} x_0 x_2) = 0,$$

where $a_1 \neq 0$. The half-discriminant curve is an ordinary elliptic curve:

$$\mathcal{D}(W) : \lambda^2\gamma + a_1\lambda\mu\gamma + a_1\sqrt{a_4}\lambda\gamma^2 + \mu^3 + a_4\mu\gamma^2 + a_6\gamma^3 = 0.$$

The curves $\widetilde{\mathcal{D}}(W), \mathrm{PB}(W), \mathcal{D}(W)$ are isomorphic; there is a separable isogeny of degree $\widetilde{\mathcal{D}}(W) \to \mathrm{Rey}(W)$ of elliptic curves corresponding to the unique non-trivial 2-torsion point on $\mathcal{D}(W)$. The curve $\mathrm{Rey}(W)$ lies in the dual plane and coincides with the Cayleyan curve of $\mathcal{D}(W)$.

(ii)

$$\lambda x_0 x_1 + \mu(x_0 x_2 + x_1^2) + \gamma(a x_0^2 + x_2^2) = 0,\ a \neq 0.$$

The discriminant curve is a cuspidal cubic:

$$\mathcal{D}(W) : \mu^3 + \lambda^2\gamma = 0.$$

The curve $\widetilde{\mathcal{D}}(W)$ consists of two smooth rational curves $C_1 + C_2$. The projection $\widetilde{\mathcal{D}}(W) \to \mathcal{D}(W)$, restricted to C_1, is the normalization map; it maps C_2 to the cusp $[0, 0, 1]$. The Steinerian curve $\mathrm{St}(W)$ is a non-reduced reducible cubic

$$x_0(ax_0^2 + x_2^2) = 0.$$

The curve $\mathrm{PB}(W) = C_1' + C_2'$ is isomorphic to $\widetilde{\mathcal{D}}(W)$. It is mapped bijectively onto $\mathrm{St}(W)$. The Reye curve consists of lines $\alpha x_0 + \beta x_1 + \gamma x_2$ satisfying the equations

$$\mathrm{Rey}(W) : \alpha\beta^2 + \gamma(\alpha^2 + a\gamma^2) = 0.$$

The line $x_0 = 0$ corresponding to the singular point $[1, 0, 0]$ of $\mathrm{Rey}(W)$ is the unique inseparable Reye line of W. Under the map $\mathrm{PB}(W) \to \mathrm{Rey}(W)$, one component is mapped to this point, and another component is a degree 2 inseparable cover of $\mathrm{Rey}(W)$.

(iii)

$$\lambda x_0^2 + \mu(x_0x_1 + a^{-1}x_1^2) + \gamma(x_2^2 + ax_0x_2 + x_1^2 + x_1(\epsilon a^{-2}x_1 + x_2)) = 0,\ a \neq 0, \epsilon = 0, 1.$$

The discriminant curve is the union of a conic and its tangent line:

$$\mathcal{D}(W) : \gamma(a^2\lambda\gamma + (\epsilon + a^2)\gamma^2 + a^2\mu^2) = 0.$$

The curve $\widetilde{\mathcal{D}}(W)$ consists of 3 smooth rational curves $C_1 + C_2 + C_3$. The projection $\widetilde{\mathcal{D}}(W) \to \mathcal{D}(W)$, restricted to $C_1 + C_2$ an isomorphism, it maps C_3 to the tangency point $[1, 0, 0]$. The Steinerian curve $\mathrm{St}(W)$ is a non-reduced reducible cubic, the union of two lines $ax_0 + x_1 = 0$ and $x_0 = 0$, the latter taken with multiplicity 2. The intersection point of the two lines is the base point of the pencil $\gamma = 0$. The Reye curve has the equation

$$\mathrm{Rey}(W) : \delta(\alpha\delta + a\beta^2 + (a + \epsilon a^{-1})\delta^2) = 0.$$

Its singular point $[1, 0, 0]$ corresponds to the line $x_0 = 0$. There are inseparable Reye lines and the map $\mathrm{PB}(W) \to \mathrm{Rey}(W)$ is an inseparable map of degree 2.

(iv)

$$\lambda x_0^2 + \mu x_1(x_0 + x_1) + \gamma(x_2^2 + ax_0x_1 + x_0x_2) = 0,\ a \neq 0.$$

The discriminant curve, the Steinerian curve and the Reye curve are the unions of three concurrent lines

$$\begin{aligned} \mathcal{D}(W) &: \gamma(\mu\gamma + \mu^2 + a^2\gamma^2) = 0, \\ \mathrm{St}(W) &: x_0x_1x_2 = 0, \\ \mathrm{Rey}(W) &: \beta(\beta^2 + \beta\delta + a\delta^2) = 0. \end{aligned}$$

(v)
$$\lambda x_0^2 + \mu x_1^2 + \gamma x_2(x_0 + x_2) = 0.$$

The discriminant curve is given by the equation:
$$\mathcal{D}(W) : \gamma^2\mu = 0.$$

The Steinerian curve is the line $x_2 = 0$ taken with multiplicity 3. The Reye curve has the equation
$$\mathrm{Rey}(W) : \delta^2\beta = 0.$$

(vi)
$$\lambda x_0^2 + \mu x_1^2 + \gamma x_2^2 = 0.$$

All curves in the net are singular.

7.4 Reye Congruences: $p \neq 2$

In this section, we specialize the discussion from Section 7.2 to the case of webs of quadrics in $\mathbb{P}^3$. We will see that, for a general web, the Reye variety $R(W)$ is a Fano model of an Enriques surface, the surface $\mathrm{PB}(W)$ is a K3 surface, and the Reye map $\mathfrak{r} : \mathrm{PB}(W) \to \mathrm{Rey}(S)$ coincides with the K3-cover.

Let $W = |L|$ be a regular web of quadrics in $\mathbb{P}^3 = |E|$. The surface $\mathcal{D}(W)$ is a quartic in W, the surface $\mathrm{St}(W)$ is a quartic in $\mathbb{P}^3$. The morphisms $\mathrm{pr}_W : \widetilde{\mathcal{D}}(W) \to \mathcal{D}(W)$ and $\mathrm{pr}_{|E|} : \mathrm{PB}(W) \to \mathrm{St}(W)$ are resolutions of singularities. Both of these surfaces are K3 surfaces, so the resolutions are minimal, hence there is an isomorphism
$$\sigma : \mathrm{PB}(W) \cong \widetilde{\mathcal{D}}(W).$$
Note that this is a speciality of the case $n = 3$ (see Remark 7.2.2).

Since the exceptional curves of the resolutions are isomorphic to $\mathbb{P}^1$, we obtain that all singular points of $\mathcal{D}(W)$ are ordinary double points. Their number is equal to 10, the degree of the locus of quadrics of corank 2. Thus, $\mathcal{D}(W)$ is a *Cayley quartic symmetroid* surface [177, 1.1.7].

Proposition 7.4.1 *The following properties are equivalent:*

(i) W is an excellent web,
(ii) $\mathcal{D}(W)$ does not contain lines.

Proof Suppose W is an excellent web. By definition, $\mathrm{St}(W)$ is a smooth quartic. Assume $\mathcal{D}(W)$ contains a line. It defines a pencil of singular quadrics in W. By Bertini Theorem, all quadrics in the pencil have a common singular point. This means that, under the Steinerian map $\mathrm{st} : \mathcal{D}(W) \to \mathrm{St}(W)$, the line is blown down to a singular point. This contradicts our assumption.

Assume that $\mathcal{D}(W)$ does not contain lines and $\mathrm{St}(W)$ has a singular point x. The image of the exceptional curve of $\mathrm{pr}_{|E|}^{-1}(x)$ under the projection $\mathrm{pr}_{|W|} : \mathrm{PB}(W) \to$

$\mathcal{D}(W)$ is a pencil of singular quadrics with common singular point. It defines a line in $\mathcal{D}(W)$. □

Note that a pencil of singular quadrics contains three reducible quadrics. Hence, each line on $\mathcal{D}(W)$ passes through three nodes.

The surface $\mathrm{PB}(W)$ is a K3 surface in $\mathbb{P}^3 \times \mathbb{P}^3$. The Reye surface is isomorphic to the quotient of $\mathrm{PB}(W)$ by a fixed-point-free involution. It is an Enriques surface embedded in $G_1(|E|)$, i.e. it is a smooth congruence in $G_1(|E|)$. It is called the *Reye congruence* associated to W. The surface $\mathrm{PB}(W)$ is isomorphic to its K3-cover.

Proposition 7.4.2 *Assume W is an excellent web. Then, no Reye line is contained in the Steinerian surface.*

Proof Suppose a Reye line is contained in $\mathrm{St}(W)$. This means that, for any point $x \in \ell$, there exists a quadric in W with a singular point at x. Suppose ℓ is not the singular line of a quadric from $|L - \ell|$. Then, there are at most two singular points of the base curve of the pencil that lie on ℓ. Therefore, there are at most two quadrics from $|L - \ell|$ that have singular points on ℓ. This shows that, for a general point x on ℓ, there exists a quadric Q not containing ℓ and containing x as its singular point. This implies that the restriction of W to ℓ is a linear series of non-reduced divisors of degree 2. Since $p \neq 2$, this is impossible.

Assume now that ℓ is the singular locus of a quadric $Q \in |L - \ell|$. Since ℓ is a Reye line, by Proposition 7.2.5, it contains two points $(x, y) \in \mathrm{PB}(W)$ such that there exists a quadric not containing ℓ but containing x as its singular point. Then, there is a pencil of quadrics with singular point at x, hence the fiber of $\widetilde{\mathcal{D}}(W) \to \mathcal{D}(W)$ is one-dimensional. This contradicts the assumption that W is an excellent web. □

According to classical terminology, the complete intersection of $G_1(\mathbb{P}^n)$ and a hypersurface of degree d in the Plücker space is called a *complex of lines of degree d*.

Lemma 7.4.3 *Let N be a net of quadrics in $\mathbb{P}^n$. Assume that N does not consist of singular quadrics, does not contain quadrics of corank ≥ 2, and its base scheme is of expected codimension 3. Then, the set of lines in $\mathbb{P}^n$ contained in some quadric from N is a cubic complex of line $M(N)$.*

Proof Take a general line in $\mathbb{G} = G_1(\mathbb{P}^n)$ represented by a pencil $\sigma_{x,\Pi}$ of lines in a general plane Π. The restriction of N to Π is a net $N(\Pi)$ of conics. If all conics are singular, then all quadrics in N are touching Π. By Bertini Theorem, all conics in $N(\Pi)$ have a common singular point x_0, hence there exists a pencil in N of quadrics with singular point at x_0. Since N does not contain quadrics of corank ≥ 2 and has the curve of singular quadrics, the set of singular points of quadrics from N is a curve in $\mathbb{P}^n$. By taking a general Π, we get a contradiction.

Thus, we may assume that not all conics in $N(\Pi)$ are singular, hence the set of singular conics is a plane cubic F in $N(\Pi)$. Taking Π away from $\mathrm{Bs}(N)$, we can further assume that $N(\Pi)$ has no base points. This easily implies that F is reduced. Take a general point x_0 in Π and consider the pencil of conics in $N(\Pi)$ that contain

x_0. Then, it is a general pencil of conics, hence it intersects F at three different points corresponding to three singular conics in the pencil. This implies that there are three singular conics passing through x. By taking x_0 general enough, we may assume that none of these conics has a singular point at x_0. Hence, x_0 belongs to three line components of singular conics from the net. This implies that a general pencil of lines in Π contains three line components of reducible conics in $N(\Pi)$. Therefore, $\sigma_{x,\Pi}$ intersects $M(N)$ at three points. This shows that the class $[M]$ of M in $A^2(\mathbb{G})$ is equal to $3[\sigma_\ell]$, i.e. $M(N)$ is a complete intersection of $\mathbb{G}$ and a cubic hypersurface.□

The cubic complex of lines $M(N)$ constructed in the lemma is called the *Montesano complex of lines*.

Theorem 7.4.4 *The Reye congruence* $\mathrm{Rey}(W)$ *is a smooth congruence of bidegree* $(7,3)$ *and sectional genus* 6.

Proof Let N be a general net in W. It easy to see that it satisfies the assumption of the previous lemma. Let M_1 and M_2 be the Montesano complexes of two general nets. They intersect along a congruence of bidegree $9[\sigma_\ell]^2 = (9,9)$. A line ℓ from $M_1 \cap M_2$ is either contained in a quadric from the pencil $N_1 \cap N_2$, or it is contained in a quadric Q_1 from N_1 and a quadric Q_2 from N_2 not belonging to $N_1 \cap N_2$. In the latter case, the line is a Reye line of W. In the former case, ℓ intersects the base curve of $N_1 \cap N_2$ at two points. Conversely, every such line is contained in a quadric from $N_1 \cap N_2$. This shows that $M_1 \cap M_2 = \mathrm{Rey}(W) \cup S$, where S is the congruence $\mathrm{Bis}(B)$ of bisecants of the base curve B of $N_1 \cap N_2$ from Example 7.1.6. We know that the order of S is equal to 2 and the class is equal to 6. This implies that $\mathrm{Rey}(W)$ is a congruence of bidegree $(9,9) - (2,6) = (7,3)$. □

By construction, $\mathrm{Rey}(W)$ is a quotient of the K3 surface $\mathrm{PB}(S)$ by a fixed-point-free involution. Thus, S is an Enriques surface. This agrees with Corollary 7.1.4. Since we assumed that $p \neq 2$, the same corollary implies that $\mathrm{Rey}(W)$ is a nodal Enriques surface. However, we see a simple confirmation of this fact: the image of an exceptional curve of $p_W : \mathrm{PB}(W) \to \mathcal{D}(W)$ under the Reye map $\mathfrak{r} : \mathrm{PB}(W) \to \mathrm{Rey}(S)$ is a smooth rational curve.

We will prove later that, if $p \neq 2$, any smooth congruence of bidegree $(7,3)$ and sectional genus 6 is equal to a Reye congruence $\mathrm{Rey}(W)$ for some regular web of quadrics in $\mathbb{P}^3$.

Let ℓ be a Reye line of W. It is spanned by the ramification points x_1, x_2 of the restriction of W to ℓ. The points x, y are the singular points of quadrics Q_1, Q_2 such that $x_i \in \mathrm{Sing}(Q_i)$. This shows that $\{x_1, x_2\} = \ell \cap \mathrm{St}(W)$, in fact, ℓ is tangent to $\mathrm{St}(W)$ at these points. It is known that the variety of bitangent lines to a normal quartic surface is a congruence of lines. Its class n is equal to the number of bitangents of a plane quartic curve, known to be equal to 28. Its order is equal to the number of bitangents passing through a general point in $\mathbb{P}^3$. It is equal to 12 (see [632, p. 283] or [14, Proposition 3.3]). Thus, $\mathrm{Rey}(W)$ is one of irreducible components of the bitangent surface $\mathrm{Bit}(\mathrm{St}(W))$. However, the focal surface of $\mathrm{Rey}(W)$ is not $\mathrm{St}(W)$.

Recall that the focal surface $\mathrm{Foc}(S)$ of a congruence S is the branch surface of the projection map $Z_S \to \mathbb{P}^3$. In our case, it consists of points $x \in \mathbb{P}^3$ such that the net of

quadrics $N(x)$ in W containing x has less than eight base points. This is equivalent to that $N(x)$ does not intersect $\mathcal{D}(W)$ transversally.

The linear system W of quadrics defines a map

$$f : |E| \to \mathbb{P}(L) = |L^\vee|,\ x \mapsto N(x). \tag{7.4.1}$$

For any point $H \in \mathbb{P}(L)$, considered as a plane in W, the fiber $f^{-1}(H)$ consists of base points of the net H. Thus, the map is of degree 8. Its branch divisor is the set of nets in W that have less than 8 base points. This happens if and only if one of the quadrics in the net has a singular point at one of the base points. This base point must be on the Steinerian surface $\mathrm{St}(W)$. Thus, we see that the branch locus $\mathsf{B}(f)$ of f is equal to $f(\mathrm{St}(W))$. Since a transversal plane section of $\mathcal{D}(W)$ has 8 base points, we obtain that $\mathsf{B}(f)$ is contained in the dual surface $\mathcal{D}(W)^*$ of $\mathcal{D}(W)$. Since $\mathrm{St}(W)$ and $\mathcal{D}(W)^*$ are irreducible,

$$\mathsf{B}(f) = \mathcal{D}(W)^*.$$

It is known that the degree of the dual surface of a quartic surface with 10 ordinary nodes is equal to 16 (see [177, 1.2.3]). The canonical class formula $K_{|E|} = \mathrm{Ram}(f) + f^*(K_{\mathbb{P}(L)})$ shows that the degree of the ramification divisor is equal to 4. Thus,

$$\mathrm{Ram}(f) = \mathrm{St}(W).$$

The restriction of W to $\mathrm{St}(W)$ is given by a linear subsystem of $|\mathcal{O}_{\mathrm{St}(W)}(2)|$. The image of the map is a surface of degree 16, this implies that the degree of the map $f : \mathrm{St}(W) \to \widetilde{\mathcal{D}}(W)^*$ is equal to 1 and therefore

$$f^{-1}(\mathcal{D}(W)^*) = \mathrm{St}(W) \cup F,$$

where $\deg F = 32 - 8 = 24$.

Remark 7.4.5 One can compute the sectional genus of $\mathrm{Rey}(W)$ without appealing to the fact that $\mathrm{Rey}(W)$ is an Enriques surface. In the notation of the proof of Theorem 7.4.4, a hyperplane section H of $\mathrm{Rey}(S) \cap \mathrm{Bis}(B)$ is a complete intersection of two hypersurfaces of degree 3 and a quadric in $\mathbb{P}^4$. Its arithmetic genus is equal to 28. On other hand, it is a reducible curve $H_1 \cup H_2$, where H_1 is a hyperplane section of $\mathrm{Rey}(W)$ and H_2 is a hyperplane section of $\mathrm{Bis}(B)$. The curves intersect at δ points, where δ is the degree of the intersection curve $\mathrm{Rey}(W) \cap \mathrm{Bis}(B)$. Let us compute the degree. Consider the restriction of W to the base curve B of a general pencil P_1 of quadrics in W. A general quadric in W intersects B at 8 points, a member of the linear system $|\mathcal{O}_B(2)|$. Any join of two intersection points is a Reye line. Take a general line in $\mathbb{P}^3$ and consider a pencil P_2 contained in $|\mathcal{O}_B(1)|$. A line from $\mathrm{Rey}(W) \cap \mathrm{Bis}(B)$ intersects B at two points that are common to a divisor from P_1 and P_2. It follows from [12, Example on p. 344] that the number of such divisors is equal to 20. We know that the genus of H_2 is equal to 3. The formula $28 = p_a(H_1 \cup H_2) = g(H_1) + g(H_2) + \delta - 1$ implies that $g(H_1) = 6$.

Proposition 7.4.6 *Assume that W is an excellent web of quadrics in $\mathbb{P}^3$. Then, the Reye congruence* $\mathrm{Rey}(W)$ *has no fundamental points and has no focal rays. Its focal*

surface $\mathrm{Foc}(\mathrm{Rey}(W))$ *coincides with the residual part* F *in* $f^{-1}(\widetilde{\mathcal{D}}(W)^*)$. *It is an irreducible and reduced surface of degree* 24.

Proof Assume that $x \in \mathbb{P}^3$ is a fundamental point of $S = \mathrm{Rey}(W)$. This means that $\dim p_S^{-1}(x) > 0$. By definition, $p_S^{-1}(x)$ is the set of Reye lines containing x. Let $N(x)$ be the net of quadrics in W containing x. The restriction of W to a Reye line ℓ containing x coincides with the restriction of $N(x)$ to ℓ. Thus, ℓ is contained in a pencil of quadrics from $N(x)$, or, equivalently, the restriction of $N(x)$ to ℓ is a set of two base points of $N(x)$ which may coincide. Thus, either all quadrics in $N(x)$ have a multiple base point at x, or the base scheme of $N(x)$ is a curve. In the former case, there exists a pencil of quadrics singular at x, contradicting the assumption that W is excellent. In the latter case, a quadric from $W \setminus N(x)$ intersects the base curve of $N(x)$ at a base point of W. This contradicts the assumptions on W. This proves that $\mathrm{Fund}(S) = \emptyset$.

Assume ℓ is a focal ray of S. Recall that this means that ℓ is a Reye line contained in the focal surface. By above, for any point $x \in \ell$, $p_S^{-1}(x)$ consists of Reye lines joining x with another base point of $N(x)$. The line ℓ is focal if, for any $x \in \ell$, the net $N(x)$ has a multiple base point on ℓ. A multiple base point of a net is a singular point of one of its member. Thus, for each $x \in \ell$, there exists a quadric in $N(x)$ singular at some point $y_x \in \ell$. Since two different $N(x)$ and $N(y)$ span W, and W has no base points, we obtain that the set of points y_x is equal to the whole ℓ. This implies that ℓ is contained in the Steinerian surface $\mathrm{St}(W)$. This contradicts Proposition 7.4.2.

We know that $\mathrm{Foc}(S)$ is the image of the ramification divisor $\mathsf{R}(S) \subset Z_S$ of the map $p_S : Z_S \to |E|$ and the projection $q_S : \mathsf{R}(S) \to S$ is of degree 2. Suppose $\mathsf{R}(S)$ is not reduced or it is reducible. Then, it contains an irreducible component mapping birationally onto S. On the other hand, it maps dominantly onto $\mathrm{Foc}(S)$ and hence onto $\mathcal{D}(W)^*$. The latter surface is birationally isomorphic to a K3 surface. Since an Enriques surfaces cannot be mapped birationally onto a K3 surface, we obtain a contradiction.

Finally, let us see that $F = \mathrm{Foc}(S)$. Let us consider the map (7.4.1). For any $x \in \mathrm{Foc}(S)$, the net $N(x)$ has less than 8 base points, hence $f(N(x)) \in \mathsf{B}(f)$. This shows that $x \in f^{-1}(\mathsf{B}(f)) = \mathrm{St}(W) \cup F$ and $\mathrm{Foc}(S) \subset \mathrm{St}(W) \cup F$. Let x' be a multiple base point of $N(x)$ and let $\ell = \langle x, x' \rangle$ be the Reye line joining the two points. If $x \in \mathrm{St}(W)$, then there exists a quadric $Q \in N(x)$ with $x \in \mathrm{Sing}(Q)$. Since $x' \in Q$, the line ℓ is contained in Q. Applying Proposition 7.2.5, we obtain that the fiber of $\mathrm{PB}(W) \to \mathbb{P}^3$ over y is one-dimensional. The number of such points is finite and it is equal to zero when the web is excellent. By the above, $\mathrm{Foc}(S)$ is an irreducible component of F. Comparing the degree of F with the degree of the focal surface found in Proposition 7.1.2, we find that $F = \mathrm{Foc}(S)$. □

Let $\mathrm{Rey}(W)^*$ be the dual congruence of lines in $\mathbb{P}(E) = |E^\vee|$. It consists of pencils $\ell^\perp$ of planes containing a Reye line ℓ. Its bidegree is equal to $(3, 7)$. Its fundamental locus consists of planes Π that contain infinitely many Reye lines (they are *focal planes* of $\mathrm{Rey}(W)$). If P is not contained in any quadric from W of corank 2, the restriction of W to Π is a web of conics. One can classify web of conics (as apolar

linear systems to pencils of conics) to check that a web of conics without base points contains only finitely many pencils with reducible base scheme. This shows that Π has only finitely many (in fact, ≤ 3) Reye lines. On the other hand, if we take Π to be an irreducible component of a corank 2 quadric, the restriction of W to Π is a net of conics without base points. Its Reye variety is isomorphic to a cubic curve. We have 20 such planes, and a Reye line from this plane will be called an *exceptional Reye line*.

Let ℓ be a Reye line and let $|L - \ell|$ be the pencil of quadrics containing ℓ, a line in W. Its base scheme contains ℓ. If ℓ is a general Reye line, the pencil contains only two singular quadrics corresponding to the tangency points of ℓ and the focal surface $\mathrm{Foc}(\mathrm{Rey}(W))$. This shows that $|L - \ell|$ intersects $\mathcal{D}(W)$ at two points instead of the expected four. It is easy to check that these points are the tangency points. Thus, the line $|L - \ell|$ is a bitangent of the Cayley quartic symmetroid $\mathcal{D}(W)$. Thus, we obtain a map

$$\nu : \mathrm{Rey}(W) \to \mathrm{Bit}(\mathcal{D}(W)). \tag{7.4.2}$$

Theorem 7.4.7 *Let* $\mathrm{Bit}(\mathcal{D}(W))$ *be the bitangent congruence of the discriminant surface* $\mathcal{D}(W)$.

1. *The bidegree of* $\mathrm{Bit}(\mathcal{D}(W))$ *is equal to* $(12, 28)$. *In particular, it is a surface of degree 40 in the Plücker space* $\mathbb{P}^5$.
2. *The surface has* 45 *singular points corresponding to lines* $\langle y_i, y_j\rangle$ *through two nodes* y_i, y_j *of* $\mathcal{D}(W)$.
3. *The singular locus of the surface is the union of* 10 *pairs of plane cubic curves* B_i, B_i' *representing the generatrices of the cubic enveloping cones of* $\mathcal{D}(W)$ *at its* 10 *nodes* $y_1, \ldots, y_{10}$.
4. *The union* $B_i \cup B_i'$ *is equal to the intersection of* $\mathrm{Bit}(\mathcal{D}(W))$ *with the plane of lines through the node* y_i.
5. *The cubic curves* B_i *and* B_i' *intersect at 9 points corresponding to the lines* $\ell_{ij} = \langle y_i, y_j\rangle$, $j \neq i$.
6. *Two cubic curves from different pairs* (B_i, B_i') *and* (B_j, B_j') *intersect at one point* ℓ_{ij}.
7. *The restriction* $\nu^*(B_i) \to B_i$ *of the map* (7.4.2) *to* $\nu^*(B_i)$ *is an unramified map of degree 2 (same for* B_i'*).*
8. *The pre-image of* ℓ_{ij} *consists of 4 points.*
9. *The map* ν *is the normalization map.*

Proof A general plane section of $\mathcal{D}(W)$ is a plane quartic curve. It has 28 bitangents. Thus, the class of the congruence $\mathrm{Bit}(\mathcal{D}(W))$, is equal to 28. Let x be a general point in $\mathbb{P}^3$. We already observed that the number of bitangents to a normal quartic surface passing through x is equal to 12. This gives the order of $\mathrm{Bit}(\mathcal{D}(W))$. We can also argue without referring to [632]. A bitangent to $\mathcal{D}(W)$ defines a pencil of quadrics in W containing only two singular quadrics. The base scheme of such a pencil is either an irreducible rational cubic and its bisecant, or a conic plus two lines. The latter corresponds to the case where the pencil contains a reducible quadric, i.e. the bitangent line passes through a singular point of $\mathcal{D}(W)$. In both cases, the line

components are Reye lines of W. Conversely, a Reye line is contained in a pencil of quadrics with a singular base scheme. Such a pencil in an excellent web W has only two singular quadrics, and the base scheme is either a line plus an irreducible cubic, or an irreducible conic plus two lines, or four lines. The latter corresponds to the case where the pencil passes through two nodes of $\mathcal{D}(W)$. In this way, we see that the map

$$\nu : \mathrm{Rey}(W) \to \mathrm{Bit}(\mathcal{D}(W))$$

from (7.4.2) is the normalization map satisfying the properties from the assertions of the theorem.

Let us see that the order of $\mathrm{Bit}(\mathcal{D}(W))$ is indeed equal to 12. Consider the regular map $f : \mathbb{P}^3 \to W^*$ given by the linear system W. The pre-image of a plane is a quadric from W. It is singular if and only if the plane is tangent to the branch locus of f. Thus, the dual hypersurface of the branch locus is $\mathcal{D}(W)$. By the duality, the branch locus is $\mathcal{D}(W)^*$, the dual hypersurface of $\mathcal{D}(W)$. A line $\mathcal{P}$ in $\mathbb{P}^3$ is bitangent to $\mathcal{D}(W)$ if and only if the line $\mathcal{P}^*$ in W^* of quadrics in W containing $\mathcal{P}$ is bitangent to $\mathcal{D}(W)^*$. In other words, the surface of bitangent lines to a surface is dual, in the sense of Grassmannians of lines, to the surface of bitangent lines of the dual surface. Under this duality, the order and the class interchange. To compute the order $\mathrm{Bit}(\mathcal{D}(W))$ we need to compute the class of $\mathrm{Bit}(\mathcal{D}(W)^*)$. A plane in W^* is a point in W, i.e. a quadric Q from W. A bitangent to $\mathcal{D}(W)^*$ contained in this plane corresponds to a Reye line contained in Q. Let $\mathcal{N}$ be a general net of quadrics which together with Q generates W. The number of Reye lines contained in Q is equal to the number of lines in $\mathcal{N}$ contained in $\mathcal{N}$ and in Q. By Lemma 7.4.3, the set of lines contained in at least one quadric from a general net of quadrics in $\mathbb{P}^3$ is a cubic line complex. Now, the set of lines in a nonsingular quadric is parametrized by the union of two disjoint conics in $G_1(\mathbb{P}^3)$. Hence, the cubic complex intersects the union at $4 \times 3 = 12$ points. □

Remark 7.4.8 Let us note that the normalization map ν is given by a linear subsystem in $|\mathcal{O}_{\mathrm{Rey}(W)}(2)|$. To see this, we may assume that $W = \mathbb{P}(E)$ is spanned by four quadrics Q_i which we represent by symmetric matrices A_i. Take a Reye line $\ell = \langle [v], [w] \rangle$, where $([v], [w]) \in \mathrm{PB}(W)$. Consider the evaluation map

$$E \cong \Bbbk^4 \to \Bbbk^3, \quad A \mapsto ({}^t vAv, {}^t wAw, {}^t vAw).$$

Its kernel consists of quadrics containing ℓ. Since ℓ is a Reye line, we have ${}^t vAw = 0$, hence the evaluation map is $\Bbbk^4 \to \Bbbk^2$, and its kernel is the pencil of quadrics containing ℓ. The Plücker coordinates of the pencil are equal to the maximal minors of the matrix

$$\begin{pmatrix} {}^t vA_1v & {}^t vA_2v & {}^t vA_3v & {}^t vA_4v \\ {}^t wA_1w & {}^t wA_2w & {}^t wA_3w & {}^t wA_4w \end{pmatrix}.$$

It is easy to see that they are expressed by quadratic polynomials in Plücker coordinates of ℓ.

Remark 7.4.9 As we saw in the proof, for any normal quartic surface X the bidegree of the bitangent surface is equal to $(12, 28)$. However, $\mathrm{Bit}(X)$ could be highly reducible. For example, when X is a Kummer quartic surface with 16 nodes, the surface $\mathrm{Bit}(X)$ consists of 6 irreducible components of bidegree $(2, 2)$ and 16 β-planes corresponding to tropes of X. Two congruences of lines are called *confocal* if they have the same focal surface. For a congruence of lines of order 2 without fundamental curves, the number of confocal surfaces is always larger than one. The common focal surface is a quartic surface with $N \geq 11$ nodes [693, Theil 2] (see also Chapter 11 of the second edition of [177]). When W is excellent, the surface $\mathrm{St}(W)$ is a nonsingular quartic, and $\mathrm{Rey}(W)$ is an irreducible component of $\mathrm{Bit}(\mathrm{St}(W))$.

Let X be an irreducible quartic surface in $\mathbb{P}^3$ with an ordinary double point $x_0 = [1, 0, 0, 0]$. We can define X by equation

$$t_0^2 A_2(t_1, t_2, t_3) + 2t_0 A_3(t_1, t_2, t_3) + A_4(t_1, t_2, t_3) = 0,$$

where A_i are homogeneous forms of degrees indicated in the subscript. Projecting from x_0, we see that X is birationally isomorphic to the double cover of $\mathbb{P}^2$ with branch curve B given by equation $A_3^2 - A_2 A_4 = 0$. The conic $C = V(A_2)$ is equal to the image of the exceptional curve over the node after we extend the projection map to a minimal resolution of the quartic surface. We assume that X does not contain lines passing through the node. Then, C is a contact conic, i.e. a conic everywhere tangent to B at nonsingular points of B. Its splits under the cover $z^2 = A_3^2 + A_2 A_4$ into two rational curves with equations $z \pm A_3 = 0$. Conversely, a double cover $f : X \to \mathbb{P}^2$ branched along a plane sextic curve B such that there exists a contact conic admits a birational model isomorphic to a quartic surface with an ordinary node. The map $X \to \mathbb{P}^3$ is given by linear system $|f^*(\ell) + C|$, where ℓ is a line in $\mathbb{P}^2$ and C is one of the two irreducible components of the split conic.

For example, the double cover of $\mathbb{P}^2$ branched along an irreducible 9-nodal plane sextic curve that admits a contact conic is isomorphic to a quartic surface with 10 nodes. Its nine nodes come from the singular points of the double cover, and the enveloping cone at tenth node is irreducible. It follows from Theorem 7.4.7 that the quartic surface is not isomorphic to a Cayley symmetroid.

The next theorem, due to Cayley, shows that the breaking of the enveloping cone at each node into two cubic cones characterizes Cayley quartic symmetroids.

Theorem 7.4.10 *A quartic surface X with 10 ordinary nodes is a Cayley quartic symmetroid if and only if one of the following conditions is satisfied:*

(i) Let $\pi : X' \to X$ be the minimal resolution of X. Let $\theta_1, \ldots, \theta_{10}$ be the divisor classes of the exceptional curves and h the class of the pre-image of a plane section of X. Then,

$$\frac{1}{2}(3h - \theta_1 - \cdots - \theta_{10}) \in \mathrm{Pic}(X').$$

(ii) The enveloping cone of each node is the union of two cubic cones.

Proof Assertion (i) follows from the theory of symmetric determinantal representations of hypersurfaces, see [177, 4.2.6]. To prove (ii) we use that the Steinerian map $\mathrm{st} : \mathcal{D}(W) \dashrightarrow \mathrm{St}(W)$ is given by the linear system of polar cubics that generate a linear subsystem of $|3c_1(\mathcal{O}_X(1)) - \theta_1 - \cdots - \theta_{10}|$. By (i), it is equal to $|\mathcal{O}_{\mathrm{St}(W)}(2)|$. Let $Q \in |\mathcal{O}_{\mathrm{St}(W)}(2)|$ be the corank 2 quadric corresponding to the singular point p_i of $\mathcal{D}(W)$. It corresponds to the polar of $\mathcal{D}(W)$ with pole at p_i. Since Q_i is the union of two planes Π_i, Π_i', the pre-image of Q_i under the Steinerian map intersects $\mathcal{D}(W)$ at the union of two cubics, each isomorphic to the residual cubic in $\Pi_i \cap \mathrm{St}(W)$ and $\Pi_i' \cap \mathrm{St}(W)$. □

Consider the map $f : \mathbb{P}^3 \to W^* = |L^\vee|$ from (7.4.1). Its restriction to a Reye line ℓ is given by the pencil obtained by restriction of the web W to ℓ. The map is a degree 2 cover onto a line ℓ^* in W^* with ramification points x, y, where $(x, y) \in \mathrm{PB}(W)$. The images of the two points are the points on the branch divisor equal to $\mathcal{D}(W)^*$. The line ℓ^* is tangent to $\mathcal{D}(W)^*$ at these points. The set of lines ℓ^* is the dual congruence of lines $\nu(\ell) = |L - \ell|$. This shows that the lines ℓ^* are parameterized by the dual congruence $\mathrm{Bit}(\mathcal{D}(W))^*$ of bidegree $(28, 12)$. The set of 28 lines ℓ^* passing through a general point $H \in W^*$ is the set of lines joining two base points of the net of quadrics in W defined by H.

Recall that we defined the Cayley surface of W. This is a subvariety of the projective space $|L^\perp| \cong \mathbb{P}^5$. Since $\mathrm{PB}(W)$ is smooth, by Theorem 7.2.12, $\mathrm{Cay}(W)$ is a smooth surface of degree 10 in $\mathbb{P}^5$. We will see later that it is not contained in a quadric, although $\mathrm{Rey}(S)$ is.

Consider the universal family $\mathcal{U} = \{(\ell, Q) \in G(2, E) \times W : \ell \subset Q\}$ of lines contained in some quadric from W and its two natural projections

$$\begin{tikzcd} & \mathcal{U} \arrow[dl, "p_1"'] \arrow[dr, "p_2"] & \\ G(2,E) & & W \end{tikzcd} \tag{7.4.3}$$

By definition of a Reye line, the first projection $p_1 : \mathcal{U} \to G(2, E)$ is isomorphic to the blow-up of $G(2, E)$ along $\mathrm{Rey}(W)$. The fiber of the second projection $p_2 : \mathcal{U} \to W$ over a quadric Q is isomorphic under the projection p_1 to the subvariety of $G(2, E)$ parameterizing lines in Q. If Q is nonsingular, then the fiber is the union of two conics. If Q is a singular quadric of corank 1 with vertex x_0, then the fiber is a conic in the plane σ_x taken with multiplicity 2, and if Q is the union of two planes $\Pi \cup \Pi'$, the fiber is the union of two planes $\sigma_\Pi \cup \sigma_{\Pi'}$ intersecting at one point $\Pi \cap \Pi'$. Using the Stein factorization, we factor p_2 as the composition of a morphism $g : \mathcal{U} \to X$ and the double cover $\pi : X \to W$ branched along the quartic symmetroid $\mathcal{D}(W)$. The variety X is singular over the 10 nodes of $\mathcal{D}(W)$. The morphism $g : \mathcal{U} \to X$ is birationally isomorphic to a conic bundle over X. Its fibers over $\pi^{-1}(W \setminus \mathcal{D}(W))$ are conics, its fibers over nonsingular points of $\pi^{-1}(\mathcal{D}(W))$ are double conics, and its fibers over the pre-images of singular points of $\mathcal{D}(W)$ are the unions of two planes. The variety X is birationally isomorphic to the *Artin–Mumford*

double solid [31]. We refer to [425] for the description of small resolutions of X (in the category of algebraic spaces) by choosing a plane σ_{Π_i} or $\sigma_{\Pi'_i}$ in each fiber consisting of two planes. Note that under the first projection the image of each plane intersects $\mathrm{Rey}(W)$ along a cubic curve F_i or F'_i.

Let $f : G(2, E) \dashrightarrow W$ be the composition of the rational maps $p_2 \circ p_1^{-1}$. The pre-image of a plane in $W = |L|$, i.e. a net N of quadrics in W, is the Montesano complex of lines of N. Thus, the map f is given by a linear system in $|\mathcal{O}_{G(2,E)}(3)|$ isomorphic to the dual space $W^* = |L^\vee|$ of W. We refer to a later Proposition 7.10.8, where this observation is used to find a resolution of the ideal sheaf of $\mathrm{Rey}(W)$ in $G(2, E)$, or, equivalently, the base scheme of the map f.

7.5 Catalecticant Quartic Symmetroid

In this section, we will briefly discuss the theory of apolarity of binary forms. The locus of binary forms of degree 4 that apolar to some binary quadratic form defines a quartic surface in $\mathbb{P}^3$ isomorphic to a Cayley quartic symmetroid.

We assume in this section that $p = 0$.

Let E be a linear space over $\Bbbk$ of dimension $n+1$. An element f of the symmetric power S^kE of E is called *apolar* to an element g from $S^mE^\vee$ if $f(g) = 0$, where S^kE is identified with a linear subspace of $(S^mE^\vee)^\vee$ by the natural extension to symmetric products of the isomorphism $E \to (E^\vee)^\vee$. Two elements f, g are called apolar if $f(g) = g(f) = 0$. Fixing a basis $(e_0, \dots, e_n)$ in E and its dual basis $(t_0, \dots, t_n)$ in $E^\vee$, we will be able to identify S^kE with the space of degree k homogeneous polynomials $\Bbbk[u_0, \dots, u_n]_k$ and $S^mE^\vee$ with the space of degree m homogeneous polynomials $\Bbbk[t_0, \dots, t_n]_m$, then we view each e_i as the differential operator $\frac{\partial}{\partial t_i}$ and apply $f(\frac{\partial}{\partial t_0}, \dots, \frac{\partial}{\partial t_n})$ to $g(t_0, \dots, t_n)$. For example, if $m = k = 2$, a quadratic form $f = \sum_{i=0}^n a_{ii}u_i^2 + \sum_{0\le i<j\le n} a_{ij}u_iu_j$ is apolar to a quadratic form $g = \sum_{i=0}^n b_{ii}t_i^2 + \sum_{0\le i<j\le n} b_{ij}t_it_j$ if and only if

$$\sum_{0\le i\le j\le n} a_{ij}b_{ij} = 0.$$

For any form $g \in S^mE^\vee$ and $k \le m$, the apolarity defines a homomorphism

$$\mathrm{ap}_g : S^kE \to S^{m-k}E,\ f \mapsto f(g)$$

(for $k > m$ such a homomorphism is obviously zero). For example, if $e \in E$ and $g \in S^mE^\vee$, the value of g at e is defined to be $e^m(g)$. If we use a basis as above and consider e as a vector in E with coordinates $(\alpha_0, \dots, \alpha_n)$ and g as a polynomial in $\Bbbk[t_0, \dots, t_n]_m$, then $e^d(g)$ is obtained by plugging in $t_i = \alpha_i$ and coincides with the value of the polynomial g at the vector e multiplied by $m!$.

In the special case $k = m$, the apolarity defines a perfect pairing $S^mE \otimes S^mE^\vee \to \Bbbk$ equal to the mth symmetric power of the canonical pairing $E \otimes E^\vee \to \Bbbk$. In particular,

it defines a canonical polarization isomorphisms

$$S^m E^\vee \to (S^m E)^\vee, \quad S^m E \to (S^m E^\vee)^\vee, \tag{7.5.1}$$

which we will constantly use in order to identify these spaces.

Now, let U denote a linear space of dimension 2 over $\Bbbk$. Elements from $S^d U^\vee$ are called *binary forms* on U of degree d. Let $\nu_d : \mathbb{P}^1 = |U| \to |S^d U| \cong \mathbb{P}^d$ be the dth *Veronese map* defined by the map $U \to S^d U, u \mapsto u^d$. Its image is the *Veronese curve* R_d of degree d. It follows from above that the Veronese map coincides with the map $|U| \to |S^d U^\vee|^* = |(S^d U^\vee)^\vee|$ given by the complete linear system $|\mathcal{O}_{|U|}(d)| = |S^d U^\vee|$. Thus, a hyperplane H in the projective space $|S^d U|$ of binary forms of degree d on $U^\vee$ can be considered as an element $[b_H]$ of $|S^d U^\vee|$ defined by a binary form $b_H \in S^d U^\vee$. Geometrically, H cuts R_d along the positive divisor $D_H = \nu_d(V(b_H))$. A hyperplane H is called an *osculating hyperplane* of R_d if the support of D_H is equal to one point. In this case $b_H = l^d$, where $l \in U^\vee$, therefore the set of osculating hyperplanes can be identified with the Veronese curve in the dual projective space $|S^d U^\vee|$, the image of the Veronese map $\nu_d^* : |U^\vee| \to |S^d U^\vee|, l \mapsto l^d$. We call it the *dual Veronese curve* and denote it by R_d^*. If we use a correlation isomorphism $c : U \to U^\vee$ defined by an isomorphism $\bigwedge^2 U \cong \Bbbk$, then the composition $\nu_d^* \circ c : |U| \to |S^d U^\vee|$ will assign to $[u] \in |U|$ the osculating hyperplane at the point $\nu_d([u])$.

The projective space $|S^2(S^d U^\vee)|$ is the complete linear system $|\mathcal{O}_{|S^d U|}(2)|$ of quadrics in $|S^d U|$. The restriction of a quadric to R_d comes from a natural (meaning $\mathrm{SL}(U)$-equivariant) homomorphism $S^2(S^d U^\vee) \to S^{2d} U^\vee$ whose kernel is the linear space $I(R_d)_2$ of elements of degree 2 in the homogeneous ideal $I(R_d)$ of R_d in $|S^d U|$. The restriction homomorphism splits and defines an isomorphism of linear representation of $\mathrm{SL}(2)$

$$S^2(S^d U^\vee) \cong S^{2d} U^\vee \oplus I(R_d)_2. \tag{7.5.2}$$

It is a special case of the *plethysm* isomorphism (see [244, §11]). Replacing U with the dual space $U^\vee$, we obtain a decomposition of linear representations of $\mathrm{SL}(U)$

$$S^2(S^d U) \cong S^{2d} U \oplus I(R_d^*)_2, \tag{7.5.3}$$

where R_d^* is the dual Veronese curve. We have

$$S^{2d} U = (S^{2d} U^\vee)^\vee = (S^2(S^d U^\vee)/I(R_d)_2)^\vee = I(R_d)_2^\perp \subset S^2(S^d U) = S^2(S^d U^\vee)^\vee.$$

Thus, the space $|S^{2d} U|$ can be naturally identified with the space of quadrics in $|S^d U^\vee| = |S^d U|^*$ *apolar* to quadrics vanishing on the Veronese curve R_d. They are also known as *harmonic quadrics* with respect to the Veronese curve.

Choose a basis (u_0, u_1) in U and the dual basis (t_0, t_1) in $U^\vee$. It defines a monomial basis $(u_0^d, u_0^{d-1} u_1, \ldots, u_1^d)$ in $S^d U$. In coordinates, the Veronese map ν_d is:

$$[\alpha u_0 + \alpha u_1] \mapsto [\sum_{k=0}^{d} \binom{d}{k} \alpha_0^{d-k} \alpha_1^k u_0^{d-k} u_1^k].$$

If we modify the monomial basis by replacing $u_0^{d-k} u_1^k$ with $\binom{d}{k} u_0^{d-k} u_1^k$, then the map is given by the familiar formula:

$$[t_0, t_1] \mapsto [t_0^d, t_0^{d-1} t_1, \dots, t_0 t_1^{d-1}, t_1^d].$$

The dual of the modified basis of $S^d U$ is $(t_0^d, t_0^{d-1} t_1, \dots, t_0 t_1^{d-1}, t_1^d)$. The dual Veronese map ν_d^* is:

$$[u_0, u_1] \mapsto [u_0^d, d u_0^{d-1} u_1, \dots, \binom{d}{k} u_0^{d-k} u_1^k, \dots, d u_0 u_1^{d-1}, u_1^d].$$

The composition $\nu_d^* \circ c : |U| \to |S^d U^\vee|$ is given now by

$$[t_0, t_1] \mapsto [-u_1, u_0] \mapsto [u_1^d, -d u_1^{d-1} u_0, \dots, (-1)^k u_1^{d-k} u_0^k, \dots, (-1)^d u_0^d].$$

If $(x_0, \dots, x_d)$ are coordinates in $|S^d U|$ with respect to the modified monomial basis, then the equations of the Veronese curve R_d are given by the 2×2-minors of the matrix

$$A = \begin{pmatrix} x_0 & x_1 & \cdots & x_{d-1} \\ x_1 & x_2 & \dots & x_d \end{pmatrix}.$$

The equations of the dual Veronese curve R_d^* are given by the 2×2-minors of the matrix A^* obtained from the matrix A by replacing x_k with $(-1)^k \binom{d}{k}^{-1} \xi_{d-k}$, where $(\xi_0, \dots, \xi_d)$ are the dual coordinates.

For example, if we take $d = 2$, the equation of the Veronese conic is $t_0 t_2 - t_1^2 = 0$ and the equation of the dual Veronese conic is $4\xi_0\xi_2 - \xi_1^2 = 0$. If $d = 3$, the equations of the Veronese cubic R_3 are:

$$x_0 x_2 - x_1^2 = x_0 x_3 - x_1 x_2 = x_1 x_3 - x_2^2 = 0,$$

and the equations of the dual Veronese cubic R_3^* are

$$9\xi_0\xi_3 - \xi_1\xi_2 = 3\xi_0\xi_2 - \xi_1^2 = 3\xi_1\xi_3 - \xi_2^2 = 0. \tag{7.5.4}$$

Thus, a quadratic form $q = \sum_{i=0}^{d} a_{ii} x_i^2 + 2 \sum_{0 \le i < j \le d} a_{ij} x_i x_j$ is harmonic with respect to R_d if and only if each 2×2-minor of the matrix A^*, considered as a differential operator in $\frac{\partial}{\partial x_0}, \dots, \frac{\partial}{\partial x_d}$, vanishes at q. This happens if and only if

$$q = \sum_{k=0}^{2d} A_k (\sum_{i+j=k} x_i x_j), \quad A_k \in \Bbbk. \tag{7.5.5}$$

The projection of this harmonic quadratic form in $(S^{2d} U)^\vee$ to $S^{2d}(U^\vee)$ is equal to $\sum_{k=0}^{2d} A_k t_0^{2d-k} t_1^k$.

A subspace Λ_{k-1} of dimension $k-1$ is called a *k-secant* of R_d if the linear system of hyperplanes with base scheme Λ_{k-1} cuts out in R_d a base-point free linear system of divisors of degree $d-k$. A 1-secant is a point on R_d, a 2-secant is a line called a *bisecant* of R_d.

Proposition 7.5.1 *Let $g \in S^{2d}U$. Under the inclusion $\iota : S^{2d}U^\vee \hookrightarrow S^2(S^dU^\vee)$ defined by* (7.5.2)*, the following conditions are equivalent:*

(i) $\mathrm{Ker}(\mathrm{ap}_g) \neq \{0\}$*;*
(ii) the quadric $V(\iota(g))$ is singular;
(iii) the point $[g] \in |S^{2d}U^\vee|$ belongs to a d-secant of the Veronese curve R_{2d}.

Proof It is known that the first condition is equivalent to the condition that g belongs to the closure of the locus of binary form that can be written as a sum $l_1^{2d} + \cdots + l_d^{2d}$ of powers of linear forms $l_i \in U^\vee$. This easily implies that conditions (i) and (iii) are equivalent. Since $\dim S^dU = d+1$, a quadratic form $Q \in S^2(S^dU^\vee)$ is degenerate if and only if it can be written as a sum of d powers of squares of linear forms.

Suppose (i) holds; to show that (ii) holds it suffices to show that, for any $l \in U^\vee$, we have $\iota(l^{2d}) = L^2$ for some $L \in S^dU^\vee$. Choosing coordinates (t_0, t_1) in U, we may assume that $l^{2d} = t_0^{2d}$. It follows from (7.5.5) that the harmonic quadric q which is projected to t_0^{2d} must coincide with $q = x_0^2$. □

Let Cat_{2d} denote the subvariety of $|S^{2d}U^\vee|$ parametrizing binary forms g which admit an apolar form $f \in S^dU$. Since ap_g is a linear map of linear spaces of the same dimension $d+1$, Cat_{2d} is a hypersurface of degree $d+1$ in $|S^{2d}U^\vee|$. It is called the *catalecticant hypersurface* of binary forms of degree $2d$. If we choose a modified monomial basis in $S^{2d}U^\vee$ to write $g = \sum_{i=0}^{2d} \binom{2d}{i} a_i t_0^{2d-i} t_1^i$, the equation of Cat_{2d} is given by the determinant of the *catalecticant matrix* (or the *Hankel matrix*) :

$$\det \begin{pmatrix} a_0 & a_1 & \dots & a_{d-1} & a_d \\ a_1 & a_2 & \dots & a_d & a_{d+1} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ a_d & a_{d+1} & \dots & a_{2d-1} & a_{2d} \end{pmatrix} = 0. \tag{7.5.6}$$

Let $\mathrm{Sec}_k(R_d)$ be the union of $(k-1)$-secants of R_d. Using Proposition 7.5.1, we can identify Cat_{2d} with the discriminant hypersurface D in the space of harmonic quadrics $|S^2(S^dU^\vee)|^{\mathrm{harm}}$. The argument from the proof of the Proposition 7.5.1 can be used to prove that

$$\mathrm{Sec}_k(R_d) = |S^2(S^dU^\vee)|_k^{\mathrm{harm}} = \mathsf{D}(d-k),$$

where $|S^2(S^dU^\vee)|_k^{\mathrm{harm}}$ is the closure of the locus of harmonic quadrics of corank $\geq d-k$.

The following proposition [323, Theorem 1.56] (that goes back to J. Sylvester and S. Gundelfinger) shows that the subspace $|S^{2d}U^\vee|$ of harmonic quadrics intersects the discriminant hypersurface $\mathsf{D}(2d)$ of quadrics in $|S^dU|$ transversally, i.e. the

catalecticant hypersurface Cat_{2d} inherits the nice properties of $\mathsf{D}(2d)$ from Theorem 7.2.11.

Proposition 7.5.2 *For any $1 \le k \le d$, the kth secant variety $\mathrm{Sec}_k(R_{2d})$ is projectively normal and Cohen–Macaulay. Its singular locus is $\mathrm{Sec}_{k-1}(R_{2d})$ and its degree is $\binom{2d-k+1}{k}$. Its homogeneous ideal is generated by the size $k+1$ minors of the catalecticant matrix.*

We specialize by letting $d = 3$. Then, Cat_6 is a quartic hypersurface in $\mathbb{P}^6 = |S^6(U^\vee)|$ equal to the intersection of the discriminant hypersurface $\mathsf{D}(3)$ of quadrics in $\mathbb{P}^3 = |S^3U|$ with the 6-dimensional hyperweb of harmonic quadrics in $|S^3U|$ with respect to the Veronese curve in the dual space $|S^3U|$. Its singular points are the intersection points with the 3-dimensional secant variety $\mathrm{Sec}_2(R_6)$ of R_6. Since the degree of the singular locus of the discriminant hypersurface $\mathsf{D}(3)$ of quadrics in $\mathbb{P}^3$ is equal to 10, we see from Theorem 7.2.11 that, for a general web of quadrics W, the surface $\mathcal{D}(W)$ has 10 nodes, as expected.

Recall that a *rational normal curve* C of degree d in $|V| = \mathbb{P}^d$ is the image of the Veronese curve R_d under an isomorphism $|S^dU| \to |V|$ defined by a linear isomorphism $S^dU \to V$. Assume that $d = 3$, the space of quadrics in $\check{\mathbb{P}}^3 = |V^\vee|$ apolar to quadrics vanishing on C is of dimension 6. A choice of a web of quadrics in this space depends on $\dim G_3(\mathbb{P}^6) = 12$ parameters. On the other hand, we know that the variety of rational normal cubics also depends on the same number of parameters. It is a natural guess that a general web W of quadrics in $\check{\mathbb{P}}^3$ is equal to the space of quadrics apolar to C and some other rational normal curve C'. This is the assertion of a classical theorem of T. Reye (see a modern proof in [207, Lemma 4.3]).

Theorem 7.5.3 (T. Reye) *For any general web of quadrics W in $\mathbb{P}^3$ there exists exactly one pair of rational normal curves C_1, C_2 such that W is equal to the space of quadrics apolar to C_1 and C_2.*

Given a reducible quadric $Q = V(q)$ in W, we can write q in the form $l_1^2 + l_2^2$, where l_1, l_2 are some linear forms. The hyperplane $Q^\perp$ of quadrics in the dual space apolar to Q contains the subspace of quadrics vanishing at the points $[l_1]$ and $[l_2]$. Hence, it contains a hyperplane H of quadrics containing the line $\ell = \langle [l_1], [l_2] \rangle$. Let $|I(C_i)_2|$ denote the net of quadrics containing C_i. The hyperplane H intersects the planes $|I(C_i)_2|$ in $Q^\perp$ along a line of quadrics vanishing on ℓ and C_i. Thus, ℓ is a common bisecant line of C_1 and C_2. We also see that the singular line of Q is the dual line of this bisecant.

The linear systems $|I(C_1)_2|$ (resp. $|I(C_2)_2|$) maps C_2 (resp. C_1) to a rational curve $\Sigma_2 \subset |I(C_2)_2|^*$ (resp. $\Sigma_1 \subset |I(C_1)_2|$) of degree 6. The images of 10 common bisecants are the 10 nodes of the sextics. Thus, we see that a quartic symmetroid defines a pair of rational plane sextics whose nodes correspond to the nodes of the symmetroid.

Remark 7.5.4 The remarkable fact that two general rational normal cubics have 10 common bisecants is originally due to Luigi Cremona. Nowadays, it easily follows from the intersection theory on the Grassmannian $G(2,4)$. The variety of bisecants

of R_3 is a congruence of lines in the cohomology class $[\sigma_x] + 3[\sigma_\Pi]$, where σ_x (resp. σ_Π) is the Schubert variety of lines through a point x (resp. lines in a plane Π). The intersection number of these two surfaces in $G(2,4)$ is equal to 10.

7.6 Reye Congruences: $p = 2$

In this section, we specialize the discussion from Section 7.3 to the case $n = 3$. We will show that a general 4-dimensional linear space of symmetric bilinear forms (resp. quadratic forms) in four variables defines a congruences of lines in $\mathbb{P}^3$ of bidegree $(7,3)$ and sectional genus 6 which isomorphic to a $\boldsymbol{\mu}_2$-Enriques surface (resp. classical Enriques surface).

Let B be a web of symmetric divisors of type $(1,1)$ in $|E| \times |E|$ and let $\mathrm{Bs}(B)$ be the scheme of base points of B. It is a complete intersection of 4 divisors of bidegree $(1,1)$. For a general B, the intersection of $\mathrm{Bs}(B)$ with the diagonal is empty, and we can define a fixed-point-free involution τ with the quotient Enriques surface embedded in the Grassmannian quadric $G(2,E)$ in $\mathbb{P}^5$ as the congruence S of lines $\ell = \langle x, y\rangle$, where $(x,y) \in \mathrm{Bs}(B)$. Let $[b] \in B$, where we identify a bilinear form b with a linear map $b : E \to E^\vee$, then the condition that $b(x,y) = 0$ for all $[b] \in B$ implies that the line ℓ is contained in the null-space of b. Thus, the congruence S coincides with the congruence of lines $\mathrm{Nul}(B) \subset G(2,E)$ of null-planes of bilinear forms from B.

We also have the same attributes as in the case $p \neq 2$. In particular, $\mathrm{PB}(B)$ is birationally isomorphic to a quartic symmetroid $\mathcal{D}(B)$ and to the Steinerian quartic surface $\mathrm{St}(B)$. Let us compute the bidegree of the congruence $\mathrm{Nul}(B)$. Take a general plane Π in $\mathbb{P}^3$. The restriction of B to Π is a 5-dimensional linear system of conics. The variety of singular conics is the half-discriminant cubic hypersurface. It is isomorphic to the quotient of $\mathbb{P}^2 \times \mathbb{P}^2$ by the switch involution. The image of our surface $X = \mathrm{Bs}(B)$ is isomorphic to the image of the complete intersection of four divisors of type $(1,1)$ in $\mathbb{P}^3 \times \mathbb{P}^3$. It consists of 3 points. Thus, the class of the congruence is equal to 3. Since the degree of S in the Plücker embedding is equal to 10, and the sum of the order and the class is equal to the degree, the order of S must be equal to 7. Therefore, the bidegree of the congruence is equal to $(7,3)$. So, we get an analog of a Reye congruence in the case $p = 2$. The Enriques surface $\mathrm{Nul}(B)$ is a $\boldsymbol{\mu}_2$-surface with smooth canonical cover $\mathrm{Bs}(B)$. As in the case were $p \neq 2$, the images in $\mathrm{Nul}(B)$ of the exceptional curves are (-2)-curves. So, we have proved the following:

Theorem 7.6.1 *Assume $p = 2$. Let B be a general web of symmetric divisors of bidegree $(1,1)$ in $\mathbb{P}^3 \times \mathbb{P}^3$. Then, the congruence of lines $\mathrm{Nul}(B)$ is of bidegree $(7,3)$ and sectional genus 6. It is isomorphic to a nodal Enriques surface.*

We have also a construction of a classical Enriques surface as a Reye congruence of a web W of quadrics in $\mathbb{P}^3 = |E|$ from Section 7.3. In this case, the linear system B_W of polar symmetric bilinear forms of quadrics from W contains the diagonal

in its base scheme $\mathrm{Bs}(B_W)$. The discriminant surface $\mathcal{D}(W)$ of singular quadrics in W is a quadric surface (taken with multiplicity 2). The set of quadrics of rank 2 in $\mathbb{P}^3$ is a 6-dimensional variety isomorphic to the quotient of $\mathbb{P}^3 \times \mathbb{P}^3$ by the involution $(x, y) \mapsto (y, x)$. It is the quotient of the Segre variety of degree $\binom{6}{3} = 20$ by the involution. Its degree in the space $|S^2(E^\vee)|$ is equal to 10. Its intersection with $W \subset |S^2(E^\vee)|$ is expected to consist of 10 points. We denote this intersection by $\mathcal{D}(W)_2$.

By analogy with the case $p \neq 2$, we say that W is an *excellent* web of quadrics if the following properties hold:

(i) $\mathrm{Bs}(W) = \emptyset$,
(ii) the polarization map $L \to \bigwedge^2 E^\vee$ is injective,
(iii) $\mathcal{D}(W)$ is a nonsingular quadric,
(iv)$\mathcal{D}(W)_2$ consists of 10 points,
(v) the projection $\widetilde{\mathcal{D}}(W) \to \mathrm{St}(W)$ is an isomorphism.

We leave it to the reader to adjust the proof of Proposition 7.4.2, to show that condition (iv) implies that no Reye line belongs to $\mathcal{D}(W)_2$ and no separable Reye line is contained in $\mathrm{St}(W)$.

From now on, we assume that W is excellent. One can show that a general web W is excellent.

We use the notation from Section 7.4. The projection $\widetilde{\mathcal{D}}(W)' \to \mathcal{D}(W)$ is a $\mathbb{P}^1$-bundle over the quadric $\mathcal{D}(W)$. It is equal to the projection of $p_S(Z_S)$, where S is the congruence $\mathrm{Nul}(B_W)$ of nullspaces $|\operatorname{Ker}(b_q)|$, $[q] \in \mathcal{D}(W)$. Since L does not contain quadratic forms with the polar bilinear form of rank 1, the polarization map $\mathfrak{p} : L \to \bigwedge^2 E^\vee$ is an inclusion. This implies that the congruence $\mathrm{Nul}(B_W)$ is equal to the intersection of $G_1(|E|)$ with the 4-dimensional linear subspace $|\mathfrak{p}(L)|$ of $|\bigwedge^2 E^\vee| = |(\bigwedge^2 E)^\vee|$. Since a nonsingular quadric cannot be mapped onto a singular quadric, we obtain that it is a nonsingular quadric and the map $\mathcal{D}(W) \to \mathrm{Nul}(B_W), V(q) \mapsto |\operatorname{Ker}(b_q)|$, is an isomorphism. Since $\deg \mathrm{Nul}(B_W) = 2$, and $\mathrm{Nul}(B_W)$ is not a plane in the Plücker space, we obtain that its bidegree is equal to $(1, 1)$. A congruence of lines of bidegree $(1, 1)$ is a quadric surface in $G_1(\mathbb{P}^3)$, hence it is equal to the base scheme of a pencil of line complexes of degree one. It contains two special complexes of lines intersecting a fixed line. This shows that the congruence is equal to the join congruence of lines $\mathrm{Join}(\ell_1, \ell_2)$ intersecting two skew lines in $\mathbb{P}^3$ from Example 7.1.6. The curve $\ell_1 \cup \ell_2$ is the fundamental curve of the congruence.

Proposition 7.6.2 *The fundamental curve of the congruence* $\mathrm{Nul}(B_W)$ *is the union of the two inseparable Reye lines of* W.

Proof Let ℓ be an inseparable Reye line. For any $x = [v], y = [w] \in \ell$ and any $V(q) \in |L - \ell|$, we have $b_q(v, w) = 0$. Also, because ℓ is inseparable, we can find a quadric $V(q_1) \notin |L - \ell|$ such that $b_{q_1}(v, w') = 0$ for all $y' = [w'] \in \ell$, and a quadric $V(q_2)$ such that $b_{q_2}(v', w) = 0$, for all $x' = [v'] \in \ell$. This shows that $b_q(v, w) = 0$ for all $q \in L$. Consider the matrix $M(x)$ from the proof of Proposition 7.3.2. Its rank is equal to 2, hence there is a pencil of quadrics Q such that x is contained in the

nullspace of Q. This shows that ℓ is an irreducible component of the fundamental curve of the congruence $\mathrm{Nul}(B_W)$. □

Note that singular lines of the ten reducible quadrics from W are rays of $\mathrm{Nul}(B_W)$, hence they intersect the two inseparable Reye lines.

Let us look at the Reye surface of W. It follows from formula (7.3.3) that the discriminant variety $\mathsf{D}(2)$ of singular plane conics is isomorphic to the cubic hypersurface in $\mathbb{P}^5$, with coordinates $(a_{11}, \ldots, a_{33})$ taken as the coefficients of a quadratic form,

$$a_{11}a_{23}^2 + a_{22}a_{22}^2 + a_{33}a_{12}^2 + a_{12}a_{13}a_{23} = 0.$$

Its singular locus is the plane $a_{12} = a_{13} = a_{23}$ of quadrics of rank 1. In fact, scheme-theoretically, it should be considered as the plane taken with multiplicity 4. It is equal to the image of the inseparable Veronese map $\mathbb{P}^2 \to \mathbb{P}^5$, $[l] \mapsto [l^2]$.

Theorem 7.6.3 *Let $p = 2$ and let W be an excellent web of quadrics in $\mathbb{P}^3$. The Reye surface* $\mathrm{Rey}(W)$ *is a smooth congruence of bidegree* $(7, 3)$ *and sectional genus six. It does not have fundamental points, nor does it contain focal rays. The surface* $\mathrm{Rey}(W)$ *is an Enriques surface.*

Proof To compute the bidegree, we use the same argument based on the Montesano complexes associated to general nets of quadrics in W. In order to do this we have to modify the proof of Lemma 7.4.3 where we used Bertini Theorem to exclude the possibility that any quadric from W restricts to a singular conic in a general plane $\Pi \subset |E|$. We used Bertini Theorem to deduce from this that the net of conics has a base point. We know from Example 7.3.11 that it is not true in characteristic 2; such a net must be given by an equation $\lambda x_0^2 + \mu x_1^2 + \gamma x_2^2 = 0$. So all conics are double lines.

Let W_Π be the net of conics obtained by restrictions of quadrics from W to the plane Π. Suppose one of the conics in W_Π is a double line 2ℓ. This happens when there exists a quadric $Q = V(q)$ touching Π along ℓ. We can choose projective coordinates to assume that $\Pi = V(t_0)$ and $q = t_0A(t_0, \ldots, t_3) + B(t_1, t_2, t_3)^2$, where A and B are linear forms. Since $\frac{\partial q}{\partial t_0} = A$, we see that the line $\ell = V(t_0) \cap V(A)$ is the null-line of Q. Since the class of the congruence $\mathrm{Nul}(B_W)$ is equal to 1, there is only one ray ℓ from $\mathrm{Nul}(B_W)$ contained in Π, hence there is only one double line in W_Π, and we get a contradiction. The rest of computation for the bidegree of $\mathrm{Rey}(W)$ goes without change.

Assume $\mathrm{Rey}(W)$ has a fundamental point x. This means that σ_x intersects $\mathrm{Rey}(W)$ along a curve, or, equivalently, the net of quadrics $N(x)$ in W containing x has one-dimensional component in its base scheme. Since W is excellent this is impossible.

Let us now show that $\mathrm{Rey}(W)$ does not have multiple rays. As we have noted in the beginning of the section, a multiple ray of $\mathrm{Rey}(W)$ must be an inseparable Reye line ℓ. We know that ℓ is a fundamental line of the congruence $\mathrm{Nul}(B_W)$. Thus, each of the 10 singular lines of reducible quadrics from W intersect ℓ. Since W is excellent, no two intersect at one point. Thus, ℓ contains ten points which are singular points of quadrics from W. Not all of these quadrics contain ℓ. Choose one such quadric Q that does not contain ℓ and intersects ℓ at a point x. Since ℓ is a multiple ray, the net

$N(x)$ has a multiple base point on ℓ. It must lie in one of the irreducible components of Q. Hence, ℓ lies in this component. This contradiction proves the assertion.

Assume ℓ is a singular point of $S = \mathrm{Rey}(W)$. For any point $x \in \ell$, the α-plane σ_x contains ℓ and intersects S at ℓ with some multiplicity. This implies that, for each $y \in \ell$, the net $N(y)$ of quadrics from W with base point at y has a multiple base point. Since ℓ is contained in the base scheme of a pencil $|L - \ell| \subset N(y)$, the multiple base point must be in ℓ. In particular, we see that ℓ is a multiple ray of the congruence $\mathrm{Rey}(W)$. As we know for each multiple base point there exists a quadric in the net which is singular at this point. This shows that ℓ is a multiple ray $\mathrm{Rey}(W)$, a contradiction.

To compute the sectional genus of $\mathrm{Rey}(W)$, we use the argument from Remark 7.4.5 to find that it is equal to 6. The rest of the assertions are proved in the same way as in the case when $p \neq 2$. □

Recall that the surface $\mathrm{PB}(W)$ is not smooth. Its singular points are the points $\tilde{x}$ such that the Reye line $\ell_{\tilde{x}}$ is a null-line of some quadric. The number of such points is equal to the number of intersection points of the Reye congruence $\mathrm{Rey}(W)$ and the congruence $\mathrm{Nul}(B_W)$ of the null-lines. Since the bidegree of $\mathrm{Nul}(B_W)$ is equal to $(1, 1)$, the number of intersection points is expected to be equal to $[\mathrm{Rey}(W)] \cdot [\mathrm{Nul}(B_W)] = (7[\sigma_x] + 3[\sigma_\Pi]) \cdot ([\sigma_x] + [\sigma_\Pi]) = 10$. The Stein factorization $\mathrm{PB}(B) \to \mathrm{PB}(W)' \to \mathrm{Rey}(W)$ gives two additional singular points on $\mathrm{PB}(W)$ corresponding to inseparable Reye lines. The cover $\mathrm{PB}(W) \to \mathrm{Rey}(W)$ is a principal $\boldsymbol{\mu}_2$-cover of $\mathrm{Rey}(W)$ that coincides with the canonical cover of the Enriques surface $\mathrm{Rey}(W)$. In particular, $\mathrm{Rey}(W)$ is a classical Enriques surface. It is expected that the canonical cover $\mathrm{PB}(W)$ is a normal surface with 12 ordinary double points. However, it could degenerate to a non-normal rational surface. This happens when $\mathrm{Rey}(W) \cap \mathrm{Nul}(B_W)$ contains one-dimensional components.

Remark 7.6.4 In the last chapter we will study in detail canonical covers of Enriques surfaces S in characteristic 2 with the canonical cover birationally isomorphic to a K3 surface. We will show that it has rational double points such that the exceptional curves of its minimal resolution of singularities generate a lattice of rank 12. The case when we have 12 points of type A_1 is a general case. Their images on S are called *canonical points*. They are singular points of simple fibers of elliptic fibrations on S. As we saw, the realization of S as a smooth Reye congruence gives a choice of two of these points corresponding to the inseparable Reye rays.

7.7 The Picard Lattice of a Reye Congruence: $p \neq 2$

In this section, we discuss special divisor classes in the Picard lattice of a nodal Enriques surface isomorphic to a Reye congruence $\mathrm{Rey}(W)$ of a web of quadrics in $|E| \cong \mathbb{P}^3$. In particular, we describe some natural non-degenerate canonical sequences of isotropic vectors in it.

Let $W = |L|$ be a regular web of quadrics in $\mathbb{P}^3 = |E|$. We know that $X = \widetilde{\mathcal{D}}(W) \cong \mathrm{PB}(W)$ are minimal resolutions of the discriminant surface $\mathcal{D}(W)$. We will identify both surfaces with a K3 surface X isomorphic to the canonical cover of the Enriques surface $\mathrm{Rey}(W)$. Let $\mathrm{Sing}(\mathcal{D}(W)) = \{q_1, \dots, q_{10}\}$, and let Θ_i be the exceptional curve over q_i. We have the following divisor classes in $\mathrm{Pic}(X)$:

$$\eta_H = \mathrm{pr}_W^*(c_1(\mathcal{O}_W(1))),\ \eta_S = \mathrm{pr}_{|E|}^*(c_1(\mathcal{O}_{|E|}(1))),\ \theta_i = [\Theta_i],\ i = 1, \dots, 10. \tag{7.7.1}$$

Let

$$\mathfrak{r} : X = \mathrm{PB}(B) \to \mathrm{Rey}(W) \subset |\bigwedge^2 E|$$

be the Reye map. We set

$$\eta = \mathfrak{r}^*(c_1(\mathcal{O}_{\mathrm{Rey}(W)}(1))).$$

We have

$$\eta_S^2 = \eta_H^2 = 4,\ \eta^2 = 20,\ \eta_S \cdot \theta_i = 1, \eta_H \cdot \theta_i = 0. \tag{7.7.2}$$

By Theorem 7.4.10,

$$2\eta_S = 3\eta_H - \theta_1 - \cdots - \theta_{10}. \tag{7.7.3}$$

We know from the proof of Proposition 7.4.1 that the singular points of $\mathrm{St}(W)$ are the images of lines in $\mathcal{D}(W)$ passing through 3 nodes.

The image Θ_i' of Θ_i under $\mathrm{pr}_{|E|}$ is a line on $\mathrm{St}(W)$. It is one of the 10 singular lines of reducible quadrics $Q_1, \dots, Q_{10}$ from W. If $\mathrm{St}(W)$ is smooth, the pre-image in X of a plane in $|E|$ containing the line Θ_i' is equal to Θ_i. Otherwise, it consists of the union of Θ_i and proper transform of the lines ℓ_i joining q_i with other two nodes. We denote the union of such curves by Z_i. The union $\Theta_i + Z_i$ forms a nodal cycle of (-2)-curves on X. It is of type A_1 if $Z_i = \emptyset$, of type A_2 (resp. A_3, resp. D_4) if Z_i consists of 1 (resp. 2, resp. 3) curves. In the latter case, $\mathcal{D}(W)$ has three lines joining q_i with 3 disjoints pairs of other nodes. Let

$$E_i = \eta_S - \theta_i - [Z_i].$$

The linear system $|E_i|$ is equal to the pre-image under $\mathrm{pr}_{|E|}$ of the linear system of plane sections of $\mathrm{St}(W)$ that contain the line Θ_i'. We have

$$E_i^2 = \eta_S^2 - 2\theta_i \cdot \eta_S + (\theta_i + [Z_i])^2 = 4 - 2 - 2 = 0.$$

This confirms that the linear system $|E_i|$ is a genus one pencil on X.

Let $\tau : X \to X$ be the canonical involution of $X = \mathrm{PB}(W)$. The corresponding birational involution

$$\tau' : \mathrm{St}(W) \dashrightarrow \mathrm{St}(W), \quad x \mapsto \cap_{Q \in W} P_x(Q),$$

permutes the tangency points of a Reye line. In coordinates, $\tau'(x)$ is the null-space of a system of 3 equations with 4 unknowns with coefficients linear forms in coordinates of x. It is clear that the coordinates of $\tau'(x)$ are cubic polynomials in the coordinates of x.

Proposition 7.7.1 *Let* $Z = \sum Z_i$.

(i) $\eta = \tau^*(\eta_S) + \eta_S$;
(ii) $\theta_i \cdot \tau^*(\theta_i) = 0,\ \theta_i \cdot \tau^*(\theta_j) = 2,\ i \neq j,\ i = 1, \dots, 10$;
(iii) $\eta_S \cdot \tau^*(\theta_i + [Z_i]) = 3,\ i = 1, \dots, 10$;
(iv) $\tau(E_i) \sim E_i$;
(v) $3\eta = E_1 + \cdots + E_{10}$;
(vi) $\eta_H = 4\eta_S - \eta - \frac{1}{3}[Z]$;
(vii) $2\eta_S - \eta_H = \tau^*(\theta_i + [Z_i]) - \theta_i - [Z_i] + \frac{1}{3}[Z],\ i = 1, \dots, 10$;
(viii) $\tau^*(\eta_H) = 2\eta - \eta_H - \frac{1}{3}([Z] + \tau^*([Z]))$.

Proof (i) The involution τ is induced by the involution of $|E| \times |E|$ which permutes the factors. Let $p_i : \mathrm{PB}(W) \to \mathbb{P}^3$ be the two projections. We can identify p_2 with the projection $\mathrm{pr}_{|E|} : \mathrm{PB}(W) \to \mathrm{St}(W)$, so that $\mathcal{O}_X(\eta_S) \cong p_2^*\mathcal{O}_{\mathbb{P}^3}(1)$. Since

$$\mathcal{O}_X(\eta) \cong p_1^*\mathcal{O}_{\mathbb{P}^3}(1) \otimes p_2^*\mathcal{O}_{\mathbb{P}^3}(1),$$

we obtain (i).

(ii) We know that no Reye line is one of the lines Θ_i' on $\mathrm{St}(W)$. This implies that, for any $x \in \Theta_i'$, the point $y = \tau(x)$ does not belong to Θ_i' (otherwise $(x, y) \in \mathrm{PB}(W)$ and Θ_i' is a Reye line). Thus, $\Theta_i' \cap \tau(\Theta_i') = \emptyset$, hence $\Theta_i \cdot \tau(\Theta_i) = 0$. On the other hand, if $i \neq j$, the intersection number $\Theta_i \cdot \tau(\Theta_j)$ is equal to the number of pairs $(x, y) = (x, \tau(x)) \in \mathrm{PB}(W)$ such that $x \in \Theta_i, y \in \Theta_j$. This is the same as the intersection number of the surfaces $\Theta_i \times \Theta_j$ and $\mathrm{PB}(W)$ in $|E| \times |E|$. It is clear that $\Theta_i \times \Theta_j$ is contained in $Z(b_i) \cap Z(b_j)$, where b_i, b_j are the polar bilinear forms of the quadrics $Q_i, Q_j \in \mathcal{D}(W)$. Thus, the intersection $\Theta_i \times \Theta_j \cap \mathrm{PB}(W)$ is equal to the intersection number of two divisors of type $(1, 1)$ on $\mathbb{P}^1 \times \mathbb{P}^1$. It is equal to 2.

(iii) The image of $x \in \Theta_i$ under the involution τ is equal to the point y such that $(x, y) \in \mathrm{PB}(W)$. If $x' = p_2(x)$, then $y' = \tau'(x')$. Since τ' is given by cubic polynomials, $\tau(\Theta_i)$ is a Veronese cubic curve. If $Z_i \neq \emptyset$, then the curve $\mathrm{pr}_W(\tau(Z_i))$ is a line, and it is blown down to a singular point of $\mathrm{St}(W)$. This gives us

$$\tau(\theta_i + [Z_i]) \cdot \eta_S = 3.$$

(iv) First, observe that

$$\eta_S \cdot \tau^*(\eta_S) = \frac{1}{2}\big((\eta_S + \tau^*(\eta_S))^2 - 2\eta_S^2\big) = \frac{1}{2}(\eta^2 - 2\eta_S^2) = 6.$$

This yields

$$E_i \cdot \tau^*(E_i) = (\eta_S - \theta_i - [Z_i]) \cdot (\tau^*(\eta_S) - \tau^*(\theta_i) - \tau^*([Z_i])) = 0.$$

Therefore, E_i and $\tau(E_i)$ define the same genus one pencil on X.

(v) We have

$$E_i \cdot E_j = (\eta_S - \theta_i - [Z_i]) \cdot (\eta_S - \theta_j - [Z_j]) = 2,$$

$$E_i \cdot \eta = 2E_i \cdot \eta_S = 2(\eta_S^2 - \theta_i \cdot \eta_S - [Z_i] \cdot \eta_S) = 6.$$

This shows that $(E_1, \dots, E_{10})$ is the inverse transform under the Reye map $\mathfrak{r}$ of a maximal isotropic sequence $(f_1, \dots, f_{10})$ on $\mathrm{Rey}(W)$ with

$$3h = f_1 + \cdots + f_{10},$$

where $h = [c_1(\mathcal{O}_{\mathrm{Rey}(W)}(1))] \in \mathrm{Num}(\mathrm{Rey}(W))$, and

$$3\eta = 3\pi^*(h) = E_1 + \cdots + E_{10}.$$

(vi) By Theorem 7.4.10 and (v),

$$2\eta_S + \theta_1 + \cdots + \theta_{10} = 2\eta_S + \Sigma(\eta_S - E_i - [Z_i]) = 12\eta_S - 3\eta - [Z] = 3\eta_H.$$

This implies that $[Z]$ is divisible by 3, and

$$\eta_H = 4\eta_S - \eta - \frac{1}{3}[Z].$$

The fact that $[Z]$ is divisible by 3 is not surprising. We already know that each exceptional curve of the map $\mathrm{pr}_{|E|} : \widetilde{\mathcal{D}}(W) \to \mathrm{St}(W)$ comes from a line on $\mathcal{D}(W)$ that passes through three singular points. This shows that each irreducible component of Z enters into exactly three divisors $\Theta_i + Z_i$, and hence enters in Z with coefficient 3.

(vii) By (vi)

$$\eta_H = 4\eta_S - \eta - \frac{1}{3}[Z] = 3\eta_S - \tau^*(\eta_S) - \frac{1}{3}[Z],\ 4\eta_S - \eta_H = \eta + \frac{1}{3}[Z].$$

Thus,

$$2\eta_S - \eta_H = \eta - 2\eta_S + \frac{1}{3}[Z] = \tau^*(\eta_S) - \eta_S + \frac{1}{3}[Z].$$

On the other hand,

$$\tau^*(\eta_S) = E_i + \tau^*(\theta_i) + \tau^*([Z_i]),$$

$$\eta_S = E_i + \theta_i + [Z_i],$$

gives, after subtracting,

$$2\eta_S - \eta_H = -\eta_S + \tau^*(\eta_S) + \frac{1}{3}[Z] = \tau^*(\theta_i + [Z_i]) - \theta_i - [Z_i] + \frac{1}{3}[Z].$$

This proves (vii).

(viii) Applying τ^* to equality (vii):

$$2\eta_S - \eta_H = \tau^*(\theta_i + [Z_i]) - \theta_i - [Z_i] + \frac{1}{3}[Z],$$

we obtain

$$2\tau^*(\eta_S) - \tau^*(\eta_H) = \theta_i + [Z_i] - \tau^*(\theta_i + [Z_i]) + \frac{1}{3}\tau^*[Z].$$

Adding up, we obtain (viii). □

Define the following divisors on the Reye surface $\mathrm{Rey}(W)$:

$$R_i = \pi(\Theta_i + Z_i),\ i = 1, \dots, 10, \tag{7.7.4}$$
$$F_i = \pi(E_i). \tag{7.7.5}$$

Applying Proposition 7.7.1, we obtain:

Theorem 7.7.2 *Let W be a regular web of quadrics in $\mathbb{P}^3$. Then, the Enriques surface $S = \mathrm{Rey}(W)$ contains ten curves R_i of degree 4 and of arithmetic genus 0, ten genus one pencils $|2F_i|$, and 45 genus one pencils $|2F_{ij}|$, where $2F_{ij} \sim R_i + R_j, i \neq j$. Let $H = c_1(\mathcal{O}_{\mathrm{Rey}(W)}(1)))$. We have:*

(i) $F_i \cdot F_j = 1$ for $i \neq j$,
(ii) $R_i \cdot F_i = 3,\ R_i \cdot F_j = 1, i \neq j$,
(iii) $R_i \cdot R_j = 2,\ i \neq j$,
(iv) $R_i \cdot F_{ij} = 0,\ R_k \cdot F_{ij} = 2,\ k \neq i, j$,
(v) $3H \sim F_1 + \cdots + F_{10}$,
(vi) $4H \sim R_1 + \cdots + R_{10}$,
(vii) $H \cdot F_i = 3,\ H \cdot F_{ij} = 4$,
(viii) $H \sim 2F_i + R_i + K_S$,
(ix) $H \equiv F_i + F_j + F_{ij}$.

Proof Properties (i) – (vii) immediately follow from the definitions and Proposition 7.7.1. Let us prove (viii). By (i) from loc. cit., we have $[p^*(H)] = \tau^*(\eta_S) + \eta_S$ on $\mathrm{PB}(W)$. Applying (ii) and (iv) from loc. cit., we obtain, by definition of the curves F_i and R_i, that $H \equiv 2F_i + R_i$. Let $F_i' \in |F_i + K_S|$. Since each curve F_i (resp. F_i') is a plane cubic, it is contained in a plane Π_i (resp. Π_i'). Obviously, this plane must lie in the Grassmannian quadric $\mathbb{G}$ containing S. Now, since $F_i \cdot F_j = F_i' \cdot F_j = 1, i \neq j$, we find that $\Pi_i \cap \Pi_j \neq \emptyset, \Pi_i' \cap \Pi_j \neq \emptyset$. This easily implies that Π_i and Π_i' belong to the same family of planes in $\mathbb{G}$ and hence intersect at one point. Then, the unique hyperplane containing Π_i and Π_i' cuts out in S an isolated curve from the linear system $|H - F_i - F_i'|$. Since $(H - F_i - F_i')^2 = -2$, this must be a nodal cycle R. If $R_i \in |H - 2F_j| \neq \emptyset$, we obtain that $2R_i \sim 2R, R_i \neq R$, that is absurd. Therefore, we have only one possibility that $R_i \sim H - F_i - F_j$. In particular, $R_i + R_j \sim 2(H - F_i - F_j)$ is divisible by 2 in $\mathrm{Pic}(S)$. So, we can define F_{ij}. It follows from (iii) that $F_{ij}^2 = 0$, thus, $|2F_{ij}| = |R_i + R_j|$ is a genus one pencil.

(ix) By (iv) we have $R_i \cdot (H - F_i - F_j) = R_j \cdot (H - F_i - F_j) = 0$. Hence, $F_{ij} \cdot (H - F_i - F_{kj}) = 0$. Since $(H - F_i - F_j)^2 = 0$, we obtain (ix). □

Recall from Theorem 7.4.7 that under the map $\nu : \mathrm{Rey}(W) \to \mathrm{Bit}(\mathcal{D}(W))$ the set of Reye lines ℓ such that the pencil $|L - \ell|$ contains one of the 10 nodes of $\mathcal{D}(W)$ is mapped to the union of two cubic curves, the union of the cones over these curves form the enveloping cone of the node. One can easily check that the set of such lines is equal to the union of two genus one curves F_i, F_i' defining a genus one pencil $|2F_i|$. On the canonical cover $\widetilde{\mathcal{D}}(W)$ the pre-images of these two curves are the curves E_i, E_i'. Under the double cover $\mathcal{D}(W) \dashrightarrow \mathbb{P}^2$ defined by the projection from the node, the curve $F_i + F_i'$ is equal to the pre-image of the branch curve.

We say that a smooth Reye congruence $\mathrm{Rey}(W)$ is *general* if $X \cong \widetilde{\mathcal{D}}(W)$ and $\mathrm{rank}\,\mathrm{Pic}(X) = 11$.

Proposition 7.7.3 *Assume* $\mathrm{Rey}(W)$ *is general. Then,* $\eta_H, \eta_S, \theta_i, 1 \leq i \leq 9$ *is an integral basis of* $\mathrm{Pic}(X)$. *There is an isomorphism of lattices*

$$\mathrm{Pic}(X) \cong \mathsf{U} \oplus \mathsf{E}_8(2) \oplus \mathsf{A}_1(2).$$

Proof Since $(\theta_1, \ldots, \theta_{10}, \eta_H)$ is a basis of $\mathrm{Pic}(X)_{\mathbb{Q}}$, for any $D \in \mathrm{Pic}(X)$, we can write

$$D = \frac{D \cdot \eta_H}{4}\eta_H - \sum_{i=1}^{10} \frac{D \cdot \theta_i}{2}\theta_i.$$

Since $\theta_{10} = 3\eta_H - 2\eta_S - \sum_{i=1}^{9} \theta_i$, we obtain

$$D = \left(\frac{D \cdot \eta_H}{4} - \frac{3D \cdot \theta_{10}}{2}\right)\eta_H + (D \cdot \theta_{10})\eta_S - \sum_{i=1}^{9} \frac{D \cdot (\theta_i - \theta_{10})}{2}\theta_i. \qquad (7.7.6)$$

We have to show that all coefficients here are integers. The images of the (-2)-curves Θ_i under the map given by the linear system $|\eta_S|$ are disjoint lines ℓ_i on a smooth quartic surface $S(W)$. For any point x outside $\ell_i \cup \ell_j$, the unique line $\ell(x)$ in $\mathbb{P}^3$ that passes through x and intersects ℓ_i and ℓ_j intersects $S(W)$ at another point x'. The rational map $x \mapsto x'$ extends to a biregular involution Φ of $S(W)$. This involution is the restriction of birational transformation $\tilde{\Phi}$ (an *Arguesian involution*) of $\mathbb{P}^3$ that assigns to a general point $x \in \mathbb{P}^3$ the point x' on the line $\ell(x)$ such that the pairs of points $\{x, x'\}$ and $\ell(x) \cap S(W)$ are harmonically conjugate on $\ell(x)$ (see [313, p. 116]). It is known that $\tilde{\Phi}$ is given by the linear system of quadric surfaces passing through the lines ℓ_i and ℓ_j. Restricting it to $S(W)$, we see that the involution Φ is given by the linear system $|2\eta_S - \theta_i - \theta_j|$ of dimension 3 and degree 4. It acts on $\mathrm{Pic}(S(W))$ by the formula

$$\Phi^*(D) = -D + \frac{(2\eta_S - \theta_i - \theta_j) \cdot D}{2}(2\eta_S - \theta_i - \theta_j).$$

In particular, we see that $D \cdot \theta_i \equiv D \cdot \theta_j \mod 2$, so the coefficients in (7.7.6) at θ_i are integers. Also, we can write $D \cdot \theta_i = k + 2n_i$, hence, applying Theorem 7.4.10, we get $3D \cdot \eta_H = 2D \cdot \eta_S + 10a + 2\sum n_i$. This shows that $\eta_H \cdot D$ is even. Thus, there exist some integers $a, b, c_1, \ldots, c_9$ such that $D = \frac{a}{2}\eta_H + b\eta_S + \sum_{i=1}^{9} c_i\theta_i$.. This gives

$$D^2 = a^2 + 2(3ab + 2b^2 + b\sum_{i=1}^{9} c_i - \sum_{i=1}^{9} c_i^2).$$

Since D^2 is even, we get that a is also even, hence all the coefficients in (7.7.6) are integers.

The last assertion follows from computing the integral Smith form of the Gram matrix of the basis $(\eta_S, \eta_H, \theta_1, \dots, \theta_9)$. It shows that the discriminant group of $\mathrm{Pic}(X)$ is isomorphic to $(\mathbb{Z}/2\mathbb{Z})^8 \oplus \mathbb{Z}/4\mathbb{Z}$. Since $\mathrm{Pic}(X)$ contains a sublattice $\pi^*(\mathrm{Pic}(\mathrm{Rey}(S))) \oplus \mathbb{Z}(\theta_i - \tau^*(\theta_i))$ of index 2 isomorphic to $\mathsf{U}(2) \oplus \mathsf{E}_8(2) \oplus \langle -4\rangle$, the assertion easily follows.

We can also argue as follows. In the notation of Proposition 7.7.1, the sublattice M_1 generated by $\eta - E_1 - E_2 - E_3, E_1, \dots, E_8$ is isomorphic to $\mathsf{E}_8(2)$. The vector $v = 2\eta_S - \eta_H$ of square norm -4 belongs to $M_1^\perp$. Finally, the sublattice M_2 spanned by $E_{10} = \eta_S - \theta_{10}$ and $E_{10} - \theta_{10}$ is isomorphic to U and it is orthogonal to $M_1 + \langle v\rangle$. Computing the determinant of the Gram matrix of the basis $(\eta_S, \eta_H, \theta_1, \dots, \theta_9)$, we find that it is equal to 2^{10}. Thus, the orthogonal sum $M_1 \oplus M_2 \oplus \langle v\rangle \cong \mathsf{E}_8(2) \oplus \mathsf{U} \oplus \langle -4\rangle$ coincides with the whole lattice. □

7.8 Smooth Congruences of Lines of Bidegree (7, 3)

By Theorem 7.4.4, the Reye surface $\mathrm{Rey}(W)$ of a regular web of quadrics W is an Enriques surface. In this section, we will prove that, if $p \neq 2$, every smooth congruence S of bidegree $(7, 3)$ and sectional genus 6 is equal to the Reye congruence of a regular web of quadrics in $\mathbb{P}^3$. By Corollary 7.1.4, S is an Enriques surface.

We will start with the following.

Lemma 7.8.1 *Let S be a smooth congruence of lines in $\mathbb{G} = G_1(\mathbb{P}^3)$ of bidegree $(7, 3)$ and sectional genus six. Let $H = c_1(\mathcal{O}_S(1))$. Then, S contains 20 plane cubic curves $F_i, i = \pm 1, \dots, \pm 10$, such that:*

(i) $F_i \cdot F_j = 1,\ i + j \neq 0$,
(ii) $F_i \cdot F_{-i} = \emptyset,\ i = 1, \dots, 10$,
(iii) $3H \sim F_1 + \cdots + F_{10}$,
(iv) $F \cdot H \geq 3$ for every genus one curve, and equality holds if and only if $F = F_i$ for some i,
(v) $|H - F_i - F_j|,\ i + j \neq 0$, consists of an isolated divisor which is either a genus one curve F_{ij} of degree 4, or the sum of some F_k $(k \neq i, j)$, and a line,
(vi) for each curve F_i, there exists a unique β-plane Λ_i such that $\Lambda_i \cap S = F_i$,
(vii) $H - F_i - F_{-i} \sim R_i$, where R_i is a nodal cycle such that $R_i \cdot H = 4, R_i \cdot F_i = 3$,
(viii) $F_i \cap F_j \cap F_k = \emptyset$ if $|i| \neq |j| \neq |k|$.

Proof We have $H^2 = 10$. Since H is an ample divisor class, by Proposition 2.4.11 from Volume I, $\Phi([H]) \geq \sqrt{10} \geq 3$, hence, $H = H_{10}$ is a Fano polarization, and there exists a non-degenerate isotropic sequence $(f_1, \dots, f_{10})$ such that $3h_{10} = f_1 + \cdots + f_{10}$.

Each isotropic vector $f_i \in \mathrm{Num}(S)$ is nef and is equal to the numerical class of half-fibers F_i and F_{-i} of a genus one fibration $|2F_i| = |2F_{-i}|$. Since $H_{10} \cdot F_i = 3$, we obtain that all 20 curves F_i are plane cubics in S. They satisfy properties (i)–(iv).

(v) We have $(H_{10} - F_i - F_j)^2 = 0$ and $H_{10} \cdot (H_{10} - F_i - F_j) = 4$. Thus, $|H_{10} - F_i - F_j| = \{D\}$ for some effective divisor D. Its nef part is a genus one curve E so that $D = F$ or $D = F + R$, where $R \cdot H_{10} = 1$, i.e. R is a line on S. In the first case, $H_{10} \cdot F = 4$; we denote this curve by F_{ij}. In the second case, $F \cdot H_{10} = 3$, so $F = F_k$ for some $k \neq i, j$.

(vi) Since each curve F_i is of degree 3, it must lie in a unique plane $\Lambda_i \subset \mathbb{G}$. The linear system $|H_{10} - F_i|$ is cut out by hyperplanes containing Λ_i. Therefore, our assertion follows from the fact that $|H_{10} - F_i|$ has no fixed components and isolated base points. Obviously, each fixed components or a base point must lie in the plane Λ_i. Assume C is a fixed component. Then, $C \cdot F \geq 3$, hence $(H_{10} - F_i - C) \cdot F_i = 3 - C \cdot F_i$ shows that $C \cdot F_i = 3$, i.e. C is a line. So $C^2 = -2$, $H_{10} \cdot C = 1$ and $(H_{10} - F_i - C)^2 = 6$. By Riemann–Roch, $\dim |H_{10} - C - F_i| \geq 3$ which is absurd. To show that $|H_{10} - F_i|$ has no isolated base points, we apply Corollary 2.6.8 from Volume I. We have to verify that for every nef divisor D with $D^2 = 0$, one has $(H_{10} - F_i) \cdot D \geq 2$.

Take F_j such that $F_j \cdot D > 0$. By (v), $H_{10} - F_i - F_j \sim F_{ij}$ or $H_{10} - F_i - F_j \sim F_k + R$. Assume $(H_{10} - F_i) \cdot D = 1$. Take $j \neq \pm i, \pm k$, then,

$$(H_{10} - F_i) \cdot D = (H_{10} - F_i - F_j) \cdot D + F_j \cdot D \geq 1$$

with the equality only if $(H_{10} - F_i - F_j) \cdot D = 0$ and $F_j \cdot D = 1$. This implies that $D = F_{ij}$, hence $(H_{10} - F_i) \cdot D = 4 - 2 = 2$.

(vii) If $i + j = 0$, then the self-intersection number of $H_{10} - F_i - F_{-i} \equiv H_{10} - 2F_i$ is equal to -2. Let us see that $h^0(H_{10} - F_i - F_{-i}) \neq 0$. Since each curve F_i is of degree 3, it must lie in a unique plane Λ_i. Recall that the quadric $\mathbb{G}$ has two families of planes, α-planes and β-planes. Two different planes from the same family intersect at one point, and two planes from different families are either disjoint or intersect along a line. If Λ_i and Λ_j intersect along a line, the linear system $|H_{10} - F_i - F_j|$ contains a pencil of curves cut out by hyperplanes through this line. If $i + j \neq 0$ this contradicts (v). If $i = \pm j$, this contradicts the equality $(H_{10} - F_i - F_j)^2 = -2$. Since $\Lambda_i \cap \Lambda_j = F_i \cap F_j \neq \emptyset$ for $i \neq \pm j$, we obtain that all 20 planes Λ_i belong to the same family. We claim that this is the family of β-planes. In fact, suppose one of these planes is an α-plane σ_x. Take a general plane Π in $\mathbb{P}^3$. Its pre-image under the map $Z_S \to S$ is a cover of degree 7. On the other hand, by property (v), the plane Λ_i intersects the cone of rays in σ_x at 3 points. This contradiction shows that the congruence S has 20 fundamental planes Π (as it is expected, since it is a Reye congruence).

In particular, we have $\Lambda_i \cap \Lambda_{-i} \neq \emptyset$. Let H be a hyperplane section of S cut out by the hyperplane $\langle \Lambda_i, \Lambda_{-i} \rangle$ in the Plücker space spanned by the planes Λ_i and Λ_{-i}. Then, $|H - F_i - F_{-i}| \neq \emptyset$ and consists of a divisor R with $R^2 = -2$ and $H_{10} \cdot R = 4$. Suppose R has a part R_1 with $R_1^2 \geq 0$. If $R_1^2 > 0$, then the Hodge Index Theorem gives $10R_1^2 < (R_1 \cdot H_{10})^2 \leq 9$, a contradiction. Thus, $R_1^2 = 0$. By moving some components of R_1 to $R - R_1$, we may assume that R_1 is nef. Then,

$10 = H_{10}^2 = 2H_{10} \cdot F_i + H_{10} \cdot R_1 + H_{10} \cdot (R - R_1)$ implies that $H_{10} \cdot R_1 = 3$ and hence R_1 coincides with some F_j and $H_{10} \cdot (R - R_1) = 1$. Thus, $R - R_1$ consists of one irreducible component with self-intersection -2 taken with multiplicity 1. Suppose $j \neq \pm i$. Then, $-2 = (R - R_1)^2 = (H_{10} - 2F_i - F_j)^2 = -4$, a contradiction. If $j = \pm i$, then $(R - R_1)^2 = (H_{10} - 3F_i)^2 = -8$, a contradiction again.

So, R consists of (-2)-curves of total degree 4 and the sublattice of $\mathrm{Num}(S)$ generated by its components is negative definite. Since $R^2 = -2$, the class of R in $\mathrm{Num}(S)$ is a root in some root system of finite type of rank ≤ 4. It is easy to list all possibilities and get that R is a nodal cycle. In general, it is an irreducible rational normal quartic in $\langle \Pi, \Lambda_{-i} \rangle \cong \mathbb{P}^4$.

(viii) Assume $F_i \cap F_j \cap F_k \neq \emptyset$ for three distinct indices with no two add up to 0. Then, the curves F_j and F_k cut out the same point on F_i, and we have an exact sequence:

$$0 \to \mathcal{O}_S(F_j - F_k - F_i) \to \mathcal{O}_S(F_j - F_k) \to \mathcal{O}_{F_i} \to 0.$$

Since $(F_j - F_k - F_i)^2 = -2$, and neither $F_j - F_k - F_i$, nor $F_i - F_k$ is effective, we have $h^1(F_j - F_k - F_i) = h^0(F_j - F_k) = 0$. Considering the exact cohomology sequence, this immediately leads to a contradiction. This proves the last assertion of the lemma. □

Let $\iota : S \hookrightarrow \mathbb{G}$ be the closed embedding, and let $\mathcal{V} = \iota^*(\mathcal{S}_{\mathbb{G}}^\vee)$ be the restriction of the dual of the universal subsheaf of the trivial locally free sheaf $\underline{E}$ on $\mathbb{G}$. We know from Section 7.1 that

$$\begin{aligned} c_1(\mathcal{V}) &= \iota^*(c_1(\mathcal{S}_{\mathbb{G}}^\vee)) = \iota^*(c_1(\mathcal{Q}_{\mathbb{G}})) = h, \\ c_2(\mathcal{V}) &= \iota^*(c_2(\mathcal{S}_{\mathbb{G}}^\vee)) = \iota^*([\sigma_\Pi]). \end{aligned}$$

Since $\mathcal{V}$ defines the Plücker embedding $S \hookrightarrow \mathbb{G}$, it is generated by global sections. The restriction of the tautological exact sequence (7.1.6) to S

$$0 \to \iota^*(\mathcal{Q}_{\mathbb{G}})^\vee \to E^\vee \otimes \mathcal{O}_S \to \mathcal{V} \to 0$$

gives $h^0(\mathcal{V}) = 4, h^i(\mathcal{V}) = 0, i > 0$.

Lemma 7.8.2 *The locally free sheaf $\mathcal{V}$ is indecomposable, i.e. it does not split into the direct sum of two invertible sheaves.*

Proof Assume that $\mathcal{V} \cong \mathcal{O}_S(D_1) \oplus \mathcal{O}_S(D_2)$. Suppose $H^0(S, \mathcal{O}_S(D_1)) = 0$. Then, every section of $\mathcal{V}$ vanishes on a curve from the linear system $|D_2|$. On the other hand, we know that the scheme of zeros of a non-zero section of $\mathcal{V}$ is equal to the intersection of S with some plane σ_Π in $\mathbb{G}$. Therefore, a general section has only three isolated zeros. Since $H^1(S, \mathcal{V}) = 0$, we obtain that $H^i(S, \mathcal{O}_S(D_1)) = H^i(S, \mathcal{O}_S(D_1)) = 0$, $i > 0$. By Riemann–Roch, $D_1^2 = 2(h^0(D_1) - 1)$. We may assume that $h^0(D_1) \leq 2$. If the equality holds, $D_i^2 = 2$. Since $c_2(\mathcal{V}) = D_1 \cdot D_2 = 3$, we get a contradiction to the Hodge Index Theorem. Thus, we may assume that

$$h^0(D_1) = 1,\ h^0(D_2) = 3,\ D_1^2 = 0, D_2^2 = 4.$$

Since $h^1(D_1) = 0$, we have $D_1 = F + R$, where F is a half-fiber of some genus one fibration and R is a nodal cycle. By the Hodge Index Theorem, $40 = H^2 D_2^2 < (H \cdot D_2)^2$. This implies $H \cdot D_2 \geq 7$. Since $H \cdot D_1 \geq 3$, we have $H \cdot D_2 = H^2 - H \cdot D_1 \leq 7$, we get $H \cdot D_2 = 7, H \cdot D_1 = 3$, and $D_1 = F$. We know that $H \cdot F = 3$ implies that $F = F_i$ for some $i = \pm 1, \ldots, \pm 10$. Thus.

$$\mathcal{V} = \mathcal{O}_S(F_i) \oplus \mathcal{O}_S(H - F_i).$$

A non-zero section of $\mathcal{O}_S(F_i)$ defines a section of $\mathcal{V}$ that vanishes on the curve $F_i \subset \Lambda_i$. The direct sum decomposition implies that every section of $\mathcal{V}$ vanishes on F_i. This is obviously absurd. □

It follows from Lemma 7.8.1 (v) that $h^0(H - F_i - F_{-i}) = h^0(H - 2F_i + K_S) \neq 0$. Since $(H - 2F_i)^2 = -2$, this implies that

$$\dim H^1(S, \mathcal{O}_S(H - 2F_i + K_S)) = \dim H^1(S, \mathcal{O}_S(-H + 2F_i)) = 1.$$

A non-zero element in $H^1(S, \mathcal{O}_S(H - F_i)) \cong \mathrm{Ext}^1(\mathcal{O}_S(H - F_i), \mathcal{O}_S(F_i))$ defines a non-split extension

$$0 \to \mathcal{O}_S(F_i) \to \mathcal{V} \to \mathcal{O}_S(H - F_i) \to 0.$$

The proof of the next result can be found in [194, Theorem 2].

Theorem 7.8.3 *In the notation from the previous lemma, for any $i = \pm 1, \ldots, \pm 10$, there is an exact non-split sequence*

$$0 \to \mathcal{O}_S(F_i) \to \mathcal{V} \to \mathcal{O}_S(H - F_i) \to 0.$$

Two such extensions are isomorphic.

Corollary 7.8.4

$$\mathcal{V} \cong \mathcal{V} \otimes \omega_S.$$

Theorem 7.8.5 *Assume $p \neq 2$. A smooth congruence of lines in $\mathbb{P}^3$ of bidegree $(7, 3)$ and sectional genus 6 is isomorphic to the Reye congruence of a regular web of quadrics in $\mathbb{P}^3$.*

Proof The tautological exact sequence (7.1.6) gives an isomorphism

$$E^\vee \cong H^0(S, \mathcal{V}).$$

Let $\pi : X \to S$ be the canonical cover of S and let τ be the corresponding fixed-point-free involution of X. Let $\tilde{V} = \pi^*(\mathcal{V})$. Applying the corollary, we find an isomorphism

$$\sigma : \tilde{V} \to \tilde{V}.$$

Considered as an automorphism of the associated projective bundles, it is an involution. This easily implies that $\mathbb{P}(\tilde{\mathcal{V}})$ has two disjoint sections corresponding to the locus of fixed points. This shows that $\tilde{\mathcal{V}}$ splits into the direct sum of invertible sheaves

$$\tilde{\mathcal{V}} \cong \mathcal{L}_+ \oplus \mathcal{L}_-,$$

where

$$\mathcal{L}_\pm \cong \tau^*(\mathcal{L}_\mp). \tag{7.8.1}$$

We have $c_1(\tilde{\mathcal{V}}) = \pi^*(c_1(\mathcal{V})) = \pi^*(h)$ and $c_2(\tilde{\mathcal{V}}) = \pi^*(c_2(\mathcal{V})) = 6$. Let η_S denote $c_1(\mathcal{L}_+)$. We have

$$\eta := \eta_S + \tau^*(\eta_S) = \pi^*(h),\ \eta_S \cdot \tau^*(\eta_S) = 6.$$

This immediately implies that $\eta_S^2 = 4$. Let

$$\tau_\pm^* : H^0(X, \mathcal{L}_\pm) \to H^0(X, \mathcal{L}_\mp)$$

be the isomorphism corresponding to the isomorphism (7.8.1). By correcting with a scalar automorphism of an invertible sheaf, we may assume that $\tau_\pm^* \circ \tau_\mp^*$ is the identity.

We have

$$H^0(X, \pi^*(\mathcal{V})) \cong H^0(S, \mathcal{V}) \oplus H^0(S, \mathcal{V} \otimes \omega_S) \cong H^0(X, \mathcal{L}_+) \oplus H^0(X, \mathcal{L}_-).$$

Using Corollary 7.8.4, we choose an isomorphism

$$E^\vee \cong H^0(X, \mathcal{L}_+).$$

Thus, we will be able to identify $H^0(X, \mathcal{L}_\pm)$ with $E^\vee$, so that $|\eta_S|$ defines a rational map $X \to |E|$ whose image is a quartic surface. Of course, this will be the Steinerian surface $\mathrm{St}(W)$.

We have $\eta^2 = 2h^2 = 20$. By Riemann–Roch, $h^0(\eta) = 12$ and

$$H^0(X, \mathcal{O}_X(\eta)) = H^0(S, \mathcal{O}_S(h)) \oplus H^0(S, \mathcal{O}_S(h + K_S)).$$

The first summand can be identified with $\bigwedge^2 E^\vee$.

Since $\eta = \eta_S + \tau^*(\eta_S)$, we have a canonical map

$$\phi : E^\vee \otimes E^\vee = H^0(X, \mathcal{L}_+) \otimes H^0(X, \mathcal{L}_-) \to H^0(X, \mathcal{O}_X(\eta)).$$

Assume that $|\eta_S|$ has no fixed components (we will prove this property later). By the Base-Point-Free Pencil Trick from [12, p. 126], for any 2-dimensional subspace V of $H^0(X, \mathcal{L}_+)$, the kernel of the restriction ϕ_V of ϕ to $V \otimes H^0(X, \mathcal{L}_-)$ is isomorphic to $H^0(X, \mathcal{L}_- \otimes \mathcal{L}_+^{-1})$. Since η is ample and $\eta \cdot (\eta_S - \tau^*(\eta_S)) = 0$, the linear system $|\eta_S - \tau^*(\eta_S)|$ is empty. This shows that the kernel is trivial. Thus, the image of ϕ contains a 8-dimensional linear subspace. Since the map is τ-equivariant, the image is τ-invariant subspace. However, the image of ϕ is obviously not τ-invariant, and

smallest τ-invariant subspace it contains must coincide with the whole space. Thus, ϕ is surjective.

The kernel of the composition of this map with the canonical projection $H^0(X, \mathcal{O}_X(\eta)) \to \bigwedge^2 E^\vee$ can be identified with $S^2E^\vee$. Thus, the kernel of ϕ is a 4-dimensional linear subspace L of $S^2E^\vee$ and the summand $H^0(S, \mathcal{O}_S(H+K_S))$ can be identified with $S^2E^\vee/L$. This shows that $|H+K_S|$ maps S to $|\mathbb{P}(S^2E^\vee/L)| \cong \mathbb{P}^5$. The image, of course, should be the Cayley model of the Reye congruence.

Now, everything is ready to finish the proof. Consider the map

$$f_{|\eta|} = f_{|\eta_S|} \times f_{|\tau^*(\eta_S)|} : X \to |E| \times |E| \subset |E \otimes E|.$$

By above its image is contained in the subspace of zeros of $\mathrm{Ker}(\phi) \subset E^\vee \otimes E^\vee$. This implies that the image of X is contained in a complete intersection of 4 divisors of type $(1,1)$ in $|E| \times |E|$. The degree of this surface in the Segre embedding is equal to 20, and we know that the map is given by the linear system of degree 20. So, the image is equal to the complete intersection. It is a K3 surface birationally isomorphic to X. Since η is ample, the surface is nonsingular. It can be identified with the surface $\mathrm{PB}(W)$, where $W = |\mathrm{Ker}(\phi)|$ is a regular web of quadrics. We leave it to the reader to see that S is the Reye congruence of W.

It remains to pay the debt and check that $|\eta_S|$ has no fixed components. Consider the exact sequence from Theorem 7.8.3 and take its pull-back to the canonical cover X. After twisting by $\mathcal{O}_X(-E_i)$, we get an exact sequence

$$0 \to \mathcal{O}_X \to \mathcal{O}_X(\eta_S - E_i) \oplus \mathcal{O}_X(\tau^*(\eta_S) - E_i) \to \mathcal{O}_X(\eta - 2E_i) \to 0.$$

The image of $1 \in H^0(X, \mathcal{O}_X)$ gives an effective divisor $D = D_1 + D_2 \sim \eta - 2E_i$, where $D_1 \sim \eta_S - E_i, D_2 \sim \tau^*(\eta_S) - E_i$. Since $\tau^*(E_i) = E_i$, we obtain $\tau^*(D_1) = D_2$. Without loss of generality, we may assume that

$$D_1 \in |\eta_S - E_i|, \ i = 1, \dots, 10.$$

We know that $D_1 + D_2 \sim \eta - 2E_i = \pi^*(\eta - E_i - E_{-i})$. By Lemma 7.8.1, $|H - F_i - F_{-i}|$ is a nodal cycle R_i. It must split under the cover into the disjoint sum of nodal cycles $\tilde{R}_i, \tau^*(\tilde{R}_i)$. Thus, we may assume that $D_1 = \tilde{R}_i$. We have $F_i \cdot \tilde{R}_i = \frac{1}{2}(F_i \cdot R_i) = 3$. Hence, $\eta_S \cdot F_i = 3, \ \eta_S \cdot \tilde{R}_i = 1$.

Suppose A is the fixed part of $|\eta_S|$. Then, each irreducible component of A is a part of E_i or $\tilde{R}_i$. Since $|\eta_S - A|$ has an irreducible divisor, by the Vanishing Theorem and Riemann–Roch, we have $(\eta_S - A)^2 = \eta_S^2 = 4$. One can easily list all possibilities for the nodal cycle R_i and verify that $(\eta_S - A)^2 > 4$ for all sums of possible irreducible components of F_i and R_i. □

7.9 Nodal Enriques Surfaces and Smooth Congruences of Lines

In this section, we will give a sufficient condition for a nodal classical Enriques surface to be isomorphic to a congruence of lines of bidegree $(7,3)$ and sectional genus 6. In particular, we will show that a general nodal surface satisfies this condition.

Suppose S is a classical Enriques surface embedded in $G_1(\mathbb{P}^3)$ as a smooth congruence of bidegree $(7,3)$. Let H be the class of a hyperplane section of S in the Plücker embedding. It follows from Lemma 7.8.1 that $R = H - F_i - F_{-i}$ is a nodal cycle, and F_i is a half-fiber of a genus one pencil with $R \cdot F_i = 3$.

Theorem 7.9.1 *Let S be a classical Enriques surface. Assume that there exists a genus one curve F not moving in a pencil and a nodal cycle R with $F \cdot R = n \geq 3$ such that $|F + R + K_S|$ has no fixed components. Then, there exists a birational map $f : S \to S'$, where S' is a surface in the Grassmann variety $\mathbb{G} = G_1(\mathbb{P}^n)$ with at most rational double points as singularities. If $H = 2F + R + K_S$ is an ample divisor, then $S \cong S'$. The map is given by the vector bundle $\mathcal{V} = f^*(i^*\mathcal{S}_{\mathbb{G}}^\vee)$ that fits in a non-split extension*

$$0 \to \mathcal{O}(F) \to \mathcal{V} \to \mathcal{O}_S(F + R + K_S) \to 0. \tag{7.9.1}$$

Proof Let F, R be as in the assertion of the theorem. Since $h^0(R) \neq 0$, by Riemann–Roch and Serre duality,

$$\dim \operatorname{Ext}^1(\mathcal{O}_S(R + K_S), \mathcal{O}_S) = \dim H^1(S, \mathcal{O}_S(-R + K_S)) = \dim H^1(S, \mathcal{O}_S(R)) = 1.$$

This implies that there exists a non-split extension (7.9.1). We will show that the vector bundle $\mathcal{V}$ defines the asserted map to the Grassmannian.

We have $(F + R + K_S)^2 = 2F \cdot R_2 = 2n - 2$. Since $h^1(\mathcal{O}_S(F)) = 0$ because $h^0(F) = 1$, taking cohomology and using Riemann–Roch, we obtain

$$h^0(\mathcal{V}) = n + 1.$$

Let us show that $\mathcal{V}$ is spanned by its global sections. Let s_F be a non-zero section of $\mathcal{O}_S(F)$. For every $s \in H^0(S, \mathcal{V})$ the section $s_F \wedge s$ is either zero, or vanishes on a curve $F + D(s) \in |\mathcal{O}_S(H)|$ for some $D(s) \in |H - F|$. Since the map

$$H^0(S, \mathcal{V}) \to H^0(S, \mathcal{O}_S(H - F)) = H^0(S, \mathcal{O}_S(F + R + K_S)$$

is surjective, we find that $\mathcal{V}$ is generated by its global sections outside the curve F unless $|H - F|$ has base points. If the latter happens, then, by Corollary 2.6.8, there exists a genus one curve G with $(H - F) \cdot G = (F + R) \cdot G = 1$. Obviously, $F \cdot G \leq 1$. If $F \cdot G = 0$, $F \equiv G$ and $R \cdot G = n > 1$. If $F \cdot G = 1$, then $R \cdot G = 0$, and R is a component of the pencil $|2G|$. But then $2F \cdot G > F \cdot R = n > 2$, which implies $F \cdot G > 1$. So, we have shown that $|H - F|$ is base-point-free, and $\mathcal{V}$ is spanned by its global sections outside F. We shall show that the same $\mathcal{V}$ can be represented by an extension:

$$0 \to \mathcal{O}(F') \to \mathcal{V} \to \mathcal{O}_S(F' + R + K_S) \to 0, \tag{7.9.2}$$

where $F' \in |F + K_S|$. Then, repeating the argument, we find that $\mathcal{V}$ is spanned by its global sections outside F'. Since $F \cap F' = \emptyset$, this would imply that $\mathcal{V}$ is spanned by its global sections everywhere. Tensoring (7.9.2) by the sheaf $\mathcal{O}_S(-F')$, we obtain an exact sequence:

$$0 \to \mathcal{O}_S(F - F') \to \mathcal{V}(-F') \to \mathcal{O}_S(R). \to 0. \tag{7.9.3}$$

Since $h^1(F - F') = h^1(K_S) = 0$, this implies that $h^0(\mathcal{V}(-F')) \neq 0$. Let $\mathcal{O}_S(F') \to \mathcal{V}$ be a non-trivial map of sheaves defined by a non-zero section of $\mathcal{V}(-F')$. Let $\mathcal{L}$ be the maximal rank 1 subsheaf of $\mathcal{V}$ which contains $\mathcal{O}_S(F')$ (identified with its image) and such that the quotient sheaf $\mathcal{V}/\mathcal{L}$ is torsion-free. Then, it is easy to see that $\mathcal{L}$ is a reflexive sheaf. Since S is a smooth surface, $\mathcal{L}$ is an invertible sheaf. We have an exact sequence:

$$0 \to \mathcal{L} \to \mathcal{V} \to \mathcal{I}_Z \otimes \mathcal{L}' \to 0, \tag{7.9.4}$$

where $\mathcal{I}_Z$ is the ideal sheaf of some 0-dimensional subscheme of S and $\mathcal{L}'$ is an invertible sheaf. This easily follows from the structure of rank 1 torsion free sheaves on a regular two-dimensional scheme. Counting the Chern classes of $\mathcal{V}$, we obtain:

$$\begin{aligned} c_1(\mathcal{V}) &= c_1(\mathcal{L}) + c_1(\mathcal{L}'), \\ c_2(\mathcal{V}) &= c_1(\mathcal{L}) \cdot c_1(\mathcal{L}') + h^0(\mathcal{O}_Z). \end{aligned}$$

On the other hand, we compute the Chern classes of $\mathcal{V}$ by using (7.9.2) to get:

$$c_1(\mathcal{V}) = H,\ c_2(\mathcal{V}) = F \cdot (H - F) = n.$$

If $\mathcal{L} = \mathcal{O}_S(F')$, then we find that $\mathcal{L}' \cong \mathcal{O}_S(H - F')$ and $Z = \emptyset$. This gives (7.9.3). Assume now that $\mathcal{L} \neq \mathcal{O}_S(F')$. Since $h^1(\mathcal{O}_S(F')) = 0$, we have $h^0(\mathcal{L}) > h^0(\mathcal{O}_S(F)) = 1$. Let $\phi\colon \mathcal{L} \to \mathcal{O}_S(H - F)$ be the composition of the inclusion $\mathcal{L} \hookrightarrow \mathcal{V}$ and the projection $\mathcal{V} \to \mathcal{O}_S(H - F)$. If ϕ is trivial, $\mathcal{L}$ is a subsheaf of $\mathcal{O}_S(F)$ which has a one-dimensional space of sections. This contradiction shows that ϕ is non-trivial, hence $\mathcal{L} \cong \mathcal{O}_S(D)$, where $D = F' + R_1$ for some effective divisor $R_1 \neq 0$, and $|H - F - D| = |H - F - F' - R| = |R - R_1| \neq \emptyset$. We have

$$n = c_2(\mathcal{V}) = D \cdot (H - D) + h^0(\mathcal{O}_Z) \geq D \cdot (H - D) = (F' + R_1) \cdot (h - F' - R_1) =$$

$$= (F' + R_1) \cdot (F + R - R_1) = F \cdot R + R_1 \cdot (R - R_1) = n + R_1 \cdot (R - R_1).$$

This implies that $R_1 \cdot (R - R_1) = 0$, $Z = \emptyset$. Since R is connected, $R = R_1$, and $D = F' + R = H - F$. So, (7.9.4) becomes

$$0 \to \mathcal{O}_S(F' + R) \to \mathcal{V} \to \mathcal{O}_S(F) \to 0.$$

However, we have

$$\mathrm{Ext}^1(\mathcal{O}_S(F), \mathcal{O}_S(F' + R)) \cong H^1(S, \mathcal{O}_S(F' - F + R)) \cong H^1(\mathcal{O}_S(R + K_F)) \cong 0.$$

This shows that $\mathcal{V}$ splits into the sum $\mathcal{O}_S(h-F) \oplus \mathcal{O}_S(F)$ in conflict with its construction.

Let $f\colon F \to \mathbb{G} = G_1(\mathbb{P}^n)$ be the regular map given by the bundle $\mathcal{V}$. Recall that it assigns to a point $x \in S$ the quotient space $\mathcal{V}_x$ of $\Gamma(\mathcal{V})$. By construction of $\mathcal{V}$, we have

$$\bigwedge^2 \mathcal{V} \cong \mathcal{O}_S(F) \otimes \mathcal{O}_S(H-F) \cong \mathcal{O}_S(H).$$

Thus, the composition of f and the Plücker embedding of $S' = f(S) \subset \mathbb{G}$ is given by the linear system $|H|$. So, the theorem will follow if we verify that $|H|$ defines the map $S \to S'$ with the asserted properties. By Theorem 2.4.16, it suffices to check that $\Phi(H) \geq 3$, i.e. for any genus one curve E one has $H \cdot E \geq 3$. We have seen already that $\Phi(H-F) \geq 2$ and $(H-F)\cdot E = 2$ implies $E \cdot F = 1$. This obviously gives that $\Phi(H) \geq 3$. This concludes the proof of the theorem. □

Corollary 7.9.2 *Let S be an Enriques surface with $K_S \neq 0$. Assume that there exists an ample divisor H with $H^2 = 4n-2$ and a genus one curve F with $h \cdot F = n \geq 3$ such that $h^0(H-2F-K_S) \neq 0$. Then, S is isomorphic to a smooth surface S in the Grassmann variety $G_1(\mathbb{P}^n)$.*

Remark 7.9.3 We refer to [128], where it is shown that there exists a linear system W of quadrics in $\mathbb{P}^n$ of dimension $\binom{n}{2}$ such that the image of S in $G_1(\mathbb{P}^n)$ is contained in the subvariety of $G_1(\mathbb{P}^n)$ parameterizing lines in $\mathbb{P}^n$ contained in a codimension 2 linear subspace of W. Note that contrary to the case $n = 3$, if $n > 3$ the linear system of quadrics must be very special in order that it defines a generalized Reye congruence.

Remark 7.9.4 Let $h_{18} = \frac{1}{2}(g_1 + \cdots + g_9)$ be a Mukai (numerical) polarization with nef vectors g_i. Suppose that it is obtained from a non-degenerate isotropic 10-sequence $(f_1, \ldots, f_{10})$ by formula $g_i = h_{10} - f_i - f_{10}, i = 1, \ldots, 9$, where h_{10} is a (numerical) Fano polarization. Let H_{18} and F_i be the lifts of h_{18} and f_i in $\mathrm{Pic}(S)$. For any $i \neq 10$, we have $f_i \cdot g_i = 2$ and $f_i \cdot g_j = 1$ if $i \neq j$. Thus, $f_i \cdot h_{18} = 5$ and the image of F_i under a map $\phi_{H_{18}}$ given by the linear system $|H_{18}|$ is a genus one curve of degree 5 spanning a $\mathbb{P}^4$. Let $F_{-i} \in |F_i + K_S|$, where we assume that $K_S \neq 0$. Then $H_{18} - 2F_i - K_S = H_{18} - F_i - F_{-i}$, hence $h^0(H_{18} - 2F_i - K_S) > 0$ if and only if $F_1 + F_{-1}, \ldots, F_{10} + F_{-10}$ span a hyperplane in $\mathbb{P}^9$. If this happens, then the Mukai model of S lies in the subvariety of the Grassmannian variety $G_1(\mathbb{P}^5)$ parameterizing lines in a 10-dimensional linear system W of quadrics in $\mathbb{P}^5$ that are contained in a codimension 2 linear subspace of W. If we compose the embedding $S \hookrightarrow G_1(\mathbb{P}^5)$ with the Plücker embedding $G_1(\mathbb{P}^5) \hookrightarrow \mathbb{P}^{14}$, then we obtain that the image of S is contained in the intersection of $G_1(\mathbb{P}^5)$ with a linear subspace of dimension 9. This is a 3-dimensional Fano variety.

Now, we assume that $K_S = 0$ and $n = 3$. Recall that S must be either a $\boldsymbol{\mu}_2$-surface or $\boldsymbol{\alpha}_2$-surface.

Lemma 7.9.5 *Let (F_1, F_2, F_3) be a representative of a non-degenerate 3-sequence (f_1, f_2, f_3). Suppose that $|F_2 + F_3 - F_1| = \emptyset$. Then, $F_1 \cap F_2 \cap F_3 = \emptyset$.*

Proof Recall that, by Proposition 4.10.6, two half-fibers have no common irreducible component. Thus, $F_i \cap F_j$ consists of one point. Consider the natural exact sequence coming from restriction of the sheaf $\mathcal{O}_S(F_1 - F_2)$ to F_3:

$$0 \to \mathcal{O}_S(F_1 - F_2 - F_3) \to \mathcal{O}_S(F_1 - F_2) \to \mathcal{O}_{F_3}(F_1 - F_2) \to 0. \tag{7.9.5}$$

We have $(F_1 - F_2 - F_3) \cdot F_1 = -2$. Since F_1 is nef, the divisor class $F_1 - F_2 - F_3$ is not effective. By assumption, $h^0(\mathcal{O}_S(K_S + F_3 + F_2 - F_1)) = 0$. Thus, by Riemann–Roch and Serre duality, $h^1(\mathcal{O}_S(F_1 - F_2 - F_3)) = 0$. Now, $h^0(\mathcal{O}_S(F_1 - F_2)) = 0$, because $(F_1 - F_2) \cdot F_1 = -1$ and F_1 is nef. Suppose $F_1 \cap F_2 \cap F_3 \neq \emptyset$, then $\mathcal{O}_{F_3}(F_1 - F_2) \cong \mathcal{O}_{F_3}$ and $h^0(\mathcal{O}_{F_3}(F_1 - F_2)) = 1$. The exact sequence of cohomology associated with exact sequence (7.9.5) gives a contradiction. □

Remark 7.9.6 Note that for any $D \in |F_2 + F_3 - F_1|$, we have $D^2 = -2$ and $D \cdot F_2 = D \cdot F_3 = 0$, so each irreducible component of D is a (-2)-curve contained in fibers of $|2F_2|$ and $|2F_3|$.

Theorem 7.9.7 *Let S be an Enriques surface with $K_S = 0$. Assume that S contains a genus one curve F and a (-2)-curve R with $F \cdot R = 3$ such that $H = 2F + R$ is ample. Then, there exists an isomorphism $f : S \to S'$, where S' is a surface in the Grassmann variety $G_1(\mathbb{P}^3)$. The map from S to the Plücker space is given by the complete linear system $|H|$.*

Proof We use the same argument from the proof of Theorem 7.9.1, only this time, instead of taking $F' \in |F + K_S|$ we take F. Let

$$3H \sim F_1 + \cdots + F_{10} \tag{7.9.6}$$

with $F_1 = F$. Since H is ample, each F_i is a half-fiber. We claim that, using the previous lemma, one can choose two F_i and F_j different from F_1 such that $F_1 \cap F_i \cap F_j = \emptyset$. Suppose $F_i \cap F_j \cap F_k \neq \emptyset$ for some $i < j < k$. Then, $|F_a + F_b - F_c| \neq \emptyset$ for $\{a, b, c\} = \{i, j, k\}$. Since $H \cdot (F_a + F_b - F_c) = 3$ and $(F_a + F_b - F_c)^2 = -2$, $|F_a + F_b - F_c| = \{R_1\}$, where R_1 is a (-2)-curve with $H \cdot R_1 = 3$, or $|F_a + F_b - F_c| = \{R_1, R_2\}$, where R_1, R_2 are two (-2)-curves with $H \cdot R_1 = 2$ and $H \cdot R_1 = 1$. In both cases, each F_i, F_j, F_k contains R_1 or $R_1 + R_2$ in one of their reducible fibers. Since $(F_i + F_j - F_k) \cdot (F_i + F_j - F_l) = -1$ if $l \notin \{i, j, k\}$, we see that $|F_i + F_j - F_k|$ or $|F_i + F_j - F_l|$ contains a (-2)-curve R' with $H \cdot R' = 1$; we call such an R' a line. Now, take $F_i = F_1$, then $1 = H \cdot L = (2F + R) \cdot L$ for any line L implies that $F \cdot L = 0$. Since H is ample, there exists F_i such that $F_i \cdot L \neq 0$. Thus, $|F_1 + F_i - F_j| = \{R_j\}$ for any $j \neq 1, i$, where R_j is a (-2)-curve with $H \cdot R_j = 3$. Since $(F_1 + F_i - F_j) \cdot (F_1 + F_i - F_k) = R_j \cdot R_k = -1$ for different $1, i, j, k$, we get a contradiction. Thus, we can always find $i \neq j \neq 1$ such that $F_1 \cap F_i \cap F_j = \emptyset$. Without loss of generality, we assume that $i = 2$, $j = 3$.

Now, we shall show that the non-split vector bundle $\mathcal{V}$ given by the exact sequence:

$$0 \to \mathcal{O}_S(F) \to \mathcal{V} \to \mathcal{O}_S(H - F) \to 0 \tag{7.9.7}$$

can be also represented by an extension:

$$0 \to \mathcal{O}_S(F_i) \to \mathcal{V} \to \mathcal{O}_S(H - F_i) \to 0 \tag{7.9.8}$$

for $i = 2$ and 3. Then, applying Lemma 7.8.1 (vii), and repeating the argument from the proof of Theorem 7.9.1 with the curve F' replaced by F_i, we find that $\mathcal{V}$ is generated by its global sections. The rest of the proof proceeds in the same way.

Twisting (7.9.7) by $\mathcal{O}_S(-F_2)$, and using that $h^0(H - F - F_2) = 1$ because $(H - F_1 - F_2)^2 = 0$ and $(H - F_1 - F_2) \cdot F_3 = 1$, we find a non-trivial map $\mathcal{O}_S(F_2) \to \mathcal{V}$. Let $\mathcal{L} = \mathcal{O}_S(D)$ be an invertible subsheaf of $\mathcal{V}$ which contains the image of $\mathcal{O}_S(F_2)$ such that the quotient sheaf $\mathcal{V}/\mathcal{L} \cong \mathcal{I}_Z(D')$ for some zero-cycle Z and an effective divisor D'. Consider the map $\phi : \mathcal{O}_S(D) \to \mathcal{O}_S(H-F)$ obtained as the composition of the inclusion $\mathcal{O}_S(D)$ in $\mathcal{V}$ and the projection map $\mathcal{V} \to \mathcal{O}_S(H - F)$. If ϕ is trivial, then $\mathcal{O}_S(D)$ is isomorphic to a subscheaf of $\mathcal{O}_S(F)$, hence $F - F_2 > 0$, a contradiction. Thus, ϕ is not trivial, hence $h - F - D$ is an effective divisor. Thus, $H - F - D \sim F + R - D \geq 0$, and we obtain $h \cdot D \leq 7$. If $h \cdot D = 7$, we have $h \cdot (F + R - D) = 0$, hence $H - F \sim F + R \sim D$. Thus, $\mathcal{V}$ fits in the extension:

$$0 \to \mathcal{O}(H - F) \to \mathcal{V} \to \mathcal{O}(F) \to 0. \tag{7.9.9}$$

Since $\mathcal{O}_S(H - F)$ is not a subsheaf of $\mathcal{O}_S(F_i)$, we see that the composition of $\mathcal{O}(H-F) \to \mathcal{V}$ and the projection to $\mathcal{O}_S(F_i)$ in extension (7.9.8) is an isomorphism. Therefore, the extension splits contrary to its construction. We have $3 = c_2(\mathcal{V}) \geq D \cdot (H - D)$, which implies that $D^2 \geq 4$ if $h \cdot D = 6$. The Hodge Index Theorem $10D^2 < (H \cdot D)^2$ gives a contradiction. Thus, $H \cdot D \leq 5$ and $D^2 \leq 2$.

We have $h^0(D) = 1$, otherwise using Proposition 2.6.1 we obtain that $\Phi(H) \leq 2$, a contradiction. If $\mathcal{L} = \mathcal{O}_S(D) \neq \mathcal{O}_S(F_j)$, the quotient sheaf is a torsion sheaf, and we obtain $h^0(L) \geq 2$, contradicting the previous remark. So, we have an exact sequence:

$$0 \to \mathcal{L} \to \mathcal{V} \to \mathcal{I}_Z(D') \to 0,$$

where $\mathcal{L} = \mathcal{O}_S(F_i)$ or $\mathcal{L} = \mathcal{O}_S(H - F_i)$. Computing the Chern classes we obtain, as in the proof of Theorem 7.9.1, that $Z = \emptyset$ and $D' \sim H - F_i$ or $D' \sim F_i$, respectively. In the latter case, the extension must split, contradicting our construction; in the former case, we obtain (7.9.8). This finishes the proof of the theorem. □

Corollary 7.9.8 *Let S be a general nodal Enriques surface in the sense of definition from Section 6.5. Then, S is isomorphic to a congruence of lines in $\mathbb{P}^3$ of bidegree* $(7, 3)$.

Proof By Theorem 6.5.5, S admits an ample Fano numerical polarization h such that there exists a (-2)-curve R with $h_{10} \cdot R = 4$. It follows from Corollary 6.3.9 that it represents the numerical class $h_{10} - 2f_i$, where $(f_1, \ldots, f_{10})$ is a non-degenerate canonical isotropic sequence such that $3h = f_1 + \cdots + f_{10}$. We have $f_i \cdot R = h \cdot f_i = 3$. Since f_i is nef, it represents a half-fiber F of a genus one fibration. Now, we can apply Theorem 7.9.1 and Theorem 7.9.7. □

Applying Proposition 7.7.3, we obtain:

Corollary 7.9.9 *Assume $p \neq 2$. The Picard lattice of the canonical cover of a general nodal Enriques surface is isomorphic to* $\mathsf{U} \oplus \mathsf{E}_8(2) \oplus \mathsf{A}_1(2)$.

Let $\mathcal{E}$ be a rank 2 vector bundle on S with $c_1(\mathcal{E}) = H$, where H is an ample Fano polarization and $c_2(\mathcal{E}) = 3$. Assume that $\mathcal{E}$ has a section s with only isolated zeros. The section s defines an exact sequence

$$0 \to \mathcal{O}_S \to \mathcal{E} \to \mathcal{I}_Z(H) \to 0,$$

where Z is a 0-dimensional closed subscheme of S with $h^0(\mathcal{O}_Z) = 3$. Suppose $\mathcal{E} = \mathcal{V}$, where $\mathcal{V}$ gives an embedding ϕ of S into the Grassmannian $\mathbb{G} = G_1(\mathbb{P}^3)$. Suppose $K_S \neq 0$. Taking global sections, we find that $h^0(S, \mathcal{I}_Z(H + K_S)) = 4$. This shows that the image of Z in $\mathbb{G}$ spans a line, a *trisecant line* of $\phi(S)$. Since $H^0(S, \mathcal{V}) = 4$, we have a $\mathbb{P}^3$ of trisecant lines. Note that a smooth Fano model admits 20 planes of trisecant lines– the lines in 20 planes spanned by the images of the half-fibers F_i from (7.9.6). See more about this in the next section.

The following theorem was proven in [194] under the assumption that $K_S \neq 0$. We can modify it along the lines of the proof of Theorem 7.9.7 to extend it to the case $K_S = 0$.

Theorem 7.9.10 *Let $\mathcal{E}$ be a rank 2 vector bundle on an Enriques surface S which admits a section with only isolated zeros. Assume $c_1(\mathcal{E})$ is an ample Fano polarization H and $c_2(\mathcal{E}) = 3$. Then, either $\mathcal{E}$ decomposes into the direct sum of invertible sheaves $\mathcal{O}_S(F_i) \oplus \mathcal{O}_S(H - F_i)$, or $\mathcal{O}_S(F_i + K_S) \oplus \mathcal{O}_S(H - F_i + K_S)$, or it is isomorphic to one of the following non-split extensions:*

(i)

$$0 \to \mathcal{O}_S(H - F_i) \to \mathcal{E} \to \mathcal{O}_S(F_i) \to 0,$$

or

$$0 \to \mathcal{O}_S(H - F_i + K_S) \to \mathcal{E} \to \mathcal{O}_S(F_i + K_S) \to 0.$$

(ii)

$$0 \to \mathcal{O}_S(F_i) \to \mathcal{E} \to \mathcal{O}_S(H - F_i) \to 0$$

or

$$0 \to \mathcal{O}_S(F_i + K_S) \to \mathcal{E} \to \mathcal{O}_S(H - F_i + K_S) \to 0.$$

In case (i), the vector bundle $\mathcal{E}$ defines an isomorphism from S to a congruence of bidegree $(7, 3)$ in $G_1(\mathbb{P}^3)$.

Remark 7.9.11 If $K_S \neq 0$, in case (i), the bundle $\mathcal{E}$ represents the unique isomorphism class of a *stable* rank 2 vector bundle with $c_1(\mathcal{E}) = H$ and $c_2(\mathcal{E}) = 3$. It is also an *exceptional vector bundle* in the sense that $\mathrm{Ext}^1(\mathcal{E}, \mathcal{E}) = \{0\}$.

More generally, it follows from [387], [388] that any exceptional rank 2 vector bundle on an Enriques surface with $c_1(\mathcal{E})^2 = 4n - 2, c_2(\mathcal{E}) = n \geq 3$ is isomorphic to a vector bundle $\mathcal{E}$ given by a non-split extension

$$0 \to \mathcal{O}_S(F) \to \mathcal{E} \to \mathcal{O}_S(F + R) \to 0,$$

where R is a nodal cycle and F is a half-fiber of a genus one fibration with $F \cdot R = n$. The bundle $\mathcal{E}$ is stable and maps S onto a generalized Reye congruence.

Definition 7.9.12 A Fano polarization H is called a *Fano–Reye polarization* if the linear system $|H|$ maps S into a nonsingular quadric Q in $\mathbb{P}^5$.

Recall from Theorem 6.3.8 that, for any Fano polarization on a nodal Enriques surface S there exists a (-2)-curve R with $R \cdot H \leq 4$. A general nodal surface admits a Fano polarization H with $H \cdot R = 4$.

Corollary 7.9.13 *Let H be a Fano polarization on a general nodal Enriques surface. Then, H is a Fano–Reye polarization if and only if there exists a smooth rational curve R on S such that $H \cdot R = 4$.*

Proof By Corollary 6.3.9, $[R] = H - 2f$, where f is a nef primitive isotropic class. Thus, we can choose its representative F such that $H - 2F \sim R$. Now, we apply the previous results. □

Remark 7.9.14 A recent paper [493] proves that a classical nodal Enriques surface always admits a birational morphism $S \to S' \subset G_1(\mathbb{P}^3)$, where S' is a (non-necessary smooth) congruence of lines of bidegree $(7, 3)$ and sectional genus six. The Reye vector bundle $\mathcal{V}$ that defines a map $S \to S' \subset G_1(\mathbb{P}^3)$ fits in an extension

$$0 \to \mathcal{O}_S(F + R_1 + \cdots + R_n) \to \mathcal{V} \to \mathcal{O}_S(H - F - R_1 - \cdots - R_n) \to 0,$$

where F is a half-fiber such that there exists a canonical isotropic sequence $(f_1 = [F], f_1 + R_1, \ldots, f_1 + R_1 + \cdots + R_n, f_2, \ldots)$ defined by H. The authors prove that such $\mathcal{V}$ always exists if S is a classical Enriques surface.

7.10 Fano–Reye Polarizations

In this section, we introduce and study special Fano polarizations of classical nodal Enriques surfaces: Fano–Reye and Fano–Cayley polarizations. We give several characterizations of such models among other Fano models of nodal surfaces.

Let $\underline{f} = (f_1, \ldots, f_{10})$ be a canonical isotropic sequence with the non-degeneracy invariant c, and

$$3h = f_1 + \cdots + f_{10}.$$

Let $f_{i_1}, \ldots, f_{i_c}$ be its nef members. Choose representatives $F_{i_1}, \ldots, F_{i_c}$ of $f_{i_1}, \ldots, f_{i_c}$ such that

$$3H \sim F_1 + \cdots + F_{10},$$

where $\mathcal{R}_{i_k} = F_{i_{k+1}} - F_{i_k}$ is a nodal cycle of type $A_{i_{k+1}-i_k-1}$ such that $F_{i_k} \cdot \mathcal{R}_{i_k} = 1$. The linear system $|H|$ maps S birationally onto a surface S' in $\mathbb{P}^5$ with singular points x_{i_k} of type $A_{i_{k+1}-i_k-1}$, the images of the nodal cycles $\mathcal{R}_{i_k}$. Recall from Section 3.5 in Volume I that the surface S' is called a *Fano model* of S. The polarization H is ample

if and only if $S \cong S'$, or, equivalently, the isotropic sequence $\underline{f}$ is non-degenerate. The images of the curves $F_{i_1}, \ldots, F_{i_c}$ are plane curves of degree 3. They span the planes $\Lambda_{i_1}, \ldots, \Lambda_{i_c}$ in $\mathbb{P}^5$. If $K_S \neq 0$, we denote by F_{-i_j} the second half-fiber in the genus one pencil $|2F_{i_j}|$. Let Λ'_{-i_j} be the plane spanning the image of F_{-i_j}. If $K_S = 0$, by definition, $F_{i_j} = F_{-i_j}, \Lambda_{i_j} = \Lambda_{-i_j}$.

Definition 7.10.1 A choice of representatives F_i, H of f_i, h in $\mathrm{Pic}(S)$, together with an order prescribed by the definition of a canonical isotropic sequence is called a *supermarking* of S. The planes $\Lambda_{i_1}, \ldots, \Lambda_{i_c}$ are called the *Fano planes* of the supermarking.

Lemma 7.10.2 *The singular point x_{i_j} lies in Λ_{i_j}. Two Fano planes Λ_{i_j} and Λ_{i_k} intersect at a unique point which is a nonsingular point of S'.*

Proof The only assertion that does not follow immediately from the definition of a canonical isotropic sequence is the assertion that $\dim \Lambda_{i_j} \cap \Lambda_{i_k} < 1$. Suppose the intersection contains a line ℓ. Then, the pre-image of the linear system of hyperplanes containing the two planes defines a linear system $|H - F_{i_j} - F_{i_k}|$ on S of dimension ≥ 1. We have $(H - F_{i_j} - F_{i_k})^2 = 0$ and $(H - F_{i_j} - F_{i_k}) \cdot H = 4$. The fixed part of $|H - F_{i_j} - F_{i_k}|$ consists of the union of nodal cycles and the moving part $|M|$ must satisfy $M^2 = 0$. Since $M \cdot H < 4$, $|M|$ cannot be equal to $|2F|$ for a genus one curve F. This shows that $\dim |M| = \dim |H - F_{i_j} - F_{i_k}| = 0$. □

Proposition 7.10.3 *Let $S' \subset \mathbb{P}^5$ be a Fano model of an Enriques surface S with $K_S \neq 0$. The following properties of S' are equivalent:*

(i) S' lies in a quadric;
(ii) S' lies in a quadric of corank ≤ 1;
(iii) the Fano planes Λ_{-i} and Λ_i intersect;
(iv) for any Fano plane Λ_i there exists a hyperplane H that cuts out S' along $(\Lambda_i \cap S') \cup (\Lambda_{-i} \cup S') \cup C$, where $\deg C = 4$ and one of the following cases occurs:

(iv)' C is a connected curve of degree 4 and arithmetic genus one spanning a $\mathbb{P}^3$ and one of the singular points of S' lies on C and coincides with $\Lambda_i \cap \Lambda_{-i}$;
(iv)" C is a connected curve of degree 4 and arithmetic genus 0 that spans a hyperplane. If C contains a singular point of S', then one of them is the point $\Lambda_i \cap \Lambda_{-i}$;

(v) if $S = S'$, then S' is isomorphic to a Reye congruence.

Proof (i) $\Leftrightarrow$ (ii) If $S \subset Q$, then every Fano plane Λ_i is contained in Q (since it cuts out a cubic curve in S). If Q contains a line in its singular locus, then two planes intersect along a line. This contradicts Lemma 7.10.2.

(ii) $\Rightarrow$ (iii) Since all planes in a singular quadric intersect, we may assume that the quadric is nonsingular. Then, the proof follows from the proof of property (v) in Lemma 7.8.1. To carry on the proof we need only that $c \geq 2$. It is covered by our assumption.

(iii) $\Rightarrow$ (iv). If $S = S'$, this was proven in Lemma 7.8.1 (vii). In this case the second possibility is not realized. In the notation of this proof, assume that $R \in |H - F_i - F_{-i}|$ contains a nef part R_1 with $R_1^2 \geq 0$. As in the proof of Lemma 7.8.1 we find that $R_1^2 = 0$ and $H \cdot (R - R_1) \leq 1$. It is maybe equal to zero instead of 1 because H is not ample anymore.

Suppose $H \cdot (R - R_1) = 1$. If S' is smooth, we deduced from this that $H \cdot R_1 = 3$, $(R - R_1)^2 = -2$ and get a contradiction. If S' is singular this only implies that R_1 coincides with one of F_j and if $R_1 \equiv F_i$ then $(R - R_1)^2 = -8, R_1 \cdot (R - R_1) = 3$. If $R_1 \not\equiv F_i$, then $(R - R_1)^2 = -4, R_1 \cdot (R - R_1) = 1$. In both cases, the image of $R - R_1$ is a line ℓ on S' passing through some singular points of S' and the image of R_1 spans one of the planes Λ_j. In the first case, $\ell \subset \Lambda_j$, and in the second case, $\ell \cap \Lambda_j \neq \emptyset$. In both cases, the image of R spans a projective subspace of dimension ≤ 3. But $h^0(H - R) = h^0(F_i + F_{-i})$, hence the image of R spans a hyperplane in $\mathbb{P}^5$, a contradiction.

Suppose $H \cdot (R - R_1) = 0$. Then, $H \cdot R_1 = 4$ and $(H - 2F_i) \cdot R_1 = R_1 \cdot (R - R_1)$ implies that either $F_i \cdot R_1 = 1, F_i \cdot (R - R_1) = 2$ and $R_1 \cdot (R - R_1) = 2$, or $F_i \cdot R_1 = 2, F_i \cdot (R - R_1) = 1$ and $R_1 \cdot (R - R_1) = 0$. It follows that $R - R_1$ is blown down to one singular point equal to $\Lambda_i \cap \Lambda_{-i}$, and this point lies on the image of R_1. The image of R_1 is a curve of degree 4 and arithmetic genus one.

If each non-empty part of R is negative definite, the argument applies without change. It shows that $R = R_1 + R_2$ is a connected curve, where R_1 is a nodal chain with $R_1^2 = -2$ mapped to a curve of degree 4 of arithmetic genus 0. Also, R_2 is empty, or a nodal chain blown down to singular points $s'_1, \ldots, s'_k$ on S'. In the latter case, we have $-2 = (R_1 + R_2)^2$ implies $R_2^2 = -2R_1 \cdot R_2 < 0$, hence $R_1 \cdot R_2 > 0$. Also, $(H - 2F_i) \cdot R_2 = -2F_i \cdot R_2 = R_1 \cdot R_2 + R_2^2 = -R_1 \cdot R_2$ implies that $R_2 \cdot F_i > 0$. Hence, $\Lambda_i \cap \Lambda_{-i}$ is one of the singular points s'_i and the image of R_1 contains all the singular points $s'_1, \ldots, s'_k$.

(iv) $\Rightarrow$ (v) This follows from Corollary 7.9.2.

(v) $\Rightarrow$ (i) Obvious. □

Any condition in the previous proposition characterizes a Fano–Reye polarization among Fano polarizations.

Suppose H is an ample Fano–Reye polarization, so that $S' = \mathrm{Rey}(W)$ for a regular web of quadrics W. Let $\mathcal{U}$ be the universal family of lines contained in quadrics from W. We cited a result from [425] that a choice of a plane component in each of the ten reducible quadrics Q_i in W defines a small resolution (in the category of algebraic spaces) of the double cover of W branched along the quartic symmetroid $\mathcal{D}(W)$. Since the Reye lines in each plane form the cubic curve F_i or F_{-i} from Lemma 7.8.1, we see that this choice is equivalent to a choice of a supermarking of the Reye polarization.

Remark 7.10.4 We do not know any examples where a Fano polarization maps S into a singular quadric.

Note that, if H is a Fano–Reye polarization and $K_S \neq 0$, then $|H + K_S|$ is not a Reye polarization. In fact, $|H - F_i - F_{-i}| \neq \emptyset$ implies that $|H + K_S - F_i -$

$F_{-i}| = |H - 2F_i| = \emptyset$. In fact, if $p \neq 2$, we have proved in Theorem 7.8.5 that any congruence of bidegree $(7, 3)$ and sectional genus six is the Reye congruence of a regular web W of quadrics. Let us identify the canonical cover $\pi : X \to S$ with the Reye map $\mathfrak{r} : \mathrm{PB}(W) \to \mathrm{Rey}(W)$. Then, in notation of Proposition 7.7.1, $\pi^*(H) = \pi^*(H + K_S) = \eta$, and $H^0(X, \mathcal{O}_X(\eta)) \cong E^\vee \otimes E^\vee$. We can identify the direct summand $\bigwedge^2 E^\vee$ with $H^0(S, \mathcal{O}_S(H))$ and the direct summand $S^2E^\vee$ with $H^0(S, \mathcal{O}_S(H + K_S))$. This shows that the linear system $|H + K_S|$ defines the Cayley map $\mathfrak{c} : S \to \mathrm{Cay}(W)$ from (7.2.7). For this reason, we call the divisor class $H + K_S$ the *Fano–Cayley polarization*.

If $p = 2$ and S is classical, we do not know whether a smooth congruence of bidegree $(7, 3)$ and sectional genus six is isomorphic to the Reye congruence $\mathrm{Rey}(W)$ of a regular web of quadrics. However, both families depend on 9 moduli.

We will continue to call $H + K_S$ a Fano–Cayley polarization if H is a Fano–Reye polarization. In the case when $S \cong \mathrm{Rey}(W)$, the linear system $|H + K_S|$ defines the Cayley map $X = \mathrm{PB}(W) \to \mathrm{Cay}(W)$ from (7.3.10).

Recall that a line ℓ in $\mathbb{P}^n$ is called a *trisecant line* of a subvariety $V \subset \mathbb{P}^n$ if it intersects V in at least three points taken with appropriately defined multiplicities. If S' is a Fano model of an Enriques surface S in $\mathbb{P}^5$ and Λ_i is a Fano plane containing a cubic curve F_i, then obviously any general line in Λ_i is a trisecant line of S'.

Proposition 7.10.5 *Assume $K_S \neq 0$. Let H be an ample Fano–Reye polarization and let S' be the image of S under $\phi_{|H+K_S|}$. Then, there is a three-dimensional family of trisecant lines of S'.*

Proof Let $\mathcal{V}$ be the vector bundle on S defined by an extension (7.9.8), where the image of F_i on S' spans a Fano plane. We have $c_1(\mathcal{V}) = H, c_2(\mathcal{V}) = 3$. Let s be a section of $\mathcal{V}$ with reduced zero scheme Z of length 3. It defines an exact sequence:

$$0 \to \mathcal{O}_S \xrightarrow{s} \mathcal{V} \to \mathcal{I}_Z(H) \to 0.$$

Tensoring the exact sequence by $\mathcal{O}_S(K_S)$ and applying Corollary 7.8.4, we obtain an exact sequence:

$$0 \to \mathcal{O}_S(K_S) \xrightarrow{s} \mathcal{V} \to \mathcal{I}_Z(H + K_S) \to 0.$$

We know that $h^0(\mathcal{V}) = 4$. Taking cohomology, we find that

$$h^0(\mathcal{I}_Z(H + K_S)) = h^0(\mathcal{V}) = 4.$$

This shows that the image Z' of Z in S' is contained in a 3-dimensional linear system of hyperplane section. The base scheme of this linear system is a line containing Z'. It is a trisecant line of S'. One can reverse the argument to show that any trisecant line $\mathfrak{t}$ of S' not passing through singular points of S' defines a non-zero section s with 0-dimensional scheme $Z(s)$. Taking $\mathfrak{t}$ to be a general line in a Fano plane we see that $Z(s)$ is reduced. Thus, we obtain a three-dimensional family of lines parametrized by an open Zariski subset of the space $|\Gamma(\mathcal{V})| \cong \mathbb{P}^3$. □

The converse is also true:

Theorem 7.10.6 *Assume $K_S \neq 0$. Let H be a Fano polarization of an Enriques surface S and $f : S \to S' \subset \mathbb{P}^5$ be a birational map given by the linear system $|H|$. Assume that there exists a line $\mathfrak{t}$ in $\mathbb{P}^5$ which intersects S' at three nonsingular points and which does not lie in one of the Fano planes Λ_i. Then, $H + K_S$ is a Fano–Reye polarization, and H is a Fano–Cayley polarization.*

Proof The pre-image in S of $\mathfrak{t} \cap S'$ is a 0-cycle Z of length 3. Consider the exact sequence

$$0 \to \mathcal{I}_Z(H) \to \mathcal{O}_S(H) \to \mathcal{O}_Z(H) \to 0.$$

Since $|H^0(\mathcal{I}_Z(H))|$ is isomorphic to the web of hyperplanes in $\mathbb{P}^5$ containing the line $\mathfrak{t}$, and $h^0(\mathcal{O}_Z(H)) = 3$, we obtain that $h^1(\mathcal{I}_Z(H)) = 1$. By Serre duality,

$$\dim H^1(S, \mathcal{I}_Z(H) \otimes \omega_S^{-1}) = \dim \mathrm{Ext}^1(\mathcal{I}_Z(H), \mathcal{O}_S),$$

hence there exists a non-split extension:

$$0 \to \mathcal{O}_S \to \mathcal{V} \to \mathcal{I}_Z(H + K_S) \to 0. \tag{7.10.1}$$

Choose a hyperplane in $\mathbb{P}^5$ which contains $\mathfrak{t}$ and one of the planes Λ_{-i}. Then, it defines an effective divisor $F_{-i} + A \sim H$, where $Z \subset A$. Adding to both sides K_S, we get a divisor $F_i + A \in |H + K_S|$. Let s be the section of $\mathcal{I}_Z(H + K_S - F_i)$ defined by the divisor A. Tensoring (7.10.1) by $\mathcal{O}_S(-F_i)$ and taking cohomology, we get an exact sequence

$$0 \to H^0(S, \mathcal{V} \otimes \mathcal{O}_S(-F_i)) \to H^0(S, \mathcal{I}_Z(H + K_S - F_i)) \to H^1(S, \mathcal{O}_S(-F_i)).$$

Since $h^1(F_i + K_S) = h^1(F_{-i}) = 0$, $H^0(S, \mathcal{V} \otimes \mathcal{O}_S(-F_i)) \neq 0$ and there is a non-zero section $\tilde{s}$ of $\mathcal{V}(-F_i)$ that extends s. Let

$$0 \to \mathcal{O}_S(F_i) \to \mathcal{V} \to \mathcal{F} \to 0$$

be the corresponding exact sequence. Suppose $\mathcal{O}_S(F_i)$ is not saturated in $\mathcal{V}$, i.e. $\mathcal{F}$ is not torsion-free. This implies that $\mathcal{V}$ contains an invertible sheaf isomorphic to $\mathcal{O}_S(F_i + C)$ for some effective divisor C such that $\mathcal{O}_S(C)$ is isomorphic to a subsheaf of $\mathcal{F}$. We get an exact sequence

$$0 \to \mathcal{O}_S(F_i + C) \to \mathcal{V} \to \mathcal{I}_{Z'}(H + K_S - F_i - C) \to 0, \tag{7.10.2}$$

where $Z \subset Z'$ and $h^0(H + K_S - F_i - C) > 0$. Replacing F_i with some other nef F_j in the isotropic sequence, we obtain a section of $\mathcal{V}(-F_j)$. We have $h^0(\mathcal{V}(-F_j)) = h^0(H + K_S - F_j) = 4$. Since $h^0(F_i + C) \leq h^0(H + K_S - F_i) = 3$, we get $h^0(F_i + C - F_j) \leq 3$. Tensoring (7.10.2) by $\mathcal{O}_S(-F_j)$ and taking cohomology, we obtain that $h^0(H + K_S - F_i - F_j - C) \neq 0$. Thus, C is a part of the unique divisor $F_{ij} \in |H + K_S - F_i - F_j|$. This implies

$$C \cdot H \leq C \cdot F_{ij} = 4, C \cdot F_k \leq F_{ij} \cdot F_k = 1, k = i, j. \tag{7.10.3}$$

Computing $c_2(\mathcal{V})$ using (7.10.2), we find

$$c_2(\mathcal{V}) = 3 = -(F_i + C)^2 + H \cdot C + h^0(\mathcal{O}_{Z'}) \geq 3 - (F_i + C)^2 + H \cdot C.$$

This gives $(F_i + C)^2 = 2F_i \cdot C + C^2 \geq H \cdot C$. Since $F_i \cdot C \leq 1$ and C does not move, hence $C^2 \leq 0$, we obtain $H \cdot C \leq 2$. In particular, the image of C on S' is a line or a conic, or it is contained in the singular locus of S'. In any case, no part of C has self-intersection equal to 0. This implies that $C^2 \leq -2$, and $H \cdot C \leq 0$. Since H is nef, we obtain $H \cdot C = 0, C^2 = -2$. Thus, C is a nodal cycle and it is blown down to a singular point of S'. Since C contains Z, and, by assumption, Z does not contain singular points, we get a contradiction.

So, we have proved that the sheaf $\mathcal{F}$ is torsion-free, hence it is isomorphic to a sheaf of the form $\mathcal{I}_{Z'} \otimes \mathcal{O}_S(D)$ for some zero cycle Z' and divisor D. Computing the Chern classes of $\mathcal{V}$, we find:

$$c_1(\mathcal{V}) = H + K_S = D + F_i,\ 3 = c_2(\mathcal{V}) = (H + K_S - F_i) \cdot F_i + h^0(\mathcal{O}_{Z'}) = 3 + h^0(\mathcal{O}_{Z'}).$$

This implies that

$$Z' = \emptyset,\ D \sim H + K_S - F_i,$$

which gives an exact sequence:

$$0 \to \mathcal{O}_S(F_i) \to \mathcal{V} \to \mathcal{O}_S(H + K_S - F_i) \to 0. \tag{7.10.4}$$

If this extension splits, every section of $\mathcal{V}$ vanishes on a subset of F_i. In particular, $\mathfrak{t}$ is contained in the image of F_i on S', hence the trisecant line $\mathfrak{t}$ lies in Λ_i. By assumption, this does not happen. Hence, (7.10.4) does not split. Therefore,

$$\mathrm{Ext}^1(\mathcal{O}_S(H+K_S-F_i), \mathcal{O}_S(F_i)) \cong H^1(S, \mathcal{O}_S(2F_i-H-K_S)) \cong H^1(S, \mathcal{O}_S(H-2F_i)) \neq \{0\},$$

and, by Riemann–Roch, $|H - 2F_i| \neq \emptyset$. Since $(H - 2F_i)^2 = -2$, $(H - 2F_i) \cdot H = 4$, it is easy to see that $|H - 2F_i| = |H + K_S + F_i + F_{-i}|$ consists of a nodal cycle R with $R \cdot E_i = 3$. Now, we can apply Theorem 7.9.1 to conclude the proof. □

Let us study in more detail a Fano model S' of an Enriques surface S. We choose a Fano polarization H and a supermarking $(F_1, \ldots, F_{10})$ such that $3H \sim F_1 + \cdots + F_{10}$. Consider the linear system $|3H|$. By Riemann–Roch, $\dim |3H| = 45$. On the other hand, $\dim |\mathcal{O}_{\mathbb{P}^5}(3)| = 55$. Thus, we see that S' is contained in the base scheme of a linear system of cubic hypersurfaces of dimension ≥ 9.

The proof of the following proposition can be found in [250].

Proposition 7.10.7 *Let $S \subset \mathbb{P}^5$ be a Fano model of an Enriques surface with $K_S \neq 0$ corresponding to an ample Fano polarization H. Then, S is 3-normal, i.e. the restriction map $|\mathcal{O}_{\mathbb{P}^5}(3)| \to |\mathcal{O}_S(3H)|$ is surjective. The homogeneous ideal of S is generated by quadrics and cubics.*

It follows from Proposition 7.10.3 that the homogeneous ideal of S is generated by cubics if the Fano polarization is not a Fano–Reye polarization. Thus, S is the base scheme scheme of a 9-dimensional linear system of cubic hypersurfaces in $\mathbb{P}^5$.

We can say a little more about the ideal of S when H is a Fano–Reye or a Fano–Cayley polarization.

Let $W = |L|$ be a regular web of quadrics in $\mathbb{P}^3 = |E|$ and let $\mathrm{Rey}(W)$ be the Reye congruence contained in the Grassmann variety $G(2, E)$ of lines in $|E|$. We first give a resolution of $\mathrm{Rey}(W)$ in $G(2, E)$.

Proposition 7.10.8 *The following sequence is a locally free resolution of the ideal sheaf $\mathcal{I}_{\mathrm{Rey}(W)}$ of the Reye congruence* $\mathrm{Rey}(W)$ *as a closed subvariety of* $G(2, E)$*:*

$$0 \to S^2(\mathcal{S}_{G(2,E)})(-3) \to L^\vee \otimes \mathcal{O}_{G(2,E)}(-3) \to \mathcal{I}_{\mathrm{Rey}(W)} \to 0.$$

Proof Restricting quadrics from $W = |L|$ to lines in $\mathbb{P}^3$ defines a surjective map of locally free sheaves $L \otimes \mathcal{O}_{G(2,E)} \to S^2(\mathcal{S}^\vee_{G(2,E)})$. Dualizing, we get a linear map $S^2(\mathcal{S}_{G(2,E)}) \to L^\vee \otimes \mathcal{O}_{G(2,E)}$. A Reye line ℓ is contained in a pencil of quadrics from W. Since this pencil intersects any net of quadrics, we see that ℓ is contained in the Montesano cubic complex of any net. In other words, the Reye congruence lies in the intersection of all cubic hypersurfaces defined by points in the dual space $\check{W} = |L^\vee|$. This defines a linear map $L^\vee \otimes \mathcal{O}_{G(2,E)}(3) \to \mathcal{I}_{\mathrm{Rey}(W)}(3)$ that gives an exact sequence:

$$0 \to S^2(\mathcal{S}_{G(2,E)})(-3) \to L^\vee \otimes \mathcal{O}_{G(2,E)}(-3) \to \mathcal{I}_{\mathrm{Rey}(W)} \to 0.$$

By Proposition 7.10.7, the homogeneous ideal I of $\mathrm{Rey}(W)$ in $\mathbb{P}^5 = \mathbb{P}(\bigwedge^2 E^\vee)$ is generated by 10 cubics. We have a 6-dimensional linear space of cubics of the form $V(ql)$, where $q = 0$ is the quadric $G(2, E)$ and $l \in E^\vee$. Together with the 4-dimensional space of Montesano cubics they generate I_3. This shows that the map $\mathcal{O}_{G(2,E)}(3) \to \mathcal{I}_{\mathrm{Rey}(W)}(3)$ is surjective and we get an exact sequence:

$$0 \to S^2(\mathcal{S}_{G(2,E)}) \to L^\vee \otimes \mathcal{O}_{G(2,E)} \to \mathcal{I}_{\mathrm{Rey}(W)} \to 0.$$

It remains to tensor it with $\mathcal{O}_{G(2,E)}(3)$. □

Now, let us look at the Fano–Cayley model. Consider the variety of singular quadrics in $\mathbb{P}^3$. It is a discriminant quartic hypersurface $\mathsf{D}(3)$ in $\mathbb{P}^9$. The variety $\mathsf{D}(3)_2$ of quadrics of corank 2 is known to be its singular locus and hence equal to the intersection of 10 cubic hypersurfaces defined by the partials of the discriminant quartic. The Cayley model is the intersection of $\mathsf{D}(3)_2$ with a 5-dimensional subspace in $\mathbb{P}^9$. So it is contained in the base scheme of a linear system of cubics of dimension ≤ 9. Taking the minimal resolution of the ideal of $\mathsf{D}(3)_2$ and restricting it to a transversal $\mathbb{P}^5$, we obtain the following resolution of the Fano–Cayley model $S = \mathrm{Cay}(W)$:

$$0 \to \mathcal{O}_{\mathbb{P}^5}(-5)^{\oplus 6} \to \mathcal{O}_{\mathbb{P}^5}(-4)^{\oplus 15} \to \mathcal{O}_{\mathbb{P}^5}(-3)^{\oplus 10} \to \mathcal{I}_S \to 0. \tag{7.10.5}$$

It shows that S is projectively normal and that the 10 partials of the discriminant cubic generate the sheaf of ideals $\mathcal{I}_S$ of S in $\mathbb{P}^5$.

Remark 7.10.9 The Fano–Cayley model is set-theoretically defined by 6 cubics, the partials of the quartic symmetroid in $\mathbb{P}^5$. The 6 cubics generate a non-saturated ideal of $\mathbf{S}(V^\vee)$, whose saturation is generated by the 10 partials of $\mathcal{D} \subset \mathbb{P}^9$ restricted to $\mathbb{P}^5$.

Let S be embedded in $\mathbb{P}^5 = |V|$ by an ample Fano polarization H and let $(\Lambda_1, \dots, \Lambda_{10})$ be the Fano planes defined by choice of a supermarking $(F_1, \dots, F_{10})$. Fix a basis of $\bigwedge^6 V \cong \mathbb{k}$ and consider the wedge-product pairing

$$\omega : \bigwedge^3 V \times \bigwedge^3 V \to \bigwedge^6 V \cong \mathbb{k}, \tag{7.10.6}$$

Choose a representative $v_i \in \bigwedge^3 V$ of a Fano plane $\Lambda_i \in G(3, V)$. Since $\Lambda_i \cap \Lambda_j \neq \emptyset$, we obtain that the 3-vectors $(v_1, \dots, v_{10})$ satisfy $v_i \wedge v_j = 0$. Suppose $K_S \neq 0$ and H is not a Fano–Reye polarization. Then, we have an opposite supermarking defined by the Fano planes $(\Lambda_{-1}, \dots, \Lambda_{-10})$. Let $(v_{-1}, \dots, v_{-10})$ be the corresponding lifts to $\bigwedge^3 V$. We have $v_i \wedge v_{-j} = 0, i \neq j$, and $v_i \wedge v_{-i} \neq 0$. Suppose we have a linear dependence $\lambda_1 v_1 + \cdots + \lambda_{10} v_{10} = 0$. Taking the wedge-product with v_{-i}, we get $\lambda_i = 0$. Thus, the 3-vectors $v_1, \dots, v_{10}$ are linearly independent. They generate a maximal isotropic subspace (a *Lagrangian subspace*) of $\bigwedge^3 V$, where $\bigwedge^3 V$ is equipped with a structure of a symplectic space by means of the wedge-product pairing. Thus, a supermarking of a Fano polarization of an Enriques surface defines a Lagrangian subspace A in the Plücker space of the Grassmannian $G(3, 6)$.

Let $V = \mathbb{C}^6$ and let A be a Lagrangian subspace of $\bigwedge^3 V$. To any $[x] = \mathbb{C}x \in \mathbb{P}(V)$, we can associate another Lagrangian subspace $x \wedge \bigwedge^2 V$ of $\bigwedge^3 V$. Following [208], one defines an *EPW-sextic hypersurface* as the degeneracy locus of points $[x]$ for which these two Lagrangian subspaces are not in general position:

$$X_A = \{[x] \in \mathbb{P}(V) : \left(x \wedge \bigwedge^2 V\right) \cap A \neq 0\}.$$

It is proven in [208] that X_A has a scheme structure of a sextic hypersurface in $\mathbb{P}(V)$ unless it is equal to the whole space. For A generic, its singular locus is the set

$$\operatorname{Sing} X_A = \{x \in X_A : \dim \left(\left(x \wedge \bigwedge^2 V\right) \cap A\right) \geq 2\}.$$

It is a smooth surface S_A of general type of degree 40 with Hilbert polynomial $126P_0 - 120P_1 + 40P_2$, where P_n stands for the Hilbert polynomial of $\mathbb{P}^n$. In [571] K. O'Grady proved that there exists a natural double cover

$$\pi : \hat{X}_A \to X_A$$

ramified along S_A which is an irreducible symplectic 4-fold, deformation equivalent to the Hilbert square of a K3 surface.

Remark 7.10.10 Note that the exact sequence

$$0 \to \Omega^2_{|V|} \to \bigwedge^3 V \otimes \mathcal{O}_{|V|} \to \Omega^3_{|V|} \to 0$$

allows one to identify the subspace $v \wedge \bigwedge^2 V$ with the fiber of $\Omega^2_{|V|}$ at the point $[v]$. Thus, one can redefine the EPW sextic as

$$X_A = \{x \in \mathbb{P}^5 : \dim \Omega^2_{|V|}(x) \cap A \geq 1\}.$$

Let

$$X_A[k] := \{x \in \mathbb{P}^5 : \dim \Omega_{|V|}(x) \cap A \geq k\}.$$

If A is not generic, we have the following proposition proved in [572].

Proposition 7.10.11 *Assume that X_A is not the whole $\mathbb{P}(V)$. Let $[v] \in X_A$. Then, X_A is smooth at $[v]$ if and only if $[v] \notin X_A[2]$ and A does not contain any decomposable form $v \wedge \omega$. In other words,* $\operatorname{Sing} X_A$ *is the union of $X_A[2]$ and the planes $\mathbb{P}(W)$, where W varies through all* 3*-planes of V such that $\bigwedge^3 W \subset A$.*

Proposition 7.10.12 *Let A be defined, as above, by a set of linearly independent Fano planes $\underline{\Lambda} = (\Lambda_1, \ldots, \Lambda_{10})$. Then, the singular locus of X_A is the union of the ten planes Λ_i from $\underline{\Lambda}$ and the degree* 40 *surface $Y := X_A[2]$.*

Finally, we consider an ample Fano polarization H of an Enriques surface S with $K_S = 0$. We know that any $\boldsymbol{\mu}_2$-surface that contains a nodal cycle R and a half-fiber F such that $F \cdot R = 3$ defines a Fano–Reye polarization $H = 2F + R$. Viewed as a smooth surface in $G_1(\mathbb{P}^3)$, it is a congruence of bidegree $(7, 3)$. We constructed such congruences in Section 7.6. Recall that the construction from Section 7.8 of such congruence using webs of quadrics in characteristic 2 leads to classical Enriques surfaces. To give explicit examples, we use that the Fano–Cayley and Fano–Reye polarizations coincide. Intersecting the variety $\mathsf{D}(3)_2$ of reducible quadrics in $\mathbb{P}^3$ by a general 5-dimensional subspace P, we obtain an Enriques surface of degree 10 in $\mathbb{P}^5$. It is contained in a nonsingular quadric in P, the intersection of P with the pfaffian hypersurface $\mathsf{D}(3)_1$ of singular quadrics in $\mathbb{P}^3$. It has an obvious étale canonical cover $X \to S$, the restriction of the double cover $\mathbb{P}^5 \times \mathbb{P}^5 \to \mathsf{D}(3)_2$.

We assume that H is not a Fano–Reye polarization. As in the above we consider S embedded in $\mathbb{P}^5 = |V|$. Since $\operatorname{Pic}(S) = \operatorname{Num}(S)$, a maximal isotropic 10-sequence defined by h_{10} defines, uniquely up to an order, a sequence $\underline{\Lambda} = (\Lambda_1, \ldots, \Lambda_{10})$ of Fano planes. The wedge-product pairing (7.10.6) becomes a symmetric bilinear pairing. We equip $\bigwedge^3 V$ with a non-degenerate quadratic form defined, with respect to a basis $e_1, \ldots, e_{10}$ by

$$q(\sum x_{i_1 i_2 i_3} e_{i_1} \wedge e_{i_2} \wedge e_{i_3}) = \sum x_{i_1 i_2 i_3} x_{j_1 j_2 j_3}, \tag{7.10.7}$$

where the summation is taken along the set of indices $i_1 < i_2 < i_3, j_1 < j_2 < j_3, i_1 < j_1, \{i_1, i_2, i_3\} \cap \{j_1, j_2, j_3\} = \emptyset$ (see [209, Lemma 8]).

Let $\mathrm{OG}(\bigwedge^3 V, q)$ be the orthogonal Grassmannian of Lagrangian (i.e. maximal isotropic) in $\bigwedge^3 V$ with respect to the quadratic form q. It consists of two irreducible

components of dimension 55; one of them consists of fibers of the vector bundle $\Omega^3_{\mathbb{P}^5}(3)$.

The following nice result can be found in [209, Theorem 8.6].

Theorem 7.10.13 *Assume $p = 2$. Let A be a general Lagrangian subspace in $OG(\bigwedge^3 V, q)$ from the irreducible component that does not contain fibers of $\Omega^3_{\mathbb{P}^5}(3)$. Then,*

$$Z_A := \{[v] \in |V| : \dim(v \wedge V) \cap A) \geq 3\}$$

is a smooth Fano model of a non-classical Enriques surface with symmetrically quasi-isomorphic resolutions

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{O}_{\mathbb{P}^5}(-6) \arrow[r] \arrow[d, equal] & \Omega^3_{\mathbb{P}^5} \arrow[r] \arrow[d] & A^\vee \otimes \mathcal{O}_{\mathbb{P}^5}(-3) \arrow[r] \arrow[d] & \mathcal{O}_{\mathbb{P}^5} \arrow[r] \arrow[d] & \mathcal{O}_{Z_A} \arrow[d, "\cong"] \\
0 \arrow[r] & \mathcal{O}_{\mathbb{P}^5}(-6) \arrow[r] & A^\vee \otimes \mathcal{O}_{\mathbb{P}^5}(-3) \arrow[r] & \Omega^3_{\mathbb{P}^5} \arrow[r] & \mathcal{O}_{\mathbb{P}^5} \arrow[r] & \omega_{Z_A}.
\end{tikzcd} \tag{7.10.8}$$

There is also the converse.

Theorem 7.10.14 *Assume $p = 2$. Let $S \subset \mathbb{P}^5 = |V|$ be a locally Gorenstein subscheme of dimension 2 and degree 10, with $\omega_S \cong \mathcal{O}_S$ and $h^i(\mathcal{O}_S) = 1$ for $i = 0, 1, 2$. Assume S is linearly normal and not contained in a quadric. Then, there exists a unique Lagrangian subspace $A \subset \bigwedge^3 V$ with respect to the quadratic form* (7.10.7) *such that $S = Z_A$ with locally free resolution* (7.10.8).

It follows from [209, Proposition 9.1] that a general surface given by this construction is a $\boldsymbol{\mu}_2$-surface and $\boldsymbol{\alpha}_2$-surfaces form a hypersurface in $\mathrm{OG}(\bigwedge^3 V, q)$ cut out by a $\mathrm{SL}(V)$-invariant hyperplane section in the Plücker embedding.

Bibliographical Notes

General facts about the Grassmannians of lines can be found in [242], [289], and [177]. We refer to classical monographs of C. Jessop [352] and R. Sturm [693] for fairly complete expositions of the theory of complexes and congruences of lines in $\mathbb{P}^3$. A new chapter of the second edition of the book [177] gives a modern exposition of this theory, including Kummer's classification of congruences of lines of $m = 1, 2$.

The Reye congruences were first discovered by G. Darboux [146] and were studied more extensively by T. Reye in [614, NeunzehnterVortrag]. In his terminology, a Reye line is a *Haupstrahl* of a Gebüsch (web) of quadrics. Curiously, A. Cayley, who introduced the quartic symmetroid surface in [115] did not notice the construction of the corresponding congruence of lines. The first modern exposition of the theory of Reye congruences can be found in [134]. Some general facts about the hyperwebs of quadrics can be found in [177] or [707]. Quartic symmetroids were used in Artin–Mumford's construction of the first examples of unirational but not rational threefolds [31]. The examples are birationally isomorphic to the double covers of $\mathbb{P}^3$ branched along a quartic symmetroid.

The extension of Reye's construction to characteristic 2 seems new.

The proof of the fact that a smooth congruence of order 7 and class 3 is isomorphic to an Enriques surface can be found in [263]. Although the fact that a general, in the sense of moduli, nodal Enriques surface is isomorphic to a Reye congruence follows from counting constants, the results from Section 7.9 are much more explicit. In particular, they show that a general nodal surface in the sense of the definition from Section 6.5 is isomorphic to a Reye congruence. Theorem 7.9.1, in a slightly weaker form, was proven in the case $p \neq 2$ by Cossec [134, Theorem 3.3.1].

In this chapter, we prove stronger forms of his result in arbitrary characteristic. The final result in this direction has been recently proved by G. Martin, G. Mezzedini and D. Veniani [493]: a classical nodal Enriques surface admits a Fano–Reye birational model.

The relationship between Reye congruences and exceptional rank 2 vector bundles on Enriques surfaces was studied in [194]. Here, one also finds a characterization of a Fano–Reye polarization among Fano polarization in terms of the variety of trisecant lines of the embedded surface.

The relationship between Fano polarizations and Lagrangian subspaces in the Plücker space of planes in $\mathbb{P}^5$ can be traced back to a paper of D. Eisenbud, S. Popescu, and C. Walters [209], where one can find a projective resolution for the homogeneous ideal of the Fano model of a non-classical Enriques surface S. Over fields of characteristic different from 2, there is a relationship between Fano models and Lagrangian subspaces that depends on a choice of a supermarking of S (see Section 5.7 in Volume 1).

Chapter 8
Automorphisms of Enriques Surfaces

Contrary to the case of K3 surfaces, a general Enriques surface has an infinite group of automorphisms, and there exist Enriques surfaces with an automorphism acting trivially on the Picard lattice. In this chapter, we discuss automorphisms of Enriques surfaces, including the description of the automorphism group of a general or a general nodal Enriques surface, automorphisms acting trivially on the Picard group, finite groups acting on Enriques surfaces by automorphisms, and Enriques surfaces with finite automorphism group.

8.1 General Facts

In this section, we collect general facts about the group scheme of automorphisms of a proper scheme over a field. We describe the identity component of the group scheme of automorphisms of an Enriques surface that turns out to be trivial if $p \neq 2$. We extend to an arbitrary characteristic the lattice-theoretical criterion for the finiteness of the automorphism group of an Enriques surface: W_S^{nod} is a subgroup of finite index of $W(\mathrm{Num}(S))$.

Let X be a proper scheme over a field $\Bbbk$. For any $\Bbbk$-scheme T, we consider the group of automorphisms $\mathrm{Aut}_T(X \times T)$ of $X_T := X \times_{\Bbbk} T$, considered as a scheme over T. It is easy to see that $T \to \mathrm{Aut}_T(X)$ defines a contravariant functor on the category of schemes over $\Bbbk$. It is known that this functor is representable by a group scheme $\mathbf{Aut}_{X/\Bbbk}$ of locally finite type over $\Bbbk$ [498]. The tangent space of $\mathbf{Aut}_{X/\Bbbk}$ is canonically identified with $\mathrm{Aut}_I(X_I)$, where $I = \Bbbk[t]/(t^2)$ is the algebra of dual numbers. There is also a canonical isomorphism of vector spaces $H^0(X, \Theta_X) \cong \mathrm{Aut}_I(X_I)$, where Θ is the tangent sheaf of X. When $\Bbbk$ is perfect, the restriction of the functor $T \to \mathrm{Aut}_T(X_T)$ to the category of reduced $\Bbbk$-schemes is representable by $(\mathbf{Aut}_{X/\Bbbk})_{\mathrm{red}}$. If $\mathrm{char}(\Bbbk) = 0$, all group schemes are reduced, so the automorphism scheme is reduced. Let $\mathbf{Aut}^0_{X/\Bbbk}$ be the connected component of identity. The group $\mathbf{Aut}_{X/\Bbbk}/\mathbf{Aut}^0_{X/\Bbbk}$ is at most countable, because X can be defined over a countable field.

Theorem 8.1.1 *Let S be an Enriques surface S. Then,* $\dim \mathbf{Aut}^0_{S/\Bbbk} = 0$. *If $p \neq 2$, or $p = 2$ and $H^0(S, \Theta_{S/\Bbbk}) = 0$ (e.g. S is a $\boldsymbol{\mu}_2$-surface), then $\mathbf{Aut}_{S/\Bbbk}$ is reduced and $\mathbf{Aut}^0_{S/\Bbbk}$ is trivial.*

Proof The second assertion follows immediately from the discussion above. In any case, we have $h^0(\Theta_{S/\Bbbk}) \leq 1$. Suppose $H^0(S, \Theta_{S/\Bbbk}) \neq 0$. If $\mathbf{Aut}_{S/\Bbbk}$ is reduced then $\mathbf{Aut}^0_{S/\Bbbk}$ is a one-dimensional connected algebraic group G over $\Bbbk$. There are three possibilities: $G = \mathbb{G}_m, \mathbb{G}_a$, or G is an elliptic curve. A connected algebraic group acts trivially on the Néron–Severi group of S. Since S has a non-trivial genus one fibration with some rational fibers, the group G preserves the set of singular fibers, and being connected, preserves any singular fiber. If G is an elliptic curve, then G must fix any point x on a singular fiber. Thus, G acts linearly on any $\mathfrak{m}^k_{S,x}/\mathfrak{m}^{k+1}_{S,x}$, and being complete, it acts trivially. This implies that G acts trivially on the completion of the local ring $\mathcal{O}_{S,x}$, hence on the ring itself, hence on its fraction ring, hence on S.

Suppose G is a linear algebraic group acting on an irreducible algebraic variety X. Then, by Rosenlicht's Theorem (see, for example, [174], Theorem 6.2), there exists a G-invariant open subset U of X such that the geometric quotient $U \to U/G$ exists and its fibers are orbits of G. In particular, when X is a surface, U, and hence X, must be a ruled surface of Kodaira dimension $-\infty$. Applying this to S, we find a contradiction. □

Next, we assume that $H^0(S, \Theta_{S/\Bbbk}) \neq \{0\}$. By Corollary 1.4.9, $\dim H^0(S, \Theta_{S/\Bbbk}) = 1$. Also, we know that this happens if and only if S is a $\boldsymbol{\alpha}_2$-surface or an exceptional classical surface (see Section 1.4 in Volume I).

If ∂ is a non-zero regular vector field on S, then S has an action of a finite group scheme G isomorphic to $\boldsymbol{\mu}_2$ or $\boldsymbol{\alpha}_2$ dependent on whether ∂ is of multiplicative or additive type. The quotient $S/G = S^\partial$ is a normal surface and the Frobenius morphism $S \to S^{(2)}$ factors through the quotient morphism $\tau : S \to S/G$. Since the Frobenius morphism is finite, the morphism $\tau' : S/G \to S^{(2)}$ is a finite inseparable morphism of degree 2.

Proposition 8.1.2 *Suppose that S is an $\boldsymbol{\alpha}_2$-surface and ∂ has only isolated zeros. Then, $(S^\partial)^{(1/2)} \to S$ is the canonical cover of S.*

Proof Let $Y = S^\partial$. Applying Proposition 0.3.14 to the morphism $\tau : S \to S^\partial$, we obtain that $\tau^*(\omega_Y) \cong \mathcal{O}_S$. On the other hand, the cover $\pi : Y \to S^{(2)} \cong S$ is an inseparable cover of degree 2 and $\omega_Y \cong \pi^*(\mathcal{L})$ for some invertible sheaf $\mathcal{L}$ which is a part of the data defining the cover. We have $(\pi \circ \tau)^*(\mathcal{L}) \cong \mathcal{O}_S$. Since $\mathrm{Pic}(S)$ has no torsion, applying Proposition 0.2.14, we obtain that $\mathcal{L} \cong \mathcal{O}_S$. Thus, π is defined by an element of $H^1(S, \boldsymbol{\alpha}_2)$, and hence it is a principal $\boldsymbol{\alpha}_2$-cover. Since it is unique, up to isomorphism, π must be the canonical cover. □

We will discuss later in Section 10.3 possible regular vector fields ∂ on normal canonical covers X of S. The quotient X^∂ could be classical or $\boldsymbol{\alpha}_2$-surfaces.

However, we have the following result proven in [491, Corollary 1.2].

Theorem 8.1.3 *Suppose S is a generic $\boldsymbol{\alpha}_2$-surface. Then, ∂ is of multiplicative type and* $\mathbf{Aut}^0_{S/\mathbb{k}} \cong \boldsymbol{\mu}_2$.

Let ∂ be a non-zero regular vector field of multiplicative type on S. Then, S must be one of the exceptional Enriques surface which we discussed in Section 6.2. Suppose S is not exceptional, then it is an $\boldsymbol{\alpha}_2$-surface. Under the isomorphism $\Theta_{S/\mathbb{k}} \to \Omega^1_{S/\mathbb{k}}$, the scheme $Z(\partial)$ is isomorphic to the scheme of zeros $Z(\omega)$ of a regular 1-form ω on S. By Proposition 1.3.8, the divisorial part of $Z(\omega)$ is not reduced. Thus, ∂ has only isolated simple zeros. Applying the previous proposition, we obtain that the canonical cover has only ordinary double points. Since $c_2(S) = 12$, there are 12 of them, counting with multiplicities. Let us record this information.

Corollary 8.1.4 *Let S be an Enriques surface with* $\mathbf{Aut}^0_{S/\mathbb{k}} \cong \boldsymbol{\mu}_2$. *Suppose it is not an exceptional Enriques surface. Then, S is an $\boldsymbol{\alpha}_2$-surface and its canonical cover is a normal surface with 12 ordinary double points, counting with multiplicities.*

Remark 8.1.5 In the case of exceptional Enriques surfaces, the group $\mathbf{Aut}^0_{S/\mathbb{k}}$ has been recently computed by G. Martin (unpublished).

1. If S is of type $\tilde{E}_6$, then $\mathbf{Aut}^0_{S/\mathbb{k}} \cong \boldsymbol{\mu}_2$.
2. If S is of type $\tilde{E}_7$, then $\mathbf{Aut}^0_{S/\mathbb{k}} \cong \boldsymbol{\alpha}_2$.
3. If S is of type $\tilde{E}_8$, then $\mathbf{Aut}^0_{S/\mathbb{k}} \cong \boldsymbol{\mu}_2$ or $\boldsymbol{\alpha}_4$.

Since $\mathbf{Aut}_{S/\mathbb{k}}(S)(\mathbb{k}) \cong (\mathbf{Aut}_{S/\mathbb{k}}(S)/\mathbf{Aut}^0_{S/\mathbb{k}})(\mathbb{k}) = \mathrm{Aut}(S)$ we will concentrate now on the structure of the automorphism group of S. Since S is a minimal surface, this group is isomorphic to the group of automorphisms of S. A useful way to study the automorphism group is to look at its representations in the orthogonal groups of $\mathrm{Pic}(S)$ and $\mathrm{Num}(S)$

$$\rho : \mathrm{Aut}(S) \to \mathrm{O}(\mathrm{Num}(S)), \tag{8.1.1}$$

$$\tilde{\rho} : \mathrm{Aut}(S) \to \mathrm{O}(\mathrm{Pic}(S)). \tag{8.1.2}$$

It is natural to ask about the structures of the image $\mathrm{Aut}(S)^*$ of ρ and the kernels of ρ and $\tilde{\rho}$. We will deal with these questions in subsequent sections.

We know that any Enriques surface has a genus one fibration $f : S \to \mathbb{P}^1$ whose jacobian fibration $j : J \to \mathbb{P}^1$ is a rational genus one surface. Let $\mathrm{MW}(j)$ be the Mordell–Weil group of j. It acts by translations on the generic fiber of f, and this action extends to a biregular action on S that preserves all fibers of the fibration.

We will be extending some results about finite automorphism groups of Enriques surfaces in characteristic 0 to the case of arbitrary characteristic of the ground field. The following result of J.-P. Serre [663] shows that nothing new appears if $p \neq 2$ and the order of the group is coprime to p.

Theorem 8.1.6 *Let $W(\mathbb{k})$ be the ring of Witt vectors with algebraically closed residue field $\mathbb{k}$, and let X be a smooth projective variety over $\mathbb{k}$, and let G be a finite automorphism group of X. Assume:*

- *#G is prime to* $\mathrm{char}(\Bbbk)$*;*
- $H^2(X, O_X) = 0$*;*
- $H^2(X, \Theta_X) = 0$*, where* Θ_X *is the tangent sheaf of* X*.*

Then, the pair (X, G) *can be lifted to* $W(\Bbbk)$*, i.e. there exists a smooth projective scheme* $\mathcal{X} \to \mathrm{Spec}\, W(\Bbbk)$ *with special fiber isomorphic to* X *and an action of* G *on* $\mathcal{X}$ *over* $W(\Bbbk)$ *such that the induced action of* G *in* X *coincides with the action of* G *on* X*.*

If $X = S$ is a classical Enriques surface, we have

$$H^2(S, \Theta_S) \cong H^0(S, \Omega^1_S(K_S)) \cong H^0(S, \Theta_S)$$

The theorem also applies to this case when $p = 2$ and $H^0(S, \Theta_S) = 0$.

Recall that, by Theorem 5.5.1 from Volume I, the fundamental domain of the group of automorphisms of a complex Enriques surface in its nef cone is a rational convex polyhedron. Equivalently, the group $W_S^{\mathrm{nod}} \rtimes \mathrm{Aut}(S)$ is of finite index in $W(\mathrm{Num}(S))$.

The following theorem from [454] extends the latter result to the case of positive odd characteristic.

Theorem 8.1.7 *Let* X *be a K3 surface over a field* $\Bbbk$ *of characteristic* $p \geq 3$*. Then,* $W_X^{\mathrm{nod}} \rtimes \mathrm{Aut}(X)$ *is a subgroup of finite index in* $\mathrm{O}(\mathrm{Pic}(X))$*.*

We shall show how their arguments can be used to extend this result to Enriques surfaces.

The following fact was proven earlier in [566] using the theory of quasi-canonical lifts of K3 surfaces to characteristic zero. A simpler proof can be found in [454, Corollary 4.2].

Proposition 8.1.8 *Let* X *be a K3 surface of finite height. Then, there exists a lifting* $\mathcal{X} \to \mathrm{Spec}\, W(\Bbbk)$ *to the ring of Witt vectors with generic geometric fiber* $X_{\bar{K}}$ *and special fiber isomorphic to* X *such that the homomorphism of specializations of the Picard groups*

$$sp : \mathrm{Pic}(\mathcal{X}_{\bar{K}}) \to \mathrm{Pic}(X) \tag{8.1.3}$$

is an isomorphism of lattices.

We apply this result to the case when $\pi : X \to S$ is the canonical cover of S. Then, $\mathrm{Pic}(X)$ contains the sublattice $\pi^*(\mathrm{Num}(S)) \cong \mathsf{E}_{10}(2)$ such that the orthogonal complement does not contain numerical divisor classes of (-2)-curves. Under specialization isomorphism (8.1.3), $\mathrm{Pic}(X_{\bar{K}})$ contains a sublattice isomorphic to $\mathsf{E}_{10}(2)$ such that the orthogonal complement does not contain vectors of square norm -2. Next, we use the following result due to J. Jang [347, Theorem 2.5] (see [383, Theorem 1] for the case $\Bbbk = \mathbb{C}$).

Proposition 8.1.9 *A K3 surface* X *of finite height over a field* $\Bbbk$ *of odd characteristic is isomorphic to the canonical cover of an Enriques surface if and only if* $\mathrm{Pic}(X)$ *contains a primitive sublattice isomorphic to* $\mathsf{E}_{10}(2)$ *that does not contain divisor classes of* (-2)*-curves in its orthogonal complement.*

It follows from this theorem that the lift of the canonical cover of an Enriques surface from Proposition 8.1.8 is the canonical cover of an Enriques surface $S_{\bar{K}}$ over $\bar{K}$. Since $p \neq 2$, the Enriques involution τ lifts to an involution $\tau_{\mathcal{X}}$ of $\mathcal{X}$ too [348, Theorem 3.2]. Since τ is fixed-point-free, the involution $\tau_{\mathcal{X}}$ is also fixed-point-free, and hence it restricts to an Enriques involution $\tau_{\bar{K}}$ of $X_{\bar{K}}$.

After embedding $\bar{K}$ into $\mathbb{C}$, we obtain that $\mathcal{X}_{\bar{K}}$ is a complex K3 surface that admits an ample lattice $\mathsf{E}_{10}(2)$ polarization. Replacing K by some finite extension L and $W(\Bbbk)$ by its normal closure A in L, we may assume that $\tau_{\bar{K}}$ is defined over K, and denote it by τ_K. Let X_K denote the generic fiber of $f : \mathcal{X}_A \to \operatorname{Spec} A$. Any non-trivial automorphism g of X_K extends to a birational automorphism of $\mathcal{X}_A$ over A. Since, $H^0(X, \Theta_X) = \{0\}$, the restriction homomorphism $\operatorname{Aut}(\mathcal{X}) \to \operatorname{Aut}(X)$ defines an injective homomorphism

$$\sigma : \operatorname{Aut}(X_{\bar{K}}) \to \operatorname{Aut}(X)$$

such that the specialization map (8.1.3) is σ-invariant with respect to the natural action on the Picard lattices [454, Theorem 2.1]. By definition of the lift, $\sigma(\tau_K) = \tau$. Passing to the quotient schemes by the involutions τ_K and τ, and the invariant sublattices $\operatorname{Num}(X_K)^{\tau_K}$ and $\operatorname{Num}(X)^{\tau}$ (both isomorphic to $\mathsf{E}_{10}(2)$), we obtain a lift $\bar{f} : \mathcal{S} \to \operatorname{Spec}(W)$ of our Enriques surface S. The group $\operatorname{Aut}(\mathcal{S}_{\bar{K}})$ (resp. $\operatorname{Aut}(S)$) coincides with the centralizer subgroup of τ_K (resp. τ). Hence, we obtain an injective homomorphism

$$\bar{\sigma} : \operatorname{Aut}(\mathcal{S}_{\bar{K}}) \hookrightarrow \operatorname{Aut}(S).$$

It is known that the specialization homomorphism (8.1.3) preserves the nef cones [454, Corollary 2.4] and hence defines an isomorphism of the reflection groups $W^{\mathrm{nod}}_{\mathcal{X}_{\bar{K}}} \to W^{\mathrm{nod}}_{X}$. Since any reflection s_r in the class of a (-2)-curve on S (resp. $\mathcal{S}_{\bar{K}}$) lifts to the product of two reflections $s_{r'} \circ s_{\tau^*(r')}$ (resp. $s_{r'} \circ s_{\tau^*_K(r')}$) on its K3-cover, we obtain that the specialization isomorphism $\operatorname{Num}(\mathcal{S}_{\bar{K}}) \to \operatorname{Num}(S)$ defines an isomorphism $W^{\mathrm{nod}}_{\mathcal{S}_{\bar{K}}} \to W^{\mathrm{nod}}_{S}$. After we embed $\bar{K}$ into $\mathbb{C}$ and apply Theorem 5.5.1 we obtain that $W^{\mathrm{nod}}_{\mathcal{S}_{\bar{K}}} \rtimes \operatorname{Aut}(\mathcal{S}_{\bar{K}})$ is a subgroup of finite index in $W_{\mathcal{S}_{\bar{K}}}$, and, therefore, $W^{\mathrm{nod}}_{S} \rtimes \operatorname{Aut}(S)$ is a subgroup of finite index in $W(\operatorname{Num}(S))$.

This finishes the proof of the following theorem under the assumption that the canonical cover $X \to S$ is of finite height.

Theorem 8.1.10 *Assume that $p \neq 2$. Then, $W^{\mathrm{nod}}_{S} \rtimes \operatorname{Aut}(S)$ is a subgroup of finite index in $W(\operatorname{Num}(S))$.*

In the case when X is supersingular the proof was provided in [731]. It uses Ogus' Global Torelli Theorem in crystalline cohomology.

Arguing as in the case $\Bbbk = \mathbb{C}$ (see in Section 5.5), we extend Corollaries 5.5.2, 5.5.3, 5.5.4 to the case of Enriques surfaces over a field of arbitrary characteristic $p \neq 2$.

Corollary 8.1.11 *Assume $p \neq 2$ or S is a $\boldsymbol{\mu}_2$-surface. Then, the following assertions are true.*

(i) $\mathrm{Aut}(S)$ *is a finitely generated group.*
(ii) In its action on the nef cone of S, the group $\mathrm{Aut}(S)$ *admits a rational convex polyhedral fundamental domain.*
(iii) The group $\mathrm{Aut}(S)$ *has only finitely many orbits on the sets of smooth rational curves, elliptic fibrations and nef divisor classes D with fixed* $D^2 > 0$.

In the next theorem, we do not need any assumption on the characteristic. Note that for the proof we use the classification of Enriques surfaces with finite automorphism group given in Sections 8.9 and 8.10.

Theorem 8.1.12 *The following assertions are equivalent:*

(i) The group $\mathrm{Aut}(S)$ *is finite.*
(ii) W_S^{nod} *is a subgroup of finite index in* $W(\mathrm{Num}(S))$.
(iii) $\mathrm{Num}(S)$ *contains a crystallographic basis formed by the classes of* (-2)*-curves.*
(iv) The set of smooth rational curves on S is finite.
(v) Any genus one fibration on S is an extremal fibration, i.e. the Mordell–Weil group of its jacobian fibration is finite.
(vi) The set of genus one fibrations on S is finite and it consists of extremal fibrations.

Proof Let $\mathbb{H}^9$ be the hyperbolic space associated with $\mathrm{Num}(S)_{\mathbb{R}}$. We know from Section 0.8 that $W(\mathrm{Num}(S)) = W_S^{\mathrm{nod}} \rtimes A(P)$, where P is a fundamental domain for the reflection group W_S^{nod}. By Proposition 0.8.18, the subgroup W_S^{nod} of $W(\mathrm{Num}(S))$ is of finite index if and only if the set of (-2)-curves on S forms a crystallographic basis. This proves the equivalence of assertions (ii) and (iii).

The assertion (iii) follows from (iv) immediately since the reflection polytope has finitely many faces, and hence it is of finite volume. By definition, if (ii) is true, the reflection polytope has only finitely many faces, and hence there exists a finite crystallographic basis of (-2)-curves. Any other such curve intersects any element of this basis non-negatively, hence belongs to the fundamental polytope (in other words, it belongs to the nef cone $\mathrm{Nef}(S)$). But this implies that its self-intersection is non-negative, a contradiction. So, we have proved that the assertions (ii), (iii), (iv) are equivalent.

By Vinberg's criterion 0.8.23, a surface with finitely many (-2)-curves contains only finitely many connected parabolic subdiagrams formed by (-2)-curves and all of them are parts of a parabolic subdiagram of the maximum possible rank. This implies that such a surface contains only finitely many genus one fibrations and all these fibrations are extremal. Thus, (iii) implies (vi).

Suppose (ii) is true. The group $\mathrm{Aut}(S)^*$ leaves invariant the set of (-2)-curves on S and hence it is contained in $A(P)$. Since W_S^{nod} is of finite index in $W(\mathrm{Num}(S))$, we see that $\mathrm{Aut}(S)^*$ is a finite group. Unfortunately, if $p = 2$, we do not know a direct proof of the implication (i) $\Rightarrow$ (ii). In the complex case this implication was Corollary 5.5.4 of the Global Torelli Theorem, and in the case $p \neq 2$, we can use Corollary 8.1.11. If $p = 2$, we prove the converse by classifying Enriques surfaces with finite automorphism group in Sections 8.9 and 8.10. The classification uses the obvious necessary condition (v) for finiteness of $\mathrm{Aut}(S)$ and starting from this finds all possible surfaces which may contain a crystallographic basis of (-2)-curves. So,

it proves that (v) implies (iii), and hence (i) implies (iii). Obviously, (vi) implies (v). □

8.2 Numerically and Cohomologically Trivial Automorphisms

Contrary to the case of a K3 surface, a complex Enriques surface may admit automorphisms that act identically on integral (resp. rational cohomology) that coincide with the Picard group $\mathrm{Pic}(S)$ (resp. $\mathrm{Pic}(S) \otimes \mathbb{Q}$). In this section, following [191], we describe Enriques surfaces over an algebraically closed field of arbitrary characteristic p that admit such automorphisms. In the case where $p \neq 2$, we give a complete classification of such surfaces.

An automorphism from $\mathrm{Ker}(\tilde{\rho})$ (resp. $\mathrm{Ker}(\rho)$) is called *cohomologically trivial* (resp. *numerically trivial*). The reason for this terminology is the fact that $\mathrm{NS}(S) \otimes \mathbb{Z}_\ell = H^2_{\text{ét}}(S, \mathbb{Z}_\ell)$ (resp. $\mathrm{NS}(S) \otimes \mathbb{Z}_p = H^2_{\text{fl}}(S, \mathbb{Z}_p[1])$. We let

$$\mathrm{Aut}_{\mathrm{ct}}(S) = \mathrm{Ker}(\tilde{\rho}), \quad \mathrm{Aut}_{\mathrm{nt}}(S) = \mathrm{Ker}(\rho).$$

We start with the following.

Proposition 8.2.1 *The groups* $\mathrm{Aut}_{\mathrm{ct}}(S)$ *and* $\mathrm{Aut}_{\mathrm{nt}}(S)$ *are finite groups.*

Proof This applies to any surface S with trivial $\mathbf{Aut}^0_{S/\Bbbk}(\Bbbk)$ and trivial $\mathbf{Pic}^0(S)$. In fact, the latter condition implies that $\mathrm{NS}(S) = \mathrm{Pic}(S)$ and $\mathrm{Num}(S)$ is the quotient of $\mathrm{NS}(S)$ by a finite group A. It follows from the theory of abelian groups that

$$\mathrm{O}(\mathrm{NS}(S)) \cong \mathrm{Hom}(\mathrm{Num}(S), \mathrm{Tors}(\mathrm{NS}(S))) \rtimes \mathrm{O}(\mathrm{Num}(S)).$$

This implies that

$$\mathrm{Aut}_{\mathrm{nt}}(S)/\mathrm{Aut}_{\mathrm{ct}}(S) \subset \mathrm{Tors}(\mathrm{NS}(S))^{\oplus \rho(S)}. \tag{8.2.1}$$

So, it is enough to prove that $G = \mathrm{Aut}_{\mathrm{ct}}(S)$ is a finite group. The group acts trivially on $\mathrm{Pic}(S)$, hence leaves invariant the isomorphism class of any very ample invertible sheaf $\mathcal{L}$. For any $g \in G$, let $\alpha_g : g^*(\mathcal{L}) \to \mathcal{L}$ be an isomorphism. Define a structure of a group on the set $\tilde{G}$ of pairs (g, α_g) by

$$(g, \alpha_g) \circ (g', \alpha_{g'}) = (g \circ g', \alpha_{g'} \circ g'^*(\alpha_g)).$$

The homomorphism $(g, \alpha_g) \to g$ defines an isomorphism $\tilde{G} \cong \Bbbk^\times \rtimes G$. The sheaf $\mathcal{L}$ admits a natural $\tilde{G}$-linearization, and hence the group $\tilde{G}$ acts linearly on the space $H^0(S, \mathcal{L})$ and the action defines a homomorphism $G \to \mathrm{Aut}(\mathbb{P}(H^0(S, \mathcal{L})))$. The group of projective transformations of S, embedded by $|\mathcal{L}|$, is a linear algebraic group that has finitely many connected components. We know that the connected component of the identity is trivial. Thus, the group G is finite. □

In the case when S is an Enriques surface, the torsion group $\mathrm{Tors}(\mathrm{NS}(S))$ is of order ≤ 2, hence we get the following.

Corollary 8.2.2 *The quotient group* $\mathrm{Aut}_{\mathrm{nt}}(S)/\mathrm{Aut}_{\mathrm{ct}}(S)$ *is a* 2*-torsion group.*

Let $f : S \to \mathsf{D}$ be a bielliptic map of degree 2 defined by a non-degenerate or degenerate U-pair as defined in 3.3. Assume that f is separable. Then, the birational deck involution extends to a biregular involution of S. We call it a *bielliptic involution*.

Assume that f is given by a non-degenerate U-pair (F_1, F_2) so that D is one of the three possible anti-canonical quartic del Pezzo surfaces

$$\begin{aligned}
\mathsf{D}_1 &: x_0^2 + x_1x_2 = x_0^2 + x_3x_4 = 0,\\
\mathsf{D}_2 &: x_0^2 + x_1x_2 = x_3x_1 + x_4(x_0 + x_2 + x_4) = 0,\\
\mathsf{D}_3 &: x_0^2 + x_1x_2 = x_3x_1 + x_4(x_2 + x_4) = 0,
\end{aligned}$$

that depend on whether S is a classical, $\boldsymbol{\mu}_2$- or an $\boldsymbol{\alpha}_2$-surface.

Recall that Proposition 0.6.25 and its proof give an explicit description of the connected component of the group of automorphisms of the surface D. For the sake of convenience, let us recall it.

- Action of $\mathrm{Aut}(\mathsf{D}_1)^0$:

$$[x_0, x_1, x_2, x_3, x_4] \mapsto [x_0, \lambda x_1, \lambda^{-1}x_2, \mu x_3, \mu^{-1}x_4]$$

- Action of $\mathrm{Aut}(\mathsf{D}_2)^0$:

$$\begin{aligned}
[x_0, x_1, x_2, x_3, x_4] \mapsto [x_0 + \alpha x_1, x_1, \alpha^2 x_1 + x_2, \beta x_0 + (\alpha\beta + \alpha^2\beta + \beta^2)x_1\\
+\beta x_2 + x_3 + (\alpha + \alpha^2)x_4, \beta x_1 + x_4]
\end{aligned}$$

- Action of $\mathrm{Aut}(\mathsf{D}_3)^0$:

$$\begin{aligned}
&[x_0, x_1, x_2, x_3, x_4] \mapsto [x_0 + \alpha x_1, x_1, \alpha^2 x_1 + x_2, (\alpha^2\beta + \beta^2)x_1 + \beta x_2 + x_3 + \alpha^2 x_4, \beta x_1 + x_4]\\
&[x_0, x_1, x_2, x_3, x_4] \mapsto [x_0, \lambda^{-1}x_1, \lambda x_2, \lambda^3 x_3, \lambda x_4]
\end{aligned}$$

By definition, any $g \in \mathrm{Aut}_{\mathrm{nt}}(S)$ leaves the divisor classes $[2F_1]$ and $[2F_2]$ invariant, and hence leaves the linear system $|2F_1 + 2F_2|$ invariant. Thus, it descends to a projective automorphism $\bar{g}$ of the surface D_i leaving invariant the pencils of conics. Also, if $g \in \mathrm{Aut}_{\mathrm{ct}}(S)$, then it preserves F_1 and F_2, hence $\bar{g}$ leaves invariant the lines on D_i that give rise to the half-fibers F_1, F_2, and hence belongs to the connected component $\mathrm{Aut}(\mathsf{D}_i)^0$.

The known information about the automorphism group of the surfaces D_i allows us to give a criterion for an automorphism to be a bielliptic involution.

Proposition 8.2.3 *Let* (F_1, F_2) *be a non-degenerate* U*-pair and let* g *be a non-trivial automorphism of* S*. Assume that* g *preserves* F_1*,* F_2 *and a* (-2)*-curve* R *with* $R \cdot F_1 = R \cdot F_2 = 0$*, which is not an irreducible component of one of the half-fibers*

F_1, F_2, F_1', F_2'. If S is an $\boldsymbol{\alpha}_2$-surface, assume additionally that g has order 2^n. Then, g is the bielliptic involution associated to the linear system $|2F_1 + 2F_2|$.

Proof Let $\phi : S \to \mathsf{D}_i$ be a bielliptic map defined by the linear system $|2F_1 + 2F_2|$. Since g leaves $|2F_1 + 2F_2|$ invariant, it descends to an automorphism of $\mathbb{P}^4 = |2F_1 + 2F_2|^*$ that leaves D_i invariant. Moreover, the induced automorphism preserves the lines on D_i by assumption. Recall that $E \cdot F_1 = E \cdot F_2 = 0$, hence $\phi(E)$ is a point P. Since E is not a component of one of the half-fibers, P does not lie on any of the lines of D. If $\mathsf{D}_i = \mathsf{D}_1$, this means that P is not on the hypersurface $x_0 = 0$ and if $\mathsf{D}_i = \mathsf{D}_2$ or $\mathsf{D}_3\}$, it means that P is not on the hypersurface $x_1 = 0$.

If $\mathsf{D}_i = \mathsf{D}_1 = 1$, the x_0 coordinate $x_0(P)$ of P is non-zero, the equations of D_1 from Corollary 0.6.14 show that all $x_i(P)$ are non-zero. The explicit description of $\mathrm{Aut}(\mathsf{D}_1)^0$ shows that the group has no fixed points outside the union of the four lines. Therefore, g coincides with the covering involution of ϕ.

If $\mathsf{D}_i = \mathsf{D}_2$ or D_3, we have $x_1(P) \neq 0$. Again, using the explicit description of $\mathrm{Aut}(\mathsf{D}_2)^0$, there is no automorphism of D_2 fixing P and preserving the lines except the identity. For D_3, we use the additional assumption to exclude the case that g acts on D_3 via $\mathbb{G}_m$. □

Remark 8.2.4 In fact, as we will see later, the failure of this criterion without the additional assumption in the $\boldsymbol{\alpha}_2$-case leads to the existence of cohomologically trivial automorphisms of odd order.

Lemma 8.2.5 *Let τ be the bielliptic involution associated to a bielliptic linear system $|2F_1 + 2F_2|$. Suppose τ is numerically trivial. Then,* $\mathrm{Num}(S)_{\mathbb{Q}}$ *is spanned by the numerical classes $[F_1], [F_2]$ and eight smooth rational curves that are contained in members of both $|2F_1|$ and $|2F_2|$.*

Proof The bielliptic map defines a degree 2 finite morphism $S' = S - \mathcal{E} \to \mathsf{D}_i' = \mathsf{D}_i - \mathcal{P}$, where $\mathcal{E}$ is the union of (-2)-curves blown down to a finite set of points $\mathcal{P}$ on D_i. We have $\mathrm{Pic}(\mathsf{D}_i')_{\mathbb{Q}} = \mathrm{Pic}(\mathsf{D}_i)_{\mathbb{Q}}$ and $\mathrm{Pic}(S')^g_{\mathbb{Q}}$ (the invariant part) $= f^*(\mathrm{Pic}(\mathsf{D}_i')_{\mathbb{Q}})$ is spanned by the restriction of F_1, F_2 to S'. Since $\mathrm{Pic}(S)$ is spanned by $\mathrm{Pic}(S')$ and the classes of components of $\mathcal{E}$, we can write any invariant numerical divisor class as a linear combination of $[F_1], [F_2]$ and invariant irreducible components of $\mathcal{E}$. In our case, all divisors classes are invariant. Since $\dim(\mathrm{Pic}(S)_{\mathbb{Q}}) - \dim(\langle [F_1], [F_2] \rangle_{\mathbb{Q}}) = 8$, $\mathcal{E}$ consists of eight (-2)-curves. □

Corollary 8.2.6 *Let (F_1, F_2) be a non-degenerate U-pair such that the bielliptic involution τ associated to $|2F_1 + 2F_2|$ is numerically trivial. Then, $|2F_1|$ and $|2F_2|$ are extremal genus one pencils on S.*

Moreover, the following hold:

(1) For every fiber D of $|2F_1|$, all but one component C of D is contained in fibers of $|2F_2|$.
(2) C is a component in the fiber of multiplicity at most 2.
(3) Neither $|2F_1|$ nor $|2F_2|$ have a singular fiber of multiplicative type with more than two components.

Proof By the previous lemma, there are eight (-2)-curves contained in fibers of both $|2F_1|$ and $|2F_2|$. Since a fiber of $|2F_1|$ cannot contain a full fiber of $|2F_2|$, this implies $8 \leq \sum_{D\in|2F|}(b_2(D)-1) \leq 8$, where $b_2(D)$ is the number of irreducible component of D. Hence, by the Shioda–Tate formula (4.3.2), $|2F_1|$ is extremal and so is $|2F_2|$. Moreover, if, for some fiber D of $|2F_1|$, two components of D are not contained in fibers of $|2F_2|$, then, by the same formula, $|2F_1|$ and $|2F_2|$ share less than eight (-2)-curves. This contradicts Lemma 8.2.5.

For (2), note that the remaining component C of multiplicity m in D satisfies $2 = D \cdot F_2 = mC \cdot F_2$. Since $C \cdot F_2 > 0$, this yields (2).

As for (3), assume that D is a singular fiber of multiplicative type with more than two irreducible components. Note that C meets distinct points on distinct irreducible components of D. The connected divisor $D' = D - C$ satisfies $D' \cdot (2F_1 + 2F_2) = 0$, hence it is contained in the exceptional locus of the bielliptic map ϕ. Since τ preserves the irreducible components of D', $\phi(C)$ is an irreducible curve with a node. But C is contained in the pencil of conics induced by $|2F_1|$. This is a contradiction. □

After these preliminary results, we are ready to prove our main results. We start with cohomologically trivial automorphisms of even order. The following result is Theorem 7.1 from [191].

Theorem 8.2.7 *Let S be an Enriques surface which is not extra-special.*

(1) If S is classical or a $\boldsymbol{\mu}_2$-surface, then $|\operatorname{Aut}_{\mathrm{ct}}(S)| \leq 2$. If S is also unnodal, then $\operatorname{Aut}_{\mathrm{ct}}(S) = \{1\}$.

(2) If S is an $\boldsymbol{\alpha}_2$-surface, then the statements of (1) hold for the 2-Sylow subgroup G of $\operatorname{Aut}_{\mathrm{ct}}(S)$.

Moreover, if a non-trivial $g \in \operatorname{Aut}_{\mathrm{ct}}(S)$ (resp. $g \in G$) exists, then g is a bielliptic involution.

Proof Let $g \in \operatorname{Aut}_{\mathrm{ct}}(S)$ and assume that g has order 2^n if S is an $\boldsymbol{\alpha}_2$-surface. We will show that there is a U-pair such that g satisfies the conditions of Proposition 8.2.3.

Take a c-degenerate isotropic 10-sequence on S with c maximal. If $3 \leq c \leq 9$, then there is a (-2)-curve R in this sequence such that $F \cdot R = 0$ for at least three half-fibers F in the sequence. Now, Lemma 6.2.2 shows that R is contained in a simple fiber of two pencils $|2F_1|$ and $|2F_2|$. By Proposition 8.2.3, g is the bielliptic involution associated to $|2F_1 + 2F_2|$. In particular, g is unique.

If $c = 10$, assume that one of the half-fibers, say F_1, is reducible. Then, by Lemma 6.2.2, for every F_i in the sequence, all but one component of F_1 is contained in simple fibers of $|2F_i|$. Hence, we find some component R with $R \cdot F_i = 0$ for at least three half-fibers and the same argument as before applies.

If $|F_i + F_j - F_k| \neq \emptyset$ or $|F_i + F_j - F_k + K_S| \neq \emptyset$ for some half-fibers F_i, F_j, F_k that occur in the sequence, then by Remark 7.9.6 after Lemma 7.9.5, there is an effective divisor D with $D \cdot F_i = D \cdot F_j = 0$ and $D^2 = -2$. Since F_i and F_j can be assumed to be irreducible, D contains a (-2)-curve which is contained in a simple fiber of both $|2F_i|$ and $|2F_j|$. Again, Proposition 8.2.3 applies.

Therefore, we can assume that all half-fibers are irreducible and $F_i \cap F_j \cap F_k = \emptyset$ by Lemma 7.9.5. This is immediate if S is unnodal. Then, g fixes all F_i pointwise. This could happen only if $p = 2$ since the locus of fixed points is smooth if $p \neq 2$. But in the case $p = 2$, the generic fiber of the genus one fibration defined by the pencil $|2F_1|$ has four fixed points, two coming from F_2 and two coming from F_3. It follows from the description of an automorphism group of an elliptic curve or a cuspidal curve that this does not happen (see the proof of Proposition 4.4.7). □

Corollary 8.2.8 *Let S be a classical extra-special surface. Then,* $\mathrm{Aut}_{\mathrm{ct}}(S)$ *is trivial unless S is extra-special of type D_8 in which case the group is of order* 2.

Proof If S is extra-special of type E_8, the assertion is proven in [192, Theorem 7.6]. Other assertions are also proved in the paper, but we give here another proof that does not involve the classification of automorphisms of rational elliptic surfaces.

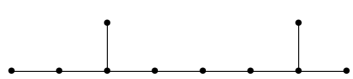
Let F_1 be a half-fiber on S. By Theorem 6.1.12, we can extend F_1 to a non-degenerate 2-sequence. Assume that there exists a non-trivial $g \in \mathrm{Aut}_{\mathrm{ct}}(S)$. Then, g acts on D_1 via its action on $|2F_1 + 2F_2|^*$. By our description of the automorphism group $\mathrm{Aut}(\mathsf{D}_1)^0$, g acts via $\mathbb{G}_m^2$ on D_1. But g has order 2 by Theorem 8.2.7, hence it acts trivially on D_1. Therefore, g is the covering involution of the bielliptic map and by Corollary 8.2.6, $|2F_1|$ is extremal. Therefore, every genus one fibration on S is extremal. In particular, by [371] Section 4, the automorphism group of S is finite. The groups $\mathrm{Aut}_{\mathrm{ct}}(S)$ of these surfaces have been calculated in [371], and the only surfaces for which the calculation of the groups depends on the specific example given in [371] are the ones of type $\tilde{D}_8$ and $\tilde{D}_4 + \tilde{D}_4$ (see Table 8.1 and Remark 8.2.18). In the latter case, there is a U-pair of fibrations with simple $\tilde{D}_8$ fibers, which share only 7 components. By Corollary 8.2.6, the corresponding bielliptic involution is not numerically trivial. Therefore, the calculation of the groups in [371] shows that the D_8-extra-special surface is the only classical Enriques surface which is not E_8-extra-special and has a non-trivial group of cohomologically trivial automorphisms (of order 2). □

In fact, not only $\mathrm{Aut}_{\mathrm{ct}}(S)$ but also $\mathrm{Aut}_{\mathrm{nt}}(S)$ in most cases is trivial if S is a classical Enriques surface in characteristic 2.

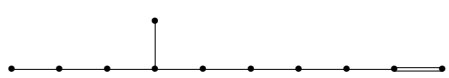
We have the following theorem (see [192, Theorem 7.4]. Its proof follows from the classification of automorphisms of rational elliptic surfaces and it is too long to reproduce it here.

Theorem 8.2.9 *Let S be a classical Enriques surface in characteristic* 2. *If* $\mathrm{Aut}_{\mathrm{nt}}(S)$ *is non-trivial, then S contains one of the following configurations of* (-2)*-curves:*

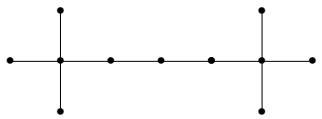
(A)



(B)



(C)



(D)

In the Cases (A), (B), and (C), these are in fact all (-2)*-curves on S.*

Corollary 8.2.10 *If S is an extra-special classical Enriques surface, then* $\mathrm{Aut}_{\mathrm{nt}}(S)$ *is trivial unless S is of type* D_8 *in which case it is isomorphic* $\mathrm{Aut}_{\mathrm{ct}}(S)$.

Let us now discuss supersingular surfaces. In [371], the groups of cohomologically trivial and numerically trivial automorphisms g of extra-special surfaces have been calculated. We give them in Table 8.1 (see also Tables 8.13, 8.14).

Type	$\mathrm{Aut}_{\mathrm{ct}}(S)$	$\mathrm{Aut}_{\mathrm{nt}}(S)$
classical of type $\tilde{E}_8$	$\{1\}$	$\{1\}$
α_2-surface of type $\tilde{E}_8$	$\mathbb{Z}/11\mathbb{Z}$	$\mathbb{Z}/11\mathbb{Z}$
classical of type $\tilde{D}_8$	$\mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}$
α_2-surface of type $\tilde{D}_8$	Q_8	Q_8
classical of type $\tilde{E}_7$	$\{1\}$	$\mathbb{Z}/2\mathbb{Z}$

Table 8.1 Numerically trivial automorphisms of extra-special surfaces

Before we start with the treatment of cohomologically trivial automorphisms of odd order of α_2-Enriques surfaces, let us collect the known examples. These surfaces have finite automorphism groups and a detailed study can be found in [371]. In Table 8.2, we give the group of cohomologically trivial automorphisms of these examples. Recently, Katsura and Schütt proved that there are no more examples [372].

Type	$\mathrm{Aut}_{\mathrm{ct}}(S)$
$\tilde{E}_8$	$\mathbb{Z}/11\mathbb{Z}$
$\tilde{E}_7^{(2)}$	$\mathbb{Z}/7\mathbb{Z}$ or $\{1\}$
$\tilde{E}_6$	$\mathbb{Z}/5\mathbb{Z}$

Table 8.2 Examples of cohomologically trivial automorphisms of odd order

The surface of type $\tilde{E}_6$ is the exceptional Enriques surface of type $\tilde{E}_6$ which we discussed in Section 6.2. The dual graph of (-2)-curves on this surface is given in (6.2.18).

Lemma 8.2.11 *Let S be an $\boldsymbol{\alpha}_2$-Enriques surface which is not E_8-extra-special and let $G \subseteq \mathrm{Aut}_{\mathrm{ct}}(S)$ be a non-trivial subgroup of odd order. Then, G is cyclic and acts non-trivially on the base of every genus one fibration of S.*

Proof Take any half-fiber F_1 and extend it to a non-degenerate 2-sequence (F_1, F_2) on S. Since G has odd order, it acts on D_3 via a finite subgroup of $\mathbb{G}_m$, hence G is cyclic. By explicit computation of $\mathrm{Aut}(\mathsf{D}_3)^0$, we know that a generator g of G acts on the image D_3 of the bielliptic map as

$$(x_0 : x_1 : x_2 : x_3 : x_4) \mapsto (x_0 : \lambda^{-1}x_1 : \lambda x_2 : \lambda^3 x_3, \lambda x_4).$$

The pencils of conics that give rise to our genus one pencils $|2F_1|$ and $|2F_2|$ are given by the equations

$$ax_3+b(ex_0+x_2+x_4) = ax_4+bx_1 = 0, \quad a(ex_0+x_2+x_4)+bx_1 = ax_3+bx_4 = 0. \quad (8.2.2)$$

Such an automorphism acts non-trivially on these pencils of conics, hence g acts non-trivially on $|2F_1|$. □

Lemma 8.2.12 *Let F be a fiber of a genus one fibration and let g be a tame automorphism of finite order that fixes the irreducible components of F. Then, the Lefschetz fixed-point formula*

$$e(F^g) = \sum_{i=0}^{2}(-1)^i Tr(g^*|H^i_{\acute{e}t}(F, \mathbb{Q}_l)) \quad (8.2.3)$$

holds for F. If F is reducible and not of type $\tilde{A}_1$, then $e(F^g) = e(F)$. If F is of type $\tilde{A}_1$, then $e(F^g) = e(F) = 2$ or $e(F^g) = 4$. The latter case can only occur if g has even order.

Proof In the case the order is equal to 2, this is proven in [178] by a case-by-case direct verification. The proof uses only the fact that a tame non-trivial automorphism of finite order of $\mathbb{P}^1$ has two fixed points. Also, note that the verification in the case F is of type $\tilde{A}_1$ and g interchanges the two singular points of F was missed, but it still agrees with the Lefschetz fixed-point formula. □

Proposition 8.2.13 *Let $g \in \mathrm{Aut}_{\mathrm{ct}}(S)$ be an automorphism of odd order. Then, every genus one pencil $|D|$ on S has one of the following combinations of singular fibers*

$$\tilde{D}_4+\tilde{D}_4,\ \tilde{D}_8+\tilde{A}_0^{**},\ \tilde{E}_6+\tilde{A}_2^{*},\ \tilde{E}_7+\tilde{A}_1^{*},\ \tilde{E}_8+\tilde{A}_0^{**},\ \tilde{A}_8+\tilde{A}_0+\tilde{A}_0+\tilde{A}_0,\ \tilde{D}_7,\ \tilde{E}_7 \quad (8.2.4)$$

The last three configurations can only occur if g has order 3.

Proof The claim is clear if S is E_8-extra-special, hence we can apply Lemma 8.2.11 and find that g acts non-trivially on all genus one pencils. Since the order of g is odd, it fixes two members F_1, F_2 of the pencil, one of which is a double fiber. Since all other fibers are moved, the set of fixed points S^g is contained in $F_1 \cup F_2$. Applying the Lefschetz fixed-point formula, we obtain

$$e(S) = 12 = e(S^g) = e(F_1^g) + e(F_2^g), \tag{8.2.5}$$

where $e()$ denotes the l-adic topological Euler–Poincaré characteristic.

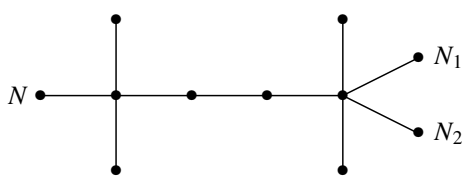
If one of the fibers, say F_1 is smooth, then, since g is of odd order and $e(F_2^g) \leq 10$, σ acts as an automorphism of order 3 on F_1. It is known that an automorphism of order 3 of an elliptic curve has three fixed points. Therefore, F_2 is of type $\tilde{A}_8$, $\tilde{D}_7$ or $\tilde{E}_7$ and g has order 3. By [431], we get the last three configurations of the Proposition.

If both fibers or the corresponding half-fibers are singular curves, then $e(F_i) = e(F_i^g)$. Indeed, for irreducible and singular curves, this follows from $e(F_2^g) \leq 10$ and for reducible fibers, this is Lemma 8.2.12 for automorphisms of odd order. The formula for the Euler–Poincaré characteristic of an elliptic surface (4.1.7) implies that F_1 and F_2 are the only singular fibers of $|D|$. In this case, denoting the number of irreducible components of F_i by m_i, we have $m_1 + m_2 \geq 8$, hence $|2F|$ is extremal and both fibers are of additive type. The classification of singular fibers of extremal rational genus one fibrations is known [431], [432], [337]. Since the types of singular fibers of a genus one fibration and of its Jacobian fibration are the same, it is straightforward to check that the list given in the proposition is complete. □

Corollary 8.2.14 *If S admits an automorphism $g \in \mathrm{Aut}_{\mathrm{ct}}(S)$ of odd order at least 5, then S is one of the surfaces in Table 8.2.*

Proof By Proposition 8.2.13, every genus one fibration on S is extremal. It follows from Theorem 8.1.12 that such an Enriques surface has finite automorphism group. Using the list of Proposition 8.2.13, the claim follows from the classification of α_2-Enriques surfaces with finite automorphism group (see Section 8.10). □

It is proven in [191, Proposition 7.9] that an Enriques surface with $\mathrm{Aut}_{\mathrm{nt}}(S)$ of order 3 must be a supersingular surface that contains the following diagram of (-2)-curves:



It shows that the surface contains a genus one fibration with a multiple fiber of type $\tilde{D}_4$ and a non-multiple fiber of the same type. The Mordell–Weil group of the associated jacobian fibration is isomorphic to $(\mathbb{Z}/2\mathbb{Z})^{\oplus 4}$, and hence cannot be an elliptic fibration. In a recent paper [372] that studies quasi-elliptic fibration on Enriques surfaces it is proven that such an Enriques surface exists and has a birational affine model given by the following equation:

$$y^2 + tx^4 + \alpha t^5 x^2 + t^7 x + t^3 = 0, \quad \alpha \in \Bbbk. \tag{8.2.6}$$

Finally, let us investigate numerically trivial automorphisms. If $K_S = 0$, $\mathrm{Aut}_{\mathrm{nt}}(S) = \mathrm{Aut}_{\mathrm{ct}}(S)$, so we only have to treat the case that $K_S \neq 0$, i.e. S is classical.

By definition, any $g \in \mathrm{Aut}_{\mathrm{nt}}(S)$ leaves invariant any genus one fibration, however, it may act non-trivially on its base, or equivalently, it may act non-trivially on the corresponding pencil $|D|$. Also, by definition, any $g \in \mathrm{Aut}_{\mathrm{ct}}(S)$ fixes the half-fibers of a genus one fibration (their difference in $\mathrm{Pic}(S)$ is equal to K_S). The following lemma proves the converse.

Lemma 8.2.15 *A numerically trivial automorphism g that fixes all half-fibers on S is cohomologically trivial.*

Proof Since g is numerically trivial, it fixes any smooth rational curve, because they are the unique representatives in $\mathrm{Pic}(S)$ of their classes in $\mathrm{Num}(S)$. By assumption, it fixes the linear equivalence class of all irreducible genus one curves. Applying Corollary 2.3.8 from Volume I, we obtain that g fixes the linear equivalence classes of all curves on S. □

Lemma 8.2.16 *Let G be a finite, tame group of automorphisms of an irreducible curve C fixing a nonsingular point x. Then, G is cyclic.*

Proof Since G is finite and tame, one can linearize the action in the formal neighborhood of the point x. It follows that the action of G on the tangent space of C at x is faithful. Since x is nonsingular, the tangent space is one-dimensional, and therefore, the group is cyclic. □

Theorem 8.2.17 *Let S be an Enriques surface and assume that $p \neq 2$. Then, $\mathrm{Aut}_{\mathrm{nt}}(S) \cong \mathbb{Z}/2^a\mathbb{Z}$ with $a \leq 2$. Moreover, if S is unnodal, then $\mathrm{Aut}_{\mathrm{nt}}(S) = \{1\}$.*

Proof By Theorem 8.2.7 and Lemma 8.2.15, any $g \in \mathrm{Aut}_{\mathrm{nt}}(S)$ has order 2 or 4, so it suffices to show that $\mathrm{Aut}_{\mathrm{nt}}(S)$ is cyclic. Since $\mathrm{Aut}_{\mathrm{nt}}(S)$ is tame, by Corollary 0.2.36, the locus of fixed points of a numerically trivial automorphism is smooth.

Assume that there is some $g \in \mathrm{Aut}_{\mathrm{nt}}(S) \setminus \mathrm{Aut}_{\mathrm{ct}}(S)$. By Lemma 8.2.15, g switches the half-fibers of some elliptic fibration $|2F_1|$ on S. The argument with the Euler–Poincaré characteristics from the proof of Proposition 8.2.13 applies and shows that one of the two fibers F', F'' of $|2F_1|$ fixed by g, say F', has at least five irreducible components. If S is unnodal, we get a contradiction, so $\mathrm{Aut}_{\mathrm{nt}}(S)$ coincides with $\mathrm{Aut}_{\mathrm{ct}}(S)$ and, by Theorem 8.2.7, the latter group is trivial.

If S is nodal, we continue the proof assuming first that F' is a reducible fiber of additive type. Then, it has an irreducible component which intersects three other components. Such a component R must be fixed pointwise by $\mathrm{Aut}_{\mathrm{nt}}(S)$, because, for any $g \in \mathrm{Aut}_{\mathrm{nt}}(S)$, it has three fixed points of g on it. Any adjacent irreducible component cannot be fixed pointwise since the locus of fixed points is smooth. Hence we can apply Lemma 8.2.16 to this component to obtain that the group $\mathrm{Aut}_{\mathrm{nt}}(S)$ is cyclic.

If F' is a reducible fiber of multiplicative type, the fixed point formula shows that F' is of type $\tilde{A}_7$ and g has four fixed points on F''. Extend F_1 to a non-degenerate

$U_{[2]}$-sequence (F_1, F_2). Since $F' \cdot F_2 = 2$, F' contains a chain of three (-2)-curves contained in a fiber D of $|2F_2|$. Now, as in the additive case, we find a (-2)-curve, which is fixed pointwise by $\mathrm{Aut}_{\mathrm{nt}}(S)$. Indeed, if D is additive, we use the same argument as before and if D is multiplicative, then some component of D meets a component of F' exactly once in a nonsingular point of F'. This component is fixed pointwise by $\mathrm{Aut}_{\mathrm{nt}}(S)$. □

Remark 8.2.18 The previous theorem is not true if $p = 2$. Indeed, there is an Enriques surface S with two multiple fibers of type $\tilde{D}_4$ with the dual graph of (-2) curves (C) from Theorem 8.2.9 that satisfies $\mathrm{Aut}_{\mathrm{nt}}(S) = (\mathbb{Z}/2\mathbb{Z})^2$ (see Theorem 8.10.7). Moreover, we have seen in the proof of Corollary 8.2.8 that $\mathrm{Aut}_{\mathrm{ct}}(S) = \{1\}$.

If $p = 2$, even though we still have the same bound on the size of $\mathrm{Aut}_{\mathrm{nt}}(S)$, the cyclic group of order 4 can not occur.

Theorem 8.2.19 *Let S be a classical Enriques surface in characteristic 2 which is not E_8-extra-special. Then,* $\mathrm{Aut}_{\mathrm{nt}}(S) \cong (\mathbb{Z}/2\mathbb{Z})^b$ *with $b \leq 2$.*

Proof By Corollary 8.2.8, $\mathrm{Aut}_{\mathrm{ct}}(S) \neq \{1\}$ if and only if S is D_8-extra-special and for such a surface we have $\mathrm{Aut}_{\mathrm{nt}}(S) = \mathrm{Aut}_{\mathrm{ct}}(S) = \mathbb{Z}/2\mathbb{Z}$. Hence, we can assume $\mathrm{Aut}_{\mathrm{ct}}(S) = \{1\}$. By Corollary 8.2.2, we have $\mathrm{Aut}_{\mathrm{nt}}(S) = (\mathbb{Z}/2\mathbb{Z})^b$ and we have to show $b \leq 2$. Suppose that $b \geq 3$ and take some half-fiber F_1. By Theorem 6.1.10, we can extend F_1 to a non-degenerate 2-sequence (F_1, F_2). Since $|\mathrm{Aut}_{\mathrm{nt}}(S)| > 4$, there is some numerically trivial involution g that preserves F_1 and F_2. Using the structure of $\mathrm{Aut}(\mathsf{D}_1)^0$, we find that such an automorphism acts trivially on D_1, hence it has to coincide with the bielliptic involution associated to $|2F_1 + 2F_2|$. Both fibrations have a unique reducible fiber F (resp. F') which has to be simple, since there is some numerically trivial involution which does not preserve F_i. By Corollary 8.2.6, F and F' are singular fibers of additive type and share 8 irreducible components. This is only possible if they are of type $\tilde{D}_8$ or $\tilde{E}_8$. If both fibers are of type $\tilde{D}_8$, then $F \cdot F' = 2$ if they share all components except one reduced component and $F \cdot F' = 8$ otherwise. Thus, $F \cdot F' \neq 4$ in this case. Similarly, we exclude the case when both F and F' are of type $\tilde{E}_8$. The only possible case when F and F' are of different types, and this leads to the D_8-extra-special surface S. We have already treated this surface. □

Corollary 8.2.20 *The following non-trivial finite groups can be realized as the groups* $\mathrm{Aut}_{\mathrm{ct}}(S)$ *(resp.* $\mathrm{Aut}_{\mathrm{nt}}(S)$*) for an Enriques surface in characteristic 2:*

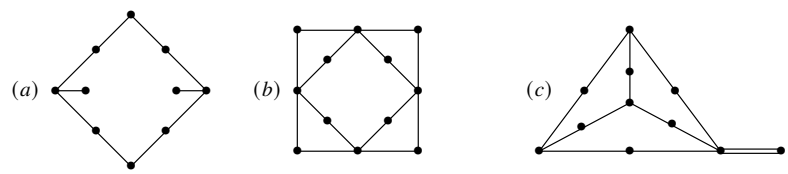
$$\mathbb{Z}/2\mathbb{Z}, \quad \mathbb{Z}/3\mathbb{Z}, \quad \mathbb{Z}/5\mathbb{Z}, \quad \mathbb{Z}/7\mathbb{Z}, \quad \mathbb{Z}/11\mathbb{Z}, \quad Q_8, \quad (\mathbb{Z}/2\mathbb{Z})^{\oplus 2} \tag{8.2.7}$$

(resp.

$$\mathbb{Z}/2\mathbb{Z}, \quad \mathbb{Z}/3\mathbb{Z}, \quad \mathbb{Z}/5\mathbb{Z}, \quad \mathbb{Z}/7\mathbb{Z}, \quad \mathbb{Z}/11\mathbb{Z}, \quad (\mathbb{Z}/2\mathbb{Z})^{\oplus 2}). \tag{8.2.8}$$

Next, we would like to classify all Enriques surfaces with non-trivial $\mathrm{Aut}_{\mathrm{nt}}(S)$ assuming that $p \neq 2$. We know that $\mathrm{Aut}_{\mathrm{nt}}(S)$ (resp. $\mathrm{Aut}_{\mathrm{ct}}(S)$) is cyclic group of order 2 or 4 (resp. 2).

Theorem 8.2.21 *Assume $p \neq 2$. Let S be an Enriques surface such that* $\mathrm{Aut}_{\mathrm{nt}}(S) \neq \{1\}$. *Then, S contains one of the following three configurations of* (-2)*-curves with the dual intersection graphs as follows:*



Proof By Theorem 8.2.17, $\mathrm{Aut}(S)$ contains a numerically trivial involution. By Theorem 8.2.7, this involution is a bielliptic involution. In Section 8.7, we will classify possible actions of bielliptic involution. The numerically trivial involutions are listed in the first three rows of Table 8.8. It follows from Table 8.7 that S^g is equal to the union of four isolated points, four (-2)-curves, and maybe one elliptic curve.

There are different bielliptic maps defining the same involution, so their branch curves W may be different.

We start with Type (a), which can be obtained by taking $W = W_0 \cup \ell$, where W_0 is an elliptic plane quintic curve with tacnodes at p_2, p_4 that passes through p_1 and has an additional cusp q. The line ℓ is the cuspidal tangent line passing through the point p_1. This is case (17) in Table 8.7. The pre-image of ℓ is a fiber F of type $\tilde{E}_8$ of the elliptic fibration $|2F_1|$. The proper transform of ℓ is its end component that enters with multiplicity 2. Together with the proper transform of the conic from the pencil $|2e_0 - e_2 - \cdots - e_5|$ passing though q this gives diagram (a).

To realize the Type (b) case, we take W to be the union of four conics, two from each pencil $|2e_0 - e_2 - e_3 - e_4 - e_5|$ and $|e_0 - e_1|$. The singular locus of W consists of eight isolated ordinary double points. They define eight disjoint (-2)-curves on S. This is case (30) from Table 8.7. The proper transform of each conic gives a fiber of the elliptic fibration $|2F_1|$ of type $\tilde{D}_4$. This gives us picture (b) of curves on S from the assertion of the theorem.

Finally, to realize the Type (c) of a cohomologically trivial involution, we use the double plane construction from case (14) of Table 8.7. The branch curve W is an irreducible rational sextic with a singular point q of type e_7 and a singular point q' of type a_1. The pre-image of the line from the pencil $|e_0 - e_1|$ passing through q gives a fiber F of type $\tilde{E}_7$ of the elliptic fibration $|2F_1|$. The proper transform of ℓ intersects the component e_2 (we use the pictures from Lemma 8.7.20). The proper transform $\bar{W}$ of W intersects the fiber at the components e_7 and e_1 and intersects ℓ at one point. The exceptional curve R over q' intersects $\bar{W}$ at two points. Next, we consider a cubic $C \in |3e_0 - e_1 - \cdots - e_5|$ with an ordinary double point at q that intersects W at this point with multiplicity 8. Its proper transform $\bar{C}$ intersects the components e_2 and e_6. Together $F, \bar{W}, \bar{C}$ form a diagram as in picture (c). □

Remark 8.2.22 In each case of Theorem 8.2.21, one can find a maximal root sublattice of rank 8: E_8 (case (a)), D_8 (case (b)), $E_7 \oplus A_1$ (case (c)). Over the complex numbers, these root lattices give Nikulin R-invariants $(E_8, \{0\})$, $(D_8, \mathbb{Z}/2\mathbb{Z})$, $(E_7 \oplus$

$A_1, \mathbb{Z}/2\mathbb{Z}$) of Enriques surfaces with numerically trivial automorphisms. To classify R-invariants of such Enriques surfaces was the main idea of Mukai and Namikawa [532].

The following theorem gives a classification of Enriques surfaces in characteristic $\neq 2$ that admit a numerically trivial involution.

Theorem 8.2.23 *Assume $p \neq 2$ and let S be an Enriques surface with non-trivial* $\mathrm{Aut}_{\mathrm{nt}}(S)$. *Then, the isomorphism class of S belongs to one of the following* 3 *two-dimensional irreducible families:*

(A) S is the double cover of a non-degenerate 4-nodal quartic del Pezzo surface D_1 with the branch curve W equal to the union of two conics intersecting at 2 points and an irreducible hyperplane section through the intersection points of the two conics. The surface contains the diagram (a) of (-2)-curves. The deck transformation is cohomologically trivial. One of the members of the family contains a numerically trivial automorphism whose square is equal to the deck transformation.

(B) S is the double cover of D_1 with the branch curve W equal to the union of two pairs of disjoint conics. The deck transformation is numerically trivial but not cohomologically trivial. The surface contains the diagram of (-2)-curves of type (b).

(C) S is the double cover of D_1 with the branch curve W equal to the union of two conics and a nodal irreducible hyperplane section. The deck transformation is numerically trivial but not cohomologically trivial. The surface contains the diagram (c) of (-2)-curves.

Any pair (S, σ) consisting of an Enriques surface S and its numerically but not cohomologically trivial involution is isomorphic to a surface from families (B), (C).

Proof Applying Theorem 8.2.21, we obtain that all surfaces that admit a bielliptic map with numerically trivial deck involution are divided into 3 classes (A), (B), (C) according to the type of one of the diagrams (a),(b),(c) of (-2)-curves lying on it.

Type (A):
We give another realization of diagram (a) by choosing construction (28) from Table 8.7. The bielliptic involution is defined by choice of a non-degenerate U-pair of elliptic fibrations, each contains a reducible fibers of type $\tilde{D}_8$. In this case, the octic branch curve on D_1 is the union of two conics and a hyperplane section that passes through the intersection points of the conics. In the plane model, they correspond to a cubic C passing through $p_1, \ldots, p_5$, a line ℓ from $|e_0 - e_1|$, and a conic K from the pencil $|2e_0 - e_1 - \cdots - e_5|$. The cubic C passes through two intersection points q_1, q_2 of the line and the conic. The curve W has two simple singular points at q_1, q_2 of type d_4.

In appropriate projective coordinates in the plane, the equation of the double plane is given by

$$x_3^2 + x_1x_2(x_1 - x_2)(x_0^2 - x_1x_2)(ax_1^2x_2 + bx_0^2x_1 + cx_0^2x_2 + dx_2^2x_1) = 0,$$

where $a + b + c + d = 0$. The points q_1, q_2 have the coordinates $[\pm 1, 1, 1]$. For example, if we take the parameters $(a, b, c, d) = (1, -1, 1, -1)$, we obtain the standard Cremona involution $T : (x_0, x_1, x_2) \mapsto (x_1x_2, x_0x_2, x_0x_1)$ that transforms F to $-(x_0^2x_1x_2)^2F$. Consider a birational model of S as a surface of degree 8 in the weighted projective space $\mathbb{P}(1, 1, 1, 4)$ given by the equations $x_3^2 + F(x_0, x_1, x_2) = 0$. Then, the formula $(x_0, x_1, x_2, x_3) \to (x_1x_2, x_0x_2, x_0x_1, ix_0^2x_1x_2)$ defines a birational automorphism of order 4 of S whose square is equal to the deck transformation σ. Since it fixes the points q_1, q_2, and the curve branch curve is invariant, it is easy to see that it is numerically trivial.

Let us show that the bielliptic involution is cohomologically trivial. Let N be the sublattice of $\mathrm{Num}(S)$ spanned by the (-2)-curves represented by all vertices of the diagram (a). We compute the discriminant of this lattice and find that it is equal to -1. Thus, the span of the curves generate $\mathrm{Pic}(S)$ and since the involution fixes these curves, it is cohomologically trivial.

Type (B):
We use the bielliptic involution from the previous theorem. A choice of two lines and two conics depend on two parameters. We fix one line and a conic as in the previous case to assume that their equations are $x_1 - x_2 = 0$ and $x_1x_2 - x_0^2 = 0$. Then, the second line has an equation $x_1 + ax_2 = 0$ and the second conic has an equation $x_1x_2 + bx_0^2 = 0$. Thus, the equation of the double plane is

$$x_3^2 + x_1x_2(x_1 - x_2)(x_1 + ax_2)(x_1x_2 - x_0^2)(x_1x_2 + bx_0^2) = 0, \tag{8.2.9}$$

where $a \neq 0, -1, b \neq -1$.

Note that in this case, the bielliptic involution is not cohomologically trivial but only numerically trivial. To see this we consider the Halphen pencil of curves of degree 6 with double points at $p_1, \ldots, p_5$ and four of the double points $q_1, \ldots, q_4$ of W no three of which lie on ℓ or ℓ'. It is the pencil $\lambda A + \mu B^2$, where $A = 0$ is the equation of the curves $A = \ell + \ell' + K + K'$ and $B = 0$ is the equation of the unique cubic B through the nine points p_i, q_i. The pre-image of this pencil on S is locally equal to $\lambda A^2 + \mu B^2 = 0$, and it general member splits into the union of two disjoint elliptic curves. It defines an elliptic fibration over the double cover of the line parameterizing the Halphen pencil. It is an elliptic fibration on S with a reducible fiber of type $\tilde{A}_7$ which we can locate on diagram (b). The deck involution permutes reducible fibers, and hence does not act trivially on its base, so it is not cohomologically trivial.

Type (C):
We could use the construction of the double plane described in the previous theorem, but we prefer to give another construction which is case (29) from Table 8.7. We take W to be the union of a line ℓ from the pencil $|e_0 - e_1|$, a conic from the pencil $|2e_0 - e_1 - \cdots - e_5|$ and a cubic from $|3e_0 - e_1 - \cdots - e_5|$ with a node q that passes through one of the intersection points q_1, q_2 of the line and the conic. This corresponds to the branch curve on D_1 equal to the union of two conics and a hyperplane section that passes through one of the intersection points and tangent to

the surface at some point. The curve W has one simple singularity of type d_4 and four ordinary double points. The pre-images of the exceptional curves of resolution of singularities of W, the proper transforms of the components of W form the diagram (c). The vertex connected with the double edge is the pre-image of the exceptional curve over the node q.

The equation of the double plane is

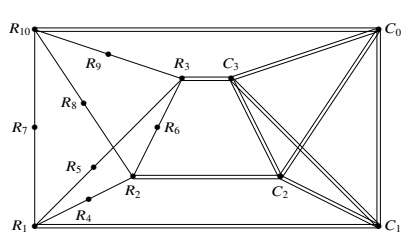
$$x_3^2+x_1x_2(x_1-x_2)(x_1x_2+x_0^2)(ax_1^2x_0+bx_2^2x_0+cx_0^2x_1+dx_0^2x_2+ex_0x_1x_2) = 0, \quad (8.2.10)$$

where $a+b+c+d+e=0$ and we have to add one more condition that the cubic is singular.

We claim that the deck transformation of the bielliptic map is numerically trivial but not cohomologically trivial. The argument is the same as in the case (A). We use a Halphen pencil $\lambda A+\mu B^2=0$, where $A=0$ is the equation of $\ell+K+C$ and $B=0$ is the equation of a unique cubic $c' \in \mathcal{P}_3$ passing through the four ordinary double points of $\ell+K+C$. □

Remark 8.2.24 Consider a surface S from family (B). The surface contains eight disjoint (-2)-curves. Their pre-images on the canonical cover X form a set of 16 disjoint (-2)-curves. If $\Bbbk=\mathbb{C}$, by Nikulin's Theorem [554], X must be birationally isomorphic to the Kummer surface associated to an abelian surface. In fact, if $p \neq 2$, one can see it directly by considering the base change of the bielliptic map $f: S \to \mathsf{D}_1$ under the cover $\mathbf{F}_0 \to \mathsf{D}_1$. It defines a degree 2 map $\tilde{f}: X \to \mathbf{F}_0 \cong \mathbb{P}^1 \times \mathbb{P}^1$ branched along the union of eight lines, four in each ruling. It is well-known and easy to see that it shows that X is birationally isomorphic to the Kummer surface $\mathrm{Kum}(E_1 \times E_2)$ of the product $E_1 \times E_2$ of two elliptic curves. Moreover, one sees that the Enriques involution τ on X is the descent to the Kummer surface of the involution $(a,b) \mapsto (a+\epsilon, -b+\eta)$, where $\epsilon \in E_1, \eta \in E_2$ are non-trivial 2-torsion points.

Consider the family of type (C) constructed in the proof of the previous theorem. We can locate additional three (-2)-curves C_1, C_2, C_3 on S equal to the proper inverse transform of a line ℓ' and the conic K' from the same pencils that pass through the node q and the conic that passes through the intersection point of the line and the cubic component of the branch sextic. We get the following diagram of (-2)-curves on S:



Here C_0 corresponds to the vertex of diagram (c) connected by the double edge.

Let $\pi : X \to S$ be the canonical cover of S. The pre-images of the curves $R_4, R_5, R_6, R_7, R_8, R_9$ are 12 disjoint (-2)-curves on X. The pre-images of the curves C_0, C_1, C_2, C_3 are eight (-2)-curves, among which we can find four mutually disjoint ones. Together with the previous set of twelve (-2)-curves. they give 16 disjoint curves on X. In the case $\mathbb{k} = \mathbb{C}$, we can apply Nikulin's Theorem to deduce that X must be birationally isomorphic to a Kummer surface associated to some abelian surface. It is shown in [532] that X is birationally isomorphic to $\mathrm{Kum}(E_1 \times E_2)$ again, but the Enriques involution is different. We refer to its description to section 2 of loc. cit..

In Section 8.9, we will classify all Enriques surfaces with finite automorphism groups over a field of characteristic $\neq 2$. We will see that the three diagrams are realized on surfaces of types I, III and V, respectively.

At present, the classification of Enriques surfaces in characteristic 2 with a numerically trivial automorphism is not known

8.3 Automorphisms of Unnodal Enriques Surfaces

In this section, we will show that the bielliptic involutions of an unnodal Enriques surface generate a subgroup isomorphic to a subgroup of finite index of the Weyl group $W_{2,3,7}$.

In the previous section, we discussed the kernel of the homomorphism

$$\rho : \mathrm{Aut}(S) \to \mathrm{O}(\mathrm{Num}(S)) \cong \mathrm{O}(\mathsf{E}_{10})$$

and it is not trivial only in very special cases.

It follows from Theorem 8.2.7 that, for any unnodal Enriques surface

$$\mathrm{Ker}(\rho) = \{1\}.$$

Let us discuss the image of ρ. From now on, we fix an isomorphism $\mathrm{Num}(S) \cong \mathsf{E}_{10}$ and identify these two lattices.

Applying Theorem 8.1.10, we obtain the following.

Proposition 8.3.1 *Let S be an unnodal Enriques surface. Assume that $p \neq 2$. Then,* $\mathrm{Aut}(S)^*$ *is a subgroup of finite index in $W(\mathsf{E}_{10})$.*

We will prove that a more precise assertion is true and also it is valid in arbitrary characteristic.

Applying Theorem 0.8.6 from Volume I, we see that the reduction homomorphism $\mathsf{E}_{10} \to \bar{\mathsf{E}}_{10} = \mathsf{E}_{10}/2\mathsf{E}_{10}$ defines a surjective homomorphism $W(\mathsf{E}_{10}) \to \mathrm{O}^+(10, \mathbb{F}_2)$. Let

$$W(\mathsf{E}_{10})(2) := \mathrm{Ker}(W(\mathsf{E}_{10}) \to \mathrm{O}^+(10, \mathbb{F}_2)).$$

It is called the 2-*level congruence subgroup* of $W(\mathsf{E}_{10})$. It is clear that under any marking $\phi : \mathrm{Num}(S) \to \mathsf{E}_{10}$, the subgroup $\phi \circ W(\mathsf{E}_{10})(2) \circ \phi^{-1}$ is equal to the subgroup $W_S(2) := \{\sigma \in W(\mathrm{Num}(S)) : \sigma \equiv \mathrm{id}_{\mathrm{Num}(S)} \mod 2\,\mathrm{Num}(S)\}$. We call it the 2-*level congruence subgroup* of the Weyl group $W(\mathrm{Num}(S))$. Over $\mathbb{C}$, it follows Theorem 5.5.1 that $\mathrm{Aut}(S)^*$ contains $W(\mathrm{Num}(S))(2)$. In the following, we will prove this result in the case of a field of arbitrary characteristic.

Proposition 8.3.2 (A. Coble) *The subgroup $W(\mathsf{E}_{10})(2)$ is the smallest normal subgroup containing the involution $\sigma = 1_{\mathsf{U}} \oplus (-1)_{\mathsf{E}_8}$ for some (hence any) orthogonal decomposition $\mathsf{E}_{10} = \mathsf{U} \oplus \mathsf{E}_8$.*

Proof Coble's proof is computational. It is reproduced in [138], Chapter 2, §10. The following nice short proof is due to Allcock [7].

Let $\Gamma = \langle\langle\sigma\rangle\rangle$ be the minimal normal subgroup containing σ. It is generated by the conjugates of σ in $W = W(\mathsf{E}_{10})$. Let $\{\mathfrak{f}, \mathfrak{g}\}$ be the standard basis of the hyperbolic plane U. If $\{\alpha_0, \dots, \alpha_7\}$ is the basis of the sublattice E_8 corresponding to the subdiagram of type E_8 of the Dynkin diagram of the Enriques lattice E_{10}, then we may take

$$\mathfrak{f} = 3\alpha_0 + 2\alpha_1 + 4\alpha_2 + 6\alpha_3 + 5\alpha_4 + 4\alpha_5 + 3\alpha_6 + 2\alpha_7 + \alpha_8,\ \mathfrak{g} = \mathfrak{f} + \alpha_9.$$

The stabilizer $W_{\mathfrak{f}}$ of $\mathfrak{f}$ in W is the semi-product $\mathsf{E}_8 \rtimes W(\mathsf{E}_8) \cong W(\mathsf{E}_9)$. The image of $v \in \mathsf{E}_8 = \mathsf{U}^\perp$ in W_f is the transformation

$$\phi_v : x \mapsto x + (x \cdot v - \frac{1}{2}(x \cdot \mathfrak{f})v^2)\mathfrak{f} - (x \cdot \mathfrak{f})v.$$

The Weyl group $W(\mathsf{E}_8)$ is a subgroup of $W(\mathsf{E}_9)$ generated by reflections in the roots $\alpha_0, \dots, \alpha_7$ that span a subdiagram of type E_8. In particular, any $w \in W(\mathsf{E}_8)$ acts identically on $\mathsf{E}_8^\perp \cong \mathsf{U}$. The image of σ in $W(\mathsf{E}_9)$ is equal to $(-1)_{\mathsf{E}_8} \in W(\mathsf{E}_8) \subset W(\mathsf{E}_9)$. Let us compute the ϕ_v-conjugates of σ. If $x \in \mathsf{E}_8$, we have

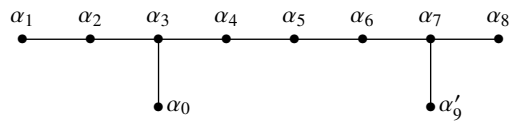
$$\phi_v \circ \sigma \circ \phi_{-v}(x) = \phi_v(\sigma(x - (v \cdot x)\mathfrak{f}))$$

$$= \phi_v(-x - (v \cdot x)\mathfrak{f}) = (-x - (v \cdot x)\mathfrak{f}) - (v \cdot x)\mathfrak{f} = -x - 2(v \cdot x)\mathfrak{f}.$$

Thus, the intersection of Γ with $W_{\mathfrak{f}}$ is equal to $\phi(2\mathsf{E}_8) \rtimes \mathbb{Z}/2\mathbb{Z}$. The quotient $W_{\mathfrak{f}}/(\Gamma \cap W_{\mathfrak{f}})$ is isomorphic to $(\mathbb{Z}/2\mathbb{Z})^8 \rtimes \mathrm{O}(8, \mathbb{F}_2)^+$, it injects into $\mathrm{O}(\bar{E}_{10}) \cong \mathrm{O}(10, \mathbb{F}_2)^+$.

Let us consider the subgroup H generated by $W_{\mathfrak{f}}$ and Γ. Since $W_{\mathfrak{f}}/(\Gamma \cap W_{\mathfrak{f}})$ injects in $\mathrm{O}(\bar{E}_{10})$, the kernel of the homomorphism $H \to \mathrm{O}(\bar{E}_{10})$ coincides with Γ. To finish the proof it suffices to show that H coincides with the pre-image $W_{\bar{\mathfrak{f}}}$ in W of the stabilizer subgroup of the image $\bar{\mathfrak{f}}$ of $\mathfrak{f}$ in $\mathrm{O}(\bar{E}_{10})$. Indeed, the kernel of $W_{\bar{\mathfrak{f}}} \to \mathrm{O}(\bar{E}_{10})$ is equal to $W(E_{10})(2)$ and hence coincides with Γ.

Consider the sublattice L of E_{10} generated by the roots $\alpha_0, \dots, \alpha_7, \alpha_8$ and $\alpha_9' = \alpha_8 + 2\mathfrak{g} - \mathfrak{f}$. The Dynkin diagram is the following:



Here all the roots, except α'_9, are orthogonal to $\mathfrak{f}$. So, H contains the reflections defined by these roots. Also, the root $\alpha_8 - \mathfrak{f}$ is orthogonal to $\mathfrak{f}$, and α'_9 is transformed to it under the group $2\mathsf{E}_8 \subset \Gamma$ stabilizing $\mathfrak{g}$. So, , H contains $s_{\alpha'_9}$ too. The Dynkin diagram contains three subdiagrams of affine types $\tilde{E}_8, \tilde{E}_8$ and $\tilde{D}_8$. The Weyl group is a crystallographic group with a Weyl chamber being a simplex of finite volume with three vertices at the boundary. This implies that H has at most three orbits of ($\pm$ pairs) of primitive isotropic vectors in L. On the other hand, $W_{\bar{\mathfrak{f}}}$ contains H and has at least three orbits of them, because the stabilizer of $\bar{\mathfrak{f}}$ in $\mathrm{O}(\bar{E}_{10})$ has three orbits of isotropic vectors (namely, $\{\bar{\mathfrak{f}}\}$, the set of isotropic vectors distinct from $\bar{\mathfrak{f}}$ and orthogonal to $\bar{\mathfrak{f}}$, and the set of isotropic vectors not orthogonal to $\bar{\mathfrak{f}}$). This implies that the set of orbits of H is the same as the set of orbits of $W_{\bar{\mathfrak{f}}}$. Since the stabilizers of $\bar{\mathfrak{f}}$ in these two groups are both equal to $W_{\mathfrak{f}}$, it follows that $H = W_{\bar{\mathfrak{f}}}$. □

Theorem 8.3.3 *Let S be an unnodal Enriques surface. Then, the image* $\mathrm{Aut}(S)^*$ *of ρ contains $W(\mathrm{Num}(S))(2)$.*

Proof Consider a bielliptic linear system $|D| = |2F_1 + 2F_2|$ with $D^2 = 8$. It defines a degree 2 bielliptic map $\phi : S \to \mathsf{D} \subset \mathbb{P}^4$, where D is one of the anti-canonical surfaces $\mathsf{D}_1, \mathsf{D}_2, \mathsf{D}_3$. Since S has no smooth rational curves, the cover is a separable map. Let σ be the deck transformation of the cover and $\sigma^* = \rho(\sigma) \in W(\mathrm{Num}(S))$. For any irreducible curve C on S, the divisor class $c = [C + \sigma^*(C)]$ is σ^*-invariant. Since $\mathrm{Pic}(\mathsf{D})_{\mathbb{Q}} \cong \mathbb{Q}^2$ and ϕ is a finite map of degree 2, we have $c = (c \cdot [F_2])[F_1] + (c \cdot [F_1])[F_2]$ in $\mathrm{Num}(S)_{\mathbb{Q}}$. This shows that σ^* acts as the identity on $\langle [F_1], [F_2] \rangle \cong \mathsf{U}$ and as the minus identity on $\langle [F_1], [F_2] \rangle^{\perp} \cong \mathsf{E}_8$.

Any conjugate of σ^* in $W(\mathrm{Num}(S))$ is also realized by some automorphism. In fact, $w \cdot \sigma^* \cdot w^{-1}$ leaves invariant $w(\langle F_1, F_2 \rangle)$, and the deck transformation corresponding to the linear system $|2w(F_1) + 2w(F_2)|$ realizes $w \cdot \sigma^* \cdot w^{-1}$.

Now, we invoke the previous proposition that says that $W(\mathsf{E}_{10})(2)$ is the minimal normal subgroup of $W(\mathsf{E}_{10})$ containing σ^*. □

Recall from Section 5.2 that, for a moduli general complex Enriques surface, the group $\mathrm{Aut}(S)^*$ coincides with $W(\mathrm{Num}(S))(2)$. We characetrized surfaces with larger automorphism groups in terms of the periods of their canonical covers.

We refer to [490] for the proof of the following result.

Theorem 8.3.4 *Let S be an unnodal Enriques surface. Suppose $p \neq 2$, or S is a $\boldsymbol{\mu}_2$-surface. Let* $\overline{\mathrm{Aut}}(S)$ *be the image of* $\mathrm{Aut}(S)$ *in* $W(\mathsf{E}_{10})/W(\mathsf{E}_{10}(2)) \cong \mathrm{O}^+(10, \mathbb{F}_2)$. *Then,* $\#\overline{\mathrm{Aut}}(S) \in \{1, 2, 4\}$. *The group is trivial if $p = 2$.*

Note that all possible cases for the order of $\#\overline{\mathrm{Aut}}(S)$ are realized. We refer to loc. cit. for examples of families of Enriques surfaces whose general member has an extra automorphism.

8.4 Automorphisms of General Nodal Surfaces

In this section, we define three different involutions of a general nodal Enriques surface S (in the sense of the definition from Section 6.5), and show that the smallest normal subgroup of $\mathrm{Aut}(S)$ containing these involutions generate a subgroup isomorphic to a normal subgroup of finite index in the Weyl group $W_{2,4,6}$.

Recall that we have defined the Reye lattice in (6.4.7)

$$\mathrm{Rey}(S) = \{x \in \mathrm{Num}(S) : x \cdot R \equiv 0 \mod 2 \text{ for any } (-2)\text{-curve } R\}.$$

Obviously, the action of $\mathrm{Aut}(S)$ on S preserves the set of (-2)-curves, hence preserves the Reye lattice. Thus, we have a homomorphism

$$\rho : \mathrm{Aut}(S) \to \mathrm{O}(\mathrm{Rey}(S)).$$

By (0.8.12), $\mathrm{O}(\mathrm{Rey}(S))' = W(\mathrm{Rey}(S))$. Thus, we may assume that the image of ρ is contained in the Weyl group $W(\mathrm{Rey}(S))$.

Proposition 8.4.1 *Assume that S is a general nodal Enriques surface, then the homomorphism ρ is injective.*

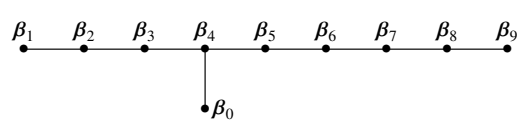
Proof If $p \neq 2$, the assertion follows from Theorem 8.2.21. If $p = 2$ and $K_S \neq 0$, the assertion follows from Theorem 8.2.9. Finally, if $K_S = 0$, Proposition 8.2.13 implies that $\mathrm{Ker}(\rho)$ does not contain non-trivial elements of odd order.

By Theorem 8.2.7 a non-trivial element of even order in $\mathrm{Aut}_{\mathrm{ct}}(S)$ must be a bielliptic involution. By Lemma 8.2.5, S cannot be a general nodal Enriques surface.□

Fix a canonical root basis in E_{10} defined by an isotropic 10-sequence $(\mathbf{f}_1, \dots, \mathbf{f}_{10})$. It defines a sublattice of E_{10} isomorphic to the Reye lattice $\mathsf{E}_{2,4,6}$. Its canonical root basis is formed by the vectors $(\boldsymbol{\beta}_0, \dots, \boldsymbol{\beta}_9)$ with

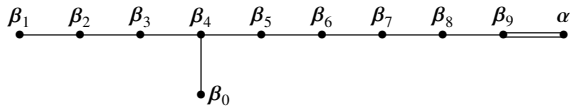
$$\begin{aligned} \boldsymbol{\beta}_0 &= \mathbf{f}_1 + \mathbf{f}_2 + \mathbf{f}_3 + \mathbf{f}_4 - \mathbf{h}_{10}, \\ \boldsymbol{\beta}_i &= \mathbf{f}_{i+1} - \mathbf{f}_i, \ i > 0. \end{aligned} \tag{8.4.1}$$

The vectors $\boldsymbol{\beta}_0, \dots, \boldsymbol{\beta}_9$ form a root basis of $\mathsf{E}_{2,4,6}$ with Dynkin diagram of type $T_{2,4,6}$



(8.4.2)

We fix this embedding $\mathsf{E}_{2,4,6} \hookrightarrow \mathsf{E}_{10}$. It satisfies the property that $\mathsf{E}_{2,4,6} = \{x \in \mathsf{E}_{10} : x \cdot \boldsymbol{\alpha} \in 2\mathbb{Z}\}$, where $\boldsymbol{\alpha} = \mathbf{h}_{10} - 2\mathbf{f}_{10}$. The root basis $(\boldsymbol{\beta}_0, \dots, \boldsymbol{\beta}_9, \boldsymbol{\alpha})$ is a crystallographic root basis of $\mathsf{E}_{2,4,6}$ with Dynkin diagram (8.4.2). Recall that we have already encountered this crystallographic basis when we considered extraspecial Enriques surfaces in characteristic 2.



(8.4.3)

Definition 8.4.2 A *Reye marking* of a general nodal Enriques surface is an isomorphism $\phi : \mathrm{Num}(S) \to \mathsf{E}_{10}$ such that $\phi(\mathrm{Rey}(S)) = \mathsf{E}_{2,4,6}$ and $\alpha = \phi^{-1}(\boldsymbol{\alpha})$ is the class of a (-2)-curve R.

By Proposition 0.8.10 from Volume I, $W(\mathsf{E}_{2,4,6}) = \langle\langle s_{\boldsymbol{\alpha}}\rangle\rangle \rtimes W_{2,4,6}$, where $W_{2,4,6}$ is the Coxeter group corresponding to the standard root basis of the lattice $\mathsf{E}_{2,4,6}$ with Dynkin diagram $T_{2,4,6}$. Using a Reye marking $\phi : \mathrm{Rey}(S) \to \mathsf{E}_{2,4,6}$, we obtain

$$W(\mathrm{Rey}(S)) = \langle\langle s_\alpha\rangle\rangle \rtimes W(\mathrm{Rey}(S))_0,$$

where $W(\mathrm{Rey}(S))_0 = \phi^{-1}(W_{2,4,6})$.

Proposition 8.4.3

$$W_S^{\mathrm{nod}} = \langle\langle s_\alpha\rangle\rangle,$$

i.e. W_S^{nod} *is a minimal normal subgroup containing* s_r.

Proof Fix a Reye marking $\phi : \mathrm{Rey}(S) \to \mathsf{E}_{2,4,6}$. Let $\beta_i = \phi^{-1}(\boldsymbol{\beta}_i)$. By Corollary 6.5.11, the canonical cover of S is birationally isomorphic to a K3 surface. Since $\beta_1, \dots, \beta_{10}$ intersect α with even multiplicity, we can apply Lemma 6.3.12 to obtain that any $W(\mathsf{E}_{2,4,6})$-conjugate of α is the class of a (-2)-curve. By Corollary 0.8.11, $(\mathsf{E}_{2,4,6})_{-2}$ has two orbits with respect to $W(\mathsf{E}_{2,4,6})$. We have shown that the orbit of α is contained in the set of effective roots (the classes of (-2)-curves). Suppose an effective root α' belongs to another obit. Then, $\alpha' = w(\boldsymbol{\beta}_1)$ for some $w \in W_{2,4,6}$. Since $\alpha' \equiv \alpha \mod 2\mathsf{E}_{10}$, applying w^{-1} to both sides we obtain that $\boldsymbol{\beta}_1 \equiv w^{-1}(\alpha) \equiv \alpha \mod 2\mathsf{E}_{10}$. We can choose a Reye marking ϕ such that $\alpha = \phi^{-1}(\boldsymbol{\alpha}) = \phi^{-1}(\mathbf{h}_{10} - 2\mathbf{f}_{10})$. Since for a general nodal Enriques surface all (-2)-curves are congruent mod $2\,\mathrm{Num}(S)$, the numerical Fano polarization $h_{10} = \phi^{-1}(\mathbf{h}_{10})$ is ample. This gives $\mathbf{h}_{10} - \mathbf{f}_{10} \equiv \mathbf{h}_{10} - \mathbf{f}_1 \mod 2\mathsf{E}_{10}$, obviously impossible. □

Proposition 8.4.4 *Let* $W(\mathrm{Rey}(S))_0$ *be the image of* $W_{2,4,6}$ *for some Reye marking. Then, the subgroup* $W(\mathrm{Rey}(S))_0$ *consists of elements from* $W(\mathrm{Rey}(S))$ *that leave invariant the nef cone* $\mathrm{Nef}(S)$ *of* S.

Proof Suppose $w(\mathrm{Nef}(S)) \neq \mathrm{Nef}(S)$ for some $w \in W(\mathrm{Rey}(S))_0$. Since $W(\mathrm{Rey}(S))_0$ normalizes W_S^{nod}, its elements permute $w'(\mathrm{Nef}(S))$, $w' \in W_S^{\mathrm{nod}}$. Thus, w must coincide with some element from W_S^{nod}. Since $W_{2,4,6} \cap \langle\langle s_{\boldsymbol{\alpha}}\rangle\rangle = \{1\}$, we get a contradiction. Since no element of W_S^{nod} leaves $\mathrm{Nef}(S)$ invariant, we obtain that $W(\mathrm{Rey}(S))_0$ is the group of isometries of $\mathsf{E}_{2,4,6}$ that leaves $\mathrm{Nef}(S)$ invariant. □

Corollary 8.4.5 *Under a Reye marking* ϕ *that sends* $\boldsymbol{\alpha}$ *to the class of a* (-2)*-curve, the image of the nef cone of* S *is equal to the union of* $W_{2,4,6}$*-translates of the set*

$$C = \{x \in (\mathsf{E}_{2,4,6})_{\mathbb{R}} : x \cdot \boldsymbol{\beta}_i \geq 0, x \cdot \boldsymbol{\alpha} \geq 0\}.$$

Recall from Section 6.4 that

$$\bar{\mathsf{E}}_{2,4,6} = \mathsf{E}_{2,4,6}/2\mathsf{E}_{2,4,6}$$

is a 10-dimensional quadratic space V with the quadratic form $q(x+2\mathsf{E}_{2,4,6}) = \frac{1}{2}x^2$. Its radical is spanned by the coset $\bar{\boldsymbol{\alpha}}$ of α and the coset $\bar{\mathsf{f}}_{E_8}$ of the generator f_{E_8} of the radical of the affine root sublattice of type $\tilde{E}_8$ in $\mathsf{E}_{2,4,6}$. Note that $\frac{1}{2}\mathsf{f}_{E_8} - \mathbf{f}_1 \in \mathsf{E}_{10}$. Also, note that $\frac{1}{2}\boldsymbol{\alpha}$ and $\frac{1}{2}\mathsf{f}_{E_8}$ generate the discriminant group of the lattice $\mathsf{E}_{2,4,6}$.

It is known that the orthogonal group $\mathrm{O}(\bar{\mathsf{E}}_{2,4,6})$ is isomorphic to $(\mathbb{Z}/2\mathbb{Z})^{\oplus 8} \rtimes \mathrm{Sp}(8,\mathbb{F}_2)$. The 2-elementary abelian group can be identified with $(V/V^\perp)^\vee$ that acts on V by

$$e_l : v \mapsto v + l(v+V^\perp)\bar{\mathsf{f}}_{E_8}.$$

The subgroup $\mathrm{Sp}(8,\mathbb{F}_2) = \mathrm{Sp}(V/V^\perp)$ can be identified with the orthogonal group of the 9-dimensional subspace equal to the orthogonal complement of $\bar{\mathsf{f}}_{E_8}$ in V.

Lemma 8.4.6 *The reduction homomorphism*

$$\mathsf{r} : W_{2,4,6} \to \mathrm{O}(\bar{\mathsf{E}}_{2,4,6}) \cong 2^8 \rtimes \mathrm{Sp}(8,\mathbb{F}_2) \tag{8.4.4}$$

is surjective. In particular,

$$W_{2,4,6}/W_{2,4,6}(2) \cong 2^8 \rtimes \mathrm{Sp}(8,\mathbb{F}_2). \tag{8.4.5}$$

Proof It is a special case of a more general result about reduction mod 2 of Coxeter groups with Coxeter diagram of type $T_{p,q,r}$ that can be found in [258]. However, for completeness sake, we provide a proof. First, the normal subgroup $\langle\langle s_\alpha\rangle\rangle$ is contained in the kernel of the action of $\mathrm{O}(\mathsf{E}_{2,4,6})$ on $\bar{E}_{2,4,6}$. Thus, it suffices to prove that the reduction homomorphism $\mathrm{O}(\mathsf{E}_{2,4,6})' = W(\mathsf{E}_{2,4,6}) \to \mathrm{O}(\bar{\mathsf{E}}_{2,4,6})$ is surjective. We can identify $\tilde{\mathsf{E}}_8$ with the affine root sublattice of $\mathsf{E}_{2,4,6}$ corresponding to the parabolic subdiagram of type $\tilde{E}_8$. We know from (0.8.17) from Section 0.8 that

$$W_{2,3,6} = W(\tilde{E}_8) = \mathsf{E}_8 \rtimes W(\mathsf{E}_8) \cong \mathbb{Z}^8 \rtimes W_{2,3,5}.$$

Formula (0.8.8) shows that, under the reduction modulo 2 homomorphism, the image of the subgroup $W(\tilde{E}_8)$ is equal to $2^8 \rtimes W(\mathsf{E}_8) \cong 2^8 \rtimes \mathrm{O}(8,\mathbb{F}_2)^+$. It is known that $\mathrm{O}(8,\mathbb{F}_2)^+$ is a maximal subgroup of $\mathrm{Sp}(8,\mathbb{F}_2)$ of index 136 (see [131]). Thus, the image of the whole group $\mathrm{O}(\mathsf{E}_{2,4,6})$ is equal to $2^8 \rtimes \mathrm{Sp}(8,\mathbb{F}_2)$. □

By Theorem 0.8.6, the reduction homomorphism from (8.4.4) is surjective. We know that $W(\mathsf{E}_{2,4,6}) = \langle\langle s_r\rangle\rangle \rtimes W_{2,4,6}$. Since r intersects any vector from $E_{2,4,6}$ with even multiplicity, its image under the reduction homomorphism is equal to zero. Thus, we have a surjective homomorphism

$$\mathsf{r} : W_{2,4,6} \to 2^8 \rtimes \mathrm{Sp}(8,\mathbb{F}_2).$$

Let $W_{2,4,6}(2) = \mathrm{Ker}(\mathsf{r})$ be the 2-level congruence subgroup of $W_{2,4,6}$. We have

$$W_{2,4,6}/W_{2,4,6}(2) \cong 2^8 \rtimes \mathrm{Sp}(8, \mathbb{F}_2) \tag{8.4.6}$$

Let

$$W'_{2,4,6} \subset W_{2,4,6} \tag{8.4.7}$$

be the normal subgroup of $W_{2,4,6}$ that is equal to the pre-image of the normal subgroup 2^8 of $\mathrm{O}(\mathsf{E}_{2,4,6}/2\mathsf{E}_{2,4,6}) \cong 2^8 \rtimes \mathrm{Sp}(8, \mathbb{F}_2)$.

Theorem 8.4.7 *Assume $p \neq 2$ or S is a $\boldsymbol{\mu}_2$-surface. Under a Reye marking,* $\mathrm{Aut}(S)$ *contains a subgroup isomorphic to $W'_{2,4,6} = W_{2,4,6} \cap W_{2,3,7}(2)$.*

Proof By Proposition 8.4.1, the homomorphism $\mathrm{Aut}(S) \to \mathrm{Aut}(S)^*$ is bijective, so we can identify $\mathrm{Aut}(X)$ with a subgroup of $W(\mathrm{Rey}(S))$ and via a marking with a subgroup of $W_{2,4,6}$, where we identify $\mathsf{E}_{2,4,6}$ with the Reye sublattice of E_{10} of vectors intersecting the fixed root r with even multiplicity. Since $W_{2,4,6}$ is generated by reflections in vectors from $E_{2,4,6} \subset \mathsf{E}_{10}$, it extends to a subgroup of $W(\mathsf{E}_{10})$. Thus, $W(\mathsf{E}_{10})(2) \cap W_{2,4,6}$ makes sense.

Applying Corollary 8.4.4 and Theorem 8.1.10, it suffices to show that this intersection coincides with $W'_{2,4,6}$. Let

$$0 \to 2\mathsf{E}_{10} \cap \mathsf{E}_{2,4,6}/2\mathsf{E}_{2,4,6} \to \mathsf{E}_{2,4,6}/2\mathsf{E}_{2,4,6} \to \mathsf{E}_{2,4,6}/2\mathsf{E}_{10} \cap \mathsf{E}_{2,4,6} \to 0$$

be the natural exact sequence of quadratic spaces over $\mathbb{F}_2$. We know from Section 6.4 that $\mathsf{E}_{2,4,6}/2\mathsf{E}_{10} \cap \mathsf{E}_{2,4,6}$ is a 9-dimensional subspace of $\bar{\mathsf{E}}_{10} = \mathsf{E}_{10}/2\mathsf{E}_{10}$ equal to the orthogonal complement to a non-isotropic vector. The subspace $2\mathsf{E}_{10} \cap \mathsf{E}_{2,4,6}/2\mathsf{E}_{2,4,6}$ is generated by an isotropic vector v_0 in the 2-dimensional radical of $\mathsf{E}_{2,4,6}/2\mathsf{E}_{2,4,6}$. We have

$$W_{2,4,6}/W_{2,4,6}(2) \cong \mathrm{O}(\mathsf{E}_{2,4,6}/2\mathsf{E}_{2,4,6})$$

and, since any element of the right-hand-side group preserves the unique isotropic vector in its radical, we have a natural homomorphism

$$W_{2,4,6}/W_{2,4,6}(2) \to \mathrm{O}(\mathsf{E}_{2,4,6}/\mathsf{E}_{2,4,6} \cap 2\mathsf{E}_{10}) = \mathrm{Sp}(8, \mathbb{F}_2).$$

By Witt's theorem, it is surjective. The kernel of this homomorphism is equal to the quotient group $W_{2,4,6} \cap W(\mathsf{E}_{10})(2)/W_{2,4,6}(2)$. This shows that $W_{2,4,6} \cap W(\mathsf{E}_{10})(2) = W'_{2,4,6}$. □

Corollary 8.4.8 *Suppose $\mathbb{k} = \mathbb{C}$. Then, there exists an open subset U of the moduli space $\mathcal{M}^{\mathrm{nod}}_{\mathrm{Enr}}$ of nodal Enriques surfaces such that for any S with the isomorphism class in U,* $\mathrm{Aut}(S) \cong W'_{2,4,6}$.

Proof We know that $\mathrm{Aut}(S)^* = \mathrm{Aut}(S)$ for any nodal surface and hence $\mathrm{Aut}(S)$ contains $W'_{2,4,6}$. Since $\mathrm{Aut}(S)^*$ acts identically on $\mathcal{M}^{\mathrm{m,nod}}_{\mathrm{Enr}}$, the kernel of the action of $W(\mathsf{E}_{10})$ on $\mathcal{M}^{\mathrm{m,nod}}_{\mathrm{Enr}}$ contains the group $W'_{2,4,6}$. Since the kernel of the action is a normal subgroup of $W(\mathsf{E}_{10})$ and the normal closure of $W_{2,4,6}$ contains $W_{2,4,6}$ with quotient $\mathrm{Sp}(8, \mathbb{F}_2)$, we obtain that either the kernel of the action coincides with $W'_{2,4,6}$ or contains $W_{2,4,6}$. Since $W^{\mathrm{nod}}_S \subset W_{2,4,6}$, this is impossible. Since the set of points in

U such that some point in its pre-image $\tilde{U}$ in $\mathcal{M}^{\mathrm{m}}_{\mathrm{Enr}}$ has a non-trivial finite stabilizer subgroup is a closed proper subset, we see that a general point in $\tilde{U}$ has a trivial stabilizer subgroup, and hence $\mathrm{Aut}(S) \cong W'_{2,4,6}$ for such points. □

For any involution σ of a lattice M, we set

$$M^+ = \{m \in M : \sigma(m) = m\}, \quad M^- = \{m \in M : \sigma(m) = -m\}.$$

It is obvious, that $(M^+)^\perp = M^-$.

Let $(\boldsymbol{\beta}_0, \dots, \boldsymbol{\beta}_9)$ be a standard basis of $\mathsf{E}_{2,4,6}$ with the Coxeter–Dynkin diagram of type $T_{2,4,6}$ as above, we immediately observe one affine subdiagram of types $\tilde{E}_7$ and one affine subdiagram of type $\tilde{E}_8$. Let

$$\mathfrak{r}_1 = 2\boldsymbol{\beta}_0 + \boldsymbol{\beta}_1 + 2\boldsymbol{\beta}_2 + 3\boldsymbol{\beta}_3 + 4\boldsymbol{\beta}_4 + 3\boldsymbol{\beta}_5 + 2\boldsymbol{\beta}_6 + \boldsymbol{\beta}_7, \tag{8.4.8}$$

$$\mathfrak{r}_2 = 3\boldsymbol{\beta}_0 + 2\boldsymbol{\beta}_2 + 4\boldsymbol{\beta}_3 + 6\boldsymbol{\beta}_4 + 5\boldsymbol{\beta}_5 + 4\boldsymbol{\beta}_6 + 3\boldsymbol{\beta}_7 + 2\boldsymbol{\beta}_8 + \boldsymbol{\beta}_9, \tag{8.4.9}$$

be their primitive isotropic vectors.

Lemma 8.4.9 *The group $W(\mathsf{E}_{2,4,6})$ acts transitively on the set of involutions σ of $\mathsf{E}_{2,4,6}$ such that $\mathsf{E}^-_{2,4,6} \cong \mathsf{E}_8$ and has two orbits on the set of involutions with $\mathsf{E}^-_{2,4,6} \cong \mathsf{E}_7$ or $\mathsf{E}^-_{2,4,6} \cong \mathsf{E}_7 \oplus \mathsf{A}_1$.*

Proof It suffices to show that the group $W(\mathsf{E}_{2,4,6})$ acts transitively (resp. with two orbits) on the set of primitive sublattices M of the lattice $\mathsf{E}_{2,4,6}$ isomorphic to E_8 (resp. E_7, resp. $\mathsf{E}_7 \oplus \mathsf{A}_1$). The lattice E_8 embeds as the direct summand $\langle \boldsymbol{\beta}_0, \boldsymbol{\beta}_2, \dots, \boldsymbol{\beta}_8 \rangle$ of $\mathsf{E}_{2,4,6}$ with the orthogonal complement $\langle \mathfrak{r}_2, \mathfrak{r}_1 - \boldsymbol{\beta}_9 \rangle$ isomorphic to $\mathsf{I}^{1,1}(2) \cong \mathsf{A}_1 \oplus \mathsf{A}_1(-1)$. It follows from [556, Proposition 1.15.1] that a primitive embedding of E_8 is unique.

Since $D(\mathsf{E}_7) \cong \mathbb{Z}/2\mathbb{Z}$ and $D(\mathsf{E}_{2,4,6}) \cong (\mathbb{Z}/2\mathbb{Z})^{\oplus 2}$, the loc. cit result of Nikulin shows that there are two primitive embedding, up to isometry, of $\mathsf{E}_{2,4,6}$. They can be represented by the sublattices $\langle \boldsymbol{\beta}_0, \boldsymbol{\beta}_1, \dots, \boldsymbol{\beta}_6 \rangle$ and $\langle \boldsymbol{\beta}_0, \boldsymbol{\beta}_2, \dots, \boldsymbol{\beta}_7 \rangle$ isomorphic to E_7. Their orthogonal complements are $\langle \boldsymbol{\beta}_8, \boldsymbol{\beta}_9, \mathfrak{r}_1 \rangle \cong \mathsf{U} \oplus \mathsf{A}_1$ and $\langle \boldsymbol{\beta}_9, \mathfrak{r}_1, \mathfrak{r}_2 \rangle \cong \mathsf{U}(2) \oplus \mathsf{A}_1$, respectively.

It follows that there are two primitive embeddings of $\mathsf{E}_7 \oplus \mathsf{A}_1$, one with orthogonal complement isomorphic to U and another with the orthogonal complement $\mathsf{U}(2)$. □

Definition 8.4.10 A *Kantor involution K* (resp. *Bertini involution B*, resp. *Geiser involution G*) of the lattice $\mathsf{E}_{2,4,6}$ is an involution with $(\mathsf{E}^-_{2,4,6}, \mathsf{E}^+_{2,4,6}) \cong (\mathsf{E}_7, \mathsf{U} \oplus \mathsf{A}_1)$ (resp. $(\mathsf{E}^-_{2,4,6}, \mathsf{E}^+_{2,4,6}) \cong (\mathsf{E}_8, \mathsf{A}_1 \oplus \mathsf{A}_1(-1))$, resp. $(\mathsf{E}^-_{2,4,6}, \mathsf{E}^+_{2,4,6}) \cong (\mathsf{E}_7 \oplus \mathsf{A}_1, \mathsf{U}(2))$).

Consider the following sublattices of $\mathsf{E}_{2,4,6}$:

- $L_1 = \langle \boldsymbol{\beta}_0, \boldsymbol{\beta}_1 \dots, \boldsymbol{\beta}_6 \rangle \cong \mathsf{E}_7$ with $L_1^\perp = \langle \mathfrak{r}_1, \boldsymbol{\beta}_8, \boldsymbol{\beta}_9 \rangle \cong \mathsf{U} \oplus \mathsf{A}_1$.
- $L_2 = \langle \boldsymbol{\beta}_0, \boldsymbol{\beta}_2, \dots, \boldsymbol{\beta}_8 \rangle \cong \mathsf{E}_8$ with $L_2^\perp = \langle \mathfrak{r}_2, \mathfrak{r}_1 - \boldsymbol{\beta}_9 \rangle \cong \mathsf{A}_1 \oplus \mathsf{A}_1(-1)$.
- $L_3 = \langle \boldsymbol{\beta}_0, \boldsymbol{\beta}_2, \dots, \boldsymbol{\beta}_7, \boldsymbol{\beta}_9 \rangle \cong \mathsf{E}_7 \oplus \mathsf{A}_1$ with $L_3^\perp = \langle \mathfrak{r}_1, \mathfrak{r}_2 \rangle \cong \mathsf{U}(2)$.

Note that we have

$$L_1 \perp L_1^\perp \cong L_2 \perp L_2^\perp \cong \mathsf{E}_{2,4,6},$$

and $L_3 \perp L_3^\perp$ is a sublattice of index 2 in $\mathsf{E}_{2,4,6}$.

The involution $-\mathrm{id}_{L_i} \oplus \mathrm{id}_{L_i^\perp}$ is a Kantor (resp. Bertini, resp. Geiser) if $i = 1$ (resp. 2, resp. 3).

It follows from the definition that B, K, G act trivially on the discriminant groups of $L_i \perp L_i^\perp$, hence can be extended to involutions of the Enriques lattice E_{10}. We will often identify these involutions with the corresponding extended involutions of E_{10}.

The main ingredient of the proof of our main result is the following analog of Proposition 8.3.2.

Theorem 8.4.11 *The group $W_{2,4,6}(2)$ (resp. $W'_{2,4,6}$) as defined in* (8.4.7) *is equal to the smallest normal subgroup of $W_{2,4,6}$ containing the involutions B and K (resp. B, K and G).*

A computational proof can be found in [138]. We refer for a conceptual proof to [7].

We will show now that the involutions B, K and G can be realized by automorphisms of a general nodal Enriques surface.

Fix a Reye marking $\phi : \mathrm{Rey}(S) \to \mathsf{E}_{2,4,6}$, and consider a geometric canonical basis $(\beta_0, \dots, \beta_9)$ with $\phi(\beta_i) = \boldsymbol{\beta}_i$. By definition of a Reye marking, $\phi^{-1}(\boldsymbol{\alpha})$ is the class of a (-2)-curve R with $[R] = r$. Using (8.4.1), we can extend the Reye marking to a marking $\tilde{\phi} : \mathrm{Num}(S) \to \mathsf{E}_{10}$ that defines an isotropic sequence $(f_1, \dots, f_{10})$ such that $R = R_{10} := h_{10} - 2f_{10}$, where h_{10} is the numerical Fano polarization defined by the isotropic sequence. Let $f_{i,j} = h_{10} - f_i - f_j, 1 < i < j \le 10$ and $R_i = h_{10} - 2f_i, i = 1, \dots, 10$.

Proposition 8.4.12 *Let K be a Kantor involution in $W(\mathsf{E}_{2,4,6})$. Then, there exists a marking $\phi : \mathrm{Num}(S) \to \mathsf{E}_{10}$ with the associated Reye marking $\phi_r : \mathrm{Rey}(S) \to \mathsf{E}_{2,4,6}$ and an automorphism g_K of S such that $K = \phi_r \circ g_K^* \circ \phi_r^{-1}$.*

Proof In the notation from above, we have $f_{8,9} \cdot f_{9,10} = 1$ and $f_{8,9} \cdot R_9 = f_{9,10} \cdot R_9 = 0$. Let $|2F_{i,j}|$ be an elliptic pencil with $[F_{i,j}] = f_{i,j}$. Let us consider the bielliptic map defined by the linear system $|D| = |2F_{8,9} + 2F_{9,10}|$ of degree $D^2 = 8$. This is a polarization of type (i) in Proposition 3.3.1 (note that $[F_{8,9} + F_{9,10} - R_9]$ can be taken as the class of an isotropic nef class). The bielliptic map $S \to \mathsf{D}_1$ blows down R_9 to a point $x_0 \in \mathsf{D}_1$, and the composition of this map with the projection from x_0 defines a regular degree 2 map $f : S \to \mathsf{C}$, where C is the Cayley 4-nodal cubic surface in $\mathbb{P}^3$. Recall from section 0.7 that it is obtained from a 4-nodal anti-canonical quartic del Pezzo surface D_1 by blowing up one point x_0 (corresponding to the point p_6 in the notation of Propositions 0.7.2, 0.7.3, and 0.7.4) from Volume I. Since R is not a component of a double fiber, the point x_0 does not lie on a line in D. It follows from Theorem 6.5.5 that no other (-2)-curve is blown down, so f is a finite separable cover of degree 2. The pencils $|2F_1|$ and $|2F_2|$ are the pre-images of pencils of conics on C and the image of R is a line on C. It is easy to see

that $f^*(\mathrm{Pic}(\mathsf{C})) = \mathrm{Pic}(S)_{\mathbb{Q}}^{\sigma_K}$ is generated over $\mathbb{Q}$ by the classes of $F_{8,9}, F_{9,10}$ and R_9. Since $2F_{ij} \sim R_i + R_j$, it is also generated by R_8, R_9, R_{10}. Let g_K be the deck transformation of this cover. The Picard group of C is of rank 3 and is generated over $\mathbb{Q}$ by the classes of $f(F_{8,9}), f(F_{9,10}), f(R_9)$. For any divisor class A, there exist rational numbers m, n, l such that

$$g_K^*(A) + A = mR_8 + nR_9 + lR_{10}.$$

Suppose $A \cdot R_9 = A \cdot R_9 = A \cdot R_{10} = 0$. Intersecting both sides with R_8, R_9, R_{10}, we obtain $m = n = l = 0$. This shows that $g_K^*(A) = -A$. Observe that the sublattice of $\mathrm{Num}(S)$ generated by R_8, R_9, R_{10} is isomorphic to $\mathsf{U}(2) \oplus \mathsf{A}_1$ and its orthogonal complement is isomorphic to L_1. More precisely, we have the following formula for g_K^*:

$$g_K^*(x) = -x + (x \cdot f_{9,10})R_8 + (x \cdot f_{8,10})R_9 + (x \cdot f_{8,9})R_{10}. \tag{8.4.10}$$

It is clear that $\phi_r \circ g_K^* \circ \phi_r^{-1}$ is the Kantor involution K. □

Proposition 8.4.13 *Let B be a Bertini involution in $W(\mathsf{E}_{2,4,6})$. Then, there exists a marking $\phi : \mathrm{Num}(S) \to \mathsf{E}_{10}$ with the associated Reye marking $\phi_r : \mathrm{Rey}(S) \to \mathsf{E}_{2,4,6}$ and an automorphism g_B of S such that $B = \phi_r \circ g_B^* \circ \phi_r^{-1}$.*

Proof We choose a Reye marking as in the previous case and consider the linear system $|4F_9 + 2R_{10}|$. It defines a bielliptic map $f : S \to \mathsf{D}'$ to an anti-canonical quartic del Pezzo surface $\mathsf{D}_1', \mathsf{D}_2'$, or D_3'. The curve R is blown down to the singular point of D' and the pencil $|2F_1|$ is the pre-image of a pencil of conics on D' that contains a double line. It follows from Propositions 0.7.2, 0.7.3 and 0.7.4 that f defines a finite map of degree two $f' : S \to \mathsf{C}'$ to a degenerate cubic symmetroid surface. The image $f'(R_{10})$ is a line on C' (blown up from the point p_6 in the notation of the Propositions). The involution B is realized by the deck transformation g_B of the double cover $f' : S \to \mathsf{D}_i'$. In fact, as above, we have $g_B^*(F_9) = F_9, g_B^*(R_{10}) = R_{10}$ and the numerical class of any g_K-invariant curve is contained in $\mathbb{Q}[F_9] + \mathbb{Q}[R_{10}] = f^*(\mathrm{Pic}(\mathsf{C}')_{\mathbb{Q}}) \cong \mathbb{Q}^2$. We have $g_B^*(x) = -x$ for any numerical class orthogonal to the sublattice $\langle [F_9], [R_{10}] \rangle = \langle f_9, h_{10} - 2f_{10} \rangle$. Intersection of this sublattice with $\mathrm{Rey}(S)$ is equal to $\langle 2f_9, h - 2f_{10} \rangle \cong \mathsf{A}_1 \oplus \mathsf{A}_1(-1)$. It is easy to see that its orthogonal complement contains the sublattice $\langle \beta_0, \beta_2, \ldots, \beta_8 \rangle \cong \mathsf{E}_8$. Since the discriminant of $\mathsf{A}_1 \oplus \mathsf{A}_1(-1) \oplus \mathsf{E}_8$ is equal to the discriminant of the whole lattice $\mathsf{E}_{2,4,6}$, we obtain that $\langle 2f_9, h - 2f_{10} \rangle^\perp \cong \mathsf{E}_8$ and $\mathsf{E}_{2,4,6} \cong \mathsf{E}_8 \oplus \mathsf{A}_1 \oplus \mathsf{A}_1(-1) \cong \mathsf{A}_1 \oplus \mathsf{E}_7 \oplus \mathsf{U}$. The involution g_B acts on $\mathrm{Num}(S)$ by

$$g_B^*(x) = -x + 2(x \cdot R_{10})f_9 + 2(x \cdot f_9)(2f_9 + R_{10}). \tag{8.4.11}$$

Thus, g_B^* realizes the Bertini involution B. □

Proposition 8.4.14 *Let G be a Geiser involution in $W(\mathsf{E}_{2,4,6})$. Then, there exists a marking $\phi : \mathrm{Num}(S) \to \mathsf{E}_{10}$ with the associated Reye marking $\phi_r : \mathrm{Rey}(S) \to \mathsf{E}_{2,4,6}$ and an automorphism g_G of S such that $G = \phi_r \circ g_G^* \circ \phi_r^{-1}$.*

Proof We have $s_{\beta_9}(R_{10}) = s_{\beta_9}(h_{10} - 2f_{10}) = h_{10} - 2f_9$. By Lemma 6.3.12, $h_{10} - 2f_9$ is effective. Since $h_{10} \cdot (h_{10} - 2f_9) = 4$, it must be the class of the (-2)-curve R_9. We have $R_9 + R_{10}$ is a member of the elliptic pencil $|2(H - F_9 - F_{10})|$ with half-fiber $F_{9,10} \sim H - F_9 - F_{10}$. Consider the bielliptic linear system $|2F_1 + 2F_{9,10}|$. It defines a bielliptic map $f : S \to \mathsf{D}_1$, where D_1 is a 4-nodal anti-canonical quartic del Pezzo surface. We have $(F_1 + F_{9,10}) \cdot R = 1$, hence any (-2)-curve intersects $F_1 + F_{9,10}$ with odd multiplicity. Thus, the linear system $|2F_1 + 2F_{9,10}|$ is ample, and the map $f : S \to \mathsf{D}_1$ is a degree 2 finite map. Let g_G be its deck transformation. Then, we argue as in the previous cases to obtain that g_G acts as the identity on $\langle f_1, f_{9,10}\rangle$ and as the minus identity on $\langle f_1, f_{9,10}\rangle^\perp$. The intersection of $\langle f_1, f_{9,10}\rangle$ with $\mathrm{Rey}(S)$ is equal to $\langle 2f_1, F_{9,10}\rangle \cong \mathsf{U}(2)$. Its orthogonal complement in $\mathrm{Rey}(S)$ is equal to the sublattice $\langle \beta_0, \beta_2, \dots, \beta_6, \beta_8\rangle \cong \mathsf{E}_7 \oplus \mathsf{A}_1$. Let $f_{9,10} = [F_{9,10}]$. We have the formula for g_G^*:

$$g_G^*(x) = -x + 2(x \cdot f_{9,10})f_1 + 2(x \cdot f_1)f_{9,10}. \tag{8.4.12}$$

Thus, g_G^* realizes the Geiser involution G. □

Corollary 8.4.15 *Let $W(\mathsf{E}_{2,4,6})^+ = \langle\langle K, B, G\rangle\rangle$ be the minimal normal subgroup of $W(\mathsf{E}_{2,4,6})$ that contains a Kantor, a Bertini and a Geiser involution. Then, there exists a Reye marking $\phi_r : \mathrm{Rey}(S) \to \mathsf{E}_{2,4,6}$ such that $\phi_r^{-1} \cdot W_{2,4,6}^+ \cdot \phi \subset \mathrm{Aut}(S)^*$.*

Proof The previous three propositions realize some representatives of the conjugacy classes of Kantor, Bertini and Geiser involution in $W(\mathsf{E}_{2,4,6})$. Since $W_{2,4,6}$ transforms the root α to a root β congruent to α modulo $2\mathsf{E}_{10}$, we can apply Lemma 6.3.12 to see that under the new Reye marking β goes to the class of a (-2)-curve on S. Thus, using the above propositions, we will be able to realize all such involutions and, of course, their products. □

Applying Theorem 8.4.11, we obtain the main result of this section.

Theorem 8.4.16 *Let S be a general nodal Enriques surface. Then, there is an injective homomorphism $\mathrm{Aut}(S) \to W_{2,4,6}$ whose image contains the normal subgroup $W'_{2,4,6}$ with the quotient group $W_{2,4,6}/W'_{2,4,6}$ isomorphic to $\mathrm{Sp}(8, \mathbb{F}_2)$.*

Corollary 8.4.17 *All smooth rational curves on a general nodal Enriques surface form one orbit with respect to the automorphism group.*

Proof By Lemma 6.5.1, for any two (-2)-curves R_1, R_2, we have $R_1 \cdot R_2 \geq 2$. This shows that $(R_1 + R_2)^2 \geq 0$, and hence the hyperplanes $H_{[R_1]}$ and $H_{[R_2]}$ in the hyperbolic space $\mathbb{H}^9$ do not intersect in $\mathbb{H}^9$. Since $W_S^{\mathrm{nod}} \rtimes W'_{2,4,6}$ is a subgroup of finite index of $W_S^{\mathrm{nod}} \rtimes W_{2,4,6} = W(\mathsf{E}_{2,4,6})$, it has a rational polyhedral fundamental domain in the nef cone. Since the representatives of the orbits of $W'_{2,4,6}$ on the set of hyperplanes H_δ are boundary faces of the polyhedron, no two of them must diverge. This contradiction proves the assertion. □

Let us compute the fibers of the forgetting map $\mathcal{M}_{\mathrm{Enr},v} \to \mathcal{M}_{\mathrm{Enr}}$ over the locus of general nodal Enriques surfaces as promised in Section 5.4.

We use Corollary 8.4.5 and its notation that describes the nef cone $\mathrm{Nef}(S)$. Let $\boldsymbol{\beta}_i^*$ be the dual vectors of the standard root basis $(\boldsymbol{\beta}_0, \dots, \boldsymbol{\beta}_9)$ of the Reye lattice $\mathsf{E}_{2,4,6}$. The inverse matrix $C^{-1} = (\boldsymbol{\beta}_i^*, \boldsymbol{\beta}_j^*)$ of the Cartan matrix $C = (\boldsymbol{\beta}_i, \boldsymbol{\beta}_j)$ is equal to the matrix

$$\begin{pmatrix} \frac{5}{2} & \frac{3}{2} & 3 & \frac{9}{2} & 6 & 5 & 4 & 3 & 2 & 1 \\ \frac{3}{2} & 0 & 1 & 2 & 3 & \frac{5}{2} & 2 & \frac{3}{2} & 1 & \frac{1}{2} \\ 3 & 1 & 2 & 4 & 6 & 5 & 4 & 3 & 2 & 1 \\ \frac{9}{2} & 2 & 4 & 6 & 9 & \frac{15}{2} & 6 & \frac{9}{2} & 3 & \frac{3}{2} \\ 6 & 3 & 6 & 9 & 12 & 10 & 8 & 6 & 4 & 2 \\ 5 & \frac{5}{2} & 5 & \frac{15}{2} & 10 & \frac{15}{2} & 6 & \frac{9}{2} & 3 & \frac{3}{2} \\ 4 & 2 & 4 & 6 & 8 & 6 & 4 & 3 & 2 & 1 \\ 3 & \frac{3}{2} & 3 & \frac{9}{2} & 6 & \frac{9}{2} & 3 & \frac{3}{2} & 1 & \frac{1}{2} \\ 2 & 1 & 2 & 3 & 4 & 3, & 2 & 1 & 0 & 0 \\ 1 & \frac{1}{2} & 1 & \frac{3}{2} & 2 & \frac{3}{2} & 1 & \frac{1}{2} & 0 & -\frac{1}{2} \end{pmatrix} \tag{8.4.13}$$

As above, we fix a Reye polarization $\phi : \mathrm{Rey}(S) \to \mathsf{E}_{10}$ that defines a root basis $(\beta_0, \dots, \beta_9)$ in $\mathrm{Rey}(S)$ and an isotropic sequence of nef vectors $(f_1, \dots, f_{10})$ such that $\phi^{-1}(\boldsymbol{\alpha}) = r$ is the class of a (-2)-curve and $r = h_{10} - 2f_{10}$. We know that $\mathrm{Nef}(S)$ is a convex span of the dual vectors β_i^* and it lies in the half-space $\{x : (x, r) \geq 0\}$. A vector in the nef cone belongs to the $W(\mathrm{Rey}(S))$-orbit of a linear combination of the vectors β_i^* with non-negative half-integer coefficients. Using matrix (8.4.13) we can find all orbits of nef divisor classes ω in $\mathrm{Num}(S)$ of degree $d = \omega^2 = 0, 2, 4, 6, 10$. Writing them as linear combinations of isotropic vectors $f_1, \dots, f_{10}$ with numerical Fano polarization $h = \frac{1}{3}(f_1 + \cdots + f_{10})$, we will be able to find those of them with maximal value of the function Φ (for $d \neq 0$).

Table 8.4 below summarizes the results of the computations. Here, G_ω stands for the stabilizer subgroup of v in $W'_{2,4,6}$ so that the degree of the map on the orbit of v is computed by formula

$$N(v) = \frac{\#\,\mathrm{Sp}(8, \mathbb{F}_2)\#G_\omega}{\#W(v^\perp)} = \frac{2^{16} \cdot 3^5 \cdot 5^2 \cdot 7 \cdot 17}{\#W(v^\perp)}.$$

We also give the expression of v as a positive linear combination of the classes of isotropic vectors $f_i, f_{i,j} = h - f_i - f_j$ and the class r of a (-2)-curve representing r from diagram (8.4.3).

Let $\omega_1(d), \dots, \omega_{k(d)}(d)$ represent the different orbits $N(\omega_i(d))$ of primitive polarizations of degree $d = 2, 4, 6, 10$ with maximal possible $\Phi(v)$ and let e_i be the corresponding numbers from the last column of the Table. Then, we have

$$\sum_{i=1}^{k(d)} e_i N(\omega_i(d)) = P(S, v),$$

where $P(S, v)$ is the degree of the forgetting map for general unnodal Enriques surfaces (see table 5.7.3 in Chapter 5). This suggests that, if $e_i = 2$, the forgetting map simply ramifies on the corresponding $W_{2,4,6}$-orbit. We do not know the reason for this.

d	orbit	$v \cdot r$	$W(v^\perp)$	G_ω	$N(v)$	e_i
0	$\beta_8^* = \mathfrak{r}_1$	0	$\mathsf{E}_7 + \mathsf{A}_1$	$2\mathsf{E}_7 \times 2$	$3 \cdot 5 \cdot 17$	1
0	$\beta_1^* = \mathfrak{r}_2$	2	$\tilde{\mathsf{E}}_8$	$2\mathsf{E}_8 \times 2$	$2^3 \cdot 17$	2
2	$\beta_2^* = f_1 + f_2$	2	$\mathsf{A}_1 + \mathsf{D}_8$	2	$2^2 \cdot 3^3 \cdot \cdot 5 \cdot 17$	2
2	$\beta_1^* + \beta_8^* = f_1 + f_{9,10}$	1	$\mathsf{A}_1 + \mathsf{E}_7$	2	$2^6 \cdot 3 \cdot 5 \cdot 17$	2
2	$2\beta_1^* + 2\beta_9^* = 2f_1 + r$	0	E_8	2	$2^3 \cdot 17$	1
2	$\beta_7^* + \beta_9^* = f_{8,10} + f_{9,10}$	0	$\mathsf{A}_1 + \mathsf{E}_7$	2	$2^6 \cdot 3 \cdot 5 \cdot 17$	1
4	$\beta_6^* = f_{6,7} + f_{8,9}$	2	$\mathsf{A}_3 + \mathsf{D}_6$	1	$2^4 \cdot 3^2 \cdot 5 \cdot 7 \cdot 17$	2
4	$\beta_0^* + \beta_9^* = f_{10} + r$	1	A_8	1	$2^9 \cdot 3 \cdot 5 \cdot 17$	2
4	$\beta_2^* + 2\beta_9^* = f_1 + f_2 + r$	0	$\mathsf{A}_1 + \mathsf{D}_7$	1	$2^5 \cdot 3^3 \cdot 5 \cdot 17$	1
6	$\beta_3^* = f_1 + f_2 + f_3$	3	$\mathsf{A}_2 + \mathsf{A}_7$	1	$2^8 \cdot 3^2 \cdot 5 \cdot 17$	2
6	$\beta_2^* + \beta_8^* = f_1 + f_2 + f_{9,10}$	2	$\mathsf{A}_1^2 + \mathsf{D}_6$	1	$2^5 \cdot 3^3 \cdot 5 \cdot 7 \cdot 17$	2
6	$2\beta_7^* = 2f_{8,9} + r$	2	$\mathsf{A}_2 + \mathsf{E}_7$	2	$2^6 \cdot 5 \cdot 17$	2
6	$\beta_1^* + \beta_7^* + \beta_9^* = f_{8,9} + f_1 + r$	1	$\mathsf{A}_1 + \mathsf{E}_6$	1	$2^8 \cdot 3 \cdot 7 \cdot 5 \cdot 17$	2
6	$2\beta_1^* + \beta_8^* + 2\beta_9^* = f_{9,10} + 2f_1 + r$	0	E_7	1	$2^6 \cdot 3 \cdot 5 \cdot 17$	1
6	$\beta_6^* + 2\beta_9^* = f_{7,8} + f_{9,10} + r$	0	$\mathsf{A}_2 + \mathsf{D}_6$	1	$2^6 \cdot 3^2 \cdot 5 \cdot 7 \cdot 17$	1
10	$2\beta_0^* = h$	4	A_9	1	$2^8 \cdot 3 \cdot 17$	2
10	$\beta_0^* + \beta_7^* = f_{8,9} + f_{10} + r$	3	$\mathsf{A}_6 + \mathsf{A}_2$	1	$2^{11} \cdot 3^2 \cdot 5 \cdot 17$	2
10	$\beta_5^* + \beta_9^* = f_{6,7} + f_{8,9} + r$	2	$\mathsf{A}_5 + \mathsf{A}_3$	1	$2^9 \cdot 3^2 \cdot 5 \cdot 7 \cdot 17$	2
10	$\beta_3^* + 2\beta_9^* = f_1 + f_2 + f_3 + r$	1	$\mathsf{A}_2 + \mathsf{A}_6$	1	$2^{11} \cdot 3^2 \cdot 5 \cdot 17$	2
10	$\beta_1^* + \beta_2^* + 3\beta_9^* = f_{1,10} + 2f_1 + r$	0	A_7	1	$2^9 \cdot 3^3 \cdot 5 \cdot 17$	1

Table 8.3 Degree of $\mathcal{M}_{\mathrm{Enr},v} \to \mathcal{M}_{\mathrm{Enr}}$ for general nodal surfaces

8.5 Automorphisms of a Cayley Quartic Symmetroid

In this section, we will explain the reason for attaching the names Kantor, Bertini, and Geiser to the involutions studied in the previous section. We will also give another proof of Theorem 8.4.11 that follows closely the ideas of A. Coble.

We assume that $p \neq 2$ and S is a general nodal surface and consider a Fano–Reye polarization H that defines an isomorphism from S onto a Reye congruence $\mathrm{Rey}(W) \subset G_1(|H|^*)$ of an excellent web of quadrics in $\mathbb{P}^3$. We know from Corollary 7.9.8 that the canonical cover X of S is isomorphic to a minimal nonsingular model $\widetilde{\mathcal{D}}(W)$ of a quartic symmetroid surface $\mathcal{D}(W)$.

We have 10 elliptic fibrations $|2F_i|$ with $[F_i] = f_i$ and 10 smooth rational curves R_i with $[R_i] = h - 2f_i$ such that $R_i + R_j \in |2F_{i,j}|, i \neq j$, where $F_{i,j} = H - F_i - F_j$ is a double fiber of the elliptic pencil $|2F_{ij}|$. The pre-images of the curves R_i on X are 20 smooth rational curves $\Theta_i, \tau^*(\Theta_i)$, where τ is the deck Enriques involution of X. We denote their divisor classes by θ_i. We use the basic relation from (7.7.3)

$$2\eta_S = 3\eta_H - \theta_1 - \cdots - \theta_{10}. \tag{8.5.1}$$

The linear system $|\eta_H|$ maps X to $\mathbb{P}^3$ onto the quartic symmetroid $\mathcal{D}(W)$ and the linear system $|\eta_S|$ maps X onto a smooth quartic surface in another $\mathbb{P}^3$. The curves Θ_i are identified with the exceptional curves of $\widetilde{\mathcal{D}}(W) \to \mathcal{D}(W)$.

We make a further assumption that $\mathrm{rank}\,\mathrm{Pic}(X) = 11$, i.e. η_H, η_S, θ_i generate $\mathrm{Pic}(X)_{\mathbb{Q}}$. Applying Proposition 7.7.3, we obtain that $\eta_S, \eta_H, \theta_1, \ldots, \theta_9$ form a basis of $\mathrm{Pic}(X)$.

Let $\mathcal{P} = \{p_1, \ldots, p_{10}\}$ be the 10 nodes of the quartic symmetroid $\mathcal{D}(W)$ and let $\mathrm{Bl}_{\mathcal{P}} \to \mathbb{P}^3$ be the blow-up of $\mathbb{P}^3$ with center at the closed reduced subscheme defined by the set $\mathcal{P}$. Let $\mathcal{E}_i = \sigma^{-1}(p_i)$ be the exceptional divisors of the blow-up. Let e_0 be the divisor class corresponding to $\sigma^*\mathcal{O}_{\mathbb{P}^3}(1)$ and let e_i be the divisor classes of the

exceptional divisors $\mathcal{E}_i$. We have

$$K_{\mathrm{Bl}_{\mathcal{P}}} = -4e_0 + 2e_1 + \cdots + 2e_{10}.$$

The proper transform of $\mathcal{D}(W)$ under the blow-up is isomorphic to X. Let

$$r : \mathrm{Pic}(\mathrm{Bl}_{\mathcal{P}}) \to \mathrm{Pic}(X)$$

be the restriction homomorphism. We have

$$\mathfrak{k} = r(\frac{1}{2}K_{\mathrm{Bl}_{\mathcal{P}}}) = -2\eta_H + \theta_1 + \cdots + \theta_{10} = -2\eta_S + \eta_H.$$

Lemma 8.5.1 *Let τ be the Enriques involution of X. Then, τ^* acts on* $\mathrm{Pic}(X)$ *as the reflection*

$$s_{\mathfrak{k}} : x \mapsto x + \frac{1}{2}(x \cdot \mathfrak{k})\mathfrak{k}.$$

Proof It is enough to check this formula on the classes θ_i, η_S, η_H. We have $\mathfrak{k}^2 = -4, \mathfrak{k} \cdot \theta_i = -2, \mathfrak{k} \cdot \eta_S = -2, \mathfrak{k} \cdot \eta_H = -8$. Thus, $\frac{1}{2}\mathfrak{k} \in \mathrm{Pic}(X)^\vee$, hence we can consider the reflection

$$s_{\mathfrak{k}} : x \mapsto x + \frac{1}{2}(x \cdot \mathfrak{k})\mathfrak{k}.$$

Applying Proposition 7.7.1, we check that

$$\begin{aligned} \tau^*(\eta_S) &= 3\eta_S - \eta_H = s_{\mathfrak{k}}(\eta_S), \\ \tau^*(\eta_H) &= 8\eta_S - 3\eta_H = s_{\mathfrak{k}}(\eta_H), \\ \tau^*(\theta_i) &= \theta_i - \mathfrak{k} = s_{\mathfrak{k}}(\theta_i). \end{aligned}$$

Now, we have to invoke the theory of *Cremona actions of Weyl groups* on the configuration spaces of points in the projective space (see [123], [193], [176]). Let e^0 be the class of the pre-image of a line in $\mathrm{Bl}_{\mathcal{P}}$ in the Chow ring $A^*(\mathrm{Bl}_{\mathcal{P}})$, and let e^i be the class of a line in the exceptional divisor $\mathcal{E}_i$. The intersection bilinear form $A^1(\mathrm{Bl}_{\mathcal{P}}) \times A^2(\mathrm{Bl}_{\mathcal{P}}) \to \mathbb{Z}$, gives us that

$$(e_i, e^j) = 0, \quad i \neq j, \quad (e_0, e^0) = 1, \quad (e_i, e^i) = -1.$$

Consider the map $\Psi : A^1(\mathrm{Bl}_{\mathcal{P}}) \to A^2(\mathrm{Bl}_{\mathcal{P}})$ defined by sending e_0 to $2e^0$ and e_i to e^i and define a quadratic lattice structure on $A^1(\mathrm{Bl}_{\mathcal{P}})$ by

$$x \cdot y = (x, \Psi(y)).$$

Obviously, it is isomorphic to the odd lattice $\langle 2 \rangle \perp \langle -1 \rangle^{\oplus 10}$. Let

$$\kappa = \Psi(\frac{1}{2}K_{\mathrm{Bl}_{\mathcal{P}}}) = -4e^0 + e^1 + \cdots + e^{10}.$$

Then,

$N_{\mathcal{P}} = K^{\perp}_{\mathrm{Bl}_{\mathcal{P}}} = \{x \in A^1(\mathrm{Bl}_{\mathcal{P}}) : (x, \kappa) = 0\} = \langle e_0 - e_1 - e_2 - e_3 - e_4, e_1 - e_2, \ldots, e_9 - e_{10}\rangle.$

Consider the following divisor classes:

$$\alpha_0 = e_0 - e_1 - e_2 - e_3 - e_4, \quad \alpha_i = e_i - e_{i+1},\ i = 1, \ldots, 9.$$

We have $\alpha_i^2 = -2$ and $(\alpha_i, \alpha_j) = -2I_{10} + \Gamma$, where Γ is the incidence matrix of the graph $T_{2,4,6}$. Thus, $\mathsf{B} = (\alpha_0, \ldots, \alpha_9)$ is a standard root basis of the lattice $K^{\perp}_{\mathrm{Bl}_{\mathcal{P}}} \cong \mathsf{E}_{2,4,6}$. The reflection group generated by the reflections s_{α_i} is isomorphic to $W_{2,4,6}$. Let $\mathcal{R}$ be the set of roots of $\mathsf{E}_{2,4,6}$, i.e. elements of the orbit of $W_{\mathsf{B}}(\mathsf{E}_{2,4,6})$ of any simple root α_i. We say that a root α is *effective* if $\Psi(\alpha)$ can be represented by the class of a curve on $\mathrm{Bl}_{\mathcal{P}}$. For example, α_i is always effective since we assume that $p_i \neq p_j$. The root α_0 is effective if and only if p_1, p_2, p_3, p_4 do not lie on a conic, in particular, they are not coplanar. We say that $\mathcal{P}$ is an *unnodal set* if all roots are not effective. One can show that the subset of $(\mathbb{P}^3)^{10}$ which consists of unnodal sets is equal to the complement of a countable set of closed proper subsets ([176], Lemma 2.2).

Let P_3^{10} be the GIT-quotient of $(\mathbb{P}^3)^{10}$ by the group SL(4) acting diagonally with respect to the linearization defined by the invertible sheaf $\mathcal{O}_{\mathbb{P}^3}(1)^{\boxtimes 10}$. Let ${}^{\mathrm{un}}P_3^{10}$ be the subset of orbits of unnodal sets. We denote by $|\mathcal{P}|$ the orbit of an unnodal set $\mathcal{P}$. The group $W_{2,4,6}$ acts on ${}^{\mathrm{un}}P_3^{10}$ as follows. Its subgroup generated by simple roots $s_{\alpha_i}, i \neq 0$, acts by permuting the points. The element s_{α_0} acts as follows. We find a representative of $\mathcal{P}$ such that the first four points are $[1, 0, 0, 0], [0, 0, 1, 0], [0, 0, 1, 0], [0, 0, 0, 1]$. Then, we consider the standard quadratic Cremona transformation T_0 that sends $[t_0, \ldots, t_3]$ to $[t_0^{-1}, t_1^{-1}, t_2^{-1}, t_3^{-1}]$ and let s_{α_0} act by sending $(p_1, \ldots, p_{10})$ to $(p_1, p_2, p_3, p_4, T_0(p_5), \ldots, T_0(p_{10}))$.

We use the basis $(e_0, \ldots, e_{10})$ to define an isomorphism of lattices

$$\phi_{\mathcal{P}} : \langle 2\rangle \oplus \langle -1\rangle^{\oplus 10} \to A^1(\mathrm{Bl}_{\mathcal{P}}).$$

We call it a *geometric marking*. Its restriction to $K^{\perp}_{\mathrm{Bl}_{\mathcal{P}}}$ defines an isomorphism of lattices $\phi'_{\mathcal{P}} : \mathsf{E}_{2,4,6} \to K^{\perp}_{\mathrm{Bl}_{\mathcal{P}}}$.

Proposition 8.5.2 *For any $w \in W_{2,4,6}$ and a geometric marking $\phi_{\mathcal{P}}$, the isomorphism*

$$\phi_{\mathcal{P}} \circ w^{-1} : \langle 2\rangle \oplus \langle -1\rangle^{\oplus 10} \to A^1(\mathrm{Bl}_{\mathcal{P}})$$

is a geometric marking defined by some unnodal set of points $\mathcal{Q}$ such that $w(|\mathcal{P}|) = |\mathcal{Q}|$.

Suppose that $w(|\mathcal{P}|) = |\mathcal{P}|$, i.e. w belongs to the stabilizer of $W_{2,4,6}$ in its action on ${}^{\mathrm{un}}P_3^{10}$. Then, Proposition 2 from [176] tells us that there exists a birational automorphism $f : \mathrm{Bl}_{\mathcal{P}} \dashrightarrow \mathrm{Bl}_{\mathcal{P}}$ which is an isomorphism outside of a closed subset of dimension 1 (a *pseudo-automorphism*) such that $w = \phi_{\mathcal{P}}^{-1} \circ f^* \circ \phi_{\mathcal{P}}$.

Observe that under the restriction homomorphism

$$r : A^1(\mathrm{Bl}_{\mathcal{P}}) = \mathrm{Pic}(\mathrm{Bl}_{\mathcal{P}}) \to \mathrm{Pic}(X),$$

we have

$$r(K_{\mathrm{Bl}_{\mathcal{P}}}^{\perp})(2) \cong \pi^*(\mathrm{Rey}(S)) = \langle \eta_H - \theta_1 - \theta_2 - \theta_3 - \theta_4, \theta_1 - \theta_2, \ldots, \theta_9 - \theta_{10} \rangle \cong \mathsf{E}_{2,4,6}(2).$$

Suppose $w \in W_{2,4,6}$ satisfies $w(|\mathcal{P}|) = |\mathcal{P}|$ and hence defines a pseudo-automorphism $f_w : \mathrm{Bl}_{\mathcal{P}} \dashrightarrow \mathrm{Bl}_{\mathcal{P}}$. It follows from the definition that w leaves invariant the linear system $|-K_{\mathbb{P}^3}|$. Now, observe that $\mathcal{P}$ consists of the nodes of a quartic symmetroid $\mathcal{D}(W)$, so $|-K_{\mathbb{P}^3}| = \{\mathcal{D}(W)\}$. This shows that $f_w|X$ is a birational transformation. Since X is a minimal surface of non-negative Kodaira dimension, it extends to an automorphism of X.

Coble shows that the Kantor, Bertini, or Geiser involution w of $\mathsf{E}_{2,4,6}$ belongs to the stabilizer of $|\mathcal{P}|$, and hence each defines an automorphism of X that commutes with τ and descends to the automorphism g_K, g_B or g_G, respectively, which we used before.

The Kantor involution coincides with the deck involution of a rational degree two map

$$\mathbb{P}^3 \dashrightarrow V \subset \mathbb{P}^6,$$

V is a projective cone over a Veronese surface in $\mathbb{P}^5$. It is defined by the six-dimensional linear system $|Q|$ of quartics with double points at the nodes $p_1, \ldots, p_7$ of the symmetroid.

Consider the net $|L|$ of quadrics with the set of base point $\{p_1, \ldots, p_7, q\}$. After we blow up the base points, we obtain an elliptic fibration $f : Y \to |L^\vee| \cong \mathbb{P}^2$. The exceptional divisor $\mathcal{E}(q)$ over the point q is its section which we use to define a group law on the generic fiber. The negation involution $a \mapsto -a$ of the generic fiber extends to a birational pseudo-automorphism of Y, called the *Kantor involution* (see [123]). The base scheme of a general pencil P in $|L|$, considered as a point in $|L^\vee|$, is a quartic elliptic curve $E(P)$ isomorphic to the fiber of f over the point P. Take a point p on a quartic elliptic curve $E(P)$ through $p_1, \ldots, p_7, q$. A quartic surface with seven nodes at $p_1, \ldots, p_7$ passing through the point p intersects $E(P)$ at one more point p'. It cuts out the divisor $2(p_1 + \cdots + p_7) + p + p'$ on $E(P)$. On the other hand a quartic with eight nodes at $p_1, \ldots, p_7, q$ cuts out the divisor $2(p_1 + \cdots + p_7) + 2q$. This shows that $p + p' \sim 2q$, so that $p' = -p$ in the group law on $E(P)$. Thus, the involution $p \mapsto p'$ on $E(P)$ is the restriction of the Kantor involution to $E(P)$. The set of fixed points of σ_K restricted to $E(P)$ is the set of four 2-torsion points. The set of fixed points of σ_K in Y is the closure of the set of 2-torsion points on a generic fiber of f. It is equal to the union of the exceptional divisor $\mathcal{E}(q)$ and the proper inverse transform of a certain surface of degree 6 with triple points at $p_1, \ldots, p_7$, the *Cayley dianodal surface* (see [123], [193]). Our quartic symmetroid surface is a quartic with 10 nodes, including the nodes at $p_1, \ldots, p_7$. It does not pass through the point q (otherwise the quadrics through $p_1, \ldots, p_8$ will cut out the net $|2\eta_H - \Theta_1 - \ldots - \Theta_8|$), however, by Riemann–Roch, this is just a pencil). Thus, the three remaining nodes of the symmetroid are fixed and this makes the symmetroid invariant under σ_K.

For any general plane Π in $\mathbb{P}^3$, the linear system $|Q|$ maps Π to a surface of degree 16 which is cut out in V by a quartic hypersurface. This shows that $\sigma_K^*(\Pi) + \Pi \sim$

$f^*(4H)$, where H is a hyperplane section in $\mathbb{P}^6$. Let $(e_0, e_1, \dots, e_7)$ be the geometric basis of $\mathrm{Pic}(\mathrm{Bl}_{p_1,\dots,p_7})$. Thus,

$$\sigma_K^*(e_0) = 4(4e_0 - 2(e_1 + \cdots + e_7)) - e_0 = 15e_0 - 8(e_1 + \cdots + e_7).$$

Similarly, we see that the proper transform of the linear system $|Q|$ to Y maps each exceptional divisor $\mathcal{E}(p_i)$ to a surface of degree 2 cut out from V by a quadric. This gives

$$\sigma_K^*(e_i) = -e_i + 2(2e_0 - e_1 - \cdots - e_7), \quad i = 1, \dots, 7.$$

Restricting to X, we get

$$\begin{aligned} \sigma_K^*(\eta_H) &= 15\eta_H - 8\theta_1 - \cdots - 8\theta_7, \\ \sigma_K^*(\theta_i) &= -\theta_i + 2(2\eta_H - \theta_1 - \cdots - \theta_7),\ i = 1, \dots, 7, \\ \sigma_K^*(\theta_j) &= \theta_j,\ j = 8, 9, 10. \end{aligned} \tag{8.5.2}$$

One can rewrite these formulas in the following way:

$$\sigma_K^*(x) = -x + (x \cdot \tilde{f}_{9,10})\theta_8 + (x \cdot \tilde{f}_{8,10})\theta_9 + (x \cdot \tilde{f}_{8,9})\theta_{10}, \tag{8.5.3}$$

where $\tilde{f}_{i,j} = \pi^*([F_{i,j}])$. Comparing this with formula (8.4.10), we find that σ_K is the lift to X of a Kantor involution on S. The involution σ_K commutes with the Enriques involution τ and descends to an involution of S whose action on $\mathrm{Num}(S)$ coincides with the action of a Kantor involution g_K. Since, by Proposition 8.4.1, no automorphism of S acts identically on $\mathrm{Num}(S)$, we obtain that it coincides with g_K.

Next, we consider the lift of a *Bertini involution*. Recall that a Bertini involution of $\mathbb{P}^2$ is defined by a choice of a set of eight points $Q = \{q_1, \dots, q_8\}$ such that its blow-up Bl_Q is a weak del Pezzo surface $\mathcal{D}$ of degree one. The linear system $|-2K_{\mathcal{D}}|$ defines a degree two finite map onto a singular quadric Q in $\mathbb{P}^3$ with a smooth canonical curve of degree 6 as the branch curve (see [177, 8.8]). The deck transformation of the cover $\mathrm{Bl}_Q \to Q$ is a birational involution of the plane, called a *Bertini involution*. The Bertini involution defines the negation involution on a general member of the pencil of cubic curves with base points $p_1, \dots, p_8, q_9$ with the group law defined by the point q_9. We see that the Kantor involution is its three-dimensional analog.

For any birational transformation $T : \mathbb{P}^n \dashrightarrow \mathbb{P}^n$ given by homogeneous polynomials $G_0, \dots, G_n$ of degree d in variables $t_0, \dots, t_n$, one defines the *dilated transformation* of $\mathbb{P}^{n+1}$ as follows. Choose homogeneous polynomials $F_{n+1}(t_0, \dots, t_n, t_{n+1}) = t_{n+1}A_1(t_0, \dots, t_n) + A_2(t_0, \dots, t_n)$ of degree $d + r$ and homogeneous polynomials $Q(t_0, \dots, t_n, t_{n+1}) = t_{n+1}B_1(t_0, \dots, t_n) + B_2(t_0, \dots, t_n)$ of degree r such that $A_1B_2 - A_2B_1 \neq 0$. Then, the transformation defined by the polynomials $(QG_0, QG_1, \dots, QG_n, F_{n+1})$ is a birational transformation $\bar{T}$ of $\mathbb{P}^{n+1}$ such that

$$\mathrm{pr}_{\mathfrak{o}} \circ \bar{T} = T \circ \mathrm{pr}_{\mathfrak{o}},$$

where $\mathfrak{o} = [0, \dots, 0, 1] \in \mathbb{P}^{n+1}$ and $\mathrm{pr}_{\mathfrak{o}} : \mathbb{P}^{n+1} \dashrightarrow \mathbb{P}^n$ is the projection from $\mathfrak{o}$ to the hyperplane $t_{n+1} = 0$ (see [123], [585]). The base scheme of the linear system defining the dilated transformation is the cone with the vertex at $\mathfrak{o}$ over the base scheme B of the linear system defining the transformation T. It follows from the definition of a dilated transformation that the multiplicity of a general member of the linear system defining $\bar{T}$ at the point $\mathfrak{o}$ is equal to $d + r - 1$. Let n_i (resp. k_i) be the minimal of multiplicities of F_{n+1} (resp. Q) at the line $\overline{\mathfrak{o}, q}$, where $q \in B$. Then, a general member of the linear system defining the base locus $\bar{B}$ of $\bar{T}$ has multiplicity at this line equal to $\min\{n_i, k_i + m_i\}$, where m_i is the multiplicity of q in B.

For example, if we take for T the standard Cremona involution $T_0 : [t_0, t_1, t_2] \mapsto [t_1t_2, t_0t_2, t_0t_1]$ and take $F_3 = t_0t_1t_2$ and $Q = t_3$, we obtain the standard cubic Cremona involution of $\mathbb{P}^3$ defined by the formula

$$\tilde{T}_0 : [t_0, t_1, t_2, t_3] \mapsto [t_1t_2t_3, t_0t_2t_3, t_0t_1t_3, t_0t_1t_2].$$

It is known that the planar Bertini transformation β acts on $\mathrm{Pic}(\mathrm{Bl}_{q_1,\dots,q_8})$ by formula

$$\begin{aligned} \beta^*(e_0) &= 17e_0 - 6(e_1 + \cdots + e_8), \\ \beta^*(e_i) &= -e_i + 6e_0 - 2(e_1 + \cdots + e_8), \end{aligned} \tag{8.5.4}$$

where $(e_0, e_1, \dots, e_8)$ is the natural basis in $\mathrm{Pic}(\mathrm{Bl}_{q_1,\dots,q_8})$ (see [177], 8.8.2). We hope that no confusion arises because we are using the same notations for a geometric basis of the blow-up of $\mathbb{P}^2$ and $\mathbb{P}^3$.

Let $p_1, \dots, p_9$ be general points in $\mathbb{P}^3$, and let $q_1, \dots, q_8$ be their projections to a general plane in $\mathbb{P}^3$ from the point p_9. Let β be the Bertini involution in the plane defined by the points $q_1, \dots, q_8$ and $\tilde{\beta}$ be its dilation defined by the point $\mathfrak{o} = p_9$. Coble shows that one can choose polynomials F_3 and Q such that the dilated transformation $\tilde{\beta}$ is given by the linear system of hypersurfaces of degree 33 vanishing at the point p_9 with multiplicity 32 and vanishing on the lines $\overline{p_9, p_i}$ with multiplicity 12. In the geometric basis of the blow-up $\mathrm{Bl}_{p_1,\dots,p_9}$, the transformation $\tilde{\beta}$ acts by the formula

$$\begin{aligned} \tilde{\beta}^*(e_0) &= 33e_0 - 32e_9 - 12(e_1 + \cdots + e_8), \\ \tilde{\beta}^*(e_i) &= 6e_0 - 2(e_1 + \cdots + e_8) - e_i - 6e_9, \ i = 1, \dots, 8, \\ \tilde{\beta}^*(e_9) &= 16e_0 - 2(e_1 + \cdots + e_8) - 15e_9. \end{aligned} \tag{8.5.5}$$

Let $\mathcal{D}(W)$ be a quartic symmetroid with nodes at $p_1, \dots, p_{10}$. One checks that $\tilde{\beta}$ leaves the linear system $|4e_0 - 2(e_1 + \dots + e_9)|$ invariant. In particular, the image of the symmetroid under $\tilde{\beta}$ is a quartic surface with nodes at $p_1, \dots, p_9, \tilde{\beta}(p_{10})$. One can show that the nine nodes of a quartic symmetroid determines the tenth one. Thus, $\tilde{\beta}(\mathcal{D}(W))$ is a quartic symmemtroid with the same set of nodes, so it must coincide with $\mathcal{D}(W)$. This shows that $\tilde{\beta}$ defines an automorphism σ_B of $\mathcal{D}(W)$ that acts on the natural basis of the Picard lattice of a minimal resolution X of $\mathcal{D}(W)$ as follows:.

$$\begin{aligned}\sigma_B^*(\eta_H) &= 33\eta_H - 32\theta_9 - 12(\theta_1 + \cdots + \theta_8),\\ \sigma_B^*(\theta_i) &= 6\eta_H - 2(\theta_1 + \cdots + \theta_8) - \theta_i - 6\theta_9,\ i = 1, \ldots, 8,\\ \sigma_B^*(\theta_9) &= 16\eta_H - 6(\theta_1 + \cdots + \theta_8) - 15\theta_9,\\ \sigma_B^*(\theta_{10}) &= \theta_{10}.\end{aligned} \tag{8.5.6}$$

One checks that this transformation of the Picard group is of order 2, so it defines an involution σ_B of X. One also checks that σ_B commutes with the Enriques involution, hence σ_B descends to an involution g_B of the Reye congruence $S = \mathrm{Rey}(W)$. Using Proposition 7.7.1, we easily find the action of g_B on $\mathrm{Num}(S)$:

$$\begin{aligned}g_B^*(h) &= 33h - 16R_9 - 6(R_1 + \cdots + R_8),\\ g_B^*(R_i) &= 12h - 2(R_1 + \cdots + R_8) - R_i - 6R_9,\ i = 1, \ldots, 8,\\ g_B^*(R_9) &= 32h - 6(R_1 + \cdots + R_8) - 15R_9,\\ g_B^*(R_{10}) &= R_{10}.\end{aligned} \tag{8.5.7}$$

It follows that $g_B^*(2f_9) = g_B^*(h - R_9) = h - R_9$. Thus, g_B^* acts identically on the sublattice $\langle 2f_9 + R_{10}, R_{10}\rangle \cong \mathsf{A}_1 \oplus \mathsf{A}_1(-1)$. We also check that it acts as the minus identity on its orthogonal complement. Therefore, g_B^* coincides with the Bertini involution.

Finally, we consider the lift of a Geiser involution to the K3-cover. Let $\mathrm{Bl}_{q_1,\ldots,q_7}(\mathbb{P}^2)$ be a weak del Pezzo surface of degree two obtained by blowing up seven points $q_1, \ldots, q_7$ in $\mathbb{P}^2$. Recall that the anti-canonical map is a degree two map onto $\mathbb{P}^2$ and its deck transformation γ is called the planar Geiser involution associated to seven points $q_1, \ldots, q_7$ on $\mathbb{P}^2$. For a general point q in the plane, $\gamma(q)$ is the ninth base point of the pencil of cubic curves passing through the points $q_1, \ldots, q_7, q$. It is known (see [177, 8.7.2]) that the planar Geiser transformation γ acts on the $\mathrm{Pic}(\mathrm{Bl}_{q_1,\ldots,q_7}(\mathbb{P}^2))$ by the following formulas:

$$\begin{aligned}\gamma^*(e_0) &= 8e_0 - 3(e_1 + \cdots + e_7),\\ \gamma^*(e_i) &= 3e_0 - (e_1 + \cdots + e_7) - e_i,\end{aligned} \tag{8.5.8}$$

where $(e_0, e_1, \ldots, e_7)$ is a geometric basis in $\mathrm{Pic}(\mathrm{Bl}_{q_1,\ldots,q_7}(\mathbb{P}^2))$. Let $p_1, \ldots, p_7, p_8$ be the first eight nodes of the quartic symmetroid. One can define a *dilated Geiser transformation* $\tilde{\gamma}$ of $\mathbb{P}^3$ that acts on the geometric basis of $\mathrm{Bl}_{p_1,\ldots,p_8}(\mathbb{P}^3)$ as follows:

$$\begin{aligned}\tilde{\gamma}^*(e_0) &= 15e_0 - 14e_8 - 6(e_1 + \cdots + e_7),\\ \tilde{\gamma}^*(e_i) &= 3e_0 - (e_1 + \cdots + e_7) - e_i - 3e_8,\ i = 1, \ldots, 7,\\ \tilde{\gamma}^*(e_8) &= 7e_0 - 3(e_1 + \cdots + e_7) - 6e_8.\end{aligned} \tag{8.5.9}$$

We immediately check that it leaves the linear system $|4e_0 - 2(e_1 + \ldots + e_8)|$ invariant and sends $\mathcal{D}(W)$ to a quartic symmetroid with nodes at $p_1, \ldots, p_8, \tilde{\gamma}(p_9), \tilde{\gamma}(p_{10})$. We refer to [**?**, p. 264] for a lengthy argument that shows that the sets $\{p_9, p_{10}\}$ and $\{\tilde{\gamma}(p_9), \tilde{\gamma}(p_{10})\}$ coincide. Thus, $\tilde{\gamma}$ leaves $\mathcal{D}(W)$ invariant and defines an involution of $\mathcal{D}(W)$.

Let σ_G be an automorphism of $\mathcal{D}(W)$ defined by the involution $\tilde{\gamma}$. It acts on $\mathrm{Pic}(\mathrm{Bl}_{p_1,\dots,p_8}(\mathbb{P}^3)$ as follows:

$$\begin{aligned}\sigma_G^*(\eta_H) &= 33\eta_H - 32\theta_9 - 12(\theta_1 + \cdots + \theta_8),\\ \sigma_G^*(\theta_i) &= 6\eta_H - 2(\theta_1 + \cdots + \theta_8) - \theta_i - 6\theta_9,\ i = 1, \dots, 8,\\ \sigma_G^*(\theta_9) &= 16\eta_H - 6(\theta_1 + \cdots + \theta_8) - 15\theta_9,\\ \sigma_G^*(\theta_{10}) &= \theta_{10}.\end{aligned} \tag{8.5.10}$$

Again we check that this transformation of the Picard group is of order 2, so it defines an involution σ_G of X. We also check that σ_G commutes with the Enriques involution, hence σ_G descends to an involution g_G of the Enriques surface $S(W)$. Using Proposition 7.7.1, we easily find the action of g_G on $\mathrm{Num}(S)$:

$$\begin{aligned}g_G^*(h) &= 33h - 16R_9 - 6(R_1 + \cdots + R_8),\\ g_G^*(R_i) &= 12h - 2(R_1 + \cdots + R_8) - R_i - 6R_9,\ i = 1, \dots, 8,\\ g_G^*(R_9) &= 32h - 6(R_1 + \cdots + R_8) - 15R_9,\\ g_G^*(R_{10}) &= R_{10}.\end{aligned} \tag{8.5.11}$$

It follows that $g_G^*(2f_9) = g_G^*(h - R_9) = h - R_9$. Thus, g_G^* acts identically on the sublattice $\langle f_9, R_{10}\rangle$. We also check that it acts as the minus identity on its orthogonal complement. Thus, g_G^* coincides with a Geiser involution.

8.6 Cyclic Groups of Automorphisms of an Enriques Surface

In this section, we will study cyclic groups of automorphisms of Enriques surfaces and classify them under the assumption that the order is prime to the characteristic.

Let G be a finite group of automorphisms of an Enriques surface S and let G^* be its image in $W(\mathrm{Num}(S)) \cong W(\mathsf{E}_{10})$. We have already studied the possible kernel of the homomorphism $G \to W(\mathrm{Num}(S))$. Let us identify $W(\mathrm{Num}(S))$ with $W(\mathsf{E}_{10})$ and consider the reduction homomorphism

$$r : W(\mathsf{E}_{10}) \to \mathrm{O}^+(10, \mathbb{F}_2).$$

Let $G_0^* = G^* \cap W(\mathsf{E}_{10})(2)$ be the kernel of the restriction of this homomorphism to G^*. The following proposition is due to D. Allcock.

Proposition 8.6.1 *Let H be a finite non-trivial subgroup contained in $W(\mathsf{E}_{10})(2)$. Then, it is a group of order 2, and all such subgroups are conjugate in $W(\mathsf{E}_{10})$.*

Proof We identify $\mathrm{Num}(S)$ with the lattice E_{10}. Suppose H contains an element σ of order 2. Then, $V = (\mathsf{E}_{10})_{\mathbb{Q}}$ splits into the orthogonal direct sum of eigensubspaces V_+ and V_- with eigenvalues 1 and -1. For any $x = x_+ + x_- \in \mathsf{E}_{10}, x_+ \in V_+, x_- \in V_-$, we have

$$\sigma(x) \pm x = (x_+ - x_-) \pm (x_+ + x_-) \in 2\mathsf{E}_{10}.$$

This implies $2x_\pm \in 2\mathsf{E}_{10}$, hence $x_\pm \in \mathsf{E}_{10}$ and the lattice E_{10} splits into the orthogonal sum of sublattices $V_+ \cap \mathsf{E}_{10}$ and $V_- \cap \mathsf{E}_{10}$. Since E_{10} is unimodular, the sublattices must be unimodular. This gives $V_+ \cap \mathsf{E}_{10} \cong \mathsf{U}$ or E_8 and $V_- \cap \mathsf{E}_{10} \cong \mathsf{E}_8$ or U, respectively. Since $\mathrm{O}(\mathsf{E}_{10}) = W(\mathsf{E}_{10}) \times \{\pm \mathrm{id}_{\mathsf{E}_{10}}\}$, only one of these possibilities occurs, say the latter one. Thus, all elements of order 2 in $W(\mathsf{E}_{10})(2)$ are conjugate to the element $\mathrm{id}_{\mathsf{U}} \oplus -\mathrm{id}_{\mathsf{E}_8}$.

Suppose H contains an element σ of odd order m. Then, $\sigma^m - 1 = (\sigma - 1)(1 + \sigma + \cdots + \sigma^{m-1}) = 0$, hence, for any $x \notin 2\mathsf{E}_{10}$ which is not σ-invariant, we have

$$x + \sigma(x) + \cdots + \sigma^{m-1}(x) \equiv mx \mod 2\mathsf{E}_{10}.$$

Since m is odd, this gives $x \in 2\mathsf{E}_{10}$, a contradiction.

Finally, we may assume that H contains an element of order $2^k, k > 1$. Then, it contains an element σ of order 4. Let $M = \mathrm{Ker}(\sigma^2 + 1) \subset \mathsf{E}_{10}$. Since $\sigma^2 = -\mathrm{id}_{\mathsf{E}_8} \oplus \mathrm{id}_{\mathsf{U}}$ for some direct sum decomposition $\mathsf{E}_{10} = \mathsf{E}_8 \oplus \mathsf{U}$, we obtain $M \cong \mathsf{E}_8$. The equality $(\sigma^2+1)(\sigma(x)) = \sigma^3(x)+\sigma(x) = -(\sigma^2+1)(x)$, implies that $\sigma(M) = M$. Consider M as a module over the principal ideal domain $R = \mathbb{Z}[t]/(t^2+1)$. Since M has no torsion, it is isomorphic to $R^{\oplus 4}$. This implies that there exists $v, w \in M$ such that $\sigma(v) = w$ and $\sigma(w) = -v$. However, this obviously contradicts our assumption that $\sigma \in W(\mathsf{E}_{10})(2)$. □

Corollary 8.6.2 *Let G be a finite group of automorphisms of an Enriques surface and let G^* be its image on $W(\mathrm{Num}(S))$. Then, G^* or its quotient by a cental subgroup of order 2 is isomorphic to a finite subgroup of* $\mathrm{O}^+(10, \mathbb{F}_2)$.

It is known that

$$\#\mathrm{O}^+(10, \mathbb{F}_2) = 2^{21} \cdot 3^5 \cdot 5^2 \cdot 7 \cdot 17 \cdot 31. \tag{8.6.1}$$

It follows that the only primes that may divide $\#G^*$ are $2, 3, 5, 7, 17$, or 31. We can also use the information about $\mathrm{Aut}_{\mathrm{nt}}(S)$ from Section 8.2 to get possible prime divisors of $\#G$.

We say that a group of automorphisms of an Enriques surface is of *translation type* if it leaves invariant a genus one fibration and is realized by a subgroup of the Mordell–Weil group of the jacobian fibration that acts on S by translation automorphisms.

The proof of the following result can be found in [574] and [646, Theorems 8.8 and 8.9].

Theorem 8.6.3 *Let $J \to \mathbb{P}^1$ be a rational jacobian elliptic surface. Then, its Mordell–Weil group is isomorphic to one of the following groups:*

$$\mathbb{Z}^r\ (1 \le r \le 8),\ \mathbb{Z}^r \oplus \mathbb{Z}/2\mathbb{Z}\ (1 \le r \le 4),\ \mathbb{Z}^r \oplus \mathbb{Z}/3\mathbb{Z}\ (1 \le r \le 2),\ \mathbb{Z}^r \oplus (\mathbb{Z}/2\mathbb{Z})^2\ (1 \le r \le 2),$$
$$\mathbb{Z} \oplus \mathbb{Z}/4\mathbb{Z},\ \mathbb{Z}/4\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z},\ (\mathbb{Z}/3\mathbb{Z})^2,\ (\mathbb{Z}/2\mathbb{Z})^2,\ \mathbb{Z}/6\mathbb{Z},\ \mathbb{Z}/5\mathbb{Z},\ \mathbb{Z}/4\mathbb{Z},\ \mathbb{Z}/3\mathbb{Z}, \mathbb{Z}/2\mathbb{Z}, \{1\}.$$

The groups containing a subgroup isomorphic to $(\mathbb{Z}/2\mathbb{Z})^2$ *(resp.* $(\mathbb{Z}/3\mathbb{Z})^2$*) can be realized only in characteristic* $\neq 2$ *(resp.* $\neq 3$*).*[1]

Corollary 8.6.4 *All the groups from the list in the previous theorem can be realized as groups of automorphisms of an Enriques surface over an algebraically closed field of characteristic* $\neq 2, 3$.

Proof Since the Mordell–Weil group of the jacobian fibration acts by translations on the torsor, we use the Ogg–Shafarevich theory (see Proposition 4.10.1 from Volume I) to construct a torsor on an Enriques surface in characteristic $\neq 2, 3$ on which these groups act. □

Proposition 8.6.5 *Suppose* G *is a group of translation type of a quasi-elliptic surface. Then* $G \cong (\mathbb{Z}/2\mathbb{Z})^{\oplus r}$, $1 \leq r \leq 4$*, and all such groups can be realized.*

Proof The Mordell–Weil group of a jacobian quasi-elliptic pencil is a p-torsion group and its structure can be found in Table (4.9) in Volume I. An Enriques surface with a quasi-elliptic fibration is a torsor of such a surface only if $p = 2$. This gives us that the possible group must be isomorphic to $(\mathbb{Z}/2\mathbb{Z})^{\oplus r}$, $1 \leq r \leq 4$. One can find a construction of the corresponding quasi-elliptic Enriques surfaces with $r \neq 4$ in Section 8.10. The case $r = 4$ corresponds to a quasi-elliptic fibrations with irreducible fibers. Their equations can be found in Remark 4.10.5. □

In the following, we will try to classify possible finite cyclic group actions on an Enriques surface. If $p \neq 2$ and the order of the group is prime to p, we can apply Theorem 8.1.6 to assume that $\Bbbk = \mathbb{C}$ and use the theory of periods of the K3-covers. However, even in this case, we would like to avoid this non-geometric approach.

Let G be a finite subgroup of $\mathrm{Aut}(S)$. Applying Lemma 6.4.8, we obtain, after identifying $\mathrm{Num}(S)$ with E_{10}, that the image G^* of G in $\mathrm{Aut}(S)^*$ is contained in some parabolic subgroup W_J of $W(\mathsf{E}_{10})$. As we explained in Example 6.4.9, W_J is a subgroup of the Weyl groups of a root lattice of one of the following types:

$$A_9,\ D_9,\ E_8{+}A_1,\ A_1{+}A_8,\ A_6{+}A_2{+}A_1,\ A_4{+}A_5,\ E_6{+}A_3,\ E_7{+}A_2,\ D_5{+}A_4. \quad (8.6.2)$$

Lemma 8.6.6 *Let* $g \in \mathrm{Aut}(S)$ *and let* $x \in \mathrm{Num}(X)$ *with* $x^2 \geq 0$ *such that* $g^*(x) = x$*. Then, there exists* $w \in W_S^{\mathrm{nod}}$ *such that* $y = w(x)$ *is nef and* $g^*(y) = y$.

Proof Since $\mathrm{Nef}(S)$ is a fundamental domain for W_S^{nod}, there exists a unique $w \in W_S^{\mathrm{nod}}$ such that $y = w(x) \in \mathrm{Nef}(S)$. We have

$$g^* \circ w \circ (g^*)^{-1}(x) = g^*(w(x)) = g^*(y) \in \mathrm{Nef}(S).$$

Since $g^* \circ w \circ (g^*)^{-1} \in W_S^{\mathrm{nod}}$, the uniqueness of w implies that $w = g^* \circ w \circ (g^*)^{-1}$. Thus,

$$g^*(y) = g^* \circ w(x) = w \circ g^*(x) = w(x) = y.$$

[1] Theorem 8.9 in [646] asserts mistakenly that Case $\mathbb{Z} \oplus \mathbb{Z}/3\mathbb{Z}$ is not realized in characteristic 2 but later, on p. 184, one finds a realization of this group in any characteristic.

Computing the orthogonal complements of the sublattices in E_{10} from (8.6.2), we obtain the following:

Corollary 8.6.7 *Let G be a finite group of automorphisms of S. Then, it preserves a nef numerical class h with $h^2 \in \{2, 4, 6, 10, 12, 18, 20, 30, 42\}$.*

Note that the set of possible values of h^2 is exactly the set of values of $\boldsymbol{\omega}^2$, where $\boldsymbol{\omega}$ is a fundamental weight of the standard root basis of E_{10} (see Proposition 1.5.3 in Volume I).

We will use the known classification of conjugacy classes of elements in the Weyl groups of root systems of finite type given in Table 8.6. According to [105] they are indexed by certain graphs. We call them *Carter graphs*. One writes each element $w \in W$ as the product of two involutions $w_1 w_2$, where each involution is the product of reflections with respect to orthogonal roots. Let $\mathcal{R}_1, \mathcal{R}_2$ be the corresponding sets of such roots. Then, the graph has vertices identified with elements of the set $\mathcal{R}_1 \cup \mathcal{R}_2$ and two vertices α, β are joined by an edge if and only if $(\alpha, \beta) \neq 0$. A connected Carter graph with no cycles is a Dynkin–Coxeter diagram. It represents the conjugacy class of the Coxeter element of the corresponding Weyl group. The (first) subscript n in the notation $A_n, D_n, E_n, A_n(a_k), D_n(a_k), E_n(a_k)$ of a Carter graph indicates the number of vertices. The notation also indicates that the conjugacy class is realized by an element of the Weyl group of the corresponding type. It may or may not be the conjugacy class of a Coxeter element of this group, if not, it has an additional notation like $E_6(a_1)$. The subscript n is also equal to the difference between the rank of the root lattice Q and the rank of its fixed sublattice $Q^{(w)}$.

The Carter graph determines the characteristic polynomial of w. In particular, it gives the trace $\mathrm{tr}_2(g^*)$ of g^* on the l-adic cohomology space $H^2(S, \mathbb{Q}_l)$. If the order of g is prime to the characteristic, we can apply the Lefschetz fixed-point formula to obtain:

$$\mathrm{tr}(g^*) := \mathrm{tr}(g^* | H^*(S, \mathbb{Q}_l)) = 2 + \mathrm{tr}_2(g^*) = e(S^g). \tag{8.6.3}$$

The following Table 8.6 gives the conjugacy classes of elements defined by connected Carter graphs.

Let H be a finite subgroup of $W(\mathsf{E}_{10})$ (or its element w). We say that H (or w) is of E_8*-type* if it is conjugate to a subgroup of the parabolic subgroup W_J defined by the set J of vertices of the subdiagram of $T_{2,3,7}$ of type $T_{2,3,5}$ (of type E_8). It is easy to see that it must coincide with the parabolic subgroup W_J, where J is the set of vertices of a subdiagram of type E_8. A finite group G of automorphisms (resp. an automorphism g) of S is said to be of E_8-type if the image G^* of G (resp. g^*) in $W(\mathrm{Num}(S))$ is of E_8-type. All conjugacy classes of elements in $W(\mathsf{E}_8)$ are listed in [105]. Note that two elements of the same type are not necessarily conjugate, for example, there are two conjugacy classes of elements of type $4A_1$ or $2A_3$.

Lemma 8.6.8 *Let G be a finite group of automorphisms of S. Assume that its image G^* in $W(\mathrm{Num}(S))$ is of E_8-type. Then, there exists a G-equivariant bielliptic map $\phi : S \to \mathsf{D}$ such that G is a lift of a group G' of automorphisms of D. The group G leaves invariant the genus one fibrations defined by the pre-images of the*

Graph	Order	Characteristic polynomial	Trace
A_k	$k+1$	$t^k+t^{k-1}+\cdots+1$	-1
D_k	$2k-2$	$(t^{k-1}+1)(t+1)$	-1
$D_k(a_1)$	l.c.m$(2k-4,4)$	$(t^{k-2}+1)(t^2+1)$	0
$D_k(a_2)$	l.c.m$(2k-6,6)$	$(t^{k-3}+1)(t^3+1)$	0
$\vdots$	$\vdots$	$\vdots$	$\vdots$
$D_k(a_{\frac{k}{2}-1})$	even k	$(t^{\frac{k}{2}}+1)^2$	0
E_6	12	$(t^4-t^2+1)(t^2+t+1)$	-1
$E_6(a_1)$	9	t^6+t^3+1	0
$E_6(a_2)$	6	$(t^2-t+1)^2(t^2+t+1)$	1
E_7	18	$(t^6-t^3+1)(t+1)$	-1
$E_7(a_1)$	14	t^7+1	0
$E_7(a_2)$	12	$(t^4-t^2+1)(t^3+1)$	0
$E_7(a_3)$	30	$(t^5+1)(t^2-t+1)$	1
$E_7(a_4)$	6	$(t^2-t+1)^2(t^3+1)$	2
E_8	30	$t^8+t^7-t^5-t^4-t^3+t+1$	-1
$E_8(a_1)$	24	t^8-t^4+1	0
$E_8(a_2)$	20	$t^8-t^6+t^4-t^2+1$	0
$E_8(a_3)$	12	$(t^4-t^2+1)^2$	0
$E_8(a_4)$	18	$(t^6-t^3+1)(t^2-t+1)$	1
$E_8(a_5)$	15	$t^8-t^7+t^5-t^4+t^3-t+1$	1
$E_8(a_6)$	10	$(t^4-t^3+t^2-t+1)^2$	2
$E_8(a_7)$	12	$(t^4-t^2+1)(t^2-t+1)^2$	2
$E_8(a_8)$	6	$(t^2-t+1)^4$	4

Table 8.4 Carter graphs and characteristic polynomials

pencils of conics on D *and a bisection defined by another genus one fibration (if* $\mathsf{D}=\mathsf{D}_1,\mathsf{D}_2,\mathsf{D}_3$*), or a smooth rational curve (if* $\mathsf{D}=\mathsf{D}_1',\mathsf{D}_2',\mathsf{D}_3'$*).*

Proof We know that $\mathsf{E}_{10}\cong\mathsf{E}_8\oplus\mathsf{U}$. Thus, G leaves invariant a hyperbolic plane U and acts identically on it. It follows from Proposition 6.1.5 that there exists a unique $w\in W_S^{\mathrm{nod}}$ such that $w(U)$ is generated by a canonical isotropic 2-sequence (f_1,f_2) that defines a biellliptic linear system $|D|$. Thus, we may assume that G leaves U invariant and acts identically on it. Let $\phi:S\to\mathsf{D}\subset|D|^*$ be the corresponding bielliptic map. Then, G acts in $|D|^*$ leaving D invariant. Thus, G is a lift of a subgroup of automorphisms of $|D|^*$ which leaves D invariant. If (f_1,f_2) are both nef, $\mathsf{D}=\mathsf{D}_1,\mathsf{D}_2,\mathsf{D}_3$, and we have two genus one pencils $|2F_1|$ and $|2F_2|$ with $[F_i]=f_i$, both fixed by G. Otherwise, f_1 is nef, and $f_2=f_1+r$, where r is the class of a smooth rational curve R. Then, $\mathsf{D}=\mathsf{D}_1',\mathsf{D}_2',\mathsf{D}_3'$, and we have one genus one fibration $|2F_1|$ with $[F_1]=f_1$ and a bisection R. □

The following useful lemma is due to J.-P. Serre.

Lemma 8.6.9 *Let V be a smooth proper connected variety of dimension n over an algebraically closed field with $H^i(V,\mathcal{O}_V)=0, i>0$. Then, any endomorphism g of V has a fixed point.*

Proof If g is an automorphism of finite order prime to p, this follows from the *Woods Hole formula* [328, Corollary 6.12]:

$$\sum_{i=0}^{n}(-1)^i \operatorname{tr}(g^*|H^i(V,\mathcal{O}_V)) = \sum_{x\in V^g} \frac{1}{\det(1-\mathrm{dg}_x)}, \tag{8.6.4}$$

where dg is the differential of g at a fixed point x (under the assumption that the set V^g is finite). So, if we assume that V^g is empty, we obtain that the right-hand side is 0 but the left-hand side is positive, a contradiction.

Without any assumption on the order of g but assuming that V^g is a finite set, we use the same argument and the following formula [328, Remark 6.12.1]:

$$\sum_{i=0}^{n}(-1)^i \operatorname{tr}(g^*|H^i(V,\mathcal{O}_V)) = \sum_{x\in V^g} \operatorname{Res}_x \frac{du_1\wedge\ldots\wedge du_n}{(u_1-g^*(u_1))\cdots(u_n-g^*(u_n))}, \tag{8.6.5}$$

where $(u_1,\ldots,u_n)$ are local coordinates at a point $x\in V^g$. □

Recall that an automorphism of a K3 surface X is called *symplectic* if it acts identically on $H^0(X,\omega_X)$.

Lemma 8.6.10 *Assume $p\neq 2$ or S is a $\boldsymbol{\mu}_2$-surface. Let g be an automorphism of S of odd order n. Then, its lift $\tilde{g}$ of order n to the canonical cover X is a symplectic automorphism of X.*

Proof We follow a nice argument from [534]. Suppose $\tilde{g}$ is not symplectic. Let $\phi : X\to Y = X/(\tilde{g})$ be the quotient map, it extends to a separable morphism $\phi' : X'\to Y'$, where X' is birational to X and Y' is a nonsingular model of Y. Since ϕ' is separable, the map $\phi'^* : H^i(Y',\omega_{Y'})\to H^i(X',\omega_{X'}), i=0,1$, is injective. Its image is contained in $H^i(X',\omega_{X'})^{\tilde{g}}=\{0\}$. Also, $\operatorname{kod}(Y')\le 0$ and $q(Y')=p_g(Y')=0$. Since an automorphism of order equal to the characteristic is symplectic, we may assume that n is prime to p. It follows from Lemma 8.6.9 that the descent of the Enriques involution τ to Y and then its lift to Y' has a fixed point. Its image on Y is a fixed point $y\in Y$. The fiber over this point has odd cardinality and it is invariant with respect to τ. Hence τ has a fixed point on the fiber, contradicting the fact that it acts on X freely. □

Proposition 8.6.11 *Assume $p\neq 2$ or S is a $\boldsymbol{\mu}_2$-surface. Let g be a non-trivial automorphism of an Enriques surface of odd order n. Then, $n=3$ or 5.*

Proof Let $\tilde{g}$ be a lift of g to a symplectic automorphism of the K3-cover. The odd order n of a symplectic automorphism of a K3 surface takes possible values $1,3,5,7$ and 11 [188]. The latter case occurs only if $p=11$ [186]. It follows from Section 8.2 that $\operatorname{Aut}_{\mathrm{nt}}(S)$ is of order 2 or 4. Applying Corollary 8.2.2 and (8.6.1), we see that $\operatorname{Aut}(S)$ has no elements of order 11.

It remains to exclude the case $n=7$. An element of order 7 in $W(\mathsf{E}_{10})$ is of E_8-type. Hence it fixes a genus one fibration and its bisection. In particular, it cannot be of translation type (the latter follows also from the fact that the Mordell–Weil group of the jacobian fibration cannot be of order 7). Thus, g acts on the base of the fibration, and since n is odd, it fixes the two double fibers. Since an element of order p fixes only one point in $\mathbb{P}^1$, we must have $p\neq 7$. It is known that in this case

a symplectic automorphism of order 7 has three fixed points. This applies to the lift $\tilde{g}$ and, since the Enriques involution leaves this set invariant, we obtain that it has a fixed point, a contradiction. □

Note that, by Theorem 8.2.17, g cannot act identically on $\mathrm{Num}(S)$, and, by Proposition 8.6.1, it does not lie in the kernel of the reduction homomorphism $W(\mathrm{Num}(S)) \to \mathrm{O}(\overline{\mathrm{Num}}(S))$.

The following result in the case $\Bbbk = \mathbb{C}$ can be found in [579, Proposition 3.1].

Theorem 8.6.12 *Assume $p \neq 2$. Let n be the order of an element $\sigma \in G^*$. Then,*

$$n \in \{1, 2, 3, 4, 5, 6, 8\}.$$

Proof We already know that an odd prime n must be equal to 3 or 5. Thus, $n = 2^a, 2^a \cdot 3$ or $2^a \cdot 5$. The classification of possible orders of elements in a finite parabolic subgroup of $W(\mathsf{E}_{10})$ shows that $a \leq 3$. Since there are no elements of order 15, the only possibilities are listed in the assertion or $n = 10, 12$.

To finish the proof it suffices to exclude the last two cases.

Assume $n = 10$. We will show later in Proposition 8.6.18 that an element of order 10 lifts to a symplectic automorphism of the canonical cover X. If $p \neq 5$, there are no such automorphisms of X. The same is true if $p = 5$ [386].

Let us assume that $n = 12$. Applying the Woods Hole formula (8.6.4) to $h = g^3$ we obtain that $\#S^h = 2$. Since $S^g \subset S^h$, this implies that $\#S^g \leq 2$.

Assume $p \neq 3$, then the same formula gives us $\#S^h = 3$, where $h = g^4$, and the differential of h acts at a fixed point with eigenvalues ϵ_3, ϵ_3^2 (this is the only possibility to make the left-hand side an integer). Since g leaves the set S^h invariant and fixes at most two points, we obtain $\#S^g = 1$ and the differential of g acts at a fixed point with eigenvalues $\epsilon_{12}, \epsilon_{12}^2$. We apply again the Woods Hole formula, and get a contradiction.

Next, we will assume that $p = 3$. First, we use that $h = g^4$ is an element of order 3. It follows from [385] that g has only one fixed point. This implies that S^g consists of one point. But now, it follows from the Woods Hole formula (8.6.4) that $\#S^{g^3} = 2$, since $S^g \subset S^{g^3}$ we see that g must fix two points, a contradiction. □

Remark 8.6.13 If $p = 2$, we do not know whether there exists an Enriques surface with an automorphism of order 10 or 12.

The following definition is due to S. Mukai:

Definition 8.6.14 An automorphism g of an Enriques surface S is called *semi-symplectic* if it acts trivially on $H^0(S, \mathcal{O}_S(2K_S)) \cong \Bbbk$. We say that a finite grpup G acts semi-symplectically if all its elements act semi-symplectically.

Note that, although $\mathcal{O}_S(2K_S) \cong \mathcal{O}_S$, the isomorphism is not canonical, so the action does not coincide with the trivial action on the constants. By duality, we have a canonical isomorphism $H^0(S, \mathcal{O}_S(2K_S)) \cong H^2(S, \mathcal{O}_S(-K_S))^\vee$. Therefore, if $K_S \cong \mathcal{O}_S$, the action on $H^0(S, \mathcal{O}_S(2K_S))$ is isomorphic to the action on $H^2(S, \mathcal{O}_S)$.

The following two propositions show that the semi-simplicity is an analog of the condition for an automorphism of a K3 surface X to be symplectic.

Proposition 8.6.15 *Assume that $p \neq 2$, or S is a $\boldsymbol{\mu}_2$-surface. Let $\pi : X \to S$ be the K3-cover. Then, an automorphism g of S is semi-symplectic if and only if one of its lifts to an automorphism of X is symplectic.*

Proof Since $\pi^*(\omega_S) \cong \mathcal{O}_X$, we have a canonical isomorphism

$$H^0(S, \mathcal{O}_S(2K_S)) \cong H^2(S, \mathcal{O}_S(-K_S))^\vee \cong H^2(X, \mathcal{O}_X)^\vee \cong H^0(X, \omega_X).$$

So, the action of two different lifts differ by the action of the canonical cover involution. If $p = 2$, both of the lifts could be symplectic. □

Proposition 8.6.16 *Suppose g is of odd order n prime to p. Then, it is semi-symplectic if and only if S^g consists of isolated fixed points and the quotient surface $Y = S/(g)$ has only rational double points of type A_{n-1}.*

Proof A section of $H^0(S, \mathcal{O}_S(2K_S))$ can be represented, locally in an affine neighborhood U of a fixed point $x \in S$, by $\phi(dx \wedge dy)^2$, where ϕ is an invertible function on U and x, y are local coordinates. Since g is semi-symplectic $\det(dg_x)^2 = 1$ and, since g is of odd order, we have $\det(dg_x) = 1$. Since $(p, n) = 1$, by Lemma 0.2.36, we can find local coordinates such that g acts on them via a diagonal matrix $\mathrm{diag}(\epsilon_n, \epsilon_n^{-1})$. The image of the point x in the quotient is locally isomorphic to the rational double point of type A_{n-1}.

Conversely, if the quotient singularity is a rational double point of type A_{n-1}, we obtain that g acts via the diagonal matrix $\mathrm{diag}(\epsilon_n, \epsilon_n^{-1})$ and hence $\det(dg_x) = 1$. This implies that one of the lifts of g is symplectic. □

Remark 8.6.17 The assumption that $(n, p) = 1$ is essential. It follows from [385] that the locus S^g of fixed points of an automorphism of S of order p could be a connected curve or an isolated point and, in the latter case the quotient may have a singular point which is not a rational double point.

It is known that the possible order of a symplectic automorphism of a K3 surface of order n prime to p satisfies $n \leq 8$, and all such values are realized (see [188], Theorem 3.3). Moreover, if $n = p$, then $p \leq 11$ (loc. cit., Theorem 2.1). The symplectic lifts of automorphisms of an Enriques surfaces satisfy a stricter condition.

Proposition 8.6.18 *Assume $p \neq 2$ or S is a $\boldsymbol{\mu}_2$-surface. Any automorphism g of order n not divisible by 4 is semi-symplectic. An element of order 8 cannot be semi-symplectic.*

Proof Let $\pi : X \to S$ be the canonical cover. It follows from Lemma 8.6.10 that any element of odd order is semi-symplectic. A lift of an automorphism of order 2 of X is either symplectic or its composition with the covering involution τ is symplectic.

Suppose $n = 2k$, where $k > 1$ is odd. Let $\tilde{g}$ be a lift of g such that $\tilde{g}^2$ is a symplectic lift of g^2. If $\tilde{g}$ is not symplectic, then it acts as -1 on $H^0(X, \omega_X)$, hence $\tilde{g} \circ \tau$ acts identically and defines a symplectic lift of g.

It remains to exclude the case $n = 8$.

Suppose a semi-symplectic automorphism g has order $n = 8$. Let $\tilde{g}$ be its symplectic lift to the canonical cover X. It is known that $\#X^{\tilde{g}} = 2$. But then S^g consists of one point. Applying the Woods Hole formula, we find a contradiction. Note that there is no contradiction with a symplectic lift of an element of order 4 since it gives us that $\#S^g = 2$ and the differential at each point has eigenvaues $\pm\sqrt{-1}$. The formula confirms that $\#S^g = 2$. □

Let g be a symplectic automorphism of order n of a K3 surface X. Assume $(n, p) = 1$. Then, X^g is finite and we have Table 8.5 below for the possible number f of fixed points (see [188, Theorem 3.3]):

n	2	3	4	5	6	7	8
f	8	6	4	4	2	3	2

Table 8.5 Number of fixed points of an automorphism of a K3 surface

Let $\tilde{g}$ be a symplectic lift of an automorphism g of S. The image of a fixed point of $\tilde{g}$ on S is a fixed point of g which is called a *symplectic fixed point*. Other fixed points of g are called *anti-symplectic fixed points*. We denote the set of symplectic (resp. anti-symplectic) fixed points by S^g_+ (resp. S^g_-). Note that, $S^g_+ \cap S^-_- = \emptyset$ because the differential $dg_{\pi(x)}$ and $d\tilde{g}_x$ are isomorphic linear representations of g and $\tilde{g}$.

Corollary 8.6.19 *Let g be a semi-symplectic automorphism of order n of S. If $p|n$, then $\#S^g = 1$ or S^g is a connected curve. If $p \nmid n$, then:*

(n=2) $\#S^g_+ = 4$ *and* $e(S^g_-) = \mathrm{tr}(g^*) - 4$*;*
(n=3) $\#S^g_+ = \#S^g = 3$*;*
(n=4) $\#S^g_+ = \#S^g_- = 2$*;*
(n=5) $\#S^g_+ = \#S^g = 2$*;*
(n=6) $\#S^g_+ = 1$ *and* $\#(S^g_-) \in \{1, 2\}$.

Proof Assume $p|n$. If $n = p$, then it is known that S^g is connected [385]. Write $n = p^k n'$, then $S^g \subset S^{g^{p^{k-1}n'}}$ and the assertion follows. Now, we assume that $(n, p) = 1$. If n is odd (i.e. 3 or 5), then $X^{\tilde{g}\circ\tau} \subset X^{\tilde{g}^2} = X^{\tilde{g}}$ so $S^g_- = S^g_+$. If $n = 4$, then $X^{\tilde{g}\circ\tau} \subset X^{\tilde{g}^2}$, so $S^g_- = S^{g^2}_+ \setminus S^g_+$. If $n = 6$, then $\#X^{\tilde{g}} = 2$. Also, $X^{\tilde{g}\circ\tau}$ and $X^{\tilde{g}}$ are subsets of the set $\#X^{\tilde{g}^2}$ of cardinality 6. This implies $\#S^g_+ = 1$ and $\#S^g_- \leq 2$. □

Example 8.6.20 If $p \neq 2$ or S is a $\boldsymbol{\mu}_2$-surface, all possible orders of semi-symplectic automorphisms can be realized by translation automorphisms of the Mordell–Weil group of the jacobian elliptic fibration. We know that these numbers are among possible orders of such groups. It remains to see that the translation automorphisms are semi-symplectic. In fact, one of the two lifts of such an automorphism is a

translation automorphism of an elliptic fibration on the K3-cover, the pre-image of the elliptic fibration on S. It is well-known that such an automorphism is symplectic. One can see it, for example, by considering the exact sequence

$$0 \to \omega_X(-F) \to \omega_X \to \mathcal{O}_F(\omega_X) \to 0,$$

where F is a general fiber of the elliptic fibration. Via the adjunction formula $\omega_F \cong \mathcal{O}_F(\omega_X)$, and the exact sequence defines a canonical isomorphism $H^0(X, \omega_X) \to H^0(F, \omega_F)$. Since the translation automorphism of an elliptic curve acts identically on $H^0(F, \omega_F)$, we obtain that it acts identically on $H^0(X, \omega_X)$.

8.7 Involutions of Enriques Surfaces

In this section, we will study involutions of Enriques surfaces and classify them in characteristic $p \neq 2$. We assume that $p \neq 2$ in this section, although some statements may also be valid in characteristic 2.

All involutions on Enriques surfaces over the field of complex numbers were classified by H. Ito and H. Ohashi in [339]. Their classification relies heavily on the Global Torelli Theorem for K3 surfaces and Nikulin's results on quadratic lattices together with an isometry of order 2. There are 18 types of involutions in the case $\Bbbk = \mathbb{C}$. We will show that the same result is true over a field of arbitrary characteristic $p \neq 2$.

We start with the following:

Lemma 8.7.1 *Let $f : S \to \mathbb{P}^1$ be a genus one fibration on an Enriques surface. Suppose it admits a separable bisection B. Then, S admits an automorphism of order 2 that preserves the fibration and the bisection B. Conversely, any automorphism of order 2 of S that preserves a genus one fibration, acts identically on the base, and leaves invariant a separable bisection arises in this way.*

Proof A bisection is a point x of degree 2 on the generic fiber S_η of f. The assumption implies that the residue field of this point is a separable extension of $\Bbbk(\eta)$. The linear system $|x|$ defines a separable degree 2 map $S_\eta \to \mathbb{P}^1_\eta$. Its deck transformation is a birational involution on S that extends to a biregular involution. The point x is the pre-image of a rational point on $\mathbb{P}^1_\eta$ and hence is preserved by the deck transformation.

Conversely, an automorphism σ of order 2 of S that preserves a genus one fibration $f : S \to \mathbb{P}^1$ and a separable bisection restricts to S_η to define a separable map $\phi : S_\eta \to C = S_\eta/(\sigma)$. Since σ leaves invariant the generic point x of the bisection, we obtain that $x = \phi^{-1}(y)$ for some point y of degree 1 on C. Hence $C \cong \mathbb{P}^1_\eta$, and σ arises in the way described above. □

Corollary 8.7.2 *Suppose S admits an elliptic fibration $f : S \to \mathbb{P}^1$. Then, the automorphism group* $\mathrm{Aut}(S)$ *contains an element of order 2.*

Proof We know that $f : S_\eta \to \mathbb{P}^1$ is a torsor of period 2 over an elliptic curve. Applying Theorem 4.6.6 and Proposition 4.6.5, we obtain that S_η contains a point x of degree 2 with a separable residue field. Then, the assertion follows from Lemma 8.7.1. □

Let $C \subset S$ be a separable bisection of an elliptic pencil $|2F|$. Applying Enriques's Reducibility Lemma (Theorem 2.3.5 from Volume I), we see that C is linearly equivalent to a positive sum of smooth rational or genus one curves. Thus, we may assume that C is one of them and the involution of S defined by C coincides with the bielliptic involution arising from a bielliptic map $|2F + 2F'|$ or $|4F + 2R|$.

Let us look how an involution g can act on $\mathrm{Num}(S)$.

According to R. Richardson [618], the conjugacy classes of involutions in a Coxeter group (W, Σ) can be described as follows. We say that a subset J of the set of Coxeter generators Σ satisfies the (-1)-condition if the Coxeter group W_J generated by J contains an element σ_J that acts as $-\mathrm{id}$ in the geometric realization $\mathbb{R}^\Sigma$ of W. In the case of Weyl groups, the root sublattice generated by J must be the orthogonal sums of root lattices of types A_1, D_{2n}, E_7, and E_8. Since E_{10} has rank 10, we have the following types of irreducible root sublattices generated by a subset satisfying the (-1)-condition:

$$\mathsf{A}_1,\ \mathsf{D}_4,\ \mathsf{D}_6,\ \mathsf{D}_8,\ \mathsf{E}_7,\ \mathsf{E}_8.$$

Other possible root sublattices (they are all primitive sublattices of E_{10}) with this property are the orthogonal sums of suitable irreducible sublattices. The following Proposition is Theorem A from [618].

Proposition 8.7.3 *Any involution in a Coxeter group is conjugate to the involution σ_J for some subset J defining a finite parabolic subgroup that satisfies the (-1)-condition. Two involutions σ_J and $\sigma_{J'}$ are conjugate if and only if the subsets J and J' are equivalent (as explained in Section 6.4 after Lemma 6.4.8).*

Applying this proposition, we find 15 non-equivalent diagrams that correspond to an involution in $W(\mathsf{E}_{10})$.

$$\begin{array}{llllllll} A_1, & 2A_1, & 3A_1, & 4A_1, & 5A_1, & D_4, & D_4 + A_1, & D_4 + 2A_1, \\ D_6, & D_6 + A_1, & D_8, & E_7, & E_7 + A_1, & E_8, & E_8 + A_1. & \end{array} \tag{8.7.1}$$

Proposition 8.7.4 *Let g be an involution of S. Then, g preserves a numerical polarization of degree* 2.

Proof Let M be one of the sublattices of E_{10} defined by a diagram from the list (8.7.1) and let N be its orthogonal complement in E_{10}. It follows from Lemma 8.6.6 that it suffices to find a vector $v \in N$ with $v^2 = 2$. For example, we can achieve this if we show that N contains a sublattice isomorphic to U.

We use the standard root basis $B = (\alpha_0, \dots, \alpha_9)$ as in (6.1.1). Let Σ be a subset of B. Suppose $\alpha_i \notin \Sigma$ and $\alpha_j \in \Sigma$ is incident to α_i such that α_i, α_j is a connected component of $\Sigma \cup \{\alpha_i\}$. Then, the elementary operation defined by α_i allows one to

replace α_i with α_j (see the definition of an elementary operation in Coxeter groups before Example 6.4.9). We leave it to the reader to check that, using such operations, one can replace Σ, not of types

$$5A_1,\ D_4+2A_1,\ D_6+A_1,\ D_8,\ A_1+E_7,\ E_8+A_1, \tag{8.7.2}$$

with a subset Σ' contained in the set $\{\alpha_0,\dots,\alpha_8\}$ which spans $M \cong \tilde{\mathsf{E}}_8$ and also is disconnected from α_9. So, the orthogonal complement contains the generator $\mathfrak{f}$ of the radical of $\langle\alpha_0,\dots,\alpha_8\rangle \cong \tilde{\mathsf{E}}_8$ which, together with α_9, generates a sublattice isomorphic to U.

If Σ is of type D_4+2A_1, then it is equivalent to the diagram with vertices $\alpha_0,\alpha_2,\alpha_3,\alpha_4,\alpha_6,\alpha_8$. Let $\alpha_{\max}^{D_8}$ be the maximal root of the sublattice of E_{10} of type D_8 spanned by the roots $\alpha_i, i\neq 1,9$. Then, $\mathfrak{f}$ and $\alpha_{\max}^{D_8}$ span a sublattice isomorphic to U which is orthogonal to all roots from Σ. The same sublattice is orthogonal to the subset $\alpha_0,\alpha_2,\alpha_3,\alpha_4,\alpha_5\alpha_6,\alpha_7$ of type D_6+A_1. It is easy to see that any diagram of type D_6+A_1 is equivalent to the diagram with this set of vertices.

It remains to consider the diagrams $5A_1, D_8, E_7+A_1$ and E_8+A_1. In the first case, the diagram is equivalent to the diagram with vertices $\alpha_1,\alpha_3,\alpha_5,\alpha_7,\alpha_9$. The vector $v=2\mathfrak{f}+\alpha_9$ is orthogonal to all these vectors, and $v^2=2$. The same is true for Σ of type E_8+A_1. If Σ is of type D_8, then the vector $v=\mathfrak{f}+\alpha_1$ is orthogonal to the sublattice spanned by Σ and satisfies $v^2=2$.

Finally, suppose that the diagram is of type E_7+A_1. Then, the orthogonal complement is spanned by the vectors $v_1=\mathfrak{f}$ and $v_2=\alpha_8+2\alpha_9$ with Gram matrix $\left(\begin{smallmatrix}0&2\\2&-6\end{smallmatrix}\right)$. We have $(2v_1+v_2)^2=2$. □

Let g be an involution of S. We set

$$\mathrm{Num}(S)_\pm := \{v\in\mathrm{Num}(S) : g^*(v)=\pm v\}.$$

From now on in this Section, we assume that $p\neq 2$.

Thanks to our assumption on the characteristic, we can apply the Lefschetz fixed point formula. Since $c_1:\mathrm{Num}(S)_{\mathbb{Q}_l}\to H^2_{\text{ét}}(S,\mathbb{Q}_l)$ is an isomorphism for $l\neq 2$, we obtain

$$e(S^g)=2+\mathrm{tr}_2(g^*), \tag{8.7.3}$$

where $e(S^g)$ is the Euler–Poincaré characteristic of the locus of fixed points of g and $\mathrm{tr}_2(g^*)$ is the trace of g^* on $\mathrm{Num}(S)_{\mathbb{Q}}$. We have

$$\mathrm{tr}_2(g^*)=10-2\,\mathrm{rank}\,\mathrm{Num}(S)_-.$$

This allows us to compute possible $e(S^g)$ to obtain the following table.

We already know that any bielliptic linear system on S defines a bielliptic involution, the deck transformation of the corresponding bielliptic map. This involution leaves invariant a general member of the pencils $|2F_i|$ and induces an involution on it such that the quotient by the involution is a conic on a 4-nodal quartic del Pezzo surface $\mathsf{D}=\mathsf{D}_1$ or D_1'.

$\mathrm{Num}(S)_-$	A_1	$2A_1$	$3A_1$	$4A_1$	D_4	$5A_1$	D_4+A_1	D_4+2A_1
$\mathrm{Num}(S)_+$	$\mathsf{E}_7+\mathsf{U}$	$\mathsf{D}_6+\mathsf{U}$	$\mathsf{D}_4+\mathsf{A}_1+\mathsf{U}$	$4\mathsf{A}_1+\mathsf{U}$	$\mathsf{D}_4+\mathsf{U}$	$4\mathsf{A}_1+\mathsf{A}_1(-1)$	$3\mathsf{A}_1+\mathsf{U}$	$3\mathsf{A}_1+\mathsf{A}_1(-1)$
$e(S^g)$	10	8	6	4	4	2	2	0

$\mathrm{Num}(S)_-$	D_6	$\mathsf{D}_6+\mathsf{A}_1$	E_7	D_8	$\mathsf{E}_7+\mathsf{A}_1$	E_8	$\mathsf{E}_8+\mathsf{A}_1$	
$\mathrm{Num}(S)_+$	$2A_1+U$	$2A_1+A_1(-1)$	A_1+U	$\left(\begin{smallmatrix}0&2\\2&2\end{smallmatrix}\right)$	$\mathsf{A}_1+\mathsf{A}_1(-1)$	U	$\mathsf{A}_1(-1)$	
$e(S^g)$	0	-2	-2	-4	-4	-4	-6	

Table 8.6 15 types of non-numerically trivial involutions of an Enriques surface

Lemma 8.7.5 *Let g be an involution of an Enriques surface leaving invariant a bielliptic linear system $|D| = |2F_1 + 2F_2|$ or $|4F_1 + 2R|$ with $D^2 = 8$. Then, g coincides with the bielliptic involution σ defined by the corresponding bielliptic map $S \to \mathsf{D}$ if and only if it leaves invariant all members of the pencils $|2F_i|$.*

Proof Each pencil is the pre-image of a pencil of conics on D, so the bielliptic involution leaves it invariant.

Conversely, since g commutes with σ, it induces a projective automorphism $\bar{g}$ of D. If it acts identically on the genus one pencils $|2F_i|$, then $\bar{g}$ acts identically on the pencils of conics that define genus one pencils on S. Since any line on D, taken with the multiplicity two, is a member of the pencil, $\bar{g}$ it acts identically on the set of lines on D. Thus, $\bar{g}$ belongs to the identity component $\mathrm{Aut}(\mathsf{D})^\circ$. We use the information about this group given in Proposition 0.6.25 in Volume I.

If $\mathsf{D} = \mathsf{D}_1$, then $\mathrm{Aut}(\mathsf{D})^\circ$ is a 2-dimensional torus that acts, in the double plane model of S, by $[x, y, z] \mapsto [\lambda, \mu y, \gamma z]$. Obviously, it does not act identically on the pencils of conic represented by lines $ax + by = 0$ or conics $axy + bz^2 = 0$.

If $\mathsf{D} = \mathsf{D}_1'$, then $\mathrm{Aut}(\mathsf{D})^\circ$ is given by transformations $[x, y, z] \mapsto [x, ay, bz + cx]$ and we come to the same conlcusion. □

Definition 8.7.6 An involution g of an Enriques surface is called an *irrational involution* if the quotient space $S/(g)$ is not a rational surface.

It follows from the classification of algebraic surfaces that the quotient $S/(g)$ is birationally isomorphic to an Enriques surface.

Example 8.7.7 Let S be an Enriques surface with an elliptic fibration $|2F|$ with a reducible fiber D of type $\tilde{D}_4$. Let R_0 be its unique component of multiplicity 2. Then, $D - 2R_0 \sim 2F - 2R_0$, hence the sum $\Sigma = D - 2R_0$ of simple irreducible components of F is divisible by 2 in $\mathrm{Num}(S)$. Hence there is a double cover $\phi : X \to S$ ramified over Σ. The ramification divisor of ϕ is the sum Σ' of four disjoint (-1)-curves. Let F' be another half-fiber of $|2F|$. We have $K_X = \phi^*(K_S + \Sigma') = \phi^*(F) - \phi^*(F') + \Sigma'$ and $e(X) = 2e(S) - e(\Sigma) = 16$. After we blow down Σ', we obtain a smooth surface Y with $K_Y^2 = 0$ and $e(Y) = 12$. It follows from the classification of algebraic surfaces that it must be an Enriques surface.

Example 8.7.8 Let $f : S \to \mathbb{P}^1$ be an elliptic fibration with reduced fibers such that its jacobian fibration $j : J \to \mathbb{P}^1$ admits a non-trivial 2-torsion section C_0. Then, the translation automorphism of S corresponding to this section is an irrational

involution. Let $C = J^\beta$ be the set of fixed points of the negation involution β. Since $p \neq 2$, it is a smooth curve. It contains, as its connected components, the zero section O and the 2-torsion section C_0. We assume that the residual curve $C_1 = C - C_0$ is irreducible. It follows from the construction of the Weierstrass model of j that C_1 is an elliptic curve and, generically, j has four reducible fibers of type A_1. The branch points of the double cover $C_1 \to \mathbb{P}^1$ induced by the fibration $j : J \to \mathbb{P}^1$ correspond to fibers of type $\tilde{A}_0^*$. The set of fixed points of the translation involution σ is the set of nodes of irreducible singular fibers. This shows that σ is an irrational involution. The quotient surface by this involution is birationally isomorphic to an Enriques surface which admits an elliptic fibration with the same types of singular fibers. The images of the reducible fibers are irreducible singular fibers of type $\tilde{A}_0^*$, and the images of irreducible fibers (after we blow-up the fixed point) are reducible fibers of type $\tilde{A}_1$. This example coincides with Example 8.7.7, if S contains another elliptic fibration $g : S \to \mathbb{P}^1$ invariant with respect to σ such that one of its half-fibers F is smooth and passes through the four fixed points of the translation automorphism. In this case the image of the fibration g on the quotient surface has a fiber of type $\tilde{D}_4$ and its pre-image on S is equal to F.

Proposition 8.7.9 *An involution g of an Enriques surface is an irrational involution if and only if S^g consists of isolated points. Moreover, the number of fixed points is equal to* 4.

Proof Suppose that S^g consists of isolated fixed points. Let g' be the unique lift of g to the blow-up $\alpha : S' \to S$ of isolated fixed points of g. Let $V' \to V$ be the minimal resolution of $k = \#S^g$ singular points of $V = S/(g)$. Then, $V' = S'/(g')$, and we have the following commutative diagram:

$$\begin{array}{ccc} S' & \xrightarrow{\phi} & V' \\ \downarrow{\scriptstyle\alpha} & & \downarrow{\scriptstyle\sigma} \\ S & \longrightarrow & V. \end{array}$$

The ramification divisor of ϕ is the union of k disjoint (-1)-curves $E_1, \ldots, E_k$, and the branch divisor is the union of k disjoint (-2)-curves $\bar{E}_1, \ldots, \bar{E}_k$ such that $\phi^*(\bar{E}_i) = 2E_i$. We have

$$K_{S'} = \alpha^*(K_S) + E_1 + \cdots + E_k = \phi^*(K_{V'}) + E_1 + \cdots + E_k,$$

hence $\phi^*(K_{V'}) = \alpha^*(K_S)$. This easily gives $h^0(K_{V'}) = 0, 2K_{V'} = 0$, hence V' is an Enriques surface. The formula $e(S') = 12 + k = 2e(V') - 2k = 24 - 2k$ shows that $k = 4$.

Conversely, let g be an irrational involution. Suppose that S^g consists of k isolated fixed points and a smooth curve C. In the notation from above, the ramification curve of ϕ is now $C + E_1 + \ldots + E_k$, where we identify C with its pre-image in S'. As above, we get

$$\alpha^*(K_S) = \phi^*(K_{V'}) + C.$$

This gives $2K_{V'} + \bar{C} = 0$, hence V' is rational, and g is not an irrational involution.□

Corollary 8.7.10 *Let g be an irrational involution of an Enriques surface. Then,*

$$\mathrm{Num}(S)_- \cong \mathsf{A}_1^{\oplus 4}, \quad \mathrm{Num}(S)_+ \cong \mathsf{A}_1^{\oplus 4} \oplus \mathsf{U}.$$

Proof In the notation of the proof of the previous proposition, $\phi^*(\mathrm{Num}(V))$ is a sublattice of $\mathrm{Num}(S')$ isomorphic to $\mathsf{E}_{10}(2)$. It is contained in $\mathrm{Num}(S')_+$ and the quotient group is isomorphic to $\langle E_1, \dots, E_4\rangle/\langle E_1 + \cdots + E_4\rangle \cong (\mathbb{Z}/2\mathbb{Z})^{\oplus 3}$. Thus, the index of the sublattice $\phi^*(\mathrm{Num}(V))$ is equal to 2^3, hence its discriminant is equal to 2^4. The discriminant groups of $\mathrm{Num}(S)_-$ and $\mathrm{Num}(S')_+$ are isomorphic, hence $\mathrm{Num}(S)_-$ has discriminant group isomorphic to the discriminant group of $\mathsf{A}_1^{\oplus 4}$. In our Table 8.6 there is only one such lattice of rank 4. □

Remark 8.7.11 Since the lattice $\mathsf{A}_1^{\oplus 4}$ is not from the list (8.7.2), an irrational involution commutes with some bielliptic involution. It is a lift of a projective involution of a surface D that preserves the pencil (or two pencils) of conics but acts non-trivially on the pencil.

Definition 8.7.12 An involution g of an Enriques surface is called a *Coble involution* if the minimal resolution of singularities of $S/(g)$ is isomorphic to a Coble surface.

For the proof of the next theorem, we will need the following result from [103, Corollary 1.3].

Lemma 8.7.13 *Let V be a weak del Pezzo surface that contains an even set of disjoint (-2)-curves. Then, there exists a birational morphism $f : V \to \mathsf{D}$, where D is quartic symmetroid surface D_1 or D_1'.*

Theorem 8.7.14 *Every involution g of an Enriques surface is either an irrational involution, a bielliptic involution or a Cobble involution.*

Proof Suppose g is not an irrational involution. It follows from the proof of Proposition 8.7.9 that S^g contains a smooth curve C and we showed that

$$2K_{V'} + \bar{C} = 0. \tag{8.7.4}$$

In particular, $\bar{C} \in 2\,\mathrm{Pic}(V')$. We also know that the branch divisor $\bar{C} + \bar{E}_1 + \cdots + \bar{E}_k \in 2\,\mathrm{Pic}(V')$. This implies that $\bar{E}_1 + \cdots + \bar{E}_k$ is an even set of $k = 2a$ disjoint (-2)-curves on V'.

We have

$$12 + 2a = e(S') = 2e(V') - e(\bar{C}) - 4a,$$

hence $12 + 6a = 2e(V') - e(\bar{C})$. Since V' is a rational surface, by Noether Theorem, $12 = K_{V'}^2 + e(V')$, and, by adjunction, $e(\bar{C}) = -2K_{V'}^2$. Hence $12 + 6a = 24$. Thus, $a = 2$ and $\bar{E}_1 + \cdots + \bar{E}_4$ is an even set of four disjoint (-2) curves.

Assume $K_{V'}^2 > 0$. Suppose E is an irreducible curve on V' with $E^2 < -2$. Intersecting with $C \in |-2K_{V'}|$, we obtain $0 \le E \cdot C = -2E \cdot K_{V'}$, contradicting

the addiction formula. This shows that V' is a weak del Pezzo surface. Applying Proposition 8.7.13, we find that $V' = \mathsf{D}_1$ or $V' = \mathsf{D}_1$, hence g is a bielliptic involution.

Assume that $K_{V'}^2 \leq 0$ and $h^0(-K_{V'}) > 0$. In the notation of the proof of Proposition 8.7.9, we get $\alpha^*(K_S) = \phi^*(K_{V'} + \bar{C})$, hence $h^0(-K_S) > 0$, a contradiction. This implies that $|-K_{V'}| = \emptyset$ and $|-2K_{V'}| \neq \emptyset$, hence V' is a Coble surface.

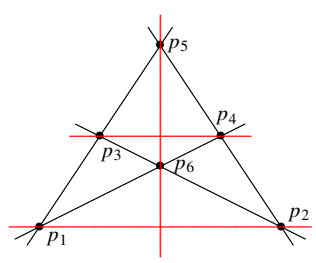
Remark 8.7.15 The fact that the number of isolated points of an involution of a complex Enriques surface is always equal to 4 follows from the holomorphic Lefschetz fixed-point-formula [35, Proposition 4.8] (one can also refer to [196] that proves the analog of this formula in the case $p \neq 2$.)

Remark 8.7.16 Let V be a Coble surface with $|-2K_{\mathsf{V}}| = \{C_1 + \cdots + C_k\}$, where C_i are (-4)-curves. Then, $K_{\mathsf{V}}^2 = -k$ and the Picard number of V is equal to $10 + k$. We have $e(S^g) = 4 + 2k$ It follows from Table 8.6 that

- $k = 1$: $\mathrm{Num}(S)_+ \cong \mathsf{D}_4 + \mathsf{A}_1 + \mathsf{U}$,
- $k = 2$: $\mathrm{Num}(S)_+ \cong \mathsf{D}_6 + \mathsf{U}$,
- $k = 3$: $\mathrm{Num}(S)_+ \cong \mathsf{E}_7 + \mathsf{U}$,
- $k = 4$:$\mathrm{Num}(S)_+ = \mathrm{Num}(S)$.

As we will see later, the same lattices are realized for bielliptic involutions. We conjecture that, in fact, any Coble involution is a bielliptic involution. This would follow if we can find a birational morphism $f : \mathsf{V}' \to \mathcal{D}$ to a weak del Pezzo surface that sends the even set of four (-2)-curves to an even set of four (-2)-curves on $\mathcal{D}$. The conjecture is true if $k = 4$, since in this case g is a numerically trivial involution and since $\mathrm{Aut}_{\mathrm{nt}}(S)$ is a subgroup of $\mathbb{Z}/4\mathbb{Z}$, we cannot have two numerically trivial involutions.

Example 8.7.17 Let $p_1, \ldots, p_6$ be the vertices of a complete quadrilateral of lines in the plane:



(8.7.5)

Choose two general points p_7, p_8 in the plane and let V be a weak del Pezzo surface of degree one obtained by blowing up the set $\{p_1, \ldots, p_8\}$. Consider the double cover $f : S' \to V'$ branched along a curve B equal to the union of a smooth sextic $\bar{C} \in |-2K_V|$ and the union of the proper transforms ℓ_i of the sides of the

quadrilateral. As in the proof of Proposition 8.7.9, we obtain that S' is the blow-up of four points on an Enriques surface S. The deck transformation of the cover defines an involution g of S. The set S^g is the union of a smooth curve of genus two and 4 isolated fixed points. The quotient $S/(g)$ is isomorphic to the anti-canonical model $\bar{V}$ of V.

Let us see that g is a bielliptic involution. Let d_{12}, d_{34}, d_{56} be the diagonals of the quadrilateral passing through $\Sigma_{12} = \{p_1, p_2\}$, $\Sigma_{34} = \{p_3, p_4\}$ and $\Sigma_{56} = \{p_5, p_6\}$, respectively. Let $\mathcal{P}_{ij}$ be a pencil of conics in the plane passing through $\{p_1, \dots, p_6\} \setminus \Sigma_{ij}$. Its pre-image in S is a σ-invariant elliptic pencil $|D_{ij}|$. Its half-fibers G_i, G_j are equal to the images on S of the proper transforms of the lines on V with divisor classes e_i and e_j. We have $G_1 \cdot G_3 = G_1 \cdot G_5 = G_3 \cdot G_5 = 1$. For example, G_1 intersects G_3 at one point equal to the point p_k such that p_1, p_3, p_k are collinear. Thus, we see that σ coincides with three bielliptic involutions $\sigma_{12}, \sigma_{34}, \sigma_{56}$ of S defined by non-degenerate U-pairs $(G_1, G_3), (G_1, G_5)$ and (G_3, G_5). The corresponding bielliptic maps ϕ_{ij} blow down three disjoint (-2)-curves. For example, ϕ_{ij} blows down the proper transform of the diagonal d_{ij} and the proper transforms Θ_7 and Θ_8 of the lines on V with divisor classes e_7, e_8. This defines a birational morphism $V \to \mathsf{D}_1$ to a quartic symmetroid surface.

The lift of the bielliptic map $\phi_{ij} : S \to \mathsf{D}_1$ to S' defines a degree 2 cover of the blow-up of a minimal resolution $\mathcal{D}_1$ blown up at three points.

Corollary 8.7.18 *Suppose g is a bielliptic involution given by a bielliptic linear system $|2F_1 + 2F_2|$ such that the corresponding bielliptic map blows down a curve R (not necessarily irreducible) to a point P on the surface D_1 which does not lie on any line. Then, g leaves invariant a non-bielliptic linear system $|F_1 + F_2 + F_3|$ of degree* 6.

Proof Our proof is essentially a rewording of the proof of Theorem 6.1.12. We know that the image of D_1 under projection from P is a 4-nodal cubic surface Y (see Section 0.7 in Volume I). The linear system $|2F_1 + 2F_2 - R|$ is a bielliptic linear system of degree 6. The involution g leaves invariant the curve $\mathcal{R}$ and coincides with the deck transformation of double cover $S \dashrightarrow Y$ defined by the linear system $|2F_1 + 2F_2 - R|$. As in the previous example, we obtain three pencils of conics on Y coming from three pencils of conics in the plane passing through four vertices of the complete quadrilateral. Two of them, say $\mathcal{P}_{12}$ and $\mathcal{P}_{34}$, define the elliptic pencils $|2F_1|$ and $|2F_2|$ on S. The proper transform of the diagonal $\langle p_5, p_6\rangle$ is the exceptional curve of the inverse map $Y \to \mathsf{D}_1$. The pre-image of the corresponding pencil $\mathcal{P}_{56}$ on S is the third elliptic pencil $|2F_3|$ on S with $F_1 \cdot F_2 = F_3 \cdot F_3 = 1$. Obviously, g leaves invariant the linear system $|F_1 + F_2 + F_3|$. □

The following example is given in [339, 5.2].

Example 8.7.19 Let S' be an Enriques sextic surface, a birational model of an Enriques surface S. Assume that it is invariant with respect to the projective involution $\tau : [x, y, z, w] \mapsto [y, x, w, z]$. Its equation must be of the form

$$(a_1(x^2 + y^2) + a_2(z^2 + w^2) + a_3xy + a_4zw + a_5(xz + yw) + a_6(xw + yz))xyzw$$

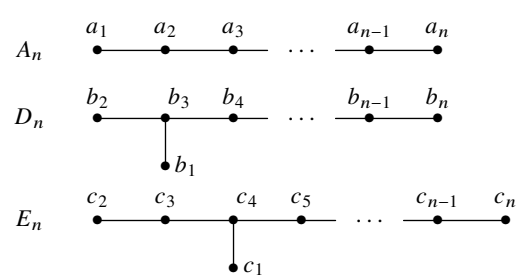
$$+(x^2y^2z^2 + x^2y^2w^2 + x^2z^2w^2 + y^2z^2w^2) = 0.$$

The set of fixed points of the involution τ in $\mathbb{P}^3$ is the union of two lines $x+y = z+w = 0$ and $x - y = z - w = 0$. Plugging these equations into the equation of S', we find that the lines intersect S' at four isolated points not lying on the coordinate tetrahedron $xyzw = 0$. Their pre-images on S are four isolated fixed points of the involution τ lifted to S. Note that a pair of opposite edges defines on S a choice of a bielliptic linear system $|2F_1 + 2F_2|$. Our involution preserves any of them. It leaves invariant the pencils $|2F_1|$ and $|2F_2|$, but either permutes both pairs of half-fibers (F_1, F_1') and (F_2, F_2') or permutes only one pair of half-fibers. This shows that it is a lift of an involution of a quartic surface D under the bielliptic map.

Now, let us study the conjugacy classes of bielliptic involutions on S.

We will use the following lemma from [671]. Because of its importance in the sequel, we supply its proof.

Lemma 8.7.20 *Assume $p \neq 2$ and let X be a smooth minimal projective surface with* $\mathrm{kod}(X) \geq 0$. *Let $f : X \to Y$ be a morphism of degree 2 onto a normal surface. Then, any connected fiber $C = f^{-1}(y)$ over a nonsingular point y of Y is a point or the union of (-2)-curves whose dual graph is of type A_n, D_n, E_n as in the following picture:*



where $E_n = E_6, E_7, E_8$. The deck transformation σ of f extends to a biregular automorphism of X. It acts on the components of C as follows

- $\sigma(a_i) = a_{n+1-i}, i = 1, \ldots, n$;
- $\sigma(b_i) = b_i$ *if n is even;*
- $\sigma(b_1) = b_2,\ \sigma(b_i) = b_i, i \neq 1, 2$ *if n is odd;*
- $\sigma(c_1) = c_1,\ \sigma(c_i) = c_{8-i}, i \neq 1$ *if* $n = 6$;
- $\sigma(c_i) = c_i$ *if* $n = 7, 8$.

Moreover, C contains k irreducible components fixed pointwise by σ, where

$$k = \begin{cases} 0, & A_n, \\ m-1, & D_{2m}, D_{2m+1}, \\ 1, & E_6, \\ 3, & E_7, \\ 4, & E_8. \end{cases}$$

Proof Let $X \to X' \to Y$ be the Stein factorization of f, where $X \to X'$ is a birational morphism and $X' \to Y$ is a finite morphism of degree 2. Since K_X is nef, any curve R blown down to a point of X' satisfies $R^2 < 0$ and $R \cdot K_X \geq 0$. By the adjunction formula, this implies that R is a (-2)-curve. By Proposition 0.4.2, the intersection matrix of the irreducible components of C is negative definite. Since $y = f(C)$ is a nonsingular point of Y, we obtain that y is a singular point of the branch curve B of the cover $f^{-1}(Y \setminus \mathrm{Sing}(Y)) \to Y \setminus \mathrm{Sing}(Y)$. Thus, the fiber $f^{-1}(y)$ coincides with C. Let $\phi(u, v) = 0$ be a local equation of B at y. Then, the pre-image x' of y in X' is a singular point locally isomorphic to $w^2 + \phi(u, v) = 0$ and $X \to X'$ is a minimal resolution of x' over an open neighborhood of x' which is equal to a minimal resolution of a rational double point of type A_n, D_n, E_n. Thus, x' is such a point and y is a simple singularity of B of the corresponding type a_n, d_n, e_n. To see how σ acts on the irreducible components of C, one resolves the singular point explicitly and observes the action of the involution $w \mapsto -w$ on the irreducible components. We leave this exercise to the reader.

Finally, we use that C^σ does not contain isolated fixed points from X^σ since $f(C)$ is a nonsingular point of Y. The intersection $X^\sigma \cap C$ consists of k irreducible components and points where the proper inverse transform $\bar{B}$ of B intersects C.

In the case of A_{2k}, we see that no component is invariant with respect to σ. In the case A_{2k+1} we see that only the central component is invariant and it contains two points on it that belong to $\bar{B}$. This implies that $k = 0$.

In the case of D_{2m}, we see that $\bar{B}$ intersects the components b_1, b_2, b_n at one point, and the components $b_3, b_5, \ldots, b_{n-1}$ must enter in X^σ. It follows that $k = m - 1$.

In the case of D_{2m+1}, $\bar{B}$ intersects the components b_3 and b_n at one point, and the components $b_4, b_6, \ldots, b_{n-1}$ must enter in X^σ. Thus, $k = m - 1$, again.

In the case of E_6, $\bar{B}$ intersects the component c_4 at one point, and the second invariant component c_1 must enter in X^σ. Thus, $k = 1$.

In the case of E_7, $\bar{B}$ intersects the components c_7 and c_1. The components c_2, c_4, c_6 enter in X^σ. This shows that $k = 3$.

In the case of E_8, $\bar{B}$ intersects the components e_1 and the components c_2, c_4, c_6, c_8 enter in X^σ. Thus, $k = 4$. □

We apply this lemma to the case where g is a bielliptic involution, the deck transformation of a bielliptic map $\phi : S \to \mathsf{D}$ given by a bielliptic linear system $|2F_1 + 2F_2|$ or $|4F_1 + 2R|$. The surface D in this case is a 4-nodal quartic del Pezzo surface D_1 or its degeneration D_1'. We described the geometry of these surfaces in Sections 0.6 and 3.3 from Volume I.

We know that g has 4 isolated fixed points. They lie over the singular points of D, the vertices of the quadrangle of lines (or the degenerate quadrangle of two lines) on D. The one-dimensional part of S^g consists of the normalization $\bar{W}$ of the branch curve $W \in |\mathcal{O}_{\mathsf{D}}(2)|$ and smooth rational curves $R_1, \ldots, R_s$ that are blown down to singular points of W. We have

$$e(S^g) = 4 + e(\bar{W}) + 2s. \tag{8.7.6}$$

Each curve blown down by f has zero intersection with F_1 and F_2, and hence must be contained in some member D_1 of $|2F_1|$ and some member of D_2 of $|2F_2|$.

Let $\mathcal{E}$ be the sublattice generated by all irreducible curves blown down by ϕ.

We denote by $\mathcal{E}(x)$ the direct summand of $\mathcal{E}$ generated by irreducible components blown down to a point $x \in \mathcal{D}$. It follows from Lemma 8.2.5 and Lemma 8.7.20 that the g^*-invariant part $\mathcal{E}(x)_+$ of $\mathcal{E}(x)$ is isomorphic to the following lattice:

$$\begin{cases} \mathsf{A}_1^{\oplus k} & \text{if } \mathcal{E}(x) \text{ is of type } A_{2k-1}, A_{2k} \\ \mathsf{D}_{2k} & \text{if } \mathcal{E}(x) \text{ is of type } D_{2k+1}, \\ \mathsf{D}_4 & \text{if } \mathcal{E}(x) \text{ is of type } E_6, \\ \mathcal{E}(x) & \text{otherwise.} \end{cases}$$

For example if $\mathcal{E}(x) \cong \mathsf{E}_6$ then $\mathcal{E}(x)_+$ has a root basis $c_1, c_4, c_3 + c_4 + c_5, c_2 + c_3 + c_4 + c_5 + c_6$ of type D_4.

Since g acts identically on the sublattice U generated by$[F_1], [F_2]$, we have $\mathcal{E}_+ \oplus U$ is a sublattice of finite index in $\mathrm{Num}(S)_+$. It follows from above that $\mathcal{E}$ is a 2-elementary sublattice. The known structure of $\mathrm{Num}(S)_+$ gives all possible cases. We list them in Table 8.7 below.

No.	d_i	g_i	δ	$\mathcal{E}$	$\mathcal{E}_+$	$\mathrm{Num}(S)_+$	m
1	(6)	(5)	0	$\{0\}$	$\{0\}$	U	10
2	(6)	(4)	1	$\mathcal{A}_1$	A_1	$U + A_1$	9
3	(6)	(3)	2	$\mathcal{A}_2$	$2A_1$	$U + 2A_1$	8
4	(6)	(2)	3	$\mathcal{A}_3$	$3A_1$	$U + 3A_1$	7
5				D_4, D_5, E_6	D_4	$U + D_4$	6
6	(6)	(1)	4	$\mathcal{A}_4$	$4A_1$	$U + D_4$	6
6*				$4A_1$	$4A_1$	$U + 4A_1$	6
7				$E_6 + \mathcal{A}_1, D_4(D_5) + \mathcal{A}_1$	$D_4 + A_1$	$U+D_4+A_1$	5
8				D_6	D_6	$U + D_6$	4
9				E_7	E_7	$U + E_7$	3
10				E_8	E_8	$U + E_8$	2
11	(6)	0	5	$\mathcal{A}_5(n)$	$5A_1$	$U+D_4+A_1$	5
12				$D_4(D_5) + \mathcal{A}_2(n)$,	D_4+2A_1	$U + D_6$	4
13				$D_6 + A_1, D_7 + A_1, D_6 + A_2$,	$D_6 + A_1$	$U + E_7$	3
14				$E_7 + A_1$	$E_7 + A_1$	$U + E_8$	2
15	(5, 1)	(1,0)	4	$\mathcal{A}_4$	$4A_1$	$U + D_6$	6
16	(5, 1)	(1,0)	5	$\mathcal{A}_5$	$5A_1$	$U + D_4 + A_1$	4
17				$E_7 + A_1$,	$E_7 + A_1$	$U + E_8$	3
18	(5,1)	(0,0)	6	$\mathcal{A}_6$	$6A_1$	$U + D_4$	3
19				$D_4 + \mathcal{A}_3$	$D_4 + 3A_1$	$U + E_7$	3
20				$2D_4$	$2D_4$	$U + E_8$	2
21				$D_6 + 2A_1$	$D_6 + 2A_1$	$U + E_8$	2
22	(3, 3)	(1, 1)	4	$\mathcal{A}_4$	$4A_1$	$U + D_4$	6
23	(3, 3)	(1, 0)	5	$\mathcal{A}_5$	$5A_1$	$U + D_4 + A_1$	4
24	(3, 3)	(1, 0)	5	$D_4(D_5) + \mathcal{A}_3(n)$	$D_4 + 2A_1$	$U + D_6$	4
25	(3, 3)	(0, 0)	6	$2D_4$	$2D_4$	$U + E_8$	2
26	(3, 2, 1)	(1, 0, 0)	6	$\mathcal{A}_6$	$6A_1$	$U + D_6$	3
27				$D_4 + 3A_1$	$D_4 + 3A_1$	$U + E_7$	3
28				$2D_4$	$2D_4$	$U + E_8$	2
29	(3, 2, 1)	(0, 0, 0)	7	$D_4 + 4A_1$	$D_4 + 4A_1$	$U + E_8$	2
30	(2, 2, 1, 1)	(0, 0, 0, 0)	8	$\mathcal{A}_8$	$8A_1$	$U + E_8$	2

Table 8.7 Singularities of the sextic branch curve W

Here, m is the number of moduli and $\mathcal{A}_k$ denotes a part of $\mathcal{E}$ of type $A_{i_1}+\cdots+A_{i_s}$, where $\sum_{j=1}^{s}[\frac{i_j+1}{2}] = k = \delta$. Thus, $\mathcal{A}_k$ combines various cases, which we do not want to list for typographical reasons. For example, $\mathcal{A}_1 = \{A_1, A_2\}$ but $\mathcal{A}_2 = \{2A_1, A_3, A_4\}$.

Let S' be the blow-up of S at the four isolated fixed points of g. The quotient surface $\mathsf{D}' = S'/(g)$ is a nonsingular model of the 4-nodal quartic surface D. The double plane model $w^2 + xyf_6(x, y, z) = 0$ of S is obtained as the composition of the quotient map $S' \to \mathsf{D}' \to \mathbb{P}^2$, where $\alpha : \mathsf{D}' \to \mathbb{P}^2$ is a birational map that sends the proper transform in D' of the branch curve of the bielliptic map $\phi : S \to \mathsf{D}$ to the part $W : f_6(x, y, z) = 0$ of the branch curve of ϕ. There is a choice of the blowing down map $\alpha : \mathsf{D}' \to \mathbb{P}^2$ that gives a different double plane model of S. Note that the curve W consists of h irreducible components W_k of degrees $d_1, \ldots, d_h$ represented in $\mathrm{Pic}(\mathsf{D})$ by the classes $d_i e_0 - m_1^{(i)} e_1 - \cdots - m_5^{(i)} e_5$ that add up to the class $6e_0 - 2e_1 - \cdots - 2e_5$. We let $(d_1, \ldots, d_h)$ be the degrees of these components and $(g_1, \ldots, g_h)$ their geometric genera. The well-known formula

$$5 = p_a(W) = \sum_{i=1}^{h} g_i + \delta - h + 1$$

gives that the total δ-invariant $\sum_{p\in W} \delta_p(W)$ is equal to $\delta = 4 + h - \sum_{i=1}^{h} g_i$.

It is not difficult but tedious to list all possible singularities of W that give the information on the lattice $\mathcal{E}$, the locus of fixed points S^g, and the action of g on $\mathrm{Num}(S)$. Table 8.7 gives the representatives of Cremona equivalence classes of possible branch curves W.

Table 8.8 below gives the classification of possible types of involutions of Enriques surfaces.

Type	$\mathrm{Num}(S)_-$	$\mathrm{Num}(S)_+$	$e(S^g)$	No. in Table 8.7	S^g
1	0	$\mathsf{E}_8 + \mathsf{U}$	12	10, 17, 27	$C^{(1)} + 4\mathbb{P}^1 + 4\mathrm{pts}$
2	0	$\mathsf{E}_8 + \mathsf{U}$	12	20, 24, 29	$4\mathbb{P}^1 + 4\mathrm{pts}$
3	0	$\mathsf{E}_8 + \mathsf{U}$	12	14, 21, 28	$4\mathbb{P}^1 + 4\mathrm{pts}$
4	A_1	$\mathsf{E}_7 + \mathsf{U}$	10	9	$3\mathbb{P}^1 + C^{(1)} + 4\mathrm{pts}$
5	A_1	$\mathsf{E}_7 + \mathsf{U}$	10	13, 19, 26	$3\mathbb{P}^1 + 4\mathrm{pts}$
6	$2\mathsf{A}_1$	$\mathsf{D}_6 + \mathsf{U}$	8	8, 17, 24	$2\mathbb{P}^1 + C^{(1)} + 4\mathrm{pts}$
7	$2\mathsf{A}_1$	$\mathsf{D}_6 + \mathsf{U}$	8	12, 15	$2\mathbb{P}^1 + 4\mathrm{pts}$
8	$3\mathsf{A}_1$	$\mathsf{D}_4 + A_1 + \mathsf{U}$	6	7	$\mathbb{P}^1 + C^{(1)} + 4\mathrm{pts}$
9	$3\mathsf{A}_1$	$\mathsf{D}_4 + A_1 + \mathsf{U}$	6	11	$\mathbb{P}^1 + 4\mathrm{pts}$
10	D_4	$\mathsf{D}_4 + \mathsf{U}$	4	5	$\mathbb{P}^1 + C^{(2)} + 4\mathrm{pts}$
11	D_4	$\mathsf{D}_4 + \mathsf{U}$	4	22	$2C^{(1)} + 4\mathrm{pts}$
12	D_4	$\mathsf{D}_4 + \mathsf{U}$	4	6	$C^{(1)} + 4\mathrm{pts}$
13	$4\mathsf{A}_1$	$4\mathsf{A}_1 + \mathsf{U}$	4	irrational	$4\mathrm{pts}$
14	$4\mathsf{A}_1$	$4\mathsf{A}_1 + \mathsf{U}$	4	6^*	$C^{(1)} + 4\mathrm{pts}$
15	$\mathsf{D}_4 + \mathsf{A}_1$	$3\mathsf{A}_1 + \mathsf{U}$	2	4	$C^{(2)} + 4\mathrm{pts}$
16	D_6	$2A_1 + \mathsf{U}$	0	3	$C^{(3)} + 4\mathrm{pts}$
17	E_7	$\mathsf{A}_1 + \mathsf{U}$	-2	2	$C^{(4)} + 4\mathrm{pts}$
18	E_8	U	-4	1	$C^{(5)} + 4\mathrm{pts}$

Table 8.8 Involutions of Enriques surfaces in characteristic $p \neq 2$

Here $C^{(g)}$ denote a smooth curve of genus g.

Remark 8.7.21 Here are some comments and hints for the above classification.

1. The first three types (1)–(3) in Table 8.8 are numerically trivial involutions. They correspond to Cases (A), (B), (C) from Corollary 8.2.23.
2. Type (13) is the only type of an involution that does not arise as a bielliptic involution. Although $\mathrm{Num}(S)_+$ and $\mathrm{Num}(S)_-$ coincide with case (6^*) in Table 8.7 the sets of fixed points are different.
3. We often use the Cremona transformation given by the linear system $|3e_0 - e_2 - e_3 - e_4 - e_5 - 2e_P|$ or by choosing appropriately an extra point P. That will allow us–sometimes– to lower the degree of components of W. If P is taken to be a triple point of W, we acquire a conic as an additional component.
4. Another useful Cremona transformation is a quadratic involution given by the linear system of conics $|2e_0 - e_2 - e_3 - e_4|$ or $|2e_0 - e_2 - e_3 - e_4|$. It exchanges the linear system of quartic $|4e_0 - 2e_1 - e_2 - e_3 - e_4 - e_5|$ with the linear system of quintics $|5e_0 - e_1 - 2e_2 - 2e_3 - 2e_4 - 2e_5|$.
5. The degrees of possible irreducible components of the branch curve on D are $(8), (6,2), (4,4)$, $(4,2,2)$, and $(2,2,2,2)$. They correspond to our cases $(6), (5,1), (3,3), (3,2,1), (2,2,1,1)$.
6. Type (6^*) occurs when the four singular points are the pre-images of the intersection points of a conic from the pencil $|e_0 - e_1|$ and a conic from the pencil $|2e_0 - e_2 - e_3 - e_4 - e_5|$.
7. The table shows that the only lattices from (8.7.1) which are not realized as the sublattice $\mathrm{Num}(S)_-$ are the exceptional lattices from (8.7.2).
8. The last case (30) corresponds to involutions g such that g^* is contained in the 2-level congruence subgroup $W(\mathrm{Num}(S))(2)$.

Following [315], we can also construct an involution on S by using the rational quadratic twist construction from Proposition 4.10.10 (see [315]).

We assume that $p \neq 2$. Start with a rational elliptic surface $j : J \to \mathbb{P}^1$ with a section O defining a group law on the set of sections. We choose two fibers J_{t_1} and J_{t_2} of multiplicative type or smooth. For simplicity of exposition, we assume that they are smooth. We also assume that there is a rational bisection C invariant with respect to the negation involution β which is tangent to the fibers J_{t_i} at their 2-torsion points $a_i \in J_{t_i}$. For example, it may arise as the pre-image of a conic under the map of J to the quotient $J/(\beta)$ isomorphic to $\mathbf{F}_2$. This conic must be tangent to the branch curve at two points and also intersect it at the images of the points a_1 and a_2.

Let $B \to \mathbb{P}^1$ be the double cover branched at t_1 and t_2. The base change $j' : X = J \times_{\mathbb{P}^1} B \to B$ is a jacobian elliptic fibration. The projection $p : X \to J$ is a double cover of J branched along the union of the two fibers J_{t_1} and J_{t_2}. It is a K3 surface which turns out to be isomorphic to the canonical cover of an Enriques surface S, the result of the rational quadratic twist construction.

The Enriques involution τ on X is constructed as follows. We additionally assume that C splits under the cover $p : X \to J$ into two (-2)-curves R_+ and R_-. This is automatic if C is smooth. They define two sections s_+ and s_- of j' that add up

to 0 in the Mordell–Weil group $\mathrm{MW}(j')$. Let σ be the deck transformation of the cover $X \to J$ and let t_{R_+} be the translation automorphism defined by the section R_+. Then, $\tau = s_+ \circ \sigma$ switches pairs of fibers X_t and $\sigma(X_t)$ if $t \neq t_1, t_2$ and restricts to a translation t_ϵ by a 2-torsion point $a_i = R_+ \cap X_{t_i}$ on X_{t_i}. So, τ is a fixed-point-free involution that creates two double fibers on the quotient $S = X/(\tau)$. The curve $R_+ + R_-$ is τ-invariant and descends to a bisection R of the elliptic fibration $f : S \to \mathbb{P}^1$ on S whose pull-back on X is the elliptic fibration $j' : X \to B$. When C is smooth, the bisection R is a special bisection. Note that $\sigma \circ \tau \circ \sigma = \sigma_+ \circ \sigma$, so the involution $s_+ \circ \sigma$ is conjugate to τ and gives the same quotient.

To define a new involution on S we consider the composition $\alpha = \iota \circ \sigma$, where ι is the negation involution of the elliptic fibration $j' : X \to B$. Then

$$\begin{aligned} \alpha \circ \tau &= (\iota \circ \sigma) \circ (s_+ \circ \sigma) = \iota \circ (\sigma \circ s_+ \circ \sigma) = \iota \circ s_- \\ &= t_{R_+} \circ \iota = t_{R_+} \circ (\sigma \circ \iota \circ \sigma) = (t_{R_+} \circ \sigma) \circ (\iota \circ \sigma) = \tau \circ \alpha. \end{aligned} \tag{8.7.7}$$

It follows that α commutes with the Enriques involution and hence descends to an involution α' of S. The involution α of X has 8 fixed points, four on each of the fibers X_{t_1} and X_{t_2}. The involution of S has 4 fixed points, two on each half-fiber of the images of X_{t_1} and X_{t_2} on S.

The quotient $X/(\alpha)$ has 8 rational double points of type A_1 and its minimal resolution is a K3 surface. The quotient $S/(\alpha')$ is a rational surface with 4 rational double points of type A_1. The composition $\beta = \tau \circ \alpha$ is a non-symplectic involution of X. The quotient Y is a smooth rational surface.

We have a commutative diagram of the quotient maps:

$$\begin{tikzcd} & X \arrow[dl, "/\tau"'] \arrow[d, "/\beta"] \arrow[dr, "/\alpha"] & \\ S \arrow[dr, "/\alpha'"] & Y \arrow[d, "/\beta'"] & X' \arrow[dl, "/\tau'"] \\ & Z & \end{tikzcd} .$$

In the case when C is smooth, the map $S \to Z$ is a bielliptic map given by the special bielliptic linear system $|4F + 2R|$, where f is given by $|2F|$. The surface $Z = \mathsf{D}_1$ is a quartic symmetroid del Pezzo surface. The map $Y \to Z$ is the double cover of the quartic symmetroid Z ramified at its singular points. The surface Y is isomorphic to a nonsingular quadric surface.

Remark 8.7.22 The two torsion points $a_i \in X_{t_i}$ define the local invariants for a torsor of $j : J \to \mathbb{P}^1$ which is isomorphic to $f : S \to \mathbb{P}^1$. The quadratic twist construction is nothing more than the usual construction of the torsor defined by a cocycle $(\sigma) = \mathrm{Aut}(B/\mathbb{P}^1) \to J'(B) \cong \mathrm{MW}(j')$.

Example 8.7.23 Let g be the covering involution of a separable bielliptic map $f : S \to \mathsf{D}$. Assume $p \neq 2$. If the branch curve W is smooth, we know that g is of type E_8 in Table 8.8. When W is irreducible with k ordinary nodes, then S^g consists of

four isolated fixed points and a curve of genus $5-k$. This gives that the trace of g^* on the cohomology is equal to $4+2-2(5-k) = 2k-4$. We also know that g^* leaves invariant a hyperbolic plane. This gives that g is of type $E_7, D_6, D_4+A_1, 4A_1, 3A_1$, respectively.

8.8 Finite Groups of Automorphisms of Mathieu Type

As was shown by Mukai [527], a finite group of symplectic automorphisms of a complex K3 surface is isomorphic to a subgroup of the Mathieu group M_{23}. In this section, we recall the definition of the Mathieu groups and, following Mukai, introduce a class of finite groups of semi-symplectic automorphisms of an Enriques with the trace $\mathrm{tr}(g^*)$ of its linear representation on $H^*(S, \mathbb{Q}_l)$ equal to the value of the character of the Mathieu group M_{11} of its natural permutation on elements of order $n = \mathrm{ord}(g)$.

Recall from [163, 6.2] that a *Steiner system* $\mathsf{S}(t,k,v)$ consists of a set Σ of cardinality v whose elements are called points and a set $\mathcal{B}$ of subsets of Σ of cardinality k whose elements are called blocks such that any subset of t points is contained in a unique block. The number b of blocks and the number r of blocks containing a fixed point satisfy:

$$bk = vr, \quad r = \frac{(v-1)(v-2)\cdots(v-t+1)}{(k-1)(k-2)\cdots(k-t+1)}. \tag{8.8.1}$$

Many interesting subgroups of $\mathrm{Sym}(\Sigma)$ are realized as the symmetry groups of a Steiner system. The most remarkable examples are the five *Mathieu sporadic simple groups* $M_{11}, M_{12}, M_{22}, M_{23}$ and M_{24}. The corresponding Steiner systems are $\mathsf{S}(4,5,11), \mathsf{S}(5,6,12), \mathsf{S}(3,6,22), \mathsf{S}(4,7,23)$ and $\mathsf{S}(5,8,24)$, respectively.

The group M_{11} can be realized as a subgroup of M_{12} that fixes one point, and the subgroups M_{22} (resp. M_{23}) are realized as subgroups of M_{24} that fix two (resp. one) points. The Mathieu group M_{12} is also isomorphic to a subgroup of M_{24}. It is a subgroup that preserves the symmetric difference of two blocks meeting two points in $\mathsf{S}(5,8,24)$.

The subgroup of M_{11} that fixes a point is a Mathieu group M_{10}. It is not a simple group but contains the group $\mathfrak{A}_6$ as a subgroup of index 2 (however, M_{10} is not isomorphic to $\mathfrak{S}_6$). It is the automorphism group of the Steiner system $\mathsf{S}(3,4,10)$.

The number $\epsilon(n)$ of fixed points of the action of $\sigma \in M_{23}$ on Σ with $v = 24$ depends only on the order n of σ. It is given in the following table:

n	1	2	3	4	5	6	7	8	11	14	15	23
$\chi_{M_{23}}(n)$	24	8	6	4	4	2	3	2	2	1	1	1

Table 8.9 Mathieu character for M_{23}

According to Mukai [527], any finite group G of symplectic automorphisms of a complex K3 surface X is isomorphic to a subgroup of M_{23} with ≥ 5 orbits in its natural action on a set Σ. This is achieved by analyzing the character χ of the action of G on the cohomology $H^*(X, \mathbb{C}) \cong \mathbb{Q}^{24}$. By the Lefschetz fixed-point formula, for any $g \in G$ of order n, we have

$$\chi(g) = |X^g| = \epsilon(n) := 24\left(n \prod_{p|n}\left(1 + \frac{1}{p}\right)\right)^{-1},$$

and this coincides with the character of M_{23} in its permutation representation on Σ given in Table 8.9. The set of one-dimensional sub-representations of G in $H^*(X, \mathbb{C})$ corresponds to the orbits of G in its action on Σ. They are spanned by the Hodge subspaces $H^{0,0}, H^{2,2}, H^{2,0}, H^{0,2}$ and the subspace of $H^{1,1}$ generated by a G-invariant Kähler class. In the case of positive characteristic $p > 0$, the same is true if we replace the complex cohomology with the l-adic ones, assume that the order of G is prime to p (this is always satisfied if $p > 11$), and X is not a supersingular K3 surface with Artin invariant 1 (see [188]).

The number of fixed points of an element $\sigma \in M_{11}$ on the subset $\Sigma^+ \subset \Sigma$ of cardinality 12 depends only on the order n of σ. It is given in the following table:

n	1	2	3	4	5	6	8	11
$\chi_{M_{11}}(n)$	12	4	3	4	2	1	2	1

Table 8.10 Mathieu character for M_{11}

Following Mukai and Ohashi, we say that a finite group G of automorphisms of an Enriques surface is of *Mathieu type* if it acts semi-symplectically and the character of χ on $H^*(X, \mathbb{Q}_\ell) \cong \mathbb{Q}_l^{12}$ satisfies

$$\mathrm{tr}(g^*) = \chi_{M_{11}}(\mathrm{ord}(g)), \quad g \in G.$$

In our case, by Theorems 8.6.12 and Proposition 8.6.15, if $p \neq 2$, we have $n \neq 8, 11$.

Note that, applying the Lefschetz fixed point formula, we obtain

$$2 + \mathrm{tr}(g^*) = \#S^g.$$

Proposition 8.8.1 *Assume $p \neq 2$. Let G be a finite group of automorphisms of an Enriques surface. Then, G is of Mathieu type if and only if it does not contain elements of order* 8*, and, in Carter's notation from Table 8.6, the conjugacy class of $g^* \neq 1$ must be one of the following types:*

$(n = 2)$ D_4 *or* $4A_1$ *with* $e(S^g) = 4$*;*
$(n = 3)$ $3A_2$ *with* $e(S^g) = 3$*;*

(n = 4) $D_4(a_1) + 2A_1$ *with* $e(S^g) = 4$*;*
(n = 5) $2A_4$ *with* $e(S^g) = 2$*;*
(n = 6) $D_4 + 2A_2$ *with* $e(S^g) = 1$.

Proof It follows from Proposition 8.6.15 that an element of order 8 is not semi-symplectic. Table 8.6 gives possible conjugacy classes of elements of order 2 (resp. 4, resp. 6) with the trace equal to 4 (resp. 4, resp. 1). In the case $n = 4$, we also eliminate those of them for which the conjugacy class of the square is not of type $4A_1$ or D_4. The conjugacy classes of elements of odd order satisfy the assumption on the trace. □

The fact that a group of automorphisms G of Mathieu type acting on an Enriques surface in characteristic $p \neq 2$ does not contain elements of order 8 and 11 gives some information about possible structure of a maximal group of this type. Using [129], Table 10.3, we find that a maximal subgroup of M_{11} must be isomorphic to one of the following groups:

$$M_{10} \cong \mathfrak{A}_6.2, \quad L_2(11), \quad M_9 : 2 \cong 3^2 : Q_8.2, \quad \mathfrak{S}_5, \quad M_8 : \mathfrak{S}_3 \cong 2^{\cdot}\mathfrak{S}_4.$$

(we employ the notation from loc. cit.). By analyzing subgroups G of these maximal groups without elements of order 8 or 11, we find that G is a subgroup of one of the following groups

$$\mathfrak{A}_6, \quad \mathfrak{S}_5, \quad N_{72}, \quad C_2 \times \mathfrak{A}_4, \quad C_2 \times C_4. \tag{8.8.2}$$

Here C_n denotes a cyclic group of order n and $N_{72} = 3^2 \rtimes D_8$.

By Lemma 6.4.8, any finite subgroup of $W(\mathsf{E}_{10})$ is conjugate to a subgroup of a finite parabolic subgroup W_J of one of the types listed in (6.4.5). The groups from (8.8.2) are not of E_8-type because they contain elements of types $3A_2$ and $2A_4$ which are not of E_8-type. The group $\mathfrak{A}_6$ is a subgroup of the Mathieu group M_{10}, it must be conjugate to a subgroup of a parabolic subgroup W_J of type A_9 with $W(A_9) \cong \mathfrak{S}_{10}$. The group $\mathfrak{S}_5$ must be a subgroup of a parabolic subgroup of type $A_4 + A_5$ with $W(A_4 + A_5) \cong \mathfrak{S}_5 \times \mathfrak{S}_6$. Finally, the group N_{72} is one of the maximal subgroups of M_{11} and occurs as a stabilizer of a 2-point set. It must be conjugate to a subgroup of a parabolic subgroup of type $A_1 + A_8$ with $W(A_1 + A_8) \cong \mathfrak{S}_2 \times \mathfrak{S}_9$.

It is known that the permutation action of the subgroups $\mathfrak{A}_6$, $\mathfrak{S}_5$, N_{72} of M_{11} on Σ^+ has orbits $(1, 1, 10), (1, 5, 6), (1, 2, 9)$, respectively. It shows that $\dim H^*(S, \mathbb{Q}_l)^G = 3$, and rank $\mathrm{Num}(S)^G = 1$. As in the proof of Corollary 8.6.7, we can compute the orthogonal complements of the root sublattices of types $A_9, A_4 + A_5$, and $A_1 + A_8$ and obtain that $\mathfrak{A}_6$, $\mathfrak{S}_5$, N_{72} must preserve a polarization of degree $10, 30, 18$, respectively.

We have also the converse.

Proposition 8.8.2 *Suppose one of the groups* $G = \mathfrak{A}_6$, $\mathfrak{S}_5$, N_{72} *acts on an Enriques surface S preserving an ample polarization of degree* $10, 30, 18$*, respectively. Then, the action of G on S is Mathieu.*

Proof It follows from Proposition 1.5.3 in Volume I that a vector $v \in \mathsf{E}_{10}$ with $v^2 \in \{10, 30, 18\}$ and $\Phi(v) \geq 3$ belongs to the orbit of the fundamental weight $\omega_0, \omega_2, \omega_4$, respectively. Its orthogonal complement is a sublattice of E_{10} isomorphic to $\mathsf{A}_9, \mathsf{A}_4 \oplus \mathsf{A}_5$ and $\mathsf{A}_1 \oplus \mathsf{A}_8$, respectively. Thus, G is embedded in the Weyl groups of these lattices.

Assume $G = \mathfrak{A}_6$. The analysis of maximal subgroups of $\mathfrak{A}_{10}$ in [129] shows that the only possible maximal subgroups which may contain $\mathfrak{A}_6$ and can be parabolic subgroups of $W(\mathsf{E}_{10})$ are the following ones: $\mathfrak{A}_9$, $\mathfrak{S}_8$, $(\mathfrak{A}_7 \times 3) : 2$, $(\mathfrak{A}_6 \times 4) : 2$, and M_{10}. Now, we use that G is not conjugate to a subgroup $W(\mathsf{A}_9)$ of a parabolic group of type E_8. Suppose $G \subset \mathfrak{S}_8$, then $\mathfrak{S}_8$ acts on 10 letters by leaving 8 letters invariant and switching a duad in the complement. Then, G is contained in a subgroup of $W(E_8)$ isomorphic to $\mathfrak{A}_8$. This discards this subgroup. The second maximal group preserves a triad of letters, so G would be embedded in $\mathfrak{A}_7$. Similar argument discards the group $(\mathfrak{A}_6 \times 4) : 2$. Suppose $G \subset \mathfrak{A}_9$. Then, G is conjugate to a parabolic subgroup of type A_8. Its orthogonal complement in E_{10} contains an isotropic vector. This would imply that $\mathfrak{A}_6$ preserves a genus one pencil, which is obviously impossible. The surviving option is the Mathieu group M_{10}, this is what we need.

Assume $G = \mathfrak{S}_5$ or N_{72}. It is conjugate to a subgroup of a parabolic subgroup of type $A_4 \times A_5$ or $A_1 \times A_8$ isomorphic to $\mathfrak{S}_5 \times \mathfrak{S}_6$ or $\mathfrak{S}_2 \times \mathfrak{S}_9$. We consider these groups as subgroups of $\mathfrak{S}_{11}$. A maximal subgroup of this group either preserves a subset of cardinality $1, 2, 3, 4, 5$ or is a subgroup of two non-conjugate subgroups isomorphic to the Mathieu group M_{11}. It is easy to see that the first possibility in our case implies that G is of E_8- type or preserves a genus one fibration. This leads to a contradiction. □

The main result of [534] is the following:

Theorem 8.8.3 *Assume $\mathbb{k} = \mathbb{C}$. A finite group G admits a Mathieu action on S if and only if it is isomorphic to a subgroup of one of the five maximal groups from* (8.8.2). *Equivalently, G is isomorphic to a subgroup of $\mathfrak{S}_6$ and its order is not divisible by* 16.

We omit proof and only provide examples of Mathieu actions of the first three maximal groups from the list (8.8.2). We refer to the examples realizing the last two groups to [533] and [535].

Example 8.8.4 ([533]) Assume $G \cong \mathfrak{S}_5$. Let X' be a surface of degree 6 in $\mathbb{P}^4$ given by the equations

$$\sum_{1 \leq i < j \leq 5} x_i x_j = \sum_{1 \leq i < j < k \leq 5} x_i x_j x_k = 0. \tag{8.8.3}$$

The surface has obvious $\mathfrak{S}_5$-symmetry. Also, it admits an involution σ defined by the standard Cremona transformation $[x_1, \ldots, x_5] \to [1/x_1, \ldots, 1/x_5]$. Its fixed points in $\mathbb{P}^4$ are the points with coordinates ± 1. They do not lie on X' unless $p = 2$ or 5.

The Cremona involution σ in $\mathbb{P}^4$ is defined by the linear system of quartic hypersurfaces passing through the points $q_1 = [1, 0, \ldots, 0], \ldots, q_5 = [0, \ldots, 0, 1]$ with

multiplicity ≥ 3. The points are the only singular points of X', they are ordinary nodes. Let X be a minimal resolution of X'. It is a K3 surface.

Assume that $p \neq 2, 5$. In this case the involution σ lifts to a fixed-point-free involution τ of X and the quotient $S = X/\langle\tau\rangle$ is an Enriques surface. The Cremona involution σ restricted to X' is defined by quartics passing through the singular points. On X, it is given by the linear system $|4h - 3(E_1 + \cdots + E_5)|$, where h is the pre-image of a hyperplane section class of X' on X and E_i are the exceptional curves of the resolution of singularities.

The linear system $|5h - 3(E_1 + \cdots + E_5)|$ is invariant with respect to the group generated by $\mathfrak{S}_5$ and τ. Thus, X admits a polarization of degree $(5h - 3(E_1 + \cdots + E_5))^2 = 25 \cdot 6 - 9 \cdot 10 = 60$. It descends to a polarization of degree 30 on S invariant with respect to the group $\mathfrak{S}_5$ acting on it. The action of $\mathfrak{S}_5$ descends to an action of this group on S. Applying Proposition 8.8.2, we obtain that the action of $\mathfrak{S}_5$ on S is Mathieu.

Note that it is not true that any action of $\mathfrak{S}_5$ must be a Mathieu action. In the next section, we will study a surface S birationally isomorphic to the quotient of the surface

$$\sum_{i=1}^{5} x_i = \sum_{i=1}^{5} \frac{1}{x_i} = 0$$

by the Cremona involution. Its full group of automorphisms is isomorphic to $\mathfrak{S}_5$ but only its subgroup $\mathfrak{A}_5$ acts in a Mathieu fashion.

Example 8.8.5 ([534]) Assume $p \neq 2$. We construct S as the quotient of a degree 8 K3 surface in $\mathbb{P}^5$ by a fixed-point free involution τ. We use the notation from Section 3.4 from Volume I. Recall that X is embedded in $\mathbb{P}^5$ by the linear system $|D'|$, where $\tilde{h}_4 = [D']$ is a τ-invariant polarization of degree 8, a lift of a polarization h_4 of degree 4 on S. The involution τ decomposes the linear space $E = H^0(X, \mathcal{O}_X(D'))$ (resp. its dual space $H^0(X, \mathcal{O}_X(D'))^\vee$) into eigensubspaces E_+, E_- (resp. $E_+^\vee = E_-^\perp, E_-^\vee = E_+^\perp$). The net $|N|$ of quadrics vanishing on X is defined by a triple (N_+, N_-, γ), where $N_\pm$ is a 3-dimensional subspace of $S^2 E_\pm$ of quadratic forms on $E_\pm^\vee$, and $\gamma : N_+ \to N_-$ is an isomorphism of linear spaces. The K3 surface X is the complete intersections of three quadrics:

$$X = \bigcap_{q \in N_+} V(q + \gamma(q)). \tag{8.8.4}$$

The discriminant curve $\Delta \subset |N|$ of $|N|$ is equal to the union $\Delta_+ + \Delta_-$ of the discriminant curves of the nets of conics $|N_\pm|$. We will deal with the case where the curves $\Delta_\pm$ are smooth cubics that intersect transversally at 9 points $p_1, \ldots, p_9$. It is known that the space $E_\pm$ is identified with the space $H^0(\Delta_\pm, \mathcal{O}_{\Delta_\pm}(\theta_\pm)(1))$, where $\theta_\pm$ is a non-trivial 2-torsion divisor class on $\Delta_\pm$ [177, 4.1.3]. To sum up, starting with a 3-dimensional linear space N, two nonsingular plane cubics $\Delta_\pm$ in $|N| = \mathbb{P}^2$ with non-trivial 2-torsion divisor classes $\theta_\pm$ on them, we can construct a K3 surface X as follows. The pairs $(\Delta_\pm, \theta_\pm)$ define nets $N_\pm$ of conics in the planes $\mathbb{P}(E_\pm)$, where $E_\pm$ as above. We consider $E = E_+ \oplus E_-$ and take X defined by equations (8.8.4). The isomorphism γ is defined by identifying N_+ and N_- with N.

Assume additionally that $p \neq 3$. Let $C = V(F_3)$ be a nonsingular plane cubic and $H(C) = V(\mathrm{Hess}(F_3))$ be the cubic surface defined by the Hessian determinant of F_3. It coincides with the discriminant curve of the net of polar conics of C, and hence defines a non-trivial 2-torsion divisor class θ in C (see [177, Chapter 3]).

In Section 4.7 and Example 4.10.13 from Volume I, we already encountered the Hesse pencil of plane cubics. If we take a nonsingular cubic to be a member of the pencil

$$C_\lambda : x^3 + y^3 + z^3 + 6\lambda xyz = 0, \tag{8.8.5}$$

then we find that the Hessian cubic $H(C_\lambda)$ is also a member $C_{\lambda'}$ of this pencil with $\lambda' = -\frac{1+2\lambda^3}{6\lambda^2}$ (see [177, 3.2.2]).

Thus, if we choose two cubics $C_\pm$ from the pencil, their Hessian cubics will give us two pairs $(\Delta_\pm, \theta_\pm)$ that will define two pencils of conics $N_\pm$. We will use this to define a net of quadrics in $\mathbb{P}^5$ and its base locus X.

The Hesse pencil contains 6 anharmonic plane cubics (i.e. cubics whose Weierstrass equation has the form $y^2 + x^3 + ax = 0$, or, equivalently, cubics with vanishing Aronhold invariant T). It follows from formula (3.10) in [177] that we find that the corresponding parameter λ in (8.8.5) is a root of the equation $8t^6 + 20t^3 - 1 = 0$. The six roots of this equation are $\lambda_i = \frac{1}{2}(-1 \pm \sqrt{3})\omega$, where $\omega^3 = 1$. Take one of these roots, say $\lambda_+ = \frac{1}{2}(-1 + \sqrt{3})$. The Hesse curve $H(C_{\lambda_+})$ coincides with C_{λ_-}, where $\lambda_- = \frac{1}{2}(-1 - \sqrt{3})$. Also, $H(C_{\lambda_-}) = C_{\lambda_+}$. The net of polar conics of a member C_λ of the Hesse pencil is spanned by conics

$$x^2 + 2\lambda yz = 0,\ y^2 + 2\lambda xz = 0,\ z^2 + 2\lambda xy = 0.$$

We choose the surface X to be defined by the data $(\Delta_\pm, \theta_\pm)$ with $\Delta_\pm = H(C_{\lambda_\pm})$. It follows that X is given by equations:

$$\begin{aligned} q_1 &= x_0^2 + 2\lambda_- x_1x_2 - (y_0^2 + 2\lambda_+ y_1y_2) = 0, \\ q_2 &= x_1^2 + 2\lambda_- x_0x_2 - (y_1^2 + 2\lambda_+ y_0y_2) = 0, \\ q_3 &= x_2^2 + 2\lambda_- x_0x_1 - (y_2^2 + 2\lambda_+ y_0y_1) = 0. \end{aligned} \tag{8.8.6}$$

Here, we use coordinates (x_0, x_1, x_2) in the plane $|E_+|$ (resp. (y_0, y_1, y_2) in the plane $|E_-|$).

The *Hessian group* G_{216} of projective automorphisms leaving invariant the Hesse pencil is a group of order 216 isomorphic to $3^2 : \mathrm{SL}(2, \mathbb{F}_3)$. The blow-up of the base points of the pencil defines an extremal rational elliptic surface $f : J \to \mathbb{P}^1$ with the Mordell–Weil group isomorphic to $3^2 := (\mathbb{Z}/3\mathbb{Z})^2$. The subgroup 3^2 acts as the group of translations (when we fix one section). The center of $\mathrm{SL}(2, \mathbb{F}_3)$ acts on a general fiber as the negation automorphism. It is given by the projective transformation $(x, y, z) \mapsto (x, z, y)$. The quotient by the center $\mathrm{PSL}(2, \mathbb{F}_3) \cong \mathfrak{A}_4$ acts on the base of the pencil as the tetrahedral group generated by the transformation σ_1 of order two defined by $\lambda \mapsto \frac{-\lambda+1}{2\lambda+1}$ and a transformation σ_2 of order three defined by $\lambda \mapsto \omega\lambda$. We check that $\sigma_1(\lambda_+) = \lambda_-$, thus, a subgroup $3^2 : 4$ of order 36 leaves

the discriminant pair invariant and acts on the surface X. In fact, one directly checks that the equations of X are invariant with respect to these transformations (acting on the variables x_i and y_i diagonally).

We would like to extend this action to an action of the group N_{72}. To do this, we consider an embedding

$$\Phi : E \hookrightarrow S^2E_+^\vee \oplus \bigwedge^2 E_-^\vee \subset E^\vee \otimes E^\vee = S^2E^\vee \oplus \bigwedge^2 E^\vee.$$

The projection map $\Phi(E) \to S^2E_+^\vee$ is equal to the composition $E \to E_+ \to S^2E_+^\vee$, where the second map is given by the net N_+ of conics in the plane $|E_+|$. The second projection $\Phi(E) \to \bigwedge^2 E_-^\vee$ is equal to the composition $E \to E_- \to \bigwedge^2 E_-^\vee$, where the second map is given by the canonical isomorphism $E_- \to \bigwedge^2 E_-^\vee$ defined by a choice of a volume form in $\bigwedge^3 E_-$. In coordinates x_i, y_i, if we identify the space S^2E_+ with the space of symmetric (3×3)-matrices, and the space $\bigwedge^2 E_-$ with the space of skew symmetric (3×3)-matrices, the map Φ is given by

$$\begin{aligned}(x_0, x_1, x_2, y_0, y_1, y_2) \mapsto A &= \begin{pmatrix} \lambda_+x_0 & x_2 & x_1 \\ x_2 & \lambda_+x_1 & x_0 \\ x_1 & x_0 & \lambda_+x_2 \end{pmatrix} + \begin{pmatrix} 0 & cy_2 & -cy_1 \\ -cy_2 & 0 & cy_0 \\ cy_1 & -cy_0 & 0 \end{pmatrix} \\ &= \begin{pmatrix} \lambda_+x_0 & x_2 + cy_2 & x_1 - cy_1 \\ x_2 - cy_2 & \lambda_+x_1 & x_0 + cy_0 \\ x_1 + cy_1 & x_0 - cy_0 & \lambda_+x_2 \end{pmatrix},\end{aligned} \tag{8.8.7}$$

where $c^2 = 1 - 4\lambda_+^2 = \frac{1}{2}\sqrt{3}$ (see [534, (2.6)]).

This defines an embedding Φ of $|E| \cong \mathbb{P}^5$ into $|S^2E_+ \oplus \bigwedge^2 E_|| \cong \mathbb{P}^8$. The image of Φ is given by three linear equations

$$a_{ii} + \lambda_+(a_{jk} + a_{kj}) = 0,\ i = 1, 2, 3. \tag{8.8.8}$$

One checks immediately that $\Phi(X) \subset \mathbb{P}^8$ is given by equations (8.8.8) and three quadratic equations:

$$a_{ii}^* + \lambda_+(a_{jk}^* + a_{kj}^*) = 0,\ i = 1, 2, 3, \tag{8.8.9}$$

where $\mathrm{adj}(A) = (a_{ij}^*)$ is the adjugate matrix satisfying $A \cdot \mathrm{adj}(A) = \det(A)I_3$.

The map $\mathrm{adj} : A \mapsto \mathrm{adj}(A)$ defines a Cremona involution in the space $\mathbb{P}^8$ of 3×3 matrices. It is given by quadrics $(a_{11}^*, \ldots, a_{33}^*)$ in coordinates a_{ij}, the partials of the determinant $\det(A)$. It blows down the determinantal hypersurface $V(\det(A))$ to the fundamental locus of adj that consists of matrices of rank 1 isomorphic, via the Segre embedding, to $\mathbb{P}^2 \times \mathbb{P}^2$.

Since adj exchanges equations (8.8.8) with equations (9.7.5), the surface $\Phi(X)$ is invariant with respect to adj. Because X is a minimal algebraic surface the restriction of adj to $\Phi(X)$ is a biregular involution.

To simplify notation, we will identify X with its image $\Phi(X)$ in $\mathbb{P}^8$. We denote by $\mathfrak{a}$ the involution of X defined by adj.

By explicit computation, one checks that the involution $\mathfrak{a}$ commutes with τ and descends to an involution on S that, together with the subgroup $3^2 : 4$, generates a group isomorphic to N_{72}.

The intersection of $\Phi(X)$ with the indeterminacy locus of adj is isomorphic to the complete intersection of three divisors of type $(1, 1)$ in $\mathbb{P}^2 \times \mathbb{P}^2$. It is a curve G_0 of arithmetic genus one and degree 6 in $\mathbb{P}^5$. Let $\tilde{H}_8$ denote the class of a hyperplane section of X embedded in $\mathbb{P}(E) \cong \mathbb{P}^5$. Since it is invariant with respect to the Enriques involution τ, it descends to a divisor class H_4 defining a polarization of degree 4 on S.

Since $\mathfrak{a} : X \to X$ is given by quadrics passing through F_0, we have $\mathfrak{a}^*(\tilde{H}_8) \equiv 2\tilde{H} - F_0$. Thus, $\tilde{H}_8 + \mathfrak{a}^*(\tilde{H}) = 3\tilde{H}_8 - F_0$ is a polarization on X of degree $(3\tilde{H}_8 - F_0)^2 = 36$. Since τ preserves $\tilde{H}$ and commutes with $\mathfrak{a}$, it descends to a polarization $H_{18} = 3H_4 - G_0$ of degree 18, where $\pi^*(G_0) = F_0$. Thus, the group N_{72} of automorphisms of S preserves a Mukai polarization of degree 18 on S. By Proposition 8.8.2, its action is Mathieu.

We will show in the next example that the same surface S admits a Mathieu action of the group $\mathfrak{A}_6$. To do this we have to introduce more geometry of S and its canonical cover X.

In the previous Example, we introduced a genus one curve F_0 of degree 6 on X. It varies in an elliptic pencil $|F_0|$. Let us describe its singular fibers,

Recall that the discriminant curve of the net $N = |\mathcal{O}_X(2)|$ of quadrics containing X is the union of two cubic curves $\Delta_\pm$ which are members of the Hesse pencil(8.8.5) with base points $p_1, \ldots, p_9$ and four triangles of lines $T_1, \ldots, T_4$.

Any line in N defines a pencil of quadrics containing X. If we take the line to be a side ℓ of one of the triangles, then all quadrics from this pencil are singular at some point in $\mathbb{P}^5$, and the quadric corresponding to three base points lying on this line are quadrics of corank 2 in the pencil. The Segre symbol of a pencil of quadrics in $\mathbb{P}^5$ containing 3 quadrics of corank 2 is $(2, 2, 2)$ (see [177, 8.6.1]). The base locus of the pencil contains a plane.

Such a quadric Q_i contains two pencils of 3-planes in $\mathbb{P}^5$ with the base locus equal to the singular line of the quadric. The restriction of N to each 3-plane is the base locus of a pencil of quadrics in this plane, hence a curve of arithmetic genus one and degree 4. Varying the 3-space in the pencil, we get a genus one pencil $|F_i|$ on X of curves of degree 4. Replacing the pencil with another pencil of 3-planes on Q_i, we get another genus one pencil $|F_{-i}|$. Since the union of two 3-planes, each from its own pencil, spans a hyperplane in $\mathbb{P}^5$, we get

$$F_i + F_{-i} \in |\tilde{H}|.$$

Recall that in the previous example, we introduced an elliptic pencil $|F_0|$, the *primary elliptic pencil* from [534].

Lemma 8.1 *Let $|F_{\pm 1}|, \ldots, |F_{\pm 9}|$ be the elliptic fibrations defined by the quadrics Q_i of corank 2. Then,*

(i) each $|F_{\pm i}|$ is τ-invariant and hence descends to an elliptic fibration $|2G_{\pm i}|$ on S;
(ii) each elliptic pencil $|2G_i|$ is of Hesse type, i.e. has 4 reducible fibers of type $\tilde{A}_2$;
(iii) $G_{\pm i} \cdot G_{\pm j} = 1$ for $|i| \neq |j|$ and $G_i \cdot G_{-i} = 2$.
(iv) $G_0 \cdot (G_i + G_{-i}) = 3$.

Proof Property (i) follows from the construction of the elliptic pencils $|F_{\pm i}|$.

(ii) We already explained that each line component of a reducible member of the Hesse pencil defines a pencil of quadric containing X with base locus containing a plane. The intersection of this plane with X is a conic. Replacing the line component with another line component of the same triangle of lines, we obtain 3 conics intersecting each other at one point (corresponding to the intersection of the lines). Thus, each triangle defines a reducible curve of arithmetic genus one and degree 6. This curve is disjoint from the curve F_0, hence it is a reducible member of the pencil $|F_0|$. Applying τ, we get 4 pairs of reducible fibers of type $\tilde{A}_2$. This shows that the pencil $|2G_0|$ descendent from $|F_0|$ on S is of Hesse type.

Consider one of the nine quadrics Q_i corresponding to a base point p_i of the Hesse pencil. For example, $p_1 = [0, 1, -1]$ defines the quadric

$$Q_1 = V(q_2 - q_3) : (x_1 - x_2)(x_1 + x_2 - 2\lambda_+ x_0) - (y_1 - y_2)(y_1 + y_2 - 2\lambda_+ y_0) = 0.$$

in the pencil N.

The base locus of the pencil generated by $V(q_2)$ and $V(q_3)$ contains two planes

$$\Pi_1 : x_1 - y_1 = x_2 - y_2 = \lambda_- x_0 - \lambda_+ y_0 = 0,$$
$$\Pi_1' : x_1 + y_1 = x_2 + y_2 = \lambda_- x_0 + \lambda_- y_0 = 0.$$

that differ by the Enriques involution

$$[x_0, x_1, x_2, y_0, y_1, y_2] \mapsto [x_0, x_1, x_2, -y_0, -y_1, -y_2].$$

They define the conic components $K_1, \tau(K_1)$ of reducible fibers of the primary pencil $|F_0|$.

The plane Π_1 is contained in a 3-plane $\Lambda \subset Q_1$:

$$(x_1 + x_2 - 2\lambda_+ x_1) - (y_1 + y_2 - 2\lambda_+ y_0) = 0,$$
$$(x_1 - x_2) - (y_1 - y_2) = 0.$$

The intersection $\Lambda \cap X$ is the union of a conic $X \cap \Pi_1$ contains the line

$$x_1 + x_2 - 2\lambda_+ x_1 = y_1 + y_2 - 2\lambda_+ y_0 = \sqrt{3}x_0 + (-1+\sqrt{3})(y_1 - y_2) = \sqrt{3}y_0 - (1+\sqrt{3})(x_1 - x_2) = 0.$$

(see [534, (2.7)]). This shows that $\Lambda \cap X$ is the union of a conic and two lines forming a reducible fiber of type $\tilde{A}_2$. There are four line-components of the triangles of lines passing through a base point of the Hesse pencil. This shows that the elliptic fibrations $|F_i|, |F_{-i}|$ corresponding a quadric Q_i has 8 fibers of type $\tilde{A}_2$ forming four orbits with respect to the action of τ.

(iii) Notice that a fiber of $|F_{\pm i}|$ and a fiber of $|F_{\pm j}|$ are contains in two different 3-planes intersecting along a line in the plane containing the base locus of the pencil generated by the quadrics Q_i and Q_j. This implies that $F_{\pm i} \cdot F_{\pm j} = 2$, hence $G_{\pm i} \cdot G_{\pm j} = 1$. Also, the union of two 3-planes from different rulings of Q_i by 3-planes is contained in a hyperplane. This implies that

$$8 = \tilde{H}_8^2 = (F_i + F_{-i}) \cdot (F_j + F_{-j}),$$

and hence $F_i \cdot F_{-i} = 4$. It follows that $G_i \cdot G_{-i} = 2$ with agreement with $H_4 \sim G_i + G_{-i}$.

(iv) We know that the primary elliptic pencil $|F_0|$ consists of curves of degree 6. This implies that $F_0 \cdot (F_i + F_{-i}) = 6$, hence $G_0 \cdot (G_i + G_{-i}) = 3$. □

Example 8.8.6 We take S with the canonical cover isomorphic to the octic surface X given by equations (8.8.6) discussed above. It has 10 elliptic fibrations $|2G_i|, i = 0, \dots, 9$, satisfying the assertions of Lemma 8.1. The jacobian fibration of each fibration is an extremal rational elliptic surface of Hesse type, and its Mordell–Weil group MW_i is isomorphic to the group $3^2 := (\mathbb{Z}/3\mathbb{Z})^{\oplus 2}$.

Let G be the subgroup of $\mathrm{Aut}(S)$ generated by these ten subgroups MW_i. Since all $|2G_i|$ are extremal elliptic fibration on S, $\mathrm{Num}(S)_{\mathbb{Q}}$ is generated by irreducible components of all fibers. A divisor class D is invariant with respect to MW_i if it intersects each irreducible fiber of $|2G_i|$ with the same multiplicity. This implies that D is invariant with respect to G if it intersects all irreducible components with the same multiplicity.

Since $G_0 \cdot (G_i + G_{-i}) = 3$, we can choose $G_1, \dots, G_9$ such that $G_0 \cdot G_i = 1$, hence $(G_0, G_1, \dots, G_{10})$ form an isotropic 10-sequence. Let

$$3H_{10} \sim G_0 + \cdots + G_{10}.$$

Each irreducible component is contained in fibers of four different fibrations $|2G_i|$. It corresponds to the fact that each base point of the Hesse configuration is contained in four sides of the Hesse triangles. Also, each side of a Hesse triangle contains three base points because each subset of three fibrations has a common irreducible component. This easily implies that H_{10} intersects each irreducible component of fibers $|2G_i|$ with the same multiplicity equal to 3. Thus, H_{10} is an invariant divisor class, hence it defines a Fano polarization invariant with respect to the group G. In particualr, G is finite group.

We also see that G acts by permuting the elliptic pencils $|2G_i|$. This realizes G as a subgrouop of $\mathfrak{S}_{10}$. It follows from above that G leaves invariant the Steiner system $\mathsf{S}(3, 4, 10)$. It consists of 30 blocks of cardinality 4 and each point is contained in 12 blocks. There are four of them because each base point of the Hesse configuration is contained in four lines of the configuration. Any subset of three fibrations is contained in a unique block because each line of the Hesse configuration has three base points.

Let us consider the action of G on the set of vectors g_i defining a homomorphism $\alpha : G \to M_{10}$. Its kernel acts identically on the set $\{h_{10}, g_0, \dots, g_9\}$ and, since this set generates $\mathrm{Num}(S)_{\mathbb{Q}}$, it consists of cohomologically trivial automorphisms

g. Such an automorphism leaves invariant components of different fibrations $|2G_i|$, for each component we have a component of another fibration that intersects it with multiplicity one at a nonsingular point of the fibers. This shows that we have at least four fixed points one each component, hence all of them are contained in S^g, an obvious contradiction. Thus, α is injective. The image cannot be the whole M_{10} because we know from Theorem 8.8.3 that M_{10} cannot be realized as a group of automorphisms of S of Mathieu type. Thus, the image of α is contained in $\mathfrak{A}_6$. Now, we use that the image contains 10 subgroups $\alpha(\mathrm{MW}_i)$, each isomorphic to 3^2. The known list of proper maximal subgroups of $\mathfrak{U}_6$ shows that none of them contains a pair of commuting elements of order 3. This shows that the image of α is the whole group $\mathfrak{U}_6$. It follows from Proposition 8.8.2 that the action of $\mathfrak{A}_6$ is Mathieu.

Remark 8.8.7 Recall that X contains a line. It follows from Remark 3.4.12 that this implies that the K3 surface X is isomorphic to a minimal resolution X' of the double cover of the plane $|N|$ branched along the union of the cubics $\Delta_\pm$. In Example 4.10.13 we explained how the data $(\Delta_\pm, \theta_\pm)$ defines a quadratic twist $f : S' \to \mathbb{P}^1$ of the Hesse elliptic fibration $j : J \to \mathbb{P}^1$ with two half-fibers over $\lambda_\pm \in \mathbb{P}^1$. This, of course, should correspond to the primary elliptic fibration $|2G_0|$ on S. We do not know whether the surfaces S and S' are isomorphic.

Remark 8.8.8 We know that each fibration $|F_{\pm 1}|, \ldots, |F_{\pm 9}|$ on X contains 8 reducible fibers, and each such a fiber contains two lines. Each line is contained in a fiber of four fibrations. This shows that X contains 36 lines. According to [149], this is the maximum possible number of lines on an octic K3 surface over $\mathbb{C}$.

Remark 8.8.9 It is known that $\mathfrak{A}_6$ is isomorphic to the *Valentiner group* of projective transformations of the plane that leaves a certain nonsingular curve of degree 6 invariant. Starting with a Hesse pencil, W. Burnside gives a geometric construction of nine more pencils projectively isomorphic to the Hesse pencil such that the corresponding subgroups 3^2 of their Hessian groups generate the Valentiner group [102]. This is an amazing analog of what we did in Example 8.8.6 to generate the group of isomorphisms of an Enriques surface isomorphic to $\mathfrak{A}_6$.

8.9 Enriques Surfaces with Finite Automorphism Group ($p \neq 2$)

In this section, we classify Enriques surfaces S with finite automorphism group $\mathrm{Aut}(S)$. Over the field of complex numbers such classification was done by the second author [409] and, via periods of their K3-covers, by V. Nikulin [559]. There are seven possible types of such surfaces. We will show, that, in positive odd characteristic p, some of these surfaces do not exist if $p = 3$ or $p = 5$.

Recall from Corollary 8.1.11 that, if $p \neq 2$ or S is a $\boldsymbol{\mu}_2$-surface, such classification is equivalent to the classification of crystallographic root bases of the classes of (-2)-curves in $\mathrm{Num}(S)$. We will see in this section that there are 7 different types I-VII of such bases in characteristic $p \neq 2$. In the next section, we shall show that in

characteristic $p = 2$ the types III, IV and V are not realized and there are additional types which go under the names

$$\tilde{E}_8,\ \tilde{E}_7^{(1)},\ \tilde{E}_7^{(2)},\ \tilde{E}_6 + \tilde{A}_2,\ \tilde{D}_8,\ \tilde{D}_4 + \tilde{D}_4,\ \text{VIII}. \tag{8.9.1}$$

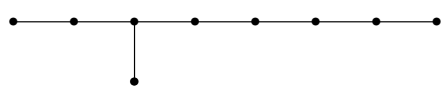
The key to the classification is the following obvious observation.

Proposition 8.9.1 *Let S be an Enriques surface with finite automorphism group. Then, the jacobian fibration of any genus one fibration on S has finite Mordell–Weil group.*

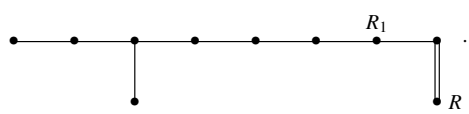
Since we know by Corollary 6.3.5 that a nodal Enriques surface has always a genus one fibration with a special bisection, we may start assuming that S has such a fibration with a special bisection R. We use the classification of extremal genus one fibrations from Section 4.8. Instead of considering many possible intersections of R with a reducible fiber, we use the quadratic twist construction from Lemma 4.10.10 to create such a bisection from a section of the Mordell–Weil group. For each case, we can find a new special elliptic fibration on S. It must be extremal, and we find new (-2)-curves among the irreducible components of its reducible fibers. If the dual graph of (-2)-curves satisfies Vinberg's criterion (Theorem 0.8.23 from Volume I), then we obtain a crystallographic basis. Otherwise, we continue this process. Finally, we get a crystallographic basis of one of seven possible types I, . . . , VII.

We illustrate this procedure in the following example.

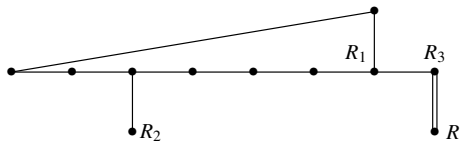
Example 8.9.2 We start with the following diagram of type $\tilde{E}_8$:



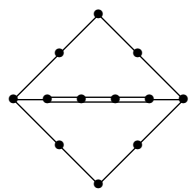
We assume that the special bisection R intersects the unique reduced component of the fiber with multiplicity 2 (a priori it may intersect an irreducible component that enters with multiplicity 2). Since $p \neq 2$, the fiber must be a simple fiber, hence R intersects it with multiplicity 2. Thus, we obtain the following diagram:



Next, consider the diagram of type $\tilde{A}_1$ which is a double fiber of a genus one fibration because it meets a bisection R_1 with multiplicity 1. This fibration is of type $\tilde{E}_7 \oplus \tilde{A}_1$ because a subdiagram of type E_7 is disjoint from the diagram of type $\tilde{A}_1$. Since $p \neq 2$, we have the following diagram:



Next, we see another parabolic diagram of type $\tilde{A}_7$ which is a double fiber of a genus one fibration because it meets a bisection R_2 with multiplicity 1. The original bisection R does not intersect any component of the curve of type $\tilde{A}_7$, and hence it must be a component of a parabolic subdiagram of type $\tilde{A}_7 \oplus \tilde{A}_1$. Since the bisection R_3 of this fibration meets R with multiplicity 2, the fiber of type $\tilde{A}_1$ is not double. Therefore, the new (-2)-curve should meet R_2 with multiplicity 2. Thus we obtain the diagram of a crystallographic basis of (-2)-curves on Kondo's surface of type I:



In [488], Martin used the same idea to extend the classification for the cases of characteristic $p \neq 2$ or $\boldsymbol{\mu}_2$-Enriques surfaces in $p = 2$. Moreover, he showed that a crystallographic diagram always contains a smaller subdiagram (which he calls a *critical* diagram). Any Enriques surface containing a critical diagram of (-2) curves has finite automorphism group.

In Nikulin's approach, one translates Proposition 8.9.1 to the classification of possible R-invariants of S.

Theorem 8.9.3 *Assume that* $\Bbbk = \mathbb{C}$. *Then,* $\mathrm{Aut}(S)$ *is finite if and only if Nikulin's R-invariant* (K, H) *is one of the following:*

$$(\mathsf{E}_8 \oplus \mathsf{A}_1, \{0\}),\ (\mathsf{D}_9, \{0\}),\ (\mathsf{D}_8 \oplus \mathsf{A}_1 \oplus \mathsf{A}_1, (\mathbb{Z}/2\mathbb{Z})^{\oplus 2}),$$

$$(\mathsf{D}_5 \oplus \mathsf{D}_5, \mathbb{Z}/2\mathbb{Z}),\ (\mathsf{E}_7 \oplus \mathsf{A}_2 \oplus \mathsf{A}_1, \mathbb{Z}/2\mathbb{Z}),\ (\mathsf{E}_6 \oplus \mathsf{A}_4, \{0\}),\ (\mathsf{A}_9 \oplus \mathsf{A}_1, \mathbb{Z}/2\mathbb{Z}).$$

Proof We give a proof for the necessity. The sufficiency follows from the examples below. Let S be an Enriques surface with the Nikulin R-invariant (K, H). Recall that K is a root lattice of rank $r = \mathrm{rank}(K)$ and H is an isotropic subgroup $H \subset K/2K$ with respect to the finite quadratic form $q_K : K/2K \to \mathbb{F}_2$. All such lattices are isomorphic to the direct sum $K_1 \oplus \cdots \oplus K_s$, where K_i are root lattices of rank r_i with $\sum r_i \leq 10$. So, by applying Lemma 6.4.6, we can list all possible root lattices K and their corresponding subgroups H and check whether the R-invariant (K, H) satisfies the following conditions:

(I) $\mathrm{rank}\, K \leq 10$. Moreover, if $\mathrm{rank}\, K = 10$, the number of minimal generators of the p-Sylow subgroup ($p \neq 2$) of $D(K)$ is at most 2.

(II) The overlattice $K_H = \{x \in K \otimes \mathbb{Q} \ : \ 2x \in H\}$ of K does not contain (-1)-vectors because otherwise S is not an Enriques surface but a Coble surface.

(III) $K/2K \mod H$ is isomorphic to $\langle \mathsf{Nod}(S) \rangle$ as quadratic forms (Remark 6.4.3). In particular $K/2K \mod H$ can be embedded in the quadratic space $\bar{E}_{10}$ which is even, regular and non-defective.

(IV) (Proposition 8.9.1) For any non-zero isotropic vector $f \in \overline{\mathrm{Num}}(S)$, the linear quadratic space generated by $\{x \in \mathsf{Nod}(S) : b_{q_K}(x, f) = 0\}$ has the maximal dimension equal to 8.

For the condition (I), recall that $K(2)$ is contained in the orthogonal complement of the transcendental lattice T_X of the canonical cover X of S in $H^2(X, \mathbb{Z})^- \cong \mathsf{E}_{10}(2) \oplus \mathsf{U}$. If rank $K = 10$, $K_H(2)$ and T_X are the orthogonal complements to each other, and hence, by comparing their discriminant quadratic forms, we have the assertion.

First of all, we show that for each root lattice K appearing in Theorem 8.9.3 the group H is uniquely determined by the above conditions. In the cases of $\mathsf{E}_8 \oplus \mathsf{A}_1$, $\mathsf{E}_6 \oplus \mathsf{A}_4$, the quadratic form q_K is regular and hence $H = \{0\}$. In the case of D_9, q_K has a one-dimensional kernel $\ker(q)$ which is represented by a (-4)-vector in K. Thus, we have $H = \{0\}$ by the condition (II). In the cases of $\mathsf{E}_7 \oplus \mathsf{A}_2 \oplus \mathsf{A}_1$, $\mathsf{A}_9 \oplus \mathsf{A}_1$, q_K is defective and has a one-dimensional kernel. The condition (III) implies $H = \mathbb{Z}/2\mathbb{Z}$. In the case of $\mathsf{D}_8 \oplus \mathsf{A}_1 \oplus \mathsf{A}_1$, q_K is defective and has a three-dimensional kernel, and hence $H = (\mathbb{Z}/2\mathbb{Z})^{\oplus 2}$ by the conditions (II), (III). In the case of $\mathsf{D}_5 \oplus \mathsf{D}_5$, q_K is non-defective and has a two-dimensional kernel, and hence $H = \mathbb{Z}/2\mathbb{Z}$ by the conditions (II), (III).

Now, we check the conditions (I), (II), (III) or (IV) for any pair (K, H). In the case that rank $K \leq 8$, the condition (IV) is not satisfied because the quadratic space q_K has at most dimension 8. Thus, it suffices to consider the cases rank $K = 9, 10$.

There are 88 possible lattices K of rank 10.

(i) The following seven K have regular even quadratic forms q_K of rank 10 such that $K(2)$ can not be embedded in $\mathsf{E}_{10}(2) \oplus \mathsf{U}$ (i.e. K does not satisfy the condition (I) for $p = 3$):

$$\mathsf{E}_6 \oplus \mathsf{A}_2^{\oplus 2},\ \mathsf{A}_5 \oplus \mathsf{A}_2^{\oplus 2} \oplus \mathsf{A}_1,\ \mathsf{A}_4 \oplus \mathsf{A}_2^{\oplus 3},\ \mathsf{A}_3 \oplus \mathsf{A}_2^{\oplus 3} \oplus \mathsf{A}_1,\ \mathsf{A}_2^{\oplus 5},\ \mathsf{A}_2^{\oplus 4} \oplus \mathsf{A}_1^{\oplus 2},\ \mathsf{A}_2^{\oplus 3} \oplus \mathsf{A}_1^{\oplus 4}.$$

(ii) In the following five cases, q_K is a regular quadratic form of odd type and of rank 10. In particular, it does not satisfy the condition (III):

$$\mathsf{A}_{10},\ \mathsf{E}_8 \oplus \mathsf{A}_2,\ \mathsf{A}_8 \oplus \mathsf{A}_2,\ \mathsf{A}_6 \oplus \mathsf{A}_4,\ \mathsf{A}_4^{\oplus 2} \oplus \mathsf{A}_2.$$

(iii) In the following nine cases, q_K is not regular but of odd type and of dimension 10. In particular, it does not satisfy the condition (III):

$$\mathsf{D}_8 \oplus \mathsf{A}_2,\ \mathsf{D}_7 \oplus \mathsf{A}_3,\ \mathsf{A}_7 \oplus \mathsf{A}_3,\ \mathsf{E}_6 \oplus \mathsf{D}_4,\ \mathsf{A}_6 \oplus \mathsf{D}_4,\ \mathsf{D}_5 \oplus \mathsf{A}_3 \oplus \mathsf{A}_2,\ \mathsf{D}_4^{\oplus 2} \oplus \mathsf{A}_2,\ \mathsf{D}_4 \oplus \mathsf{A}_4 \oplus \mathsf{A}_2,\ \mathsf{D}_4 \oplus \mathsf{A}_3^{\oplus 2}.$$

(iv) In the following 50 cases, H is non-trivial by the condition (III) but K_H contains a (-1)-vector, i.e. it does not satisfy the condition (II):

$$\mathsf{D}_{10},\ \mathsf{D}_9 \oplus \mathsf{A}_1,\ \mathsf{E}_8 \oplus \mathsf{A}_1^{\oplus 2},\ \mathsf{A}_8 \oplus \mathsf{A}_1^{\oplus 2},\ \mathsf{E}_7 \oplus \mathsf{A}_3,\ \mathsf{E}_7 \oplus \mathsf{A}_1^{\oplus 3},\ \mathsf{D}_7 \oplus \mathsf{A}_2 \oplus \mathsf{A}_1,\ \mathsf{D}_7 \oplus \mathsf{A}_1^{\oplus 3},\ \mathsf{A}_7 \oplus \mathsf{A}_1^{\oplus 3},$$

$\mathsf{E}_6\oplus\mathsf{A}_3\oplus\mathsf{A}_1$, $\mathsf{E}_6\oplus\mathsf{A}_2\oplus\mathsf{A}_1^{\oplus 2}$, $\mathsf{E}_6\oplus\mathsf{A}_1^{\oplus 4}$, $\mathsf{D}_6\oplus\mathsf{D}_4$, $\mathsf{D}_6\oplus\mathsf{A}_4$, $\mathsf{D}_6\oplus\mathsf{A}_2^{\oplus 2}$, $\mathsf{D}_6\oplus\mathsf{A}_1^{\oplus 4}$, $\mathsf{A}_6\oplus\mathsf{A}_3\oplus\mathsf{A}_1$, $\mathsf{A}_6\oplus\mathsf{A}_2\oplus\mathsf{A}_1^{\oplus 2}$, $\mathsf{A}_6\oplus\mathsf{A}_1^{\oplus 4}$, $\mathsf{D}_5\oplus\mathsf{A}_5$, $\mathsf{D}_5\oplus\mathsf{D}_4\oplus\mathsf{A}_1$, $\mathsf{D}_5\oplus\mathsf{A}_4\oplus\mathsf{A}_1$, $\mathsf{D}_5\oplus\mathsf{A}_2^{\oplus 2}\oplus\mathsf{A}_1$, $\mathsf{D}_5\oplus\mathsf{A}_2\oplus\mathsf{A}_1^{\oplus 3}$, $\mathsf{D}_5\oplus\mathsf{A}_1^{\oplus 5}$, $\mathsf{A}_5\oplus\mathsf{D}_4\oplus\mathsf{A}_1$, $\mathsf{A}_5\oplus\mathsf{A}_3\oplus\mathsf{A}_2$, $\mathsf{A}_5\oplus\mathsf{A}_2\oplus\mathsf{A}_1^{\oplus 3}$, $\mathsf{A}_5\oplus\mathsf{A}_1^{\oplus 5}$, $\mathsf{D}_4\oplus\mathsf{A}_4\oplus\mathsf{A}_1^{\oplus 2}$, $\mathsf{D}_4\oplus\mathsf{A}_3\oplus\mathsf{A}_2\oplus\mathsf{A}_1$, $\mathsf{D}_4\oplus\mathsf{A}_2^{\oplus 3}$, $\mathsf{D}_4\oplus\mathsf{A}_2^{\oplus 2}\oplus\mathsf{A}_1^{\oplus 2}$, $\mathsf{D}_4\oplus\mathsf{A}_2\oplus\mathsf{A}_1^{\oplus 4}$, $\mathsf{D}_4\oplus\mathsf{A}_1^{\oplus 6}$, $\mathsf{A}_4^{\oplus 2}\oplus\mathsf{A}_1^{\oplus 2}$, $\mathsf{A}_4\oplus\mathsf{A}_3^{\oplus 2}$, $\mathsf{A}_4\oplus\mathsf{A}_3\oplus\mathsf{A}_2\oplus\mathsf{A}_1$, $\mathsf{A}_4\oplus\mathsf{A}_3\oplus\mathsf{A}_1^{\oplus 3}$, $\mathsf{A}_4\oplus\mathsf{A}_2^{\oplus 2}\oplus\mathsf{A}_1^{\oplus 2}$, $\mathsf{A}_4\oplus\mathsf{A}_2\oplus\mathsf{A}_1^{\oplus 4}$, $\mathsf{A}_4\oplus\mathsf{A}_1^{\oplus 6}$, $\mathsf{A}_3^{\oplus 3}\oplus\mathsf{A}_1$, $\mathsf{A}_3^{\oplus 2}\oplus\mathsf{A}_1^{\oplus 4}$, $\mathsf{A}_3\oplus\mathsf{A}_2^{\oplus 2}\oplus\mathsf{A}_1^{\oplus 3}$,

$$\mathsf{A}_3 \oplus \mathsf{A}_2 \oplus \mathsf{A}_1^{\oplus 5},\ \mathsf{A}_3 \oplus \mathsf{A}_1^{\oplus 7},\ \mathsf{A}_2^{\oplus 2} \oplus \mathsf{A}_1^{\oplus 6},\ \mathsf{A}_2 \oplus \mathsf{A}_1^{\oplus 8},\ \mathsf{A}_1^{\oplus 10}.$$

(v) The following is the list of the remaining twelve cases of K except the five types

$$\mathsf{D}_8 \oplus \mathsf{A}_1 \oplus \mathsf{A}_1,\ \mathsf{D}_5 \oplus \mathsf{D}_5,\ \mathsf{E}_7 \oplus \mathsf{A}_2 \oplus \mathsf{A}_1,\ \mathsf{E}_6 \oplus \mathsf{A}_4,\ \mathsf{A}_9 \oplus \mathsf{A}_1$$

in Theorem 8.9.3:

$\mathsf{A}_7\oplus\mathsf{A}_2\oplus\mathsf{A}_1$, $\mathsf{D}_6\oplus\mathsf{A}_3\oplus\mathsf{A}_1$, $\mathsf{D}_6\oplus\mathsf{A}_2\oplus\mathsf{A}_1^{\oplus 2}$, $\mathsf{A}_6\oplus\mathsf{A}_2^{\oplus 2}$, $\mathsf{D}_5\oplus\mathsf{A}_3\oplus\mathsf{A}_1^{\oplus 2}$, $\mathsf{A}_5\oplus\mathsf{A}_5$, $\mathsf{A}_5\oplus\mathsf{A}_4\oplus\mathsf{A}_1$,

$$\mathsf{A}_5 \oplus \mathsf{A}_3 \oplus \mathsf{A}_1^{\oplus 2},\ \mathsf{D}_4^{\oplus 2} \oplus \mathsf{A}_1^{\oplus 2},\ \mathsf{D}_4 \oplus \mathsf{A}_3 \oplus \mathsf{A}_1^{\oplus 3},\ \mathsf{A}_3^{\oplus 2} \oplus \mathsf{A}_2^{\oplus 2},\ \mathsf{A}_3^{\oplus 2} \oplus \mathsf{A}_2 \oplus \mathsf{A}_1^{\oplus 2}.$$

These cases do not satisfy condition (IV). To show this, we use the following lemma (for the proof, e.g. [560, Corollary 4.4]).

Lemma 8.9.4 *Let R be an irreducible root lattice and let A_1 be a sublattice of R. Then, the root sublattice R_0 of the orthogonal complement of A_1 in R is given as follows:*

(i) $R = \mathsf{A}_n$: $R_0 = \mathsf{A}_{n-2}$ $(n \geq 3)$, $R_0 = \emptyset$ $(n = 1, 2)$.
(ii) $R = \mathsf{D}_n$: $R_0 = \mathsf{A}_1 \oplus \mathsf{D}_{n-2}$ $(n \geq 5)$, $R_0 = \mathsf{A}_1^{\oplus 3}$ $(n = 4)$.
(iii) $R = \mathsf{E}_n$: $R_0 = \mathsf{A}_5$ $(n = 6)$, $R_0 = \mathsf{D}_6$ $(n = 7)$, $R_0 = \mathsf{E}_7$ $(n = 8)$. □

For example, in the case $K = \mathsf{A}_5 \oplus \mathsf{A}_5$, q_K has rank 9 and dimension 10. By the property (III), $H = \mathbb{Z}/2\mathbb{Z}$ and $q_K \mod H$ is defective and of dimension 9. Consider the sum of a root in each factor A_5 of K which gives a non-zero isotropic vector in the quadratic form q_K. The root sublattice of its orthogonal complement in K is isomorphic to $\mathsf{A}_3^{\oplus 2} \oplus \mathsf{A}_1^{\oplus 2}$ (Lemma 8.9.4) whose image in $K/2K \mod H$ has dimension 7. Thus, this case does not satisfy the condition (IV).

We remark that many cases as above do not satisfy not only one but also two or more conditions among the conditions (I)–(IV).

The proof for the case rank $K = 9$ is similar and is easier. Almost all cases do not satisfy the condition (IV). We leave the details to the reader. □

It follows that there are seven types of surfaces with finite automorphism group over $\mathbb{C}$. It is a surprising result of Martin's classification that there are no new types of crystallographic basis in any characteristic $p \neq 2$, although some of surfaces with finite automorphism group in characteristic 0 do not live in an arbitrary characteristic.

To summarize, we exhibit the final result of classification in arbitrary characteristic $p \neq 2$ in Table 8.11. Here D_8 is the dihedral group of order 8.

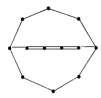
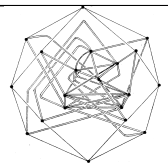
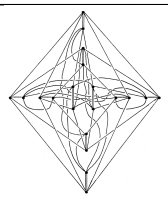
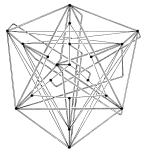
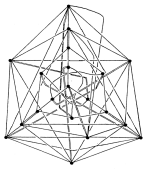
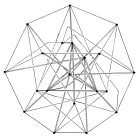
Type	Dual Graph of (-2)-curves	Aut	Aut_{nt}	$\mathrm{Char}(k)$	Moduli
I		D_8	$\mathbb{Z}/2\mathbb{Z}$	any	$\mathbb{A}^1 - \{0, -256\}$
II		$\mathfrak{S}_4$	$\{1\}$	any	$\mathbb{A}^1 - \{0, -64\}$
III		$(\mathbb{Z}/4\mathbb{Z} \times (\mathbb{Z}/2\mathbb{Z})^2) \rtimes D_8$	$\mathbb{Z}/2\mathbb{Z}$	$\neq 2$	unique
IV		$(\mathbb{Z}/2\mathbb{Z})^4 \rtimes (\mathbb{Z}/5\mathbb{Z} \rtimes \mathbb{Z}/4\mathbb{Z})$	$\{1\}$	$\neq 2$	unique
V		$\mathfrak{S}_4 \times \mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}$	$\neq 2, 3$	unique
VI		$\mathfrak{S}_5$	$\{1\}$	$\neq 3, 5$	unique
VII		$\mathfrak{S}_5$	$\{1\}$	$\neq 2, 5$	unique

Table 8.11 Enriques surfaces with finite automorphism group ($p \neq 2$)

We refer to Table 8.12 in the next section that shows which of these surfaces exist in characteristic $p = 2$.

In the following, we will give some detailed description of the surfaces of Types I–VII.

Type I

These surfaces realize the crystallographic basis whose Coxeter graph has 12 vertices and is given in Figure 8.1 below.

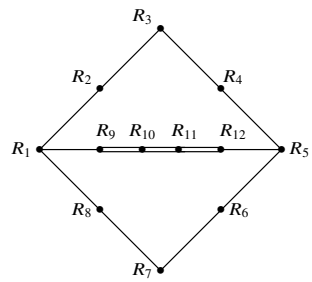


Fig. 8.1 Crystallographic basis of type I

We know that these surfaces belong to the two-dimensional family (A) of surfaces which admit non-trivial numerically trivial automorphisms which we classified in Theorem 8.2.23. We constructed them as non-degenerate Enriques double planes with the branch curve of degree 8 whose part of degree 6 is equal to the union of a line, a conic, and a cubic passing through the intersection points q_1, q_2 of the line and the conic. The curves R_{10} and R_{11} are the proper transforms of the lines $\ell_1 = \langle p_2, q_1 \rangle$ and $\ell_2 = \langle p_4, q_2 \rangle$, where p_2, p_4 are the tacnode singularities of the Enriques octic branch curve. The special property of our family is that one of the lines, say ℓ_1 is tangent to the cubic at the point q_1. The other line will have the same property automatically.

We know from the proof of Theorem 8.2.23 from which we borrow the notation that the equation of the 2-dimensional family of double planes is

$$x_3^2 + x_1x_2(x_1 - x_2)(x_0^2 - x_1x_2)(ax_1^2x_2 + bx_0^2x_1 + cx_0^2x_2 + dx_2^2x_1) = 0, \qquad (8.9.2)$$

where $a + b + c + d = 0$. The condition that the line $\ell_1 = V(x_0 - x_1)$ is tangent to the cubic at q_1 is $(a + c)^2 - 4bd = 0$. The same condition guarantees that the line $\ell_2 = V(x_0 + x_1)$ is tangent to C at q_2. Thus, we have a one-parameter family of branch curves B satisfying our assumptions.

The canonical cover X of the surface S is birationally isomorphic to the double cover of $\mathbf{F}_0 \cong \mathbb{P}^1 \times \mathbb{P}^1$ branched along the pre-image $\tilde{B}$ of the curve B. The curve $\tilde{B}$ is equal to the union of a quadrangle T of lines, the pre-images of the conics, and a curve C of bi-degree $(2, 2)$, the pre-image of a hyperplane section. It passes through the vertices of T. The pre-images of the lines ℓ_1, ℓ_2 are two conics Q_1, Q_2 on $\mathbb{P}^1 \times \mathbb{P}^1$ passing through the opposite vertices of T and tangent to C there. They intersect at two points fixed under the involution σ such that $\mathbf{F}_0/(\sigma) = \mathsf{D}_1$.

Proposition 8.9.5 *A crystallographic basis of* (-2)*-curves of type* I *can be realized in any characteristic* $p \neq 2$*. Any Enriques surface realizing this basis is birationally*

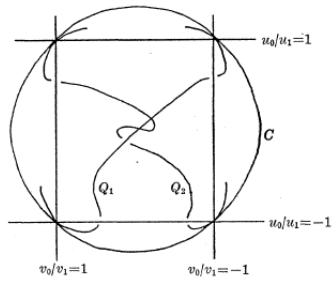


Fig. 8.2 The branch curve for the canonical cover of type I surfaces

isomorphic to the double plane given by equation (8.9.2)*, where* $a+b+c+d=0$, $(a+b)^2=4cd$. *Moreover,*

$$\mathrm{Aut}(S) \cong D_8.$$

Proof The only new assertion here is about the structure of the automorphism group. We already know that the surface admits a cohomologically trivial involution g_0. It can be realized by the deck transformation of the bielliptic cover $S \to \mathsf{D}_1$ defined by the degenerate U-pair (f, R), where f is the class of a half-fiber in the elliptic fibration with a fiber of type $\tilde{E}_8$ and R is its special bisection which we easily locate in the diagram from Figure 8.1.

Consider the action of $\mathrm{Aut}(S)$ on the diagram. We will show that the image of $\mathrm{Aut}(S)$ is the group of symmetries of the diagram isomorphic to $(\mathbb{Z}/2\mathbb{Z})^2$. Together with g_0, this will generate a group isomorphic to D_8.

The symmetry of the graph which switches R_3 with R_7 is realized by the Mordell–Weil group of the jacobian fibration of one of the two elliptic fibrations with a fiber of type $\tilde{E}_7$. The other symmetry which switches R_1 with R_5 is realized by the Mordell–Weil group of the jacobian fibration of one of the two elliptic fibrations with a fiber of type $\tilde{D}_8$. □

A straightforward computation proves the following.

Proposition 8.9.6 *A surface of type* I *has genus one fibrations with reducible fibers of the following types:*

$$\tilde{E}_8\ (4), \quad \tilde{D}_8\ (2), \quad \tilde{E}_7 + 2\tilde{A}_1\ (2), \quad 2\tilde{A}_7 + \tilde{A}_1\ (1),$$

where the number in the brackets is the number of fibrations of given type. Let $|2F_i|, i = 1,2,3,4$ *(resp.* $|2F_5|, |2F_6|$*, resp.* $|2F_7|, |2F_8|$*, resp.* $|2F_9|$*) be the genus one pencils of the first (resp. the second, resp. the third, resp. the forth) type. The intersection matrix of the corresponding primitive isotropic vectors* f_i *is equal to*

$$\begin{pmatrix} 0 & 3 & 4 & 5 & 1 & 2 & 1 & 3 & 2 \\ 3 & 0 & 5 & 4 & 2 & 1 & 1 & 3 & 2 \\ 4 & 5 & 0 & 3 & 1 & 2 & 3 & 1 & 2 \\ 5 & 4 & 3 & 0 & 2 & 1 & 3 & 1 & 2 \\ 1 & 2 & 1 & 2 & 0 & 1 & 1 & 1 & 1 \\ 2 & 1 & 2 & 1 & 1 & 0 & 1 & 1 & 1 \\ 1 & 1 & 3 & 3 & 1 & 1 & 0 & 2 & 1 \\ 3 & 3 & 1 & 1 & 1 & 1 & 2 & 0 & 1 \\ 2 & 2 & 2 & 2 & 1 & 1 & 1 & 1 & 0 \end{pmatrix}$$

In particular, $\mathsf{nd}(S) = 4$.

Proposition 8.9.7 *The Fano root invariant of a surface with crystallographic basis of* (-2)*-curves of type* I *is equal to* $(\mathsf{E}_8 \oplus \mathsf{U}(4), \{0\})$. *The Nikulin* R*-invariant is equal to* $(\mathsf{E}_8 \oplus \mathsf{A}_1, \{0\})$. *The quadratic space* $\langle \mathsf{Nod}(S) \rangle$ *is defective of rank 8 and dimension 9. The Reye lattice is isomorphic to* $\mathsf{E}_8(4) \oplus \mathsf{A}_1 \oplus \mathsf{A}_1(-1)$.

Proof Let f_1 be the class of an elliptic fibration with half-fiber of type $\tilde{\mathsf{A}}_7$, f_2 the class of a half-fiber of the elliptic fibration with fiber of type $\tilde{\mathsf{D}}_8$ which contains R_9, R_8, R_6, R_{12} as its components with multiplicity 1, and f_3 the class of a half-fiber of the elliptic fibration with fiber of type $\tilde{\mathsf{E}}_7$ which contains R_4, R_6 as its components with multiplicity 1. Let us consider the Fano polarization h defined by the following canonical isotropic sequence:

$$(f_1, f_1 + R_9, f_1 + R_9 + R_1, \ldots, f_1 + R_9 + R_1 + \ldots + R_4, f_2, f_2 + R_7, f_2 + R_7 + R_6, f_3).$$

We find that $h_{10} \cdot R_i = 0, i \neq 5, 8, 10, 11, 12$, $h_{10} \cdot R_5 = h_{10} \cdot R_{11} = 1, h_{10} \cdot R_8 = h_{10} \cdot R_{12} = 2, h_{10} \cdot R_{10} = 5$. Thus, we see that $\mathsf{N}_S^{h_{10}}$ is spanned by all R_i's except R_{10}. Computing the integral Smith normal form of the intersection matrix, we find that the discriminant group is isomorphic to $(\mathbb{Z}/4\mathbb{Z})^{\oplus 2}$. It is generated by $\frac{1}{4}([R_6 + R_8 + R_9 + R_{12}])$ and $\frac{1}{4}([R_2 + R_4 + R_9 + R_{12}])$. The discriminant quadratic form is u_2. This implies that $\mathsf{N}_S^{h_{10}} \cong \mathsf{E}_8 \oplus \mathsf{U}(4)$.

If we add $\delta_{R_{10}}$, we obtain that K contains a sublattice M isomorphic to $\mathsf{E}_8 \oplus \mathsf{A}_1$. Since $R_{11} + R_{10}$ is a simple fiber, $\delta_{R_{11}} \in M$. Also, we know that $\delta_{R_{12}}, \delta_{R_1}, \ldots, \delta_{R_8}$ are linearly dependent, so M is of finite index in K and does not contain vectors of norm -1. However, the discriminant group $D(M)$ does not contain isotropic vectors. So, $M = K$ and the Nikulin R-invariant is $(\mathsf{E}_8 \oplus \mathsf{A}_1, \{0\})$.

Applying Lemma 6.4.6, we obtain the assertion about the quadratic space $\langle \mathsf{Nod}(S) \rangle$. The Reye lattice is the pre-image of a one-dimensional defective quadratic space in $\overline{\mathsf{Num}}(S)$. This easily implies that $\mathrm{Rey}(S) \cong \mathsf{E}_8(4) \oplus \mathsf{A}_1 \oplus \mathsf{A}_1(-1)$. □

Type II

Here the crystallographic basis defines the following diagram from Figure 8.3 of twelve (-2)-curves on S:

Let us construct such surfaces, they will depend on one parameter. Choose a non-degenerate U-pair formed by two half-fibers $F_1 = R_1 + R_2 + R_9 + R_{10}$ and

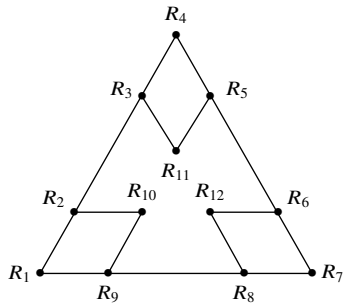
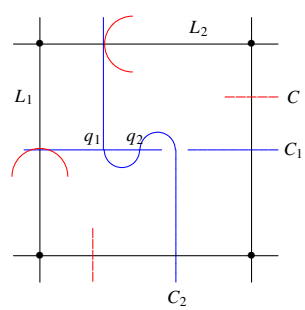


Fig. 8.3 Crystallographic basis of Type II

$F_2 = R_3+R_4+R_5+R_{11}$ of type $\tilde{A}_3$. Each elliptic fibration $|2F_i|$ has an additional simple fiber $R_4+R_{11}+2R_5+2R_6+R_7+R_{12}$, $R_1+R_{10}+2R_9+2R_8+R_{12}+R_7$, respectively, of type $\tilde{D}_5$. Let $f : S \to \mathsf{D}_1$ be the bielliptic map defined by the pair (F_1, F_2). The two elliptic fibrations share eight irreducible components $R_1, R_4, R_5, R_7, R_9, R_{10}, R_{11}, R_{12}$ that span the lattice isomorphic to $\mathsf{A}_3^{\oplus 2} \oplus \mathsf{A}_1^{\oplus 2}$. Comparing with Table 8.7, we find the corresponding involution in row 26. Note that, since F_1 and F_2 are double fibers, they are mapped to two non-skew lines L_1 and L_2 on D_1. The branch curve on D_1 consists of two conics C_1, C_2 from different pencils $|e_0 - e_1|$ and $|2e_0 - e_2 - e_3 - e_4 - e_5|$ and a hyperplane section C that are tangent to C_1 and C_2 at one of the intersection points of C_1 and C_2 with lines L_1, L_2 on D_1. In the plane model, C_1 is a line ℓ, C_2 is a conic K, and C is a cubic curve C' tangent ℓ at $\ell \cap \langle p_2, p_4 \rangle$ and tangent to K at the exceptional curve over p_3. The branch sextic $\ell + K + C'$ has additionally two double points at the intersection points of ℓ with K and two tacnodes. It is a rational curve. These points become the two tacnodes of W. The ordinary nodes are the intersection points q_1, q_2 of the conics. The curves R_2 and R_3 are mapped to the lines L_1, L_2. The conics are the images of R_6 and R_8. The exceptional points over their intersection points q_1, q_2 are the curves R_7 and R_{12}. The rest of the curves are the exceptional curves over the tacnodes of the branch curve on D_1.

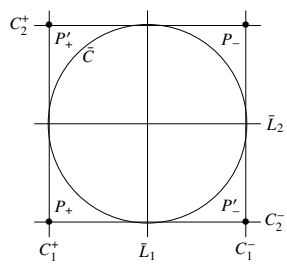
Here is the picture of the branch curve on D_1:



(8.9.3)

Here, C_1, C_2 are conics, the pre-images of ℓ and K. The curve C is a hyperplane section. The conics intersect at two (different) points q_1, q_2.

Picture (8.9.4) below describes the pre-image of the branch curve to the quadric Q. It becomes the branch curve of the double cover $X \to Q$. The pre-image of the plane section C is $\bar{C}$. The pre-images of the conic C_1, C_2 are the curves $C_1^{\pm}, C_2^{\pm}$. The pre-images of the points q_1, q_2 are the points $q_1^{\pm}, q_2^{\pm}$. The pre-images of the two lines that contain the intersection points of $C \cap C_1, C \cap C_2$ are the lines $\bar{L}_1, \bar{L}_2$. They correspond to the reducible half-fibers on X of the two elliptic fibrations lifted from the elliptic fibrations on S.



(8.9.4)

To find the equation of the double plane, we find a line through p_1 and a cubic through $p_1, \ldots, p_5$ that are tangent at a point on the line $V(x_0) = \langle p_2, p_4 \rangle$. They equation of the line must be $ax_2 + dx_1 = 0$ and the equation of the cubic is $ax_1^2x_2 + bx_0^2x_1 + cx_0^2x_2 + dx_1x_2^2 = 0$, where $a^2 = d^2$. Now, we find a conic $x_1x_2 + tx_0^2 = 0$ through $p_2, \ldots, p_4$ that intersects the cubic with multiplicity 4 at p_2. It corresponds to the parameter $t = b/a$. Replacing x_1 with $-x_1$, we may assume $a = -d = 1$, so the equation of the double plane is

$$x_3^2 + x_1x_2(x_1 - x_2)(x_1x_2 - x_0^2)(x_1x_2(x_1 - x_2) + x_0^2(bx_1 + cx_2)) = 0 \qquad (8.9.5)$$

The isomorphism class of the surface is uniquely defined, via scaling the variables, by the point $[b, c] \in \mathbb{P}^1 \setminus \{0, \infty\}$.

Proposition 8.9.8 *Any Enriques surface realizing a crystallographic basis of type* II *admits a birational model as an Enriques double plane given by equation* (8.9.5). *Moreover,*

$$\mathrm{Aut}(S) \cong \mathfrak{S}_4.$$

Proof It remains only to describe the automorphism group.

It follows from our description of cohomologically trivial automorphisms that the group $\mathrm{Aut}(S)$ acts faithfully on the set of (-2)-curves. The group of symmetries of the diagram is isomorphic to the symmetric group

$$\mathfrak{S}_4 \cong (\mathbb{Z}/2\mathbb{Z})^2 \rtimes \mathfrak{S}_3.$$

Each of the three commuting involutions switches curves in two of the three pairs (R_1, R_{10}), (R_4, R_{11}), (R_7, R_{12}) of curves in the diagram. They are realized by the deck transformations corresponding to the bielliptic linear system defined by a pair of elliptic fibration $|F_i|, i = 1, 2, 3$ with reducible fibers of type $\tilde{A}_3$ and $\tilde{D}_5$. It can

be also realized by the action of the Mordell–Weil group of the jacobian fibration of the elliptic fibration on S with a reducible fiber of type $\tilde{D}_8$.

The Mordell–Weil group of the elliptic fibration with a reducible fiber of type $\tilde{A}_8$ is isomorphic to the group $\mathbb{Z}/3\mathbb{Z}$. It realizes elements of order 3 in $\mathfrak{S}_4$. The Mordell–Weil group of the elliptic fibration $|F_i|$ is isomorphic to $\mathbb{Z}/4\mathbb{Z}$. It realizes elements of order 4 in $\mathfrak{S}_4$. Together these elements generate $\mathfrak{S}_4$. □

Proposition 8.9.9 *A surface of type* II *has genus one fibrations with reducible fibers of the following types:*

$$2\tilde{A}_8\ (4),\ \tilde{A}_8\ (4),\ \tilde{D}_8\ (6),\ \tilde{D}_5 + 2\tilde{A}_3\ (3).$$

The group $\mathrm{Aut}(S)$ *acts transitively on each group. Let* $|2F_i|, i = 1, \dots, 4$ *(resp.* $i = 5, 6, 7, 8$*, resp.* $i = 9, \dots, 14$*, resp.* $i = 15, 16, 17$*) be the genus one pencils of the first (resp. the second, resp. the third, resp. the fourth) type. One can reorder the fibrations in the first group such that intersection matrix of the corresponding primitive isotropic vectors* f_i *is equal to*

$$\begin{pmatrix}
0 & 4 & 4 & 4 & 1 & 1 & 1 & 3 & 2 & 2 & 2 & 4 & 4 & 4 & 2 & 2 & 2 \\
4 & 0 & 4 & 4 & 1 & 1 & 3 & 1 & 2 & 4 & 2 & 2 & 4 & 2 & 2 & 2 & 2 \\
4 & 4 & 0 & 4 & 1 & 3 & 1 & 1 & 4 & 2 & 4 & 4 & 2 & 2 & 2 & 2 & 2 \\
4 & 4 & 4 & 0 & 3 & 1 & 1 & 1 & 4 & 4 & 2 & 2 & 2 & 4 & 2 & 2 & 2 \\
1 & 1 & 1 & 3 & 0 & 1 & 1 & 1 & 1 & 1 & 2 & 2 & 2 & 1 & 1 & 1 & 1 \\
1 & 1 & 3 & 1 & 1 & 0 & 1 & 1 & 1 & 2 & 1 & 1 & 2 & 2 & 1 & 1 & 1 \\
1 & 3 & 1 & 1 & 1 & 1 & 0 & 1 & 2 & 1 & 1 & 2 & 1 & 2 & 1 & 1 & 1 \\
3 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 2 & 2 & 2 & 1 & 1 & 1 & 1 & 1 & 1 \\
2 & 2 & 2 & 2 & 1 & 1 & 2 & 2 & 0 & 3 & 3 & 3 & 3 & 3 & 1 & 2 & 1 \\
2 & 2 & 2 & 2 & 1 & 2 & 1 & 2 & 3 & 0 & 3 & 3 & 3 & 3 & 1 & 1 & 2 \\
2 & 2 & 2 & 2 & 2 & 1 & 1 & 2 & 3 & 3 & 0 & 3 & 3 & 3 & 2 & 1 & 1 \\
4 & 2 & 4 & 2 & 2 & 1 & 2 & 1 & 3 & 3 & 3 & 0 & 3 & 3 & 1 & 1 & 2 \\
4 & 4 & 2 & 2 & 2 & 2 & 1 & 1 & 3 & 3 & 3 & 3 & 0 & 3 & 1 & 2 & 1 \\
4 & 2 & 2 & 4 & 2 & 2 & 2 & 1 & 3 & 3 & 3 & 3 & 3 & 0 & 2 & 1 & 1 \\
2 & 2 & 2 & 2 & 1 & 1 & 1 & 1 & 1 & 1 & 2 & 1 & 1 & 2 & 0 & 1 & 1 \\
2 & 2 & 2 & 2 & 1 & 1 & 1 & 1 & 2 & 1 & 1 & 1 & 2 & 1 & 1 & 0 & 1 \\
2 & 2 & 2 & 2 & 1 & 1 & 1 & 1 & 1 & 2 & 1 & 2 & 1 & 1 & 1 & 1 & 0
\end{pmatrix}$$

In particular, $\mathsf{nd}(S) = 7$.

Proposition 8.9.10 *The Fano nodal lattice* $\mathsf{N}_S^{h_{10}}$ *of a surface of Type II coincides with* N_S*. The Nikulin R-invariant is* $(D_9, \{0\})$*. The quadratic space* $\langle \mathsf{Nod}(S) \rangle$ *is an even quadratic space of rank 8 and dimension 9. The Reye lattice is isomorphic to* $\mathsf{E}_8(4) \oplus \mathsf{U}$.

Proof Let h_{10} be the Fano polarization corresponding to the following canonical isotropic sequence:

$$(f_1, f_1 + R_3, f_1 + R_3 + R_4, f_2, f_2 + R_6, f_2 + R_6 + R_7, f_3, f_3 + R_9, f_3 + R_9 + R_{10}, f_4),$$

where f_1, f_2, f_3 are the classes of half-fibers of type $\tilde{A}_3$ that contain R_1, R_3, R_6 as its components, respectively, and f_4 is the class of a half-fiber of a genus one fibration

with simple fiber of type $\tilde{A}_8$ with components $R_1, \ldots, R_9$. We find that $h_{10} \cdot R_i = 0$ for $i = 1, 3, 4, 6, 7, 9, 10$ and $h_{10} \cdot R_i = 1$ otherwise. Thus, $\mathsf{N}_S^{h_{10}}$ is freely generated by all R_i. By the Enriques Reducibility Lemma, $\mathrm{Num}(S)$ is generated by N_S and the classes of half-fibers of elliptic fibrations. The only half-fibers that is not composed of (-2)-curves are the half-fibers of elliptic fibrations of type $\tilde{D}_8$.

Thus, we see that $2\,\mathrm{Num}(S) \subset \mathsf{N}_S$ and $\mathrm{Num}(S)$ is generated by N_S and $\frac{1}{2}(R_1 + R_{10} + R_7 + R_{12})$, $\frac{1}{2}(R_1 + R_{10} + R_4 + R_{11})$, $\frac{1}{2}(R_7 + R_{12} + R_4 + R_{11})$. This shows that $\mathrm{Num}(S)/\mathsf{N}_S \cong (\mathbb{Z}/2\mathbb{Z})^{\oplus 2}$ is a maximal isotropic subgroup of $D(\mathsf{N}_S)$. We also know that A_8 and D_8 are realized as sublattices of N_S. Looking at possible sublattices with the previous properties we find that the only one that passes all the tests is $\mathsf{A}_1(-2) \oplus \mathsf{D}_9 \cong \mathsf{U}(4) \oplus \mathsf{E}_8$.

Let us compute the Nikulin R-invariant. Applying Lemma 6.4.11 (iii) to the set $R_1, \ldots, R_9$, we obtain that K contains a sublattice M of K isomorphic to D_9. Now, note that $R_{10} + R_{12} + R_1 + R_7 \in 2\,\mathrm{Num}(S)$ because they are components of a simple fiber of type $\tilde{D}_5$. We have also similar relations involving $R_{10} + R_{11}$ and $R_{11} + R_{12}$. This shows that K is generated by M and $\delta_{R_{10}}$. Replacing the set $R_1, \ldots, R_8, R_9$ with $R_2, R_3, \ldots, R_8, R_{10}$, we obtain that K contains another sublattice isomorphic to D_9. No root lattice of rank 10 has this property. So, the two sublattices coincide and are equal to K.

Applying Lemma 6.4.6, we obtain that $\langle \mathsf{Nod}(S) \rangle$ is an even quadratic space of rank 8 and dimension 9. Thus, the image of the Reye lattice in $\overline{\mathrm{Num}}(S)$ is a one-dimensional space with trivial quadratic form. The Reye lattice is isomorphic to $\mathsf{E}_8(4) \oplus \mathsf{U}$. □

Type III

The crystallographic basis consists of 20 curves with the intersection graph of (-2)-curves given in Figure 8.4 below.

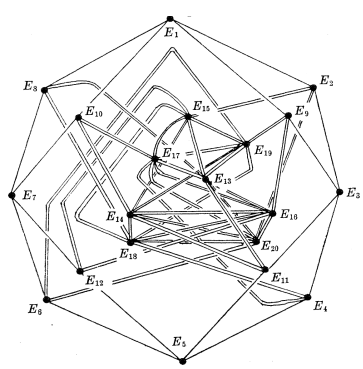


Fig. 8.4 Crystallographic basis of Type III

The vertices $E_1, \dots, E_{12}$ form a diagram of type (b) in Theorem 8.2.21. By Corollary 8.2.23, the surfaces containing such a diagram form a 2-dimensional family. They are obtained as double covers $S \to \mathsf{D}_1$ branched along the union of four conics, two from each of two pencils. The equation (8.2.9) of the double plane

$$x_3^2 + x_1x_2(x_1 - x_2)(x_1 + ax_2)(x_0^2 - x_1x_2)(x_0^2 + bx_1x_2) = 0$$

must be special in order to obtain the remaining 8 (-2)-curves $E_{13}, \dots, E_{20}$. Let $\ell_i \cap C_j = \{q_{ij}, q'_{ij}\}, i = 1, 2$. The choice of this equation is uniquely determined by the property that there are 4 lines each passing through p_2 and a pair of points $\{q_{11}, q'_{22}\}, \{q'_{11}, q'_{22}\}, \{q_{12}, q_{21}\}, \{q'_{12}, q'_{21}\}$ and 4 lines each passing through p_4 and 4 pairs of points $\{q_{11}, q_{22}\}, \{q_{11}, q_{22}\}, \{q'_{12}, q'_{21}\}, \{q_{12}, q_{21}\}$ each passes through two intersection points in $\ell_i \cap K_j$. The pre-image of the pencil of lines through p_1 defines an elliptic fibration on S with two fibers of type $\tilde{D}_4$, the pre-images of the lines ℓ_1, ℓ_2. The pre-image of the pencil of conics through p_2, p_3, p_4, p_5 defines an elliptic fibration on S with two fibers of type $\tilde{D}_4$, the pre-images of the conics K_1, K_2. The components of these four fibers correspond to curves $E_1, \dots, E_{12}$ in the diagram. The remaining curves $E_{13}, \dots, E_{20}$ correspond to the proper transforms of the eight lines.

Thus, we have the following equation of the double plane model of the surface:

$$x_3^2 + x_1x_2(x_0^4 - x_1^2x_2^2)(x_1^2 - x_2^2) = 0. \tag{8.9.6}$$

Proposition 8.9.11 *Let S be an Enriques surface realizing a crystallographic basis of type* III. *Then, S is unique up to isomorphism and coincides with the surface birationally isomorphic to the double plane given by equation* (8.9.6). *Moreover,* $\mathrm{Aut}_{\mathrm{nt}}(S) \cong \mathbb{Z}/2\mathbb{Z}$, *and*

$$\mathrm{Aut}(S) \cong ((\mathbb{Z}/4\mathbb{Z}) \times (\mathbb{Z}/2\mathbb{Z})^{\oplus 2}) \rtimes D_8, \quad \mathrm{Aut}(S)/\mathrm{Aut}_{\mathrm{nt}}(S) \cong (\mathbb{Z}/2\mathbb{Z})^{\oplus 3} \rtimes D_8.$$

Proof The uniqueness follows from the equation (8.9.6). One can also argue as follows. The elliptic fibration $|D_1| = |2(E_1 + E_2 + E_3 + E_9)|$ has two double fibers of types $\tilde{A}_3$. This shows that it is obtained from its jacobian extremal fibration by choosing local invariants at two reducible fibers of types $\tilde{A}_3$. According to our classification of such fibrations, it is unique, up to isomorphism. It is easy to see that the Mordell–Weil group of this fibration isomorphic to $\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/4\mathbb{Z}$ acts transitively on these local invariants. Thus, the corresponding torsor is defined uniquely up to isomorphism.

Let us describe the group of automorphisms of S. It is easy to see that the group of symmetries $\mathrm{Sym}(\Gamma)$ of the diagram Γ from 8.4 is isomorphic to the group of symmetries of the subdiagram with vertices corresponding to the curves $E_1, \dots, E_{12}$. It is isomorphic to the group $2^4 \rtimes D_8$. The 2-elementary subgroup P consists of transformations that switch the curves in some of the pairs $(E_2, E_9), (E_4, E_{11}), (E_6, E_{12}), (E_8, E_{10})$. We used to determine a double plane model of S by taking the numerically trivial bielliptic involution g_0 corresponding to the U-pair of two fibrations, each with two reducible fibers of type $\tilde{D}_4$. Any automor-

phism that acts non-trivially on Γ by an element from P preserves the pair, and hence arises as an automorphism of the 4-nodal quartic D_1 and as an automorphism of the double plane. Suppose g acts on Γ as a switch of one pair, say (E_2, E_9). Then, it defines an automorphism of the double plane that preserves one of the pencil, say the pencil of lines through p_1 and fixes one of the lines that defines the reducible fiber with the components E_2, E_9. We may assume that the line is $V(x_1 - x_2)$ with 2 pairs of singular points of the branch sextic $q_1 = [1, 1, 1], q_2 = [1, -1, -1]$ and $q_1' = [1, i, i], q_2' = [1, -i, i]$ lying on the intersection of the line with two branch conics. The transformation g must switch only one pair of points. It is obvious that this is impossible. On the other hand, the group of projective automorphisms generated by the transformations $[x_0, x_1, x_2] \mapsto [x_0, ix_1, ix_2], [x_0, -x_1, x_2], [x_0, x_2, x_1]$ generates the subgroup $\tilde{H} \cong \mathbb{Z}/4\mathbb{Z} \times (\mathbb{Z}/2\mathbb{Z})^{\oplus 2}$ whose image in $\mathrm{Sym}(\Gamma)$ is equal to $(\mathbb{Z}/2\mathbb{Z})^{\oplus 3}$. A generator of the first summand defines an automorphism of the double plane $[x_0, x_1, x_2, x_3] \mapsto [x_0, ix_1, ix_2, ix_3]$ whose square is the numerically trivial automorphism g_0.

So, it remains to realize the symmetries of Γ that belong to the subgroup G isomorphic to the dihedral group D_8 of order eight.

To realize a generator of G of order four, we use an elliptic fibration $|E_1 + \cdots + E_8|$ with two simple reducible fibers of type $\tilde{A}_7$ and $\tilde{A}_1$. The Mordell–Weil group of its jacobian fibration is isomorphic to $\mathbb{Z}/4\mathbb{Z}$. Finally, we realize a generator of order two of D_8 by taking an elliptic fibration $|E_1 + E_2 + E_3 + E_8|$ with two reducible fibers of type $\tilde{A}_3$ and two reducible fibers of type $\tilde{A}_1$. The Mordell–Weil group of the jacobian fibration is isomorphic to $\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/4\mathbb{Z}$. It contains an element that switches E_1, E_3 and E_5, E_7. □

Proposition 8.9.12 *The Fano nodal invariant of Enriques surface of type* III *is* $(\mathsf{N}_{h_{10}}, \mathsf{G}_{h_{10}}) = (\mathsf{D}_8 \oplus \mathsf{A}_1 \oplus \mathsf{A}_1(-1), (\mathbb{Z}/2\mathbb{Z})^{\oplus 2})$. *The Nikulin R-invariant of an Enriques surface of type III is equal to* $(\mathsf{D}_8 \oplus \mathsf{A}_1^{\oplus 2}, (\mathbb{Z}/2\mathbb{Z})^{\oplus 2})$. *The quadratic space* $\langle \mathsf{Nod}(S) \rangle$ *is a defective quadratic space of rank 6 and dimension 8. The Reye lattice is isomorphic to* $\mathsf{E}_8(4) \oplus \mathsf{U}(2)$.

Proof First, we claim that the curves E_i with $13 \leq i \leq 20$ form two cosets modulo $2\,\mathrm{Num}(S)$. Two curves are in the same cosets if and only if they intersect. Observe that the divisors $F_1 = E_1 + E_2 + E_3 + E_9$ and $F_1' = E_5 + E_6 + E_7 + E_{12}$ are two half-fibers of type $\tilde{A}_3$ of a genus one fibration. Similarly, we have a genus one fibration with half-fibers $F_2 = E_3 + E_4 + E_5 + E_{11}$ and $F_2' = E_7 + E_8 + E_1 + E_{10}$. The divisors $E_{14} + E_{18}$ and $E_{13} + E_{17}$ are simple fibers of $|2F_1|$ of type $\tilde{A}_1$. Similarly, $E_{15} + E_{19}$ and $E_{16} + E_{20}$ are simple fibers of $|2F_2|$. This shows that $E_{14} - E_{18}, E_{13} - E_{17}, E_{15} - E_{19}, E_{16} - E_{20} \in 2\,\mathrm{Num}(S)$. Observe also that, if two different (-2)-curves are in the same coset modulo $2\,\mathrm{Num}(S)$ then they must intersect. For example, E_{13} and E_{14} are in different cosets. The divisor $E_{13} + E_{15}$ is disjoint from the divisor $F_3 = E_1 + E_9 + E_3 + E_4 + E_5 + E_6 + E_7 + E_8$ that forms a half-fiber of type $\tilde{A}_7$. The divisor $F_4 = E_2 + 2E_3 + E_4 + E_9 + E_{11}$ is a simple fiber of type $\tilde{D}_4$ such that $F_4 \cdot F_3 = 2$. Since $(E_{13} + E_{15}) \cdot F_4 = 4$, we obtain that $E_{13} + E_{15}$ is a simple fiber of type $\tilde{A}_1$ of the genus one-fibration $|2F_1|$. Thus, $E_{13} - E_{15} \in 2\,\mathrm{Num}(S)$. Similarly, we prove that $E_{13} - E_{19} \in 2\,\mathrm{Num}(S)$. This implies that the coset of E_{13} modulo

$2\,\mathrm{Num}(S)$ consists exactly of the curves $E_i, i \geq 14$ that intersect E_{13}, so the claim has been proved.

Now, we see that N_S is generated by the curves $E_1, \ldots, E_{12}, E_{13}, E_{14}$. Let $\pi : X \to S$ be the K3-cover. Consider the sublattice M_1 of $\mathrm{Pic}(X)$ generated by $\delta_{E_i}, i = 2, \ldots, 9$, and $\delta_{E_{13}}, \delta_{E_{14}}$. It is isomorphic to $\mathsf{D}_8 \oplus \mathsf{A}_1^{\oplus 2}$. Consider the sublattice M_2 of $\mathrm{Pic}(C)$ spanned by $\delta_{E_i}, i = 1, 2, 4, \ldots, 9, 13, 14$. By Lemma 6.4.11, it is isomorphic to $\mathsf{D}_8 \oplus \mathsf{A}_1^{\oplus 2}$. Both lattices are of rank 10; hence they must be of finite index in the root lattice K defined by the Nikulin R-invariant of S. Suppose $M_1 \neq K$. Since the discriminant groups of M_1 and M_2 are isomorphic to $(\mathbb{Z}/2\mathbb{Z})^{\oplus 4}$, the discriminant group of K must be isomorphic to $(\mathbb{Z}/2\mathbb{Z})^{\oplus 2}$ or $\{0\}$. Since there are no unimodular root lattice of rank 10, only the first case is possible. However, in this case, it follows from the Borel–De Siebethal–Dynkin algorithm that there is only one lattice E_8 contains only one sublattice isomorphic to D_8. It is obtained by deleting the vertex α_1 from the extended Dynkin diagram of type $\tilde{E}_8$. Thus, in any case, we see that $M_1 = M_2$ and $\delta_{E_1} \in M_1$. Similarly, we show that $\delta_{E_{10}}, \delta_{E_{11}}$ and $\delta_{E_{12}} \in M_1$. This shows that $M_1 = K$. Since the cosets of E_{13} and E_{14} modulo $2\,\mathrm{Num}(S)$ belong to the radical of the quadratic space $\langle \mathsf{Nod}(S) \rangle$ we see that $\mathrm{rank}\langle \mathsf{Nod}(S) \rangle \leq 8$ and hence $\dim\langle \mathsf{Nod}(S) \rangle \leq 8$. Thus, the finite part H of the Nikulin invariant is $(\mathbb{Z}/2\mathbb{Z})^a$, where $a \geq 2$. On the other hand, the discriminant quadratic form of K is $\mathsf{u}_1 \oplus \mathsf{w}_{2,1}^{-1} \oplus \mathsf{w}_{2,1}^{-1}$, hence its maximal isotropic subgroup is isomorphic to $(\mathbb{Z}/2\mathbb{Z})^2$. This shows that $a \leq 2$, and we conclude that $a = 2$. The assertion about the Nikulin R-invariant has been proved. We also proved that $\langle \mathsf{Nod}(S) \rangle$ is a defective quadratic space of rank 6 and dimension 8.

Let h_{10} be a Fano polarization defined by a canonical isotropic sequence

$$(f_1, f_1{+}E_{10}, f_1{+}E_{10}{+}E_1, f_1{+}E_{10}{+}E_1{+}E_2, f_2, f_2{+}E_7, f_2{+}E_7{+}E_6, \ldots, f_2{+}E_7{+}\cdots{+}E_4, E_6{+}E_{19},$$

where $2f_1$ is the numerical class of a simple fiber of type $\tilde{A}_7$, and $2f_2 = [2E_1 + E_2 + E_9 + E_8 + E_{10}]$ is the class of a simple fiber of type $\tilde{D}_4$. Computing the intersection numbers $h_{10} \cdot E_i$, we find that $h_{10} \cdot E_i \leq 4$ for all $i = 1, \ldots, 14$. This implies that $\mathsf{N}_S^{h_{10}} = \mathsf{N}_S$. Using our information about $\langle \mathsf{Nod}(S) \rangle$, the only possibility is that N_S is isomorphic to $\mathsf{D}_8 \oplus \mathsf{A}_1 \oplus \mathsf{A}_1(-1)$.

It remains to prove the assertion about the Reye lattice $\mathrm{Rey}(S)$. It is equal to the pre-image of a 2-dimensional defective quadratic subspace in $\overline{\mathrm{Num}}(S)$. It must be isomorphic to $\mathsf{E}_8(4) \oplus \mathsf{U}(2)$. □

Proposition 8.9.13 *A surface of type* III *has the following elliptic fibrations:*

$$\begin{aligned} &2\tilde{A}_7 + 2\tilde{A}_1\ (8), \quad \tilde{A}_7 + \tilde{A}_1\ (8), \quad \tilde{D}_8\ (16), \quad \tilde{D}_4 + \tilde{D}_4\ (2), \\ &\tilde{D}_6 + 2\tilde{A}_1 + 2\tilde{A}_1\ (8), \quad 2\tilde{A}_3 + 2\tilde{A}_3 + \tilde{A}_1 + \tilde{A}_1\ (2). \end{aligned} \tag{8.9.7}$$

The group $\mathrm{Aut}(S)$ *acts transitively on each set of fibrations. Two half-fibers from the first or the second group intersect each other with multiplicity* $2, 4, 6$ *or* 8. *Two half-fibers from the third group (resp. the fourth, resp. the fifth, resp. the sixth) group intersect each other with multiplicity* 2 *(resp.* 1*, resp.* 2*, resp.* 1*). We have* $\mathsf{nd}(S) = 8$ *and a canonical isotropic* 8*-sequence realizing* $\mathsf{nd}(S)$ *can be formed*

by four fibrations from the second group, two fibrations from the fourth group, one fibration from the fifth group and one fibration from the sixth group[2].

Remark 8.9.14 One can realize an Enriques surface of type III as the quotient of the Kummer surface of the self-product A of the elliptic curves $E_{\sqrt{-1}}$ with complex multiplication of order 4. To do this, we consider two involutions of A

$$\iota : (x_1, x_2) \mapsto (-x_1, -x_2),$$
$$\sigma : (x_1, x_2) \mapsto (-x_1 + \alpha, x_2 + \alpha),$$

where α is a non-zero 2-torsion point on $E_{\sqrt{-1}}$. The K3-cover of the surface S is isomorphic to the minimal resolution $\widetilde{\mathrm{Kum}}(A)$ of the Kummer surface $\mathrm{Kum}(A)$ and the surface S is its quotient by the fixed-point involution descended from σ.

Type IV

The crystallographic basis consists of 20 curves with the dual graph given in the Figure 8.5 below.

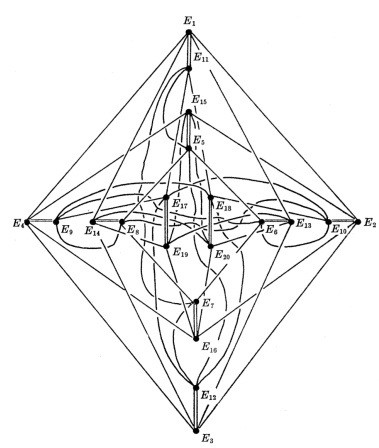


Fig. 8.5 Crystallographic basis of Type IV

Let us consider a non-degenerate U-pair formed by $F_1 = E_1 + E_{11} \equiv E_3 + E_{12}$ and $F_2 = E_4 + E_9 \equiv E_2 + E_{10}$. The two elliptic fibrations $|2F_1|$ and $|2F_2|$ have two simple fibers of types $\tilde{A}_3$ and two double fibers of type $\tilde{A}_1$ or $\tilde{A}_1^*$. The other reducible fibers are $D_1 = E_6 + E_8 + E_9 + E_{10}$, $D_1' = E_{15} + E_{16} + E_{19} + E_{20} \in |2F_1|$ and $D_2 = E_5 + E_{11} + E_{12} + E_7$, $D_2' = E_{13} + E_{14} + E_{19} + E_{20} \in |2F_2|$. They share six common irreducible fiber components $E_9, E_{10}, E_{11}, E_{12}, E_{19}, E_{20}$. This implies that the curves E_1, E_2, E_3, E_4 are mapped to the lines on D_1 and the curves $E_{11}, E_{12}, E_9, E_{10}$ are

[2] The correct value of the non-degeneracy invariant $\mathsf{nd}(S)$ in the proposition was pointed out by Moschetti, Rota, Schaffler [**?**]

mapped to points q_1, q_2, q_3, q_4, respectively, lying on the lines, with q_1, q_2 and q_3, q_4 lying on the opposite lines of the quadrangle of lines.

This also means that the components C_1, C_2 of the branch curve on D_1, together with the union of 4 lines belong to the same pencil cut out by hyperplanes passing through the points $q_1, \ldots, q_4$. Recall that the equations of the lines on D_1 in $\mathbb{P}^4$ are $x_i = x_j = 0, i \in \{1, 2\}, j \in \{3, 4\}$ so $q_1 = [0, 1, 0, 0, a_1],\ q_2 = [0, 1, 0, a_2, 0],\ q_3 = [1, 0, 0, a_3, 0],\ q_4 = [1, 0, 0, 0, a_4]$, where $a_i \neq 0$. The pairs q_1, q_3 and q_3, q_4 lie on opposite sides of the quadrangle of lines. The condition that these points are linearly dependent is $a_2a_3 - a_1a_4 = 0$. We can use projective automorphisms of D_1 to assume that

$$q_1 = [0, 1, 0, 0, 1],\ q_2 = [0, 1, 0, 1, 0],\ q_3 = [1, 0, 0, 1, 0],\ q_4 = [1, 0, 0, 0, 1].$$

We also see that the curves E_{19}, E_{20} are mapped to the singular points of C_1, C_2, Thus, the images of C_1, C_2 are the curves E_{17}, E_{18}. The fibers D_1' and D_2' share the components E_{19}, E_{20}, hence their images are conics K_1, K_2 from two different pencils that pass through the singular points of C_1, C_2. The other fibers D_1, D_2 are mapped to conics K_3, K_4 that pass through the points q_1, q_3 and q_2, q_4.

In the plane model, we may assume that two cubics are tangent at p_1 with tangent direction $x_1 - x_2 = 0$ and intersect the line $x_0 = 0$ at the point $[0, 1, 1]$ so that the conics K_3, K_4 are mapped to the line $V(x_1 - x_2)$. The two cubics belong to the pencil that contains the cubic $x_0x_1x_2 = 0$. Thus, their equations are

$$ax_1x_2(x_1 - x_2) + bx_0^2(x_1 - x_2) + cx_0x_1x_2 = 0,$$

$$ax_1x_2(x_1 - x_2) + bx_0^2(x_1 - x_2) + dx_0x_1x_2 = 0,$$

The conic $V(x_1x_2 - \lambda x_0^2)$ intersects the cubics at the points p_2, p_4 with multiplicity 3 if and only if $a = -b$ and $\lambda = 1$. Substituting $x_2 = tx_1$, we find that the line $x_2 - tx_1 = 0$ intersects both of these curves with multiplicity 2 at some point only if and only if $d = -c$ (here we see why do we need the assumption $p \neq 2$). We check that both curves are invariant under the standard Cremona transformation corresponding to an automorphism of D_1 that switches the skew lines. Thus, the singular point must be one of the fixed points of this transformation, and we check that indeed the point $[-1, 1, \pm 1]$ is the singular point if and only if $c = \pm 4a$. Thus, we may assume that the equation of the curves correspond to $(a, c) = (1, -4), (1, 4)$.

Now, we know the equation of the double plane:

$$x_3^2 + x_1x_2((x_1x_2 - x_0^2)^2(x_1 - x_2)^2 - 16x_0^2x_1^2x_2^2) = 0 \tag{8.9.8}$$

Remark 8.9.15 The K3-cover of S is the double cover of $\mathbb{P}^1 \times \mathbb{P}^1 \subset \mathbb{P}^3$ branched over the pre-image of the branch curve on D_1. The pre-image of the each of the two nodal conics are a pair of conics intersecting at two points. The pre-images the four conics are four lines, two in each ruling.

Our other remark is that, after we blow-up D_1 at the points q_1, q_2, q_3, q_4, we obtain a rational extremal elliptic surface V with two reducible fibers of type $\tilde{A}_7$

and $\tilde{A}_1$. The first fiber is the pre-image of the quadrangle of lines, and the second fiber is the pre-image of the union of the conics K_3, K_4. The surface has also two irreducible nodal fibers, the pre-images of the hyperplane sections C_1, C_2. There is only one, up to isomorphism, such surface. The surface S is birationally isomorphic to the double cover of V branched along the union of two irreducible singular fibers and four disjoint components of the fiber of type $\tilde{A}_7$.

Proposition 8.9.16 *Let S be an Enriques surface realizing the crystallographic basis of type* IV. *It exists only if $p \neq 2$, and in this case S is unique up to isomorphism and its double plane model is given in* (8.9.8). *Moreover,*

$$\mathrm{Aut}(S) \cong (\mathbb{Z}/2\mathbb{Z})^{\oplus 4} \rtimes N,$$

where $N = \mathbb{Z}/4\mathbb{Z} \ltimes \mathbb{Z}/5\mathbb{Z}$ is isomorphic to the normalizer of a cyclic subgroup of order 5 in $\mathfrak{S}_5$.

Proof Let Γ be the diagram of type IV. It follows from Theorem 8.2.23 that classifies surfaces admitting a non-trivial cohomologically trivial automorphism that our surface does not have such a property. Hence the natural homomorphism from $\mathrm{Aut}(S)$ to the group $\mathrm{Sym}(\Gamma)$ of symmetries of Γ is injective. Observe that S has 5 elliptic pencils D_i with fibers of type $\tilde{A}_3, \tilde{A}_3, 2\tilde{A}_1, 2\tilde{A}_1$. So, , the group $\mathrm{Aut}(S)$ acts on these five fibration $|D_i|$ and defines a homomorphism $r : \mathrm{Aut}(S) \to \mathfrak{S}_5$. It is easy to see that the kernel K of the homomorphism $r : \mathrm{Sym}(\Gamma) \to \mathfrak{S}_5$ is isomorphic to $(\mathbb{Z}/2\mathbb{Z})^4$. It is generated by an even number of permutations of two singular fibers of type $\tilde{A}_1$ in each of the five fibrations. The elements of the group K are realized by the elements of the Mordell–Weil group of the jacobian fibrations with two fibers of types $\tilde{D}_4$, for example $|2E_4 + E_1 + E_{15} + E_{16} + E_3|$. The image of $\mathrm{Aut}(S)$ contains an element of order 4 defined by the Mordell–Weil group of the jacobian fibration of any of the five elliptic fibrations $|D_i|$. It also contains an element of order 5 that is defined by the Mordell–Weil group of the jacobian fibration of any of the elliptic fibrations with 2 reducible fibers of types $\tilde{A}_4$, for example, $|E_3+E_4+E_{13}+E_{16}+E_{19}|$. Since the group $r(\mathrm{Sym}(\Gamma))$ does not have an element of order 3 and does not have any transposition, the assertion follows from the classification of subgroups of $\mathfrak{S}_5$.

Remark 8.9.17 One can show that the K3-cover of Enriques surfaces of type IV is birationally isomorphic to the same Kummer surface as the K3-cover of the surface of type III [409, Proposition (3.4.2)]. However the action of the Enriques involution is different. This is an example of the different fixed-point free involutions which may occur on a K3 surface birationally isomorphic to a Kummer surface studied in [578].

Proposition 8.9.18 *The Nikulin R-invariant of an Enriques surface of type IV is equal to* $(\mathsf{D}_5^{\oplus 2}, \mathbb{Z}/2\mathbb{Z})$. *The quadratic space $\langle \mathsf{Nod}(S) \rangle$ is an even quadratic space of dimension 9 and rank 8. The Reye lattice is isomorphic to* $\mathsf{E}_8(4) \oplus \mathsf{U}$.

Proof Consider the genus one fibrations of type $\tilde{D}_5 + 2\tilde{A}_3$ with irreducible components E_5, E_6, E_7, E_{12} and $E_1, E_2, E_4, E_{14}, E_{19}, E_{20}$. The curve E_9 is its bisection.

Applying Lemma 6.4.11, we find that K contains a sublattice isomorphic to $\mathsf{D}_5 \oplus \mathsf{D}_5$. Its discriminant group is $\mathsf{w}_{2,4}^{-5} \oplus \mathsf{w}_{2,4}^{-5}$. Its maximal isotropic subgroup A is of order 2. Suppose $K \neq M$, then K must be an overlattice with discriminant quadratic form $\mathsf{w}_{2,1}^{-1}$. There are no such root lattices. By Lemma 6.4.6, $K/2K$ is an even quadratic space of dimension 10 and rank 8. Since its image cannot be the whole space, the kernel H of the reduction map $K/2K \to \overline{\mathrm{Num}}(S)$ is not trivial. Since the 2-torsion of the discriminant group of K is of order 2, we obtain that $H \cong \mathbb{Z}/2\mathbb{Z}$. The quadratic space must be of even type, of dimension 9 and rank 8. □

Proposition 8.9.19 *A surface of type* IV *has genus one fibrations of the following types:*

$$\tilde{D}_4+\tilde{D}_4\ (10),\ \tilde{D}_5+2\tilde{A}_3\ (40),\ \tilde{A}_4+\tilde{A}_4\ (16),\ 2\tilde{A}_4+2\tilde{A}_4\ (16)\ \tilde{A}_3+\tilde{A}_3+2\tilde{A}_1+2\tilde{A}_1\ (5).$$

The intersection number of half-fibers of two fibrations from the first (the second, the third, the fourth) group is equal to 2 *(resp.* 2*, resp.* 2*, resp.* 1*). The non-degeneracy invariant* $\mathsf{nd}(S)$ *is equal to* 10*.*

Type V

This surface has the following crystallographic basis formed by (-2)-curves given in Figure 8.6.

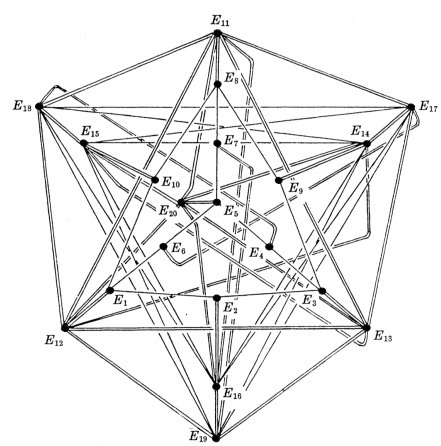


Fig. 8.6 Crystallographic basis of type V

First we locate a genus one fibration

$$|D| = |2(E_1 + E_2 + E_3 + E_8 + E_9 + E_{10})| = |2(E_5 + E_{20})| = |E_{17} + E_{18} + E_{19}|$$

with two double fibers of type $\tilde{A}_5$ and $\tilde{A}_1$ and one fiber of type $\tilde{A}_2$. It shows that such diagram cannot be realized if $p = 2$. In fact, the fibration contains two double fibers, hence $K_S \neq 0$. Since the half-fiber is of multiplicative type, it is impossible.

So, , we assume, as everywhere in this section, that $p \neq 2$.

We consider a non-degenerate U-pair formed by the elliptic fibrations

$$|E_{10} + E_2 + 2E_1 + 2E_5 + 2E_6 + E_4 + E_7| = |2(E_9 + E_{14})| = |E_{11} + E_{13}|,$$

$$|E_2 + E_9 + 2E_3 + 2E_4 + 2E_5 + E_6 + E_7| = |2(E_{10} + E_{15})| = |E_{11} + E_{12}|,$$

with reducible simple fibers of type $\tilde{D}_6, \tilde{A}_1$ and a double fiber of type $\tilde{A}_1$.

We easily locate a subdiagram of type (c) from Theorem 8.2.21. Thus, our surface is of type (C) from Theorem 8.2.23. We use a construction of this surface as a double plane indicated in the proof of the theorem.

Recall that the branch curve on D_1 is the union of two conics from different pencils and a nodal hyperplane section passing through one of the intersection points of the two conics. In the plane, the hyperplane section is a cubic C, and the conics are a line ℓ through p_1 and a conic K through $p_2, \dots, p_5$.

Let

$$\ell \cap K = \{q_1, q_2\},\ \ell \cap C = \{q_1, q_3\},\ C \cap K = \{q_1, q_4\}.$$

The point q_1 is a singular point of type d_4 of the branch curve W. The remaining singular points are q_2, q_3, q_4 and a node q of C.

The pre-image of the line ℓ (resp. the conic K) is the fiber of the type $\tilde{D}_6$ of the first (resp. the second) fibration. The pre-image of the line $\ell' = \langle p_1, q_4\rangle$ (resp. the conic K' through q_3) is a simple fiber of type $\tilde{A}_1$ of the first (resp. the second) fibration.

To realize the double fibers we have to assume additionally the following:

(1) the point q_3 lies on the line $\langle p_2, p_4\rangle$;
(2) the point q_4 lies on the line on D_1 equal to the exceptional curve over the point p_3.

So far, we see all curves E_i except $E_8, E_{16}, E_{17}, E_{18}, E_{19}, E_{20}$. The curve E_8 is the proper transform of the cubic C, the exceptional curve over its node is the curve E_{11}. The curve E_{20} is a hyperplane section that contains the point q and has a node at the points q_1. Counting constants, we find it as soon as we find our cubic C. So, its existence does not impose any new conditions. We also see that E_{16} is represented by a line $\langle q_2, q\rangle$, the same condition we used for a general member of the 2-dimensional family (C) from Theorem 8.2.23.

We observe that E_{17}, E_{18}, E_{19} are connected to E_6, E_4, E_7 by double edges. This means that the corresponding curves are tangent to l, K, C, respectively at the point q_1. We also observe that their images on D_1 are curves of degree 3. The curves E_{18}, E_{19} also are joined to E_{11} that represents the exceptional curve over the node q of C. This prompts us to put the following conditions:

(3) there is a conic from $|2e_0 - e_2 - e_3 - e_4|$ that is tangent to ℓ at q_1 and passes through q (this will give us E_{17});

(4) the tangent line to K at q_1 passes through q (this will give us E_{18});
(5) the tangent line to C at q_1 contains $x_2 = [0, 1, 0]$ (this will give us the curve E_{19}).

After straightforward computations, we find the equation of the double plane.

$$x_3^2 + x_1x_2(x_1 - x_2)(x_1x_2 - x_0^2)(x_1x_2(x_1 - x_2) + x_0^2(-9x_1 + x_2) - 8x_0x_1x_2) = 0. \quad (8.9.9)$$

The point q_1 has coordinates $[1, 1, -1]$, the singular point q of C has coordinates $[1, -3, 1]$.

We see that the construction works only if $p \neq 2, 3$. We will show in the next section that indeed these surfaces do not exist in characteristic 2 and it follows from [488, Theorem 7.1] that they do not exist in characteristic 3.

Remark 8.9.20 The canonical cover of our Enriques surface is the minimal resolution of the double cover S' of Q branched along the curve $\tilde{W} = C_+ + C_- + L_{1,+} + L_{1,-} + L_{2,+} + L_{2,-}$. The curve $C_+ + C_-$ is the pre-image of the nodal cubic C, it splits in the cover. The remaining curves are the pre-images of ℓ and K. In Figure 8.7 below borrowed from [409], we see also the curves L_3 and $F_{1,\pm}, F_{2,\pm}$. They are the pre-images of ℓ' and K'.

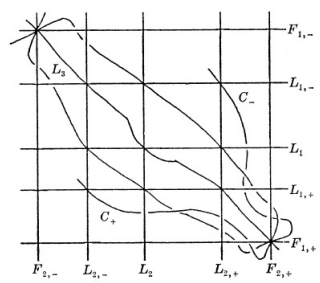


Fig. 8.7 Branch curve for the canonical cover of surfaces of type V

Proposition 8.9.21 *Let S be an Enriques surface realizing the crystallographic basis with diagram V. It exists only if $p \neq 3$ and its double plane model is given in equation* (8.9.9). *Moreover,*

$$\mathrm{Aut}(S) \cong \mathbb{Z}/2\mathbb{Z} \times \mathfrak{S}_4.$$

Proof We have only to explain the structure of the group of automorphisms. First of all, we know that our surface admits a numerically trivial involution of type (c) from Theorem 8.2.21. This is the bielliptic involution σ corresponding to our double plane model.

Next, as in the previous examples, we consider a homomorphism $\mathrm{Aut}(S) \to \mathrm{Sym}(\Gamma)$ defined by the action of $\mathrm{Aut}(S)$ on the diagram Γ of (-2)-curves. We immediately see the group $\mathrm{Sym}(\Gamma)$ is isomorphic to $\mathfrak{S}_4$. Consider the subset Σ of

vertices E_1, E_3, E_5, E_8. Any symmetry of Γ that fixes this set pointwise is the identity. Thus, $\mathrm{Sym}(\Gamma)$ is isomorphic to a subgroup of $\mathfrak{S}_4$. It is easy to see that it coincides with this group.

Let us prove that each symmetry of Γ is realized by an automorphism of S. We use that the Mordell–Weil group of the jacobian fibration of any elliptic fibration on S acts by translations on S. Thus, if we take an elliptic fibration with reducible fibers of type $\tilde{A}_5, \tilde{A}_2$ and $\tilde{A}_1$, we obtain the group $\mathbb{Z}/6\mathbb{Z}$ acting on S. For example, take the fiber of type $\tilde{A}_5$ to be $E_1, E_2, E_3, E_9, E_8, E_{10}$. Then, the cube of its generator fixes the vertices $E_{11}E_{12}, E_{13}$, hence fixes the vertices E_8, E_1, E_3. Hence it fixes all the vertices of the hexagon defining the fiber. This also implies that it fixes all vertices of the graph and hence defines the numerically trivial automorphism σ of S. Thus, we locate an element of order 3 in $\mathrm{Aut}(S)/(\sigma)$. Now, we look at an elliptic fibration with reducible fibers of type $\tilde{D}_6, \tilde{A}_1, \tilde{A}_1$. The Mordell–Weil of the jacobian fibration is isomorphic to $A = (\mathbb{Z}/2\mathbb{Z})^{\oplus 2}$. It generates the normal subgroup of $\mathfrak{S}_4$. Finally, we consider an elliptic fibration with a fiber of type $\tilde{A}_7$, for example, we take $|E_1 + E_2 + E_3 + E_9 + E_8 + E_7 + E_5 + E_6|$. The Mordell–Weil group of the jacobian fibration is a cyclic group of order 4. The square of the generator interchanges E_6 with E_7 and E_2 with E_9. It defines an element of the Mordell–Weil group of the elliptic fibration $|E_6 + E_7 + E_2 + E_9 + 2E_3 + 2E_4 + 2E_5|$ of type $\tilde{D}_6$. So, we obtain a generator of order 2 and 3 in $\mathfrak{S}_4/A \cong \mathfrak{S}_3$, this proves the claim. □

Proposition 8.9.22 *The Nikulin R-invariant of an Enriques surface of type* V *is equal to* $(\mathsf{E}_7 \oplus \mathsf{A}_2 \oplus \mathsf{A}_1, \mathbb{Z}/2\mathbb{Z})$. *The quadratic space* $\langle \mathsf{Nod}(S) \rangle$ *is a defective quadratic space of dimension 9 and rank 8. The Reye lattice is isomorphic to* $\mathsf{E}_8(4) \oplus \mathsf{A}_1 \oplus \mathsf{A}_1(-1)$.

Proof The divisor $E_1 + E_2 + E_3 + E_8 + E_9 + E_{10}$ is a half-fiber of type $\tilde{A}_5$. Applying Lemma 6.4.11, we find that the divisor classes of $\delta_{E_i}, i = 1, 2, 3, 8, 9, 10$ together w ith the class δ_{E_7} generate a sublattice of K isomorphic to E_7. Together with the classes $\delta_{E_{17}}, \delta_{E_{18}}$ and $\delta_{E_{20}}$ they generate a sublattice L isomorphic to $\mathsf{E}_7 \oplus \mathsf{A}_2 \oplus \mathsf{A}_1$. Its discriminant group is isomorphic to $(\mathbb{Z}/2\mathbb{Z})^{\oplus 2} \oplus \mathbb{Z}/3\mathbb{Z}$. If $K \neq L$, then its discriminant group must be isomorphic to $\mathbb{Z}/3\mathbb{Z}$. Thus, $H = \{0\}$ and the image of K or N_S in $\overline{\mathrm{Num}}(S)$ is the whole space. However, the diagram shows that the image $[E_{20}] \cdot x \in 2\mathbb{Z}$ for any $x \in \mathsf{N}_S$, hence its image belongs to the radical of the image of N_S. This contradiction proves that $K = L$. Since $\dim\langle \mathsf{Nod}(S) \rangle < 10$, we see that $H \neq \{0\}$, hence it must coincide with the only isotropic subgroup of K isomorphic to $\mathbb{Z}/2\mathbb{Z}$. It follows from Lemma 6.4.6 that the r-invariant $\langle \mathsf{Nod}(S) \rangle$ is a 9-dimensional defective quadratic space of rank 8. The Reye lattice is the pre-image in $\mathrm{Num}(S)$ of $\langle \mathsf{Nod}(S) \rangle^{\perp}$ and must be isomorphic to $2\mathsf{E}_8 \oplus \mathsf{A}_1 \oplus \mathsf{A}_1(-1)$. □

Proposition 8.9.23 *A surface of type* V *has genus one fibrations of the following types:*

$$2\tilde{A}_5 + \tilde{A}_2 + 2\tilde{A}_1(4),\ \tilde{E}_7 + 2\tilde{A}_1\ (12),\ \tilde{D}_6 + 2\tilde{A}_1 + \tilde{A}_1(6),\ \tilde{A}_7 + \tilde{A}_1(3),\ \tilde{E}_6 + \tilde{A}_2\ (4).$$

The group $\mathrm{Aut}(S)$ *has two orbits on each group except the last one. One of the fibrations in the first group is invariant. In particular,* $\mathsf{nd}(S) = 7$.

Type VI

The crystallographic basis consists of 20 curves with the dual graph given in the following Figure 8.8

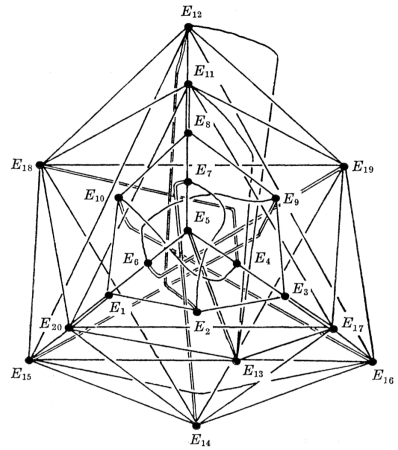


Fig. 8.8 Crystallographic basis of Type VI

Observe that the diagram contains a subdiagram isomorphic to the Petersen graph in Figure 6.1. Each vertex is connected by a double edge to one of the remaining 10 vertices. They form a symmetric incidence configuration (10_6). The presence of the Petersen graph prompts us to search for a construction of this surface as an Enriques surface of Hessian type which we studied in Example 6.4.20.

Recall that the most special Hessian surface is the Hessian surface of the Clebsch diagonal cubic surface. The involutions defined by the projections from the nodes descend to the involutions on the Enriques surface that generate a group of automorphisms isomorphic to the symmetric group $\mathfrak{S}_5$. If we index the nodes as in the Petersen graph, the projection involutions τ_{ab} act on the (-2)-curves U_{ab} as transpositions [181, Corollary 4.2]. We can also interpret 10 elliptic pencils $|F_{ab}|$, $1 \le a < b \le 5$, defined by the double edges in the diagram. They correspond to pencils of cubic curves cut out by a pencil of planes through each of the 10 edges L_{ab} of the Sylvester pentahedron Π. It vertices are the 10 nodes of the Hessian surface.

With all this in mind, let us construct an Enriques surface of Type VI as the quotient of the Hessian surface of the Clebsch diagonal cubic surface. It is given by the equation

$$\sum_{i=1}^{5} t_i = t_1t_2t_3t_4t_5 \sum_{i=1}^{5} \frac{1}{t_i} = 0, \tag{8.9.10}$$

in $\mathbb{P}^4$. The fixed-point-free involution τ is the standard Cremona cubic transformation in $\mathbb{P}^4$ given by the formula

$$\sigma : [t_1, \dots, t_5] \mapsto \left[\frac{1}{t_1}, \dots, \frac{1}{t_5}\right].$$

The involution has no fixed points on X if $p \neq 3, 5$. It has 5 fixed points if $p = 3$ and one fixed point if $p = 5$.

When $p \neq 2$, the surface given by these equations is isomorphic to the Hessian surface of the *Clebsch diagonal cubic surface*

$$\mathsf{C}_3 : \sum_{i=1}^{5} t_i = \sum_{i=1}^{5} t_i^3 = 0$$

(see [177]). If $p = 2$, the same is true only the equation of the Clebsch diagonal surface is the following:

$$\mathsf{C}_3 : \sum_{i=1}^{5} t_i = \sum_{1 \leq i < j < k \leq 5} t_i t_j t_k = 0. \tag{8.9.11}$$

If $p \neq 3$, the surface X' has 10 ordinary double points P_{ab} with coordinates $t_i = t_j = t_k = 0$ for $i, j, k \notin \{a, b\}$. They are the vertices of the *Sylvester pentahedron* Π formed by the planes $t_i = 0$. We also have 30 lines

$$L'_{ab} : t_a = t_b = t_c + t_d + t_e = 0,\ \{a, b, c, d, e\} = \{1, 2, 3, 4, 5\},$$
$$L_{cde}(\omega)' : t_a + t_b = 0,\ \omega t_c - t_d = \omega^2 t_c - t_e = 0,\ \{a, b, c, d, e\} = \{0, 1, 2, 3, 4\},$$

where $\omega^3 = 1, \omega \neq 1$. The first 10 lines appear on any Hessian surface, they are the edges of the Sylvester pentahedron. The additional lines $L_{cde}(\omega)'$ are absent in the case of the Hessian surface of a general cubic surface. They correspond to splitting of the conics cut out by planes tangent to the surface along a line L'_{ab} and passing through opposite node P_{ab} with coordinates $t_c = t_d = t_e = 0$ into two lines.

Let R_{ab} be the exceptional curve of the minimal resolution $X \to X'$ over the singular point P_{ab} and let $L_{ab}, L_{cde}(\omega)$ be the proper transforms of the lines in X'. Together we have 40 (-2)-curves on X.

The Cremona involution σ lifts to a biregular automorphism of the minimal resolution of singularities X of X' and switches the curves L_{ab} with the exceptional curve R_{cde} over the point p_{cde} such that $\{a, b\} \cap \{c, d, e\} = \emptyset$.

The involution σ has 20 orbits on the set of 40 curves $R_{ab}, L_{ab}, L_{cde}(\omega)$. If $p \neq 2$, the quotient map $X \to S = X/\langle\sigma\rangle$ is the canonical cover of an Enriques surface. The images of the 40 (-2) curves on S is the set of 20 (-2)-curves which form the crystallographic basis of type VI. To see this, we first observe that the incidence graph of the images of the orbits L_{ab}, R_{ab} is the Petersen graph 6.1. The curves $E_1, \dots, E_{10}$ correspond to these curves in the diagram of type VI. The remaining 10 curves $E_{11}, \dots, E_{20}$ correspond to 10 lines $L_{abc}(\omega)$.

Next, we will find an explicit equation of the surface as a double plane model corresponding to a choice of a bielliptic involution. This will prove the uniqueness of the surface up to an isomorphism and also will show that the surface exists only under the assumption that $p \neq 3, 5$.

We have mentioned that the surface has 10 elliptic pencils $|2F_{ab}|$ indexed by the vertices of the Petersen graph corresponding to the nodes of the Hessian surface, and also to the orbits U_{ab} of an edge and the vertex of the Sylvester pentahedrons. There are two kinds of U-pairs $(|2F_{ab}|, |2F_{cd}|)$ formed by these fibrations corresponding to whether the two subsets $\{a, b\}, \{c, d\}$ are disjoint or not. In the latter case, the bielliptic involution defined by the pair coincides with the projection involution defined by the complementary subset of $\{a, b, c, d\}$ in $\{1, 2, 3, 4, 5\}$ [181, Lemma 4.5].

Let us, for the sake of convenience, reindex the curves $E_1, \dots, E_{10}$ by subsets $\{a, b\}$ as in the Petersen graph:

$$\begin{pmatrix} E_1 & E_2 & E_3 & E_4 & E_5 & E_6 & E_7 & E_8 & E_9 & E_{10} \\ U_{45} & U_{12} & U_{34} & U_{15} & U_{24} & U_{13} & U_{35} & U_{14} & U_{25} & U_{23} \end{pmatrix}.$$

Let us choose the U-pair

$$|2F_{34}| = |E_3 + E_4 + E_5 + E_7 + E_8 + E_9| = |E_{12} + E_{16} + E_{19}| = |2(E_1 + E_{20})|,$$

$$|2F_{45}| = |E_1 + E_6 + E_5 + E_7 + E_8 + E_{10}| = |E_{12} + E_{15} + E_{18}| = |2(E_3 + E_{17})|.$$

The biellliptic involution acts on the diagram as the transposition (12). We find that the curves $E_1, E_3, E_5, E_7, E_8, E_{12}$ are common irreducible components of both elliptic pencils. They span a sublattice of $\mathrm{Num}(S)$ isomorphic to $\mathsf{A}_3 + \mathsf{A}_1^{\oplus 3}$ and they are blown down to singular points of the branch curve that consist of one tacnode and three ordinary nodes. The branch curve can be found in Row 23 in Table 8.7. It consists of two hyperplane sections tangent at one point and also intersecting at two points lying on two non-skew lines on the 4-nodal anti-canonical quartic del Pezzo surface D_1, one of them has a node. In the double plane model, the branch sextic curve is the union of two cubics C_1 and C_2. They intersect at a point q_1 on the line $\langle p_2, p_4 \rangle = V(x_0)$ and at some point p_3' infinitely near to p_3. One of them, say C_1, has a node q_2 and C_1, C_2 are tangent at some point q_3.

It is easy to see from the diagram that the node q_2 is the image of E_{12}, the tacnode q_3 is the image of $E_5 + E_7 + E_8$ where E_7 is invariant with respect to the involution, and the other components are switched. The curve E_1 (resp. E_3) is mapped to the point p_3' (resp. q_1). The curves E_{20} (resp. E_{17}) is mapped to the line $V(x_0)$ (resp. is blown down to the point q_2). The curve E_2 is mapped to the cubic C_1. The reducible fiber of type $\tilde{A}_5$ of the pencil $|2F_{34}|$ is mapped to the line ℓ_1 which passes through p_1, q_1, q_3 and its proper transform on S splits under the cover into the curves $E_9 + E_4$. Similarly, the fiber of the same type of $|2F_{45}|$ is mapped to a conic K_1 that passes through $p_2, p_3, p_4, p_5, p_3', q_3$. Its proper transform splits into $E_6 + E_{10}$. The curves E_1 and E_3 are the pre-images of the exceptional curves over q_1, p_3'. The pairs (E_{15}, E_{18}) and (E_{16}, E_{19}) are mapped to lines $\ell_2 \in |e_0 - e_1|$ and the conic

$K_2 \in |2e_0 - e_2 - e_3 - e_4 - e_5|$ that pass through the node q_2. They are tangent to the cubic C_2 and split under the cover.

So far, all curves E_i have been accounted for, except the curves E_{11}, E_{13}, E_{14}. We observe that the curve E_{14} intersects $2(F_{34} + F_{45})$ with multiplicity 4. Since it is invariant under the involution, it is mapped to a conic on D_1 passing through two opposite nodes of D_1. In the plane model it is the line through p_4 (not p_2 because E_{14} intersects both E_{20} and E_{17}) that tangent to C_1, C_2 at the point q_3. The curves E_{11} and E_{13} are switched by the involution and mapped to the cubic curve C_3 with a double point at q_3 and containing q_2. Since E_{11}, E_{13} are connected in the diagram, the cubic C_3 must be tangent to C_2 at some point.

Now, all 20 curves are accounted for, and we may proceed to write down the equation of the double plane.

First, we observe that the involution of the surface defined by the transposition (35) leaves invariant the U-pair and hence descends to a Cremona involution T of the plane that leaves invariant the branch curve. Since it switches the two pencils, it corresponds to an automorphism of the surface D_1 that fixes two opposite vertices of the quadrangle of lines. It follows from the proof of Proposition 0.6.25, that we may assume that T is given by formula

$$T : [x_0, x_1, x_2] \mapsto [x_0^2, x_1x_2, x_0x_1].$$

Our cubics C_1, C_2 are invariant cubics from the linear system $|-K_{\mathsf{D}_1}|$ that intersect at the same point on $\langle p_2, p_4\rangle = V(z)$. They must be given by equations

$$x_1x_2(ax_2 - x_1) + x_0^2(b_ix_1 - x_2) + c_ix_0x_1x_2 = 0, i = 1, 2. \tag{8.9.12}$$

Because of the invariance, the cubics automatically pass through the same point q_2 infinitely near to x_3.

The line $\ell = V(ax_2 - x_1)$ intersects C_1, C_2 at their tangency point q_3. Since the fixed locus of T is equal to $V(x_1^2 - x_0^2)$, we may assume that $q_3 = [a, 1, a]$.

Computing the partial derivatives, we obtain the condition that the tangent line is of the form $x_0 + kx_2 = 0$ is $b_i = -(c_i - 1)/a$. They will be automatically tangent at this point.

Next, we use the condition that C_1 has a singular point at some point $[-1, A, 1]$ on the other fixed line. This gives $A = 3/a, c_1 = -8$. Then, we use that the line $V(x_2)$ is tangent to C_2 at p_1 This gives $c_2 = 8/3$. Thus, the equations of C_1, C_2 and C_1 become

$$C_1 : x_1x_2(ax_2 - x_1) + (\frac{9}{a}x_1 - x_2)x_0^2 - 8x_0x_1x_2 = 0,$$

$$C_2 : x_1x_2(ax_2 - x_1) + (-\frac{5}{3a}x_1 - x_2)x_0^2 + \frac{8}{3})x_0x_1x_2 = 0.$$

The change of variables $y' = ay$ allows us to assume that $a = 1$. We verify that the condition for the existence of the cubic C_3 representing E_{13} is automatic. This gives the equation of the double plane:

$$x_3^2+x_1x_2(x_1x_2(x_2-x_1)+(9x_1-x_2)x_0^2-8x_0x_1x_2)(x_1x_2(x_2-x_1)+(-\frac{5}{3}x_1-x_2)x_0^2+\frac{8}{3}x_0x_1x_2 = 0. \tag{8.9.13}$$

We see that the surface exists if and only if $p \neq 3, 5$. It is indeed proven in [488, Corollary 7.5] that such surfaces do not exist in characteristic 3 and 5.

Proposition 8.9.24 *Let S be an Enriques surface with a crystallographic basis with the diagram of type* VI. *Then, it exists only if $p \neq 3, 5$, it is unique, up to isomorphism, and*

$$\mathrm{Aut}(S) \cong \mathfrak{S}_5.$$

Proof We have already explained that the Hessian surface admits a group of symmetries isomorphic to $\mathfrak{S}_5$. It is inherited from the group of symmetries of the Clebsch diagonal surface and descends to the group of automorphisms of the Enriques surface. The group of symmetries of the diagram is isomorphic to $\mathfrak{S}_5$ and is induced by the symmetries of the Petersen subgraph. Since the diagram has no subdiagrams of type (a), (b), (c) from Theorem 8.2.21, S has no non-trivial numerically trivial automorphisms. We conclude that $\mathrm{Aut}(S) \cong \mathfrak{S}_5$. □

Remark 8.9.25 If $\Bbbk = \mathbb{C}$, it is known that the Kummer surface associated with the Jacobian of a curve of genus two is isomorphic to the irreducible Hessian surface of a nonsingular cubic surface [318]. The coefficients $(a_1, \dots, a_5)$ in the standard equation of the Hessian surface satisfy a cubic equation

$$\sum_{i=1}^{5} a_i^3 - 2 \sum_{1\le i<j<k\le 5} a_i^2 a_j a_k + \sum_{1\le i<j\le 5} a_i a_j a_k = 0.$$

(see [624, Theorem 7.1]). In particular, taking $a_1 = a_2 = a_3 = a_4 = a_5 = 1$, we see that the Hessian surface of the Clebsch diagonal cubic surface is not a Kummer surface. It is known that the lattice of transcendental lattice of our Hessian surface is of rank 2 defined by the matrix $\left(\begin{smallmatrix} 4 & 1 \\ 1 & 4 \end{smallmatrix}\right)$ (see [409, p. 277]).

Proposition 8.9.26 *The Nikulin R-invariant of an Enriques surface of type* VI *is equal to* $(\mathsf{E}_6 \oplus \mathsf{A}_4, \{0\})$. *The quadratic space* $\langle \mathsf{Nod}(S)\rangle$ *coincides with* $\overline{\mathrm{Num}}(S)$. *The Reye lattice coincides with* $2\,\mathrm{Num}(S)$.

Proof The curves $E_3, E_4, E_5, E_6, E_7, E_8$ span a sublattice of N_S isomorphic to E_6. Thus, $\delta_{E_i}, 3 \le i \le 8$, span a sublattice of K isomorphic to E_6. The divisor $E_{12} + E_{15} + E_{16}$ is a half-fiber of type $\tilde{A}_2$ of the genus one fibration with a simple fiber of type $\tilde{E}_6$. It has a special bisection E_{20} which is disjoint from $E_3, E_4, E_5, E_6, E_7, E_8$. Applying Lemma 6.4.11, we find that K contains a sublattice isomorphic to $\mathsf{E}_6 \oplus \mathsf{A}_4$. Its discriminant quadratic group has no isotropic vectors. Thus, $K = \mathsf{E}_6 \oplus \mathsf{A}_4$. Note that the projection of $\mathsf{E}_6 \oplus \mathsf{A}_4$ to the space $\langle \mathsf{Nod}(S)\rangle$ is the whole space. One confirms again following Lemma 6.4.6 that it is a regular even space of dimension 10. □

Proposition 8.9.27 *A surface of type* VI *has genus one fibrations of the following types:*

$$2\tilde{A}_4 + \tilde{A}_4\ (12),\ \tilde{A}_5 + \tilde{A}_2 + 2\tilde{A}_1\ (10),\ \tilde{D}_5 + \tilde{A}_3\ (15),\ \tilde{E}_6 + 2\tilde{A}_2\ (10).$$

The group $\mathrm{Aut}(S)$ *acts transitively on each group. The non-degeneracy invariant* $\mathsf{nd}(S)$ *is equal to* 10.

The last assertion follows from the fact that ten half-fibers corresponding to the double edges in the diagram form a non-degenerate isotropic 10-sequence. Note that the surface is a special case of an Enriques surface of Hessian type, and, as follows from (6.4.18), any such surface has the non-degeneracy invariant equal to 10.

Type VII

The crystallographic basis of (-2)-curves is given in the following Figure 8.10.

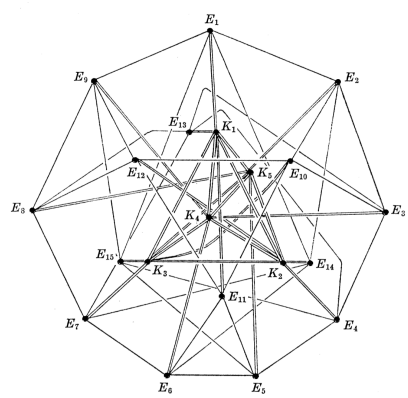


Fig. 8.9 Crystallographic basis of Type VII

We can redraw this graph differently exhibiting more explicitly its $\mathfrak{S}_5$-symmetry as in Figure **??**:

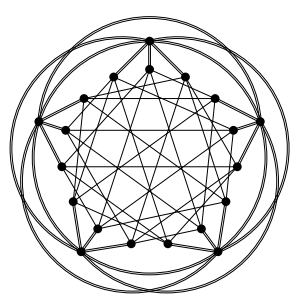


Fig. 8.10 Crystallographic basis of Type VII (different view)

Here, the five outside vertices represent smooth rational curves $K_1, \dots, K_5$, they are joined by double edges.

Assume that such a surface exists. We first determine its automorphism group. First we observe that the group of symmetries of the diagram Γ is isomorphic to $\mathfrak{S}_5$ via its action on the complete subgraph $K(5)$ with 10 double edges. Next, looking at parabolic subdiagrams we observe the following:

Proposition 8.9.28 *A surface of type* VII *has genus one fibrations of the following types:*

$$\tilde{A}_8\ (20),\ \tilde{A}_7 + 2\tilde{A}_1\ (15),\ \tilde{A}_4 + \tilde{A}_4\ (12),\ \tilde{A}_5 + 2\tilde{A}_2 + \tilde{A}_1\ (10).$$

The group $\mathrm{Aut}(\Gamma)$ *acts transitively on the corresponding maximal parabolic subdiagrams.*

Observe that the numbers $20, 15, 12, 10$ are equal to the number of conjugacy classes in $\mathfrak{S}_5$ of cyclic subgroups of order $3, 4, 5, 6$, respectively. Looking at the Mordell–Weil group of each fibration we find that they are cyclic group of order $3, 4, 5, 6$, respectively. So, we can realize these symmetries by translation transformations. Since the diagram Γ has no subdiagram described in Theorem 8.2.21, we obtain that $\mathrm{Aut}_{\mathrm{nt}}(S)$ is trivial. This proves the following:

Theorem 8.9.29 *Assume $p \neq 5$. Let S be an Enriques surface with crystallographic basis of (-2)-curves of type* VII. *Then*

$$\mathrm{Aut}(S) \cong \mathfrak{S}_5.$$

Example 8.9.30 To realize a crystallographic basis of type VII we consider, following Ohashi, the surface X' of degree 6 in $\mathbb{P}^4$ given by the equations

$$\sum_{1 \le i < j \le 5} x_i x_j = \sum_{1 \le i < j < k \le 5} x_i x_j x_k = 0. \tag{8.9.14}$$

We have already encountered this surface in Example 8.8.4, where we have shown that the group $\mathfrak{S}_5$ acts on S in a Mathieu fashion.

The surface has the obvious $\mathfrak{S}_5$-symmetry. It also has an involution σ defined by the standard Cremona transformation $[x_1, \dots, x_5] \to [1/x_1, \dots, 1/x_5]$. The surface has 5 nodes that form an orbit of $\mathfrak{S}_5$ of the point $[1, 0, 0, 0, 0]$. The hyperplane sections $A_i' : x_i = 0$ are curves of degree 6 and of arithmetic genus 4. Each of these curves contains 4 nodes. The proper transforms A_i of these curves on a minimal resolution X, which is a K3 surface, are smooth rational curves. The lift τ of the Cremona involution to X interchanges the exceptional curves of the resolution with the curves A_i.

Assume that $p \neq 5$. In this case the involution σ has no fixed points on X and the quotient $S = X/\langle\sigma\rangle$ is an Enriques surface. The images of the curves A_i on S is a set of 5 curves $K_1, \dots, K_5$ whose dual graph is a complete graph Γ_1 with 5 vertices and double edges.

For each even involution $\sigma = (ij)(kl)$ in $\mathfrak{S}_5$, we have two lines $\ell_\sigma^\pm$. If $\sigma = (12)(34)$, the line $\ell_\sigma^\pm$ is the span of the points $[1, -1, \pm\sqrt{-1}, \mp\sqrt{-1}, 0]$ and

$[0, 0, 0, 0, 1]$. The remaining lines are obtained by applying permutations of the coordinates.

Their proper inverse transforms of all these lines to X forms the set of 30 smooth rational curves. The involution τ acts on this set via switching ℓ_σ^+ with ℓ_σ^-. The images of these curves on S form the set of 15 curves $E_1, \ldots, E_{15}$ whose incidence graph is a 4-regular graph Γ_2. The exceptional curve over the singular point $[1, 0, 0, 0, 0]$ is mapped under the Cremona involution to the proper transform on X of the hyperplane section $H_1 = X' \cap V(x_1)$. It is a curve of degree 6 in $\mathbb{P}^3$ with 4 nodes $[1, 0, 0, 0], \ldots, [0, 0, 0, 1]$. It intersects with multiplicity 2 the exceptional curves over the other four nodes of X'. The orbit of this curve under the Cremona involution is mapped to the curve K_1 in the graph. In this way, we realize the curves $K_1, \ldots, K_5$ in the diagram.

Remark 8.9.31 A different construction of an Enriques surface with a crystallographic basis of type VII was given by G. Fano [227], and reproduced in [409, (3.7)].

The surface is obtained by a quadratic twist construction from an extremal rational elliptic surface $j : J \to \mathbb{P}^1$ with reducible fibers of types $\tilde{A}_7$ and $\tilde{A}_1$. It can be realized as the blow-up of the base points of a pencil of cubic curves given in (4.9.14), but the blowing-down is different. In the notation of Figure 4.5, the exceptional configuration $E_0 + R_0 + \cdots + R_7$ is blown down to one base point x_1 of the pencil. The image of R_8 is a plane cubic C with a node at x_1. In suitable coordinates, the equation of C is $x_0x_1x_2+x_1^3+x_2^3 = 0$. All other cubics from the pencil are tangent to one of two branches of C with multiplicity 8. Using the parametrization $x_0 = u^3 - v^3, x_1 = -vu^2, y = uv^2$ we find the conditions for a cubic intersect C as above. This gives us the equation of the pencil

$$C_{\lambda,\mu} = \lambda(x_1^3 - x_2^3 - x_0x_1x_2) + \mu(x_0^2x_1 + x_1^2x_2 + x_2^2x_0) = 0.$$

The double cover of the rational elliptic surface defined by this pencil branched along two nonsingular fibers is a nonsingular model of a Cayley quartic symmetroid. If we choose the fibers $F_\pm$ corresponding to the parameters $[\lambda, \mu] = [1 \pm \sqrt{-1}, 1]$, then the lines $(1 \pm \sqrt{-1})x_1 + x_2 = 0$ define two bisections tangent to $F_\pm$. The corresponding quadratic twist defines an Enriques surface of type VII.

Assume $p = 5$. Then, the ordinary node $[1, 1, 1, 1, 1]$ of X' is the only fixed point of the Cremona transformation. When we resolve this point and take the quotient, we obtain a rational Coble surface. We will discuss such surfaces in Section 9.8.

Proposition 8.9.32 *The Nikulin R-invariant of an Enriques surface of type* VII *is equal to* $(\mathsf{A}_9 \oplus \mathsf{A}_1, \mathbb{Z}/2\mathbb{Z})$. *The quadratic space* $\langle \mathsf{Nod}(S) \rangle$ *is a hyperplane of rank* 8 *in* $\overline{\mathrm{Num}}(S)$. *Since* $\langle \mathsf{Nod}(S) \rangle$ *is the same as those of type* I *and type* V, *the Reye lattice is* $\mathsf{E}_8(4) \oplus \mathsf{A}_1 \oplus \mathsf{A}_1(-1)$.

Proof The nodal cycle $E_1 + \cdots + E_8 + E_{12}$ is of type A_9 and the sublattice of K generated by δ_{E_i} is isomorphic to A_9. Together with δ_{K_4} it generates a sublattice M of K isomorphic to $\mathsf{A}_9 + \mathsf{A}_1$. □

Proposition 8.9.33 *Recall that a surface of type* VII *has genus one fibrations of the following types (Proposition 8.9.28):*

$$\tilde{A}_8\ (20),\ \tilde{A}_7 + 2\tilde{A}_1\ (15),\ \tilde{A}_4 + \tilde{A}_4\ (12),\ \tilde{A}_5 + 2\tilde{A}_2 + \tilde{A}_1\ (10).$$

The group $\mathrm{Aut}(S)$ *acts transitively on each group. The non-degeneracy invariant* $\mathsf{nd}(S)$ *is equal to 10 and can be realized by taking 6 elliptic fibrations of type* $\tilde{A}_4 + \tilde{A}_4$ *and 4 elliptic fibrations* $|K_1 + K_i|, i = 2, 3, 4, 5$.

Unfortunately, we were not able to find an explicit formula for a double plane model of a surface of type VII. Nevertheless, it is true that all such surfaces are isomorphic (see [488, Corollary 9.5]).

Proposition 8.9.34 *An Enriques surface with finite automorphism group of type* VII *is unique up to an isomorphism.*

Proof This is certainly true in the case $\mathbb{k} = \mathbb{C}$. The proof follows from the knowledge of the Nikulin R-invariant of the surface. However, if $p > 0$, we need another argument to prove the assertion. We use the Ogg–Shafarevich theory from Section 4.7 in Volume I. It follows from this theory that a torsor of a jacobian elliptic fibration on a rational surface is uniquely determined by the local invariants that consist of a choice of a collection of fibers that become multiple fibers in the torsor and a choice of a torsion point in the connected component of the identity of the Picard group.

Choose an elliptic fibration on S with a double fiber of type $\tilde{A}_1$ and a simple fiber of type $\tilde{A}_7$. Then, S is isomorphic to a torsor over the corresponding jacobian fibration $j : J \to \mathbb{P}^1$. It is defined by data $(J_1, \epsilon_1, J_2, \epsilon_2)$, where J_1 is a smooth fiber, J_2 is the fiber of type $\tilde{A}_1$, and ϵ_i are non-trivial 2-torsion points in the connected component of the identity of $\mathrm{Pic}(J_i)$. The pair (F_2, ϵ_2) is unique but there are three choices for ϵ_1. However, the Mordell–Weil group acts with two orbits on the set of non-trivial 2-torsion points on J_1. One orbit corresponds to an elliptic fibration with a special bisection (it is easy to construct it using a double plane model of S), and another one corresponds to our elliptic fibration with a special bisection. Thus, our surface is unique up to an isomorphism. □

Note that the argument from the proof can be used to prove the uniqueness of all surfaces of types III–VII and the dependence on one parameter for surfaces of types I and II.

Since we have not been able to find an explicit equation for a double plane model as in the previous cases, we have to prove the following.

Proposition 8.9.35 *A surface of type* VII *does not exist if* $p = 5$.

Proof We take an elliptic fibration with two reducible fibers of type $\tilde{A}_4$. For example, we take the fibers $E_1 + E_2 + E_{10} + E_{12} + E_9$ and $E_4 + E_5 + E_6 + E_{13} + E_7$. Then, the permutation $(13)(24)$ defines an automorphism g of the surface that permutes these two fibers. If $p = 5$, the jacobian fibration has one irreducible singular cuspidal fiber. It must be invariant and define two isolated fixed points, one of them is the

cusp. The trace of the action of g^* on the l-adic cohomology $H^*_{\text{ét}}(S, \mathbb{Q}_l)$ is equal to 4, so, applying the Lefschetz fixed-point-formula we obtain $e(S^g) = 4$. On the other hand, there is another invariant fiber which must be smooth. The involution g has no fixed points on it, or has four isolated fixed points, or identical on it, and we get a contradiction. □

8.10 Enriques Surfaces with Finite Automorphism Group ($p = 2$)

In Section 8.9, we discussed Enriques surfaces with finite automorphism group in characteristic $p \neq 2$. Table 8.11 summarizes all the information about the existence and the number of parameters for these surfaces. In this section, we will deal with the case $p = 2$.

First, we check whether a surface with a crystallographic basis of type I – –VII exists in characteristic 2, and then we discuss whether there are new types of crystallographic bases in characteristic two.

We will use the following easy observation that follows from Theorem 4.10.3:

Lemma 8.10.1 *Suppose S has an elliptic fibration with a singular half-fiber of multiplicative type. Then, S is a $\boldsymbol{\mu}_2$-surface. Moreover, if there are two half-fibers of this type, then S does not exist.*

The crystallographic basis of (-2)-curves on the surface allows one to check whether a fiber is multiple or not and determine its type. However, in some cases we cannot distinguish between the types $\tilde{A}_1$ (resp. $\tilde{A}_2$) or $\tilde{A}_1^*$ (resp. type $\tilde{A}_2^*$). The following corollary gives an easy conclusion.

Corollary 8.10.2 *1. A surface with a crystallographic basis of type* I, II *or* VI *may exist only as a $\boldsymbol{\mu}_2$-surface.*
2. A surface with a crystallographic basis of type III, IV *or* V *does not exist.*
3. A surface with a crystallographic basis of type VII *cannot be a $\boldsymbol{\mu}_2$-surface.*

Proof (i) By Proposition 8.9.6, a surface with a crystallographic basis of (-2)-curves of type I contains a genus one fibration with a double fiber of type $\tilde{A}_7$. So, it could exist only as a $\boldsymbol{\mu}_2$-surface. By Proposition 8.9.9, a surface with a crystallographic basis of (-2)-curves of type II contains a genus one fibration with a double fiber of type $\tilde{A}_3$, so we have same conclusion. By Proposition 8.9.27, a surface with a crystallographic basis of (-2)-curves of type VI contains a genus one fibration with a double fiber of type $\tilde{A}_4$, so it could exist only as a $\boldsymbol{\mu}_2$-surface.

(ii) By Propositions 8.9.13, 8.9.19, 8.9.23, a surface with a crystallographic basis of (-2)-curves of type III, IV, or V contains a genus one fibration with two double fibers, and hence it is classical. Moreover, it has a genus one fibration with a half-fiber of multiplicative type. So, , the surface does not exist in characteristic 2.

(iii) By Proposition 8.9.33, a surface with a crystallographic basis of (-2)-curves of type VII contains a genus one fibration with reducible fibers of types $\tilde{A}_5, \tilde{A}_2(\tilde{A}_2^*)$

and $\tilde{A}_1(\tilde{A}_1^*)$ and the fiber of type $\tilde{A}_2(\tilde{A}_2^*)$ is a double fiber. However, the classification of extremal genus one fibrations on a rational surface shows that this fiber is of additive type $\tilde{A}_2^*$. So, the surface cannot be a $\boldsymbol{\mu}_2$-surface. □

The next task is to realize surfaces with a crystallographic basis of type I, II, or VI as a $\boldsymbol{\mu}_2$-surfaces.

Example 8.10.3 Assume that a surface contains a crystallographic basis of type I. Recall that, if $p \neq 2$, we have constructed a one-dimensional family of such surfaces as a double cover of the 4-nodal quartic del Pezzo surface D_1 branched along the union of two conics and a hyperplane section C passing through their two intersection points. We also have two conics $Q_1 \in |e_0 - e_2|$ and $Q_2 \in |e_0 - e_4|$, each passing through one of the intersection points of C_1 with C_2.

To extend this construction to the case $p = 2$, we consider the anti-canonical quartic Del Pezzo surface D_2 instead of D_1. Let $C_1 \in |e_0 - e_1|$ and $C_2 \in |2e_0 - e_2 - e_3 - e_4 - e_5|$ be two conics, and let C be a hyperplane section passing through their two intersection points q_1, q_2 (everything as in the case $p \neq 2$). We also consider two conics $Q_1, Q_2 \in |e_0 - e_2|$, each passing through one of the points q_1, q_2. Now, we consider the Artin–Schreier double cover of D_2 locally given by equation $z_U^2 + a_U z + b_U = 0$, where (a_U) defines a section of $\mathcal{L} \cong \mathcal{O}_{\mathsf{D}_2}(e_0 - e_3 - e_5)$ vanishing on $C_1 + C_2$ and (b_U) define a section of $\mathcal{L}^{\otimes 2}$ vanishing on $C + C_1 + C_2$. At each point q_1, q_2, the local equation of the cover is $z^2 + uvz + uv(u + v) = 0$. So, the cover has two singular points of type $D_4^{(1)}$ over q_1, q_2. After we resolve them, we find that the dual of the intersection graph of the eight exceptional curves and the pre-images of C_1 and C_2 coincides with the diagram given in Theorem 8.2.21 (a). The additional two (-2)-curves are the pre-images of Q_1 and Q_2. Note that the elliptic fibration of type $2\tilde{A}_7 + \tilde{A}_1$ is of type $2\tilde{A}_7 + \tilde{A}_1^*$.

Note that we can obtain the K3-cover of S as an Artin–Schreier cover of the quadric $\mathbf{F}_0$ as in [368, 4.1]. The surface D_2 is the quotient of $\mathbf{F}_0$ by the involution σ that acts as $(x, y) \mapsto (x^{-1}, y^{-1})$ in affine toric coordinates on $\mathbf{F}_0$. The pre-image of the pencil $|e_0 - e_2|$ on $\mathbf{F}_0$ is the pencil of conics that are tangent at the unique fixed point $(1, 1)$ of σ. Each passes through two opposite vertices of the triangle T of lines equal to the pre-image of $C_1 + C_2$. The branch curve is the union of T and the pre-image of C. We get the picture as in Figure 8.2. The only difference is that two conics Q_1, Q_2 are tangent at one point.

Example 8.10.4 Recall that, in characteristic $p \neq 2$, a surface of Type II can be obtained as a double cover of D_1 branched along the union of two conics C_1, C_2, as in the previous example, and a hyperplane section $C \in |-K_{\mathsf{D}_1}|$ that is tangent to C_1 and C_2 at points q_3 and q_4 lying on two non-skew lines L_1, L_2 on D_1.

To extend this construction to characteristic two, we consider, again, as in the previous example, the surface D_2. We take the split Artin–Schreier cover of D_2 locally given by $z^2 + a_u z + b_u = 0$, where (a_U) is a section of $\mathcal{L}$ vanishing on C and (b_U) is a section of $\mathcal{L}^{\otimes 2}$ vanishing on C_1 and C_2. Locally at the tangency points q_1, q_2 of C with C_1 and C_2, the local equation is $z^2 + zu + (u + v^2)u = 0$. In the first case, by the change of coordinates

$$t = z + \omega u + v^2, \quad s = z + \omega^2 u + v^2, \quad v = v$$

($\omega^3 = 1$, $\omega \neq 1$), then we have $v^4 + ts = 0$ which gives a rational double point of type A_3. This gives two singular points of type A_3 as in the case of the cover in characteristic $p \neq 2$. On the other hand, at a point of intersection of C_1 with C_2, we have a local equation $z^2 + uv = 0$. This gives two singular points of type A_1, again as in the case $p \neq 2$. Comparing with Figure 8.3, we find that the proper transform of L_1 (resp. L_2) on S is R_9 (resp. R_5). The proper transform of C_1 (resp. C_2) is R_3 (resp. R_5). The pre-image of the exceptional curve at q_1 (resp. q_2) is $R_1 + R_2 + R_{10}$ (resp. $R_6 + R_7 + R_{12}$). The pre-images of the exceptional curves at q_3, q_4 are R_4 and R_{11}. So, , all (-2)-curves are accounted for. As in the case $p \neq 2$, a choice of C_1, C_2, C depends on one parameter.

We can also find the lift of the Artin–Schreier cover $S \to \mathsf{D}_2$ to an Artin–Schreier cover $X \to \mathbf{F}_0$, where $X \to S$ is the K3-cover of S as it is constructed in [368]. The pre-images of curves C_1, C_2, L_1, L_2, C are exhibited in Figure 8.9.4.

Example 8.10.5 To construct a $\boldsymbol{\mu}_2$-surface with a crystallographic basis of type VI we use the same construction of this surface as in characteristic $p \neq 2, 3, 5$. Although the Hessian surface of the Clebsch cubic surface degenerates, we can still consider the corresponding quartic surface given by equations

$$t_0 + \cdots + t_4 = \sum_{i=0}^{4} t_i^{-1} = 0.$$

The standard Cremona involution inversing the coordinates acts freely on this surface and the quotient is an Enriques surface. We discover the same configuration of (-2) curves on it forming a crystallographic basis of type VI.

In Table 8.12, we summarize the existence or non-existence of an Enriques surface with the dual graph of type I, . . . , VII. In this table, ◯ means the existence and × means the non-existence

Type	I	II	III	IV	V	VI	VII
$\boldsymbol{\mu}_2$-surface	◯	◯	×	×	×	◯	×
classical	×	×	×	×	×	×	◯
$\boldsymbol{\alpha}_2$-surface	×	×	×	×	×	×	◯

Table 8.12 Surfaces of types I-VII in characteristic 2

The only lacking justification of this table is the existence of classical and $\boldsymbol{\alpha}_2$-surfaces with a crystallographic basis of type VII.

In any characteristic, except for classical and $\boldsymbol{\alpha}_2$-Enriques surfaces in characteristic 2, Enriques surfaces of type I and II form a 1-dimensional family and each of the other types is unique. On the other hand, classical and $\boldsymbol{\alpha}_2$-Enriques surfaces of

type VII form a 1-dimensional family in which a special fiber is an $\boldsymbol{\alpha}_2$-surface and all other fibers are classical.

In the following, we introduce the classification of classical and $\boldsymbol{\alpha}_2$-Enriques surfaces with finite automorphism group. Since any double fiber of a genus one fibration on a classical or $\boldsymbol{\alpha}_2$-Enriques surface is of additive type, contrary to the case of characteristic $p \neq 2$ or the case of a $\boldsymbol{\mu}_2$-Enriques surface, possible crystallographic bases of (-2)-curves are completely different. Only the dual graph of type VII is common between classical/$\boldsymbol{\alpha}_2$-surfaces and the other cases. On the other hand, in the case of classical and $\boldsymbol{\alpha}_2$-surfaces, the canonical covers have singularities which are invariants. We use the classification of possible conductrices due to Ekedahl and Shepherd-Barron [213]. We take a possible conductrix and fix it, and then determine the possible crystallographic basis of (-2)-curves by using Proposition 8.9.1. Thus, we have the following:

Theorem 8.10.6 *Let X be an $\boldsymbol{\alpha}_2$-Enriques surface in characteristic* 2.

1. *X has a finite group of automorphisms if and only if the dual graph of all (-2)-curves on X is one of the graphs in Table 8.13*[3].
2. *There exists an example of an $\boldsymbol{\alpha}_2$-Enriques surface or a family of $\boldsymbol{\alpha}_2$-Enriques surfaces with finite automorphism group of each type. Table 8.13 also gives the automorphism groups and the dimensions of the families.*

Theorem 8.10.7 *Let X be a classical Enriques surface in characteristic* 2.

1. *X has a finite group of automorphisms if and only if the dual graph of all (-2)-curves on X is one of the graphs in Table 8.14.*
2. *There exists an example of a family of classical Enriques surfaces with finite automorphism group of each type. Table 8.14 also gives the automorphism groups and the dimensions of the families.*

Remark 8.10.8 In [371], the authors only gave examples, no classification was known. In particular, the last three columns were the dimensions and the automorphism groups of the examples. Very recently, Katsura and Schütt [372] have shown that no other such Enriques surfaces exist, that is, the above Tables give the classification. The outline of their proof is as follows. They give the classification of the normal forms of quasi-elliptic fibrations on Enriques surfaces. For each type of the dual graphs except three cases $\tilde{E}_6 + \tilde{A}_2$, VII and VIII, there exists a quasi-elliptic fibration on such an Enriques surface which determines the dual graph of all (-2)-curves on it. Thus, the normal form confirms that the dimension of the families of such Enriques

[3] In [371], a surface of type $\tilde{E}_7^{(1)}$ is called of type $\tilde{E}_7 + \tilde{A}_1^{(2)}$ and the one of type $\tilde{E}_7^{(2)}$ is called of type $\tilde{E}_7 + \tilde{A}_1^{(1)}$.

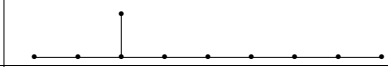
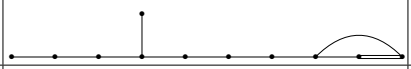
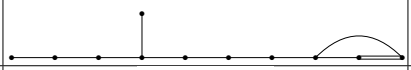
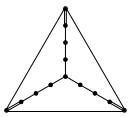
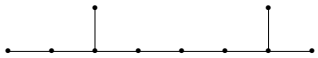
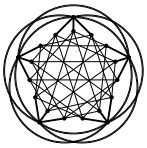
Type	Dual Graph of (-2)-curves	$\mathrm{Aut}(X)$	$\mathrm{Aut}_{\mathrm{nt}}$	dim
$\tilde{E}_8$		$\mathbb{Z}/11\mathbb{Z}$	$\mathbb{Z}/11\mathbb{Z}$	0
$\tilde{E}_7^{(2)}$		$\mathbb{Z}/2\mathbb{Z}$	$\{1\}$	1
$\tilde{E}_7^{(2)}$		$\mathbb{Z}/14\mathbb{Z}$	$\mathbb{Z}/7\mathbb{Z}$	0
$\tilde{E}_6 + \tilde{A}_2$		$\mathbb{Z}/5\mathbb{Z} \rtimes \mathfrak{S}_3$	$\mathbb{Z}/5\mathbb{Z}$	0
$\tilde{D}_8$		Q_8	Q_8	1
VII		$\mathfrak{S}_5$	$\{1\}$	0

Table 8.13 α_2-Enriques surfaces with finite automorphism group

surfaces is the same as that of the example, and hence the example is complete. For type VII, this Enriques surface has only elliptic fibrations and the canonical covering is the supersingular K3 surface X with Artin invariant one. The second author [418] classified Enriques surfaces whose canonical coverings are isomorphic to X. This classification implies that the example of type VII is also complete. Finally, in case of type VIII or of type $\tilde{E}_6 + \tilde{A}_2$, Katsura and Schütt start from the normal form of a quasi-elliptic fibration and then determine a subfamily of Enriques surfaces with finite automorphism group which turns out to be the one given in the example.

Also, we remark that the examples of α_2-Enriques surfaces of type $\tilde{E}_7^2$ form a one-dimensional family with a parameter $a \in k^*$, but their automorphism may become larger when $a^7 = 1$. Thus, this family is non-isotrivial. The family of classical Enriques surfaces of type $\tilde{E}_7^2$ is non-isotrivial because these surfaces specialize to classical Enriques surfaces of type $\tilde{E}_7^1$. The families of classical Enriques surfaces of type $\tilde{E}_6 + \tilde{A}_2$, of type VII, of type VIII and of type $\tilde{D}_4 + \tilde{D}_4$ are also non-isotrivial. In the first three cases, Enriques surfaces have an elliptic fibration with a double fiber F such that the j-invariant of F varies. In the last case, their Jacobian quasi-elliptic fibrations are non-isotrivial. For the remaining cases, non-isotriviality is not known.

We refer for the proof of the previous theorems to [371]. Here, we only give examples of the families of classical and α_2-surfaces in each case from the Theorems 8.10.6, 8.10.7. The idea of the construction of the examples is as follows. First, we take a special genus one fibration on the dual graphs and consider a genus one fibration π on a rational surface with the same singular fibers. Then, after taking the

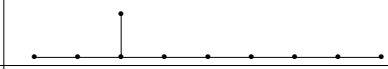
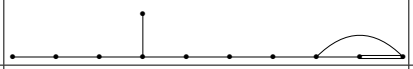
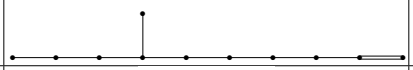
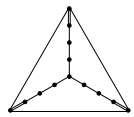
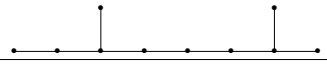
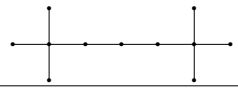
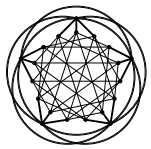
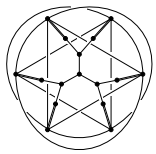
Type	Dual Graph of (-2)-curves	$\mathrm{Aut}(X)$	$\mathrm{Aut}_{\mathrm{nt}}$	dim
$\tilde{E}_8$		$\{1\}$	$\{1\}$	1
$\tilde{E}_7^{(2)}$		$\mathbb{Z}/2\mathbb{Z}$	$\{1\}$	2
$\tilde{E}_7^{(1)}$		$\mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}$	1
$\tilde{E}_6+\tilde{A}_2$		$\mathfrak{S}_3$	$\{1\}$	1
$\tilde{D}_8$		$\mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}$	2
$\tilde{D}_4+\tilde{D}_4$		$(\mathbb{Z}/2\mathbb{Z})^{\oplus 3}$	$(\mathbb{Z}/2\mathbb{Z})^{\oplus 2}$	2
VII		$\mathfrak{S}_5$	$\{1\}$	1
VIII		$\mathfrak{S}_4$	$\{1\}$	1

Table 8.14 Classical Enriques surfaces with finite automorphism group

Frobenius base change of π, we get a new genus one fibration $\tilde{\pi}$. We define a suitable rational derivation on the total space of $\tilde{\pi}$ whose quotient is birational to an Enriques surface with the desired configuration of (-2)-curves.

Example 8.10.9 : **Type** VII**.**

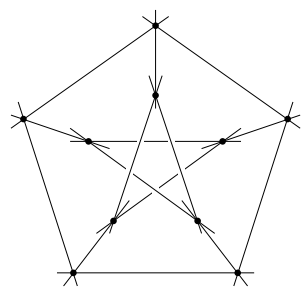
First we consider surfaces of type VII. Recall that an Enriques surface with a crystallographic basis of (-2)-curves of type VII has an elliptic fibration with simple reducible fibers of type $\tilde{A}_4+\tilde{A}_4$ and with ten special bisections. For example, the linear system

$$|E_1+E_2+E_{10}+E_{12}+E_9| = |E_4+E_5+E_6+E_7+E_{13}|$$

gives such a fibration with bisections $E_3, E_8, E_{11}, E_{14}, E_{15}, K_1, ..., K_5$ in Figure 8.10.

Now, assume that a classical or $\boldsymbol{\alpha}_2$-Enriques surface S in characteristic 2 contains fifteen (-2)-curves which are represented by the lines in Figure 8.10.1. Each (-2)-

curve passes through two dotted points (we will discuss this fact in Lemma 10.2.9 later). Note that there exist two disjoint pentagons of (-2)-curves which form two singular fibers of type $\tilde{A}_4$ of an elliptic fibration $f : S \to \mathbb{P}^1$, and the remaining five (-2)-curves are bisections of this fibration. The dual graph of these fifteen (-2)-curves on S coincides with the one of $E_1, ..., E_{15}$ in Figure 8.10.



(8.10.1)

The fibration f has simple singular fibers of type $\tilde{A}_4 + \tilde{A}_4 + \tilde{A}_0^* + \tilde{A}_0^*$ (see 4.7 in Volume I) and the twelve singular points of singular fibers are the images of the twelve ordinary double points of the canonical cover $\pi : X \to S$ (we will prove this later in Corollary 10.2.7). Assume that there exist additionally five (-2)-curves which are special bisections of the fibration. Moreover, we assume that they pass through the singular points of singular fibers of type $\tilde{A}_0^*$ and intersect transversely at these two points (see Lemma 10.2.9). The dual graph of the five (-2)-curves coincides with the complete graph (with double edges) in Figure 8.10 defined by (-2)-curves $K_1, ..., K_5$. Thus, if the additional five (-2)-curves meet with fifteen (-2)-curves in Figure 8.10.1 correctly, then we have an Enriques surface with the crystallographic basis of type VII.

Now, let Y be the minimal resolution of singularities of the canonical cover X of S and R_i, $1 \le i \le 12$, the exceptional curves. Then, the fibration f induces an elliptic fibration $p : Y \to \mathbb{P}^1$ which has two singular fibers of type $\tilde{A}_9$ and two of type $\tilde{A}_1$, and the curves R_i are irreducible components of these singular fibers. Moreover, p has ten sections which are pull-backs of ten bisections of f. Then, the Shioda–Tate formula implies that Y is the supersingular K3 surface with Artin invariant 1. Instead of the canonical cover X of S, we consider a rational divisorial derivation ∂ on Y such that the quotient Y^∂ of Y by ∂ is the blow-up of twelve points on S which are the images of singular points of type A_1 on X. We denote by $\varphi : Y^\partial \to S$ the contraction morphism.

$$\begin{array}{ccc} Y & \longrightarrow & X \\ \pi \downarrow & & \downarrow \\ Y^\partial & \xrightarrow{\varphi} & S \end{array}$$

Since K_Y and K_S are numerically trivial, Proposition 0.3.14 implies that

$$0 = K_Y = \pi^*(K_{Y\partial}) + D = \pi^*(\varphi^*(K_S) + \sum_{i=1}^{12} R_i) + D = \pi^*\varphi^*(\sum R_i) + D, \quad (8.10.2)$$

where D is the divisor of the derivation ∂. Since $\pi(R_i)^2 = -1$, by Proposition 0.3.19, R_i is integral with respect to ∂. Therefore $\pi^*(\pi(R_i)) = R_i$ and hence ∂ has poles along each R_i with multiplicity 1. So, we should first find an elliptic fibration $p : Y \to \mathbb{P}^1$ on the supersingular K3 surface Y with the Artin invariant equal to 1 which has singular fibers of type $\tilde{A}_9 + \tilde{A}_9 + \tilde{A}_1 + \tilde{A}_1$ and ten sections. Next, find a rational divisorial vector field ∂ on Y such that twelve disjoint (-2)-curves R_i in fibers are integral and $D = -\sum_{i=1}^{12} R_i$. In Chapter 10, we will discuss Enriques surfaces in characteristic 2 whose canonical covers are birational to supersingular K3 surfaces. For the divisor D, see Lemma 10.3.13.

To get such an Enriques surface, we start with a rational elliptic surface with Weierstrass equation

$$y^2 + sxy + y + x^3 + x^2 + s = 0.$$

The change of variables $(x, y, s) \mapsto (x + t + 1, y, s = t + 1)$ transforms this equation to the Weierstrass equation from Table 4.7 for an elliptic fibration with two fibers of type $\tilde{A}_4$ and two irreducible singular fibers $\tilde{A}_0^*$. We see that the discriminant is equal to $(s + 1)^5(s^2 + s + 1)$ and the j-invariant is equal to $\frac{s^{12}}{(s+1)^5(s^2+s+1)}$. It has two reducible fibers of type $\tilde{A}_4$ over $s = 1, \infty$ and two singular fibers of type $\tilde{A}_0^*$ over $s = \omega, \omega^2$, where $\omega^3 = 1, \omega \neq 1$. By taking the Frobenius base change $t^2 = s$, we get an elliptic fibration with Weierstrass equation

$$y^2 + t^2xy + y + x^3 + x^2 + t^2 = 0. \quad (8.10.3)$$

Let $p : Y \to \mathbb{P}^1$ be the relatively minimal non-singular model of the fibration (8.10.3). The discriminant of the elliptic surface Y is equal to $h = (t + 1)^{10}(t^2 + t + 1)^2$ and the j-invariant is given by $j = t^{24}/(t + 1)^{10}(t^2 + t + 1)^2$.

Hence, the elliptic fibration p has two reducible fibers of type $\tilde{A}_9$ over $t = 1, \infty$ and two singular fibers of type $\tilde{A}_1$ over $t = \omega, \omega^2$. The relatively minimal elliptic surface $p : Y \to \mathbb{P}^1$ defined by (8.10.3) has ten sections s_i, m_i $(i = 0, 1, 2, 3, 4)$ given as follows:

s_0 : the zero section
$s_1 : x = 1, y = t^2,$
$s_2 : x = t^2, y = t^2,$
$s_3 : x = t^2, y = t^4 + t^2 + 1,$
$s_4 : x = 1, y = 1,$
$m_0 : x = \frac{1}{t^2}, y = \frac{1}{t^3} + \frac{1}{t^2} + t,$
$m_1 : x = t^3 + t + 1,\ y = t^4 + t^3 + t,$
$m_2 : x = t,\ y = t^3,$
$m_3 : x = t,\ y = 1,$
$m_4 : x = t^3 + t + 1,\ y = t^5 + t^4 + t^2 + t + 1.$

Then, the Shioda–Tate formula implies that Y is the supersingular K3 surface with Artin invariant 1.

The Weierstrass model W is singular at the point $P = (x, y, t) = (1, 1, 1)$ and the fiber F_1 over $t = 1$ is an irreducible nodal curve. The singular points is a rational double points of type A_8 and the fiber of the resolution of singularities map $Y \to W$ over this point consists of nine (-2) curves $E_{1,i}$ $(i = 1, 2, \ldots, 9)$. We index the components of the decagon $F_1 E_{1,1} E_{1,2} \ldots E_{1,9}$ in a clockwise manner. Here, we denote by the same symbol F_1 the proper transform of F_1 on Y. The blow-up at the singular point P gives two exceptional curves $E_{1,1}$ and $E_{1,9}$, and they intersect each other at a singular point of the obtained surface. The blow-up at the singular point again gives two exceptional curves $E_{1,2}$ and $E_{1,8}$. The exceptional curve $E_{1,2}$ (resp. $E_{1,8}$) intersects $E_{1,1}$ (resp. $E_{1,9}$) transversely. Exceptional curves $E_{1,2}$ and $E_{1,8}$ intersect each other at a singular point, and so on. By successive blowing-ups, the exceptional curve $E_{1,5}$ finally appears to complete the resolution of singularity at the point P, and it intersects $E_{1,4}$ and $E_{1,6}$ transversely. Summarizing these results, we see that F_1 intersects $E_{1,1}$ and $E_{1,9}$ transversely, and that $E_{1,i}$ intersects $E_{1,i+1}$ $(i = 1, 2, \ldots, 8)$ transversely. We choose $E_{1,1}$ to be the irreducible component which intersects the section m_2.

Let F_∞ be the fiber over the point defined by $t = \infty$. The open subset of W over this point is given by the equation

$$y^2 + xy + t'^6 y + x^3 + t'^4 x^2 + t'^{10} = 0,$$

where $t' = 1/t$. Similarly, we have nine exceptional curves $E_{\infty,i}$ $(i = 1, 2, \ldots, 9)$ over the singular point $P_\infty = [x, y, t'] = [0, 0, 0]$ of the surface (8.10.3), and F_∞ and these nine exceptional curves make a decagon $F_\infty E_{\infty,1} E_{\infty,2} \ldots E_{\infty,9}$ clockwisely. F_∞ intersects $E_{\infty,1}$ and $E_{\infty,9}$ transversely, and that $E_{\infty,i}$ intersects $E_{\infty,i+1}$ $(i = 1, 2, \ldots, 8)$ transversely.

The singular fiber of $p : Y \to \mathbb{P}^1$ over the point defined by $t = \omega$ (resp. $t = \omega^2$) consists of two irreducible components F_ω and E_ω (resp. F_{ω^2} and E_{ω^2}), where F_ω (resp. F_{ω^2}) is the proper transform of the fiber over the point $t = \omega$ (resp. $t = \omega^2$).

Then, the 10 sections above intersect singular fibers of elliptic surface $p : Y \to \mathbb{P}^1$ as in Table 8.15 below.

sections	s_0	s_1	s_2	s_3	s_4	m_0	m_1	m_2	m_3	m_4
$t = 1$	F_1	$E_{1,8}$	$E_{1,6}$	$E_{1,4}$	$E_{1,2}$	$E_{1,5}$	$E_{1,3}$	$E_{1,1}$	$E_{1,9}$	$E_{1,7}$
$t = \infty$	F_∞	$E_{\infty,6}$	$E_{\infty,2}$	$E_{\infty,8}$	$E_{\infty,4}$	$E_{\infty,5}$	$E_{\infty,1}$	$E_{\infty,7}$	$E_{\infty,3}$	$E_{\infty,9}$
$t = \omega$	F_ω	F_ω	F_ω	F_ω	F_ω	E_ω	E_ω	E_ω	E_ω	E_ω
$t = \omega^2$	F_{ω^2}	F_{ω^2}	F_{ω^2}	F_{ω^2}	F_{ω^2}	E_{ω^2}	E_{ω^2}	E_{ω^2}	E_{ω^2}	E_{ω^2}

Table 8.15 Intersection between sections and fibers

Now, consider a rational derivation $\partial = \partial_{\alpha,\beta}$ defined by

$$\partial = \frac{1}{(t+1)} \left((t+1)(t+\alpha)(t+\beta) \frac{\partial}{\partial t} + (1 + t^2 x) \frac{\partial}{\partial x} \right),$$

where $\alpha, \beta \in \Bbbk, \alpha + \beta = \alpha\beta, \alpha \neq 1$. Note that the invariant differential form ω on an elliptic curve given in the Weierstrass form $y^2 + a_1 y + a_3 y + x^3 + a_2 x^2 + a_4 x + a_6 = 0$ is given by

$$\omega = \frac{dx}{2y - a_1 x + a_3} = \frac{dy}{a_1 y + 3x^2 + 2a_2 x + a_4}.$$

This shows that the restriction of the second summand δ_2 of ∂ to the general fiber coincides with the invariant differential form on it. So, we modify this form by adding the first summand δ_1.

Lemma 8.10.10 *(i) $\partial^2 = \alpha\beta\partial$, namely, ∂ is 2-closed and ∂ is of additive type if $\alpha = \beta = 0$ and of multiplicative type otherwise.*
(ii) On the surface Y, the divisorial part D of ∂ is given by

$$D = -\left(F_1 + F_\infty + \sum_{i=1}^{4}(E_{1,2i} + E_{\infty,2i}) + E_\omega + E_{\omega^2}\right),$$

and $D^2 = -24$.
(iii) The integral curves with respect to ∂ in the fibers of $p : Y \longrightarrow \mathbb{P}^1$ are the following: the smooth fibers over $t = \alpha, \beta$ (in the case $\alpha = \beta = 0$, the smooth fiber over $t = 0$) and

$$F_1, F_\infty, E_{1,2i},\ E_{\infty,2i}\ (1 \leq i \leq 4),\ E_\omega,\ E_{\omega^2}.$$

Proof We check this statement only on the complement of the fiber over ∞. We need to blow up the singular point. To do this, first change the parameters $u = t + 1, x = X + 1, y = Y + 1$, so the equation of the Weierstrass surface W becomes

$$F = Y^2 + XY(u^2 + 1) + u^2 Y + X^3 + u^2 X = 0.$$

The singular point now is $(0, 0, 0)$. Blowing up this point, we introduce new coordinates $X = uX', Y = uY'$ and get a new equation

$$F' = X'Y' + Y'^2 + u^2 X'Y' + u(X'^3 + X' + Y') = 0.$$

We see that the exceptional divisor $u = 0$ is given by $Y'(X' + Y') = 0$ and consists, as was observed before, of two components intersecting at a singular point. In new coordinates,

$$\frac{\partial}{\partial t} = \frac{\partial}{\partial u}, \quad \frac{\partial}{\partial X} = u^{-1}\frac{\partial}{\partial X'}$$

and $\frac{1}{t+1}(1 + t^2 x^2)\frac{\partial}{\partial x}$ transforms to $\epsilon u^{-1}\frac{\partial}{\partial X'}$, where ϵ is a unit in a neighborhood of the new singular point. Differentiating F' in X', we find that it has a pole of order 1 at the component given by $Y = 0$. We choose this component for $E_{1,2}$. Multiplying ∂ by u we see that ∂ is equivalent to a vector field whose normal component along the component $F_1, E_{1,2i}, E_\omega, E_{\omega^2}$ is equal to zero. Thus, these components are integral curves of ∂.

Similarly, we deal with singular points $(\omega, 1, \omega)$ (and its conjugate) and find that D has a pole of order 1 on the exceptional curve of the minimal resolution. □

By Proposition 0.3.18 from Volume I, we have

$$c_2(Y) = \deg\langle Z\rangle - K_Y \cdot D - D^2,$$

where $\langle Z\rangle$ is the scheme of non-divisorial zeros of ∂. Using the facts $c_2(Y) = 24, K_Y \cdot D = 0, D^2 = -24$, we obtain $Z = 0$ and the quotient surface $Y^{\partial_{\alpha,\beta}}$ is non-singular (Theorem 0.3.9). It has twelve exceptional curves of the first kind which are the images of the above integral (-2)-curves. By contracting these curves we get a non-singular surface $S_{\alpha,\beta}$. It follows from the formula (8.10.2) that K_S is numerically trivial. Since the quotient morphism $Y \to Y^{\partial}$ is finite and purely inseparable, $b_2(Y^{\partial}) = b_2(Y) = 22$ and hence $b_2(S_{\alpha,\beta}) + 12 = 22$. Thus, $S_{\alpha,\beta}$ is an Enriques surface.

The elliptic fibration $p : Y \to \mathbb{P}^1$ induces an elliptic fibration $f : S_{\alpha,\beta} \to \mathbb{P}^1$, it has two singular fibers of type $\tilde{A}_4$ and two singular fiber of type $\tilde{A}_0^*$. If $\alpha \neq 0$, then the images of two smooth integral curves are double fibers of the elliptic fibration, and hence $S_{\alpha,\beta}$ is classical. If $\alpha = \beta = 0$, then the fibration has one double fiber, and hence $S_{\alpha,\beta}$ is an $\boldsymbol{\alpha}_2$-surface (it can not be a $\boldsymbol{\mu}_2$-surface because the canonical cover is supersingular). The surface $S_{\alpha,\beta}$ contains 20 (-2)-curves which are the images of ten components of the fibers of p over $t = 0, \infty$ and ten sections. The incidence relation between components of fibers of p and sections given in Table 8.15 shows that the dual graph of 20 (-2)-curves on $S_{\alpha,\beta}$ is the crystallographic basis of type VII. There exist twelve points on $S_{\alpha,\beta}$ which are the images of twelve integral (-2)-curves. Each (-2)-curve on $S_{\alpha,\beta}$ passes through two points from the twelve points (see Figure 8.10.1).

Thus, we have proved the following theorem:

Theorem 8.10.11 *There exists a* 1*-dimensional family* $\{S_{\alpha,\beta}\}$ *of classical and* $\boldsymbol{\alpha}_2$*-Enriques surfaces* $S_{\alpha,\beta}$*, where* $\alpha, \beta \in k$*,* $\alpha + \beta = \alpha\beta$*,* $\alpha^3 \neq 1 \neq \beta^3$*. The K3-cover of* $S_{\alpha,\beta}$ *has twelve nodes and the resolution of singularities is the supersingular K3 surface* Y *with Artin invariant* 1*. If* $\alpha = \beta = 0$*, then* $S_{\alpha,\beta}$ *is an* $\boldsymbol{\alpha}_2$*-surface, and otherwise classical. Each* $S_{\alpha,\beta}$ *contains* 20 (-2)*-curves whose dual graph coincides with the crystallographic basis of type* VII*. In particular, the automorphism group* $\mathrm{Aut}(S_{\alpha,\beta})$ *is finite and isomorphic to* $\mathfrak{S}_5$*.*

The next proposition is an extension of Proposition 8.9.28 to characteristic 2.

Proposition 8.10.12 *There exist exactly four types of elliptic fibrations on* S *as follows:*

$$\tilde{A}_4 + \tilde{A}_4 + \tilde{A}_0^* + \tilde{A}_0^*, \quad \tilde{A}_5 + \tilde{A}_1 + 2\tilde{A}_2^*, \quad \tilde{A}_7 + 2\tilde{A}_1^*, \quad \tilde{A}_8 + \tilde{A}_0^* + \tilde{A}_0^* + \tilde{A}_0^*.$$

Next, we discuss the problem of the existence of crystallographic bases of (-2)-curves of different types. The known examples are the crystallographic bases of extra-special Enriques surfaces of types $\tilde{E}_8, \tilde{D}_8$ and $\tilde{E}_7$ with $\mathsf{nd}(S) \leq 2$ from Theorem 6.2.3.

Example 8.10.13 **Type $\tilde{E}_6 + \tilde{A}_2$: α_2-surfaces.**

We consider a rational elliptic surface $\pi : R \to \mathbb{P}^1$ with the Weierstrass model

$$y^2 + sy = x^3.$$

It has a singular fiber of type $\tilde{A}_2^*$ over $s = 0$ and of type $\tilde{E}_6$ over $s = \infty$. By taking the Frobenius base change $t^2 = s$, we have a rational elliptic surface $\tilde{\pi} : \tilde{R} \to \mathbb{P}^1$ with the Weierstrass model

$$y^2 + t^2 y = x^3.$$

It has a singular fiber of type $\tilde{E}_6$ over $t = 0$ and of type $\tilde{A}_2^*$ over $t = \infty$. Note that π and $\tilde{\pi}$ are isomorphic by changing the coordinate $x' = x/s^2,\ y' = y/s^3,\ s = 1/t$. Let $E_{0,1} + E_{0,2} + E_{0,3} + 2(E_{0,4} + E_{0,5} + E_{0,6}) + 3E_{0,7}$ be the singular fiber of $\tilde{\pi}$ of type $\tilde{E}_6$ and let $E_{\infty,1} + E_{\infty,2} + E_{\infty,3}$ be the singular fiber of type $\tilde{A}_2^*$. We assume that $E_{0,i}$ meets with $E_{0,i+3}$ $(i = 1, 2, 3)$. There exist three sections denoted by S_1, S_2, S_3 which do not intersect. We assume that S_i intersects $E_{0,i}$ and $E_{\infty,i}$.

Now, we discuss what kind of a derivation we need. Assume that $\tilde{R}$ is birational to a resolution of the normalization of the canonical cover of the desired Enriques surface S. Then, $\tilde{\pi}$ should induce a special elliptic fibration on S with singular fibers of type $\tilde{A}_2^*$ over the point corresponding to $s = 0$ and of type $\tilde{E}_6$ over the point corresponding to $s = \infty$. This means that four components of the fiber of type $\tilde{E}_6$ over $t = 0$ map to a point on S to get a singular fiber of type $\tilde{A}_2$ and the fiber of $\tilde{A}_2^*$ over $t = \infty$ should be blowing up to get a fiber of type $\tilde{E}_6$ on S. So, , we first blow up $\tilde{R}$ at the singular point of the singular fiber of type $\tilde{A}_2^*$ and denote by $E_{\infty,4}$ the exceptional curve. Then, blow up the three points which are the intersection of $E_{\infty,4}$ and the proper transforms of three components of the original fiber. Denote by $E_{\infty,5}$, $E_{\infty,6}$, $E_{\infty,7}$ the three exceptional curves and by the same symbols $E_{\infty,1}$, $E_{\infty,2}$, $E_{\infty,3}$ their proper transforms under these blow-ups. We assume that $E_{\infty,i}$ meets $E_{\infty,i+4}$ $(i = 1, 2, 3)$. Then,

$$E_{\infty,1}^2 = E_{\infty,2}^2 = E_{\infty,3}^2 = E_{\infty,4}^2 = -4, \quad E_{\infty,5}^2 = E_{\infty,6}^2 = E_{\infty,7}^2 = -1.$$

Also,

$$E_{0,i}^2 = -2 \ (i = 1, 2, \ldots, 7), \quad S_1^2 = S_2^2 = S_3^2 = -1.$$

We denote by Y the surface obtained by these blowing ups.

Now, assume that there exists a derivation ∂ on Y without isolated zeros such that the integral curves with respect to ∂ are given by

$$E_{0,4},\ E_{0,5},\ E_{0,6},\ E_{\infty,1},\ E_{\infty,2},\ E_{\infty,3},\ E_{\infty,4}. \tag{8.10.4}$$

Then, by taking the quotient by ∂ we get a smooth surface Y^∂ and a purely inseparable double covering $p : Y \to Y^\partial$. Since $2(p(C))^2 = C^2$ if C is integral and $(p(C))^2 = 2C^2$ otherwise, we have

$$E_{\infty,i}^2 = S_j^2 = -2 \ \ (i = 1, 2, \ldots, 7; j = 1, 2, 3),$$

$$E_{0,k}^2 = -4 \ \ (k = 1, 2, 3, 7), \ \ E_{0,l}^2 = -1 \ \ (l = 4, 5, 6).$$

Now, contract three (-1)-curves $E_{0,4}, E_{0,5}, E_{0,6}$ and then contract the image of $E_{0,7}$. Then, we have a smooth surface S and the contraction map $q : Y^\partial \to S$. By construction, the corresponding elliptic fibration has a singular fiber of type $\tilde{A}_2^*$ consisting of the images of $E_{0,1}, E_{0,2}, E_{0,3}$ and a singular fiber of type $\tilde{E}_6$. Since $S_i, E_{0,i}, i = 1, 2, 3$ are non-integral and $E_{\infty,i}$, $i = 1, 2, 3$ are integral, the images of S_i are tangent to $E_{0,i}$ at one point and S_i meets $E_{\infty,i}$ at one point transversally. Therefore, S_i are bisections of the fibration, the fiber of type $\tilde{E}_6$ is double and the one of type $\tilde{A}_2^*$ is simple. Thus, we have 13 (-2)-curves whose dual graph is a crystallographic basis of type $\tilde{E}_6 + \tilde{A}_2$ in Theorem 8.10.6. Since the Euler number of $\tilde{R}$ is 12 and p is purely inseparable double cover, the Euler number of Y and Y^∂ is 16 and hence that of S is 12.

Obviously,

$$K_{Y^\partial} = q^*(K_S) + 2(E_{0,4} + E_{0,5} + E_{0,6}) + E_{0,7}.$$

Since $E_{0,4}, E_{0,5}, E_{0,6}$ are integral and $E_{0,7}$ is not, it follows that

$$K_Y = p^* K_{Y^\partial} + D = p^* q^*(K_S) + 2(E_{0,4} + E_{0,5} + E_{0,6} + E_{0,7}) + D,$$

where D is the divisor of ∂. On the other hand, Y is obtained by blowing ups 4 points of the rational elliptic surface and hence

$$K_Y = -(E_{\infty,1} + E_{\infty,2} + E_{\infty,3}) - 2(E_{\infty,4} + E_{\infty,5} + E_{\infty,6} + E_{\infty,7}).$$

Assume that S is an Enriques surface. Then, $p^* q^*(K_S) = 0$. This gives another condition:

$$D = -2(E_{0,4}+E_{0,5}+E_{0,6}+E_{0,7}+E_{\infty,4}+E_{\infty,5}+E_{\infty,6}+E_{\infty,7}) - (E_{\infty,1}+E_{\infty,2}+E_{\infty,3}). \tag{8.10.5}$$

We conclude that ∂ should satisfy two conditions (8.10.4) and (8.10.5) to get the desired Enriques surface S.

Now, we consider the following derivation:

$$\partial = \frac{\partial}{\partial t} + t^2 \frac{\partial}{\partial x}.$$

Then, $\partial^2 = 0$, that is, ∂ is 2-closed. We denote by the same symbol ∂ the induced derivation on Y. The following lemma follows from local calculations:

Lemma 8.10.14 *(1) The derivation ∂ on Y satisfies the two conditions (8.10.4) and (8.10.5).*

(2) $D^2 = -12, \quad (D, K_Y) = -4.$

Remark 8.10.15 We blow up $\tilde{R}$ at four points. These points are nothing but the isolated singular points of the derivation on $\tilde{R}$ induced by ∂.

Lemma 8.10.16 *The derivation ∂ is divisorial.*

Proof Let Z be the scheme of non-divisorial zeros of ∂. By using the facts $c_2(Y) = 16$ and $D^2 = -12$, $(D, K_Y) = -4$, we have

$$16 = c_2(Y) = \deg(Z) - (K_Y, D) - D^2 = \deg(Z) + 16.$$

Hence, $\deg(Z) = 0$ which means ∂ is divisorial. □

It follows that Y^∂ is smooth. Then, applying the previous argument we can see S is the desired Enriques surface. Since $\partial^2 = 0$, S is an $\boldsymbol{\alpha}_2$-surface.

Theorem 8.10.17 *The surface S is an $\boldsymbol{\alpha}_2$-Enriques surface with a crystallographic basis of type $\tilde{E}_6 + \tilde{A}_2$.*

Proposition 8.10.18 *There are exactly one elliptic fibration with singular fibers of type $2\tilde{E}_6 + \tilde{A}_2^*$ and three quasi-elliptic fibrations with singular fibers of type $\tilde{E}_7 + 2\tilde{A}_1^*$.*

Proof For the elliptic fibration the assertion is clear. Note that both S_1 and $E_{0,1}$ are non-integral and meet at one point transversally on Y, their images on S are tangent together. Twice the sum of these two curves defines a genus one fibration with a double fiber of type $\tilde{A}_1^*$. By the classification of extremal genus one fibrations, this fibration is quasi-elliptic. □

Remark 8.10.19 To construct S we started from the elliptic surface $y^2 + t^2 y = x^3$ and the derivation $\partial = \frac{\partial}{\partial t} + t^2 \frac{\partial}{\partial x}$. Then, we can determine the function field $k(x, y, t)^\partial$ of the quotient surface and have the following equation of the surface birationally isomorphic to S:

$$Y^2 + TY + TX^4 + X^3 + T^3 X + T^7 = 0. \tag{8.10.6}$$

For more details we refer to [371].

Theorem 8.10.20 *The automorphism group* $\mathrm{Aut}(S)$ *is isomorphic to* $\mathfrak{S}_3 \times \mathbb{Z}/5\mathbb{Z}$ *in which* $\mathbb{Z}/5\mathbb{Z}$ *is cohomologically trivial.*

Proof Note that the symmetry group of the dual graph of (-2)-curves is isomorphic to the symmetric group $\mathfrak{S}_3$ of degree 3. It is easy to prove the existence of automorphisms generating $\mathfrak{S}_3$ by using the Jacobian fibrations of genus one fibrations on S. On the other hand, we can determine the automorphism group of the surface defined by 8.10.6 preserving the fibration defined by the projection $(X, Y, T) \to T$ as follows:

$$\varphi : (X, Y, T) \to (\zeta^4 X, \zeta Y, \zeta T), \quad \psi : (X, Y, T) \to (X, Y + T, T)$$

where ζ is a primitive fifth root of unity. Then, φ, ψ and an automorphism of order 3 in $\mathfrak{S}_3$ generate $\mathfrak{S}_3 \times \mathbb{Z}/5\mathbb{Z}$. There are no symmetries of order 5 of the dual graph of (-2)-curves, hence φ is cohomologically trivial. □

Remark 8.10.21 By construction, the canonical cover of S has a rational double point of type D_4. On the other hand, any bisection of a quasi-elliptic fibration on an Enriques surface is contained in the conductrix. Therefore, the conductrix of S

is non-empty. In fact, the support of the conductrix is the union of 7 components of the singular fiber of type $\tilde{E}_6$. In our construction, we take a non-singular model of the normalization of the canonical cover of S, and hence the canonical cover itself does not appear here.

Example 8.10.22 **Type $\tilde{E}_6 + \tilde{A}_2$: Classical Case.**

In this case we start with a rational elliptic surface associated with the Weierstrass model

$$y^2 + xy + sy + x^3 = 0.$$

It has a singular fiber of type $\tilde{A}_2$ over $s = 0$, a singular fiber of type $\tilde{A}_0^*$ over $s = 1$ and a singular fiber of type $\tilde{E}_6$ over $s = \infty$. Then, by taking the Frobenius base change $t^2 = s$ we have a rational elliptic surface $\tilde{\pi} : \tilde{R} \to \mathbb{P}^1$ associated with the Weierstrass equation which has a singular fiber of type $\tilde{A}_5$ over $t = 0$, a singular fiber of type $\tilde{A}_1$ over $t = 1$, and a singular fiber of type $\tilde{A}_2^*$ over $t = \infty$.

Blowing up of the fiber over $t = \infty$ four times, we get a rational surface Y with the same configuration of curves in the fiber over $t = \infty$ as that of the $\boldsymbol{\alpha}_2$-surface. We use the same symbols:

$$E_{\infty,1},\ E_{\infty,2},\ E_{\infty,3},\ E_{\infty,4},\ E_{\infty,5},\ E_{\infty,6},\ E_{\infty,7}.$$

We also denote by $E_{0,1}, ..., E_{0,6}$ the component of the fiber over $t = 0$ such that $E_{0,i} \cdot E_{0,i+1} = 1$ ($i \in \mathbb{Z}/6\mathbb{Z}$), and by $E_{1,1}, E_{1,2}$ the components of the fiber over $t = 1$. We may assume that the three sections S_i meet $E_{0,2i-1}, E_{1,1}, E_{\infty,i}$ ($i = 1, 2, 3$).

Now, we consider the following derivation:

$$\partial_a = (t + a)\frac{\partial}{\partial t} + (x + t^2)\frac{\partial}{\partial x}, \tag{8.10.7}$$

where $a \in k, a \neq 0, 1$. Then, $\partial_a^2 = \partial_a$, that is, ∂_a is 2-closed. By calculations we have the following two lemmas:

Lemma 8.10.23 (1) *The divisor D_a of ∂_a is given by*

$$-(E_{0,2} + E_{0,4} + E_{0,6} + E_{1,2} + E_{\infty,1} + E_{\infty,2} + E_{\infty,3}) - 2(E_{\infty,4} + E_{\infty,5} + E_{\infty,6} + E_{\infty,7}).$$

(2) $D_a^2 = -12, \quad (D_a, K_Y) = -4.$

(3) *The derivation ∂_a has no isolated zeros.*

Lemma 8.10.24 *The integral curves with respect to ∂_a are the smooth fiber over $t = a$ and*

$$E_{0,2},\ E_{0,4},\ E_{0,6},\ E_{1,2},\ E_{\infty,1},\ E_{\infty,2},\ E_{\infty,3},\ E_{\infty,4}.$$

Now, we contract four (-1)-curves on Y^{∂_a}, the images of $E_{0,2},\ E_{0,4},\ E_{0,6},\ E_{1,2}$ and denote the obtained surface by S_a. By combining these two lemmas and by the same argument as in the case of the $\boldsymbol{\alpha}_2$-Enriques surface, we obtain the following theorem.

Theorem 8.10.25 *The surfaces S_a form a 1-dimensional family of classical Enriques surfaces with a crystallographic basis of type $\tilde{E}_6 + \tilde{A}_2$.*

Proof Since the fiber of $\tilde{\pi}$ over $t = a$ is integral, its image on S_a is a double fiber. Thus, the fibration on S_a has two double fibers (another one is the fiber of type $\tilde{E}_6$), S_a is classical. □

The dual graph of (-2)-curves shows that the following proposition holds.

Proposition 8.10.26 *There are exactly one elliptic fibration with singular fibers of type $2\tilde{E}_6 + \tilde{A}_2$ and three quasi-elliptic fibrations with singular fibers of type $\tilde{E}_7 + 2\tilde{A}_1^*$.*

Theorem 8.10.27 *The automorphism group* $\mathrm{Aut}(S_a)$ *is isomorphic to* $\mathfrak{S}_3$.

Proof It suffices to see that there are no numerically trivial automorphisms. If $g \in \mathrm{Aut}(S_a)$ is numerically trivial, it preserves each of 13 (-2)-curves. In particular, since g fixes three points on each irreducible component of the singular fiber of type $\tilde{A}_2$, it fixes each component pointwise. Now, consider a quasi-elliptic fibration p. Since an irreducible component of the fiber of type $\tilde{A}_2$ is a special bisection of p, g acts trivially on the base of p. Since p has two special bisections, g fixes three points on a general fiber F of p and hence g acts trivially on F. □

Example 8.10.28 **Type** VIII**.**

We take a parabolic subdiagram of type $\tilde{D}_5 + \tilde{A}_3$ in the dual graph of type VIII. We consider an extremal rational elliptic fibration $\pi : R \to \mathbb{P}^1$ with the Weierstrass model

$$y^2 + sxy = x^3 + s^2x.$$

It has a singular fiber of type $\tilde{D}_5$ over $s = 0$ and a singular fiber of type $\tilde{A}_3$ over $s = \infty$. By the Frobenius base change $t^2 = s$ we have a rational elliptic fibration $\tilde{\pi} : \tilde{R} \to \mathbb{P}^1$ with the Weierstrass model

$$y^2 + t^2xy = x^3 + t^4x.$$

It has a singular fiber of type $\tilde{A}_1^*$ over $t = 0$ and a singular fiber of type $\tilde{A}_7$ over $t = \infty$. The elliptic fibration $\tilde{\pi}$ has four sections:

$$S_1 : \text{the zero section}, \quad S_2 : x = y = 0, \quad S_3 : x = t, y = 0, \quad S_4 : x = 0, y = t.$$

Moreover, there exist two bisections defined by

$$B_1 : x + y = x^2 + tx + t = 0, \quad B_2 : x + y + tx + t = x^2 + tx + t = 0,$$

both of which pass the singular point of the fiber of type $\tilde{A}_1^*$. We refer the reader to Figure 10 in [371] for the incidence relation between components of fibers and sections, bisections.

To get the dual graph of type VIII, we blow up four points on the fiber over $t = 0$ and contract four disjoint components of the fiber over $t = \infty$. Let $E_{0,1}, E_{0,2}$ be components of the fiber of type $\tilde{A}_1^*$ and let $E_{\infty,1}, \ldots, E_{\infty,8}$ the components of the fiber

of type $\tilde{A}_7$ such that $E_{\infty,i} \cdot E_{\infty,i+1} = 1$ ($i \in \mathbb{Z}/8\mathbb{Z}$). We first blow up the singular point of the fiber of type $\tilde{A}_1^*$, and then blow up at the point of the intersection of the proper transforms of $E_{0,1}$ and $E_{0,2}$. Denote by $E_{0,3}$ the exceptional curve of the first blow-up and by $E_{0,4}$ the exceptional curve of the second blowing-up. We also use the same symbols $E_{0,i}$, S_j, B_k for their proper transforms. Now, three curves $E_{0,1}, E_{0,2}, E_{0,3}$ are disjoint. Note that B_1 and B_2 meet $E_{0,3}$. Blow up the two points $B_1 \cap E_{0,3}$, $B_2 \cap E_{0,3}$. The obtained surface is denoted by Y.

Now, we consider the following derivation:

$$\partial_a = t(at+1)\frac{\partial}{\partial t} + (x+1)\frac{\partial}{\partial x}, \tag{8.10.8}$$

where $a \in k, a \neq 0$. Then, $\partial_a^2 = \partial_a$, that is, ∂_a is 2-closed. By calculations we have the following two lemmas.

Lemma 8.10.29 (1) *The divisor D_a of ∂_a is given by*

$$-(E_{0,1} + E_{0,2} + E_{0,3} + 2E_{0,4} + E_{\infty,2} + E_{\infty,4} + E_{\infty,6} + E_{\infty,8}).$$

(2) $D_a^2 = -12, \quad (D_a, K_Y) = -4.$
(3) *The derivation ∂_a has no isolated zeros.*

Lemma 8.10.30 *The integral curves with respect to ∂_a are the smooth fiber over $t = 1/a$ and*

$$E_{0,1},\ E_{0,2},\ E_{0,3},\ E_{\infty,2},\ E_{\infty,4},\ E_{\infty,6},\ E_{\infty,8}.$$

Let S_a be the surface obtained by contracting four (-1)-curves on Y^{∂_a} which are the images of $E_{\infty,2}$, $E_{\infty,4}$, $E_{\infty,6}$, $E_{\infty,8}$. By combining these two lemmas and by the same argument as in the case of the $\boldsymbol{\alpha}_2$-Enriques surface of type $\tilde{E}_6$, we have:

Theorem 8.10.31 *The surfaces S_a form a one-dimensional family of classical Enriques surfaces with a crystallographic basis of type* VIII *from Table 8.14.*

Proof Since the fiber of $\tilde{\pi}$ over $t = 1/a$ is integral, its image on S is a double fiber. Thus, the fibration on X has two double fibers (another one is the fiber of type $\tilde{D}_5$), so S is classical. □

The following proposition follows from inspecting the dual graph of (-2)-curves and calculations of the intersection numbers.

Proposition 8.10.32 *There are exactly three elliptic fibrations with singular fibers of type $2\tilde{D}_5 + \tilde{A}_3$, three quasi-elliptic fibrations with singular fibers of type $\tilde{D}_6 + 2\tilde{A}_1^* + 2\tilde{A}_1^*$, four elliptic fibrations with singular fibers of type $2\tilde{E}_6 + 2\tilde{A}_2^*$ and four elliptic fibrations with singular fibers of type $\tilde{E}_6 + \tilde{A}_2 + \tilde{A}_0^*$.*

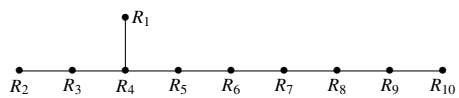
Using an argument similar to one from the proof of Theorem 8.10.27, we have the following:

Theorem 8.10.33 *The automorphism group* $\mathrm{Aut}(S)$ *is isomorphic to* $\mathfrak{S}_4$.

Remark 8.10.34 By construction, the canonical cover of S has four rational double point of type A_1. The support of the conductrix of S is the union of four (-2)-curves each of which appears as a component with multiplicity 2 of a singular fiber of type $\tilde{D}_6$.

Example 8.10.35 **Type $\tilde{E}_8$: α_2-surface.**

Recall that the crystallographic root basis of type $\tilde{E}_8$ has the following Dynkin diagram:



(8.10.9)

On such an Enriques surface S, there exists a unique quasi-elliptic fibration with a bisection R_{10}. First we discuss how to find a rational surface and a derivation on it whose quotient is birational to S. Assume that the canonical cover has a rational double point of type D_4 over the point $P = R_9 \cap R_{10}$. First blow up P and denote the exceptional curve by E_1. Then, blow up the intersection points of E_1 and the proper transforms of R_9 and R_{10} and denote the exceptional curves by E_2 and E_3 respectively. Also, blow up one point on E_1 not lying on R_9 and R_{10} and denote the exceptional curve by E_4. We denote by S' the obtained surface. Assume that a resolution Y of the normalization of the canonical cover of S is an inseparable double cover $\pi : Y \to S'$, that is, $S' = Y^{\partial}$ for a derivation ∂ on Y.

We denote by $\tilde{R}_i$, $\tilde{E}_i$ the pre-images on Y of R_i, E_i. Note that the cycle $\tilde{E}_1 + \tilde{E}_2 + \tilde{E}_3 + \tilde{E}_4$ on Y is the exceptional curve corresponding to the rational double point of type D_4. Since $\tilde{E}_i^2 = -2$ $(i = 1, 2, 3, 4)$ and $E_1^2 = -4$, $E_2^2 = E_3^2 = E_4^2 = -1$, $\tilde{E}_2, \tilde{E}_3, \tilde{E}_4$ should be integral and $\tilde{E}_1$ not with respect to the derivation ∂. We assume that $\tilde{R}_2$, $\tilde{R}_4$, $\tilde{R}_6$ and $\tilde{R}_8$ are integral with respect to ∂, and other $\tilde{R}_i$ not. Then,

$$\tilde{R}_2^2 = \tilde{R}_4^2 = \tilde{R}_6^2 = \tilde{R}_8^2 = -4, \quad \tilde{R}_1^2 = \tilde{R}_3^2 = \tilde{R}_5^2 = \tilde{R}_7^2 = -1, \quad \tilde{R}_9^2 = \tilde{R}_{10}^2 = -2.$$

Now, we contract (-1)-curves and then contract new (-1)-curves except $\tilde{E}_4$ successively. Finally, we get the projective plane $\mathbb{P}^2$. Thus, Y should be obtained from $\mathbb{P}^2$ by blowing up 13 times.

On the other hand, as in the case of the α_2-Enriques surface of type $\tilde{E}_6 + \tilde{A}_2$, the triviality of the canonical bundle K_S implies some conditions on the divisor D of ∂ and K_Y. Assume $K_S = 0$. Then, $K_{S'} = E_1 + 2(E_2 + E_3 + E_4)$. Since $\tilde{E}_1$ is not integral and other $\tilde{E}_i$ are integral, we have

$$K_Y = \pi^* K_{S'} + D = 2(\tilde{E}_1 + \tilde{E}_2 + \tilde{E}_3 + \tilde{E}_4) + D.$$

This holds if

$$D = -(5\tilde{E}_4 + 2\tilde{R}_{10} + 6\tilde{E}_3 + 8\tilde{E}_1 + 7\tilde{E}_2 + 4\tilde{R}_9 + 3\tilde{R}_8 + 2\tilde{R}_2 + 4\tilde{R}_6 + 5\tilde{R}_4 + 6\tilde{R}_7 + 8\tilde{R}_5 + 4\tilde{R}_1 + 6\tilde{R}_3)$$

and

$$K_Y = -(3\tilde{E}_4+2\tilde{R}_{10}+4\tilde{E}_3+6\tilde{E}_1+5\tilde{E}_2+4\tilde{R}_9+3\tilde{R}_8+2\tilde{R}_2+4\tilde{R}_6+5\tilde{R}_4+6\tilde{R}_7+8\tilde{R}_5+4\tilde{R}_1+6\tilde{R}_3).$$

Moreover, we can see that $D^2 = -12$, $K_Y \cdot D = -4$, and hence ∂ has no isolated zeros. Thus, we need to find a rational surface Y and a derivation ∂ on Y satisfying these conditions.

Now, we consider the affine plane $\mathbb{A}^2 \subset \mathbb{P}^2$ with a coordinate (x, y) and the following derivation:

$$\partial = \frac{1}{x^5}\left((xy^6 + x^3)\frac{\partial}{\partial x} + (x^6 + y^7 + x^2y)\frac{\partial}{\partial y}\right).$$

Then, $\partial^2 = 0$, that is, ∂ is 2-closed. Note that ∂ has a pole with order 5 along the line ℓ defined by $x = 0$ and ℓ is integral with respect to ∂. Moreover, ∂ has a unique isolated singularity at $P = (0, 0)$. We blow up at the point P. Then, the exceptional curve is not integral with respect to the induced derivation. The induced derivation has a pole of order 2 along the exceptional curve and has a unique isolated singularity at the intersection of the exceptional curve and the proper transform of ℓ. Continue this process until the induced derivation has no isolated singularities. The resulting surface is denoted by Y which, together with the induced derivation, satisfies the desired conditions. Here, the proper transform of ℓ corresponds to $\tilde{E}_4$ and the exceptional curve of the first blowing-up corresponds to $\tilde{R}_{10}$. Thus, we have proved the following theorem.

Theorem 8.10.36 *The surface S is an $\boldsymbol{\alpha}_2$-Enriques surface with a crystallographic basis of type $\tilde{E}_8$ from Table . There exists a unique quasi-elliptic fibration which has a singular fiber of type $2\tilde{E}_8$.*

Remark 8.10.37 As in the case of the $\boldsymbol{\alpha}_2$-Enriques surface of type $\tilde{E}_6 + \tilde{A}_2$, we can determine the function field $k(x, y)^{\partial}$ of the quotient surface and have the following equation of the surface birationally isomorphic to S:

$$Y^2 + TX^4 + X + T^7 = 0. \tag{8.10.10}$$

Theorem 8.10.38

$$\mathrm{Aut}(S) = \mathrm{Aut}_{ct}(S) \cong \mathbb{Z}/11\mathbb{Z}.$$

Proof Note that the symmetry group of the dual graph of (-2)-curves is trivial and hence any automorphism is cohomologically trivial. We can determine the automorphism group of the surface defined by equation (8.10.10) preserving the fibration defined by the projection $(X, Y, T) \to T$ as follows:

$$\varphi : (X, Y, T) \to (\zeta^7 X, \zeta^9 Y, \zeta T),$$

where ζ is a primitive 11th root of unity. □

Remark 8.10.39 By construction, the canonical cover of S has a rational double point of type D_4. The support of the conductrix is the union of ten (-2)-curves on S.

Example 8.10.40 **Type $\tilde{E}_8$: Classical Case.** A classical Enriques surface of type $\tilde{E}_8$ has the same dual graph as in Figure 8.10.9. In this case, the quasi-elliptic fibration has two double fibers. We denote by F_0 the irreducible one. The difference from the $\boldsymbol{\alpha}_2$-surface is that the canonical cover has 4 rational double points of type A_1 instead of a rational double point of type D_4. So, , we blow up at $F_0 \cap R_{10}$, $R_{10} \cap R_9$, a nonsingular point on F_0 and a point on R_9. We denote by E_1, E_2, E_3, E_4 the exceptional curves, respectively. We use the same notation as in the case of the $\boldsymbol{\alpha}_2$-Enriques surface of type $\tilde{E}_8$. We also assume that $\tilde{E}_i, (1 \leq i \leq 4)$, $\tilde{R}_2$, $\tilde{R}_4$, $\tilde{R}_6$ and $\tilde{R}_8$ are integral with respect to a derivation ∂. The pull-back of F_0 to Y, denoted by $\tilde{F}_0$, is a nonsingular rational curve.

Obviously,

$$\tilde{E}_1^2 = \tilde{E}_2^2 = \tilde{E}_3^2 = \tilde{E}_4^2 = -2, \quad \tilde{R}_2^2 = \tilde{R}_4^2 = \tilde{R}_6^2 = \tilde{R}_8^2 = -4,$$

$$\tilde{F}_0^2 = \tilde{R}_1^2 = \tilde{R}_3^2 = \tilde{R}_5^2 = \tilde{R}_7^2 = -1, \quad \tilde{R}_9^2 = \tilde{R}_{10}^2 = -2.$$

Now, we contract (-1)-curves and then contract new (-1)-curves except $\tilde{E}_3, \tilde{E}_4, \tilde{R}_{10}$ successively. Finally we get a nonsingular quadric surface $\mathbb{P}^1 \times \mathbb{P}^1$. Thus, Y is obtained from $\mathbb{P}^1 \times \mathbb{P}^1$ by blowing up 12 times.

Let us consider the affine open set $\mathbb{A}^1 \times \mathbb{A}^1$ of $\mathbb{P}^1 \times \mathbb{P}^1$ with coordinates (x, y) and the following derivation:

$$\partial_a = \frac{1}{x^3y^2}\left(x^4y^2\frac{\partial}{\partial x} + (x^2 + ax^4y^4 + y^4)\frac{\partial}{\partial y}\right), \quad (a \neq 0 \in k) \tag{8.10.11}$$

We get $\partial_a^2 = \partial_a$, that is, ∂_a is 2-closed. Note that ∂_a has a pole with order 3 along the divisor defined by $x = 0$, a pole of order 1 along the divisor defined by $x = \infty$ and a pole of order 2 along the divisor defined by $y = 0$. Moreover, ∂_a has two isolated singularities at $(x, y) = (0, 0), (\infty, 0)$. As in the case of the $\boldsymbol{\alpha}_2$-Enriques surface of type $\tilde{E}_8$, we blow up at the singular points of ∂_a and those of the induced derivations successively. The resulting surface is denoted by Y and the configuration of curves satisfies the conditions in the above. Here, the proper transform of curves defined by $x = 0$, $x = \infty$ or $y = 0$ is $\tilde{E}_4$, $\tilde{E}_3$ or $\tilde{R}_{10}$, respectively.

Lemma 8.10.41 *Let D_a be the divisor of ∂_a. Then,*

$$D_a = -(2\tilde{R}_{10}+3\tilde{E}_4+\tilde{E}_3+2\tilde{E}_1+4\tilde{E}_2+4\tilde{R}_9+3\tilde{R}_8+6\tilde{R}_7+4\tilde{R}_6+8\tilde{R}_5+5\tilde{R}_4+6\tilde{R}_3+2\tilde{R}_2+4\tilde{R}_1),$$

and

$$K_Y = -(2\tilde{R}_{10}+2\tilde{E}_4+\tilde{E}_1+3\tilde{E}_2+4\tilde{R}_9+3\tilde{R}_8+6\tilde{R}_7+4\tilde{R}_6+8\tilde{R}_5+5\tilde{R}_4+6\tilde{R}_3+2\tilde{R}_2+4\tilde{R}_1).$$

It follows from the formula $K_Y = \pi^* K_{Y\partial_a} + D_a$ that $K_{Y\partial_a} = E_1 + E_2 + E_3 + E_4$ because all $\tilde{E}_i$ are integral. Moreover, we can see that $D_a^2 = -12$, $K_Y \cdot D_a = -4$, and hence D_a has no isolated zeros. By contracting (-1)-curves $E_1, ..., E_4$ on Y^{∂_a} we get a smooth surface S_a which we were looking for. We have proved the following theorem.

Theorem 8.10.42 *The surface S_a is a classical Enriques surface with a crystallographic basis of type $\tilde{E}_8$. There exists a unique quasi-elliptic fibration which has a singular fiber of type $2\tilde{E}_8$.*

Remark 8.10.43 As in the case of the $\boldsymbol{\alpha}_2$-Enriques surface of type $\tilde{E}_6 + \tilde{A}_2$, we can determine the function field $k(x, y)^{\partial_a}$ of the quotient surface and have the following equation of the surface birationally isomorphic to S_a:

$$Y^2 + TX^4 + bT^3X + T^3 + T^7 = 0, \quad b = a^{\frac{-5}{4}}. \tag{8.10.12}$$

Theorem 8.10.44 *The automorphism group* $\mathrm{Aut}(S_a)$ *is trivial.*

Proof Note that the symmetry group of the dual graph of (-2)-curves is trivial and hence any automorphism is numerically trivial. We can see that the automorphism group of the surface defined by (8.10.12) preserving the fibration defined by the projection $(X, Y, T) \to T$ is trivial. □

Remark 8.10.45 By construction, the canonical cover of S has four rational double points of type A_1. The support of the conductrix is the union of 10 (-2)-curves on S.

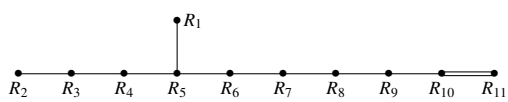
Remark 8.10.46 We have already constructed a classical extra-special surfaces of type $\tilde{E}_8$ in Example 6.2.10. The equation (8.10.12) is transformed to equation (6.2.6) by the following change of variables:

$$T = \frac{t_1}{t_2},\ X = \frac{t_0}{t_2},\ Y = \frac{t_3}{t_2^4}.$$

It uses the unique (degenerate) bielliptic linear system on S and constructs a birational model of S as an inseparable double plane. It follows from this construction that all surfaces of this type form an irreducible one-dimensional family and hence coincide with the surfaces constructed in the previous example.

Example 8.10.47 **Type $\tilde{E}_7^{(1)}$.**

Recall that the crystallographic root basis of type $\tilde{E}_7^{(1)}$ has the following Dynkin diagram:



(8.10.13)

On such an Enriques surface S, there exists a quasi-elliptic fibration with singular fibers of type $2\tilde{E}_7 + 2\tilde{A}_1^*$ and a bisection R_9. Assume that the canonical cover has rational double points of type A_1 over two points of $R_2 \setminus R_3$ and two points of $R_{11} \setminus R_{10}$. First blow up these four points and denote by the exceptional curves over the points of $R_2 \setminus R_3$ by E_1, E_2 and those over the points of $R_{11} \setminus R_{10}$ by E_3, E_4, respectively. We denote by S' the obtained surface. Assume that a resolution Y of the

normalization of the canonical cover of S is an inseparable double cover $\pi : Y \to S'$, that is, $S' = Y^{\partial}$ for a derivation ∂ on Y. We denote the pre-images on Y of R_i, E_i by $\tilde{R}_i$, $\tilde{E}_i$. Note that $\tilde{E}_1, \tilde{E}_2, \tilde{E}_3, \tilde{E}_4$ on Y are the exceptional curves corresponding to the four rational double points of type A_1 on S. Assume that $\tilde{E}_1$, $\tilde{E}_2$, $\tilde{E}_3$, $\tilde{E}_4$, $\tilde{R}_3$, $\tilde{R}_5$, $\tilde{R}_7$ and $\tilde{R}_9$ are integral with respect to a derivation ∂ on Y. Then,

$$\tilde{E}_i^2 = -2 \quad (i = 1, 2, 3, 4),\ \tilde{R}_2^2 = \tilde{R}_{11}^2 = -2,\ \tilde{R}_3^2 = \tilde{R}_5^2 = \tilde{R}_7^2 = \tilde{R}_9^2 = -4,$$

$$\tilde{R}_1^2 = \tilde{R}_4^2 = \tilde{R}_6^2 = \tilde{R}_8^2 = \tilde{R}_{10}^2 = -1.$$

Now, we contract (-1)-curves and then contract new (-1)-curves successively as in the following order:

$$\tilde{R}_1,\ \tilde{R}_4,\ \tilde{R}_6,\ \tilde{R}_8,\ \tilde{R}_{10},\ \tilde{R}_5,\ \tilde{R}_7,\ \tilde{R}_3,\ \tilde{R}_{11}.$$

On the other hand, we contract two (-2)-curves $\tilde{E}_1,\ \tilde{E}_2$. Finally, we get a surface Y with two rational double points of type A_1. The quasi-elliptic fibration on S induces a conic bundle structure on Y which has singular fibers $\tilde{E}_3 + \tilde{E}_4$, $2\tilde{R}_2$ and a bisection $\tilde{R}_9$. Two rational double points of R lie on $\tilde{R}_2$.

To get an Enriques surface, we need the following condition:

$$D = -(2\tilde{E}_1 + 2\tilde{E}_2 + \tilde{E}_3 + \tilde{E}_4 + 2\tilde{R}_1 + 2\tilde{R}_2 + 2\tilde{R}_3 + 4\tilde{R}_4 + 3\tilde{R}_5 + 4\tilde{R}_6 + 2\tilde{R}_7 + 2\tilde{R}_8 + \tilde{R}_9),$$

where D is the divisor of ∂ and

$$K_Y = -(\tilde{E}_1 + \tilde{E}_2 + 2\tilde{R}_1 + 2\tilde{R}_2 + 2\tilde{R}_3 + 4\tilde{R}_4 + 3\tilde{R}_5 + 4\tilde{R}_6 + 2\tilde{R}_7 + 2\tilde{R}_8 + \tilde{R}_9).$$

Now, we consider a surface Y defined by the following equation:

$$aSX_0^2 + T(X_1^2 + aX_1X_2) = 0 \quad (a \in k, a \neq 0).$$

The surface has a structure of a conic bundle $Y \to \mathbb{P}^1$ with $(S, T) \in \mathbb{P}^1$. The fiber over $(S, T) = (0, 1)$ is a union of two lines (corresponding to $\tilde{E}_3$, $\tilde{E}_4$) defined by $X_1(X_1 + aX_2) = 0$ and the fiber over $(S, T) = (1, 0)$ is a double line (corresponding to $\tilde{R}_2$) defined by $X_0^2 = 0$. The line defined by $X_2 = 0$ is a bisection (corresponding to $\tilde{R}_9$) of the fiber space. The surface has two rational double points $((X_0, X_1, X_2), (S, T)) = ((0, 0, 1), (1, 0)), ((0, a, 1), (1, 0))$ of type A_1 (corresponding to $\tilde{E}_1$, $\tilde{E}_2$). Let $(x = X_0/X_2,\ y = X_1/X_2,\ s = S/T)$ be affine coordinates. Set

$$\partial = \frac{1}{s}\left((s^2 + 1)\frac{\partial}{\partial x} + s^2x^2\frac{\partial}{\partial y}\right).$$

Then, $\partial^2 = \partial$, that is, ∂ is 2-closed. A calculation shows that ∂ has two isolated singularities at the intersection of the bisection and the two fibers over $(S, T) = (1, 0), (0, 1)$. As in the previous cases, we blow up the two rational double points and isolated singularities of ∂ successively, and finally get the surface Y and a derivation

which we were looking for. Since R has a parameter a, we denote by S_a the obtained Enriques surface.

Theorem 8.10.48 *The surface S_a is a classical Enriques surface with a crystallographic basis of type $\tilde{E}_7^{(1)}$.*

Using the dual graph of (-2)-curves, we can prove the following proposition:

Proposition 8.10.49 *There are exactly one quasi-elliptic fibration with a singular fiber of type $\tilde{E}_8$ and one quasi-elliptic fibration with singular fibers of type $2\tilde{E}_7+2\tilde{A}_1^*$.*

Theorem 8.10.50 *The automorphism group* $\mathrm{Aut}(S_a)$ *is isomorphic to $\mathbb{Z}/2\mathbb{Z}$ which is numerically trivial.*

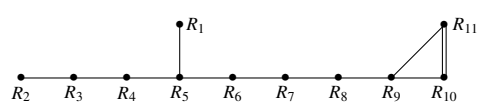
Proof Since there are no non-trivial symmetries of the dual graph of (-2)-curves, all automorphisms are numerically trivial. Let $\pi_1 : S_a \to \mathbb{P}^1$ be the quasi-elliptic fibration with a singular fiber of type $\tilde{E}_8$ and let $\pi_2 : S_a \to \mathbb{P}^1$ be the quasi-elliptic fibration with singular fibers of type $2\tilde{E}_7+2\tilde{A}_1^*$. Let F_1, F_2 be the double fibers of π_1. Note that F_1, F_2 are bisections of π_2. If $g \in \mathrm{Aut}(S_a)$ preserves each F_i, then g fixes three points on F_i (the cusp and the two intersection points with the two double fibers of π_2), and hence acts trivially on F_i. Therefore, g fixes three points on a general fiber of π_2, that is, the cusp and two intersection points with F_1, F_2. Thus, g fixes a general fiber of π_2 pointwise and hence g is trivial. Thus, we have $|\mathrm{Aut}(S_a)| \leq 2$. On the other hand, the Mordell–Weil group $\mathbb{Z}/2\mathbb{Z}$ of the Jacobian fibration of π_2 acts on S_a. This proves the assertion. □

Remark 8.10.51 By construction, the canonical cover of S has four rational double points of type A_1. The support of the conductrix is the union of nine (-2)-curves $R_1, \ldots, R_9$.

Remark 8.10.52 We have already given a construction of classical extra-special surfaces of type $\tilde{E}_7^{(1)}$ in Example 6.2.11. It uses a unique non-degenerate bielliptic map of S to the 4-nodal anti-canonical quartic del Pezzo surface D_1. It follows from this construction that all surfaces of this type form an irreducible one-dimensional family and hence coincide with the family of surfaces constructed in the previous example.

Example 8.10.53 **Type $\tilde{E}_7^{(2)}$: α_2-surface.**

Recall that the crystallographic root basis of type $\tilde{E}_7^{(2)}$ has the following Dynkin diagram.



(8.10.14)

We use the same idea for finding a suitable rational surface and a derivation as in the case of classical Enriques surfaces of type $\tilde{E}_7^{(1)}$ (Example 8.10.47). Consider the quasi-elliptic fibration with a singular fiber of type $2\tilde{E}_7$ and with a bisection R_9. Assume that the canonical cover of S has a rational double point of type D_4 over a

point P of $R_2 \setminus R_3$. First blow up P and denote the exceptional curve by E_1. Then, blow up the intersection point of E_1 and the proper transform of R_2, and denote the exceptional curve by E_2. And blow up two points on E_1 and denote the exceptional curves by E_3, E_4. We denote by S' the obtained surface. Assume that a resolution Y of the normalization of the canonical cover of S is an inseparable double cover $\pi : Y \to S'$, that is, $S' = Y^{\partial}$ for a derivation ∂ on Y. We denote by $\tilde{R}_i$, $\tilde{E}_i$ the pre-images on Y of R_i, E_i, respectively. Note that $\tilde{E}_1, \tilde{E}_2, \tilde{E}_3, \tilde{E}_4$ on Y is the exceptional curves over the rational double point of type D_4. Assume that $\tilde{E}_2$, $\tilde{E}_3$, $\tilde{E}_4$, $\tilde{R}_3$, $\tilde{R}_5$, $\tilde{R}_7$ and $\tilde{R}_9$ are integral with respect to a derivation ∂ on Y. Then,

$$\tilde{E}_i^2 = -2 \quad (i = 1, 2, 3, 4),\ \tilde{R}_2^2 = -2, \quad \tilde{R}_3^2 = \tilde{R}_5^2 = \tilde{R}_7^2 = \tilde{R}_9^2 = -4,$$

$$\tilde{R}_1^2 = \tilde{R}_4^2 = \tilde{R}_6^2 = \tilde{R}_8^2 = \tilde{R}_{10}^2 = \tilde{R}_{11}^2 = -1.$$

Note that the pre-image of the fiber $R_{10}+R_{11}$ of type $\tilde{A}_1^*$ is a union of two (-1)-curves $\tilde{R}_{10}$, $\tilde{R}_{11}$ meeting at one point transversally.

Next, we contract (-1)-curves and then contract new (-1)-curves successively as in the following order:

$$\tilde{R}_1,\ \tilde{R}_4,\ \tilde{R}_6,\ \tilde{R}_8,\ \tilde{R}_5,\ \tilde{R}_7,\ \tilde{R}_3,\ \tilde{R}_2,\ \tilde{E}_2.$$

We also contract two (-2)-curves $\tilde{E}_3$, $\tilde{E}_4$. Thus, we get a surface Z with two rational double points of type A_1. The quasi-elliptic fibration on S induces a conic bundle structure on Y which has two singular fibers $\tilde{R}_{10} + \tilde{R}_{11}$, $2\tilde{E}_1$ and a bisection $\tilde{R}_9$. Two rational double points of Z lie on $\tilde{E}_1$.

To get an Enriques surface, we need the following condition:

$$D = -(4\tilde{E}_1 + 4\tilde{E}_2 + 3\tilde{E}_3 + 3\tilde{E}_4 + 2\tilde{R}_1 + 2\tilde{R}_2 + 2\tilde{R}_3 + 4\tilde{R}_4 + 3\tilde{R}_5 + 4\tilde{R}_6 + 2\tilde{R}_7 + 2\tilde{R}_8 + \tilde{R}_9),$$

where D is the divisor of ∂ and

$$K_Y = -(2\tilde{E}_1 + 2\tilde{E}_2 + \tilde{E}_3 + \tilde{E}_4 + 2\tilde{R}_1 + 2\tilde{R}_2 + 2\tilde{R}_3 + 4\tilde{R}_4 + 3\tilde{R}_5 + 4\tilde{R}_6 + 2\tilde{R}_7 + 2\tilde{R}_8 + \tilde{R}_9).$$

Now, we consider a surface Z defined by the following equation:

$$S(X_0^2 + a^3 X_2^2) + T(X_1^2 + X_1 X_2 + a^2 X_0 X_2) = 0 \quad (a \in k, a \neq 0).$$

The surface has a structure of a conic bundle $Z \to \mathbb{P}^1$ with $(S, T) \in \mathbb{P}^1$. The fiber over $(S, T) = (1, 0)$ is a double line and the fiber over $(S, T) = (a^4, 1)$ is a union of two lines. The line defined by $X_2 = 0$ is a bisection of the fiber space. The surface has two rational double points $((X_0, X_1, X_2), (S, T)) = ((\alpha, \beta_i, 1), (1, 0))$ of type A_1, where $\alpha = \sqrt{a^3}$ and β_i is a root of the equation $z^2 + z + a^3\sqrt{a} = 0$.

Let $(x = X_0/X_2,\ y = X_1/X_2,\ s = S/T)$ be affine coordinates. Set

$$\partial = (s^2 + a)\frac{\partial}{\partial x} + (x^2 + a^2 s^2)\frac{\partial}{\partial y}.$$

Then, $\partial^2 = 0$, that is, ∂ is 2-closed. A calculation shows that ∂ has an isolated singularity at the intersection of the bisection and the fiber over the point $(S, T) = (1, 0)$. As in the previous cases, we blow up the two rational double points and the isolated singularity of ∂ successively, and finally get the surface Y and a desired derivation. Since R has a parameter a, we denote by S_a the obtained Enriques surface.

Theorem 8.10.54 *The surface S_a is an $\boldsymbol{\alpha}_2$-Enriques surface with a crystallographic basis of type $\tilde{E}_7^{(2)}$.*

By the dual graph of (-2)-curves, we can prove the following proposition.

Proposition 8.10.55 *There are exactly one quasi-elliptic fibration with singular fibers of type $2\tilde{E}_7 + \tilde{A}_1^*$ and two quasi-elliptic fibrations with a singular fiber of type $\tilde{E}_8$.*

Remark 8.10.56 As in the case of the $\boldsymbol{\alpha}_2$-Enriques surface of type $\tilde{E}_6 + \tilde{A}_2$, we can determine the function field $k(x, y, s)^{\partial}$ of the quotient surface and have the following equation of the surface birationally isomorphic to S_a:

$$Y^2 + Y + TX^4 + (T + a^4)^7 = 0. \tag{8.10.15}$$

By using this equation we can determine the automorphism group.

Theorem 8.10.57 *If $a^7 \neq 1$, then the automorphism group $\mathrm{Aut}(S_a)$ is isomorphic to $\mathbb{Z}/2\mathbb{Z}$ which is not numerically trivial. If $a^7 = 1$, then $\mathrm{Aut}(S_a)$ is isomorphic to $\mathbb{Z}/14\mathbb{Z}$ and $\mathrm{Aut}_{nt}(S_a)$ is isomorphic to $\mathbb{Z}/7\mathbb{Z}$.*

Proof Here we give only a generator σ of $\mathrm{Aut}(S_a)$ in terms of the equation (8.10.15): In the case $a^7 \neq 1$,

$$\sigma(T, X, Y) = (T,\ X,\ Y + 1).$$

In the case $a^7 = 1$,

$$\sigma(T, X, Y) = (\zeta T, \zeta^{-2}X + (1+\zeta^{-2})a^6 + (\zeta+\zeta^{-2})a^2T, Y + 1 + (1+\zeta^2)a^6T^2 + (1+\zeta^3)a^2T^3)$$

where ζ is a primitive 7th root of unity. □

Remark 8.10.58 By construction, the canonical cover of S has a rational double point of type D_4. The support of the conductrix is the union of nine (-2)-curves $R_1, \ldots, R_9$.

Example 8.10.59 **Type $\tilde{E}_7^{(2)}$: Classical Case.**

A classical Enriques surface S of type $\tilde{E}_7^{(2)}$ has the same dual graph as in diagram 8.10.14. We use the same idea to find a suitable rational surface and a derivation as in the case of classical Enriques surfaces of type $\tilde{E}_7^{(1)}$ (Example 8.10.47). In fact, the type $\tilde{E}_7^{(1)}$ is a degeneration of this case. In this case the quasi-elliptic fibration with a singular fiber of type $2\tilde{E}_7$ and with a bisection R_9 has two double fibers. We denote

by F_0 the irreducible double fiber. Assume that the canonical cover of S has four rational double points of type A_1 over two points of $R_2 \setminus R_3$ and two points of $F_0 \setminus R_9$. First blow up these four points and denote by the exceptional curves over the points of $R_2 \setminus R_3$ by E_1, E_2 and those over the points of $F_0 \setminus R_9$ by E_3, E_4, respectively. We denote by S' the obtained surface. Assume that a resolution Y of the normalization of the canonical cover of S is an inseparable double cover $\pi : Y \to S'$, that is, $S' = Y^{\partial}$ for a derivation ∂ on Y. We denote by $\tilde{R}_i$, $\tilde{E}_i$, $\tilde{F}_0$ the pre-images on Y of R_i, E_i, F_0, respectively. Note that $\tilde{E}_1, \tilde{E}_2, \tilde{E}_3, \tilde{E}_4$ on Y is the exceptional curves corresponding to four rational double points of type A_1. Assume that $\tilde{E}_1$, $\tilde{E}_2$, $\tilde{E}_3$, $\tilde{E}_4$, $\tilde{R}_3$, $\tilde{R}_5$, $\tilde{R}_7$ and $\tilde{R}_9$ are integral with respect to the derivation ∂ on Y. Then,

$$\tilde{E}_i^2 = -2 \quad (i = 1, 2, 3, 4), \ \tilde{R}_2^2 = -2, \quad \tilde{R}_3^2 = \tilde{R}_5^2 = \tilde{R}_7^2 = \tilde{R}_9^2 = -4,$$

$$\tilde{R}_1^2 = \tilde{R}_4^2 = \tilde{R}_6^2 = \tilde{R}_8^2 = \tilde{R}_{10}^2 = \tilde{R}_{11}^2 = -1.$$

Note that the pre-image of the fiber $R_{10}+R_{11}$ of type $\tilde{A}_1^*$ is a union of two (-1)-curves $\tilde{R}_{10}, \tilde{R}_{11}$ meeting at one point transversally. Also, the pre-image of the cuspidal curve F_0 is a nonsingular rational curve $\tilde{F}_0$ with $\tilde{F}_0^2 = -1$.

Now, we contract (-1)-curves and then contract new (-1)-curves successively as in the following order:

$$\tilde{R}_1, \ \tilde{R}_4, \ \tilde{R}_6, \ \tilde{R}_8, \ \tilde{R}_5, \ \tilde{R}_7, \ \tilde{R}_3, \ \tilde{F}_0, \ \tilde{E}_3.$$

We also contract two (-2)-curves $\tilde{E}_1$, $\tilde{E}_2$. Finally, we get a surface Z with two rational double points of type A_1. The quasi-elliptic fibration on S induces a conic bundle structure on Y which has two singular fibers $\tilde{R}_{10} + \tilde{R}_{11}$, $2\tilde{R}_2$ and a bisection $\tilde{R}_9$. Two rational double points of Z lie on $\tilde{R}_2$.

To get an Enriques surface, we need the following condition:

$$D = -(2\tilde{E}_1 + 2\tilde{E}_2 + \tilde{E}_3 + \tilde{E}_4 + 2\tilde{R}_1 + 2\tilde{R}_2 + 2\tilde{R}_3 + 4\tilde{R}_4 + 3\tilde{R}_5 + 4\tilde{R}_6 + 2\tilde{R}_7 + 2\tilde{R}_8 + \tilde{R}_9),$$

where D is the divisor of ∂ and

$$K_Y = -(\tilde{E}_1 + \tilde{E}_2 + 2\tilde{R}_1 + 2\tilde{R}_2 + 2\tilde{R}_3 + 4\tilde{R}_4 + 3\tilde{R}_5 + 4\tilde{R}_6 + 2\tilde{R}_7 + 2\tilde{R}_8 + \tilde{R}_9).$$

Let us consider a surface Z defined by the following equation:

$$S(aX_0^2 + bX_2^2) + T(X_1^2 + aX_1X_2 + bX_0X_2) = 0 \quad (a, b \in k, a \neq 0, b \neq 0).$$

The surface has a structure of a conic bundle $Z \to \mathbb{P}^1$ with $(S, T) \in \mathbb{P}^1$. The fiber over $(S, T) = (0, 1)$ is nonsingular (corresponding to the above $\tilde{E}_4$), the fiber over $(S, T) = (1, 0)$ is a double line and the fiber over $(S, T) = (b^2, a^3)$ is a union of two lines. The line defined by $X_2 = 0$ is a bisection of the fibration. The surface has two rational double points $((X_0, X_1, X_2), (S, T)) = ((\alpha, \beta_i, 1), (1, 0))$ of type A_1, where $\alpha = \sqrt{b/a}$ and β_i is a root of the equation $z^2 + az + \sqrt{b^3/a} = 0$.

Let $(x = X_0/X_2, \ y = X_1/X_2, \ s = S/T)$ be affine coordinates. Set

$$\partial = \frac{1}{s}\left(a(s^2+c)\frac{\partial}{\partial x} + (as^2x^2+bc)\frac{\partial}{\partial y}\right) \quad (b \neq a^2c), \tag{8.10.16}$$

where c is a root of the equation of $z^2 + (b/a)z + 1 = 0$. Then, $\partial^2 = a\partial$, that is, ∂ is 2-closed. A calculation shows that ∂ has two isolated singularities at the intersection of the bisection and the two fibers over $(S,T) = (1,0),(0,1)$. As in the previous cases, we blow up at the two rational double points and isolated singularities of ∂ successively, and finally get the surface Y and a desired derivation. Since R has parameters a, b, we denote by $S_{a,b}$ the obtained Enriques surface.

Theorem 8.10.60 *The surface $S_{a,b}$ is a classical Enriques surface with a crystallographic basis of type $\tilde{E}_7^{(2)}$.*

By inspecting the dual graph of (-2)-curves, we get the following proposition:

Proposition 8.10.61 *There are exactly one quasi-elliptic fibration with singular fibers of type $2\tilde{E}_7 + \tilde{A}_1^*$ and two quasi-elliptic fibrations with a singular fiber of type $\tilde{E}_8$.*

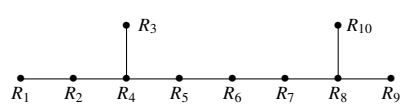
Theorem 8.10.62 *The automorphism group* $\mathrm{Aut}(S_{a,b})$ *is isomorphic to $\mathbb{Z}/2\mathbb{Z}$ which is not numerically trivial.*

Proof First, we show that there are no numerically trivial involutions. Let $\pi_1 : S_{a,b} \to \mathbb{P}^1$ be a quasi-elliptic fibration with a singular fiber of type $\tilde{E}_8$ and let $\pi_2 : S_{a,b} \to \mathbb{P}^1$ be the quasi-elliptic fibration with singular fibers of type $2\tilde{E}_7 + \tilde{A}_1^*$ and with a bisection R_9. Let F_1, F_2 be the double fibers of π_1 and let F_0 be the remaining double fiber of π_2. Note that F_1, F_2 (resp. F_0) are bisections of π_2 (resp. π_1). If $g \in \mathrm{Aut}(S_a)$ is numerically trivial, then g fixes four points on F_0 (the cusp and the three intersection points with F_1, F_2 and R_9). Hence g fixes F_0 pointwise. Then, g fixes three points on F_i $(i = 1,2)$ (the cusp and the two intersection points with F_0 and R_2) and hence fixes F_i pointwise. Therefore, g fixes three points on a general fiber of π_2, that is, the cusp and two intersection points with F_1, F_2. Thus, g fixes a general fiber of π_2 pointwise, and hence, g is trivial. Thus, we have $|\mathrm{Aut}(S_{a,b})| \leq 2$. On the other hand, the Mordell–Weil group $\mathbb{Z}/2\mathbb{Z}$ of the Jacobian fibration of π_2 acts on $S_{a,b}$. This proves the assertion. □

Remark 8.10.63 By construction, the canonical cover of $S_{a,b}$ has four rational double points of type A_1. The support of the conductrix is the union of nine (-2)-curves $R_1, \ldots, R_9$.

Example 8.10.64 **Type $\tilde{D}_8$: α_2-surface.**

Recall that the crystallographic root basis of type $\tilde{D}_8$ has the following Dynkin diagram:



(8.10.17)

We use the same idea to find a suitable rational surface and a derivation as in the case of the α_2-Enriques surface of type $\tilde{E}_8$ (Example 8.10.35). On such an Enriques surface S, there exists a quasi-elliptic fibration with a singular fiber of type $2\tilde{D}_8$ and with a bisection R_1. Assume that the canonical cover has a rational double point of type D_4 over the point $P = R_1 \cap R_2$. First blow up P and denote the exceptional curve by E_1. Then, blow up the intersection points of E_1 and the proper transforms of R_1 and R_2 and denote the exceptional curves by E_2 and E_3 respectively. And blow up one point on E_1 not lying on R_1 and R_2 and denote the exceptional curve by E_4. We denote by S' the obtained surface. Assume that a resolution Y of the normalization of the canonical cover of S is an inseparable double cover $\pi : Y \to S'$, that is, $S' = Y^{\partial}$ for a derivation ∂ on Y. We denote by $\tilde{R}_i$, $\tilde{E}_i$ the pre-images on Y of R_i, E_i. Note that the cycle $\tilde{E}_1 + \tilde{E}_2 + \tilde{E}_3 + \tilde{E}_4$ on Y is the exceptional divisor corresponding to the rational double point of type D_4. Since $\tilde{E}_i^2 = -2$ $(i = 1, 2, 3, 4)$ and $E_1^2 = -4$, $E_2^2 = E_3^2 = E_4^2 = -1$, $\tilde{E}_2, \tilde{E}_3, \tilde{E}_4$ should be integral and $\tilde{E}_1$ not with respect to the derivation ∂. We assume that $\tilde{R}_4$, $\tilde{R}_6$ and $\tilde{R}_8$ are integral with respect to ∂. Then,

$$\tilde{R}_4^2 = \tilde{R}_6^2 = \tilde{R}_8^2 = -4, \quad \tilde{R}_3^2 = \tilde{R}_5^2 = \tilde{R}_7^2 = \tilde{R}_9^2 = \tilde{R}_{10}^2 = -1, \quad \tilde{R}_1^2 = \tilde{R}_2^2 = -2.$$

Now, we contract (-1)-curves and then contract new (-1)-curves except $\tilde{E}_4$ successively. Finally, we get the projective plane $\mathbb{P}^2$. Thus, Y is obtained from $\mathbb{P}^2$ by blowing up 13 times..

To get an Enriques surface S, we need the following condition:

$$D = -(2\tilde{R}_1 + 6\tilde{E}_2 + 8\tilde{E}_1 + 7\tilde{E}_3 + 5\tilde{E}_4 + 4\tilde{R}_2 + 3\tilde{R}_4 + 2\tilde{R}_3 + 4\tilde{R}_5 + 2\tilde{R}_6 + 2\tilde{R}_7 + \tilde{R}_8),$$

where D is the divisor of ∂ and

$$K_Y = -(2\tilde{R}_1 + 4\tilde{E}_2 + 6\tilde{E}_1 + 5\tilde{E}_3 + 3\tilde{E}_4 + 4\tilde{R}_2 + 3\tilde{R}_4 + 2\tilde{R}_3 + 4\tilde{R}_5 + 2\tilde{R}_6 + 2\tilde{R}_7 + \tilde{R}_8).$$

Let us consider the affine plane $\mathbb{A}^2 \subset \mathbb{P}^2$ with affine coordinates (x, y) and the following derivation:

$$\partial_a = \frac{1}{x^5}\left(x(x^4 + x^2 + y^6)\frac{\partial}{\partial x} + (ax^6 + y(x^4 + x^2 + y^6))\frac{\partial}{\partial y}\right),$$

where $a \in k$, $a \neq 0$. Then, $\partial_a^2 = 0$, that is, ∂_a is 2-closed. Note that ∂_a has a pole with order 5 along the line ℓ defined by $x = 0$ (corresponding to $\tilde{E}_4$) and ℓ is integral with respect to ∂_a. Moreover, ∂_a has a unique isolated singularity at $P = (0, 0)$. We blow up the point P. Then, the exceptional curve is not integral with respect to the induced derivation. The induced derivation has a pole of order 2 along the exceptional curve and has a unique isolated singularity at the intersection of the exceptional curve and the proper transform of ℓ. Then, continue this process until the induced derivation has no isolated singularities. The resulting surface is denoted by Y and the configuration of curves is as desired. Then, contracting exceptional curves on $S' = Y^{\partial}$ described above, we obtain an Enriques surface S_a.

Theorem 8.10.65 *The surface S_a is an $\boldsymbol{\alpha}_2$-Enriques surface with a crystallographic basis of type $\tilde{D}_8$. There exist exactly one quasi-elliptic fibration with a singular fiber of type $2\tilde{D}_8$ and two elliptic fibrations with a singular fiber of type $\tilde{E}_8$.*

Proof The non-trivial assertion is that any genus one fibration with singular fiber of type $\tilde{E}_8$ is an elliptic fibration. This follows from the fact that the conductrix is contained in the fiber of type $\tilde{E}_8$ (see Lemma 3.3 in [371]). □

Remark 8.10.66 As in the case of the $\boldsymbol{\alpha}_2$-Enriques surface of type $\tilde{E}_6 + \tilde{A}_2$, we can determine the function field $k(x, y)^{\partial_a}$ of the quotient surface and have the following equation of the surface birationally isomorphic to S_a:

$$Y^2 + TX^4 + TX^2 + aX + T^7 = 0. \tag{8.10.18}$$

Using this equation, we can determine the automorphism group.

Theorem 8.10.67 *The automorphism group* $\mathrm{Aut}(S_a)$ *is isomorphic to the quaternion group Q_8 of order* 8 *which is cohomologically trivial.*

Proof Here we give only a generator $\{\sigma_{\omega,\alpha}\}$ of $\mathrm{Aut}(S_a)$ in terms of the equation (8.10.18):

$$\sigma_{\omega,\alpha}(T, X, Y) = (T + \omega,\ X + \alpha + \omega^2 T,\ Y + \omega^2 X^2 + \omega^2 X + \omega^2 T^3 + \sqrt{a\alpha} + \sqrt{a}),$$

where ω is a primitive cube root of unity and α is a root of the equation $z^2 + z + \omega\sqrt{a} + 1 = 0$. □

Remark 8.10.68 By construction, the canonical cover of S_a has a rational double point of type D_4. The support of the conductrix is the union of eight (-2)-curves $R_1, \ldots, R_8$ on S_a

Example 8.10.69 **Type $\tilde{D}_8$: Classical Case.**

A classical Enriques surface of type $\tilde{D}_8$ has the same dual graph as in diagram 8.10.17. We use the same idea to find a suitable rational surface and a derivation as in the case of classical Enriques surfaces of type $\tilde{E}_8$ (Example 8.10.40). In this case, a quasi-elliptic fibration with a singular fiber of type $2\tilde{D}_8$ has two double fibers. We denote by F_0 the irreducible one. The difference from the $\boldsymbol{\alpha}_2$-surface is that the canonical cover has four rational double points of type A_1 instead of a rational double point of type D_4. So, , we blow up $F_0 \cap R_1$, $R_1 \cap R_2$, a nonsingular point on F_0 and a point on R_2. We denote by E_1, E_2, E_3, E_4 the exceptional curves, respectively. We use the same notation as in the case of the $\boldsymbol{\alpha}_2$-Enriques surface of type $\tilde{D}_8$. We also assume that $\tilde{E}_i$, $(1 \le i \le 4)$, $\tilde{R}_4$, $\tilde{R}_6$ and $\tilde{R}_8$ are integral with respect to a derivation ∂. The pull-back of F_0 to Y, denoted by $\tilde{F}_0$, is a nonsingular rational curve. Obviously

$$\tilde{E}_1^2 = \tilde{E}_2^2 = \tilde{E}_3^2 = \tilde{E}_4^2 = \tilde{R}_1^2 = \tilde{R}_2^2 = -2,\ \tilde{R}_4^2 = \tilde{R}_6^2 = \tilde{R}_8^2 = -4,$$

$$\tilde{F}_0^2 = \tilde{R}_3^2 = \tilde{R}_5^2 = \tilde{R}_7^2 = \tilde{R}_9^2 = \tilde{R}_{10}^2 = -1.$$

Now, we contract (-1)-curves and then contract new (-1)-curves except $\tilde{R}_1, \tilde{E}_3, \tilde{E}_4$successively. Finally, we get a nonsingular quadric surface $\mathbb{P}^1 \times \mathbb{P}^1$. Thus, Y is obtained from $\mathbb{P}^1 \times \mathbb{P}^1$ by blowing up 12 times. To get an Enriques surface, we need the following condition:

$$D = -(2\tilde{E}_1 + 4\tilde{E}_2 + \tilde{E}_3 + 3\tilde{E}_4 + 2\tilde{R}_1 + 4\tilde{R}_2 + 2\tilde{R}_3 + 3\tilde{R}_4 + 4\tilde{R}_5 + 2\tilde{R}_6 + 2\tilde{R}_7 + \tilde{R}_8),$$

where D is the divisor of ∂ and

$$K_X = -(\tilde{E}_1 + 3\tilde{E}_2 + 2\tilde{E}_4 + 2\tilde{R}_1 + 4\tilde{R}_2 + 2\tilde{R}_3 + 3\tilde{R}_4 + 4\tilde{R}_5 + 2\tilde{R}_6 + 2\tilde{R}_7 + \tilde{R}_8).$$

Let us consider an affine open set $\mathbb{A}^1 \times \mathbb{A}^1$ of $\mathbb{P}^1 \times \mathbb{P}^1$ with affine coordinates (x, y) and the following derivation:

$$\partial_{\alpha,\beta} = \frac{1}{xy^2}\left(\alpha x^2 y^2 \frac{\partial}{\partial x} + (x^4 y^4 + \beta y^4 + x^2 y^2 + x^2)\frac{\partial}{\partial y}\right), \quad (\alpha, \beta) \neq (0, 0). \tag{8.10.19}$$

Then, $\partial_{\alpha,\beta}^2 = \alpha\partial_{\alpha,\beta}$, that is, $\partial_{\alpha,\beta}$ is 2-closed. Note that $\partial_{\alpha,\beta}$ has a pole with order 1 along the divisor defined by $x = 0$ (corresponding to $\tilde{E}_3$), a pole of order 3 along the divisor defined by $x = \infty$ (corresponding to $\tilde{E}_4$) and a pole of order 2 along the divisor defined by $y = 0$ (corresponding to $\tilde{R}_1$). Moreover, $\partial_{\alpha,\beta}$ has two isolated singularities at $(x, y) = (0, 0), (\infty, 0)$. As in the case of the $\boldsymbol{\alpha}_2$-Enriques surface of type $\tilde{E}_8$, we blow up the singular points of $\partial_{\alpha,\beta}$ and those of the induced derivations successively. The resulting surface is Y and the configuration of curves is as desired. Now, after contracting exceptional curves on $S' = Y^{\partial_{\alpha,\beta}}$ we have an Enriques surface $S_{\alpha,\beta}$.

Theorem 8.10.70 *The surface $S_{\alpha,\beta}$ is a classical Enriques surface with a crystallographic basis of type $\tilde{D}_8$. It has exactly one quasi-ellliptic fibration with a singular fiber of type $2\tilde{D}_8$ and two elliptic fibrations with a singular fiber of type $\tilde{E}_8$.*

Remark 8.10.71 As in the case of the $\boldsymbol{\alpha}_2$-Enriques surface of type $\tilde{E}_6 + \tilde{A}_2$, we can determine the function field $k(x, y)^{\partial_{\alpha,\beta}}$ of the quotient surface and have the following equation of the surface birationally isomorphic to $S_{a,b}$:

$$Y^2 + TX^4 + \alpha T^3 X^2 + \beta T^5 X + T^3 + T^7 = 0. \tag{8.10.20}$$

We remark that the equation (12.6) in [371] corresponding to (8.10.20) and the following generator σ of $\mathrm{Aut}(S_{a,b})$ in the proof of [371][Theorem 12.11] are wrong.

Theorem 8.10.72 *The automorphism group* $\mathrm{Aut}(S_{\alpha,\beta})$ *is* $\mathbb{Z}/2\mathbb{Z}$ *which is numerically trivial.*

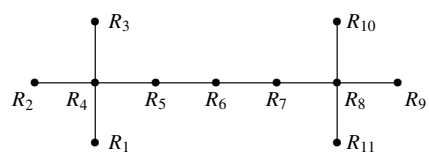
Proof Here, we give only a generator σ of $\mathrm{Aut}(S_{a,b})$ in terms of the equation (8.10.20):

$$\sigma(T, X, Y) = (T,\ X + \sqrt{\alpha}T,\ Y + \sqrt[4]{\alpha}\sqrt{\beta}T^3)$$

Remark 8.10.73 By construction, the canonical cover of $S_{\alpha,\beta}$ has four rational double points of type A_1. The support of the conductrix is the union of eight (-2)-curves $R_1, \ldots, R_8$ on $S_{\alpha,\beta}$.

Example 8.10.74 **Type $\tilde{D}_4 + \tilde{D}_4$.**

Recall that the crystallographic root basis of type $\tilde{D}_4 + \tilde{D}_4$ has the following Dynkin diagram:



(8.10.21)

We use the same idea to find a suitable rational surface and a derivation as in the case of classical Enriques surfaces of type $\tilde{E}_8$ (Example 8.10.40). In this case, the canonical cover of the desired Enriques surface S has four rational double points of type A_1. So, , we blow up $R_5 \cap R_6$, $R_6 \cap R_7$, a nonsingular point on R_5 and a point on R_7. We denote by E_1, E_2, E_3, E_4 the exceptional curves, respectively. We denote by S' the obtained surface. Assume that a resolution Y of the normalization of the canonical cover of S is an inseparable double cover $\pi : Y \to S'$, that is, $S' = Y^{\partial}$ for a derivation ∂ on Y. We denote by $\tilde{R}_i$, $\tilde{E}_i$ the pre-images on Y of R_i, E_i. Note that the cycle $\tilde{E}_1, \tilde{E}_2, \tilde{E}_3, \tilde{E}_4$ on Y is the exceptional curves corresponding to rational double points of type A_1. We assume that $\tilde{E}_i, (1 \le i \le 4)$, $\tilde{R}_4$ and $\tilde{R}_8$ are integral with respect to the derivation ∂. Then, we have

$$\tilde{E}_1^2 = \tilde{E}_2^2 = \tilde{E}_3^2 = \tilde{E}_4^2 = -2, \quad \tilde{R}_4^2 = \tilde{R}_8^2 = -4,$$

$$\tilde{R}_1^2 = \tilde{R}_2^2 = \tilde{R}_3^2 = \tilde{R}_9^2 = \tilde{R}_{10}^2 = \tilde{R}_{11}^2 = -1, \quad \tilde{R}_5^2 = \tilde{R}_6^2 = \tilde{R}_7^2 = -2.$$

Now, we contract (-1)-curves and then contract new (-1)-curves except $\tilde{E}_3, \tilde{E}_4, \tilde{R}_6$ successively. Finally, we get a nonsingular quadric surface $\mathbb{P}^1 \times \mathbb{P}^1$. Thus, Y should be obtained from $\mathbb{P}^1 \times \mathbb{P}^1$ by blowing up 12 times.

To get an Enriques surface, we need the following condition:

$$D = -(3\tilde{E}_1 + 3\tilde{E}_2 + 2\tilde{E}_3 + 2\tilde{E}_4 + \tilde{R}_4 + 2\tilde{R}_5 + 2\tilde{R}_6 + 2\tilde{R}_7 + \tilde{R}_8),$$

where D is the divisor of ∂ and

$$K_Y = -(2\tilde{E}_1 + 2\tilde{E}_2 + \tilde{E}_3 + \tilde{E}_4 + \tilde{R}_4 + 2\tilde{R}_5 + 2\tilde{R}_6 + 2\tilde{R}_7 + \tilde{R}_8).$$

We consider an affine open set $\mathbb{A}^1 \times \mathbb{A}^1$ of $\mathbb{P}^1 \times \mathbb{P}^1$ with affine coordinates (x, y) and the following derivation:

$$\partial_{\alpha,\beta} = \frac{1}{x^2y^2}\left(\beta x^3y^2\frac{\partial}{\partial x} + (\alpha x^2y^2 + x^2 + x^4y^4 + y^4 + \beta x^2y^3)\frac{\partial}{\partial y}\right), \quad \beta \neq 0.$$

Then, $\partial_{\alpha,\beta}^2 = \beta\partial_{\alpha,\beta}$, that is, $\partial_{\alpha,\beta}$ is 2-closed. Note that $\partial_{\alpha,\beta}$ has poles of order 2 along the divisors defined by $x = 0$ (corresponding to $\tilde{E}_3$), $x = \infty$ (corresponding to $\tilde{E}_4$) and $y = 0$ (corresponding to $\tilde{R}_6$). Moreover, $\partial_{\alpha,\beta}$ has two isolated singularities at $(x, y) = (0, 0), (\infty, 0)$. The divisors defined by $x = 0$ and $x = \infty$ are integral with respect to $\partial_{\alpha,\beta}$. We first blow up $(x, y) = (0, 0), (\infty, 0)$. The induced derivation has poles of order 3 along the two exceptional curves and has isolated zeros at the intersection points of the exceptional curves and the proper transforms of the divisors defined by $x = 0$ and $x = \infty$. The two exceptional curves are integral. As in the case of the $\boldsymbol{\alpha}_2$-Enriques surface of type $\tilde{E}_8$, we blow up the singular points of the induced derivations successively. The resulting surface is Y and the configuration of curves is as desired. After contracting exceptional curves on $Y^{\partial_{\alpha,\beta}}$ we obtain an Enriques surface $S_{\alpha,\beta}$.

Theorem 8.10.75 *The surface $S_{\alpha,\beta}$ is a classical Enriques surface with a crystallographic basis of type $\tilde{D}_4 + \tilde{D}_4$.*

Remark 8.10.76 As in the case of the $\boldsymbol{\alpha}_2$-Enriques surface of type $\tilde{E}_6 + \tilde{A}_2$, we can determine the function field $k(x, y)^{\partial_{\alpha,\beta}}$ of the quotient surface and have the following equation of the surface birationally isomorphic to $S_{\alpha,\beta}$:

$$Y^2 + TX^4 + \alpha T^3 X^2 + \beta T^4 X + T^3 + T^7 = 0. \tag{8.10.22}$$

Theorem 8.10.77 *The automorphism group* $\mathrm{Aut}(S_{\alpha,\beta})$ *is* $(\mathbb{Z}/2\mathbb{Z})^3$ *and* $\mathrm{Aut}_{nt}(S_{\alpha,\beta})$ *is isomorphic to* $(\mathbb{Z}/2\mathbb{Z})^2$.

Proof We give only a generator $\{\sigma_\alpha, \tau\}$ of $\mathrm{Aut}(S_{\alpha,\beta})$ that acts on surface (8.10.22) as follows:

$$\sigma_a(T, X, Y) = (T,\ X + aT,\ Y), \quad \tau(T, X, Y) = (1/T,\ X/T^2,\ Y/T^5),$$

where a is a root of the equation $z^3 + \alpha z + \beta = 0$. □

Remark 8.10.78 There is a unique quasi-elliptic fibration p with singular fibers of type $2\tilde{D}_4 + 2\tilde{D}_4$ and nine elliptic fibrations with a singular fiber of type $\tilde{D}_8$. By construction, the canonical cover of $S_{\alpha,\beta}$ has four rational double point of type A_1. The support of the conductrix is the union of five (-2)-curves $R_4, \ldots, R_8$ on $S_{\alpha,\beta}$.

Remark 8.10.79 Salomonsson [633] studied Enriques surfaces of type $\tilde{E}_8, \tilde{E}_7^{(1)}, \tilde{E}_7^{(2)}$, $\tilde{E}_6 + \tilde{A}_2$ in a different point of view and gave equations of such Enriques surfaces.

Bibliographical Notes

Enriques himself knew that the group of birational automorphism of a general Enriques surface must be infinite because it contains elliptic pencils [219].

The discovery of the precise structure of the automorphism group of a general Enriques surface had to wait until the arrival of the transcendental methods based on the theory of periods of

K3 surfaces developed in the work of I. Pyatetsky-Shapiro and I. Shafarevich [598]. Using these methods W. Barth and C. Peters [42], and independently, V. Nikulin [557] proved Theorem 8.3.3 that asserts that this group is isomorphic to the 2-level congruence subgroup of the Weyl group $W_{2,3,7}$. The first geometric proof of this result, based on the ideas of A. Coble, was given by the first author in [172]. The assumption that $p \neq 2$ was eliminated later in [104]. The structure of the automorphism group of a general nodal Enriques surface over the complex numbers can also be deduced from Nikulin's paper. The first geometric proof of Theorem 8.4.7 was given by F. Cossec and the first author [137]; the details of the proof are given for the first time in Sections 8.4 and 8.5 of this book. Both cases rely on Coble's lattice theoretical description of natural involutions of Enriques surface [123] given by him, essentially, without proof. The computational proofs of these results given in [138] are now replaced by conceptual proofs provided by D. Allcock [6], [7]. Based on the known structure of the automorphism group, Cossec computed the degrees of the forgetting map from the moduli space of polarized general nodal Enriques surfaces to the moduli space of Enriques surfaces which we reproduced in Table 8.4.

The first example of a numerically trivial automorphism of an Enriques surface was constructed by D. Lieberman in 1976 [453]. The second one was found by Barth and Peters [42] and the first author [171]. It was erroneously claimed by Mukai and Namikawa in [532] that there are no more examples. However, the second author had found a third example in [409]. It was later proven by Mukai that there are no more examples over the field of complex numbers [528]. One of the main tools of the Mukai–Namikawa classification is the Global Torelli Theorem for K3-covers of Enriques surfaces. The absence of these tools in the case of characteristic $p > 0$ requires different methods. A paper [178] by the first author was the first attempt to extend the work of Mukai and Namikawa to this case. Although the main result of the paper is correct when $p \neq 2$, some arguments were not complete and the analysis of possible groups in characteristic 2 was erroneous and far from giving a classification of possible groups. In section 8.2, we follow a paper by the first author and G. Martin [191] which gives possible structures of groups of numerically or cohomologically trivial automorphisms of Enriques surfaces over fields of arbitrary characteristic. The extension of Mukai-Namikawa classification to the case of positive odd characteristic given in Theorem 8.2.23 is new.

A systematic study of possible groups of automorphisms of complex Enriques surfaces was undertaken by S. Mukai and Ohashi [533],[534], [535]. The goal here is to find them in terms of their Nikulin R-invariant. In his influential work, Mukai classified finite automorphisms groups of K3 surfaces that act trivially on the space of regular 2-forms [527]. They turned out to be subgroups of the Mathieu group M_{23}. By analogy, Mukai introduces finite groups of automorphisms of Enriques surfaces of Mathieu type, and together with Ohashi, they classify Enriques surfaces admitting such a group of automorphisms [534]. We give an exposition of their results in Section 8.8. The classification of involutions of complex Enriques surfaces was given by H. Ito and H. Ohashi [339]. In Section 8.7 we reproduce their classification in any characteristic except two by using geometric methods that do not rely on the theory of periods of K3 surfaces.

In his fundamental paper [219], Enriques asked whether the sextic model of an Enriques surface could degenerate in such a way that the group of its birational automorphisms becomes finite. An example of a family of such surfaces was given by the first author in [171]. It is now the family of type I in the Kondo–Nikulin classification. It was later discovered by the first author that, much earlier, Gino Fano had given another example, although his construction lacked rigor and his claim about the structure of the automorphism group was wrong [227]. Fano's example is a surface of type VII with the group of automorphisms isomorphic to $\mathfrak{S}_5$ (instead of $\mathfrak{S}_3$ claimed by Fano).

V. Nikulin was the first to classify complex Enriques surfaces with finite automorphisms in terms of their periods expressed by his R-invariant [559]. One of the cases was missing, and no geometric construction was given. In [409] the second author gave a geometric classification of such surfaces. Both results rely on the theory of periods of Enriques surfaces and some of the surfaces do not exist over fields of positive characteristic. The classification of Enriques surfaces with finite automorphism group in positive characteristic $p \neq 2$ was given by G. Martin [488]. Together with T. Katsura and the second author, they finished the classification in characteristic 2 [369], [371].

In some special cases, the infinite groups of automorphisms of Enriques surfaces were computed in terms of generators and relations. We will discuss this in the last chapter.

Chapter 9
Rational Coble Surfaces

Coble surfaces are smooth projective rational surfaces which appear as degenerations of Enriques surfaces and depend on one less parameter. We studied these surfaces over the complex numbers in Chapter 5 of Volume I. In this chapter, we continue this study without making any assumption on the characteristic. Many results discussed in this example are analogs of the previously discussed results on Enriques surfaces. In particular, we will classify Coble surfaces with finite automorphism groups.

9.1 Rational Coble Surfaces of K3 type

In Section 5.4, we discussed Coble surfaces over the field of complex numbers. In this section, we present some general facts about these surfaces which are valid in any characteristic.

Recall that a Coble surface is a rational surface V with $|-K_{\mathsf{V}}| = \emptyset$ and $|-2K_{\mathsf{V}}| \neq \emptyset$. We will be concerned only with *Coble surfaces of K3 type* for which we assume additionally that $|-2K_{\mathsf{V}}| = \{C\}$, where C is a reduced divisor. It follows from [195, Lemma 1.4] that C is the reduced union of smooth rational curves forming a simple normal crossing divisor. Blowing up its singular points, we obtain a Coble surface V' with smooth anti-bicanonical divisor C'. Such a Coble surface of K3 type is called *terminal*. Its irreducible components $C_1, \dots, C_n$ are smooth rational curves with self-intersection -4 ((-4)-curves for short). They are called the *boundary components* of V. We have

$$K_{\mathsf{V}}^2 = -n, \tag{9.1.1}$$

where n is the number of connected components of C. By Noether's formula.

$$e(\mathsf{V}) = 12 + n, \quad \rho(\mathsf{V}) = b_2(\mathsf{V}) = 10 + n. \tag{9.1.2}$$

Let

$$\pi : X \to \mathsf{V} \tag{9.1.3}$$

be the simple double cover defined by pair $(\mathcal{L}, s)$, where $\mathcal{L} = \mathcal{O}_{\mathsf{V}}(-K)$ and $s \in H^0(\mathsf{V}, \mathcal{L}^{\otimes 2})$ with $Z(s) = C$. We call X and π, the *canonical cover* of V.

Contrary to the case of Enriques surfaces, the surface X is always smooth if V is terminal.

Proposition 9.1.1 *Suppose V is a terminal Coble surface of K3 type. Then, the canonical cover X is a K3 surface.*

Proof If $p \neq 2$, the canonical class formula for the $\boldsymbol{\mu}_2$ cover $X \to \mathsf{V}'$ branched over C shows that $\omega_X \cong \mathcal{O}_X$. The ramification divisor of the cover is the disjoint union of n smooth rational curves, hence X is smooth. We also have

$$e(X) = 2e(X) + 2e(C) = 2(12 + n) - 2n = 24.$$

The classification of algebraic surfaces gives that X is a K3 surface.

If $p = 2$, the cover is an inseparable $\boldsymbol{\mu}_2$-cover.

Proposition 0.2.12 from Volume I implies that $\omega_X \cong \mathcal{O}_X$. By Proposition 0.2.10 in Volume I, X is smooth if $H^0(\mathsf{V}, \Omega^1_{\mathsf{V}/\Bbbk}(C)) = 0$. The boundary map in the exact sequence

$$0 \to \Omega^1_{\mathsf{V}/\Bbbk} \to \Omega^1_{\mathsf{V}/\Bbbk}(C) \xrightarrow{\text{res}} \mathcal{O}_C \to 0 \tag{9.1.4}$$

is a homomorphism $\delta : H^0(\mathsf{V}, \mathcal{O}_C) \to H^1(\mathsf{V}, \Omega^1_{\mathsf{V}/\Bbbk})$ whose image at $1 \in H^0(\mathsf{V}, \mathcal{O}_{C_i})$ is equal to the Chern class of the divisor class of C under the map

$$c_1 : H^1(\mathsf{V}, \mathcal{O}^*_{\mathsf{V}}) \to H^1(\mathsf{V}, \Omega^1_{\mathsf{V}/\Bbbk}).$$

This homomorphism corresponds to the map

$$\mathcal{O}^*_{\mathsf{V}} \to \Omega^1_{\mathsf{V}/\Bbbk},\ a \mapsto da/a.$$

It follows that δ is injective, hence

$$H^0(\mathsf{V}, \Omega^1_{\mathsf{V}/\Bbbk}(C)) \cong H^0(\mathsf{V}, \Omega^1_{\mathsf{V}/\Bbbk}) = 0.$$

Recall that a rational surface Z is called a *basic rational surface* if it admits a birational morphism to $\mathbb{P}^2$. An example of a non-basic rational surface is a minimal ruled surface $\mathbf{F}_n, n \neq 1$.

Lemma 9.1.2 *For any smooth rational curve E with negative E^2 on a terminal Coble surface of K3 type,*

$$E^2 \in \{-1, -2, -4\}.$$

If $E^2 = -4$, then E is an irreducible component of the anti-bicanonical curve C.

Proof This follows immediately from the adjunction formula $E^2 = -2 - E \cdot K_{\mathsf{V}} = -2 + \frac{1}{2}(E \cdot C)$. □

Proposition 9.1.3 *A terminal Coble surface V of K3 type is a basic rational surface. The image of the anti-bicanonical curve C under a birational morphism $f : \mathsf{V} \to \mathbb{P}^2$*

is a curve W of degree 6 with only double singular points (maybe infinitely near). Each irreducible component of W is a rational curve.

Proof Suppose V is not basic and let $\pi : \mathsf{V} \to \mathbf{F}_n, n \neq 1$, be a birational morphism to a minimal ruled surface. We choose n minimal possible. Obviously, $n \neq 1$. Also, $n \neq 0$, because otherwise π factors as $\mathsf{V} \to \mathrm{Bl}_x(\mathbf{F}_0) \to \mathbf{F}_0$, where $\mathrm{Bl}_x(\mathbf{F}_0)$ is the blow-up of some point $x \in \mathbf{F}_0$, and we use that there is a birational morphism $\mathrm{Bl}_x(\mathbf{F}_0) \to \mathbb{P}^2$. Let $x \in \mathbf{F}_n$ be one of the fundamental points of π^{-1}. Suppose that x does not belong to the exceptional section $\mathfrak{e}$ with $\mathfrak{e}^2 = -n$. Then, π factors through the blow-up $\mathrm{Bl}_x(\mathbf{F}_n)$ of $\mathbf{F}_n$ at x. The proper transform of the fiber of $\mathbf{F}_n \to \mathbb{P}^1$ passing through x is a (-1)-curve on $\mathrm{Bl}_x(\mathbf{F}_n)$ that intersects the proper transform of $\mathfrak{e}$. After we blow it down, we obtain a birational morphism $\mathrm{Bl}_x(\mathbf{F}_n) \to \mathbf{F}_{n-1}$ whose composition with $\mathsf{V} \to \mathrm{Bl}_x(\mathbf{F}_n)$ gives a birational morphism $\mathsf{V} \to \mathbf{F}_{n-1}$, contradicting the choice of n.

Thus, we may assume that π is an isomorphism over the complement of $\mathfrak{e}$. The proper transform $\bar{\mathfrak{e}}$ of $\mathfrak{e}$ on V is a smooth rational curve with $m = \bar{\mathfrak{e}}^2 \leq -3$. Applying Lemma 9.1.2, we obtain that $\bar{\mathfrak{e}}$ is an irreducible component of C and $m = -4$. Let $\mathcal{E}_1, \dots, \mathcal{E}_k$ be the exceptional configurations of $\pi : \mathsf{V} \to \mathbf{F}_n$. It follows from the same lemma that each irreducible component of $\mathcal{E}_i$ is a $(-l)$-curve with $l = 1, 2$, or 4. Since $\bar{\mathfrak{e}}$ does not intersect a (-2) or (-4)-curve, it intersects $\mathcal{E}_i$ at a point of one of (-1)-curves contained in $\mathcal{E}_i$. Blowing down all (-1)-curves that do not intersect $\bar{\mathfrak{e}}$ we may assume that each $\mathcal{E}_i$ is a chain of (-2)-curves that ends with a (-1)-curve that intersects $\bar{\mathfrak{e}}$. This means that $\mathsf{V} \to \mathbf{F}_n$ is obtained by blowing up a point on $\mathfrak{e}$, followed by the blow-up of the point where the proper transform of $\mathfrak{e}$ intersects the exceptional curve, and so on, and then doing the same with other fundamental points of π^{-1} on $\mathfrak{e}$. It follows that $-4 = \bar{\mathfrak{e}}^2 = -n - s$, where s is the number of the blow-ups in the factorization of π as a composition of blow-ups. Since $s = K^2_{\mathbf{F}_n} - K^2_{\mathsf{V}} > 8$, we get a contradiction.

So, V is a basic rational surface. The image of C under a birational morphism $f : \mathsf{V} \to \mathbb{P}^2$ is a curve $W \in |-2K_{\mathbb{P}^2}|$ of degree 6. By Lemma 9.1.2, each exceptional configuration of f is a chain of smooth rational curves $R_1 + \cdots + R_k + E_k$, where R_i are (-2)-curves and the end curve E_k is a (-1)-curve. This implies that all singular points of W including infinitely near are double points. Since each irreducible component of W is equal to the image of an irreducible compontent of C, it must be a rational curve. □

Proposition 9.1.4 *Any terminal Coble surface of K3 type is isomorphic to the blow-up of a Halphen surface of index* 1 *or* 2 *at singular points of two simple reduced fibers in the first case or one simple reduced fiber in the latter case. If the fiber is of type $\tilde{A}_1^*$ or $\tilde{A}_2^*$, we need to blow up twice or four times, respectively.*

Proof Choose a geometric basis $(e_0, e_1, \dots, e_{n+9})$ of $\mathrm{Pic}(\mathsf{V})$ corresponding to a birational morphism $f : \mathsf{V} \to \mathbb{P}^2$. Since $C \in |-2K_{\mathsf{V}}|$,

$$[C] = 6e_0 - 2e_1 - \cdots - 2e_{n+9}.$$

Note that $n \geq 1$ since $K_{\mathsf{V}}^2 < 0$. We use induction on $m = 9 - K_{\mathsf{V}}^2 = 9 + n$. The minimal possible value of m is equal to 10. In this case C is a (-4)-curve, and any (-1)-curve E intersects it with multiplicity 2. Let $\pi : \mathsf{V} \to X$ be the blowing down of E and $\bar{C} = \pi(C)$. Then, $K_X^2 = 0$ and $\bar{C} \in |-2K_X|$. It is an irreducible nodal or cuspidal curve of genus one. By Riemmann–Roch, $h^0(-K_X) \neq 0$ and $|-K_X|$ contains a curve F of arithmetic genus one such that $2F$ and $\bar{C}$ generate a pencil in $|-2K_X|$ which is base-point-free because $\bar{C}$ is an irreducible and $\bar{C}^2 = 0$. The pencil is an Halphen pencil of index 2 with double fiber F. The curve C is proper transform of $\bar{C}$ at its double point.

Assume now that $n > 1$. Following [195], we say that a Coble surface V is *minimal* if for any (-1)-curve E on V, the image of V under the blowing down of E is not a Coble surface. For example, if $n = 1$, the surface is minimal.

Suppose V is not minimal, and let E be a (-1)-curve on V such that its blowing down is a Coble surface V'. The curve E intersects C with multiplicity 2, hence it either intersects one of its irreducible components C_i with multiplicity 2 or intersects two irreducible components C_i, C_j, each with multiplicity 1. We say that E is of the first type in the first case and of the second type in the second case. Let $\pi : \mathsf{V} \to \mathsf{V}'$ be the blowing down of E. By induction, V' is the blow-up of a Halphen surface of index ≤ 2. The formula $K_{\mathsf{V}} = \pi^*(K_{\mathsf{V}'}) + E$ easily implies that C_i is the proper transform of a simple irreducible fiber if E is of the first kind and C_i and C_j are the proper transforms of two irreducible components of a reducible simple fiber intersecting each other.

Suppose V is a minimal Coble surface. Let $\pi : \mathsf{V} \to X$ be the blow-down of a (-1)-curve E on V. The argument from above shows $h^0(-K_X) \geq 2$ and we write $|-K_X| = |M| + G$, where $|M|$ is the mobile part and G the fixed part. Since V is minimal the arithmetic genus $p_a(C + 2E) = h^1(\mathcal{O}_{C+2E})$ is equal to one [195, Lemma 2.2]. This implies that E is of the first kind and the image $\bar{C}$ of C on X contains an irreducible genus one curve $\bar{C}_i$ as its connected component. All other connected components are (-4)-curves. It follows that $|M| = |C_i|$ and $|M|$ is a genus one pencil on X, and X is the blow-up of a Halphen surface. A (-1)-curve on X intersects M with multiplicity at most 2, so the index of the Halphen pencil is at most 2. The composition of the blow-up with π gives a birational morphism $f : \mathsf{V} \to \mathsf{H}$ to a Halphen surface H of index 1 or 2. The formula for the canonical class of the blow-up shows that f is the blow-up as from the assertion of the proposition. □

Note that a rational Coble surface which is of K3 type may not be a basic rational surface. A complete classification of all possible Coble surfaces as well as the proof of the results from above can be found in [195].

Corollary 9.1.5 *Let* V *be a Coble surface of K3 type. Then,*

$$K_{\mathsf{V}}^2 \geq -10.$$

In particular, the number n of boundary components of a terminal Coble surface of K3 type is at most 10.

Proof Over the complex numbers, we know this from Table 5.1 in Volume I. We use that V is obtained by blowing up the singular points of one reduced singular fiber (resp. two such fibers) of a Halphen surface H of index two (resp. one). We may assume that V is terminal. In the first case the fiber must be of type $\tilde{A}_n$ and in the second case the two fibers are of types $\tilde{A}_k, \tilde{A}_m$ (or their degenerations $\tilde{A}_k^*$). The classification of singular fibers of a rational elliptic surface gives $n \leq 8$ and $k+m \leq 8$. This gives that the maximal number of singular points is equal to 9 in the first case, and 10 in the second case. □

Remark 9.1.6 If a Halphen surface has a non-reduced singular fiber, we can blow up points on a non-reduced component as many times as we want to obtain a Coble surface which is not of K3 type. So, for an arbitrary Coble surface V, there is no lower bound for K_{V}^2.

Example 9.1.7 The case of a terminal Coble surface of K3 type with $n = 1$ was the case originally considered by A. Coble. In this case, the image of the anti-bicanonical curve C is a *rational nodal sextic*, i.e. an irreducible plane sextic $W = V(F_6)$ with 10 ordinary nodes or cusps (maybe infinitely near). If we choose nine of them, then we will find a plane cubic curve $V(F_3)$ that passes through them. The pencil $V(\lambda F_6 + \mu F_3^2)$ is a Halphen pencil of index 2. The minimal resolution of its base points is a Halphen surface of index 2. The Coble surface V is obtained by blowing up a singular point of an irreducible simple fiber of the pencil. If we blow-up a singular point of a reducible simple fiber F instead, we obtain a non-terminal Coble surface of K3 type with $n = 1$. Blowing up the singular points of its connected anti-bicanonical curve equal to the proper transform of the fiber, we obtain a terminal Coble surface of K3 type with n equal to the number of components of F.

For future use, we will need the following:

Proposition 9.1.8 *Assume $p \neq 2$. Let V be a terminal Coble surface of K3 type with $K^2 = -n$. Then,*

$$h^i(\Theta_{\mathsf{V}/\mathbb{k}}) = \begin{cases} 10 + 2n & \text{if } i = 1, \\ 0 & \text{otherwise}. \end{cases}$$

Proof Let $\pi : X \to \mathsf{V}$ be the canonical double cover. By Proposition 9.1.1, the surface X is a K3 surface, and, by Rudakov-Shafarevich [626], $H^0(X, \Theta_{X/\mathbb{k}}) = 0$. We use exact sequence (0.2.9) from Volume I:

$$0 \to \Theta_{X/\mathbb{k}} \to \pi^*\Theta_{\mathsf{V}/\mathbb{k}} \to \Theta_{X/\mathsf{V}} \to 0.$$

Since $\pi_* O_X \cong O_{\mathsf{V}} \oplus O_{\mathsf{V}}(K)$, by the projection formula, for any coherent sheaf $\mathcal{F}$ on V we have

$$H^i(X, \pi^*\mathcal{F}) \cong H^i(\mathsf{V}, \mathcal{F}) \oplus H^i(\mathsf{V}, \mathcal{F}(K_{\mathsf{V}})).$$

This gives

$$H^0(X, \pi^*\Theta_{\mathsf{V}/\Bbbk}) \cong H^i(\mathsf{V}, \Theta_{\mathsf{V}/\Bbbk}) \oplus H^i(\mathsf{V}, \Theta_{\mathsf{V}/\Bbbk}(K_{\mathsf{V}})) \cong H^i(\mathsf{V}, \Theta_{\mathsf{V}/\Bbbk}) \oplus H^{2-i}(\mathsf{V}, \Omega^1_{\mathsf{V}/\Bbbk}),$$

where we used the Serre Duality Theorem.

In particular, we get

$$H^i(X, \pi^*\Theta_{\mathsf{V}/\Bbbk}) = H^i(\mathsf{V}, \Theta_{\mathsf{V}/\Bbbk}),\ i = 0, 2$$

and

$$H^i(X, \pi^*\Theta_{\mathsf{V}/\Bbbk}) = H^1(\mathsf{V}, \Theta_{\mathsf{V}/\Bbbk}) \oplus H^1(\mathsf{V}, \Omega^1_{\mathsf{V}/\Bbbk}).$$

By Corollary 0.2.9, we have a projective resolution

$$0 \to \pi^*\mathcal{O}_{\mathsf{V}}(-K_{\mathsf{V}}) \to \pi^*\mathcal{O}_{\mathsf{V}}(-2K_{\mathsf{V}}) \to \Theta_{X/\mathsf{V}} \to 0.$$

By the projection formula, we easily find that

$$h^0(\Theta_{\mathsf{V}/\Bbbk}) = h^0(\mathcal{O}_{\mathsf{V}}(-2K_{\mathsf{V}})) - h^0(\mathcal{O}_{\mathsf{V}}) = 0.$$

Thus, using the computations from the above, we conclude that

$$H^0(\mathsf{V}, \Theta_{\mathsf{V}/\Bbbk}) = 0.$$

We know that $H^2(X, \Theta_{X/\Bbbk}) = 0$; similar computations show that

$$H^2(\mathsf{V}, \Theta_{\mathsf{V}/\Bbbk}) = 0.$$

The last equality $h^1(\mathsf{V}, \Theta_{\mathsf{V}/\Bbbk}) = 10 + 2n$ follows from the Riemann–Roch Theorem applied to the rank 2 locally free sheaf $\Theta_{\mathsf{V}/\Bbbk}$. We have

$$\chi(\Theta_{\mathsf{V}/\Bbbk}) = K_{\mathsf{V}}^2 - c_2(\mathsf{V}) + 2\chi(\mathcal{O}_{\mathsf{V}}) = -n - (12 + n) + 2 = -10 - 2n.$$

Remark 9.1.9 Note that $h^1(\mathsf{V}, \Theta_{\mathsf{V}/\Bbbk})$ coincides with the number of moduli of $9 + n$ points in $\mathbb{P}^2$ modulo projective equivalence. To get the number of moduli for Coble surfaces with n boundary components, we have to use the number $h^1(\Theta_{\mathsf{V}/\Bbbk}(-C))$. Taking the dual of the logarithmic exact sequence (9.1.4), we obtain an exact sequence

$$0 \to \Theta_{\mathsf{V}/\Bbbk}(-C) \to \Theta_{\mathsf{V}/\Bbbk} \to \mathcal{E}xt^1(\mathcal{O}_C, \mathcal{O}_{\mathsf{V}}) \to 0.$$

By Serre duality,

$$\mathcal{E}xt^1(\mathcal{O}_C, \mathcal{O}_{\mathsf{V}}) \cong \mathcal{E}xt^1(\mathcal{O}_C, \omega_{\mathsf{V}}) \otimes \omega_{\mathsf{V}}^{-1} \cong \omega_C \otimes \omega_{\mathsf{V}}^{-1},$$

and by the adjunction formula,

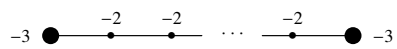
$$\omega_C \otimes \omega_{\mathsf{V}}^{-1} \cong \mathcal{O}_C(C).$$

Since $\mathcal{O}_{C_i}(C_i) \cong \mathcal{O}_{\mathbb{P}^1}(-4)$, we get

$$h^1(\Theta_{\mathsf{V}/\Bbbk}(-C)) = h^1(\Theta_{\mathsf{V}/\Bbbk}) - 3n = 10 + 2n - 3n = 10 - n.$$

For example, if $n = 1$, we get $h^1(\Theta_{\mathsf{V}/\Bbbk}(-C)) = 9$ that agrees with our computations of the number of moduli of complex Coble surfaces in Section 5.4 in Volume I. It also agrees with the number of moduli for general Halphen surfaces of index two.

Let V be a Coble surface of K3 type with boundary components $C_1, \dots, C_s$. Let H be a nef and big divisor such that $H \cdot C_i = 0$. We can choose H such that the map $\mathsf{V} \to X$ defined by the linear system $|H|$ is an isomorphism outside C. It blows down each C_i to a singular point $x_i \in X$. It follows from Proposition 9.1.4 that the exceptional curve over each point x_i is either a (-4)-curve or a chain of rational curves $R_1 + R_2 + \cdots + R_k$, where $R_1^2 = R_k^2 = -3, R_i^2 = -2, i \neq 1, k$ and $R_i \cdot R_{i+1} = 1$.



A singular point of this type is a special case of a *toric singularity* of class $\mathcal{T}$. It is denoted by $\frac{1}{dn^2}(1, dna - 1)$. If $(p, dn) = 1$, It is a special cyclic quotient singularities and the notation agrees with the standard one. We will be interested only in singularities $\frac{1}{4k}(1, 2k - 1)$ which have exceptional curves with k irreducible components as above. In arbitrary characteristic they are isomorphic to the quotient of a rational double point of type A_k by $\boldsymbol{\mu}_2$. Their Gorenstein index is equal to 2. The speciality of toric singularities of class $\mathcal{T}$ is that they admit $\mathbb{Q}$-Gorenstein smoothings [449, Theorem 4.7].

Theorem 9.1.10 *Let Y be a normal projective surface over $\Bbbk$ with only toric singularity of type $\mathcal{T}$ besides rational double points. Assume that Y satisfies the following conditions*

(a) $H^2(Y, \Theta_{Y/\Bbbk}) = 0$.
(b) $H^2(Y, \mathcal{O}_Y) = 0$.

Then, there is a deformation $f : \mathcal{Y} \to T$ of Y over nonsingular algebraic curve T defined over $\Bbbk$ such that

(i) The morphism f is projective and it is smooth over $T \setminus \{t_0\}$.
(ii) The fiber $\mathcal{T}_{t_0}$ over t_0 is isomorphic to Y.
(iii) The scheme $\mathcal{Y}$ is normal and, for some integer r, the divisor class $rK_{\mathcal{Y}}$ is Cartier.
(iv) If r is the Gorenstein index of Y, then $\mathcal{O}_{\mathcal{Y}}(rK_{\mathcal{Y}}) \otimes \mathcal{O}_{\mathcal{Y}_{t_0}} \cong \mathcal{O}_Y(rK_Y)$.
(v) For any $t \neq t_0$, $K^2_{\mathcal{Y}_t} = K^2_Y, H^i(\mathcal{Y}_t, \mathcal{O}_{\mathcal{Y}_t}) \cong H^i(Y, \mathcal{O}_Y) = 0$.
(vi For any $t \neq t_0$, $H^2(\mathcal{Y}_t, \Theta_{\mathcal{Y}_t/\Bbbk}) = 0$.

We want to apply this theorem to our situation, where Y is obtained from a terminal Coble surface of K3 type by blowing down the boundary components $C_1, \dots, C_n$. To blow down, we use a linear system $|4D + \sum_{i=1}(D \cdot C_i)C_i|$, where D is any very ample divisor on V. By [449, Proposition 2.20], $H^2(Y, \Theta_{Y/\Bbbk}) \cong H^2(\mathsf{V}, \Theta_{\mathsf{V}/\Bbbk})$ if $p \neq 2$, and, applying Proposition 9.1.8, we verify conditions (a) and (b) in the theorem.

Corollary 9.1.11 *Assume $p \neq 2$. Let V be a Coble surface of K3 type. Then, there exists a projective flat morphism $f : \mathcal{X} \to T$ over a nonsingular algebraic curve T defined over $\Bbbk$ such that $\mathcal{X}_{t_0}$ is isomorphic to the blow-down of the boundary components of V and $\mathcal{X}_t$ is an Enriques surface.*

Let $\sigma : \mathsf{V} \to \mathsf{H}$ be a morphism from a Coble surface of K3 type to a Halphen surface H. Let $E_1, \dots, E_l$ be the (-1)-curves that are blown down to singular points $y_1, \dots, y_l$ of a fiber F of type $\tilde{A}_m$ on H. Then, the proper transform of F on V is the union of l disjoint (-4)-curves.

We know from Section 5.4 that the moduli space of complex Coble surfaces with n boundary components is irreducible of dimension $10 - n$ if $n \neq 8$ and consists of two irreducible components when $n = 8$. This follows from the computation of the Picard lattice of their canonical K3-covers. Let us give a construction of a general member V of an irreducible family of dimension $10 - n$ for each $n = 1, \dots, 10$.

In the first examples, we exhibit V as the blow-up of the plane at double points of a plane sextic curve W with only nodes as singularities.

$n = 1$: W is an irreducible rational sextic with 10 nodes.

$n = 2$: W is the union of two nodal plane cubics intersecting at 9 distinct points.

$n = 3$: W is the union of three conics intersecting pairwise at 4 distinct points.

$n = 4$: W is as in the previous case only one of the conics is reducible.

For $5 \leq n \leq 10$ we exhibit V as the blow-up of singular points of the union of five conics on an anti-canonical del Pezzo surface $\mathcal{D}_5$ of degree 5 taken from different five pencils of conics on V. The proper transforms of the irreducible components of the conics is the boundary of V.

Recall that $\mathcal{D}_5$ is isomorphic to the blow-up of four points $p_1, \dots, p_4$ in the plane no three of which are collinear. We use the standard geometric basis $(e_0, e_1, \dots, e_4)$ of $\mathrm{Pic}(\mathcal{D}_5)$ defined by the blow-up. The five pencils L_i of conics are the linear systems $|e_0 - e_1|, \dots, |e_0 - e_4|, |2e_0 - e_1 - e_2 - e_3 - e_4|$. The surface has ten lines which we index by duads (ab) from $[1, 5]$. They are the exceptional curves E_{a5} over the points p_a and the proper transforms of lines $\ell_{ab} \in |e_0 - e_c - e_d|$, where $\{a, b, c, d\} = \{1, 2, 3, 4\}$.

$n = 5$: $K_i \in L_i, i = 1, \dots, 5$, are irreducible. The image of $K_i, i \neq 5$, in the plane is a lines passing through p_i and not passing through other points p_j. The image of K_5 is an irreducible conic containing exactly four points p_i.

$n = 6$: One of the conics K_i is reducible. There are three such conics in each pencil. If $i = 5$, the image of K_5 is equal to the proper transform of two lines $\ell_{a,b}, \ell_{cd}$. In this way, we see V as the blow-up of intersection points of six lines in the general linear position. If $i \neq 5$, the image of K_i is equal to the exceptional curve E_i and a line whose image in the plane is a line ℓ_{ij}. There are five projectively non-equivalent, but Cremona equivalent, ways to exhibit $\mathcal{D}_5$ as the blow-up of four points in the plane and hence $5! = \#\,\mathrm{Aut}(\mathcal{D}_5)$ ways to define a geometric basis in $\mathrm{Pic}(\mathcal{D}_5)$. All different choices of one reducible conic are equivalent under a change of a geometric basis. In another words, if we consider a family of marked del Pezzo surfaces, there will be $15 = 3 \times 5$ different choices of a reducible conic, but if we forget about the marking there will be only one choice.

$n = 7$: Two conics are reducible. We can choose them from two pencils $L_i, L_j, i, j \neq 5$, say from L_1 and L_2. The image will be two lines $\ell_{1,i}, \ell_{2,j}$, two general lines through p_i, p_j and a smooth conic through p_1, p_2, p_3, p_4. Applying a quadratic Cremona transformation with fundamental points at p_1, p_i, p_j we may assume that one of the reducible conics is taken from L_5. In this case, the image of the five conics in the plane is the union of six lines with one triple point.

$n = 8$: Three reducible conics. We may take them from the pencils L_1, L_2, L_3. The image of the conics could be the union of four lines $\ell_{12}, \ell_{2,3}, \ell_{34}$, a general line through p_4 and a smooth conic through p_1, p_2, p_3, p_4. Applying a quadratic Cremona transformation with fundamental points at p_2, p_3, p_4, we may assume that one of the reducible conics is taken from L_5. We can take one of them from L_5 and two from L_1, L_2. The image of the five conics in the plane could be the union of the lines ℓ_{13}, ℓ_{24}, two general lines through p_3, p_4 and the union of two lines $\ell_{a,b}$ and ℓ_{cd}. There will be two different choices: (a) the two lines are ℓ_{14} and ℓ_{23} or (b) one of these lines passes through p_3, p_4. We can draw the following two pictures which display the two different choices:

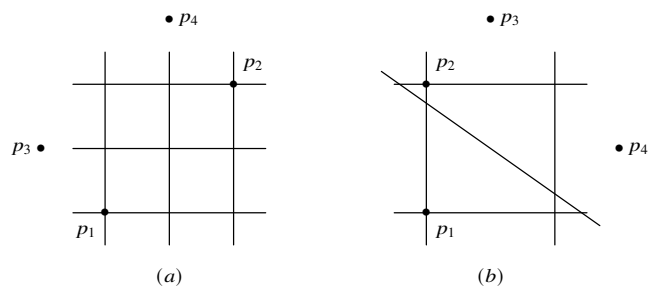


Fig. 9.1 Two families of Coble surfaces with eight boundary components

Here, the parallel lines meet at one of the points p_3, p_4. In the second picture, one of the lines is the line at infinity passing through p_3, p_4.

By using the rational map $\mathbb{P}^2 \to Q = \mathbb{P}^1 \times \mathbb{P}^1$ defined by the linear system of conics through p_3, p_4, we may consider a surface of type (a) as the blow-up of the intersection points of eight lines on Q, four from each family of lines and a surface of type (b) as the blow-up of the intersection points of six lines, three from each family, and a conic passing through one of the intersection points. We also blow-up the infinitely near points corresponding to the tangent directions of the two lines and the conic at this point.

We know from Table 5.1 in Volume I that the moduli space of complex Coble surfaces with eight boundary components consists of two irreducible components. It is a natural guess that the two different choices (a) and (b) lead to different components. To show this, we compute the Picard lattice of the canonical K3-covers X of these surfaces. Using the model of V as the blow-up of points on a quadric, we locate two elliptic pencils on X coming from the two rulings on Q. In the case (a), it has four fibers of type $\tilde{D}_4$ and four disjoint sections. The Picard lattice is isomorphic

to $\mathsf{U} \oplus \mathsf{D}_8 \oplus \mathsf{D}_8$. In the case (b), it has two fibers of type $\tilde{D}_6$, two fibers of type $\tilde{A}_1$ and 4 sections. The Picard lattice is isomorphic to $\mathsf{U} \oplus \mathsf{E}_7 \oplus \mathsf{E}_7 \oplus \mathsf{A}_1 \oplus \mathsf{A}_1$. The discriminant group in both cases is $(\mathbb{Z}/2\mathbb{Z})^{\oplus 4}$ but the discriminant form is even in case (a) and odd in case (b).

$n = 9$: four conics are reducible. We may take for the images of the five conics the union of four lines $\ell_{12}, \ell_{23}, \ell_{34}, \ell_{14}$ and a smooth conic through the four points.

$n = 10$: all conics are reducible. The surface is the blow-up of 15 intersection points of 10 lines on V. The image of the union of the five conics is the complete quadrangle of lines and its two diagonals. We will discuss this surface later in Example 9.2.7.

9.2 Coble–Mukai Lattice

In this section, we introduce the Coble–Mukai lattice of a terminal Coble surface of K3-type. It substitutes the Enriques lattice and allows one to define effective roots which are analogs of the classes of (-2)-curves on an Enrique surface. We will prove that, over the complex numbers, as an abstract lattice, the Coble–Mukai lattice is isomorphic to the Enriques lattice.

From now on, we will consider only terminal Coble surfaces of K3 type. Let V be such a surface and let $C = C_1 + \cdots + C_n$ be its anti-bicanonical curve. We denote by β_i the divisor class of C_i. We have $\beta_i^2 = -4, K_{\mathsf{V}} \cdot \beta_i = 2$. Let $\widetilde{\mathrm{Pic}}(\mathsf{V})$ be the $\mathbb{Z}$-submodule of the quadratic vector $\mathbb{Q}$-space $\mathrm{Pic}(\mathsf{V})_{\mathbb{Q}}$ generated by $\mathrm{Pic}(\mathsf{V})$ and $\frac{1}{2}\beta_1, \dots, \frac{1}{2}\beta_n$. Following S. Mukai, we introduce the following quadratic lattice.

$$\mathrm{CM}(\mathsf{V}) := \{x \in \widetilde{\mathrm{Pic}}(\mathsf{V}) : x \cdot \beta_i = 0, i = 1, \dots, n\}. \tag{9.2.1}$$

Let us see that it is indeed a quadratic lattice. Any $x \in \mathrm{CM}(\mathsf{V})$ may be expressed as $y + \frac{1}{2}\sum_{i=1}^k r_i\beta_i$, where $y \in \mathrm{Num}(\mathsf{V})$ and $r_1, \dots, r_k \in \mathbb{Z}$. Rewriting this as $y = x - \frac{1}{2}\sum r_i\beta_i$, expressing $x' \in \mathrm{CM}(\mathsf{V})$ similarly, and using $x, x' \in \beta_i^\perp$ gives

$$y \cdot y' = x \cdot x' + \frac{1}{4}\sum r_i r_i' \beta_i^2 = x \cdot x' - \sum r_i r_i'$$

This proves $x \cdot x' \in \mathbb{Z}$.

We call $\mathrm{CM}(\mathsf{V})$ the *Coble–Mukai lattice* of V. If $n = 1$, the lattice coincides with $K_{\mathsf{V}}^\perp$ and hence it is isomorphic to the Enriques lattice E_{10}. We will show later that the Coble–Mukai lattice is always isomorphic to E_{10} under the assumption $\Bbbk = \mathbb{C}$ (see Theorem 9.2.14).

Obviously, the automorphism group $\mathrm{Aut}(\mathsf{V})$ leaves the set of curves $\{C_1, \dots, C_n\}$ invariant and hence acts on the lattice $\mathrm{CM}(\mathsf{V})$.

By Proposition 9.1.4, there exists a birational morphism $\phi : \mathsf{V} \to \mathsf{H}$, where H is a Halphen surface of index 1 or 2. Let $\pi : \mathsf{H} \to \mathbb{P}^2$ be a birational morphism and let $(e_0, e_1, \dots, e_9)$ be a geometric basis of $\mathrm{Pic}(\mathsf{H})$ (see Section 0.3). Let $(e_0, e_1, \dots, e_9, r_1, \dots, r_n)$ be a geometric basis of $\mathrm{Pic}(\mathsf{V})$.

Let $W_{\mathsf{V}}^{\mathrm{nod}}$ be the *reflection group* of V, the subgroup of $\mathrm{O}(\mathrm{CM}(\mathsf{V}))$ generated by reflections s_α, where $\alpha \in \mathrm{CM}(\mathsf{V})$ is an effective divisor class with $\alpha^2 = -2$. We call such a divisor class an *effective root*. We say that an effective root α is *irreducible* if $|\alpha - \beta| = \emptyset$ for any other effective root β.

Lemma 9.2.1 *Let α be an effective irreducible root. Then, α is either the divisor class of a (-2)-curve or the $\mathbb{Q}$-divisor class of an effective root of the form $2e + \frac{1}{2}\beta_j + \frac{1}{2}\beta_k$, where e is the class of a (-1)-curve E that intersects two different components C_j, C_k of C.*

Proof Writing an effective class R from $|\alpha|$ as a sum of irreducible curves and the halves of the boundary components, we may assume that

$$R = \frac{1}{2}\sum_{i \in I} C_i + D,$$

where D has no (-2)-curves among its irreducible components. Suppose $\#I = 1$. We have $R \cdot K_{\mathsf{V}} = 1 + D \cdot K_{\mathsf{V}} = 0$ and $-2 = R^2 = -1 + C_1 \cdot D + D^2 = -1 - 2D \cdot K_{\mathsf{V}} + D^2 = 1 + D^2$, hence $D^2 = -3, D \cdot K_{\mathsf{V}} = -1$. Applying Lemma (9.1.2), we can write D as a sum of (-1)-curves and curves N with $N \cdot K_{\mathsf{V}} \leq 0$ and $N^2 \geq 0$, we get a contradiction.

Assume now that $a = \#I \geq 2$. Arguing as above, we obtain $D \cdot K_{\mathsf{V}} = -a$ and $D^2 = -2 - a$.

Write

$$[D] = [m_1 E_1 + \cdots + m_t E_t + D'] = \alpha - \frac{1}{2}\sum_{i \in I} \beta_i, \tag{9.2.2}$$

where E_i are all (-1)-components of D and D' is a sum of curves with non-negative self-intersections. Intersecting with K_{V}, we get $a = \sum m_i - D' \cdot K_{\mathsf{V}} \geq \sum m_i$. We also have $D^2 = -2 - a \geq -\sum m_i^2$. Thus, we get the following inequalities:

$$\sum_{i=1}^{t} m_i \leq a, \quad \sum_{i=1}^{t} m_i^2 \geq a + 2.$$

If all m_i's are equal to 1, these inequalities are not satisfied. Thus, we may assume that $m_1 \geq 2$. Intersecting with C_i, we obtain $0 = R \cdot C_i \geq -2 + 2E_1 \cdot C_i$. This implies that we can find C_i and C_j such that $E_1 \cdot C_i = E_1 \cdot C_j = 1$. Thus, we find an irreducible effective root $\frac{1}{2}(C_i + C_j) + 2E_1$ as a part of R. By assumption of irreducibility, R must be equal to this effective root. □

Let $W_{\mathsf{V}}^{\mathrm{nod}}$ be the subgroup of the orthogonal group $\mathrm{O}(\mathrm{CM}(\mathsf{V}))$ of Coble–Mukai lattice generated by reflections in the classes of effective roots.

Let

$$\mathrm{CM}(\mathsf{V})^+ = \{x \in \mathrm{CM}(\mathsf{V}) : x^2 \geq 0, x \cdot h > 0 \text{ for some ample divisor } h\}.$$

By Riemann–Roch on V, $\mathrm{CM}(\mathsf{V})^+$ consists of effective divisor classes with non-negative self-intersection.

Proposition 9.2.2 *A divisor class $x \in \mathrm{CM}(\mathsf{V})^+$ is nef if and only if $x \cdot r \geq 0$ for any effective root. In other words, the intersection of the nef cone in $\mathrm{Pic}(\mathsf{V})_{\mathbb{Q}}$ with $\mathrm{CM}(\mathsf{V})^+$ is a fundamental domain for the group $W_{\mathsf{V}}^{\mathrm{nod}}$ in $\mathrm{CM}(\mathsf{V})^+$.*

Proof Suppose $x \in \mathrm{CM}(\mathsf{V})^+$. Then, x is nef if and only if $x \cdot r \geq 0$ for every effective divisor class $r \in \mathrm{Pic}(\mathsf{V})$ with $r^2 = -1$ or -2. We may assume that r is the class of a (-1)-curve or of a (-2)-curve. Let x be a nef class in $\mathrm{CM}(\mathsf{V})^+$. Applying Lemma 9.2.1, it suffices to show that $x \cdot r \geq 0$ for any effective root of the form $r = 2e + \frac{1}{2}(\beta_i + \beta_j)$. Since $x \cdot \beta_i = 0$, and $x \cdot e \geq 0$, this is obvious. Thus, x belongs to the fundamental domain of $W_{\mathsf{V}}^{\mathrm{nod}}$. Conversely, suppose x belongs to the fundamental domain. By definition, $x \cdot r \geq 0$ for any divisor class of a (-2)-curve and $x \cdot r \geq 0$ for any positive root of the form $2e + \frac{1}{2}(\beta_i + \beta_j)$. Since $x \cdot \beta_i = 0$, we obtain $x \cdot e \geq 0$. Thus, for any (-1)-curve E that intersects two different C_i, we have $x \cdot [E] \geq 0$. Suppose E is a (-1)-curve that intersects only one C_i with multiplicity 2. Let $f = [E] + \frac{1}{2}\beta_i$. Then, $f^2 = 0$ and $f \in \mathrm{CM}(\mathsf{V})^+$. Let $\pi : \mathsf{V} \to X$ be the blowing-down of E to a point $q \in X$. Then, the divisor class $2f = [2E + C_i] \in \mathrm{Pic}(\mathsf{V})$ is equal to $\pi^*(\bar{C}_i)$, where $\bar{C}_i = \pi(C_i)$ is an irreducible curve with $\bar{C}_i^2 = 0$ and a node at q. Suppose $x \cdot E < 0$, then $x \cdot 2f = 2x \cdot E < 0$ and we obtain $x \cdot 2f = \pi_*(x) \cdot \bar{C}_i < 0$. Since $[\bar{C}_i]$ is obviously nef, we get a contradiction. □

We say that a Coble surface of K3 type is *unnodal* if it does not contain effective roots.

Proposition 9.2.3 *A Coble surface V of K3-type is unnodal if and only if $n = 1$ and it has no (-2)-curves.*

Proof Using Lemma 9.2.1, it suffices to show that $n = 1$ if V is unnodal. Suppose $n > 1$. By the same lemma, any (-1)-curve E intersects one component of C with multiplicity 2. Thus, no component C_i of C is blown down to a point under the morphism $\pi : \mathsf{V} \to \mathbb{P}^2$, and the image B_i of C_i is a smooth rational curve of degree ≤ 6. Since $n > 1$, the curve $B = B_1 + \cdots + B_n$ is reducible, and any two components intersect transversally (because V does not contain (-2)-curves). The exceptional curve R over an intersection point will intersect two components of C contradicting the assumption. □

Remark 9.2.4 One can show (see [104]) that a Coble surface is unnodal if and only if it is obtained by blowing up a set $\{p_1, \ldots, p_{10}\}$ of ten points in the plane satisfying the following 496 conditions:

(i) no points among the ten points are infinitely near;
(ii) no three points are collinear;
(iii) no six points lie on a conic;
(iv) no plane cubic passes through eight points with one of them being a singular point of the cubic;

(v) no plane quartic curve passes through the ten points with one of them being a triple point.

In Section 3.5 of Volume I, we studied Fano models of Enriques surfaces. They are normal surfaces of degree 10 in $\mathbb{P}^5$. One can also introduce the Fano models of Coble surfaces.

Assume $n = 1$. As we remarked before, $\mathrm{CM}(\mathsf{V}) \cong \mathsf{E}_{10}$. Let

$$f_i = 3e_0 - e_1 - \cdots - e_{10} + e_i,\ i = 1, \ldots, 10,$$
$$h = \frac{1}{3}(f_1 + \cdots + f_{10}) = 10e_0 - 3(e_1 + \cdots + e_{10}).$$

We have $f_i^2 = 0, f_i \cdot f_j = 1, h^2 = 10, h \cdot f_i = 3$. The classes f_i represent the proper transforms of cubic curves passing through the points $p_j, j \neq i$. Since the plane sextic W is irreducible, all f_i are nef divisors, and $(f_1, \ldots, f_{10})$ is a maximal non-degenerate isotropic sequence of vectors in $\mathrm{CM}(\mathsf{V})$. The linear system $|h|$ defines a birational map from V onto a surface of degree 10 in $\mathbb{P}^5$. We call it the *Fano model* of a Coble surface. The images of effective divisors F_i representing f_i are plane cubics. The anti-bicanonical curve C is blown down to a singular point of type $\frac{1}{4}(1, 1)$ of the Fano model.

We can also consider the adjoint linear system $|h + K_{\mathsf{V}}|$ representing the divisor class $7e_0 - 2(e_1 + \cdots + e_{10})$. This time, the linear system is very ample and its image is a surface F of degree 9 in $\mathbb{P}^5$. The images of F_i are still cubic curves, but the image of C is a curve of degree 2. The union of F and the plane spanned by the image of C is a reducible surface of degree 10 in $\mathbb{P}^5$. We call it the *adjoint Fano model*.

Example 9.2.5 Assume V is a general Coble surface (as always terminal and of K3 type) with $n = 2$ and let $|-2K_{\mathsf{V}}| = \{C_1 + C_2\}$. Let $\pi : \mathsf{V} \to \mathbb{P}^2$ be a birational morphism. Let $W_i = \pi(C_i)$ and let d_i be the degree of W_i. We have the following possibilities $\{d_1, d_2\} = \{1, 5\}, \{2, 4\}, \{3, 3\}$. We assume $d_1 \leq d_2$, then there are three possibilities: W_1 is a line intersecting a 6-nodal quintic W_2 at 5 distinct points, or W_1 is a smooth conic intersecting a 3-nodal quartic W_2 at eight points, or W_1 and W_2 are nodal cubics intersecting at nine distinct points.

Suppose $(d_1, d_2) = (1, 5)$. Let T be a quadratic Cremona transformation with three nodes of W_2 as its fundamental points. Then, $T(W_1)$ is a conic and $T(W_2)$ is a 3-nodal quartic. Thus, the cases $(1, 5)$ and $(2, 4)$ can be reduced to each other by a Cremona transformation. This means that we get isomorphic Coble surfaces, but the birational morphisms to $\mathbb{P}^2$ are different. We can also reduce the case $(2, 4)$ to the case $(3, 3)$ by taking a quadratic Cremona transformation with two fundamental points at two nodes of the quartic and the third fundamental point taken from $W_1 \cap W_2$.

Thus, we may assume that W_1, W_2 are nodal cubics and V is obtained by blowing up their intersection points $p_1, \ldots, p_9$ and the nodes p_{10}, p_{11} of W_1 and W_2. Let H be the rational elliptic surface obtained by blowing up $p_1, \ldots, p_9$. Then, the proper transforms of W_1, W_2 are two nodal fibers of the elliptic fibration, and V is obtained by blowing up the singular points of these two fibers.

Following Mukai, we consider a birational map $\mathbb{P}^2 \dashrightarrow \mathbb{P}^1 \times \mathbb{P}^1 \subset \mathbb{P}^3$ given by the linear system of conics through the points p_{10} and p_{11}. The composition map

$\mathsf{V} \to \mathbb{P}^2 \dashrightarrow \mathbb{P}^1 \times \mathbb{P}^1$ is a birational morphism that maps W_1 (resp. W_2) onto a smooth rational curve $\bar{W}_1$ of bidegree $(1,3)$ (resp. $\bar{W}_2$ of bidegree $(3,1)$) intersecting at 10 points. The surface V is obtained by blowing up the set $\{q_1, \ldots, q_9, q_{10}\} = \bar{W}_1 \cap \bar{W}_2$, where q_{10} is equal to the image of the line $\langle p_{10}, p_{11}\rangle$.

Let $(h, h', e_1, \ldots, e_{10})$ be a basis of $\mathrm{Pic}(\mathsf{V})$ formed by the pre-images of a canonical basis of $\mathrm{Pic}(\mathbb{P}^1 \times \mathbb{P}^1)$ and the divisor classes of the exceptional curves. In this basis, the boundary components are

$$\beta_1 = [C_1] = 3h + h' - (e_1 + \cdots + e_{10}), \quad \beta_2 = [C_2] = h + 3h' - (e_1 + \cdots + e_{10}).$$

We have $\beta_1 + \beta_2 = -2K_{\mathsf{V}}$ and $\beta_1 - \beta_2 = 2(h - h')$. Thus, we can write $\beta_1 = -K_{\mathsf{V}} + h - h'$ and $\beta_2 = -K_{\mathsf{V}} + h' - h$. This shows that $\mathrm{CM}(\mathsf{V})$ contains a sublattice of index 2 equal to the orthogonal complement of $\langle K_{\mathsf{V}}, h - h'\rangle$ in $\mathrm{Pic}(\mathsf{V})$. We have $K_{\mathsf{V}}^{\perp} \cong \mathsf{E}_{11}$ with canonical root basis $(\alpha_0 = e_1 - e_2, \alpha_1 = h' - h, \alpha_2 = h - e_1 - e_2, \alpha_3 = e_2 - e_3, \ldots, \alpha_{10} = e_9 - e_{10})$. The orthogonal complement of $h - h'$ in this lattice has a basis $(h + h' - e_1 - e_2 - e_3 - e_4, e_1 - e_2, \ldots, e_9 - e_{10})$. This is a canonical root basis of the lattice $\mathsf{E}_{2,4,6}$ of discriminant 4. Since $\mathrm{CM}(\mathsf{V})$ contains this lattice as a sublattice of index 2, it must be a unimodular lattice. Let

$$v = \frac{1}{2}(h + h' - K_{\mathsf{V}}) - e_1 = \frac{1}{2}\beta_1 + h' - e_1.$$

We have $v \cdot (\beta_1 - \beta_2) = v \cdot (h - h') = 0$ and $v \cdot (\beta_1 + \beta_2) = -2v \cdot K_{\mathsf{V}} = 0$, hence $v \in \mathrm{CM}(\mathsf{V})$. We have $v^2 = 0$. This shows that $\mathrm{CM}(\mathsf{V})$ is an even lattice, and hence

$$\mathrm{CM}(\mathsf{V}) \cong \mathsf{E}_{10}.$$

Let

$$\begin{aligned} f_i &= 3(h + h') - (e_1 + \cdots + e_{10}) - 2e_i, \ i = 1, \ldots, 10, \\ h_{10} &= 5(h + h') - 2(e_1 + \cdots + e_{10}). \end{aligned}$$

We have $f_i^2 = 0$, $f_i \cdot f_j = 4$, $h_{10}^2 = 10$, $h_{10} \cdot f_i = 6$. Moreover, we have

$$6h_{10} = f_1 + \cdots + f_{10}$$

with a complete analogy with the Fano polarization on an Enriques surface. The classes f_i represent the proper transforms of plane quintics curves passing through $p_1, \ldots, p_9, p, q$ with double points at p, q and a triple point at p_i. The linear system $|h_{10}|$ defines a birational morphism onto a surface of degree 10 in $\mathbb{P}^5$ which blows down each boundary component to a quotient singularity of type $\frac{1}{4}(1,1)$. We will call the image surface a *Fano–Mukai model* of V. We may also consider the adjoint linear system $|H_{10} + K_{\mathsf{V}}|$, where $H_{10} + K_{\mathsf{V}} = 3(h + h') - (e_1 + \cdots + e_{10})$. It maps V onto a surface of degree 8 in $\mathbb{P}^5$. This time, the images of the boundary components are curves of degree 2, each spanning a plane. The union of the planes and the octic surface is called an *adjoint Fano–Mukai model* of V.

One can follow the definition of a canonical isotropic sequence in the Enriques lattice and define a canonical isotropic sequence in the lattice $\mathrm{CM}(\mathsf{V})$. Thus, we can speak about non-degenerate (degenerate) U-pair (f_1, f_2) of such vectors. Similarly to the case of an Enriques surface, it defines a regular degree 2 map onto one of the anti-canonical quartic del Pezzo surfaces $\mathsf{D} = \mathsf{D}_1, \mathsf{D}_2, \mathsf{D}_3$ (resp. $\mathsf{D}'_1, \mathsf{D}'_2, \mathsf{D}'_3$). We still call it a *bielliptic map*. The difference in the case of Coble surfaces is that this map is never finite. It blows down each boundary component to a singular point of D. If $p \neq 2$, the branch curve W of the bielliptic map is cut out by a quadric passing through k singular points of D.

Figure 9.2 below describes the branch locus of a bielliptic map for a Coble surface with one boundary component on the weak del Pezzo surface $\tilde{\mathsf{D}}_1$. It consists of a curve cut out by a quadric and three (-2)-curves. They are colored blue.

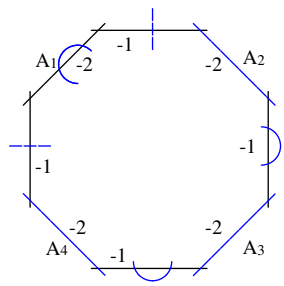


Fig. 9.2 Branch curve of a bielliptic map for Coble surfaces ($p \neq 2$)

The equation of the double plane model of a Coble surface in $\mathbb{P}(1, 1, 1, 4)$ obtained from a bielliptic map (if $p \neq 2$) to D_1 is the following:

$$x_3^2 + x_1(x_0^4 A_1 + x_0^3 x_1 B_1 + x_0^2 x_1 A_2 + x_0 x_1^2 x_2 C_1 + x_1^2 x_2 B_2) = 0, \tag{9.2.3}$$

where A_1, B_1, C_1 are linear forms in x_1, x_2 and A_2, B_2 are quadratic forms in x_1, x_2. The branch curve is equal to the union of the line $V(x_1)$ and a quintic curve from the linear system $|5e_0 - e_1 - 2e_2 - 2e_3 - e_4 - e_5|$.

Example 9.2.6 Let us consider the previous example and let I_1 and I_2 be two subsets of cardinality 7 among $p_1, \ldots, p_9$ with $\#I_1 \cap I_2 = 6$. Let $f_k = 3e_0 - e_{10} - e_{11} - \sum_{i \in I_k} e_i$, $k = 1, 2$. Then, $f_k^2 = 0$, $f_1 \cdot f_2 = 1$, $f_k \cdot C_1 = f_k \cdot C_2 = 0$. The linear system $|2f_I + 2f_J|$ defines a bielliptic map $\phi : \mathsf{V} \to \mathsf{D}_1$. It blows down C_1, C_2 to two different nodes of D_1.

On the other hand, let us consider another model of V where $C_1 \in |e_0 - e_7 - e_8 - e_9 - e_{10} - e_{11}|$ and $C_2 \in |5e_0 - 2(e_1 + \cdots + e_6) - (e_7 + \cdots + e_{11})|$. Let F_I be a cubic curve passing through the points $p_1, \ldots, p_6$ and a subset I of 3 points among $p_7, \ldots, p_{11}$. Let f_I be the divisor class of its proper inverse transform on V. Then, f_I belongs to the Coble–Mukai lattice $\mathrm{CM}(\mathsf{V})$. We have $f_I \cdot f_J = 3 - \#I \cap J \in \{1, 2\}$.

Let us choose I, J with $\#I \cap J = 2$, say $I = \{7, 8, 9\}$, $J = \{7, 8, 10\}$. Then, (f_I, f_J) is a non-degenerate U-pair and we can define the corresponding bielliptic map

$\phi : \mathsf{V} \to \mathsf{D}_1$ given by the linear system $|2F_I + 2F_J|$. It blows down the curves C_1, C_2 and E_{10} to one of the four nodes of D_1, say q_1. Let $\tilde{\mathsf{D}}_1$ be the corresponding weak del Pezzo surface. The proper transform $\bar{B}$ of the branch curve B on $\tilde{\mathsf{D}}_1$ is tangent to R at some point. The pre-image of R splits into the union of two (-3)-curves intersecting transversally at one point q'. After we blow-up this point,we obtain the surface V and the exceptional curve over q' is equal to E_{10}.

Let us now extend Definition 6.4.2 of Nikulin R-invariant for Enriques surfaces to the case of Coble surfaces.

Let V be a Coble surface with n boundary components $C_1, \dots, C_n$. For the convenience of notation we denote $\mathrm{CM}(\mathsf{V})$ by Λ. Let $\pi : X \to \mathsf{V}$ be the canonical double cover branched along $C_1, \dots, C_n$, and ι the covering transformation. For any curve C on V, we denote by $\tilde{C}$ the proper transform of C on X. Define $L^+ = \pi^*(\Lambda)$. Denote by L^- the orthogonal complement of L^+ in $\mathrm{Pic}(X)$. Note that $\tilde{C}_1, \dots, \tilde{C}_n$ are contained in L^+. Define

$$h^\pm = \{\delta^\pm \in L^\pm \ : \ \exists\, \delta^\mp \in L^\mp,\ (\delta^\pm)^2 = -4,\ \frac{\delta^+ + \delta^-}{2} \in \mathrm{Pic}(X)\}. \tag{9.2.4}$$

Let $\langle h^- \rangle$ be the sublattice of L^- generated by h^-. Then, $\langle h^- \rangle = K(2)$, where K is a root lattice. We have a homomorphism

$$\gamma : K/2K \cong K(2)/2K(2) \to \Lambda/2\Lambda \cong L^+/2L^+, \tag{9.2.5}$$

$$\gamma(\delta^- \bmod 2) = \delta^+ \bmod 2.$$

We define a subgroup H of $K/2K$ by the kernel of γ which is an isotropic subgroup with respect to q_K. In this case, we allow that the overlattice $K_H = \{x \in K \otimes \mathbb{Q} \ : \ 2x \bmod 2K \in H\}$ contains (-1)-vectors contrary to the case of Enriques surfaces (see Section 6.4). We call the pair (K, H) the R-invariant of V as in the case of Section 6.4.

Recall that, by Lemma 9.2.1, there are two types of effective irreducible roots. One is a (-2)-curve and the other is $2E + \frac{1}{2}C_i + \frac{1}{2}C_j$, where E is a (-1)-curve that intersects C_i and C_j. If C is a (-2)-curve, then $\pi^*(C) = \tilde{C}^+ + \tilde{C}^-$, where $\tilde{C}^+, \tilde{C}^-$ are disjoint (-2)-curves on X with $\iota(\tilde{C}^+) = \tilde{C}^-$. If C is a (-1)-curve, it intersects the boundary at two points and hence $\pi^*(C) = \tilde{C}$. If $C = C_i$, then $\pi^*(C_i) = 2\tilde{C}_i$. Therefore, $\pi^*(2E + \frac{1}{2}C_i + \frac{1}{2}C_j) = 2\tilde{E} + \tilde{C}_i + \tilde{C}_j$. We associate $\delta^+ = \tilde{C}^+ + \tilde{C}^-$ and $\delta^- = \tilde{C}^+ - \tilde{C}^-$ to a (-2)-curve C, and we associate $\delta^+ = 2\tilde{E} + \tilde{C}_i + \tilde{C}_j$ and $\delta^- = \tilde{C}_i \pm \tilde{C}_j$ to an effective irreducible root $2\tilde{E} + \frac{1}{2}C_i + \frac{1}{2}C_j$. As mentioned above, K_H contains (-1)-vectors corresponding to $\tilde{C}_1, \dots, \tilde{C}_n$ when $n \geq 2$.

Note that this definition of $\delta^\pm$ is very similar to the one for Enriques surfaces in characteristic 2 whose canonical covers are supersingular $K3$ surfaces with rational double points (see Example 10.6.8).

Example 9.2.7 Here we consider an example of a Coble surface with ten boundary components obtained as the quotient of a K3-surface studied by E. Vinberg in [721]. We will describe explicitly its automorphism group.

We start with a quintic del Pezzo surface $\mathcal{D}_5$ which we used earlier to construct Coble surfaces. The surface $\mathcal{D}_5$ contains 10 lines whose dual graph is the Petersen graph given in Figure 6.1. Note that each line on $\mathcal{D}_5$ meets exactly three lines. Let V be the surface obtained from $\mathcal{D}_5$ by blowing up 15 intersection points of the 10 lines. They are represented by the edges in the Petersen graph. Thus, V contains 15 (-1)-curves which are exceptional curves of the blow-up $\mathsf{V} \to \mathcal{D}_5$ and ten (-4)-curves which are the proper transforms of 10 lines. The Picard number of V is equal to 20. We denote by C_{ij} $(1 \le i < j \le 5)$ the ten (-4)-curves. Here, we use the same subscripts as we used it for the vertices U_{ab} given in Figure 6.1. Then, $|-2K_V| = \{\sum_{1\le i<j\le 5} C_{ij}\}$. Let $\pi : X \to \mathsf{V}$ be the double cover branched along $\sum_{1\le i<j\le 5} C_{ij}$ which is a K3 surface with Picard number 20. Let ℓ_{ij} be the line in $\mathbb{P}^2$ passing through p_i, p_j. The surface X is birationally isomorphic to the double cover of the plane branched over the union of lines ℓ_{ij}. The blow-up of seven singular points of the branch curve is a weak del Pezzo surface of degree 2. The anti-canonical linear system defines a double cover of the plane branched along a complete quadrilateral of lines (see Figure 8.7.5). This shows that the surface X is birationally isomorphic to the quartic surface

$$w^4 + xyz(x + y + z) = 0$$

as asserted in [721, Theorem 2.5]. It has six rational double points of type A_3.

Let $C_{ab}, 1 \le a, b \le 4$ be the proper transform of the line $\ell_{cd} = \langle p_c, p_d\rangle$ with $\{ab\} \cap \{cd\} = \emptyset$, and let C_{i5} be the exceptional curve over p_i. The curves $C_{12}+C_{45}+C_{23}+C_{15}+C_{34}$ and $C_{13}+C_{14}+C_{24}+C_{35}+C_{25}$ span a pencil on V with 5 base points defined by the edges $(U_{12}, U_{35}), (U_{34}, U_{25}), (U_{15}, U_{24}), (U_{45}, U_{13})$. The pencil originates from the pencil of plane cubics generated by $\ell_{34} + \ell_{12} + \ell_{14}$ and $\ell_{24} + \ell_{13} + \ell_{23}$ on $\mathbb{P}^2$. After we blow up the base points, we obtain an elliptic fibration on X with singular fibers of type $\tilde{A}_9 + \tilde{A}_9$ and five sections. The Shioda–Tate formula implies that the discriminant of $\mathrm{Pic}(X)$ is equal to $-10^2/5^2 = -2^2$. Since $\mathrm{Pic}(X)$ is the invariant part of $H^2(X, \mathbb{Z})$ under the action of the covering transformation σ of $X \to \mathsf{V}$, $\mathrm{Pic}(X)$ is a 2-elementary lattice. Therefore, $\mathrm{Pic}(X)$ is isomorphic to $\mathsf{U} \oplus \mathsf{E}_8^{\oplus 2} \oplus \mathsf{A}_1^{\oplus 2}$, and hence the transcendental lattice isomorphic to $\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$. This K3 surface is one of the two K3 surfaces called *most algebraic K3 surfaces*. It is known that the automorphism group $\mathrm{Aut}(\mathcal{D}_5)$ is isomorphic to $\mathfrak{S}_5$ of degree 5 (see [177, Theorem 8.5.8], the proof does not use any assumption on the characteristic). It induces an action on V. Then, this action can be lifted to an action on X. The involution σ acts on $\mathrm{Pic}(X)$ trivially and acts on the transcendental lattice as -1, and hence it is contained in the center of $\mathrm{Aut}(X)$. Thus, $\mathrm{Aut}(\mathsf{V}) \cong \mathrm{Aut}(X)/\langle\sigma\rangle$.

The K3 surface X has 25 (-2)-curves which are the pre-images of 15 (-1)-curves and ten (-4)-curves on V. We depict them in Figure9.3. Here the red vertices are the pre-images of the (-4)-curves.

Additionally, Vinberg found five (-4)-classes $c_1, ..., c_5$ in $\mathrm{Pic}(X)$, each of which defines a reflection of $\mathrm{Pic}(X)$, and showed that this reflection is represented by an involution of X. Let us explain how these classes arise.

We may assume that four points $p_1, ..., p_4$ are the reference points

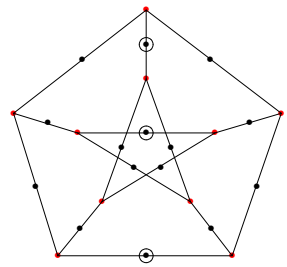


Fig. 9.3 Twenty five (-2)-curves on Vinberg's most algebraic K3 surface

$$p_1 = [1,0,0], \quad p_2 = [0,1,0], \quad p_3 = [0,0,1], \quad p_4 = [1,1,1].$$

Then, six lines joining them by pairs form a complete quadrangle with the diagonal points $q_1 = [1,1,0]$, $q_2 = [1,0,1]$, $q_3 = [0,1,1]$. The quadratic Cremona transformation with fundamental points at q_1, q_2, q_3 is given by

$$T : [x,y,z] \mapsto [(x-y+z)(x-y-z), (x+y-z)(x-y-z), (z-x-y)(x-y+z)]. \quad (9.2.6)$$

As is well-known, it acts on the blow-up of the fundamental points as a reflection in the vector $e_0 - e_1 - e_2 - e_3$, where e_1, e_2, e_3 are the classes of the exceptional curves over q_1, q_2, q_3. Since V is the blow-up of a set of points in $\mathbb{P}^2$ including the points q_1, q_2, q_3, the class $e_0 - e_1 - e_2 - e_3$ is lifted to a divisor class δ in $\mathrm{Pic}(\mathsf{V})$ of square-norm -2. The involution T also lifts to an involution of V. There are exactly 5 different linear systems on $\mathcal{D}_5$ that define a birational morphism $\mathcal{D}_5 \to \mathbb{P}^2$ [177, 8.5]. So, in this way, we obtain 5 classes $\delta_1, \dots, \delta_5$ in $\mathrm{Pic}(\mathsf{V})$. Their lifts to the K3-cover X are the five classes $c_1, \dots, c_5$ of Vinberg. It follows from the definition that each δ_i intersects three (-1)-curves on $\mathcal{D}_5$ corresponding to the diagonal points. So, each c_i intersects three (-2)-curves on X. For example, we marked one of these sets of three points in Figure 9.3.

Theorem 9.2.8 *Assume that* $\Bbbk = \mathbb{C}$. *Then,* $\mathrm{Aut}(X)$ *is isomorphic to a central extension of*

$$\mathrm{UC}(5) \rtimes \mathfrak{S}_5$$

by $\mathbb{Z}/2\mathbb{Z}(= \langle\sigma\rangle)$. *The subgroup* $\mathrm{UC}(5)$ *is generated by the lifts of five Cremona involutions* T_i *that act on* $\mathrm{Pic}(X)$ *as reflections in* c_i. *The subgroup* $\mathfrak{S}_5$ *is the lift of the group of automorphisms of the del Pezzo surface* $\mathcal{D}_5$ *of degree* 5.

Recall that we use the notation $\mathrm{UC}(k)$ for the free product of k cyclic groups of order two (the universal Coxeter group with k generators).

As a corollary, we now have:

Corollary 9.2.9 *Assume that* $\Bbbk = \mathbb{C}$. *Then,* $\mathrm{Aut}(\mathsf{V})$ *is isomorphic to*

$$\mathrm{UC}(5) \rtimes \mathfrak{S}_5.$$

The five Cremona involutions $T_1, \dots, T_5$ define five involutions of V (and X), and generate the subgroup $\mathrm{UC}(5)$ of $\mathrm{Aut}(\mathsf{V})$. Any automorphism of $\mathrm{Aut}(\mathcal{D}_5) \cong \mathfrak{S}_5$ lifts to an automorphisms of V that permutes the classes $\delta_1, \dots, \delta_5$. This defines a subgroup $\mathrm{UC}(5) \rtimes \mathfrak{S}_5$ of $\mathrm{Aut}(\mathsf{V})$.

We extend the result for $\mathrm{char}(\Bbbk) \neq 2$. To do this, we show that the Coble–Mukai lattice of V contains 20 (-2)-classes forming a crystallographic basis of type VII. Recall that the diagram in Figure 8.10 is the union of the complete graph $K(5)$ and the dual of the Petersen graph whose vertices are the edges in the Petersen graph and the edges are its vertices. Each vertex of the first subgraph is joined by a double edge to three vertices of the second subgraph. This is visible in the Figure **??**. Let

$$\alpha_{ab,cd} = 2E + \frac{1}{2}(C_{ab} + C_{cd})$$

be an effective root where E is the exceptional curve over the intersection point of two lines ℓ_{ab} and ℓ_{cd} $(1 \leq a, b, c, d \leq 5,\ \{a, b\} \cap \{c, d\} = \emptyset)$. We immediately check that the intersection graph of $\alpha_{ab,cd}$'s is the dual Petersen graph, and the intersection graph of $\delta_1, \dots, \delta_5$ is the complete graph $K(5)$ and each K_i intersects with multiplicity 2 exactly three roots $\alpha_{ab,cd}$ (note that in Figure 9.3, Vinberg normalized the norms of vectors corresponding to vertices as (-1)-vectors). Thus, the twenty roots $\alpha_{ab,cd}, \delta_i$ form the crystallographic basis of type VII. The five involutions of V work as the reflections associated with five roots $\delta_1, \dots, \delta_5$

This gives us the following:

Theorem 9.2.10 *Assume that* $\mathrm{char}(\Bbbk) \neq 2$. *Then,* $\mathrm{Aut}(\mathsf{V})$ *is isomorphic to*

$$\mathrm{UC}(5) \rtimes \mathfrak{S}_5.$$

In $\mathrm{char}(\Bbbk) = 2$, we will show that five classes $\delta_1, \dots, \delta_5$ are represented by (-2)-curves and thus $\mathrm{Aut}(\mathsf{V})$ is isomorphic to $\mathfrak{S}_5$ (see Theorem 9.8.15).

Next, we calculate the R-invariant of the surface from this example.

Proposition 9.2.11 *Assume that* $\Bbbk = \mathbb{C}$. *Then, the* R*-invariant* (K, H) *of* V *is* $(\mathsf{D}_{10}, \mathbb{Z}/2\mathbb{Z})$.

Proof Since ten $\tilde{C}_{ij}$ generate a root lattice $\mathsf{A}_1^{\oplus 10}$, K is a sublattice of $\mathsf{A}_1(\frac{1}{2})^{\oplus 10}$. Since K is generated by (-2)-vectors $\tilde{C}_{ij} \pm \tilde{C}_{kl}$ $(\{i, j\} \cap \{k, l\} = \emptyset)$, K is a root sublattice of $\mathsf{A}_1(\frac{1}{2})^{\oplus 10}$ of index 2, and hence isomorphic to D_{10}. Since $\tilde{C}_{ij}$ gives a non-zero element in H, we conclude that $H = \mathbb{Z}/2\mathbb{Z}$. □

Remark 9.2.12 The most algebraic K3 surface X from the previous example is a special case of the following family of K3 surfaces. Choose one conic from each pencil of conics on a del Pezzo surface $\mathcal{D}_5$ of degree 5 and consider the double cover of V branched over the union of the five conics. The blow-up of ten intersection points of the five conics is a Coble surface with 5 boundary components. Our surface X is obtained when each conic is the union of two lines. The canonical covers of the Coble surfaces form a 5-dimensional family of K3 surfaces that contains a codimension one family of surfaces birationally isomorphic to a 15-nodal quartic surface [182].

Let V be a Coble surface with n boundary components. It is obvious that the Coble–Mukai lattice of V is isomorphic to the Enriques lattice E_{10} if $n = 1$. We also saw in Example 9.2.5 that it is true if $n = 2$. In Example 6.4.20, we considered Coble surfaces with $n \leq 4$ boundary components constructed as the quotients of the Hessian quartic surfaces of cubic surfaces with n nodes by the Cremona involution. It was proven in [8, Lemma 2.4] that the Coble–Mukai lattice is isomorphic to the Enriques lattice in these cases too. The isomorphism also holds in the case $n = 10$. Indeed, as we saw in Example 9.2.7, the twenty roots $\alpha_{ab,cd}, \delta_i$ form a crystallographic basis of type VII. Since on the corresponding Enriques surface with finite automorphism group the roots from this basis represent all (-2)-curves on S, and, by Theorem 2.3.5, they generate $\mathrm{Num}(S)$, we see that $\mathrm{CM}(\mathsf{V})$ contains a sublattice isomorphic to E_{10} and hence coincides with it.

All these examples lead to the following conjecture.

Conjecture 9.2.13 The Coble–Mukai lattice of a Coble surface is isomorphic to the Enriques lattice.

In the following we prove that this conjecture is true under the assumption that $\Bbbk = \mathbb{C}$.

Theorem 9.2.14 *Assume $\Bbbk = \mathbb{C}$. Them the Coble–Mukai lattice of a Coble surface is isomorphic to the Enriques lattice E_{10}.*

Proof Assume V is general in the irreducible component of the moduli space of such surfaces. Let X be its canonical K3 cover. Then, Table 5.1 from section 5.4 in Volume I shows that the Picard lattice of X is uniquely determined by the number n unless $n = 8$.

As we remarked earlier, the assertion is true if $n \leq 4$ and $n = 10$. We use the construction of the irreducible families with $n = 5, \ldots, 10$ from the end of the previous section. Let V_n be a general, in the sense of moduli, with boundary components $C_1, \ldots, C_n$. We know that it is isomorphic to the blow-up of a quintic del Pezzo surface $\mathcal{D}_5$ at the singular points of the union of 5 conics $K_1, \ldots, K_5$. We may assume that the first $10 - n$ of them are irreducible and the remaining $n - 5$ are reducible. We can specialize V_n to V_{n+1} by replacing K_{10-n} with a reducible conic $L_1 + L_2$ from the same pencil.

The Picard lattice of V_n is generated by the classes of $5 + n$ exceptional curves E_i of the blow-up $\pi_n : \mathsf{V}_n \to V$ and $\pi^*(\mathrm{Pic}(V))$. The $\mathbb{Q}$-rational lattice $\widetilde{\mathrm{Pic}}(\mathsf{V}_n)$ is generated by $\mathrm{Pic}(\mathsf{V}_n)$ and $\frac{1}{2}[C_1], \ldots, \frac{1}{2}[C_n]$. Without loss of generality, we may assume that C_1 is the proper transform of a conic from the pencil K_{10-n}. Consider the isometry embedding

$$\iota : \widetilde{\mathrm{Pic}}(\mathsf{V}_n) \hookrightarrow \widetilde{\mathrm{Pic}}(\mathsf{V}_{n+1})$$

which is the identity on the set of classes of exceptional curves E_i, on $\pi^*(\mathrm{Pic}(\mathcal{D}_5))$, and on the classes $\frac{1}{2}[C_i], i \neq 1$. It sends the class $\frac{1}{2}[C_1]$ to the class $\frac{1}{2}[L_1]$. The image of this embedding is equal to the orthogonal complement of $\frac{1}{2}[L_2]$. It follows

from the definition of the Coble–Mukai lattice that the embedding ι maps $\mathrm{CM}(\mathsf{V}_n)$ isomorphically onto $\mathrm{CM}(\mathsf{V}_{n+1})$. Since $\mathrm{CM}(\mathsf{V}_{10}) \cong \mathsf{E}_{10}$, we are done. □

Remark 9.1 In a recent paper [713], Giancarlo Urzuà uses Corollary 9.1.11 to give another proof of the theorem without appealing to its K3 cover. It allows him to extend the result to a positive characteristic $\neq 2, 3$.

Remark 9.2.15 In the case where $n = 8$ (a) (see Figure 9.1), the Coble–Mukai lattice contains 40 roots that span a remarkable polytope with denoted by Γ_{MII} (see Section 10.6). Recall that a general member V_8 in this family can be obtained by the blow-up of the intersection points of 8 lines ℓ_i, ℓ'_i $(i = 1, 2, 3, 4)$ on a smooth quadric Q with $\ell_i \cdot \ell'_j = 1$. Denote by p_{ij} the intersection point of ℓ_i and ℓ'_j and let E_{ij} be the exceptional curve over p_{ij}. Let C_i, C'_j be the proper transforms of ℓ_i, ℓ'_j which are the boundary curves of V, and let h, h' be the total transforms of ℓ_i, ℓ'_j. Then, $\mathrm{CM}(\mathsf{V}_8)$ contains 16 effective roots $\alpha_{ij} = 2E_{ij} + \frac{1}{2}(C_i + C'_j)$ and 24 non-effective roots $\delta_\sigma = h + h' - (E_{1\sigma(1)} + E_{2\sigma(2)} + E_{3\sigma(3)} + E_{4\sigma(4)})$ indexed by $\mathfrak{S}_4$. The 40 roots $\{e_{ij}, \delta_\sigma\}$ are the vertices of the polytope Γ_{MII}. The subgraph with 16 vertices e_{ij} is dual to the complete bipartite graph $BK(4)$ on two sets of cardinality four (see the left-hand side of Figure 10.8). Let H_0 be the 2-elementary subgroup of $\mathfrak{S}_4$ generated by $(12)(34), (13)(24)$. Consider the coset decompositions $\mathfrak{A}_4 = H_0 + H_1 + H_2$ and $\mathfrak{S}_4 \setminus \mathfrak{A}_4 = H_3 + H_4 + H_5$. Then, $\delta_\sigma \cdot \delta_{\sigma'} = 2$, for any $\sigma, \sigma' \in H_i$ $(0 \leq i \leq 4)$, and $\delta_\sigma \cdot \delta_{\sigma'} = 1$ for any $\delta, \delta' \in \mathfrak{A}_4$ or $\delta, \delta' \notin \mathfrak{A}_4$ which belong to different cosets. Thus, the subgraph with vertices $\{\delta_\sigma\}$ is the union of two sets of three complete graphs $K(4)$ with double edges. In each set of three subgraphs, each vertex of one subgraph is joined to all other vertices from the two remaining subgraphs (see the right-hand side of Figure 10.8).

9.3 Quadratic Twist Construction

In this section, we will extend the construction of a quadratic twist for Enriques surfaces from Section 4.8 to Halphen surfaces of index 2. This will be applied to a construction of special Coble surfaces. We assume that $p \neq 2$.

Let $f : \mathsf{H} \to \mathbb{P}^1$ be the elliptic fibration on a Halphen surface H of index 2 and let R be its smooth rational bisection. The double cover $f : R \to \mathbb{P}^1$ ramifies at two points $p_0 \in F_0, p \in F$, where F_0 is the unique half-fiber, and F is another fiber. For our application to Coble surfaces, it would be enough to assume that F is nonsingular or of type $\tilde{A}_n$. We also assume that F_0 is of type $\tilde{A}_m$ (smooth if $m = 0$). Two possibilities may arise: p is a nonsingular point or p is a double point of F. If $\mathsf{V} \to \mathsf{H}$ is the blow-up of singular points of F and $f' : \mathsf{V} \to \mathbb{P}^1$ is the induced elliptic fibration from f, then the proper inverse transform of R on V is a bisection of f'. It is a (-1)-curve if p is nonsingular, and a (-2)-curve otherwise. Let us say that a bisection on H (resp. on V) is of the *first kind* if p is nonsingular and of the *second kind* otherwise.

Since $F \sim 2F_0$, we can consider the double cover $\pi' : X' \to \mathsf{H}$ branched over F. If F is singular, then X' has ordinary double points over the singular points of F. Let $\pi : X \to X' \to \mathsf{H}$ be the composition of π' with the minimal resolution of singularities of X'. The pre-image $\tilde{F}$ of F on X is a singular fiber of type $\tilde{A}_{2n+1}$. The Hurwitz type formula for the canonical class implies that X is a K3 surface.

Assume R is a bisection of the first kind. Then, R is tangent to the branch curve at its nonsingular point $p \in F$, hence its pre-image on X splits into two components R_+ and R_- intersecting at the point $\pi^{-1}(p)$. Each curve $R_\pm$ is a (-2)-curve and a section of the elliptic fibration $\tilde{f} : X \to \mathbb{P}^1$ lifted from f. As in the case of Enriques surfaces, we define an involution $\sigma_+ = t_{R_-} \circ \tau$, where τ is the covering involution, and t_{R_-} is the translation involution by R_- with respect to the group law with zero section R_+ on the set of sections of $\tilde{f}$. The involution τ acts identically on $n+1$ components of $\tilde{F}$ and acts as an involution on each other component with two fixed points. The involution t_{R_-} fixes the point $\pi^{-1}(p) = R_+ \cap R_-$ on one of the components of the ramification locus. This implies that σ_+ acts identically on this component and hence identically on $n+1$ components of $\tilde{F}$ including the pre-image of the component on H that intersects R. As in the Enriques case, we see that σ_+ acts identically on $m+1$ components of the half-fiber $\tilde{F}_0$ of type $\tilde{A}_{2m+1}$. The quotient surface $X/(\sigma_+)$ is smooth. After we blow down $m+1$ components on the image of $\tilde{F}_0$ and $n+1$ components of the image of $\tilde{F}$, we obtain a rational jacobian elliptic surface $j : J \to \mathbb{P}^1$ with fibers $\bar{F}_0$ and $\bar{F}$ of types $\tilde{A}_m$ and $\tilde{A}_n$. As in the Enriques case, $\sigma_+(R_-) = R_-$ and $\sigma_+(R_+) = t_{R_-}(R_-) = R'_-$. The curve R_- descends to a section of j which we can choose as the zero section C_0 in the Mordell–Weil group. Since $R_+ \cap R_- = \{\pi^{-1}(p)\}$, the curve $R_+ + R'_-$ descends to a rational bisection C of j that intersects the zero section at one point and at this point it is tangent to the fiber passing through this point.

Suppose R is a bisection of $f : \mathsf{H} \to \mathbb{P}^1$ of the second kind. In the previous notation, R_+ and R_- are now disjoint on X and intersect the same component of $\tilde{F}$ which does not belong to the locus of fixed points of σ_+. Everything else remains unchanged and we arrive at a rational jacobian elliptic surface $j : J \to \mathbb{P}^1$ such that X is a minimal resolution of singularities of two fibers $\bar{F}_0$ and $\bar{F}$ of multiplicative types. The curve R_- descends to a section C_0 of j and the curves $R_+ + R'_-$ descends to a rational bisection C of j that is tangent to $\bar{F}_0$ and $\bar{F}$ at their nonsingular points. In both cases, $f : \mathsf{V} \to \mathbb{P}^1$ is a torsor of $j : J \to \mathbb{P}^1$ defined by the cocycle $\mathrm{Gal}(C/\mathbb{P}^1) \to J_\eta(C_\eta)$ with the image equal to C_η. The Halphen fibration f is reconstructed from j by the Weil descent.

The bisection C that gives the inverse construction is defined similar to the case of Enriques surfaces. We start with a jacobian rational surface $j : J \to \mathbb{P}^1$, then choose two fibers F_{t_1}, F_{t_2}, smooth or of multiplicative type. Then, we fix a section C_0 to define a group law on the set of sections with the zero section C_0. Next we look for a rational curve C satisfying conditions 1)–5) from Section 4.9 where we defined an Enriques bisection. If R is a bisection of the first kind, we replace property 3 with the following property:

(3') C intersects C_0 at one point over a point t_2 with multiplicity 2.

We will call such a bisection C a *Halphen bisection*. Let $X \to J$ be the double cover branched at the fibers F_{t_1}, F_{t_2} which in the following we assume to be smooth for brevity of notation. The pre-image of C in X splits into R_1 and R_2. We consider the translation automorphisms t_{R_1} with respect to the group law on the pre-image of the elliptic fibration on X with zero section R_0 equal to the pre-image of C_0. As in the case of an Enriques surface, we have $R_1 \boxplus R_2 = 0$. The involution $\sigma_1 = t_{R_1} \circ \tau$ acts identically on the pre-image $\tilde{F}_{t_1}$ of the fiber F_{t_1} and acts as a translation by a 2-torsion point on the pre-image $\tilde{F}_{t_2}$ of the fiber F_{t_2}. The quotient $\mathsf{H} = X/(\sigma_1)$ is a Halphen surface of index 2 with the double fiber equal to the image $\tilde{F}_{t_2}$ on H. We have $\sigma_1(R_0) = R_1$ and the image of $R_0 + R_1$ is a smooth rational bisection R on H. Since C is tangent to the fiber F_{t_1} at its nonsingular point where it intersects C_0, we obtain that R_0 intersects R_1 at one point on the fiber $\tilde{F}_{t_1}$. Thus, R is tangent to the image of $\tilde{F}_{t_1}$ on H at a nonsingular point.

To find C, we look at the quotient $J \to J/(\iota) \cong \mathbf{F}_2$ by the negation involution defined by the zero section C_0. Then, we look for a section E of $\mathbf{F}_2 \to \mathbb{P}^1$ from the linear system $|3\mathfrak{f}+\mathfrak{e}|$ that intersects transversally the branch curve B at one nonsingular point and intersects the exceptional section transversally at another point. We have $E \cdot B = (3\mathfrak{f}+\mathfrak{e}) \cdot (6\mathfrak{f}+3\mathfrak{e}) = 9$, so in order the pre-image of E on J be a rational curve we must require that E intersects B with multiplicity 2 at four additional points.

Now, suppose V is a Coble surface obtained from a Halphen surface H of index 2 by blowing up singular points of a singular simple fiber F of type $\tilde{A}_n$. Let R' be a smooth bisection of the elliptic fibration on V lifted from a bisection R of the elliptic fibration on H. If R is of the first kind, then R' is (-1)-curve, otherwise R' is a (-2)-curve. We assume that R is tangent to F at one nonsingular point. This means that R' is tangent to one of the boundary components at a nonsingular point. In particular, R' is a (-1)-curve. Using the image of R, we can apply the rational quadratic twist construction to H.

Example 9.3.1 Here, we consider an analog of Example 4.10.13 from Section 4.8 that gives a construction of a Coble surface. In this example, $j : J \to \mathbb{P}^1$ is defined by the Hesse pencil of cubic curves and has 4 singular fibers of type $\tilde{A}_2$. We fix the section defined by the base point $p_0 = [0, 1, -1]$ and consider the Weierstrass model with the curve B coming from the curve

$$V(u_2^3 + 12u_1(u_0^3 - u_1^3)u_2 + 2(u_0^6 - 20u_0^3u_1^3 - 8u_1^6)) \subset \mathsf{Q} \cong \mathbb{P}(1, 1, 2).$$

It has four cusps c_1, c_2, c_3, c_4:

$$[u_0, u_1, u_2] = [0, 1, -2],\ \left[1, -\frac{1}{2}, \frac{3}{2}\right],\ \left[1, -\frac{1}{2}\omega, \frac{3}{2}\omega^2\right],\ \left[1, -\frac{1}{2}\omega^2, \frac{3}{2}\omega\right].$$

We want to construct a Halphen torsor with a nonsingular half-fiber F_1 and a bisection of the first kind. To do this, we need to consider a rational curve K of degree 3 on Q that passes through the vertex of Q, passes through all cusps and intersects B transversally at one point b_1 that does not lie over a branch point of $B \to \mathbb{P}^1$. Since $\dim |3\mathfrak{f} + \mathfrak{e}| = 5$, we can find a unique K satisfying these conditions. We find K as the residual curve in the intersection of Q with a quadric in $\mathbb{P}^3$ passing through the

vertex of Q and containing a fiber of $\mathsf{Q} \dashrightarrow \mathbb{P}^1$ such that the residual curve is a curve from $3\mathfrak{f} + \mathfrak{e}$ satisfying our conditions. The linear system of such curves is a pencil generated by the curves

$$6u_0^2u_1 - 6u_0u_1^2 + 6u_1^3 + (3u_0 - u_1)u_2 = 0$$

and

$$3u_0^3 - 6u_0^2u_1 - 12u_0u_1^2 + 4u_1u_2 = 0.$$

So, we can choose a unique curve K that passes through an additional point b_1 on B. Note that choosing the point b_1 determines uniquely the intersection point with the exceptional section, and hence the second fiber F_2.

Next, let us construct H with a singular half-fiber F_1. There are two possible bisections on H which we need to do this job. Assume first that it is of the first kind and F_2 is nonsingular. Then, we look for K as above but require that it intersects B at one of the cusps with multiplicity 3. If it intersects the exceptional section at the point lying on the ruling containing another cusp, we get F_2 singular too.

Applying this to Coble surfaces, we have to construct a Halphen surface with a singular non-multiple fiber. So, the previous construction gives us a bisection on the Coble surface which intersects one of the singular fibers at two points in different irreducible components. We choose one of these fibers to blow up its singular points and get a Coble surface V. The proper transform of the bisection is a (-1)-curve with class e that intersects two boundary components β_1, β_2. It gives rise to an effective root $\alpha = 2e + \frac{1}{2}\beta_1 + \frac{1}{2}\beta_2$ on V.

9.4 Self-Projective Rational Nodal Plane Sextics

In this section, following [737], we classify rational irreducible plane curves W of degree 6 that admit a non-trivial group of projective automorphisms. We assume that the singular points of such a sextic and infinitely near to them are ordinary nodes or cusps and refer to such curves as *nodal sextics*.

Under the assumption, the blow-up of $\mathbb{P}^2$ at the singular points is a Coble surface with one boundary component.

The interest in this classification is partially explained by the following:

Lemma 9.4.1 *Let V be a Coble surface with one boundary component C. Then, a finite group G of automorphisms of V preserves a Fano polarization if and only if V is isomorphic to the blow-up of $\mathbb{P}^2$ at ten double points of an irreducible nodal plane sextic that admits a group of projective automorphisms isomorphic to G.*

Proof Suppose $G \subset \mathrm{Aut}(\mathsf{V})$ preserves a Fano polarization h. Let $3h = F_1 + \cdots + F_{10}$, where $f_i = [F_i]$ form a canonical isotropic sequence in $K_{\mathsf{V}}^{\perp} \cong \mathsf{E}_{10}$. Let $e_i = f_i + K_{\mathsf{V}}$. Then, $e_i^2 = -1$, $e_i \cdot K_{\mathsf{V}} = -1$. By Riemann–Roch, e_i is effective and linearly equivalent to a sum of irreducible curves $Z_1, \ldots, Z_s$. Assume F_i is nef. Since $F_i \cdot e_i = 0$ each Z_i is either (-1) or a (-2)-curve. Since $-2K_{\mathsf{V}} \cdot E = 2$ for any (-1)-curve, there is only

one (-1)-curve and the rest is a chain of (-2)-curves. In other words, e_i is the class of an exceptional configuration $\mathcal{E}_i$. If F_i is not nef, then we know that $F_i \sim F_j + \mathcal{R}_{ij}$ for some nef F_j and a chain of (-2)-curves $\mathcal{R}_{ij}$. Thus, $F_i + K_{\mathsf{V}} \sim \mathcal{E}_i = \mathcal{E}_j + \mathcal{R}_{ij}$ is an exceptional configuration extending the exceptional configuration $\mathcal{E}_j$. This shows that we can blow down $\mathcal{E}_1, \dots, \mathcal{E}_{10}$ to points $p_1, \dots, p_{10}$ (maybe infinitely near) in $\mathbb{P}^2$. The image of C is a plane sextic with double points at $p_1, \dots, p_{10}$. One checks that

$$e_0 = h + 3K_{\mathsf{V}} = \frac{1}{3}(f_1 + \cdots + f_{10}) + 3K_{\mathsf{V}}$$

satisfies $e_0^2 = 1, e_0 \cdot e_i = 0$, hence it is equal to the pre-image of the class of a line in the plane. Since G leaves e_0 invariant, it descends to a group of projective transformations of $\mathbb{P}^2$ that leaves W invariant.

Conversely, assume that a rational nodal plane sextic C admits a non-trivial group G of projective symmetries. Then, G lifts to an automorphism of V that leaves invariant the classes $e_0, e_1, \dots, e_{10}$. Thus, it leaves invariant the divisor class h. □

Definition 9.4.2 Let S be an Enriques or a Coble surface. We say that S is *Fano-symmetric* with respect to a finite group G and a Fano polarization h if G is isomorphic to a subgroup of $\mathrm{Aut}(S)$ that leaves invariant h.

Let C be a rational plane sextic admitting a group G as its group of projective automorphisms. This means that G admits a faithful projective representation $\rho_1 : G \to \mathrm{Aut}(\mathbb{P}^2) \cong \mathrm{PGL}_3(\Bbbk)$ such that W is invariant (since $\deg C > 1$, the restriction of the action to C is also faithful). The action of G on C induces a faithful action on its normalization $\bar{C} \cong \mathbb{P}^1$ and defines a projective representation $\rho_2 : G \to \mathrm{Aut}(\mathbb{P}^1) \cong \mathrm{PGL}_2(\Bbbk)$. The map $\mathbb{P}^1 \cong \bar{C} \to C \subset \mathbb{P}^2$ is a rational parametrization

$$s : [t_0, t_1] \mapsto [P_1(t_0, t_1), P_2(t_0, t_1), P_3(t_0, t_1)]. \tag{9.4.1}$$

Here $P_i(t_0, t_1)$ are mutually coprime binary forms of degree 6. By construction, the map s is G-equivariant. Let $\mathbb{P}^1 = |U| = \mathbb{P}(U^\vee)$ and $\mathbb{P}^2 = |V| = \mathbb{P}(V^\vee)$. A G-equivariant map s originates from a morphism of projective representations $|V| \to |S^6 U^\vee|$.

We assume that G is a tame group, i.e.

$$(p, \#G) = 1.$$

As is well-known, a finite tame subgroup G of $\mathrm{PGL}_2(\Bbbk)$ is isomorphic to one of the *polyhedral groups* given by the following proposition due to Felix Klein.

Proposition 9.4.3 *Each finite tame subgroup of* $\mathrm{PGL}_2(\Bbbk)$ *is isomorphic to one of the following groups:*

- *a cyclic group C_n of order n;*
- *a dihedral group $\mathbb{D}_{2n}$ of order $2n$;*
- *a tetrahedral group $\mathbb{T}$ isomorphic to $\mathfrak{A}_4$;*
- *an octahedral group $\mathbb{O}$ isomorphic to $\mathfrak{S}_4$;*

- *an icosahedral group* $\mathbb{I}$ *isomorphic to* $\mathfrak{A}_5$.

Two isomorphic subgroups are conjugate in $\mathrm{PGL}_2(\Bbbk)$. *A cyclic group* G *of odd order lifts isomorphically to a subgroup of* $\mathrm{SL}_2(\Bbbk)$. *All other groups lift to a central extension* $\tilde{G} = 2.G$, *a subgroup of* $\mathrm{SL}_2(\Bbbk)$.

The group $\tilde{G} = 2.G$ is called a *binary polyhedral group*. Recall that any projective representation $G \to \mathrm{PGL}_n(\Bbbk)$ of a tame group G can be obtained from a linear representation $\rho' : G' \to \mathrm{GL}_{n+1}(\Bbbk)$, where $G' = K.G$ is a central extension of G with a group K, called the group of Schur multipliers. It is isomorphic to $H^2(G, \Bbbk^*)$. It is known that any prime divisor p of $\#K$ is equal to the order of a non-cyclic p-Sylow subgroup of G [334, 11.2]. Applying this to a polyhedral group G, we find that K is a cyclic group of order 2 or trivial if G is cyclic. Thus, every projective representation of G originates from a linear representation of $\tilde{G}$. In particular, our rational parametrization $\mathbb{P}^1 = |U| \to |V| = \mathbb{P}^2$ is given by a linear representation $S^6U \to V$ of $\tilde{G}$. Passing to the duals, we get a linear 3-dimensional summand $V^\vee$ of the natural representation of $\tilde{G}$ in the space $S^6U^\vee$ of binary forms of degree 6. Note that the center of $\tilde{G}$ acts trivially on $S^6U^\vee$, hence $V^\vee$ is a 3-dimensional linear sub-representation of G in $S^6U^\vee$. Note that two linear representations define isomorphic projective representations if and only if they differ by a one-dimensional character of G.

Our task is to find all such representations that define a rational parametrization whose image is a rational nodal plane sextic.

First, let us remind some known facts about irreducible linear representations of a binary polyhedral (non-cyclic) group. The group $\tilde{G}$ can be generated by elements g_1, g_2, g_3 of orders $2p, 2q, 2r$, where

$$(p,q,r) = \begin{cases} (2,2,n) & G = \mathbb{D}_{2n}, \\ (2,3,3) & G = \mathbb{T}, \\ (2,3,4) & G = \mathbb{O}, \\ (2,3,5) & G = \mathbb{I}. \end{cases}$$

The generators satisfy the following basic relations

$$g_1^p = g_2^q = g_3^r = g_1 g_2 g_3 = -1. \tag{9.4.2}$$

The conjugacy classes are represented by the elements 1, -1, g_1^a $(0 < a < p)$, g_2^b $(0 < b < q)$, g_3^c $(0 < c < r)$.

Let $\Gamma(p,q,r)$ be the Dynkin diagram of an affine root system of one of the types $\tilde{D}_{n+2}, \tilde{E}_6, \tilde{E}_7, \tilde{E}_8$, where

$$(p,q,r) = (2,2,n),\ (2,3,3),\ (2,3,4),\ (2,3,5), \tag{9.4.3}$$

respectively. In the last three cases, we set

$$(c,b,a) = (3,3,3),\ (2,4,4),\ (2,3,6)$$

for the lengths of the arms of these diagrams.

The group $\tilde{G}$ has $a + b + c - 2$ irreducible characters indexed by vertices of the diagram. Their degrees coincide with the multiplicities of the irreducible components entering in a fiber of an elliptic fibrations with the dual graph of components equal to $\Gamma(p, q, r)$ (see Figure 4.1).

All of this is well-known and a part of the *McKay correspondence* (see [395] and other references therein).

The character of the standard two-dimensional representation of $\tilde{G}$ in U corresponds to the neighbor of the vertex used to extend a finite type diagram to the affine one. We denote this character by x. We have the following McKay rule. For any character χ_v corresponding to a vertex v of the Dynkin diagram, a vertex v' incident to v enters into the tensor product $x \otimes \chi_v$ with multiplicity equal to the degree of $\chi_{v'}$.

The following Figure 9.4 summarizes the known facts about irreducible representations of binary polyhedral groups $\tilde{G}$. The squares correspond to representations that factor through a representation of G.

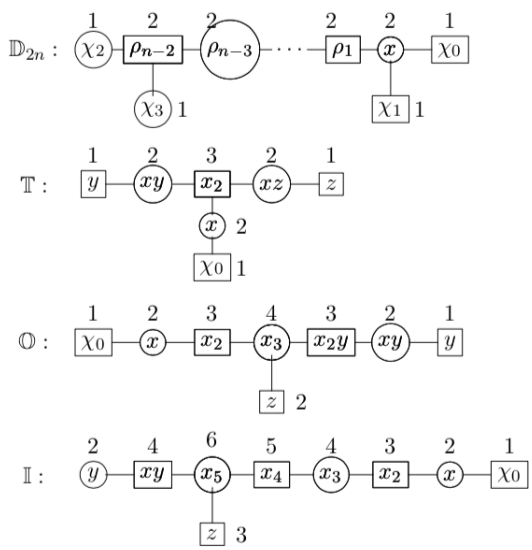


Fig. 9.4 Representations of binary polyhedral groups

The characters ρ_k of two-dimensional representations of the binary dihedral groups are defined on generators

$$r = \begin{pmatrix} \epsilon_{2n} & 0 \\ 0 & \epsilon_{2n}^{-1} \end{pmatrix}, \quad s = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$$

(in the standard 2-dimensional representation of these groups) by the matrices

$$\begin{pmatrix} \epsilon_{2n}^{k+1} & 0 \\ 0 & \epsilon_{2n}^{-k-1} \end{pmatrix}, \quad \begin{pmatrix} 0 & i^{k+1} \\ i^{k+1} & 0 \end{pmatrix}.$$

Here, ϵ_k denotes a generator of a cyclic subgroup of order k of $\mathbb{k}^*$. The one-dimensional characters χ_k are defined by their values on the generators as follows:

$$\chi_0 = (1,1),\ \chi_1 = (1,-1),\ \chi_2 = (-1,1),\ \chi_3 = (-1,-1).$$

The characters y and z are uniquely determined by the McKay rule for decomposition of the character $x\chi_v$.

We start our classification of self-projective rational nodal plane sextics with the case when the group G is a cyclic group.

- $G = C_n = (g)$ is a cyclic group of order n.

We may assume that it acts faithfully on $|U|$ and lifts to a linear representation of a cyclic group $\tilde{G} = (\tilde{g})$ of order $2n$ on U such that $\tilde{g}$ acts by $(t_0, t_1) \mapsto (\epsilon_{2n} t_0, \epsilon_{2n}^{-1} t_1)$.

The monomials $t_0^k t_1^{6-k}$ are eigenvectors with eigenvalues $\epsilon_{2n}^{2k-6} = \epsilon_n^{k-3}$. For brevity of notation, we give a rational parametrization

$$(x_0, x_1, x_2) = (P_0(t_0,t_1), P_1(t_0,t_1), P_2(t_0,t_1))$$

in terms of the affine coordinate $t = t_1/t_0$.

If $n > 6$, the eigenvalues are different, and, the curve is a bimonomial curve. It is a not a nodal sextic. It is easy to see, that up to a projective transformation of $\mathbb{P}^2$, the parametrization is given by

$$(x_0, x_1, x_2) = (1, t, t^6)$$

and the curve is $V(x_2^6 - x_1 x_0^5)$.

So, we may assume that $n \leq 6$. To choose a parametrization, we may use a projective transformation of the plane and a Moebius transformation of t. We also use that the polynomials t_0 and t_1 do not divide all $P_i(t_0, t_1)$, hence one of the dehomogenized polynomials must be equal to $t^6 + \cdots$ and another (maybe the same) to $1 + \cdots$. We also use that the curve is not monomial.

<u>$G = C_2$</u>

In this case $S^6 U^\vee$ is the direct sum of two eigensubspaces of $\tilde{g}$. They are $V_0 = \langle t_1^6, t_0^2 t_1^4, t_0^4 t_1^2, t_0^6 \rangle$ and $V_1 = \langle t_0 t_1^5, t_0^3 t_1^3, t_0^5 t_1 \rangle$ with eigenvalues -1 and 1, respectively. Thus, we can choose one parametrization of the form

$$(x_0, x_1, x_2) = (t^6 + a_1 t^4 + a_2 t^2 + a_3, a_4 t^4 + a_5 t^2 + a_6, a_7 t^5 + a_8 t^3 + a_9 t), \quad (9.4.4)$$

where $(a_3, a_6) = (1,0)$ or $(0,1)$ and another one of the form

$$(x_0, x_1, x_2) = (t^6 + a_1 t^4 + b_2 t^2 + 1, a_3 t^5 + a_4 t^3 + a_5 t, a_6 t^3 + a_7 t). \quad (9.4.5)$$

To write an explicit equation, one uses any computer algebra package (we used MAPLE). In the cyclic case, the equations are usually too long to display, so we give

only the equations of some subfamilies

$$4(a-2)^4xy^5 + a(a-2)^4y^4z^2 - 4(a-2)^4x^3y^3 - (a^2-3a-8)(a-1)^2xyz^4 \\ + (5a^2-5a-2)(a-2)x^2y^2z^2 - x^5y + x^4z^2 + (a-1)^2z^6 = 0. \tag{9.4.6}$$

in the case of parametrization (9.4.4), where $a_1 = a_4 = a_6 = 0, a_2 = a, a_3 = a_6 = 1, a_7 = 2$, and

$$(a^3-1)^2y^6 + a^5x^2y^4 - a^5x^4yz - 6a(a-1)y^5z + a(4a^3-5)x^2y^2z^2 + (2a^3-1)x^2y^3z \\ - 2(a^3-1)y^3z^3 + 9a^2y^4z^2 - 5a^2x^2yz^3 + 6ay^2z^4 - 6az^6 = 0, \tag{9.4.7}$$

in the case of parametrization (9.4.5), where $a_1 = a_2 = a_4 = a_5 = 0, a_3 = a_6 = 1, a_7 = a$.

$\underline{G = C_3}$

We have three eigensubspaces $\langle t_1^6, t_0^3t_1^3, 1\rangle$, $\langle t_0t_1^5, t_0^4t_1^2\rangle$, $\langle t_0^2t_1^4, t_0^5t_1\rangle$, with eigenvalues $1, \epsilon_3, \epsilon_3^2$, respectively. We must take one of the polynomials P_i from the first eigensubspace and get the following three possible cases:

$$(x_0, x_1, x_2) = (t^6 + a_1t^3 + 1, a_2t^5 + a_3t, a_4t^4 + a_4t^2),$$
$$(x_0, x_1, x_2) = (t^6 + a_1t^3, a_2t^3 + 1, a_3t^4 + a_4t),$$
$$(x_0, x_1, x_2) = (t^6 + at^3 + 1, t^4, t).$$

Computer computations show that, in the last two cases, the curve has a triple point, and hence, these cases must be omitted. The explicit equation is the following:

$$((a+2)^3y^6 - (a+2)^2(a+4)xy^4z + 2(a^2+20)x^2y^2z^2 - 8x^3y^3 + 8x^4yz \\ + (a-4)(a-2)^2xyz^4 - 8x^3z^3 + (a^2-4)^2y^3z^3 - (a-2)^3z^6 = 0, \tag{9.4.8}$$

where we take $a_1 = a, a_2 = a_3 = a_4 = 1, a_5 = -1$.

$\underline{G = C_4}$

We have four eigensubspaces $\langle t_0^3t_1^3\rangle$, $\langle t_0t_1^5, t_0^5t_1\rangle$, $\langle t_1^6, t_0^4t_1^2\rangle$, $\langle t_0^2t_1^4, t_0^6\rangle$, with eigenvalues $1, -1, i, -i$. We must take two of the polynomials P_i to be equal to $t^6 + at^2$ and $bt^4 + c$ with $c \neq 0$. This leaves us with two possibilities:

$$(x_0, x_1, x_2) = (t^6 + a_1t^2, a_2t^4 + 1, t^3),$$
$$(x_0, x_1, x_2) = (t^6 + a_1t^2, a_2t^4 + 1, a_3t^5 + a_4t).$$

Again, we check, using MAPLE, that the coefficient a_2 must be nonzero, so we may assume that $a_2 = 1$. In the second equation a_2 or a_3 (but not both) could be equal to zero. The explicit equations are the following:

$$a^3y^4z^2 - x^3y^3 - (a+3)(a-1)^2xyz^4 - (a^3-3)x^2y^2z^2 + x^4z^2 + (a-1)^4z^6 = 0, \tag{9.4.9}$$

in the first case, and

$$a^5y^4z^2+a^2(5-3a)x^2y^2z^2-a(a-5)(a-1)^2xyz^4-x^5y+(a-1)^4z^6=0, \quad (9.4.10)$$

in the second case, where $a_1=a, a_2=a_3=1, a_4=0$.

$\underline{G=C_5}$

We have five eigensubspaces $\langle t_0^3t_1^3\rangle$, $\langle t_0^4t_1^2\rangle$, $\langle t_1^6, t_0^5t_1\rangle$, $\langle t_0^6, t_0t_1^5\rangle$, $\langle t_0^2t_1^4\rangle$ with eigenvalues $1, \epsilon_5, \epsilon_5^2, \epsilon_5^3, \epsilon_5^4$, respectively. This leaves us with the following representatives of three isomorphism classes of parametrizations:

$$\begin{aligned}(x_0,x_1,x_2)&=(t^6+at, bt^5+1, t^2),\\ (x_0,x_1,x_2)&=(t^6+at, bt^5+1, t^3),\\ (x_0,x_1,x_2)&=(t^6+at, bt^5+1, t^4).\end{aligned}$$

Here, $b\neq 0$ in the first two cases, otherwise the curve is a not a nodal sextic. So, we may assume that $b=1$ in these cases. The equations are the following:

$$-a^2y^5z+x^2y^4-(a-1)^2(3a+2)xy^2z^3+a^2(a+4)(a-1)x^3yz^2-x^5z-(a-1)^5z^6=0. \quad (9.4.11)$$

$$x^5z-x^3y^3-(a-1)(a^2+a+3)x^2y^2z^2+(2a+3)(a-1)^3xyz^4-a^3y^5z-(a-1)^5z^6=0. \quad (9.4.12)$$

$$-a^4y^5z++a^2(ab-1)(ab+4)xy^3z^2+x^4y^2-b^2x^5-(3ab+2)(ab-1)^2x^2yz^3-(ab-1)^5z^6=0. \quad (9.4.13)$$

Substituting $b=0$ in the last equation, we obtain

$$-a^4y^5z+x^4y^2+z^6-4a^2xy^3z^2-2x^2yz^3=0, \quad a\neq 0. \quad (9.4.14)$$

The curve has five infinitely near cusps at $[1,0,0]$ (i.e. a simple curve singularity of type a_{10}) and five ordinary nodes at the point $[\alpha^2, 2\alpha^3, a^2]$, where $2\alpha^2+a^2=0$.

$\underline{G=C_6}$

We have 6 eigensubspaces $\langle t_0^3t_1^3\rangle$, $\langle t_0^6, t_1^6\rangle$, $\langle t_0t_1^5\rangle$, $\langle t_0^5t_1\rangle$, $\langle t_0^2t_1^4\rangle$, $\langle t_0^4t_1^2\rangle$ with eigenvalues $1, -1, \epsilon_3, \epsilon_3^2, \epsilon_6, \epsilon_6^{-1}$. We have to take one of the polynomials equal to $t_0^6+t_1^6$. After scaling, we may assume that $a=b=1$. Up to switching t_0 and t_1, we have the following parametrizations:

$$(x_0,x_1,x_2)=(1+t^6, t, t^a), \ a=2,3,4,5,$$

and

$$(x_0,x_1,x_2)=(1+t^6, t^2, t^3).$$

Using MAPLE, we check that only one of them gives a nodal sextic:

$$(x_0,x_1,x_2)=(1+t^6, t, t^5)$$

with equation

$$y^6 - x^4yz + 4x^2y^2z^2 - 2y^3z^3 + z^6 = 0. \tag{9.4.15}$$

It has 10 ordinary double points and has the dihedral group $\mathbb{D}_{12}$ as its group of projective symmetries.

Before we go to non-cyclic groups, let us prove the following.

Lemma 9.4.4 *Suppose that G is not a cyclic group. Then, the action of G in $|V|$ has only isolated fixed points.*

Proof Suppose we have a line ℓ of fixed points. Let η be its generic point. Since G is tame, it is mapped injectively into the group of automorphisms of the local ring $\mathcal{O}_{\mathbb{P}^2,\eta}$ and acts trivially on its residue field. Thus, it acts by sending a local parameter u to $\chi(g)u$, where $\chi : G \to \Bbbk^*$ is an injective homomorphism. Since G is not cyclic, this is impossible. □

The lemma shows that all characters of V enter with multiplicity 1.

We use Figure 9.4 and the paragraphs before the figure to describe such representations. We have to find all homomorphisms of linear representations

$$\rho : V \otimes \chi \to S^6U,$$

where $\chi : G \to \mathbb{G}_m$ is a one-dimensional representation of G. Since $U \cong U^\vee$, we find an expression of the character x_6 of $S^6U^\vee$ as a sum of irreducible characters and take $V^\vee$ to be the direct sum of three irreducible characters up to tensoring them simultaneously by a one-dimensional character of G. We also have to take care that the rational parametrization given by V defines a nodal sextic.

Next, we consider the case of a dihedral group.

- $G = \mathbb{D}_{2n}$.

It follows from the classification of rational nodal sextics with a cyclic group of symmetries that $n \in \{2, 3, 4, 5, 6\}$.

$\underline{G = \mathbb{D}_4}$

In this case, the character x_6 of $S^6U^\vee$ decomposes as $x_6 = \chi_0 + 2\chi_1 + 2\chi_2 + 2\chi_3$ with irreducible summands $\chi_0 = \langle t_0t_1^5 - t_0^5t_1\rangle, 2\chi_1 = \langle t_0t_1^5 + t_0^5t_1, t_0^3t_1^3\rangle, 2\chi_2 = \langle t_1^6 - t_0^6, t_0^2t_1^4 - t_0^4t_1^2\rangle, 2\chi_3 = \langle t_1^6 + t_0^6, t_0^2t_1^4 + t_0^4t_1^2\rangle$. One of the summands of V must be χ_2 or χ_3. We have the following possibilities:

$$(x_0, x_1, x_2) = (t^6 + 1 + a(t^4 + t^2), t^5 - t, b(t^5 + t) + ct^3), \tag{9.4.16}$$

$$(x_0, x_1, x_2) = (t^6 + 1 + a(t^4 + t^2), b(t^6 - 1) + c(t^2 - t^4), d(t^5 + t) + et^3), \tag{9.4.17}$$

$$(x_0, x_1, x_2) = (t^6 + 1 + a(t^4 + t^2), b(t^6 - 1) + c(t^2 - t^4), t^5 - t)$$

corresponding to the sum of characters $\chi_3 + \chi_0 + \chi_1, \chi_3 + \chi_2 + \chi_1$ and $\chi_3 + \chi_2 + \chi_0$, respectively. If $e = 0$, the last two parametrizations become isomorphic under the change $t \mapsto \epsilon_8 t$.

The explicit equation for parametrization (9.4.16) is the following:

$$y^6+2(a^3-3a^2+12)y^4z^2-3x^2y^4+(a+1)(a-2)^2(a^3-3a^2+12)y^2z^4+6(a^3-4)x^2y^2z^2+3x^4y^2$$
$$+6(a^2+2)x^4z^2-3a^2(3a^2-8)x^2z^4-x^6+4(a+1)^2(a-2)^2z^6=0, \tag{9.4.18}$$

where we used $b=e=1, c=d=0$, and for parametrization (9.4.17) the equation is the following:

$$4(a+1)^2x^2y^4-x^4y^2-2(a+1)(a-3)y^2z^4+(a+1)^2(a-3)^2y^4z^2+z^6+4ax^2y^2z^2=0, \tag{9.4.19}$$

where $b=0, c=1$.

$\underline{G=\mathbb{D}_6}$

We have two 2-dimensional characters x and ρ_1. The representation x_6 decomposes as $x_6=2\rho_1+\chi_0+2\chi_1$, where $2\rho_1=\langle t_0t_1^5, t_0^5t_1\rangle+\langle t_0^4t_1^2, t_0^2t_1^4\rangle$, $2\chi_1=\langle t_0^3t_1^3, t_1^6+t_0^6\rangle$, $\chi_0=\langle t_1^6-t_0^6\rangle$.

We have two possible decompositions of V, namely $V=\rho_1+\chi_0$, or $V=\rho_1+\chi_1$. This gives us two possible isomorphism classes of parametrizations:

$$(x_0,x_1,x_2)=(at^5+bt^2, bt^4+at, t^6+1+ct^3), \tag{9.4.20}$$
$$(x_0,x_1,x_2)=(at^5+bt^2, bt^4+at, t^6-1). \tag{9.4.21}$$

However, if $c=0$, they become isomorphic under the change $t\mapsto it$. The explicit equation for parametrization (9.4.20) with $a=0$ is the following:

$$x^6+y^6-(b^4-4b^2+2)x^3y^3+2(b^2+2)x^2y^2z^2+4b(x^4yz+xy^4z)-xyz^4=0. \tag{9.4.22}$$

$\underline{G=\mathbb{D}_8}$

In this case, $x_6=2\rho_1+\chi_1+\chi_2+\chi_3$, where $2\rho_1=\langle t_0^4t_1^2, t_0^2t_1^4\rangle+\langle t_0^6, t_1^6\rangle$, $\chi_1=\langle t_0^3t_1^3\rangle$, $\chi_2=\langle t_0t_1^5-t_0^5t_1\rangle$, $\chi_3=\langle t_0t_1^5+t_0^5t_1\rangle$. We obtain three possible parametrizations $V=\rho_1+\chi_i, i=1,2,3$.

$$(x_0,x_1,x_2)=(t^6+at^2, at^4+1, t^3), \tag{9.4.23}$$
$$(x_0,x_1,x_2)=(t^6+at^2, at^4+1, t^5+t), \tag{9.4.24}$$
$$(x_0,x_1,x_2)=(t^6+at^2, at^4+1, t^5-t). \tag{9.4.25}$$

The projective representations in the last two cases are isomorphic under the change $t\mapsto\epsilon_8t$. Note that the first two curves are special members of the family of curves with $\mathbb{D}_4$-symmetry defined in (9.4.16). They correspond to parameters d or e equal to zero. Also, the second family from (9.4.17) can also specialize to a member of the first family with $\mathbb{D}_8$-symmetry. We give to parameters the values $b=0, e=1$ and $a=-3$ and change the generator g_1 to the transformation $(t_0,t_1)\mapsto(\frac{1}{\sqrt{2}}(t_0-it_1), \frac{1}{\sqrt{2}}(t_0+it_1))$. The explicit equations of the two families are the following:

$$z^6-x^4y^2+4(a+1)^2x^2y^4+4ax^2y^2z^2+(a+1)^2(a-3)^2y^4z^2-x^3y^3+2(a+1)(a-3)y^2z^4=0. \tag{9.4.26}$$

and

$$(a+1)^4z^6+(a^2+8a+9)x^2y^2-x^5y-2x^3y^3-xy^5-2(a+3)(a+1)^2xyz^4+a(x^4+y^4)z^2=0. \tag{9.4.27}$$

$\underline{G=\mathbb{D}_{10}}$

Here, for the future use, we slightly change our generators of the binary dihedral group by taking

$$s=\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}.$$

We find that $x_6=\rho_1+2\rho_3+\chi_1$, where $\rho_1=\langle t_0^2t_1^4, t_0^4t_1^2\rangle$, $2\rho_3=\langle t_0^6,t_1^6\rangle+\langle t_0t_1^5,t_0^5t_1\rangle$, and $\chi_1=\langle t_0^3t_1^3\rangle$. There is only one isomorphism class of parametrizations $V=\rho_3+\chi_1$ represented by

$$(x_0,x_1,x_2)=(t^6+at,-at^5+1,t^3). \tag{9.4.28}$$

We have the following explicit equation

$$(a^2+1)^5z^6+(2a^2-3)(a^2+1)^3xyz^4+(a^2+1)(a^4-a^2+3)x^2y^2z^2+a^3x^5z-a^3y^5z-x^3y^3=0. \tag{9.4.29}$$

$\underline{G=\mathbb{D}_{12}}$

We know that there is only one isomorphism class of rational nodal sextics with cyclic symmetry of order 6 given by equation (9.4.11). It has additional projective symmetry given by interchanging y with z.

Before we go to the remaining cases, when $G=\mathbb{T},\mathbb{O}$, or $\mathbb{I}$, we would need to know how the character x_k of $S^kU^\vee$ decomposes into irreducible summands. Let $m_k(v)=\langle\chi_v,x_k\rangle$ be the multiplicity of the character χ_v corresponding to the vertex v of the extended graph of $\Gamma(p,q,r)$ from Table 9.4 in the representation S^kU with character x_k. Let

$$P_v(T)=\sum_{k=0}^{\infty}m_k(v)T^k.$$

The proof of the following result can be found in [684, 4.2].

Proposition 9.4.5 *Assume $G=\mathbb{T},\mathbb{O}$ or $\mathbb{I}$. Let a be the length of the arm of the Dynkin diagram that contains the trivial representation χ_0. The Poincaré series $P_v(T)$ is a rational function with denominator*

$$(1-T^{2a})(1-T^{4a-4}).$$

Denoting the numerator of $P_v(T)$ corresponding to the characters χ_v equal to $x_h, 0\le h\le a-1$, $yx_i, 0\le i\le b-2$ and $zx_j, 0\le j\le c-2$ by X_h, Y_i, Z_j, we have

$$X_h(T) = T^h + T^{6a-6-h} + (T^{2a-h} + T^{4a-4-h})\frac{(1-T^{2h})}{1-T^2}, \quad v = x_h,$$
$$Y_i(T) = T^{a+b-i-2}(1+T^{2a-2})\frac{(1-T^{2a})(1-T^{2i+2})}{(1-T^{2b})(1-T^2)}, \quad v = yx_i,$$
$$Z_j(T) = T^{a+c-j-2}(1+T^{2a-2})\frac{(1-T^{2a})(1-T^{2j+2})}{(1-T^{2c})(1-T^2)}, \quad v = zx_j.$$

A straightforward computation gives the following:

Corollary 9.4.6 *The character x_6 is equal to*

$$x_6 = \begin{cases} \chi_0 + 2x_2 & \text{if } G = \mathbb{T}, \\ x_2 + y + x_2 y & \text{if } G = \mathbb{O}, \\ xy + z & \text{if } G = \mathbb{I}. \end{cases}$$

Using these tools, we can proceed.

Without loss of generality, we may assume that $\Bbbk = \mathbb{C}$. Each such group is conjugate to a finite subgroup of $\mathrm{SU}(2)$. Let $\mathbb{H}^*$ be the multiplicative group $\mathbb{H}^*$ of quaternions. We use an isomorphism

$$\mathbb{H}^* \to \mathrm{SU}(2), \quad (a+bi) + (c+di)j \mapsto \begin{pmatrix} a+bi & c+di \\ -c+di & a-bi \end{pmatrix}.$$

This allows one to realize $\tilde{G}$ as a finite subgroup of $\mathbb{H}^*$.

Let $g_1 = \mathbf{i}$, $g_2 = \frac{1}{2}(1+\mathbf{i}+\mathbf{j}+\mathbf{k}), g_3 = \frac{1}{2}(1+\mathbf{i}+\mathbf{j}-\mathbf{k})$. They form a set of standard generators of $\tilde{G}$ satisfying $g_1^2 = g_2^3 = g_3^3 = g_1 g_2 g_3 = -1$. We will use other generators

$$g_1 = \begin{pmatrix} \epsilon_4 & 0 \\ 0 & \epsilon_4^{-1} \end{pmatrix}, \quad \mathbf{k} = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, \quad g_3 = \begin{pmatrix} \frac{1+i}{2} & \frac{1+i}{2} \\ \frac{-1+i}{2} & \frac{1-i}{2} \end{pmatrix}.$$

We see that the subgroup generated by $\mathbf{i}$ and $\mathbf{k}$ is isomorphic to the binary dihedral group $\tilde{D}_4$.

The transformation g_3 acts on the space $S^6U^\vee$ with a basis $(t_0^6, t_0^5t_1, t_0^4t_1^2, t_0^3t_1^3, t_0^2t_1^4, t_0t_1^5, t_1^5)$ via the matrix

$$S = 1/8 \cdot \begin{pmatrix} 1 & -i & -1 & i & 1 & -i & -1 \\ 6 & -4i & -2 & 0 & -2 & 4i & 6 \\ 15 & -5i & 1 & -3i & -1 & -5i & -15 \\ 20 & 0 & 4 & 0 & 4 & 0 & 20 \\ 15 & 5i & 1 & 3i & -1 & 5i & -15 \\ 6 & 4i & -2 & 0 & -2 & -4i & 6 \\ 1 & i & -1 & -i & 1 & i & -1 \end{pmatrix} \tag{9.4.30}$$

We start with the case

- $G = \mathbb{T}$

By Corollary 9.4.6, $x_6 = \chi_0 + 2x_2$. Hence, V is given by the character x_2. The group contains a subgroup isomorphic to $\mathbb{D}_4$. We have two families of rational nodal sextics

with $\mathbb{D}_4$-symmetry. We check that no member of the second family given in row 9 of Table 9.1 is invariant with respect to g_3. Using the matrix S, we check that a member of the first family with parameters (a, b, c, d, e) is invariant with respect to g_3 if and only if the following relations are satisfied:

$$(2d+e)a = 3e-10d, \quad (2d-e)c = (3e+10d)b, \quad b(15-a) = c(a+1). \quad (9.4.31)$$

The third relation follows from the first two relations. Also, the conditions $b = 0$, $e = 2d$ and $a = -1$ are equivalent.

It follows that we have a one-parameter family of tetrahedral rational nodal sextics. Assume that $b \neq 0, d \neq 0$, hence we can take $b = d = 1$.

We check that g_3 changes the basis of V via a scalar multiple of the matrix

$$\begin{pmatrix} 0 & \frac{i(2+e)}{e-2} & 0 \\ 0 & 0 & \frac{2-e}{8} \\ \frac{8i}{e+2} & 0 & 0 \end{pmatrix}.$$

This can be applied only if $e^2-4 \neq 0$. As we remarked before, under our assumption, $e \neq 2$ and also $e = -2$ does not satisfy the first of relations (9.4.31).

After scaling the variables $(x, y, z) \mapsto (-8i(e-2)x, 8(e+2)y, (e^2-4)z)$, we obtain the following equation:

$$\begin{aligned}&(e^2-4)^2(x^6+y^6+z^6) + (e+2)(3e^3-6e^2+36e-8)(x^4y^2+x^2z^4+y^4z^2)\\ &+ (e-2)(3e^3+6e^2+36e+8)(x^4z^2+x^2y^4+y^2z^4) - 3(7e^4+40e^2-16)x^2y^2z^2 = 0.\end{aligned} \quad (9.4.32)$$

Note that the equation has a full $\mathfrak{S}_3$-symmetry if and only if $e = 0$. In this case, we get a rational nodal sextic

$$x^6+y^6+z^6-(x^4y^2+x^2z^4+y^4z^2+x^4z^2+x^2y^4+y^2z^4)+3x^2y^2z^2 = 0 \quad (9.4.33)$$

with an octahedral symmetry. The plane sextic curve has ten ordinary nodes forming 4 orbits of points $[\pm 1, 1, 1], [-1, 1, 1], [\pm 1, 0, 1]$.

Assume now that $b = 0$. In this case, $a = -1, e = 2d$ and we may also take $c = d = 1$. The transformation g_3 is given by the matrix

$$\begin{pmatrix} 0 & \frac{i}{4} & 0 \\ 0 & 0 & 2 \\ -2i & 0 & 0 \end{pmatrix}.$$

So, we take a basis $(t^6+1-t^2-t^4, t^2-t^4, t^5+t+2t^3)$ in V and obtain the following equation of the rational nodal sextic:

$$x^4y^2+64y^4x^2-x^2z^4+12x^2y^2z^2 = 0.$$

After scaling $(x, y, z) \mapsto (-2ix, \frac{1}{2}y, z)$, we get the equation

$$x^4y^2 + y^4x^2 + x^2z^4 - 3x^2y^2z^2 = 0. \tag{9.4.34}$$

It has an obvious tetrahedral symmetry.

The curve has three tacnodes at the $\mathfrak{S}_3$-orbit of $[1, 0, 0]$ and four ordinary nodes at the $\mathfrak{S}_3$-orbit of $[\pm 1, 1, 1]$ and $[\pm 1, 1, 0]$.

Finally, we assume that $d = 0$. In this case $V = \langle t^6 + 1 + 3(t^2 + t^4), t^6 - 1 + 3(t^2 - t^4), t^3\rangle$. The action of the transformation g_3 is given by the following matrix

$$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & \frac{1}{8} \\ -8i & 0 & 0 \end{pmatrix}.$$

After scaling the basis, we get the following equation

$$x^6 + y^6 + z^6 - 21x^2y^2z^2 + 3(x^4y^2 + x^4z^2 + x^2y^4 + x^2z^4 + y^2z^2 + y^2z^4) = 0. \tag{9.4.35}$$

The curve has ten ordinary nodes forming the $\mathfrak{S}_3$-orbits of the points $[\pm 1, 1, 1]$ and $[0, \pm i, 1]$.

Next we consider the case

- $G = \mathbb{O}$.

It follows from Corollary 9.4.6 that there are two isomorphism classes of octahedral rational sextics realizing the irreducible representations of $\mathbb{O} \cong \mathfrak{S}_4$ with characters x_2 and x_2y. We have already found both of them as special members of the tetrahedral family. They correspond to the parameters $d = 0$ and $e = 0$. They are also special members of the family of nodal sextics given by (9.4.23) with $a = -5$ and in (9.4.23) with $a = 3$. Finally, we come to the case

- $G = \mathbb{I}$.

The group $\tilde{G}$ has generators from SU(2)

$$g_1 = \frac{1}{2}(\lambda^{-1}\mathbf{i} + \mathbf{j} + \mathbf{k}), \quad g_2 = \frac{1}{2}(1 + \mathbf{i} + \mathbf{j} + \mathbf{k}) \quad g_3 = \frac{1}{2}(\lambda + \lambda^{-1}\mathbf{i} + \mathbf{j})$$

satisfying $g_1^2 = g_2^3 = g_3^5 = g_1g_2g_3 = -1$.

However, we will use Klein's generators [391, p.213])

$$S = \begin{pmatrix} \epsilon_{10}^3 & 0 \\ 0 & \epsilon_{10}^{-3} \end{pmatrix}, \quad T = \frac{1}{\sqrt{5}}\begin{pmatrix} \epsilon_5 - \epsilon_5^4 & \epsilon_5^2 - \epsilon_5^3 \\ \epsilon_5^2 - \epsilon_5^3 & -\epsilon_5 + \epsilon_5^4 \end{pmatrix}, \quad U = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

of orders 10, 4 and 4.

By Corollary 9.4.6, $x_6 = xy + z$. Thus, $V \cong z$ and there is only one isomorphism class of nodal rational plane sextics with icosahedral projective symmetry. Since $\mathbb{D}_{10}$ is a subgroup of $\mathbb{I}$, our plane sextic belongs to the one-parameter family given in row (14) in Table 9.1. Applying transformation T, we see that the parametrization $t_1^6 + at_0^5t_1, -at_0t_1^5 + t_0^6, t_0^3t_1^3$ is invariant if and only if $a = 3$. If we multiply z by $5i$, we obtain the equation

$$32x^6 + 27xy^5 - 120x^4yz + 150x^2y^2z^2 + 5y^3z^3 + 27xz^5 = 0, \tag{9.4.36}$$

which can be found in [738, §2].

We summarize our classification in Table 9.1.

	G	V	x	y	z	Equation
(1)	C_2	$\epsilon_2 \oplus \epsilon_2 \oplus 1$	$t^6 + a_1t^4 + a_2t^2$	$a_3t^4 + a_4t^2 + 1$	$a_5t^5 + a_6t^3 + a_7t$	(9.4.6)
(2)	C_2	$\epsilon_2 \oplus 1 \oplus 1$	$t^6 + a_1t^4 + a_2t^2 + 1$	$a_3t^5 + a_4t^3 + a_5t$	$a_6t^3 + a_7t$	(9.4.7)
(3)	C_3	$1 \oplus \epsilon_3 \oplus \epsilon_3^2$	$t^6 + a_1t^3 + 1$	$a_2t^5 + a_3t^2$	$a_4t^4 + a_5t$	(9.4.8)
(4)	C_4	$\epsilon_4 \oplus \epsilon_4^3 \oplus 1$	$t^6 + at^2$	$t^4 + 1$	t^3	(9.4.9)
(5)	C_4	$\epsilon_4 \oplus \epsilon_4^3 \oplus \epsilon_2$	$t^6 + a_1t^2$	$a_2t^4 + 1$	$a_3t^5 + a_4t$	(9.4.10)
(6)	C_5	$\epsilon_5^2 \oplus \epsilon_5^3 \oplus \epsilon_5$	$t^6 + at$	$t^5 + 1$	t^2	(9.4.11)
(7)	C_5	$\epsilon_5^2 \oplus \epsilon_5^3 \oplus 1$	$t^6 + at$	$t^5 + 1$	t^3	(9.4.12)
(8)	C_5	$\epsilon_5^2 \oplus \epsilon_5^3 \oplus \epsilon_5^4$	$t^6 + at$	$t^5 + 1$	t^4	(9.4.13)
(9)	$\mathbb{D}_4$	$\chi_3 \oplus \chi_2 \oplus \chi_1$	$t^6 + 1 + a(t^4 + t^2)$	$b(t^6 - 1) + c(t^4 - t^2)$	$d(t^5 + t) + et^3$	(9.4.18)
(10)	$\mathbb{D}_4$	$\chi_3 \oplus \chi_1 \oplus \chi_0$	$t^6 + 1 + a(t^4 + t^2)$	$b(t^5 + t) + ct^3t^5 - t$		(9.4.19)
(11)	$\mathbb{D}_6$	$\rho_1 \oplus \chi_1$	$t^5 + at^2$	$at^4 + t$	$t^6 + bt^3 + 1$	(9.4.22)
(12)	$\mathbb{D}_8$	$\rho_1 \oplus \chi_1$	$t^6 + at^2$	$at^4 + 1$	t^3	(9.4.26)
(13)	$\mathbb{D}_8$	$\rho_1 \oplus \chi_3$	$t^6 + at^2$	$at^4 + 1$	$t^5 + t$	(9.4.27)
(14)	$\mathbb{D}_{10}$	$\rho_3 \oplus \chi_1$	$t^6 - at$	$at^5 + 1$	t^3	(9.4.29)
(15)	$\mathbb{D}_{12}$	$\rho_3 \oplus \chi_2$	t	t^5	$t^6 + 1$	(9.4.15)
(16)	$\mathbb{T}$	x_2	$t^6 + 1 + a(t^4 + t^2)$	$b(t^6 - 1) + c(t^2 - t^4)$	$d(t^5 + t) + et^3 = 0$	(9.4.31), (9.4.32)
(17)	$\mathbb{O}$	x_2y	$t^6 + 3t^2$	$3t^4 + 1$	t^3	(9.4.33)
(18)	$\mathbb{O}$	x_2	$t^6 - 5t^2$	$-5t^4 + 1$	$t^5 + t$	(9.4.35)
(19)	$\mathbb{I}$	z	$t^6 + 3t$	$3t^5 - 1$	$-5t^3$	(9.4.36)

Table 9.1 Symmetric rational nodal sextics

Remark 9.4.7 A homogeneous real polynomial $P \in \mathbb{R}[x_0, \ldots, x_n]$ is called positive semi-definite (psd) if it takes non-negative values on $\mathbb{R}^{n+1}$. Let $P_{n+1,d}$ denote the set of psd polynomials and $\Sigma_{n+1,d}$ its subset of polynomials which are equal to the sum of squares of real polynomials. Obviously, d must be even. Hilbert proved that $P_{3,4} = \Sigma_{3,4}$ and $P_{3,6} \neq \Sigma_{3,6}$. It is clear that any zero of a psd polynomial must be its critical point, and it is known that the number of zeros of $P \in P_{3,6}$ is at most 10, and every $P \in P_{3,6} \setminus \Sigma_{3,6}$ with tenzeros is extremal in the sense that it is not equal to a sum of other psd polynomials.

It is not surprising that we can find some of psd ternary sextics in Winger's list of self-projective rational nodal sextics.

The first explicit examples of a psd ternary sextics were the *Motzkin ternary sextic*

$$M = x^4y^2 + x^2y^4 - 3x^2y^2z^2 + z^6,$$

the *Robertson ternary sextic*

$$R = x^6 + y^6 + z^6 - (x^4y^2 + x^2z^4 + y^4z^2 + x^4z^2 + x^2y^4 + y^2z^4) + 3x^2y^2z^2 = 0,$$

and the *Choi–Lam ternary sextic*

$$S = x^4y^2 + x^2z^4 + y^4z^2 - 3x^2y^2z^2.$$

We find them in our list of self-projective nodal sextics. Thus, $M = 0$ is a member of the family of sextics with $\mathbb{D}_4$-symmetry given in (9.4.19), where we substitute $a = 3$ and change the variables $(x, y, z) \mapsto (2x, iy/4, z)$. Its real roots are its critical points, 4 ordinary nodes at the $\mathfrak{S}_3$-orbits of $[\pm 1, 1, 1]$ and two simple singular points $[0, 1, 0]$, $[1, 0, 0]$ of type a_3.

The polynomial has a larger projective symmetry group isomorphic to $\mathbb{D}_8$. If we use the change of variables $(x, y, z) \mapsto (\epsilon_8(-x - iy), \epsilon_8(ix + y), \frac{z}{2})$, where $\epsilon_8 = \frac{\sqrt{2}}{2}(1+i)$, the family given by equation (9.4.26) is transformed to a family given by the equation

$$\begin{aligned} &16(x^4y^2 + x^2y^4) + (a+3)^2(x^4 + y^4)z^2 + 2(a+3)(a+1)^2(x^2 + y^2)z^4 \\ &+ 2(a^2 + 14a + 9))x^2y^2z^2 + (a+1)^4z^6 = 0. \end{aligned} \tag{9.4.37}$$

If we substitute $a = -3$, the left-hand side becomes equal to $16M$, where M is the Motzkin polynomial. After the substitution $a = \frac{3-2t^2}{2t^2-1}$, we get the following one-parameter family:

$$\begin{aligned} M_t(x, y, z) &= (1 - 2t^2)(x^4y^2 + x^2y^4) + t^4(x^4 + y^4)z^2 \\ &- 2t^2(x^2 + y^2)z^4 - (3 - 8t^2 + 2t^4)x^2y^2z^2 + z^6 = 0 \end{aligned} \tag{9.4.38}$$

with $M_0 = M$. It was shown by B. Reznick that for $t^2 < 1$ all members of this family are psd polynomials with tencritical points [617]. So, they are not sums of squares and extremal among psd polynomials.

The Robertson polynomial and the Choi–Lam polynomials can be found in the family of tetrahedral nodal sextics corresponding to the values of the parameter $e = 0$ and $b = 0$, respectively. Substituting $e = \frac{2-2t^2}{t^2+1}$ in equation (9.4.32), we obtain another family: of psd ternary sextics discovered by Reznick

$$\begin{aligned} S_t(x, y, z) &= t^4(x^6 + y^6 + z^6) + (1 - 2t^6)(x^4y^2 + y^4z^2 + x^2z^4) \\ &+ (t^8 - 2t^2)(x^2y^4 + y^2z^4 + z^2x^4) - 3(1 - 2t^2 + t^4 - 2t^6 + t^8)x^2y^2z^2. \end{aligned} \tag{9.4.39}$$

We have $S_1 = R$ and $S_0 = S$.

Note that, by *Timofte's test of positivity*, a homogeneous symmetric polynomial $P(x_1, \ldots, x_d)$ is a psd polynomial if and only if it takes positive values for $x_1 = \cdots = x_k = s, x_{k+1} = \cdots = x_d = t$, where $2k < d$ (see [619]).

Remark 9.4.8 The equation of the icosahedral sextic can be expressed as

$$5A^3 + 27F = 0$$

where $A = x_0x_1 + x_2^2$ and $F = x_2(x_0^5 + x_1^5 + x_2^5 + 5x_0^2x_1^2x_2 - 5x_0x_1x_2^3)$ are fundamental invariants of the icosahedron group of degree 2 and 6 (see [737], [177, 9.5.4]).

9.5 Quartic Symmetroids with Projective Symmetry

In Section 7.5 we discussed a construction of quartic symmetroids as catalecticant quartic surfaces. In this section, we use the classification of self-projective rational nodal plane sextics to give examples of quartic symmetroids with projective self-symmetry.

Let G be a polyhedral group. In the previous section we found a 3-dimensional subrepresentation V of G in $S^6U^\vee$ that defines a G-equivariant rational parametrization of a Coble sextic. Let

$$S^6U^\vee = V \oplus L,$$

where L is a 4-dimensional linear representation of G. The linear space $S^6U^\vee$ admits a unique $\mathrm{SL}(U)$-invariant symmetric bilinear form such that the two summands V and L become orthogonal summands.

Let us recall the definition of this bilinear form from Section 7.5. The linear representations of $\mathrm{SL}(U)$ in U and its dual representation in $U^\vee$ are isomorphic via the correlation isomorphism $c : U \to U^\vee$ defined by choosing a volume form $\Omega \in \bigwedge^2 U \cong \Bbbk$. If we choose a basis (e_0, e_1) of U and the dual basis (t_0, t_1) of $U^\vee$, correlation isomorphism is given by $(t_0, t_1) \mapsto (-e_1, e_0)$. It defines an isomorphism of the linear representations $S^6U^\vee \to S^6U$ of $\mathrm{SL}(U)$ whose composition with the polarization isomorphism $S^6U \to (S^6U^\vee)^\vee$ is an $\mathrm{SL}(U)$-invariant symmetric bilinear form on $S^6(U^\vee)$. In coordinates, it is given by

$$(\sum_{i=0}^{6} a_i t_0^i t_1^{6-i}, \sum_{i=0}^{6} b_i t_0^i t_1^{6-i}) = \sum_{i=0}^{6} (-1)^i \binom{6}{i} a_i b_{6-i}.$$

It corresponds to the quadratic invariant

$$I_2 = a_0a_6 - 6a_1a_5 + 15a_2a_4 - 10a_3^2$$

of binary sextics $\sum_{i=0}^{6} \binom{6}{i} a_i t_0^i t_1^{6-i}$. Of course, all of this applies to binary forms of any even degree (in odd degree we obtain a skew-symmetric bilinear form).

The linear space $L = V^\perp \subset S^6U^\vee$ defines a web $W = |L|$ of harmonic quadrics and its discriminant quartic is a quartic symmetroid $\mathcal{D}(W)$ with $G \subset \mathrm{Aut}(\mathcal{D}(W))$. In this section, we will find its equation for each possible polyhedral group G and a linear 3-dimensional representation of G in $V \subset S^6U^\vee$ that leads to a rational nodal sextic. We will restrict ourselves to non-cyclic groups G.

Recall from Section 7.5 that there exists a $\mathrm{SL}(U)$-equivariant isomorphism from the space $S^3((S^3U)^\vee)_{\mathrm{harm}}$ of harmonic quadratic forms on S^3U and the space $S^6U^\vee$ of binary sextics. Using the explicit definition of harmonic forms given in Section

7.5, we can choose a basis in $S^3(V^\vee)_{\mathrm{harm}}$ formed by the following polynomials:

$$p_1 = xw+9yz,\ p_2 = 2xz+3y^2,\ p_3 = 2yw+3z^2,\ p_4 = x^2,\ p_5 = w^2,\ p_6 = xy,\ p_7 = zw.$$

(see equation (7.5.4)). They correspond to the following basis in $S^6U^\vee$:

$$10t_0^3t_1^3,\ 5t_0^4t_1^2,\ 5t_0^2t_1^4,\ t_0^6,\ t_1^6,\ t_0^5t_1,\ t_0t_1^5.$$

Let us identify the representation L with a linear subspace of harmonic quadratic form. Then, the associated quartic symmetroid $\mathcal{D}(W)$ admits G as its group of projective automorphisms. If the web of harmonic quadrics has no base points, the quotient of the minimal nonsingular model of the symmetroid by the Reye involution is an Enriques surface that admits a Fano–Reye polarization with the group of symmetry isomorphic to G.

We use the classification of symmetric Coble sextics from Table (9.1).

We start with the group

- $G = \mathbb{D}_4$

There are two families corresponding to Coble sextics in rows 9 and 10 of the table with V given in (9.4.16) and (9.4.17). Let us consider the first case. We have

$$V = \langle t^6 + 1 + a(t^4 + t^2), b(t^6 - 1) + c(t^4 - t^2), d(t^5 + t) + et^3\rangle.$$

Assume first that $ac \neq 0$. We can choose the following basis of $V^\perp$:

$$e_1 = t^5-t,\ e_2 = -3e(t^5+t)+20dt^3,\ e_3 = a(t^6+1)+15(t^2+t^4),\ e_4 = c(t^6-1)+15b(t^4-t^2).$$

The action of g_1 and g_2 is given by diagonal matrices $(-1, -1, 1, 1)$ and $(-1, 1, 1, -1)$. The web of harmonic quadrics is

$$x_0(p_7-p_6)+x_1(-3e(p_7+p_6)+2dp_1)+x_2(a(p_4+p_5)-3(p_2+p_3))+x_3(c(p_5-p_4)-3b(p_3-p_2)) = 0.$$

Using MAPLE, we find the following equation of the discriminant quartic:

$$\begin{aligned}
&x_0^4 + (e^2 - 4d^2)^2x_1^4 + 16(a + 1)^2x_2^4 + 16b^2(b + c)^2x_3^4 - 2(e^2 - 4d^2)x_0^2x_1^2 + 8(a + 1)x_0^2x_2^2\\
&- 8b(b + c)x_0^2x_3^2 + 8(-2a^2d^2 + 4ade + ae^2 - 6d^2 - 4de - e^2)x_1^2x_2^2\\
&+ 8(6b^2d^2 - 4b^2de + b^2e^2 + 4bcde - bce^2 + 2c^2d^2)x_1^2x_3^2 + 16(-a^2b^2 + 6ab^2 - 4b^2\\
&+ 6bc - c^2)x_2^2x_3^2 + 16(2abd - abe + 4bd + 4cd + ce)x_0x_1x_2x_3 = 0.
\end{aligned} \tag{9.5.1}$$

If $a = 0, c \neq 0$, we may assume $c = 1$ and take the following basis of $V^\perp$

$$e_1 = t^5 - t,\ e_2 = -3e(t^5 + t) + 20dt^3,\ e_3 = t^4 - t^2,\ e_4 = (t^6 - 1) + 15b(t^2 + t^4).$$

The equation of the web of harmonic quadrics is the following:

$$x_0(p_7-p_6)+x_1(-3e(p_6+p_7)+2dp_1)+x_2(p_3-p_2)+x_3(a(p_5-p_4)+3b(p_2+p_3)) = 0.$$

The equation of the discriminant quartic is given by:

$$\begin{aligned}
&x_0^4 + 81(e^2-4d^2)^2x_1^4 + 16x_2^4 + 1296b^2(b^2-1)x_3^4 - 18(e^2-4d^2)x_0^2x_1^2 - 8x_0^2x_2^2\\
&+ (72b^2x_0^2x_3^2 + 72(6d^2-4de+e^2))x_1^2x_2^2 + 648(-6b^2d^2-4b^2de-b^2e^2+2d^2))x_1^2x_3^2\\
&+ 144(-2b^2+1)x_2^2x_3^2 - 576bdx_0x_1x_2x_3 + 2592b^2x_2x_3^3 + 96x_2^3x_3 - 432b(2d+e)x_0x_1x_3^2\\
&- 24x_0^2x_2x_3 + 216e(4d-e)x_1^2x_2x_3 = 0.
\end{aligned} \tag{9.5.2}$$

If $c = 0$, we take $d = 1$ and find the following basis of $V^\perp$:

$$e_1 = t^5 - t,\ e_2 = -3e(t^5+t) + 20t^3,\ e_3 = t^4 - t^2,\ e_4 = a(t^5+t) - 30t^2.$$

The equation of the web of harmonic quadrics is the following:

$$x_0(p_7-p_6) + x_1(-3e(p_6+p_7) + 2p_1) + x_2(p_3-p_2) + x_3(a(p_2+p_3) - 6p_2) = 0.$$

The equation of the discriminant quartic is:

$$\begin{aligned}
&x_0^4 + 81(e^2-4d^2)^2x_1^4 + 16x_2^4 + (4a^2-6a-9)(4a^2+6a+9)x_3^4 - 18(e^2-4d^2)x_0^2x_1^2\\
&+ 2(4a^2+9)x_0^2x_3^2 + 72(e^2-4de+6d^2)x_1^2x_2^2 - 4(8a^2-27)x_2^2x_3^2 - 192adx_0x_1x_2x_3\\
&- 8x_0^2x_2^2 + 18(-24a^2d^2-16a^2de-4a^2e^2+108d^2-9e^2)x_1^2x_3^2 + 72(6a^2d+2a^2e-9d)x_1x_2^3\\
&+ 96ax_0x_2x_3^2 + 648d(e^2-4d^2)x_1^3x_3 + 144(e-3d)x_1x_2^2x_3 - 72dx_0^2x_1x_3 = 0.
\end{aligned} \tag{9.5.3}$$

In particular, if $a = c = 0$, we take $d = 1$ and get the equation

$$\begin{aligned}
&x_0^4 + 81(e^2-4d^2)^2x_1^4 + 16x_2^4 + 81x_3^4 - 18(e^2-4d^2)x_0^2x_1^2 + 18x_0^2x_3^2\\
&+ 72(e^2-4de+6d^2)x_1^2x_2^2 + 108x_2^2x_3^2 - 8x_0^2x_2^2 + 162(12d^2-e^2)x_1^2x_3^2\\
&+ 648dx_1x_2^3 + 648d(e^2-4d^2)x_1^3x_3 + 144(e-3d)x_1x_2^2x_3 - 72dx_0^2x_1x_3 = 0.
\end{aligned} \tag{9.5.4}$$

Next, we consider the second family with $\mathbb{D}_4$-symmetry.

This time, $V = \langle t^6 + 1 + a(t^4+t^2), t^5 - t, b(t^5+t) + ct^3\rangle$ and we choose a basis of $V^\perp$ formed by

$$e_1 = t^6 - 1,\ e_2 = t^2 - t^4,\ e_3 = 3c(t^5+t) - 10bt^3,\ e_4 = a(t^6+1) - 15(t^2+t^4).$$

The group acts by diagonal matrices $(1, 1, -1, 1)$ and $(-1, -1, 1, 1)$. The web of harmonic quadrics is:

$$x_0(p_5-p_4)+x_1(p_2-p_3)+x_2(-3c(p_6+p_7)-bp_1)+x_3(a(p_4+p_5)-3(p_2+p_3)) = 0.$$

The equation of the discriminant quartic (after scaling of the unknowns) is:

$$(b^2-c^2)^2x_2^4+(a+1)^2x_3^4+x_1^4+(b^2-c^2)x_2^2x_3^2+(3b^2+4bc+2c^2)x_1^2x_2^2-x_0^2x_3^2$$
$$-(a^2-6a+2)x_1^2x_3^2+2c(2b+c)x_0x_1x_2^2+b^2x_0^2x_2^2-6x_0x_1x_3^2+x_0^2x_1^2-2x_0x_1^3=0. \tag{9.5.5}$$

Next, we consider the case

- $G=\mathbb{D}_6$

In this case, $V=\langle t^5+at^2, at^4+t, t^6+1+bt^3\rangle$ and $V^\perp$ has a basis

$$e_1=5t^4+2at,\ e_2=2at^5+5t^2,\ e_3=b(t^6+1)+40t^3,\ e_4=t^6-1.$$

The web of harmonic quadrics is

$$x_0(p_3+2ap_6)+x_1(p_2+2ap_7)+x_2(b(p_4+p_5)+4p_1)+x_3(p_5-p_4)=0.$$

To simplify the equation, we replace the generators Q_0,Q_1,Q_2 with $Q_2+\epsilon_3^2Q_1+\epsilon_3Q_0,\ Q_2+\epsilon_3Q_1+\epsilon_3^2Q_0,\ Q_2+Q_1+Q_0$, where ϵ_3 is a primitive third root of unity. In this way, the generator g_1 of the group $\mathbb{D}_6$ acts as a cyclic permutation of the coordinates x_0,x_1,x_2. The generator g_2 acts in the projective space by permuting x_0 and x_1. The equation of the discriminant hypersurface becomes

$$(a^2-2a+15b-29)(a^2+2a-21b-41)(x_0^4+x_1^4+x_2^4)+657(x_0x_1+x_0x_2+x_1x_2)x_3^2$$
$$+3(a^4+6a^2b-2a^2-648b^2-24a+2b+2593)(x_0^2x_1^2+x_0^2x_2^2+x_1^2x_2^2)++315(x_0^2+x_1^2+x_2^2)x_3^2$$
$$+(-2a^4+3a^2b-68a^2+72ab-1287b^2-12a+3b+5074)(x_0^3(x_1+x_2)+x_1^3(x_0+x_2)$$
$$+x_2^3(x_0+x_1)+9(-2a^2b+24a^2-24ab-435b^2+8a-2b+1764)(x_0^2x_1x_2+x_0x_1^2x_2$$
$$+x_0x_1x_2^2)-9\sqrt{-3}(a^2-1)(x_0^2(x_1-x_2)+x_1^2(x_2-x_0)+x_2^2(x_0-x_1)x_3=0. \tag{9.5.6}$$

- $G=\mathbb{D}_8$

We have two families of those corresponding to rows (12) and (13) in the table. We start with the first case. It is a specialization of the first case with $G=\mathbb{D}_4$ by taking $d=0, e=1, a=c, b=1$. We have $V=\langle t^6+at^2, at^4+1, t^3\rangle$ and take the following basis of $V^\perp$:

$$e_1=t^5,\ e_2=t,\ e_3=a-15t^4,\ e_4=at^6-15t^2,\ \ .$$

The web of harmonic quadrics is given by:

$$x_0p_6+x_3p_7+x_0(ap_4-3p_3)+x_1(ap_5-3p_2)=0.$$

The equation of the discriminant quartic is

$$1296a(x_2^4+x_3^4)+1296(a^2+1)x_2^2x_3^2+x_0^2x_1^2+36a(x_0^2+x_1^2)x_2x_3-72x_0x_1x_2x_3=0. \quad (9.5.7)$$

The transformation g_1 is given by the diagonal matrix $(1,-1,i,-i)$ and the transformation g_2 switches x_0,x_1 and x_2,x_3. Without destroying the symmetry, we can scale the variables x_1 and x_2 to obtain equation

$$a(x_2^4+x_3^4)+(a^2+1)x_2^2x_3^2+x_0^2x_1^2+a(x_0^2+x_1^2)x_2x_3-2x_0x_1x_2x_3=0. \quad (9.5.8)$$

Let us look at the second family with $\mathbb{D}_8$-symmetry. It is a specialization of the first family with D_4-symmetry obtained by taking $a=c, b=1, d=1, e=0$. In this case, $V=\langle t^6+1+a(t^4+t^2), t^6-1+a(t^2-t^4), t^5+1\rangle$ and $V^\perp$ has a basis

$$e_1=t^5-t,\ e_2=t^3,\ e_3=at^6-15t^2,\ e_4=-15t^4+a.$$

The transformation g_1 acts via a diagonal matrix $(i,-i,-1,1)$ and the transformation g_2 multiplies x_0 by -1 and switches the last two coordinates.

The web of harmonic quadrics is:

$$x_0(p_7-p_6)+x_1p_1+x_2(ap_5-3p_2)+x_3(ap_4-3p_3)=0.$$

The equation of the discriminant quartic (after scaling the first two variables) is:

$$x_0^4+x_1^4+a(x_2^4+x_3^4)+(a^2+1)x_2^2x_3^2+2x_0^2x_1^2-(a^2+3)x_0^2x_2x_3+2(a+1)(x_0x_1x_3^2+x_1^2x_2x_3-x_0x_1x_2^2)=0. \quad (9.5.9)$$

- $G=\mathbb{D}_{10}$

We have only one family with $V=\langle t^6+at, at^5-1, t^3\rangle$ and use a basis of $V^\perp$

$$e_1=6t^5+a,\ e_2=6t-at^6,\ e_3=t^4,\ e_4=t^2.$$

The web of harmonic quadrics is

$$x_0(6p_7+ap_4)+x_1(ap_5+6p_6)+x_2p_3+x_3p_2=0.$$

The equation of the discriminant quartic (after scaling the last two variables) is:

$$(a^2-2)x_0x_1x_2x_3+ax_1x_3^3+ax_0^3x_3+ax_0x_2^3+ax_1^3x_2+x_2^2x_2^2+x_0^2x_1^2=0. \quad (9.5.10)$$

- $G=\mathbb{D}_{12}$

Again, we have only one isomorphism class of rational nodal sextics with $\mathbb{D}_{12}$-symmetry. The linear space V is spanned by t^6+1, t^5, t and $V^\perp=\langle t^6-1, t^2, t^3, t^4\rangle$. The web of harmonic quadrics is:

$$x_0(p_4-p_5)+x_1p_1+x_2p_2+x_3p_3=0.$$

The equation of the symmetroid (after an equivariant scaling of the variables) is the following:

$$x_0x_3^3 + x_2^2x_3^2 - x_0^2x_2x_3 - 3x_1^2x_2x_3 + x_0^2x_1^2 - x_0x_2^3 + x_1^4 = 0. \tag{9.5.11}$$

- $G = \mathbb{T}$

  Here,

$$V = \langle t^6 + 1 + a(t^4 + t^2), b(t^6 - 1) + c(t^2 - t^4), d(t^5 + t) + et^3\rangle,$$

where the parameters satisfy the relations

$$(2d + e)a = 3e - 10d, \quad (e - 2d)c = (3e + 10d)b. \tag{9.5.12}$$

Let us first omit the special cases $b = 0$, $d = 0$. Thus, we may assume that $b = d = 1$. The relations imply that $e \neq \pm 2$, and

$$a = \frac{3e - 10}{e + 2}, \ c = \frac{3e + 10}{e - 2}.$$

If $e \neq 0$, we can take the following basis of $V^\perp$:

$$e_1 = t^5 - t, \ e_2 = 10t^3 - 3et, \ e_3 = 15(3e^2+20)t^2 + 240et^4 - (9e^2-100)t^6, \ e_4 = 16e + 15(e^2-4)t^2 - (3e^2+20)t^6.$$

If $e = 0$, we take

$$e_1 = t^5 - t, \ e_2 = t^3, \ e_3 = 3t^2 + t^6, \ e_4 = 3t^4 + 1.$$

Let us postpone the latter case until later.

We check that g_1, g_2 and g_3 act on the basis via the following matrices:

$$g_1 = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \ g_2 = \begin{pmatrix} -1 & -3e & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \frac{3e^2+20}{16e} & \frac{e^2-4}{16e} \\ 0 & 0 & \frac{100-9e^2}{16e} & -\frac{3e^2+20}{16e} \end{pmatrix}.$$

Using (9.4.30), we also check that g_3 acts via the matrix

$$g_3 = \begin{pmatrix} 1 & \frac{3e}{2} & 6ie(3e+10) & 6ie(e-2) \\ 0 & 0 & -4i(3e+10) & -4i(e-2) \\ 0 & \frac{e-2}{128e} & -\frac{i(e+2)(3e-10)}{32e} & \frac{-i(e+2)^2}{32e} \\ 0 & -\frac{3e+10}{128e} & \frac{i(3e-10)^2}{32e} & \frac{i(e+2)(3e-10)}{32e} \end{pmatrix}.$$

We would like to change a basis of $V^\perp$ so that the group $\mathbb{T} \cong \mathfrak{A}_4$ would act by even permutations of coordinates. Let

$$C = \begin{pmatrix} 1 & -\frac{3e}{2} & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & e+2 & 2-e \\ 0 & 0 & 10-3e & 3e+10 \end{pmatrix}.$$

We check that $C^{-1}g_2C$ is the diagonal matrix $(-1,1,1,-1)$ and

$$g_3' = C^{-1}g_3C = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & -128ie & 0 \\ 0 & 0 & 0 & -i \\ 0 & -\frac{1}{128e} & 0 & 0 \end{pmatrix}.$$

Replacing C with the product $C\cdot J$, where J is the diagonal matrix $(1,1,\frac{i}{128e},-\frac{i}{128e})$, we obtain

$$C^{-1}g_3C = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{pmatrix}.$$

Therefore, we achieve our goal if we take a new basis

$$f_1 = e_1,\ f_2 = e_2 - \frac{3e}{2}e_1,\ f_3 = \frac{i}{128e}((e+2)e_3 + (10-3e)e_4),\ f_4 = \frac{1}{128e}((e-2)e_3 - (3e+10)e_4).$$

The web of harmonic quadrics is now given by

$$\begin{aligned} &x_0(p_7 - p_6) + x_1(p_1 - \frac{3e}{2}(p_6 + p_7)) + \frac{i}{8}x_2((3e-10)(p_4+p_5) - 3(e+2)(p_2+p_3)) \\ &+ \frac{1}{8}x_3((3e+10)(p_5 - p_4) + 3(e-2)(p_3 - p_2)) = 0. \end{aligned} \tag{9.5.13}$$

After equivariant scaling $(x_0,x_1,x_2,x_3) \mapsto (ix_0, x_1/3, x_2/3, x_3/3)$, we obtain the following equation of the quartic symmetroid:

$$\begin{aligned} &x_0^4 + (e^2-4)^2(x_1^4 + x_2^4 + x_3^4) + 2(e^2-4)x_0^2(x_1^2 + x_2^2 + x_3^2) \\ &- (e^4 - 40e^2 - 112)(x_1^2x_2^2 + x_2^2x_3^2 + x_1^2x_2^2) + 16(3e^2+4)x_0x_1x_2x_3 = 0. \end{aligned} \tag{9.5.14}$$

Let us now consider the case $e = 0$. Here, we may assume that $b = d = 1$ and it follows from relations (9.5.12) that $a = c = -5$. We have a basis $V^\perp$ formed by $t^5 - t,\ t^3,\ t^6 + 3t^2,\ 3t^4 + 1$. The matrices g_2, g_3 and C change to

$$g_2 = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \quad g_3 := \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & -4i & -4i \\ 0 & \frac{1}{8} & \frac{i}{2} & \frac{-i}{2} \\ 0 & -\frac{1}{8} & \frac{i}{2} & \frac{-i}{2} \end{pmatrix}.$$

In a new basis

$$f_1 = e_1,\ f_2 = e_2,\ f_3 = \frac{i}{8}(e_3 + e_4),\ f_4 = \frac{i}{8}(e_3 - e_4),$$

the equation of the discriminant quartic is

$$x_0^4+x_1^4+x_2^4+x_3^4-2x_0^2(x_1^2+x_2^2+x_3^2)+7(x_1^2x_2^2+x_2^2x_3^2+x_1^2x_2^2)+8x_0x_1x_2x_3=0. \quad (9.5.15)$$

It is obtained from equation (9.5.14) by substituting $e = 0$ and replacing x_1, x_2, x_3 with $\frac{1}{2}x_1, \frac{1}{2}x_2, \frac{1}{2}x_3$. The equation has obvious $\mathfrak{S}_3$-symmetry in coordinates x_1, x_2, x_3, also it admits a symmetry group $\mathbb{D}_4$ of transformations defined by diagonal matrices $(1, \pm 1, \pm 1, \pm 1)$.

The total symmetry is the octahedral group $\mathbb{O} \cong \mathfrak{S}_4$. We see that it is also a specialization of the second family of surfaces with dihedral symmetry $\mathbb{D}_8$. It corresponds to the parameter $a = -5$.

The surface has ten ordinary nodes at $[1, 2, 1, -1]$, $[1, 2, -1, 1]$, $[1, -2, 1, 1]$, $[1, -2, -1, -1]$ and the $\mathfrak{S}_4$ orbits of the points $[1, 0, 1, 0]$, $[1, 0, -1, 0]$.

In the second omitted case, we have $b = 0$. We may assume that $c = 1$ and relations (9.5.12) imply that $e = 2d$, so we may also assume that $d = 1$ and $e = 2$. We have

$$V = \langle t^6 + 1 - (t^2 + t^4), t^2 - t^4, (t^5 + t) + 2t^3 \rangle.$$

We start with the following basis of $V^\perp$

$$e_1 = t^5 - t,\ e_2 = -5t^3 + 3t,\ e_3 = t^6 - 1,\ e_4 = 2t^6 + 15(t^2 + t^4).$$

The transformation g_3 acts in this basis via the following matrix:

$$C := \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2i \\ 0 & -2 & -\frac{i}{2} & -i \\ 0 & -2 & \frac{i}{4} & \frac{i}{2} \end{pmatrix}.$$

In a new basis

$$f_1 = e_1,\ f_2 = 2i(3e_1 + 2e_2),\ f_3 = -4ie_3,\ f_4 = -e_3 + e_4,$$

the equation of the web of harmonic quadrics is:

$$x_0(p_7 - p_6) + ix_1(-2p_1 + 6p_6 + 6p_7) - 4ix_2(p_5 - p_4) + x_3(3p_2 + 3p_3 + p_4 + p_5) = 0.$$

The equation of the quartic symmetroid (after an equivariant scaling of the variables) is

$$x_0^4 + x_1^2x_3^2 + x_2^2x_3^2 + x_1^2x_2^2 + 4x_0x_1x_2x_3 = 0 \quad (9.5.16)$$

The surface has four ordinary nodes at the $\mathfrak{S}_4$ orbit of $[1, 1, 1, -1]$ and three rational double points of type A_2 at $[0, 1, 0, 0]$, $[0, 0, 1, 0]$, $[0, 0, 0, 1]$.

Finally, assume $d = 0$. Then, relations (9.5.12) let us also assume that $b = e = 1, a = -5$. We take a basis

$$e_1 = t^5 - t,\ e_2 = t^5 + t,\ e_3 = t^6 - 5t^2,\ e_4 = -5t^2 + 1$$

and find that g_3 acts in this basis via the following matrix:

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & -2i & -2i \\ 0 & -\frac{1}{4} & -\frac{i}{2} & \frac{i}{2} \\ 0 & \frac{1}{4} & -\frac{i}{2} & \frac{i}{2} \end{pmatrix}.$$

In a new basis

$$v_1 = e_1,\ v_2 = -ie_2,\ v_3 = \frac{i}{4}(e_3 - e_4),\ v_4 = \frac{i}{4}(e_3 + e_4),$$

the transformation g_3 acts as cyclic permutation (234) of the basis.

The web of harmonic quadrics in the new basis is:

$$x_0(p_7 - p_6) - 2ix_1(p_7 + p_6) + \frac{i}{2}x_2(p_3 - p_2 + p_5 - p_4) + \frac{1}{2}x_3(p_4 + p_5 - p_2 - p_3) = 0.$$

The equation of the discriminant quartic is:

$$x_0^4 + x_1^4 + x_2^4 + x_3^4 + 2x_0^2(x_1^2 + x_2^2 + x_3^2) - (y^2z^2 + y^2w^2 + z^2w^2) = 0. \qquad (9.5.17)$$

The surface has ten ordinary nodes at $[0, 1, 1, 1]$ and $\mathfrak{S}_3$ orbits of point $[1, \pm i, 0, 0]$ (acting on the last tree coordinates).

- $G = \mathbb{O}$

We know that there are two non-isomorphic dihedral rational nodal sextics given in rows (17) and (18) in Table 9.1. They are obtained as specializations of the families of dihedral sextics from rows (12) and (13) by taking the parameters $a = -5, 3$, respectively. As we saw in above, they are also special members of the tetrahedral family by taking $d = 0$ or $e = 0$. Their equations are given in (9.5.17) and (9.5.15). In the first case, the linear representation of the group $\mathbb{O} \cong \mathfrak{S}_4$ in $V^\perp$ is isomorphic to the standard 4-dimensional representation twisted by the sign character. Note that, in the second case, $V^\perp$ is isomorphic to the standard 4-dimensional permutation representation of $\mathfrak{S}_4$.

Let P_1 be the permutation matrix of the cyclic permutation (1234). We have

$$S^{-1}g_1S = iP_1,$$

where

$$S = \begin{pmatrix} -1 & 1 & -1 & 1 \\ 1 & 1 & 1 & 1 \\ -i & 1 & i & -1 \\ i & 1 & -i & -1 \end{pmatrix}$$

is the inverse of a matrix of eigenvectors of P_1. We have

$$h_2 := S^{-1}g_2S = \frac{1}{2}\begin{pmatrix} -1 & 1 & 1 & 1 \\ 1 & 1 & 1 & -1 \\ 1 & 1 & -1 & 1 \\ 1 & -1 & 1 & 1 \end{pmatrix}$$

and

$$h_3 := S^{-1}g_3S = \frac{1}{16}\begin{pmatrix} 4-i & 4-31i & 4-i & 4+33i \\ -4 & 4+8i & 12 & 4-8i \\ 4+i & 4-33i & 4+i & 4+31i \\ 12 & 4+8i & -4 & 4-8i \end{pmatrix}.$$

A matrix A commuting with P_1 must be of the form

$$\begin{pmatrix} a & b & c & d \\ d & a & b & c \\ c & d & a & b \\ b & c & d & a \end{pmatrix}.$$

If we take its first column to be an eigenvector of h_3 with eigenvalue 1, then we obtain that

$$P_2 := A^{-1}h_2A = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \quad P_3 := A^{-1}h_3A = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}.$$

The eigensubspace of h_3 with eigenvalue 1 is two-dimensional and spanned by vectors $(41+4i, 16-4i, 0, 17)$ and $(1,1,1,1)$. A choice of an eigenvector such that the matrix A is invertible defines a symmetric form of the symmetroid. However, the coefficients of the equation may be complex numbers. However, if we take A such that

$$M = SA = \begin{pmatrix} -8i & 8i & -8i & 8i \\ 1 & 1 & 1 & 1 \\ -1-i & 1-i & 1+i & -1_i \\ -1+i & 1+i & 1-i & -1_i \end{pmatrix},$$

the equation of the discriminant quartic becomes

$$\begin{aligned} F &= 407569(x_0^4+x_1^4+x_2^4+x_3^4) - 689852(x_0^3x_1+\cdots) + 791142(x_0^2x_1^2+\cdots) \\ &+ 368076(x_0^2x_1x_2+\cdots) - 2355816x_0x_1x_2x_3 = 0. \end{aligned} \tag{9.5.18}$$

Finally, let us consider the icosahedral case.

- $G = \mathbb{I}$

The surface is a member of the one-dimensional family (9.5.10) corresponding to the value of the parameter $a = 3$. We have $V = \langle t^6+3t, 3t^5-1, t^3\rangle$ and $V^\perp = \langle 2t^5+1, 2t-t^6, t^2, t^4\rangle$. The web of harmonic quadrics is:

$$x_0(2p_7 + p_4) + x_1(2p_6 - p_5) + x_2p_2 + x_3p_3 = 0.$$

We get the following equation of the discriminant quartic:

$$-3x_0^3x_2 + x_0^2x_1^2 - 11x_0x_1x_2x3 - 3x_0x_3^3 + 3x_1^3 * x_3 + 3x_1x_2^3 + x_2^2x_3^2 = 0. \qquad (9.5.19)$$

Remark 9.5.1 It is shown in [180, Section 5] that, after a change of variables, the equation of the icosahedral symmetroid can be reduced to the form

$$30\sum_{i=1}^{5} y_i^4 - 7(\sum_{i=1}^{5} y_i^2)^2 = y_1 + \cdots + y_5 = 0. \qquad (9.5.20)$$

It becomes a member of the Hashimoto pencil of quartic surfaces with icosahedron symmetry (see [296]).

Remark 9.5.2 The advantage of the symmetrical form (9.5.18) is that we can apply Timofte's positivity test from Remark 9.4.7. Substituting these vectors in the polynomial F from (9.5.18), we find

$$F(s, s, s, t) = (369s^2 - 1098st + 1129t^2)(-19t + 39s)^2 \geq 0$$

and

$$F(s, s, t, t) = 16(17s - 7t)^2(7s - 17t)^2 \geq 0.$$

The same test applied to the equation from the previous remark shows that the polynomial

$$F = 30\sum_{i=0}^{3} x_i^4 + 30(x_0 + x_1 + x_2 + x_3)^4 - 7(\sum_{i=0}^{3} x_i^2 + (x_0 + x_1 + x_2 + x_3^2)^2)^2$$

is non-negative.

Another example of a non-negative polynomial is given in (9.5.16). It is known that it is not a sum of squares of real polynomials and, in fact, it is one of the first examples of quaternary quartic forms with this property [616]. The polynomials from equations (9.5.17) and (9.5.17) defining two symmetroids with octahedral symmetry seem to be non-negative too.

Remark 9.5.3 We know from Reye Theorem 7.5.3 in Section 7.5 that a general quartic symmetroid admits a unique pair (C_1, C_2) of apolar rational normal curves. This means that the linear system of apolar quadrics to the web W defining the symmetroid is spanned by the nets of quadrics $|I(C_1)_2|$ and $|I(C_2)_2|$ vanishing on the curves. If W is a linear representation of a group G then the pair (C_1, C_2) must be invariant and a subgroup of index ≤ 2 of G leaves the subspaces $I(C_1)_2$ and $I(C_2)_2$ invariant. However, in our cases the web $W = |L|$ is not a general web so the Reye Theorem may not apply. In fact, this happens in some of our cases. By definition of a

harmonic quadric, one of the curves C_i must coincide with the dual Veronese curve R_3^*. The trouble could be in finding another rational cubic curve.

For example, if $G = \mathbb{D}_{12}$,

$$L = \langle x^2 - w^2,\ xw + 9yz, 2xz + 3y^2,\ 2yw + 3z^2\rangle$$

and there is a unique decomposition of linear representations of G in the space of apolar quadrics $L^\perp$

$$L^\perp = \langle x^{*2} + y^{*2}, x^*z^*, y^*w^*\rangle \oplus I(R_3^*)_2,$$

where x^*, y^*, z^*, w^* are the dual coordinates. The first summand is isomorphic to V and the discriminant quartic of the corresponding net $\lambda(x^{*2} + y^{*2}) + \mu x^*z^* + \gamma y^*w^*$ of quadrics is equal to the union of two double lines $\mu^2 = 0$ and $\gamma^2 = 0$. However, the discriminant curve of a net of quadrics vanishing on a rational cubic curve is a smooth conic taken with multiplicity 2.

On the other hand, if we take $G = \mathbb{D}_{10}$, then everything is as expected. We find that

$$L^\perp = \langle -az^*w^* + 3x^{*2}, -ax^*y^* + 3w^{*2}, 9x^*w^* - a^2y^*z^*\rangle \oplus I(R_3^*)_2$$

and

$$I(C_2)_2 = \langle z^*w^* + 3x^{*2}, -ax^*y^* + 3w^{*2}, -a^2y^*z^* + 9x^*w^*\rangle.$$

The discriminant curve of the net

$$\lambda(z^*w^* + 3x^{*2}) + \mu(-ax^*y^* + 3w^{*2}) + \gamma(-a^2y^*z^* + 9x^*w^*) = 0$$

is a smooth conic $\lambda\mu - 9\gamma^2 = 0$ taken with multiplicity 2. Although this does not imply in general that the net coincides with the net $|I(C)_2|$ for some rational normal curve C, this is true in this case and

$$C = \{[-auv^2, -27u^3, v^3, 3au^2v], [u, v] \in \mathbb{P}^1\}.$$

In the case $G = \mathbb{O}$, the linear system W contains a unique G-invariant net of quadrics whose discriminant curve is a double conic but singular quadrics in the net are all reducible. This must be considered as a degenerate case of the Reye Theorem.

The composition of the Veronese map $\nu_3^* : |U| \to R_3^*$ with the map $R_3^* \to |V'^\vee| \cong |V|$ is the rational parametrization of a rational nodal sextic from which we derived our G-symmetric symmetroid. This reconstructs the sextic from the symmetroid.

In the case, when the Reye Theorem holds for W, the two nets $|I(C_1)|$ and $I(R_3^*)|$ define the same self-dual rational nodal sextics.

9.6 Automorphisms of Unnodal Coble Surfaces

In Section 8.3 we computed the group of automorphisms of a general Enriques surface. In this section, we do the same for an unnodal Coble surface. It is an amazing fact that the group of automorphisms of a general Coble surface is isomorphic to the same group.

Theorem 9.6.1 *Let* V *be an unnodal Coble surface. Then, the homomorphism* $\rho : \mathrm{Aut}(\mathsf{V}) \to \mathrm{O}(\mathrm{CM}(\mathsf{V}))$ *is injective and the image* $\mathrm{Aut}(\mathsf{V})^*$ *of* $\mathrm{Aut}(\mathsf{V})$ *contains the* 2*-congruence subgroup* $W(\mathsf{E}_{10})(2)$ *of* $W(\mathsf{E}_{10})$.

Proof The proof is similar to the proof of Theorem 8.3.3. We skip some details referring to the full proof in [104]. Let V be obtained by blowing up ten nodes of an irreducible plane sextic. An automorphism g of V that acts identically on $\mathrm{CM}(\mathsf{V}) = K_{\mathsf{V}}^{\perp}$ sends the geometric basis $(e_0, \dots, e_{10})$ defined by the blowing down map $\mathsf{V} \to \mathbb{P}^2$ to itself. This implies that g originates from a projective symmetry of the tenpoints that preserves the ordering of the set of points. Clearly this is impossible.

Let F_i be a member of $|3e_0 - (e_1 + \cdots + e_{10}) + e_i| = |-K_{\mathsf{V}} + e_i|$. It is represented by the unique plane cubic passing through the points $p_j \neq p_i$. We have $F_i^2 = 0$ and $F_i \cdot F_j = 1$. The divisor classes $f_i = [F_i]$ and $f_j = [F_j]$ generate a hyperbolic plane and define a primitive embedding $\mathsf{U} \hookrightarrow K_{\mathsf{V}}^{\perp} \cong \mathsf{E}_{10}$. Conversely, any such embedding defines two divisor classes f_i, f_j with $f_i^2 = f_j^2 = 0$, $f_i f_j = 1$. By Riemann–Roch, any isotropic vector $f \in K_{\mathsf{V}}^{\perp}$ is effective. Let $f = [F]$, where F is an effective divisor. Let $|-2K_{\mathsf{V}}| = \{C\}$. Write F as a sum of irreducible curves $C_i \neq C$ and some multiple mC of C. Since all effective primitive isotropic vectors in E_{10} are in the same orbit of the reflection group $W(\mathsf{E}_{10})$, there exists $w \in W(\mathsf{E}_{10})$ such that $w(f) = f_i$. One can show that the image of any effective curve on V under an element $w \in W(\mathsf{E}_{10})$ is effective (see Lemma 3.4 in [104]). Since F_i is an irreducible curve, we obtain that f is represented by an irreducible curve F of arithmetic genus one.

The short exact sequence:

$$0 \to \mathcal{O}_{\mathsf{V}}(F) \to \mathcal{O}_{\mathsf{V}}(2F) \to \mathcal{O}_F(2F) \to 0,$$

together with the adjunction formula $\mathcal{O}_F(2F) \cong \mathcal{O}_F(-2K_{\mathsf{V}}) \cong \mathcal{O}_F$, shows that $\dim |2F| = 1$, thus $|2F|$ is an irreducible pencil of genus one curves. For each $\mathsf{U} \hookrightarrow K_{\mathsf{V}}^{\perp}$ we thus find two genus one pencils $|2F|$ and $|2F'|$ with $F \cdot F' = 1$.

Next, as in section 9.2, we consider a bielliptic map $f : \mathsf{V} \to \mathbb{P}^4$ given by the linear system $|2F + 2F'|$ of degree 8. One shows that the image of this map is a 4-nodal anti-canonical del Pezzo surface D_1. Since V has no (-2)-curves, the map f must be a separable map of degree 2. Since $F \cdot C = F' \cdot C = 0$, the image of the curve C is one of the nodes. The deck transformation defines an involution on V and we apply Proposition 8.3.2 to conclude the proof. □

As we see, the deck involution defined by a pair of genus one curves F, F' with $F \cdot F' = 1$ is similar to the one we used for the description of automorphism group

of an unnodal Enriques surface. When V is unnodal, each such pair is defined by a primitive embedding $j : U \hookrightarrow K_{\mathsf{V}}^{\perp}$. If $p \neq 2$, the double cover $\mathsf{V} \to \mathsf{D}$ factors through a degree 2 map $\mathsf{V} \to \mathsf{D}'$, where D' is the blow-up of one of the nodes of D, the image of the anti-bicanonical curve C. The branch curve is a smooth curve of genus 4 equal to a proper inverse transform of a curve from $|O_{\mathsf{D}}(2)|$ passing through a node. The fixed points of the deck involution is the union of the pre-image of this curve on V and three isolated fixed points over the nodes of D'.

In a geometric basis of the blow-up of five points p_1, p_2, p_3, p_4, p_5 (see Section 0.4), the branch curve becomes the union of the curve from $|e_0 - e_3 - e_4|$ and a curve from $|5e_0 - e_1 - e_2 - e_3 - 2p_4 - 2p_5|$. So, its image in the plane is the union of a line through p_1, p_2 and a quintic curve of geometric genus 4 with tacnode at p_4 which intersects the line at p_1 and is tangent to the line at p_2.

Remark 9.6.2 Assume $p \neq 2$. Let $p_1, \ldots, p_{10}$ be the ten nodes of a rational plane sextic B. Let $g_{ij} \in \mathrm{Aut}(\mathsf{V})$ be the automorphism of V defined by the deck transformation of the double cover $f : \mathsf{V} \to \mathsf{D}$ defined by the linear system $|2F_i + 2F_j|$, where F_i, F_j are the proper transforms on V of the cubic curves passing through points $p_1, \ldots, p_{10}$ excluding p_i, p_j. Let $\mathsf{V}_{ij} \to \mathbb{P}^2$ be the blow-up of the points $p_k, k \neq i, j$. The surface V_{ij} is a weak del Pezzo surface of degree 1, the linear system $|-2K_{\mathsf{V}_{ij}}|$ defines a degree 2 map $\phi_{ij} : \mathsf{V}_{ij} \to Q \subset \mathbb{P}^3$ onto a quadratic cone Q with the branch curve W cut out by a cubic surface (see [177], Theorem 8.3.2). The proper transform $\tilde{B}$ of the curve B on V_{ij} belongs to this linear system, and hence it is equal to the pre-image of a hyperplane section C of Q. The restriction of ϕ_{ij} to $\tilde{B}$ is a degree 2 map

$$\phi_{ij} : \tilde{B} \to C \cong \mathbb{P}^1.$$

Since each point p_i, p_j (we identify it with a point on V_{ij}) is a double point of $\tilde{B}$, the hyperplane section C is tangent to the branch curve of ϕ_{ij} at the images of these points in C. This implies that the points p_i, p_j are on the ramification curve of ϕ_{ij}. Let β_{ij} be the deck transformation of the map ϕ_{ij} which defines, via the blow-up map, a birational *Bertini involution* of $\mathbb{P}^2$ (see [177], 8.8.2). The points p_i, p_j are among its fixed points. Thus, β_{ij} extends to an automorphism τ_{ij} of V. We claim that it coincides with the deck transformation g_{ij} of the map $f : \mathsf{V} \to \mathsf{D}$ defined by the linear system $|2F_i + 2F_j|$. To see this, it is enough to check that g_{ij} and τ_{ij} act identically on $\mathrm{Pic}(\mathsf{V})$. In fact, it is known that g_{ij}^* acts as the minus identity on the sublattice $K_{\mathsf{V}_{ij}}^{\perp}$ of $\mathrm{Pic}(\mathsf{V}_{ij})$. In the geometric basis $(e_0, e_1, \ldots, e_{10})$ of $\mathrm{Pic}(\mathsf{V})$, this lattice is identified with $(3e_0 - (e_1 + \cdots + e_{10}) + e_i + e_j)^{\perp}$. This shows that $\tau_{ij}^*(F_k) = F_k, k \neq i, j$, and τ_{ij}^* acts identically on $\mathbb{Z}[F_i] + \mathbb{Z}[F_j]$. This implies that $\tau_{ij}^* = g_{ij}^*$.

Note that the Bertini involution β_{ij} has one isolated fixed point $q_i \in E_i, q_j \in E_j$ on each of the exceptional curves. The quotient D' of V by this involution has three ordinary double points, the image of the base point of the pencil $|-K_{\mathsf{V}_{ij}}|$ and the images of the points q_i, q_j. One can show that the surface D' is obtained from the 4-nodal anti-canonical quartic del Pezzo surface D by blowing up one of its four ordinary double points. The image of the curve $\tilde{B}$ is the exceptional curve R over this point. We have $\tilde{B}^2 = -4$ and $R^2 = -2$, so, this agrees. It is known that the

ramification map of the map $f_{ij} : \mathsf{V}_{ij} \to Q$ is the proper transform $\tilde{Z}$ on V_{ij} of a curve Z of geometric genus 4, of degree 9 with triple points at $p_1, \ldots, p_8$. It passes through the points p_i, p_j, hence $\tilde{Z} \cdot \tilde{B} = 54 - 6 \cdot 8 - 2 \cdot 2 = 2$. The intersection points are the ramification points of the map $\tilde{B} \to R$.

Example 9.6.3 Assume $\Bbbk = \mathbb{C}$. Consider the Coble surface V obtained by blowing up the singular points of a 10-nodal plane sextic W with projective self-symmetry isomorphic to the group $\mathfrak{A}_5$ (see the bottom row in Table 9.1).

Let X be the canonical cover of V. The group $\mathfrak{A}_5$ lifts isomorphically to a finite group acting symplectically on X. It acts on the cohomology $H^*(X, \mathbb{C})$ and acts trivially on the 6-dimensional subspace generated by the divisor class of the pre-image of an ample divisor on V, the divisor class of the ramification (-2)-curve, and the part $H^{0,0} \oplus H^{2,2} \oplus H^{0,2} \oplus H^{2,0}$ of the Hodge decomposition. According to the classification of such groups due to S. Mukai [527], the dimension of the subspace of invariant cohomology classes for an action of a symplectic group isomorphic to $\mathfrak{A}_5$ on a K3 surface is equal to 6. This is proven by analyzing the Mathieu character of the action of the group on the cohomology. Assume that V has a (-2)-curve. Then, X has one additional $\mathfrak{A}_5$-invariant divisor class, namely, the sum D of the curves in the orbit of any (-2) curve whose image on V is a (-2)-curve. Note that this class is different from the orbit of an ample class which we considered before. In fact, the sum of the curves in the new orbit does not intersect the ramification (-2)-curve, but the sum of the curves in the old orbit does intersect this curve. This contradicts Mukai's computation.[1] So, our surface V is unnodal, hence its group of automorphisms contains the group $W(\mathsf{E}_{10})(2)$ and the group $\mathfrak{A}_5$. Applying Proposition 8.6.1, we find that the group generated by these two subgroups is isomorphic to the semi-direct product $W(\mathsf{E}_{10})(2) \rtimes \mathfrak{A}_5$. This should be compared with Theorem 8.3.4, where it is shown that, for an unnodal Enriques surface S, the order of $\mathrm{Aut}(S)^*/W(\mathrm{Num}(S))(2)$ is at most 4.

9.7 Enriques and Coble Surfaces of Hessian Type

In Example 6.4.20 from Section 6.4, we discussed Enriques surfaces of Hessian type associated to a nonsingular cubic surface. When the cubic surface acquires $k \leq 4$ nodes, the quotient by the switch involution acquires k quotient singularities of type $\frac{1}{4}(1,1)$. After resolving these singularities, we obtain a Coble surface with k boundary components. This allows one to study simultaneously the automorphism group of Enriques surfaces and Coble surfaces of this Hessian type. This what we will do in this section.

We assume that $p = 0$, and discuss the case of positive characteristic at the end of this section.

[1] We thank H. Ohashi for this argument.

Recall that an Enriques surface of the Hessian type is defined to be the quotient of the minimal resolution X of the Hessian quartic surface $H(C)$ of a nonsingular cubic surface $C = V(F)$ by the switch involution that exchanges the proper transforms of lines on $H(C)$ with the exceptional curves over the nodes. If the cubic surface acquires k nodes, the Hessian surface acquires them as its nodes. The switch involution τ fixes the exceptional (-2)-curve over the new nodes pointwise. The quotient surface $X/(\tau)$ becomes a Coble surface with k boundary components. We call a Coble surface obtained in this way a *Coble surface of Hessian type*.

The moduli space of Coble surfaces of Hessian type with k boundary components is of dimension $4-k$. It is isomorphic to the moduli space of Sylvester non-degenerate k-nodal cubic surfaces. It is shown in [181, Proposition 5.1] that the moduli space of Coble surfaces of Hessian type with one boundary component (together with a geometric marking defined by a blowing down $\mathsf{V} \to \mathbb{P}^2$) is naturally isomorphic to the moduli space of *Desargues configurations* of 10 lines and 10 points in the projective plane.

The group of automorphisms of a general Enriques surface of Hessian type was described in terms of generators and relations by I. Shimada [672]. Let us describe his result.

The group of birational automorphisms of the Hessian surface of a general cubic surface was described in [187]. Some of its generators are the deck transformations of the projections from the 10 nodes p_α. It is proven in loc. cit. that they commute with the involution τ and hence descend to biregular involutions j_α of the Enriques quotient surface S.

Theorem 9.7.1 *Let X be a minimal resolution of the Hessian surface $H(C)$ of a general cubic surface C and let $S = X/(\tau)$ be the Enriques surface of Hessian type. The ten involutions j_α generate the group* $\mathrm{Aut}(S)$. *The defining relations are of the following type:*

- $j_\alpha^2 = \mathrm{id}$*;*
- $(j_\alpha \circ j_\beta \circ j_\gamma)^2 = \mathrm{id}$ *for each triple of collinear nodes;*
- $(j_\alpha \circ j_\beta)^2 = \mathrm{id}$ *for each pair of nodes not lying on the same line contained in $H(C)$.*

In his proof, Shimada describes a fundamental polytope for the action of the automorphism group in the nef cone. It coincides with the reflection polytope of the crystallographic root basis of (-2)-curves of type VI from Figure 8.8. The ten vertices $E_1, \dots, E_{10}$ correspond to outer walls, i.e. walls lying on the boundary of the nef cone. The remaining ten nodes correspond to non-effective primitive vectors of square norm -2 such that the projection involution j_α acts as the reflection in these vectors.

A similar description of the nef cone can be given for any Enriques or Coble surface of Hessian type. The following theorem is proven in [8, Lemma 2.4 and Theorem 3.2].

Theorem 9.7.2 *Let* $\Lambda = \mathrm{Num}(X)$ *(resp.* $\mathrm{CM}(X)$*) if* X *is an Enriques surface (resp.* X *is a Coble surface) of Hessian type. Let* G_0 *be the subgroup of the Weyl group* $W(\Lambda)$ *generated by* j_α^**, where* p_α *is not an Eckardt point of the cubic surface* C*. Then,* $\mathrm{Nef}(X) \cap \Lambda_{\mathbb{R}}$ *is the closure* Q *of the union of* G_0*-images of the polytope* P *whose faces are orthogonal to* (-2)*-vectors* $E_1, \ldots, E_{20}$ *with the intersection diagram from Figure 8.8. The vectors* $E_1, \ldots E_{10}$ *are the classes of* (-2)*-curves* U_{ab} *and the number of vectors* $E_{11}, \ldots, E_{20}$ *represented by the classes of* (-2)*-curves is equal to the number of Eckardt points on* C*.*

Recall that the group of projective automorphisms of a general cubic surface is trivial. The classification of automorphism groups of nonsingular cubic surfaces in arbitrary characteristic can be found in [185]. When the cubic is Sylvester non-degenerate and acquires a non-trivial automorphism group, the corresponding Enriques surface becomes special and its automorphism group changes.

Here, we consider the family of cubic surfaces with automorphism group isomorphic to $\mathfrak{S}_4$. The cubic surfaces with this group of symmetry can be characterized by the property that they contain six Eckardt points. Each such cubic surface is isomorphic to a member of the following pencil of cubic surfaces:

$$C_t : y_1^3 + y_2^3 + y_3^3 + y_4^3 + t(y_1 + y_2 + y_3 + y_4)^3 = 0 \tag{9.7.1}$$

The Hessian surfaces $H_t = H(C_t)$ is defined by the following equation:

$$\frac{1}{y_1} + \frac{1}{y_2} + \frac{1}{y_3} + \frac{1}{y_4} - \frac{t}{y_1 + y_2 + y_3 + y_4} = 0. \tag{9.7.2}$$

The group of automorphism of a general member C_t of the pencil is isomorphic to $\mathfrak{S}_4$. There are three values of the parameter $t \in \mathbb{P}^1 \setminus \{\infty\}$ when the surface acquires a singular point or it is still smooth but the automorphism group changes.

1. $t = 1$: the cubic surface C_1 is isomorphic to the Clebsch diagonal cubic surface with $\mathrm{Aut}(C_1) \cong \mathfrak{S}_5$. The Hessian surface acquires the same group of projective automorphisms. As we saw in Section 8.9, the corresponding Enriques surface is of type VI with finite automorphism group isomorphic to $\mathfrak{S}_5$.
2. $t = \frac{1}{16}$: the cubic surface $C_{\frac{1}{16}}$ acquires one node. The corresponding quotient $X/(\tau)$ is a Coble surface with one boundary component. The surface $\mathsf{V}_{\frac{1}{16}}$ is obtained by blowing up ten nodes of an irreducible plane rational sextic that admits a group $\mathsf{S}_4 \cong \mathbb{O}$ of projective symmetries. We found its equation (9.4.35) in Section 9.4. It follows from Remark 9.4.7 that one can choose a more symmetric equation given by the Robertson ternary sextic (9.4.33).
3. $t = \frac{1}{4}$: the cubic surfaces has four nodes. The corresponding quotient $X/(\tau)$ is a Coble surface with four boundary components. The surface $\mathsf{V}_{\frac{1}{4}}$ is obtained by blowing up $\mathbb{P}^2$ at the singular points of a reducible sextic with four irreducible components:

$$(y + z)(y + z + 4x)(xy + yz + xz)(xy + yz + xz + 4x^2) = 0$$

(see [181, 5.5]).

It turns out that the group of automorphisms of any Enriques or Coble surface in the pencil (9.7.2) are isomorphic to the same group unless the automorphism group becomes finite and isomorphic to $\mathfrak{S}_5$. The proof of the following theorem can be found in [8].

Theorem 9.7.3 *Let S_t be an Enriques or Coble surface of Hessian type associated to the Hessian surface $H_t, t \neq 1$, from* (9.7.2)*. Then,*

$$\mathrm{Aut}(S_t) \cong \mathrm{UC}(4) \rtimes \mathfrak{S}_4,$$

where $\mathrm{UC}(4)$ *is the free product of four copies of the group of order* 2*. The generators of* $\mathrm{UC}(4)$ *are the projection involutions corresponding to four of the original nodes of the Hessian surface H_t. The remaining six projection involutions correspond to transpositions (ab) in $\mathfrak{S}_4$. The action of* $\mathrm{Aut}(S_t)$ *in the nef cone of S_t admits a fundamental domain as in the case of a general Enriques surface of Hessian type, however its has only six vertices corresponding to non-effective roots.*

One can find various interesting realizations of the group $\mathrm{UC}(4) \rtimes \mathfrak{S}_4$ in [8]. One of them realizes the group as a group of isometries of the Euclidean space $\mathbb{R}^3$ generated by the isometries of a regular tetrahedron and the reflections across its faces.

Remark 9.7.4 In [126] A. Coble discussed the following problem. Let V be a Coble surface with one boundary component and $C \in |-2K_{\mathsf{V}}|$. Then, there is a natural restriction homomorphism

$$r : \mathrm{Aut}(\mathsf{V}) \to \mathrm{Aut}(C) \cong \mathrm{PGL}(2).$$

What is the kernel and the image of this homomorphism ? Coble conjectured that the kernel is trivial for a general Coble surface. The answer for the question is known for the Coble surface $\mathsf{V}_{\frac{1}{16}}$: the kernel is trivial and the image is conjugate to the subgroup of isometries of a regular tetrahedron.

A slightly different family of Enriques quotients of quartic surfaces with $\mathfrak{S}_4$ symmetry is discussed in [535]. The family is the following:

$$ts_2^2 + ks_4 + ls_1s_3 = 0, \tag{9.7.3}$$

where s_i are elementary symmetric polynomials in four variables y_1, y_2, y_3, y_4 and t, k, l are parameters. Note that the subfamily with $t = 0$ coincides with our family of quartic Hessian surfaces with $\mathfrak{S}_4$-symmetry. The four points x_i with coordinates $[1, 0, 0, 0], \ldots, [0, 0, 0, 1]$ are the common nodes of the surfaces in the family. The Enriques involution of the general member of the family is defined by the standard Cremona transformation $y_i \mapsto y_i^{-1}$. It has eight fixed points in $\mathbb{P}^3$ with coordinates $[1, \pm 1, \pm 1, \pm 1]$. If the surface does not contain any of these points, the switch

involution has no fixed points and the quotient is an Enriques surface $S(t,k,l)$. This condition is equivalent to that

$$-36t + k + 16l \neq 0, \quad 4l + k \neq 0, \quad 4t + k \neq 0.$$

The group $\mathfrak{S}_4$ acts on the set of 8 fixed points with the following orbits:

$$\begin{aligned}
&\{[1,1,1,1]\}, \quad -36t + k + 16l = 0,\\
&\{[1,-1,-1,1],[1,1,-1,-1],[1,-1,1,1]\}, \quad 4t + k = 0,\\
&\{[1,-1,1,1],[1,1,-1,1],[1,-1,-1,1],[1,-1,1,-1]\}, \quad k + 4l = 0.
\end{aligned}$$

This gives examples of Coble surfaces with $n = 1, 3, 4$ boundary components which admit the group $\mathfrak{S}_4$ as its group of automorphisms. If $t = 0, k \neq 0$, we obtain our Coble surfaces $S_{\frac{1}{16}}$ and $S_{\frac{1}{4}}$. Note that the Coble surfaces with one boundary component define a one-parameter family of ten-nodal plane sextics with $\mathfrak{S}_4$-Cremona isometry. Only one of them with parameter $t = 0$ has the projective $\mathfrak{S}_4$-symmetry.

The four projection involutions of nodal members of the family descend to four involutions of the quotient surface $X/(\tau)$. They generate the free product $\mathrm{UC}(4)$ and this gives an embedding

$$\iota : \mathrm{UC}(4) \rtimes \mathfrak{S}_4 \hookrightarrow \mathrm{Aut}(S) \tag{9.7.4}$$

for all members of the family. When $t = 0$, the four nodes specialize to the four nodes of the Hessian surface that define non-projective involutions coming from the projections from the nodes. The bijectivity of ι in the case $t = 0, k \neq 0$ is the assertion of Theorem 9.7.3. Mukai and Ohashi prove the following:

Theorem 9.7.5 *Suppose $t = 1, l = 0$ and $k \neq 0, 4, -36$. Then, the homomorphism ι is bijective.*

We do not know the structure of the automorphism group of a general member of the family of surfaces $S(t,k,l)$.

Note that the Kummer surface $\mathrm{Kum}(\mathrm{Jac}(C))$ of the Jacobian of a curve C of genus two admits a projective embedding in $\mathbb{P}^3$ isomorphic to the Hessian of a cubic surface [318]. The equation of the Hessian surface birationally isomorphic to $\mathrm{Kum}(\mathrm{Jac}(C))$ is of the form

$$A_0x_0^{-1} + A_1x_1^{-1} + A_2x_2^{-1} + A_3x_3^{-1} + (A_0x_0 + A_1x_1 + A_2x_2 + A_3x_3)^{-1} = 0$$

(see [318], [187, Theorem 4.1]). Note that after coordinate change $y_i = -A_ix_i$, the equation becomes the equation of the Hessian surface of the cubic surface

$$A_0^2x_0^3 + A_1^2x_1^3 + A_2^2x_2^3 + A_3^2x_3^3 + (x_0 + x_1 + x_2 + x_3)^3 = 0.$$

It follows that our pencil (9.7.2) contains only one member birationally isomorphic to some Kummer surface $\mathrm{Kum}(\mathrm{Jac}(C))$. It corresponds to the parameter $(\lambda : \mu) = (1 : 1)$. Note that the Hessian of the Clebsch diagonal cubic corresponds to the parameter

$(\lambda : \mu) = (1 : -1)$, and hence, it is not birationally isomorphic to a jacobian Kummer surface. We noted this fact in Section 8.9 while discussing surfaces of type VI with finite automorphism group. We do not know the structure of the lattice of transcendental cycles of this Kummer surface.

Note that the family (9.7.3) of quartic surfaces studied by Mukai and Ohashi also contains some members birationally isomorphic to a Kummer surface. They correspond to parameters satisfying $t = 1, (k-4)(l-4) = 0$ and, as we noticed before $t = 0, k = l$.

Finally, let us see what happens if we assume that the characteristic is positive. Although the Hessians of cubic surfaces in characteristic $p = 2, 3$ can degenerate, we may still consider the quartic surfaces given by the equations $\sum_{i=0}^{4} \frac{1}{a_i x_i} = \sum_{i=0}^{4} x_i = 0$ and continue to call them Hessian quartic surfaces. The pencil (9.7.2) and the Mukai-Ohashi family still makes sense. In characteristic $p = 2$, the Cremona involution has only one fixed points $[1, 1, 1, 1, 1]$ and it lies on the member $S(t, k, l)$ of the family if and only if $k = 0$. In particular, in the pencil (9.7.2) the two Coble surfaces degenerate to reducible surfaces $s_1 s_3 = 0$. However, if $p = 3$ they just coincide and contain five boundary components corresponding to 5 nodes $[1, 1, 1, 1], [-1, 1, 1, 1], [1, -1, 1, 1], [1, 1, -1, 1], [1, 1, 1, -1]$. Also, if $p = 5$, we see that the Clebsch diagonal cubic surface and its Hessian surface has a singular point $[1, 1, 1, 1]$ and the corresponding Enriques surface of Hessian type specializes to the Coble surface $S_{\frac{1}{4}} = S_1$ with one boundary component.

9.8 Coble Surfaces with Finite Automorphism Group

In this section, we will extend the classification of complex Enriques surfaces with finite automorphisms to Coble surfaces and also give examples of such surfaces.

The following theorem extends Theorem 8.1.10 to the case of Coble surfaces. Its proof uses Proposition 9.2.2 and is left to the reader.

Theorem 9.8.1 *Let* V *be a Coble surface. Suppose* $W_{\mathsf{V}}^{\mathrm{nod}}$ *is of finite index in* $\mathrm{O}(\mathrm{CM}(\mathsf{V}))$. *Then,* $\mathrm{Aut}(\mathsf{V})$ *is finite.*

Over the field of complex numbers, the proof of the converse is similar to the analogous statement for Enriques surfaces. It is achieved by applying the Global Torelli Theorem of lattice polarized K3 surfaces. Recently, Katsura and the second author classified Coble surfaces with finite automorphism group in any characteristic [370], [419]. By the classification and the argument given in the proof of Theorem 8.1.12, the converse is also true. Later we mention the classification in Theorems 9.8.4 and Theorem 9.8.5.

Another general fact about Coble surfaces with finite automorphism group is given in the following.

Let $\sigma : \mathsf{V} \to \mathsf{H}$ be the blow-up of a Halphen surface H. Let $\mathrm{MW}(\sigma)$ be the Mordell–Weil group of the jacobian elliptic (quasi-elliptic) surface of $f : \mathsf{H} \to \mathbb{P}^1$. Then, it acts on H by translations leaving invariant the set of singular points of fibers

of f. A subgroup of finite index of $\mathrm{MW}(\sigma)$ will fix the fundamental points of σ^{-1}, and hence could be lifted to $\mathrm{Aut}(\mathsf{V})$. Of course, in general, an automorphism of V does not descend to any H obtained by a blowing down morphism σ.

Proposition 9.8.2 *Suppose* $\mathrm{Aut}(\mathsf{V})$ *is finite. Let* $\sigma : \mathsf{V} \to \mathsf{H}$ *be a birational morphism onto a Halphen surface* $f : \mathsf{H} \to \mathbb{P}^1$. *Then, the jacobian genus one fibration of* f *has finite Mordell–Weil group.*

It follows from this proposition that a Coble surface with finite automorphism group is obtained as the blow-up of an extremal rational elliptic or quasi-elliptic surface.

Next, we first assume that $\Bbbk = \mathbb{C}$ and give a characterization of Coble surfaces with finite automorphisms in terms of their Nikulin R-invariant defined in Section 9.2. It is similar to what we did for Enriques surfaces in Section Theorem 8.9.3 in Section 8.9.

Theorem 9.8.3 *Assume that* $\Bbbk = \mathbb{C}$. *Then,* $\mathrm{Aut}(\mathsf{V})$ *is finite if and only if the* R-*invariant* (K, H) *is isomorphic to one of the following:*

$$(\mathsf{E}_8 \oplus \mathsf{A}_1, \{0\}),\ (\mathsf{E}_8 \oplus \mathsf{A}_1^{\oplus 2},\ \mathbb{Z}/2\mathbb{Z}),\ (\mathsf{D}_9, \{0\}).$$

Proof The proof is similar to the one of Theorem 8.9.3.

The sufficiency follows from examples below where we construct Coble surfaces with finite automorphism group and compute their Nikulin R-invariants. Let us show the necessity.

Let $C_1, \dots, C_n$ be the boundary components of a Coble surface V and $\tilde{C}_1, \dots, \tilde{C}_n$ their proper transforms under the canonical cover $\pi : X \to \mathsf{V}$.

First, we assume $n = 1$. Then, any effective irreducible root is a (-2)-curve which is orthogona to the boundary component $C_1 \in |-2K_{\mathsf{V}}|$. This implies that rank $K \le 9$, and hence this case is reduced to the case of Theorem 8.9.3. The same proof shows that (K, H) is isomorphic to $(\mathsf{E}_8 \oplus \mathsf{A}_1, \{0\}),\ (\mathsf{D}_9, \{0\})$.

Next, we consider the case $n \ge 2$. The divisor classes

$$\tilde{C}_1 \pm \tilde{C}_2, \quad \tilde{C}_2 + \tilde{C}_3, \quad \dots, \quad \tilde{C}_{n-1} + \tilde{C}_n$$

generate the lattice $\mathsf{D}_n(2)$, where we set $\mathsf{D}_2 := \mathsf{A}_1^{\oplus 2}$ and $\mathsf{D}_3 := \mathsf{A}_3$. Since any (-2)-curve on V is orthogonal to $C_1, \dots, C_n$, the root lattice $K = \mathsf{D}_n \oplus K_0$, where K_0 is generated by (-2)-curves.

Let $\gamma : K/2K \to \mathrm{CM}(\mathsf{V})/2\,\mathrm{CM}(\mathsf{V})$ be the homomorphism defined in (9.2.5). Since $\gamma(\tilde{C}_1 \pm \tilde{C}_2) = 2E + \tilde{C}_1 + \tilde{C}_2$, $2\tilde{C}_1$ is contained in $\mathrm{Ker}(\gamma)$, it suffices to determine the sublattice $K' = \mathsf{A}_{n-1} \oplus K_0$, where $\mathsf{A}_{n-1}(2)$ is generated by $\tilde{C}_1 + \tilde{C}_2, \tilde{C}_2 + \tilde{C}_3, \dots, \tilde{C}_{n-1} + \tilde{C}_n$. Since rank $K' \le 9$, again the proof is reduced to the similar case from Theorem 8.9.3. The proof of the theorem implies that $K_0 = \mathsf{E}_8$, and $\mathsf{A}_{n-1} = \mathsf{A}_1$, that is, $K' = \mathsf{A}_1 \oplus \mathsf{E}_8$. Thus, we have $(\mathsf{E}_8 \oplus \mathsf{A}_1^{\oplus 2},\ \mathbb{Z}/2\mathbb{Z})$ (note that $n \ge 2$ implies that K' contains an orthogonal summand A_{n-1}, and hence the case D_9 can be excluded). □

Let us outline how we proceed with the classification. Applying Theorem 9.8.1 and arguing as in the case of Enriques surfaces, we can determine possible crystallographic bases. Using some geometric arguments, this will allow us to determine the number of boundary components. Since Coble surfaces are rational, we can construct Coble surfaces with such crystallographic basis as a blowing up of $\mathbb{P}^2$. This also allow us to determine the moduli of Coble surfaces with finite automorphism group.

We now give the following theorems of the classification of Coble surfaces with finite automorphism group. For more details, we refer the reader to [419], [370].

Theorem 9.8.4 *Coble surfaces* V *with finite automorphism group in characteristic* $p \neq 2$ *are classified in the following Table 9.2. Each type is unique. Here* n *is the number of boundary components and* k *is the number of irreducible effective roots.*

Type	p	n	k	$\mathrm{Aut}(\mathsf{V})$	R-invariant	Example
I	any	2	12	D_8	$(E_8 \oplus A_1^{\oplus 2}, \mathbb{Z}/2\mathbb{Z})$	Ex. 9.8.6
I	any	1	12	D_8	$(E_8 \oplus A_1, \{0\})$	Ex. 9.8.7
II	any	1	12	$\mathfrak{S}_4$	$(D_9, \{0\})$	Ex. 9.8.8
V	3	2	20	$\mathfrak{S}_4 \times \mathbb{Z}/2\mathbb{Z}$	$(E_7 \oplus A_2 \oplus A_1^{\oplus 2}, (\mathbb{Z}/2\mathbb{Z})^2)$	Thm. 9.8.9
VI	3	5	20	$\mathfrak{S}_5$	$(E_6 \oplus D_5, \mathbb{Z}/2\mathbb{Z})$	Thm. 9.8.10
VI	5	1	20	$\mathfrak{S}_5$	$(E_6 \oplus A_4, \{0\})$	Thm. 9.8.12
VII	5	1	20	$\mathfrak{S}_5$	$(A_9 \oplus A_1, \mathbb{Z}/2\mathbb{Z})$	Thm. 9.8.13
MI	3	2	40	$\mathrm{Aut}(\mathfrak{S}_6)$	$(A_5^{\oplus 2} \oplus A_1^{\oplus 2}, (\mathbb{Z}/2\mathbb{Z})^3)$	Thm. 10.5.18
MII	3	8	40	$(\mathfrak{S}_4 \times \mathfrak{S}_4) \cdot \mathbb{Z}/2\mathbb{Z}$	$(D_8 \oplus A_2^{\oplus 2}, (\mathbb{Z}/2\mathbb{Z})^2)$	Thm. 10.6

Table 9.2 Coble surfaces with finite automorphism group ($p \neq 2$)

Theorem 9.8.5 *Coble surfaces with finite automorphism group in characteristic* 2 *are classified in Table* 9.3 *below. The moduli space of Coble surfaces with* n *boundary components of each type is irreducible and of dimension given in the fifth column in the table. Every Coble surface is a specialization of classical Enriques surfaces with finite automorphism group of the same type.*

Type	n	k	$\mathrm{Aut}(\mathsf{V})$	dim	Example
$\tilde{E}_8$	1	10	$\mathbb{Z}/5\mathbb{Z}$	0	Ex. 9.8.18
$\tilde{E}_7^1$	2	11	$\mathbb{Z}/2\mathbb{Z}$	0	Ex. 9.8.19
$\tilde{E}_7^2$	1	11	$\mathbb{Z}/2\mathbb{Z}$	1	Ex. 9.8.20
$\tilde{E}_6 + \tilde{A}_2$	3	13	$\mathfrak{S}_3$	0	Ex. 9.8.21
$\tilde{E}_6 + \tilde{A}_2$	1	13	$\mathfrak{S}_3$	0	Ex. 9.8.22
$\tilde{D}_8$	1	10	$\mathbb{Z}/2\mathbb{Z}$	1	Ex. 9.8.23
VII	10	20	$\mathfrak{S}_5$	0	Thm. 9.8.15
VII	2	20	$\mathfrak{S}_5$	0	Thm. 9.8.16
VIII	4	16	$\mathfrak{S}_4$	0	Ex. 9.8.24

Table 9.3 Coble surfaces!finite automorphism group ($p = 2$)

In this section, we give examples of Coble surfaces with finite automorphism group of each type except types MI and MII. The examples of surfaces of type MI and MII will be given in the next chapter (see Theorems 10.5.18 and 10.6).

The following examples show that the isomorphism class of a Coble surface with finite automorphism group is represented by a boundary point in a one-dimensional family of Enriques surfaces with finite automorphism group of types I and II.

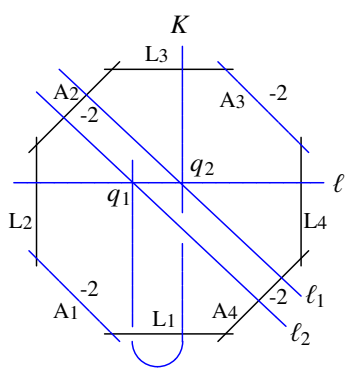
The following example was first given in [426, §6].

Example 9.8.6 This surface lies in the boundary of the one-dimensional family of surfaces with crystallographic basis of (-2)-curves of Type I.

We use the equation of the double plane model of the family of Enriques surfaces with finite automorphism group of Type I given in (8.9.2) and specialize the parameters by taking $a + c = b = d = 0$. The equation becomes

$$x_3^2 + x_1x_2^2(x_1 - x_2)(x_1x_2 - x_0^2)(x_1^2 - x_0^2) = 0.$$

After normalization, we get rid of x_2^2 and get a special case of the equation of a double plane for a Coble surface given in (9.2.3). We see that the branch curve is equal to the union of a line ℓ passing through $p_1 = [1, 0, 0]$ representing the divisor class $e_0 - e_1$ on $\tilde{\mathsf{D}}_1$, a conic K through $p_2 = [0, 0, 1]$, $p_3 \succ p_2$, $p_4 = [0, 1, 0]$, $p_5 \succ p_4$ representing the divisor class $2e_0 - e_2 - e_3 - e_4 - e_5$, and two lines ℓ_1, ℓ_2 passing through p_4 and each passing through one of the intersection points $q_1 = [-1, 1, 1]$, $q_2 = [1, 1, 1] \in \ell \cap K$. The points q_1, q_2 are ordinary triple points of the quintic branch curve. The double cover $Y \to \tilde{\mathsf{D}}_1$ has two singular points of type D_4 over q_1, q_2. The picture of curves on the minimal resolution $\tilde{\mathsf{D}}_1$ of the branch curve is as follows:



Here, we kept the notations for the proper transforms of the curves ℓ, ℓ_1, ℓ_2, K. The branch curve is drawn in blue. The curves $L_1, \ldots, L_4$ are (-1)-curves, the proper transforms of the four lines on D_1. The (-2)-curves A_2, A_3 represent the divisor classes $e_2 - e_3$ and $e_0 - e_3 - e_4$.

Since the branch curve has ordinary triple points at q_1, q_2, the double plane has singular points of type D_4 over these points. Let $2R + R_1 + R_2 + R_3$ and $2R' + R_1' + R_2' + R_3'$ be the fundamental cycles of the minimal resolution X of the double cover over these points.

The reduced pre-images C_1 and C_2 of A_2 and A_4 are (-4)-curves. The pre-images of A_1 and A_3 are (-1)-curves. Let V be the blow-down of these curves to points P_1, P_2. The surface V is a Coble surface with two boundary components, the images of C_1 and C_2 under the blow-down. We identify them with C_1 and C_2.

The reduced pre-image of A_2 in X is a chain $R_4 + R_4' + R_4''$ of (-2)-curves, where $2R_4$ is the proper transform of A_2 and R_4', R_4'' are the pre-images of the exceptional (-1) curves over $\ell_1 \cap A_2$ and $\ell_2 \cap A_2$.

After we blow down the last two curves to points x_1, x_2, we obtain a Coble surface V. The images of C_1 and C_2 are the two boundary components. The images $\tilde{L}_1, \dots, \tilde{L}_4$ on V of the pre-images of the (-1)-curves $L_1, \dots, L_4$ on X are (-1)-curves.

We denote the images in V of the pre-images of ℓ_1, ℓ_2, ℓ, K on $\tilde{\mathsf{V}}$ by L_1, L_2, L_3, L_4, respectively. We have the following picture on V:

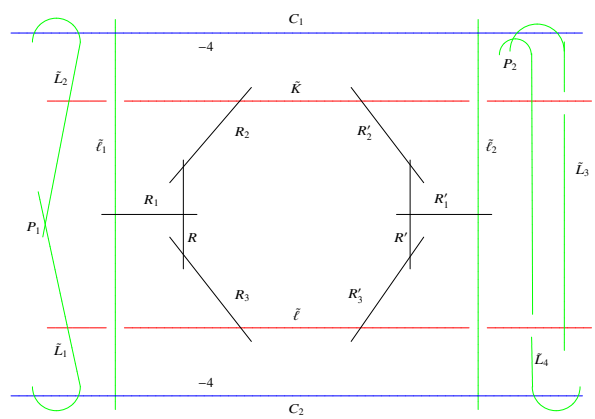


Fig. 9.5 Coble surface with crystallographic basis of (-2)-curves of Type I and two boundary components

We draw (-1)-curves in green, the pre-images $\tilde{\ell}, \tilde{K}$ of the line ℓ and the conic K are drawn in red. The curves drawn in black are (-2)-curves.

Let $\alpha_1 = \frac{1}{2}(C_1 + C_2) + 2\tilde{\ell}_1, \alpha_2 = \frac{1}{2}(C_1 + C_2) + 2\tilde{\ell}_2$ be two irreducible effective roots in $\mathrm{CM}(\mathsf{V})$. We have $\alpha_1 \cdot \alpha_2 = 2$. The effective roots

$$\alpha_1,\ \alpha_2,\ \tilde{\ell},\ \tilde{K},\ R,\ R_1,\ R_2,\ R_3,\ R',\ R_1',\ R_2',\ R_3'$$

form the crystallographic basis of type I.

Applying Theorem 9.8.1, we obtain that $\mathrm{Aut}(\mathsf{V})$ is finite. Let us find its automorphism group. The covering involution g_0 fixes all 12 effective roots, and, as in the case of Enriques surfaces of type I, defines a cohomologically trivial involution of V. The projective transformation $[x_0, x_1, x_2] \mapsto [-x_0, x_1, x_2]$ defines another involution of V. It switches the points q_1, q_2 and the lines ℓ_1, ℓ_2. It acts identically on the effective crystallographic basis of type I from Figure (8.1); however, it does not act

identically on $\mathrm{Pic}(\mathsf{V})$. This is the peculiarity of our Coble surface because two of its irreducible effective roots are not (-2)-curves. As in the case of Enriques surfaces (Proposition 8.9.5), there exists an automorphism of order 4 whose square is g_0. The group of automorphisms of V is isomorphic to D_8.

Let us calculate the R-invariant of V. Let $\pi : X \to \mathsf{V}$ be the canonical cover of V and $\pi^*(C_i) = 2\tilde{C}_i$. The classes $\tilde{C}_1 + \tilde{C}_2, \tilde{C}_1 - \tilde{C}_2$ generate the lattice $\mathsf{A}_1(2) \oplus \mathsf{A}_1(2)$. There are (-2)-curves orthogonal to $\tilde{C}_1, \tilde{C}_2$ which generate $\mathsf{E}_8(2)$. Thus, K contains $\mathsf{E}_8 \oplus \mathsf{A}_1 \oplus \mathsf{A}_1$. Since there are no root lattices containing $\mathsf{E}_8 \oplus \mathsf{A}_1 \oplus \mathsf{A}_1$ properly, we have $K = \mathsf{E}_8 \oplus \mathsf{A}_1 \oplus \mathsf{A}_1$. The sum of $\tilde{C}_1 + \tilde{C}_2$ and $\tilde{C}_1 - -\tilde{C}_2$ gives a generator of $H = \mathbb{Z}/2\mathbb{Z}$. Thus, the R-invariant of this surface is: $(\mathsf{E}_8 \oplus \mathsf{A}_1^{\oplus 2},\ \mathbb{Z}/2\mathbb{Z})$.

Example 9.8.7 We can also degenerate an Enriques surface S with a crystallographic basis of type I to a Coble surface with one boundary component. For this type we specialize equation (8.9.2) of the double plane model of S by taking $c = 0, a = -b - d, a^2 - 4bd = 0$. Thus, we can take $a = -2, b = d$, and, after normalization, we obtain the following equation of the double plane:

$$x_3^2 + x_2(x_1 - x_2)(x_0^2 - x_1x_2)(-2x_1x_2 + x_0^2 + x_2^2) = 0.$$

Now, we see that the branch quintic becomes the union of a line ℓ, a conic K (as in the previous example), and an irreducible conic $K' = V(-2x_1x_2 + x_0^2 + x_2^2)$ passing through $\{q_1, q_2\} = \ell \cap K$ which replaces the lines ℓ_1, ℓ_2 from the previous example. In the picture of the branch curve on $\tilde{\mathsf{D}}_1$, we have to replace ℓ_1, ℓ_2 with K' that intersects A_1 at two points and intersects two (-1)-curves. Its pre-image on $\tilde{\mathsf{V}}$ is a (-1)-curve. The lines $V(x_0 \pm x_2)$ are tangent lines to the conic K' at one of the points q_1, q_2. Their pre-images on $\tilde{\mathsf{V}}$ are (-4)-curves, and their pre-images on V are two (-2)-curves intersecting at two points. They intersect the curves R, R' with multiplicity 2. Now, we see Figure 8.1 of the crystallographic basis of (-2)-curves of type I.

After we blow down any (-1)-curve E, we obtain a Halphen surface of index 2 with an elliptic fibration whose type depends on a choice of E. For example, if we take E equal to the proper transform of K', we obtain an elliptic fibration with fibers of types $\tilde{A}_7$ and $\tilde{A}_1$.

The group of automorphism does not change and is isomorphic to the dihedral group D_8 as in the case of Enriques surfaces. The deck transformation defined by the double plane is a cohomologically trivial involution.

Since all effective irreducible roots are the classes of (-2)-curves and we have the same crystallographic basis, the R-invariant of V is the same as for the Enriques surfaces of type I, i.e. $(\mathsf{E}_8 \oplus \mathsf{A}_1, \{0\})$.

Example 9.8.8 Let us consider a degeneration of an Enriques surface with crystallographic basis of (-2)-curves of type II to a Coble surface.

In equation (8.9.5) of the double plane model of an Enriques surface of type II,

$$x_3^2 + x_1x_2(x_1 - x_2)(x_1x_2 - x_0^2)(x_1x_2(x_1 - x_2) + x_0^2(bx_1 + cx_2)) = 0. \qquad (9.8.1)$$

We take $b = 0$ (or $c = 0$). After normalization, we obtain the following equation:

$$x_3^2 + x_1(x_1 - x_2)(x_1x_2 - x_0^2)(x_1(x_1 - x_2) + x_0^2) = 0. \tag{9.8.2}$$

The branch quintic curve (in the plane model) is the union of the line $\ell = V(x_1 - x_2)$, and two conics $K = V(x_1x_2 - x_0^2)$ and $K' = V(x_1(x_1 - x_2) + x_0^2)$. The conic K' intersects K at the point $[0, 0, 1]$ with multiplicity 4. The proper transfers of K and K' in $\tilde{\mathsf{D}}_1$ are tangent at the point $q_3 \in L_1$. The conic K' is also tangent to ℓ at the points $q_3 = [0, 1, 1]$, whose pre-image on $\tilde{\mathsf{D}}_1$ is a point $q_4 \in L_2$.

Figure 9.6 below depicts the branch curve on the weak del Pezzo surface $\tilde{\mathsf{D}}_1$. We keep the notation for the proper transforms of ℓ, K, and K' under the blow-up of the node of D_1 corresponding to the line $V(x_2)$. The slanting line exhibits the exceptional curve A_4 over the node a_4.

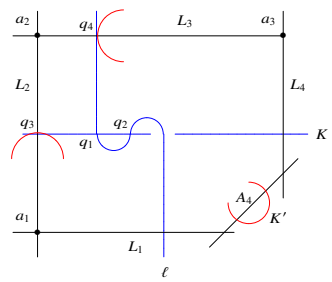


Fig. 9.6 The branch curve of the bielliptic map for Coble surfaces with crystallographic basis of (-2)-curves of type II

The double cover $Y \to \tilde{\mathsf{D}}_1$ has two ordinary double points over q_1, q_2 and two rational double points of type A_3 over q_3 and q_4. It is ramified over the union of ℓ, K, K' and the exceptional (-2)-curvesd over a_1, a_2, a_3. Let $X \to Y$ be a minimal resolution of singularities of Y.

The reduced pre-images of the (-2)-curves A_1, A_2 and A_3 are (-1)-curves on Y. After we blow them down to points P_1, P_2, P_3, we obtain a Coble surface V. It has a unique boundary component C, the pre-image of the (-2)-curve A_4. The pre-image $\tilde{L}_1$ and $\tilde{L}_2$ of L_1 and L_2) in X are (-2)-curves. Let $R_1 + R_2 + R_3$ and $R_1' + R_2' + R_3'$ be the exceptional curves over q_3 and q_4. The pre-image of $\tilde{\ell}$ (resp. $\tilde{K}$ of ℓ (resp. K) intersects R_2 (resp. R_2'). The pre-image $\tilde{L}_1$ (resp. $\tilde{L}_2$) of L_1 (resp. L_2) is a (-2)-curve that intersects R_1 and R_3 (resp. R_1' and R_2'). We see now two quadrangles from Figure 8.3. The third one is equal to the union of $\tilde{\ell}, \tilde{K}$ and the exceptional curves over q_1, q_2.

So, we obtain a crystallographic basis of (-2)-curves on V.

The group of automorphisms of the Coble surface is finite. It contains a subgroup $(\mathbb{Z}/2\mathbb{Z})^2$ generated as in the previous example by the deck transformation g_1 and the transformation $g_2 : [x_0, x_1, x_2] \mapsto [-x_0, x_1, x_2]$. This time the deck transformation does not act identically on the Picard lattice. It switches the exceptional curves R_1, R_{10} and R_4, R_{11}, where we use the notation from Figure 8.3. The second transformation

switches the points q_1, q_2 and hence the exceptional (-2)-curves R_7, R_{12} over them. The exceptional curve over $p_1 = [1, 0, 0]$ is invariant with respect to g_2, its pre-image on V is a (-1)-curve $\bar{C}$. When we blow it down, we obtain a Halphen surface of index 2 with an elliptic fibration with reducible fibers of types $\tilde{D}_5$ and $\tilde{A}_3$ and one nodal fiber, the image of the boundary component of V. The Mordell–Weil group of translations is isomorphic to $\mathbb{Z}/4\mathbb{Z}$ and lifts to a cyclic group of automorphisms of order 4 of V. The square of its generator is equal to g_2. This transformation does not originate from an automorphism of D_1 that switches the lines L_1, L_2 and fixes the node that it is contained in the branch curve. The surface also has an elliptic fibration with a reducible fiber of type $\tilde{A}_8$. It has also a fiber which consists of the union of the (-4)-curve and the (-1)-curve $\bar{C}$. We can blow down $\bar{C}$ to obtain a Halphen surface with an elliptic fibration with singular fiber of type $\tilde{A}_8$ and three irreducible nodal fibers. The Mordell–Weil group of translations of this fibration is isomorphic to $\mathbb{Z}/3\mathbb{Z}$ and lifts to an automorphism of V. Thus,

$$\mathrm{Aut}(\mathsf{V}) \cong \mathfrak{S}_4.$$

Since all effective irreducible roots are the classes of (-2)-curves and we have the same crystallographic basis the R-invariant is the same as in the case of Enriques surfaces. The root invariant is $(D_9, \{0\})$.

Note that all three examples of Coble surfaces with finite automorphism groups over a field of characteristic 0 are obtained as specialization of one-parameter families of Kondo surfaces which are necessarily of type I or II. Although, by Corollary 9.1.11, any Coble surface is obtained as a specialization of a one-parameter family of Enriques surfaces, the general member of this family is expected to have an infinite automorphism group.

Let us now discuss the case of positive characteristic. If $p \neq 2$, we look at the construction of an Enriques surface of type V–VII which are not realized in some characteristic $p = 3$ or 5, and then we show that the constriction still holds in these characteristics but gives a Coble surface instead of an Enriques surface.

If $p = 2$, we use that some of the surfaces with finite automorphism group vary in a family and we try to find a Coble surface on the boundary of these families

We will start with constructions of Coble surfaces with finite automorphism group in positive characteristic $p \neq 2$.

- type V specialization in characteristic $p = 3$.

We use equation (8.9.9) for an Enriques surface with finite automorphism group of type V. When $p = 3$, it degenerates, after normalization, to a double plane model of a Coble surface given by the following equation:

$$x_3^2 + x_1(x_1 - x_2)(x_1x_2 - x_0^2)(x_1(x_1 - x_2) + x_0^2 + x_0x_1) = 0. \tag{9.8.3}$$

Note that it is very similar to equation (9.8.2). The only difference is that the conics K and K' intersect at $p_2 = [0, 0, 1]$ with multiplicity 3 instead of 4. The conic K

intersects ℓ at $q_1 = [-1, 1, 1]$ and $q_2 = [1, 1, 1]$. The conic K' intersects ℓ and $q_3 = [0, 1, 1]$. Since $p = 3$, the conic K' is tangent to $V(x_2)$ at the point $q = [1, 1, 0]$. The proper transforms of K and K' in $\tilde{\mathsf{D}}_1$ intersect at $q_4 \in L_2$.

Figure 9.7 depicts the branch curve on D blown up at the node a_4 corresponding to the line $V(x_2)$ in Figure 9.7.

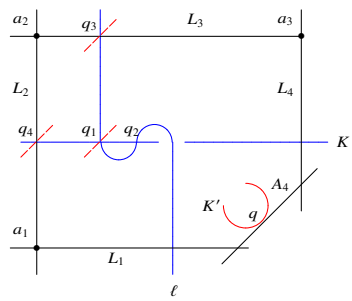


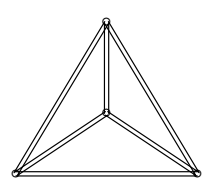
Fig. 9.7 The branch curve of the bielliptic map of a specialization of type V

Since $V(x_2)$ is tangent to K' at $q = [1, 1, 0]$, it splits into two (-3)-curves on the double cover. They intersect at one point which we identify with the point q. After, we blow-up this point and resolve the singularities, and blow-down the reduced pre-images of the (-2)-curves A_1, A_2, A_3, we obtain a terminal Coble surface V with two boundary components C_1, C_2. Their images on $\tilde{\mathsf{D}}_1$ is the curve A_2. The pre-images of the lines L_1, L_4 on D_1 that pass through the node a_4 are two (-1)-curves $\tilde{L}_1, \tilde{L}_4$ on V. Let E be the exceptional curve over q. It intersects both C_1, C_2. Thus, we have three irreducible effective roots two effective roots

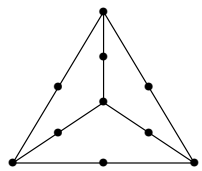
$$\alpha_1 = 2\tilde{L}_1 + \frac{1}{2}(C_1 + C_2), \quad \alpha_2 = 2\tilde{L}_4 + \frac{1}{2}(C_1 + C_2), \quad \alpha_3 = 2E + \frac{1}{2}(C_1 + C_2)$$

with $\alpha_i \cdot \alpha_j = 2$.

The proper transform of the line $V(x_0 + x_1) = \langle q_1, p_2 \rangle$ on $\tilde{\mathsf{D}}_1$ is a (-1)-curve. Its pre-image on the double cover is a (-2)-curve. Since it intersects A_2, its image in V is a (-1)-curve L. Since $V(x_0 - x_1)$ intersects $V(x_2)$, L intersects C_1 and C_2 with multiplicity 1. This gives one more irreducible effective root $\alpha_4 = 2L + \frac{1}{2}(C_1 + C_2)$ with $\alpha_4 \cdot \alpha_i = 2$, $i = 1, 2, 3$, Now, we see that $\alpha_1, \ldots, \alpha_4$ form a subset of roots with the Dynkin diagram:

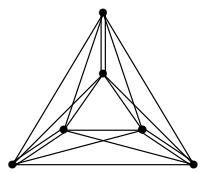


Next, we look at the pre-images of the curves ℓ, K, K'. It is easy to see that they define a subdiagarm with ten vertices:



Here, the middle vertex corresponds to the triple point q_2 of the branch curve.

Finally, we have a subdiagram with six vertices:



It is obtained from the lines $V(x_0 - x_2) = \langle p_2, q_2 \rangle$, $V(x_0 - x_1 - x_2) = \langle q_1, q \rangle$, and $V(x_0 + x_2) = \langle p_2, q_1 \rangle$, which are split under the cover.

Therefore, we obtain twenty irreducible effective roots that form a crystallographic basis of type V in $\mathrm{CM}(\mathsf{V})$.

It follows that the surface V has a finite automorphism group. As in the case of Enriques surfaces, we obtain that $\mathrm{Aut}(\mathsf{V})$ contains an cohomologically trivial element σ of order 2 and the quotient $\mathrm{Aut}(\mathsf{V})/(\sigma)$ is a subgroup of $\mathrm{Sym}(\Gamma)$. As in the case of Enriques surface (Proposition 8.9.21), $\mathrm{Aut}(\mathsf{V}) \cong \mathbb{Z}/2\mathbb{Z} \times \mathfrak{S}_4$.

Let us record this example.

Theorem 9.8.9 *Under specialization to characteristic 3, an Enriques surface with a crystallographic basis of type* V *specializes to a Coble surface* V *with two boundary components that contains a crystallographic basis of the same type of irreducible effective roots in its Coble–Mukai lattice. Its group of automorphism is isomorphic to* $\mathbb{Z}/2\mathbb{Z} \times \mathfrak{S}_4$. *Its canonical cover X is a supersingular K3 surface with Artin invariant* $\sigma = 1$ *and the R-invariant is* $(K, H) = (\mathsf{E}_7 \oplus \mathsf{A}_2 \oplus 2\mathsf{A}_1, (\mathbb{Z}/2\mathbb{Z})^2)$.

Proof We need to prove only the last assertion. One can see that there exists a quasi-elliptic fibration on X with two singular fibers of type $\tilde{E}_6$, four singular fibers of type $\tilde{A}_2$ and with nine sections (in the case of the Enriques surface of type V, the covering K3 surface contains an elliptic fibration with two singular fibers of type $\tilde{E}_6$ and two singular fibers of type $\tilde{A}_2$). Each of the pre-images of two boundary components is a component of additional two singular fibers. The Shioda–Tate formula implies that the determinant of the Picard group of X is equal to $-3^6/9^2 = -3^2$, that is, X is the supersingular K3 surface with Artin invariant 1. For K, we find $\mathsf{E}_7 \oplus \mathsf{A}_2$ from (-2)-curves on V as in the case of the Enriques surface and $2\mathsf{A}_1$ from two boundary components. Since V has two boundary components, the Coble–Mukai lattice Λ is isomorphic E_{10} (Example 9.2.5). By considering the map $\gamma : K/2K \to \Lambda/2\Lambda$ given in (9.2.5) and the fact $\dim \mathrm{rad}(q_{K/2K}) = 3$, we have $H = (\mathbb{Z}/2\mathbb{Z})^2$. □

In Remark 8.9.20, we mentioned the canonical cover of the Enriques surface of type V. Similar description of the Coble surface of type V is given in [419, Example 4].

- Type VI specialization in characteristic $p = 3$.

In Section 8.9, we considered Enriques surfaces with a crystallographic basis of type VI as the quotient of the Hessian surface of the Clebsch diagonal cubic surface by an involution defined by cubic Cremona involution σ in the ambient space. We assumed in the construction that $p \neq 3, 5$ in order that the Cremona involution does not have fixed points. In this and the next example, we consider the excluded cases $p = 3$ (resp. $p = 5$) and will show that the quotient becomes a Coble surface with five (resp. one) boundary components.

So let us assume that $p = 3$. The Hessian surface H is still defined as a quartic surface but it is not isomorphic to the Hessian of any cubic surface. The surfaces still has 10 nodes with the same coordinates as in the case $p \neq 2, 3, 5$. The Cremona involution σ has five fixed points which form the $\mathfrak{S}_5$-orbit of the point $[-1, 1, 1, 1, 1]$. They are additional ordinary nodes of H. Let X be a minimal resolution of H. Then, the Cremona involution of H lifts to the switch involution τ of X that fixes pointwise the exceptional curves R_i over the new nodes. The quotient surface is a Coble surface V with five boundary components.

We refer for the notations of what follows to Section 8.9. The lines $L_{abc}(\omega)$ disappear and instead we have ten lines $\ell'_{ab} : x_a + x_b = 0, x_c = x_d = x_e$. Each such line passes through two of the new node. For example, the line ℓ'_{12} passes through $[-1, 1, 1, 1, 1]$ and $[1, -1, 1, 1, 1]$. Their proper transforms on X are ten (-2)-curves ℓ_{ab} invariant under σ and intersecting two of the exceptional curves R_i. Their images in V are ten (-1)-curves E_{ab}, each intersecting two boundary components. This defines irreducible effective roots $\alpha_{ab} = 2E_{ab} + \frac{1}{2}(C_a + C_b)$. We have $\alpha_{ab} \cdot \alpha_{cd} = 1$ if $\#\{a, b\} \cap \{c, d\} = 1$ and zero if the sets are disjoint. Thus, we see that each α_i intersects 6 other α_j's. They play the role of $E_{11}, \dots, E_{20}$ in the crystallographic basis of type VI. The edges L'_{ab} of the Sylvester pentahedron survive and define, as in the case of Enriques surfaces ten (-2)-curves R_{ab} on V whose intersection diagram is the Petersen graph. The ten curves play the role $E_1, \dots, E_{10}$ in the crystallographic basis of type VI. Each ℓ'_{ab} passes through one node and intersects the opposite edge. This shows that each α_i intersects one of intersects one R_{ab} with multiplicity 2. So, we see that 20 effective roots α_{ab}, R_{ab} form the crystallographic basis of type VI.

Let us now give a construction of the Coble surface V as the blow-up of 15 points in the plane. Recall that in Example 9.2.7 we constructed a Coble surface with ten boundary components whose K3 cover was one of Vinberg's most algebraic K3 surfaces. It was obtained by blowing up the intersection points of 10 lines on the quintic del Pezzo surface $\mathcal{D}$ of degree 5. This time we blow up one point on each line not equal to one of 15 intersection points. We index the lines by duads $\{a, b\} \subset [1, 5]$. As a result we obtain a surface with ten (-2)-curves R_{ab} that form the Petersen graph.

We also have ten (-1)-curves E_{ab}, the exceptional curves over the points p_{ab} lying on one of the lines. Suppose we find 5 conics C_i from different five pencils of conics on $\mathcal{D}$, each passing through four points p_{ab}. Then, the proper transform of C_i would be a (-4)-curve intersecting four E_{ab}'s and each E_{ab} would intersect two of the C_i's. It is easy to see that the curves R_{ab} and the effective roots $2E_{ab}+\frac{1}{2}(C_a+C_b)$ form the crystallographic basis of type VI. It is an amazing fact that we can find such conics (uniquely) only if $p = 3$.

We use a model of the quintic del Pezzo surface $\mathcal{D}$ as the blow up of four points $p_1 = [1, 0, 0], p_2 = [0, 1, 0], p_3 = [0, 0, 1], p_4 = [1, 1, 1]$ in the plane. The ten lines obtained as the proper transforms of the six lines $V(t_i)$ and $V(t_i - t_j)$, and four exceptional curves over these lines corresponding to the tangent directions at the points p_i. The five pencils of conics on $\mathcal{D}$ come from the pencil of lines with base point p_i and the pencil of conics with base points p_1, p_2, p_3, p_4. Thus, our 5 conics originate from four lines

$$V(a_1x_0 + a_2x_1),\ V(b_1x_0 + b_2x_2),\ V(c_1x_1 + c_2x_2),\ V(d_1(x_0 - x_2) + d_2(x_0 - x_1))$$

and the conic $V(ax_0x_1 + bx_1x_2 + cx_0x_2)$.

A straightforward computation shows that the conditions on the intersection points force the conic to have the equation $x_0x_1 + x_0x_2 + x_1x_2 = 0$ and the lines to have the equations $x_i + x_j = 0, x_0 + x_1 + x_2 = 0$. The nodal sextic is the image of the five curves C_i in the plane which is the union of the four lines and the conic. The set of its 14 double points consists of 10 points

$$[0, 1, -1],\ [1, -1, 0],\ [1, 0, -1],\ [1, 1, -1],\ [1, -1, 1],$$

$$[-1, 1, 1],\ [1, 0, 0],\ [0, 1, 0],\ [0, 0, 1],\ [1, 1, 1],$$

and four infinitely near points to the last four ones that correspond to the tangent directions defined by the lines $V(t_i)$. By solving the linear equations, we find this is possible only if $p = 3$.

It is known that $\mathrm{Aut}(\mathcal{D}) \cong \mathfrak{S}_5$ (see [177], 8.5.4). The blowing-up morphism $\mathsf{V} \to \mathcal{D}$ is obviously $\mathfrak{S}_5$-invariant and the lift of the action of $\mathfrak{S}_5$ on V coincides with the action of $\mathfrak{S}_5$ descended from its action on X. By analyzing its action on the crystallographic basis of V, we find that

$$\mathrm{Aut}(\mathsf{V}) \cong \mathfrak{S}_5.$$

Let us record this example.

Theorem 9.8.10 *Under specialization to characteristic 3, an Enriques surface with a crystallographic basis of type* VI *specializes to a Coble surface with five boundary components that contains a crystallographic basis of the same type of irreducible effective roots in its Coble–Mukai lattice. Its group of automorphism is isomorphic to* $\mathfrak{S}_5$. *Its canonical cover X is a supersingular K3 surface with Artin invariant* $\sigma = 1$ *and the R-invariant is* $(K, H) = (\mathsf{E}_6 \oplus \mathsf{D}_5, \mathbb{Z}/2\mathbb{Z})$.

Proof One can see that there exists a quasi-elliptic fibration on X with three singular fibers of type $\tilde{E}_6$, one singular fiber of type $\tilde{A}_2$ and with three sections. The Shioda–Tate formula implies that the determinant of the Picard group of X is equal to $-3^4/3^2 = -3^2$, that is, X is again a supersingular K3 surface with Artin invariant one. For K, we find E_6 from (-2)-curves on V and D_5 from five boundary components. Since rank $H \geq 1$ and the dimension of the kernel of $q_{K/2K}$ is 1, we have $H = \mathbb{Z}/2\mathbb{Z}$.□

Remark 9.8.11 This is a follow-up of Remark 9.2.12. In this remark, we discussed Coble surfaces with five boundary components obtained by blowing up 10 intersection points of five conics C_a from different pencils on $\mathcal{D}$. This 5-dimensional family of Coble surfaces lives in any characteristic. The ten exceptional curves E_{ab} and five conics define ten irreducible effective roots $\alpha_i = 2E_{ab} + \frac{1}{2}(C_a + C_b)$. Its intersection graph is isomorphic to the subgraph of the graph of type VI formed by the vertices $E_{11}, \ldots, E_{20}$. So, we see our surface as a specialization to characteristic 3 of a five-dimensional family of Coble surfaces with five boundary components whose general member has infinite automorphism group.

Note that, when we choose the conics to be pairs of lines, we obtain a Coble surface with 10 boundary components which is not terminal. Blowing up five singular points, we obtain a terminal Coble surface whose canonical cover is Vinberg's most algebraic K3 surface.

- Type VI specialization in characteristic $p = 5$.

Now, we specialize the same Enriques surface S of type VI to characteristic 5. The point $q = [1, 1, 1, 1, 1]$ is a unique fixed point of σ on the Hessian surface H. This time H is the Hessian surface of the Clebsch diagonal cubic surface in characteristic 5. We have the same 30 lines $L(a, b, c)(\omega)$ and $L(a, b)$ as in the case $p = 0$. Also, we have the exceptional curves R_{ab} over the 10 singular points p_{cde} and one additional (-1)-curve E_q.

Note that the residual component of the plane section $x_i = x_j = 0$ of H containing an edge of the pentahedron formed by the coordinate hyperplanes $x_i = 0$ is a cubic curve with a double point at q. Its proper inverse transform on X is a (-2)-curve C_{ij} intersecting the exceptional curve E_q at two points. The $\mathfrak{S}_5$-orbit of E_{ij} consists of 10 disjoint curves on X. Since $\rho(X') = 20$, we obtain that $\rho(X) = 22$, so that X is a supersingular K3 surface.

The quotient $\mathsf{V} = X/(\tau)$ of the minimal resolution X of H is a Coble surface V with $K_{\mathsf{V}}^2 = -1$ with an irreducible isolated curve R_0 in $|-2K_{\mathsf{V}}|$, the image of the curve E_q.

The surface V contains 20 (-2)-curves $E_1, \ldots, E_{20}$ which form a crystallographic basis of type VI in $K_{\mathsf{V}}^{\perp}$. The curves $E_1, \ldots, E_{10}$ are the images of the edges of the pentahedron. Their intersection graph is the Peterson graph. The curves $E_{11}, \ldots, E_{20}$ are the images of the curves $L(a, b, c)(\omega)$. Their intersection graph is the incidence graph of the configuration $(10_6, 30_2)$. The images of the curves C_{ij} are disjoint (-1)-curves S_{ij} on V. Pick up one of them, say C_{01} and blow it down to obtain a surface Y with $K_Y^2 = 0$. It is a straightforward computation that Y is a Halphen rational elliptic surface of index two. The image of E_q on Y is an irreducible nodal fiber on Y. The

images of $C_{ij} \neq C_{01}$ are 9 disjoint 2-sections. The fibration has 3 reducible fibers of type $\tilde{A}_5, \tilde{A}_2, \tilde{A}_1$. The fiber of type $\tilde{A}_5$ is formed by curves $E_2, E_3, E_5, E_6, E_7, E_9$. The fiber of type $\tilde{A}_2$ is formed by the curves E_{14}, E_{18}, E_{19}. The fiber of type $\tilde{A}_2$ is formed by E_{10}, E_{16}.

The automorphism group of V is again $\mathfrak{S}_5$.

Theorem 9.8.12 *Under specialization to characteristic 5, an Enriques surface with a crystallographic basis of type* VI *specializes to a Coble surface with one boundary component that contains a crystallographic basis of the same type of irreducible effective roots in its Coble–Mukai lattice. Its group of automorphism is isomorphic to* $\mathfrak{S}_5$. *Its canonical cover X is a supersingular K3 surface with Artin invariant* $\sigma_0 = 1$ *and the R-invariant is* $(K, H) = (\mathsf{E}_6 \oplus \mathsf{A}_4, \{0\})$.

Proof Since there is only one boundary component, the R-invariant is the same as for an Enriques surface from the family. We already know that X is a supersingular K3 surface. Note that $\pi^*(\mathrm{Num}(S)) \oplus K(2)$ has the determinant $2^{20} \cdot 3 \cdot 5$ and by definition of R-invariant there exists a sublattice N of $\mathrm{Pic}(X)$ such that $\pi^*(\mathrm{Num}(S)) \oplus K(2) \subset N$ and N is of rank 20 and of the determinant 15. Let T be the orthogonal complement of N in $\mathrm{Pic}(X)$. Then, its rank is equal to 2. Since $\mathrm{Pic}(X)$ contains $N \oplus T$ as a sublattice of finite index, the Artin invariant is equal to one. □

- Type VII specialization in characteristic $p = 5$.

Consider the surface X' given by equations (8.9.14). If $p = 5$, the Cremona involution σ has a unique fixed point which is an ordinary node of X'. The quotient is not an Enriques surface but rather a Coble surface.

We know from Example 8.9.30 that an Enriques surface of type VII contains 15 curves (-2)-curves L_σ, the images of 15 pairs of lines $\ell_\sigma^\pm \subset X'$ spanned by points $[1, -1, \pm i, \mp i, 0]$ and $[0, 0, 0, 0, 1]$, where σ is one of 10 even involutions in $\mathfrak{S}_5$. It also contains five (-2)-curves K_i, the images of the exceptional curves on X over the five nodes of X. Together, they form the diagram of type VII.

Consider a hyperplane in $\mathbb{P}^4$ spanned by two lines $\ell_\sigma^\pm$ and the new singular point $q = [1, \ldots, 1]$. The intersection of X' with this hyperplane is a curve of bidegree $(3, 3)$ on a quadric that contains its two lines intersecting at a point. The residual curve is of bidegree $(2, 2)$, and since it has a double point at q, it must be a rational (-2)-curve. Its proper transform on the K3 surface X is a new (-2)-curve. So, we have additionally fifteen (-2)-curves on X. Each curve is invariant with respect to the switch involution τ. We denote its image on the Coble surface V by R_σ. This is a (-1)-curve on V. The images of each line-pair $\ell_\sigma^\pm$ is a (-2)-curve L_σ on V. The image of the exceptional curve over q on V is a (-4)-curve $C \in |-2K_{\mathsf{V}}|$.

One checks that two curves L_{σ_1} and L_{σ_2} intersect if and only if they share a duad. Two curves R_{σ_1} and R_{σ_2} intersect with multiplicity one if and only if σ_1 and σ_2 do not fix the same element in $[1, 5]$. The curve L_{σ_1} intersects R_{σ_2} with multiplicity two if and only if σ_1 and σ_2 fix the same element in $[1, 4]$. Otherwise, they intersect with multiplicity one.

Consider the following nine curves:

$$L_{12,35}, \quad L_{35,24}, \quad L_{24,13}, \quad L_{13,25}, \quad L_{25,14}, \quad L_{14,23}, \quad L_{23,45}, \; L_{12,45}.$$

It follows from above that their sum is a nodal cycle L of type $\tilde{A}_8$. They realize the curves $E_1, \ldots, E_9$, respectively, from the Coxeter diagram of the crystallographic basis of type VII. The disjoint (-1)-curves $R_{23,45}, R_{24,35}, R_{25,34}$ intersect the components E_1, E_4, E_7. Consider the blowing down $\mathsf{V} \to \mathbb{P}^2$ of the disjoint exceptional configurations $R_{23} + E_1 + E_2 + E_{10}$, $R_{24,35} + E_4 + E_5$, and $R_{25,34} + E_7 + E_8$. The image of the (-4)-curve is a curve of degree 6 with 10 double points $p_4 \succ p_3 \succ p_2 \succ p_1, p_7 \succ p_6 \succ p_5, p_{10} \succ p_9 \succ p_8$. The surface is obtained by blowing up the singular point of an irreducible fiber of a Halphen elliptic surface with one double fiber of type $\tilde{A}_8$. The images of the curves $R_{23,45}, R_{24,35}, R_{25,34}$ are three disjoint bisections. The image of the nodal cycle L is the triangle of lines with vertices at the points p_1, p_5, p_8.

The surface V has the obvious symmetry with the group $\mathfrak{S}_5$. The only effective roots in the Coble–Mukai lattice are the classes of (-2)-curves. Using Theorem 9.8.1, we see that $\mathrm{Aut}(\mathsf{V}) \cong \mathfrak{S}_5$.

Theorem 9.8.13 *Under specialization to characteristic* 5*, an Enriques surface with a crystallographic basis of type* VII *specializes to a Coble surface with one boundary component that contains a crystallographic basis of the same type of irreducible effective roots in its Coble–Mukai lattice. Its group of automorphism is isomorphic to* $\mathfrak{S}_5$*. Its canonical cover* X *is a supersingular K3 surface with Artin invariant* $\sigma_0 = 1$ *and the* R*-invariant is* $(K, H) = (\mathsf{A}_9 \oplus \mathsf{A}_1, \mathbb{Z}/2\mathbb{Z})$.

Proof Since the number of boundary components is equal to one, as we already used it before, the Nikulin R-invariant does not change when we specialize Enriques surfaces to a Coble surface. The Picard number of the canonical cover is at least 21 $(= 10 + 10 + 1)$ and hence X is a supersingular K3 surface. Since $\pi^*(\mathrm{Num}(S)) \oplus K(2)$ has the determinant $2^{22} \cdot 5$ and its orthogonal complement in $\mathrm{Pic}(X)$ has the rank 2, the Artin invariant is equal to one. □

We will give two examples of Coble surfaces in characteristic 3 with two and eight boundary components and with finite automorphism group in Theorems 10.5.18, 10.6.

Next, we give examples of Coble surfaces in characteristic 2 with finite automorphism group.

- Type VII specialization in characteristic $p = 2$ with ten boundary components.

In Theorems 9.2.8, 9.2.10, we discussed a Coble surface with ten boundary components that lives in any characteristic $p \neq 2$. Here, we show that this surface in characteristic two is a Coble surface with ten boundary components and with the crystallographic basis in $\mathrm{CM}(\mathsf{V})$ of type VII given in Figure 8.10.

In the proof of Theorem 9.2.10, we showed that the Coble–Mukai lattice contains 20 (-2)-classes forming a crystallographic basis of type VII in which 15 classes are effective and the remaining five classes correspond to five involutions. Thus, it

suffices to prove that the five classes are represented by effective classes. We consider the projective plane $\mathbb{P}^2(\mathbb{F}_4)$. We use the same coordinates as in Example 9.2.7 for the complete quadrangle and its vertices p_1, p_2, p_3, p_4 and the diagonal points q_1, q_2, q_3. As in this example, we consider the surface obtained by blowing up the intersection points of ten lines on the quintic del Pezzo surface $\mathcal{D}$, the blow-up of p_1, p_2, p_3, p_4. The specific of characteristic 2 is that the proper transforms of the following four conics K_i each passing through the set $\Sigma_i = \{p_1, p_2, p_3, p_4\} \setminus \{p_i\}$ and tangent at these points to the lines $\langle p_i, p_j \rangle, j \in \Sigma_i$:

$$K_1 : x^2 + yz = 0,\ K_2 : y^2 + xz = 0,\ K_3 : z^2 + xy = 0,\ K_4 : xy + yz + xz = 0.$$

We also let K_5 be the line $x + y + z = 0$ that passes through q_1, q_2, q_3. The proper transform of each K_i is a rational curve with self-intersection 1 that passes through three intersection points of the ten lines. Thus, their proper transforms on V are five (-2)-curves which we will continue to denote by $K_1, \dots, K_5$. Therefore, the twenty effective roots $\alpha_{ab,cd}, K_i$ form the crystallographic basis of type VII.

Note that three lines given in (9.2.6) degenerate to $x+y+z = 0$ in characteristic two. Thus, (-2)-classes defining five reflections in Vinberg's most algebraic K3 surface are now represented by effective curves and hence do not define an involution of the surface.

This example was obtained independently by S. Mukai.

As mentioned in Example 9.2.8, V dominates an extremal elliptic surface with singular fibers of $\tilde{A}_4$, $\tilde{A}_4$, $\tilde{A}_0^*$, $\tilde{A}_0^*$. Let $\pi : X \to \mathsf{V}$ be the canonical cover defined by the invertible sheaf $\mathcal{L} = \omega_{\mathsf{V}}^{-1}$ and the section of $\mathcal{L}^{\otimes 2}$ with the zero scheme equal to the boundary $\sum C_{ab}$. We have $\omega_X \cong \mathcal{O}_X$ and Proposition 0.2.10 tells that $\mathrm{Sing}(X)$ is a finite subscheme Z with $h^0(\mathcal{O}_Z) = 12$. Since X has an ordinary double point over singular points of fibers of type $\tilde{A}_4$ and of $\tilde{A}_0^*$, we infer that X has exactly 12 double points and its minimal resolution Y is a supersingular K3 surface. It has an elliptic fibration with singular fibers of type $\tilde{A}_9, \tilde{A}_9, \tilde{A}_1, \tilde{A}_1$ and with 10 sections, and the Shioda–Tate formula implies that the Artin invariant of Y is equal to one.

Next, we show that the above Coble surface is a specialization of the Enriques surfaces of type VII given in §8.9, Example 8.10.9. We use the same notation given there. Recall that we considered the elliptic fibration $p : Y \to \mathbb{P}^1$ on the minimal resolution Y of the canonical cover X of S which is the minimal model of the elliptic surface defined by the following Weierstrass equation:

$$y^2 + t^2xy + y = x^3 + x^2 + t^2.$$

The elliptic fibration p has two reducible fibers of type $\tilde{A}_9$ over $t = 1, \infty$ and two singular fibers $F_\omega + E_\omega, F_{\omega^2} + E_{\omega^2}$ of type $\tilde{A}_1$ over $t = \omega, \omega^2$. The elliptic surface f has 10 sections s_i, m_i $(i = 0, 1, \dots, 4)$. To obtain Enriques surfaces, we considered the following derivation defined by

$$\partial = \frac{1}{t+1}\left((t+1)(t+\alpha)(t+\beta)\frac{\partial}{\partial t} + \frac{(1+t^2x)}{t+1}\frac{\partial}{\partial x}\right)$$

with $\alpha, \beta \in k,\ \alpha + \beta = \alpha\beta,\ \alpha^3 \neq 1$.

Now, we put $\beta = \alpha/(\alpha + 1)$ in ∂, multiple ∂ by $\alpha + 1$ and put $\alpha = 1$. Then, we obtain the derivation $\partial_0 = (t+1)\frac{\partial}{\partial t}$. A direct calculation shows the following lemma.

Lemma 9.8.14 (i) *The integral curves with respect to ∂_0 are $E_{1,1}, E_{1,3}, E_{1,5}, E_{1,7}, E_{1,9}$, $E_{\infty,1}$, $E_{\infty,3}$, $E_{\infty,5}$, $E_{\infty,7}, E_{\infty,9}, E_\omega, E_{\omega^2}$, s_0, s_1, s_2, s_3, s_4.*

(ii) *Let D be the divisor of ∂_0. Then,*

$D = F_1 + E_{1,2} + E_{1,4} + E_{1,6} + E_{1,8} + F_\infty + E_{\infty,2}, E_{\infty,4} + E_{\infty,6} + E_{\infty,8} + E_\omega + E_{\omega^2}$.

(iii) $D^2 = -24$.

Let Y^{∂_0} be the quotient of Y by ∂_0 and $\pi : Y \to Y^{\partial_0}$ the canonical map. By the same argument as in the case of Enriques surfaces, Y^{∂_0} is smooth. It follows from the canonical divisor formula that $\pi^* K_{Y^{\partial_0}} = -D$. For a curve C on Y, we denote by $\bar{C}$ the image of C on Y^{∂_0}. It follows that the integral curves in Lemma 9.8.14, (i) are (-1)-curves and the other are (-4)-curves. Thus, we have $2K_{Y^{\partial_0}} = -(\bar{F}_1 + \bar{E}_{1,2} + \bar{E}_{1,4} + \bar{E}_{1,6} + \bar{E}_{1,8} + \bar{F}_\infty + \bar{E}_{\infty,2} + \bar{E}_{\infty,4} + \bar{E}_{\infty,6} + \bar{E}_{\infty,8} + 2\bar{E}_\omega + 2\bar{E}_{\omega^2})$. By contracting (-1)-curves $\bar{E}_\omega, \bar{E}_{\omega^2}$, we obtain a non-singular surface V with $|-2K_{\mathsf{V}}| = \{\bar{F}_1 + \bar{E}_{1,2} + \bar{E}_{1,4} + \bar{E}_{1,6} + \bar{E}_{1,8} + \bar{F}_\infty + \bar{E}_{\infty,2}, \bar{E}_{\infty,4} + \bar{E}_{\infty,6} + \bar{E}_{\infty,8}\}$. The images of five sections $m_0, m_1, \ldots, m_4$ are (-2)-curves forming a complete graph of five vertices with double edges. The images of 15 integral curves in Lemma 9.8.14, (i) are 15 (-1)-curves which define 15 irreducible effective roots. One can check that the dual graph of 20 roots is nothing but the one of type VII. Thus, we have a Coble surface V with the dual graph of type VII and with ten boundary components.

The following theorem summarizes our discussion:

Theorem 9.8.15 *The surface V is a Coble surface in characteristic 2 with ten boundary components and with the crystallographic basis of type* VII *which is a specialization of the one dimensional family $\{S_{\alpha,\beta}\}$ $(a, b \in k,\ a + b = ab,\ a^3 \neq 1)$ of Enriques surfaces given in Theorem 8.10.11. The automorphism group* $\mathrm{Aut}(\mathsf{V})$ *is isomorphic to $\mathfrak{S}_5$.*

- Type VII specialization in characteristic $p = 2$ with two boundary components.

In this example, we construct a Coble surface in characteristic 2 with two boundary components C_1, C_2 and with the same crystallographic basis as in the previous example. It is obtained as a specialization of the one-dimensional family of Enriques surfaces of type VII given in Theorem 8.10.11.

We use the notation from the previous example. Observe that the curves $K_1, \ldots, K_5$ have two points $Q(\omega)$ in common with coordinates $[1, \omega, \omega^2]$, $[1, \omega^2, \omega]$, where $\omega^3 = 1, \omega \neq 1$. Looking at the Petersen graph, we immediately locate two pentagons (e.g. the interior 5 vertices and the exterior five vertices) each representing a cycle of five (-1)-curves on $\mathcal{D}$. Each vertex of one pentagon is connected to one vertex of the second pentagon. We blow up the corresponding set of five intersection points $P_1, \ldots, P_5$ of the ten lines, we obtain a surface V' with two disjoint cycles of (-2)-curves and five sections $E_1, \ldots, E_5$ coming from the exceptional curves over the five points. They define a structure of a minimal rational elliptic surface on V'

with two fibers of type $\tilde{A}_4$. This pencil contains two irreducible nodal fibers with nodes $Q(\omega)$ (identified with points on V').

Note that $\#\mathcal{D}(\mathbb{F}_4) = (\#\mathbb{P}^2(\mathbb{F}_4) - 4) + 4\#\mathbb{P}^1(\mathbb{F}_4) = 17 + 20 = 37$. The set of ten lines contains $15 + 10\#(\mathbb{P}^1(\mathbb{F}_4) - 3) = 35$ points. Since the points $Q(\omega)$ belong to $\mathcal{D}(\mathbb{F}_4)$ and do not lie on lines, we obtain a beautiful fact:the two points $Q(\omega)$ do not depend on a choice of two disjoint pentagons and coincide with the set of singular points of two irreducible singular fibers of the elliptic fibrations defined by the two pentagons. Another equivalent fact is that any automorphism g of order five of $\mathcal{D}$ has the same set of two fixed points.

Next, we blow up the points $Q(\omega)$ and obtain a Coble surface with two boundary components C_1 and C_2 equal to the proper transforms of the two irreducible singular fibers. We know that the rational elliptic surface V' is an extremal rational elliptic surface with the Mordell–Weil group isomorphic to $\mathbb{Z}/5\mathbb{Z}$. Each of the curves E_i is a section intersecting two nodal fibers. This defines five irreducible effective roots

$$\alpha_i = 2E_i + \frac{1}{2}(C_1 + C_2).$$

In the previous example, we used that the proper transforms of the curves $K_1, \dots, K_5$ on $\mathcal{D}$ are curves with self-intersection 1. Each of these curves contains one point P_i and hence its proper transform on V' is a curve of self-intersection 0 passing through the singular points of irreducible singular fibers and one singular point of each reducible fiber. The proper transforms of K_i on V are (-2)-curves.

Now again, we have 15 classes of (-2)-curves which are components of reducible fibers and five curves $K_1, \dots, K_5$. Its intersection graph is the dual Petersen graph. We also have five classes of effective irreducible roots α_i with $\alpha_i \cdot \alpha_j = 2$. As in the previous example, we check the 20 divisor classes in $\mathrm{CM}(\mathsf{V})$ form the crystallographic basis of type VII.

Now, we show that this Coble surface is a specialization to characteristic two of the Enriques surfaces of type VII given in Example 8.10.9. We use the same notation given there. Recall that we considered the elliptic fibration $p : Y \to \mathbb{P}^1$ on the minimal resolution Y of the canonical cover X of S which is the minimal model of the elliptic surface defined by the following Weierstrass equation:

$$y^2 + t^2xy + y = x^3 + x^2 + t^2.$$

The fibration p has two reducible fibers of type $\tilde{A}_9$ over $t = 1, \infty$ and two singular fibers $F_\omega + E_\omega, F_{\omega^2} + E_{\omega^2}$ of type $\tilde{A}_1$ over $t = \omega, \omega^2$. We consider the rational derivation ∂ defined by:

$$\partial = \frac{1}{(t+1)}\left((t+1)(t+\omega)(t+\omega^2)\frac{\partial}{\partial t} + (1+t^2x)\frac{\partial}{\partial x}\right),$$

where $\omega^3 = 1, \omega \neq 1$. One can prove that the divisor D of ∂ is equal to

$$D = E_\omega + E_{\omega^2} - \left(F_1 + F_\infty + \sum_{i=1}^{4}(E_{1,2i} + E_{\infty,2i})\right), \tag{9.8.4}$$

and $F_\omega, F_{\omega^2}, F_1, F_\infty, E_{1,2i}, E_{\infty,2i}$ $(1 \le i \le 4)$ are integral with respect to ∂ (compare this with Lemma 8.10.10). Using an argument from Example 8.10.9, we see that ∂ has no isolated zeros, the quotient map $\pi : Y \to Y^\partial$ is inseparable and Y^∂ is non-singular. By contracting the (-1)-curves which are images of $E_{1,2i}, E_{\infty,i}$ $(1 \le i \le 4)$, we obtain a birational morphism $\varphi : Y^\partial \to \mathsf{V}'$. Let $\bar{E}_\omega, \bar{E}_{\omega^2}, \bar{F}_\omega, \bar{F}_{\omega^2}$ be the images of $E_\omega, E_{\omega^2}, F_\omega, F_{\omega^2}$. Then, $\bar{E}_\omega^2 = \bar{E}_{\omega^2}^2 = -4, \bar{F}_\omega^2 = \bar{F}_{\omega^2}^2 = -1$. Note that the non-singular surface V' is obtained by blowing ups of an elliptic surface at two singular points of two fibers of type $\tilde{A}_0^*$. Then

$$0 = 2K_Y = \pi^*(2K_{Y^\partial}) + 2D = \pi^*(\varphi^*(2K_{\mathsf{V}'}) + 2\sum_{i=1}^{4}(E_{1,2i} + E_{\infty,2i})) + 2D.$$

Since E_ω, E_{ω^2} are not integral, we have $\bar{E}_\omega + \bar{E}_{\omega^2} \in |-2K_{\mathsf{V}'}|$. Thus, V' is a rational surface and hence a Coble surface with two boundary components. Let $\bar{s}_i, \bar{m}_i$ $(0 \le i \le 4)$ be the images of sections s_i, m_i respectively. Since $s_i \cdot F_\omega = s_i \cdot F_{\omega^2} = m_i \cdot E_\omega = m_i \cdot E_{\omega^2} = 1$, $\bar{s}_i^2 = -2$ and $\bar{m}_i^2 = -1$ (these imply that m_i is integral and s_i is not). Now, it is easy to see that the surface V' is isomorphic to the Coble surface V obtained above.

Thus, we have the following theorem:

Theorem 9.8.16 *The surface* V *is a Coble surface in characteristic 2 with two boundary components and with the crystallographic basis of type* VII *which is a specialization* $a = \omega, b = \omega^2$ *of the one dimensional family* $\{S_{\alpha,\beta}\}$ $(a, b \in k,\ a + b = ab,\ a^3 \neq 1)$ *of Enriques surfaces given in Theorem 8.10.11. The automorphism group* $\mathrm{Aut}(\mathsf{V})$ *is isomorphic to* $\mathfrak{S}_5$.

Remark 9.8.17 In the above example, the canonical cover X of V is obtained by contracting twelve (-2)-curves appeared in the divisor D given in (9.8.4). The induced derivation on X has two isolated zeros at the images of E_ω, E_{ω^2} because ∂ has zeros along E_ω, E_{ω^2}. We will explain the reason why we can get a Coble surface in Proposition 10.3.3.

Finally, we give examples of Coble surfaces in characteristic $p = 2$ with finite automorphism group, and with a crystallographic basis of the remaining types that appeared in the classification of classical Enriques surfaces given in Table 8.14 except the case of type $\tilde{D}_4 + \tilde{D}_4$. The latter case is not realized for Coble surfaces. For the proof of this fact, we refer the reader to [370, the proof of Theorem 5.5]. All examples are specializations of the corresponding families of classical Enriques surfaces with finite automorphism group discussed in §8.10. We can also construct them as a blowing up of a plane sextic curve. This allows us to determine the moduli of Coble surfaces with finite automorphism group. We omit the details (see [370, §6]).

Example 9.8.18 In this example, we give a Coble surface V in characteristic 2 with one boundary component B and with the same crystallographic basis of type $\tilde{E}_8$ given in Figure **??**. It is obtained as a specialization of the one-dimensional family of classical Enriques surfaces of type $\tilde{E}_8$ given in Example 8.10.40 by putting $a = 0$ in the equation (8.10.11) of the derivation ∂_a. In the following, we use the notation as in Figure **??**. The Coble surface V has a quasi-elliptic fibration f with a multiple singular fiber of type $\tilde{E}_8$ and the cuspidal curve R_{10}. The only difference with Enriques surfaces is that it is obtained from a Halphen surface by blowing up the cusp of an irreducible fiber. Thus, B is the proper transform of the fiber and there exists a (-1)-curve E meeting R_{10} transversally and B with multiplicity 2 at the pre-image of the cusp. By contracting $E, R_{10}, R_9, \dots, R_2$ successively, we obtain a projective plane $\mathbb{P}^2$. The image of B is a plane sextic curve C with a unique singular point P and the image of R_1 is a line ℓ meeting B only at P. We may assume that $P = [0, 0, 1]$ and ℓ is defined by $x = 0$. Then, by an elementary but long calculation, C is given by $y^6+x^5y+x^2z^4 = 0$. Note that the pencil $y = \alpha x$ of lines passing through p gives a purely inseparable covering $C \setminus \{p\} \to \mathbb{A}^1$ by $x^2(\sqrt[4]{\alpha^6 + \alpha x} + z)^4 = 0$, and hence C is rational. The quasi-elliptic fibration f is induced from the pencil of sextic curves given by

$$\{sx^6 + t(y^6 + x^5y + x^2z^4)\}_{[s,t]\in\mathbb{P}^1}.$$

The automorphism group $\mathrm{Aut}(\mathsf{V})$ is isomorphic to $\mathbb{Z}/5\mathbb{Z}$ generated by

$$[x, y, z] \to [x, \zeta y, \zeta^{-1}z], \quad \zeta \neq 1, \zeta^5 = 1.$$

The curve is a specialization in characteristic 2 of a member of the family of rational nodal sextic with cyclic symmetry of order 5 given by equation (9.4.14). By construction, a Coble surface with the crystallographic basis of type $\tilde{E}_8$ is unique and isomorphic to this example.

Example 9.8.19 Here we give a Coble surface V in characteristic 2 with two boundary components B_1, B_2 and with the crystallographic basis of type $\tilde{E}_7^{(1)}$ given in Figure 8.10.13. It is obtained as a specialization of the one-dimensional family of Enriques surfaces of the same type given in Example 8.10.47 by putting $b = c = 0$ in the equation (8.10.16) of the derivation ∂. Note that the case $b = 0, c = 1$ gives Enriques surfaces of type $\tilde{E}_7^{(1)}$. In the following we use the notation as in Figure 8.10.13. The surface V has a quasi-elliptic fibration f with a multiple singular fiber of type $\tilde{E}_7$. It is obtained from a Halphen surface by blowing up the singular point (and the infinitely near point) of the fiber F of type $\tilde{A}_1^*$. Thus, B_1, B_2 are the proper transforms of the two components of F, R_9 is the proper transform of the cuspidal curve and R_{10} is the one of the exceptional curve of the first blowing up. Let E be the exceptional curve of the second blowing up. Then, $R_{11} = 2E + \frac{1}{2}B_1 + \frac{1}{2}B_2$. Note that there exists a genus one fibration f' with a singular fiber of type $\tilde{E}_8$ which is a quasi-elliptic one with the cuspidal curve R_2 and is Jacobian because E is a section. Let E_1, E_2 be (-1)-curves such that $2E_1 + B_1, 2E_2 + B_2$ are two fibers of f' containing the boundary components. Since R_2 is a 2-section of f', $R_2 \cdot E_1 = R_2 \cdot E_2 = 1$. Now,

by contracting $E, R_{10}, R_9, \ldots, R_2$ and E_1, E_2 successively, we obtain a projective plane $\mathbb{P}^2$. The image of B_i is a plane cubic curve C_i with a cusp p_i the image of E_i ($i = 1, 2$). Two cubics C_1 and C_2 meet at a point p_0 with multiplicity 9. The image of R_1 is a line ℓ_1 meeting C_i at p_0 with multiplicity 3, and the image of R_2 is a line ℓ_2 passing through p_0, p_1, p_2. We may assume that $p_0 = [0, 0, 1]$, $p_1 = [1, 0, 0]$, $p_2 = [1, 0, 1]$, ℓ_1 is defined by $x = 0$ and ℓ_2 by $y = 0$. Then,

$$C_1 : y^3 + xz^2 = 0, \quad C_2 : x^3 + y^3 + xz^2 = 0.$$

The quasi-elliptic fibration f is induced from the pencil of sextic curves given by

$$\{s(x^2y)^2 + t(y^3 + xz^2)(x^3 + y^3 + xz^2)\}_{[s,t]\in\mathbb{P}^1}.$$

The automorphism group $\mathrm{Aut}(\mathsf{V})$ is isomorphic to $\mathbb{Z}/2\mathbb{Z}$ generated by

$$[x, y, z] \to [x, y, x + z].$$

By construction, Coble surface with the crystallographic basis of type $\tilde{E}_7^{(1)}$ is unique and isomorphic to this example.

Example 9.8.20 We give a one-dimensional irreducible family of Coble surfaces in characteristic 2 with one boundary component and with the crystallographic basis of type $\tilde{E}_7^{(2)}$ given in Figure 8.10.14. They are obtained as specializations of the two-dimensional family of Enriques surfaces of the same type given in Example 8.10.59 by putting $b \neq 0, c = 0$ in the equation (8.10.16) of the derivation ∂. The Coble surface has a quasi-elliptic fibration f with a multiple singular fiber of type $\tilde{E}_7$. It is obtained from a Halphen surface H with singular fibers of type $\tilde{E}_7$, $\tilde{A}_1^*$ by blowing up the singular point of an irreducible fiber F. Note that if we blow up H at the singular point (and the infinitely near point) of the fiber of type $\tilde{A}_1^*$, then we obtain the previous Example 9.8.19. Thus, we can use the same pencil of sextic curves given in the previous Example. Since the point blown up of the Halphen surface depends on the choice of the fiber F, these Coble surfaces form an irreducible one-dimensional family.

Example 9.8.21 We give an example of a Coble surface in characteristic 2 with three boundary components B_1, B_2, B_3 and with the crystallographic basis of type $\tilde{E}_6 + \tilde{A}_2$ given in Table 8.14. It is obtained as a specialization of the one-dimensional family of Enriques surfaces of the same type given in Example 8.10.22 by putting $a = 0$ in the equation (8.10.7) of the derivation ∂_a. It has a unique elliptic fibration f with a multiple singular fiber of type $\tilde{E}_6$ which is obtained from a Halphen surface H with three singular fibers of type $\tilde{E}_6, \tilde{A}_2, \tilde{A}_0$ by blowing up the three singular points of the fiber F of type $\tilde{A}_2$. Thus, B_1, B_2, B_3 are the proper transforms of the three components of F. To get the Coble surface, we consider the following curves and points in $\mathbb{P}^2$:

$$\ell : x + y + z = 0,\ \ell_1 : y + z = 0,\ \ell_2 : x + z = 0,\ \ell_3 : x + y = 0,$$

$$Q_1 : y^2+z^2+xy+yz+zx = 0,\ Q_2 : x^2+z^2+xy+yz+zx = 0,\ Q_3 : x^2+y^2+xy+yz+zx = 0,$$

$$p_0 : [1,1,1],\ p_1 = [1,0,0],\ p_2 = [0,1,0],\ p_3 = [0,0,1],$$

$$p_{12} = [1,1,0],\ p_{13} = [1,0,1],\ p_{23} = [0,1,1].$$

Two conics Q_i and Q_j meet at p_{ij} with multiplicity 4, ℓ passes through three points p_{12}, p_{13}, p_{23}. The three lines ℓ_1, ℓ_2, ℓ_3 meet at p_0, each line ℓ_i tangents to Q_i at p_i and to Q_j, Q_k at p_{jk} where $\{i, j, k\} = \{1, 2, 3\}$. Now, blowing up p_{12}, p_{13}, p_{23} and their infinitely near points, we obtain the following configuration as in Figure 9.8.

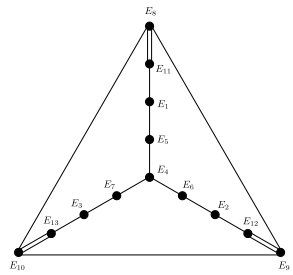


Fig. 9.8

The boundary component B_i is the proper transform of Q_i and E_4 in the Figure 9.8 is the proper transform of ℓ. For example, E_5, E_1, E_{11} are the exceptional curves over p_{12}. If we denote F_{ij} the exceptional curve of the last blow-up over p_{ij}, then $E_8 = 2F_{12} + \frac{1}{2}B_1 + \frac{1}{2}B_2$. The fibration f is induced from the pencil of sextics curves given by

$$\{C_{[s,t]} = s\ell^6 + tQ_1Q_2Q_3\}_{[s,t]\in\mathbb{P}^1}, \tag{9.8.5}$$

where ℓ, Q_i means their equations. The proper transform of ℓ_i is a (-1)-curve, denoted by R_i, and $B_i + 2R_i$ is a fiber of a quasi-elliptic fibration with a singular fiber of type $\tilde{E}_7$. Thus, we obtain a Coble surface V with three boundary components and with the crystallographic basis of type $\tilde{E}_6 + \tilde{A}_2$. The automorphism group $\mathrm{Aut}(\mathsf{V})$ is isomorphic to $\mathfrak{S}_3$ induced from the permutations of coordinates of $\mathbb{P}^2$. One can prove that a Coble surface with the crystallographic basis of type $\tilde{E}_6 + \tilde{A}_2$ is unique and isomorphic to this example.

Example 9.8.22 Here, we give an example of a Coble surface V in characteristic 2 with one boundary component B and with the crystallographic basis of type $\tilde{E}_6 + \tilde{A}_2$. It is obtained as a specialization of the one-dimensional family of Enriques surfaces of the same type given in Example 8.10.22 by putting $a = 1$ in the equation (8.10.7) of the derivation ∂_a. It is also obtained from the Halphen surface H with three singular fibers of types $\tilde{E}_6, \tilde{A}_2, \tilde{A}_0^*$, given in the previous Example 9.8.21, by blowing up the singular point of the fiber F' of type $\tilde{A}_0^*$. Here, we use the same notation as in the

previous example. Note that the member $C_{[1,1]}$ in the pencil of sextic curves given by the equation (9.8.5) has a node at p_0 and three singularities at p_{12}, p_{13}, p_{23}. The pre-mage of $C_{[1,1]}$ on the Halphen surface H is a singular fiber of type $\tilde{A}_0$. We blow up H at the point corresponding to the intersection p_0 of three lines ℓ_i. Then, we obtain a Coble surface V with one boundary component B which is the proper transform of the curve $C_{[1,1]}$. In Figure 9.8, E_8, E_9, E_{10} are the proper transforms of Q_1, Q_2, Q_3, and those of ℓ_1, ℓ_2, ℓ_3 are E_{11}, E_{12}, E_{13}. This Coble surface is also unique and $\mathrm{Aut}(\mathsf{V}) \cong \mathfrak{S}_3$.

Example 9.8.23 We give a one-dimensional family of Coble surfaces V in characteristic 2 with one boundary component B and with the crystallographic basis of type $\tilde{D}_8$ given in Figure 8.10.17. It is obtained as a specialization of the two-dimensional family of Enriques surfaces of type $\tilde{D}_8$, given in Example 8.10.69, by putting $b = 0$ in the equation (8.10.19) of the derivation $\partial_{a,b}$. Now, we give this Coble surface by blowing up $\mathbb{P}^2$. Consider the following curves in $\mathbb{P}^2$:

$$\ell : x = 0,\ Q : y^2 + xz = 0,\ C : x^6 + (y^2 + xz)^2(ax + z)z = 0\ (a \neq 0 \in k).$$

Note that C is a rational sextic curve with a node at $q = [0, 1, 0]$ and a singular point at $p = [0, 0, 1]$. The line ℓ meets Q at the point p with multiplicity 2 and C at p with multiplicity 4 and at q with multiplicity 2. The non-singular conic Q meets C only at the point p with multiplicity 12. Consider the pencil of sextic curves defined by

$$\{sx^2(y^2 + xz)^2 + t(x^6 + (y^2 + xz)^2(ax + z)z)\}_{[s,t]\in\mathbb{P}^1}.$$

By blowing up $\mathbb{P}^2$ at p (and their infinitely near points) and q, we get ten (-2)-curves forming the dual graph given in Figure 8.10.17. The proper transform of C is the boundary component B, that of Q is R_3 and that of L is R_{10}. The curves $R_1, R_2, R_4, \ldots, R_9$ are exceptional curves over p. Let E be the exceptional curve over q. Then, $2E + B$ is a fiber of an elliptic fibration with singular fiber of type $\tilde{E}_8$. The automorphism group $\mathrm{Aut}(\mathsf{V})$ is isomorphic to $\mathbb{Z}/2\mathbb{Z}$ generated by

$$[x, y, z] \to [x, \sqrt{a}x + y, ax + z].$$

By elementary, but long calculation, one can prove that Coble surfaces with the crystallographic basis of type $\tilde{D}_8$ form a one-dimensional irreducible family isomorphic to this example.

Example 9.8.24 Here, we give an example of a Coble surface V in characteristic 2 with four boundary components $B_1, \ldots, B_4$ and with the same crystallographic basis of type VIII. The dual graph is given by the following Figure 9.9:
The surface is obtained as a specialization of the one-dimensional family of Enriques surfaces of type VIII given in Example 8.10.28 by putting $a = 0$ in the equation (8.10.8) of the derivation ∂_a. In Figure 9.9, the ten vertices $E_1, \ldots, E_{10}$ are represented by (-2)-curves and $E_{11}, \ldots, E_{16}$ are represented by effective irreducible roots. Denote by $e_{11}, \ldots, e_{16}$ the six (-1)-curves satisfying

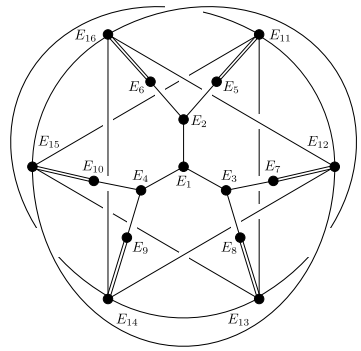


Fig. 9.9

$$E_{11} = 2e_{11} + \frac{1}{2}B_1 + \frac{1}{2}B_3,\ E_{12} = 2e_{12} + \frac{1}{2}B_1 + \frac{1}{2}B_2,\ E_{13} = 2e_{13} + \frac{1}{2}B_3 + \frac{1}{2}B_4,$$

$$E_{14} = 2e_{14} + \frac{1}{2}B_2 + \frac{1}{2}B_3,\ E_{15} = 2e_{15} + \frac{1}{2}B_1 + \frac{1}{2}B_4,\ E_{16} = 2e_{16} + \frac{1}{2}B_2 + \frac{1}{2}B_4.$$

By contracting e_{13}, e_{15}, e_{16} and B_4, we obtain a Halphen surface H with a multiple fiber $E_5 + E_7 + E_9 + 2(E_2 + E_3 + E_4) + 3E_1$ of type $\tilde{E}_6$ and a fiber $\bar{B}_1 + \bar{B}_2 + \bar{B}_3$ of type $\tilde{A}_2^*$ where $\bar{B}_i$ is the image of B_i on H. Then, by contracting $e_{11}, e_{12}, e_{14}, E_5, E_7, E_9, E_2, E_3, E_4$ successively, we obtain a projective plane $\mathbb{P}^2$. Denote by $\ell, \ell_1, \ell_2, \ell_3, Q_1, Q_2, Q_3$ the images of $E_1, E_6, E_8, E_{10}, B_1, B_2, B_3$, respectively. Then, ℓ, ℓ_i are lines and Q_i are non-singular conics. Let p_0, p_1, p_2, p_3 be the images of $e_{11}, e_{12}, e_{14}, B_4$ on $\mathbb{P}^2$. We may assume that

$$p_0 : [1, 1, 1],\ p_1 = [0, 1, 1],\ p_2 = [1, 0, 1],\ p_3 = [1, 1, 0].$$

Then, we have

$$\ell : x + y + z = 0,\ \ell_1 : y + z = 0,\ \ell_2 : x + z = 0,\ \ell_3 : x + y = 0.$$

An easy calculation shows that

$$Q_1 : x^2 + y^2 + z^2 + yz = 0,\ Q_2 : x^2 + y^2 + z^2 + zx = 0,\ Q_3 : x^2 + y^2 + z^2 + xy = 0.$$

Two conics Q_i and Q_j meet at p_k with multiplicity 3 ($\{i, j, k\} = \{1, 2, 3\}$) and intersect transversally at p_0. The Halphen fibration is induced from the pencil of sextics curves given by:

$$\{C_{[s,t]} = s\ell^6 + tQ_1Q_2Q_3\}_{[s,t]\in\mathbb{P}^1}. \tag{9.8.6}$$

This gives us a Coble surface V with four boundary components and with the crystallographic basis of type VIII. One can prove that the automorphism group

$\mathrm{Aut}(\mathsf{V})$ is isomorphic to $\mathfrak{S}_4$ by using the Mordell–Weil groups of the Jacobian fibrations of genus one fibrations on V. A Coble surface with the crystallographic basis of type VIII is unique and isomorphic to the surface from this example.

Bibliographical Notes

The notion of a Coble surface arises from the work of A. Coble on the group of birational transformation of the projective plane that leaves a rational curve of degree 6 invariants [123], [125]. Although not expressed explicitly by Coble, the group of such automorphisms can be interpreted as the group of biregular automorphisms of the blow-up of the plane with center at the ten nodes of the plane sextic. The first study of such surfaces which were appropriately called Coble surfaces was undertaken in unpublished notes by M. Miyanishi, which were based on his joint work with R. Miranda and P. Murthy during his stay at the University of Chicago in 1980. The important role played by Coble surfaces in the theory of automorphisms of rational surfaces was explained in an influential paper by M. Gizatullin [251]. A more general definition of a Coble surface as a rational surface V with $|-K_{\mathsf{V}}| = \emptyset$ and $|-2K_{\mathsf{V}}| \neq \emptyset$ was given in [195]. This paper contains a complete classification of Coble surfaces. The classical Coble surfaces constructed from rational ten-nodal plane sextics were studied in [104], where it was proved that they are the only rational surfaces whose automorphism group is represented in the corresponding hyperbolic Weyl group by a subgroup of finite index.

In this chapter, we study a particular class of Coble surfaces that are related to the theory of K3 and Enriques surfaces. Over the field of complex numbers, they are related to K3 surfaces with a 2-elementary Picard lattice classified by V. Nikulin [557]. We discussed the moduli space of complex Coble surfaces in terms of the periods of their K3-covers in Section 5.4 in Volume I.

In the case when the boundary of a Coble surface consists of only one irreducible component, the orthogonal complement of its canonical class is isomorphic to the Enriques lattice. This is one of the main ingredients of a connection between the theories of Enriques and Coble surfaces. The analog of the Enriques lattice for a Coble surface that we discuss in section 9.2 was communicated to us by S. Mukai. The fact that over $\mathbb{C}$ the Coble–Mukai lattice is isomorphic to the Enriques lattice seems new. Another proof of this fact is based on the theory of $\mathbb{Q}$-Gorenstein deformations of singular rational surfaces, was recently given by G. Urzùa [713].

The classification of self-projective rational plane curves of degree 6 was given in a paper of R. Winger [737] of 1916. The details were lacking and we provide them in section 9.4. The Coble surfaces isomorphic to the blow-up of $\mathbb{P}^2$ at singular points of some of these rational nodal sextics were studied in [181] and [180]. The observation that many examples of rational nodal sextics from Winger's list gives examples of positive semi-definite real ternary forms of degree 6 which are not sums of squares seems new. There is extensive research on this topic related to Hilbert's 17th Problem, for which we refer to [616].

The connection between quartic symmetroids and rational nodal sextics is based on the work of Coble [123], [125]. We discussed it at the end of Section 7.5. The fact that this connection allows one to classify quartic symmetroids with projective symmetry seems new. Some of the examples from our list give examples of positive semi-definite quartic quaternary real forms which are not sums of squares.

The fact that the groups of automorphisms of a general Coble and a general Enriques surfaces are isomorphic was proven by using the Global Torelli Theorem for K3 surfaces in section 5.5 of Volume I. Its extension to arbitrary characteristic was first given in [104].

The group of automorphisms of a general Hessian quartic surface was first studied in [187]. It was proven there that the ten projection involutions descend to the quotient. The fact that these involutions generate the automorphism group of the quotient was proven by I. Shimada [672] and S. Mukai. The study of Coble surfaces of Hessian type was first undertaken in [181] and [180].

The automorphism groups of Enriques and Coble surfaces of Hessian type with the octahedron symmetry were described in [8].

The classification of complex Coble surfaces with finite automorphism groups, based on an analog of Nikulin's R-invariant was given in an unpublished work of S. Mukai. Very recently, a classification of Coble surfaces with finite automorphism group has been finished by the second author [419] in characteristic $p \neq 2$, and by T. Katsura and the second author [370] in characteristic 2. We give examples of such surfaces in this section. Some of them were communicated to us by S. Mukai.

Chapter 10
Supersingular K3 Surfaces and Enriques Surfaces

The canonical cover of an Enriques or a Coble surface in characteristic $p = 2$, when it is inseparable and the surface has only rational double points, is birationally isomorphic to a supersingular K3 surface. In this chapter, after introducing some basic facts from the theory of supersingular K3 surfaces, we will study some constructions of Enriques and Coble surfaces with interesting groups of automorphisms as quotients of a supersingular K3 surface by a rational vector field.

10.1 Supersingular K3 Surfaces

In this section, we recall some general facts about supersingular K3 surfaces.

Recall Definition 1.1.10: a K3 surface Y is called Shioda-supersingular if $b_2(Y) = 22$. Also, in Remark 0.10.27, we defined a supersingular variety satisfying two equivalent conditions: the height of the formal Brauer group is infinite or the slopes of the Frobenius acting on the crystalline cohomology $H^2(Y/W)$ are equal to 1. The Igusa-Artin–Mazur equality (0.10.65) implies that a Shioda-supersingular K3 surface is always supersingular. The converse follows from the Tate Conjecture for K3 surfaces (see Proposition 0.10.28 in Volume I). Recently, the conjecture was proven in [500], [483] if $p \neq 2$ and [390, Appendix A] if $p = 2$. A partial explanation for the name supersingular is the following property (however weaker than the supersingularity) which defines supersingular elliptic curves and supersingular Enriques surfaces.

Proposition 10.1.1 *The Frobenius endomorphism* $\mathbf{F}$ *acts trivially on* $H^2(Y, \mathcal{O}_Y)$.

Proof It follows from Section 0.10 (after Example 0.10.18) that the Dieudonne module $\mathbb{D}(\Phi^2_{Y/\Bbbk})$ of the formal Brauer group is isomorphic to $H^2(Y, W\mathcal{O}_Y)$. Since $\Phi^2_{Y/\Bbbk} \cong \hat{\mathbb{G}}_a$, Example 0.1.19 shows that $\mathbf{F}(H^2(Y, W\mathcal{O}_Y)) = 0$. In particular $\mathbf{F}(H^2(Y, \mathcal{O}_Y)) = 0$. □

Recall that, in Section 0.10 in Volume I, we assigned to any variety Y the crystalline cohomology $H^i(Y/W)$ and defined a structure of a crystal or a Dieudonne W-module. This structure consists of a σ-linear endomorphism $\Phi : H^i(Y/W) \to H^i(Y/W)$, i.e. an endomorphism of abelian groups satisfying $\Phi(wx) = \sigma(w)\Phi(x)$, where $\sigma : W \to W$ is the automorphism of Frobenius acting on the ring of Witt vectors W.

Definition 10.1.2 A *K3-crystal* is a crystal (H, B, Φ), where H is a W-module, $B : H \otimes H \to W$ is a symmetric bilinear form, and $\Phi : H \to H$ is a σ-linear endomorphism satisfying the following properties:

(i) H is a free W-module of rank 22;
(ii) B is a perfect pairing, i.e. the map $H \to H^\vee$ defined by B is an isomorphism.
(iii) For $x, y \in H$ we have $B(\Phi(x), \Phi(y)) = p^2(x, y)$.
(iv) The rank of $\Phi \otimes \Bbbk : H \otimes \Bbbk \to H \otimes \Bbbk$ is equal to 1.

Theorem 10.1.3 *Let Y be a supersingular $K3$ surface over an algebraically closed field $\Bbbk$. Then, the crystalline cohomology $H^2(Y/W)$ is a $K3$-crystal.*

Proof (i) By the universal coefficient formula (0.10.43), $H = H^2(Y/W)$ has no torsion. By (0.10.42), rank $H^2(Y/W) = b_2(Y) = 22$.

(ii) This follows from the Poincaré duality for crystalline cohomology (0.10.41).

(iii) This follows from the fact that the Frobenius acts on $H^4(Y/W)$ by multiplication by p^2 (see Example 0.10.10).

(iv) Since $H^2(Y/W)$ has no torsion,

$$H^2(Y/W) \otimes_W \Bbbk \cong H^2_{\mathrm{DR}}(Y/\Bbbk).$$

Under this isomorphism, the map $\Phi \otimes \Bbbk$ is the action of the Frobenius on the de Rham cohomology. We use the Hodge versus de Rham spectral sequence (0.10.35). It defines a filtration $0 \subset F_2 \subset F_1 \subset F_0 = H^2_{\mathrm{DR}}(Y/\Bbbk)$ with

$$F_2 = H^0(Y, \Omega^2_{Y/\Bbbk}),\ F_1/F_2 = H^1(Y, \Omega^1_{Y/\Bbbk}),\ F_0/F_1 = H^2(Y, \mathcal{O}_Y).$$

Since $\mathbf{F}$ acts trivially on differential forms (because $\mathbf{F}(da) = d(a^p) = 0$), we see that $\mathbf{F}^*(F_1) = \{0\}$. This shows that the rank of $\mathbf{F}^*$ is less than or equal to one.

Suppose that $\mathbf{F}^*$ is the zero map. This means that $\Phi(H) \subset pH$. It follows from (0.10.75) in Volume I that the first Chern class map $c_1 : \mathrm{NS}(Y) \otimes \mathbb{Z}_p \to H^2_{\mathrm{fl}}(Y, \mathbb{Z}_p) \to H^2(Y/W)$ is injective and its image lies in the Tate module

$$T_H = \{x \in H^2(Y/W) : \mathbf{F}(x) = px\} \subset H^2(X, W\Omega^{\geq 1}_X).$$

(The latter is a $\mathbb{Z}_p$-module, but not a W-module.) Let us prove that

$$c_1(\mathrm{NS}(Y) \otimes W) = T_H.$$

First of all, the rank of T_H as a $\mathbb{Z}_p$-module cannot be greater than 22. In fact, suppose we have $n > 22$ linearly independent elements in T_H, then they must be linearly

independent over W. Indeed, in any linear combination $\sum \alpha_i x_i = 0$ with coefficients in W, we may assume that there exists a coefficient, say α_1, which is not divisible by p. Applying the Frobenius $\mathbf{F}$ and subtracting, we obtain that a new linear combination with $\alpha_1 = 0$ and all coefficients divisible by p. Continuing in this way, we obtain that the coefficients must be equal to zero. Since the rank of $H^2(Y/W)$ is equal to 22, the rank of T_H over $\mathbb{Z}_p$ must be equal to 22. The fact that $H^2(Y/S)/c_1(\mathrm{NS}(Y) \otimes \mathbb{Z}_p)$ is torsion-free gives the claimed equality $c_1(\mathrm{NS}(Y) \otimes W) = T_H$.

Let $\Phi' = p^{-1}\Phi : H \to H$. Restricting Φ' to T_H, we obtain the identity map. Since T_H is of finite index in H, this would imply that Φ' is the identity on H (see the proof of Proposition 10.1.4 below). Therefore, $T_H = H$, and $c_1(\mathrm{NS}(Y) \otimes W) = H$. It follows from property (ii) that $\mathrm{NS}(Y)$ is a unimodular even lattice. This contradicts Proposition 0.8.8 that claims that there are no hyperbolic unimodular lattices of rank 22. □

Recall that a non-degenerate quadratic lattice is called p-elementary if its discriminant group is a p-elementary abelian group.

Proposition 10.1.4 *The Picard lattice* $\mathrm{Pic}(Y) = \mathrm{NS}(Y)$ *is a p-elementary lattice with discriminant equal to* $-p^{2\sigma_0}$.

Proof As was explained in Remark 0.10.31 in Volume I, the Poincaré duality for l-adic cohomology implies that $\mathrm{NS}(Y) \otimes \mathbb{Q}_l = \{0\}$ for $l \neq p$. Since $\mathrm{NS}(Y)$ is an even hyperbolic lattice of rank 22, we obtain that its discriminant is equal to $-p^s$ for some s. It follows from the proof of Theorem 10.1.3 that the image of $c_1 : \mathrm{NS}(Y) \to H^2_{\mathrm{fl}}(Y, \mathbb{Z}_p) \to H^2(Y/W)$ is W-submodule of finite index d that coincides with the Tate module T_H. This implies that $\mathrm{discr}(NS(Y) \otimes W) = d^2\,\mathrm{discr}(H^2(Y/W))$. By property (ii) of a K3-crystal, $\mathrm{discr}(H^2(Y/W)) = 1$. Thus, we obtain that $\mathrm{discr}(\mathrm{NS}(Y) \otimes W) = \mathrm{discr}(\mathrm{NS}(Y))$ is a square, hence s is even.

It remains to prove that $\mathrm{NS}(Y)$ is p-elementary. Since Y is supersingular, the Kummer sequence (0.10.61) shows that $\mathrm{NS}(Y) \otimes \mathbb{F}_p = H^2_{\mathrm{fl}}(Y, \mathbb{Z}_p)/pH^2_{\mathrm{fl}}(Y, \mathbb{Z}_p)$. This implies that $\mathrm{NS}(Y)/p\,\mathrm{NS}(Y) \cong H^2_{\mathrm{fl}}(Y, \mu_p)$. Using (0.10.70), we have

$$H^2_{\mathrm{fl}}(Y, \mu_p) \cong H^1_{\text{ét}}(Y, \mathcal{O}_Y^*/\mathcal{O}_Y^{*p}) \overset{dlog}{\hookrightarrow} H^2(Y, \Omega_Y^{\geq 1}) \hookrightarrow H^2_{\mathrm{DR}}(Y/\mathbb{k}). \tag{10.1.1}$$

The last inclusion follows from the Hodge versus de Rham spectral sequence (0.10.36). It follows from the inclusion $\mathrm{NS}(Y)/p\,\mathrm{NS}(Y) \hookrightarrow H^2(Y/W)/pH^2(Y/W)$ that the quotient of $H^2(Y/W)_{\mathbb{Z}_p}$ by $\mathrm{NS}(Y)_{\mathbb{Z}_p}$ has no torsion.

Let

$$K = \{x \in H^2(Y/W) : \Phi^n(x) \in p^n H^2(Y/W), \text{ for all } n \geq 0\} \subset H^2(Y/W).$$

It contains T_H, and hence, it contains $c_1(\mathrm{NS}(Y)_{\mathbb{Z}_p})$. Obviously, the rank of K over W is equal 22. Consider the endomorphism $\phi = p^{-1} \circ \Phi$. We have 22 linearly independent elements x_i in K coming from T_H such that $\Phi(x_i) = x_i$. Suppose $e_1, \ldots, e_{22}$ is a W-basis of K, and let C be the matrix expressing x_i in terms of e_i' and let A be the matrix of ϕ in the basis $e_1, \ldots, e_{22}$. Then, $C = \sigma(C) \cdot A$. We

know that σ is an automorphism of W, hence the (p)-adic evaluation of $\det(C)$ and $\det(\sigma(C))$ are equal. This implies that $\det(A) \in W^\times$, hence $A \in \mathrm{GL}(22, W)$. Since the homomorphism $r : \mathrm{GL}(22, W) \to \mathrm{GL}(22, \Bbbk)$ is surjective, we can find an invertible matrix C_0 such that $r(A) = C_0 \cdot \mathbf{F}(C_0)^{-1}$ and then lift it to an invertible matrix $B \in \mathrm{GL}(22, W)$ such that $A = B \cdot \sigma(B)^{-1}$. This implies that the matrix of $\phi = p^{-1}\mathbf{F}$ in the basis $e_1, \ldots, e_{22}$ is the identity too, hence $K = T_H = c_1(\mathrm{NS}(Y) \otimes \mathbb{Z}_p)$.

Suppose we prove that $\Phi^n(H) \subset p^{n-1}H$ for all $n \geq 1$. Then, this would imply that $pH \subset K$ and hence $pH \subset \mathrm{NS}(Y) \otimes W$. Passing to the dual W-module, we obtain $\mathrm{NS}(Y)^\vee \subset (pH)^\vee = p^{-1}H^\vee = p^{-1}H$. This shows that $p\,\mathrm{NS}(Y)^\vee \subset H$. Since $p\,\mathrm{NS}(Y)^\vee$ is invariant with respect to Φ and belongs to K, we obtain that $p\,\mathrm{NS}(Y)^\vee \subset \mathrm{NS}(Y)$. This means that $\mathrm{NS}(Y)$ is a p-elementary lattice.

So, it remains to prove that $\mathbf{F}^n(H) \subset p^{n-1}H$ for all $n \geq 1$. The proof is by induction on n. It is obviously true for $n = 1$. Suppose it is true for $n = k$. Let $\phi = p^{-k+1}\Phi$. Then, $\Phi(x) = px$ if and only if $\phi(x) = px$. This implies that $\phi^N(x) = p^{N+1}x$ and taking N to go to ∞, we see that $\bar{\phi} = \phi \otimes \Bbbk$ is a nilpotent endomorphism of $H \otimes \Bbbk$. Since it commutes with $\bar{\Phi} = \Phi \otimes \Bbbk$ which is of rank 1 by property (iv) of a K3-crystal, we obtain that $\bar{\phi} \circ \bar{\Phi} = 0$. Thus, $\phi \circ \Phi^k(x) = p^{-k+1}\Phi^{k+1}(x) \in pH$, hence $\Phi^{k+1}(x) \in p^kH$ and the assertion is true for $n = k + 1$. □

We have already introduced the Artin invariant of a supersingular surface in Remark 0.10.31 from Volume I, but, in view of its importance in the sequel, let us recall it.

Definition 10.1.5 The *Artin invariant* of a supersingular K3 surface Y is the number σ_0 such that $D(\mathrm{NS}(Y)) \cong (\mathbb{Z}/p\mathbb{Z})^{2\sigma_0}$.

Thanks to the work of Nikulin [556],[557] and Rudakov and Shafarevich [628], [627], the classification of p-elementary hyperbolic lattices is known.

Theorem 10.1.6 *Let M be a p-elementary even hyperbolic lattice of rank $n > 2$.*

1. *If $p \neq 2$, the isomorphism class of M is uniquely determined by the rank n and the order 2^s of its discriminant group. Such a lattice exists if and only if:*
   - $n \equiv 0 \mod 2$,
   - $n \equiv 2 \mod 4$ *if* $s \equiv 0 \mod 2$,
   - $p \equiv (-1)^{\frac{1}{2}(n-2)} \mod 4$ *if* $s \equiv 1 \mod 2$,
   - $n > s > 0$ *if* $n \equiv 2 \mod 8$.
2. *If $p = 2$, the isomorphism class of M is uniquely determined by the rank $n > 4$, the order 2^s of its discriminant group and the type. A lattice is of type* I *if the discriminant quadratic form takes values in $\mathbb{Z}/2\mathbb{Z}$, and the remaining lattices are of type* II. *A lattice of type* I *exists if and only if:*
   - $s \equiv 0 \mod 2, n \equiv 2 \mod 4$ *(the discriminant quadratic space is isomorphic to a regular quadratic space $(\mathbb{F}_2^s, q)$, where q is of even type);*
   - $n > s > 0$ *if* $n \not\equiv 2 \mod 8$.

   A lattice of type II *exists if and only if:*

- $s > 0, n \equiv s \mod 2$;
- *if* $s = 2$, *then* $n \not\equiv 6 \mod 8$,
- *if* $s = 1$, *then* $n \equiv 1 \mod 8$ *or* $n \equiv 3 \mod 8$.

Proposition 10.1.7 *The Néron–Severi lattice of a supersingular K3 surface over a field of characteristic* 2 *is of type* I.

Proof Since there is only one isomorphism class of a 2-elementary hyperbolic lattice of given type, rank and discriminant, we can exhibit all of them which are of Type II. Since in our case $n = 22 \equiv 6 \mod 8$, it follows from Theorem 10.1.6 that $s \neq 2$ and $s = 2\sigma_0$ is even. The lattice

$$M = \mathsf{U} \oplus \mathsf{A}_1^{\oplus 2\sigma_0 - 2} \oplus \mathsf{D}_{22-2\sigma_0}. \tag{10.1.2}$$

is 2-elementary with $n = 22$ and $s = 2\sigma_0 \neq 2$. It must be of type II and every 2-elementary hyperbolic lattice of type II must be isomorphic to M. Let f be a primitive isotropic vector from the summand U. Applying an element from W_Y^{nod}, we may assume that it is nef. Hence it defines a genus one fibration. The other summands show that the fibration has one reducible fiber of type $\tilde{D}_{22-2\sigma_0}$ and $2(\sigma_0 - 1)$ fibers of type $\tilde{A}_1$ or $\tilde{A}_1^*$.

Suppose f is a an elliptic fibration. Adding up the Euler-Poincar'e characteristics of fibers, we obtain that the sum is greater than 24 if $\sigma_0 > 2$. Thus, we may assume that $\sigma_0 = 2$. Recall that the isomorphism class of any even 2-elementary hyperbolic lattice is uniquely determined by the discriminant quadratic form and its type I or II (Nikulin [557], Theorem 4.3.2). This implies that $M \cong \mathsf{U} \oplus E_8^{\oplus 2} \oplus A_1^{\oplus 4}$. Again, by considering a primitive isotropic vector f in the summand U, we have a genus one fibration with two fibers of type $\tilde{E}_8$ and four fibers of type $\tilde{A}_1$. We assume that this fibration is also an elliptic fibration. Then, the sum of the Euler–Poincaré characteristics of fibers is greater than 24, giving a contradiction.

So, we may assume that f defines a quasi-elliptic fibration. Now, we use a nice argument from [628, Proposition, §5]. Passing to the jacobian fibration, we may assume that it has a section which we fix. Let $\mathfrak{C}$ be the curve of cusps. It intersects each fiber of type $\tilde{A}_1^*$ at its cusp. Write $[\mathfrak{C}] = c_u + \sum_{i=1}^{2\sigma_0-2} a_i[R_i] + c_d$, where $c_u \in \mathsf{U}, c_d \in \mathsf{D}_{22-2\sigma_0}$, $a_i \in \mathbb{Z}$ and R_i is the component of the fiber of type $\tilde{A}_1^*$ that does not intersect the zero section. Intersecting both sides with $[R_1]$, we obtain $1 = -2a_1$, a contradiction. □

Corollary 10.1.8 *The Artin invariant* σ_0 *satisfies*

$$1 \leq \sigma_0 \leq 10.$$

Proof Since the rank of $\mathrm{NS}(Y)$ is equal to 22, we have $\sigma_0 \leq 11$. If $\sigma_0 = 11$, then $\mathrm{NS}(Y)(\frac{1}{p})$ is a unimodular lattice. In the case where $p \neq 2$, this lattice is even. Since there are no even unimodular hyperbolic lattices of rank 22, we get a contradiction. In the case where $p = 2$, by the classification of indefinite unimodular lattices, $\mathrm{NS}(Y) = \mathsf{A}_1(-1) \oplus \mathsf{A}_1^{\oplus 21}$ which is of type II. This contradicts Proposition 10.1.7. For the same reason, we have $\sigma_0 > 0$. □

Corollary 10.1.9 *Let Y be a supersingular K3 surface in characteristic 2 with Artin invariant σ_0. Then, $\mathrm{NS}(Y)$ is isomorphic to one of the following lattices:*

$$1 \le \sigma_0 \le 5 : \mathsf{U} \oplus \oplus_{i=1}^{\sigma_0} \mathsf{D}_{4n_i}, \quad \sum_{i=1}^{\sigma_0} n_i = 5,$$

$$10 > \sigma_0 > 5 : \mathsf{U} \oplus \oplus_{i=1}^{10-\sigma_0} \mathsf{D}_{4n_i}^{\vee}(2), \quad \sum_{i=1}^{10-\sigma_0} n_i = 5,$$

$$\sigma_0 = 10 : \mathsf{E}_{10}(2) \oplus M_{12} \cong \mathsf{U}(2) \oplus \mathsf{D}_{20}^{\vee}(2),$$

where $M_{12} = \langle e_1, \dots, e_{12}, \frac{1}{2}(e_1 + \cdots + e_{12})\rangle,\ e_i \cdot e_j = -2\delta_{ij}$.

Remark 10.1.10 Let M be a p-elementary hyperbolic lattice of rank $2r$ and discriminant group of rank $2t$. For $p = 2$, we assume that M is of type I. Since M is p-elementary, $pM^{\vee} \subset M$ and hence $M^{\vee}(p)$ is a lattice. Let G_M be the Gram matrix of some basis of M, so that pG_M^{-1} is the Gram matrix $G_{M^{\vee}(p)}$ of the dual basis of $M^{\vee}$. Then, $\det G_{M^{\vee}(p)} = p^{2r} \det G_{M^{-1}} = -p^{2r-2t}$. Since $p(pG_M^{-1})^{-1} = G_M$ is integral, $M^{\vee}(p)$ is p-elementary. Now, assume that p is odd. Then, for any $x \in M^{\vee}$, $px \in M$ and hence $\langle x, x\rangle$ is even. Similarly, in the case where $p = 2$, it follows from the assumption M being of type I that $M^{\vee}(p)$ is even and of type I. Thus, $M^{\vee}(p)$ is a p-elementary hyperbolic lattice of rank r with the discriminant group of rank $2r - 2t$, and is of type I for $p = 2$.

Applying this to the case when $M = \mathrm{NS}(Y)$ for a supersingular K3 surface of Artin invariant σ_0, we find a duality between families of supersingular K3 surfaces of Artin invariant σ_0 and $11 - \sigma_0$ [420]. One of the features of this duality is that, if a primitive isotropic vector $v \in \mathrm{NS}(Y)$ defines a jacobian genus one fibration on Y, then the corresponding vector in $\mathrm{NS}(Y')^{\vee}(2)$ defines a genus one fibration without a section on the dual K3 surface.[1] For example, it is known that every genus one fibration on a supersingular K3 surface with Artin invariant $\sigma_0 = 1$ is jacobian [215, Theorem 1.2]. Thus, any genus one fibration on a supersingular K3 surface with Artin invariant $\sigma_0 = 10$ has no sections. Passing to the jacobian fibration, we obtain a supersingular K3 surface with Artin invariant 9 (see [461, Proposition 3.8])

Although we are not going to use it, let us mention the following fundamental fact due to Rudakov and Shafarevich [627].

Theorem 10.1.11 *A supersingular K3 surface in characteristic 2 is unirational.*

It is not known whether it is true in other positive characteristics (the proof of this claim in [461] contains a gap). In the case where $p > 2$, the unirationality is known only the following cases $p = 3, \sigma_0 \le 6$ [627] and $p = 5, \sigma_0 \le 3$ [594]).

Next, we discuss the moduli problem of supersingular K3 surfaces.

[1] It follows from Theorem 4.7.2 in Volume I that the p-torsion part of the Tate–Shafarevich group of a jacobian K3 surface is killed by p and hence a non-jacobian genus one fibration on a supersingular K3 surface has always a p-section.

We know from Theorem 1.4.4 in Volume I that a K3 surface has no non-zero regular vector fields. The theory of local deformations from Section 5.11 in Volume I implies that the functor $\underline{\mathrm{Def}}_{Y/\Bbbk}$ is pro-representable with tangent space $H^1(Y, \Theta_{Y/\Bbbk}) \cong \Bbbk^{20}$. The formal local universal deformation scheme is isomorphic to the algebra $R = \Bbbk[[t_1, \dots, t_{20}]]$. If we choose an invertible sheaf $\mathcal{L} \not\cong \mathcal{O}_Y$, then one can consider the local deformation functor $\underline{\mathrm{Def}}_{Y/\Bbbk,\mathcal{L}}$ for the pair $(Y, \mathcal{L})$ and prove that it is pro-representable and the forgetting morphism $\underline{\mathrm{Def}}_{Y/\Bbbk,\mathcal{L}} \to \underline{\mathrm{Def}}_{Y/\Bbbk}$ is a closed embedding defined by one equation (the latter because $h^2(\mathcal{O}_Y) = 1$). A theorem of Deligne from [155] asserts that this equation is non-trivial (i.e. it is not divisible by p) and the local versal deformation space is a formal local scheme of dimension 19. More generally, one can choose a finite set Σ of invertible sheaves spanning a subspace of dimension r in the image of $c_1 : N = \mathrm{Pic}(Y) \to H^1(Y, \Omega^1_Y)$, defined by the first Chern class map $c_1 : \mathrm{NS}(Y) \to H^2_{\mathrm{DR}}(Y/\Bbbk)$, and to construct the local versal deformation of the functor $\underline{\mathrm{Def}}_{Y/\Bbbk,\Sigma}$. Its dimension is less than or equal to $20 - r$ (see Section 5.11). Thus

$$20-r = \dim H^1(Y, \Omega^1_Y)/c_1(N) \le \dim F^1 H_{\mathrm{DR}}(Y)/c_1(N) \le \dim H^2(Y/W)/c_1(N)-1.$$

However, for a 2-elementary lattice N we have

$$\dim H^2(Y/W)/c_1(N) \otimes \Bbbk = \dim H^2(Y/W)/c_1(N) \otimes W = \sigma_0.$$

This gives $20 - r \le \sigma_0 - 1$.

A more delicate argument using the formal Brauer groups proves that, in fact, we have the equality [628, §9].

Theorem 10.1.12 *Let $\mathcal{X} \to \mathrm{Spec}\, R$ be the versal formal deformation of a supersingular K3 surface Y. Let N be a p-elementary sublattice of $\mathrm{NS}(Y)$ of rank 22 and discriminant $-p^{2\sigma_0}$. Then, the formal versal deformation of the functor $\underline{\mathrm{Def}}_{X,\Sigma}$, where Σ is a basis of N, is a smooth subscheme of $\mathrm{Spec}\, R$ of dimension $\sigma_0 - 1$.*

Let us discuss the global moduli problem for supersingular K3 surfaces. First we fix one of p-elementary lattices N (of type I if $p = 2$) and discriminant $-p^{2\sigma_0}$ and consider a lattice N polarization of Y as defined in Section 5.2. We also use the notion of an ample polarization and the notion of a family of lattice N polarized K3 surfaces.

Theorem 10.1.13 *Let $\underline{K}_N$ be the moduli stack of families of lattice N polarized K3 surfaces. Then, $\underline{K}_N$ is representable by a smooth locally separated algebraic space K_N of dimension $\sigma_0 - 1$.*

Proof Let $\underline{K}_N$ be the moduli stack of families of lattice N polarized K3 surfaces. It is proven in [575, Proposition 2.3] that for any families $(\mathcal{Y}/T, \phi : N \to \mathrm{Pic}(\mathcal{Y}/T))$ and $(\mathcal{Y}'/T, \phi' : N \to \mathrm{Pic}(\mathcal{Y}'/T))$ of lattice N polarized K3 surfaces over an algebraic base space T, the functor $\underline{\mathrm{Isom}}_{(\mathcal{Y}/T,\phi),(\mathcal{Y}'/T,\phi')}$ is represented by a locally closed algebraic subspace of T. Together with Lemma 8.1.8 from [581], this implies that $\underline{K}_N$ is an algebraic (or Artin) stack. Since $H^0(Y, \Theta_{Y/\Bbbk}) = \{0\}$ and $\mathrm{Aut}_T((\mathcal{Y}/T, \phi)) = \{1\}$

[628, §8, Proposition 3], we obtain that $\underline{K}_N$ is represented by an algebraic space [581, Corollary 8.5.3]. Applying Theorem 10.1.12, we obtain that the stack $\underline{K}_N$ is smooth and of dimension $\sigma_0 - 1$. □

So, now we may consider families of lattice N amply polarized supersingular K3 surfaces $(f : \mathcal{Y} \to T, j : N_T \to \mathrm{Pic}_{\mathcal{Y}/T})$, where, by definition, $j_t(N) \subset \mathrm{Pic}_{\mathcal{X}_t/\Bbbk}$ always contains an ample class. This allows one to construct a Deligne–Mumford stack $\mathcal{K}_N^\circ$ of ample lattice N polarized supersingular K3 surfaces. We have seen already that $\mathrm{NS}(Y)_W := \mathrm{NS}(Y) \otimes W$ is a free submodule of finite index p^{σ_0} in $H^2(Y/W)$. Let Y be a lattice N-polarized supersingular K3 surface. Let $N_W = N \otimes W \cong W^{2\sigma_0}$. The inclusions $N_W \subset \mathrm{NS}(Y)_W \subset H^2(Y/W) \subset \mathrm{NS}(Y)_W^\vee \subset N_W^\vee$ imply (as in the case of quadratic lattices) that $H^2(Y/W)/N_W$ is an isotropic subspace of dimension p^{σ_0} in the quadratic space $D(N) \otimes \Bbbk$ of dimension $2\sigma_0$. This defines the *period* of lattice N-polarized supersingular K3 surface Y, a point in the orthogonal Grassmann variety $\mathrm{OG}(\sigma_0, 2\sigma_0)$ of maximal isotropic subspaces in $D(\mathrm{NS}(Y)) \otimes \Bbbk$. We also have shown that $\mathbf{F}_{Y/W}$ defines a linear map $H^2(Y/W) \otimes \Bbbk \to H^2(Y/W) \otimes \Bbbk$ of rank 1. This shows that the period L satisfies the property $\dim L \cap \mathbf{F}^*(L) = \sigma_0 - 1$. We denote by Ω_N the open subset of $\mathrm{OG}(\sigma_0, 2\sigma_0)$ consisting of periods and call it the *period space*. The standard computations show that $\dim \mathrm{OG}(\sigma_0, 2\sigma_0) = \frac{1}{2}\sigma_0(\sigma_0 - 1)$. The period space is a closed smooth subvariety of dimension $\sigma_0 - 1$.

Let V be a quadratic space over $\mathbb{F}_p$ of dimension $2\sigma_0$. A maximal isotropic subspace L of V is called a *characteristic subspace* if $L \cap \mathbf{F}^*(L)$ is a hyperplane in L.

For any $\mathbb{F}_p$-algebra A, let $\mathcal{M}_V(A)$ be the set of direct summands $\mathcal{L} \subset V \otimes_{\mathbf{F}_p} A$ such that $\mathbf{F}^*(\mathcal{L}) \cap \mathcal{L}$ is a direct summand of $\mathcal{L}$ of rank $\sigma_0 - 1$.

We refer for the proof of the next proposition to [576, Proposition 4.6], [628, §10, Theorem 1].

Proposition 10.1.14 *The functor $\mathcal{M}_V : A \mapsto \mathcal{M}_V(A)$ is representable by a smooth projective variety M_V of dimension $\sigma_0 - 1$ over $\mathbb{F}_p$. The variety $M_V \otimes \bar{\mathbb{F}}_p$ consists of two disjoint copies interchanged by the Frobenius isomorphism. Each copy is a unirational variety.*

Example 10.1.15 If $\sigma_0 = 1$, $M_V \cong \mathrm{Spec}\,\mathbb{F}_{p^2}$. Hence $M_V \otimes \bar{\mathbb{F}}_p$ is isomorphic to the disjoint union of two copies of $\mathrm{Spec}\,\bar{\mathbb{F}}_p$. If $\sigma_0 = 2$, $M_V \cong \mathbb{P}^1_{\mathbb{F}_{p^2}}$ and $M_V \otimes \bar{\mathbb{F}}_p$ is isomorphic to the disjoint union of two copies of $\mathbb{P}^1_{\mathbb{F}_p}$ (see [575, Examples 4.7]). The Frobenius morphism exchanges the two copies.

If $\sigma_0 = 3$, M_V is isomorphic to the Fermat surface

$$x^{p+1} + y^{p+1} + z^{p+1} + w^{p+1} = 0$$

in $\mathbb{P}^3_{\mathbb{F}_{p^2}}$ (see [628, §10]).

We apply this to our case and take $V = N_0$, where

$$N_0 := pN^\vee/pN \cong D(N) \cong \mathbb{F}_p^{2\sigma_0}$$

and set

$$\Omega_N := M_{N_0}.$$

By the above, this is a smooth projective subvariety (not geometrically connected) of dimension $\sigma_0 - 1$ over $\mathbb{F}_p$.

Next, we discuss the extension of the Global Torelli Theorem for complex K3 surfaces to the case of supersingular K3 surfaces. Recall that the fine moduli space of complex marked K3 surfaces exists in the category of analytic spaces but not separated. The reason is that the marking may not be ample. The same reason explains that our space K_N is only locally separated. Since Ω_N is separated, we have to deal first with this problem.

Let W_N be the (-2)-reflection group of N and let us choose an open fundamental domain C_N for the action of W_N on the positive convex cone $V(N_\mathbb{R})^+$ in $N_\mathbb{Q}$. For each characteristic subspace K in N_0, we have an over-lattice $N \subset N_K$ and C_K is equal to the union of fundamental chambers of W_{N_K} in $N_\mathbb{Q}$. Since the number of over-lattices is finite, the number of such chambers in C_N is finite too. Now, suppose we have a family of characteristic subspaces $K_T \in \mathcal{M}_{N_0}(T)$. Let

$$N(t) := N_{K(t)} := \{x \in N_\mathbb{Q} : px \in N, \bar{px} \in K_t\}$$

be the over-lattice of N corresponding to K_t, where the bar denotes the coset of modulo pN. Following Ogus, the *ample cone of* K_T is a choice $\alpha(t)$ for each $t \in T$ of a fundamental chamber of $W_{N(t)}$ lying in C_N. We require that $\alpha(t) \subset \alpha(t')$ whenever t specializes to t'.

Let $\widetilde{\mathcal{M}}_{N_0}$ be the functor that assigns to each T the set of pairs (K_T, σ_0) that consist of a choice of $K_T \in \mathcal{M}_{N_0}(T)$ and an ample cone of K_T.

We have the following proposition (1.16) from [575]:

Proposition 10.1.16 *The functor $\widetilde{\mathcal{M}}$ is represented by a $\Bbbk$-scheme $\tilde{\Omega}_N$ which is locally of finite type and the natural forgetting map $\widetilde{M}_{N_0} \to M_{N_0} = \Omega_N$ is étale and surjective.*

Now, suppose we have a family $(\mathcal{Y}/T, \phi)$ of N-polarized K3 surfaces. For each $t \in T$, we have an ample cone $\mathrm{Amp}(\mathcal{Y}_t)$ in $\mathrm{NS}(\mathcal{X}_t)_\mathbb{Q} = N(t)_\mathbb{Q}$. We define the period map

$$\mathfrak{p}_N : \mathsf{K}_N \to \tilde{\Omega}_N$$

by assigning to a family $(\mathcal{Y}/T, \phi)$ the characteristic subspace $K_T \in \mathcal{M}_N(T)$ and its ample cone which is the image of $\prod_{t \in t} \mathrm{Amp}(\mathcal{Y}_t)$.

The following theorem is a supersingular analog of the Global Torelli Theorem for Kähler K3 surfaces of Burns–Rapoport [100]. We refer for the proof to [576, §3] in the case $p > 3$ and in [90] in the case $p = 3$. We expect that the same theorem is true if $p = 2$ but, as of today, nobody has written up a proof.

Theorem 10.1.17 *The period map $\mathfrak{p}_N$ is an isomorphism.*

Corollary 10.1.18 *Let K_N^0 be the subspace of ample lattice N polarized supersingular K3 surfaces. Then, the composition of the restriction of the period map $\mathfrak{p}_N$ to K_N^0 with the projection $\tilde{\Omega}_N \to \Omega_N$ defines an isomorphism*

$$\mathfrak{p}_N^0 : \mathsf{K}_N^0 \cong \Omega_N^0,$$

where Ω_N^0 is an open subset of the period domain Ω_N.

Remark 10.1.19 Over the complex numbers, it follows from the Torelli type theorem that a K3 surface X is the canonical cover of an Enriques surface if and only if the Picard lattice $\mathrm{Pic}(X)$ contains $\mathsf{E}_{10}(2)$ as a primitive sublattice and its orthogonal complement in $\mathrm{Pic}(X)$ does not contain any (-2)-class. In Section 5.3 of Volume 1 we discussed the finiteness of the number of Enriques surfaces covered by a given K3 surface X up to isomorphisms (equivalently, the number of the conjugacy classes of fixed-point-free involutions of X) and Ohashi's estimate of the number of such Enriques quotients.

One may ask the same questions in positive characteristic. Using Ogus' Torelli theorem, J. Jang [346] proved that if $p > 23$ or $p = 19$, a supersingular K3 surface admits a fixed-point-free involution if and only if its Artin invariant σ_0 is less than 6. Recently, Behrens [54] proved that, if a K3 surface X is of finite height, then the number of Enriques quotients of X is finite. He proves this result by applying a result of Lieblich–Maulik [454]. Behrens also extends Ohashi's estimate for supersingular K3 surfaces in characteristic $p \neq 2$. For example, in the case where $p = 3$ and $\sigma_0 = 1$, there are exactly two Enriques quotients. They are isomorphic to Enriques surfaces with finite automorphism group of type III and of type IV. In characteristic $p = 2$ and $\sigma_0 = 1$, the second author [418] proved that there are three types of Enriques quotients (see Remark 10.6.12). We will give examples of such Enriques surfaces later.

10.2 Supersingular K3-Covers of Enriques Surfaces

In this section, we study canonical covers $\pi : X \to S$ of classical and $\boldsymbol{\alpha}_2$-Enriques surfaces in characteristic two which are birationally isomorphic to a K3 surface. It turns out that X is always supersingular, and we study possible singularities of X and the moduli space of such supersingular K3 surfaces.

Assume that the canonical cover is inseparable. This happens if and only if $p = 2$ and $\mathrm{Pic}^{\tau}_{S/\Bbbk}$ is $\mathbb{Z}/2\mathbb{Z}$ or $\boldsymbol{\alpha}_2$, i.e. a unipotent group scheme of order two, or, equivalently, when S is simply connected.[2]

Assume that X is birationally isomorphic to a K3 surface, i.e. it has only rational double points as its singularities. We call such a surface a *RDP-K3 surface*. By Theorem 1.3.5 in Volume I, its minimal resolution $p : Y \to X$ is a Shioda-supersingular K3-surface. We denote by $\tilde{\pi} : Y \to S$ the composition of $\pi \circ p$.

[2] In [214] such surfaces are called *unipotent*.

We also know from Section 4.9 in Volume I that S has no quasi-elliptic fibrations, and simple fibers of any elliptic fibration are reduced. If the half-fibers are smooth, then the singular points of X lie over singular points of fibers. They are locally isomorphic to $z^2 + f(x, y) = 0$ (because locally a principal cover is given by $z^2 = \epsilon(x, y)$, where ϵ is a unit, changing z to $z+\epsilon(0, 0)$, we get the claim). These are Zariski singularities which we encountered in Section 4.4. If a fiber is of multiplicative type, then the singular points lying over its singular points are of type A_1. If a fiber is of additive type $\tilde{A}_0^{**}$ then the singular point over its singular point is of possible type D_4^0, E_7^0, or E_8^0. If a fiber is of additive type $\tilde{A}_1^*$ then the singular point over its singular point is of possible type $D_n^0, n \geq 6$. Finally, if a fiber is of additive type $\tilde{A}_2^*$ then the singular point over its singular point is of possible types D_4^0 or E_7^0.

Proposition 10.2.1 *The singular locus of X consists of rational double points of types A_n, D_n^0, E_n^0 with the total index (i.e. sum of the subscripts) equal to 12* [3]

Proof Since the minimal resolution Y of X is a K3 surface, all singular points of X are rational double points. Since X is homeomorphic to S in étale topology, we obtain that $e(X) = 12$ and hence $e(Y) - e(X) = 12$ and clearly this number coincides with the total index of singularities. We have already observed that the singular points are Zariski singularities. It follows from Artin's classification of rational double points in characteristic 2 given in Proposition 0.4.13 that all of them are of type A_n, D_n^0, E_n^0.□

The classification of possible configurations of singular points on K3-covers X of simply connected Enriques surfaces birationally isomorphic to a K3 surface is due to Ekedahl, Hyland and Shepherd-Barron [214, Corollary 6.16] (see also [494, Corollary 1.6]).

Theorem 10.2.2 *Under the previous assumptions:*

- *If S is classical,* $\mathrm{Sing}(X)$ *is one of the following collections of singular points:*

$$12A_1,\ 8A_1 + D_4^0,\ 6A_1 + D_6^0,\ 5A_1 + E_7^0.$$

- *If S is an $\boldsymbol{\alpha}_2$-surface,* $\mathrm{Sing}(X)$ *is one of the following collections of singular points:*

$$12A_1,\ 3D_4^0,\ D_4^0 + D_8^0,\ D_4^0 + E_8^0,\ D_{12}^0.$$

All such possibilities can be realized.

Example 10.2.3 Assume $\sigma_0 = 10$. Then, the sublattice $\mathrm{Exc}(Y)$ generated by exceptional curves of $p : Y \to X$ is contained in $\pi^*(\mathrm{NS}(S))^\perp \cong M_{12}$. The embedding of the lattices is defined by an isotropic subgroup A of the discriminant group of $\mathrm{Exc}(Y)$ such that $A^\perp/A \cong D(M_{12}) \cong (\mathbb{Z}/2\mathbb{Z})^{10}$. The classification of possible singularities shows that the only possibility is $\mathrm{Exc}(Y) = \mathsf{A}_1^{\oplus 12}$.

[3] It does not coincide with the Milnor number, which may not be defined, but twice the index coincides with the Tyurina number [639, Proposition 3.3].

Remark 10.2.4 Schröer [640] and Matsumoto [494] gave examples of Enriques surfaces whose canonical cover has a non rational double point (an elliptic double point) of Arnold's type E_{12} (i.e. formally isomorphic to the singularity $z^2 + x^3 + y^7 = 0$). Such singular points do not occur on the canonical covers of classical Enriques surfaces [494, Proposition 3.2].

Definition 10.2.5 Let S be an Enriques surface whose canonical cover is a RDP-K3 surface X. The image of a singular point of X on S is called a *canonical point*.

Proposition 10.2.6 *Let $f : S \to \mathbb{P}^1$ be an elliptic fibration on S. Let $Z = (\omega)_0$ be the 0-cycle of zeros of a non-zero regular 1-form ω generating $H^0(S, \Omega^1_{S/\Bbbk})$. Then, the support of Z is equal to the set of canonical points and $i^*\mathcal{I}_Z = \operatorname{adj} F$, where $i : F \hookrightarrow S$ is the inclusion morphism of a fiber or a half-fiber of f and $\operatorname{adj} F$ is the adjoint ideal of F.*

Proof The first assertion follows from Proposition 1.3.8. Let F be a fiber or a half-fiber of f. Since X is birationally isomorphic to a K3 surface, it is reduced. The exact sequence of sheaves of differentials

$$0 \to \mathcal{I}_F/\mathcal{I}_F^2 \to \Omega^1_{S/\Bbbk} \otimes_{\mathcal{O}_S} \mathcal{O}_F \to \Omega^1_{F/\Bbbk} \to 0$$

defines a homomorphism of sheaves $\Omega^1_{S/\Bbbk} \otimes_{\mathcal{O}_S} \mathcal{O}_F \to \Omega^1_{F/\Bbbk}$ which induces an isomorphism of the spaces of global sections.

Let ω generate $H^0(S, \Omega^1_{S/\Bbbk})$. By Proposition 0.2.10 it vanishes at a finite set of points which is the support of a 0-cycle Z of degree $12 = c_2(\Omega^1_{S/\Bbbk})$. Let $i : F \hookrightarrow S$ be the closed embedding of a fiber or a half-fiber of f. The exact sequence of sheaves of differentials

$$0 \to \mathcal{I}_F/\mathcal{I}_F^2 \to i^*(\Omega^1_{S/\Bbbk}) \to \Omega^1_{F/\Bbbk} \to 0$$

shows that the restriction map $H^0(S, \Omega^1_{S/\Bbbk}) \to H^0(S, \Omega^1_{S/\Bbbk} \otimes_{\mathcal{O}_S} \mathcal{O}_F) \to H^0(F, \Omega^1_{F/\Bbbk})$ is an isomorphism and hence $i^*(\omega)$ generates $H^0(F, \Omega^1_{F/\Bbbk})$. If F is smooth, then $\Omega^1_{F/\Bbbk} \cong \omega_F$ and $i^*(\omega)$ has no zeros. Thus, $Z \cap F = \emptyset$. If F is singular, $\Omega^1_{S/\Bbbk}/\text{Torsion}$ is equal to $\operatorname{adj}_F \otimes \omega_F$, where adj_F is the adjoint ideal of F generated locally at a point by the partial derivatives ϕ_u, ϕ_v of the local equation $\phi = 0$ of F at this point. This follows from considering a natural homomorphism $\Omega^1_{F/\Bbbk} \otimes \mathcal{I}_F/\mathcal{I}_F^2 \to \bigwedge^2 i^*(\Omega^1_{S/\Bbbk})$ that defines a homomorphism

$$\Omega^1_{F/\Bbbk} \to i^*(\Omega^1_{S/\Bbbk}) \otimes (\mathcal{I}_F/\mathcal{I}_F^2)^\vee \cong \omega_F.$$

Its kernel is the torsion subsheaf $\mathcal{T}$ of $\Omega^1_{F/\Bbbk}$ and its image is equal to $\operatorname{adj}_F \otimes \omega_F$. The image of $i^*(\omega)$ in $H^0(F, \operatorname{adj}_F \otimes \omega_F)$ is equal to zero, hence $i^*(\omega) \in H^0(S, \mathcal{T})$. It is known that $h^0(F, \mathcal{T}) = h^0(S, \mathcal{O}_S/J_F)$, where J_F is the jacobian ideal of F locally generated by $\phi, \partial_u, \partial_v$ [742]. Since singular points of F satisfy $\phi \subset (\partial_u, \partial_v)$, the jacobian ideal coincides with the adjoint ideal, we obtain that $h^0(\mathcal{O}_{Z\cap F}) = h^0(F, \mathcal{T})$. This implies that $i^*\mathcal{I}_Z$ is equal to the adjoint ideal of F. □

Corollary 10.2.7 *Let F be a fiber or a half-fiber of an elliptic fibration on S over a point t. Then, X has a singular point over each singular point x of F and the sum of indices of these singular points is equal to $\nu_t(h)$, where h is the discriminant of the jacobian fibration of f. In particular, if F is of multiplicative type with s irreducible components, then X has s singular points of type A_1.*

Proof It follows from the previous proposition that the length of $F \cap (\omega)_0$ is equal to the sum of Milnor numbers of singular points of F. It is known that this number is equal to the total number of vanishing cycles of f that coincides with the contribution $e(S) + \delta_t$ of F in the formula (4.1.12) for the Euler–Poincaré characteristic of S (see [152]). It remains to use that, by Corollary 4.4.10, this local contribution is equal to the order $\nu_x(h)$ of the discriminant h of the jacobian fibration. □

Example 10.2.8 We can use the classification of rational jacobian elliptic surfaces in characteristic 2 from [438] to determine possible singularities of the canonical cover of S. For example, the Weierstrass equation $y^2 + t^3y + x^3 + c_1tx^2 + c_2tx + tc_3 = 0$ defines a fibration with one singular fiber of type $\tilde{A}_1^*$. Since the j-invariant of this fibration is equal to 0, any Enriques torsor S must be an $\boldsymbol{\alpha}_2$-surface. We see that the canonical cover X of S has a singular point of type D_{12} or E_{12}. We do not know whether the latter case can be excluded. Another example of a rational elliptic surface with one singular fiber (of type $\tilde{A}_0^{**}$) is given by the Weierstrass equation $y^2+ty+x^3+c_1tx^2+c_2tx+tc_3 = 0$. It gives the same kind of singularity on X. However, if we take an Enriques torsor (again necessarily $\boldsymbol{\alpha}_2$-surface) with this singular fiber, we may obtain a normal rational canonical surface with an elliptic double point of type E_{12} (see Remark 10.2.4).

Lemma 10.2.9 *Let R be a (-2)-curve on S. Then, R contains two canonical points x_1, x_2 and its proper transform on Y is a (-2)-curve that intersects one of the irreducible component of the exceptional curve over each point x_1, x_2 with multiplicity* 1.

Proof We use the intersection theory on normal algebraic surfaces [537]. Let $\phi : V' \to V$ be a minimal resolution of singularities of such a surface and $\tilde{C}$ the full pre-image of an irreducible curve C on V. We can write it as $\tilde{C} = \bar{C} + \sum_{i\in I} m_iE_i$, where $(E_i)_{i\in I}$ is the set of irreducible components of the exceptional curve of ϕ. Since $C \sim D$ for some Weil divisor on V that does not contain singular points in its support, we have $E_i \cdot \tilde{C} = 0$ for all $i \in I$. Since the matrix $A = (E_i \cdot E_j)$ is negative definite, we can solve for the coefficients m_i in $\mathbb{Q}$ in terms of the integer numbers $\bar{C} \cdot E_i$ and compute $\tilde{C}^2$. By definition $C^2 = \tilde{C}^2$. In the same way one defines the intersection numbers of any two Weil divisors and checks that it depends only on its linear equivalence class and prove that the intersection theory satisfies the usual property for finite maps $f : V' \to V$ of degree n, i.e. $f^*(D) \cdot f^*(D') = n(D \cdot D')$. Let us now apply this to the case when $V = X$ and $V' = Y$ with $\phi = p : Y \to X$. Let R be a (-2)-curve on S. Since the restriction of the principal cover $\pi : X \to S$ to R is trivial, $\pi^*(R) = 2C$ for some curve C on X. The intersection theory gives $C^2 = -1$, hence

$$p^*(C)^2 = C^2 = -1 = p^*(C) \cdot \bar{C} = \bar{C}^2 + \sum_{i \in I} m_i \bar{C} \cdot E_i.$$

This implies that $\bar{C}^2$ must be negative, and since it is an irreducible curve, we get $\bar{C}^2 = -2$, hence

$$\sum_{i \in I} m_i \bar{C} \cdot E_i = 1. \tag{10.2.1}$$

Since our singularities are rational double points, the inverse of the intersection matrix $A = (E_i \cdot E_j)$ has non-positive entries. Also, the classification of singularities on X from Theorem 10.2.2 shows that these entries belong to $\frac{1}{2}\mathbb{Z}$. It follows that $m_i \geq 0$ and $m_i \in \frac{1}{2}\mathbb{Z}$. Thus, the only solutions of (10.2.1) are:

- $\#I = 2, m_1 = m_2 = \frac{1}{2}, \bar{C} \cdot E_1 = \bar{C} \cdot E_2 = 1,$
- $\#I = 1, m_1 = \frac{1}{2}, \bar{C} \cdot E_1 = 2.$

In the latter case, the image of $\bar{C}$ on X is a singular curve that cannot be equal to $\pi^*(R)_{\mathrm{red}} \cong R$. So, only the first equality holds that implies that $\bar{C}$ intersects two irreducible exceptional components with multiplicity 1. If these components are blown down to one point, the image R of $\bar{V}$ is a singular curve. Thus, the image of $\bar{C}$ contains two canonical points. This proves the assertion. □

It follows from the previous Lemma that, if a (-2)-curve contains a canonical point of type D_{12}, then its proper transform on Y intersects one of the components of multiplicity 2. But then its image cannot be a smooth rational curve. Hence, S is an unnodal Enriques surface in this case. The same is true if the singularity is of type E_{12}.

Let $\mathcal{M}_{\mathrm{En,CV}}$ be the stack of Cossec–Verra polarized Enriques surface. Recall from Section 5.11 in Volume I that it is a smooth quasi-separated Deligne–Mumford stack over $\Bbbk$. It consists of two irreducible components $\mathcal{M}^{\boldsymbol{\mu}_2}_{\mathrm{En,CV}}$ and $\mathcal{M}^{\mathrm{cl}}_{\mathrm{En,CV}}$, both of which are smooth algebraic stacks of dimension 10. They intersect transversally along the 9-dimensional smooth stack $\mathcal{M}^{\boldsymbol{\alpha}_2}_{\mathrm{En,CV}}$ of $\boldsymbol{\alpha}_2$-surfaces. The geometric points of the complement $\mathcal{M}^{\boldsymbol{\mu}_2}_{\mathrm{En,CV}} \setminus \mathcal{M}^{\boldsymbol{\alpha}_2}_{\mathrm{En,CV}}$ classifies classical Enriques surfaces.

Let $N = \mathsf{E}_{10}(2) \oplus M_{12}$ be the Picard lattice of supersingular K3 surfaces with Artin invariant $\sigma_0 = 10$. Let $N_0 = 2N^\vee/2N \cong D(N) \cong \mathbb{F}_2^{20}$. Let Ω_N be the subvariety of Grassmann variety $G(10, N_0 \otimes \Bbbk)$ that parameterizes totally isotropic subspaces L of dimension 10 such that $\dim L \cap \mathbf{F}(L) = 9$. It consists of two irreducible components of dimension 9.

Let $c_1 : \mathrm{NS}(X) \to H^2(X/W)$ be the first Chern class homomorphism with values in crystalline cohomology. We proved in the previous section (10.1.1) that it is injective and its composition with the reduction modulo p defines an injective map

$$\bar{c}_1 : \mathrm{NS}(X)/p\,\mathrm{NS}(X) \to H^2_{\mathrm{DR}}(X/\Bbbk).$$

The following theorem is stated in [214, Theorem 6.12] (with reference to the Global Torelli Theorem for supersingular K3 surfaces whose proof in the case $p = 2$ is still unavailable).

Theorem 10.2.10 *There is an algebraic space K_N which is a fine moduli space of N lattice polarized supersingular K3 surfaces. The period map $\mathsf{K}_N \to \Omega_N$ is étale and surjective of degree* 1. *It is an isomorphism on the open subspace K_N^0 of ample polarized surfaces and its complement is a divisor. Let $\mathcal{K}_N \to \mathsf{K}_N$ be the universal family and $\mathcal{K}_N^0$ its pre-image over K_N^0. Then, there is a contraction $\mathcal{K}_N^0 \to \mathcal{X}_N$ over K_N^0 such that each fiber $(\mathcal{X}_N)_t$ over a geometric point is a K3 surface with rational double points of total index 12 such that* $\mathrm{Exc}((\mathcal{K}_N)_t)$ *is contained in the orthogonal complement of the image of* $\mathsf{E}_{10}(2)$ *in* $\mathrm{NS}((\mathcal{K}_N)_t)$.

10.3 Quotients of a Supersingular K3 Surface by a Vector Field

In this section, we will show that an Enriques with canonical cover X birationally isomorphic to a supersingular K3 surface Y is isomorphic to the quotient of X by a nowhere-vanishing regular vector field. We study such vector fields, and also the types of possible singularities of X. The types of these singularities define a lattice polarization on Y, and we study the corresponding moduli spaces.

We keep the notations $p : Y \to X, \pi : X \to S$ from the previous section. In this section, we assume that the characteristic of the ground field $\Bbbk$ is equal to two.

Proposition 10.3.1 *Assume that X is a RDP-K3 surface with Zariski singularities. Then the tangent sheaf Θ_X is free of rank 2.*

Proof First, we show Θ_X is locally free. So, let A be the local ring at a singular point of X. We know that it is locally isomorphic to $B/(f)$, where $f = z^2 + g(x, y) = 0$ and $B = \Bbbk[[x, y, z]]$. The standard exact sequence of modules of differentials gives an exact sequence

$$(f)/(f^2) \to \Omega^1_{B/\Bbbk} \otimes_B A \to \Omega^1_{A/\Bbbk} \to 0.$$

Passing to the duals, we get an exact sequence

$$0 \to \Theta_{A/\Bbbk} \to A^3 \to A \to \mathrm{Ext}^1_A(\Omega^1_{A/\Bbbk}, A) \to 0.$$

The assertion will follow if we show that the A-module $T^1_A := \mathrm{Ext}^1_A(\Omega^1_{A/\Bbbk}, A)$ is of finite projective dimension (it will be automatically less than or equal to 2). It follows from the exact sequence that $T^1_{A/\Bbbk} = B/J$, where $J = (g_x, g_y, f)$. Since the singular point is isolated, it is a module of finite length (called the *Tyurina number*). We know that $A_0 = \Bbbk[x, y] = A^G$, where $G = \boldsymbol{\mu}_2$ or $\boldsymbol{\alpha}_2$ that makes A a flat A_0-module. Let $J_0 = (g_x, g_y)$. Obviously $J_0 A = J$, and the projective dimension of B-module $A/J = A/J_0 A = (A_0/J_0 A_0) \otimes_{A_0} A$ coincides with the projective

dimension of A_0-module A_0/J_0. By the Auslander–Buchsbaum formula, it is equal to $\dim B - \operatorname{depth}(A_0/J_0) \leq 2$. □

Note that, by [214, Corollary 7.3] or [494, Theorem 1.4], all vector fields $\partial \in H^0(X, \Theta_X)$ are 2-closed.

Theorem 10.3.2 *Let X be the canonical cover of an Enriques surface S which is a RDP-K3 surface. For any section of Θ_X that does not vanish at singular points, the quotient X^∂ is an Enriques surface whose canonical cover is isomorphic to X.*

Proof Let $j : X' = X \setminus \operatorname{Sing}(X) \hookrightarrow X$ and let S' be the image of this set in S. The cover $Y' \to X' \to S'$ is a principal cover of smooth varieties. Applying (0.3.2), we get an exact sequence

$$0 \to \mathcal{O}_{X'} \to \Theta_{X'/\Bbbk} \to \pi^*(\Theta_{S'/\Bbbk}) \to \mathcal{O}_{X'} \to 0.$$

Since the kernel of $\pi^*(\Theta_{S'/\Bbbk}) \to \mathcal{O}_{X'}$ is locally free and the determinant of $\pi^*(\Theta_{S'/\Bbbk})$ is equal to $\pi^*\omega_{S'} \cong \mathcal{O}_{X'}$, we obtain an exact sequence

$$0 \to \mathcal{O}_{X'} \to \Theta_{X'/\Bbbk} \to \mathcal{O}_{X'} \to 0.$$

Since $\Theta_{X'/\Bbbk}$ is the restriction of a locally free sheaf $\Theta_{X/\Bbbk}$ on a Cohen–Macaulay scheme X, we have $\Theta_{X/\Bbbk} = j_*\Theta_{X'/\Bbbk} = \Theta_X$ and $j_*\mathcal{O}_{X'} = \mathcal{O}_X, R^1 j_*\mathcal{O}_{X'} = 0$. Thus, applying j_* to the exact sequence, we get an exact sequence

$$0 \to \mathcal{O}_X \to \Theta_{X/\Bbbk} \to \mathcal{O}_X \to 0.$$

Since $\operatorname{Ext}^1_{\mathcal{O}_X}(\mathcal{O}_X, \mathcal{O}_X) = H^1(X, \mathcal{O}_X) = 0$ because X is K3-like, the exact sequence splits. □

Now, we see that

$$H^0(X, \Theta_{X/\Bbbk}) \cong \Bbbk^2.$$

Since $S = X/G$, where G is a group scheme of order 2, S is isomorphic to the quotient of X by a regular global vector field ∂. It is of additive (resp. multiplicative) type if and only if $G \cong \boldsymbol{\alpha}_2$ (resp. $\boldsymbol{\mu}_2$). The surface X^∂ is an Enriques surface if and only if the scheme of fixed points of G is empty, or, equivalently, the projection $X \to X^\partial$ is a non-trivial principal G-cover.

Proposition 10.3.3 *Suppose $\partial \in H^0(X, \Theta_X)$ is of multiplicative type and vanishes at singular points $p_1, \ldots, p_k$ of type A_1. Then, the quotient X^∂ is a rational surface whose minimal resolution is a Coble surface with k boundary components.*

Proof Let y_i be the image of the singular point p_i on X^∂. We may assume that p_i is formally isomorphic to a singular point $xy + z^2 = 0$. Then, the derivations $x\partial_x + y\partial_y, \partial_z$ form a basis of the module of derivations [640, Proposition 2.3]. We lift these derivations to the derivations of $\Bbbk[[x, y, z]]$ that leave the ideal $(xy + z^2)$ invariant. According to Proposition 0.3.10 from Volume I, the subring $\Bbbk[[x, y, z]]^\partial$ is equal to the Veronese ring $\Bbbk[[x^2, y^2, z^2, xy, xz, yz]]$. It follows that $\operatorname{Spec} R^\partial$ is

formally isomorphic to the singular point of the vertex of the affine cone over a hyperplane section of the Veronese surface $\nu_2(\mathbb{P}^2) \subset \mathbb{P}^5$. The exceptional curve of its minimal resolution of singularities is a smooth rational curve C_i with self-intersection -4. Let V be the minimal resolution of X^∂. The composition map $Y \to X \to X^\partial$ extends to a map $\phi : Y \to \mathsf{V}$ equal to the projection map $Y \to Y^D$, where D is a rational derivation of Y with the divisor of zeros equal to the union of exceptional curves over the zeros of ∂. The formula for the canonical class of purely inseparable covers shows that $2K_{\mathsf{V}} + C_1 + \cdots + C_k \sim 0$. This shows that V is a Coble surface with k boundary components. □

Remark 10.3.4 Note that any derivation $\partial \in H^0(X, \Theta_X)$ of additive type does not vanish at isolated fixed points of type A_1 and hence the quotient X^∂ is an Enriques surface [214, Lemma 7.5]. This explains the fact that isomorphism classes of Coble surfaces lie in the closure of the moduli space of classical Enriques surfaces.

We have already used [639, Proposition 2.3] for the description of the module of differentials of the ring $\Bbbk[[x, y, z]]/(z^2 + xy)$. In fact, the description holds for any Zariski singularity defined by the ring $\Bbbk[[x, y, z]]/(z^2 + f(x, y))$. According to Proposition 2.4 from loc. cit., the module of derivations has a basis formed by ∂_z and $f_y\partial_x + f_x\partial_y$

Proposition 10.3.5 *Let $\partial = u\partial_z + v(f_y\partial_x + f_x\partial_y)$ be a 2-closed derivation of $R = \Bbbk[[x, y, z]]/(z^2 + f(x, y))$. Assume that $u \in R^\times$. Then, the corresponding action of $G \cong \boldsymbol{\mu}_2$ or $\boldsymbol{\alpha}_2$ is free and R^∂ is a regular ring.*

Corollary 10.3.6 *Let X be the canonical cover of an Enriques surface S birationally isomorphic to a K3 surface. Then, there exists a line $\ell_{\mathrm{sing}} \subset H^0(X, \Theta_X)$ such that $\partial \in H^0(X, \Theta_X)$ defines a free action of the group scheme G if and only if it does not belong to ℓ_{sing}.*

Proof For any singular point x of X, let

$$r_x : H^0(X, \Theta_X) \to \mathrm{Der}_{\Bbbk}(\hat{\mathcal{O}}_{X,x}) \cong \mathrm{Der}_{\Bbbk}\, \Bbbk[[x, y, z]]/(z^2 + f(x, y))$$

be the restriction homomorphism. We define ℓ_{sing} to be the linear span of the pre-image of ∂_z. One checks that this is well-defined and that the line does not depend on the choice of a singular point x (see the proof of [639, Proposition 2.5]. □

Another line in $H^0(X, \Theta_X)$ controls whether the quotient Enriques surface is classical or not. We refer to [494, Theorem 1.4] for the proof of the following theorem.

Theorem 10.3.7 *Let $X \to S$ be the canonical cover of a simply connected Enriques surface which is birationally isomorphic to a K3 surface. Let $\mathfrak{g} = H^0(X, \Theta_X)$. Assume that the singular points of X are either $12A_1$, or $8A_1 + D_4^{(0)}$, or $6A_1 + D_6^{(0)}$, or $5A_1 + E_7^{(0)}$. Then, the subset ℓ_{add} of $\partial \in \mathfrak{g}$ with $\partial^2 = 0$ is a line. In the remaining cases it coincides with the whole $\mathfrak{g}$. Moreover,*

(1) If all singular points are of type A_1, then $\ell_{add} \neq \ell_{\mathrm{sing}}$ and $\partial \in \ell_{add}$ if and only if X^∂ is an $\boldsymbol{\alpha}_2$-Enriques surface.
(2) If the singular points are of other types and ℓ_{add} is a line, then it coincides with the line ℓ_{sing}.
(3) If $\ell_{add} = \mathfrak{g}$, then all quotients X^∂ for $\partial \notin \ell_{\mathrm{sing}}$ are $\boldsymbol{\alpha}_2$-Enriques surfaces.

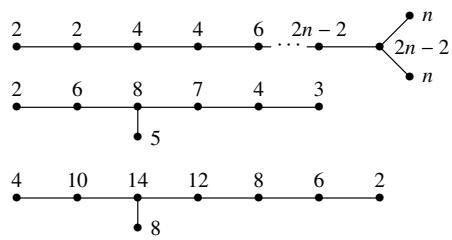
If X is a normal rational surface, then it has a unique minimal elliptic double point of type E_{12} and all quotients X^∂ are $\boldsymbol{\alpha}_2$-Enriques surfaces.

Recall from Section 5.11 that there is a stack $\mathcal{E}_{\mathrm{uni}} : \mathrm{Sch}/\mathbb{F}_2 \to$ (Groupoids) that assigns to each scheme B over $\mathbb{F}_2$ morphisms of algebraic spaces $(S, \phi) \to B$ whose generic fibers S_b are simply connected (=unipotent) Enriques surfaces with a marking $\phi : \mathsf{E}_{10,B} \to \mathrm{Num}(S)$ such that $\phi_b : \mathsf{E}_{10} \to \mathrm{Num}(S_b)$ maps the chamber $\mathcal{D}_0$ in $\mathsf{E}_{10} \otimes \mathbb{R}$ defined by simple roots $\alpha_0, \dots, \alpha_9$ into the ample cone of S_b. Let $\mathcal{E}_{\mathrm{uni},K3}$ be the open and dense substack of $\mathcal{E}_{\mathrm{uni}}$ of those Enriques surfaces whose canonical cover is birationally K3 surface. Let $\mathcal{K}_{\mathrm{EnrK3}}$ be the stack of families $(f : X \to T, \phi)$ of algebraic spaces whose geometric fibers are RDP-K3 surfaces with singular points of total index 12 and trivial tangent bundle together with a marking $\phi : \mathsf{E}_{10}(2) \to \mathrm{Pic}(X)$ such that $\phi(\mathcal{D}_0)$ lies in the ample cone. It follows from the Global Torelli Theorem (although its proof in characteristic 2 is not available at this time) that the stack $\mathcal{K}_{\mathrm{EnrK3}}$ has a fine moduli space K_{EnrK3} in the category of algebraic spaces which admits a universal lattice $\mathsf{E}_{10}(2)$ polarized K3 surface $\mathfrak{f} : \mathcal{X} \to K_{\mathrm{EnrK3}}$. Let $\mathrm{PK}_{\mathrm{EnrK3}} = \mathbb{P}(\mathfrak{f}_*\Theta^\vee_{\mathcal{X}/K_{\mathrm{EnrK3}}})$ be the projective line bundle over K_{EnrK3} whose fibers are projectivized spaces of regular vector fields on the fibers of $\mathfrak{f}$.

Suppose $(X, \phi) \in \mathcal{K}_{\mathrm{EnrK3}}(\mathrm{Spec}\ \Bbbk)$, and let $p : Y \to X$ be its minimal resolution. Then, $\tilde{\phi} := p^* \circ \phi : \mathsf{E}_{10}(2) \to \mathrm{NS}(Y)$ defines a lattice $\mathsf{E}_{10}(2)$ polarization. It is obviously non-ample since the exceptional curves of p lie in the orthogonal complement of $\tilde{\phi}(\mathsf{E}_{10}(2))$. Let us extend the lattice to get an ample polarization. We set $N = \mathsf{E}_{10}(2) \oplus M_{12}$ to be the 2-elementary lattice isomorphic to the Néron–Severi lattice of a supersingular surface with Artin invariant $\sigma_0 = 10$. Let $e_1, \dots, e_{12}$ be the natural generators of M_{12} and $\rho_{12} = \frac{1}{2}(e_1 + \cdots + e_{12})$ as in Corollary 10.1.9. Note that ρ_{12} coincides with the half-sum of positive roots in M_{12} that coincides with the Weyl vector, i.e. the vector whose inner product with each positive root is equal to -1. We encountered such vectors in the Appendix in Volume I.

The following lemma follows immediately by applying the Borel–de Siebenthal–Dynkin algorithm to embed the root lattice $A_1^{\oplus r}$ into an irreducible root lattice of type A, D, E. We leave its proof to the reader (also see [214, Lemma 6.5]).

Lemma 10.3.8 *Let M be one of the root lattices of type D_{2n}, E_7, E_8. Then, M contains $r = \mathrm{rank}\ M$ orthogonal positive roots such that their sum is equal to the sum of simple roots taken with coefficients indicated in the following diagrams:*



Corollary 10.3.9 *Let* $\mathrm{Exc}(Y)_s, s = 1, \ldots, k$, *be an orthogonal direct summand of* $\mathrm{Exc}(Y)$ *generated by the divisor classes of a connected component of the exceptional curve of* $\pi : Y \to X$. *The set of generators* $e_1, \ldots, e_{12}$ *of the lattice* M_{12} *can be split into disjoint subsets* $I_1, \ldots, I_k$, *one for each component* $\mathrm{Exc}(Y)_s$, *such that there exists a lattice embedding* $j_s : \oplus_{i \in I_s} \mathbb{Z} e_i \cong A_1^{\# I_s} \hookrightarrow \mathrm{Exc}(Y)_s$ *with the images of* e_i *being orthogonal roots in* $\mathrm{Exc}(Y)_s$ *whose sum is indicated in the diagrams from the previous lemma.*

Let

$$j = \oplus_{s=1}^k j_s : \oplus_{s=1}^k A_1^{\oplus \# I_s} \cong A_1^{\oplus 12} \hookrightarrow \mathrm{Exc}(Y) = \oplus_{s=1}^k \mathrm{Exc}(Y)_s \tag{10.3.1}$$

be the lattice embedding obtained from the previous corollary. Note that $\sum_{i \in I_s} j_s(e_i)$ intersects each simple root with multiplicity ± 2, therefore $\frac{1}{2} \sum_{i \in I_s} j_s(e_i) \in \mathrm{Exc}(Y)_s^\vee$. This defines an embedding

$$j : M_{12} \hookrightarrow \mathrm{Exc}(Y)^\vee. \tag{10.3.2}$$

We know that Y is a specialization of a supersingular K3 surface Y' with Artin invariant $\sigma_0 = 10$. Thus, the lattice $\mathrm{NS}(Y') \cong \mathsf{E}_{10}(2) \oplus M_{12}$ admits an embedding into the lattice $\mathrm{NS}(Y)$.

Definition 10.3.10 Let $N = \mathsf{E}_{10}(2) \oplus M_{12}$. Let Y be a supersingular K3 surface which is birational to the canonical covering of a simply connected Enriques surface. A lattice N polarization of Y is a lattice embedding $\phi : N \hookrightarrow \mathrm{NS}(Y)$ such that $\phi(E_{10}(2))$ is a primitive sublattice and the restriction of ϕ to M_{12} coincides with the embedding (10.3.2).

Let $\phi : N \hookrightarrow \mathrm{NS}(Y)$ be a lattice N polarization. Fix a root chamber C in the positive cone $V(N_{\mathbb{R}})^+$ of N. The Weyl group $W(N)$ acts transitively on the set of chambers. We may choose C such that the intersection of its closure with $\mathsf{E}_{10}(2)_{\mathbb{R}}$ contains the fundamental chamber $\mathcal{D}$ in E_{10} spanned by the fundamental weights $\omega_0, \ldots, \omega_9$ dual to the simple roots $\alpha_0, \ldots, \alpha_9$.

A lattice polarization $\phi : N \to \mathrm{NS}(Y)$ is called an *ample polarization* if $\phi(N)$ meets the ample cone of Y. Following [214, §6], we say that (Y, ϕ) is *good* if $\mathrm{NS}(Y) \cap \phi(\mathsf{E}_{10}(2))^\vee$ does not contain a root.

Suppose r is the class of a (-2)-curve on Y. Since $\mathrm{NS}(Y) \subset \phi(N)^\vee$, we can write $r = r_1 + r_2$, where $r_1 \in \phi(\mathsf{E}_{10}(2))^\vee \cong \mathsf{E}_{10}(\frac{1}{2})$ and $r_2 \in M_{12}^\vee = \mathsf{I}^{12}(-\frac{1}{2})$. By the assumption $\mathcal{D} \subset C$, $r_1 \neq 0$. Since $r_1^2 \in \mathbb{Z}$, we have $r_1^2 = r_2^2 = -1$ or $r_1^2 = -2$. So, if

(Y, ϕ) is good, the latter case does not occur. Suppose Y is realized as the minimal resolution of the canonical cover $\pi : X \to S$ and we take lattice N polarization such that the restriction of ϕ to $\mathsf{E}_{10}(2)$ is equal to the composition $\tilde{\pi}^* \circ j$, where $j : \mathsf{E}_{10} \to \mathrm{Num}(S)$ is an ample lattice E_{10} polarization of S, and $\tilde{\pi}$ is the composition of π and the minimal resolution $p : Y \to X$. Assume that $r \notin \mathrm{Exc}(Y)$. Then, $2r_1 = \tilde{\pi}^*(R)$, where R is a (-2)-curve on S, so that $\tilde{\pi}^*(R) = 2r + 2r_2$. It follows from Lemma 10.2.9 that $2r_2$ is the sum of the classes of two disjoint irreducible components of the minimal resolution. If we choose a vector η in E_{10} such that $j(\eta)$ is an ample class, we obtain that $\phi(\tilde{\pi}^*(j(\eta)))$ intersects positively r, hence $\phi(\mathcal{D})$ meets the ample cone of Y. The converse is also true since $\tilde{\pi}^*(\mathrm{Pic}(S)) = p^*(\mathrm{Pic}(X))$ is equal to the orthogonal complement of $\mathrm{Exc}(Y)$, and hence, it coincides with its primitive closure.

This proves the following:

Proposition 10.3.11 *Let $\phi : N \hookrightarrow \mathrm{NS}(Y)$ be a lattice N polarization of the canonical cover birationally isomorphic to a K3 surface Y of a simply connected Enriques surface in characteristic 2. Then, the polarization is ample.*

Let $\mathcal{E}_{\mathrm{uni}}$ be the moduli stack of simply connected (= unipotent) Enriques marked surfaces (S, j) such that the image of the cone $\mathcal{D}_0 \subset (\mathsf{E}_{10})_{\mathbb{R}}$ spanned by the fundamental weights $\omega_0, \ldots, \omega_9$ lies in the ample cone. We introduced this moduli space in the last section of Chapter 5. The image of the fundamental weight ω_1 of square-norm 4 corresponds to a Cossec–Verra polarization. There is a forgetting map

$$\mathcal{E}_{\mathrm{uni}} \to \mathcal{M}_{\mathrm{Enr,CV}}$$

to the Deligne–Mumford stack of Enriques surfaces with Cossec–Verra polarization (not necessarily ample). Its image lies in the closure of the component $\mathcal{M}^{\mathrm{cl}}_{\mathrm{Enr,CV}}$ parameterizing classical Enriques surfaces.

The following proposition complements Theorem 10.2.10. whose notations we will keep and its proof can be found in [214, Theorem 6.1.2 (5)].

Proposition 10.3.12 *There are invertible sheaves $\mathcal{L}_0, \ldots, \mathcal{L}_9$ in $\mathrm{Pic}(\mathcal{X}_N)$ that are equal to the images of $\omega_0, \ldots, \omega_9$ under the polarization map.*

Let $\mathcal{E}^0_{\mathrm{uni}}$ be substack of $\mathcal{E}_{\mathrm{uni}}$ parameterizing simply connected Enriques surfaces whose canonical cover is a RDP K3-surface. Let $\mathcal{E}^0_{\mathrm{uni}} \to \mathsf{K}_N$ be the map of stacks defined by taking the minimal resolution of the canonical cover. We know that its image lies in K^0_N. Also, we know that the lattice N polarization on the images is good. By [214, Lemma 6.11], the complement is a divisor.

Let $\Theta_{\mathcal{X}_N/\mathsf{K}_N}$ be the relative tangent sheaf. Let K^{00}_N be the open subset of K^0_N where this sheaf is trivial. By taking the canonical cover, we get a map $\mathcal{E}^0_{\mathrm{uni}} \to \mathsf{K}^{00}_N$ whose images are families of RDP K3-surfaces with trivial tangent bundle. Let $\mathcal{K}^{00}_N$ be the restriction of the universal family over K^{00}_N. Let $\mathbb{P}(\Theta_{\mathcal{K}^{00}_N/\mathsf{K}^{00}_N}) \cong \mathsf{K}^{00}_N \times \mathbb{P}^1$ be the projectivization of the relative tangent sheaf and let $\mathbb{P}(\Theta_{\mathcal{K}^{00}_N/\mathsf{K}^{00}_N})^0$ be the open subset whose fibers correspond to free derivations. By taking the corresponding free $\boldsymbol{\mu}_2$ or

α_2 action, we can reconstruct the Enriques surface. This defines an isomorphism of the moduli functors

$$\mathcal{E}^0_{\mathrm{uni}} \to \mathbb{P}(\Theta_{\mathcal{K}^{00}_N/\mathsf{K}^{00}_M})^0.$$

The composition of this map with the projection to K^{00}_N defines a map

$$\Phi : \mathcal{E}^0_{\mathrm{uni}} \to \mathsf{K}^{00}_N.$$

By Corollary 10.3.6, its fibers are isomorphic to affine lines.

Let $\mathsf{K}^0_{N,\mathcal{R}}$ be the subset of K^0_N such the sublattice $\mathrm{Exc}((\mathcal{K}_N)_t)$ of the Néron–Severi lattice of the fibers generated by the exceptional curves is isomorphic to one of the lattices $\mathcal{R}$ from Theorem 10.2.2. For example, $\mathsf{K}_{N,12A_1}$ parameterizes fibers of $\mathcal{X}_N \to \mathsf{K}^0_N$ with 12 ordinary nodes. Let $\mathcal{E}_{\mathrm{uni},\mathcal{R}} \subset \mathcal{E}_{\mathrm{uni}}$ be its pre-image under Φ.

Let $\mathcal{E}_{\mathrm{uni},\alpha_2}$ be the locus of α_2-Enriques surfaces. Applying Theorem 10.3.7, we see that:

- $\Phi : \mathcal{E}_{\mathrm{uni},12A_1} \to \mathsf{K}_{N,12A_1} \times \mathbb{A}^1$ is bijective.
- $\Phi(\mathcal{E}_{\mathrm{uni},12A_1} \cap \mathcal{E}_{\mathrm{uni},\alpha_2})$ is a section of $\mathsf{K}_{N,12A_1} \times \mathbb{A}^1 \to \mathsf{K}_{N,12A_1}$.
- If $\mathcal{R} \cong \mathsf{A}_1^{\oplus 8} \oplus \mathsf{D}_4$, or $\mathsf{A}_1^{\oplus 5} + \mathsf{E}_7$, or $\mathsf{A}_1^{\oplus 6} + \mathsf{D}_6$, then $\mathcal{E}_{\mathrm{uni},\mathcal{R}} \cap \mathcal{E}_{\mathrm{uni},\alpha_2} = \emptyset$.
- If $\mathcal{R}$ is one of the remaining lattices then $\mathcal{E}_{\mathrm{uni},\mathcal{R}} \subset \mathcal{E}_{\mathrm{uni},\alpha_2}$ and $\Phi : \mathcal{E}_{\mathrm{uni},\mathcal{R}} \to \mathsf{K}_{N,\mathcal{R}} \times \mathbb{A}^1$ is bijective.

Let (Y, ϕ) be a lattice N polarization of a supersingular K3 surface Y such that the embedding $M_{12} \hookrightarrow \phi(\mathsf{E}_{10}(2))^{\perp}$ is a natural isomorphism if all singular points of the corresponding RDP K3-surface X are of type A_1 or else given by one of the diagrams from Lemma 10.3.8.

The following lemma is proven in [214, Lemma 6.5].

Lemma 10.3.13 *Suppose there exists a 2-closed vector field ∂ of X that does not vanish at a singular point whose exceptional curve defines an irreducible root lattice of type A_1, D_n, E_n. Then, there exists a rational derivation D of Y with the divisor of poles is given by one of the diagrams from Lemma 10.3.8.*

10.4 The Cremona–Richmond Polytope

In this section, we recall one of the most fascinating objects in classical algebraic geometry: the *Cremona–Richmond abstract symmetric configuration* (15_3) and its various geometric realizations. We will show that it defines a convex polytope of finite volume with 40 facets in the hyperbolic space $\mathbb{H}^9$ associated with $(\mathsf{E}_{10})_{\mathbb{R}}$. It will appear several times in our constructions of Enriques surfaces as quotients of a supersingular K3 surface with Artin invariant one in characteristic two, which we will study in the next section.

An *abstract configuration* is a triple $\{\mathcal{A}, \mathcal{B}, R\}$, where $\mathcal{A}, \mathcal{B}$ are non-empty finite sets and $R \subset \mathcal{A} \times \mathcal{B}$ is a relation such that the cardinality of the set $R(a) = \{b \in \mathcal{B} : (a, b) \in R\}$ (resp. of the set $R(b) = \{a \in \mathcal{A} : (a, b) \in R\}$) does not depend

on $a \in \mathcal{A}$ (resp. $b \in \mathcal{B}$). Elements of $\mathcal{A}$ are called *points*, elements of $\mathcal{B}$ are called *blocks*. If $a \in R(b)$, we say that a belongs to b. If

$$u = \#\mathcal{A}, \quad v = \#\mathcal{B}, \quad r = \#R(a), \quad s = \#R(b),$$

then a configuration is said to be an (v_r, u_s)-configuration. A *symmetric configuration* is a configuration with $u = v$. Since $ur = vs$, this is equivalent to $r = s$. It is said to be a (v_r)-configuration. Replacing the relation $R \subset \mathcal{A} \times \mathcal{B}$ with the dual relation $R^* \subset \mathcal{B} \times \mathcal{A}$, we obtain the definition of the *dual abstract configuration* of type (v_s, u_r).

Note that we consider only a particular case of the notion of an abstract configuration studied in combinatorics. Ours are *tactical configurations* [314].

Each abstract configuration (u_r, v_s) defines a bipartite graph with the set of vertices equal to the union of the sets $\mathcal{A}$ and $\mathcal{B}$ where each point a from $\mathcal{A}$ is joined by an edge to a block b from $\mathcal{B}$ if $(a, b) \in R$. This graph is called the *Levi graph* of the configuration.

Let $[1, 6] = \{1, 2, 3, 4, 5, 6\}$. A subset of two elements is a *duad*, a partition of $[1, 6]$ into three duads is a *syntheme*. A *total* is a set of five synthemes that contains all duads. There are six totals with a bijection to $[1, 6]$ such that a duad (ab) corresponds to a common syntheme in the corresponding totals. If one views a duad as a transposition in the permutation group $\mathfrak{S}_6$ and a syntheme as the products of three commuting transpositions, then the map from the set of duads to the set of synthemes defined by five totals defines an outer automorphism of the group $\mathfrak{S}_6$. It is known that the quotient of the group of outer automorphisms by the group of inner automorphisms is a cyclic group of order 2. Thus, the group $\mathfrak{S}_6$ acts transitively on the set of sixtuples of totals, each can be identified with an outer automorphism.

Following [349], we choose an outer automorphism $\iota \in \mathrm{Out}(\mathfrak{S}_6)$ satisfying the property

$$\iota((ab)) = (ij, kl, mn) \leftrightarrow (ab) = \iota((ij)) \cap \iota((kl)) \cap \iota((mn)). \tag{10.4.1}$$

It gives the following bijection between duads and synthemes:

(12)	(15,26,34)	(23)	(16,23,45)	(35)	(14,26,35)
(13)	(13,25,46)	(24)	(14,25,36)	(36)	(15,24,36)
(14)	(16,24,35)	(25)	(13,24,56)	(45)	(15,23,46)
(15)	(12,36,45)	(26)	(12,35,46)	(46)	(13,26,45)
(16)	(14,23,56)	(34)	(12,34,56)	(56)	(16,25,34)

Table 10.1 Outer automorphism bijection between duads and synthemes

To reconstruct a total from this table, we collect all synthemes which correspond to five duads sharing a common number from $[1, 6]$. The six totals obtained in this way are the totals (R, Q, Q', P, R', P') in the list of totals from [39, Vol. II, Note II, p. 221] and the totals $(T_2, T_3, T_4, T_5, T_1, T_6)$ in the list of totals from [177, 9.4.3]).

It is clear that the sets of 15 duads and 15 synthemes is an abstract configuration of type (15_3). It is called, for reasons which will become clear later, the *Cremona–Richmond configuration*.

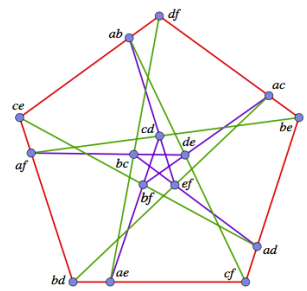


Fig. 10.1 Cremona–Richmond configuration

The Levi graph of the Cremona–Richmond configuration is known as the *Tutte–Coxeter graph*.

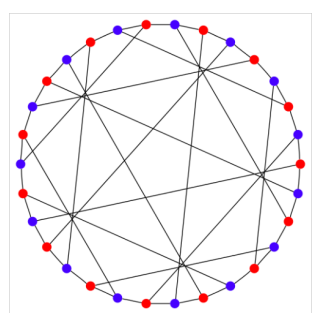


Fig. 10.2 The Tutte–Coxeter graph

Note that there are many non-isomorphic abstract configurations of type (15_3), ours is of symmetry type 1, i.e. its points and blocks represent one orbit with respect to the symmetry group of the configuration (see [280, 2.9]).

The set $\mathfrak{D}$ of duads, together with the empty set, can be naturally endowed with the structure of a 4-dimensional symplectic linear space over $\mathbb{F}_2$. In fact, the set $2^{[1.6]}$ of subsets of $[1,6]$ can be identified with the linear space $\mathbb{F}_2^6$ with the addition law defined by the symmetric difference. The subspace of sets of even cardinality modulo the one dimensional subspace spanned by the set $[1,6]$ is isomorphic to $\mathbb{F}_2^4$. Its elements can be identified with subsets of even cardinality modulo taking the complementary subset. Choosing a representative of cardinality 2 or 0, we obtain a

structure of a linear space on $\bar{\mathfrak{D}} = \mathfrak{D} \cup \emptyset$ isomorphic to the symplectic space $J_2 = \mathbb{F}_2^4$. The symplectic structure can be defined by

$$\langle ab, cd \rangle = \#\{a, b\} \cap \{c, d\} \mod 2.$$

In this way, one can identify $\mathfrak{D}$ with the set of non-zero elements of $\bar{\mathfrak{D}}$ and the set $\mathfrak{S}$ of synthemes with the set of isotropic planes with the zero vector deleted. This shows that the Cremona–Richmond (15_3)-configuration is isomorphic to the configuration where X is the set of non-zero vectors in a symplectic space $\bar{\mathfrak{D}}$ and Y is the set of its isotropic planes with the zero vector deleted.

Recall that the group of symplectic automorphisms of J_2 is denoted by $\mathrm{Sp}(4, \mathbb{F}_2)$. It is isomorphic to the permutation group $\mathfrak{S}_6$ via the natural action of the latter on the set of maps from $[1, 6]$ to $\mathbb{F}_2$. This shows that the Cremona–Richmond configuration (15_3) has the group $\mathfrak{S}_6$ as its group of its symmetries. An *outer automorphism* of a configuration $(\mathcal{A}, \mathcal{B}, R)$ of type (15_3) is a bijective map $\alpha : \mathcal{A} \to \mathcal{B}$ such that $(x, y) \in R$ if and only if $(\alpha^{-1}(y), \alpha(x)) \in R$. The bijection between the set of duads and the set of synthemes defined by the outer automorphism of $\mathfrak{S}_6$ is the outer automorphism of the configuration $(\mathfrak{D}, \mathfrak{S})$. Thus, the group $\mathrm{Aut}(\mathfrak{S}_6)$ is realized as the group generated by the automorphisms and outer automorphisms of the Cremona–Richmond configuration.

We have already seen a geometric realization of the Cremona–Richmond configuration (15_3) over the field of two elements. Let us give an example of its self-dual geometric realization over any field.

Example 10.4.1 Let $(p_1, \ldots, p_6)$ be an ordered set of six points in $\mathbb{P}^2(\Bbbk)$ with no three points are collinear. Recall that the linear system of plane cubic curves passing through the six points defines a birational map $f : \mathbb{P}^2 \dashrightarrow \mathbb{P}^3$ from the plane to a cubic surface S in $\mathbb{P}^3$. If not all of the points lie on a conic, then the map is an isomorphism from the blow-up V of $\mathbb{P}^2$ at the six points. Otherwise, the map blows down the proper transform of the conic to the unique ordinary double point of S.

Let $\ell_{ab} = \langle p_a, p_b \rangle$ be the line spanned by the points p_a, p_b. Its image under the map f is a line on S not passing through the node, if S has a node. Thus, we have 15 lines in $\mathbb{P}^3$. This is our set $\mathcal{A}$. The lines $\ell_{ab}, \ell_{cd}, \ell_{ef}$ whose set of indices form a syntheme define a plane in $\mathbb{P}^3$ that cuts S along the three lines $(ab), (cd), (ef)$. It is called a *tritangent plane*. The set of these 15 tritangent planes is our set $\mathcal{B}$. The relation R is of course the incidence relation. This realization of the configuration (15_3) appears in the study of cubic surfaces by L. Cremona.

Another frequently used in modern literature realization of the Cremona–Richmond configuration appears in the study of the Segre cubic hypersurface in $\mathbb{P}^4$ and its dual, the Castelnuovo–Richmond quartic hypersurface. We refer for this to [39, Volume 4, Chapter V] or [177, 9.8].

Let (abc) be a triad with complementary set (def), we denote by $\mathfrak{D}(abc)$ (resp. $\mathfrak{S}(abc)$) the set of six duads (resp. six synthemes) not appeared in the entries (resp. determined by the determinantal terms) of the matrix

$$\begin{pmatrix} (ad) & (ae) & (af) \\ (bd) & (be) & (bf) \\ (cd) & (ce) & (cf) \end{pmatrix}.$$

Definition 10.4.2 We define the *extended double Cremona–Richmond diagram* Γ_{CR} as follows. Its set of vertices is the union of the set $\mathfrak{D}$ of 15 duads (ab), the set $\mathfrak{S}$ of 15 synthemes (ij, kl, mn) and the set $\mathfrak{T}$ of 10 triads (abc). Their edges are described as follows:

- The subgraph with vertices $\mathfrak{D}$ or $\mathfrak{S}$ is the dual graph of the complete graph $K(6)$.
- Each vertex from $\mathfrak{D}$ is connected by a double edge with a vertex from $\mathfrak{S}$ and vice versa forming the Levi graph of the Cremona–Richmond configuration where all edges are doubled.
- Two vertices from $\mathfrak{T}$ are joined by a double edge.
- Each vertex (abc) from $\mathfrak{T}$ is joined by a double edge with the set of vertices from $\mathfrak{D}(abc)$ and $\mathfrak{S}(abc)$.

Recall from Section 0.8 in Volume I that a finite set of vectors $v_i, i \in I$, of square norm -2 in $V = (\mathsf{E}_{10})_{\mathbb{R}}$ defines a polytope Π in the hyperbolic space $\mathbb{H}^9$ associated to the quadratic vector space V with the Sylvester signature $(1, 9)$. It is the closure of a fundamental domain for the reflection group Γ generated by reflections with respect to the vectors v_i. It is equal to a connected component of the complement of the set of the hyperplanes $H(v_s)$ orthogonal to the vectors v_s. The group Γ is a Coxeter group generated by the reflections r_{v_i} with the Coxeter matrix (m_{ij}), where $m_{ij} = \infty$ if $v_i \cdot v_j \geq 2$, or 3 if $v_i \cdot v_j = 1$, or 2 if $v_i \cdot v_j = 0$. We assume that there exists a set of forty (-2)-vectors in V forming the diagram Γ_{CR}. We call the polytope Π in $\mathbb{H}^9$ defined by these forty vectors the *Cremona–Richmond polytope*.

Theorem 10.4.3 *The Cremona–Richmond polytope in $\mathbb{H}^9$ is of finite volume. Its symmetry group is isomorphic to the group of automorphisms of $\mathfrak{S}_6$ isomorphic to $\mathfrak{S}_6 \cdot 2$.*

Proof Any symmetry of Γ_{CR} must be a symmetry of the Cremona–Richmond configuration. The group of such symmetries is obviously $\mathrm{Aut}(\mathfrak{S}_6)$. We see immediately that it extends to the group of symmetries of the whole diagram.

The polytope is of finite volume in $\mathbb{H}^9$ if and only if the intersection matrix $(v_s, v_{s'})$ satisfies Vinberg's criterion from Theorem 0.8.23 in Volume I. To check this, we exhibit all parabolic subdiagrams of Γ_{CR} of maximal rank. They are the following:

$$\tilde{A}_2 + \tilde{A}_2 + \tilde{A}_2 + \tilde{A}_2, \quad \tilde{A}_3 + \tilde{A}_3 + \tilde{A}_1 + \tilde{A}_1, \quad \tilde{A}_4 + \tilde{A}_4, \quad \tilde{A}_5 + \tilde{A}_2 + \tilde{A}_1. \tag{10.4.2}$$

For example, take the parabolic diagram of type $\tilde{A}_1$ defined by two vectors $\mathfrak{c}_{abc}, \mathfrak{c}_{a'b'c'}$. Without loss of generality, we may assume that the triads are (123) and (124). We check that they are not connected to the vertices corresponding to duads $(15), (16), (25), (26), (34)$ and the vertices corresponding to synthemes

$$(12, 36, 45), \quad (14, 23, 56), \quad (13, 24, 56), \quad (12, 35, 46), \quad (12, 34, 56).$$

The first four vertices in each set define two disjoint parabolic subdiagrams of type $\tilde{A}_3$ and the pair $(34), (12, 34, 56)$ defines another parabolic subdiagram of type $\tilde{A}_1$.

We refer for the rest of the proof to [368, Lemma 7.2]. □

In the next section, we will see the relevance of this diagram Γ_{CR} and the Coxeter-Richmond polytope to our examples of Enriques and Coble surfaces obtained as quotients of supersingular K3 surfaces by a vector field.

Remark 10.4.4 Let us identify a duad (ab) with a transposition (ab), a syntheme (ij, kl, mn) with the product of three transposition, and a triad (abc) with the product or two commuting cyclic subgroups of order 3. Then, one easily checks the following.

Two duads or synthemes do not commute if and only if they are incident in the extended Cremona–Richmond diagram.

A duad or a synthemes centralizes a triad if and only if they are incident in the extended Cremona–Richmond diagram.

10.5 Quotients of the Supersingular K3 Surface with $\sigma_0 = 1$: Type MI

In this section, we assume that $p = 2$ if not otherwise mentioned. We will construct a family of Enriques surfaces S (resp. a unique Coble surface V) that realize the extended Cremona–Richmond diagram as the intersection diagram of 40 divisor classes in the lattice $\mathrm{Num}(S)$ (resp. $\mathrm{CM}(\mathsf{V})$) with square-norm -2. We will refer to these surfaces as *surfaces of Type* MI. Their canonical covers have 12 (resp. 6) ordinary double points and their minimal resolution are supersingular K3 surface with the Artin invariant one. Finally, we give an example of a Coble surface of type MI in characteristic three.

We start with the finite plane $\mathbb{P}^2(\mathbb{F}_4)$. It has 21 points and 21 lines. Each line contains five points and each point is contained in five lines. This gives an example of a symmetric configuration (21_5).

Let us fix a *six-arc* $\mathcal{P} = \{p_1, ..., p_6\}$ in $\mathbb{P}^2(\mathbb{F}_4)$ (an *n-arc* in a finite plane is a subset of n points no three of which are collinear). It is known that there are 168 six-arcs and the group $\mathrm{PGL}(3, \mathbb{F}_4)$ acts transitively on the set of six-arcs with the stabilizer subgroup isomorphic to $\mathfrak{A}_6$ (see [131, p. 23]).

We have 15 lines in $\mathbb{P}^2(\mathbb{F}_4)$ joining pairs of these points. Each point in $\mathcal{P}$ is contained in 5 such lines. This shows that there are six lines $T_1, \ldots, T_6$ which do not contain any points from $\mathcal{P}$. None of the points in T_i belongs to $\mathcal{P}$. Let $\mathcal{P}^c$ be the set of 15 intersection points of these lines in $\mathbb{P}^2(\mathbb{F}_4)$ and let

$$\pi_{\mathcal{P}} : \mathsf{V}(\mathcal{P}) \to \mathbb{P}^2$$

be the blow-up of this set. Each line $\ell_{ij} = \langle p_i, p_j \rangle$ contains three points from the set $\mathcal{P}^c$, hence its proper transform on $\mathsf{V}(\mathcal{P})$ is a (-2)-curve which we denote by R_{ij}. On

the other hand, each line T_i contains 5 points from $\mathcal{P}^c$, hence its proper transform is a (-4)-curve C_i. Thus, we see that $\mathsf{V}(\mathcal{P})$ is a Coble surface with 6 boundary components.

Remark 10.5.1 Let C be the blow-up of any six-arc $\mathcal{P}$ in the finite plane $\mathbb{P}^2(\mathbb{F}_4)$. It is isomorphic to a smooth cubic surface in $\mathbb{P}^3$. Since the stabilizer subgroup of $\mathcal{P}$ in $\mathrm{PGL}(3, \mathbb{F}_4)$ is isomorphic to $\mathfrak{A}_6$, we see that $\mathfrak{A}_6 \subset \mathrm{Aut}(\mathsf{C})$. The classification of possible automorphism groups of smooth cubic surfaces from [185] shows that C is isomorphic to the Fermat cubic.

Every triad (abc) defines two triangles with sides $\ell_{ab}, \ell_{ac}, \ell_{bc}$ and $\ell_{de}, \ell_{df}, \ell_{ef}$. The triangles intersect at 9 points in $\mathcal{P}^c$. We denote the complementary set of 6 points in this set by $Q(abc)$ (as before $Q(abc) = Q(def)$). We call it a *cardinal set* of 6 points. Let $\mathfrak{c}_{abc}$ be the divisor class $2e_0 - \sum_{p \in Q(abc)} e_p$, where we use the standard notation for a geometric basis of a blow-up of a set of points in the plane. We have $\mathfrak{c}_{abc}^2 = -2$. Since a six-arc does not lie on a conic (necessarily defined over $\mathbb{F}_4$)), $\mathfrak{c}_{abc}$ is not an effective class.

Theorem 10.5.2 *Let $\mathfrak{C}$ be the union of the set of* 15 *classes α_{ab} of the (-2)-curves R_{ab},* 15 *effective roots $\alpha_{ij,kl,mn} = 2E_p + \frac{1}{2}(C_q + C_r)$, where p corresponds to the syntheme (ij, kl, mn) that is common to the totals T_q and T_r, and the set of 10 divisor classes $\mathfrak{c}_{abc}$. Then, the intersection diagram of the* 40 *classes in the Coble–Mukai lattice* $\mathrm{CM}(\mathsf{V}(\mathcal{P}))$ *is the extended Cremona–Richmond diagram.*

Proof Every line $\ell_{ab}, a, b \in \mathcal{P}$, contains three points from $\mathcal{P}^c$, each point is the intersection of two lines T_i. In this way, each duad (ab) defines a syntheme (ij, kl, mn). Since each T_i contains five points in $\mathcal{P}^c$, we see each T_i can be taken as a total. It is immediately seen, that the set of 15 lines ℓ_{ab} and 15 points $\mathcal{P}^c$ form a symmetric (15_3)-configuration invariant with respect to the group $\mathfrak{S}_6$ permuting the points in $\mathcal{P}$. Thus, they form a Cremona–Richmond configuration with the set $\mathcal{A} = \mathcal{P}^c$ of points associated to synthemes and the set $\mathcal{B}$ of lines associated to duads. We have also defined a bijection $\iota : \mathcal{B} \to \mathcal{A}$. It is immediate to check that the intersection diagram of the divisor classes α_{ab} (resp. $\alpha_{ij,kl,mn}$) corresponding to duads (resp. synthemes) is equal to the Levi graph of the Cremona–Richmond configuration.

First of all, $\mathfrak{c}_{abc} \cdot \mathfrak{c}_{a'b'c'} = 2$ for different triads (abc) and $(a'b'c')$. Indeed, replacing (abc) with the complementary set, we may assume that a, b, c and a', b', c' have one common element. Without loss of generality, we can take $(abc) = (123)$ and $(a'b'c') = (145)$, and check that they define two common synthemes $(16, 25, 34)$ and $(16, 24, 35)$ from $Q(abc)$. This shows that $\mathfrak{c}_{123} \cdot \mathfrak{c}_{145} = 2$.

It follows from the definition of a cardinal set $Q(abc)$ that the six lines $\ell_{ij}, i, j \in \{a, b, c\}$ or $\{d, e, f\}$ do not contain any point from this set. Thus, $\mathfrak{c}_{abc} \cdot \alpha_{ij} = 2$. Any other line ℓ_{ij} passes through two points in $\mathcal{P}$, each from different triangle defined by (abc). It also passes through the intersection point of the opposite sides of the triangles. It follows that ℓ_{ij} contains 2 points from the set $Q(abc)$, and hence $\mathfrak{c}_{abc} \cdot \alpha_{ij} = 0$.

Each point in $\mathcal{A} \setminus \mathcal{Q}(abc)$ is the intersection point of two lines ℓ_{ab}, hence it is the intersection point of 2 lines T_i and one line ℓ_{ij} passing through the vertex of the opposite side of the triangle containing this point. This defines the incidence correspondence of type $(9_2, 6_3)$. It follows that each T_i passes through three points not in $\mathcal{Q}(abc)$, hence passes through 2 points from this set. This shows that $\mathfrak{c}_{abc} \cdot C_i = 0$ (as expected because $\mathfrak{c}_{abc}^2 = -2$). Now, $\mathfrak{c}_{abc}$ intersects with multiplicity one each $E_p, p \in \mathcal{Q}(abc)$, and hence it intersects the corresponding effective root with multiplicity two. All other such roots it does not intersect. We leave it to the reader (for example, by taking $(abc) = (123)$ and using Table 10.1) to check that the synthemes are (ai, bj, ck), where $\{i, j, k\} = \{d, e, f\}$.

Finally, if ℓ_{ab} corresponds to the duad (ab), then it intersects all lines corresponding to totals (in 3 pairs intersecting at one point on the line). This shows that α_{ab} intersects three effective roots $\alpha_{ij,kl,mn}$ with multiplicity two, where the three synthemes correspond to (ab) in the Cremona–Richmond configuration. Similarly, each point corresponding to a syntheme (ij, kl, mn) lies on three lines $\ell_{ij}, \ell_{kl}, \ell_{mn}$, and hence $\alpha_{ij,kl.mn}$ intersect $\alpha_{ij}, \alpha_{kl}, \alpha_{mn}$. These are the only non-zero intersections between the divisor classes corresponding to duads and synthemes. This agrees with our definition of the extended Cremona–Richmond diagram.

Remark 10.5.3 Note that the divisor classes $\mathfrak{c}_{abc}$ are never effective. In fact, any irreducible conic in plane $\mathbb{P}(\mathbb{F}_4)$ contains exactly five points in this plane, and hence neither $\mathcal{P}$ nor $\mathcal{Q}(abc)$ lie on a conic.

We know that each triad (abc) defines a pair of triangles of lines with vertices a, b, c and d, e, f intersecting at the set B of 9 points from $\mathcal{P}^c$. We also know that each point in $\mathcal{B}$ lies on two lines T_i and each line passes through two points in $\mathcal{B}$. This defines a symmetric configuration. The only possible Levi graphs of this configuration is a hexagon with vertices in $\mathcal{Q}(abc)$ or the disjoint sum of two triangles. Thus, we see that a choice of (abc) defines two triangles with vertices in $\mathcal{Q}(abc)$.

Example 10.5.4 Choose $(abc) = (123)$. Then, $\mathcal{Q}(123)$ consists of 6 points corresponding to synthemes

$$(14)(25)(36),\ (14)(26)(35),\ (15)(24)(35),\ (15)(26)(34),\ (16)(25)(34),\ (16)(24)(35).$$

They define two triangles with sides (T_1, T_2, T_4) and (T_3, T_5, T_6).

It follows from above that a choice of a triad (abc) defines four triangles of lines in $\mathbb{P}^2(\mathbb{F}_4)$, considered as plane cubic in $\mathbb{P}^2$ each pair intersecting at the same set of points B. The four triangles are the four reducible members of the Hesse pencil of cubic curves (4.9.33) which we discussed in Section 4.9 in Volume I and Section 8.8. It is defined uniquely, up to projective transformation, over any field containing three distinct third roots of unity. So, we can choose coordinates to assume that its equation is:

$$\lambda(x^3 + y^3 + z^3) + \mu xyz = 0. \tag{10.5.1}$$

We can order the coordinates of the nine base points of the Hesse pencil as follows:

$$\begin{aligned} &p_1 = [0,1,1], \quad p_2 = [0,1,\epsilon], \quad p_3 = [0,1,\epsilon^2], \\ &p_4 = [1,0,1], \quad p_5 = [1,0,\epsilon^2], \quad p_6 = [1,0,\epsilon], \\ &p_7 = [1,1,0], \quad p_8 = [1,\epsilon,0], \quad p_9 = [1,\epsilon^2,0], \end{aligned} \tag{10.5.2}$$

where ϵ is a primitive 3rd root of unity.

The four triangles are $\Delta_\infty = V(xyz)$ and the triangles $\Delta_{\epsilon^i}, i = 0,1,2$, with sides $x + \epsilon^a y + \epsilon^b z, a + b \equiv i \mod 3$. They correspond to members of the pencil with $t = \mu/\lambda = \infty, 1, \epsilon, \epsilon^2$.

Let $L_{0,i}$ (resp. $L_{1,i}$, resp. $L_{2,i}$) be the lines that join $[1,0,0]$ (resp. $[0,1,0]$, resp. $[0,0,1]$) with three base points $[0,1,\epsilon^i]$ (resp. $[1,0,\epsilon^i]$, resp. $[1,\epsilon^i,0]$). Three of the lines $L_{0,\epsilon^i}, L_{1,\epsilon^j}, L_{1,\epsilon^k}$ intersect at a point $p_{i,j,k}$ if and only if $i + j + k \equiv 0 \mod 3$. Thus, we have a symmetric configuration of type (9_3) of nine lines and nine intersection points $p_{i,j,k}$. It is a special case of the *Ceva configuration* Ceva(n) with $n = 3$.

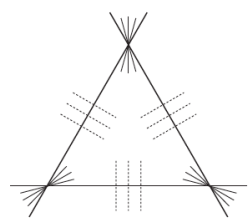


Fig. 10.3 Nine sections and nine bisections on H

For any base point p_i, the polar conic of a general member of the Hesse pencil with pole at p_i is equal to the union of the tangent line at this point and a line ℓ_i that does not depend on the parameter. It is classically known as a *harmonic line* of the Hesse pencil (see [16]). In our case where $p = 2$, the nine harmonic lines coincide with the nine lines $L_{0,i}, L_{1,j}, L_{2,k}$. The nine points $p_{i,j,k}$ are the vertices of the triangles Δ_{ϵ^i}. So, we see that each harmonic line ℓ_i passes through 4 vertices, one in each triangle and intersect the opposite sides at the base point p_i of the pencil. Note that this is specific to the characteristic 2 case. In other characteristics (different from 3), the harmonic lines intersect the opposite side of each triangle at a point different from the base point.

We denote by $\pi_{abc} : \mathsf{H} \to \mathbb{P}^2$ the blow-up of the set B. Let $f_{abc} : \mathsf{H} \to \mathbb{P}^1$ be the corresponding relatively minimal rational elliptic surface. It is one of the extremal rational elliptic surfaces in characteristic 2 which we classified in Section 4.8 in Volume I. We have the blowing down morphism $p_{abc} : \mathsf{V}(\mathcal{P}) \to \mathsf{H}$ that blows up the set of six singular points of two fibers whose image in the plane is the union of six lines T_i corresponding to the totals.

Each base point defines a section of the elliptic fibration on the surface H obtained by blowing up the set B of base points. It is the exceptional curve E_p, $p \in B$. Fixing such a section E_p, we have the standard Bertini involution whose restriction to a general fiber is the negation involution. Its set of fixed points consists of the section and the harmonic line corresponding to the base point p. Each harmonic line defines an inseparable bisection of the pre-image of the Hesse pencil on H. It intersects each singular fiber at one of its singular points.

Let $\sigma_{abc} : \mathsf{H}' \to \mathsf{H}$ be the blow-up of the 12 vertices of the four triangles in H. This is of course the same as the blow-up of the set $\mathbb{P}^2(\mathbb{F}_4)$ in $\mathbb{P}^2$, or the blow-up of the set $\mathcal{P}$ on the Coble surface $\mathsf{V}(\mathcal{P})$. The pre-image of the Hesse pencil is an elliptic pencil on H' (not relatively minimal). It has four reducible fibers formed by a hexagon of smooth rational curves. Three mutually non-incident vertices correspond to (-1)-curves, they enter with multiplicity 2. The other such set corresponds to (-4)-curves. The pre-image of a harmonic line is a (-4)-curve intersecting the double component of a fiber. The pre-image of a section on H is a (-1)-curve that intersects the simple component of a fiber.

The Frobenius base change of the Hesse elliptic fibration $f_{abc} : \mathsf{H} \to \mathbb{P}^1$ defines a surface X with 12 singular points lying over the singular points of the fibers of f_{abc}. Locally, at each such point the cover $X \to \mathsf{H}$ is given by $z^2 = xy$, hence all twelve points are ordinary double points. Let $\sigma' : Y \to X$ be the minimal resolution of singularities. The proper transform of an irreducible component of singular fibers of the Hesse pencil on Y are (-2)-curves, as follows from the intersection theory on a normal surface. This shows that the pre-image of the Hesse pencil on Y is a relatively minimal elliptic fibration $f : Y \to \mathbb{P}^1$ with four fibers of type $\tilde{A}_5$. The pre-image of any section of the Hesse pencil is a (-2)-curve which is a section of f. The formula for the canonical class of an elliptic surfaces tells us that Y is a K3 surface. It is easy to see that the surface Y is the minimal resolution of singularities of the split inseparable cover $\mathbb{P}^2$ defined by the sheaf $\mathcal{L} = \mathcal{O}_{\mathbb{P}^2}(3)$ and a section of $\mathcal{L}^{\otimes 2}$ defined by the union of any two members of the Hesse pencil. By Proposition 0.2.10 in Volume I, the inseparable cover the set $\mathbb{P}^2(\mathbb{F}_4)$ of 21 ordinary double points. The surface Y is the minimal resolution of singularities of an inseparable finite map of degree 2 of the surface H' (see [189]). It is a supersingular surface with Artin invariant 1. We refer to loc. cit. for different birational models of Y.

It follows from above that the four reducible fibers F_α of $f : Y \to \mathbb{P}^1$ over the points $\alpha = 1, \epsilon, \epsilon^2, \infty$ are the unions of two sets F_α^+, F_α^-, each consists of three disjoint components. We may assume that F_α^+ consists of exceptional curves over the singular points of the Hesse pencil, and F_α^- consists of the proper transforms of fibers of the Hesse pencil. In our duad-syntheme notation, F_α^+ can be indexed by synthemes, and F_α^- by duads. In Figure 10.4, the components F_α^+ (resp. F_α^-) are drawn in blue (resp. red).

The proper transforms of sections of $\mathsf{H} \to \mathbb{P}^1$ are of course sections of $f : Y \to \mathbb{P}^1$. They intersect the components F_α^+. In Figure 10.4, they are drawn in red. The proper transforms of harmonic lines of the Hesse pencil are also sections. They intersect the components F_α^-. They are drawn in blue. Altogether we obtain 18 sections which generate the Mordell–Weil group isomorphic to $(\mathbb{Z}/3\mathbb{Z})^{\oplus 2} \oplus \mathbb{Z}/2\mathbb{Z}$.

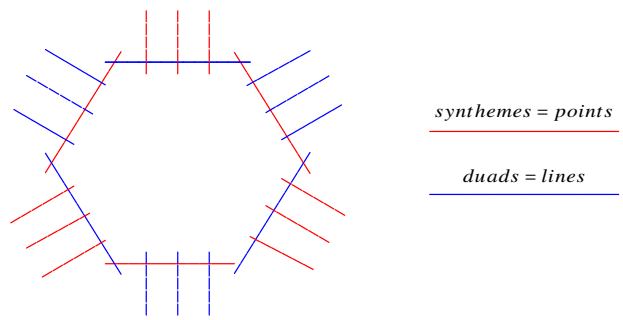


Fig. 10.4 Fibers and sections of $f : Y \to \mathbb{P}^1$

The unique element of order 2 in the Mordell–Weil group of $f : Y \to \mathbb{P}^1$ translates the normal subgroup of index 2 to its coset. Thus, it sends the blue sections corresponding to base points of the Hesse pencil to red sections corresponding to harmonic lines. It also translates the 12 blue components of reducible fibers to 12 red components. The involution of Y defined by the order two section is called a *switch* [189]. Also it defines a correlation between $\mathbb{P}^2(\mathbb{F}_4)$ and the dual finite plane.

Remark 10.5.5 It is shown by Mukai that the surface Y admits a birational model isomorphic to the intersection of two divisors of type $(2, 1)$ and $(1, 2)$ in $\mathbb{P}^2 \times \mathbb{P}^2$:

$$x_0^2 y_0 + x_1^2 y_1 + x_2^2 y_2 = y_0^2 x_0 + y_1^2 x_1 + y_1^2 x_2 = 0.$$

(see [189]). The switch involution is induced by the interchanging the factors in $\mathbb{P}^2 \times \mathbb{P}^2$. Its set of fixed points is its intersection with the diagonal which is a supersingular elliptic curve in characteristic 2. It is the unique smooth supersingular fiber of the elliptic fibration of $f : Y \to \mathbb{P}^1$ arising from the Hesse pencil on which the translation by the section of order 2 acts identically.

We know that a harmonic line passes through one of the base points of the Halphen pencil. It follows that the corresponding blue and red sections intersect transverselly at one point. Again, this is special to positive characteristic, since it follows from Proposition 4.2.1 that sections of finite order prime to the characteristic do not intersect.

Let $\pi : X \to \mathsf{V}(\mathcal{P})$ be the canonical cover. We have $\omega_X \cong \mathcal{O}_X$ and Proposition 0.2.10 tells us that $\mathrm{Sing}(X)$ is a finite subscheme Z with $h^0(\mathcal{O}_Z) = 12$. Since X has an ordinary double point over singular points of fibers of type $\tilde{A}_2$, we infer that X has exactly 12 double points and its minimal resolution Y is a supersingular K3 surface. It has an elliptic fibration with four fibers of type $\tilde{A}_5$ and with 18 sections, and hence the Shioda–Tate formula implies that the Artin invariant of Y is equal to 1.

Applying Theorem 10.3.2, we obtain the following:

Theorem 10.5.6 *The canonical cover of the Coble surface $\mathsf{V}(\mathcal{P})$ has 12 ordinary double points and its minimal resolution is the supersingular K3 surface Y with Artin invariant one.*

The Coble surface $\mathsf{V}(\mathcal{P})$ is called the *Coble surface of type* MI.

Next, we show that the surface Y admits a family of rational vector field $\partial_{\alpha,\beta}$, where $\alpha + \beta = \alpha\beta = a$ such that the quotient by the derivation is birationally isomorphic to an Enriques surface if $a \neq 1$ and to the Coble surface $\mathsf{V}(\mathcal{P})$ otherwise.

Let

$$y^2 + sxy + y + x^3 + 1 + s^3 = 0 \tag{10.5.3}$$

be the Weierstrass equation of the Hesse pencil (see (4.9.34) in Volume I), where $s = \mu/\lambda$. Its discriminant is equal to $(1 + s^3)^3$.

We will construct the surfaces $S_{\alpha,\beta}$ as the quotients of Y by a rational vector field and will describe a certain configuration of 30r (-2)-curves on $S_{\alpha,\beta}$ that is equal to the image of the double Hesse configuration of 42 (-2) curves on Y. The construction of the vector field is similar to the constructions in Section 8.10 and we will omit the details.

Replacing s with t^2 in (10.5.3), we obtain the Weierstrass equation of the pre-image of the Hesse pencil on Y:

$$y^2 + t^2xy + y + x^3 + 1 + t^6 = 0 \tag{10.5.4}$$

Following Example 8.10.9, we consider the rational derivation on Y

$$\partial_{\alpha,\beta} = \frac{1}{t+1}\left((t+1)(t+\alpha)(t+\beta)\frac{\partial}{\partial t} + (1+t^2x)\frac{\partial}{\partial x}\right),$$

where $\alpha, \beta \in \Bbbk,\ \alpha + \beta = \alpha\beta,\ \alpha^3 \neq 1$. The choice of the Weierstrass model defines a choice of an irreducible component in each singular fiber that is intersected by the zero section. If $t \neq \infty$, it does not enter in the divisor of poles of $\partial_{\alpha,\beta}$ on Y and it does enter if $t = \infty$. A direct calculation following Lemma 10.3.13 shows that the extension of $\partial_{\alpha,\beta}$ to Y will have poles on blue components.

Lemma 10.5.7 (i) $\partial^2_{\alpha,\beta} = \alpha\beta \cdot \partial_{\alpha,\beta}$, *hence,* $\partial_{\alpha,\beta}$ *is* 2*-closed. Moreover, it is of additive type if* $\alpha = \beta = 0$ *and of multiplicative type otherwise.*

(ii) *On the surface* Y*, the divisor* D *of* ∂ *is given by*

$$D = -(F^+_\epsilon + F^+_{\epsilon^2} + F^-_1 + F^-_\infty),$$

and $D^2 = -24$.

(iii) *The integral curves with respect to* $\partial_{\alpha,\beta}$ *are the smooth fibers over* $t = \alpha, \beta$ *(in the case where* $\alpha = \beta = 0$*, the smooth fiber over* $t = 0$*) and* $F^-_1, F^-_\infty, F^+_\epsilon, F^+_{\epsilon^2}$.

Since K_Y is trivial and $D^2 = -24$, it follows formula (0.3.4) that

$$24 = c_2(Y) = \deg(Z) - \langle K_Y, D\rangle - D^2,$$

where Z is the scheme of non-divisorial zeros of $\partial_{\alpha,\beta}$. Therefore, $\deg(Z) = 0$ and ∂ has no isolated zeros. Thus, the quotient surface Y^∂ by $\partial_{\alpha,\beta}$ is non-singular. Denote by $\pi' : Y \to Y^\partial$ the quotient map. It is a finite inseparable map of degree 2.

The canonical bundle formula from Proposition 0.3.14in Volume I gives

$$K_Y = \pi'^* K_{Y^\partial} + D.$$

If a (-2)-curve C is integral with respect to ∂, then applying Proposition 0.3.19, we obtain that $\pi'(C)$ is a (-1)-curve on Y^∂.

Let $\sigma' : Y^\partial \to S_{\alpha,\beta}$ be the contraction of the sum $\mathcal{E}$ of the twelve (-1)-curves. Note that $\sigma'^*(\mathcal{E}) = -D$. We have $K_{Y^\partial} = \sigma'^*(K_{S_{\alpha,\beta}}) + \mathcal{E}$, and hence

$$0 = D + \pi'^*(\sigma'^*(K_{S_{\alpha,\beta}})) + \pi'^*(\mathcal{E}) = D + \pi'^*(\sigma'^*(K_{S_{\alpha,\beta}})) - D = \pi'^*(\sigma'^*(K_{S_{\alpha,\beta}})).$$

Thus, $\pi'^*(\sigma'^*(K_{S_{\alpha,\beta}})) = 0$, hence $K_{S_{\alpha,\beta}}$ is numerically trivial. Since π is finite and purely inseparable, $b_2(Y^\partial) = b_2(Y) = 22$, and hence $b_2(S_{\alpha,\beta}) = 10$. Thus, $S_{\alpha,\beta}$ is an Enriques surface. Let $\pi' \circ \sigma' = \sigma \circ \pi$, where $\sigma : Y \to X$ is a birational morphism and $\pi : X \to S_{\alpha,\beta}$ is a finite inseparable cover. We immediately see σ blows down the integral components of D to ordinary double points and $\pi : X \to S_{\alpha,\beta}$ is the canonical cover of $S_{\alpha,\beta}$.

We will refer to the surface $S_{\alpha,\beta}$ as an *Enriques surface of Type MI in characteristic 2*.

To summarize, we have the following commutative diagram:

$$\begin{tikzcd}
\mathsf{H}' \arrow[d, "\sigma_{abc}"'] & Y \arrow[l, "q'"'] \arrow[r, "\pi'"] \arrow[d, "\sigma"'] & Y^\partial \arrow[d, "\sigma'"] \\
\mathsf{H} \arrow[d, "f_{abc}"'] & X \arrow[l, "q"'] \arrow[r, "\pi"] \arrow[d, "f"] & S_{\alpha,\beta} \arrow[d, "f'"] \\
\mathbb{P}^1 & \mathbb{P}^1 \arrow[l, "\mathbf{F}"'] \arrow[r, "\mathbf{F}"] & \mathbb{P}^1
\end{tikzcd} \tag{10.5.5}$$

where $\mathbf{F}$ is the Frobenius morphism.

It follows from Lemma 10.5.7 that the integral curves in fibers over $1, \infty$ (resp. ϵ, ϵ^2) originate from lines ℓ_{ab} (resp. synthemes (ij, kl, mn)), they are drawn in red (resp. blue) in Figure 10.4 below. When we blow down their images in Y^∂, we obtain Figure 10.5 below which describes the reducible fibers of the Hesse type elliptic fibration on $S_{\alpha,\beta}$.

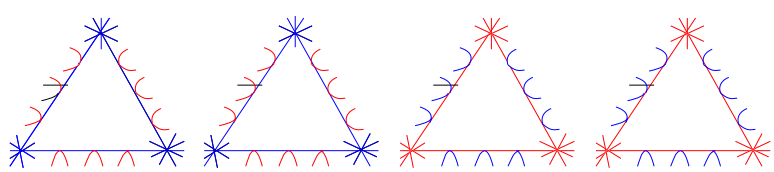


Fig. 10.5 Fibers of the Hesse type elliptic fibration on $S_{\alpha,\beta}$

There are two fibers for each type. All in all, we see 30 (-2)-curves. Twelve are components of fibers, they are divided into two sets of six, blue and red. The rest are 18 special bisections divided into two sets of 9, blue and red. The set of 30 curves is the union of a set $\mathcal{A}$ of blue curves and the set $\mathcal{B}$ of red curves. The incidence relation between $\mathcal{A}$ and $\mathcal{B}$ is the Cremona–Richmond configuration (15_3). However, the intersection diagram has the double edges of the corresponding Levi graph. If we identify the vertices of the blue triangles with the set $[1, 6]$ and the vertices of the red triangles with the set of totals, then we see that the curves from the set $\mathcal{A}$ (resp. resp. $\mathcal{B}$) intersecting at $a \in [1, 6]$ (resp. at T_i), correspond to duads $(ab), b \neq a$, (resp. the five synthemes entering in T_i). This shows that the intersection graph of $\mathcal{A}$ and $\mathcal{B}$ is the dual of the complete graph $K(6)$ on 6 vertices.

Let $\mathfrak{c}_{ijk}$ be the divisor class on the surface H' equal to $2e_0 - \sum_{p \in Q(ijk)} e_p$ (we identify it with the corresponding class on $\mathsf{V}(\mathcal{P})$). We immediately check that its pre-image in Y is a divisor class with self-intersection -4 which is orthogonal to the 12 components of fibers that are blown down to singular points of X under the map σ. Its image $\bar{\mathfrak{c}}_{abc}$ in $S_{\alpha,\beta}$ is a divisor class of square norm -2 that intersects six red curves (corresponding to the points from $Q(abc)$) and six blue curves (corresponding to harmonic lines) with multiplicity two.

There are 12 canonical points on $S_{\alpha,\beta}$ which are the images of $F^-_\epsilon, F^-_{\epsilon^2}, F^+_1, F^+_\infty$. These 12 points are indexed by the set of six points and the set of six totals. In Figure 10.6 below, the black (resp. white) circles are canonical points corresponding to six points (resp. six totals). Each line passing through two black circles (resp. white circles) is the (-2)-curve corresponding to the duad (resp. syntheme) containing two points (resp. contained in the two totals).

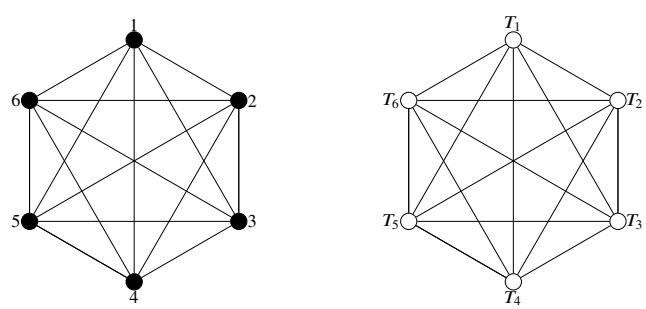


Fig. 10.6 Six points (six totals) and 15 duads (15 synthemes)

This shows that the intersection diagram of the set of 40 divisor classes is the same as the extended Cremona–Richmond diagram.

Theorem 10.5.8 *The Enriques surface $S_{\alpha,\beta}$ contains a set of* 30 (-2)*-curves, the union of two sets $\mathcal{A}$ and $\mathcal{B}$ of 15 divisor classes, and it also contains a set $\mathcal{C}$ of 10 divisor classes of square norm* -2*. The set $\mathcal{A}$ (resp. $\mathcal{B}$) is the union of two subsets $\mathcal{A}_{\text{fib}}$ and $\mathcal{A}_{bis}$ (resp. $\mathcal{B}_{\text{fib}}$ and $\mathcal{B}_{bis}$) of cardinalities* 6 *and* 9.

- *The intersection graph of the sets $\mathcal{A}$ and $\mathcal{B}$ are the dual graphs of the complete graph $K(6)$.*
- *Each element from $\mathcal{A}_{\text{fib}}$ (resp. $\mathcal{B}_{\text{fib}}$) intersects three elements from the set $\mathcal{B}_{bis}$ (resp. $\mathcal{A}_{bis}$) at one point with multiplicity 2. Each element from $\mathcal{A}_{bis}$ (resp. $\mathcal{B}_{bis}$) intersects two elements from the set $\mathcal{B}_{\text{fib}}$ (resp. $\mathcal{A}_{\text{fib}}$) at one point with multiplicity 2 and also intersects one element from $\mathcal{B}_{bis}$ (resp. $\mathcal{B}_{bis}$) at one point with multiplicity 2.*
- *The incidence relation between the sets $\mathcal{A}$ and $\mathcal{B}$ defines a symmetric configuration (15_3) isomorphic to the Cremona–Richmond configuration.*
- *The divisor classes from $\mathcal{C}$ intersect each other with multiplicity 2.*
- *Each divisor class from $\mathcal{C}$ intersects six divisor classes from $\mathcal{B}$ and 6 divisor classes from $\mathcal{A}$ with multiplicity 2.*
- *The surface contains 10 elliptic fibrations of Hesse type indexed by the set of triads (up to complementary set). The curves from the sets $\mathcal{A}_{bis}$ and $\mathcal{B}_{bis}$ are inseparable special bisections. Each fibration has 18 inseparable special bisections divided into two subsets $\mathcal{A}_s$ and $\mathcal{B}_s$ of nine of the sets $\mathcal{A}$ and $\mathcal{B}$.*
- *The set of 12 singular points of singular fibers of the fibration is the set of canonical points on $S_{\alpha,\beta}$.*
- *There is a bijection between the set $\mathcal{A}$ and the set of duads and the set $\mathcal{B}$ with the set of synthemes.*

The intersection diagram of the total set of 40 divisor classes is the extended Cremona–Richmond diagram.

We have ten elliptic fibrations $|2F_{abc}|$ of Hesse type corresponding to 10 triads. The next lemma follows easily from Theorem 10.5.8, and we leave its proof to the reader.

Lemma 10.5.9 *Let $|2F_{abc}|$ be the 10 elliptic fibrations on $S_{\alpha,\beta}$ of Hesse type corresponding to 10 triads.*

(i) Each of 45 pairs of the fibrations form a non-degenerate U-pair. In particular, it defines a bielliptic involution.
(ii) If we fix one of the 45 pairs of the Hesse elliptic fibrations, then each fiber of the first one has a unique common component with some fiber from the second one.
(iii) A component of a reducible fiber of one fibration either does not intersect a fiber of another fibration, or it is tangent to a component of its reducible fiber, or it passes through a singular point of its reducible fiber.

It follows from the lemma that 45 pairs of the elliptic fibrations define 45 bielliptic maps. Since the surface $S_{\alpha,\beta}$ does not admit a quasi-elliptic fibrations, this map is separable, and hence, defines an involution of the surface.

Proposition 10.5.10 *Let τ_{ij} be the bielliptic involution defined by a pair of the Hesse elliptic fibrations. Then, it fixes the four common components of two fibers pointwise and permutes by pairs the other components in the same fiber. It acts on the set of 30 (-2)-curves as the product of two transpositions.*

Proof Without loss of generality, we may assume that our pair of Hesse elliptic fibrations corresponds to triads (123) and (124). The four fibers of the first fibration correspond to two triples of duads and two triples of synthemes:

$$\begin{array}{ll} (12), (23), (13) & (15, 26, 34), (16, 23, 45), (13, 25, 46) \\ (45), (46), (56) & (15, 23, 46), (13, 26, 45), (16, 25, 34). \end{array}$$

The four fibers of the second fibration correspond to two triples of duads and two triples of synthemes:

$$\begin{array}{ll} (12), (24), (14) & (15, 26, 34), (14, 25, 36), (16, 24, 35) \\ (35), (36), (56) & (14, 26, 35), (15, 24, 36), (16, 25, 34). \end{array}$$

The four common components $R_1, \ldots, R_4$ are (-2)-curves corresponding to duads (12), (56) and synthemes (15, 26, 34), (16, 25, 34). The bielliptic map $\phi : S_{\alpha,\beta} \to \mathsf{D}$ blows them down to nonsingular points in D ($\mathsf{D} = \mathsf{D}_1$ or D_3, depending whether $S_{\alpha,\beta}$ is a classical or a supersingular surface). We may assume that the image of a fiber is a line in the plane model of D. The proper transform of this line splits in the cover into two components of the fiber that are different from R_i. This shows that the bielliptic involution τ acts on the set of 20 components of the two fibrations as the involution (12)(56).

At the end of the previous section we listed the remaining 10 curves from the set $\mathcal{A} \cup \mathcal{B}$. They are the following:

$$\begin{array}{llllll} (15), & (16), & (25), & (26), & (34), & (12, 36, 45), \\ (14, 23, 56), & (13, 24, 56), & (12, 35, 46), & (12, 34, 56). & & \end{array} \tag{10.5.6}$$

This set is invariant with respect to the involution (12)(56). Each of these curves intersects the half-fibers F_{123} and F_{124} of the elliptic fibrations $|2F_{123}|$ and $|F_{124}|$ with multiplicity one. Thus, the bielliptic map defines a bijective map from each curve to a rational quartic curve in $\mathbb{P}^4$. Its image under the involution splits into two curves from the set of ten curves from above. This shows that τ leaves the set $\mathcal{A} \cup \mathcal{B}$ invariant and acts on this set as the involution (12)(56). □

The products of two commuting transpositions generate the subgroup $\mathfrak{A}_6$ of $\mathfrak{S}_6$. We see the group G generated by the 10 involution acts on the set $\mathcal{A} \cup \mathcal{B}$ as a group isomorphic to $\mathfrak{A}_6$. Since the 30 (-2)-curves generate $\mathrm{Num}(S_{\alpha,\beta})$, we obtain the kernel of $G \to \mathfrak{A}_6$ consists of numerically trivial automorphisms.

Proposition 10.5.11 *Let S be an Enriques or a Coble surface of type* MI. *Then,* $\mathrm{Aut}_{\mathrm{nt}}(S)$ *is trivial.*

Proof A numerically trivial automorphism preserves any of the ten Hesse pencils on S. It leaves all their fibers invariant. Thus, it acts identically on the pencil, and hence, acts identically on each of the 18 inseparable bisections (or nine sections if S is a Coble surface). This shows that it fixes too many points on the generic fiber, and hence acts identically on it, and hence identically on the surface. □

Corollary 10.5.12 *The group of automorphisms of $S_{\alpha,\beta}$ generated by 45 bielliptic involutions τ_{ij} is isomorphic to $\mathfrak{A}_6$.*

Recall from Remark 10.5.1 that the Coble surface of type MI is obtained from the Fermat cubic surface C by blowing up the set of nine points, the base points of a Hesse pencil $f_{abc} : \mathsf{V}(\mathcal{P}) \to \mathbb{P}^1$. Here, we fix an isomorphism from C to the blow-up of the set $Q(abc)$ of six points.

It is known that the group $\mathrm{Aut}(\mathsf{C})$ is isomorphic to the subgroup $\mathrm{PSU}_4(2)$ of $\mathrm{PGL}_4(\mathbb{F}_4)$ that leaves invariant the Hermitian form $x^3 + y^3 + z^3 + w^3 = 0$ over $\mathbb{F}_4$ [185]. It contains the group $\mathfrak{S}_6$ as a maximal subgroup of index 36 isomorphic to the stabilizer subgroup of a double-sixer on the cubic surface. The subgroup $W(\mathsf{E}_6)^+$ of $W(\mathsf{E}_6)$ generated by the product of two simple reflections is of index 2 and it is isomorphic to the group $\mathrm{PSU}_4(2)$. All elements in the coset are realized on the surface as the compositions of an automorphism and the Frobenius endomorphism. The orbits of $\mathrm{Aut}(\mathsf{C})$ on linear subspaces of $\mathbb{P}^3(\mathbb{F}_4)$ containing the set $\mathsf{C}(\mathbb{F}_4)$ are known [185, Lemma 5.3]. One orbit is the set $\mathsf{C}(\mathbb{F}_4)$ that consists of 45 points and coincides with the set of Eckardt points on C. Its stabilizer subgroup is a maximal subgroup of $W(\mathsf{E}_6)^+$ of index 45. There is also an orbit of 40 planes; each cuts out a smooth supersingular elliptic curve E on C. Its stabilizer subgroup is one of the two maximal subgroups of $W(\mathsf{E}_6)^+$ of index 40 (another one is the stabilizer of a point in $\mathbb{P}^3(\mathbb{F}_4) \setminus \mathsf{C}(\mathbb{F}_4)$). It is isomorphic to an extension of the binary tetrahedral group with the normal subgroup isomorphic to the Heisenberg group of order 3^3. The quotient by the center is isomorphic to the Hesse group G_{216} of automorphisms of a Hesse pencil. In our case, it acts on the base of the fibration as the affine group $\mathbb{A}^1(\mathbb{F}_4)$ and the kernel is isomorphic to the group $(\mathbb{Z}/3\mathbb{Z})^{\oplus 2}$ of translations and the Bertini involution.

In fact, the plane contains nine Eckardt points lying on the cubic, and the blowing up this set, we obtain our Coble surface $\mathsf{V}(\mathcal{P})$. The subgroup $3^2 \rtimes \mathfrak{S}_3$ of G_{216} of order 54 is realized as a subgroup of $\mathrm{Aut}(\mathsf{C})$ preserving one of the ten Hesse elliptic fibrations. We know that $\mathfrak{A}_6$ acts transitively on the lifts of these fibrations on $\mathsf{V}(\mathcal{P})$, Thus, the intersection of G_{216} with $\mathfrak{A}_6$ is a subgroup of order 36.

The group $\mathrm{Aut}(\mathsf{C})$ contains two conjugacy classes of involutions, type 2A and type 2B in terminology of the ATLAS. The Hesse subgroup contains an involution of type 2A; it acts on the Hesse pencil fixing pointwise its unique smooth supersingular fiber. In our situation, it is realized by an involution from $\mathfrak{A}_6$ that preserves the triad (abc), for example the product $(ab)(de)$ of two transpositions. For such a triad there exists a unique other triad (ijk) that is invariant with respect to the same involution. For example, if $(abc) = (123)$ and the involution is as above, then $(ijk) = (126)$. This shows that all involutions of type $2A$ that occur in $\mathrm{Aut}(\mathsf{V}(\mathcal{P}))$ originate from the bielliptic involutions on the Coble surface similar to ones we studied in Proposition 10.5.10.

An involution g of type 2B is realized on C as an involution that leaves invariant a double-sixer and switches its sixers. If $p : \mathsf{C} \to \mathbb{P}^2$ is the blowing down of one sixer, then g descends to a Cremona transformation $\bar{g}$ that blows down six conics through five of its six fundamental points to the fundamental points. Note that, if we order the fundamental points $p_1, \ldots, p_6$ and denote by C_k the conic through all points

except p_k, then $\bar{g}(C_k) = p_{\sigma(k)}$, where σ is the product of three transpositions. The involution $\bar{g}$ is given by the homaloidal linear system of plane quintics with double points at the fundamental points. In the standard geometric basis, g acts as the composition of the involution with respect to $2e_0 - e_1 - \cdots - e_6$ and the product of three commuting transpositions of the e_i's. Its set of fixed points is a line on C.

The known properties of the Fermat cubic in characteristic 2 show that each line on C contains 5 Eckardt points and we have altogether 45 Eckardt points. In our case, the set of 27 lines consists of nine lines

$$(ad),\ (ae),\ (af),\ (bd),\ (be),\ (bf),\ (cd),\ (ce),\ (cf),$$

six lines $T_1, \ldots, T_6$, six conics through five points in $Q(abc)$, and six exceptional curves $E_p, p \in Q(abc)$.

Recall that an element of the Weyl group $W(\mathsf{E}_6)$ that switches the ordered sixers in a double-sixer is a reflection s_α with respect to the vector $\alpha = 2e_0 - e_1 - \cdots - e_6$. Since this reflection does not belong to $W(\mathsf{E}_6)^+$, it is never realized by an automorphism. However, its composition with an odd permutation from the subgroup $\mathfrak{S}_6$ of $W(\mathsf{E}_{10})$ could be realized by an automorphism. In other words, a cubic surface may contain an automorphism that switches the sixers but does not preserve the orders of the set of six lines. Since $\mathrm{Aut}(\mathsf{C}) \cong W(E_6)^+$, we see that in our case such an automorphism exists. It is of type 2B in the terminology of the ATLAS. Its fixed locus is a line on the cubic surface. As a Cremona transformation, it is given by a homaloidal linear system of quintics with double points at points from the set $\mathcal{A}$.

We apply this by choosing a cardinal set $Q(abc)$ of six points. Since $\mathfrak{A}_6$ is a subgroup of $\mathrm{Aut}(\mathsf{C})$, a choice of the quintic Cremona involution is unique up to composition with an even involution from $\mathfrak{A}_6$ that commutes with it.

Example 10.5.13 We take for $Q(abc)$ the following set of six points in $\mathbb{P}^2$:

$$p_1 = [1, 0, 0],\ p_2 = [0, 1, 0],\ p_3 = [0, 0, 1],\ p_4 = [1, 1, 1],\ p_5 = [1, \epsilon, \epsilon^2],\ p_6 = [1, \epsilon^2, \epsilon].$$

They are the vertices of two triangles:

$$V(x_0x_1x_2), \quad V((x_0 + x_1 + x_2)(x_0 + \epsilon x_1 + \epsilon^2 x_2)(x_0 + \epsilon^2 x_1 + \epsilon x_2)).$$

Consider the following quintic Cremona involution g:

$$\begin{aligned}&[x_0, x_1, x_2] \mapsto [x_0^3x_1^2 + x_0^2x_1x_2^2 + x_0x_1^3x_2 + x_1^2x_2^3,\\ &x_0^3x_1x_2 + x_0^2(x_1^3 + x_2^3) + x_0x_1^2x_2^2,\ x_0^3x_2^2 + x_0^2x_1^2x + x_0x_1x_2^3 + x_1^3x_2^2].\end{aligned} \tag{10.5.7}$$

We leave it to the reader to verify that it is indeed a Cremona involution. Let C_i be the conic through the points $p_1, \ldots, p_6$ except p_i. Then, g blows down $(C_1, \ldots, C_6)$ to $(p_1, p_2, p_3, p_4, p_5, p_6)$. Its action on $\mathrm{Pic}(\mathsf{C})$ in the geometric basis defined by the ordered six-arc A is given by the matrix

$$\begin{pmatrix} 5 & 2 & 2 & 2 & 2 & 2 & 2 \\ -2 & -1 & 0 & -1 & -1 & -1 & -1 \\ -2 & 0 & -1 & -1 & -1 & -1 & -1 \\ -2 & -1 & -1 & 0 & -1 & -1 & -1 \\ -2 & -1 & -1 & -1 & 0 & -1 & -1 \\ -2 & -1 & -1 & -1 & -1 & 0 & -1 \\ -2 & -1 & -1 & -1 & -1 & -1 & 0 \end{pmatrix}.$$

We see that its trace is equal to 3, and this agrees with the fact this transformation defines an involution of the cubic surface of type 2B. Its acts as the composition of two reflections with respect to the roots $2e_0 - e_1 - \cdots - e_6$ and $e_1 - e_2$.

We check that it switches the sides $V(x_0)$ and $V(x_1)$ of the first triangle and leaves its remaining side pointwise fixed. It leaves invariant the sides of the second triangle. It leaves invariant the 10 base points of the Hessian pencil spanned by the two triangles. It transforms the remaining 6-arc $\mathcal{P}$ of points $[\epsilon^i, 1, 1]$, $[1, \epsilon^i, 1]$, $[1, 1, \epsilon^i 1], i = 1, 2$, to itself by switching three pairs $[\epsilon, 1, 1], [\epsilon^2, 1, 1]$, etc.. The permutation is odd because g does not come from a projective automorphism of the plane. The set $\mathcal{P}$ is the set of vertices of the remaining two triangles of the Hesse pencil. The image of the Hesse pencil under the involution is a Hesse pencil. The images of each of the two triangles with vertices at $\mathcal{P}$ is the union of three plane quintic curves with double points from the set $\mathcal{A}$. The other two triangles are invariant with respect to g.

The centralizer of g in $\mathrm{Aut}(\mathsf{C})$ is the subgroup isomorphic to $(\mathbb{Z}/2\mathbb{Z})^{\oplus 2}$ generated by the even involutions of the points p_3, p_4, p_5, p_6. The product of g with any such transformation is an involution of type 2A of the cubic surface C.

Remark 10.5.14 Over the complex numbers, Mukai constructed an Enriques surface which contains 30 (-2)-curves with the same dual graph as in the above example. The canonical cover of the Mukai's example is the intersection of three quadrics given by the equations:

$$x^2 - (1 + \sqrt{3})yz = u^2 - (1 - \sqrt{3})vw,$$

$$y^2 - (1 + \sqrt{3})xz = v^2 - (1 - \sqrt{3})uw,$$

$$z^2 - (1 + \sqrt{3})xy = w^2 - (1 - \sqrt{3})uv.$$

See [534, Remark 2.7]. We discussed this surface in Example 8.8.5. We do not know whether it admits a specialization to characteristic 2.

The following seems to be a natural conjecture:

Conjecture 10.5.15

$$\mathrm{Aut}(S_{\alpha,\beta}) \cong \mathrm{UC}(10) \rtimes \mathfrak{A}_6,$$

where $\mathrm{UC}(10)$ is the universal Coxeter group with 10 generators identified with the sets of triads and the group $\mathfrak{A}_6$ acts on this group as its natural action on the sets of triads.

The group of automorphisms of the K3 surface Y was described in [189]. It is generated by the group $\mathrm{PSL}(3,\mathbb{F}_4)$ that acts on Y via its action on $\mathbb{P}^2(\mathbb{F}_4)$, 168 involutions that act on $\mathrm{Pic}(Y)$ as a lift of a quintic Cremona involution associated to six-arc, and a switch. It is known that $\mathrm{PSL}(3,\mathbb{F}_4)$ (denoted by $L_3(4)$ in [131]) contains three conjugacy classes of maximal subgroups isomorphic to $\mathfrak{A}_6$ (there is one conjugacy class in its extension $\mathrm{PGL}(3,\mathbb{F}_4)$). It also contains one conjugacy class of maximal subgroup of order 72, the subgroups of index 3 of the Hesse group G_{216} of automorphisms of the Hesse pencil, or, equivalently, the automorphism group of the rational elliptic surface H. Another maximal subgroup is the Klein group $\mathrm{PSL}_2(\mathbb{F}_7)$ of order 168. There are three conjugacy classes of such subgroups. One of these subgroups is $\mathrm{PSL}_3(\mathbb{F}_2)$. It acts transitively on the set of six-arcs in the finite plane $\mathbb{P}^2(\mathbb{F}_2)$ with the stabilizer subgroup of order 24.

Note that there are other types of elliptic fibrations on $S_{\alpha,\beta}$. We have listed in (10.4.2) all maximal parabolic subdiagrams of rank 8 of Γ_{CR}. Some of them contain a connected component with a vertex represented by a non-effective root, so this connected component does not represent a reducible fiber. Taking this into account, it is easy to prove the following.

Proposition 10.5.16 *An elliptic fibration on surface $S_{\alpha,\beta}$ is one of the following types:*

$$\tilde{A}_2+\tilde{A}_2+\tilde{A}_2+\tilde{A}_2,\quad \tilde{A}_3+\tilde{A}_3+2\tilde{A}_1^*,\quad \tilde{A}_4+\tilde{A}_4+\tilde{A}_0^*+\tilde{A}_0^*,\quad \tilde{A}_5+2\tilde{A}_1+\tilde{A}_2^*.$$

Finally, in this section, we show that the Coble surface of type MI is the specialization of surfaces $S_{\alpha,\beta}$ of type MI when (α,β) becomes the excluded pair equal to (ϵ,ϵ^2).

We use the rational derivation $\partial_{\epsilon,\epsilon^2}$. By a direct calculation, we check that the divisor of $\partial_{\epsilon,\epsilon^2}$ is given by

$$D = F_\epsilon^+ + F_{\epsilon^2}^+ - F_1^- - F_\infty^-$$

and the integral curves on the fibers over $t=\epsilon,\epsilon^2$ are $F_\epsilon^-, F_{\epsilon^2}^-$. Compare these with Lemma 10.5.7. By contracting six (-2)-curves appearing as components of D, we have a surface X with twelve rational double points of type A_1. The induced derivation has isolated zeros at six singular points which are the images of $F_\epsilon^-, F_{\epsilon^2}^-$. By applying Proposition 10.3.3, we can see that the quotient surface of Y by $\partial_{\epsilon,\epsilon^2}$ is non-singular and, by contracting six exceptional curves on the fibers over the points $t=1,\infty$, we obtain the Coble surface V discussed above.

Theorem 10.5.17 *The Coble surface $\mathsf{V}(\mathcal{P})$ of type* MI *is a specialization $\alpha=\epsilon,\beta=\epsilon^2$ of the one-dimensional family $\{S_{\alpha,\beta}\}$ $(\alpha,\beta\in\Bbbk,\ \alpha+\beta=\alpha\beta,\ \alpha^3\neq 1)$ of Enriques surfaces of type* MI. *It has* 30 *effective roots and* 10 *non-effective* (-2)*-classes. To each non-effective* (-2)*-class we associate an involution induced from a quintic Cremona transformation. The automorphism group* $\mathrm{Aut}(\mathsf{V})$ *is isomorphic to* $\mathrm{UC}(10)\rtimes\mathfrak{A}_6$.

In the following example we construct a Coble surface of type MI in characteristic three with finite automorphisms that was included in Theorem 9.8.4 as a surface of type MI. The description of the finite set of (-2)-curves on the surface uses the combinatorics of the extended Cremona–Richmond diagram.

- Example of a Coble surface in characteristic $p = 3$ with two boundary components and finite automorphism group.[4]

Let X be the Fermat quartic surface

$$x_0^4 + x_1^4 + x_2^4 + x_3^4 = 0. \tag{10.5.8}$$

The equation is defined by a Hermitian form over $\mathbb{F}_9$ and hence the unitary group $\mathrm{PGU}(4, \mathbb{F}_9)$ acts on X by projective automorphisms. It is known that X contains 112 lines. Let ℓ, ℓ' be two skew lines on X. They define two pencils of cubic curves $|\mathcal{O}_X(1) - \ell|$ and $|\mathcal{O}_X(1) - \ell'|$. The linear system $|\mathcal{O}_X(2) - \ell - \ell'|$ gives a map

$$\phi : X \to \mathbb{P}^1 \times \mathbb{P}^1 \tag{10.5.9}$$

of degree 2. Its branch divisor is the union of two curves C and C' of bidegree $(1, 3)$ and $(3, 1)$, the images of the lines ℓ and ℓ'. The Fermat quartic X contains 10 lines $\ell_1, \dots, \ell_{10}$ that intersect ℓ and ℓ'. They are blown down to the singular points of the branch curve. The deck transformation of the double cover defines a biregular involution $s_{\ell,\ell'}$ of X. This is a well-known classical involution of any smooth quartic surface containing two skew lines. It extends a birational involution of X that assigns to a point $x \notin \ell \cup \ell'$ the residual point on the unique line through x intersecting ℓ and ℓ'. The locus of fixed point $X^{s_{\ell,\ell'}}$ is equal to $\ell \cup \ell'$. The quotient

$$\mathsf{V} = X/(s_{\ell,\ell'})$$

is isomorphic to the blow-up of $\mathbb{P}^1 \times \mathbb{P}^1$ at the 10 intersection points $C \cap C'$. This is a Coble surface with two boundary components considered in Example 9.2.5.

For each line ℓ on X, there are exactly 30 lines meeting ℓ, and hence the number of pairs of two skew lines on X is $112 \cdot 81/2 = 2^4 \cdot 3^4 \cdot 7$. The stabilizer subgroup of a pair of two skew lines is $\mathrm{PGL}(2, \mathbb{F}_9) \times \mathbb{Z}/2\mathbb{Z}$ of order $2^5 \cdot 3^2 \cdot 5$. Thus, the group $\mathrm{PGU}(4, \mathbb{F}_9)$ acts transitively on the set of pairs of two skew lines and a subgroup of order $720 = 2^4 \cdot 3^2 \cdot 5$ descends to the symmetry group of the quotient surface V.

In coordinates $([u_0, u_1], [v_0, v_1])$ on $Q = \mathbb{P}^1 \times \mathbb{P}^1$, we may take the branch curves of (10.5.9) to be

$$C : u_0 v_0^3 = u_1 v_1^3, \quad C' : u_0^3 v_0 = u_1^3 v_1. \tag{10.5.10}$$

Denote by ζ a primitive eighth root of unity with $\zeta^2 = \sqrt{-1}$. The curves C and C' meet at 10 points

$$p_1 : [[1, \zeta], [1, -\zeta]],\ p_2 : [[1, \zeta^2], [1, \zeta^2]],\ p_3 : [[1, \zeta^3], [1, -\zeta^3]],$$

[4] The existence of this example was communicated to us by S. Mukai.

$$p_4 : [[1,-1],[1,-1]],\ p_5 : [[1,-\zeta],[1,\zeta]],\ p_6 : [[1,-\zeta^2],[1,-\zeta^2]],$$
$$p_7 : [[1,-\zeta^3],[1,\zeta^3]],\ p_8 : [[1,1],[1,1]],\ p_9 : [[1,0],[0,1]],\ p_{10} : [[0,1],[1,0]].$$

Consider the following set of twenty lines on Q, 10 from each of the two rulings:

$$C_i : u_1 = \zeta^i u_0 \quad (1 \le i \le 8), \quad C_9 : u_1 = 0, \quad C_{10} : u_0 = 0,$$
$$C'_i : v_1 = \zeta^i v_0 \quad (1 \le i \le 8), \quad C'_9 : v_1 = 0, \quad C'_{10} : v_0 = 0.$$

Then, C_i (resp. C'_i) meets C (resp. C') at one of the above 10 points with multiplicity 3. Each splits in the cover into a pair of lines on X. There are also 30 curves of bidegree $(1,1)$ on Q that four of the points p_i. Each splits in the cover (10.5.9) into the union of two lines.

Thus, we have 112 $(= 2 + 10 + 40 + 60)$ lines on X, which are (-2)-curves.

There are 30 (-2)-curves on V that are identified with the duads and synthemes in the following way. In the parentheses, we indicate four points among 10 points through which the curve passes.

$(12) : u_0v_0 - u_1v_1 = 0 \quad (p_4, p_8, p_9, p_{10})$,
$(13) : u_0v_0 - \zeta^3(u_0v_1 + u_1v_0) + \zeta^2u_1v_1 = 0 \quad (p_2, p_3, p_4, p_7)$,
$(14) : u_0v_0 + \zeta(u_0v_1 + u_1v_0) - \zeta^2u_1v_1 = 0 \quad (p_1, p_2, p_5, p_8)$,
$(15) : u_0v_0 + \zeta^2u_1v_0 - u_1v_1 = 0 \quad (p_3, p_5, p_6, p_9)$,
$(16) : u_0v_0 + \zeta^2u_0v_1 - u_1v_1 = 0 \quad (p_1, p_6, p_7, p_{10})$,
$(23) : u_0v_0 - \zeta(u_0v_1 + u_1v_0) - \zeta^2u_1v_1 = 0 \quad (p_1, p_4, p_5, p_6)$,
$(24) : u_0v_0 + \zeta^3(u_0v_1 + u_1v_0) + \zeta^2u_1v_1 = 0 \quad (p_3, p_6, p_7, p_8)$,
$(25) : u_0v_0 - \zeta^2u_1v_0 - u_1v_1 = 0 \quad (p_1, p_2, p_7, p_9)$,
$(26) : u_0v_0 - \zeta^2u_0v_1 - u_1v_1 = 0 \quad (p_2, p_3, p_5, p_{10})$,
$(34) : u_0v_0 + u_1v_1 = 0 \quad (p_2, p_6, p_9, p_{10})$,
$(35) : u_0v_0 + u_0v_1 + u_1v_1 = 0 \quad (p_1, p_3, p_8, p_{10})$,
$(36) : u_0v_0 + u_1v_0 + u_1v_1 = 0 \quad (p_5, p_7, p_8, p_9)$,
$(45) : u_0v_0 - u_0v_1 + u_1v_1 = 0 \quad (p_4, p_5, p_7, p_{10})$,
$(46) : u_0v_0 - u_1v_0 + u_1v_1 = 0 \quad (p_1, p_3, p_4, p_9)$,
$(56) : u_0v_1 - u_1v_0 = 0 \quad (p_2, p_4, p_6, p_8)$,
$(12, 34, 56) : u_0v_1 + u_1v_0 = 0 \quad (p_1, p_3, p_5, p_7)$,
$(12, 35, 46) : u_0v_0 + u_0v_1 - u_1v_0 + u_1v_1 = 0 \quad (p_2, p_5, p_6, p_7)$,
$(12, 36, 45) : u_0v_0 - u_0v_1 + u_1v_0 + u_1v_1 = 0 \quad (p_1, p_2, p_3, p_6)$,
$(13, 24, 56) : u_0v_0 - \zeta^2u_1v_1 = 0 \quad (p_1, p_5, p_9, p_{10})$,
$(13, 25, 46) : u_0v_0 - \zeta^3u_0v_1 + \zeta^2u_1v_1 = 0 \quad (p_5, p_6, p_8, p_{10})$,
$(13, 26, 45) : u_0v_0 - \zeta^3u_1v_0 + \zeta^2u_1v_1 = 0 \quad (p_1, p_6, p_8, p_9)$,
$(14, 23, 56) : u_0v_0 + \zeta^2u_1v_1 = 0 \quad (p_3, p_7, p_9, p_{10})$,
$(14, 25, 36) : u_0v_0 + \zeta u_0v_1 - \zeta^2u_1v_1 = 0 \quad (p_3, p_4, p_6, p_{10})$,
$(14, 26, 35) : u_0v_0 + \zeta u_1v_0 - \zeta^2u_1v_1 = 0 \quad (p_4, p_6, p_7, p_9)$,
$(15, 23, 46) : u_0v_0 - \zeta u_0v_1 - \zeta^2u_1v_1 = 0 \quad (p_2, p_7, p_8, p_{10})$,
$(15, 24, 36) : u_0v_0 + \zeta^3u_0v_1 + \zeta^2u_1v_1 = 0 \quad (p_1, p_2, p_4, p_{10})$,
$(15, 26, 34) : u_0v_0 - \zeta^2(u_0v_1 - u_1v_0) - u_1v_1 = 0 \quad (p_1, p_4, p_7, p_8)$,
$(16, 23, 45) : u_0v_0 - \zeta u_1v_0 - \zeta^2u_1v_1 = 0 \quad (p_2, p_3, p_8, p_9)$,
$(16, 24, 35) : u_0v_0 + \zeta^3u_1v_0 + \zeta^2u_1v_1 = 0 \quad (p_2, p_4, p_5, p_9)$,

$(16, 25, 34) : u_0v_0 + \zeta^2(u_0v_1 - u_1v_0) - u_1v_1 = 0 \quad (p_3, p_4, p_5, p_8)$.
We can also identify 10 exceptional curves E'_i with 10 elements in $\mathfrak{S}_6$ as follows:

$$(123)(456) : E'_4,\ (124)(356) : E'_8,\ (125)(346) : E'_9,\ (126)(345) : E'_{10},\ (134)(256) : E'_2,$$

$$(135)(246) : E'_3,\ (136)(245) : E'_7,\ (145)(236) : E'_5,\ (146)(235) : E'_1,\ (156)(234) : E'_6.$$

Now, we define 10 effective roots by

$$E_i = \frac{1}{2}C + \frac{1}{2}C' + 2E'_i.$$

The dual graph of $E_1, ..., E_{10}$ is the complete graph with double edges. It is now easily to see that the dual graph of forty effective roots $\{(ij), (ij, kl, mn), E_i\}$ is the same as that of forty (-2)-divisors on Enriques surfaces of type MI. Since the forty (-2)-classes are effective, by Vinberg's criterion 0.8.23, $\mathrm{Aut}(\mathsf{V})$ is finite. The symmetry group of this dual graph is the automorphism group $\mathrm{Aut}(\mathfrak{S}_6)(\cong \mathfrak{S}_6 \cdot \mathbb{Z}/2\mathbb{Z})$ of the symmetric group $\mathfrak{S}_6$. On the other hand, a subgroup of $\mathrm{Aut}(Q)$ of order 1440 generated by $\mathrm{PGL}(2, \mathbb{F}_9)$ and an involution $([u_0, u_1], [v_0, v_1]) \to ([v_0, v_1], [u_0, u_1])$ preserves B, B' and hence acts on V as automorphisms. Thus we have $\mathrm{Aut}(\mathsf{V}) \cong \mathrm{Aut}(\mathfrak{S}_6)$. We now have the most of the following theorem:

Theorem 10.5.18 *The surface* V *is a Coble surface of type* MI *in characteristic 3 with two boundary components which is a double quadric model of the Fermat quartic surface. It has 30* (-2)*-curves forming duads and synthemes, and* 10 *effective irreducible roots. The automorphism group* $\mathrm{Aut}(\mathsf{V})$ *is isomorphic to* $\mathrm{Aut}(\mathfrak{S}_6)$. *The R-invariant* (K, H) *is* $(\mathsf{A}_5^{\oplus 2} \oplus \mathsf{A}_1^{\oplus 2}, (\mathbb{Z}/2\mathbb{Z})^3)$.

Proof It remains only to prove the last assertion. The 30 (-2)-curves define an orthogonal summand of the root lattice isomorphic $\mathsf{A}_5 \oplus \mathsf{A}_5$. The boundary components define another orthogonal summand isomorphic to $\mathsf{A}_1 \oplus \mathsf{A}_1$. Since the Coble–Mukai lattice is isomorphic to E_{10} (Example 9.2.5), $\dim H \geq 3$. Since the dimension of the kernel of $q_{K/2K}$ is equal to 3, we obtain $H \cong (\mathbb{Z}/2\mathbb{Z})^{\oplus 3}$. □

10.6 Quotients of the Supersingular K3 Surface with $\sigma_0 = 1$: Type MII

In this section, we will construct a family of Enriques surfaces S and a unique Coble surface V that realizes a diagram similar to the Cremona–Richmond diagram. The diagram consists of 40 divisor classes in the lattice $\mathrm{Num}(S)$ or $\mathrm{CM}(\mathsf{V})$ with square-norm -2. We will refer to these surfaces as *surfaces of type* MII. Their canonical covers have 12 ordinary double points, and their minimal resolution are the supersingular K3 surface with Artin invariant 1. We also give an example of Coble surface of type MII in characteristic 3. Finally, we will introduce R-invariants of Enriques and Coble surfaces in characteristic 2 whose canonical covers are birationally isomorphic

to supersingular K3 surfaces, and calculate the nodal R-invariants of Enriques and Coble surfaces of type VII and of type MII. If not stated otherwise, we continue to assume that $p = 2$. First of all, we recall the classification of genus one fibrations on the supersingular K3 surface Y with Artin invariant one. They are classified into 18 types (see [215], [420]).

The following is the list of reducible singular fibers of genus one fibrations on Y:

Elliptic fibrations:

$$\tilde{A}_5 + \tilde{A}_5 + \tilde{A}_5 + \tilde{A}_5, \quad \tilde{A}_7 + \tilde{A}_7 + \tilde{D}_5, \quad \tilde{A}_9 + \tilde{A}_9 + \tilde{A}_1 + \tilde{A}_1, \quad \tilde{A}_{11} + \tilde{D}_7,$$

$$\tilde{A}_{11} + \tilde{A}_3 + \tilde{E}_6,\ \tilde{E}_6 + \tilde{E}_6 + \tilde{E}_6, \quad \tilde{A}_{15} + \tilde{D}_5,\ \tilde{A}_{17} + \tilde{A}_1 + \tilde{A}_1 + \tilde{A}_1.$$

Quasi-elliptic fibrations:

$$\tilde{D}_4+\tilde{D}_4+\tilde{D}_4+\tilde{D}_4+\tilde{D}_4, \quad \tilde{D}_6+\tilde{D}_6+\tilde{D}_6+\tilde{A}_1^*+\tilde{A}_1^*, \quad \tilde{D}_4+\tilde{D}_8+\tilde{D}_8, \quad \tilde{D}_6+\tilde{E}_7+\tilde{E}_7,$$

$$\tilde{D}_{10}+\tilde{E}_7+\tilde{A}_1^*+\tilde{A}_1^*+\tilde{A}_1^* \quad \tilde{D}_8+\tilde{D}_{12}, \quad \tilde{D}_4+\tilde{D}_{16}, \quad \tilde{D}_{12}+\tilde{E}_8, \quad \tilde{D}_4+\tilde{E}_8+\tilde{E}_8,\ \tilde{D}_{20}.$$

Moreover, it is shown in [215] that these 18 types are unique up to automorphism of the surface. Any genus one fibration on Y has a section. The existence of a quasi-elliptic fibration with a singular fiber of type $\tilde{D}_{20}$ shows that $\mathrm{Pic}(Y)$ is isomorphic to $\mathsf{U} \oplus \mathsf{D}_{20}$.

Recall that in the previous section we have constructed a rational vector field $\partial_{\alpha,\beta}$ with quotient birationally isomorphic to an Enriques surface $S_{\alpha,\beta}$ by choosing one of these fibrations, namely the (double) Hesse type with four singular fibers of type $\tilde{A}_5$. The surface $S_{\alpha,\beta}$ has 12 canonical points, and its canonical cover has 12 ordinary double points. A natural question is whether we can construct other examples by choosing different fibration. It turns out that only two more fibrations work, namely types $\tilde{A}_7+\tilde{A}_7+\tilde{D}_5$ and $\tilde{A}_9+\tilde{A}_9+\tilde{A}_1+\tilde{A}_1$ [418]. The last case leads to a construction of an Enriques surface of type VII in characteristic 2 which we have already discussed in Section 8.10. In this section, we will discuss the remaining construction.

We start with a construction of an elliptic fibration of type $\tilde{A}_7 + \tilde{A}_7 + \tilde{D}_5$ on Y.

Let $Q = \mathbb{P}^1 \times \mathbb{P}^1$, considered as a quadric over $\mathbb{F}_4$. The set $Q(\mathbb{F}_4)$ consists of 25 points (q_i, q_j), where $(q_1, q_2, q_3, q_4, q_5) = (0, 1, \epsilon, \epsilon^2, \infty)$. Let $F_{i,a} = \pi_i^{-1}(a)$, where $\pi_i : Q \to \mathbb{P}^1$ are the two projections and $a \in \mathbb{P}^1(\mathbb{F}_4)$. Let $P = (\infty, \infty)$. The quadric contains $85 = \#\mathbb{P}^3(\mathbb{F}_4)$ conics, among them 25 are reducible and correspond to tangent hyperplanes at points in $Q(\mathbb{F}_4)$. We consider irreducible conics passing through P. There are $12 = \#\mathbb{P}^2(\mathbb{F}_4) - 9$ of them that correspond to planes passing through P and not containing $F_{1,\infty}, F_{2,\infty}$. Since there are three tangent directions defined over $\mathbb{F}_4$ and different from $F_{1,\infty}, F_{2,\infty}$, they are divided into the union of three sets, two conics from the same set are tangent at P. Among these 12 conics one is the diagonal of $\mathbb{P}^1 \times \mathbb{P}^1$ that passes through (q_i, q_i). All others are obtained from this conic by applying elements of $\mathrm{PGL}_2(\mathbb{F}_4) \times \mathrm{PGL}_2(\mathbb{F}_4)$ that fix (∞, ∞). We may assume that a conic passes through the points $(q_1, q_{\sigma(1)}), \ldots, (q_4, q_{\sigma(4)})$, where σ is a permutation if $\{1, 2, 3, 4\}$. Since elements of the $\mathrm{PGL}_2(\mathbb{F}_4)_\infty$ define even

permutations of the point $0, 1, \epsilon, \epsilon^2$, we see that the conics correspond to elements of $\mathfrak{A}_4$.

We say that the union of two fibers of π_1 and two fibers of π_2 is a *quadrangle*. Fix two disjoint quadrangles T_1 and T_2. Let $B = \{(q_i, q_j) \ : 1 \le i, j \le 4\}$. The blow-up $\mathsf{V} \to Q$ of B is a Coble surface with eight boundary components. The Coble surface V is called the *Coble surface of type MII in characteristic 2*.

Besides the boundary components, it contains 28 curves, the proper transforms of the conics and the exceptional curves. We associate to each exceptional curve E_{ij} over $(q_i, q_j) \in B$, the corresponding effective irreducible root $\alpha_{ij} = 2E_{ij} + \frac{1}{2}(F_{1,i} + F_{2,j})$. We also have 12 non-effective roots

$$\alpha_{i_1,i_2,i_3,i_4} = h - e_{b_{1,i_1}} - e_{b_{2,i_2}} - e_{b_{3,i_3}} - e_{b_{4,i_4}},$$

where h is the divisor class of a hyperplane section and $b_{k,i_k} = F_{1,k} \cap F_{2,i_k} \in B$. Since the roots are not effective (i_1, i_2, i_3, i_4) is an odd permutation of $\{1, 2, 3, 4\}$.

Let us denote the classes of 12 conics by $\alpha_\sigma, \sigma \in \mathfrak{A}_4$, the classes of 12 non-effective roots by $\beta_\sigma, \sigma \in \mathfrak{S}_4 \setminus \mathfrak{A}_4$, and the classes of irreducible effective roots by $\alpha_{ij}, 1 \le i, j \le 4$.

Let Γ_{MII} be the intersection diagram of the 40 divisor classes. It is determined by the following properties:

- $\beta_\sigma \cdot \beta_\tau = 1$ if $(\sigma^{-1} \circ \tau)^3 = 1$ and 2 if otherwise.
- $\beta_\sigma \cdot \alpha_{ab} = 2$ if $\sigma = (ab)$ or $(\sigma \circ (ab))^3 = 1$, 0 otherwise.
- $\beta_\sigma \cdot \alpha_\tau = 2$ if $(\sigma \circ \tau)^2 \neq 1$ and 0 otherwise.
- $\alpha_{ab} \cdot \alpha_\sigma = 2$ if $(\sigma \circ (ab))^2 = 1$ and 0 otherwise.
- $\alpha_{ab} \cdot \alpha_{ij} = 1$ if $\#(ab) \cap (ij) = 1$ and 0 otherwise.
- $\alpha_\sigma \cdot \alpha_\tau = 2$ if $(\sigma \circ \tau)^3 = 1$ and 1 otherwise.

Note that the subgraph with 16 vertices α_{ab} is dual to the complete bipartite graph $BK(4)$ on two disjoint sets of cardinality 4.

As we remarked earlier, the 12 conics are grouped in subsets such that conics in the same subsets are tangent at P and conics from different subsets intersect with multiplicity one at P. In our notation in terms of substitutions, we may assume that the three groups are the following:

$$\{1, (12)(34), (13)(24), (14)(23)\}, \quad \{(123), (134), (142), (243)\}$$

$$\{(132), (124), (143), (234)\}.$$

The subgraph with vertices α_σ is the union of three complete graphs $K(4)$ on four vertices with double edges, each vertex of one subgraph is joined to all other vertices from two remaining subgraphs.

One can directly check that maximal affine subdiagrams in Γ_{MII} are the following:

$$\tilde{A}_7 \oplus \tilde{A}_1, \quad \tilde{A}_5 \oplus \tilde{A}_2 \oplus \tilde{A}_1, \quad \tilde{A}_3^{\oplus 2} \oplus \tilde{A}_1^{\oplus 2}, \quad \tilde{A}_2^{\oplus 4},$$

and any affine subdiagram is contained in a maximal one. In other words, Γ_{MII} satisfies the condition of Theorem 0.8.23 from Volume I.

Let us see that one can also associate an automorphism of V to any non-effective roots β_σ, $\sigma \in \mathfrak{S}_4 \setminus \mathfrak{A}_4$. To do this, we consider the linear system $|3h - p_{1,\sigma(1)} - p_{2,\sigma(2)} - p_{3,\sigma(3)} - p_{4,\sigma(4)}|$ of curves of bi-degree $(3,3)$ with double points at the four points in $B(\sigma) = \{p_{1,\sigma(1)}, p_{2,\sigma(2)}, p_{3,\sigma(3)}, p_{4,\sigma(4)}\}$. One can choose a basis in this linear system such that this linear system defines a birational automorphism ϕ of Q. The map blows down the four conics that pass through three of the fundamental points to a subset of four points defining another root β_τ. Composing it with the automorphism of Q coming from the group $\mathrm{Sym}(\Pi_{\mathrm{MII}})$ of symmetries of Π_{MII}, we may assume that the image of each conic is a fundamental point. This shows that ϕ lifts to the blow-up of $B(\sigma)$ and hence defines an automorphism g_σ of V. Let g_σ^* be the action of g_σ on $\mathrm{Pic}(\mathsf{V})$. We choose a geometric basis formed by the divisor classes h_1, h_2 of two rulings and the classes e_b of the exceptional curves $E_b, b \in B$ (ordered in some way). Then,

$$\begin{aligned} g_\sigma^*(h) &= s_{\beta_\sigma}(h) = 3h - 2\sum_{b\in B(\sigma)} e_b,\\ g_\sigma^*(h_1) &= s_{\beta_\sigma}(h_1) = 2h_1 + h_2 - \sum_{b\in B(\sigma)} e_b,\\ g_\sigma^*(h_2) &= s_{\beta_\sigma}(h_2) = h_1 + 2h_2 - \sum_{b\in B(\sigma)} e_b. \end{aligned}$$

However, $s_{\beta_\sigma}(e_b) = h + e_b - \sum_{b\in B(\sigma)} e_b$ may differ from g_σ^* by a permutation τ of B leaving invariant $B(\sigma)$ coming from $(\mathfrak{A}_4 \times \mathfrak{A}_4) \rtimes 2 \subset \mathrm{Aut}(\mathsf{V})$. Since g leaves invariant $2h - \sum_{b\in B(\sigma)} e_b = -K_{\mathcal{D}_4}$, it comes from an automorphism of the quartic del Pezzo surface $\mathcal{D}_4$ that together with the subgroup $\mathfrak{A}_4$ generated $\mathfrak{A}_5$. This shows that we can also choose τ to be an odd involution of the set $B(\sigma)$ such that its composition with s_{β_σ} is an even involution. Thus, we can always choose g_σ to be an involution. We now can state the following:

Theorem 10.6.1 *Let* V *be the Coble surface of type* MII *in characteristic 2 with eight boundary components. It has twelve* (-2)*-curves, sixteen effective irreducible roots and twelve non-effective* (-2)*-classes satisfying the condition in Theorem 0.8.23. The automorphism group* $\mathrm{Aut}(\mathsf{V})$ *is generated by* $(\mathfrak{A}_4 \times \mathfrak{A}_4) \rtimes \mathbb{Z}/2\mathbb{Z}$ *and the group* $G(12)$ *generated by twelve involutions* g_σ, $\sigma \in \mathfrak{S}_4 \setminus \mathfrak{A}_4$.

Let $Z \to \mathsf{V}$ be the canonical cover of V defined by the invertible sheaf ω_{V}^{-1} and the section of ω_{V}^{-2} whose scheme of zeros is the boundary $C_1 + \cdots + C_8$. Applying Proposition 0.2.10, we find that it is expected to have four ordinary double points. Since $\omega_Z = \mathcal{O}_Z$ and $b_2(Z) = b_2(\mathsf{V}) = 18$, we see that Z birationally isomorphic to a supersingular K3 surface Y. We can see this in another way. Consider the pencil of quartic elliptic curves on the quadric spanned by the two quadrangles of lines on the quadric. It contains another reducible member, the union of two conics tangent at P and passing through the set of eight base points. The proper transform of this pencil on Y is an elliptic pencil of type $\tilde{A}_7 + \tilde{A}_7 + \tilde{D}_5$. It follows from the Shioda–Tate formula that the surface Y is a supersingular K3 surface with the Artin invariant one.

This agrees with the known list of possible elliptic fibrations on such surfaces which we reproduced in the beginning of this section.

So, we now have a situation similar to the one we discussed in the previous section. It suggests the existence of a family of Enriques surfaces S which are quotients of Y by a rational vector field. We now proceed to do this. Unfortunately, the Coble surface V is not a specialization of this family. This is because the canonical covering of Enriques surfaces of type MII has a rational double point of type D_4, on the other hand, the canonical cover of any Coble surface has only rational double points of type A_1 ([370, Proposition 4.7]).

Recall that the supersingular K3 surface Y with the Artin invariant one is the minimal resolution of a purely inseparable double cover of $\mathbb{P}^2$ which has 21 nodes over $\mathbb{P}^2(\mathbb{F}_4)$, as we mentioned in Section 10.5. Let ℓ be a line on $\mathbb{P}^2(\mathbb{F}_4)$ and $p_1, \ldots, p_5$ the five $\mathbb{F}_4$-rational points of ℓ. For $i = 1, 2$, we denote by ℓ_{ij} ($j = 1, \ldots, 4$) the four lines in $\mathbb{P}^2(\mathbb{F}_4)$ passing through p_i except ℓ (see Figure 10.7).

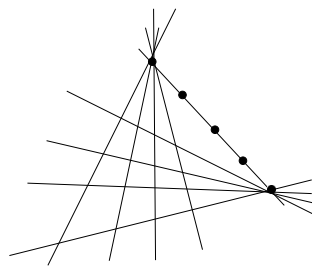


Fig. 10.7 Nine lines ℓ, ℓ_{ij} in $\mathbb{P}^2(\mathbb{F}_4)$

Let L, L_{ij} be the proper transforms of ℓ, ℓ_{ij} on Y. Also denote by E_3, E_4, E_5 the exceptional curves over the points p_3, p_4, p_5. Let X be the surface obtained by contracting L_{ij}, L, E_3, E_4, E_5 which has eight rational double points of type A_1 and one rational double point of type D_4. We shall construct a classical Enriques surface S whose canonical cover is X. The Enriques surface S contains 28 (-2)-curves. Sixteen of them are the images of the sixteen exceptional curves E_{ij} on Y over the sixteen intersection points of ℓ_{1i} and ℓ_{2j}, and twelve of them are the images of the twelve lines on $\mathbb{P}^2(\mathbb{F}_4)$ through p_3, p_4, or p_5. The sixteen (-2)-curves are the straight lines on the left-hand side in Figure 10.8 below and the eight black circles are the eight canonical points which are the images of eight singular points of type A_1. The twelve (-2)-curves are the curves on the right-hand side in Figure 10.8 which pass through the black circle corresponding to the other canonical point, i.e. the image of a singular point of type D_4. These 12 (-2)-curves are divided into three groups, each of which consists of four (-2)-curves touching each other at the point corresponding to the black circle.

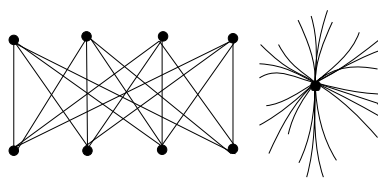


Fig. 10.8 $(8+1)$ canonical points and 28 irreducible effective roots

Thus, the canonical cover of the desired Enriques surface S has $8A_1 + D_4$ as singularities, and hence, S is classical (Theorem 10.2.2).

To construct S, we consider a rational elliptic surface defined by:

$$y^2 + xy + s(s+1)y = x^3 + s(s+1)x^2. \tag{10.6.1}$$

It has two singular fibers of type $\tilde{A}_3$ over $s = 0, 1$ and a singular fiber of type $\tilde{A}_2^*$ over $s = \infty$ (see Table 4.7). By taking the Frobenius base change $s = t^2$, we have an elliptic fibration on $p : Y \to \mathbb{P}^1$ defined by

$$y^2 + xy + t^2(t+1)^2 y = x^3 + t^2(t+1)^2 x^2. \tag{10.6.2}$$

The fibration p has two singular fibers of type $\tilde{A}_7$ over $t = 0, 1$ and a singular fiber of type $\tilde{D}_5$ over $t = \infty$. In the following, we may assume that the fibration defined by the linear system

$$|L_{11} + E_{11} + L_{21} + E_{21} + L_{12} + E_{22} + L_{22} + E_{12}|,$$

induced from $|\ell_{11} + \ell_{21} + \ell_{12} + \ell_{22}|$, which has another fiber $L_{13} + E_{33} + L_{23} + E_{43} + L_{14} + E_{44} + L_{24} + E_{34}$ of type $\tilde{A}_7$ and a fiber of type $\tilde{D}_5$ containing $L, E_3, E_4, E_5, F_1, F_2$, where F_1, F_2 are the pull-backs of lines on $\mathbb{P}^2(\mathbb{F}_4)$ passing through a point among p_3, p_4, p_5.

Now, let us consider a rational derivation $\partial_{\alpha,\beta}$ on Y induced by

$$\frac{1}{t(t+1)}\left(t(t+1)(t+\alpha)(t+\beta)\frac{\partial}{\partial t} + \alpha\beta(x + t^2(t+1)^2)\frac{\partial}{\partial x}\right), \tag{10.6.3}$$

where $\alpha, \beta \in \mathbb{k}^*,\ \alpha + \beta = 1$. A direct calculation shows the following:

Lemma 10.6.2 (i) $\partial_{\alpha,\beta}^2 = \partial_{\alpha,\beta}$, *namely,* $\partial_{\alpha,\beta}$ *is* 2*-closed and of multiplicative type.*
(ii) *On the surface* Y*, the divisor* D *of* $\partial_{\alpha,\beta}$ *is given by*

$$D = -(L_{11} + L_{12} + L_{21} + L_{22} + L_{13} + L_{14} + L_{23} + L_{24} + 2(L + E_3 + E_4 + E_5))$$

and $D^2 = -24$.

(iii) *The integral curves with respect to* $\partial_{\alpha,\beta}$ *in the fibers of* $p : Y \longrightarrow \mathbb{P}^1$ *are the following:*

$$L_{11},\ L_{12},\ L_{21},\ L_{22},\ L_{13},\ L_{14},\ L_{23},\ L_{24},\ E_3,\ E_4,\ E_5,$$

and also the smooth fibers over the points $t = \alpha, \beta$.

Now, by the same argument as in the previous example, the quotient surface $Y^{\partial_{\alpha,\beta}}$ has 11 exceptional curves of the first kind which are the images of the above integral curves. By contracting these curves and then contracting the image of L we get a smooth Enriques surface $S = S_{\alpha,\beta}$. The elliptic fibration $p : Y \to \mathbb{P}^1$ induces an elliptic fibration $\pi : S_{\alpha,\beta} \to \mathbb{P}^1$ which has two singular fibers of type $\tilde{A}_3$ and a singular fiber of type $\tilde{A}_1^*$ consisting of the images of F_1 and F_2. Since the images of two smooth integral elliptic curves over $t = \alpha, \beta$ are double fibers of the elliptic fibration, $S_{\alpha,\beta}$ is a classical Enriques surface. It is not difficult to see that $S_{\alpha,\beta}$ contains 28 (-2)-curves as in Figure 10.8.

We will refer to the surface $S_{\alpha,\beta}$ as an *Enriques surface of type* MII *in characteristic 2* because it corresponds to the fundamental polytope Π_{MII} (see Remark 10.6.5).

Theorem 10.6.3 *There exists a one-dimensional family* $\{S_{\alpha,\beta}\}$ *of classical Enriques surfaces* $S_{\alpha,\beta}$ *of type* MII *where* $\alpha, \beta \in \mathbb{k}^*$, $\alpha + \beta = 1$. *The canonical cover* X *has eight nodes and one rational double point of type* D_4. *Its resolution of singularities is the supersingular K3 surface* Y *with Artin invariant* 1. *The surface* $S_{\alpha,\beta}$ *contains* 28 (-2)*-curves as in Figure 10.8.*

Proposition 10.6.4 *An elliptic fibration on the surface* $S_{\alpha,\beta}$ *is one of the following types:*

$$\tilde{A}_3 + \tilde{A}_3 + 2\tilde{A}_1^*,\ \tilde{A}_5 + 2\tilde{A}_1^*,\ \tilde{A}_5 + \tilde{A}_1 + 2\tilde{A}_2^*,\ 2\tilde{A}_2^* + 2\tilde{A}_2^* + \tilde{A}_2^*,\ \tilde{A}_7 + \tilde{A}_1^*.$$

It follows that $S_{\alpha,\beta}$ *exists only as a classical Enriques surface.*

Remark 10.6.5 Let W be the subgroup of W_S^{nod} generated by reflections associated with 28 (-2)-curves as mentioned above. Then, W is not of finite index in $\mathrm{O}(\mathrm{Num}(S_{\alpha,\beta}))$. In fact, the automorphism group $\mathrm{Aut}(S_{\alpha,\beta})$ is infinite. However, there exist 12 non-effective (-2)-classes in $\mathrm{Num}(S_{\alpha,\beta})$ such that the classes of 28 (-2)-curves and these 12 (-2)-classes satisfy Vinberg's Criterion 0.8.23, that is, the subgroup generated by reflections associated with these 40 (-2)-classes is of finite index in $\mathrm{O}(\mathrm{Num}(S_{\alpha,\beta}))$. These 12 (-2)-classes bijectively correspond to 12 (-4)-classes among 168 (-4)-classes which are perpendicular to $A_1^{\oplus 8} \oplus D_4$ generated by exceptional curves over eight nodes and a rational double point of type D_4 on the canonical cover of S. The symmetry group of the dual graph of 40 (-2)-classes is $(\mathfrak{A}_4 \times \mathfrak{A}_4) \rtimes \mathbb{Z}/2\mathbb{Z}$ (see Figure 10.8). This remarkable diagram of (-2)-classes was

first discovered by Mukai in the case where $\Bbbk = \mathbb{C}$ and S is covered by the Kummer surface associated with a curve of genus two (see Example 10.7.11).

The following example was found independently by Mukai and Ohashi.

- An example of a Coble surface with finite automorphism group and eight boundary components in characteristic three.

Let E be a supersingular elliptic curve in characteristic 3 (for example, given by equation $y^2 + x^3 - x = 0$). Let $\mathrm{Kum}(E \times E)$ be the Kummer surface associated with the product $E \times E$. It is known that $\mathrm{Kum}(E \times E)$ is the supersingular K3 surface with Artin invariant 1. The involution $1_E \times (-1_E)$ of $E \times E$ descends to an involution σ of $\mathrm{Kum}(E \times E)$ which fixes eight (-2)-curves pointwise. The quotient of $E \times E$ by the group $\langle 1_E \times (-1_E),\ (-1_E) \times 1_E \rangle (\cong (\mathbb{Z}/2\mathbb{Z})^2)$ is isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$. Then, there is a morphism from the quotient of $\mathrm{Kum}(E \times E)$ by σ to $\mathbb{P}^1 \times \mathbb{P}^1$ which contracts sixteen (-1)-curves on $\mathrm{Kum}(E \times E)/\langle \sigma \rangle$. Let $\pi : \mathsf{V} \to \mathbb{P}^1 \times \mathbb{P}^1$ be the blow-up the sixteen $\mathbb{F}_3$-rational points of $\mathbb{P}^1 \times \mathbb{P}^1$ and let E_{ij} be the exceptional curve over (p_i, p_j). Let $p_1 = (1, 0),\ p_2 = (1, 1),\ p_3 = (1, -1),\ p_4 = (0, 1)$ be $\mathbb{F}_3$-rational points of $\mathbb{P}^1$ and let F_i, G_i be the proper transforms of $\mathbb{P}^1 \times \{p_i\}, \{p_i\} \times \mathbb{P}^1$ $(1 = 1, \ldots, 4)$, respectively. Then

$$-2K_{\mathsf{V}} = \left\{ \sum_{i=1}^{4} (F_i + G_i) \right\}.$$

Thus, V is a Coble surface with eight boundary components $\sum_{i=1}^{4}(F_i + G_i)$ and isomorphic to $\mathrm{Kum}(E \times E)/\langle \sigma \rangle$. We have sixteen effective roots

$$e_{ij} = \frac{1}{2}F_i + \frac{1}{2}G_j + 2E_{ij} \quad (1 \le i, j \le 4).$$

Since $e_{ij} \cdot e_{kl} = 1$ if $i = k$ or $j = l$ and $e_{ij} \cdot e_{kl} = 0$ for other $\{i, j\} \neq \{k, l\}$, the dual graph of $\{e_{ij}\}$ coincides with the one of sixteen (-2)-curves of the left-hand side in Figure 10.8. On the other hand, the pull-backs of the following 24 smooth curves of bidegree $(1, 1)$ in $\mathbb{P}^1 \times \mathbb{P}^1$ are (-2)-curves on V:

$C_1 :\ u_0v_1 + u_1v_0 - u_1v_1 = 0, \quad C_2 :\ u_0v_0 - u_1v_0 - u_1v_1 = 0,$
$C_3 :\ u_0v_0 + u_0v_1 + u_1v_0 = 0, \quad C_4 :\ u_0v_0 - u_0v_1 - u_1v_1 = 0,$
$C_5 :\ u_0v_0 - u_1v_1 = 0, \quad C_6 :\ u_0v_1 + u_1v_0 = 0,$
$C_7 :\ u_0v_0 - u_0v_1 + u_1v_0 + u_1v_1 = 0, \quad C_8 :\ u_0v_0 + u_0v_1 - u_1v_0 + u_1v_1 = 0,$
$C_9 :\ u_0v_0 - u_0v_1 - u_1v_0 = 0, \quad C_{10} :\ u_0v_0 + u_0v_1 - u_1v_1 = 0,$
$C_{11} :\ u_0v_0 + u_1v_0 - u_1v_1 = 0, \quad C_{12} :\ u_0v_1 + u_1v_0 + u_1v_1 = 0.$
$D_1 :\ u_0v_0 - u_0v_1 - u_1v_0 - u_1v_1 = 0, \quad D_2 :\ u_0v_0 + u_0v_1 + u_1v_0 - u_1v_1 = 0,$
$D_3 :\ u_0v_0 + u_1v_1 = 0, \quad D_4 :\ u_0v_1 - u_1v_0 = 0,$
$D_5 :\ u_0v_0 - u_0v_1 + u_1v_1 = 0, \quad D_6 :\ u_0v_0 + u_0v_1 - u_1v_0 = 0,$
$D_7 :\ u_0v_0 + u_1v_0 + u_1v_1 = 0, \quad D_8 :\ u_0v_1 - u_1v_0 + u_1v_1 = 0,$
$D_9 :\ u_0v_0 - u_0v_1 + u_1v_0 = 0, \quad D_{10} :\ u_0v_0 + u_0v_1 + u_1v_1 = 0,$
$D_{11}\ u_0v_0 - u_1v_0 + u_1v_1 = 0, \quad D_{12} :\ u_0v_1 - u_1v_0 - u_1v_1 = 0.$

Denote by c_i and d_j the pull-back of C_i and D_j on V, respectively. We can divide 24 (-2)-curves into six groups

$$\mathcal{C}_1 = \{c_i\}_{1\le i\le 4},\ \mathcal{C}_2 = \{c_i\}_{5\le i\le 8},\ \mathcal{C}_3 = \{c_i\}_{9\le i\le 12},$$

$$\mathcal{D}_1 = \{d_i\}_{1\le i\le 4},\ \mathcal{D}_2 = \{d_i\}_{5\le i\le 8},\ \mathcal{D}_3 = \{d_i\}_{9\le i\le 12}.$$

One can easily check the following:

$$c_i \cdot c_j = 2\ if\ c_i\ and\ c_j \in \mathcal{C}_k\ (i \ne j)\ and\ c_i \cdot c_j = 1\ if\ c_i \in \mathcal{C}_k\ and\ c_j \in \mathcal{C}_l\ (k \ne l),$$

$$d_i \cdot d_j = 2\ if\ d_i\ and\ d_j \in \mathcal{D}_k\ (i \ne j)\ and\ d_i \cdot d_j = 1\ if\ d_i \in \mathcal{D}_k\ and\ d_j \in \mathcal{D}_l\ (k \ne l),$$

and the intersection number of c_i and d_j is 0 or 2. Thus, the dual graphs of twelve (-2)-curves $\{c_i\}$ and $\{d_j\}$ are the same as the one of twelve (-2)-curves of the right-hand side in Figure 10.8.

We summarize this example in the following theorem.

Theorem 10.6.6 *The surface* V *is a Coble surface of type* MII *in characteristic three with eight boundary components whose double cover is the supersingular* $K3$ *surface with Artin invariant* 1. *The surface* V *has* 24 *effective* (-2)*-curves and sixteen effective irreducible roots. The automorphism group* $\mathrm{Aut}(\mathsf{V})$ *is isomorphic to* $(\mathfrak{S}_4 \times \mathfrak{S}_4) \cdot \mathbb{Z}/2\mathbb{Z}$. *The* R*-invariant* (K, H) *is* $(\mathsf{D}_8 \oplus \mathsf{A}_2^{\oplus 2}, (\mathbb{Z}/2\mathbb{Z})^2)$.

Proof We has to prove the last assertion. The orthogonal summand D_8 comes from the eight boundary components. The orthogonal summand $\mathsf{A}_2 \oplus \mathsf{A}_2$ comes from twenty four (-2)-curves. Since $\mathrm{rank}\, H \ge 2$, and $\dim \mathrm{Ker}(q_{K/2K}) = 2$ we get $H \cong (\mathbb{Z}/2\mathbb{Z})^2$. □

Remark 10.6.7 The Enriques surfaces of type MII in characteristic 2 have $(16 + 12)$ (-2)-curves and 12 non-effective roots. The above Coble surface V contains 16 effective roots and $(12 + 12)$ (-2)-curves. This explains the difference of the symmetries $(\mathfrak{A}_4 \times \mathfrak{A}_4) \cdot \mathbb{Z}/2\mathbb{Z}$ and $(\mathfrak{S}_4 \times \mathfrak{S}_4) \cdot \mathbb{Z}/2\mathbb{Z}$.

We have defined the notion of the Nikulin's nodal R-invariant for Enriques surfaces which are étale quotients of K3 surfaces (see Section 6.4). In the following we extend this to the case that the canonical covers of Enriques surfaces have only rational double points. Let S be a classical or an α_2-Enriques surface and let $\pi : X \to S$ be its canonical cover. Assume that X has only rational double points. Denote by $\phi : Y \to X$ the minimal resolution. By Theorem 1.3.5, Y is a supersingular K3 surface. Put $\bar{\pi} = \pi \circ \phi$. Let $\mathrm{Pic}(Y)^+ = \bar{\pi}^*(\mathrm{Pic}(S))$ and let $\mathrm{Pic}(Y)^-$ be the orthogonal complement of $\mathrm{Pic}(Y)^+$ in $\mathrm{Pic}(Y)$. We denote by R the root sublattice of $\mathrm{Pic}(Y)^-$ generated by exceptional curves $E_1, \dots, E_{12}$ of the minimal resolution $\phi : Y \to X$. Note that R is of finite index in $\mathrm{Pic}(Y)^-$. Define

$$h^\pm = \{\delta^\pm \in \mathrm{Pic}(Y)^\pm\ :\ \exists\, \delta^\mp \in \mathrm{Pic}(Y)^\mp,\ (\delta^\pm)^2 = -4,\ \frac{\delta^+ + \delta^-}{2} \in \mathrm{Pic}(Y)\}. \quad (10.6.4)$$

Let $\langle h^- \rangle$ be the sublattice of $\mathrm{Pic}(Y)^-$ generated by h^-. Then, $\langle h^- \rangle = K(2)$, where K is a root lattice. We have a homomorphism

$$\gamma : K(2)/2K(2) \to \mathrm{Pic}^+(Y)/2\mathrm{Pic}^+(Y) \quad (10.6.5)$$

defined by

$$\gamma(\delta^- \bmod 2) = \delta^+ \bmod 2.$$

We let H to be the kernel of γ. We call the pair (K, H) the R-invariant of S as in the case of Section 6.4.

Example 10.6.8 Let C be a (-2)-curve that passes through the images $\bar{p_1}, \dots, \bar{p_k}$ of singular points $p_1, \dots, p_k$ of X. We assume that all these points are of type A_1 and denote by $E_i \subset Y$ the exceptional curve over p_i. Since the restriction of the K3 cover to C is trivial, $\pi^*(C) = 2C'$. By Proposition 0.4.19 in Volume I, X is $\mathbb{Q}$-factorial. Applying the intersection theory of $\mathbb{Q}$-factorial surfaces, we find that $C'^2 = -1$. Let $\tilde{C}$ be the strict transform of C' in Y. Then, the full transform of $2C' \in \mathrm{Pic}(X)$ in $\mathrm{Pic}(Y)$ is equal to $D = 2\tilde{C} + \sum_{i=1}^k a_i E_i$, where $D \cdot E_i = 0$. Since C' is nonsingular, $\tilde{C}$ intersects each E_i with multiplicity 1. This gives $a_i = 1$, and $-4 = (2C')^2 = D^2 = -8 + 4k - 2k = -2(4 - k)$. This gives $k = 2$. Thus

$$\bar{\pi}^*(C) = \phi^*(2C') = 2\tilde{C} + E_1 + E_2.$$

Therefore, we may take $\delta^+ = \bar{\pi}^*(C)$, and $\delta^- = -(E_1 + E_2)$ or $\delta^- = E_1 - E_2$ to obtain that $\frac{1}{2}(\delta^+ + \delta^-) = \tilde{C} \in \mathrm{Pic}(Y)$ or $\frac{1}{2}(\delta^+ + \delta^-) = \tilde{C} + E_1 \in \mathrm{Pic}(Y)$. Thus $K(2)$ contains $\mathsf{A}_1(2) \oplus \mathsf{A}_1(2)$ generated by $E_1 + E_2$ and $E_1 - E_2$, and H contains $\mathbb{Z}/2\mathbb{Z}$ generated by $2E_1$ modulo $2K(2)$.

Example 10.6.9 Let $p \in S$ and let $\bar{p} \in X$ with $\pi(\bar{p}) = p$. Assume that X has a rational double point of type D_4 at $\bar{p}$, and denote by $E_i \subset Y$ $(i = 0, 1, 2, 3)$ the exceptional curves over $\bar{p}$ with $E_0 \cdot E_i = 1$, $i = 1, 2, 3$. Let C be a (-2)-curve on S through p. As before, let C' be the reduced pre-image of C in X and let $\tilde{C}$ be the proper inverse transform of C' in Y. By Proposition 0.4.19 in Volume I, $2C' \in \mathrm{Pic}(X)$, so $\phi^*(2C') = 2\tilde{C} + \sum a_i E_i \in \mathrm{Pic}(Y)$. Since C' is nonsingular, $\tilde{C}$ intersects only one component with multiplicity one. Suppose this component is E_0. Since $\phi^*(2C') \cdot E_i = 0$, we obtain $-2a_i + a_0 = 0, i \neq 0$ and $2 - 2a_0 + a_1 + a_2 + a_3 = 0$, and hence $\phi^*(2C') = 2\tilde{C} + 2(2E_0 + E_1 + E_2 + E_3)$. This gives $-4 = \pi^*(2C')^2 = -8 - 8 + 16 = 0$, a contradiction. So, we may assume that $\tilde{C} \cdot E_1 = 1$. Similar computations to the above show that

$$\bar{\pi}^*(C) = 2\tilde{C} + 2E_0 + 2E_1 + E_2 + E_3,$$

that agrees with the fact that $\bar{\pi}^*(C)^2 = -4$. This shows that we may take $\delta^+ = \bar{\pi}^*(C)$ and $\delta^- = -(E_2 + E_3)$ or $\delta^- = E_2 - E_3$ to obtain $\frac{1}{2}(\delta^+ + \delta^-) = \tilde{C} + E_0 + E_1 \in \mathrm{Pic}(Y)$ or $\frac{1}{2}(\delta^+ + \delta^-) = \tilde{C} + E_0 + E_1 + E_2 \in \mathrm{Pic}(Y)$. Thus $K(2)$ contains $\mathsf{A}_1(2) \oplus \mathsf{A}_1(2)$ generated by $E_2 + E_3$ and $E_2 - E_3$, and H contains $\mathbb{Z}/2\mathbb{Z}$ generated by $2E_2 \mod 2K(2)$.

Example 10.6.10 Suppose S has no (-2)-curves. Then, $h^+ = \emptyset$, hence $h^- = \emptyset$ and the R-invariant is equal to $\{0\}$. On the other hand, suppose that S is a general nodal surface. By Theorem 6.5.5, there exists an elliptic fibration with irreducible fibers. Since the lattice $\bar{\pi}^* \mathrm{Pic}(S)^\perp$ is 2-elementary, all singular fibers are nodal curves and the number of them is equal to 12. All singular points of X are ordinary

double points. By Example 10.6.8, any (-2)-curve R passes through the projections $\bar{p}, \bar{q}$ of two singular points p, q of X. By Theorem 6.5.5, any other (-2) curve R' is f-equivalent to R, i.e. there exists a sequence $R_1, \dots, R_k$ of (-2)-curves with $R \cdot R_1 = R_1 \cdot R_2 = \cdots = R_{k-1} \cdot R_k = R_k \cdot R' = 2$. Since $R + R_1$ is a fiber of an elliptic fibration, there are two singular points in X over the intersection points R with R_1. Since R_1 passes through only two such points, we see that $R \cap R_1 = \{\bar{p}, \bar{q}\}$. Continuing in this way, we see that all $R \cap R_i = R \cap R' = \{\bar{p}, \bar{q}\}$. Thus, we get a remarkable fact that the 12 singular points have a distinguished subset of two points. We will find an explanation of this in Section 8.6 by realizing S as a Reye congruence of lines in $\mathbb{P}^3$. It follows from Example 10.6.8 that the R-invariant (K, H) is equal to $(\mathsf{A}_1 \oplus \mathsf{A}_1, \mathbb{Z}/2\mathbb{Z})$, contrary to $(\mathsf{A}_1, \{0\})$ in the case $p \neq 2$.

In the following proposition, we calculate the nodal R-invariants of Enriques surfaces of type MI, MII and of type VII in characteristic 2.

Proposition 10.6.11 *Let S be an Enriques surface of type* VII*, of type* MI *or of type* MII*. Then, its R-invariant is* $(\mathsf{D}_{10} \oplus \mathsf{A}_1^{\oplus 2}, (\mathbb{Z}/2\mathbb{Z})^{\oplus 3})$*,* $(\mathsf{D}_6^{\oplus 2}, (\mathbb{Z}/2\mathbb{Z})^{\oplus 3})$ *or* $(\mathsf{D}_8 \oplus \mathsf{A}_3, (\mathbb{Z}/2\mathbb{Z})^{\oplus 3})$*, respectively.*

Proof Let (K, H) be the R-invariant of S. In the case where the surface is of type VII, there exist nine nodal curves in Figure 8.10.1 passing 10 canonical points and forming a dual graph of type A_9. On the other hand, there exists a nodal curve passing the remaining two canonical points. This gives us twelve (-4)-vectors in h^- generating $(\mathsf{D}_{10} \oplus \mathsf{A}_1^{\oplus 2})(2)$ as mentioned in Example 10.6.8.

We know that there are no (-2)-curves containing a point among the first 10 canonical points and a point among the second two canonical points. This implies that there are no roots in K connecting D_{10} and $\mathsf{A}_1^{\oplus 2}$. Since $\mathsf{D}_{10} \oplus \mathsf{A}_1^{\oplus 2}$ has the maximal rank 12 and there are no root lattices of rank 10 (resp. rank 2) containing D_{10} (resp. $\mathsf{A}_1^{\oplus 2}$), we have $K = \mathsf{D}_{10} \oplus \mathsf{A}_1^{\oplus 2}$. Since $\frac{1}{2}(E_1 + \cdots + E_{12}) \in \mathrm{Pic}(Y)$ (see [214, Lemmas 3.14 and 6.5]) by using the calculation in Example 10.6.8, we get $H = (\mathbb{Z}/2\mathbb{Z})^3$.

In the case where S is of type MI, for example, consider the 10 (-2)-curves on S corresponding to the following five duads and five synthemes:

$$\begin{aligned} &(12), \quad (23), \quad (34), \quad (45), \quad (56), \quad (14, 25, 36), \\ &(15, 26, 34), \quad (14, 23, 56), \quad (15, 24, 36), \quad (14, 26, 35). \end{aligned} \tag{10.6.6}$$

As mentioned in Example 10.6.8, these curves define twelve (-4)-vectorsin h^-. They generate the lattice $(\mathsf{D}_6 \oplus \mathsf{D}_6)(2)$. Note that this lattice is of maximal rank 12. There are no (-2)-curves passing through a canonical point corresponding to a number in $\{1, ..., 6\}$ and a point corresponding to a total (e.g. [418, Lemma 3.3, (2)]). This fact implies that there are no roots in K connecting two components of $\mathsf{D}_6 \oplus \mathsf{D}_6$. Since there are no root lattices of rank 6 containing D_6, we have $K = \mathsf{D}_6 \oplus \mathsf{D}_6$. Since $(E_1 + \cdots + E_{12})/2 \in \mathrm{Pic}(X)$, we have $(K, H) = (\mathsf{D}_6 \oplus \mathsf{D}_6, (\mathbb{Z}/2\mathbb{Z})^3)$.

In the case where the type is MII, there are seven nodal curves among 16 nodal curves on the left-hand side in Figure 10.8 passing 8 canonical points and forming a dual graph of type A_7 (see Example 10.6.8). On the other hand, there are two nodal

curves among 12 nodal curves on the right-hand side in Figure 10.8 passing one canonical point p_0 and forming a dual graph of type A_2 (see Example 10.6.9). Note that there are no (-2)-curves passing through p_0 and another canonical point. Thus, we have $\mathsf{D}_8 \oplus \mathsf{A}_3$ which has the maximal dimension 11 because we get at most a root lattice of rank 3 from p_0 (see Example 10.6.9). The only root lattice containing $\mathsf{D}_8 \oplus \mathsf{A}_3$ properly is $\mathsf{E}_8 \oplus \mathsf{A}_3$. Assume that $R = \mathsf{E}_8 \oplus A_3$. Then, $\mathsf{E}_8(2)$ is contained in $\mathsf{A}_1^{\oplus 8}$ generated by the exceptional curves over the eight canonical points, but both have the determinant 2^8, which is a contradiction. Thus, we have $R = \mathsf{D}_8 \oplus \mathsf{A}_3$. Since the sum of the eight components of $\mathsf{A}_1^{\oplus 8}$ is divisible by 2 (see [214, Lemma 3.14 and Lemma 6.5]), we have $H = (\mathbb{Z}/2\mathbb{Z})^3$. □

Remark 10.6.12 Any Enriques surface S in characteristic 2 covered by the supersingular K3 surface Y with Artin invariant one has the same dual graph of (-2)-curves as the one of the three Enriques surfaces of type MI, MII and of type VII (see [418]). All three examples have 20 or 40 (-2)-classes which generate the reflection group of finite index in $\mathrm{O}(\mathrm{Num}(S))$ (by Vinberg's Criterion 0.8.23 in Volume I). Enriques surfaces with finite automorphism group are nothing but those with a crystallographic basis of (-2)-curves. We will discuss these examples in Section 10.7 from the point of view of the Leech lattice.

Finally, we calculate the nodal R-invariants of the Coble surfaces V from Theorems 9.8.15, 9.8.16, 10.5.17, 10.6.1. We use the definition of the nodal R-invariants for Coble surfaces in characteristic 2 by mixing the definition for Coble surfaces in characteristic $p \neq 2$ and the one above for classical and $\boldsymbol{\alpha}_2$-surfaces.

Proposition 10.6.13 *Let* V *be the Coble surface of type* VII *with* 10 *boundary components, of type* VII *with two boundary components, of type* MI *or of type* MII *in characteristic 2. The R-invariant* (K, H) *of* V *is* $(\mathsf{D}_{10} \oplus \mathsf{A}_1^{\oplus 2}, (\mathbb{Z}/2\mathbb{Z})^{\oplus 3})$, $(\mathsf{D}_{10} \oplus \mathsf{A}_1^{\oplus 2}, (\mathbb{Z}/2\mathbb{Z})^{\oplus 3})$, $(\mathsf{D}_6^{\oplus 2}, (\mathbb{Z}/2\mathbb{Z})^{\oplus 3})$ *or* $(\mathsf{D}_8 \oplus \mathsf{A}_3, (\mathbb{Z}/2\mathbb{Z})^{\oplus 3})$, *respectively.*

Proof The situation is very similar to the case of Enriques surfaces in Proposition 10.6.11. The minimal resolutions of the canonical covers are the supersingular K3 surfaces Y with Artin invariant one, and we consider the same elliptic fibration on Y with singular fibers $\tilde{A}_9, \tilde{A}_9, \tilde{A}_1, \tilde{A}_1$, with singular fibers $\tilde{A}_5, \tilde{A}_5, \tilde{A}_5, \tilde{A}_5$, or with singular fibers $\tilde{A}_7, \tilde{A}_7, \tilde{D}_5$. The only difference is that we are replacing some of the canonical points with the boundary components. Thus, the proof of Proposition 10.6.11 works in this case, too. □

10.7 Enriques Surfaces and the Leech Lattice

In this section, we will illustrate by several examples a method for computing the group G of automorphisms of a K3 surface and an Enriques surfaces. It follows from Corollary 8.1.11 and Theorem 8.1.10 that the action of the group G in the nef cone of the surface has a finite polyhedral fundamental domain Π. By the well-known

method from the combinatorial group theory, the knowledge of the fundamental domain allows one to describe G by means of generators and relations [724, Chapter 2, §1]. As we know, the nef cone is a fundamental domain for the Weyl group of the numerical lattice of the surface. We also know from (0.8.12) in Volume I that the orthogonal group of a lattice is equal to the semi-direct product of its reflection group and the group of symmetries of its fundamental polytope. In his paper [79], R. Borcherds introduced a method for computing the fundamental polytope for the reflection group of a hyperbolic lattice M by embedding it into the unique even unimodular hyperbolic lattice $\mathrm{II}_{1,25}$ of rank 26 and using the description of its reflection polytope given by J. Conway. Since the reflections into divisor classes of (-2)-curves on the surface are reflections of $\mathrm{II}_{1,25}$, the intersection of the Conway (infinite) polytope with $M_{\mathbb{R}}$ will define a reflection polytope contained in the nef cone of the surface and it serves as a fundamental domain Π of G in the nef cone.

In [411], the second author applied this method for computation of the automorphism group of the K3 surface birationally isomorphic to the Kummer surface of a general curve of genus two. In a subsequent paper of Borcherds [83], which he refers to Kondō, this method was applied for computation of the automorphism group of some K3 surfaces with the rank of the Picard lattice equal to 20 and small discriminant. For this reason by suggestion of the first author), we will refer to this method of computing the automorphism groups as the *Borcherds-Kondō method*. We will see in this section that the Borcherds-Kondō method can be also applied to the computation of the automorphism groups of Enriques surfaces.

First, let us recall the case of K3 surfaces.

The lattice $\mathrm{II}_{1,25}$ is an even unimodular quadraticlattice of signature $(1, 25)$. By Milnor Theorem 0.8.8, its isomorphism class is unique. We fix an isomorphism

$$\mathrm{II}_{1,25} \cong \mathsf{U} \oplus \Lambda \tag{10.7.1}$$

where Λ is the *Leech lattice* , the unique even negative definite lattice with no vectors of square-norm -2. For typographical reasons, we redenote $\mathrm{II}_{1,25}$ by L. For $x \in L$, we write $x = (m, n, \lambda)$, where $\lambda \in \Lambda$ and $x-\lambda = mf+ng \in \mathsf{U}$ with (f, g) being a standard basis of U. We have $x^2 = 2mn + \lambda^2$. We fix an isotropic vector $\rho = (1, 0, 0) \in L$ called the *Weyl vector*. Note that there are vectors in L_{-2} orthogonal to ρ because $\Lambda_{-2} = \emptyset$. A vector $r \in L_{-2}$ is called a *Leech root* if $\langle r, \rho\rangle = 1$. We denote by L_{-2}^{Leech} the set of all Leech roots. Note that there is a bijective correspondence between Λ and L_{-2}^{Leech} given by:

$$\Lambda \ni \lambda \longleftrightarrow (-1 - \frac{1}{2}\lambda^2, 1, \lambda) \in L_{-2}^{\mathrm{Leech}}.$$

Since Λ is a unimodular even lattice, all its roots are (-2)-vectors and the reflection group $\mathrm{Ref}(L)$ coincides with its Weyl group $W(L) = \mathrm{Ref}_2(L)$ (see Section 0.8 in Volume I). The following theorem of J. Conway asserts that the Leech roots form a root basis of $W(L)$ [130].

Theorem 10.7.1 *The set*

$$C_L = \{x \in P^+(L \otimes \mathbb{R}) \ : \ \langle x, r\rangle > 0 \text{ for any } r \in L_{-2}^{\text{Leech}}\}$$

is a fundamental domain of $W(L)$.

Our first example of applying the Borcherds-Kondo method is the case when $X = \widetilde{\text{Kum}}(\text{Jac}(C))$ is the K3 surface of Kummer type associated with the Jacobian of a general curve of genus two.

- *Automorphisms of the Kummer K3 surface* $X = \widetilde{\text{Kum}}(\text{Jac}(C))$*, where* C *is a general curve of genus 2.*

Let $R = 6\mathsf{A}_1 + \mathsf{A}_3$. Then, there is an embedding of R in L whose image is generated by Leech roots and its orthogonal complement in L is isomorphic to the Picard lattice of X (Note: R is not primitive in L, but an overlattice of R of index 2 is primitive.) Denote by S_X the Picard lattice of X. The primitive embedding of S_X in L induces an embedding of the positive cone $P^+(S_X \otimes \mathbb{R})$ in $P^+(L \otimes \mathbb{R})$. Let $\Pi(X)$ be the restriction of C_L to $P^+(S_X \otimes \mathbb{R})$ and $\overline{\Pi}(X)$ its closure. The condition that R contains a Leech root implies that $\overline{\Pi}(X)$ is a finite polytope. We call a face of $\overline{\Pi}(X)$ of codimension one a *facet*. It is cut out by a hyperplane $H_\alpha = |\alpha^\perp|, \alpha^2 < 0$ in the hyperbolic space $\mathbb{H}^{25}$ associated with L.

By using the geometry of the Leech lattice, we have the following:

Proposition 10.7.2 *The set of facets of* $\Pi(X)$ *are cut out by the following hyperplanes:*

- 32 *hyperplanes* $H_\alpha,\ \alpha^2 = -2$,
- 32 *hyperplanes* $H_\alpha,\ \alpha^2 = -4$,
- 60 *hyperplanes* $H_\alpha,\ \alpha^2 = -4$,
- 192 *hyperplanes* $H_\alpha,\ \alpha^2 = -12$.

The projection ρ_0 *of the Weyl vector* ρ *is in* $\Pi(X)$ *with* $\rho_0^2 = 8$.

The facets can be interpreted in terms of the geometry of the Kummer surface $\text{Kum}(\text{Jac}(C))$. For the convenience of the notation, we denote $\text{Jac}(C)$ by A. It is a principally polarized simple abelian surface. Let C be given by the following (affine) equation:

$$y^2 = \prod_{i=1}^{6}(x - \xi_i), \tag{10.7.2}$$

Let $q_i = (\xi_i, 0), i = 1, \ldots, 6$, be the ramification points of the unique g_2^1 on C. Then, the divisor classes $\mu_{(i6)} = [q_i - q_6]$ and $\mu_{(ij)} = [q_i + q_j] - K_C, 1 \le i < j \le 5$, are the 2-torsion points in A. We identify the indices with duads from $[1, 6]$ considered as vectors in the symplectic $\mathbb{F}_2$-space of pairs of complementary subsets of $[1, 6]$ of even cardinality (see Section 10.4).

Let Θ be the symmetric theta divisor on $\text{Jac}(C)$, the image of the Abel-Jacobi map

$$C \to A = \text{Pic}^0(C), x \mapsto [2x] - K_C,$$

and $\Theta_\alpha = \Theta + \mu_\alpha$ be its translates. The incidence relation between $\{\mu_i, \mu_{ij}\}$ and $\{\Theta_i, \Theta_{ij}\}$ is as follows:

$$\mu_\alpha \in \Theta_\beta \Leftrightarrow \alpha + \beta \in \{0, (16), (26), (36), (46), (56)\}.$$

The negation involution ι of A leaves Θ invariant and acts linearly on linear spaces $H^0(A, \mathcal{O}_A(m\Theta))$. The linear system $|2\Theta|$ is ι-invariant and defines a map of A to $\mathbb{P}^3$ that factors through an embedding

$$\mathrm{Kum}(A) \hookrightarrow \mathbb{P}^3$$

with the image the *Kummer quartic surface*. The images of the points μ_α are the sixteen nodes n_α of the Kummer quartic surface, and the images of Θ_β are the trope-conics. They are cut out by a plane section with multiplicity two. The incidence relation becomes the *Kummer abstract* (16_6)*-configuration*.

The linear subsystem of $|4\Theta| = |H^0(A, \mathcal{O}_A(4\Theta)^-|$ of anti-invariant sections has the set ${}_2A$ of 2-torsion points as its set of base points. It defines a map of the blow-up $\mathrm{Bl}_{{}_2A}(A)$ to $\mathbb{P}^5$ that factors through a closed embedding

$$j : X = \widetilde{\mathrm{Kum}}(A) \hookrightarrow \mathbb{P}^5.$$

The image is a K3 surface of degree 8 isomorphic to the surface given by the following quadratic equations:

$$\sum_{i=1}^{6} z_i^2 = \sum_{i=1}^{6} \xi_i z_i^2 = \sum_{i=1}^{6} \xi_i^2 z_i^2 = 0, \tag{10.7.3}$$

where $[z_1, \ldots, z_6]$ are homogeneous coordinates of $\mathbb{P}^5$ and ξ_i is given in (10.7.2) (see [177, 10.3]). The six hyperplane sections $V(z_i)$ are the six quartic del Pezzo surfaces realized as confocal congruences of lines with the focal quartic surface isomorphic to $\mathrm{Kum}(A)$ (see Example 7.4.9).

We denote by N_α the image of the exceptional curve over μ_α and by T_α the image of the proper transform of Θ_α in X. The curves N_α are the exceptional curves of the minimal resolution

$$\pi : \widetilde{\mathrm{Kum}}(A) \to \mathrm{Kum}(A).$$

The curves T_α are the proper transforms of the trope-conics in $\widetilde{\mathrm{Kum}}(A)$.

The both sets $\{N_\alpha\}$ and $\{T_\alpha\}$ consist of sixteen disjoint (-2)-curves and from an abstract configuration isomorphic to the Kummer (16_6)-configuration.

The group $G = (\mathbb{Z}/2\mathbb{Z})^5$ acts on the octic surface (10.7.3) by projective transformations

$$[z_1, z_2, z_3, z_4, z_5, z_6] \to [\epsilon_1 z_1, \epsilon_2 z_2, \epsilon_3 z_3, \epsilon_4 z_4, \epsilon_5 z_5, \epsilon_6 z_6], \tag{10.7.4}$$

where $\epsilon_i = \pm 1$. The subgroup

$$G_0 = \mathrm{Ker}(G \to \mu_2) = \{\epsilon = (\epsilon_1, \dots, \epsilon_6) \in G : \prod_{i=1}^{6} \epsilon_i = 1\}$$

acts symplectically with the quotient isomorphic to $\widetilde{\mathrm{Kum}}(A)$. Its elements t_ϵ are called the *translation involutions* because they are induced by translation involutions of A lifted to the blow-up. The quotient of $\widetilde{\mathrm{Kum}}(A)$ by G is isomorphic to the Coble surface $\mathsf{V} = \mathrm{Bl}_{\mathcal{P}}(\mathbb{P}^2)$, where $\mathcal{P}$ is the set of intersection points of the six lines $V(t_0 + \xi t_1 + \xi^2 t_2)$ in $\mathbb{P}^2$.

The double cover

$$X \cong X/G_0 \to X/G = \mathsf{V}$$

is the canonical cover of V branched over the boundary components.

An involution s_ϵ, where $(\epsilon_1, \dots, \epsilon_6) \notin G_0$ is called a *switch involution* . It sends the set of $\{N_\alpha\}$ to the set $\{T_\alpha\}$. Each switch involutions represents a coset with respect to the subgroup of translation involutions. The quotient $X/(s_\epsilon)$, where $\epsilon = (\epsilon_1, \dots, \epsilon_6)$ has only one ϵ_i equal to -1, is isomorphic to a quartic Pezzo surface $\mathcal{D}$. The double cover $X \to X/\langle s_\epsilon \rangle = \mathcal{D}$ is branched along a curve $B \in |-2K_{\mathcal{D}}|$. In the realization of $\mathcal{D}$ as a congruence of lines S of bidegree $(2,2)$, the surface X is identified with the ramification divisor $R(S)$ of the map $p : Z_S \to \mathbb{P}^3$ branched along $\mathrm{Kum}(A)$. The quotient $X/(s_\epsilon)$, where ϵ contains three coordinates equal to -1, is an Enriques surface. The double cover $X \to X/(s_\epsilon)$ is the canonical cover. Note that one can choose an isomorphism $G_0 \cong {}_2A$ of symplectic spaces such that elements s_ϵ is identified with the 2-torsion point $\mu_{(ab)}$, where $(ab) \neq (i6)$. The theta divisor $\Theta + \mu_{(ab)}$ correspond to an even theta characteristic $\vartheta_{ab} = q_a + q_b - q_6$ on C. It is called an even theta divisor.

Consider a set $\{n_\alpha, n_\beta, n_\gamma, n_\delta\}$ of four nodes. It is called a *Göpel tetrad* (resp. *Rosenhain tetrad*) if no three nodes in the set lie on a trope-conic (resp. if any three nodes lie in a trope-conic). For example, $\{n_0, n_1, n_{25}, n_{34}\}$ is a Göpel tetrad and $\{n_0, n_{13}, n_{15}, n_{35}\}$ is a Rosenhain tetrad. There exist sixty Göpel tetrads and eighty Rosenhain tetrads. A set of six nodes is called a *Weber hexad* if it is the symmetric difference of a Göple tetrad and a Rosenhain tetrad. For example, $\{n_1, n_{13}, n_{15}, n_{25}, n_{34}, n_{35}\}$ is a Weber hexad which is the symmetric difference of $\{n_0, n_1, n_{25}, n_{34}\}$ and $\{n_0, n_{13}, n_{15}, n_{35}\}$. There exist 192 Weber hexads (see [312], [187]).

Let us now return to Proposition 10.7.2. Let $h_4 = c_1(\pi^* \mathcal{O}_{\mathrm{Kum}(A)}(1))$ and s_ϵ be one of the switches.

One can identify 32 vectors $\alpha \in L_{-2}$ in Proposition 10.7.2 with 32 (-2)-curves N_α, T_α on $\mathrm{Kum}(A)$, 32 vectors $\alpha \in L_{-4}$ with the divisor classes $h_4 - 2N_\alpha$ and $s_\epsilon(h - 2N_\alpha)$ (the latter set does not depend on the choice of s_ϵ), 60 vectors $\alpha \in L_{-4}$ with the divisor classes $h_4 - N_\alpha - N_\beta - N_\gamma - N_\delta$, and 192 vectors $\alpha \in L_{-12}$ with the divisor classes $3h_4 - 2\sum_{\alpha \in W} N_\alpha$. respectively, where $\{n_\alpha, n_\beta, n_\gamma, n_\delta\}$ is a Göpel tetrad and W is a Weber hexad.

The projection ρ_0 of ρ into $S_X \otimes \mathbb{Q}$ coincides with the class

$$h_8 = \frac{1}{4}\sum_{\alpha}(N_\alpha + T_\alpha)$$

equal to the divisor class of a hyperplane section of the optic surface (10.7.3).

It is classically known that a general quartic Kummer surface $\mathrm{Kum}(A)$ admits the followimng set of birational involutions that regularize on X:

- 16 involutions p_α associated with the projections from the sixteen nodes n_α. They act on S_X as the reflections in vectors $h_4 - 2N_\alpha$.
- 16 involutions $p_\alpha \circ s_\epsilon$,
- 60 involutions defined by the cubic Cremona involution with fundamental points in a Göpel tetrad [319],
- 192 Cremona transformations associated with 192 Weber hexads [318].

The symmetry group of the finite polytope $\Pi(X)$ is isomorphic to $G \cdot \mathfrak{S}_6$ and $\mathfrak{S}_6$ is the symmetric group of degree 6 acting on the set of six Weierstrass points q_i of C. The following theorem describes the group $\mathrm{Aut}(X) \cong \mathrm{Bir}(\mathrm{Kum}(A))$:

Theorem 10.7.3 *Assume that C is a general curve of genus two, then* $\mathrm{Aut}(X)$ *is generated by the* 2*-elementary group $G \cong 2^5$ of translation and switch involutions,* 16 *projection involutions p_α,* 60 *Cremona transformations associated with* 60 *Göpel tetrads and* 192 *Cremona transformations associated with* 192 *Weber hexads.*

Remark 10.7.4 The second author [411] used 192 automorphisms of infinite order constructed by Keum [384]. It was pointed out by Ohashi [578] that one can use the 192 Cremona transformations instead.

Mukai [529] observed that there exist three kinds of fixed-point-free involutions of X and then Ohashi [578] proved that no other such an involution exists.

Theorem 10.7.5 *Assume that C is generic. Then, there are exactly* $31(= 10+15+6)$ *fixed-point-free involutions up to conjugacy in* $\mathrm{Aut}(X)$. *They are*

- 10 *switches associated with even theta characteristics,*
- 15 *Cremona involutions associated with Göpel tetrads,*
- *six Cremona involutions associated with Weber hexads.*

Later we will discuss Enriques surfaces isomorphic to quotients of X by fixed-point-free involutions.

- *Automorphisms of the supersingular K3 surface with Artin invariant 1 in characteristic 2.*

Let R be the root lattice D_4. It admits a primitive embedding in L whose image is generated by Leech roots and its orthogonal complement in L is isomorphic to the Picard lattice of the supersingular K3 surface Y with the Artin invariant one in characteristic two mentioned in Section 10.5. Denote by S_Y the Picard lattice of Y. The primitive embedding of S_Y in L induces an embedding of the positive cone $P^+(S_Y \otimes \mathbb{R})$ in $P^+(L \otimes \mathbb{R})$. Let $\Pi(Y)$ be the restriction of C to $P^+(S_Y \otimes \mathbb{R})$. In this case, we have the following:

Proposition 10.7.6 *The facets of* $\Pi(Y)$ *are cut out by* 42 *hyperplanes* $H_\alpha,\ \alpha \in L_{-2}$ *and* 168 *hyperplanes* $H_\alpha,\ \alpha \in L_{-4}$.

Recall that Y is obtained as the minimal resolution of a purely inseparable double cover of $\mathbb{P}^2$. The inseparable double cover has 21 nodes over 21 points of $\mathbb{P}^2(\mathbb{F}_4)$, and thus Y contains 42 (-2)-curves divided into two sets $\mathcal{A}$ and $\mathcal{B}$ both of which consist of 21 disjoint curves. A curve in $\mathcal{A}$ (resp. in $\mathcal{B}$) is an exceptional curve over a node (resp. a proper transform of a line on $\mathbb{P}^2(\mathbb{F}_4)$). The incidence relation of 42 curves is an abstract $(21)_5$-configuration. One can identify 42 vectors α in Proposition 10.7.6 with forty two (-2)-curves in $\mathcal{A} \cup \mathcal{B}$, and 168 vectors $\alpha \in L_{-4}$ with the divisor class $2\ell - N_1 - \cdots - N_6$ respectively, where ℓ is the pull-back of the class of a line on $\mathbb{P}^2$ and $\{N_1, \ldots, N_6\}$ are (-2)-curves in $\mathcal{A}$ corresponding to a six-arc $\mathbb{P}^2(\mathbb{F}_4)$. The symmetry group of the finite polytope $\Pi(Y)$ is isomorphic to $\mathrm{PGL}(3, \mathbb{F}_4) \cdot (\mathbb{Z}/2\mathbb{Z})^2$ where $(\mathbb{Z}/2\mathbb{Z})^2$ is generated by a switch σ, and the involution ι induced from the Frobenius automorphism of $\mathbb{F}_4$. Thus, we have the following theorem, already mentioned in Section 10.5:

Theorem 10.7.7 *The automorphism group of the supersingular K3 surface with Artin invariant* $\sigma_0 = 1$ *in characteristic* 2 *is generated by the finite projective linear group* $\mathrm{PGL}(3, \mathbb{F}_4)$*, the switch and* 168 *Cremona transformations associated with* 168 *sets of six points in* $\mathbb{P}^2(\mathbb{F}_4)$ *in general position.*[5]

Very recently Brandhorst and Shimada [91] have found the following theorem.

Theorem 10.7.8 *There exist exactly* 17 *primitive embeddings of* $\mathsf{E}_{10}(2)$ *in* L.

In Table 10.2 we give seventeen primitive embeddings of $\mathsf{E}_{10}(2)$ in L and seventeen polytopes obtained by restricting Conway's fundamental domain C_L to the positive cone of the Enriques lattice. For typographical reasons, we write nM for the direct sum of n copies of a lattice M.

The name is the symbol of the polytope from the loc. cit. paper of Brandhorst and Shimada. The column named facets gives the number of facets of the polytope. It is remarkable that all facets are defined by (-2)-vectors. For example, 12A and 12B are two polytopes both of which have 12 facets. The column root indicates the root lattice, which is the sublattice generated by (-2)-vectors in the orthogonal complement of $\mathsf{E}_{10}(2)$ in L.

In the cases of the Nos. 5 and 6, or the Nos. 10 and 11, or the Nos. 12 and 13, or the Nos. 15 and 16, their polytopes coincide, but the embeddings are different. The last column gives the Enriques surface, where the polytope is realized geometrically. The symbols I, ..., VII mean the types of Enriques surfaces with finite automorphism group and crystallographic basis of (-2)-curves of types I, ..., VII) and VII, MI, MII in char 2 are type VII, type MI, type MII given in Sections 10.5 and 10.6. The complex Enriques surfaces of type MII and MIII are obtained by Mukai

[5] Each of 168 (-4)-vectors defines a reflection of the Picard lattice S_Y of Y However the action of the corresponding Cremona transformation on S_Y does not coincide with the one of the reflection. It coincides with the composite of the actions of the Cremona transformation and ι. This fact was pointed out by S. Mukai. See also Example 10.5.13

No.	name	facets	root	Enriques surface
1	$12A$	12	D_8	I
2	$12B$	12	A_7	II
3	$20A$	20	$\mathsf{D}_4 + \mathsf{D}_5$	V
4	$20B$	20	$2\mathsf{D}_4$	III
5	$20C$	20	$10\mathsf{A}_1 + \mathsf{D}_6$	VII in char 2 (Example 8.10.9)
6	$20D$	20	$\mathsf{A}_3 + \mathsf{A}_4$	VII
7	$20E$	20	$5\mathsf{A}_1 + \mathsf{A}_5$	VI
8	$20F$	20	$2\mathsf{A}_3$	IV
9	$40A$	40	$4\mathsf{A}_1 + 2\mathsf{A}_3$	MIII (Example 10.7.9)
10	$40B$	40	$8\mathsf{A}_1 + 2\mathsf{D}_4$	MII in char 2 (Theorem10.6.3)
11	$40C$	40	$6\mathsf{A}_1 + \mathsf{A}_3$	MII (Example 10.7.11)
12	$40D$	40	$12\mathsf{A}_1 + \mathsf{D}_4$	MI in char 2 (Theorem10.5.8)
13	$40E$	40	$2\mathsf{A}_1 + 2\mathsf{A}_2$	MI (Remark 10.5.14)
14	$96A$	96	$8\mathsf{A}_1$	
15	$96B$	96	$16\mathsf{A}_1$	
16	$96C$	96	$4\mathsf{A}_1$	
17	infty	–		

Table 10.2 Brandhorst–Shimada polytopes

as the quotient of the Kummer surface associated with a curve of genus two by a switch involution and by a Cremona transformation associated with a Göpel tetrad, respectively (see Examples 10.7.9, 10.7.11). The complex Enriques surface of type MI is obtained by Mukai as the quotient of the K3 surface mentioned in Remark 10.5.14. In the following we will give more details. , ,

In the cases of the Nos. 1, . . . , 4, 6, 7, 8 in Table 10.2, the polytope coincides with that of Enriques surfaces defined over the complex numbers with finite automorphism group of the same type. In the case of No. 5, the root lattice $10\mathsf{A}_1 + \mathsf{D}_6$ contains a sublattice $12\mathsf{A}_1 + \mathsf{D}_4$ of index 2. Recall that the orthogonal complement of the root lattice D_4 in L is isomorphic to the Picard lattice of the supersingular K3 surface Y with the Artin invariant one in characteristic two. The remaining $12\mathsf{A}_1$ in $12\mathsf{A}_1 + \mathsf{D}_4$ can be considered as the exceptional curves of 12 A_1-singularities of the canonical covering of the Enriques surfaces in characteristic 2 of type VII. Thus, the polytope from No. 5 is obtained as a restriction of the polytope $\Pi(Y)$ given in Proposition 10.7.6. Similarly, in the cases of Nos. 10 and 12, the root lattices contain D_4 as a direct summand and the remaining summands are $12\mathsf{A}_1$ or $8\mathsf{A}_1 + \mathsf{D}_4$ which correspond to $12A_1$- or $8A_1 + D_4$-singularities of the canonical covers of Enriques surfaces in characteristic 2 of type MI or of type MII. The polytopes Nos. 10 and 12 are obtained as restrictions of the polytope $\Pi(Y)$ given in Proposition 10.7.6. For 40 (-2)-vectors defining the facets of the polytopes in these cases, see Theorem 10.5.8, Remark 10.6.5.

The following Example is due to Mukai.

Example 10.7.9 In this example, we realize the polytope 40A with 40 facets. Let $X = \widetilde{\mathrm{Kum}}(\mathrm{Jac}(C))$ and let $\Pi(X)$ be the fundamental polytope given in Proposition 10.7.2. Let $\mathcal{T} = \{n_{\alpha_0}, n_{\beta_0}, n_{\gamma_0}, n_{\delta_0}\}$ be a Göpel tetrad and $c_{\mathcal{T}}$ the associated Cremona transformation. The Enriques surface of type MIII is defined to be the quotient

$$S = X/\langle c_{\mathcal{T}}\rangle.$$

The polytope Π_{MIII} of type MIII is realized as the fundamental polytope $\Pi(S)$ for the action of $\mathrm{Aut}(S)$ in the nef cone. We describe its 40 facets by bounding hyperplanes H_r, where $r \in A$, where $A \subset \mathrm{Num}(S)_{-2}$ is divided into four sets A_1, A_2, A_3, A_4 of cardinalities $12, 8, 12, 8$, respectively. The divisor class r will be described in terms of their pre-images under the map

$$\pi^* : \mathrm{Num}(S)_{\mathbb{R}} \to \mathrm{Pic}(X)_{\mathbb{R}}.$$

It is known how $c_{\mathcal{T}}$ acts on the generating set $\{H, N_\alpha, T_\beta\}$ of $\mathrm{Pic}(X)$ [384]. For a given Göpel tetrad $\mathcal{T}$, there exist exactly four trope-conics that do not contain any node from $\mathcal{T}$. We denote these trope-conics by $\{T_{\alpha_0'}, T_{\beta_0'}, T_{\gamma_0'}, T_{\delta_0'}\}$. Then, $c_{\mathcal{T}}$ preserves the set of twenty four (-2)-curves N_α $(\alpha \neq \alpha_0, \beta_0, \gamma_0, \delta_0)$, T_α $(\alpha \neq \alpha_0', \beta_0', \gamma_0', \delta_0')$.

The set A_1 consists of twelve (-2)-curves R_r such that

$$\pi^*(R_r) \in \{N_\alpha + c_G(N_\alpha)\}_{\alpha\neq\alpha_0,\beta_0,\gamma_0,\delta_0} \bigcup \{T_\alpha + c_G(T_\alpha)\}_{\alpha\neq\alpha_0',\beta_0',\gamma_0',\delta_0'}.$$

The set A_2 consists of eight (-2)-curves R_r such that

$$\pi^*(R_r) \in \{N_\alpha + c_G(N_\alpha)\}_{\alpha=\alpha_0,\beta_0,\gamma_0,\delta_0}, \bigcup \{T_\alpha + c_G(T_\alpha)\}_{\alpha=\alpha_0',\beta_0',\gamma_0',\delta_0'}.$$

The set A_3 consists of twelve non-effective divisor classes D_r such that

$$\pi^*(D_r) = h_4 - N_\alpha - N_\beta - N_\gamma - N_\delta$$

which is orthogonal to $h_4 - N_{\alpha_0} - N_{\beta_0} - N_{\gamma_0} - N_{\delta_0}$.

The set A_4 consists of eight non-effective divisor classes D_r such that

$$\pi^*(D_r) \in \{h_4 - 2N_\alpha,\ \sigma(h_4 - 2N_\alpha)\}_{\alpha=\alpha_0,\beta_0,\gamma_0,\delta_0}.$$

For example, if we take

$$\mathcal{T} = \{n_{\alpha_0}, n_{\beta_0}, n_{\gamma_0}, n_{\delta_0}\} = \{n_0, n_1, n_{25}, n_{34}\},$$

then $\{\alpha_0', \beta_0', \gamma_0', \delta_0'\} = \{23, 24, 35, 45\}$ and $c_{\mathcal{T}}$ acts on N_α, T_α as follows:

$$N_2 \longleftrightarrow N_{15},\ N_3 \longleftrightarrow N_{14},\ N_4 \longleftrightarrow N_{13},\ N_5 \longleftrightarrow N_{12},\ N_{23} \longleftrightarrow N_{45},\ N_{24} \longleftrightarrow N_{35};$$

$$T_0 \longleftrightarrow T_1,\ T_2 \longleftrightarrow T_5,\ T_3 \longleftrightarrow T_4,\ T_{12} \longleftrightarrow T_{15},\ T_{13} \longleftrightarrow T_{14},\ T_{25} \longleftrightarrow T_{34};$$

$$N_0 \longleftrightarrow H - N_1 - N_{25} - N_{34}, \quad N_1 \longleftrightarrow H - N_0 - N_{25} - N_{34},$$

$$N_{25} \longleftrightarrow H - N_0 - N_1 - N_{34}, \quad N_{34} \longleftrightarrow H - N_0 - N_1 - N_{25},$$

and $c_{\mathcal{T}}$ acts on T_α ($\alpha \in \{23, 24, 35, 45\}$) as the reflection associated with the (-4)-vector $H - N_0 - N_1 - N_{25} - N_{34}$.

The twelve vectors orthogonal to $H - N_0 - N_1 - N_{25} - N_{34}$ correspond to twelve Göpel tetrads which have exactly two common nodes with $\mathcal{T}$. They are the following tetrads:

$$\{0, 1, 24, 35\},\ \{0, 1, 23, 45\},\ \{0, 2, 15, 34\},\ \{0, 5, 12, 34\},\ \{0, 3, 14, 25\},\ \{0, 4, 13, 25\},$$

$$\{23, 25, 34, 35\},\ \{24, 25, 34, 35\},\ \{1, 5, 12, 25\},\ \{1, 2, 15, 25\},\ \{1, 4, 13, 34\},\ \{1, 3, 14, 34\}.$$

If C is a curve of genus two defined by $y^2 = x(x^4 - 1)$, then Mukai showed that the third set also represented by (-2)-curves. By this description, Mukai and Ohashi [531] proved the following theorem:

Theorem 10.7.10 *Assume C is given by $y^2 = x(x^4 - 1)$. Then,* $\mathrm{Aut}(\mathrm{Kum}(C)/\langle c_G \rangle)$ *is isomorphic to* $\mathrm{UC}(8) \rtimes H_{192}$*, where the first factor is the free product of* 8 *cyclic groups of order* 2 *generated by* 8 *projection and correlation involutions, and H_{192} is a finite group $(\mathbb{Z}/2\mathbb{Z})^3 \rtimes \mathfrak{S}_4$ of order* 192*. The group H_{192} is a subgroup of index* 4 *of the automorphism group of the polytope of type* MIII.

The next example is also due to Mukai.

Example 10.7.11 In this example, we realize the polytope 40C with 40 facets. Consider a switch $\sigma = s_\epsilon$ associated with an even theta characteristic which is a fixed-point-free involution (Theorem 10.7.5). There exist ten switches. The surface of type MII is the quotient

$$S = X/\langle \sigma \rangle.$$

In this case the restriction of the polytope of $\Pi(X)$ given in Proposition 10.7.2 coincides with fundamental polytope $\Pi(S_{\alpha,\beta})$ from Theorem 10.6.3. For example, we can take a switch σ associated with μ_{12} satisfying

$$\sigma(N_i) = T_{i+12}, \quad \sigma(N_{ij}) = T_{ij+12}.$$

Thus, the Enriques surface $X/\langle \sigma \rangle$ contains 16 (-2)-curves whose dual graph coincides with the dual graph of 16 (-2)-curves given in Figure 10.8 (the left-hand side 16 curves). Moreover, there are 24 Göpel tetrads (e.g. $\mathcal{T} = \{0, 25, 14, 3\}$) such that the associated the (-4)-divisors are invariant under the action of σ and hence descend to (-2)-divisors on the Enriques surface. Among 24 Göpel tetrads, the following twelve give twelve (-2)-classes forming the dual graph in Figure 10.8 (the right-hand side of 12 curves):

$$\{0, 4, 13, 25\},\ \{1, 3, 14, 34\},\ \{12, 15, 23, 35\},\ \{1, 4, 15, 45\},\ \{0, 5, 14, 23\},\ \{2, 3, 25, 35\},$$

$$\{12, 13, 24, 34\},\ \{1, 4, 15, 45\},\ \{0, 3, 15, 24\},\ \{12, 14, 25, 45\},\ \{1, 4, 13, 34\},\ \{2, 5, 23, 35\}.$$

The remaining twelve (-2)-classes form the same dual graph. The dual graph of the $40 (= 16 + 24)$ (-2)-classes coincides with the polytope 40C.

The next example is due to Shimada [672].

Example 10.7.12 In Section 9.7 we discussed Enriques surface S of Hessian type obtained as the quotients of the Hessian surface of a cubic surface C by a fixed-point-free involution. We referred to a theorem of Shimada who described the group $\mathrm{Aut}(S)$ in the case of a general cubic C and found that the fundamental polytope $\Pi(S)$ coincides with the polytope 20E. When C is the Clebsch diagonal surface, this polytope is the Coxeter polytope of the crystallographic basis of (-2)-curves the Enriques surface of type VI. Another case considered in Section 9.7 was the case of Enriques surfaces coming from cubic surfaces with $\mathfrak{S}_4$-symmetry. They realize the same polytope.

It is known that the Kummer quartic surface is isomorphic to the Hessian quartic surface of a cubic surface [318], [187]. This model is defined by the linear system

$$2h_4 - \sum_{\alpha \in \mathcal{H}} N_\alpha,$$

where $\mathcal{H}$ is a Weber hexad. The corresponding fixed-point-free involution of the Hessian surface corresponds to one of the involutions $c_{\mathcal{H}}$ associated with a Weber hexad.

For example, if we take $\{n_0, n_{14}, n_{15}, n_{23}, n_{25}, n_{34}\}$ as a Weber hexad $\mathcal{H}$, then $c_{\mathcal{H}}$ exchanges the following 20 (-2)-curves:

$$N_1 \longleftrightarrow T_{23},\ N_2 \longleftrightarrow T_{14},\ N_3 \longleftrightarrow T_{15},\ N_4 \longleftrightarrow T_{25},\ N_5 \longleftrightarrow T_{34},$$

$$N_{12} \longleftrightarrow T_5,\ N_{13} \longleftrightarrow T_4,\ N_{24} \longleftrightarrow T_3,\ N_{35} \longleftrightarrow T_2,\ N_{45} \longleftrightarrow T_1.$$

We can easily see that the dual graph of the images of these curves is the Petersen graph 6.1. The remaining 10 (-2)-vectors correspond to the following $c_{\mathcal{H}}$-invariant (-4)-classes:

$$h_4 - N_0 - N_1 - N_{25} - N_{34}, \quad h_4 - N_0 - N_2 - N_{15} - N_{34}, \quad h_4 - N_0 - N_3 - N_{14} - N_{25},$$

$$h_4 - N_0 - N_4 - N_{15} - N_{23}, \quad H - N_0 - N_5 - N_{14} - N_{23}, \quad h_4 - N_{12} - N_{14} - N_{23} - N_{34},$$

$$h_4 - N_{13} - N_{15} - N_{23} - N_{25}, \quad h_4 - N_{14} - N_{15} - N_{24} - N_{25}, \quad h_4 - N_{14} - N_{15} - N_{34} - N_{35},$$

$$h_4 - N_{23} - N_{25} - N_{34} - N_{45}.$$

With the help of computer computations, Brandhorst and Shimada checked that in each case in Table 10.2, except the last one, there exists a complex Enriques surface S with the fundamental polytope of given type [92]. It would be interesting to give a geometric construction of Enriques and Coble surfaces corresponding to polytopes from rows 14, 15, and 16. The surfaces corresponding to the remaining rows are given in the last column of the Table.

Bibliographical Notes

The first example of a supersingular K3 surface (the Fermat quartic surface in characteristic $p \equiv 3 \mod 4$) was given by J. Tate [700] in 1965. The terminology is due to T. Shioda [677], who determined which Kummer surfaces are supersingular. In [25] M. Artin gave another definition of a supersingular K3 surface based on the notion of the formal Brauer group (see Section 0.10 in Volume 1). He proved that the two definitions coincide if the surface carries an elliptic pencil.

The two definitions are equivalent if the surface is unirational. One conjectures that this is true for all p, but it is known only if $p = 2$ [627] (the proof of this conjecture in [461] contains a gap). If $p \neq 2$, the coincidence of the two definitions follows from the Tate Conjecture for K3 surfaces that was recently proved in [500] and [483].

The relationship between simply connected Enriques surfaces in characteristic 2 and supersingular K3 surfaces was first studied by W. Lang [432]. He was the first to ask questions about possible singularities and possible Artin invariant of the canonical cover of an Enriques surface. Lang also proved that the Artin invariant is equal to 10 if the surface is general.

The first systematic study of supersingular covers of Enriques surfaces was undertaken in the preprint of T. Ekedahl, J. Hyland, and N. Shepherd-Barron [214]. Some of their results have been reproved and extended by S. Schröer [640] and Matsumoto [494]. In particular, Schröer gave the first example of an Enriques surface whose K3 cover is normal and has one non-rational singular point.

A construction of algebraic surfaces in characteristic $p > 0$ as quotients by rational vector fields was pioneered by T. Katsura and Y. Takeda [373]. A systematic study of inseparable maps defined by quotients by a rational vector field was done in a paper by A. Rudakov and I. Shafarevich [626]. The constructions of Enriques surfaces as quotients of supersingular K3 surfaces with the Artin invariant one, which we discuss in this chapter is based on the work of the second author and T. Katsura [368], [369], [418]. It was also successfully used in the classification of Enriques surfaces in characteristic 2 with finite automorphism group, which we discussed in Section 8.9.

We already explained in Section 10.7 the contribution of R. Borcherds and S. Kondō to computations of automorphism groups of K3 surfaces. The extension of this method to Enriques surfaces was given by S. Brandhorst and I. Shimada in [91], [92].

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