Endomorphisms of complex abelian varieties, Milan, February 2014

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Contents

Introduction

The following is an extended version of my lecture notes for a Ph.D. course at the University of Milan in February, 2014. The goal of the course was to relate some basic theory of endomorphisms of complex abelian varieties to the theory of K3 surfaces and classical algebraic geometry. No preliminary knowledge of the theory of complex abelian varieties or K3 surfaces was assumed.

It is my pleasure to thank the audience for their patience and Professor Bert van Geemen for giving me the opportunity to give the course.

Lecture 1

Complex abelian varieties

The main references here are to the book [67], we briefly remind the basic facts and fix the notation.

Let $A = V/\Lambda$ be a complex torus of dimension g over $\mathbb{C}$. Here V is a complex vector space of dimension $g > 0$ and Λ is a discrete subgroup of V of rank $2g$.[1] The tangent bundle of A is trivial and is naturally isomorphic to $A \times V$. Thus the complex space V is naturally isomorphic to the tangent space of A at the origin or to the space of holomorphic vector fields $\Theta(A)$ on A. It is also isomorphic to the universal cover of A. The group Λ can be identified with the fundamental group of A that coincides with $H_1(A, \mathbb{Z})$. The dual space V^* is naturally isomorphic to the space $\Omega^1(A)$ of holomorphic 1-forms on A, the map

$$\alpha : \Lambda = H_1(A, \mathbb{Z}) \to \Omega^1(A)^* = V, \quad \alpha(\gamma) : \omega \mapsto \int_\gamma \omega,$$

can be identified with the embedding of Λ in V. Let $(\gamma_1, \ldots, \gamma_{2g})$ be a basis of Λ and let $(\omega_1, \ldots, \omega_g)$ be a basis of V^*. The map $H_1(A, \mathbb{Z}) \to V$ is given by the matrix

$$\Pi = \begin{pmatrix} \int_{\gamma_1} \omega_1 & \int_{\gamma_2} \omega_1 & \ldots & \int_{\gamma_{2g}} \omega_1 \\ \int_{\gamma_1} \omega_2 & \int_{\gamma_2} \omega_2 & \ldots & \int_{\gamma_{2g}} \omega_2 \\ \vdots & \vdots & \vdots & \vdots \\ \int_{\gamma_1} \omega_g & \int_{\gamma_2} \omega_g & \ldots & \int_{\gamma_{2g}} \omega_g \end{pmatrix}, \tag{1.1}$$

called the *period matrix* of A. The columns of the period matrix are the coordinates of $\gamma_1, \ldots, \gamma_{2g}$ in the dual basis $(e_1, \ldots, e_g)$ of the basis $(\omega_1, \ldots, \omega_g)$, i.e. a basis of V. The rows of the period matrix are the coordinates of $(\omega_1, \ldots, \omega_g)$ in terms of the dual basis $(\gamma_1^*, \ldots, \gamma_{2g}^*)$ of $H^1(A, \mathbb{C})$.

Let $W = \Lambda_\mathbb{R} := \Lambda \otimes_\mathbb{Z} \mathbb{R}$. We can view W as the vector space V considered as a real vector space of dimension $2g$ by restriction of scalars. A *complex structure* on V is defined by the $\mathbb{R}$-linear operator $I : W \to W, w \mapsto iw$, satisfying $I^2 = -1$. The space $W_\mathbb{C} := W \otimes_\mathbb{R} \mathbb{C}$ decomposes into the direct sum $V_i \oplus V_{-i}$ of eigensubspaces with eigenvalues i and $-i$. Obviously, $V_{-i} = \bar{V}_i$.

[1] A subgroup Γ of V is discrete if for any compact subset K of V the intersection $K \cap \Gamma$ is finite, or, equivalently, Γ is freely generated by r linearly independent vectors over $\mathbb{R}$, the number r is the rank of Γ.

We can identify V_i with the subspace $\{w - iI(w), w \in W\}$ and V_{-i} with $\{w + iI(w), w \in W\}$ (since $I(w \pm iI(w)) = I(w) \mp iw = \mp i(w \pm iI(w))$). The map $V_i \to V, w - iI(w) \to w$, is an isomorphism of complex linear spaces. Thus a complex structure $V = (W, I)$ on W defines a decomposition $W_{\mathbb{C}} = V \oplus \bar{V}$.

The space V (resp. $\bar{V}$) can be identified with the holomorphic part $T^{1,0}$ (resp. anti-holomorphic part $T^{0,1}$) of the complexified tangent space of the real torus W/Λ at the origin. Passing to the duals, and using the De Rham Theorem, we get the Hodge decomposition

$$H^1_{\mathrm{DR}}(A, \mathbb{C}) \cong H^1(A, \mathbb{C}) = W^*_{\mathbb{C}} = H^{1,0}(A) \oplus H^{0,1}(A), \tag{1.2}$$

where $H^{1,0}(A) = \Omega^1(A) = V^*$ (resp. $H^{0,1}(A) = \bar{V}^*$) is the space of holomorphic (resp. anti-holomorphic) differential 1-forms on A. Note that $H^{1,0}(A)$ embeds in $H^1(A, \mathbb{C})$ by the map that assigns to $\omega \in \Omega^1(A)$ the linear function $\gamma \mapsto \int_\gamma \omega$. If we choose the bases $(\gamma_1, \ldots, \gamma_{2g})$ and $(\omega_1, \ldots, \omega_g)$ as above, then $H^{1,0}$ is a subspace of $H^1(A, \mathbb{C})$ spanned by the vectors $\omega_j = \sum_{i=1}^{2g} a_{ij}\gamma_i^*$, where $(\gamma_1^*, \ldots, \gamma_{2g}^*)$ is the dual basis in $H^1(A, \mathbb{C})$, and (a_{ij}) is equal to the transpose ${}^t\Pi$ of the period matrix (1.1).

A complex torus is a *Kähler manifold*, a Kähler form Ω is defined by a Hermitian positive definite form H on V. In complex coordinates $z_1, \ldots, z_g$ on V, the Kähler metric is defined by $\sum h_{ij} z_i \bar{z}_j$, where (h_{ij}) is a positive definite Hermitian matrix. The Kähler form Ω of this metric is equal $\frac{i}{2} \sum h_{ij} dz_j \wedge d\bar{z}_i$. Its cohomology class $[\Omega]$ in the De Rham cohomology belongs to $H^2(A, \mathbb{R})$.

A complex torus is called an *abelian variety* if there exists an ample line bundle L on A, i.e. a line bundle such that the holomorphic sections of some positive tensor power of L embed A in a projective space. In our situation this means that the restriction of the imaginary part $\mathrm{Im}(H)$ to $\Lambda \times \Lambda$ takes integer values. By *Kodaira's Theorem*, this is equivalent to that one can find a Kähler form Ω on A with $[\Omega] \in H^2(A, \mathbb{Z})$. A choice of an ample line bundle is called a *polarization* of A. Two polarizations L, L' are considered equivalent if $c_1(L) = c_1(L')$ (in this case we say that the line bundles are algebraically equivalent).

Recall that a Hermitian form $H : V \times V \to \mathbb{C}$ on a complex vector space can be characterized by the properties that its real part $\mathrm{Re}(H)$ is a real symmetric bilinear form on the corresponding real space W and its imaginary part $\mathrm{Im}(H)$ is a skew-symmetric bilinear form on W. The form H is positive definite if $\mathrm{Re}(H)$ is positive definite and $\mathrm{Im}(H)$ is non-degenerate (a *symplectic form*). Using the isomorphism

$$H^2(A, \mathbb{Z}) \cong \bigwedge^2 H^1(A, \mathbb{Z}) = \bigwedge^2 \Lambda^\vee,$$

we can identify $\mathrm{Im}(H)$ with $c_1(L)$, where L is an ample line bundle on A. Explicitly, a line bundle L trivializes under the cover $\pi : V \to V/\Lambda$ and it is isomorphic to the quotient of the trivial bundle $V \times \mathbb{C}$ by the action of Λ defined by

$$\lambda : (z, t) \mapsto (z + \lambda, e^{\pi H(z,\lambda) + \frac{\pi}{2} H(\lambda,\lambda)} \chi(\lambda) t),$$

where $\chi : \Lambda \to U(1)$ is a *semi-character* of Λ, i.e. a map $\Lambda \to U(1)$ satisfying $\chi(\lambda\lambda') = \chi(\lambda)\chi(\lambda')e^{\pi i \mathrm{Im}(H(\lambda,\lambda'))}$. It follows that

$$\mathrm{Pic}^0(A) := \mathrm{Ker}(c_1 : \mathrm{Pic}(A) \to H^2(A, \mathbb{Z})) \cong \mathrm{Hom}(\Lambda, U(1)).$$

Note that the Hermitian form H can be uniquely reconstructed from the restriction of $\mathrm{Im}(H)$ to $\Lambda \times \Lambda$, first extending it, by linearity, to a real symplectic form E on W, and then checking that

$$H(x,y) = E(ix,y) + iE(x,y). \tag{1.3}$$

In fact, $H(x,y) = A(x,y) + iE(x,y)$ implies

$$H(ix,y) = A(ix,y) + iE(ix,y) = iH(x,y) = iA(x,y) - E(x,y),$$

hence, comparing the real and imaginary parts, we get $A(x,y) = E(ix,y)$. Since $H(x,y) = H(ix,iy)$ and its real part is a positive definite symmetric bilinear form, we immediately obtain that E satisfies

$$E(ix,iy) = E(x,y), \quad E(ix,y) = E(iy,x), \quad E(ix,x) > 0,\ x \neq 0. \tag{1.4}$$

We say that a complex structure (W,I) on W is *polarized* with respect to a symplectic form E on W if E satisfies (1.4) (where $ix := I(x)$).

We can extend E to a Hermitian form $H_{\mathbb{C}}$ on $W_{\mathbb{C}}$, first extending E to a skew-symmetric form $E_{\mathbb{C}}$, by linearity, and then setting

$$H_{\mathbb{C}}(x,y) = \tfrac{1}{2} i E_{\mathbb{C}}(x,\bar{y}). \tag{1.5}$$

Let $x = a + ib, y = a' + ib' \in W_{\mathbb{C}}$. We have

$$H_{\mathbb{C}}(a+bi, a'-ib') = \tfrac{1}{2}(-E_{\mathbb{C}}(b,a') + E_{\mathbb{C}}(a,b')) + \tfrac{1}{2} i (E_{\mathbb{C}}(a,a') + E_{\mathbb{C}}(b,b')).$$

The real part of $H_{\mathbb{C}}$ is symmetric and the imaginary part is alternating, so $H_{\mathbb{C}}$ is Hermitian. Also, by taking a standard symplectic basis $e_1, \ldots, e_{2g}$ of W and a basis $(f_1, \ldots, f_g, \bar{f}_1, \ldots, \bar{f}_g)$ of $W_{\mathbb{C}}$, where $f_k = e_k + ie_{k+g}, \bar{f}_k = e_k - ie_{k+g}$, we check that $H_{\mathbb{C}}$ is of signature (g,g).

Now, if $x = w - iI(w), x' = w' - iI(w') \in V$,

$$H_{\mathbb{C}}(x,x) = \tfrac{1}{2} i E_{\mathbb{C}}(w - iI(w), w + iI(w)) = E(I(w), w) > 0$$

and

$$E_{\mathbb{C}}(x,x') = E_{\mathbb{C}}(w - iI(w), w' - iI(w'))$$
$$= E_{\mathbb{C}}(w,w') - E_{\mathbb{C}}(I(w), I(w')) - i(E_{\mathbb{C}}(I(w), w') + E_{\mathbb{C}}(w, I(w')) = 0.$$

Thus $V = (W,I)$ defines a point in the following subset of the Grassmann variety $\mathsf{G}(g, W_{\mathbb{C}})$:

$$\mathsf{G}(g, W_{\mathbb{C}})_E := \{V \in \mathsf{G}(g, W_{\mathbb{C}}) : H_{\mathbb{C}}|V > 0, E_{\mathbb{C}}|V = 0\}. \tag{1.6}$$

It is obvious, that V and $\bar{V}$ are orthogonal with respect of $H_{\mathbb{C}}$ and $H_{\mathbb{C}}|\bar{V} < 0$.

Conversely, let us fix a real vector space W of dimension $2g$ that contains a lattice Λ of rank $2g$, so that W/Λ is a real torus of dimension $2g$. Suppose we are given a symplectic form $E \in \bigwedge^2 W^\vee$ on W. We extend E to a skew-symmetric form $E_{\mathbb{C}}$ on $W_{\mathbb{C}}$, by linearity, and define the Hermitian form of signature (g,g) by using (1.5).

Suppose $V = (W, I) \in \mathsf{G}(g, W_{\mathbb{C}})_E$. It is immediate to check that $E_{\mathbb{C}}(\bar{x}, y) = \overline{E_{\mathbb{C}}(x, \bar{y})}$. Thus, $H(\bar{x}, \bar{x}) = -H(x, y) < 0$. This implies that $V \cap \bar{V} = \{0\}$, hence $W_{\mathbb{C}} = V \oplus \bar{V}$. Now $W = \{v + \bar{v}, v \in V\}$ and the complex structure I on W defined by $I(w) = iv - i\bar{v}$ is isomorphic to the complex structure on V via the projection $W \to V, v + \bar{v} \to v$. It is easy to check that $E_{\mathbb{C}}$ restricted to W is equal to E, and $E(I(w), w) > 0, E(I(w), I(w)) = E(w, w)$. We obtain that the set of complex structures on W polarized by E is parameterized by (1.6).

The group $\mathsf{Sp}(W, E) \cong \mathsf{Sp}(2g, \mathbb{R})$ acts transitively on $\mathsf{G}(g, W_{\mathbb{C}})_E$ with isotropy subgroup of V isomorphic to the unitary group $\mathsf{U}(V, H_{\mathbb{C}}|V) \cong U(g)$. Thus

$$\mathsf{G}(g, W_{\mathbb{C}})_E \cong \mathsf{Sp}(2g, \mathbb{R})/U(g)$$

is a Hermitian symmetric space of type III in Cartan's classification. Its dimension is equal to $g(g+1)/2$.

Remark 1.1. According to Elie Cartan's classification of Hermitian symmetric spaces there are 4 classical types I,II, III and IV and two exceptional types E_6 and E_7. We will see type IV spaces later when we discuss K3 surfaces and other classical types when we will discuss special subvarieties of the moduli spaces of abelian varieties. So far, the exceptional types have no meaning as the moduli spaces of some geometric objects.

So far, we have forgot about the lattice Λ in the real vector space W. The space $\mathsf{G}(g, W_{\mathbb{C}})_E$ is the *moduli space of complex structures* on a real vector space W of dimension $2g$ which are polarized with respect to a symplectic form E on W or, in other words, it is the *moduli space of complex tori* equipped with a Kähler metric H defined by a symplectic form $E = \mathrm{Im}(H)$. Now we put an additional *integrality condition* by requiring that

$$\mathrm{Im}(H)(\Lambda \times \Lambda) \subset \mathbb{Z}.$$

Recall that a skew-symmetric form E on a free abelian group of rank $2g$ can be defined in some basis by a skew-symmetric matrix

$$J_{\mathsf{D}} = \begin{pmatrix} 0_g & \mathsf{D} \\ -\mathsf{D} & 0_g \end{pmatrix},$$

where D is the diagonal matrix $\mathrm{diag}[d_1, \ldots, d_g]$ with $d_i | d_{i+1}, i = 1, \ldots, g-1$. The sequence $(d_1, \ldots, d_g)$ defines the skew-symmetric form uniquely up to a linear isomorphism preserving the skew-symmetric form. In particular, if E is non-degenerate, the product $d_1 \cdots d_g$ is equal to the determinant of any skew-symmetric matrix representing the form. If H is a positive definite Hermitian form defining a polarization on A, the sequence $(d_1, \ldots, d_g)$ defining $\mathrm{Im}(H)|\Lambda \times \Lambda$ is called the *type of the polarization*. A polarization is called *primitive* if $(d_1, \ldots, d_g) = 1$. It is called *principal* if $(d_1, \ldots, d_g) = (1, \ldots, 1)$.

Choose a basis $\underline{\gamma} = (\gamma_1, \ldots, \gamma_{2g})$ of Λ such that the matrix of the symplectic form $E|\Lambda \times \Lambda$ is equal to the matrix J_{D}.

We know that the matrix $(E(i\gamma_a, \gamma_b))_{g+1 \le a,b \le 2g}$ is positive definite. This immediately implies that the $2g$ vectors $\gamma_a, i\gamma_a, a = g+1, \ldots, 2g$, are linearly independent over $\mathbb{R}$, hence we may take $\frac{1}{d_1}\gamma_{g+1}, \ldots, \frac{1}{d_g}\gamma_{2g}$ as a basis $(e_1, \ldots, e_g)$ of V. It follows that the period matrix Π in this basis of V and the basis $(\gamma_1, \ldots, \gamma_{2g})$ of Λ is equal to a matrix $(\tau\ \mathsf{D})$. Write $\tau = X + iY$, where $X = \mathrm{Re}(\tau)$

and $Y = \mathrm{Im}(\tau)$ are real matrices. Then $\gamma_k = \sum_{s=1}^{g} x_{ks}e_s + \sum y_{ks}ie_s, k = 1, \ldots, g$, and the matrix of E on $W = \Lambda_{\mathbb{R}}$ in the basis $(e_1, \ldots, e_g, ie_1, \ldots, ie_g)$ of W is equal to

$$ {}^t\begin{pmatrix} X & D \\ Y & 0 \end{pmatrix}^{-1} J_{\mathsf{D}} \begin{pmatrix} X & D \\ Y & 0 \end{pmatrix}^{-1} = {}^t\begin{pmatrix} X & \mathsf{D} \\ Y & 0 \end{pmatrix}^{-1} J_{\mathsf{D}} \begin{pmatrix} 0 & Y^{-1} \\ \mathsf{D}^{-1} & -\mathsf{D}^{-1}XY^{-1} \end{pmatrix} $$

$$ = \begin{pmatrix} 0 & -Y^{-1} \\ {}^tY^{-1} & -{}^tY^{-1}(X - {}^tX)Y^{-1} \end{pmatrix}. $$

Since $E(e_i, e_j) = E(ie_i, ie_j) = \frac{1}{d_i d_j} E(\gamma_{g+i}, \gamma_{g+j}) = 0$ and $(E(ie_i, e_j))$ is a symmetric positive definite matrix, we obtain that Y is a symmetric positive definite matrix, and X is a symmetric matrix. In particular, $\tau = X + iY$ is a symmetric complex matrix.

We have proved one direction of the following theorem.

Theorem 1.2 (Riemann-Frobenius conditions)**.** *A complex torus $A = V/\Lambda$ is an abelian variety admitting a polarization of type D if and only if one can choose a basis of Λ and a basis of V such that the period matrix Π is equal to the matriz $(\tau\ \mathsf{D})$, where*

$$ {}^t\tau = \tau, \quad \mathrm{Im}(\tau) > 0. $$

We leave the proof of the converse to the reader.

Note that the matrix of the Hermitian form H in the basis $e_1, \ldots, e_g$ as above is equal to $S = (E(ie_a, e_b))$. Since

$$ d_b\delta_{ab} = E(\gamma_a, \gamma_{g+b}) = \sum_{k=1}^{g} E((x_{ka} + iy_{ka})e_k, d_b e_b) $$

$$ = \sum_{k=1}^{g} y_{ka} E(ie_k, d_b e_b) = \sum_{k=1}^{g} E(ie_b, d_b e_k) y_{ka} = d_b \sum_{k=1}^{g} E(ie_b, e_k) y_{ka}, $$

we obtain that

$$ S = \mathrm{Im}(\tau)^{-1}. \tag{1.7} $$

So, we see that we can choose a special basis $(\gamma_1, \ldots, \gamma_{2g})$ such that the period matrix Π of A is equal to $(\tau\ \mathsf{D})$, where τ belongs to the *Siegel upper-half space of degree g*

$$ \mathcal{Z}_g := \{\tau \in \mathrm{Mat}_n(\mathbb{C}) : {}^t\tau = \tau, \mathrm{Im}(\tau) > 0\}. $$

Every abelian variety with a polarization of type D is isomorphic to the complex torus

$$ A \cong \mathbb{C}^g/\tau\mathbb{Z}^g + \mathsf{D}\mathbb{Z}^g. $$

Note that $\mathcal{Z}_g \cong \mathsf{G}(g, \mathbb{C}^g)_E$, where $E : \mathbb{R}^{2g} \times \mathbb{R}^{2g} \to \mathbb{R}$ is a symplectic form defined by the matrix D. However, the isomorphism depends on a choice of a special basis in $\mathbb{R}^{2g}$. One must view $\mathcal{Z}_g$ as the moduli space of polarized complex structures on a symplectic vector space W of dimension $2g$ equipped with a linear symplectic isomorphism $\mathbb{R}^{2n} \to W$, where the symplectic form $\mathbb{R}^{2n}$ is defined by the matrix D.

Two such special bases are obtained from each other by a change of a basis matrix that belongs to the group

$$\mathrm{Sp}(J_{\mathsf{D}},\mathbb{Z}) = \{X \in \mathrm{Sp}(2g,\mathbb{Q}) : X \cdot J_{\mathsf{D}} \cdot {}^tX = J_{\mathsf{D}}\}.$$

If $X = \begin{pmatrix} A_1 & A_2 \\ A_3 & A_4 \end{pmatrix}$, where A_1, A_2, A_3, A_4 are square matrices of size g, then $X \in \mathrm{Sp}(J_{\mathsf{D}},\mathbb{Z})$ if and only if

$$A_1\mathsf{D}^tA_2 = A_2\mathsf{D}^tA_1, \quad A_3\mathsf{D}^tA_4 = A_4\mathsf{D}^tA_3, \quad A_1\mathsf{D}^tA_4 - A_2\mathsf{D}^tA_3 = \mathsf{D}.$$

Thus, we obtain that the coarse moduli space for the isomorphic classes of abelian varieties with polarization of type D is isomorphic to the orbit space

$$\mathcal{A}_{g,\mathsf{D}} = \mathcal{Z}_g/\mathrm{Sp}(J_{\mathsf{D}},\mathbb{Z}).$$

The group $\mathrm{Sp}(J_{\mathsf{D}},\mathbb{Z})$ acts on $\mathcal{Z}_g$ by

$$\tau \mapsto (\tau A_1 + A_2)(A_3\tau + A_4)^{-1}\mathsf{D}.$$

If $J_{\mathsf{D}} = J$, then we denote $\mathrm{Sp}(J_{\mathsf{D}},\mathbb{Z})$ by $\mathrm{Sp}(2g,\mathbb{Z})$ and $\mathcal{A}_{g,\mathsf{D}}$ by $\mathcal{A}_g$ and get

$$\mathcal{A}_g = \mathcal{Z}_g/\mathrm{Sp}(2g,\mathbb{Z}).$$

So far, the geometry of abelian varieties is reduced to linear algebra. One can pursue it further by interpreting in these terms the intersection theory on A. It assigns to any line bundles $L_1,\ldots,L_g$ an integer $(L_1,\ldots,L_g)$ that depends only on the images of L_i under the first Chern class map. Of course, it is also linear in each L_i with respect to the tensor product of line bundles. Let $c_1(L_i) = \alpha_i \in \bigwedge^2\Lambda^\vee$ and

$$\alpha_1 \wedge \cdots \wedge \alpha_g \in \bigwedge^{2g}\Lambda^\vee.$$

A choice of a basis in Λ defines an isomorphism $\bigwedge^{2g}\Lambda^\vee \cong \mathbb{Z}$. This isomorphism depends only on the orientation of the basis. We choose an isomorphism such that $L^g := (L,\ldots,L) > 0$ if L is an ample line bundle. For example, if L corresponds to a polarization of type D, we have $\alpha = \sum d_i\gamma_i \wedge \gamma_{i+g}$ and

$$L^g = g!d_1\cdots d_g.$$

By constructing explicitly a basis in the space of holomorphic sections of an ample line bundle L in terms of *theta functions*, one can prove that

$$h^0(L) = \frac{L^g}{g!} = \mathrm{Pf}(\alpha),$$

where $\mathrm{Pf}(\alpha)$ is the pfaffian of the skew-symmetric matrix defining α. More generally, for any line bundle L, the Riemann-Roch Theorem gives

$$\chi(L) = \sum_{i=0}^{g}(-1)^i \dim H^i(A,L) = \frac{L^g}{g!}.$$

Let us now define a duality between abelian varieties. Of course, this should correspond to the duality of the complex vector spaces.

Let $A = V/\Lambda$ be a complex g-dimensional torus. Consider the Hodge decomposition (1.2), where we identify the space $H^{1,0}(A)$ with $V^\vee$. Using the Dolbeault's Theorem, one can identify $H^{0,1}(A)$ with the cohomology group $H^1(A, \mathcal{O}_A)$. The group $H^1(A,\mathbb{Z}) = \Lambda^\vee$ embeds in $H^1(A,\mathbb{C})$ and its projection to $H^{0,1}$ is a discreet subgroup Λ' of rank $2g$ in $H^{0,1}$. The inclusion $H^1(A,\mathbb{Z}) \to H^1(A,\mathcal{O}_A)$ corresponds to the homomorphism derived from the exponential exact sequence

$$0 \to \mathbb{Z} \to \mathcal{O}_A \xrightarrow{e^{2\pi i}} \mathcal{O}_{\hat{A}} \to 0$$

by passing to cohomology. It also gives an exact sequence

$$H^1(A,\mathcal{O}_A)/\Lambda' \to H^1(A,\mathcal{O}_A^*) \stackrel{c_1}{\to} H^2(A,\mathbb{Z}),$$

where the group $H^1(A,\mathcal{O}_A^*)$ is isomorphic to $\mathrm{Pic}(A)$. Thus, we obtain that the group of points of the complex torus $H^1(A,\mathcal{O}_A)/\Lambda'$ is isomorphic to the group $\mathrm{Pic}^0(A)$. It is called the *dual complex torus* of A and will be denoted by $\hat{A}$.

Now, we assume that A is an abelian variety equipped with a polarization L of type D. The corresponding Hermitian form H defines an isomorphism from the space V to the space $\bar{V}^\vee$ of $\mathbb{C}$-antilinear functions on V (where $\bar{V}$ is equal to V with the complex structure $I(v) = -iv$).[2] Considered as a vector space over $\mathbb{R}$, it is isomorphic to the real vector space $W^\vee = \mathrm{Hom}_\mathbb{R}(V,\mathbb{R})$ by means of the isomorphism

$$\bar{V}^\vee \to W^\vee,\ l \mapsto k = \mathrm{Im}(l)$$

with the inverse defined by $k \to -k(iv) + ik(v)$. We may identify $\bar{V}^\vee$ with $H^{0,1}(A)$. We have

$$\Lambda' = \Lambda^\vee := \{l \in \bar{V}^\vee : l(\Lambda) \subset \mathbb{Z}\},$$

so that

$$\hat{A} = \bar{V}^\vee/\Lambda^\vee.$$

Also, $\mathrm{Im}(H)$ defines a homomorphism $\Lambda \to \Lambda^\vee$. Composing it with the homomorphism $\Lambda^\vee = H^1(A,\mathbb{Z}) \to \Lambda' \subset H^{0,1}(A)$, we obtain a homomorphism $\Lambda \to \Lambda'$. Let

$$\phi_L : A \to \hat{A} \tag{1.8}$$

be the homomorphism defined by the maps $V \to H^{0,1}$ and $\Lambda \to \Lambda'$. It is a finite map, and

$$K(L) := \mathrm{Ker}(\phi_L) \cong \Lambda^\vee/\Lambda \cong (\mathbb{Z}^g/\mathsf{D}\mathbb{Z}^g)^2 \cong \bigoplus_{i=0}^{g} (\mathbb{Z}/d_i\mathbb{Z})^2.$$

In particular, ϕ_L is an isomorphism if L is a principal polarization. The dual abelian variety can be defined over any field as the Picard variety $\mathbf{Pic}^0(A)$ and one can show that an ample line bundle L defines a map (1.8) by using the formula

$$\phi_L(a) = t_a^*(L) \otimes L^{-1},$$

[2] It also defines an isomorphism of complex vector spaces $\bar{V} \to V^\vee$

where t_a denotes the translation map $x \mapsto x + a$ of A to itself.

If we identify $\hat{A}$ with A by means of this isomorphism, then the map ϕ_L corresponding to the polarization L of type $(d, \ldots, d)$ can be identified with the multiplication map $[d] : x \to dx$. Its kernel is the subgroup $A[d]$ of d-torsion points in A. Let e_L be the exponent of the group K_L, i.e. the smallest positive integer that kills the group, then $\hat{A} \cong A/K_L$ and the multiplication map $[e_L] : A \to A$ is equal to the composition of the map $\phi_L : A \to \hat{A}$ and a finite map $\hat{A} \to A$ with kernel isomorphic to the group $(\mathbb{Z}/e_L\mathbb{Z})^{2g}/K_L$ of order $\frac{d_g^{2g-2}}{(d_1 \cdots d_{g-1})^2}$. Abusing the notation, we denote this map by ϕ_L^{-1}. So, by definition, $\phi_L^{-1} \circ \phi_L = [e_L]$. In the ring $\mathrm{End}(A)_{\mathbb{Q}}$ the element ϕ_L^{-1} is the inverse of $\frac{1}{e_L}\phi_L$.

Lecture 2

Endomorphisms of abelian varieties

A morphism $f : A = V/\Lambda \to A' = V'/\Lambda'$ of complex tori that sends zero to zero is called a *homomorphism* of tori. One can show that this is equivalent to that f is a homomorphism of complex Lie groups. Obviously, it is defined by a linear $\mathbb{C}$-map $f_a : V \to V'$ (called an *analytic representation* of f) and a $\mathbb{Z}$-linear map $f_r : \Lambda \to \Lambda'$ (called a *rational representation* of f) such that the restriction of f_a to Λ coincides with f_r.

Let $\mathrm{End}(A)$ be the set of *endomorphisms* of an abelian variety $A = V/\Lambda$, i.e. homomorphisms of A to itself. As usual, for any abelian group, it is equipped with a structure of an associative unitary ring with multiplication defined by the composition of homomorphisms and the addition defined by value by value addition of homomorphisms. By above, we obtain two injective homomorphisms of rings

$$\rho_a : \mathrm{End}(A) \to \mathrm{End}_{\mathbb{C}}(V) \cong \mathrm{Mat}_g(\mathbb{C}), \quad \rho_r : \mathrm{End}(A) \to \mathrm{End}_{\mathbb{Z}}(\Lambda) \cong \mathrm{Mat}_{2g}(\mathbb{Z}).$$

They are called the *analytic* and *rational* representations, respectively.

We fix a polarization L_0 on A of type $\mathsf{D} = (d_1, \ldots, d_g)$. The corresponding Hermitian form on H_0 and the symplectic form $E_0 = \mathrm{Im}(H_0)$ on Λ allow us to define the involutions in the rings $\mathrm{End}_{\mathbb{C}}(V)$ (resp. $\mathrm{End}_{\mathbb{Z}}(\Lambda)$) by taking the adjoint operator with respect to H_0 (resp. $\mathrm{Im}(H_0)$).[1] Using the representations ρ_a and ρ_r, we transfer this involution to $\mathrm{End}(A)$. It is called the *Rosati involution* and, following classical notation, we denote it by $f \mapsto f'$. One can show that the Rosati involution can be defined as

$$f' = \phi_{L_0}^{-1} \circ f^* \circ \phi_{L_0} : A \to \hat{A} \to \hat{A} \to A.$$

Here $(f^*)_a : \bar{V}^\vee \to \bar{V}^\vee$ is the transpose of f. If we view $\hat{A}$ as the Picard variety, then f^* is the usual pull-back map of line bundles on A.

For any $f \in \mathrm{End}(A)$, let

$$P_a(f) = \det(tI_g - f_a) = \sum_{i=0}^{g} t^{g-i}(-1)^i c_i^a$$

[1] Recall that the *adjoint operator* of a linear operator $T : V \to V$ of complex spaces equipped with a non-degenerate Hermitian form H is the unique operator T^* such that $H(T(x), y) = H(x, T^*(y))$ for all $x, y \in V$.

be the characteristic polynomial of f_a and

$$P_r(f) = \det(tI_{2g} - f_r) = \sum_{i=0}^{2g}(-1)^i c_i^r t^{2g-i}$$

be the characteristic polynomial of f_r. It is easy to check that

$$P_a(f') = \overline{P_a(f)},$$

so all eigenvalues of f'_a are conjugates of the eigenvalues of f_a.

We have

$$(f_r)_{\mathbb{C}} = f_a \oplus \bar{f}_a,$$

where $(f_r)_{\mathbb{C}}$ is considered as a linear operator on $\Lambda_{\mathbb{C}}$ (see Proposition (5.1,2) in [67]). In particular,

$$P_r(t) = P_a(f)P_a(\bar{f}).$$

An endomorphism $f \in \mathrm{End}(A)$ is called *symmetric* if $f = f'$. Let $\mathrm{End}^s(A)$ denote the subring of symmetric endomorphisms. It follows from above that, if $f \in \mathrm{End}^s(A)$, then f_a is a self-adjoint operator with respect to H_0, and its eigenvalues are real numbers. Also, we see that $P_r(f) = P_a(f)^2$.

Let $\mathrm{NS}(A) = \mathrm{Pic}(A)/\mathrm{Pic}^0(A)$ be the *Néron-Severi group* of A. We define a homomorphism

$$\alpha : \mathrm{NS}(A) \to \mathrm{End}(A), \quad L \mapsto \phi_{L_0}^{-1} \circ \phi_L.$$

If f is in the image, then $\phi_L = \phi_{L_0} \circ f$. This means that $H_0(f_a(z), z') = H(z, z')$ for some Hermitian form H and $\mathrm{Im}(H)(\Lambda \times \Lambda) \subset \mathbb{Q}$. Since $H(z, z') = \overline{H(z', z)}$, this means that the operator f_a is self-adjoint, hence f is symmetric. This easily implies that α defines an isomorphism of $\mathbb{Q}$-linear spaces

$$\alpha : \mathrm{NS}(A)_{\mathbb{Q}} \to \mathrm{End}^s(A)_{\mathbb{Q}}.$$

If L_0 is a principal polarization, we can skip the subscript $\mathbb{Q}$ [67], 5.2.1.

Note that $\alpha(L_0) = \mathrm{id}_A$, hence the subgroup generated by L_0 is mapped isomorphically to the subgroup of $\mathrm{End}^s(A)$ of endomorphisms of the form $[m], m \in \mathbb{Z}$. Also, it follows from the definition that $\alpha(L)$ is an isomorphism if and only if L is a principal polarization.

If we identify $\mathrm{NS}(A)$ with the space of Hermitian forms H such that $\mathrm{Im}(H)(\Lambda \times \Lambda) \subset \mathbb{Z}$, then the inverse map α^{-1} assigns to f the Hermitian form

$$H = H_0(f_a(z), z'). \tag{2.1}$$

Suppose $f \in \mathrm{End}(A)$ and f_a is given by a complex matrix M of size g. Then we must have

$$M \cdot (\tau|\mathsf{D}) = (\tau|\mathsf{D}) \cdot N, \tag{2.2}$$

where the matrix

$$N = \begin{pmatrix} A_1 & A_3 \\ A_2 & A_4 \end{pmatrix} \in \mathrm{Mat}_{2g}(\mathbb{Z})$$

defines f_r. Thus we get

$$M = (\tau \cdot A_3 + \mathsf{D}A_4)\mathsf{D}^{-1},$$

hence :

$$M\tau = (\tau \cdot A_3 + \mathsf{D}A_4)\mathsf{D}^{-1}\tau = \tau A_1 + \mathsf{D}A_2. \tag{2.3}$$

Thus the period matrix τ must satisfy a "quadratic equation". Now assume, additionally, that $f \in \mathrm{End}^s(A)$ is a symmetric endomorphism. This means that f_r and f_r' considered as linear operators on $W = \Lambda_\mathbb{R}$ are adjoint operators with respect to the alternating form $E = \mathrm{Im}(H)$ defined by the matrix J_D. Thus the matrix N must satisfy ${}^tN \cdot J_\mathsf{D} = -J_\mathsf{D}{}^t \cdot N$. This gives

$${}^tA_1\mathsf{D} = \mathsf{D}A_4, \quad {}^tA_2\mathsf{D} = -\mathsf{D}A_2, \quad {}^tA_3\mathsf{D} = -\mathsf{D}A_3. \tag{2.4}$$

If $\mathsf{D} = I_g$, then

$$N = \begin{pmatrix} A & B \\ C & {}^tA \end{pmatrix}, \tag{2.5}$$

where B and C are skew-symmetric matrices of size $g \times g$.

The coefficients of the characteristic polynomial have the following geometric meaning.

For any $f = \alpha(L) \in \mathrm{End}^s(A)$,

$$dc_i^a = \frac{(L_0^{g-i}, L^i)}{(g-i)!i!}, \; i = 0, \ldots, g, \tag{2.6}$$

where $d = d_1 \cdots d_g$ [67], (5.2.1). In particular, L is ample if and only if all eigenvalues of f_a are positive.[2] In the last statement, we use that a line bundle L is ample if and only if $(L_0^{g-i}, L^i) > 0$ for all $i = 0, \ldots, g$.

A homomorphism $f : A \to A'$ of abelian varieties of the same dimension is called an *isogeny* if its kernel is a finite group. The order of the kernel is called the *degree* of the isogeny and is denoted by $\deg(f)$. It is equal to the topological degree of the map. Equivalently, f is an isogeny if its image is equal to A'. An example of an isogeny is a map $\phi_L : A \to \hat{A}$, where L is an ample line bundle. The *inverse isogeny* is the map $g : A' \to A$ such that $g \circ f = [e]$, where e is the exponent of the kernel of f. For example, ϕ_L^{-1} is the inverse isogeny of ϕ_L. One checks that the isogeny is an equivalence relation on the set of isomorphism classes of abelian varieties.

Suppose $\alpha(L)$ defines $f \in \mathrm{End}^s(A)$ which is an isogeny. By definition, $\phi_{L_0} \circ f = \phi_L$. It follows that $\deg(\phi_{L_0})\deg(f) = \deg(\phi_L)$. We know that $\deg(\phi_{L_0}) = d = \det \mathsf{D}$ and $\deg(\phi_L) = d' = \det \mathsf{D}'$, where D' is the type of L. This gives $\deg(f) = d'/d$. Applying (2.6) with $i = g$, we obtain

$$c_g^a = \frac{d'g!}{g!d} = \deg(f). \tag{2.7}$$

One can also compute the coefficients c_i in the characteristic polynomial $P^a_{f \circ f'}$

$$c_i = \tbinom{g}{i}\frac{(f^*(L_0)^i, L_0^{g-i})}{(L_0^g)} \tag{2.8}$$

[2] This follows from Sturm's theorem relating the number of positive roots with the number of changes of signs of the coefficients of a polynomial.

(see [67], (5.1.7)). We set

$$\mathrm{Tr}(f)_a = c_1^a,\ \mathrm{Tr}_r = c_1^r,\ \mathrm{Nm}(f)_a = c_g^a,\ \mathrm{Nm}(f)_r = c_g^r.$$

We have

$$\mathrm{Tr}(f\circ f') = \frac{2}{(g-1)!}\frac{(f^*(L_0), L_0^{g-1})}{(L_0^g)}, \quad \mathrm{Nm}(f\circ f') = \frac{(f^*(L_0)^g)}{(L_0^g)}. \tag{2.9}$$

The first equality implies that the symmetric form $(f,g)\to \mathrm{Tr}(f\circ g')$ on $\mathrm{End}(A)$ is positive definite.

We know that $\mathrm{End}(A)_{\mathbb{Q}}$ is isomorphic to a subalgebra of the matrix algebra and hence it is finite-dimensional algebra over $\mathbb{Q}$. A finite-dimensional associative algebra over a field F is called a *simple algebra* if it has no two-sided ideals and its center coincides with F. An algebra is called *semi-simple* if it is isomorphic to the direct product of simple algebras. The center of a simple algebra is a field, if it coincides with F, the algebra is called a central simple algebra. An example of a central simple algebra is a matrix algebra $\mathrm{Mat}_n(F)$. Another example of a simple algebra is a *division algebra* or *skew field*, an algebra where every nonzero element is invertible (a *skew field*).

An example of a non-commutative central division algebra is the *quaternion algebra*

$$H = \left(\frac{a,b}{F}\right) = F + F\mathbf{i} + F\mathbf{j} + F\mathbf{k},$$

were $\mathbf{i}^2 = a\neq 0, \mathbf{j}^2 = b\neq 0, \mathbf{k} = \mathbf{ij} = -\mathbf{ji}$. It is equipped with an anti-involution $x = x_0 + x_1\mathbf{i} + x_2\mathbf{j} + x_3\mathbf{k}\mapsto x' = x_0 - x_1\mathbf{i} - x_2\mathbf{j} - x_3\mathbf{k}$ such that $\mathrm{Nm}(x) := xx' = x_0^2 - ax_1^2 - bx_2^2 + abx_3^2\in F$. If $\mathrm{Nm}(x)\neq 0$ for any $x\neq 0$, then $\frac{1}{\mathrm{Nm}(x)}x$ is the inverse of x, so H is a skew field. A quaternion algebra H over a number field K is called *totally definite* if for every real embedding $\sigma: K\hookrightarrow\mathbb{R}$, the $\mathbb{R}$-algebra $H_\sigma = H\otimes_\sigma\mathbb{R}$ obtained by the change of scalars H_σ is a skew field. If, for any σ as above, the algebra H_σ acquires zero divisors, hence become isomorphic to $\mathrm{Mat}_4(\mathbb{R})$, it is called *totally indefinite*. If K is the center of a skew field D, then the degree of D over K is always a square. This is proved by showing that over some finite extension L of K, the algebra $R_L = R\otimes_K L$ *splits*, i.e. becomes isomorphic to a matrix algebra over L. For example, for the quaternion algebra $H = \left(\frac{a,b}{\mathbb{Q}}\right)$, a splitting field is $L = \mathbb{Q}+\mathbb{Q}\mathbf{i}\cong\mathbb{Q}(\sqrt{a})$, so that $H = L + L\mathbf{j}$. One can write any element in H as $x = m + n\mathbf{j}$, where $m,n\in L$. The rule of multiplication becomes

$$(m+n\mathbf{j})(m'+n'\mathbf{j}) = mm' + n\bar{n}'b + (mn' + n\bar{m}')\mathbf{j},$$

in particular, for any $m\in L$, we have $m\mathbf{j} = \mathbf{j}\bar{m}$. The map

$$m + n\mathbf{j}\mapsto \begin{pmatrix} m & n\\ b\bar{n} & \bar{m}\end{pmatrix}$$

defines an isomorphism from $f: H_L\to \mathrm{Mat}_2(L)$. Observe that $\overline{m+n\mathbf{j}} = \bar{m} - n\mathbf{j}$ and $\overline{xy} = \bar{y}\bar{x}$. We see that under this isomorphism the trace $\mathrm{tr}(x)$ (resp. the norm $\mathrm{Nm}(x)$) corresponds to the usual trace (resp. the determinant) of a matrix. Also observe that $f(\bar{x}) = J\cdot {}^tf(x)\cdot J^{-1}$, where $J = \left(\begin{smallmatrix} 0 & -1\\ 1 & 0\end{smallmatrix}\right)$.

A simple algebra R over a field F is isomorphic to the matrix algebra $\mathrm{Mat}_r(D)$ with coefficients with some division algebra D over F. In particular, its dimension over F is always a square of some number. If R is central, then this number is called the *reduced degree* of R and is denoted by $[R : F]_{\text{red}}$. If R is not central with the center K, then the reduced degree is defined to be $[R : K]_{\text{red}}[K : F]$. The reduced degree of a semi-simple algebra is defined to be the product of the reduced degrees of its simple factors.

A simple central algebra comes equipped with the *trace* F-bilinear map $R \times R \to F$ defined $(x, y) \mapsto \text{Tr}(xy)$, where $\text{Tr}(r)$ is the trace of the linear operator $R \to R, x \mapsto xr$. We can also define a *reduced trace* and *reduced norm* of a central simple algebra by embedding R into matrix algebra $\text{Mat}_r(L)$ over a splitting field L, and taking the usual trace and norm of a matrix. Note that this does not depend on the choice of L and the values of the reduced trace and the reduced norm belong to K.

The possible structure of the $\mathbb{Q}$-algebra $\text{End}(A)_{\mathbb{Q}}$ is known. It is a finite-dimensional associative algebra R admitting an anti-involution[3] $x \to x^*$ and a symmetric bilinear form $\text{Tr} : R \times R \to \mathbb{Q}$ such that the quadratic form $x \mapsto \text{Tr}(xx^*)$ is positive definite. An equivalent definition is that R is a semi-simple algebra over $\mathbb{Q}$ admitting a positive definite anti-involution. Such algebras have been classified by A. Albert and G. Scorza in the beginning of the last century. Assume that R is a central simple algebra over $\mathbb{Q}$. Let K be the center of R, it is a field admitting an involution σ, the restriction of the anti-involution of R. Let $K_0 = K^\sigma$ be the subfield of invariants. Then K_0 is a totally real algebraic number field and $K = K_0$ is an imaginary quadratic extension of K_0. Since R is semi-simple, its dimension over K is equal to n^2 for some number n. Let $e = [K : \mathbb{Q}], e_0 = [K_0 : \mathbb{Q}]$. Each such algebra is isomorphic to the product of simple central algebras.

An abelian variety is called *simple* if it is not isogenous to the product of positive-dimensional abelian varieties. An equivalent definition uses *Poincaré Reducibility Theorem* and asserts that an abelian variety is simple if and only if it does not contain an abelian subvariety of dimension $0 < k < g$. The endomorphism algebra $R = \text{End}(A)_{\mathbb{Q}}$ of a simple abelian variety A is a skew-field. We have four possible cases for a simple algebra:

- I $n = 1$, $R = K$ is a totally real field, $e = e_0 = \rho$, $e|g$;
- II $n = 2$, R is totally indefinite quaternion algebra over K, $e = e_0$, $\rho = 3e, 2e|g$;
- III $n = 2$, R is totally definite quaternion algebra over K, $e = e_0 = \rho$, $2e|g$;
- IV $K_0 \neq K, e = 2e_0, \rho = e_0 d^2, e_0 d^2|g$.

It is known that any finite-dimensional central simple algebra over $\mathbb{R}$ is isomorphic to either $\text{Mat}_r(\mathbb{R})$, or $\text{Mat}_r(\mathbb{C})$, or $\text{Mat}(\mathbb{H})$, where $\mathbb{H}$ is the usual quaternion algebra $\left(\frac{-1,-1}{\mathbb{R}}\right)$. By embedding R into $R_{\mathbb{R}}$, we can identify the anti-involution $x \mapsto x^*$ with taking the transpose ${}^t x$ of the matrix in the first case, and with taking the adjoint ${}^t\bar{x}$ of the matrix in the remaining two cases. Since the $\mathbb{Q}$-subalgebra of symmetric elements in $\text{End}(A)_{\mathbb{Q}}$ is isomorphic to the subalgebra of R of elements x such that $x = x^*$, this explains the information in above about the possible Picard number of A.

If A is not simple, its endomorphism algebra is not a skew-field, it is a simple or a semi-simple central algebra.

[3]An anti-involution means an involutive isomorphism from the algebra to the opposite algebra, i.e. the algebra with the same abelian group but with the multiplication law $x \cdot y := y \cdot x$.

Note that we always have

$$[\mathrm{End}(A)_{\mathbb{Q}} : \mathbb{Q}]_{\mathrm{red}} \leq 2g \tag{2.10}$$

since $\mathrm{End}(A)_{\mathbb{Q}}$ embeds in $\mathrm{Mat}_{2g}(\mathbb{Q})$ via its rational representation. If A is a simple abelian variety, we have $[\mathrm{End}(A)_{\mathbb{Q}} : \mathbb{Q}]_{\mathrm{red}}|g$ if A is of types I-III, and $[\mathrm{End}(A)_{\mathbb{Q}} : \mathbb{Q}]_{\mathrm{red}}|2g$ if A is of type IV. In the latter case the equality in (2.10) occurs if and only if A is of type IV with $e_0 = g$. If A is not necessary simple, the equality occurs if and only if A is isogenous to the product of simple abelian varieties A_i with $[\mathrm{End}(A)_{\mathbb{Q}} : \mathbb{Q}]_{\mathrm{red}} = 2\dim A_i$. We say in this case that A is of *CM-type*. We will study such varieties in details later.

Lecture 3

Elliptic curves

An *elliptic curve* is a one-dimensional abelian variety $A = \mathbb{C}/\Lambda$. We can find a special symplectic basis in Λ of the form $(\tau, 1)$, where $\tau \in \mathbb{H}$. The matrix of the symplectic form E on Λ with respect to this basis is the matrix $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. Since $i = -\frac{x}{y} + \frac{1}{y}\tau$, we get $E(i, 1) = \frac{1}{y}$. By (1.7), the corresponding Hermitian form is equal to $\frac{1}{y}z\bar{z'}$ in agreement with (1.7). The Hermitian form H defines a principal polarization on E. It is defined by a line bundle L_0 of degree 1. We will always consider E as a one-dimensional principally polarized abelian variety.

Note that $\mathrm{Sp}(2, \mathbb{Z}) \cong \mathrm{SL}(2, \mathbb{Z})$, so the moduli space of elliptic curves is

$$\mathcal{A}_1 = \mathbb{H}/\mathrm{SL}(2, \mathbb{Z}),$$

where $\mathbb{H} = \{\tau \in \mathbb{C} : \mathrm{Im}(\tau) > 0\}$. The quotient space is known to be isomorphic to $\mathbb{C}$, the isomorhism is defined by a holomorphic function $j : \mathbb{H} \to \mathbb{C}$ which is invariant with respect to $\mathrm{SL}(2, \mathbb{Z})$. It is called the *absolute invariant*. If τ is the period of E, then $j(\tau)$ is called the absolute invariant of E. We refer to the explicit definition of j to any (good) text-book on functions of one complex variable.

Let f be an endomorphism of A, then f_a is a complex number z and $f_r : \Lambda \to \Lambda$ is the map $\lambda \mapsto z\lambda$. In the basis $(\tau, 1)$ of Λ, the transformation f_r is given by an integral matrix $N = \left(\begin{smallmatrix} a_1 & a_3 \\ a_2 & a_4 \end{smallmatrix}\right)$ so that we have $(z\tau, z) = (a_1\tau + a_2, a_3\tau + a_4)$. This gives $z = a_3\tau + a_4$ and $(a_3\tau + a_4)\tau = a_1\tau + a_2$, and hence a quadratic equations for τ

$$a_3\tau^2 + (a_4 - a_1)\tau - a_2 = 0. \tag{3.1}$$

It agrees with (2.2). The discriminant of the quadratic equation (2.2) is equal to

$$D = (a_4 - a_1)^2 + 4a_2a_3 = (a_1 + a_4)^2 - 4(a_1a_4 - a_2a_3) = \mathrm{Tr}(N)^2 - 4\det(N). \tag{3.2}$$

Since $\mathrm{Im}(\tau) > 0$, we must have $a_3 \neq 0, D < 0$ or $a_3 = a_4 - a_1 = a_2 = 0$. In the latter case, the matrix N is a scalar matrix, and the endomorphism is just the multiplication $[a_1]$ and there is no condition on τ. In the former case

$$\tau = \frac{a_1 - a_4 + i\sqrt{-D}}{2a_3}.$$

It shows that $\tau \in \mathbb{Q}(\sqrt{D})$, i.e. it is an an imaginary quadratic algebraic number. Also

$$z = a_3\tau + a_4 = \frac{1}{2}(a_1 + a_4 + i\sqrt{-D})$$

belongs to the same field. For this reason an elliptic curve A is called an *elliptic curve with complex multiplication* by $K = \mathbb{Q}(\sqrt{D})$.

Multiplying (3.1) by a_3, we obtain that $a_3\tau$ and, hence z, satisfies a monic equation over $\mathbb{Z}$, hence belongs to the ring $\mathfrak{o}_K$ of integers of the field K. Note that formula (3.2) shows that, D is divisible by 4 if $\mathrm{Tr}(N) = a_1 + a_4$ is even, and $D \equiv 1 \mod 4$ otherwise.

Recall that, if D is square-free, then $\mathfrak{o}_K$ has a basis, as a module over $\mathbb{Z}$, equal to $(1, \frac{1}{2}(1+\sqrt{D})$ if $D \equiv 1 \mod 4$ or $1, \sqrt{D}$ otherwise. If $D = m^2 D_0$, where D_0 is square-free, then $\mathrm{End}(E)$ is an order in K and D is its *discriminant*. The order is equal to $\mathbb{Z} + m\mathfrak{o}_K$ (see [12]). In any case, $\mathrm{End}(E)_{\mathbb{Q}} \cong K$, so we are in case IV of classification of endomorphism rings of abelian varieties. Also, we see that $\mathrm{End}(A)$ is an order $\mathfrak{o}$ in K. The lattice Λ must be a module over $\mathfrak{o}$, in fact, one can show that it is a projective module of rank 1. Conversely, if we take Λ to be such a module over an order $\mathfrak{o}$ in K, we obtain an elliptic curve $A = \mathbb{C}/\Lambda$ with $\mathrm{End}(A) \cong \mathfrak{o}$. In this way one can show that there is a bijective correspondence between isomorphism classes of elliptic curves with $\mathrm{End}(A) = \mathfrak{o}_K$ and the *class group* of K (i.e. the group of classes of ideals in $\mathfrak{o}_K$ modulo principal ideals, or, in a scheme-theoretical language, the Picard group of Spec $\mathfrak{o}_K$. The number of such classes is called the *class number* of K.

Note that $\mathrm{Aut}(E) = \mathrm{End}(E)^*$ can be larger than $\{\pm 1\}$ only if E admits complex multiplication with Gaussian integers (i.e. $D = -1$) or Eisenstein integers (i.e. $D = -3$). In fact, if $D \equiv 1 \mod 4$, an invertible algebraic integer $a + \frac{1}{2}b(1+\sqrt{D}), a, b \in \mathbb{Z}$ must satisfy $\mathrm{Nm}(\frac{1+\sqrt{D}}{2}) = \pm 1$. This implies $D = -3$. Similarly, if $D \not\equiv 1 \mod 4$, we obtain $a^2 - Db^2 = \pm 1$ implies $D = -1$. If C is a birational model of E as a nonsingular plane cubic, then C is a harmonic cubic if $D = -1$ and equianharmonic cubic otherwise.

Remark 3.1. Let E be an elliptic curve with complex multiplication $\mathrm{End}(E)_{\mathbb{Q}} = K$. Recall that E admits a Weierstrass equation

$$y^2 = x^3 + a_4 x + a_6,$$

and the isomorphism class of E is determined by the value of the absolute invariant

$$j(E) = 1728 \frac{4a_4^3}{4a_4^3 + 27a_6^2}.$$

The curve E has complex multiplication by Gaussian numbers (resp. Eisenstein number) if and only if $a_6 = 0$ (resp. $a_4 = 0$).

According to the *Theorem of Weber and Fuerter*, the j-invariant $j(E)$ of an elliptic curve with complex multiplication is an algebraic integer such that $[K(j(E)) : K] = [\mathbb{Q}(j(E)) : \mathbb{Q}]$ and the field $K(j(E))$ is the *class field* of K, i.e. a maximal unramified extension of K (see [100], Chapter 2). Assume that $j(E) \in \mathbb{Q}$, by the class fields theory this implies that the class number of K is equal to 1. Also, it is known that $j(E) \in \mathbb{Q}$ if and only if E can be defined over $\mathbb{Q}$. There are exactly nine imaginary quadratic fields K with class number 1. They are the fields $\mathbb{Q}(\sqrt{-d})$, where

$$d \in \{1, 2, 3, 7, 11, 19, 43, 67, 163\}.$$

The corresponding values of the absolute invariants are

$$2^6\cdot 3^3,\quad 2^6\cdot 5^3,\quad 0,\quad -3^3\cdot 5^3,\quad -2^{15},\quad -2^{15}\cdot 3^3,\quad -2^{18}\cdot 3^3\cdot 5^3,\ -2^{15}\cdot 3^3\cdot 5^3\cdot 11^3,$$

$$-2^18\cdot 3^3\cdot 5^3\cdot 23^3,\quad 2^3\cdot 3^3\cdot 11^3,\quad 2^4\cdot 3^3\cdot 5^3,\quad 3^3\cdot 5^3\cdot 17^3,\ -3\cdot 2^{15}\cdot 5^3.$$

Let $f: E\to E$ be an endomorphism of E of finite degree $n>0$. By Hurwitz' formula, the map f is an unramified finite cover of degree n. Its kernel is a finite subgroup T of order n of E. The group $E[n]$ of n-torsion elements of $E=\mathbb{C}/\Lambda$ is isomorphic to $\frac{1}{n}\Lambda/\Lambda\cong(\mathbb{Z}/n\mathbb{Z})^2$. Assume that f_r is defined by a matrix N whose entries are mutually coprime (otherwise the endomorphism a composition of an endomorphism g with g_r satisfying this property and multiplication by an integer). The theory of elementary divisors allows us to find two bases (γ_1,γ_2) and (γ_1',γ_2') in Λ such that $(f_r(\gamma_1),f_r(\gamma_2))=(n\gamma_1',\gamma_2')$. Since $j(\tau)$ depends only on Λ, we obtain that $j(\tau)=j(n\tau)$. It is known that there exists a polynomial $\Phi_n(X,Y)$ with integer coefficients such that $\Phi(j(\tau),j(n\tau))\equiv 0$ for any $\tau\in\mathbb{H}$. The equation $\Phi_n(X,Y)=0$ is called the *modular equation* of level n. Thus the number of elliptic curves admitting an endomorphism of degree n is equal to the number of solutions of the equation $\Phi_n(x,x)=0$. It has been computed by R. Fricke and it is equal to $h_0(-n)+h_0(-4n)$ if $n\equiv 2,3\mod 4$, and $h_0(-4n)$ if $n\equiv 1\mod 4$. Here $h_0(-d)$ is the class number of primitive quadratic integral positive definite forms with discriminant equal to $-d$.

Let $f: E\to E'$ be an *isogeny* of elliptic curves and $g: E'\to E$ be its inverse, i.e. $g\circ f=[n]$, where n is the degree of f. Let f_a be given by a complex number z and g be given by a complex number z'. Then $zz'=d$. Also we know that $|z|^2=\det f_r=d$. Thus, we obtain that $z'=\bar{z}$ is the complex conjugate of z.

Let $A=E_1\times\cdots\times E_g$ be the product of g isogenous elliptic curves. We assume that $\mathrm{End}(E_i)=\mathbb{Z}$. Let α_{ij} be an isogeny $E_i\to E_j$ of minimal degree so that any isogeny $E_i\to E_j$ can be written in form $[d_{ij}]\circ\alpha_{ij}$ (which we denote, for brevity, by $d_{ij}\alpha_{ij}$) for some integer d_{ij} and a complex number α_{ij}. Obviously $\alpha_{ii}=1$.

The analytic representation of an endomorphism $f: A\to A$ is given by a matrix

$$M=\begin{pmatrix} d_{11} & d_{12}\alpha_{12} & \dots & d_{1g}\alpha_{ig}\\ d_{21}\overline{\alpha_{12}} & d_{22} & \dots & d_{2g}\alpha_{2g}\\ \vdots & \vdots & \vdots & \vdots\\ d_{g1}\overline{\alpha_{1g}} & d_{g2}\overline{\alpha_{2g}} & \dots & d_{gg}\end{pmatrix}.$$

We may choose the period matrix of A to be equal to the diagonal matrix $\mathrm{diag}[\tau_1,\dots,\tau_g]$, where $\tau_i=x_i+\sqrt{-1}y_i$ is the period of E_i. Let us choose a principal polarization L_0 on A to be the reducible one coming from the principal polarizations on the curves E_i. Its Hermitian form is given by the diagonal matrix $\mathrm{diag}[y_1^{-1},\dots,y_g^{-1}]$. Assume that A has another principal polarization L and M is a symmetric endomorphism corresponding to L. By (2.1), the matrix of the Hermitian form H corresponding to L is equal to the matrix

$$M'=\mathrm{diag}[y_1^{-1},\dots,y_g^{-1}]\cdot M \tag{3.3}$$

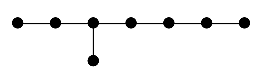
In particular, this implies that $y_id_{ij}=y_jd_{ji}$.

Assume now that $E_1 = \ldots = E_g = E$ and $\mathrm{End}(E) = \mathbb{Z}$. Since E has no complex multiplications, $\alpha_{ij} = 1$, hence M is a symmetric integral matrix. It follows from (2.2) that f_r is given by the matrix $N = \begin{pmatrix} M & 0 \\ 0 & M \end{pmatrix}$. Since we are looking for f defined by a principal polarization, f must be an isomorphism, hence $\det M = 1$. We know also that the coefficients of its characteristic polynomial are positive rational numbers. This implies that M is positive definite. Let $(\gamma_1, \ldots, \gamma_{2g}) = (\tau e_1, \ldots, \tau e_g, e_1, \ldots, e_g)$ be a basis of $\Lambda_{\mathbb{R}}$. It follows from (3.3) that the matrix of the symplectic form corresponding to H in this basis is equal to (a_{ij}), where $a_{ij} = y^{-1}\mathrm{Im}(H(\gamma_i, \gamma_j))$. We get for $1 \leq i < j \leq g$

$$a_{ij} = y^{-1}\mathrm{Im}(H(e_i, e_j)|\tau|^2) = 0, \ a_{i,j+g} = y^{-1}\mathrm{Im}(H(e_i, e_j)\tau) = d_{ij}.$$

This implies that the type D of the polarization L is equal to the matrix (d_{ij}) (reduced to the diagonal form).

It is known that a positive definite symmetric matrix of rank $g \leq 7$ with determinant ± 1 and some odd diagonal entries can be reduced over $\mathbb{Z}$ to the identity matrix. By above this implies that the only principal polarization on an abelian variety $A = E^g$ is of the form $\sum_{i=1}^{g} p_i^*(\text{point})$, where p_i is the projection to the i-th factor. In particular, A cannot be isomorphic to the Jacobian variety of a curve of genus g. However, if $g = 8$, there is a positive definite symmetric matrix with determinant 1 that cannot be reduced to the identity matrix. This matrix is equal to $2I_8 - P_{E_8}$, where P_{E_8} incidence matrix of the Dynkin diagram of type E_8

E_8 

Remark 3.2. It is known that the rank of any positive definite symmetric matrix with determinant ± 1 and even diagonal entries is divisible by 8 (see [96], 2.3). Thus, if E has no complex multiplication, the product of r copies of E does not admit a principal polarization unless r is divisible by 8. Note that there is only one isomorphism class of positive definite unimodular quadratic lattices of rank 16 not isomorphic to $E_8 \oplus E_8$ and there are 24 non-isomorphic such lattices of rank 24, the Leech lattice is among them. So we have 2 (resp. 24) principally polarized abelian varieties isomorphic to E^8 (resp. E^{12}), where E is an elliptic curve. Do they have any geometric meaning, e.g. being the Prym or Jacobian varieties?

Example 3.3. Let M be a *quadratic lattice*, i.e. a free abelian group of finite rank equipped with a symmetric bilinear form $B : M \times M \to \mathbb{Z}$. Assume that the rank of M is an even number $2k$ and the bilinear form is positive definite (when tensored with $\mathbb{R}$). Assume also that the orthogonal group of M (i.e. the subgroup of $\mathrm{Aut}(M)$ that preserves the symmetric form) contains an element ι such that $\iota^2 = -\mathrm{id}_M$. Then we can use ι to define a complex structure on $W = M_{\mathbb{R}}$ and define a hermitian form H by taking $E(x, y) := -B(\iota(x), y)$ so that $E(\iota(x), y) = B(x, y)$ is symmetric and positive definite, and

$$E(y, x) = -B(\iota(y), x) = -B(x, \iota(y)) = -B(\iota(x), \iota^2(y)) = B(\iota(x), y) = -E(x, y)$$

is skew-symmetric, obviously non-degenerate.

Let us consider M as a module over $\mathbb{Z}[i]$ by letting i act on M as the isometry ι. Since $\mathbb{Z}[i]$ is a principal ideal domain, we get $M \cong \mathbb{Z}[i]^k$ and we have an isomorphism $(M_{\mathbb{R}}, \iota) \cong \mathbb{C}^k$, so that M

can be identified with the lattice Λ with a basis equal to the union of k copies of the basis $(i, 1)$. Obviously, the abelian variety $A = \mathbb{C}^k/M$ becomes isomorphic to the product $E_{\sqrt{-1}}^k$, where $E_{\sqrt{-1}}$ is the elliptic curve with complex multiplication by $\mathbb{Z}[i]$. On the other hand, if we take M to be an even unimodular lattice of rank $2k$, then our Hermitian form H defines an indecomposable principal polarization. As we remarked before such lattices M exist only in dimension divisible by 8. So, k is divisible by 4.

If $k = 4$, there exists a unique such lattice, the E_8-lattice M. The abelian 4-fold $A = \mathbb{C}^4/M$ is remarkable for many reasons. For example, it is isomorphic to the intermediate Jacobian of a Weddle quartic double solid, i.e. a nonsingular model of the double cover of $\mathbb{P}^3$ branched along a Weddle quartic surface with 6 nodes (see [107]). Another remarkable property of A is that the theta function corresponding to its indecomposable principal polarization has maximal value of critical points (equal to 10 in dimension 4 for simple abelian varieties which are not isomorphic to the Jacobian variety of a hyperelliptic curve) (see [24]).

Recall that the *Jacobian variety* $\mathrm{J}(C)$ of a compact Riemann surface C of genus g (or, equivalently, nonsingular complex projective curve of genus g) is an abelian variety whose period matrix is equal to

$$\Pi = (\int_{\gamma_i} \omega_j),$$

where $\omega_1, \ldots, \omega_g$ is a basis of holomorphic 1-forms on C and $\gamma_1, \ldots, \gamma_{2g}$ is a basis of $H_1(C, \mathbb{Z})$. One can always choose a basis of $H_1(C, \mathbb{Z})$ and a basis in $\Omega^1(C)$ such that the period matrix $\Pi = [\tau I_g]$, where $\tau \in \mathcal{Z}_g$. In particular, $\mathrm{J}(C)$ has always a principal polarization L_0. The unique nonzero section of L_0 has the divisor of zeros equal to the image of the symmetric product $C^{(g-1)} = C^g/\mathfrak{S}_{g-1}$ under the *Abel-Jacobi map*

$$C^{(g-1)} \to \mathrm{J}(C), \quad \sum_{k=1}^{g-1} c_k \mapsto \sum_{k=1}^{g-1} (\int_{p_k}^{c_k} \omega_1, \ldots, \int_{p_k}^{c_k} \omega_g) \mod \Pi\mathbb{Z}^{2g},$$

where $p_1, \ldots, p_{g-1}$ are fixed points on C.

Let $\mathrm{Pic}^0(C)$ be the group of linear equivalence classes of divisors of degree 0 on C, or, equivalently, the group of isomorphism classes of line bundles of degree 0 on C. The *Abel-Jacobi Theorem* asserts that the map

$$\iota_p : C \to \mathrm{J}(C), \quad x \mapsto (\int_p^x \omega_1, \ldots, \int_p^x \omega_g) \mod \Pi\mathbb{Z}^{2g}$$

extends by linearity to an isomorphism of groups $\mathrm{Pic}^0(C) \to \mathrm{J}(C)$.

Example 3.4. Following [40], let us give an example of the Jacobian of a curve of genus 2 isomorphic to the product of two isomorphic elliptic curves. Let $K = \mathbb{Q}(-m)$ be an imaginary quadratic field and $\mathfrak{o}$ be its ring of integers. We assume that the class number of K is greater than 1 and choose a non-principal ideal $\mathfrak{a}$ in $\mathfrak{o}$. For example, we can take $m = 5$. Since $-5 \equiv 3 \mod 4$, the ring $\mathfrak{o}$ is generated over $\mathbb{Z}$ by 1 and $\omega = \sqrt{-5}$. We may take for $\mathfrak{a}$ the ideal generated by $(2, 1 + \omega)$. In fact, $\mathrm{Nm}(\mathfrak{a}) = (\mathrm{Nm}(2), \mathrm{Nm}(1+\omega)) = (4, 6) = (2)$ and since the equation $\mathrm{Nm}(x+y\omega) = x^2+5b^2 = 2$ has no integer solutions, we obtain that the ideal $\mathfrak{a}$ is not principal. Let

$$E = \mathbb{C}/\mathfrak{o} = \mathbb{C}/\mathbb{Z} + \mathbb{Z}\omega.$$

Consider a homomorphism $\phi : E \to E \times E$ defined by $x \mapsto (2x, (1+w)x)$. Let E' be the image of this homomorphism. Let $E_1 = E \times \{0\}, E_2 = \{0\} \times E$, and Δ be the diagonal. Let us compute the intersection indices of E' with these three curves.

Suppose $\phi(x) \in E_1$, then $x(1+\omega) \in \mathfrak{o}$, hence there exists $m, n \in \mathbb{Z}$ such that

$$x = \frac{m + n\omega}{1+\omega} = \frac{1}{6}(m + 5 + (m-n)\omega) \in \mathbb{Z}\frac{1+\omega}{6} + \mathbb{Z}.$$

This shows that there are 3 intersection points $(0,0), (\frac{1+\omega}{3}, 0), (\frac{2(1+\omega)}{3}, 0)$.

Suppose $\phi(x) = (0, (\omega+1)x) \in E_2$, then $2x \in \mathfrak{o}$, hence there are two intersection points $(0,0), (0, \frac{1}{2}(1+\omega))$.

Suppose $\phi(x) = (2x, (1+\omega)x) \in \Delta$, then $(1-\omega)x = 2x - (1+w)x \in \mathfrak{o}$. This implies that $x \in \frac{1+\omega}{6}\mathbb{Z} + \mathbb{Z}$, hence there are 3 intersection points $(0,0), (\frac{1+\omega}{3}, \frac{1+\omega}{3}), (\frac{2(1+\omega)}{3}, \frac{2(1+\omega)}{3})$.

Now we consider the divisor

$$C = 2\Delta + E' + E_1 - 2E_2.$$

We have $C \cdot \Delta = 2, C \cdot E' = 5, C \cdot E_1 = 3, C \cdot E_2 = 5, C^2 = 2$. By Riemann-Roch, C is an effective divisor class, so we may assume that C is a curve of arithmetic genus 2. If C is reducible, then $C = C_1 + C_2$ is the sum of two elliptic curves with $C_1 \cdot C_2 = 1$, and we may assume that one of its components, say C_1, intersects Δ and E_1 with multiplicity 1. We have $C_2 = C - C_1 \sim 2\Delta + E' + E_1 - 2E_2 - C_1$. Intersecting with C_1, we get $1 = 4 - 2(E_2 \cdot C_1)$, a contradiction. Thus C is an irreducible curve of arithmetic genus 2. It is known that this implies that C is a smooth curve of genus 2 and $A \cong \mathrm{J}(D)$. [1] and as we remarked before, it must be a nonsingular curve of genus 2, and $A = E \times E$ is isomorphic to $\mathrm{J}(C)$.

[1] To see this use one considers the normalization map $\bar{D} \to A$ and the dual map $\hat{A} \to \mathrm{J}(\bar{D})$ and proves that it is injective, hence the geometric genus coincides with the arithmetic genus.

Lecture 4

Humbert surfaces

Let A be an abelian variety of dimension 2, i.e. an abelian surface. The Poincaré duality equips the group $H^2(A,\mathbb{Z}) = \mathbb{Z}^6$ with a structure of a unimodular quadratic lattice of signature $(3,3)$. It is an even lattice, i.e. its values are even integers. By Milnor's theorem, $H^2(A,\mathbb{Z}) \cong U \oplus U \oplus U$, where U is a hyperbolic plane over $\mathbb{Z}$, i.e. its quadratic form could be defined by a matrix $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, and the direct sum is the orthogonal direct sum. Let T_A be the orthogonal complement of $\mathrm{NS}(A)$ in $H^2(A,\mathbb{Z})$. It is a quadratic lattice of signature $(2, 4-\rho)$, and we have an orthogonal decomposition of quadratic lattices

$$H^2(A,\mathbb{Z}) = \mathrm{NS}(A) \oplus T_A.$$

The quadratic form on $\mathrm{NS}(A)$ is defined by the intersection theory of curves on an algebraic surface. For any irreducible curve C on A, the adjunction formula $C^2 + C \cdot K_A = C^2 = -2\chi(\mathcal{O}_C)$, together with the fact that A has no rational curves,[1] gives $C^2 \geq 0$ and $C^2 = 0$ if and only if C is a smooth elliptic curve. By writing any effective divisor as a sum of irreducible curves, we obtain that $D^2 \geq 0$ on the cone $\mathrm{Eff}(A)$ in $\mathrm{NS}(A)_{\mathbb{R}}$ of classes of effective divisors modulo homological equivalence. By Hodge's Index Theorem, we have $D \cdot C \geq 0$ for any effective divisors D and C. This implies that $\mathrm{Eff}(A)$ coincides with the cone $\mathrm{Nef}(A)$ of nef divisor classes. The latter is known to be the closure of the cone $\mathrm{Amp}(A)$ of ample divisor classes. By Riemann-Roch and the vanishing Theorem, $h^0(D) = D^2/2$ for any ample divisor D. Thus the restriction of the trace quadratic form on $\mathrm{End}(A)$ to $\mathrm{Amp}(A)$ is equal to twice of the restriction of the intersection form to $\mathrm{Amp}(A)$.

Suppose A is a simple abelian surface with $\mathrm{End}(A) \neq \mathbb{Z}$. According to the classification of possible endomorphism algebras, we have three possible types:

(i) $\mathrm{End}(A)_{\mathbb{Q}}$ is a totally real quadratic field K and $\rho = 2$;

(ii) $\mathrm{End}(A)_{\mathbb{Q}}$ is a totally indefinite quaternion algebra over $K = \mathbb{Q}$ and $\rho = 3$;

(iii) $\mathrm{End}(A)_{\mathbb{Q}}$ is a totally imaginary quadratic extension K of a real quadratic field K_0 and $\rho = 2$.

[1] A morphism of a rational curve C to a complex torus $T = \mathbb{C}^g/\Lambda$ can be composed with the normalization morphism $\tilde{C} \to C$ and then lifted to a holomorphic map of the universal covers $\mathbb{P}^1(\mathbb{C}) \to \mathbb{C}^g$. The latter map is obviously a constant map.

Observe that we have intentionally omitted the cases when $\mathrm{End}(A)_{\mathbb{Q}}$ is a definite quaternion algebra and when $\mathrm{End}(A)_{\mathbb{Q}}$ is a totally imaginary quadratic extension of $\mathbb{Q}$. These types of algebras occur for a non-simple abelian surface. In the former case it must be the product of two elliptic curves with complex multiplication by $\sqrt{-1}$ (see [67], Chapter 9, Example 9.5.5 and Exercises 1). In the latter case, $\mathrm{End}(A)_{\mathbb{Q}}$ must be isomorphic to an indefinite quaternion algebra (loc.cit. Exercise 4 in Chapter 4).[2]

Let

$$\tau = \begin{pmatrix} z_1 & z_2 \\ z_2 & z_3 \end{pmatrix}$$

be the period matrix of A. We assume that $A = \mathbb{C}^2/\mathbb{Z}^2 + \mathsf{D}\mathbb{Z}^2$ has a primitive polarization of degree n. Its type is defined by the diagonal matrix $\mathsf{D} = \mathrm{diag}[1, n]$. Let $f \in \mathrm{End}^s(A)$, where f_a is defined by a matrix M and f_r is defined by a matrix N as in (2.2). Since f is symmetric, N satisfies (2.4). We easily obtain that

$$\begin{pmatrix} A_1 & A_3 \\ A_2 & A_4 \end{pmatrix} = \begin{pmatrix} a_1 & na_2 & 0 & nb \\ a_3 & a_4 & -b & 0 \\ 0 & nc & a_1 & na_3 \\ -c & 0 & a_2 & a_4 \end{pmatrix}.$$

By (2.2) and (2.3), we have

$$M = (\tau A_3 + \mathsf{D}A_4)\mathsf{D}^{-1}, \quad M\tau = \tau A_1 + \mathsf{D}A_2,$$

and

$$(\tau A_3 + \mathsf{D}A_4)\mathsf{D}^{-1}\tau = \tau A_1 + \mathsf{D}A_2.$$

The left-hand side in the second equality is equal to

$$\begin{pmatrix} 0 & b(-z_2^2 + z_1 z_3) \\ b(z_2^2 - z_1 z_3) & 0 \end{pmatrix} + \begin{pmatrix} a_1 z_1 + a_3 z_2 & a_1 z_2 + a_3 z_3 \\ na_2 z_1 + a_4 z_2 & na_2 z_2 + a_4 z_3 \end{pmatrix}$$

$$= \begin{pmatrix} a_1 z_1 + a_3 z_2 & b(-z_2^2 + z_1 z_3) + a_1 z_2 + a_3 z_3 \\ b(z_2^2 - z_1 z_3) + na_2 z_1 + a_4 z_2 & +na_2 z_2 + a_4 z_3 \end{pmatrix}.$$

The right-hand side is equal to

$$\begin{pmatrix} a_1 z_1 + a_3 z_2 & na_2 z_1 + a_4 z_2 + nc \\ a_1 z_2 + a_3 z_3 - nc & na_2 z_2 + a_4 z_3 \end{pmatrix}.$$

Comparing the entries of the matrices in each side, we find a relation

$$b(z_2^2 - z_1 z_3) + a_2 n z_1 + (a_4 - a_1) z_2 - a_3 z_3 + nc = 0.$$

We rename the coefficients to write it in the classical form to obtain what G. Humbert called the *singular equation* for the period matrix τ:

$$naz_1 + bz_2 + cz_3 + d(z_2^2 - z_1 z_3) + ne = 0. \tag{4.1}$$

[2] See solutions to these exercises in [90].

We also assume that $(a,b,c,d,e)=1$. In this new notations, the matrix $N_0 = N - a_1 I_4$ representing $(f_0)_r = (f - a_1\mathrm{id})_r$ can be rewritten in the form

$$N_0 = -a_1 I_4 + N = \begin{pmatrix} 0 & na & 0 & nd \\ -c & b & -d & 0 \\ 0 & ne & 0 & -nc \\ -e & 0 & a & b \end{pmatrix}. \tag{4.2}$$

and $(f_0)_a$ is represented by the matrix

$$M_0 = \begin{pmatrix} -dz_2 & dz_1 - c \\ -dz_3 + na & dz_2 + b \end{pmatrix}. \tag{4.3}$$

We have

$$\mathrm{Tr}(N_0) = 2\mathrm{Tr}(M_0) = 2b,\ \det(N_0) = \det(M_0)^2 = n^2(ac+ed)^2.$$

Thus f_0 satisfies a quadratic equation

$$t^2 - bt + n(ac+ed) = 0, \tag{4.4}$$

so that 1 and f_0 generate a subalgebra $\mathfrak{o}$ of rank 2 of $\mathrm{End}^s(A)$ isomorphic to

$$\mathfrak{o} \cong \mathbb{Z}[t]/(t^2 - bt + n(ac+ed)). \tag{4.5}$$

The discriminant Δ of the equation (4.4) is equal to

$$\Delta = b^2 - 4n(ac+ed). \tag{4.6}$$

It is called the *discriminant* of the singular equation. Note that, if b is even, $\Delta \equiv 0 \mod 4$, otherwise $\Delta \equiv 1 \mod 4$.

Since we know that the eigenvalues of M are real numbers,

$$\Delta > 0. \tag{4.7}$$

Thus if Δ is not a square, the algebra $\mathfrak{o}$ is an order in the real quadratic field $\mathbb{Q}(\sqrt{\Delta})$. On the other hand, if Δ is a square, then the algebra $\mathfrak{o}$ has zero divisors defined by the integer roots $\frac{1}{2}(b \pm \sqrt{\Delta})$ of equation (4.4).

Note that, replacing t with $t+\alpha$, we may assume that $b=0$ if b is even, or $b=1$, otherwise.

Let L_Δ be the line bundle that is mapped to f_0 inder $\alpha : \mathrm{NS}(A) \to \mathrm{End}^s(A)$. Applying (2.6), we obtain that

$$(L_0, L_\Delta) = nb = \tfrac{1}{2}(L_0^2)b, \quad (L_\Delta^2) = \tfrac{1}{2}n(b^2 - \Delta). \tag{4.8}$$

Thus the sublattice $\langle L_0, L_\Delta\rangle$ of $\mathrm{NS}(A)$ generated by L_0, L_Δ has discriminant equal to $(L_0)^2(L_\Delta^2) - (L_0, L_\Delta)^2 = -n^2\Delta$.

Recall that a finite R algebra over $\mathbb{Z}$ of degree n can be considered as a quadratic lattice with associated symmetric bilinear form defined by

$$(x,y) = \mathrm{Tr}(xy), \tag{4.9}$$

where Tr : $R \to \mathbb{Z}$ is the $\mathbb{Z}$-linear function whose value on an element $x \in R$ is equal to the trace of the endomorphism $\alpha_x : r \mapsto xr$ (the coefficient at $(-\lambda)^{n-1}$ in the characteristic polynomial). The discriminant of the corresponding quadratic form is called the *discriminant* of R (the last coefficient of the characteristic polynomial of α_x).

In our case when $R = \mathfrak{o}$ from (4.5), we take the basis $(1, \bar{t})$ of $\mathfrak{o}$, where $\bar{t}$ is the coset of t, and obtain that the matrix of the bilinear form (4.9) is equal to

$$\begin{pmatrix} 2 & -b \\ -b & b^2 - n(ac+ed) \end{pmatrix} = \begin{pmatrix} 2 & -b \\ -b & \frac{1}{2}(b^2 - \Delta) \end{pmatrix}.$$

Comparing this with the sublattice $\langle L_0, L_\Delta \rangle$ of NS(A) we obtain that, there is an isomorphism of quadratic lattices

$$\langle L_0, L_\Delta \rangle \cong \mathfrak{o}(n), \tag{4.10}$$

where (n) means that we multiply the values of the quadratic form by n.

When L_Δ is ample, we can also determine the type of the polarization defined by L_Δ. It is equal to the type of the alternating form given by the matrix

$${}^tN_0 J_D = \begin{pmatrix} 0 & na & 1 & nd \\ -c & b & -d & n \\ -1 & ne & 0 & -nc \\ -e & -n & a & b \end{pmatrix}. \tag{4.11}$$

Let $\mathcal{A}_{2,n} = \mathcal{Z}_2/\mathrm{Sp}(J_{\mathsf{D}}, \mathbb{Z})$ be the coarse moduli space of abelian surfaces with polarization of type $(1, n)$. We denote by $\mathcal{H}_\Delta$ the set of period matrices $\tau \in \mathcal{Z}_2$ satisfying a singular modular equation with discriminant Δ. Let

$$\mathrm{Hum}_n(\Delta) = \mathcal{H}_\Delta/\mathrm{Sp}(J_{\mathsf{D}}, \mathbb{Z})$$

be the image of $\mathcal{H}_\Delta$ in $\mathcal{A}_{2,n} := \mathcal{A}_{2,\mathsf{D}}$. This is the locus of isomorphism classes of abelian surfaces with primitive polarization of degree n that admit an embedding of a quadratic algebra $\mathbb{Z}[t]/(t^2 + \alpha t + \beta)$ with discriminant $\Delta = \alpha^2 - 4\beta$ in End(A). It is called the ***Humbert surface*** of discriminant Δ.

Suppose $\tau \in \mathcal{H}_\Delta$ and let $\tau' = M \cdot \tau$ for some $M \in \mathrm{Sp}(4, \mathbb{Z})$. If τ satisfies a singular equation (4.1), then the matrix N_0 defining an endomorphism of $\mathbb{C}^2/\Lambda_\tau$ changes to ${}^tM^{-1} \cdot N_0 \cdot M$ ([67], 8.1). Thus τ' satisfies another singular equation although with the same discriminant.

We will prove later the following theorem, which is in the case $n = 1$ due to G. Humbert.

Theorem 4.1. *Every irreducible component of the Humbert surface* Hum$_n(\Delta)$ *is equal to the image in* $\mathcal{Z}_2/\mathrm{Sp}(J_D, \mathbb{Z})$ *of the surface given by the equation*

$$nz_1 + bz_2 + cz_3 = 0, \tag{4.12}$$

where $\Delta = b^2 - 4nc, 0 \le b < 2n$. *The number of irreducible components is equal to*

$$\#\{b \mod 2n : b^2 \equiv \Delta \mod 4n\}.$$

Consider the quadratic $\mathbb{Z}$-algebra $\mathfrak{o}$ from (4.5). Let $K = \mathfrak{o} \otimes \mathbb{Q}$. If Δ is not a square, then K is a real quadratic extension of $\mathbb{Q}$. Let $\Delta = m^2\Delta_0$, where Δ_0 is square-free. Then $K = \mathbb{Q}(\sqrt{\Delta_0})$. If m is odd, then the order $\mathfrak{o}$ is generated by 1 and $\frac{1}{2}m(1+\sqrt{\Delta_0})$. If m is even, then $\mathfrak{o}$ is generated by 1 and $m\sqrt{\Delta_0}$ if $\Delta_0 \equiv 2,3 \mod 4$, and by 1 and $\frac{1}{2}m(1+\sqrt{\Delta_0})$ otherwise. Note that the *discriminant* of the order $\mathfrak{o}$ is equal to Δ.

Let $\sigma_1, \sigma_2 : K \to \mathbb{R}$ be two embedding of K into the field of real numbers.

If $\Delta = k^2$ is a square, then $\mathfrak{o} = \mathbb{Z}[\omega]$ is just an order in $K = \mathfrak{o} \otimes \mathbb{Q}$. Under the isomorphism

$$\mathfrak{o}_{\mathbb{Q}} \to \mathbb{Q} \oplus \mathbb{Q},\ x + y\omega \mapsto (x + y\alpha_+, x + y\alpha_-),$$

where $\alpha_\pm = \frac{1}{2}(b \pm k)$, the order $\mathfrak{o}$ becomes isomorphic to an order in $\mathbb{Z} \oplus \mathbb{Z}$. We denote by σ_1, σ_2 be the projections from $K \otimes \mathbb{R} \cong \mathbb{R} \oplus \mathbb{R} \to \mathbb{R}$.

Let $\mathrm{SL}_2(\mathfrak{o})$ be the group of matrices with determinant 1 with entries in $\mathfrak{o}$. Consider its action on the product $\mathbb{H} \times \mathbb{H}$ of the upper-half planes

$$(z_1, z_2) \mapsto (\frac{\sigma_1(\alpha)z_1 + \sigma_1(\gamma)}{\sigma_1(\beta)z_1 + \sigma_1(\delta)}, \frac{\sigma_2(\alpha)z_1 + \sigma_2(\gamma)}{\sigma_2(\beta)z_1 + \sigma_2(\delta)}).$$

Let $R = \begin{pmatrix} 1 & -\frac{1}{2}(b-\sqrt{\Delta}) \\ -1 & \frac{1}{2}(b+\sqrt{\Delta}) \end{pmatrix}$. Write Δ in the form $\Delta = b^2 - 4nc$. Consider the map

$$\mathbb{H} \times \mathbb{H} \to \mathcal{Z}_2,\ (z_1, z_2) \mapsto {}^tR \begin{pmatrix} z_1 & 0 \\ 0 & z_2 \end{pmatrix} R.$$

Then the image of the map is equal to the set of matrices $\begin{pmatrix} w_1 & w_2 \\ w_2 & w_3 \end{pmatrix} \in \mathcal{Z}_2$ satisfying equation (4.12). Let $\Phi : \mathrm{SL}_2(\mathfrak{o}) \to \mathrm{Sp}(J_{\mathsf{D}}, \mathbb{Z})$ be the homomorphism of groups defined by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \begin{pmatrix} {}^tR & 0 \\ 0 & R^{-1} \end{pmatrix} \cdot \begin{pmatrix} a & 0 & b & 0 \\ 0 & a & 0 & b \\ c & 0 & d & 0 \\ 0 & c & 0 & d \end{pmatrix} \cdot \begin{pmatrix} {}^tR^{-1} & 0 \\ 0 & R \end{pmatrix}.$$

One checks that the map $\mathbb{H}^2 \to \mathcal{Z}_2$ is equivariant with respect to the action of $\mathrm{SL}_2(\mathfrak{o})$ on $\mathbb{H}^2$ and the action of $\mathrm{Sp}(J_{\mathsf{D}}, \mathbb{Z})$ on $\mathcal{Z}_2$. This defines a morphism

$$\Phi : \mathbb{H}^2/\mathrm{SL}_2(\mathfrak{o}) \to \mathrm{Hum}_n(\Delta).$$

If $b \not\equiv 0 \mod n$, then the morphism Φ is of degree 1. Otherwise, Φ is of degree 2 and factors through the involution τ that switches the factors in $\mathbb{H}^2$ (see [106], IX, Proposition 2.5).

The quotient $\mathbb{H}^2/\mathrm{SL}_2(\mathfrak{o})$ (resp. $\mathbb{H}^2/(\mathrm{SL}_2(\mathfrak{o}), \tau)$) is a special case of a *Hilbert modular surface* (resp. *symmetric Hilbert modular surface*).

Lecture 5

Δ is a square

Let $i : B \hookrightarrow A$ be an abelian subvariety of an abelian variety A with primitive polarization L_0 of degree n. Let $L_0' = i^*(L_0)$ be the induced polarization of B and $\phi_B : B \to \hat{B}$ be the isogeny defined by L_0'. Consider the composition

$$\mathrm{Nm}_B := i \circ \phi_{L_0'}^{-1} \circ i^* \circ \phi_{L_0} : A \to \hat{A} \to \hat{B} \to B \to A.$$

It is called the *norm-endomorphism* associated to B. It is a symmetric endomorphism corresponding to the Hermitian form obtained by restricting the Hermitian form of L_0 to $H_1(B, \mathbb{C}) \subset H_1(A, \mathbb{C})$ and then extending it to $H_1(A, \mathbb{C})$ by zero. Also it is easy to see that $\mathrm{Nm}_B^2 = e(L_0')\mathrm{Nm}_B$. Taking $f = \mathrm{Nm}_B$ and $d = e(L_0')$, we obtain that f satisfies the equation $f^2 - df = 0$.

Let us go back to abelian surfaces and assume that $\Delta = k^2$ is a square. Then the minimal polynomial defining the corresponding endomorphism has roots $\alpha_\pm = \frac{1}{2}(b \pm k)$. Since $\Delta \equiv b^2 \mod 4n$, $\alpha_\pm \in \mathbb{Z}$. The equation

$$0 = (f - \alpha_+ \mathrm{id}_A)(f + \alpha_- \mathrm{id}_A) = 0$$

shows that the endomorphisms $g_\pm = f - \alpha_\pm \mathrm{id}_A$ satisfy the equations

$$g_\pm^2 = \pm k g_\pm, \quad g_+ \circ g_- = 0. \tag{5.1}$$

Let $E_\pm = g_\pm(A) \subset A$. These are elliptic curves on A, and we have exact sequences of homomorphisms of abelian varieties:

$$0 \to E_+ \longrightarrow A \xrightarrow{g_-} E_- \to 0, \quad 0 \to E_- \longrightarrow A \xrightarrow{g_+} E_+ \to 0$$

Note that $g_\pm | E_\pm = [\pm k]$, hence $E_+ \cdot E_- = \#\mathrm{Ker}([k]) = k^2$. Since the kernel of the isogeny

$$E_+ \times E_- \to A, (x, y) \mapsto x + y$$

is the group $E_+ \cap E_-$, we obtain that its degree is equal to k^2.

Suppose $A = \mathrm{J}(C)$ for some curve C of genus 2 and the polarization $L_0 \cong \mathcal{O}_A(C)$ is the principal polarization defined by C embedded in $\mathrm{J}(C)$ via the Abel-Jacobi map. Since k is equal to the trace of the characteristic equation for g_+, formula (2.8) and the projection formula imply that

$$\mathrm{Tr}(g_+^2) = \mathrm{Tr}(kg_+) = k\mathrm{Tr}(g_+) = k^2 = (g_+^*(C), C) = (C, (g_+)_*(C)) = d_+ C \cdot E_+ = d_+ d_-,$$

where $d_\pm$ is the degree of the projection $g_\pm|C : C \to E_\pm$. Since $d_+, d_- \le k$, we get $d_+ = d_- = k$. Obviously, $k > 1$ since C is not isomorphic to an elliptic curve.

Thus we obtain the following.

Theorem 5.1. *Suppose a period τ of $\mathrm{J}(C)$ satisfies a singular equation with discriminant $\Delta = k^2 > 1$, then C is a degree k cover of an elliptic curve.*

Conversely, assume that there exists a degree k cover $q : C \to E$, where E is an elliptic curve. Then the cover is ramified, hence the canonical map $q^* : E = \mathrm{J}(E) \to A = \mathrm{J}(C)$ is injective. We identify its image with E. Let $N : \mathrm{J}(C) \to \mathrm{J}(E) = E$ be the norm map (defined on divisors by taking q_*). Then $N \cdot q^* : E \to E$ is the map $[k]$. Let $g = \mathrm{Nm}_E : A \to A$. Then, it follows from the definition of the norm-endomorphism that $g^2 = kg$. Arguing as above, we find that the symmetric endomorphism Nm_E defines a singular equation for a period of $\mathrm{J}(C)$ whose discriminant is equal to k^2.

Example 5.2. Assume that a period of $A = \mathrm{J}(C)$ satisfies a singular equation with $\Delta = 4$, so that C is a bielliptic curve, i.e. there exists a degree 2 cover $\alpha : C \to E$. Let $\iota : C \to C$ be the deck transformation of this cover. If C is given by the equations

$$y^2 - f_6(x) = 0 \tag{5.2}$$

then, we may choose (x, y) in such a way that ι is given by $(x, y) \mapsto (y, -x)$ and $f_6(x) = g_3(x^2)$. Let

$$v^2 - g_3(u) = 0$$

be the equation of an elliptic curve E. The map $(x, y) \to (x^2, v)$ defines the degree 2 cover $\alpha : C \to E$. Let du/v be a holomorphic 1-form on E, then $\alpha^*(du/v) = xdx/y$ is a holomorphic 1-form on C. The involution ι^* acts on the space of holomorphic 1-forms on C spanned by dx/y and xdx/y, and decomposes it into two eigensubspaces with eigenvalues $+1$ and -1. Consider the involution $\iota' : (x, y) \mapsto (-y, -x)$. The field of invariants is generated by y^2, xy, x^2. Again $f_6 = g_3(x^2)$ and we get the equation $(xy)^2 = x^2 g_3(x^2)$. Thus the quotient $C/(\iota')$ is another elliptic curve with equation

$$v^2 - ug_3(u) = 0.$$

The map $\alpha' : C \to E'$ is given by $(u, v) \mapsto (x^2, xy)$. We have $\alpha'^*(du/v) = 2dx/y$. Thus any hyperelliptic integral $\int \frac{a+bdx}{y}$ can be written as a linear combination of elliptic integrals. This was one of the motivation for the work of G. Humbert.

One may ask how to find whether a hyperelliptic curve given by equation (5.2) admits a degree 2 map onto an elliptic curve in terms of the coefficients of the polynomial f_6. The answer was known in the 19th century. Let us explain it. First let us put a *level* on the curve by ordering the Weierstrass points $(0, x_i), f_6(x_i) = 0, i = 1, \ldots, 6$. By considering the Veronese map $\nu : \mathbb{P}^1 \to \mathbb{P}^2$ we put these 6 points $(x_i, 1)$ on a conic K in $\mathbb{P}^2$. Let $p_i = \nu(x_i)$. Applying Proposition 9.4.9 from [28], we obtain that the following is equivalent:

- there exists an involution σ of $\mathbb{P}^1$ with orbits $(x_1, x_2), (x_3, x_4), (x_5, x_6)$;
- the lines $\overline{p_1, p_2}, \overline{p_3, p_4}, \overline{p_5, p_6}$ are concurrent;

- the three quadratic polynomial $(x-x_1)(x-x_2),(x-x_3)(x-x_4),(x-x_5)(x-x_6)$ are linearly dependent;
- if $a_it_0+b_it_1+c_it_2=0$ are the equations of the three lines, then

$$D_{12,34,56}=\det\begin{pmatrix}a_1 & b_1 & c_1\\ a_2 & b_2 & c_2\\ a_3 & b_3 & c_3\end{pmatrix}=\det\begin{pmatrix}1 & x_1+x_2 & x_1x_2\\ 1 & x_3+x_4 & x_3x_4\\ 1 & x_5+x_6 & x_5x_6\end{pmatrix}=0.$$

(see [28], p. 468). Let

$$I=\prod_{\sigma\in\mathfrak{S}_6}D_{\sigma(1)\sigma(2),\sigma(3)\sigma(4),\sigma(5)\sigma(6)}.$$

The stabilizer subgroup of $(D_{12,34,56})^2$ in $\mathfrak{S}_6$ is generated by the transpositions $(12),(34),(56)$ and permutations of three pairs $(12),(34),(56)$. It is a subgroup of order 48. Thus, after symmetrization, I defines the *Clebsch skew invariant* I_{15} of degree $6!/48=15$ in coefficients of the binary form.[1] Recall that the algebra of SL_2-invariants of binary forms of degree 6 is generated by *Clebsch invariants* $I_2,I_4,I_6,I_{10},I_{15}$ (in Salmon's notation they are A,B,C,D,E) of degrees indicated in the subscript. These invariants satisfy the basic relation

$$I_{15}^2=P(I_2,I_4,I_6,I_{10}), \tag{5.3}$$

where P is a weighted homogenous polynomial of degrees 15 explicitly given by the expression

$$P=\det\begin{pmatrix}\frac{1}{2}(I_2A_4+18A_6) & 4(A_2^2+3I_2A_6) & 2A_{10}\\ 4(2A_4^3+3I_2A_6) & 2A_{10} & 288(A_4^3+2I_2A_4A_6+9A_4^2)\\ 2A_{10} & 288(A_4^3+2I_2A_4A_6+9A_4^2) & 72(A_4A_{10}+48A_4^2I_6+72I_2I_6^2)\end{pmatrix},$$

where

$$\begin{aligned}12A_2 &= I_2^2-36A_4,\\ 216A_6 &= 108I_2I_4+54I_6,\\ 3125A_{10} &= 9D-384I_2^5+12000I_2^2(I_2A_4+5A_6)-75000A_4(I_2A_4+6A_6).\end{aligned}$$

Here D is the *discriminant* of a binary form of degree 6 . We have

$$-\frac{1}{2\cdot 3^4}D=3\cdot 2^7I_2^5-3\cdot 2^4\cdot 5^3I_2^3I_4-2^4\cdot 5^4I_2^2I_6+150(I_2I_4^2+I_4I_6)+3^2\cdot 5^5I_{10}.$$

Remark 5.3. Note that, if one does not assume that the 6 points $p_1,\ldots,p_6$ are on a conic, the last two conditions define an irreducible component of the moduli space of marked cubic surfaces with an Eckardt point (see [28], 9.4.5).

Remark 5.4. Explicitly, suppose the characteristic equation of f_0 and N_0 is equal to $t^2-bt+(ac+ed)=0$. Suppose that $\Delta=b^2-4((ac+ed)=k^2$. The matrix N_0 in its action on Λ has two eigensublattices $\Lambda_\pm$ of Λ with eigenvalues $\alpha_\pm$. They are generated by

$$v_1^\pm=(d,0,-c,\alpha_\pm),\quad v_2^\pm=(0,d,b-\alpha_\pm,-a),$$

[1] Its explicit formula occupies 14 pages of Salmon's book [92], Appendix.

where the coordinates are taken with respect to the basis $(\gamma_1,\gamma_2,e_1,e_2)$ of $\Lambda=\tau\mathbb{Z}^2+\mathbb{Z}^2$. So, we can write

$$v_1^{\pm}=(dz_1-c,dz_2+\alpha_\pm),\ v_2^{\pm}=(dz_2+b-\alpha_\pm,dz_3-a).$$

The endomorphism f_0 represented by the matrix M_0 has the eigenvalues $\alpha_\pm$ with one-dimensional eigensubspaces $V_\pm$ generated by the vectors $w_\pm=v_1^\pm$, the vectors $v_1^\pm,v_2^\pm$ are proportional over $\mathbb{C}$ with the coefficient proportionality equal to

$$\tau_\pm=\frac{dz_2+\alpha_\pm}{dz_3-a}=\frac{dz_1-c}{dz_2+b-\alpha_\pm}.$$

Let

$$E_\pm=V_\pm/\Lambda_\pm\cong\mathbb{C}/\mathbb{Z}\tau_\pm+\mathbb{Z}.$$

The embedding $\Lambda_\pm\hookrightarrow\Lambda$ defines a homomorphism $E_\pm\to A$. Its kernel is equal to the torsion of the group $\Lambda/\Lambda_\pm$. We have

$$v_1^\pm\wedge v_2^\pm=(d^2,d(b-\alpha_\pm),-ad,cd,d\alpha_\pm,ed)$$

is equal to d times a vector with mutually coprime coordinates. More precisely,

$$av_1^\pm+\alpha_\pm v_2^\pm=(da,d\alpha_\pm,-ac+\alpha_\pm(b-\alpha_\pm),0)=d(a,\alpha_\pm,e,0)=dg_\pm.$$

This shows that the torsion is of degree d.

Let $\Lambda_\pm'=\Lambda_\pm+\mathbb{Z}g_\pm$. Then $E_\pm'=V_\pm/\Lambda_\pm'$ embeds in A. We have $E(v_1^\pm,g_\pm)=(b-2\alpha_\pm)=k$, where $k^2=\Delta$.

Then we have homomorphism of the complex tori:

$$E_+\times E_-=V_+\oplus V_-/\Lambda_+'\oplus\Lambda_-'\to A=V_+\oplus V_-/\Lambda.$$

Its kernel is a finite group $\Lambda/\Lambda_+'\oplus\Lambda_-'$ of order equal to the determinant of the 4×4-matrix with columns v_1^+,v_1^-,v_2^+,v_2^- divided by d^2. Computing the determinant, we find that it is equal to $d^2\Delta$.

Remark 5.5. We know from Example 3.4 that the Jacobian variety $\mathrm{J}(C)$ of a curve of genus 2 could be isomorphic to the product of two isogenous elliptic curves $E_1\times E_2$. Let k_1,k_2 be the degrees of the projections of $C\to E_i$. Fix an embedding $E_i\hookrightarrow E_1\times E_2$ and consider the corresponding norm-endomorphisms g_i. Then, we obtain that the period matrix of A satisfies two singular equations with discriminants k_1^2 and k_2^2. We have two isogenies

$$E_1\times E_1'\to E_1\times E_2,\quad E_2\times E_2'\to E_1\times E_2$$

of degrees k_1^2 and k_2^2.

Remark 5.6. (see [82]). Consider the abelian variety A defined by the period matrix

$$\tau=\begin{pmatrix}z_1&1/k\\1/k&z_3\end{pmatrix}\tag{5.4}$$

Let $p : \mathbb{C}^2 \to \mathbb{C}^2$ be the linear map $(a, b) \mapsto (0, kb)$. Then $p(\gamma_1) = e_2, p(\gamma_2) = k\gamma_2 - e_1, p(e_1) = 0, p(e_2) = ke_2$. Thus p defines an endomorphism of A with

$$f_a = \begin{pmatrix} 0 & 0 \\ 0 & k \end{pmatrix}, \; f_r = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & k & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & k \end{pmatrix}$$

We have $p(\Lambda) = \mathbb{Z}1 + \mathbb{Z}kz_3 = \mathbb{C}/\Lambda_1$ and $\mathrm{Ker}(p) \cap \Lambda = \mathbb{Z}(k\gamma_1 - e_2) + \mathbb{Z}e_1$. We see that the matrix is a special case of the matrix N_0 from (4.2). We get $a = c = d = 0, b = k, e = -1$. Thus τ satisfies the singular equation $kz_2 = 1$, of course, this was obvious from the beginning. The discriminant of this equation is equal to k^2. This shows that p defines a surjective homomorphism to the complex 1-torus $E = \mathbb{C}/\mathbb{Z} + \mathbb{Z}kz_3$ and its kernel is the complex torus $E' = \mathbb{C}/\mathbb{Z} + \mathbb{Z}kz_1 = \mathbb{C}/\Lambda_2$ embedded in A by the map $z \mapsto (z, 0)$ that sends 1 to e_1 and kz_1 to $k\gamma_1 - e_2$. We also can embed E' in A by the map $\mathbb{C} \to \mathbb{C}^2$ that sends 1 to e_2 and kz_3 to $k\gamma_2$. The determinant of the matrix of the map $\Lambda_1 \oplus \Lambda_2 \to \Lambda$ is equal to k^2, thus we have an isogeny $E \times E' \to A$ of degree k^2.

Example 5.7. Assume $k = 3$. Let $f : C \to E$ be a degree 3 map onto an elliptic curve E. Assume that $\mathrm{J}(C)$ contains only one pair of one-dimensional subgroups E, E' with $E \cdot E' = k^2$ and that E is not isomorphic to E'. Let σ be the hyperelliptic involution of C and $\phi : C \to C/(\sigma) = \mathbb{P}^1$ be the canonical degree 2 cover. By our assumption, the subfield of the field of rational functions on C contains a unique subfield isomorphic to the field of rational functions on E. This shows that σ leaves this field invariant and hence induces an involution $\bar{\sigma}$ on E such that we have a commutative diagram

$$\begin{CD} C @>\sigma>> C \\ @VfVV @VVfV \\ E @>\bar{\sigma}>> E \end{CD}\;.$$

First we assume that the map $f : C \to E$ ramifies at two distinct points. This is a general case, in a special case we may have one ramification point of index 3. Let x be one of the Weierstrass points, a fixed point of σ. We have $f(x) = f(\sigma(x)) = \bar{\sigma}(f(x))$. Thus, by taking $f(x)$ to be the origin of a group law on E, we may assume that $\bar{\sigma}$ is an order 2 automorphism of E. Obviously, it has four fixed points, the 2-torsion points on E. This shows that f defines a map of a set W of 6 Weierstrass points to the set $F = E^{\bar{\sigma}}$ of 4 fixed points $a_1, \ldots, a_4$ of $\bar{\sigma}$. If a is one of these fixed points and $f(x) = a$, then $f(\sigma(x)) = a$, hence σ preserves the fiber $f^{-1}(a)$ (considered as an effective divisor of degree 3 on C). Since σ is of order 2, it must fix one of the points or the whole fiber. The latter case happens if one of the points of the fiber is a ramification point of f. Thus the fibers of the map $W \to F$ have cardinalities $(3, 1, 1, 1)$ or $(2, 2, 1, 1)$. In the latter case, both ramification points of f are over four points from F. Let us consider the commutative diagram

$$\begin{CD} C @>\phi>> \mathbb{P}^1 \\ @VfVV @VV\bar{f}V \\ E @>\bar{\phi}>> \mathbb{P}^1 \end{CD}\;,$$

In the case $(2,2,1,1)$, the composition $\bar{\phi}\circ f : C \to \mathbb{P}^1$ has four branch points. On the other hand, the equal composition $f\circ\phi : C \to \mathbb{P}^1$ has at least 6 branch points because ϕ has 6 branch points. Thus the case $(2,2,1,1)$ is not realized. Let us consider the case $(3,1,1,1)$. We assume that $f^{-1}(a_1)$ consists of three points in W. Let $y_i = \bar{\phi}(a_i)$. The map $\bar{\phi}\circ f : C \to \mathbb{P}^1$ ramifies at the 3 preimage of each point $y_i \in \bar{\phi}(F)$ with index ramification equal to 2, and ramifies at 2 points over the image b in $\mathbb{P}^1$ of the two branch points of $C \to E$.

Using the commutative diagram, we see that the branch points of the map $\bar{f} : \mathbb{P}^1 \to \mathbb{P}^1$ are three points $y_2, y_3, y_4 \in \bar{\phi}(F)$. The fiber $\bar{f}^{-1}(y_i)$ contains one point from $\phi(W)$, the other point in the this fiber is a ramification point.

Now, we see that the set of Weierstrass points W splits into a disjoint set of triples of points $A+B$, where $f(A) = a \in F$ and $f(B) = F \setminus \{a\}$. We choose a group law on E to assume that $a_1 = \{0\}$. We know that Ker(J(C) $\to E$) = Ker(Nm : J(C) $\to E$). Since $\mathrm{Nm}(x+y+z) = 0$, we obtain that $\{x+y+z\}$ is contained in E'. The image $\phi(A)$ of A in $\mathbb{P}^1$ is a fiber of the map $\bar{f} : \mathbb{P}^1 \to \mathbb{P}^1$ over $y_1 = \bar{\phi}(0)$. The image of each point in B under ϕ is contained in a fiber over a point y_2, y_3, y_4 complementary to the ramification point over y_2, y_3, y_4.

Thus we come to the following problem. Let $C : y^2 - F_6(x) = 0$. The polynomial F_6 should be written as the product $\Phi_3\Psi_3$ of two cubic polynomials such that there exists a degree 3 map $\bar{f} : \mathbb{P}^1 \to \mathbb{P}^1$ such that the zeros of Φ_3 form one fiber, and each zero of Ψ_3 is of index of ramification equal to 1 and is contained in a fiber over a critical point of $\bar{f}$.

We follow the argument of E. Goursat [32] and H. Burhardt [14], in a nice exposition due to T. Shaska [99].

Let $F(u,v) = 0$ be the binary form of degree 6 defining the ramification points of $\phi : C \to \mathbb{P}^1$. We seek for a condition that $F(u,v) = \Phi(u,v)\Psi(u,v)$, where the cubic binary forms satisfy the following conditions.

Let $G(u,v)$ be a binary cubic and

$$J(u,v) = J(G,\Phi) = \det\begin{pmatrix} G'_u & G'_v \\ \Phi'_u & \Phi'_v \end{pmatrix}$$

be the Jacobian of G, Φ. Its zeros are the four ramification points of the map $\phi : \mathbb{P}^1 \to \mathbb{P}^1$ given by (G,Φ). Let

$$K = K(u,v;u'v') = \det\begin{pmatrix} G(u,v) & \Phi(u,v) \\ G(u',v') & \Phi(u',v') \end{pmatrix} /(uv' - u'v)$$

be the anti-symmetric bi -homogeneous form of bidegree $(2,2)$ on $\mathbb{C}^2\times\mathbb{C}^2$ expressing the condition that two points (u,v) and (u',v') are in the same fiber of ϕ. Its set of zeros $(u:v) = (u':v')$ consists of 4 ramification points of ϕ. In other words,

$$K(u,v;u',v') = J(G,\Phi).$$

Consider K as a polynomial in u', v' with coefficients in $\mathbb{C}[u,v]$. Let

$$R(u,v) = R(K(u,v;u',v'), J(u',v'))$$

be the resultant. Its vanishing expresses the condition that K and J have a common zero. It is a binary form of degree 7 in u, v. Obviously, $J(u,v)$ divides this polynomial. Let $R(u,v) = J(u,v)\Psi(u,v)$. Then the hyperelliptic curve $y^2 - \Phi(u,v)\Psi(u,v) = 0$[2] admits a map of degree 3 to E. The equation of E is $y^2 - \psi(x) = 0$, where the zeros of $\psi(x)$ are the images of the zeros of the zeros of the polynomials $R(u,v)$.

To see this, we consider the map $\bar{f} : \mathbb{P}^1 \to \mathbb{P}^1$ that ramifies over the zeros of $\Psi(u,v)$ and over the infinity point $infty$. Let $\bar{\sigma} : E \to \mathbb{P}^1$ be the double cover of E e ramified over ∞ and the zeroes of $\Psi(u,v)$. Then the normalization of the base change $E \times_{\mathbb{P}^1} \mathbb{P}^1 \to \mathbb{P}^1$ is a double cover of $\mathbb{P}^1$ branched over the zeroes of $\Phi(u,v)$ and the zeros of $\Psi(u,v)$. It must be isomorphic to oour hyperelliptic curve C

Using the projective transformations of (u,v) and a linear transformation of G, Φ, one may assume that $G(u,v) = u^2v$. We can also assume that $\Phi(u,v) = u^3 + au^2v + buv^2 + v^3$. Then we find that

$$F(u,v) = (u^3 + au^2v + buv^2 + v^3)(4u^3 + b^2 + 2bx + 1),$$

so that a, b are two parameters on which our hyperelliptic curves

$$y^2 - (u^3 + au^2v + buv^2 + v^3)(4u^3 + b^2 + 2bx + 1) = 0 \tag{5.5}$$

depend. The only condition on a, b is the condition

$$\Delta = (4a^3 + 27 - 18ab - a^3b^2 + 4b^3)^2(b^3 - 27) \neq 0,$$

where Δ is the discriminant of $F(u,v)$. From above, to find the equation of the elliptic curve

$$E : y^2 - \psi(x) = y^2 - (\alpha x^3 + \beta x^2 + \gamma x + \delta),$$

where

$$\psi(\Phi(u,v), u^2v) = \alpha\Phi(u,v)^3 + \beta\Phi(u,v)^2u^2v + \gamma\Phi(u,v)u^4v^2 + \delta u^6v^3,$$

and comparing the coefficients we find

$$\begin{aligned}
\alpha &= 4, \\
\beta &= b^2 - 12a, \\
\gamma &= 12a^2 - 2ab^2 - 18b, \\
\delta &= a^2b^2 - 4a^3 - 4b^3 + 18ab - 27 = \Delta/(b^3 - 27).
\end{aligned}$$

Next we assume that the map $f : C \to E$ ramifies at one point. In this case, the map $\bar{f} : \mathbb{P}^1 \to \mathbb{P}^1$ ramifies at three points with one of the ramification point q_1 of index 3. Following the previous analysis, we find that the unique ramification point of $f : C \to E$ is a Weierstrass point and, under the map $\phi : C \to \mathbb{P}^1$, it is mapped to the point q_1. Its image under $\bar{f} : \mathbb{P}^1 \to \mathbb{P}^1$ is one of the branch points of $\bar{\phi} : E \to \mathbb{P}^1$, say the infinity point ∞. Three of the remaining Weierstrass points are mapped to a fiber of $\bar{f}$ over some branch point of $\bar{\phi} : E \to \mathbb{P}^1$, say the point 0. fone of the ramification points of $\bar{\phi} : E \to \mathbb{P}^1$, say the point 0. The remaining two points are mapped to

[2] One views this equation as a curve in $\mathbb{P}(1,1,2)$.

non-ramification points of $\mathbb{P}^1 \to \mathbb{P}^1$ over two critical points, the remaining two branch points of $E \to \mathbb{P}^1$.

Let t be the coordinate function on $\mathbb{P}^1$ with the zero at 0 and the pole at ∞. The rational function $\bar{f}^*(t)$ vanishes at the images of three Weierstrass points defined by a binary cubic form $\Phi(u, v)$. By affine transformation of the coordinates u, v, we may assume that $\Phi(u, v) = u^3 + uv^2 + cv^3$. Continuing as in the previous case, we obtain that the equation of C is

$$y^2 = (u^3 + u + c)(3u^2 + 4), \tag{5.6}$$

where $c^2 \neq -\frac{4}{27}$. The elliptic curve E has the equation

$$y^2 = x(27x^2 - 54cx + 4 + 27c^2).$$

Let

$$y^2 = a_6u^6 + a_5u^5v + a_4u^4v^2 + a_3u^3v^3 + a_2u^2v^4 + a_1uv^5 + a_0v^6$$

be a curve of genus 2. Recall that the (even part) of the algebra of $\mathrm{SL}(2, \mathbb{C})$-invariants for binary forms of degree 6 is generated by polynomials J_2, J_4, J_6, J_{10} of degrees indicated in the subscript. The *absolute invariants* are $\mathrm{GL}(2, \mathbb{C})$-invariants

$$j_1 = 144\frac{J_4}{J_2^2},\ j_2 = -1728\frac{J_2J_4 - 3J_6}{J_2^3},\ j_3 = 486\frac{J_{10}}{J_2^5}. \tag{5.7}$$

The expressions for J_2, J_4, J_6 and J_{10} can be found in many classical sources as well as in [**?**]. If one puts

$$A = ab,\ B = b^3,$$

where a, b are parameters in (**??**), then, according to [99], the explicit expressions of j_1, j_2, j_3 in terms of A, B show that $\mathbb{C}(A, B)$ is a degree 2 extension of $\mathbb{C}(j_1, j_2, j_3)$. It is also a degree 2 extension of the field $\mathbb{C}(r_1, r_2)$, where r_1, r_2 are the invaria

Remark 5.8. A cover of an elliptic curve ramified at one point is a translation covering (see [**?**]).According to a theorem of E. Gutkin and C. Judge [**?**], the family of curves (**??**) is an example of a *Teichmüller curve* in the moduli space $\mathcal{M}_2$ of curves of genus 2. It is a totally geodesic curve with respect to the Teichmüller metric in $\mathcal{M}_2$.

One may ask to describe the set of all degree N covers $f : C \to E$ of a fixed elliptic curve E. To describe this set one introduces a functor (the *Hurwitz functor*) that assigns to a scheme T the family of normalized T-covers $f : \mathcal{C}/T \to (E \times T)/T$ such that, for each $t \in T$, the cover $\mathcal{C}_t \to E \times \{t\}$ is a normalized degree N cover of a genus 2 curve.[3] According to E. Kani [53] this functor is represented by an open subscheme of the modular curve $X(N)$ of level N.

Finally, we refer to [14] and [99] for an explicit invariant of binary sextics defining the locus $\mathrm{Hum}(9)$. In [98] one can find a treatment of the case $k = 5$.

Remark 5.9. A generalization of a problem of finding the conditions that a map $C \to E$ of degree k exists is the following problem.

[3] A cover is normalized if it is not a composition of a cover $C \to E$ and an isogeny $E \to E$.

A principally polarized abelian variety P is called a *Prym-Tyurin variety of exponent* e if there exists a curve C and an embedding of $P \hookrightarrow \mathrm{J}(C)$ such that the principal polarization of C induces a polarization of type $(e, \ldots, e)$ on P. Prym-Tyurin varieties of exponent 2 are the Prymians of covers $C \to D$ of degree 2 with at most 2 branch points. A generalization of the Prym constructions is a symmetric correspondence T on C such that $(T-1)(T+e-1) = 0$ in the ring of correspondences (see later in Lecture 16). The associated Prym variety of exponent e is the image of $T-1$.

For example, the existence of a degree k cover $C \to E$ gives a realization of E as a Prym-Tyurin variety of exponent k. So, the problem is the following. Fix a ppav P of dimension p and a positive number e. Find all curves C of fixed genus g such that $P \subset \mathrm{J}(C)$ and the principal polarization induces a polarization of type $(e, \ldots, e)$ on P.

For example, assume that $p = 2$ and $g = 3$. Then $\mathrm{J}(C)$ should be isogenous to the product $P \times E$, where E is an elliptic curve.

Let $X(k)$ be the compactification of $\mathbb{H}/\Gamma(k)$, where $\Gamma(k)$ is the principal congruence subgroup of $\mathrm{SL}_2(\mathbb{Z})$. Let $G_k = \mathrm{SL}_2(\mathbb{Z})/\Gamma(k)$ be the quotient group. For $\epsilon \in (\mathbb{Z}/k\mathbb{Z})^*$ denote by α_ϵ the automorphism of G_k induced by the conjugation with the matrix $\begin{pmatrix} \epsilon & 0 \\ 0 & 1 \end{pmatrix}$. It sends the matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_n(\mathbb{Z}/k\mathbb{Z})$ to the matrix $\begin{pmatrix} a & \epsilon b \\ \epsilon^{-1} c & d \end{pmatrix} \in \mathrm{SL}_n(\mathbb{Z}/k\mathbb{Z})$. We define the diagonal modular surface

$$Z(k;\epsilon) := X(k) \times X(k)/G_k,$$

where G_k acts by $g \cdot (x, y) = (g(x), \alpha_\epsilon(g)(y)$.

The following theorem was proved by E. Kani [52].

Theorem 5.10. *Let $\tilde{Z}(k;\epsilon)$ be a minimal desingularization of $Z(k;\epsilon)$. It is a regular surface with Kodaira dimension* $\min(2, p_{g,\epsilon})$*, where $p_{g,\epsilon}$ is the geometric genus of the surface. We have*

(a) $\tilde{Z}(k;\epsilon)$ is a rational surface if and only if $k \leq 5$, or

$$(k, \epsilon) = (6, 1), (7, 1), (8, 1).$$

(b) $\tilde{Z}(k;\epsilon)$ is birationally elliptic K3 if and only if

$$(k, \epsilon) = (6, 5), (7, 3), (8, 3), (8, 5), (9, 1), (12, 1).$$

(c) $\tilde{Z}(k;\epsilon)$ is of Kodaira dimension 1 with $p_g = 2$ if and only if

$$(k, \epsilon) = (8, 7), (9, 2), (10, 1), (10, 3), (9, 1), (11, 1).$$

(d) $\tilde{Z}(k;\epsilon)$ is of general type with $p_g \geq 3$ if and only if $k \geq 13$, or

$$(k, \epsilon) = (11, 2), (12, 5), (12, 7), (12, 11).$$

Let $\mathcal{M}_g^{\text{ell}}(k)$ be the moduli space of curves of genus g that admit a finite map of degree k onto an elliptic curve. If $g = 2$, then any such curve admits two maps onto an elliptic curve, hence $\mathcal{M}_2^{\text{ell}}(k)$ is a double cover of the Humbert surface $\text{Hum}(k^2)$.

The following nice observation is due to E. Kani.

Theorem 5.11. *$\mathcal{M}_2^{\text{ell}}(k)$ is an open subvariety of of $Z(k,-1)$. In particular, it is rational if and only if $k \leq 5$, K3 if and only if $k = 6, 7$, elliptic if and only if $k = 8, 9, 10$ and it is of general type otherwise.*

Proof. Recall that a principally polarized abelian surface A defines a point in $\text{Hum}(k^2)$ if and only if there exists a pair of elliptic curves (E, E') on A such that $E \times E' \to A$ is an isogeny of degree k^2. Let U be an open subset of $\text{Hum}(k^2)$ of abelian surfaces for which such a pair of curves is unique. Let U' be its pre-image under the natural map $\mathcal{M}_2^{\text{ell}}(k) \to \text{Hum}(k^2)$. The canonical inclusions $\phi : E \cap E' \hookrightarrow E$ and $\phi' : E \cap E' \hookrightarrow E'$ define an isomorphism $\phi^{-1} \circ \phi' : E'[k] \to E[k]$. One can show that this isomorphism is compatible with the Weil pairing on E' and the Weil pairing multiplied by -1 on $E[k]$. If we fix a full k-level structure on E', i.e. an isomorphism of the standard symplectic group $(\mathbb{Z}/k\mathbb{Z})$ to $E'[k]$, then the composition with $\phi' \circ \phi^{-1}$ defines a full k-level structure on E. This defines a point in $X(k) \times X(k)$. To get rid of the levels we have to divide $X(k) \times X(k)$ by $G_k(-1)$. □

Corollary 5.12. *The Humbert surfaces* $\text{Hum}(k^2)$ *are rational for* $k \leq 10$.

Proof. If $Z(k;-1)$ is rational, then the quotient is rational. Suppose $Z(k;\epsilon)$ is birationally K3 surface. The fixed locus of the rational cover involution $Z(k;-1) \to \text{Hum}(k^2)$ consists of $G_k(-1)$-orbits of pairs $((E,\alpha),(E,\alpha^{-1}))$, where ϕ is the full k-level. It is a curve R isomorphic to $X(k)$. Let $\tilde{Z}(k;-1)] \to X$ be a birational morphism of a resolution of $Z(k;-1)$ to a minimal K3 or elliptic surface. The image of R is a curve on a K3 surface that is fixed under the induced biregular involution induced by the switch involution. If X is a K3 surface, then this implies, as is well-known, that the quotient is a rational surface. If X is of Kodaira dimension 1, then the involution preserves the elliptic fibration, and, since $X(k)$ is never an elliptic curve, we obtain that it intersects the general fiber at 4 points. Since X is regular, the quotient is birationally isomorphic to a rational ruled surface. □

Remark 5.13. One should compare this with results about rationality of Humbert surfaces $\text{Hum}_n(\Delta)$, where D is square-free. For example, when $D = p \equiv 1 \mod 4$, it is known that the corresponding Hilbert modular surface is rational for $p = 5, 13, 17$, a K3 surface if $p = 29, 37, 41$ and an elliptic surface for $p = 53, 61, 73$ [43]. As before one proves that the quotient by $\mathfrak{S}_2$ is rational for all these primes.

Corollary 5.14. *Let* $\text{Hum}(k^2)'$ *be the closed subvariety of* $\text{Hum}(k^2)$ *parameterizing principally polarized abelian surfaces A for which there exists an isogeny $E \times E \to A$ of degree k^2. Then* $\text{Hum}(k^2)'$ *is a rational curve.*

Proof. It follows from the proof of the previous corollary that $\text{Hum}(k^2)'$ is isomorphic to the quotient $X(k)/G_k \cong \mathbb{P}^1$. □

Remark 5.15. A recent thesis of Robert Auffarth [1] gives some conditions in terms of the Néron-Severi group for the existence of an elliptic curve on an abelian variety of arbitrary dimension.

We will see more examples of Humbert surfaces with square discriminant in Lecture 10.

Lecture 6

$\triangle$ is not a square

Let us study the Humbert surface $\mathrm{Hum}(\Delta) := \mathrm{Hum}_1(\Delta)$, where Δ is not a square. We will see the speciality of abelian surfaces belonging to the Humbert surface $\mathrm{Hum}(\Delta)$ in terms of the associated Kummer surface.

For any abelian variety A, the quotient space by the cyclic group generated by the involution $\iota = [-1]_A$ is denoted by $\mathrm{Kum}(A)$ and is called the *Kummer variety* associated to A. The fixed points of the involution ι are 2-torsion points of A. In local coordinates $z_1, \ldots, z_g$ at such a point, the involution acts as $z_i \mapsto -z_i$. Thus the image of a 2-torsion point in $\mathrm{Kum}(A)$ is a singular point whose local ring is isomorphic to the local ring of the vertex of the affine cone over the second Veronese variety V_2^{g-1}, the image of $\mathbb{P}^{g-1}$ in $\mathbb{P}^{\frac{1}{2}g(g+1)-1}$ under the Veronese map given by quadratic forms in $z_1, \ldots, z_g$.

Let A be a principally polarized abelian surface and let $\mathrm{Kum}(A)$ be the associated Kummer surface. Let L be a principal polarization of A. The involution ι is a symmetric endomorphism corresponding to L^{-1}. Then ι^* acts on $H^1(A, \mathbb{Z})$ as the multiplication by -1, hence its acts on $H^2(A, \mathbb{Z})$ identically. This shows that $c_1(L) = c_1(\iota^*(L))$, hence $M = \iota^*(L) \otimes L$ satisfies $\iota^*(M) = M$ (such line bundles are called *symmetric*) and $c_1(M) = 2c_1(L)$, or, equivalently, M defines a polarization of type $(2,2)$ with $(M, M) = 4(L, L) = 8$. By Riemann-Roch, $\dim H^0(A, M) = 4$, and the linear system $|M|$ defines a regular map $f : A \to \mathbb{P}^3$ that factors through a degree 2 quotient map

$$\phi : A \to \mathrm{Kum}(A)$$

and a map $\psi : \mathrm{Kum}(A) \to X \subset \mathbb{P}^3$. If the polarization is irreducible, ψ is an isomorphism onto a quartic surface X. Otherwise, the map ψ is a degree 2 map onto a nonsingular quadric Q, with the branch divisor equal to the union of 8, four from one ruling. Assume that the polarization L is irreducible. It follows from above that X has 16 singular points which are locally isomorphic to the singular point of a quadratic cone in $\mathbb{C}^3$, i.e. an ordinary double point. Then $A \cong \mathrm{J}(C)$ for some smooth genus 2 curve $C \subset A$ and A can be identified with the subgroup $\mathrm{Pic}^0(C)$ of divisor classes of degree 0. By translating C by a point in A, we may assume that C is the divisor of zeros of a section of L. For any 2-torsion point $e \in A$, let C_e denote the translation of C by the point e. We have $2(C_e) \in |L^{\otimes 2}|$. Let us identify $\mathrm{Kum}(A)$ with the quartic surface X and let T_e be the image $f(C_e)$ in X. Then $f^{-1}(2T_e) = 2(C_e)$, hence $2T_e$ is equal to $X \cap H_e$ for some plane H_e in $\mathbb{P}^3$.

Since plane sections of X are plane curves of degree 4, we see that T_e must be a conic. The plane H_e (or the conic C_e) is called a *trope*.

Note that the map $C_e \to T_e$ is given by the linear system $|L^{\otimes 2}|C_e|$ of degree 2 on $C_e \cong C$. It defines a degree 2 map $C_e \to T_e$, so T_e is a smooth conic. Thus we have 16 nodes $p_e \in X$ and 16 tropes T_e. The 6 ramification points of the map $C_e \to T_e$ are fixed points of ι. Hence, they are 2-torsion points lying on C_e. Thus each trope passes through 6 nodes. It is clear that the number of tropes containing a given node does not depend on the node (use that nodes differ by translation automorphism of A descent to X). By looking at the incidence relation $\{(C_e, e') : e' \in C_e\}$, we obtain that each node is contained in 6 tropes. Thus we get a combinatorial configuration (16_6) expressing the incidence relation between two finite sets. This is the famous *Kummer configuration*.

To obtain a minimal resolution of $\mathrm{Kum}(A)$, we lift the involution ι to an involution $\tilde{\iota}$ of the blow-up $\tilde{A} \to A$ of the set $A[2]$. The quotient $\tilde{X} = \tilde{A}/(\tilde{\iota})$ has the projection to $A/(\iota) = \mathrm{Kum}(A)$ which is a minimal resolution of the 16 nodes of $\mathrm{Kum}(X)$.

$$\begin{array}{ccc} \tilde{A} & \xrightarrow{\tilde{\phi}} & \tilde{X} \\ \downarrow{\scriptstyle \tilde{\sigma}} & & \downarrow{\scriptstyle \sigma} \\ A & \xrightarrow{\phi} & X \end{array}$$

Since ι acts as -1 on the tangent space $T_0(A)$, it acts identically on the exceptional curves R_i' of $\tilde{\sigma}$. Thus the quotient $\tilde{A}/\tilde{\iota}$ is nonsingular and the projection $\tilde{p}$ is a degree 2 cover of nonsingular surfaces ramified over 16 curves R_i' isomorphic to $\mathbb{P}^1$. Using the known behaviour of the canonical class under a blow-up, we obtain $K_{\tilde{A}} = \sum R_i'$. The Hurwitz formula $K_{\tilde{A}} = \tilde{p}^*(K_{\tilde{X}}) + \sum R_i'$ implies that $K_{\tilde{X}} = 0$. Since $\tilde{\iota}$ acts on $H^1(\tilde{A}, \mathbb{Q})$ as -1, we obtain that $H^1(\tilde{X}, \mathbb{Q}) \subset H^1(\tilde{A}, \mathbb{Q})^{\tilde{p}} = \{0\}$ must be trivial. Thus $b_1(\tilde{X}) = 0$, and we obtain that $\tilde{X}$ is a K3 usrace (see more about K3 surfaces in Lecture 9).

Let p be one of the 16 nodes of X. Projecting from this point, we get a morphism $X \setminus \{p\} \to \mathbb{P}^2$ of degree 2. Let us choose coordinates in $\mathbb{P}^3$ such that $p = [1, 0, 0, 0]$. Then the equation of X can be written in the form

$$t_0^2 F_2(t_1, t_2, t_3) + 2t_0 F_3(t_1, t_2, t_3) + F_4(t_1, t_2, t_3) = 0, \tag{6.1}$$

where $F_k(t_1, t_2, t_3)$ is a homogeneous form of degree indicated by the subscript. It is clear that the pre-image of a point $[x_1, x_2, x_3]$ on the plane consists of two points which coincide when

$$F = F_3(t_1, t_2, t_3)^2 - F_2(t_1, t_2, t_3) F_4(t_1, t_2, t_3) = 0.$$

We see that X is birationally isomorphic to the double cover of $\mathbb{P}^2$ with branch curve $B : F = 0$ of degree 6. Note that the conic $F_2 = 0$ is the image of the tangent cone at p and it is tangent to B at all its intersection points with it. Of course, this is true for any irreducible quartic surface with a node p. In our case we get more information about the branch curve B. Let $C_1, \ldots, C_6$ be the six tropes containing p. Then any line in the plane T_i spanned by C_i intersects the surface at one points besides p. This implies that the projection of C_i, which is a line ℓ_i in the plane, must be contained in B. Thus, we obtain that B is the union of 6 lines $\ell_1, \ldots, \ell_6$. Obviously, they intersect at $15 = \binom{6}{2}$

points, the images of the remaining 15 nodes on X. So, we obtain that X is birationally isomorphic to a surface in $\mathbb{P}(3,1,1,1)$ given by the equation

$$x_0^2 = l_1 \cdots l_6,$$

where $l_1,\ldots,l_6$ are linear forms in variables x_1,x_2,x_3. The corresponding lines $\ell_1,\ldots,\ell_6$ are in general linear position. However, they are not general 6 lines in the plane since they satisfy an additional condition that there exists a smooth conic K that touches each line.

Conversely, one can show that equation (6.1) defines a surface birationally isomorphic to the Kummer surface corresponding to the hyperelliptic curve of genus 2 isomorphic to the double cover of K branched at the tangency points. One uses that the pre-image of K under the cover splits into the sum of two smooth rational curves K_1+K_2 intersecting at 6 points. Let h be the pre-image of a general line in the plane. Then $h\cdot K_1 = h\cdot K_2 = 2$ and $(h+K_1)^2 = 2+4-2 = 4$. The linear system $|h+K_1|$ maps the double plane to a quartic surface in $\mathbb{P}^3$ with 16 nodes, fifteen of them are the images of the intersection points of the lines, and the sixteenth is the image of K_2.

In the following we will follow the paper of C. Birkenhake and H. Wilhelm [10]. Applying Lemma 4.1, we may assume that $b=0,1$ and $\Delta = b+4m$. Recall from (4.8) that $A\in \mathrm{Hum}(\Delta)$ contains a line bundle L_Δ such that

$$(L_\Delta^2) = \tfrac{1}{2}(b^2-\Delta) = -2m, \quad (L_0, L_\Delta) = b.$$

Suppose

$$\Delta = 8d^2+9-2k,$$

where $k\in\{4,6,8,10,12\}$ and $d\geq 1$. We have $(L_\Delta^2) = -(4d^2+4-k)$. Let $L = L_0^{\otimes d}\otimes L_\Delta$. We easily compute

$$(L^2) = 4d(d+1)+k-4, \quad (L,L_0) = 4d+1.$$

Using formula (4.11), we find that the type of the polarization defined by L is equal to $(1, 2d(d+1)+\frac{k}{2}-2)$. After tensoring L with some line bundle from $\mathrm{Pic}^0(A)$, we may assume that L is symmetric, i.e. $[-1]^*(L) = L$.[1] For any symmetric line bundle L defining a polarization of type (d_1,d_2), $[-1]_A$ acts on $H^0(L)$ decomposing it into the direct sum of linear subspaces $H^0(L)^\pm$ of eigensubspaces of dimensions $\frac{1}{4}((L^2)-\#X_2^\mp(L))+2$, where

$$X_2^\pm(L) = \{x\in A[2] : [-1]_A|L(x) = \pm 1\}.$$

It is known that

$$X_2^+(L) \in \begin{cases} \{8,16\} & \text{if } d_1 \text{is even},\\ \{4,8,12\} & \text{if } d_1 \text{is odd and } d_2 \text{is even},\\ \{6,10\} & \text{if } d_2 \text{ is odd.}\end{cases}$$

(see [67], 4.7.7 and 4.14). Since in our case $d_1 = 1$, we can choose L such that $k = \#X_2(L)^+$ and $\dim H^0(L)^- = d(d+1)+1$. By counting constants, we can choose a divisor $D\in|L|$ such that

[1]We use that $[-1]_A$ acts as $[-1]$ on $\mathrm{Pic}^0(A)$, since $M = [-1]^*(L)\otimes L^{\otimes -1}\in \mathrm{Pic}^0(A)$, we write $M = N^{\otimes 2}$ and check that $[-1]^*(L\otimes N)\cong L\otimes N$.

$\mathrm{mult}_0 D \geq 2d+1$ (the number of conditions is $d(d+1)$). The geometric genus $g(D)$ of D is equal to $1 + \frac{1}{2}D^2 - d(2d+1) = d + \frac{k-2}{2}$. Let

$$\phi : A \to \mathrm{Kum}(A) = A/([-1]_A) \subset \mathbb{P}^3$$

be the map from A to the Kummer surface given by the linear system $|L_0^{\otimes 2}|$. It extends to a map $\tilde{A} \to X$ from the blow-up of 16 2-torsion points of A to a minimal nonsingular model of $\mathrm{Kum}(A)$. The divisor D is invariant with respect to the involution $[-1]_A$. The normalization $\bar{D}$ of D is mapped $(2:1)$ onto the normalization $\bar{C}$ of $C = \phi(D)$ and ramifies at $k-1$ points and some point in the pre-image of 0. The Hurwitz formula applied to the map $\bar{D} \to \bar{C}$ gives

$$g(\bar{D}) = d + \frac{k-2}{2} = -1 + 2g(\bar{C}) + \frac{k-1+r}{2}, \tag{6.2}$$

where r is the number of ramification points over 0 (one can show that C is smooth outside $\phi(0)$, see [10], Proposition 6.3). We may obtain $\bar{D}$ by blowing up 0 and taking the proper inverse transform of D. The preimage of 0 consists of $2d+1$ points that are fixed under the involution $[-1]_A$ extended to $\tilde{A}$. This shows that $r = 2d+1$ and (6.2) gives $g(\bar{C}) = 0$. Thus C is a rational curve and the proper transform of $\phi(C)$ in the blow-up of $\phi(0)$ intersects the exceptional curve with multiplicity $2d+1$. Since $(L_0, L) = 4d+1$, the image C' of C under the proejction $\pi : X \dashrightarrow \mathbb{P}^2$ from $\phi(0)$ is a plane curve of degree $4d+1-(2d+1) = 2d$ that passes through $k-1$ intersection points $\ell_i \cap \ell_j$. Also note that, if C intersects one of the six tropes T_i corresponding to the lines ℓ_i at a point q with multiplicity m, then C' intersect ℓ_i at $\bar{q} = \pi(q)$ with multiplicity $2m$. This follows from the projection formula $(\pi(C), \ell_i)_{\bar{q}} = (C, \pi^*(\ell_i))_q = 2(C, T_i)_q$.

So, we obtain the following theorem.[2]

Theorem 6.1. *Suppose $\Delta = 8d^2 + 9 - 2k$, where $d \geq 1$ and $k \in \{4, 6, 8, 10, 12\}$. If (A, L_0) is an abelian surface with an irreducible principal polarization L_0 belonging to* $\mathrm{Hum}(\Delta)$*, then the double plane model of* $\mathrm{Kum}(A)$ *defined by 6 lines $\ell_1, \ldots, \ell_6$ has the property that there exists a rational curve C of degree $2d$ with nonsingular points at $k-1$ intersection points $\ell_i \cap \ell_j$ and intersecting the lines at the remaining intersection points with even multiplicity.*

Similarly, Birkenhake and Wilhelm prove the following.

Theorem 6.2. *Suppose $\Delta = 8d(d+1) + 9 - 2k$, where $d \geq 1$ and $k \in \{4, 6, 8, 10, 12\}$. If (A, L_0) is an abelian surface with an irreducible principal polarization L_0 belonging to* $\mathrm{Hum}(\Delta)$*, then the double plane model of* $\mathrm{Kum}(A)$ *defined by 6 lines $\ell_1, \ldots, \ell_6$ has the property that there exists a rational curve C of degree $2d+1$ with nonsingular points at k intersection points $\ell_i \cap \ell_j$ and intersecting the lines at the remaining intersection points with even multiplicity.*

The following is the special case considered by G. Humbert.

Example 6.3. Take $\Delta = 5, d = 1, k = 6$. Then C is a conic passing through 5 intersection points $p_i = \ell_i \cap \ell_{i+1}, i = 1, \ldots, 4$ and $p_5 = \ell_1 \cap \ell_5$ forming the set of 5 vertices of a 5-sided polygon Π with sides $\ell_1, \ldots, \ell_5$ and touching the sixth line ℓ_6.

[2] We omitted some details justifying, for example, why C can be chosen irreducible or why its singular point at 0 is an ordinary point of multiplicity $2d+1$.

Together with the conic K touching all 6 lines, the pentagon is the *Poncelet pentagon* for the pair of conics K, C (i.e. K is inscribed in Π and C is circumscribed around Π).

It is easy to see that an abelian surface with real multiplication by $\mathbb{Q}(\sqrt{5})$ admits a principal polarization. A general such surface is the Jacobian of a curve C of genus 2. We may assume that its period τ satisfies a singular equation with $b = 1$. It follows from (4.7) that A admits a divisor class D with $D^2 = -2$ and $C \cdot D = 1$. Let $C' = C + D$ so that $C'^2 = 2$ and $C \cdot C' = 3$. The linear system $|C + C'|$ defines a map $A \to \mathbb{P}^4$ onto a surface of degree 10. An abelian surface of degree 10 in $\mathbb{P}^4$ was first studied by A. Comessatti [22]. We refer to [66] for a modern account of Comessatti's paper. There is a huge literature devoted to these surfaces, for example, exploring the relationship between such surfaces and the geometry of the *Horrocks-Mumford rank 2 vector bundle* over $\mathbb{P}^4$ whose sections vanish on Comessatti surfaces (see [46]).

Example 6.4. Take $\Delta = 13, d = 1, k = 6$. The only possibility is the following. Let $p_1 = \ell_1 \cap \ell_2, p_2 = \ell_2 \cap \ell_3, p_3 = \ell_1 \cap \ell_3$. Take $p_4 = \ell_1 \cap \ell_4, p_5 = \ell_2 \cap \ell_5, p_6 = \ell_3 \cap \ell_6$. Then there must be a plane rational cubic passing through $p_1, \ldots, p_6$ and touching ℓ_4, ℓ_5, ℓ_6.

These two theorems deals with the case when $\Delta \equiv 1 \mod 4$ (although they do not cover all possible Δ's. The next theorem treats the cases with $\Delta \equiv 0 \mod 4$

Theorem 6.5. *Suppose* $\Delta = 8d^2 + 8 - 2k$ *(resp.* $8d(d+1) + 8 - 2k$*, where* $d \geq 1$ *and* $k \in \{4, 6, 8, 10, 12\}$*. If* (A, L_0) *is an abelian surface with an irreducible principal polarization* L_0 *belonging to* $\mathrm{Hum}(\Delta)$*, then the double plane model of* $\mathrm{Kum}(A)$ *defined by 6 lines* $\ell_1, \ldots, \ell_6$ *has the property that there exists a rational curve* C *of degree* $2d$ *(resp.* $2d+1$*) with nonsingular points at* k *(resp.* $k-1$*) intersection points* $\ell_i \cap \ell_j$ *and intersecting the lines at the remaining intersection points with even multiplicity.*

Remark 6.6. It follows from the Teichmüller theory that any holomorphic differential on a Riemann surface X of genus g defines an immersion of $\mathbb{H}$ in $\mathcal{M}_g$ such the image is a complex geodesic with respect to the Techmüller metric. According to C. McMullen [72], the closure of the image of $\mathbb{H}$ in $\mathcal{M}_2$ is either a curve, or a Humbert surface $\mathrm{Hum}(\Delta)$, where Δ is not a square, or the whole $\mathcal{M}_2$.

Lecture 7

Fake elliptic curves

We will discuss abelian surfaces with the endomorphism ring of the third type, i.e. imaginary quadratic extensions of a real quadratic field later. They are examples of abelian varieties of CM-type. In this lecture we will consider *fake abelian surfaces*, i.e. abelian surfaces with the ring $\mathrm{End}(A)_{\mathbb{Q}}$ isomorphic to an order in an indefinite quaternion algebra.

For the following properties of quaternion algebras we refer to [108] or [109]. Let $H = \left(\frac{a,b}{\mathbb{Q}}\right)$ be a quaternion algebra over $\mathbb{Q}$. An *order* in H is a subring $\mathfrak{o}$ of $\mathbb{H}$ containing $\mathbb{Z}$ whose elements have integral trace and norm, and $\mathfrak{o} \otimes \mathbb{Q} \cong H$. An order is maximal if it is not contained in a strictly larger order. Considered as a $\mathbb{Z}$-module, an order has a basis u_1, u_2, u_3, u_4. The determinant of the matrix $(\mathrm{tr}(u_i u_j'))$ generates an ideal which is a square of an ideal generated by a positive integer which is called the *discriminant* of $\mathfrak{o}$ and is denoted by $D(\mathfrak{o})$.

For any prime p or the real point ∞ of $\mathbb{Q}$, the algebra $H_p = H \otimes \mathbb{Q}_p$ ($H_\infty = H_{\mathbb{R}}$) is either isomorphic to the matrix algebra $\mathrm{Mat}_4(\mathbb{Q}_p)$ or to a unique (up to isomorphism) division algebra over H_p. We say that a prime number p *ramifies* or H *splits* over p in H if H_p is a division algebra. If $p \neq 2$, the quaternion division algebra over $\mathbb{Q}_p$ is isomorphic to the algebra $\left(\frac{e,p}{\mathbb{Q}_p}\right)$, where e is any element in $\mathbb{Z}_p$ that does not reduce to a square $\mod p$. If $p = 2$, the quaternion division algebra over $\mathbb{Q}_2$ is isomorphic to the algebra $\left(\frac{-1,-1}{\mathbb{Q}_2}\right)$. It is known that any extension L that splits H ramifies at the set of primes over which H ramifies.

The discriminant of a maximal order of H is equal to the product of primes over which H ramifies. In fact, this property characterizes maximal orders. It is equal to the discriminant of the algebra and it determines the algebra uniquely up to isomorphism. The discriminant of any order is equal to the product of powers of primes over which the algebra ramifies.

For example, the discriminant of the order $\mathfrak{o} = \mathbb{Z} + \mathbb{Z}\mathbf{i} + \mathbb{Z}\mathbf{j} + \mathbb{Z}\mathbf{k}$ in $\left(\frac{-1,-1}{\mathbb{Q}}\right)$ is equal 4. It is contained in a maximal order generated by $\mathfrak{o}$ and $q = \frac{1}{2}(1 + \mathbf{i} + \mathbf{j} + \mathbf{k})$. Its discriminant is equal to 2.

Let us identify $H_{\mathbb{R}}$ with $\mathrm{Mat}_2(\mathbb{R})$ and consider a linear $\mathbb{R}$-isomorphism

$$\phi : H_{\mathbb{R}} \to \mathbb{C}^2, \quad X \mapsto X \cdot z,$$

where $z \in \mathbb{P}^1 \setminus \mathbb{P}^1(\mathbb{R})$. Let $\Lambda_z = \phi(\mathfrak{o})$. The complex torus $A_z = \mathbb{C}^2/\Lambda_z$ is an abelian variety. To define a polarization, we would like to use the symmetric form $(x, y) \mapsto \mathrm{tr}(xy')$ on H. However, it is not positive definite. Let us can change it as follows. Since H is totally indefinite, one of the numbers $a, b, -ab$ must be positive. Permuting $\mathbf{i}, \mathbf{j}, \mathbf{k}$, we may assume that $a, b > 0$, hence $\mathbf{k}^2 = -ab < 0$. Define $x^* = \mathbf{k}^{-1}\bar{x}\mathbf{k}$. If $x = \alpha + \beta\mathbf{i} + \gamma\mathbf{j} + \delta\mathbf{k}$, then

$$x^* = \mathbf{k}^{-1}(\alpha - \beta\mathbf{i} - \gamma\mathbf{j} - \delta\mathbf{k})\mathbf{k} = \mathbf{k}^{-1}(\alpha + \beta\mathbf{i} + \gamma\mathbf{j} + \delta\mathbf{k}) = \alpha + \beta\mathbf{i} + \gamma\mathbf{j} - \delta\mathbf{k}.$$

The map $x \to x^*$ is an anti-involution on H. For any $x \neq 0$,

$$\mathrm{tr}(xx^*) = \alpha^2 + a\beta^2 + b\gamma^2 + ab\delta^2 > 0.$$

This defines a positive definite symmetric form on $\Lambda_z \otimes \mathbb{R}$. Since $\bar{\mathbf{k}} = -\mathbf{k}$, we have

$$E(x, y) := \mathrm{tr}(\mathbf{k}xy^*) = \mathrm{tr}(-yx^*\mathbf{k}) = -\mathrm{tr}(\mathbf{k}yx^*) = -E(y, x), \tag{7.1}$$

hence E is a skew-symmetric form on $\Lambda_z \otimes \mathbb{R}$. Obviously it takes integral values on the lattice Λ_z. This defines a polarization on the torus $\mathbb{C}^2/\Lambda_z$. In fact, formula (7.1) defines a skew-symmetric form if we replace $\mathbf{k}$ with any $h \in \mathfrak{o}$ satisfying

$$\mathrm{tr}(h) = 0, \quad \mathrm{tr}(hx) \in \mathbb{Z} \text{ for all } x \in \mathfrak{o}. \tag{7.2}$$

The corresponding symmetric form $\mathrm{tr}(xx^*)$ is positive definite if $h^2 < 0$.

Note that $A_z \cong A_{z'}$ if and only if there exists a unit u from $\mathfrak{o}$ such that $\phi(u)(z) = z'$. We can find u with $\mathrm{Nm}(u) = -1$ such that $\mathrm{Im}(z') > 0$, and then obtain that z is defined uniquely up to the action of the group $\Gamma = \phi(\mathfrak{o}_1^*)/\{\pm\} \subset \mathrm{PSL}_2(\mathbb{R})$, where $\mathfrak{o}_1^*$ is the group of elements in $\mathfrak{o}$ with $\mathrm{Nm}(u) = 1$. The group Γ is a discrete subgroup of $\mathrm{PSL}_2(\mathbb{R})$, a *Fuchsian group of the first kind* (a discrete subgroup Γ of $\mathrm{PSL}_2(\mathbb{R})$ such that the quotient $\mathbb{H}/\Gamma$ is isomorphic to the complement of finitely many points on a compact Riemann surface). It is known that Γ is a cocompact, i.e. the quotient $\mathbb{H}/\Gamma$ is a compact Riemann surface. It is also an arithmetic group.[1] Such quotients are called the *Shimura curves*. Conversely, any point on the curve $\mathbb{H}/\Gamma$ defines a polarized abelian surface with endomorphism algebra containing $\mathfrak{o}$ for some order in a H. The curve $\mathbb{H}/\Gamma$ is the coarse moduli space of such abelian surfaces.

Let us give an example of a fake elliptic curve from [9], [38], . Let H be an indefinite quaternion algebra over $\mathbb{Q}$ and $\mathfrak{o}_B$ be the maximal order in H. By definition, $H_{\mathbb{R}} \cong \mathrm{Mat}_2(\mathbb{R})$. Let $x \mapsto x^*$ be the involution in H induced by the transpose involution of $\mathrm{Mat}_2(\mathbb{R})$. The trace bilinear form $\mathrm{Tr}(xy^*)$ restricted to the symmetric part $H^s = \{x \in H : x = x^*\}$ of H defines a structure of a positive definite lattice on $\mathfrak{o}_H^s := H^s \cap \mathfrak{o}_H$ of rank 3. The discriminant of H is equal to the discriminant of the lattice $\mathfrak{o}_H^s$.[2]

Let us choose $H = \left(\frac{-6,2}{\mathbb{Q}}\right)$. The maximal order $\mathfrak{o}_H$ has a basis

$$(\alpha_1, \ldots, \alpha_4) = (1, \tfrac{1}{2}(\mathbf{i} + \mathbf{j}), \tfrac{1}{2}(\mathbf{i} - \mathbf{j}), \frac{1}{4}(2 + 2\mathbf{j} + \mathbf{k})).$$

[1] This means that its preimage in $\mathrm{SL}_2(\mathbb{R})$ contains a subgroup of finite index whose elements are matrices with entries in an algebraic number field.

[2] They are also called abelian surfaces with *quaternionic multiplication*, or *QM-surfaces*, for short.

Note that $\mathbf{i}, \mathbf{j}, \mathbf{k}/2 = (\mathbf{i}-\mathbf{j})(\mathbf{i}+\mathbf{j})/4 - 1 \in \mathfrak{o}_H$. The discriminant is equal to the determinant of the matrix $(\mathrm{Tr}(\alpha_i\bar{\alpha}_j))$, it is equal to -6. The embedding of $\mathbb{H}_{\mathbb{R}}$ in $\mathrm{Mat}_2(\mathbb{R})$ is given by

$$\mathbf{i} \mapsto \begin{pmatrix} 0 & -1 \\ 6 & 0 \end{pmatrix}, \quad \mathbf{j} \mapsto \begin{pmatrix} \sqrt{2} & 0 \\ 0 & -\sqrt{2} \end{pmatrix}.$$

We consider the isomorphism $\phi_z : H_{\mathbb{R}} \to \mathbb{C}^2$ given by $X \mapsto X \cdot \left(\begin{smallmatrix} z \\ 1 \end{smallmatrix}\right)$, where $z \in \mathbb{C}$ and consider the abelian surface A_z. Let $\omega_i = \phi_z(\alpha_j) \in \mathbb{C}^2$. One computes the matrix of the alternating form E_z in this basis to obtain that it is equal to

$$\begin{pmatrix} 0 & -1 & -1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}.$$

If we put $\omega_1' = -\omega_3, \omega_2' = \omega_4, \omega_3' = -\omega_1, \omega_4' = \omega_3 - \omega_2$, we obtain a standard symplectic basis defined by the matix J. We easily compute the period matrix

$$\tau_z = \begin{pmatrix} z_1 & z_2 \\ z_2 & z_3 \end{pmatrix} = \begin{pmatrix} \frac{3z}{2} - \frac{1}{4z} & -\frac{3\sqrt{2}z}{4} - \frac{1}{2} - \frac{\sqrt{2}}{8z} \\ -\frac{3\sqrt{2}z}{4} - \frac{1}{2} - \frac{\sqrt{2}}{8z} & \frac{3z}{4} - \frac{1}{2} - \frac{1}{8z} \end{pmatrix}.$$

One finds that the period matrix τ_z satisfies the following 2-parametrical family of singular equations:

$$-(\lambda+\mu)z_1 + \lambda z_2 + (\lambda + 2\mu)z_3 + \lambda(z_2^2 - z_1z_3) + \mu = 0.$$

Its discriminant is equal to

$$\Delta = \lambda^2 + 4(\lambda+\mu)(\lambda+2\mu) - 4\lambda\mu = 5\lambda^2 + 8\mu(\lambda+\mu).$$

Taking $(\lambda, \mu) = (1, 0)$ and $(0, 1)$, we obtain that the image of τ lies in the intersection of two Humbert surfaces $\mathrm{Hum}(5)$ and $\mathrm{Hum}(8)$ which we discussed in the previous lecture. It will turn out that the family of genus 2 curves whose endomorphism rings contains H is given by the following formula.

$$y^2 = x(x^4 - px^3 + qx^2 - rx + 1),$$

where

$$p = -2(s+t), r = -2(s-t), q = \frac{(1+2t^2)(11-28t^2+8t^4)}{3(1-t^2)(1-4t^2)},$$

and $g(s,t) = 4s^2t^2 - s^2 + t^2 + 2 = 0$.

The base is the elliptic curve given by the affine equation $g(s,t) = 0$. The Shimura curve is of genus 0, the quotient of the base by the subgroup generated by the involutions $(t,s) \mapsto (-t, \pm s), (x,y) \mapsto (-x, iy), (x^{-1}, yx^{-3})$.

Some other examples can be found in [6].

Lecture 8

Periods of K3 surfaces

A K3 surface is a complex projective algebraic surface X with $K_X = 0$ and $b_1(X) = 0$. The *Noether formula*

$$12\chi(X, \mathcal{O}_X) = K_X^2 + c_2,$$

where $\chi(X, \mathcal{O}_X) = 1 - q(X) + p_g(X) := 1 - \dim H^0(X, \Omega_X^1) + \dim H^0(X, \Omega_X^2)$ and c_2 is the second Chern class of X equal to the Euler-Poincaré characteristic of X, gives us that $c_2(X) = 24$ and $b_2(X) = 22$. The cohomology $H^2(X, \mathbb{Z}) \cong \mathbb{Z}^{22}$ and the Poincaré duality equips it with a structure of a unimodular indefinite quadratic lattice.[1] Its signature is equal to $(3, 19)$. The lattice $H^2(X, \mathbb{Z})$ is an even unimodular lattice, and as such, by a theorem of J. Milnor, must be unique, up to isomorphism. We can choose a representative of the isomorphism class to be the lattice

$$L_{K3} := U \oplus U \oplus U \oplus E_8(-1) \oplus E_8(-1).$$

(sometimes referred to as the *K3-lattice*). Here the direct sum is the orthogonal direct sum, U is an integral hyperbolic plane that has a basis (f, g) with $f^2 = g^2 = 0, f \cdot g = 1$ (called a *canonical basis*) and E_8 is the negative definite unimodular lattice of rank 8 (the lattice E_8 that we discussed in Lecture 3 with the quadratic form multiplied by -1).

The first Chern class map $c_1 : \text{Pic}(X) \to H^2(X, \mathbb{Z})$ is injective, and its image is a sublattice S_X of $H^2(X, \mathbb{Z})$ which is, by Hodge Index Theorem is of signature $(1, \rho)$, where $\text{Pic}(X) \cong \mathbb{Z}^\rho$. Note that the Poincaré duality allows us to identify $H^2(X, \mathbb{Z})$ with $H_2(X, \mathbb{Z})$. Applying this to S_X, gives the identification between cohomology classes defined by line bundles via the first Chern class and divisor classes defined by their meromorphic sections. So we will identify S_X with the subgroup of algebraic cycles $H_2(X, \mathbb{Z})_{\text{alg}}$ of $H_2(X, \mathbb{Z})$.

Let $T_X = (S_X)^\perp$ be the *transcendental lattice*. We have the *Hodge decomposition*

$$H^2(X, \mathbb{C}) = H^{2,0} \oplus H^{1,1} \oplus H^{0,2} \cong \mathbb{C} \oplus \mathbb{C}^{20} \oplus \mathbb{C},$$

[1] The assumption that $b_1(X) = 0$ implies that the group $H_1(X, \mathbb{Z})$ is finite. Any its nonzero element defines a finite unramified cover $f : X' \to X$ of some degree $d > 1$ with $K_{X'} = f^*(K_X) = 0$, hence $p_g(X') = 1$ and $c_2(X') = dc_2(X) = 24d$ giving a contradiction to the Noether formula. This shows that $H_1^2(X, \mathbb{Z})$ and hence, by universal coefficient formula, $H^2(X, \mathbb{Z})$ have no torsion. A much more non-trivial fact is that $\pi_1(X) = 0$.

and $(S_X)_{\mathbb{C}} \subset H^{1,1}$. Thus $(T_X)_{\mathbb{C}}$ has a decomposition

$$(T_X)_{\mathbb{C}} = H^{2,0} \oplus H^{1,1}_0 \oplus H^{0,2} \cong \mathbb{C}^{22-\rho},$$

where $H^{1,1}_0 = (T_X)_{\mathbb{C}} \cap H^{1,1}$. The complex line $\mathfrak{p}(X) := (H^{2,0} \subset (T_X)_{\mathbb{C}})$, viewed as a point in the projective space $|(T_X)_{\mathbb{C}}|$ of lines in $(T_X)_{\mathbb{C}}$ is called the *period* of X. If we choose a basis ω in $H^{2,0}(X) = \Omega^2(X)$, then we have a complex valued linear function on $H_2(X, \mathbb{Z})$ defined by $\gamma \mapsto \int_\gamma \omega$. Integrating over an algebraic cycle coming from S_X, we get zero (because our form is of type $(2,0)$ and an analytic cycle has one complex coordinate z), so the function can be considered as a linear function on $(H_2(X, \mathbb{C})/S_X)$, i.e. an element from $(T_X)_{\mathbb{C}}$. This explains the name period.

The Poincaré Duality on $H^2(X, \mathbb{C})$ corresponds via the de Rham Theorem, to the exterior product of 2-forms. Since ω is a form of type $(2,0)$, we get $w \wedge w = 0$. Thus $\mathfrak{p}(X)$ belongs to a quadric Q_T in $|(T_X)_{\mathbb{C}}|$ defined by the quadratic form of the quadratic lattice $H^2(X, \mathbb{Z})$ restricted to T_X. Also, $\omega \wedge \bar{\omega}$ is a form of type $(2,2)$ which is proportional to the volume form generating $H^4(X, \mathbb{R})$. Since its sign does not depend on a scalar multiple of ω, we may choose an orientation on the 4-manifold X to assume that it is positive. Thus we get a second condition $\omega \wedge \bar{\omega} > 0$. This defines an open (in the usual topology) subset Q^0 of Q. So, we see that the period $\mathfrak{p}(X)$ defines a point on the manifold Q^0 of dimension $20 - \rho(X)$. Our manifold Q^0 obviously depends on X, so we have to find some common target for the map $X \mapsto \mathfrak{p}(X)$.

We fix an even quadratic lattice S of signature $(1, r)$ and a primitive embedding $S \hookrightarrow L_{K3}$ (primitive means that the quotient group has no torsion). Then we repeat everything from above, replacing S_X with S, and denoting by T its orthogonal complement in L_{K3}. The signature of the lattice T is $(2, 19-r)$. Then we obtain a quadric Q_T in the projective space $|T_{\mathbb{C}}| \cong \mathbb{P}^{20-r}$ defined by the quadratic form of T. We also obtain its open subset Q^0_T defined by the condition $x \cdot \bar{x} > 0$. Now we fix a manifold $\mathcal{D}_T := Q_T$ which is called the *period domain* defined by the lattice T. Of course, as a manifold it depends only on its dimension $19 - r$. When, its dimension is positive, it consists of two connected components, each is a Hermitian symmetric domain of orthogonal type, or of type IV in Cartan's classification of such spaces. We have

$$\mathcal{D}_T \cong \mathrm{O}(2, 19-r)/\mathrm{SO}(2) \times O(19-r), \quad \mathcal{D}^0_T \cong \mathrm{SO}(2, 19-r)/\mathrm{SO}(2) \times \mathrm{SO}(19-r),$$

where $\mathcal{D}^0_T$ denotes one of the connected components.

A choice of an isomorphism of quadratic lattices $\phi : H^2(X, \mathbb{A}) \to L_{K3}$ (called a *marking*) and a primitive embedding $j : S \hookrightarrow S_X$ such that $\phi \circ j : S \hookrightarrow L_{K3}$ coincides with a fixed embedding $S \hookrightarrow L_{K3}$ (called a *lattice S polarization*) defines a point $\phi(\mathfrak{p}(X)) \in \mathcal{D}_T$. For some technical reasons one has additionally to assume that the image of S in S_X contains a semi-ample divisor class, i.e. the class D such that $D^2 > 0$ and $D \cdot R \geq 0$ for every irreducible curve on X. A different choice of (ϕ, j) with the above properties replaces the point $\phi(\mathfrak{p}(X))$ by the point $g \cdot \phi(\mathfrak{p}(X))$, where g belongs to the group

$$\Gamma_S := \{g \in \mathrm{O}(L_{K3}) : g|S = \mathrm{id}_S\}.$$

Let $A_T = T^\vee/T$ be the *discriminant group*, quadratic lattice!discriminant group where T embeds in its dual group $T^\vee = \mathrm{Hom}(T, \mathbb{Z})$ via viewing the symmetric bilinear form on T as a homomorphism $\iota : S \to \mathrm{Hom}(S, \mathbb{Z})$ such that $\iota(s)(s') = s \cdot s'$. It is a finite abelian group defined by a symmetric matrix representing the quadratic form on T in some basis of T. Its order is equal to the absolute value of the discriminant of the quadratic form. The discriminant group is equipped with a quadratic map

$$q_T : A_T \to \mathbb{Q}/2\mathbb{Z}, \quad x^* \mapsto x^{*2} \mod 2, \tag{8.1}$$

where $x^* \in T^\vee$ is a representative of a coset in A_T, and we extend the quadratic form q of T to $T^\vee \subset T_\mathbb{Q}$ and then check that the definition is well defined on cosets.

We have a natural homomorphism

$$\rho : \mathrm{O}(T) \to \mathrm{O}(A_T, q_T).$$

Its kernel consists of orthogonal transformations of T that can be lifted to an orthogonal transformation σ of L_{K3} such that $\sigma|S = \mathrm{id}_S$. Thus we obtain that

$$\Gamma_T \cong \mathrm{Ker}(\rho).$$

Now we can consider the quotient space

$$\mathcal{M}_{K3,T} := \mathcal{D}_T/\Gamma_T.$$

It is a quasi-projective algebraic variety of dimension $20 - \rho$. The *Global Torelli Theorem* of I. Pyatetsky-Shapiro and I. Shafarevich asserts that assigning to X its period point $\mathfrak{p}$ defines a point in $\mathcal{D}_T$ that does not depend on marking ϕ and two S-poilarzed surfaces are isomorphic preserving the polarization if and only if the images are the same. One can use this to identify the quotient with the coarse moduli space $\mathcal{M}_{K3,S}$ of S-polarized K3 surfaces.

For any vector $\delta \in T$, let $\delta^\perp$ denote the orthogonal complement of $\mathbb{C}\delta$ in $T_\mathbb{C}$. This is a hyperplane in the projective space $|T_\mathbb{C}|$ defined by a linear function with rational coefficients. Let $H_\delta = \mathcal{D}_T \cap \delta^\perp$ be the subset of the period domain $\mathcal{D}_T$. If $\delta^2 < 0$, then the signature of the lattice $(\mathbb{R}\delta)^\perp \subset T_\mathbb{R}$ is equal to $(2, 18 - r)$, so H_δ is a domain of the same type. For any positive integer N consider

$$\mathcal{H}(N) = \bigcup_{\delta, \delta^2 = -N} H_\delta.$$

The group Γ_T acts on the set of δ's with $\delta^2 = -N$ and we denote by $\mathrm{Heeg}(N)$ the image of $\mathcal{H}(N)$ in the quotient space $\mathcal{M}_{K3,S}$. It is empty or a hypersurface in $\mathcal{M}_{K3,S}$. It is denoted by $\mathrm{Heeg}(S; N)$ and is called the *Heegner divisor* in the moduli space of lattice S polarized K3 surfaces.

In the next lecture we will compare the Heegner divisors

$$\mathrm{Heeg}_n(N) := \mathrm{Heeg}(S; N), \tag{8.2}$$

where $S = E_8 \oplus E_8 \oplus \langle -2n \rangle$ with the Humbert surfaces $\mathrm{Hum}_n(\Delta)$, where $N = \Delta/2n$.

Lecture 9

Shioda-Inose K3 surfaces

Let Kum(A) be the Kummer surface of an abelian surface A and X be its minimal resolution of singularities obtained as the quotient of the blow-up $\tilde{A}$ of A at its set of 2-torsion points by the lift $\tilde{\iota}$ of the involution $\iota = [-1]_A$ of A. The cover $\tilde{\phi} : \tilde{A} \to X$ is a degree two cover with the branch divisor equal to the sum $R = R_1 + \cdots + R_{16}$ of exceptional curves of the resolution $\sigma : X \to$ Kum(A).

In general, let $S' \to S$ be a double cover of smooth surfaces branched over a curve (necessary smooth) B on S. Let $\psi_U = 0$ be a local equation of B in an affine open subset U, then the preimage of U in S' is isomorphic to the hypersurface in $V = U \times \mathbb{C}$ given by the equation $z_U^2 - \psi_U = 0$. Thus, locally the ring $\mathcal{O}(V)$ of regular functions on V is a free module of rank 2 over the ring $\mathcal{O}(U)$ of functions on U generated by 1 and z_U. Let $\mathcal{O}(U)z_u$ be the submodule of rank 1. One checks that, taking an affine cover of S, the $\mathcal{O}(U)$-modules $\mathcal{O}(U)z_u$ are glued together to define a line bundle L such that $L^{\otimes -2}$ is isomorphic to the line bundle $L(B) = \mathcal{O}_S(B)$ associated to the curve B. It may not have sections but its tensor square has a section with the zero divisor equal to B. In particular, we see that the divisor class of B is divisible by 2 in the Picard group Pic(S). Conversely, if B is a smooth curve on S such that its divisor class $[B]$ is divisible by two in Pic(S) there exists a double cover of smooth surfaces $S' \to S$ with the branch divisor B. The set of isomorphism classes of such covers is bijective to the set of square roots of $[B]$ in Pic(S). It is a principal homogeneous space over the group Pic$(S)[2]$ of 2-torsion points in Pic(S).

Let us return to our example. We see that the sum $R = R_1 + \cdots + R_{16}$ must be divisible by 2 in Pic$(\tilde{X})$. Since $\tilde{X}$ is a K3 surface, we have Tors(Pic$(\tilde{X})) = 0$, hence $[R] = 2[R_0]$ for a unique divisor class R_0. Since $R^2 = 16(-2) = -32$, we obtain $R_0^2 = -8$. It is easy to see that the line bundle $\mathcal{O}_{\tilde{X}}(R_0)$ has no sections but its tensor square has a unique section (up to a constant multiple) vanishing on R.

Suppose we have a disjoint set of (-2)-curves $E_1, \ldots, E_k$ on a K3 surface Y, we ask whether there exists a double cover $Y' \to Y$ with branch divisor equal to $E = E_1 + \cdots + E_k$. Since $E^2 = -2k = 4D^2$ for some divisor D and D^2 is even, we obtain that $k \in \{4, 8, 12, 16\}$ (it cannot be larger since the classes $[E_i]$ are linearly independent in $H^2(Y, \mathbb{Q}) = \mathbb{Q}^{22}$). Let $f : Y' \to Y$ be the double cover with the branch divisor E and let $R = R_1 + \cdots + R_k$ be the ramification divisor on Y'. Since $f^*(E_i) = 2R_i$, we have $R_i^2 = -1$. The standard Hurwitz formula gives us that $K_{Y'} = f^*(K_Y) + R = R$. Since each R_i is an exceptional curve of the first kind, we can blow down R to obtain a surface Y with $K_Y = 0$. It is known that a surface with trivial canonical class is either an abelian surface or a K3 surface. Now the standard topological formula gives us that $e(Y') = 2e(X) - e(R) = 48 - 2k = e(Y) + k$. This gives $e(Y) = 48 - 3k$. If Y is an abelian surface, we obtain $k = 16$. If Y is a K3 surface, we obtain $k = 8$.

Note that a theorem of V. Nikulin asserts that any disjoint sum of sixteen (-2)-curves on a K3 surface is divisible by 2 in the Picard group and hence defines a double cover birationally isomorphic to an abelian surface A. It is easy to see that it implies that X is birationally isomorphic to $\mathrm{Kum}(A)$.

In the case $k = 8$, we have more possibilities. A set of eight disjoint (-2)-curves on a K3 surface Y is called an *even eight*, if the divisor class of the sum is divisible by 2 in $\mathrm{Pic}(Y)$.

Let $E_1, \ldots, E_8$ be an even eight on a K3 surface Y and $\pi : \tilde{Y} \to Y$ be the corresponding double cover. Let $\bar{E}_1 + \cdots + \bar{E}_8$ be the ramification divisor on $\tilde{Y}$. We have $\pi^*(E_i) = 2\bar{E}_i$, hence $4\bar{E}_i^2 = 2E_i^2 = -4$, hence $\bar{E}_i^2 = -1$. Also $\bar{E}_i \cong E_i$, hence $\bar{E}_i \cong \mathbb{P}^1$. Thus $\bar{E}_i$ is an exceptional curve of the first kind, hence can be blown down to a smooth point of a surface. Let $\sigma : \tilde{Y} \to Y'$ be the blow-down of the eight exceptional curves $\bar{E}_i$. As above, we obtain that $e(\tilde{Y}) = 2e(Y) - e(\bar{E}) = 48 - 16 = 32$. This shows that $e(Y') = 32 - 8 = 24$. Also, we have $K_{\tilde{Y}} = \sigma^*(K_Y) + \bar{E} = \bar{E}$, hence $K_{Y'} = 0$. Together with the Noether formula this implies that $b_1(Y') = 0$, hence Y' is a K3 surface. Let $\tilde{\tau}$ be the deck transformation of the cover σ, it descents to an involution (= an automorphism of order 2) τ of Y'. It has 8 fixed points, the images of the curves $\bar{E}_i$ on Y'. The quotient $Y'/(\tau)$ is a surface $\bar{Y}$ with 8 ordinary double points. The rational map $\pi \circ \tilde{\pi} \circ \tilde{\sigma}^{-1} : Y \to \bar{Y}$ is a minimal resolution of the surface $\bar{Y}$. We have the following commutative diagram of regular maps:

$$\begin{array}{ccc} \tilde{Y} & \xrightarrow{\tilde{\sigma}} & Y \\ \downarrow{\scriptstyle \tilde{\pi}} & & \downarrow{\scriptstyle \pi} \\ Y' & \xrightarrow{\sigma} & \bar{Y} \end{array} .$$

Thus we obtain that each even eight on a K3 surface Y defines a K3 surface Y' and an involution τ on Y' such that Y is isomorphic to a minimal resolution of the singular surface $Y/(\tau)$. One can show that any involution on a K3 surface that acts identically on a holomorphic 2-form (a *symplectic involution*) has 8 fixed points , and its quotient has a minimal resolution of singularities isomorphic to a K3 surface with exceptional curves forming an even eight. A K3 surface obtained in this way is called a *Nikulin K3 surface*. In general, one expects that a Nikulin surface has the Picard number equal to 9 and the moduli spaces of polarized Nikulin surfaces have dimension equal to 11.

We will be interested in Nikulin surfaces isomorphic to a nonsingular minimal model X of the Kummer surface $\mathrm{Kum}(A)$.

Let $E = E_1 + \cdots + E_8$ be an even eight on X. We know that $E \sim 2E_0$, where $E_0^2 = -4$. Let N be the sublattice of $\mathrm{Pic}(X)$ generated by E_0 and $E_1, \ldots, E_8$. It is a negative definite even lattice of rank 8, called the *Nikulin lattice*. It contains the sublattice spanned by $E_1, \ldots, E_8$ isomorphic to $\langle -2 \rangle^{\oplus 8}$, where $\langle a \rangle$ denotes the lattice of spanned by a vector v with $v^2 = a$. This lattice is of index 2 in the lattice N, hence the elementary theory of finite abelian groups tells us that the discriminant group of N is equal $(\mathbb{Z}/2\mathbb{Z})^6$. The inclusion $N \hookrightarrow S_X$ is a primitive embedding. Thus each Nikulin surface must contain a primitive sublattice isomorphic to the Nikulin lattice.

One can show that the Nikulin involution τ acts on $H^2(Y', \mathbb{Z}) \cong L_{K3} = U^{\oplus 3} \oplus E_8^{\oplus 2}$ as the identity on $U^{\oplus 3}$ and by sending a vector in E_8 to the same vector in the other copy of E_8. Let $H^\tau \cong U^{\oplus 3} \oplus E_8(2)$ be the sublattice of invariant elements and $H_\tau \cong E_8(2)$ be the sublattice of anti-invariant elements (i.e. $\tau^*(\gamma) = -\gamma$).[1] Note that τ acts identically on $\Omega^2(X') \cong \mathbb{C}$, since otherwise the quotient has no regular 2-forms, so it must be a symplectic involution. Thus, for any cycle $\gamma \in H_\tau$, we have

$$0 = \int_{\gamma+\tau^*(\gamma)} \omega = \int_\gamma \omega + \int_{\tau^*(\gamma)} \tau^*(\omega) = 2\int_\gamma \omega.$$

By Lefschetz, this implies that $\gamma \in S_X = H^2(X, \mathbb{Z})_{\text{alg}}$. Since H_τ and H^τ are obviously orthogonal to each other, we obtain

$$E_8(2) \cong H_\tau \subset S_X, \quad T_X \subset H^\tau \cong U^{\oplus 3} \oplus E_8(2).$$

Nikulin shows that the converse is true:if S_Y contains a primitive sublattice S isomorphic to $E_8(2)$, then there exists a Nikulin involution τ on Y' such that $S \subset H_\tau$.

Note that under the rational cover $f : Y' \to Y'/(\tau) \overset{\pi^{-1}}{\dashrightarrow} Y$, we have

$$f^*(T_Y) \cong T_Y(2) \subset T_{Y'}.$$

Now suppose $Y = \widetilde{\text{Kum}(A)}$. One can show that, under the pre-image map $A \dashrightarrow Y$, we have $T_Y \cong T_A(2)$.

Suppose $\text{Kum}(A)$ is a quotient of a K3 surface Y by a Nikulin involution. Then, by above, $T_Y(2) = T_A(4) \subset T_Y$. One can show that this inclusion comes from an inclusion $T_Y \subset T_A$ with quotient $(\mathbb{Z}/2\mathbb{Z})^\alpha$, where $0 \leq \alpha \leq 4$ (for any $x \in T_A$ we have $2x \in 2T_A \subset T_X$). Conversely, if there exists a primitive embedding $T_Y \hookrightarrow T_A$ with such a quotient, then Y admits a Nikulin involution with quotient isomorphic to $\text{Kum}(A)$. If $\alpha = 0$, then we have $T_Y \cong T_A$, and, in this case we say that X admits a *Shioda-Inoue structure*. Even eights with $\alpha \neq 0$ were studiied in [73].

Note that $T_A \subset H^2(A, \mathbb{Z}) \cong U^{\oplus 3}$ thus

$$T_Y \subset U^3 \subset L_{K3} = U^3 \oplus E_8 \oplus E_8$$

and, we obtain that $E_8 \oplus E_8 \subset S_Y$. Here we have to use some lattice theory to check that all primitive embeddings of T_Y in L_{K3} are equivalent under orthogonal transformations of L_{K3}. Conversely, a theorem of D. Morrison [79] asserts that the condition that $E_8 \oplus E_8$ primitively embeds in S_X is necessary and sufficient in order X admits a Shioda-Inose structure. We express this structure by the triangle of rational maps

$$\begin{array}{ccc} X & & A\,. \\ & \searrow\!\!\!\cdot\; \swarrow\!\!\!\cdot & \\ & \text{Kum}(A) & \end{array} \tag{9.1}$$

[1] For any quadratic lattice M we denote by $M(k)$ the quadratic lattice obtained from M by multiplying its quadratic form by an integer k.

Example 9.1. Suppose A has a principal polarization L_0. We have $(L_0)^2 = 2$, hence

$$T_A \subset \langle L_0\rangle^\perp = U \oplus U \oplus \langle -2\rangle$$

(we embed $h = c_1(L_0)$ in one copy of $U = \mathbb{Z}f + \mathbb{Z}g$ with the image of h equal to $f + g$, where f, g is a canonical basis of U). Let Y be a K3 surface with $T_Y \cong T_A$. Then $T_Y \cong T_A \subset U \oplus U \oplus \langle -2\rangle$ and $S_Y = (T_Y)^\perp$ contains $(U \oplus U \oplus \langle -2\rangle)^\perp = E_8 \oplus E_8$. Hence Y is related to A by a Shioda-Inose structure. Let us construct such a surface Y.

Let B be a curve on $Q = \mathbb{P}^1 \times \mathbb{P}^1$ of bidegree $(4, 4)$ which is the union of a curve B_0 of bidegree $(3, 3)$ and some fibers F_1 and F_1' of the projections $Q \to \mathbb{P}^1$. We assume that B is invariant with respect to the involution $\alpha : (x, y) \mapsto (y, x)$ of Q, and B_0 has a cusp p at a point $q \in F_1$ with local equation $u^2 + v^3 = 0$, where $u = 0$ is a local equation of the fiber F_1 at q.

We assume that B_0 has no other singular points besides q and q'. Let $\pi : X' \to Q$ be the double cover of Q branched along B. It is a singular surface with singularities over q and $q' = \alpha(q)$ locally given by equations $z^2 = u(u^2 + v^3)$. This type of singularity is known as a *double rational point of type* E_7. Let X be a minimal resolution of X'. Its exceptional curve over q (resp. over q') is reducible and consists of 7 irreducible components which are (-2)-curves. Its intersection matrix is equal to the Coxeter matrix of type E_7 (multiplied by -1). The surface X is a K3 surface. Fix the first projection $Q \to \mathbb{P}^1$. The composition of the projections $X \to X' \to Q \to \mathbb{P}^1$ gives an elliptic fibration on X with two degenerate fibers of type II^* and III^* in Kodaira's notation. We also have 6 other irreducible singular fibers isomorphic to a nodal cubic curve. They correspond to 6 ordinary ramification points of the cover $\tilde{B}_0 \to B_0 \hookrightarrow Q \to \mathbb{P}^1$ of the normalization $\tilde{B}_0$ of B_0. The fiber of the first type lies over the fiber F_1 and the fiber of type III^* lies over the fiber F_2 of the same projection that passes through the point q'. The preimage of F_1' on X defines a section S of the fibration. If we take the sublattice generated by one fiber f and the section s, we obtain a sublattice given by a matrix $\left(\begin{smallmatrix} 0 & 1 \\ 1 & -2 \end{smallmatrix}\right)$. By changing a basis $f \to f + s, f \to f$, we reduce this matrix to the form $\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$. Thus this sublattice is isomorphic to U. The orthogonal complement to U in Pic(X) contains the classes of irreducible components of fibers that are disjoint from s. We easily find that this lattice is isomorphic to $E_8 \oplus E_7$. Thus, we obtain a lattice embedding (in fact, a primitive embedding)

$$U \oplus E_8 \oplus E_7 \hookrightarrow \text{Pic}(X).$$

It is easy to see that E_8 primitively embeds in $U \oplus E_7$, thus $E_8 \oplus E_8$ embeds in Pic(X), and, by Morrison, X has a Shioda-Inose structure with

$$T_A \cong T_X \subset U \oplus U \oplus \langle -2\rangle.$$

Since $(U \oplus U \oplus \langle -2\rangle)^\perp_{H^2(A,\mathbb{Z}} = \langle 2\rangle \subset \text{NS}(A)$, we obtain that A admits a principal polarization.

Let $Q \to \mathbb{P}^2$ be the quotient map of $Q \to Q/(s)$. The quadric Q is a cover of $\mathbb{P}^2$ branched along a conic K. Since B was invariant under the switch involution s, we see that it is equal to the preimage of the plane curve under this cover. The plane curve is a cuspidal cubic C plus a line ℓ which is tangent to the conic K and intersects C at the cusp with multiplicity 3. Assume that C intersects K at 6 distinct points. One can show that the double cover of $\mathbb{P}^2$ branched along the union $K + C + \ell$ has a minimal resolution isomorphic to the Kummer surface Kum(A), where $A \cong \text{J}(C)$ for some curve C of genus 2. We have the following diagram of rational maps

$$\begin{array}{ccc} X & \xrightarrow{\pi} & \mathbb{P}^1 \times \mathbb{P}^1 \\ \downarrow{\scriptstyle \phi} & & \downarrow \\ \widetilde{\mathrm{Kum}(A)} & \xrightarrow{\phi} & \mathbb{P}^2 \end{array}.$$

One can see explicitly the even eight on $\mathrm{Kum}(A)$ defining the rational double cover $\phi : X \to \mathrm{Kum}(A)$ (see [73]).

Note that the six points $C \cap K$ define a curve C' of genus 2 which is in general not isomorphic to C. This curve is birationally isomorphic to the curve B_0. It comes with an additional structure. We have $3q \sim 2q' + a$ and $3q' \sim 2q + a'$, where $a + a' \sim K_{C'}$. This implies $3K_{C'} \sim 5q + a \sim 5q' + a'$. There are 16 pairs (q, q') with such property on a curve of genus 2. This implies that the Shioda-Inose construction gives a rational self-map from $\mathcal{M}_2$ to $\mathcal{M}_2$ of degree 16.

We see that X admits an elliptic fibration $|f|$ with two singular fibres

$$f_1 = 3R_0 + 2R_1 + 4R_2 + 6R_3 + 5R_4 + 4R_5 + 3R_6 + 2R_7 + R_8$$

and

$$f_2 = 2N_0 + N_1 + 2N_2 + 3N_3 + 4N_4 + 3N_5 + 2N_6 + N_7$$

of type $\tilde{E}_8$ and $\tilde{E}_7$. It is also has a section S. The fixed locus of τ consists of smooth rational curves $R_1, R_3, R_5, R_7, N_2, N_4, N_6, S$ and a genus 2 curve W which intersects R_0, N_0, N_7 with multiplicity 1. The switch involution lifts to an involution σ on X that transforms the elliptic fibration defined by the first projection $Q \to \mathbb{P}^1$ to the elliptic fibration $|f'|$ defined by the second projection. Its singular fibers are

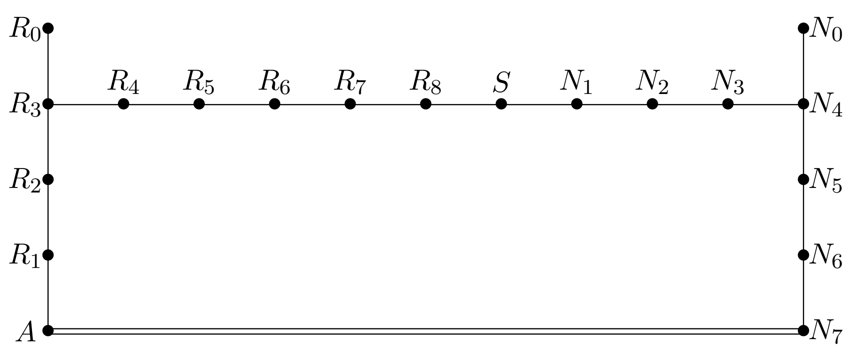
$$F_1' = 3N_0 + R_8 + 2S + 3N_1 + 4N_2 + 5N_3 + 6N_4 + 4N_5 + 2N_6$$

and

$$F_2' = 2R_0 + A + 2R_1 + 3R_2 + 4R_3 + 3R_4 + 2R_5 + R_6$$

of type $\tilde{E}_8$ and $\tilde{E}_7$. The curve R_7 is a section. The involution σ induces the hyperelliptic involution on W. Its set of fixed points are 2 points on the curve R_8 and 6 points on W. Also note that σ maps the fibration $|f|$ to the fibration $|f'|$.

We have altogether 19 (-2) curves whose incidence graph is the following



One can prove that these are all (-2)-curves on X.

Let $p = W \cap R_0, q = W \cap N_0, a = W \cap N_7, a' = W \cap A$. We have $3p \sim 2q + a$ and the fibration defines a g_3^1 on W spanned by the divisors $3p$ and $2q + a$.

It is easy to see that $q = \sigma(p), a' = \sigma(a)$. This gives $3p \sim 2K_W - 2p + a$, hence $5p \sim 2K_W + a$, or, equivalently, $3K_W \sim 5p + a'$.

Consider the divisor class $D = R_0 + R_1 + R_2 + R_3 + A + W$. We have $D^2 = 4$ and $D \cdot R_i = 0, i = 5, 6, 7, 8$ and $D \cdot N_i = 0, i = 1, \ldots, 6$. The linear system $|D|$ maps X to a quartic surface in $\mathbb{P}^3$ and blows down the four curves R_i (resp. 11 curves N_i) as above to double rational points of type A_5 (resp. A_{11}). [2] Its equation can be found in [17]

$$X(\alpha,\beta,\gamma,\delta) : y^2zw - 4x^3z + 3\alpha xzw^2 + \beta zw^3 + \beta zw^3 + \gamma xz^2w - \tfrac{1}{2}(\delta z^2w^2 + w^4) = 0. \quad (9.2)$$

Here $\alpha, \beta, \gamma, \delta$ are complex parameters with $\gamma, \delta \neq 0$. The surfaces $X(\alpha, \beta, \gamma, \delta)$ and $X(\alpha', \beta', \gamma', \delta')$ are birationally isomorphic if and only if there exists a nonzero number c such that $(\alpha', \beta', \gamma', \delta') = (c^2\alpha, c^3\beta, c^5\gamma, c^6\delta)$. It follows that $\mathcal{M}_{K3,U+E_8+E_7}$ is isomorphic to an open subset of the weighted projective space $\mathbb{P}(2, 3, 5, 6)$ known to be isomorphic to a compactification of $\mathcal{A}_2$.

The explicit correspondence between Kummer surfaces associated to curves of genus 2 and the Shioda-Inose K3 surfaces was given in [63], Theorem 11. Recall from Lecture 5 that a genus 2 curve $y^2 = f(x)$ is determined by the Clebsch invariants I_2, I_4, I_6, I_{10} of the binary form $f(x, y)$. We also recall that a K3-surface admitting an elliptic fibration with a section is birationally isomorphic to its *Weierstrass model*, a surface of degree 12 in $\mathbb{P}(1, 1, 4, 6)$

$$w^2 = z^3 + a(x, y)z + b(x, y),$$

where $a(x, y)$ and $b(x, y)$ are binary forms of degrees 8 and 12.

We have the following result.

Theorem 9.2. *Let*

$$y^2 = f_6(x, y)$$

be a nonsingular genus 2 curve, and I_2, I_4, I_6, I_{10} be the Clebsch invariants of the binary form $f_6(x, y)$. Then the Shioda-Inose surface associated to Kum$(J(C))$ *is an elliptic K3 surface with Weierstrass equation*

$$w^2 = z^3 - t_0^4t_1^3(t_0 + \frac{I_4}{12}t_1)z + t_0^5(\frac{I_2}{24}t_0^2 + \frac{I_2I_4 - 3I_6}{108}t_0t_1 + \frac{I_{10}}{4}t_1^2). \quad (9.3)$$

Let X be a K3 surface admitting a Shioda-Inose structure with the corresponding rational map of degree 2 $X \dashrightarrow$ Kum(A). A theorem of Shouhei Ma [70] asserts that a minimal resolution Y of Kum(X) admits a Nikulin involution τ such that a minimal resolution of $Y/(\tau)$ is isomorphic to X. We say that Kum(A) is *sandwiched* between X. A geometric realization of the sandwich structure can be often seen as follows. One finds an elliptic fibration $\pi : X \to \mathbb{P}^1$ on X such that the Mordell-Weil group of its sections contains a non-zero 2-torsion section S so that the translation

[2] A singular point of type A_k is a surface singularity locally isomorphic to the singularity $u^2 + u^{k+1} = 0$.

automorphism t_S defines a Nikulin involution with quotient birationally isomorphic to $\mathrm{Kum}(A)$. Let β be the involution of X that induces the involution $[-1]_E$ on each smooth fiber of the elliptic fibration. The fixed locus of β consists of some irreducible components of fibers and a horizontal divisor $S_0 + S + T$, where S_0 is the zero section and T is a 2-section. The intersection of $S + T$ with a smooth fiber coincides with the set of non-trivial 2-torsion points. The curve T is invariant with respect to β and its image on $\mathrm{Kum}(A)$ defines a 2-torsion section $\bar{T}$ on the image of the elliptic fibration on X to $\mathrm{Kum}(A)$. The Nikulin involution defined by the translation $t_{\bar{T}}$ has the quotient birationally isomorphic to X. To see this one should restrict the action of t_S on the generic fiber E_η of π and observe that the composition of $t_{\bar{T}} \circ t_S$ is the map $E_\eta \to E_\eta/E_\eta[2] \cong E_\eta$.

In the previous example, the Nikulin involution is defined by the translation t_S, where S is a 2-torsion section of the elliptic fibration with singular fiber F of type $\tilde{D}_{14}$ equal to

$$F = R_0 + R_2 + 2(R_3 + \cdots + R_8 + S + N_1 + N_2 + N_3 + N_4) + N_0 + N_5.$$

We may take S_0 to be equal to R_1 and S to be equal to N_6. The curve T coincides with W.

Remark 9.3. In this and the previous lectures we compared properties of abelian surfaces with the properties of the associated Kummer or Shioda-Inose K3 surfaces. There is also connections to cubic surfaces in $\mathbb{P}^3$. Recall that a nonsingular cubic surface in $\mathbb{P}^3$ is isomorphic to the blow-up of 6 points in the plane, no three of which are on a line, and not all of them are on a conic. The birational map is given by the linear system of plane cubics through the six points. When we allow the six points to lie on a conic, the cubic become singular, the image of the conic is its ordinary double point. A set of 6 distinct points on a conic defines a genus 2 curve C, and the Kummer surface $\mathrm{Kum}(\mathrm{J}(C))$ has a double plane model with the branch curve equal to the union of the tangents to the conic at the six points. It is interesting to investigate for which Δ the property that $\mathrm{J}(C) \in \mathrm{Hum}(\Delta)$ is the restriction of some divisor in the moduli space $\mathcal{M}_{\text{cub}}$ of cubic surfaces. We have already remarked that this is so for $\Delta = 4$. The divisor in $\mathcal{M}_{\text{cub}}$ is the locus of cubic surfaces with an Eckardt point.

There is also another way to consider $\mathrm{J}(C)$ as a divisor in $\mathcal{M}_{\text{cub}}$. The Hessian surface $H(F)$ of a general cubic surface F is a quartic surface with 10 nodes (see [28]). Its minimal resolution is a K3 surface. It is known since R. Hutchinson [41] that there is a divisor in $\mathcal{M}_{\text{cub}}$ such that the Hessian surface of a general surface from this divisor is birationally isomorphic to a Kummer surface of a curve of genus 2. It is defined by vanishing of the invariant $I_8 I_{24} + 8I_{32}$ of degree 32 of cubic surfaces (see [23], 6.6). Which properties of Kummer surfaces are special properties of Hessians of cubic surfaces? One answer in this direction is given in [86] where it is proven that $\mathrm{J}(C) \in \mathrm{Hum}(5)$ implies that the Hessian quartic surface admits an additional ordinary double point.

It is known that every K3 surface X with $\rho(X) = 19$ admits a Shioda-Inose structure (see [79]). Let T_X be the transcendental lattice of X. Suppose T_X contains a direct summand isomorphic to the hyperbolic plane U. Then $T_X \cong T_n := U \oplus \langle -2n \rangle$, where $2n$ is the discriminant of T_X. Let $M_n = (T_n)^{\perp}_{L_{K3}} \cong U^{\oplus 2} \oplus E_8^{\oplus 2} \oplus \langle 2n \rangle$. The moduli space $\mathcal{M}_{K3,M_n}$ is isomorphic to a non-compact modular curve $\mathbb{H}/\Gamma_0^+$ (see [29]). The loc.cit. paper contains a construction of K3 surfaces from this moduli space for some small n. The corresponding abelian surfaces are isogenous to the product of two isogenous elliptic curves.

Suppose $T_X \cong T$ does not contain an isotropic vector. Then the moduli space $\mathcal{M}_{K3,T^\perp}$ is known to be a compact Shimura curve. The corresponding abelian surfaces are fake elliptic curves. In general, they are simple abelian surfaces. We refer to K. Hashimoto [38] and A. Sarti [93], [94] for description of some of these transcendental lattices and the families of the corresponding K3 surfaces.

Lecture 10

Humbert surfaces and Heegner divisors

Let is explain an exceptional isomorphism between two Hermitian spaces of dimension 3, the Siegel space $\mathcal{Z}_2$ and a type IV domain associated to a 3-dimensional quadric. Recall that $\mathcal{Z}_2$ is isomorphic to an open subset of the Grassmannian $G(2,4)$ represented by complex 2×4-matrices of the form $[\tau \mathsf{D}]$, where τ is symmetric and $\mathrm{Im}(\tau) > 0$. In the Plücker embedding $G(2,4) \hookrightarrow \mathbb{P}^5$, the Grassmannian becomes isomorphic to a nonsingular quadric, the *Klein quadric* given by the *Plücker equation*

$$p_{12}p_{34} - p_{13}p_{24} + p_{14}p_{24} = 0. \tag{10.1}$$

We will see that the open subset $\mathcal{Z}_2$ coincides with a Hermitian symmetric space of orthogonal type which we used to construct the coarse moduli space of lattice polarized K3 surfaces. We will also see that the modular group $\mathrm{Sp}(J_{\mathsf{D}}, \mathbb{Z})$ is isomorphic to the group Γ_T acting on $\mathcal{D}_{T_n}$, where the lattice T_n is determined by $\mathsf{D} = \mathrm{diag}[1, n]$, namely

$$T_n \cong U \oplus U \oplus \langle -2n \rangle.$$

The cohomology group $H^1(A, \mathbb{Z})$ is a free abelian group H of rank 4 and $H^2(A, \mathbb{Z}) \cong \bigwedge^2 \mathsf{H}$. The group $H^2(A, \mathbb{Z})$ is a quadratic lattice with respect to the natural pairing

$$\bigwedge^2 H^1(A, \mathbb{Z}) \times \bigwedge^2 H^1(A, \mathbb{Z}) \to \bigwedge^4 H^1(A, \mathbb{Z}) \cong H^4(A, \mathbb{Z}) \cong \mathbb{Z},$$

where we fix an isomorphism $H^4(A, \mathbb{Z}) = \bigwedge^4 \mathbb{Z}^4 \to \mathbb{Z}$, called an *orientation* on H. The quadratic lattice $H^2(A, \mathbb{Z})$ is a unimodular even lattice of signature $(3,3)$. It is isomorphic to the orthogonal sum $U^{\oplus 3}$ of three hyperbolic planes U.

A choice of a basis (e_1, e_2, e_3, e_4) in $H^1(A, \mathbb{Z}) \cong \mathbb{Z}^4$ defines a basis $e_1 \wedge \cdots \wedge e_4$ of $\bigwedge^4 \mathsf{H}$, hence an orientation on H. We fix this choice.

We can choose a basis $(\gamma_1, \ldots, \gamma_4)$ of $H_1(A, \mathbb{Z})$ and a basis (ω_1, ω_2) of $\Omega^1(A)$ such that

$$\omega_1 = (z_1, z_2, 1, 0), \quad \omega_2 = (z_2, z_3, 0, n).$$

Here $\tau = \left(\begin{smallmatrix} z_1 & z_2 \\ z_2 & z_3 \end{smallmatrix}\right)$, $\mathsf{D} = \mathrm{diag}[1, n]$. In the Plücker embedding the plane spanned by ω_1, ω_2 is the point

$$\mathfrak{p} = \omega_1 \wedge \omega_2 = (z_1 z_3 - z_2^2) w_1 - z_2 (w_2 - n w_5) - n z_1 w_3 - z_3 w_4 + n w_6,$$

where

$$(w_1, \dots, w_6) = (e_1 \wedge e_2, e_1 \wedge e_3, e_1 \wedge e_4, e_2 \wedge e_3, e_2 \wedge e_4, e_3 \wedge e_4).$$

is the corresponding basis in $\bigwedge^2 \mathsf{H}$. Let

$$(f_1, g_1, f_2, g_2, k) := (-w_1, w_6, -w_3, w_4, nw_5 + w_2).$$

We see that $f_i^2 = g_i^2 = 0, f_i \cdot g_i = 1$ and and

$$h_0 = w_2 - nw_5, \quad h_0^2 = 2n.$$

We can rewrite

$$\mathfrak{p} = (z_2^2 - z_1 z_3) f_1 + n g_1 + n z_1 f_2 + z_3 g_2 + z_2 k.$$

and check that

$$\mathfrak{p} \in (\mathbb{Z} h_0)^\perp, \quad \mathfrak{p}^2 = 0$$

Thus $\mathfrak{p}$ defines a point $[\mathfrak{p}]$ in the quadric $Q_{T_n} = Q \cap \mathbb{P}((T_n)_\mathbb{C})$. One checks that the condition $\mathrm{Im}(\tau) > 0$ translates into the condition that $[\mathfrak{p}]$ belongs to one of the two connected components of $Q_{T_n}^0$, which we fix and denote it by $\mathcal{D}_{T_n}^0$.[1]

Note that the matrix of the quadratic form in the basis (f_1, g_1, f_2, g_2, e) is equal to

$$\begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -2n \end{pmatrix}$$

So, this confirms that the lattice $\langle 2n \rangle^\perp$ in $H^2(A, \mathbb{Z}) \cong U \oplus U \oplus U$ is isomorphic to $U \oplus U \oplus \langle -2n \rangle$.

Let L_0 be a polarization on A of degree $2n$ with $h_0 = c_1(L) \in H^2(A, \mathbb{Z})$. We have $h_0^2 = 2n$. Choose a basis $(e_1, \dots, e_4)$ of H and let

$$(w_1, \dots, w_6) = (e_1 \wedge e_2, e_1 \wedge e_3, e_1 \wedge e_4, e_2 \wedge e_3, e_2 \wedge e_4, e_3 \wedge e_4)$$

be the corresponding basis in $\bigwedge^2 \mathsf{H}$. The intersection form is defined by the exterior product and the choice of an orientation. The matrix in this basis is equal to

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

In the dual basis $(p_{12}, p_{13}, p_{14}, p_{23}, p_{24}, p_{34})$, the quadratic form is equal to

$$q = 2(p_{12} p_{34} - p_{13} p_{24} + p_{14} p_{24}). \tag{10.2}$$

[1] If we write $z_i = x_i + \sqrt{-1} y_i$, then the condition $\mathrm{Im}(z_1) > 0$ chooses a connected component and the condition $y_1 y_3 - y_2^2 > 0$ makes sure that the point lies on the open subset $Q_{T_n}^0$ of the quadric and hence defines the period point of a marked polarized K3 surface.

The equation $q = 0$ is the Plücker equation (10.1). It is known that the Grassmannian contains two families of planes corresponding to lines through a fixed point or lines in a fixed plane. Any automorphism of $G(2,4)$ preserving each of the families, originates from a projective automorphism of $|\mathsf{H}_{\mathbb{C}}|$ by taking the wedge square of the corresponding linear map. There is also an integral version of this isomorphism. The integral analog of plane in $G(2,\mathbb{H}_{\mathbb{C}})$ is a maximal isotropic sublattice F of rank 3 in $\bigwedge^2 \mathsf{H}$.[2] The homomorphism

$$\sigma : \mathsf{GL}(\mathsf{H})_0 \to \mathsf{GL}(\bigwedge^2 \mathsf{H}), \quad \phi \mapsto \phi \wedge \phi,$$

has the image equal to the index 2 subgroup $\mathsf{O}_0(\bigwedge^2 \mathsf{H})$ of the orthogonal group $\mathsf{O}(\bigwedge^2 \mathsf{H})$ of the lattice $\bigwedge^2 \mathsf{H}$. It consists of isometries preserving a family of maximal isotropic sublattices (see [7], Lemma 4).

Let h_0' be a primitive vector with $h_0'^2 = 2n$.It follows from Lemma 10.3 below that there exists an isometry $\sigma : \bigwedge^2 \mathsf{H} \to \bigwedge^2 \mathsf{H}$ that sends h_0' to h_0. Replacing $(w_1,\ldots,w_6)$ with $(\phi(w_1),\ldots,\phi(w_6))$ we may assume that

$$h_0 = w_2 - nw_5.$$

Let $\mathsf{O}(T_n)$ denote the orthogonal group of the lattice T_n. Let $D_{T_n} = T_n^*/T_n$ be the discriminant group equipped with the quadratic map (8.1). Let $\mathsf{O}(T_n)^*$ be the kernel of the natural homomorphism $r : \mathsf{O}(T_n) \to \mathsf{O}(A_{T_n}, q_{A_{T_n}})$. We know from Lecture 8 that the orbit space

$$\mathcal{D}_{T_n}/\mathsf{O}(T_n)^* \cong \mathcal{D}^0_{T_n}/\mathsf{O}_0(T_n)^*$$

is isomorphic to the coarse moduli space $\mathcal{M}_{K3,M_n}$ of pairs (X,j), where j is a fixed primitive embedding of the lattice $M_n = T_n^\perp$ into $\mathrm{Pic}(X)$ (or, equivalently, a primitive embedding $T_X \hookrightarrow T_n$) (with some additional technical conditions formulated in terms of the Picard lattice $\mathrm{Pic}(X)$ of X (see [29])).

In our case $A_{T_n} = \mathbb{Z}/2n\mathbb{Z}$ and the value of the discriminant quadratic form q at its generator is equal to $-\frac{1}{2n} \mod 2\mathbb{Z}$. The group $\mathsf{O}(G(T_n))$ is isomorphic to the group $(\mathbb{Z}/2\mathbb{Z})^{p(n)}$, where $p(n)$ is the number of distinct prime factors of n and the homomorphism $r : \mathsf{O}(T_n) \to \mathsf{O}(A_{T_n})$ is surjective (see [95], Lemma 3.6.1).

Recall that we have defined earlier a surjective homomorphism $\sigma : \mathsf{SL}(\mathsf{H}) \to \mathsf{O}_0(\bigwedge^2 \mathsf{H})$, where $\mathsf{O}_0(\bigwedge^2 \mathsf{H})$ is a subgroup of index 2 of $\mathsf{O}(\bigwedge^2 \mathsf{H})$. Consider h_0 as an element of $\bigwedge^2 \mathsf{H}^\vee = \bigwedge^2 H_1(A,\mathbb{Z})^\vee$. Then the stabilizer subgroup of h_0 in $\mathsf{O}(\bigwedge^2 \mathsf{H})_0$ is equal to the image under σ of the subgroup of $\mathsf{SL}(\mathsf{H})$ that preserves the symplectic form h_0. It is isomorphic to the group $\mathsf{Sp}(J_{\mathsf{D}},\mathbb{Z})$. where $\mathsf{D} = \mathrm{diag}[1,n]$. This gives an isomorphism

$$\mathsf{Sp}(J_{\mathsf{D}},\mathbb{Z})/(\pm 1) \cong \mathsf{O}_0(\bigwedge^2 \mathsf{H})_{h_0} \cong \mathsf{O}_0(T_n)^*. \tag{10.3}$$

The latter isomorphism comes from the interpretation of the group $\mathsf{O}(T_n)^*$ as a subgroup of T_n of isometries that lift to an isometry of $\bigwedge^2 \mathsf{H}$ leaving h_0 invariant. Note that the subgroup $\mathsf{Sp}(J_{\mathsf{D}},\mathbb{Z})$

[2] A sublattice of a lattice is called isotropic if the restriction of the quadratic form to the sublattice is identically zero.

is conjugate to a subgroup Γ_n of $\mathrm{Sp}(4,\mathbb{Q})$ by the conjugation map $g \mapsto R^{-1}gR$, where R is the diagonal matrix $\mathrm{diag}(1,1,1,n)$ ([47], p. 11).

So, let us record the previous information in the following.

Theorem 10.1. *There is an isomorphism of coarse moduli spaces*

$$\mathcal{A}_{2,n} \cong \mathcal{M}_{K3,M_n}.$$

Example 10.2. Consider the surface $A = E \times E$, where E has a complex multiplication by $\mathfrak{o} = \mathbb{Z} + \mathbb{Z}\omega, \omega = \sqrt{-5}$ from Example 3.4. Let us compute its lattice T_A. We have $\mathrm{End}^s(A) = \{M \in \mathrm{Mat}_2(\mathfrak{o}) : {}^t\bar{M} = M\}$. As a $\mathbb{Z}$-module, it has a basis that consists of four matrices

$$\begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix}, \begin{pmatrix}0 & 0\\ 0 & 1\end{pmatrix}, \begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}, \begin{pmatrix}0 & \omega\\ \bar{\omega} & 0\end{pmatrix}. \tag{10.4}$$

Under the isomorphism $\mathrm{NS}(A) \to \mathrm{End}(A)$, the first three matrices correspond to the divisors $E_1 = E \times \{0\}, E_1 = \{0\} \times E$ and the class $\Delta - E_1 - E_2$. The last matrix corresponds to some divisor D. Consider the basis $(\gamma_1,\gamma_2,\gamma_3,\gamma_4) = (\omega e_1, e_1, \omega e_2, e_2)$ of the lattice Λ. The reducible principal polarization H_0 is given in the basis (e_1, e_2) by the matrix $y^{-1}I_2$, where $y = \mathrm{Im}(\omega) = \sqrt{5}$. The corresponding symplectic form is defined by $h_0 = \gamma_1^* \wedge \gamma_2^* + \gamma_3^* \wedge \gamma_4^*$. The Hermitian forms corresponding to the four endomorphisms (10.4) are obtained by multiplying these matrices by y^{-1}. We give the alternating forms defining the first Chern class in terms of the dual basis $(\gamma_1^*, \ldots, \gamma_4^*)$.

$$\gamma_1^* \wedge \gamma_2^*,\ \gamma_3^* \wedge \gamma_4^*,\ \gamma_1^* \wedge \gamma_4^* - \gamma_2^* \wedge \gamma_3^*,\ 5\gamma_1^* \wedge \gamma_3^* + \gamma_2^* \wedge \gamma_4^*.$$

To find the intersection matrix we choose the volume form

$$h_0 \wedge h_0 = \gamma_1^* \wedge \gamma_2^* \wedge \gamma_3^* \wedge \gamma_4^*$$

and compute the exterior products. The result is the intersection matrix

$$\begin{pmatrix}0 & 1 & 0 & 0\\ 1 & 0 & 0 & 0\\ 0 & 0 & -2 & 0\\ 0 & 0 & 0 & -10\end{pmatrix}$$

The transcendental lattice is a rank 2 positive lattice isomorphic to $\langle 2\rangle \oplus \langle 10\rangle$.

Let us see the meaning of the singular equation (4.1) in terms of the period $[\mathfrak{p}]$ of a K3-surface. Consider the vector

$$\delta = ef_1 + dg_1 + cf_2 + ag_2 + \frac{b}{2n}k \in T_n^* \subset T(d)_{\mathbb{Q}}, \tag{10.5}$$

Using the singular equation (4.1), we have

$$\mathfrak{p} \cdot \delta = naz_1 + bz_2 + cz_3 + d(z_2^2 - z_1z_3) + nk = 0.$$

Finally, we get

$$\delta^2 = -\frac{b^2}{2n} + 2(ac + ed) = -\frac{\Delta}{2n}. \tag{10.6}$$

We obtain that $\mathrm{End}^s(A) \neq \mathbb{Z}$ if and only if the period of the corresponding K3 surface lies on a hyperplane $H_\delta := \delta^\perp = \mathbb{P}((\mathbb{C}\delta)^\perp) \cap \mathcal{D}_T$.

We use the following result from the theory of quadratic lattices (see [95], Propositiuon 3.7.3).

Lemma 10.3. *Let L be an even lattice such that it contains $U \oplus U$ as a primitive sublattice. Let $v, w \in L^*$ be two primitive vectors with $v^2 = w^2$. Then there exists $\sigma \in \mathrm{O}(L)$ such that $\sigma(v) = w$ if and only if the images of v, w in $L^\vee/L$ coincide.*

We apply this to our case where $L = T_n = U \oplus U \oplus \langle -2n \rangle$, where $\langle -2n \rangle$ is generated by a vector e with $e^2 = -2n$. We have $L^*/L \cong \mathbb{Z}/2n\mathbb{Z}$ and the generator $e^* = \frac{1}{2n}e + L$. We have $e^{*2} = (2ne^*)^2/2n = -1/2n$. Let $x = re^*$, then $x^2 = -r^2/2n$. Thus x^2 is determined by $r^2 \mod 4n$. Suppose we have a singular equation defined by the vector δ from (10.5). So we obtain that the number of orbits of hyperplanes H_δ with $-2n\delta^2 = \Delta$ with respect to the group $\mathrm{O}_0(T(d))^*$ is equal to

$$\mu(\Delta; n) := \#\{r \in \mathbb{Z}/2n\mathbb{Z} : \Delta \equiv r^2 \mod 4n\}. \tag{10.7}$$

This number [3] is equal to the number of irreducible components of the Humbert surface $\mathrm{Hum}(\Delta; D)$ in $\mathcal{A}_{2,n}$. In particular, the Humbert surface $\mathcal{A}_{2,n}(\Delta)$ is irreducible if $n = 1$. If $n = 2$, we get two components corresponding to $r = 1, r = 3 \mod 4$ and $\Delta \equiv 1 \mod 8$. For $n = 3$ we have four irreducible components corresponding to $r = 1, 2, 4, 5 \mod 6$ and $\Delta \equiv 1, 4 \mod 12$.

Applying Proposition 10.1, we obtain a proof of Humbert's Lemma 4.1. In fact, assume that $\Delta \equiv 0 \mod 4$. We write $\Delta = 4m$ and choose $\delta = mf_3 - f_4$ and obtain the singular equation $mz_1 - z_3 = 0$. If $\Delta = 4m + 1$, we choose $\delta = f_2 - f_5 + 2(f_3 - mf_4)$ to obtain the singular equation $mz_1 - z_2' - z_3' = 0$.

The divisors in the moduli spaces of lattice polarized K3 surfaces defined by requiring that the periods belong to the orthogonal complement of some vector with negative norm are called the *Heegner divisors*. The following theorem follows from the previous discussion.

Theorem 10.4. *Under the isomorphism $\mathcal{A}_{2,n} \cong \mathcal{M}_{K3,T_n}$, the image of the Humbert surface $\mathrm{Hum}_n(\Delta)$ is equal to the Heegner divisor $\mathrm{Heeg}_n(\delta)$, where*

$$\delta = -\frac{\Delta}{2n}.$$

Let A belongs to $\mathrm{Hum}_n(\Delta)$. Let $\mathfrak{o}_\Delta$ be the quadratic ring with a fixed basis such that it can be identified with the algebra (4.5), where $b = 0, 1$. Let $\mathfrak{o}_\Delta(n)$ be the corresponding quadratic lattice. We know that it is isomorphic to the sublattice $\langle L_0, L_\Delta$ of $\mathrm{NS}(A)$ from (4.8). Let T_A be the lattice of transcendental cycles of A. It is contained in the orthogonal complement of $\mathfrak{o}_\Delta(n)$ in $U \oplus U \oplus U$. It is a lattice of signature $(2, 1)$ with discriminant group (together with the discriminant quadratic form) isomorphic to the discriminant group of $\mathfrak{o}_\Delta(-n)$. Let X be an Inose-Shioda K3-surface with $T_A \cong T_X$. Then its Néron-Severi lattice is isomorphic to the orthogonal complement of T_A in $E_8^{\oplus 2} \oplus U^{\oplus 3}$. Its discriminant lattice is isomorphic to the discriminant lattice of $\mathfrak{o}_\Delta(n)$. An example of such a lattice is the lattice $E_8^{\oplus 2} \oplus \mathfrak{o}_\Delta(n)$. It follows from [85], Corollary 1.13.3 that

[3] If we write $\Delta = Df^2$, where D is square-free, then this number is equal to the number of $\mathrm{SL}(2, \mathbb{Z})$-nonequivalent primitive representations of n by all binary quadratic forms of discriminant D.

the isomorphism class of a quadratic lattice with such discriminant group consists of one element. Thus, we obtain

Theorem 10.5. *There is an isomorphism of coarse moduli spaces*

$$\mathrm{Hum}_n(\Delta) \cong \mathcal{M}_{K3,S_\Delta},$$

where

$$S_\Delta = E_8 \oplus E_8 \oplus \mathfrak{o}_\Delta(n).$$

Recall that $\mathrm{Hum}_n(\Delta)$ may consist of several irreducible components. They correspond to different embedding of the lattice S_Δ in $\mathrm{NS}(X)$.

Example 10.6. Let $n = 1$. We have

$$\mathrm{Hum}(1) \cong \mathrm{Heeg}(-1/2) \cong \mathcal{M}_{K3,E_8\oplus E_8\oplus \mathfrak{o}_1},$$

where $\mathfrak{o}_1$ is defined by the matrix $\left(\begin{smallmatrix} 2 & 1 \\ 1 & 0 \end{smallmatrix}\right)$. Obviously, it is isomorphic to U. Thus

$$\mathrm{Hum}(1) \cong \mathcal{M}_{K3,E_8\oplus E_8\oplus U}.$$

These lattice polarized K3 surfaces contain an elliptic pencil with a section and two reducible fibers of type $\tilde{E}_8$ (of type II^* in Kodaira's notation). These surfaces are studied in [16], [51], [102], [63]. The surface admits a birational model isomorphic to the quartic surface

$$y^2zw - 4x^3z + 3axzw^2 - 12(z^2w^2 + w^4) + bzw^3 = 0.$$

The equation is a special case of equation (9.2). It admits a birational model isomorphic to the double cover of $\mathbb{P}^2$ branched along the union of a cuspidal cubic C, the cuspidal line L and the union of two lines intersecting at a point on L. Note that the Heegner divisor $\mathrm{Heeg}(-1/2)$ is equal to an irreducible component of the *discriminant* in $\mathcal{M}_{K3,T_1^\perp}$ corresponding to non-ample lattice polarized K3 surfaces.

Example 10.7. Similarly, we get that $\mathfrak{o}_4 \cong \langle 2\rangle \oplus \langle -2\rangle$, hence

$$\mathrm{Hum}(4) \cong \mathcal{M}_{K3,E_8\oplus E_8\oplus\langle 2\rangle\oplus\langle -2\rangle}.$$

Note that,

$$E_8 \oplus E_8 \oplus \langle 2\rangle \oplus \langle -2\rangle \cong U \oplus E_8 \oplus E_7 \oplus \langle -2\rangle.$$

A surface polarized with this lattice admits an elliptic fibration with a section and four singular fibers of Kodaira's types I_2, III^*, II^*, I_1. These surfaces define the second irreducible component of the discriminant.

In our example, the Weierstrass equation of the genus 2 curve could be chosen in the form

$$y^2 = x^3 - t_0^4t_1^3\left(\frac{3f-e^2}{3}t_1 - t_0\right) + t_0^5t_1^5\left(fgt_1^2 - \frac{54g+9ef-2e^3}{27}t_0t_1 + \frac{3g+ef}{3f}t_0^2\right) = 0,$$

where e, f, g are some constants (see [63]). Two such collections of scalars (e, f, g) and (e', f', g') define isomorphic surfaces if and only if there exists $\lambda \neq 0$ such that $(e', f', g') = (\lambda^2 e\lambda^4 f, \lambda^6 g)$.

This shows that the moduli space of such surfaces is isomorphic to the weighted projective plane $\mathbb{P}(1,2,3)$. Thus $\mathrm{Hum}(4)\cong\mathbb{P}(1,2,3)$ that confirms Corollary 5.11.

Comparing with Kumar's Theorem 9.2, we obtain that this surface is the Shioda-Inose surface associated with the Kummer surface of the Jacobian of the curve $y^2 = f_6(x,y)$ with Clebsch invariants

$$(I_2,I_4,I_6,I_{10}) = (8(3s+r)/r, -4(3r-1), -4(6rs-8s+5r^2-2r)/r, 4rs),$$

where $r = f/e^2, s = g/e^3$ (see [64], 3.2). One can plug in these values of the invariants in the formula (5.3) to obtain that $I_{15} = 0$ to agree with Example 5.2.

One can find in [64] a similar explicit description of the Humbert surfaces of discriminants k^2 for $k\le 11$.

Example 10.8. Let us look at the Humbert surface $\mathrm{Hum}_2(1)\subset\mathcal{A}_{2,2}$. Then $\delta^2 = -1/4$ and we have 2 components corresponding to $\delta^* = 1,3 \mod 4$. In the former case, we may represent δ by a generator $\frac{1}{4}e$, where $e\in T_2$ generates $\langle -4\rangle$. Then $\delta^\perp\cong U^2$, so $\mathrm{Heeg}_2(1)\cong\mathcal{M}_{K3,U\oplus U}$ as in the case of $n=1$. In the latter case we may represent δ by $\frac{1}{4}(3e+4f-4g)\in T_2^*$. We have $\delta^\perp\cong U\oplus\langle f+g, 2e+3f+3g\rangle\cong U\oplus\langle 2\rangle\oplus\langle -2\rangle$. So, we obtain that the second irreducible component of $\mathrm{Hum}_2(1)$ is isomorphic to $\mathcal{M}_{K3,M}$, where $M\cong U\oplus E_7\oplus\langle -2\rangle$. It is isomorphic to an irreducible component of the discriminant variety in $\mathcal{M}_{K3,T_1^\perp}$. Thus we obtain that the Humbert surface $\mathrm{Hum}_2(1)$ is isomorphic to the discriminant of $\mathcal{M}_{K3,M_1}$.

Example 10.9. The lattice $\mathfrak{o}_5$ could be defined by the matrix $\left(\begin{smallmatrix}2 & 1\\ 1 & -2\end{smallmatrix}\right)$. We have

$$\mathrm{Hum}(5)\cong\mathrm{Heeg}(-5/2)\cong\mathcal{M}_{K3,E_8^{\oplus 2}\oplus\mathfrak{o}_5}.$$

The Humbert surface $\mathrm{Hum}(5)$ admits a compactification $\overline{\mathrm{Hum}}(5)$ (isomorphic to the symmetric Hilbert surface for the field $\mathbb{Q}(\sqrt{5})$. It has been explicitly constructed by F. Hirzebruch [42] (see also [56]). The ring of Hilbert modular forms (whose projective spectrum is isomorphic to $\mathrm{Hum}(5)$) is generated by four forms A,B,C,D of weights $2,6,10,15$ with a relation of degree 30

$$-144D^2 - 1728B^5 + 720AB^3C - 80A^2BC^2 + 64A^3(5B^2-AC)^2 + C^3 = 0.$$

According to F. Klein [55], II, 4,§3, this ring is isomorphic to the ring of invariants of the icosahedron group $\mathfrak{A}_5$ acing in its irreducible 3-dimensional linear representation. The projective spectrum is isomorphic to the weighted projective plane $\mathbb{P}(1,3,5)$. The surface $\mathrm{Hum}(5)$ is isomorphic to the complement of one point $[1,0,0]$. The symmetric Hilbert modular surface corresponding to a principal congruence subgroup of the Hilbert modular group associated to the ring of integers $\mathfrak{o}$ in $\mathbb{Q}(\sqrt{5}$ and the principal ideal $\mathfrak{a}$ generated by $\sqrt{5}$ has a natural action by the group $\mathfrak{o}/\mathfrak{a}\cong\mathfrak{A}_5$. According to F. Hirzebruch [44], it is $\mathfrak{A}_5$-equivariantly isomorphic to $\mathbb{P}^2$. So this explains the isomorphism $\overline{\mathrm{Hum}}(5)\cong\mathbb{P}^2/\mathfrak{A}_5$.

The projective representation of $\mathfrak{A}_5$ in $\mathbb{P}^2$ has a minimal 0-dimensional orbit that consists of 6 points, called the *fundamental points*. The blow-up of the plane at these points is isomorphic to the *Clebsch diagonal surface* $\mathcal{C}$ with automorphism group isomorphic to $\mathfrak{S}_5$ (see [28], 9.5.4). The Hilbert modular surface corresponding to the pair $(\mathfrak{o},\mathfrak{a})$ is isomorphic to the double cover of $\mathbb{P}^2$ branched along the curve of degree 10 defined by the invariant of degree 10. It has 6 singular points, the pre-images of the fundamental points under the cover. Its minimal resolution is isomorphic to the blow-up of $\mathcal{C}$ at its 10 Eckardt points.

Lecture 11

Modular forms

Let us remind some definitions and known facts about modular forms on the Siegel half-space $\mathcal{Z}_g$.

A holomorphic function $\Phi : \mathcal{Z}_g \to \mathbb{C}$ is called a *Siegel modular form of weight* w with respect to a discrete group $\Gamma \subset \mathrm{Sp}(2g, \mathbb{R})$ of automorphisms of $\mathcal{Z}_g$ if it satisfies the following functional equation

$$\Phi((A\tau + B)(C\tau + D)^{-1}) = \det(C\tau + D)^w \Phi(\tau), \quad \sigma = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \Gamma.$$

Let $\mathsf{M}_k(g, \Gamma)$ denote the complex linear space of such forms. The multiplication of functions defines the graded algebra over $\mathbb{C}$

$$\mathsf{M}(g; \Gamma) = \bigoplus_{k=0}^{\infty} \mathsf{M}_k(g, \Gamma).$$

It is called the algebra of Siegel modular forms.zeros

For example, when $\Gamma = \mathrm{Sp}(4, \mathbb{Z})$, the even part $\mathsf{M}(g; \Gamma)^{(2)}$ of this algebra is freely generated by four forms $E_4, E_6, \chi_{10}, \chi_{12}$ of weights indicated by the subscripts. The whole algebra is generated by $\mathsf{M}(g; \Gamma)^{(2)}$ and a form of degree χ_{35} of weight 35. Here

$$E_w(\tau) = \sum_{(C,D)} \det(C\tau + D)^{-w}$$

is an *Eisenstein series*, where the summation is taken over all representatives of all inequivalent block-rows of elements of $\mathrm{Sp}(4, \mathbb{Z})$ with respect to left multiplication by matrices from $\mathrm{SL}(2, \mathbb{Z})$. The other forms are expressed in terms of the Eisenstein series

$$\chi_{10} = E_4 E_6 - E_{10}, \quad \chi_{12} = 3^2 7^2 E_4^3 + 50 E_6^2 - 691 E_{12}$$

(see [48], p. 195). Thus we may also say that the graded ring $\mathsf{M}(g; \Gamma)^{(2)}$ is generated by the Eisenstein series of degrees 4,6,10, and 12.

Another way to define modular forms is by using *theta constants*. Recall that a *theta function with characteristic* $(\mathbf{m}, \mathbf{m}')$ is a holomorphic function on $\mathcal{Z}_g$ defined by the infinite series

$$\theta\left[\begin{smallmatrix}\mathbf{m}\\ \mathbf{m}'\end{smallmatrix}\right](\mathbf{z};\tau) = \sum_{\mathbf{r}\in\mathbb{Z}^g} e^{2\pi i(\frac{1}{n}\mathbf{m}+\mathbf{r})\cdot\tau\cdot{}^t(\frac{1}{n}\mathbf{m}+\mathbf{r})+2(\mathbf{z}+\frac{1}{2}\mathbf{m}')\cdot(\frac{1}{2}\mathbf{m}+\mathbf{r}))},$$

where $(\mathbf{m}, \mathbf{m}') \in (\mathbb{Z}/n\mathbb{Z})^g \times (\mathbb{Z}/n\mathbb{Z})^g$, $\mathbf{z} \in \mathbb{C}^g$ (we identify in matrix multiplication a row vector with a column vector). The corresponding *theta constant* $\theta\left[\begin{smallmatrix}\mathbf{m}\\ \mathbf{m}'\end{smallmatrix}\right](\tau)$ is the value of this function at $(0;\tau)$. One assumes here that $\mathbf{m}\cdot\mathbf{m}' = 0$, otherwise the constant is equal to zero. The main property of theta constants is the following functional equation ([50], p.176 and p.182):

$$\theta\left[\sigma\cdot\left[\begin{smallmatrix}\mathbf{m}\\ \mathbf{m}'\end{smallmatrix}\right]\right](\sigma\cdot\tau) = \kappa(\sigma)e^{2\pi i\phi_{(\mathbf{m},\mathbf{m}')}(\sigma)}\det(C\tau+D)^{\frac{1}{2}}\theta\left[\begin{smallmatrix}\mathbf{m}\\ \mathbf{m}'\end{smallmatrix}\right], \tag{11.1}$$

where $\sigma = \left(\begin{smallmatrix}A & B\\ C & D\end{smallmatrix}\right) \in \mathrm{Sp}(2g,\mathbb{Z})$, and

$$\begin{aligned}
\sigma\cdot\left[\begin{smallmatrix}\mathbf{m}\\ \mathbf{m}'\end{smallmatrix}\right] &= ((\mathbf{m},\mathbf{m}')\cdot\sigma^{-1}+\tfrac{1}{2}(C\cdot{}^tD)_0(A\cdot{}^tB)_0),\\
\phi_{(\mathbf{m},\mathbf{m}')}(\sigma) &= -\tfrac{1}{2}(\mathbf{m}\cdot{}^tD\cdot B\cdot{}^t\mathbf{m}-2\mathbf{m}\cdot{}^tB\cdot C\cdot{}^t\mathbf{m}'+\mathbf{m}'\cdot{}^tC\cdot A\cdot\mathbf{m}')\\
&\quad+\tfrac{1}{2}(\mathbf{m}\cdot{}^tD-\mathbf{m}'\cdot{}^t(A\cdot{}^tB)_0,\\
\kappa(\sigma)^8 &= 1.
\end{aligned}$$

where $()_0$ denotes the vector of diagonal elements of a square matrix. Let

$$\Gamma = \Gamma_g(n) := \{\left(\begin{smallmatrix}A & B\\ C & D\end{smallmatrix}\right) \in \mathrm{Sp}(2g,\mathbb{Z}) : B \equiv C \equiv\equiv 0 \mod n, A \equiv D \equiv I_g \mod n\}.$$

Then, $\sigma\cdot\left[\begin{smallmatrix}\mathbf{m}\\ \mathbf{m}'\end{smallmatrix}\right] = \left[\begin{smallmatrix}\mathbf{m}\\ \mathbf{m}'\end{smallmatrix}\right], \phi_{(\mathbf{m},\mathbf{m}')} = 0$, and we obtain

$$\theta\left[\begin{smallmatrix}\mathbf{m}\\ \mathbf{m}'\end{smallmatrix}\right](\sigma\cdot\tau) = \kappa(\sigma)\theta\left[\begin{smallmatrix}\mathbf{m}\\ \mathbf{m}'\end{smallmatrix}\right](\tau),$$

and $\kappa(\sigma)^2 = e^{\frac{gn}{2}\pi i}$. This implies that $\theta\left[\begin{smallmatrix}\mathbf{m}\\ \mathbf{m}'\end{smallmatrix}\right](\tau)^2$ is a modular form of weight 1 if $gn \equiv 0 \mod 4$.

A *level n-structure* on an abelian variety A is a symplectic isomorphism

$$\phi : (\mathbb{Z}/n\mathbb{Z}^{2g}, J_{\mathsf{D}}) \to H_1(A, \mathbb{Z}/2\mathbb{Z}),$$

where $H_1(A,\mathbb{Z}/2\mathbb{Z})$ is equipped with a symplectic form on $\mathrm{Im}(H)|\Lambda\times\Lambda)$ taken modulo n. The moduli space of abelian varieties with level n and polarization of type D is denoted by $\mathcal{A}_{g,\mathsf{D}}(n)$. We have

$$\mathcal{A}_{g,\mathsf{D}}(n) \cong \mathcal{Z}_g/\mathrm{Sp}(J_{\mathsf{D}},\mathbb{Z})\cap\Gamma_g(n).$$

If $D = I_g$, we set $\mathcal{A}_{g,\mathsf{D}}(n) = \mathcal{A}_g(n)$.

Let $C : y^2 = f_{2g+2}(x_0, x_)$ be an equation of a hyperelliptic curve of genus g. It is known that a choice of an order on the zeros of the binary form f_6 is equivalent to an isomorphism of symplectic spaces $\mathbb{F}_2^{2g} \to \mathrm{J}(C)[2]$. This defines a point in $\mathcal{A}_g(2)$. For $g = 2$, we have the following *Rosenhain formula* expressing the zeros of $f_6(x_0, x_1)$ in terms of theta constants. We order the zeros of f_6 to assume that they are $(0,1), (1,0), (1,1), (1,\lambda), (1,\mu), (1,\gamma)$. Then

$$\lambda = \frac{\theta\left[\begin{smallmatrix}0&0\\0&0\end{smallmatrix}\right]^2\theta\left[\begin{smallmatrix}0&0\\1&0\end{smallmatrix}\right]^2}{\theta\left[\begin{smallmatrix}0&0\\1&1\end{smallmatrix}\right]^2\theta\left[\begin{smallmatrix}0&0\\0&1\end{smallmatrix}\right]^2},\quad \mu = \frac{\theta\left[\begin{smallmatrix}0&0\\1&0\end{smallmatrix}\right]^2\theta\left[\begin{smallmatrix}1&1\\1&0\end{smallmatrix}\right]^2}{\theta\left[\begin{smallmatrix}0&0\\0&1\end{smallmatrix}\right]^2\theta\left[\begin{smallmatrix}1&1\\1&1\end{smallmatrix}\right]^2},\quad \gamma = \frac{\theta\left[\begin{smallmatrix}0&0\\0&0\end{smallmatrix}\right]^2\theta\left[\begin{smallmatrix}1&1\\1&0\end{smallmatrix}\right]^2}{\theta\left[\begin{smallmatrix}0&0\\1&1\end{smallmatrix}\right]^2\theta\left[\begin{smallmatrix}1&1\\1&1\end{smallmatrix}\right]^2}. \tag{11.2}$$

Let $V(2g+2)$ be the space of binary forms of degree $2g+2$, the group $\mathrm{SL}(2)$ acts naturally on the vector space $V(6)$ and we denote by $\mathsf{A}(2,2g+2)$ the ring of invariants $S^\bullet(V(2g+2)^*)^{\mathrm{SL}(2)}$. The relationship between the graded algebra of modular forms $\mathsf{M}(g;\mathsf{Sp}(2g,\mathbb{Z}))$ and the graded algebra of polynomial invariants $\mathsf{A}(2,2g+2)$ is given by the following theorem of Igusa [49], Theorem 4:

Theorem 11.1. *Suppose $g=2,4$ or g is odd. There exists a ring homomorphism*

$$\rho : \mathsf{M}(g;\Gamma_2(1)) \to \mathsf{A}(2,2g+2)$$

such that $\rho(\mathsf{M}(g;\mathsf{Sp}(2g,\mathbb{Z})_w) \subset \mathsf{A}(2,2g+2)_{\frac{1}{2}wg}$. If $g=2$, the homomorphism defines an isomorphism of the fields of fractions.

For example, assume $g=1$. Then it is known that $\mathsf{M}(1;\mathsf{Sp}(2,\mathbb{Z}))$ is generated by the Eisenstein series g_4,g_6 (the coefficients of the Weierstrass equation) and the ring of invariants is generated by the invariants of degree 2 and 3.

It is known since A. Clebsch and P. Gordan that the ring $\mathsf{A}(2,6)$ is generated by invariants $I_2,I_4,I_6,I_{10},I_{15}$ with a basic relation of the form $I_{15}^2 = F(I_2,I_4,I_6,I_{10})$ (see [92]). We have already encountered with the skew-invariant I_{15} in Lecture 5. We have (again up to multiplicative constants):

$$\begin{aligned}\rho(E_4) &= I_4, \quad \rho(E_6) = I_2I_4 - 3I_6, \quad \rho(\chi_{10}) = I_{10},\\ \rho(\chi_{12}) &= I_2I_{10}, \quad \rho(\chi_{35}) = I_{10}^2I_{15}\end{aligned}$$

(see [11], [49], p.848).

Note that I_{10} is equal to the discriminant of a binary sextic. Thus χ_{10} does not vanish on the jacobian locus of $\mathcal{A}_2$. It vanishes on the locus $\mathcal{A}_2^{\mathrm{decom}}$ of decomposable abelian varieties $E\times E'$ with decomposable principal polarization. We see that the divisor of zeros of χ_{35} is equal to $2\mathcal{A}_2^{\mathrm{decom}} + \mathrm{Hum}(4)$.

Let P_1^{2g+2} be the GIT-quotient of $(\mathbb{P}^1)^{2g+2}$ by the group $\mathrm{PGL}(2)$ with respect to the linearization defined by the invertible sheaf $\mathbb{L} = \mathcal{O}_{\mathbb{P}^1}^{\boxtimes 6}$ (see [26]). Its points are minimal closed orbits of ordered sets of points $(p_1,\ldots,p_{2g=2}$ on $\mathbb{P}^1$ with no more than $g+1$ points coincide. We have

$$\mathsf{P}_1^{2g+2} = \mathrm{Proj}\ \mathsf{R}_1^{2g+2},$$

where

$$\mathsf{R}_1^{2g+2} = \bigoplus_{n=0}^{\infty} H^0((\mathbb{P}^1)^{2g+2},\mathbb{L}^{\otimes n})^{\mathrm{SL}(2)}.$$

The permutation group $\mathfrak{S}_{2g+2}$ acts on $(\mathbb{P}^1)^{2g+2}$, and via this action, acts on the ring R_1^{2g+2}. The ring of invariants is isomorphic to the graded ring $\mathsf{A}(2,2g+2)$. Thus, we obtain

$$\mathsf{S}_1^{2g+2} := \mathrm{Proj}\ \mathsf{A}(2,2g+2) \cong \mathsf{P}_1^{2g+2}/\mathfrak{S}_{2g+2}.$$

By taking the double cover ramified along an unordered set of $2g+2$ points on $\mathbb{P}^1$, we can identify the hyperelliptic locus $\mathcal{H}_g$ in $\mathcal{M}_g$ with an open subset of S_1^{2g+2} of orbits of unordered sets of $2g+2$

distinct points. The pre-image of this open subset in P_1^{2g+2} can be identified with the moduli space $\mathcal{H}_g(2)$ of hyperelliptic curves together with a 2-level on its Jacobian variety. The group $\mathfrak{S}_{2g+2}$ is a subgroup of $\mathrm{Sp}(2g+2,\mathbb{F}_2)$ that acts on $\mathcal{H}_g^{2g+2}$ via changing the 2-level structure.

From now on we assume that $g=2$. By computing explicitly the algebra of invariants R_1^6 one finds that it is generated by the subspace $(\mathsf{R}_1^6)_1 = H^0((\mathbb{P}^1)^6,\mathbb{L})^{\mathrm{SL}(2)}$ of dimension 4 with a defining cubic relation that defines an $\mathfrak{S}_6$-equivariant isomorphism between P_1^6 and the *Segre cubic primal*, a cubic 3-fold in $\mathbb{P}^5$ given by equations

$$\sum_{i=0}^{5} t_i = \sum_{i=0}^{5} t_i^3 = 0$$

in $\mathbb{P}^5$. The group $\mathfrak{S}_6$ acts by permuting the variables. The Segre cubic is characterized among all cubic threefolds with at most ordinary nodes as singularities by the property that it has maximal number of nodes equal to 10. The singular points is the $\mathfrak{S}_6$-orbit of the point $[1,1,1,-1,-1,-1]$. It also has 15 planes forming the $\mathfrak{S}_6$-orbit of the plane $t_0+t_1=t_2+t_3=t_4+t_5=0$. Each plane contains 4 singular points and each singular point is contained in 6 planes. The smooth part $\mathcal{S}_3'$ of $\mathcal{S}_3$ parameterizes orbits of ordered sets of points with no more than two points coincide. As it is explained in [28], 9.4.4, the intersection of each plane with $\mathcal{S}_3'$ parameterizes the sets of points with two points coincide. The singular points represent the minimal closed orbits of sets of points where three points coincide.

The discriminant invariant I_{10} of binary forms of degree 6 is a SL(2)-invariant homogeneous polynomials in the coefficients of degree 10. If we write it in terms of roots as the product of bracket functions $(ij)^2, i<j$, we obtain an $\mathfrak{S}_6$-invariant section from $(\mathsf{R}_1^6)_{10}$. In the coordinates t_i in $\mathbb{P}^4$ it is defined by a hypersurface of degree 10. Its divisor of zeros on $\mathcal{S}_3$ is a surface of degree 30 equal to the union D of 15 planes of $\mathcal{S}_3$ taken with multiplicity 2.

It is a remarkable fact that the dual hypersurface of the Segre cubic primal $\mathcal{S}_3$ is isomorphic to Proj $\mathsf{M}(g;\Gamma_2(2))$, a compactification $\overline{\mathcal{A}_2(2)}$ of the moduli space $\mathcal{A}_2(2)$ of abelian surfaces with a 2-level structure. In fact, according to J. Igusa [49], the ring of modular forms $\mathsf{M}(g;\Gamma_2(2))$ is generated by fourth powers of 10 theta constants $\theta\left[\begin{smallmatrix}\mathbf{m}\\ \mathbf{m}'\end{smallmatrix}\right](\tau)$ generating the 5-dimensional space of modular forms of weight 2. The generators satisfy an $\mathfrak{S}_6$-invariant quartic relation such that, in appropriate choice of a basis, defines an isomorphism between $\overline{\mathcal{A}_2(2)}$ and the quartic 3-fold $\mathcal{I}_4$ defined by the following equations in $\mathbb{P}^5$:

$$\sigma_1 = \sigma_2^2 - 4\sigma_4 = 0,$$

where σ_k denote the k-th power-sums symmetric polynomials in variables x_i (see [49], [106]). The group $\mathrm{Sp}(4,\mathbb{Z})/\Gamma_2(2) \cong \mathfrak{S}_6$ acts on $\mathcal{I}_4$ by permuting the unknowns.

We have

$$H^0(\mathcal{I}_4,\mathcal{O}_{\mathcal{I}_4}(n)) \cong \mathsf{M}(2,\Gamma_2(2)_{2n}.$$

Considered as a hypersurface in $\mathbb{P}^4$, the quartic $\mathcal{I}_4$ of degree 4 in $\mathbb{P}^4$ that was classically known as the dual hypersurface of the Segre cubic primal. It was called the *Castelnuovo quartic*, but nowadays, because of the moduli interpretation, it is called the *Igusa quartic* (in [28] it is called the *Castelnuovo-Richardson quartic*). The duality map

$$\Phi : \mathcal{S}_3 \dashrightarrow \mathcal{I}_4 \tag{11.3}$$

is given by the polar quadrics of $\mathcal{S}_3$ defined by linear combinations of partial derivatives of the equation of $\mathcal{S}_3$ in $\mathbb{P}^4$

$$F_3 = t_0^3 + t_1^3 + t_2^3 + t_3^3 - (t_0 + t_1 + t_2 + t_3 + t_4)^3 = 0.$$

Let $P_i = \frac{1}{3}\frac{\partial F_3}{\partial t_i} = 3t_i^2 - 3L^2$, where $L = t_0 + t_1 + t_2 + t_3 + t_4$. If we put

$$Q_i = P_i - \frac{1}{3}(P_0 + P_1 + P_2 + P_3), \quad i = 0, \dots, 4, \tag{11.4}$$

$$Q_5 = -(t_1 + \cdots + t_4), \tag{11.5}$$

than we observe that the action of the group $\mathfrak{S}_6$ on the variables $t_0, \dots, t_4$ defines the action on the polynomials $Q_0, \dots, Q_5$ by permuting the set $\{0, \dots, 5\}$. The usual Plücker formula implies that the dual of $\mathcal{S}_3$ is a quartic hypersurface (see [28], 1.2.3). Thus the image of Φ is equal to a quartic 3-fold given by the equations $\sigma_1 = \sigma_2^2 + \lambda\sigma_4 = 0$ in variables $x_0, \dots, x_5$. Observe that

$$x_i - x_j = Q_i - Q_j = t_i^2 - t_j^2, \quad 0 \le i, j \le 5. \tag{11.6}$$

This shows that the image of the plane $t_0 + t_1 = t_2 + t_3 = t_4 + t_5 = 0$ in $\mathcal{S}_3$ is equal to the line $x_0 - x_1 = x_2 - x_3 = x_4 - x_5 = x_0 + \cdots + x_5 = 0$. After plugging in these relations in the equation of the dual hypersurface, we find that $\lambda = -4$. This gives us the equation of the Igusa quartic.

We also check that the 15 lines on $\mathcal{I}_4$ equal to the images of the 15 planes under the map Φ are the double lines. Also each line contains 3 points, and each point lies on three lines.

Via the moduli interpretation, the restriction of the map Φ to the complement of the 15 planes should be viewed as the *Torelli map* that assigns to a hyperelliptic curve of genus 2 with an order on its Weierstrass points its Jacobian variety with a 2-level structure defined by the order.

The map Φ extends to a resolution of singularities of $\mathcal{S}_3$ with exceptional divisors isomorphic to quadrics. They are mapped isomorphically to 10 quadrics contained in $\mathcal{I}_4$, taken with multiplicity 2 that are cut out by 10 hyperplanes. The intersection of the 10 quadrics with the open subset $\mathcal{A}_2(2)$ is the locus of abelian surfaces with 2 level structure that are isomorphic to the product of two elliptic curves. To find the equations of the quadrics we use the following fact about the duality map. Suppose X is a hypersurface of degree d with an isolated ordinary point x_0 of multiplicity $d-1$. Choose coordinates such that $x_0 = [1, 0, \dots, 0]$, so that the equation of X can be written in the form

$$F = x_0 F_d(x_1, \dots, x_n) + F_d(x_1, \dots, x_n) = 0.$$

Then the dual map is not defined at x_0, but the image of the exceptional divisor under the lift of the duality map to the blow-up of x_0 is equal to the hyperplane in the dual projective space corresponding to the partial derivative $\frac{\partial F}{\partial x_0} = F_d(x_1, \dots, x_n)$. Applying this to our case, by taking the singular point $[1, 1, 1, -1, -1, -1]$ of $\mathcal{S}_3$ we obtain that the image of the exceptional divisor is cut out by the hyperplane $x_0 + x_1 + x_2 = 0$. Plugging in this equation in the equation of $\mathcal{I}_4$, we easily obtain

$$(x_0x_1 + x_0x_2 + x_1x_2 + x_3x_4 + x_3x_5 + x_4x_5)^2 = 0. \tag{11.7}$$

This shows that the hyperplane cuts out $\mathcal{I}_4$ along a quadric surface taken with multiplicity 2.

Now we are ready to see an invariant-theoretical interpretation of Igusa modular forms $\chi_{10}, \chi_{12}, \chi_{35}$ when they are considered as modular forms with respect to the congruence subgroup $\Gamma_2(2)$.

Let $\mathrm{Hum}(\Delta; n)$ denote the set-theoretical preimage of the Humbert surface $\mathrm{Hum}(\Delta)$ under the cover $\mathcal{A}_2(n) \to \mathcal{A}_2$.

Considered as $\mathrm{SL}(2)$-invariant sections of the line bundle $\mathbb{L}$ on $(\mathbb{P}^1)^6$, the functions $t_i - t_j$ are expressed in terms of the bracket functions (up to a constant multiple) by the formula

$$t_i - t_j = [ab, cd, ef], \tag{11.8}$$

where $[ab, cd, ef] = (ad)(cf)(be) - (bc)(df)(fa)$ vanish on the orbits of point sets in $\mathcal{H}_2(2) \subset P_1^6$ representing bielliptic curves ([28], Proposition 9.4.9 and (9.44)). The sums $t_i + t_j$ are expressed in terms of the bracket functions by the formula

$$t_i + t_j = (ab)(cd)(ef) \in (\mathsf{R}_1^6)_1. \tag{11.9}$$

They vanish only on the union D of the 15 planes. Formulas (11.6) show that the pre-image of the hyperplane sections $x_i - x_j = 0$ of $\mathcal{I}_4$ in $\mathcal{S}_3$ is equal to the union of a plane and an irreducible component of the locus representing bielliptic curves.

Let us consider the $\mathfrak{S}_6$-invariant polynomial

$$D = \prod_{0 \le i < j < k \le 5} (x_i + x_j + x_k). \tag{11.10}$$

Since $\sigma_1 = 0$ on $\mathcal{I}_4$, when restricted to $\mathcal{I}_4$, it becomes a square of a section s_D of $\mathcal{O}_{\mathcal{I}_4}(10)$. The divisor of zeros of s_D is equal to the union of 10 quadric surfaces representing $\mathrm{Hum}(1;2)$ taken with multiplicity 2. The subgroup of $\mathfrak{S}_6$ stabilizing each irreducible component is isomorphic to $H = \mathfrak{S}_3 \times \mathfrak{S}_3$. It acts on the quadric Q defined by equation (11.7) via permuting $(0,1,2)$ and $(3,4,5)$. The ring of invariant polynomials for the action of $\mathfrak{S}_3 \times \mathfrak{S}_3$ on $\mathbb{C}[x_0, \ldots, x_5]$ is generated by $\sigma_1, \sigma_2, \sigma_3, \sigma_1, \sigma_2, \sigma_3$, where σ_i (σ_i') is an elementary symmetric polynomial in x_0, x_1, x_2 (x_3, x_4, x_5). This easily implies that the quotient Q/H is isomorphic to $\mathbb{P}(2,3,3)$. This is a compactification of $\mathrm{Hum}(1)$. The boundary is equal to the union of two lines $z_1 = 0$ and $z_2 = 0$ intersecting at the unique singular point of $\mathbb{P}(2,3,3)$.

The pre-image of the section s_D under the map Φ is a $\mathfrak{S}_6$-invariant section of $\mathbb{L}^{\otimes 20}$ that vanishes on the union of 15 planes with multiplicity 4 (since the pre-image of each $x_i + x_j + x_k$ is a polar quadric of a singular point that vanishes on 6 planes containing the point). As we remarked earlier, the discriminant invariant I_{10} vanishes on the same set with multiplicity 2. This shows that

$$\Phi^*(s_D) = I_{10}^2.$$

Recall that the divisor of zeros of s_D on $\mathcal{I}_4$ is the union of 10 quadric surfaces taken with multiplicity 2. Applying Theorem 11.1, we find that χ_{10}, considered as a modular form with respect to $\Gamma_2(2)$ vanishes on the union of the ten quadrics with multiplicity 1. If we consider χ_{10} as a section of $\mathcal{O}_{\mathcal{I}_4}(5)$, we get the equality (up to a scalar factor) of sections of $\mathcal{O}_{\mathcal{I}_4}^{10}$

$$\chi_{10}^2 = s_D.$$

It is known that

$$\chi_{10} = \Delta_5^2,$$

where

$$\Delta_5 := \prod_{\left[\begin{smallmatrix}\mathbf{m}\\ \mathbf{m}'\end{smallmatrix}\right]} \theta\left[\begin{smallmatrix}\mathbf{m}\\ \mathbf{m}'\end{smallmatrix}\right](\tau)^2.$$

However, Δ_5 does not represent a modular form, it is a modular form up to a non-trivial character taking values ± 1.

Let

$$H = \prod_{0\le i<j\le 5} (x_i - x_j). \tag{11.11}$$

The square H^2 is a $\mathfrak{S}_6$-invariant polynomial, the discriminant of a general equation of degree 6 with roots $x_0,\ldots,x_5$. Let s_H be the corresponding section of $\mathcal{O}_{\mathcal{I}_4}(30)$. It follows from (11.8) and (11.9) that the divisor of zeros of s_H is equal to the closure $\overline{\mathrm{Hum}}(4;2)$ of the surface $\mathrm{Hum}(4;2)$ in $\overline{\mathcal{A}_2(2)}$. It consists of 15 irreducible components cut out by the hyperplanes $x_i - x_j = 0$. It is easy to see from the formulas that each such component is isomorphic to the *Steiner quartic surface* in $\mathbb{P}^3$ with three concurrent double lines (see [28], p. 70). The boundary $\overline{\mathrm{Hum}}(4;2)\setminus\mathrm{Hum}(4;2)$ consists of the union of the three lines. The group $\mathfrak{S}_6$ permutes the 15 components with stabilizer subgroup isomorphic to $\mathfrak{S}_4$. The normal subgroup of $\mathfrak{S}_4$ generated by the products of two commuting transpositions acts identically on the component. Thus we obtain

$$\overline{\mathrm{Hum}}(4) \cong \overline{\mathrm{Hum}}(4;2)/\mathfrak{S}_3.$$

It is known that the equation of a Steiner quartic surface can be reduced to the form

$$t_0t_1t_2t_3 + t_1^2t_2^2 + t_1^2t_3^2 + t_2^2t_3^2 = t_0s_3 + (s_2^2 - 2s_3s_1),$$

where s_i are elementary symmetric functions in t_1,t_2,t_3. The group $\mathfrak{S}_3$ acts by permuting the variables t_1,t_2,t_3. This shows that $\overline{\mathrm{Hum}}(4)$ is isomorphic to a hypersurface of degree 4 in the weighted projective space $\mathbb{P}(1,1,2,3)$ given by the equation $z_3(z_0-2z_1)+z_2^2=0$. The union of the three singular lines in $\overline{\mathrm{Hum}}(4;2)$ has the equation $t_1t_2t_3=0$. Its image in $\overline{\mathrm{Hum}}(4)$ is given by the equation $z_3=0$. The complement is isomorphic to the affine plane $\mathbb{C}^2$.

The pre-image of S_H under the map Φ is a SL(2)-invariant section of $\mathbb{L}^{60}$ which is invariant with respect to $\mathfrak{S}_6$. It vanishes on the union of the locus of bielliptic curves with multiplicity 2 and on the union of 15 planes with multiplicity 6. We know that the invariant I_{15} vanishes on the locus of bielliptic curves and the discriminant invariant I_{10} vanishes on the union of planes with multiplicity 2. This implies that

$$\Phi^*(s_H) = I_{10}^3 I_{15}^2.$$

Comparing with Theorem 11.1, we find that

$$\chi_{35}^2 = \chi_{10}s_H.$$

Taking the square root we obtain

$$\chi_{35} = \Delta_5 \cdot \prod_{0\le i<j\le 5}(x_i - x_j)$$

As we saw before, this gives the irreducible component of (see [106], (8.3)). Note that each factor is not a modular form, but the product is.

Let us now see the surface $\mathrm{Hum}(5;2)$. We refer for the proofs to [106], 8.4. The surface $\overline{\mathrm{Hum}}(5;2)$ consists of 6 irreducible components $H_i, i = 0, \ldots, 5$. Each component H_i is given by an additional equation

$$2(\sum_{j\neq i} x_j)^2 - \sum_{j\neq i} x_j^2 = 0.$$

It contains 5 of the 15 triple points of $\mathcal{I}_4$ no two of which are on a double line. For example, H_5 contains the points $[1,1,1,1,-2.-2], [1,1,1,-2,1,-2], \ldots, [-2,1,1,1,1,-2]$. The complement to these five points is $\mathrm{Hum}(5;2)$. The plane Π_{ijk} spanned by three points is contained in one of the hyperplanes $x_i + x_j + x_k = 0$. For example, the first three points in above are contained in $x_2 + x_3 + x_5 = 0$. Thus the intersection of $\Pi_{ijk} \cap \overline{\mathrm{Hum}}(5;2)$ is a conic contained in one of the 10 quadric surfaces cut out by a hyperplane $x_i + x_j + x_k = 0$.

Consider the following divisor on $\mathcal{A}_2(2)$

$$\mathsf{G}_\Delta = \sum_{d\geq 1, d^2|\Delta} v(\Delta/d^2)_2 \mathrm{Hum}(\Delta/v^2;2),$$

where $v(k)_2 = 1/2$ if $k = 1$ and 1 otherwise.

Theorem 11.2. *The divisor G_Δ is the divisor of zeros of a Siegel modular form g_Δ of weight $-60H(2,\Delta)$.*

Here

$$\sum_{k=0}^{\infty} H(2,4k)e^{2\pi i 4kz} + \sum_{k=0}^{\infty} H(2,4k+1)e^{2\pi i(4k+1)z}$$

is a certain modular form in one variable of weight $5/2$ with respect to the group $\Gamma_0(4)$. Its first 8 nonzero coefficients $H(2,N)$ are given by $-120H(2,N) = 10, 70, 48, 120, 250, 240, 240$ for $N = 1, 4, 5, 8, 9, 12, 13$, respectively. For example, we have $\mathsf{G}_4 = \frac{1}{2}\mathrm{Hum}(1;2) + \mathrm{Hum}(4;2)$ is the divisor of the image of χ_{35} in $\mathsf{M}(2,\Gamma(2))$. The coefficient $1/2$ is explained by the fact that the map $\mathcal{A}_2(2) \to \mathcal{A}_2$ is ramified along H_1.

If $\Delta = 1$, the modular form g_1 with respect $\Gamma_2(2)$ is the discriminant Δ_5, a square root of χ_{10}. One can construct a modular form on $\mathcal{Z}_2$ that vanishes exactly on a Humber surface $\mathrm{Hum}(\Delta)$ for every Θ (see [105]).

Lecture 12

Bielliptic curves of genus 3

Let A be an abelian surface with primitive polarization L_0 of degree $n = 2$. We have $(L_0^2) = 4$ and $h^0(L_0) = 2$. We assume that $|L_0|$ has no fixed components (this could happen only if $L_0 \cong \mathcal{O}_A(E + 2F)$, where E, F are elliptic curves). Then $|L_0|$ has 4 simple base points and its general member is a smooth curve of genus 3. Translating C by some point in A, we may assume that C is symmetric in the sense that it is invariant with respect to the involutions $[-1]_A$. This implies that all members of the pencil $|L_0|$ are invariant (obviously, ι preserves the tangent directions at the base points, hence its lift to the blow-up of the base points fixes the exceptional curves pointwisely, hence it acts identically on the base of the fibration defined by the pencil). The base points are among fixed points of $\iota : C \to C$. It follows from the Hurwitz's formula that there are no more fixed points and the quotient $C/(\iota)$ is an elliptic curve. A smooth projective curve is called *bielliptic* if it admits a degree 2 cover of an elliptic curve. Conversely, suppose $\pi : C \to E$ is a degree 2 cover of an elliptic curve by a smooth curve of genus 3. Then $A = \mathrm{J}(C)/\pi^*E$ is an abelian surface. Choose a point $c_0 \in C$ and consider the composition $\tau : C \to A$ of the Abel-Jacobi embedding $i_{c_0} : C \hookrightarrow \mathrm{J}(C)$ and the projection $\mathrm{J}(C) \to A$. It follows from [5], Proposition (1.8) that this composition is a closed embedding. By the adjunction formula, $\tau(C)^2 = 4$ and $L_0 = \mathcal{O}_A(\tau(C))$ defines a primitive polarization of degree 2 on A.

Note that one can also consider the Prym variety $\mathrm{Prym}(C/E)$ defined to be the connected component of the kernel of the norm map $\mathrm{J}(C) \to E$. It is proven in loc. cit., Proposition (1.12) that it is the dual abelian surface $\hat{A}$.

Counting constants, we expect that bielliptic curves of genus 3 depend on 4 moduli, i.e. they form a subvariety of codimension 2 in $\mathcal{M}_3$. Since a general curve has at most one bielliptic involution, we see that the locus of bielliptic curves is birationally isomorphic to a $\mathbb{P}^1$-bundle over $\mathcal{A}_{2,2}$. In particular, it is a rational variety (see for another proof of this fact in [4]).

Let C be a canonical curve of genus 3 over $\mathbb{C}$ with a bielliptic involution $\tau : C \to C$. In its canonical plane model, τ is induced by a projective involution $\tilde{\tau}$ whose set of fixed points consists of a point x_0 and a line ℓ_0. The intersection $\ell_0 \cap C$ are the fixed points of τ on C.

Theorem 12.1 (S. Kowalevskaya [62])**.** *The point x_0 is the intersection point of four distinct bitangents of C. Conversely, if a plane quartic has four bitangents intersecting at a point x_0, then there* *exists a bielliptic involution τ of C such that the projective involution $\tilde{\tau}$ has x_0 as its isolated fixed point.*

Proof. Choose the projective coordinates such that $\tilde{\tau}$ is defined by the formula $(x,y,z)\mapsto(x,y,-z)$. The isolated fixed point is $x_0=(0,0,1)$ and the line of fixed points is $z=0$. Since C is invariant with respect to $\tilde{\tau}$, its equation $f(x,y,z)=0$ of C can be written in the form

$$f(x,y,z)=z^4-2a_2(x,y)z^2+a_4(x,y)=(z^2-a_2(x,y))^2+(a_4(x,y)-a_2(x,y)^2)=0. \quad (12.1)$$

The equation $(a_4(x,y)-a_2(x,y)^2=0$ is the equation of the union of four lines $\ell_1,\ldots,\ell_4$ passing through the point $x_0=(0,0,1)$. Each line $\alpha_i x-\beta_i y_i=0$ is tangent to C at two points $p_i^{\pm}=(\beta_i,\alpha_i,\pm\sqrt{a_2(\beta_i,\alpha_i)})$. Note that the four lines are distinct since otherwise the curve has a singular point at some point $((\beta_i,\alpha_i,\pm\sqrt{a_2(\beta_i,\alpha_i)})$. Also note that, if $a_2(\beta_i,\alpha_i)=0$, then the point $p_i^+=p_i^-$ is the undulation point, i.e. the two tangency points coincide. The quartic curves with an undulation point is hypersurface in the space of quartics given by the known undulation invariant I_{60} of degree 60 (see [20] and [88]).

Conversely, suppose that four bitangents $\ell_1,\ldots,\ell_4$ intersect at a point x_0. By Proposition 6.1.4 from [28], any three of the lines form a syzygetic triad of bitangents, i.e. the corresponding six tangency points lie on a conic. This implies that all eight tangency points lie on a conic. Choose coordinates so that $x_0=(0,0,1)$. Let $\ell_i: l_i=0$ and $B_2(x,y,z)=0$ be the equation of the conic K passing through the eight tangency points. Then the curves $V(B_2^2)$ and $V(l_1\cdots l_4)$ cut out the same divisor on C, hence the equation of C can be written in the form $F=B_2^2+l_1l_2l_3l_4=0$, where $\ell_i=V(l_i)$ and $B_2=a_0z^2+2a_1(x,y)z+a_2(x,y)$. If $a_1\neq 0$, we replace z with $a_0z+a_1(x,y)$ to assume that $a_1(x,y)=0$. Now the equation of C is reduced to the form (12.1). The involution $(x,y,z)\mapsto(x,y,-z)$ is the bielliptic involution of C. □

Here is another characterization of bielliptic quartic curves.

Theorem 12.2. *C is bielliptic if and only if the following conditions are satisfied:*

(i) There exists a line ℓ intersecting C at four distinct points $p_1,\ldots,p_4$ such that the tangent lines ℓ_i at the points p_i intersect at one point p_0.

(ii) Let $P_{p_0}(C)$ be the cubic polar of C with respect to the point p_0 and let Q be the conic component of $P_{p_0}(C)$ (note that the line ℓ from above is a line component of $P_{p_0}(C)$). Then ℓ is the polar line of Q with respect to p_0.

Proof. Suppose C is bielliptic. Applying the previous theorem, we may assume that it is given by the equation (12.1). The polar cubic $P_{x_0}(C)$ has the equation $q=z(z^2-a_2(x,y))=0$. It is the union of the line $\ell_0=V(z)$ and the conic $Q=V(z^2-a_2(x,y))$. The line ℓ_0 intersect C at the points $(\beta_i,\alpha_i,0)$, where $a_4(\beta_i,\alpha_i)=0$. By the main property of polars, $P_{x_0}(C)$ intersects C at the points p such that the tangent line of C at p contains the point x_0. Thus the tangent lines of C at the intersection points of ℓ_0 with C pass through the point x_0. This verifies the first property. Let us check the second one. Using the equation, we compute the line polar $P_{x_0^3}(C)=V(\frac{\partial^3}{\partial z^3}(F))$ of C. It coincides with the line ℓ_0. On other hand

$$P_{x_0^3}(C)=P_{x_0^2}(P_{x_0}(C))=P_{x_0^2}(qz)=P_{x_0}(q+P_{x_0}(q)z)=2P_{x_0}(q)+P_{x_0^2}(q)z=z$$

(where we identify the polar curves with the corresponding partial derivatives). This implies that $V(P_{x_0}(q)) = V(z) = \ell_0$. This checks property (ii).

Let us prove the converse. Choose coordinates to assume that $\ell = V(z)$ and the intersection point of the four tangent lines is $x_0 = (0,0,1)$. The cubic polar $P_{x_0}(C)$ must contain the line component equal to ℓ. Write the equation of C in the form

$$a_0z^4 + a_1(x,y)z^3 + a_2(x,y)z^2 + a_3(x,y)z + a_4(x,y) = 0.$$

We get

$$P_{x_0}(C) = V(4a_0z^3 + 3a_1(x,y)z^2 + a_2(x,y)z + a_3(x,y)),$$

$$P_{x_0^2}(C) = V(12a_0z^2 + 6a_1(x,y)z + a_2(x,y)),\ P_{x_0^3}(C) = 24a_0z + 6a_1(x,y)$$

Since z divides the equation of the cubic polar, we obtain that $a_3(x,y) = 0$. If $a_0 = 0$, then $x_0 \in C$ and the line polar $P_{x_0^3}(C)$ vanishes at x_0. But this polar is the tangent line of C at x_0. This implies that C is singular at x_0. So, we may assume that $a_0 \neq 0$. Thus the first condition implies that C can be written in the form

$$z^4 + a_1(x,y)z^3 + a_2(x,y)z^2 + a_4(x,y) = 0.$$

Now as in the first part of the proof, we obtain that $a_1(x,y) = 0$ if and only if condition (ii) is satisfied. Thus C can be written in the form (12.1), and hence it is a bielliptic curve. □

For any general line ℓ, let $\ell_1,\dots,\ell_4$ be the tangents of C at the points $C \cap \ell$. Let $\ell_i \cap C = 2a_i + c_i + d_i$. adding up, we see that $\sum(c_i + d_i) \sim 4K_C - 2\sum a_i \sim 4K_C - 2K_C = 2K_C$. This shows that there exists a conic $S(\ell)$ that cuts out on C the divisor $\sum(c_i + d_i)$ of degree 8. This conic is called the *satellite conic* of ℓ (see [19]). The map $S : \mathbb{P}^2 \dashrightarrow \mathbb{P}^5, \ell \mapsto S(\ell)$ is given by polynomials of degree 10 whose coefficients are polynomials in coefficients of C of degree 7. Since $2\ell + S(\ell)$ and $T = \ell_1 + \cdots + \ell_4$ cut out on C the same divisor, we obtain that the equation of C can be written in the form

$$F = l_1 \cdots l_4 + l^2 q = 0,$$

where $\ell_i = V(l_i), \ell = V(l)$, and $S(\ell) = V(q)$.

Assume that ℓ has the property

(*) the four tangents ℓ_i intersect at a common point x_ℓ.

Choose the coordinates such that $x_0 = (0,0,1)$ and $l = z$. Then the equation of C is of the form

$$F = z^2(a_0z^2 + a_1(x,y)z + a_2(x,y)) + a_4(x,y) = 0.$$

It is a bielliptic curve if and only if $a_1(x,y) = 0$. This is equivalent to that $P_{x_0}(S(\ell)) = \ell$. Thus we obtain

Theorem 12.3. *Suppose a line ℓ satisfies the property (*) from above. Then C is a bielliptic curve if and only if the polar line of the satellite conic $S(\ell)$ with respect to the point x_ℓ coincides with ℓ.*

Let ℓ be a line satisfying (*). The polar cubic of $P_{x_\ell}(C)$ passes through $C \cap \ell$, hence it contains ℓ as an irreducible component. In particular, $P_{x_\ell}(C)$ is singular. Recall that the locus of points $x \in \mathbb{P}^2$ such that $P_x(C)$ is a singular cubic is the Steinerian curve $\mathrm{St}(C)$ [28], 1.1.6. If C is a general enough, the degree of $\mathrm{St}(C)$ is equal to 12 and it has 24 cusps and 21 nodes. The cusps correspond to points such that the polar cubic is cuspidal, the nodes correspond to points such that the polar cubic is reducible. The line components define the set of 21 lines satisfying property (*). In [19] the 21 lines are described as singular points of multiplicity 4 of the curve of degree 24 in the dual plane parameterizing lines ℓ such that the tangents to C at three intersection points of C and ℓ are concurrent.

According to [20], the equation of the satellite conic $S(\ell)$ is equal to

$$SC_{7,2,10} + lC_{7,1,9} + l^2C_{7,0,8} = 0,$$

where $C_{a,b,c} \in S^a(S^4(V^*)^*) \otimes S^b(V) \otimes S^c(V^*)$ is a commitant of degree a in coefficients of C, of degree b in coordinates in the plane and the degree c in the dual coordinates. Thus the vanishing of $a_1(x,y)$ from above is equivalent to the vanishing of the commitant $C_{7,1,9}$. The loc. cit. paper of Cohen gives an explicit equation of $C_{7,1,9}$.

Theorem 12.4. *C is bielliptic if and only if $C_{7,1,9}$, considered as a map $\mathbb{P}(V) \dashrightarrow \mathbb{P}(V^*)$ has one of the 21 lines corresponding to the nodes of $\mathrm{St}(C)$ as its indeterminacy point. The rational map is given by polynomials of degree 9 with polynomial coefficients in coefficients of C of degree 7. So, this gives in principle, the equations of the locus of bielliptic curves.*

Next we assume that C is a hyperelliptic curve of genus 3. It is given by an equation in $\mathbb{P}(1,1,4)$

$$z^2 - f_8(x,y) = 0,$$

where f_8 is a binary form of degree 8 without multiple zeros. Any involution of C different from the hyperelliptic involution $\iota_h : (x,y,z) \mapsto (x,y,-z)$ defines an involution of $\mathbb{P}^1$. After choosing an appropriate coordinates (x,y), it can be written in the form $(x,y) \mapsto (x,-y)$. In these coordinates, the binary octic, being invariant, must be of the form $f_8 = g_4(x^2,y^2)$, where $g_4(u,v)$ is a binary quartic. Since f_8 has no multiple roots, the fixed point $0,\infty$ are not among its zeros (otherwise f_8 is divisible by x or y and cannot be written in the form $g_4(x^2,y^2)$).

The involution $\iota_b : (x,y,z) \mapsto (x,-y,z)$ has four fixed points $(0,1,\pm 1$ and $(1,0,\pm 1)$. The quotient is an elliptic curve with equation $w^2 = g_4(u,v)$. The involution $\iota_h \circ \iota_e : (x,y,z) \mapsto (x,-y,-z)$ has no fixed points. The quotient is a curve D of genus 2. I believe that $\mathrm{Prym}(C/D) \cong E$.

We have already remarked in the previous Lecture that putting an order on the set of zeros of f_8 is equivalent to putting a level 2-structure on $\mathrm{J}(C)$. Let $\mathcal{A}_3(2)$ be the moduli space of principally polarized abelian 3-folds with level 2-structure and let $\mathcal{H}yp_3$ in be the hypersurface in $\mathcal{A}_3$ of Jacobians of hyperelliptic curves of genus 3. Its pre-image in $\mathcal{A}_3(2)$ splits in $36 = [\mathrm{Sp}(8,\mathbb{F}_2) : S_8]$ irreducible components permuted by $\mathrm{Sp}(8,\mathbb{F}_2)$. One of this components $\mathcal{H}_3(2)^0$ corresponds to a special symplectic basis in $H_1(C,\mathbb{Z})$ defined by the Weierstrass points of C. It is isomorphic to the GIT-quotient Y_8 of the variety of 8 distinct ordered 8 points in $\mathbb{P}^1$ modulo the group $\mathrm{PGL}(2)$. An involution $(x,y) \mapsto (x,-y)$ divides the set of zeros of f_8 into four orbits that belong to the same

g^1_2 on $\mathbb{P}^1$. As we know from Example 5.2, the condition is that the four binary forms defining these orbits are linearly dependent. Let $\pi : Y_8 \to Y_6$ be the projection $(p_1,\ldots,p_8) \mapsto (p_1,\ldots,p_6)$. The pre-image of a set of points corresponding to 3 pairs of points defining a bielliptic curve of genus 2 is isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$.

Let us identify the points $(p_1,\ldots,p_8)$ with the images in $\mathbb{P}^2$ under the Veronese map $\mathbb{P}^1 \to \mathbb{P}^2$. The four pairs (p_i, p_{i+1}) define a bielliptic curve of genus 3 if and only the lines $\langle p_i, p_{i+1}\rangle$ intersect at one point. Thus the locus $\mathcal{H}yp_3^{\mathrm{biel}}(2)^0$ of bielliptic curves in $\mathcal{H}yp_3(2)^0$ is projected to the variety $\mathcal{M}_2^{\mathrm{biel}}$ of bielliptic curves of genus 2 with a level 2 structure on its Jacobian. The fibers are isomorphic to the pre-image of a line under the map $\mathbb{P}^1 \times \mathbb{P}^1 \to (\mathbb{P}^1)^{(2)} \cong \mathbb{P}^2$. They are conic in $\mathbb{P}^1 \times \mathbb{P}^1 \subset \mathbb{P}^3$. It is known that the GIT-compactification P_1^6 of Y_6 is isomorphic to the Segre cubic primal S_3. From the previous Lecture we learn that that the condition that six points define a bielliptic curve is that the product of the differences $x_i - x_j$ is equal to zero. It consists of 15 irreducible components transitively permuted under $\mathfrak{S}_6$. Each irreducible component is isomorphic to a hyperplane section of S_3. It is isomorphic to a rational cubic surface. This shows that $\mathcal{H}yp_3^{\mathrm{biel}}(2)^0$ consists of 15 irreducible components each isomorphic to a conic fibration over a rational surface. It implies that $\mathcal{H}yp_3^{\mathrm{biel}}$ is birationally isomorphic to each such component and hence is a rational variety. An algebraic proof of this fact can be found in [98].

Remark 12.5. Let $tf_4(x,y,z) + g_2(x,y,z)^2 = 0$ be a pencil of plane quartics, where $V(f_4)$ is a nonsingular quartic curve and $V(g_2)$ is a nonsingular conic. For each t corresponding to a smooth quartic, we have 28 bitangents. When t goes to zero, these bitangents go to 28 chords connecting 8 intersection points $V(f_4) \cap V(g_2)$ (see [15], 5.3). This relates the Kowalevskaya's Theorem with the previous characterization of hyperelliptic bielliptic curves of genus 3.

Let $C = V(f_4(x,y,z))$ be a nonsingular plane quartic. The quartic surface X given by the equation

$$w^4 + f_4(x,y,z) = 0$$

is a nonsingular K3 surface. It admits an automorphism σ of order 4, a generator of the group of deck transformations of the cover. The surface X can be also viewed as the double cover of the del Pezzo surface S of degree 2 given by the equation

$$u^2 + f_4(x,y,z) = 0.$$

Since S is isomorphic to the blow-up of 7 points in the plane, $\mathrm{Pic}(S) \cong I^{1,7}$, the standard odd unimodular hyperbolic lattice. This easily implies that $\mathrm{Pic}(X) \cong S := \langle 2\rangle \oplus \langle -2\rangle^{\oplus 7}$. Using Nikulin's results [85], one can show that

$$T_X \cong T := \langle 2\rangle^{\oplus 2} \oplus D_4^{\oplus 3}.$$

The automorphism σ acts on T_X and equips it with a structure of a quadratic lattice L of rank 7 over the ring of Gaussian integers $\mathbb{Z}[i]$. It is isomorphic to the lattice T where $i = \sqrt{-1}$ acts preserving each direct summand and equal to the direct sum of the operators given by the following matrices

$$J_1 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad J_2 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 \\ -1 & 1 & 2 & 1 \end{pmatrix}.$$

Using this action one equips the 14-dimensional linear space $L_{\mathbb{R}}$ with a structure of a complex linear space V of dimension 7. Let

$$\mathbb{B}_6 = \{[z] = [z_0, \ldots, z_6] \in |V| : z_1^2 + \cdots + z_7 < z_0^2, z_0 \neq 0\} \cong \{(z_1, \ldots, z_6) \in \mathbb{C}^6 : z_1^2 + \cdots + z_6 < 1\}.$$

It is a complex ball of dimension 6. The moduli space $\mathcal{M}_{K3,S,\phi}$ of lattice S polarized K3 surfaces together with an isomorphism $\phi : T \to L$ of $\mathbb{Z}[i]$-lattices is isomorphic to the quotient

$$\mathbb{B}_6/\Gamma,$$

where Γ is a certain arithmetic group acting discretely on the ball (see [59]). For any primitive vector $\delta \in L^\vee \subset V^\vee$ one defines a hyperplane

$$H_\delta = \{z \in |V| : \delta(z) = 0\} \cap \mathbb{B}_6.$$

The image of the union of H_δ with fixed $r = \delta^2 = -2n$ in $\mathcal{M}_{K3,S,\phi}$ is denoted by $\mathrm{Heeg}(n)$ and is called the *Heegner divisor*.

Let $\Lambda(\delta) = \langle \delta, \sigma^*(\delta) \rangle$. One checks that $\Lambda(\delta) \cong \langle -2n \rangle^{\oplus 2}$. It is clear that $H_\delta = H_{\sigma^*(\delta)}$, os that H_δ is described by a primitive embedding of Λ_δ in L. Suppose that the period point of X belongs to H_δ. Then $S \oplus \Lambda_\delta$ primitively embeds in Pic(X), so that means that X acquires two additional linearly independent cycles. All vectors δ with fixed $\delta^2 = -2n$ are divided into two types according to whether $\frac{1}{2}\delta$ belongs to $L^\vee$ or not (types 1 and 2, respectively). They exists for any n and any type ([3], Proposition 3.4). We denote by $\mathrm{Heeg}(n)_i$ the image of the union of hyperplanes $H(\delta)$ with $\delta^2 = -2n$ and δ is of type $i = 1, 2$.

For example, $\mathrm{Heeg}(1)$ consists of two irreducible components $\mathrm{Heeg}(1)_1$ and $\mathrm{Heeg}(1)_2$. They parameterize, accordingly, the nodal quartic curves and the locus of hyperelliptic curves.

An irreducible plane curve D is called a *splitting curve* (cf. [3], Definition 4.4) if under the cover $X \to \mathbb{P}^2$ its pre-image splits in the union of four irreducible components. For example, a line intersecting $W = V(f_4)$ at one point is a splitting line. The main result of Artebani's paper is the following.

Theorem 12.6 (M. Artebani [3])**.** *If X belongs to $\mathrm{Heeg}(n)_i, n > 1$, then the quartic $C = V(f_4)$ admits a rational splitting curve of minimal degree $2(n-1)$ if $i = 1$ and degree $n - 2$ if $i = 2$. Moreover, C admits a splitting curve of odd degree if and only if X belongs to some $\mathrm{Heeg}(n)_2$.*

Here are examples:

- $\mathrm{Heeg}(3)_2$ is the locus of quartics admitting a hyperflex (i.e. a line intersecting the quartic at one point).
- $\mathrm{Heeg}(2)_1$ is the locus of quartics admitting a splitting conic.

Note that $\mathrm{Heeg}(3)_2$ is given by vanishing of an invariant of degree 60 on the space of quartics (see [20], [88]). We do not know whether it corresponds to the zero divisor of some automorphic form on the ball $\mathbb{E}_6$. However, S. Kondō [61] constructs such automorphic forms for the Heegner divisors $\mathrm{Heeg}(1)_1$ and $\mathrm{Heeg}(1)_2$.

Remark 12.7. Every C admits a splitting curve of degree 4. To see this, take D be defined by the equation $l^4 + f_4(x, y, z) = 0$, where l is a linear form. Then D intersects C at four points $V(l) \cap C$. The pre-image of D on X splits in four plane sections $w^4 + l^4 = 0$. However, this obviously does not give rise to a Heegner divisor, see [3], Remark 4.11.

Suppose $A = \mathrm{J}(C)$ for some curve C of genus 3. It is easy to see from the description of moduli spaces of abelian varieties with the given type of endomorphsms that the condition that $\mathrm{End}(A) \neq \mathbb{Z}$ is not divisorial. However, it is interesting to investigate whether one can express this condition as the intersection of Heegner divisors.

Lecture 13

Complex multiplications

We have already discussed elliptic curves with complex multiplication. This time, as promised, we go to higher dimension. Let us start with an example. Suppose A admits an automorphism g of order m. Let $\Phi_m(x)$ be the cyclotomic polynomial, a minimal polynomial of the cyclotomic field $\mathbb{Q}(\zeta_m)$. Then

$$\mathbb{Q}(\zeta_m) \cong \mathbb{Q}[x]/(\Phi_m(x)) \hookrightarrow \text{End}(A)_{\mathbb{Q}}, \quad x \mapsto g.$$

The Galois group of $\mathbb{Q}(\zeta_m)$ is isomorphic to the group of invertible elements in the ring $\mathbb{Z}/m\mathbb{Z}$ and its order is equal to $\phi(m)$. Let $m = p$ be an odd prime, the field $\mathbb{Q}(\zeta_{p^k})$ is a cyclic extension of $\mathbb{Q}$. It is a quadratic extension of a totally real subfield $\mathbb{Q}(\eta), \eta = \zeta_{p^k} + \zeta_{p^{-k}}$, by a complex number ζ_{p^k}. If $p = 2$ and $k > 2$, then $\mathbb{Q}(\zeta_{p^k}) = \mathbb{Q}(\eta, \sqrt{-1})$ and the Galois group is the direct product of two cyclic groups of orders 2^{k-2} and 2.

A cyclotomic field is an example of a ***CM-field***, an imaginary quadratic extension K of a totally real field K_0. This means that $K = K_0(\alpha)$, where $\rho(\alpha^2) < 0$ for all embedding $\rho : K \hookrightarrow \mathbb{C}$. Equivalently, a CM-field K can be characterized by the property that there exists a non-trivial automorphism ι of K (called the *conjugation*) that commutes with any embedding $\rho : E \hookrightarrow \mathbb{C}$. The Galois closure of a CM-field in any larger field is known to be a CM-field.

We say that a simple abelian variety has a *complex multiplication* if its endomorphism ring $\text{End}(A)_{\mathbb{Q}}$ belongs to the forth type, i.e the Rosati involution acts non-trivially on the center K. In this case $e = 2e_0$ and $e_0 d^2 | g$. If, additionally, $e_0 = g$, then A is of CM-type (see Lecture 2). It follows from the classification of endomorphism algebras of simple abelian varieties that in this case $\text{End}(A)_{\mathbb{Q}}$ is a CM-field. An abelian variety of CM-type is characterized by the property that

$$[\text{End}(A)_{\mathbb{Q}} : \mathbb{Q}]_{\text{red}} = 2 \dim A.$$

This is equivalent to that A is isogenous to the product of simple abelian varieties of CM-type, or, equivalently, that $\text{End}(A)_{\mathbb{Q}}$ is the product of CM-fields. In Chapter 3, we have already studied one-dimensional abelian varieties of CM-type.

Obviously, abelian varieties of CM-type admit real multiplication by a field of degree g. In particular, in the case $g = 2$, their isomorphism classes are points in the Humbert surface.

Example 13.1. Suppose a simple abelian variety A admits an automorphism of prime order $p > 2$. Then $e_0 \leq g$, hence the degree $\frac{1}{2}(p-1)$ of the real subfield of $\mathbb{Q}(\zeta_p)$ is less than or equal to g, hence $p \leq 2g+1$. For example, a simple abelian surface does not have automorphisms of prime order > 5. An example of an abelian variety of dimension g admitting an automorphism of order $p = 2g+1$ is the Jacobian of the hyperelliptic curve

$$C_p : y^2 = x^p - 1. \tag{13.1}$$

The Jacobian of the curve C_5 defines one of the 19 isomorphism classes of principally polarized abelian surfaces of CM-type defined over $\mathbb{Q}$ (see [83], [112]). The corresponding CM-fields are

$$\mathbb{Q}(\sqrt{-(2+\sqrt{2})}),\ \mathbb{Q}(\sqrt{-(5+2\sqrt{5})}),\ \mathbb{Q}(\sqrt{-(13+2\sqrt{13})}),\ \mathbb{Q}(\sqrt{-(29+2\sqrt{29})}),$$
$$\mathbb{Q}(\sqrt{-(37+6\sqrt{37})}),\ \mathbb{Q}(\sqrt{-(53+2\sqrt{53})}),\ \mathbb{Q}(\sqrt{-(61+6\sqrt{61})}),\ \mathbb{Q}(\sqrt{-5(2+\sqrt{2})}),$$
$$\mathbb{Q}(\sqrt{-(5+\sqrt{5})}),\ \mathbb{Q}(\sqrt{-13(5+2\sqrt{5})}),\ \mathbb{Q}(\sqrt{-17(5+2\sqrt{5})}),\ \mathbb{Q}(\sqrt{-(13+3\sqrt{13})}),$$
$$\mathbb{Q}(\sqrt{-5(13+2\sqrt{13})}).$$

The field $\mathbb{Q}(\sqrt{-(5+2\sqrt{5})})$ is equal to $\mathbb{Q}(\zeta_5)$ and corresponds to the curve C_5. Note that the Shioda-Inose K3 surface associated to this curve admits a non-symplectic automorphism of order 5. The surface admits an elliptic fibration with two reducible fibers of types $\tilde{E}_8$ and $\tilde{E}_7$ with Weierstrass equation

$$y^2 = x^3 + t^3x - t^7 = 0$$

[58]. Note that the rank of the Mordell-Weil group of this fibration is equal to 1 (and not 0 as was in the case of Example 9.1). The surface can be also given by the following equation in the weighted projective space $\mathbb{P}(5,7,8,20)$

$$x^8 + xy^5 + z^5 + w^2 = 0.$$

The group of order 5 acts by $(x,y,z,w) \mapsto (x,y,\zeta_5 z,w)$. The Picard lattice is isomorphic to $E_8^{\oplus 2} \oplus \left(\begin{smallmatrix} -2 & 1 \\ 1 & 2 \end{smallmatrix}\right)$ (see [8], [68]). Finally, note that the isomorphism class of the associated abelian surface belongs to the Humbert surface $\mathrm{Hum}(5)$.

More generally, for any prime p and $0 < a < p$, the Jacobian of the normalization of the curve

$$y^p = x^a(x^{p^{e-1}} - 1), \tag{13.2}$$

is a simple abelian variety of dimension $p^{e-1}(p-1)/2$ with complex multiplication by $\mathbb{Q}(\zeta_{p^e})$ [2].

One defines a *CM-algebra* to be a finite product of CM-fields. A not necessary simple abelian variety is called of CM-type if $\mathrm{End}(A)_\mathbb{Q}$ contains an étale subalgebra of dimension $2\dim A$ (it will be a CM-algebra). Equivalently, $[\mathrm{End}(A)_\mathbb{Q}) : \mathbb{Q}]_{\mathrm{red}} = 2\dim A$. If A is isogenous to the products $\prod_{i=1}^k A_i^{n_i}$, where $A_1,\ldots,A_k$ are simple not pairwise isogenous abelian varieties, then A is of CM-type if and only each A_i is of CM-type. In this case, $\mathrm{End}(A)_\mathbb{Q} \cong \prod_{i=1}^k \mathrm{Mat}_{n_i}(K_i)$, where $K_i = \mathrm{End}(A_i)_\mathbb{Q}$ is a CM-field. The converse is also true (see [75]).

Let R be a CM-algebra and $\iota : R \to R$ be an automorphism that induces the conjugation on each CM-field component. The set of $\mathbb{Q}$-homomorphisms a CM-algebra R to $\mathbb{C}$ consists of pairs

$(\rho, \iota \circ \rho)$. A choice of one element in each pair gives a set Φ of homomorphisms and the pair (R, Φ) is called a *CM-type* of R. One can construct an abelian variety of CM-type as follows. Let (R, Φ) be a CM-type. Choose a lattice L in R, i.e. a free $\mathbb{Z}$-submodule of R of rank equal to $[R : \mathbb{Q}]$. Let $\mathfrak{o} = \{x \in R : x \cdot L \subset L\}$. This is an order in R. We have a natural pairing

$$R \times \Phi \to \mathbb{C},\ (r, \rho) \mapsto \rho(r).$$

It defines an isomorphism

$$R_{\mathbb{R}} \to \mathbb{C}^{\Phi} \cong \mathbb{C}^{\frac{1}{2}[R:\mathbb{Q}]}.$$

The image of L is a lattice Λ in $\mathbb{C}^{\Phi}$, we set $A = \mathbb{C}^{\Phi}/\Lambda$. Let $\mathfrak{o} = \{x \in R : x \cdot L \subset LK\}$. This is an order in R, and $\mathfrak{o} \subset \mathrm{End}(A)$ so that $R \subset \mathrm{End}(A)_{\mathbb{Q}}$. To define a polarization we consider the following bilinear form on R

$$E : R \times R \to \mathbb{Q}, \quad (x, y) \mapsto \mathrm{Tr}_{R/\mathbb{Q}}(\alpha x \bar{y}),$$

where $\alpha \in R^*$ satisfies

(i) $\bar{\alpha} = -\alpha$,

(ii) $\mathrm{Im}(\rho(\alpha) > 0$ for all $\rho \in \Phi$.

One can always find such α. It follows from (i) that E is a skew-symmetric bilinear form. Also it implies

$$E(x, y) = \sum_{\rho \in \Phi} \mathrm{Tr}_{\mathbb{C}/\mathbb{R}}(\rho(\alpha x \bar{y})) = \rho(\alpha)(x\bar{y} - \bar{x}y),$$

hence,

$$E(ix, iy) = \sum_{\rho \in \Phi} \mathrm{Tr}_{\mathbb{C}/\mathbb{R}}(\rho(\alpha i x \bar{i y})) = E(x, y),$$

and, using (i) and (ii),

$$E(ix, y) = \sum_{\rho \in \Phi} \mathrm{Tr}_{\mathbb{C}/\mathbb{R}}(\rho(\alpha i x \bar{y})) = i\rho(\alpha)(x\bar{y} + \bar{x}y) > 0,$$

Thus, E defines a polarization on A. Its type is equal to the discriminant of the bilinear form E restricted to the lattice L.

Abelian varieties of CM-type do not vary in families. They are isolated points in $\mathcal{A}_{g,n}$. However, less restrictive condition that the endomorphism algebra is of type IV, allow one to construct the moduli space. We refer to [67], Chapter 9 for the general theory. Note that in this case the Hermitian symmetric spaces of unitary type appear.

Lecture 14

Hodge structures and Shimura varieties

Let V be a finite-dimensional vector space over $\mathbb{R}$. Recall that a *Hodge structure* on V consists of a direct sum decomposition

$$V_{\mathbb{C}} = \bigoplus_{p,q\in\mathbb{Z}} V^{p,q} \tag{14.1}$$

such that $\overline{V^{p,q}} = V^{q,p}$. We say that the Hodge structure is of *weight* n if $V^{p,q} = 0$ for $p + q \neq n.$.

One restates the definition of a Hodge structure of weight n by introducing the *Hodge filtration*

$$V = F^0 \supset F^1 \supset \cdots \supset F^p \supset \{0\}$$

where $F^p = \oplus_{p'\geq p} V^{p',q}, \quad p = 0, \ldots, n$. We require that $F^p \oplus \bar{F}^{n-p+1} = V$ and put $F^{p+1}/F^p \cong V^{p,q}$.

A *polarized Hodge structure* consists of a Hodge structure on V and a a non-degenerate bilinear form $Q : V \times V \to \mathbb{R}$ satisfying the following properties

(i) the conjugation map $V_{\mathbb{C}} \to V_{\mathbb{C}}$ maps induces an isomorphism $\bar{V}^{p,n-p} \cong V^{n-p,p}$;

(ii) $Q(a, b) = (-1)^n Q(b, a)$;

(iii $Q_{\mathbb{C}}(V^{p,q}, V^{p',q'}) = 0, \quad p' \neq n - p$;

(iv) $i^{p-q} Q_{\mathbb{C}}(V^{p,n-p}, V^{n-p,p}) > 0$.

A *rational (integral) polarized Hodge structure* of weight n is defined by an additional choice of a $\mathbb{Q}$-vector space L (a lattice of rank equal to $\dim V$) such that $L_{\mathbb{R}} = V$ and Q is obtained from a bilinear form on L after tensoring with $\mathbb{R}$.

One can define the *category of rational poilarized Hodge structures* by taking for morphisms linear maps that preserve the Hodge filtrations and are compatible with the bilinear forms. One also put Hodge structure on the tensor product $V \otimes W$ by setting

$$(V \otimes W)^{p,q} = \bigoplus_{r+r'=p, s+s'=q} V^{r,s} \otimes W^{r',s'}.$$

and on the dual space by setting $F^p(V^*) = (V/F^{-p})^* = (F^{-p})^\perp$. In this way the standard pairing $V \otimes V^* \to \mathbb{R}$, where $\mathbb{R} = \mathbb{R}^{0,0}$ becomes a morphism of Hodge structures. In particular, $(V^*)^{p,q} = V^{-p,-q}$.

Example 14.1. Define the Hodge structure $\mathbb{Z}(m)$ of weight $-2m$ on $\mathbb{R}$ by setting $V_\mathbb{C} = V^{-m,-m}$ with the polarization form $Q(x,y) = xy$. It has an integral structure with respect to the lattice $\mathbb{Z}$ in $\mathbb{R}$. Let $(V^{p,q})$ be a Hodge structure of weight n on a vector space V. Then $V(m) := (V^{p,q}) \otimes \mathbb{Z}(m)$ is isomorphic to the Hodge structure $('V^{p,q})$ of weight $n-2m$ on V with $V(m)^{p,q} = V^{p-m,q-m}$. If $(V^{p,q})$ admits an integral structure defined by a lattice L in V, then $(V^{p,q}(m))$ admits an integral structure with $L(m) = L \otimes_Z \mathbb{Z} \cong L$ and $Q' = Q$. Note that, in particular,

$$V(m)^{0,0} = V^{m,m}.$$

The bilinear form Q defines an isomorphism $V^{p,q} \to (V^{n-p,p})^\vee = (V^*)^{p-n,-p} = V^\vee(n)^{p,q}$. Thus we may view the polarization Q as a non-degenerate bilinear form of Hodge structures of weights n

$$V \times V \to \mathbb{R}(-n),$$

or as a tensor $q \in (V^\vee \otimes V^\vee)(-n)$ of type $(0,0)$.

Example 14.2. Let (V,J) be a complex structure on a real vector space V and $V_\mathbb{C} = V_i \oplus V_{-i}$ be the eigensubspace decomposition with respect to J. Putting $V^{-1,0} = V_i, V^{0,-1}$ defines a Hodge structure on W of weight -1. If Q is a symplectic form on V such that the complex structure is polarizable with respect to E, then the Hodge structure becomes Q-polarizable. The converse is also true, a Q-polarizable Hodge structure of weight -1 defines a Q-polarizable Hodge structure on V. Thus the homology space $H_1(A,\mathbb{R})$ of an abelian variety acquires a polarizable Hodge structure of weight -1. The dual space $H^1(A,\mathbb{R})$ acquires the dual Hodge structure of weight 1.

Example 14.3. Let X be any nonsingular complex algebraic variety of dimension n and $h_0 \in H^2(X,\mathbb{Z})$ be the cohomology class of an ample divisor on X. The cup product $c \mapsto c \cup h_0$ defines a $\mathbb{Q}$-linear map $L : H^k(X,\mathbb{Q}) \to H^{k+2}(X,\mathbb{Q})$ and, by the Hard Lefschetz Theorem, for every positive $k \le n$,

$$L^{n-k} : H^k(X,\mathbb{Q}) \to H^{2n-k}(X,\mathbb{Q})$$

is an isomorphism. One defines the *primitive cohomology* by setting

$$H^k(X,\mathbb{Q})_{\text{prim}} = \text{Ker}(L^{n-k+1} : H^k(X,\mathbb{Q}) \to H^{2n+2-k}).$$

The primitive cohomology $H^k(X,\mathbb{Q})_{\text{prim}}$ admit a Hodge decomposition of weight k

$$H^k(X,\mathbb{C})_{\text{prim}} = \bigoplus_{p+q=k,p,q\ge 0} H^{p,q}(X),$$

which is polarizable with respect to the bilinear form

$$Q(\psi,\eta) = (-1)^{k(k-1)/2} \int_X h_0^{n-k} \wedge \psi \wedge \eta.$$

Note that in the case when X is a polarized abelian variety this agrees with the definition on the Hodge structure on $H^1(A,\mathbb{Q}) = H^1(A,\mathbb{Q})_{\text{prim}}$. We have

$$h_0^{n-1} \in H^{2n-2}(A,\mathbb{Q}) \cong H_2(A,\mathbb{Q}) = \bigwedge^2 H_1(A,\mathbb{Q})^*$$

and we can consider the integral in the above as the value of the corresponding symplectic form on $\psi, \eta \in H^1(A,\mathbb{Q})$.

Let $f : X \to S$ be a smooth projective morphism of complex manifolds. Then it defines a variation of rational Hodge structures $(H^n(X_s,\mathbb{Q})_{\text{prim}}, H^{p,q}(X_s))$. We refer for the details to any exposition of the Hodge Theory on algebraic varieties (e.g. [110]).

Let $f : X \to M$ be a morphism of complex manifolds over a connected complex manifold M equipped with a closed embedding in the projective bundle over M. Then the cohomology $H^n(X_t,\mathbb{Z})$ of fibers form a local coefficient system on M that give rise to a *variation of integral polarized Hodge structures* which we shall now define.

Let M be a connected smooth complex manifold and $\mathcal{V}$ be a locally constant sheaf (in usual topology) of real vector spaces on M equipped with a decreasing filtration $(\mathcal{F}^p)$ satisfying $\mathcal{F}^p \oplus \bar{\mathcal{F}}^{n-p+1} = \mathcal{V}$. Let $\underline{\mathcal{V}} := \mathcal{V} \otimes \mathcal{O}_M$ be associated locally free sheaf of $\mathcal{O}_M$-modules, where $\mathcal{O}_M$ is the sheaf of holomorphic functions on M. We require that the sheaves $\mathcal{F}^p$ generate locally free submodules $\underline{\mathcal{F}}^p$ of $\underline{\mathcal{V}}$ (or, in terms of vector bundles, holomorphic subbundles of the holomorphic vector bundle $\underline{\mathcal{V}}$). Let

$$\Delta : \underline{\mathcal{V}} \to \underline{\mathcal{V}} \otimes \Omega^1_M$$

be the flat connection defined by differentiation of local trivializations of $\underline{\mathcal{V}}$ (it globalizes because the transition matrices have constant entries). We require that the following *transversality condition* is satisfied

$$\Delta(\underline{\mathcal{F}}^p) \subset \underline{\mathcal{F}}^{p-1} \otimes \Omega^1_M.$$

In the case when M is simply connected so that $\mathcal{V}$ trivializes, we can restate this condition in a simpler way. We consider the map

$$\phi : M \to G(f_p, V), \quad x \mapsto \mathcal{F}^p_x$$

to the Grassmannian of subspaces of V of dimension equal to $\dim \mathcal{F}^p_s$ (which does not depend on $s \in M$). The map is holomorphic and the image of the differential map of the tangent spaces of complex manifolds

$$d\phi_x : T^{\text{hol}}_{M,x} \to T^{\text{hol}}_{G(f_p,V),\mathcal{F}^p_x} = \text{Hom}(\mathcal{F}^p_x, V/\mathcal{F}^p_x),$$

is contained in the subspace $\text{Hom}(\mathcal{F}^p_x, \mathcal{F}^{p-1}_x/\mathcal{F}^p_x)$. One can also express the transversality by saying that the differential map of the tangent spaces of smooth manifolds equipped with a complex structure is compatible with the Hodge structure (i.e. map the $(-1,0)$-component $T^{\text{hol}}_{M,x}$ to the $(-1,0)$ component of $\text{Hom}(\mathcal{F}^p_x, V/\mathcal{F}^p_x)$).

Following P. Deligne [25], one can reformulate the definition of a Hodge structure in the following way. Let

$$\mathbb{S} = \text{Res}_{\mathbb{C}/\mathbb{R}}(\mathbb{G}_{m,\mathbb{C}})$$

be the algebraic group over $\mathbb{R}$ obtained by Weil's restriction of scalars. It represents the functor $K \to (\mathbb{C} \otimes_{\mathbb{R}} K)^*$. It is easy to see that

$$\mathbb{S} = \text{Spec } \mathbb{R}[X, Y, T]/((X^2 + Y^2)T - 1).$$

For any algebra K over $\mathbb{R}$, we have a natural bijection

$$\mathbb{S}(K) = \{\left(\begin{smallmatrix} a & b \\ -b & 0 \end{smallmatrix}\right) : a^2 + b^2 \neq 0\} \subset \text{GL}_2(K).$$

In particular, we have a natural isomorphisms of groups

$$\mathbb{S}(\mathbb{R}) \to \mathbb{C}^*, \quad \left(\begin{smallmatrix} a & b \\ -b & 0 \end{smallmatrix}\right) \to a + bi$$

and

$$\mathbb{S}(\mathbb{C}) \to (\mathbb{C}^*)^2, \quad \left(\begin{smallmatrix} a & b \\ -b & 0 \end{smallmatrix}\right) \to (a + bi, a - bi).$$

Under these isomorphism $\mathbb{S}(\mathbb{R})$ embeds in $\mathbb{S}(\mathbb{C})$ via $z \mapsto (z, \bar{z})$. Also note the isomorphism of complex algebraic groups

$$\mathbb{S}_{\mathbb{C}} \to \mathbb{S} \otimes_{\mathbb{R}} \mathbb{C} \cong \mathbb{G}_{m,\mathbb{C}}^2 \cong \text{Spec}(\mathbb{C}[Z, \bar{Z}, T/(Z\bar{Z}T - 1)).$$

The last isomorphism is of course defined by $Z = X + iY, \bar{Z} = X - iY$. The group $\mathbb{G}_{m,\mathbb{R}}(\mathbb{C})$ embeds in $\mathbb{S}(\mathbb{C})$ diagonally $z \mapsto (z, z)$. Let

$$\mathbb{U}(1) = \text{Spec } \mathbb{R}[U, V]/(U^2 + V^2 - 1)$$

be the real algebraic group with $\mathbb{U}(1)(\mathbb{R}) \cong U(1) = \{z \in \mathbb{C} : |z| = 1\}$ and $\mathbb{U}(\mathbb{C}) \cong \mathbb{C}^*$. It is obviously a subgroup of $\mathbb{S}$ isomorphic to the kernel of the *norm homomorphism*

$$\text{Nm} : \mathbb{S} \to \mathbb{G}_{m,\mathbb{R}}, \quad \left(\begin{smallmatrix} a & b \\ -b & a \end{smallmatrix}\right) \mapsto a^2 + b^2.$$

Also it is isomorphic to the quotient $\mathbb{S}/\mathbb{G}_{m,\mathbb{R}}$ via the homomorphism $(z_1, z_2) \mapsto z_1/\bar{z}_2$.

Let $\rho : \mathbb{S} \to \text{GL}(V)$ be an injective homomorphism of real algebraic groups (a faithful real linear representation). It defines a complex linear representation $\rho_{\mathbb{C}} : \mathbb{S}(\mathbb{C}) \to \text{GL}(V_{\mathbb{C}})$. Restricting it to the subgroup $\mathbb{S}(\mathbb{R})$, we obtain an eigensubspace decomposition

$$V_{\mathbb{C}} = \oplus V^{p,q}, \quad V^{p,q} = \{v \in V_{\mathbb{C}} : \rho_{\mathbb{C}}(z_1, z_2) \cdot v = z_1^{-p} \bar{z}_2^{-q} v\}. \tag{14.2}$$

Obviously, $\overline{V^{p,q}} = V^{q,p}$, so $V^n := \oplus_{p+q=n} V^{p,q}$ is a Hodge structure on V^n of weight n. Any $z \in \mathbb{S}(\mathbb{R})$ acts on $V^{p,q}$ by multiplication $v \mapsto z^{-p}\bar{z}^{-q}$. In particular, any element in $\mathbb{G}_{m,\mathbb{R}}(\mathbb{R}) \subset \mathbb{S}(\mathbb{R})$ acts by scalar multiplication $v \mapsto \lambda^{-(p+q)}v$, and hence belongs to the center of $\text{GL}(V)$. So, the action of $\mathbb{G}_{m,\mathbb{R}}$ decomposes V into the direct sum of eigensubspaces V^n with eigencharacter $\lambda \mapsto \lambda^{-n}$ each of which is equipped with a Hodge structure of weight n.

Let $i = \sqrt{-1}$ be considered as an element of $\mathbb{S}(\mathbb{R})$ and let $C = \rho(i)$. It is clear that C acts as i^{q-p} on $H^{p,q}$ and C^2 acts on V^n as the multiplication by $(-1)^n$. To get a polarized Hodge structure on V^n we require that the exists a bilinear form $Q : V \times V \to \mathbb{R}$ such that $Q(C(x), C(y)) = Q(x, y)$ (this implies that $Q(x, y) = (-1)^n Q(y, x)$ if $x, y \in V^n$), and $Q(C(x), y) > 0, x, y \neq 0$. It is immediately checked that all properties of a polarized Hodge structure are satisfied. Conversely, a polarized Hodge structure on V defines a representation $\rho : \mathbb{S} \to \mathrm{GL}(V)$ as above.

In the following we will be using the theory of *real forms* of complex algebraic groups. Let us remind some basic construction of this theory. First we start with real forms of complex finite-dimensional Lie algebras $\mathfrak{g}$. We denote by $\mathfrak{g}^{\mathbb{R}}$ the real Lie algebra obtained from $\mathfrak{g}$ by restriction of scalars. By definition, a real form of $\mathfrak{g}$ is a real Lie subalgebra $\mathfrak{b}$ of $\mathfrak{g}^{\mathbb{R}}$ such that there exists an isomorphism $\alpha : \mathfrak{b}_{\mathbb{C}} \cong \mathfrak{g}$ of complex Lie algebras. The conjugation automorphism $x + iy \mapsto x - iy$ of $\mathfrak{b}_{\mathbb{C}}$ defines, via α, an anti-involution θ of $\mathfrak{g}$, i.e. θ is an involution of $\mathfrak{g}^{\mathbb{R}}$ that satisfies $\theta(\lambda z) = \bar{\lambda}\theta(z)$, for any $z \in \mathfrak{g}$ and any $\lambda \in \mathbb{C}$. Conversely, any such involution θ defines a real form $\mathfrak{b}$ of $\mathfrak{g}$ by setting $\mathfrak{b} = \mathfrak{g}^{\theta} := \{z \in \mathfrak{g} : \theta(z) = z\}$. It is easy to check that this construction defines a bijection between the set of isomorphism classes of real forms of $\mathfrak{g}$ and the set of conjugacy classes of anti-involutions of $\mathfrak{g}$.

A real Lie algebra $\mathfrak{b}$ is called *compact* if it admits a positive definite bilinear B form which is invariant with respect to the adjoint representation (i.e. $B([x, y], z) = B(x, [y, z])$ for any $x, y, z \in \mathfrak{b}$). Recall that $\mathfrak{b}$ has always an invariant bilinear form, the *Killing form* defined by $B(x, y) = \mathrm{Tr}(\mathrm{ad}(x) \circ \mathrm{ad}(y))$, where $ad(x) : y \mapsto [x, y]$ is the adjoint representation of $\mathfrak{b}$. This form is non-degenerate if and only if $\mathfrak{g}$ is semi-simple. Since any invariant bilinear form on a semi-simple Lie algebra is a scalar multiple of the Killing form, we see that a compact Lie algebra is semi-simple if and only if its Killing form is definite (in fact, negative definite). Every semi-simple complex Lie algebra admits a unique, up to isomorphism, compact real form. The corresponding involution is called a *Cartan involution*.

Example 14.4. Let $\mathfrak{g} = \mathfrak{sl}_2(\mathbb{C})$ generated over $\mathbb{C}$ by the matrices $h = \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right), e = \left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right), f = \left(\begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix}\right)$. It admits a non-compact real from $\mathfrak{sl}_2(\mathbb{R})$ generated by the same matrices over $\mathbb{R}$ and a compact real form $\mathfrak{su}_2$ generated by the matrices $\left(\begin{smallmatrix} i & 0 \\ 0 & -i \end{smallmatrix}\right), e = \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right), f = \left(\begin{smallmatrix} 0 & i \\ i & 0 \end{smallmatrix}\right)$. The corresponding anti-involutions are defined by $A \mapsto \bar{A}$ and $A \mapsto -{}^t\bar{A}$. The similar formulae define the anti-involutions on $\mathfrak{sl}_n(\mathbb{C})$ with a non-compact real form $\mathfrak{sl}_n(\mathbb{R})$ and a compact real form $\mathfrak{su}_n$. Note that any commutative Lie algebra is obviously compact. The Lie algebra $\mathfrak{gl}_n(\mathbb{C}$ has a compact form $\mathfrak{u}_n$.

The notions of a real form and a Cartan involution extends to algebraic groups. An algebraic group defined over $\mathbb{R}$ is a real form of a complex algebraic group G if $H_{\mathbb{C}} \cong G$. According to the general nonsense about Galois cohomology, the group H is determined uniquely by an element of $H^1(\mathrm{Gal}(\mathbb{C}/\mathbb{R}), \mathrm{Aut}(G(\mathbb{C}))$ defined by an automorphism α of $G(\mathbb{C})$ such that $\alpha^{-1} = \bar{\alpha}$. The group H is reconstructed from this automorphism as an algebraic group with the group $H(K)$ of K-points equal to $G^{\alpha}(K) := \{g \in G^{\mathbb{R}}(K) : \alpha(g) = \bar{g}\}$, where $G^{\mathbb{R}} := \mathrm{Res}_{\mathbb{C}/\mathbb{R}}G$ is the Weil restriction of scalars functor on the category of complex algebraic groups which admits a natural action of $\mathrm{Gal}(\mathbb{C}/\mathbb{R})$ via the conjugation isomorphism $K \otimes_{\mathbb{R}} \mathbb{C} \to K \otimes_{\mathbb{R}} \mathbb{C}$. The involution α as above is called the *Cartan involution* of G. There is a natural bijection between the set of real forms of G and the conjugacy classes of Cartan involutions. The Lie algebra of the real Lie group $H(\mathbb{R})$ is a real form of the complex Lie algebra of the complex Lie group $G(\mathbb{C})$ and the converse is true if one additionally assumes that $G^{\mathbb{R}}$ is generated by H and its connected component of the identity.

A real algebraic group H is called *compact* if the real Lie group $H(\mathbb{R})$ is compact. The Lie algebra $\mathrm{Lie}(H(\mathbb{R}))$ of $H(\mathbb{R})$ is compact, a positive definite invariant symmetric form can be obtained by integral average over $H(\mathbb{R})$ of any positive symmetric bilinear form on $\mathrm{Lie}(H(\mathbb{R}))$. Every semi-simple complex algebraic group admits a unique, up to isomorphism, compact real form. The involutive automorphism α of G that defines a compact real form is called a *Cartan involution*.

Example 14.5. The complex multiplicative group $G = \mathbb{G}_{m,\mathbb{C}}$ has two non-isomorphic real forms: a non-compact form $\mathbb{G}_{m,\mathbb{R}}$ and a compact form $\mathbb{U}(1)$ which we introduced earlier. The first one corresponds to the involution $z \mapsto \bar{z}$, the second one corresponds to the involution $z \mapsto z^{-1}$.

The group SU_n is a compact form of $\mathrm{SL}_{n,\mathbb{C}}$ defined by the Cartan involution $A \mapsto {}^tA^{-1}$. A non-compact form is isomorphic to either $\mathrm{SL}_{n,\mathbb{R}}$, or $\mathrm{SU}_{p,n-p}$, or, if $n = 2m$, to the group $\mathrm{SL}_m(\mathbb{H})$ defined by the involutions $A \mapsto A$, or $A \mapsto I_{p,n-p}{}^tAI_{p,n-p}^{-1}$, or $A \mapsto J_n{}^tAJ_n^{-1}$, where $I_{p,n-p} = \begin{pmatrix} I_p & 0 \\ 0 & -I_{n-p} \end{pmatrix}$ and J_n is the matrix of the standard symplectic structure on $\mathbb{R}^{2m}$. The group $\mathrm{SU}_{p,n-p}$ consists of complex matrices with determinant 1 preserving the Hermitian form $|z_1|^2 + \cdots + |z_p|^2 - |z_{p+1}|^2 - \cdots - |z_n|^2$. The group $\mathrm{SL}_m(\mathbb{H})$ consists of matrices of determinant 1 preserving a structure on $\mathbb{C}^{2m}$ of a module of rank m over the algebra of quaternions $\mathbb{H}$ (by viewing $(z_1, \ldots, z_{2m}) \in \mathbb{C}^{2m}$ as a vector $(z_1 + z_{m+1}j, \ldots, z_m + z_{2m}j) \in \mathbb{H}^m$).

Let G be a reductive algebraic group over $\mathbb{Q}$ and

$$h : \mathbb{S} \to G_{\mathbb{R}}$$

be an injective homomorphism. For any faithful linear representation $\sigma : G \to \mathrm{GL}(V)$ of G in a $\mathbb{Q}$-vector space V, the composition $\rho = \sigma \circ h : \mathbb{S} \to \mathrm{GL}(V_{\mathbb{R}})$ defines a Hodge structure on V. Suppose V admits a polarization Q which is invariant with respect to the representation σ. Let $\theta = h(i) \in G$ so that Q satisfies the symmetry and positivity conditions with respect to $C = \sigma(\theta)$. Suppose G acts on V via σ leaving Q invariant. Then $h(i) \in G(\mathbb{R})$ is an element whose square is mapped via σ to $-\mathrm{id}_V$. It follows that $h(i)$ belongs to the center of $G(\mathbb{R})$. It is known that the conjugation automorphism $\mathrm{Ad}(h(i))$ of $G_{\mathbb{C}}$ is a Cartan involution if and only if there exists a (equivalently, any) linear representation $\sigma : G(\mathbb{R}) \to \mathrm{GL}(V)$ preserves a bilinear form Q which is symmetric and positive with respect to $C = \sigma(h(i))$. Also the condition that G leaves the polarization on V invariant implies that G is a reductive group (see [25], Proposition 1.1.14).

Let D be a connected component of the conjugacy class of a homomorphism $h_0 : \mathbb{S} \to G_{\mathbb{R}}$ of algebraic groups over $\mathbb{R}$. We say that the pair (G, D) is a *Shimura data* if the following properties hold:

(S1) For any $h \in \mathsf{D}$, $h(\mathbb{G}_{m,\mathbb{R}})$ belongs to the center Z of $G_{\mathbb{C}}$ and the induced action of $U(1)$ on $\mathrm{Lie}(G^{\mathrm{ad}})_{\mathbb{C}}$ is via the characters $z, 1, \bar{z}$;

(S2) $\mathrm{Ad}(h(i))$ is a Cartan involution θ on $G^{\mathrm{ad}}_{\mathbb{C}} := G_{\mathbb{C}}/Z$;

(S3) G^{ad} has no $\mathbb{Q}$-factors on which the projection of h is trivial.

Let $\sigma : G_{\mathbb{R}} \to \mathrm{GL}(V)$ be a faithful linear representation of G as above. For any $h \in X$, consider the composition $\rho_h = \sigma \circ h : \mathbb{S} \to \mathrm{GL}(V)$. It follows from the condition (S1) that the grading $V = \oplus_{n \in \mathbb{Z}} V^n$ defined by the action of $\mathbb{G}_{m,\mathbb{R}}$ on V does not depend on h. The condition $(S2)$ implies that G is a reductive algebraic group and that the stabilizer subgroup K_0 of h_0 is a maximal compact subgroup K of $G(R)$. The image $h(U(1))$ of $(\mathbb{S}/\mathbb{G}_{m,\mathbb{R}}(\mathbb{R})$ is a subgroup of K_0. Let D be a connected component of $X = G(\mathbb{R})/K_0$. For any point $x \in \mathsf{D}$, the group $U(1)$ acts on the tangent space TD_x and defines a complex structure. In this way, D becomes equipped with a structure of a hermitian symmetric space. Condition (S3) implies that D is of non-compact type.

Fix a representation $\rho : G_{\mathbb{R}} \to \mathrm{GL}(V)$ and assume that it is defined over $\mathbb{Q}$ and preserves a bilinear for Q on V. So we obtain a *variation of Hodge structures* on V parameterized by $X = G(\mathbb{R})/K_0$.

Example 14.6. Let (V, J) be a complex structure on a real vector space V and $V_{\mathbb{C}} = V_i \oplus V_{-i}$ be the eigensubspace decomposition with respect to J. Putting $V^{-1,0} = V_i, V^{0,-1}$ defines a Hodge structure on W of weight -1. If Q is a symplectic form on V such that the complex structure is polarizable with respect to Q, then the Hodge structure becomes Q-polarizable. The converse is also true, a Q-polarizable Hodge structure of weight -1 defines a Q-polarizable Hodge structure on V. Thus the homology space $H_1(A, \mathbb{R})$ of an abelian variety acquire a polarizable Hodge structure of weight -1.

Let $G = \mathrm{CSp}(V; Q) \cong \mathrm{CSp}(2n)$ be the reductive group over $\mathbb{Q}$ whose set of K-points is equal to the set of linear maps $f : V \to V$ such that $Q(f(x), f(y)) = \lambda Q(x, y)$ for some $\lambda \in K^*$. Its center is isomorphic to $\mathbb{G}_m$ and the quotient G^{ad} is a simple algebraic group $\mathrm{Sp}(V; Q)$ of Q-isometries of V. Let $h : \mathbb{S} \to G$ be defined by sending $a + bi \in \mathbb{S}(\mathbb{R})$ to $aI_{2n} + bJ$. Then the stabilizer of $h(\mathbb{S})$ in $G^{\mathrm{ad}}(\mathbb{R})$ consists of matrices $X = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right) \in \mathrm{Sp}(2n, \mathbb{R})$ such that $X^{-1}JX = J$. The fact that $X \in \mathrm{Sp}(2n, \mathbb{R})$ means that $A = D$ and $B = -C$. The second condition means that ${}^tBA - {}^tAB = 0$ and ${}^tAA + {}^tBB = I_n$. The map $X \mapsto \left(\begin{smallmatrix} A & B \\ -B & A \end{smallmatrix}\right) \to A + iB$ defines an isomorphism from the stabilizer subgroup to the unitary group $U(n)$ of complex matrices Z such that ${}^t\bar{Z}Z = I_n$. Thus

$$\mathsf{D} \cong \mathcal{Z}_n \cong \mathrm{Sp}(2n, \mathbb{R})/U(n).$$

The Q-polarized Hodge structures on V of weight -1 correspond to the natural representation of G in V. This means that $z \in \mathbb{S}(\mathbb{R})$ acts on $V^{-1,0}$ by $x \mapsto zx$ and on $V^{0,-1}$ by $x \mapsto \bar{z}x$. This implies that it acts on the real vector space$V$ by $v \mapsto zv$.

We may also consider other linear representations of G, for example $\bigwedge^k V$ preserving the bilinear form $\bigwedge^k Q$. They define polarized Hodge structures on $\bigwedge^k V$ of weight $-k$. We could also consider the dual representation V^* and the dual symplectic form Q^{-1} (where Q is considered as an invertible linear map $V \to V^*$). The Hodge structure on V^* is of weight 1. We can view it as a Hodge structure on cohomology $H^1(A, \mathbb{R})$ of an abelian variety $\mathbb{C}^g/\Lambda$, where $Q(\Lambda \times \Lambda) \subset \mathbb{Z}$. The Hodge structure on the exterior product $\bigwedge^k V^*$ is the Hodge structure on the cohomology $H^k(A, \mathbb{R})$. Its Hodge decomposition is $\oplus_{i=0}^k H^{k-i,i}$, where

$$h^{k-i,i} = \dim H^{k-i,i} = \binom{g}{i}^2.$$

One can also introduce other z objects of the category of Hodge structures. For example, suppose $g = 2k + 1 > 1$ is odd. The polarization class $h \in H^2(A, \mathbb{R})$ defined by Q belongs to the piece $H^{1,1}(A)$ of the Hodge structure. Consider the linear map

$$\Phi : H^1(A, \mathbb{R}) \to H^{2k+1}(A, \mathbb{R}),\ x \mapsto x \wedge h^{\wedge k}.$$

The quotient Hodge structure $H^{2k+1}(A,\mathbb{R})/\mathrm{Im}(\Phi)$ is of weight g and has the Hodge decomposition as in (14.2) with

$$h^{g-i,i} = \begin{cases} \binom{g}{i}^2 & \text{if } i \neq 1, g-1, \\ g^2 - g & \text{otherwise.} \end{cases}$$

I do not know whether there exists an algebraic variety whose Hodge structure on cohomology is naturally isomorphic to this Hodge structure.

Let D be a connected component of $X = G(\mathbb{R})/K$ regarded as a symmetric domain. The connected component of the group of holomorphic automorphisms of D is isomorphic to a connected component $G(\mathbb{R})^+$ of the identity of $G(\mathbb{R})$. A subgroup Γ of $G(\mathbb{Q})$ (a reductive algebraic group over $\mathbb{Q}$) is called a *congruence subgroup* if there exists a linear faithful representation $G \hookrightarrow \mathrm{GL}_n$ over $\mathbb{Q}$ such that the image of Γ contains a subgroup of finite index

$$\Gamma(N) = G(\mathbb{Q}) \cap \{g \in \mathrm{GL}_n(\mathbb{Z}) : g \equiv I_n \mod N\}.$$

A subgroup of $G(\mathbb{Q})$ is called *arithmetic* if it is commensurable with $\Gamma(1)$ (i.e. contains a subgroup of finite index in both of them). It is known that any arithmetic subgroup Γ acts discretely on D and the quotient $\Gamma\backslash\mathsf{D}$ has a structure of a quasi-projective algebraic variety. A *connected Shimura variety* $\mathrm{Sh}^o(G,\mathsf{D})$ is the inverse system of locally symmetric spaces $\Gamma\backslash\mathsf{D}$ where Γ runs the set of torsion-free arithmetic subgroups of $G^{\mathrm{ad}}(\mathbb{Q})$ whose preimage in $G(\mathbb{Q})$ is a congruence subgroup.

Let $\mathbb{A}_f$ be the ring of *finite adèles* of $\mathbb{Q}$, i.e. the subring of the product of the fields of p-adic numbers $\mathbb{Q}_p$ where all components except finitely many belong to the ring of integer p-adic numbers $\mathbb{Z}_p$. We use the p-adic topology on $\mathbb{Q}_p$ in which a base of open subsets of 0 is formed by the fractional ideals $p^\nu\mathbb{Z}_p$, where $\nu \in \mathbb{Z}$. For example, any element $x \in \mathbb{Q}_p$ contains an open compact neighborhood of the form $\{y \in \mathbb{Q}_p : y - x \in p^n\mathbb{Z}_p\}$. This topology make $\mathbb{Q}_p$ a locally compact field. One equips $\mathbb{A}_f$ with a topology whose base of open sets consist of subsets of the form $\prod_{p\in S} U_p \times \prod_{p\notin S}\mathbb{Z}_p$, where S is a finite set of prime numbers and U_p is an open subset of $\mathbb{Q}_p$. This topology, called the *adéle topology*. It is stronger than the product topology on $\mathbb{A}_f$. For an algebraic group G over $\mathbb{Q}$ one defines $G(\mathbb{A}_f)$ to be the subgroup of the product of groups $G(\mathbb{Q}_p)$ where all components except finitely many belong to $G(\mathbb{Z}_p)$. For example, when $G = \mathbb{G}_{m,\mathbb{Q}}$, we obtain the group of *idèles* $\mathbb{A}_f^*$. There is a canonical injection $G(\mathbb{Q}) \to G(\mathbb{A}_f)$ defined by the homomorphisms $\mathbb{Q} \to \mathbb{Q}_p$. One can show that, for any compact subgroup K of $G(\mathbb{A}_f)$ the intersection $G(\mathbb{Q}) \cap K$ is a congruence subgroup of $G(\mathbb{Q})$ ([76], Proposition 4.1). It follows that the induced topology on $G(\mathbb{Q})$ is a topology defined by a basis of open subsets equal to congruence subgroups. The *Strong Approximation Theorem* asserts that $G(\mathbb{Q})$ is dense in $G(\mathbb{A}_f)$ if G is a semi-simple simply-connected with $G(\mathbb{R})$ of non-compact type.

The adèlic definition of the Shimura variety is based on an isomorphism

$$\Gamma\backslash\mathsf{D} \cong G(\mathbb{Q})\backslash\mathsf{D} \times G(\mathbb{A}_f)/K,$$

where K is a compact open subgroup of $G(\mathbb{A}_f)$ such that $K \cap G(\mathbb{Q}) = \Gamma$, acting on $G(\mathbb{A}_f)$ on the right and $G(\mathbb{Q})$ acts on the product $\mathsf{D} \times G(\mathbb{A}_f)$ diagonally on the left so that

$$q \cdot (x,a) \cdot k = (qx, qak), \quad q \in G(\mathbb{Q}), x \in \mathsf{D}, a \in G(\mathbb{A}_f), k \in K.$$

In this way $\mathrm{Sh}^o(G,\mathsf{D})$ becomes the inductive limit of $G(\mathbb{Q})\backslash\mathsf{D}\times G(\mathbb{A}_f)/K$, where K runs the set of open compact subgroups of $G(\mathbb{Q})$.

Let B be a semi-simple $\mathbb{Q}$-algebra with positive anti-involution $b\mapsto b'$ and let (V,Q) be a symplectic space that is a $(B,')$-module. This means that $Q(bx,y)=Q(x,b'y)$. Then one considers a reductive Q-subgroup G of $\mathrm{CSp}(V,Q)$ that acts on V preserving the structure of a $(B,;)$-module. Then there exists a homomorphism $h:\mathbb{S}\to G(\mathbb{R})$ such that $h(\bar{z})=h(z)'$ and $Q(h(i)x,y)$ is a positive symmetric bilinear form. The conjugacy class of such h defines a Shimira data (G,X) that is mapped to the modular Shimura data $(\mathrm{CSp}(V,Q),\mathcal{Z}_g)$. When the algebra B is simple and the involution is the identity on the center, then this Shimura variety is called of *PLE-type*.

Let (G,D) be a Shimura data and $h\in X$ and (V,h) be a Hodge structure defined by h. The *Mumford-Tate group* $\mathrm{MT}(h)$ is defined to be the smallest algebraic subgroup H of G defined over $\mathbb{Q}$ such that $G_{\mathbb{R}}$ contains $h(\mathbb{S})$. Let $\sigma:G\to\mathrm{GL}(V)$ be a faithful linear representation defined over $\mathbb{Q}$. We denote the image of $\rho=\sigma\circ h$ of $\mathrm{MT}(h)$ in $\mathrm{GL}(V)$ by $\mathrm{MT}(\sigma,h)$ (or $\mathrm{MT}(V,h)$ if no confusion arises). It is called the *Mumford-Tate group of the Hodge structure* on V defined by ρ. It follows from the definition that the group $\mathrm{MT}(\mathsf{D},h)$ is a connected reductive group over $\mathbb{Q}$ equal to the $\mathbb{Q}$-closure of $h(\mathbb{S})$ in $\mathbb{G}_{\mathbb{C}}$. Moreover, the Mumford-Tate group of a polarized Hodge structure is a semi-simple group (because it preserves a non-degenerate bilinear form).

One can also define the *Hodge group* or *special Mumford-Tate group* by considering the restriction map $h':\mathbb{U}(1)\to G^{\mathrm{ad}}$ and setting $\mathrm{Hg}(h)$ to be the smallest algebraic subgroup H over $\mathbb{Q}$ of G such that $H_{\mathbb{R}}$ contains $h'(\mathbb{U}(1))$. The groups $\mathrm{MT}(\mathsf{D},h)$ and $\mathrm{Hg}(\mathsf{D},h)\times\mathbb{G}_{m,\mathbb{Q}}$ are isogenous algebraic groups over $\mathbb{Q}$. Similarly, one defines the subgroup $\mathrm{Hg}(h,V)$ of $\mathrm{GL}(V)$ which is contained in $\mathrm{SL}(V)$.

For any positive integers a and b let $T^{a,b}=V^{\otimes a_i}\otimes V^{*\otimes b_i}$ be the tensor product space equipped with the tensor product Hodge structure of weight $a-b$. A rational vector $t\in\oplus_i T^{a_i,b_i}$ is of type $(0,0)$ if and only if it is invariant with respect to $\mathrm{MT}(V,h)$. In fact, if t is of type $(0,0)$, then $h(\mathbb{S})$ leaves it invariant, hence the smallest algebraic subgroup that leaves it invariant must contain $\mathrm{MT}(V,h)$. Conversely, if t is invariant with respect to the Mumford-Tate group, then it is in particular invariant with respect to $h(\mathbb{S})$, hence it is of type $(0,0)$.

Since morphisms $V\to V$ preserving the Hodge structure correspond to tensors in $V\otimes V^*$ of type $(0,0)$, we obtain another equivalent definition of the Mumford-Tate group $\mathrm{MT}(V,h)$, it is the smallest algebraic $\mathbb{Q}$-subgroup of $\mathrm{GL}(V)$ such that

$$\mathrm{End}(V,h)=\mathrm{End}(V)^{\mathrm{MT}(V,h)}.$$

More generally, if T is a subspace of the tensor algebra $T(V)\otimes T(V^\vee)$ with the inherited Hodge structure, then $\mathrm{MT}(V)$ acts in T and any $\mathbb{Q}$-subspace W of T is a Hodge substructure if and only if it is invariant under $\mathrm{MT}(V)$. In particular, if (V',h') is a Hodge substructure of (V,h), then $\mathrm{MT}(V',h')\subset\mathrm{MT}(V,h)$.

In the case of the Hodge structure on cohomology of an algebraic variety X, we may consider Cartesian product X^d so that, by Künneth formula, their cohomology are isomorphic to the direct sums of tensor products $H^{a_1}(X,\mathbb{Q})\otimes\ldots\otimes H^{a_d}(X,\mathbb{Q})$. The cocycles of type (p,p) can be viewed as vectors of type $(0,0)$ in the tensor product with $\mathbb{Q}[r]$ for some r, where $\mathbb{Q}[r]$ is the one-dimensional space equipped with Hodge structure of type $(-r,-r)$. Such classes are called *Hodge classes* (by

Hodge's Conjecture they must be algebraic classes). Thus, in this situation, the Mumford-Tate group is the smallest algebraic subgroup of $\mathrm{GL}(H^*(X,\mathbb{Q})$ that leaves invariant Hodge classes.

Example 14.7. Let A be an abelian variety and $V = H_1(A,\mathbb{Q})$ with the Hodge structure of weight -1. The set of Hodge classes in $\mathrm{End}(V) = V^\vee \otimes V$ is equal to the set $\mathrm{End}(A)_\mathbb{Q}$. Hence

$$\mathrm{End}(A)_\mathbb{Q} = \mathrm{End}(V)^{\mathrm{MT}(V)}.$$

Using the notion of the Mumford-Tate group one can characterize abelian varieties of CM-type by the property that its Mumford-Tate group is commutative.

Proposition 14.8. *An abelian variety A is of CM type if and only if the Mumford-Tate group of the Hodge structure on $V = H^1(A,\mathbb{Q})$ is commutative (hence isomorphic to $\mathbb{G}_{m,\mathbb{Q}}^r$).*

Proof. Suppose A is of CM-type. Let R be the CM-algebra acting on V. Its center is a $\mathbb{Q}$-subalgebra K of dimension $2\dim A = \dim V$, so that V is a vector space of dimension 1 over K. The action of $K \otimes_\mathbb{Q} \mathbb{R}$ on V commutes with the complex structure, hence $\mathbb{C}$ is contained in the centralizer of $K \otimes_\mathbb{Q} \mathbb{R}$ in $\mathrm{End}_\mathbb{R}(V)$ and hence coincides with it. This shows that the Mumford-Tate group is a subgroup of the torus $\mathbb{C}^*$ considered as an algebraic group $\mathrm{Res}_{K/\mathbb{Q}}\mathbb{G}_{m,K}$ over $\mathbb{Q}$.

Conversely, assume that $\mathrm{MT}(V) \subset \mathrm{GL}(V)$ is a torus T. Let R be the subalgebra of $\mathrm{End}_\mathbb{Q}(V)$ of endomorphisms that are endomorphisms of the $\mathrm{MT}(V)$-module V. Since $\mathrm{MT}(V)$ contains $\mathbb{C}^* = \mathbb{S}(\mathbb{R})$ that acts on V by $v \mapsto zv, z \in \mathbb{C}^*$, we see that R is isomorphic to a sublalgebra of $\mathrm{End}(A)_\mathbb{Q}$. Since $G = \mathrm{MT}(V)$ is a diagonalizable commutative algebraic group, we have $[R:\mathbb{Q}]_{\mathrm{red}} = \dim_\mathbb{Q} V$. In fact, we have decomposition of V into eigenspaces $V = \oplus_{\chi\in\mathcal{X}(G)} V_\chi$, hence $R = \prod_\chi \mathrm{End}_\mathbb{Q}(V_\chi)$. This implies that the reduced degree of R over $\mathbb{Q}$ is equal to $\sum_\chi \dim_\mathbb{Q} V_\chi = \dim_\mathbb{Q} V$. Thus $\mathrm{End}(A)_\mathbb{Q}$ contains a central algebra of reduced degree equal to $\dim A$. This algebra is a CM-algebra, and hence A is of CM-type. □

A smooth projective algebraic variety is called of *CM-type* if the Mumford-Tate group of the Hodge structure on its cohomology $H^*(X,\mathbb{Q})$ is commutative. It is conjectured that such a variety can be defined over a field of algebraic numbers [97].

Example 14.9. Let A be an abelian variety with polarization skew-symmetric form Q. Let $D = \mathrm{End}(A)_\mathbb{Q}$ and $\mathrm{CSp}_D(V,\mathbb{Q}) := \mathrm{CSp}(V,\mathbb{Q})^D$. Since this group is a $\mathbb{Q}$-group containing $h(\mathbb{S})$, we have

$$\mathrm{MT}(A) \subset \mathrm{CSp}_D(V,\mathbb{Q}). \tag{14.3}$$

When A is an elliptic curve, we get $\mathrm{Sp}(2) \cong \mathrm{SL}(2)$ and $\mathrm{CSp}(V,\mathbb{Q}) \cong \mathrm{GL}_{2,\mathbb{Q}}$. If $D = \mathbb{Q}$, then $\mathrm{MT}(A) = \mathrm{GL}_{2,\mathbb{Q}}$. If $D = \mathbb{Q}(\sqrt{-d})$, then $\mathrm{MT}(A)$ is conjugate to a subgroup of $\mathrm{GL}_{2,\mathbb{Q}}$ with the group of K-points equal to a subgroup of $\mathrm{GL}_2(K)$ of matrices of the form $\left(\begin{smallmatrix} x & y \\ -dy & x\end{smallmatrix}\right), x,y \in K$. The Hodge group $\mathrm{Hdg}(A)$ is defined by an additional condition that $x^2 + dy^2 = 1$.

Suppose A is a simple abelian surface. Then we have the equality in (14.3). If $D = \mathbb{Q}$, then $\mathrm{MT}(A) \cong \mathrm{CSp}_{4,\mathbb{Q}}$. If $D = \mathbb{Q}(\sqrt{d})$ is a real quadratic field, then $\mathrm{MT}(A)$ is a subgroup of $\mathrm{Res}_{D/\mathbb{Q}}\mathrm{GL}_{2,D}$ whose group of K-points is equal to $\{g \in \mathrm{GL}_2(D\otimes_\mathbb{Q} K) : \det(g) \in K^*\}$. If D is an indefinite quaternion algebra, then $\mathrm{MT}(A)$ is the group of unites of the opposite algebra

D, i.e. $\mathrm{MT}(A)(K) = (D^{\mathrm{op}} \otimes_{\mathbb{Q}} K)^*$. Finally, if D is a CM-field, then $\mathrm{MT}(A)$ is a subgroup of $\mathrm{Res}_{D/\mathbb{Q}}\mathbb{G}_{m,D}$ with $\mathrm{MT}(A)(K) = \{x \in (D \otimes K)^* : x\bar{x} \in K^*\}$. Note that, when $\dim A \geq 4$, the group $\mathrm{MT}(A)$ could be a proper subgroup of $\mathrm{CSp}_D(V, Q)$. For example, this happens in Mumford's example 17.1 of an abelian variety of dimension 4.

Example 14.10. Let X be a K3-surface that admits a non-symplectic automorphism σ of order m that acts identically on $\mathrm{Pic}(X)$. All such K3 surfaces have been classified in [58] and [111]. It is known that $\phi(m)$ divides the rank of the transcendental lattice T_X. All possible values of m are known. Assume that $m = \phi(m)$. Then $m \in \{12, 28, 36, 42, 44, 66\}$ if T_X is unimodular and $m \in \{3, 5, 7, 9, 11, 13, 17, 19, 25, 27\}$ otherwise. There is only one isomorphism class of such a surface. Their lattices T_X are computed in [68].

The cyclotomic field $\mathbb{Q}(\zeta_m)$ acts on $(T_X)_{\mathbb{Q}}$ and hence equips it with a structure of a one-dimensional vector space over $\mathbb{Q}(\zeta_m)$. The proof of the previous proposition extends to this case and shows that the Mumford-Tate group of the Hodge structure on $(T_X)_{\mathbb{Q}}$ induced by the Hodge structure on $H^*(X, \mathbb{Q})$ is of CM type.

Example 14.11. Let

$$X_m^1 : x_0^m + x_1^m + x_2^m = 0$$

be the *Fermat curve* of degree $m > 2$. For any pair (r, s) with $1 \leq r, s, r + s \leq m - 1$, let

$$n = m/g.c.d.(m, r, s).$$

Let $\langle a \rangle$ denote the unique representative of $a \mod m$ between 0 and $m - 1$ and let $H_{r,s}$ be the subset of $(\mathbb{Z}/n\mathbb{Z})^*$ of elements h such that

$$\langle rs \rangle, \langle hr \rangle \leq m - 1.$$

It is easy to see that $H_{r,s}$ coincides with the set of representatives of the subgroup $\{\pm 1\}$ of $(\mathbb{Z}/n\mathbb{Z})^*$. Let us fix the standard isomorphism

$$\phi : (\mathbb{Z}/n\mathbb{Z})^* \to \mathrm{Gal}(\mathbb{Q}(\zeta_n)/\mathbb{Q}), \quad h \to \sigma_h : \zeta_n \to \zeta_n^h.$$

Define a lattice $\Lambda_{r,s}$ in $\mathbb{C}^{\phi(n)/2}$ to be the span of the vectors

$$\sigma_h(\omega_1, \ldots, \omega_{\phi(n)}), h \in H_{r,s},$$

where $(\omega_1, \ldots, \omega_{\phi(n)})$ is a basis of the ring of integers $\mathbb{Z}[\zeta_n]$. Since $H_{r,s} = hH_{\langle hr \rangle, \langle hs \rangle}$ for any $h \in H_{r,s}$, we obtain $\Lambda_{r,s} = \Lambda_{\langle hr \rangle, \langle hs \rangle}$. We say that the two pairs (r, s) and (r', s') related in this way are equivalent. Let $A_{r,s} = \mathbb{C}^{\phi(n)/2}/\Lambda_{r,s}$. There is an isogeny

$$\prod_{\{r,s\}} A_{r,s} \to \mathrm{J}(X_m^1), \tag{14.4}$$

where the product is taken over equivalence classes of pairs (r, s) as above [57]. Note that each variety $A_{r,s}$ is of dimension $\phi(n)/2$ and has multiplication by $\mathbb{Q}(\zeta_n)$, hence it is of CM type. This implies that $\mathrm{J}(C)$ is of CM type.

For example, take $m = p$ to be prime. Then $n = p$, we have $p - 2$ equivalence classes of pairs (r, s) and obtain that $\mathrm{J}(C)$ is isogenous to the product of $p - 2$ copies of an abelian variety of dimension $\frac{1}{2}(p - 1)$ with complex multiplication by ζ_p.

Note that not all factors $J_{r,s}$ are simple abelian varieties, also some of the factors could be isomorphic. This is investigated in [57].

Example 14.12 (T. Katsura, T. Shioda [54])**.** Let X_m^r denote the Fermat hypersurface of degree m and dimension r:

$$x_0^m + \cdots + x_{r+1}^m = 0.$$

We will show that it is of CM type. The assertion is true for $r = 1$, since we already know that the Jacobian variety of the Fermat curve is of CM-type. Let us consider the following rational map

$$X_m^r \times X_m^s \dashrightarrow X_m^{r+s},$$

defined by

$$\big([x_0, \ldots, x_{r+1}], [y_0, \ldots, y_{s+1}]\big) \mapsto [z_0, \ldots, z_{r+s+1}],$$

where

$$z_i = x_i y_{s+1}, \quad i = 0, \ldots, r, \quad z_{r+1+j} = \epsilon_{2m} x_{r+1} y_j,\ j = 0, \ldots, s,$$

and $\epsilon_{2m} = e^{\pi i/m}$. It is clear that the map is dominant and its set $Z_m^{r,s}$ of indeterminacy points consists of the product $V(x_{r+1}) \times V(y_{s+1}) \cong X_m^{r-1} \times X_m^{s-1}$. After we blow up $Z_m^{r,s}$, we obtain a morphism $f : Y_m^{r,s} \to X_m^{r+s}$. Let Y_0 (resp. Y_∞) be the proper inverse transform of $X_m^r \times V(y_{s+1}) \cong X_m^r \times X_m^{s-1}$ (resp. $V(x_{r+1}) \times X_m^s \cong X_m^{r-1} \times X_m^s$). The restriction morphism

$$\tilde{f} : U_m^{r,s} := Y_m^{r,s} \setminus (Y_0 \cup Y_\infty) \to X_m^{r+s} \setminus X_m^{r-1} \cup X_m^{s-1}$$

is a finite morphism. It is an étale map outside of the pre-image of the divisor $B = V(z_0^m + \cdots + z_r^m)$.

The group μ_m of mth roots of unity acts on $X_m^r \times X_m^s$ by multiplying the last coordinate in each factor by a root from μ_m. The locus of fixed points is the subvariety $Z_m^{r,s}$. The extended action of μ_m to the blow-up $Y_m^{r,s}$ has the locus of fixed points equal to the smooth exceptional divisor of the blow-up. This implies that the quotient $\tilde{X}_m^{r+s} = Z_m^{r,s}/\mu_m$ is a nonsingular variety. The map $\tilde{f} : Z_m^{r,s} \to X_m^{r+s}$ factors as the composition of the quotient morphism $p : Z_m^{r,s} \to \tilde{X}_m^{r+s}$ and the blow-up $\phi : \tilde{X}_m^{r+s} \to X_m^{r+s}$ of $f(Y_0 \cup Y_\infty) = X_m^{s-1} \cup X_m^{r-1}$ in X_m^{r+s}.

$$\begin{array}{ccccccc}
 & & Z_m^{r+s} & \xrightarrow{/\mu_m} & \tilde{X}_m^{r+s} & \hookleftarrow & \mathbb{P}^r \times X_m^{s-1} \cup X_m^{r-1} \times \mathbb{P}^{s-1} \\
 & & \downarrow & \searrow^{\tilde{f}} & \downarrow & & \downarrow \\
X_m^{r-1} \times X_m^{s-1} & \hookrightarrow & X_m^r \times X_m^s & \overset{f}{\dashrightarrow} & X_m^{r+s} & \hookleftarrow & X_m^{s-1} \cup X_m^{r-1}
\end{array}$$

For example, take $m = 3, r = s = 1$, so that we have a map $E \times E \dashrightarrow X$ of the self-product of the Fermat plane cubic onto the Fermat cubic surface X_3^2. The open subset $U_m^{r,s}$ is equal to the complement of three fibers E_1, E_2, E_3 and E_1', E_2', E_3' of each projection $E \times E \to E$. The set $Z_3^{1,1}$ is the union of 9 intersection points $p_{ij} = E_i \cap E_j$. The curves E_i, E_j' are blown down to 6 points q_i, q_i' on X_3^2 lying on two lines $\ell : z_0 = z_1 = 0$ and $\ell' : z_2 = z_3 = 0$. The images of the exceptional curves R_{ij} over p_{ij} are the 9 lines on the cubic surface that join a point on one line to a point on another one. The rational map $E \times E$ is given by the linear subsystem $|\sum E_i + \sum E'_i - \sum p_{ij}|$ of curves in the complete linear system $|\sum E_i + \sum E'_i|$ that pass through the points p_{ij}. The inverse transform of this linear system on the blow-up $Y_3^{1,1}$ is the complete linear 3-dimensional system $|\tilde{D}|$ with $\tilde{D}^2 = 9$. It defines a morphism of degree 3 onto the cubic surface X_3^2. The morphism $\tilde{f} : Y_3^{1,1} \to X_3^2$ factors through a finite Galois map $Y_3^{1,1} \to \tilde{X}_3^2$ of degree 3 and the blow-up $\tilde{X}_3^2 \to X_3^2$ of the six points q_i, q'_i. The branch divisor of the first map is the disjoint union of nine smooth rational curves R_{ij} with self-intersection equal to -3. They are the proper inverse transform of the 9 lines $\langle q_i, q'_j \rangle$ on the cubic surface X_3^2 to the blow-up $\tilde{X}_3^2$.

Note that one can show that the existence of 9 lines and 6 points on a cubic surface forming a configuration $(6_3, 9_2)$ characterizes the Fermat cubic surface.

Applying inductively the construction, we obtain a rational map

$$(X_m^1)^r \dashrightarrow X_m^r$$

of the self-product of the Fermat plane curve X_m^1 to the Fermat hypersurface X_m^r.

We have already observed that the Mumford-Tate group of a Hodge substructure is a subgroup of the Mumford-Tate group of the Hodge structure. The following lemmas (see [89], Lemma 7.1.4, Lemma 7.1.5) allow us to conclude that the Fermat hypersurface X_m^r is of CM type.

Lemma 14.13. *Let (V, h) and (V', h') be rational polarized Hodge structures. Then* $\mathrm{MT}(V \otimes v", h \otimes h')$ *is commutative if and only* $\mathrm{MT}(V, h)$ *and* $\mathrm{MT}(V', h')$ *are commutative.*

Lemma 14.14. *Let Y be the blow-up of a smooth variety X along a smooth subvariety Z of codimension 2. Then the Mumford-Tate group of $H^k(Y, \mathbb{Q})$ is commutative if and only if the Mumford-Tate groups of $H^k(X, \mathbb{Q})$ and of $H^{k-2}(Z, \mathbb{Q})$ are commutative.*

From the previous example we know that all Fermat curves $\mathrm{J}(X_m^1)$ are of CM type (see [57]). Applying this to Katsura-Shioda construction, we obtain that a Fermat hypersurface X_m^n has commutative Mumford-Tate group of $H^n(X_m^n, \mathbb{Q})$ (and hence for all other cohomology since it is a hypersurface).

Remark 14.15. A generalization of a Fermat hypersurface is a *Delsarte hypersurface* defined to be a hypersurface in $\mathbb{P}^{r+1}$ given by a homogeneous polynomial of degree m equal to the sum of $r+2$ monomials $x_0^{a_{j0}} \cdots x_{r+1}^{a_{jr+1}}, j = 0, \ldots, r+1$, such that the matrix $A = (a_{ij})$ is nondegenerate and all its rows add up to m. One also assumes that each columns contains at least one zero entry. An example of a Delsarte surface is a surface

$$x_0 x_1^{m-1} + x_1 x_2^{m-1} + x_2^{m-1} + x_3^m = 0.$$

Let A^* be the adjugate matrix of the matrix A, i.e. $AA^* = \det(A) I_{r+2}$. Let δ be the greatest common divisor of the entries a^*_{ij} of A^*, and $d = \det(A)/\delta$ so that $B = dA^{-1} = \delta^{-1} A^*$ is an integral matrix. One constructs a dominant rational map from the Fermat hypersurface X_d^r to a Delsarte hypersurface of degree d defined by the formulas

$$(x_0, \ldots, x_{r+1}) \to (\prod_{j=0}^{r+1} y_j^{b_{0j}}, \ldots, \prod_{j=0}^{r+1} y_j^{b_{r+1j}}),$$

where $B = (b_{ij})$.

One can use this to prove that Delsarte hypersurfaces are of CM type. Finally note that one can also consider a weighted homogenous version of a Delsarte hypersurface by giving the weights to the unknowns x_i. They are finitely covered covered by Delsarte polynomials. One uses this method in [68] to prove that some K3 surfaces are of CM type.

Remark 14.16. One can generalize the constriction from Example 14.12 as follows. Let $F(x_0,\dots,x_r)$ be a weighted homogeneous polynomial of degree d with weights $q_0,\dots,q_r$ and $G(y_0,\dots,y_s)$ be a weighted homogeneous polynomial of degree m with weightes $q'_0,\dots,q'_s$. Consider the hypersurfaces $X = V(F+x_{r+1}^m), Y = V(G+y_{s+1}^m)$ and $Z = V(F(z_0,\dots,z_r)+G(z_{r+1},\dots,z_{r+s}))$ in the weighted projective spaces $\mathbb{P}(q_0,\dots,q_r,1), \mathbb{P}(q'_0,\dots,q'_s,1)$ and $\mathbb{P}(q_0,\dots,q_r,q'_{r+1},\dots,q'_{r+s+1})$, respectively. Then the rational map

$$X \times Y \dashrightarrow Z,$$

given by the same formula as in Example 14.12 is a dominant map of finite degree defined over the complement of $V(x_{r+1}) \times V(y_{s+1}$. In particular, any smooth surface of degree m in $\mathbb{P}^3$ defined by an equation $f(x,y) + g(z,w) = 0$ can be finitely rationally covered by the product of two smooth plane curves of degree m.

Example 14.17 (Yu. Zarhin [113])**.** Let X be an algebraic K3 surface. Let Hdg(X) and MT(X) be the Hodge group and the Mumford-Tate group of the rational Hodge structure on $H^2(X,\mathbb{Q})$. It fixes algebraic cycles and preserves the intersection form on the lattice of transcendental cycles T_X, hence

$$\text{Hdg}(X) \subset \text{SO}(T_{X,\mathbb{Q}}).$$

Let C be the *Weil operator* on $T_{X,\mathbb{R}}$, it acts as -1 on $\{(x,\bar{x}) \in H^{2,0}(X) \oplus H^{0,2}(X)\}$ and as 1 on $H^{1,1}(X) \cap T_{X,\mathbb{R}}$. Thus the form $(x,y) \mapsto \langle x, Cy\rangle$ is a positive definite symmetric form on $T_{X,\mathbb{R}}$. This defines a polarized rational Hodge structure on $T_{X,\mathbb{Q}}$. This implies that MT(X) and Hdg(X) are reductive algebraic groups over $\mathbb{Q}$. Consider $V = T_{X,\mathbb{Q}}$ as a linear $\mathbb{Q}$-representation of Hdg(X). Then it is an irreducible representation (it is true for any surface with $p_g = 1$) ([113], Theorem 1.4.1). Let

$$E_X = \text{End}_{\text{Hdg}(X)}(V).$$

Since V is a simple Hdg(X)-module, E_X is a division algebra. In fact, it is a commutative field, a totally real field or an imaginary quadratic extension of a totally real field E_0. To show that it is commutative one considers a natural non-trivial homomorphism $E_X \to \text{End}(H^{2,0}(X))$. Since it sends 1 to 1, it an injective homomorphism, hence E_X is commutative. The assertion about the structure of the field E_X follows from the existence of a positive anti-involution $x \mapsto x'$ on E_X defined by the taking the adjoint operator with respect to the bilinar form $\langle x,y\rangle = (x,Cy)_X$.

For any $x, y \in E_X$, consider the linear function $E_X \to \mathbb{Q}$ defined by $e \mapsto (ex,y)_X$. Since the bilinear form $(a,b) \mapsto \text{tr}_{E_X/\mathbb{Q}}(ab)$ is non-degenerate, there exists $\alpha_{x,y} \in E_X$ such that $(ex,y) = \text{tr}_{E_X/\mathbb{Q}}(e\alpha_{x,y})$. Define a bilinear form by setting

$$\Phi : V \times V \to E_X, \quad (x,y) \mapsto \alpha_{x,y}.$$

Since $(ex,y)_X = (x,e'y)_X = (e'y,x)_X$, we obtain that $\Phi(x,y) = \Phi(y,x)'$. Also, it is easy to see that $\Phi(ex,y) = e\Phi(x,y)$. In particular, if E_X is a totally real field (resp. a CM-field), then Φ is a symmetric (resp. Hermitian) bilinear form on the E_X-vector space V. Since Hdg_X commutes with E and preserves the intersection form on X, we see that Hdg_X preserves Φ.

The main result of Zarhin's paper is the following.

$$\mathrm{Hdg}_X = \mathrm{SO}(T_{X,\mathbb{Q}}, \Phi),$$

if E_X is a totally real field, and

$$\mathrm{Hdg}_X = U(T_{X,\mathbb{Q}}, \Phi),$$

otherwise. In the former (resp. the latter case) the dimension of the Hodge group is equal to $\frac{t_X^2 - t_X}{2}$ (resp. $\frac{t_X}{4}$, where $t_X = \mathrm{rank} T_X = 22 - \rho(X)$.

For example, when $E_X = \mathbb{Q}$, $\mathrm{Hdg}_X \cong \mathrm{SO}(T_{X,\mathbb{Q}})$.

Lecture 15

Endomorphisms of Jacobian varieties

Let C be a nonsingular projective curve of genus $g > 1$. We are interested in a question when End(J$(C)) \neq \mathbb{Z}$. Of course, easy examples are given by curves admitting non-trivial group of automorphisms or admitting a degree d cover to a curve of lower genus $g' > 0$. In the latter case, and in most of the former cases J(C) is not a simple abelian variety. We also saw in the previous lectures many examples of curves of genus 2 with real or complex multiplication with simple Jacobian.

Let L be a line bundle on the product $C \times C$. For any point $x \in C$, let $L(x) = i_x^*(L) \in$ Pic(C), where $i_x : C \to C \times C$ be the closed embedding map $c \mapsto (x, c)$. We will prefer to switch from line bundles. Extending this map by linearity, we obtain a homomorphism

$$u_L : \text{J}(C) \to \text{J}(C),$$

where J(C) is identified, via the Abel-Jacobi map, with the group of divisor classes of degree 0 on C. Let T be the subgroup of Pic$(C \times C)$ generated by line bundles of the form $p_1^*(M), p_2^*(M)$, where $p_i : C \times C \to C$ are the two projections. It is easy to see that $u_L = 0$ for any $L \in T$. Applying the Seesaw Theorem ([28], Appendix), one shows that any L with $u_L = 0$ belongs to T.

Thus we obtain an injective homomorphism of abelian groups

$$\mathfrak{u} : \text{Corr}(C) := \text{NS}(C \times C)/T \to \text{End}(\text{J}(C)), \quad L \mapsto u_L.$$

An element of the group Corr(C) is called a *correspondence* on C.

Remark 15.1. One can interpret this homomorphism as follows. First, via the inclusion $C \hookrightarrow$ J(C) we identify $H^1(C, \mathbb{Z})$ with $H^1(\text{J}(C), \mathbb{Z})$. This is compatible with the Hodge structures on $H^1(C, \mathbb{C})$ and $H^1(\text{J}(C), \mathbb{C})$. Using the principal polarization, we can identify J(C) with the dual abelian variety $H^{0,1}(C, \mathbb{C})/H^1(C, \mathbb{Z})$. The Küneth Formula and the Poicaré Duality, give a homomorphism

$$H^2(C \times C, \mathbb{Z}) \cong H^1(C, \mathbb{Z}) \otimes H^1(C, \mathbb{Z}) \cong \text{End}(H_1(C, \mathbb{Z}))$$

Using the Hodge decomposition we obtain a map

$$\text{NS}(C \times C) = H^{1,1}(C) \cap H^2(C \times C, \mathbb{Z}) \to H^{1,0}(C) \otimes H^{0,1}(C) \cong \text{End}(H^{0,1}(C)).$$

This defines a rational and algebraic representation of the endomorphism ϕ_L, where $c_1(L) \in H^{1,1}(C \times C) \cap H^2(C \times C, \mathbb{Z})$.

Let us use divisor classes on $C\times C$ instead of line bundles, so for example we write u_D instead of u_L. Since the sum F_1+F_2 of two fibers of the projections $C\times C\to C$ is an ample divisor on the surface $C\times C$, adding some multiple of it, we may assume that a correspondence is represented by an effective divisor. Also, replacing some positive multiple D by a linearly equivalent divisor, we may assume that a correspondence is represented by a divisor of the form $\frac{1}{r}Z$, where Z is an irreducible curve. One may consider Z as a map $C\to C^{(d_1)}, x\mapsto Z\cap\{x\}\times C$, where d_1 is the degree of the projection $p_1:Z\to C$. Similarly, Z defines a map $C\to C^{(d_2)}, x\mapsto Z\cap C\times\{x\}$, where d_2 is the degree of the projection $p_2:Z\to C$. The switch of the factors automorphism $C\times C\to C\times C$ is an involution $D\to D'$ on $\mathrm{Corr}(C)$. Note that the numbers (d_1,d_2) can be defined for any divisor class on $C\times C$, but are not well-defined for correspondences. However, the following number

$$t(D)=d_1+d_2-(D,\Delta), \tag{15.1}$$

where Δ is the diagonal, is a well-defined linear function on $\mathrm{Corr}(C)$. We have

$$t(D)=\mathrm{tr}((u_D)_r).$$

To prove this we apply the Lefschetz fixed-point formula for correspondences (see [37], Example 16.1.15) that gives

$$(D,\Delta)=\mathrm{tr}(u_D^*|H^0(C,\mathbb{Q}))+\mathrm{tr}(u_D^*|H^2(C,\mathbb{Q}))-\mathrm{tr}(u_D^*|H^1(C,\mathbb{Q})).$$

It is easy to see that $d_1=\mathrm{tr}(u_D^*|H^0(C,\mathbb{Q})), d_2=\mathrm{tr}(u_D^*|H^2(C,\mathbb{Q}))$ and $\mathrm{tr}(u_D^*|H^1(C,\mathbb{Q}))=\mathrm{tr}(u_D^*|H^1(\mathrm{J}(C),\mathbb{Q}))=\mathrm{tr}((u_D))_r)$.

One defines the inverse of the map $\mathfrak{u}$ as follows. Recall that $\mathrm{J}(C)$ comes with a natural principal polarization defined by the class in $\mathrm{NS}(\mathrm{J}(C))$ of a *theta divisor* Θ, the image of the symmetric product $C^{(g-1)}$ in $\mathrm{J}(C)$ under the Abel-Jacobi map. As a divisor this image depends on a choice of points $(p_1,\ldots,p_{g-1})$ on C. One can always choose a theta divisor Θ to be symmetric, i.e. satisfy $[-1]^*_{\mathrm{J}(C)}(\Theta)=\Theta$. It is still not defined uniquely. One can show that there exists a divisor class ϑ of degree $g-1$ satisfying $2\vartheta=K_C$ (a *theta characteristic*) such that

$$\Theta+\vartheta:=\{\vartheta+D, D\in\mathrm{J}(C)\}=\{\text{effective divisor classes on } C \text{ of degree } g-1\}.$$

Fix a symmetric theta divisor Θ and embedding $\iota_c:C\hookrightarrow\mathrm{J}(C)$ via the Abel-Jacobi map defined by a choice of a point $c\in C$. For any $u\in\mathrm{End}(\mathrm{J}(C))$, consider the map

$$d_u:C\times C\to\mathrm{J}(C),\quad (x,y)\mapsto u(\iota_c(x))-\iota_c(y),$$

and define

$$\beta(u)=d_u^*(\Theta)\mod T.$$

In other terms, let $\Theta_u=\{(a,b)\in\mathrm{J}(C)\times\mathrm{J}(C):u(a)-b\in\Theta\}$, then $\beta(u)=(\iota_c\times\iota_c)^*(\Theta_u)$. It is clear that choosing different c, replaces the image of $C\times C$ in $\mathrm{J}(C)\times\mathrm{J}(C)$ by a translate by some point in the abelian variety $\mathrm{J}(C)\times\mathrm{J}(C)$, hence replaces $\beta(u)$ by an algebraically equivalent divisor on $C\times C$.

We refer to [67], Chapter 11, §5 for the proof of the fact that β is the inverse of $\mathfrak{u}$ making an isomorphism

$$\mathfrak{u}:\mathrm{Corr}(C)\cong\mathrm{End}(\mathrm{J}(C)). \tag{15.2}$$

Note that β gives a natural section of NS$(C \times C) \to$ Corr(C), we can call the corresponding divisor class $\beta(u) \in$ NS$(C \times C)$ a *canonical correspondence* associated to u. Fixing ϑ and a point $c \in C$, we can even choose a representative of $\beta(u)$ in Div$(C \times C)$. Note that

$$d_1(\beta(u)) = (C, \Theta), \quad d_2(\beta(u)) = (u(C), \Theta).$$

It is known that $(C, \Theta) = g$ ([67], 11.2.2).

In fact, the isomorphism (15.2) is an isomorphism of rings, where the ring structure is defined by the composition of correspondences

$$D \Diamond D' = (p_{13})_*(p_{12}^*(D) \cdot p_{23}^*),$$

where $p_{ij} : C \times C \times C \to C \times C$ are the natural projections. One easily checks that the multiplication law is well-defined on Corr(C). The homomorphism $\mathfrak{u}$ becomes an isomorphism of rings.

Next we define a symmetric bilinear form on Corr(C) by setting

$$\sigma(D, D') = d_1 d_2' + d_1' d_2 - (D, D').$$

Obviously, the radical of the form contains the subgroup of divisors algebraically equivalent to zero. It also contains the subgroup T. Thus it defines a symmetric bilinear form on the group Corr(C). The famous *Castelnuovo inequality* asserts that the corresponding quadratic form

$$\sigma(D) := \sigma(D, D) = 2d_1 d_2 - (D, D) \tag{15.3}$$

is positive definite. An exercise on p. 368 of Hartshorne's book sketches a proof.

Note that our trace function (15.1) can be expressed in terms of the symmetric form σ

$$t(D) = \sigma(D, \Delta).$$

Let $D \to D'$ be the involution on Corr(C) defined by the switch of the factors of $C \times C$. Under the isomorphism (15.2), it corresponds to the Rosati involution $f \mapsto f'$ [67], 11.5.3. Considering effective correspondences D_1, D_2 as multi-valued maps $C \to C$, one checks that

$$d_1(D_1 \Diamond D_2') = n_1(D_1) n_2(D_2),\ n_2(D_1 \Diamond D_2') = n_2(D_1) n_1(D_2),\ (D_1, D_2) = (D_1 \Diamond D_2', \Delta).$$

(cf. [37], Chapter 16, Examples 16.3.3 and 16.3.4). This implies that

$$\sigma(D_1, D_2) = n_1(D_1 \Diamond D_2') + n_2(D_1 \Diamond D_2') - (D_1 \Diamond D_2', \Delta) = \mathrm{tr}(\alpha(D_1 \Diamond D_2')).$$

Thus the symmetric bilinear form $\sigma(D_1, D_2)$ coincides, under the isomorphism α, with the symmetric form tr$(f\phi')$ on End(J(C)).

It is known that, under the isomorphism $\mathfrak{u}$, the symmetric form becomes the symmetric form Tr$(\phi\psi')$ defined by the Rosati involution. Note that, taking D to be the diagonal Δ in $C \times C$, we obtain that $\sigma(D) = 2 - (2 - 2g) = 2g$ and $\phi(\Delta) = \mathrm{id}_{\mathrm{J}(C)}$, so the formulas agree.

Note, that, applying the adjunction formula, we have $D^2 = 2p_a(D) - 2 - (2g-2)(d_1 + d_2)$, so we may rewrite (15.3) in the form

$$\sigma(D) = 2d_1 d_2 + (2g-2)(d_1 + d_2) - 2p_a(D) + 2. \tag{15.4}$$

A correspondence $D \in \mathrm{Corr}(C)$ such that $u_D = [-\nu]_{\mathrm{J}(C)}$ is called a correspondence with *valence* ν. In this case, it can be represented by a curve Z in $C \times C$ with the class $[C]$ in $H^2(C \times C, \mathbb{Z})$ equal to

$$(d_1 + \nu)[C \times \{x\}] + (d_2 + \nu)[\{x\} \times C] - \nu[\Delta]$$

So, $\mathrm{End}(\mathrm{J}(C)) \neq \mathbb{Z}$ if and only if C admits a correspondence without valence. Many classical enumerative problems are solved by constructing correspondence with valence and applying the *Brill-Noether formula* that expresses the valence in terms of the number (D, Δ) of *united points* of the correspondence.

$$(D, \Delta) = d_1 d_2 - 2\nu g,$$

where g is the genus of C (see [28], Corollary 5.5.2).

A correspondence D is called *symmetric* if $D' = D$. It follows from above that the subgroup $\mathrm{Corr}(C)^s$ of symmetric correspondences is isomorphic to the group of symmetric endomorphisms of $\mathrm{J}(C)$, and hence to the Néron-Severi group $\mathrm{NS}(\mathrm{J}(C))$. Note that a canonical representative $D = \beta(u)$ of a symmetric endomorphism u must satisfy $d_1 = d_2 = g$, hence $(D, \Delta) = 2g - t(D) = 2g - \mathrm{tr}(u_r)$.

An example of a symmetric correspondence with valence -1 on a curve of genus g is the *Scorza correspondence* R_θ with $d_1 = d_2 = (g, g)$ defined by a choice of a non-effective theta characteristic ϑ (see [28], 5.5). It is equal

$$\beta_\theta(\mathrm{id}_{\mathrm{J}(C)}) = \{(x, y) \in C \times C : x - y \in \Theta_\theta\}.$$

Note that it does not depend on a choice of an embedding of C in $\mathrm{J}(C)$.

Example 15.2. Let $f : C \to C'$ be a finite map of curves. It defines a correspondence

$$\Gamma(f) = C \times_X C = \{(x, y) : f(x) = f(y)\}.$$

It follows from the definition that $\mathfrak{u}(\Gamma(f))$ maps a divisor class $d = \sum x_i \in \mathrm{Pic}(C)^0$ to the divisor class $\sum f^*(f(x_i)) \in \mathrm{Pic}^0(C)$. Obviously, it is equal to $f^*(\mathrm{Nm}(d))$, where $\mathrm{Nm} : \mathrm{J}(C) \to \mathrm{J}(C')$ is the norm map and $f^* : \mathrm{J}(C) \to \mathrm{J}(C')$ is the pull-back map. Since the norm map is surjective, we obtain that the image of the endomorphism $u = \phi(\Gamma(f))$ is equal to $f^*(\mathrm{J}(C'))$. Thus, if $g(C') > 0$, the endomorphism u coincides with the norm map of the abelian subvariety $f^*(\mathrm{J}(C'))$ of $\mathrm{J}(C)$.

Example 15.3. Let $f : C \to C$ be an automorphism of C and $D = \Gamma_f$ be its graph. Then $d_1(D) = d_2(D) = 1$ and $p_a(D) = g$. Applying (15.4), we get $\sigma(\Gamma_f) = \sigma(\Delta) = 2 - (2 - 2g) = 2g$. Let $\nu = (\Gamma_f, \Delta)$ be the number of fixed points of f. Thus $\sigma(\Gamma_f, \Delta) = 2 - (\Gamma_f, \Delta) = 2 - \nu$. Since the quadratic form σ is positive definite we must have

$$\sigma(\Gamma_f)\sigma(\Delta) - \sigma(\Gamma_f, \Delta)^2 = 4g^2 - (2 - \nu)^2 = (2g - 2 + \nu)(2g + 2 - \nu) > 0$$

unless $\Gamma_f = m\Delta$ in $\mathrm{Corr}(C)$. If $g = 1$, we get that the latter is possible only if $\nu = 0$ or $\nu = 4$, i.e. f is a translation by a point or the quotient by (f) is $\mathbb{P}^1$. If $g > 1$, the latter happens only if $\nu = 2g + 2$. Since the eigenvalues of $f^* : H^1(C, \mathbb{C}) \to H^1(C, \mathbb{C})$ are roots of unity, we have $|\mathrm{tr}(f^*)| \leq 2g$. The Lefschetz fixed-point formula gives us that $\nu = 2 - \mathrm{tr}(f^*) \leq 2 + 2g$ with the equality taking place if and only if $f^* = -\mathrm{id}$. This happens only if f is an involution with quotient isomorphic to $\mathbb{P}^1$, hence C is a hyperelliptic curve and f is its hyperelliptic involution.

Observe that the graph of an automorphism f is symmetric if and only if $f = f^{-1}$, i.e. f is an involution. Thus, if f is of order > 2, the corresponding automorphism of $\mathrm{J}(C)$ is not symmetric. For example, it can never define a real multiplication of $\mathrm{J}(C)$.

Example 15.4 (I. Shimada [101])**.** Let $f : C \to C'$ be a finite cover of curves and let G be its Galois group. The Galois theory of finite covers provides us with a finite map $\phi : X \to C$ such that the composition $f \circ \phi : X \to C'$ is a Galois cover with the Galois group G and $C \cong X/H$ for some subgroup H of C. We assume that H is not a normal subgroup, or, equivalently, the cover f is not a Galois cover. Let $g \in G$ be such that $H' = gHg^{-1} \neq H$. Then the map $g : X \to X$ induces an isomorphism $\alpha_g : C = X/H \to X/H'$, hence defines a map $(\phi, \alpha_g) : X \to C \times C$. Let S be the correspondence defined by the image of this map. It consists of points $(\phi(x), \phi(g(x)), x \in X$. The curve S is birationally isomorphic to X, it is isomorphic to X if no element of the double coset HgH has a fixed point on X. We have $d_1(S) = d_2(S) = d = [H : H' \cap H]$ and S is symmetric if and only if g is an involution.

We have

$$(S, \Delta) = \sum_{h \in H} X^{hg},$$

where X^{hg} denotes the set of fixed points of hg. Thus

$$t(u_S) = 2d - (S, \Delta) = 2[H : H' \cap H] - \sum_{h \in H} X^{hg}. \tag{15.5}$$

suppose $u_S = [m]_{\mathrm{J}(C)}$ for some $m \in \mathbb{Z}$. Then $t(u_S) = 2gm$, so we can construct a correspondence without valence if the right-hand side of (15.5) is not a multiple of $2g$. For example, suppose f is an unramified cover, so that its Galois closure is unramified too. Then H acts without fixed points. Hence, if $\#H < 2g$, we obtain $0 < t(u_S) < 2g$, so we get a non-trivial endomorphism.

In the case $C' = \mathbb{P}^1$, Shimada gives another criterion when u_S has no valence:the Galois group acts 2-transitively on fibers of $C \to C'$.

Lecture 16

Curves with automorphisms

Let G be a finite group and $\mathbb{Q}[G]$ be the group algebra of G. It is a semi-simple algebra over $\mathbb{Q}$ over the center $Z(G)$ generated by elements $c_i = \sum_{g\in C_i} g$, where $C_1, \ldots, C_k$ are conjugacy classes of G. The group algebra $\mathbb{Q}[G]$ decomposes into the direct sum $Q_1 \times \cdots \times Q_r$ of simple algebras corresponding to irreducible rational representations of G. A simple factor Q_i is isomorphic to a right ideal in $\mathbb{Q}[G]$ generated by an element e_i from $Z(G)$ satisfying $e_i^2 = e_i$ (a *central idempotent*). Being a simple $\mathbb{Q}$-algebra, each Q_i is isomorphic to a matrix algebra $M_{n_i}(D_i)$ over some skew field D_i.

Example 16.1. Assume G is an abelian group. First, we decompose G into the direct sum of cyclic groups G_i of orders m_i. Then $\mathbb{Q}[G] \cong \prod_i \mathbb{Q}[G_i]$. Assume $G = (g)$ is cyclic of order m generated by. Then

$$\mathbb{Q}[G] \cong \prod_{d|m} \mathbb{Q}[t]/(\Phi_d(t)),$$

where $\Phi_d(t)$ is an irreducible cyclotomic polynomial of degree $\phi(d)$. Each direct factor is a cyclotomic field of degree $\phi(m)$ over $\mathbb{Q}$. It is generated by the central idempotent $f_d = \phi_d(g)$, where $\phi_d(t) \in (\Phi_d(t))$ and $1 = \sum_{d|m} \phi_i(t)$. For example, if $m = 3$, we may take $f_1 = (1-g)(2+g)/3$ and $f_3 = (1 + g + g^2)/3$.

The group G acts on each summand $Q_i \cong \mathbb{Q}(\zeta_d)$ considered as a linear space over $\mathbb{Q}$ of dimension $\phi(d)$. This is an irreducible rational representation of G. Of course, considered as a complex representation it splits into the direct sum of one-dimensional representations.

Suppose that G acts faithfully on an abelian variety A. Then the action defines a homomorphism

$$\rho : \mathbb{Q}[G] \to \mathrm{End}(A)_{\mathbb{Q}}.$$

Recall that it defines two homomorphisms

$$\rho_a : \mathbb{Q}[G] \to \mathrm{End}(V), \quad \rho_r : \mathbb{Q}G \to \mathrm{End}(\Lambda_{\mathbb{Q}}),$$

where $A = V/\Lambda$. In particular, $\Lambda_{\mathbb{Q}}$ splits into the direct sum of irreducible representations of G. Let e_W be the central idempotent corresponding to an irreducible rational representation W contained in $\Lambda_{\mathbb{Q}}$. Let n be the smallest integer such that $\rho(ne_W) \in \mathrm{End}(A)$. Then the image of ne_W is an abelian subvariety A_W of $\mathrm{J}(C)$. We have

$$A_W \cong V_W/\Lambda_W,$$

where $V_W = \rho_a(e_W)$ and $\Lambda_W = \mathrm{Im}(\rho_r(e_W)) \cap \Lambda$, where the intersection is taken in $\Lambda_{\mathbb{Q}}$. If the inclusion $\Lambda_W \to \Lambda$ is given by a matrix P, then the type of the polarization of J_W is equal to the type of the symplectic form defined by the matrix ${}^tPJP$, where J_{D} is the symplectic matrix defining a polarization on A.

An abelian variety A with a faithful G-action is called *G-simple* if it does not contain proper G-invariant subvarieties. Similar to the case when $G = \{1\}$, one constructs an isogeny

$$A_1 \times \cdots \times A_k \to A,$$

where A_i are G-simple abelian varieties (see [67], 13.6). The varieties A_i are called *isotypical components* of A. Each isotypical component is G-isomorphic to a subvariety of A_W for some W. For example, if $G = \{1, g\}$ is of order 2, then A decomposes into a g-invariant and g-anti-invariant parts corresponding to the idempotents $\frac{1}{2}(1+g)$ and $\frac{1}{2}(1-g)$.

Example 16.2 (V. Popov, Yu. Zarhin [87])**.** Let G be an irreducible finite subgroup of $\mathrm{GL}(V)$, i.e. the image $\mathbb{C}G$ of $\mathbb{C}[G]$ in $\mathrm{End}(V)$ coincides with $\mathrm{End}(V)$. In other words, the representation V is an irreducible representation of G. In this case $\mathbb{Q}G$ (the image of $\mathbb{Q}[G]$ in $\mathrm{GL}(V)$) is a simple central algebra over the center $Z(G)$ of $\mathbb{Q}[G]$. The field $Z(G)$ coincides with the subfield $\mathbb{Q}(\chi_V)$ of $\mathbb{C}$ generated by the values of the character χ_V of the representation V on elements of G.

Let $\mathbb{Q}G \cong \mathrm{Mat}_r(D)$ be an isomorphism with the matrix algebra over some skew field D over $Z(G)$. In particular,

$$n^2 = [\mathbb{Q}G : Z(G)] = r^2[D : Z(G)].$$

The dimension $[D : Z(G)]$ of D over $Z(G)$ is equal $m(\chi_V)^2$, where $m(\chi_V)$ is the *Schur index* of χ_V.[1]

Suppose $\Lambda \subset V$ is a lattice of rank $2\dim V$ which is G-invariant, so that G acts on the complex torus $T = V/\Lambda$.

Since $Z(G)$ belongs to the center of $\mathbb{Q}G$, it acts on V by scalar multiplication. This implies that, for any nonzero $v \in \Lambda$, the ring $(Z(G) \cap \mathbb{Z}[G])v \subset \mathbb{C}v \cap \Lambda \subset \mathbb{Z}v + iv\mathbb{Z}$. Thus $[Z(G) : \mathbb{Q}] \leq 2$, and the equality happens only $iv \in \Lambda$, i.e. $\mathbb{C}v/\mathbb{C}v \cap \Lambda$ is an elliptic curve with complex multiplication by $\mathbb{Z}[i]$ and $Z(G)$ is an imaginary quadratic field.

Suppose $Z(G) = \mathbb{Q}$. It is known that $m(\chi) \leq 2$ if the value of the character χ are real numbers. Thus $D = Z(G)$ if $\mu(\chi_V) = 1$ and $[D : \mathbb{Q}] = 4$ otherwise. In the first case $\mathbb{Q}G \cong M_d(\mathbb{Q})$. The projector operators in the matrix algebra define norm-endomorphisms in A with isomorphic images. This shows that A is isogenous to the product of E^n, where E is an elliptic curve. In the second case D is a quaternion algebra. It is indefinite (resp. definite) if and only if V admits a G-invariant non-degenerate symmetric (resp. skew-symmetric) bilinear form. In this case, T is isogenous to

[1] It is equal to the minimum of the degrees $[F : Z(G)]$, where the representation V of G can be realized over a finite extension F of $\mathbb{Q}$.

a self-product of a 2-dimensional torus T_1 with multiplication by an order in D. It is always an abelian variety if D is indefinite quaternion algebra.

Note that V. Popov and Yu. Zarhin prove the converse: a G-invariant lattice of rank $2n$ in V exists if if one of the following conditions is satisfied

- $Z(G) = \mathbb{Q}$ or an imaginary quadratic and $\mu(\chi_V) = 1$;
- $Z(G) = \mathbb{Q}$ and $\mu(\chi_V) = 2$.

Note that it agrees with the following sufficient condition from [30] for a Jacobian variety $\mathrm{J}(C)$ to be isogenous to a product of elliptic curve:

Let χ be the character of G in its representation on $H^0(C, K_C)$. Then any irreducible representation V contained in $H^0(C, K_C)$ has multiplicity 1. Also the field $\mathbb{Q}(\chi_V)$ is $\mathbb{Q}$ or imaginary quadratic extension of $\mathbb{Q}$, and the Schur index $m(\chi_V) = 1$.

Example 16.3. Let $f : C \to \mathbb{P}^1$ be a cover of nonsingular curves with cyclic Galois group G of order m. Let $B = \{p_1, \ldots, p_{r+1}\}$ be the set of its branch points, and let e_i be the ramification index of a ramification point lying over p_i, so that we have m/e_i ramification points over p_i. We assume that the genus of C is larger than 0, this implies that $r \geq 3$. Let $U = \mathbb{P}^1 \setminus B$ and $\gamma_1, \ldots, \gamma_{r+1}$ be standard generators of the fundamental group $\pi_1(U)$ satisfying the relation $\gamma_1 \cdots \gamma_{r+1} = 1$. The cover defines a surjective homomorphism $\tau : \pi_1(U) \to \mathbb{Z}/m\mathbb{Z}$. Let $\chi(\gamma_i) = a_i \mod m$. Since $\tau(\gamma_i^{e_i}) = 1$, we must have $e_i a_i \equiv 0 \mod m$ and $\sum a_i \equiv 0 \mod m$. Since τ is surjective,

$$(a_1, \ldots, a_{r+1}) = 1 \mod m.$$

Let $\bar{\mathbf{e}}_i$ be the images of the unit vectors in $\mathbb{Z}^{r+1}$ in $(\mathbb{Z}/m\mathbb{Z})^{r+1}$ and $\bar{e} = \bar{e}_1 + \cdots + \bar{e}_{r+1}$. We can factor τ through a surjective homomorphisms

$$\sigma : \pi_1(U) \to A_{m,r} := (\mathbb{Z}/m\mathbb{Z})^{r+1}/(\bar{e}),$$

that sends γ_j to $\bar{\mathbf{e}}_j$. Let $X \to C$ be the Galois cover corresponding to the homomorphism σ. Its Galois group is equal to $H = \mathrm{Ker}(\sigma)$.

Let

$$H^1(X, \mathbb{C}) = \oplus_\chi H^1(X, \mathbb{C})_\chi$$

be the decomposition of $H^1(C, \mathbb{C})$ into direct sum of eigensubspaces with characters $\chi \in \mathrm{Hom}(A_{d,r}, \mu_m)$. We have

$$H^1(C, \mathbb{C}) \cong H^1(X, \mathbb{C})^H = \oplus_{\chi, \chi|H=1} H^1(X, \mathbb{C})_\chi. \tag{16.1}$$

The group $\mathcal{X}$ of characters whose restriction to H is the identity is a cyclic group generated by the character $\chi_{\boldsymbol{\mu}}$ that sends $\bar{\mathbf{e}}_j \in A_{m,r}$ to $e^{2\pi i a_j/m}$. Here we use $\boldsymbol{\mu}$ to denote the vector

$$\boldsymbol{\mu} = (\frac{a_1}{m}, \ldots, \frac{a_{r+1}}{m}).$$

It satisfies the condition that $|\boldsymbol{\mu}| = \mu_1 + \cdots + \mu_{r+1} \in \mathbb{Z}$. Any other character in $\mathcal{X}$ is a power $\chi_{\boldsymbol{\mu}}^n, n = 0, \ldots, m-1$. It corresponds to the vector

$$\boldsymbol{\mu}^n := (\frac{(na_1)}{m}, \ldots, \frac{(na_{r+1})}{m}),$$

where the round brackets denote the remainder of the number for the division by m. We set

$$d_n = |\boldsymbol{\mu}^n| := \frac{1}{m}\sum_{i=1}^{r+1}(na_i).$$

The curve X is easy to describe by equations. For convenience, let us choose projective coordinates on $\mathbb{P}^1$ such that

$$p_i = [1, x_i], \quad p_{r+1} = [0, 1]$$

and consider a linear embedding

$$\alpha : \mathbb{P}^1 \to \mathbb{P}^r, \quad [t_0, t_1] \mapsto [x_1t_0 - t_1, \dots, x_rt_0 - t_1, t_0].$$

Let $r_m : \mathbb{P}^r \to \mathbb{P}^r$ be the cover given by raising the coordinates in mth power. Then X is isomorphic to the pull-back of the cover s to C, i.e. we have a commutative diagram

$$\begin{array}{ccc} X & \longrightarrow & \mathbb{P}^r \\ \Big\downarrow{\scriptstyle \phi} & & \Big\downarrow{\scriptstyle s} \\ C & \xrightarrow{\alpha} & \mathbb{P}^r \end{array}$$

It follows that X is isomorphic to the complete intersection of Fermat hypersurfaces

$$F_i = \sum_{i=0}^{r} \alpha_{ij}y_i^m = 0, \quad j = 1, \dots, r-1,$$

where $M = (\alpha_{ij})$ is a matrix of size $(r-1) \times 2$ and rank $r-1$ satisfying

$$\begin{pmatrix} 1 & 1 & \dots & 1 & 0 \\ x_1 & x_2 & \dots & x_r & 1 \end{pmatrix} \cdot M = 0.$$

The curve C is explicitly computed by using the action of $A_{m,r}$ on X. It is birationally isomorphic to the curve

$$y^m = (x - x_1)^{a_1} \cdots (x - x_r)^{a_r}, \tag{16.2}$$

where $e_i = m/(m, a_i)$ with $a_{r+1} \equiv -(a_1 + \cdots + a_r) \mod m$.[2] Note that, for any k prime to m, the curve (16.2) is isomorphic to the curve with the same branch points but with $(a_1, \dots, a_r)$ replaced by $(ka_1, \dots, ka_r) \mod m$.

Applying the Hurwitz's formula, we obtain that the genus of C is equal to

$$g = 1 + \frac{m(r-1) - \sum_{i=1}^{r+1}(m, a_i)}{2}. \tag{16.3}$$

We have

$$H^1(C, \mathbb{C})_{\chi_{\boldsymbol{\mu}}^n} = H^{1,0}(C)_{\chi_{\boldsymbol{\mu}}^n} \oplus H^{0,1}(C)_{\chi_{\boldsymbol{\mu}}^n},$$

[2] One can also view C as the normalization of the cover defined by a line bundle $L = \mathcal{O}_{\mathbb{P}^1}(\frac{1}{m}\sum_{i=1}^{r+1} a_ix_i)$ on $\mathbb{P}^1$ and a section of $L^{\otimes m}$ with the divisor of zeros equal to $\sum_{i=1}^{r+1} a_ix_i$.

and the well-known formula due to Hurwitz and Chevalley-Weil gives:

$$\dim H^{1,0}(C)_{\chi_{\boldsymbol{\mu}}^n} = d_n, \quad \dim H^{1,0}(C)_{\chi_{\boldsymbol{\mu}}^n} = d_{m-n}, \quad n = 1, \ldots, m-1, \tag{16.4}$$

and $H^1(C,\mathbb{C})_1 = \{0\}$. There are many proofs of this formula. For example, one computes the cohomology $H^1(X,\mathbb{C})$ as a representation of $A_{N,r}$ (see [104]):

$$H^1(X,\mathbb{C}) \cong \mathbb{C}[T_0, \ldots, T_r, \lambda_1, \ldots, \lambda_{r-1}]/J,$$

where J is the ideal generated by partial derivatives in y_j and λ_k of the equation $F(y,\lambda) = \sum_{i=1}^{r-1} \lambda_i F_i$. Here each coset of a monomial $y_0^{s_0} \cdots y_r^{s_r}$ is an eigenvector of $A_{N,r}$ with eigenvalue $-r + \sum_{j=0}^{r} s_j/N$.[3]

The case $|\boldsymbol{\mu}| = 2$ is special since it gives a one-dimensional part $H^{1,0}(C)_{\chi_{\boldsymbol{\mu}}}$ of $H^{1,0}(C)$ that allows one to construct an *eigenperiod map* for curves C with varying $(x_1, \ldots, x_r)$ with values in a complex ball. For some spacial $\boldsymbol{\mu}$ one relates this period map with the period map of certain families of K3 surfaces (see [27]).

The cyclic group G acts on C and hence acts on its Jacobian variety $\mathrm{J}(C)$. For example, if $m = p$ is prime, we get

$$g = \tfrac{1}{2}(r-1)(p-1),$$

and taking $r = 2$, i.e. a cover with 3 branch points, we obtain that $g = \frac{1}{2}(p-1)$ and $\mathrm{J}(C)$ has multiplication by the CM field $\mathbb{Q}(\zeta_p)$. This agrees with *Belyi's Theorem* that any cover of $\mathbb{P}^1$ ramified over 3 points is defined over $\mathbb{Q}$.

Suppose $m'|m$, then the surjective homomorphism of cyclic group $G_m = (\mathbb{Z}/m\mathbb{Z}) \to G_{m'} = (\mathbb{Z}/m'\mathbb{Z})$ defines a Galois cover $C \to C'$ with the cyclic Galois group of order m/m'. It is easy to see that

$$C' : y^{m'} = (x-x_1)^{\bar{a}_1} \cdots (x-x_r)^{\bar{a}_r}, \tag{16.5}$$

where $0 \le \bar{a}_i < m', a_i \equiv \bar{a}_i \mod m'$.

[3] This method extends to a similar computation for cyclic covers of projective spaces branched over an arrangement of hyperplanes, see [27].

Lecture 17

Special families of abelian varieties

Let $f : \mathcal{A} \to T$ be a smooth family of polarized abelian varieties over a smooth manifold T. This means that there exists a relatively line bundle $\mathcal{L}$ on $\mathcal{A}$ such that its restriction to each fiber defines a polarization of type D that does not depend on $t \in T$. The family is called *special* if it contains a dense set of points $t \in T$ such that the fiber $\mathcal{A}_t$ is of CM type. For example, a constant family $\mathcal{A} = A \times T \to T$ is special if A is of CM type.

Similarly, using the variation of polarized rational Hodge structures one can define a special family of families of polarized algebraic varieties.

Passing to the universal cover of S we obtain a family $\tilde{\mathcal{X}} \to \tilde{T}$ such that we can identify the lattice $\Lambda_{\tilde{t}} = H_1(\mathcal{A}_{\tilde{t}}, \mathbb{Z}))$ with a fixed lattice $\Lambda = \mathbb{Z}^{2g}$ so that the cohomology $H_1(\mathcal{A}_{\tilde{t}}, \mathbb{R})$ can be identified with $W = \Lambda_{\mathbb{R}}$. We can also fix the symplectic form E on Λ. The E-polarized complex structure on $\Lambda_{\mathbb{R}}$ defined by the complex structure on $\mathcal{A}_{\tilde{t}}$ defines a point in $\mathsf{G}(g, \Lambda_{\mathbb{C}})_E$ and gives rise to the (marked) period map

$$\tilde{p} : \tilde{T} \to \mathsf{G}(g, \Lambda_{\mathbb{C}})_E \cong \mathcal{Z}_g.$$

Its composition with the projection to $\mathcal{A}_g$ defines the period map

$$p : T \to \mathcal{A}_g.$$

Fix a connected algebraic group $G_{\mathbb{Q}}$ defined over $\mathbb{Q}$ and a faithful homomorphism

$$\rho : G_{\mathbb{Q}} \to \mathrm{Sp}(W, E)_{\mathbb{Q}},$$

where $\mathrm{Sp}(W, E)_{\mathbb{Q}}$ is the algebraic group over $\mathbb{Q}$ whose points in a $\mathbb{Q}$-algebra K is the group of linear transformations of the symplectic form (W_K, E_K). In particular, ρ defines a homomorphism

$$\rho_{\mathbb{R}} : G_{\mathbb{Q}}(\mathbb{R}) \to \mathrm{Sp}(W, E)_{\mathbb{Q}}(\mathbb{R}) = \mathrm{Sp}(W, E) \cong \mathrm{Sp}(2g, J_{\mathsf{D}}).$$

We also fix an arithmetic subgroup Γ of $G_{\mathbb{Q}}$ such that $\rho(\Gamma)$ leaves $\Lambda \subset W$ invariant. A E-polarized complex structure on W is determined by a homomorphism

$$\phi : U(1) \to \mathrm{Sp}(W, E)$$

that sends $e^{i\theta}$ to the multiplication by $e^{i\theta}$ in W. It can be viewed as the restriction of the unique $h_\phi : \mathbb{S} \to \mathrm{Sp}(W,E)_\mathbb{Q}$ by passing to real points and restricting h to the subgroup of $\mathbb{S}^1$ of $\mathbb{S}$ such that $\mathbb{S}^1(\mathbb{R}) = U(1) \subset \mathbb{C}^*$. We fix one complex structure ϕ_0 and consider the orbit of ϕ_0 under the action of $\rho(G), g \mapsto \rho(g)\phi_0\rho(g)^{-1}$. The orbit is a homogeneous space $G_\mathbb{R}/K^0_\mathbb{R}$, where $K_\mathbb{R}$ is the connected component of the stabilizer of ϕ_0. It can be shown that the orbit defines a family of polarized abelian varieties

$$\mathcal{X}(G,\Gamma,\rho,\phi_0) \to T = \Gamma\backslash G_\mathbb{R}/K^0_\mathbb{R}. \tag{17.1}$$

The base of the family is a Hermitian symmetric domain if the following condition is satisfies:

$$\rho(G(\mathbb{R})) \text{ is normalized by } \phi_0(U(1)). \tag{17.2}$$

Let $\mathrm{MT}(h_\phi) \subset \mathrm{CSp}(W,E)_\mathbb{Q}$ be the Mumford-Tate group and $\mathrm{Hg}(\phi) \subset \mathrm{Sp}(W,E)$ be the Hodge group. In above take $G = \mathrm{Hg}(\phi_0)$ to obtain a family $\mathcal{X}(\mathrm{Hg}(\phi_0),\Gamma,\rho,\phi_0)$ of fixed Hodge type. For example, if ϕ_0 defines an abelian variety of CM type, then $\mathrm{Hg}(\phi_0)$ is commutative, hence it must coincide with the centralizer subgroup of ϕ_0, hence $G = K$, and the orbit consists of one point. Now, one shows that any family of Hodge type contains an abelian variety of CM type (a CM-point in the base of the family). The idea is simple (see [81]): any G as above contains a maximal torus defined over $\mathbb{Q}$. One takes a maximal torus T of the stabilizer subgroup K of ϕ_0, then for some $g \in G$, its conjugate gTg^{-1} must be contained in a maximal torus T' of G defined over $\mathbb{Q}$, then $\mathrm{Hg}(g\phi_0g^{-1})$ must be contained in T', hence must be commutative. Thus the point $g\phi_0g^{-1}$ defines a CM-point. In fact, we have a dense set of CM-points since $G_\mathbb{Q}(\mathbb{Q})$ is a dense subset of $G_\mathbb{Q}(\mathbb{R})$, so we take the $G_\mathbb{Q}(\mathbb{Q})$-orbit of a CM-point to a get a dense subset of CM points.

Thus, we see that families of Hodge type are examples of special families. Note that the converse is true, namely a family (17.1) with a CM-point is isomorphic to a family of Hodge type.

Example 17.1. In dimension ≤ 3 all Hodge familes are determinde by a Hodge class that defines a special endomorphism of the abelian variety. The following is an example of Mumford [81] of a Hodge family not determined by a special property of endomorphism algebra of its members.

Let K/K^0 be a finite extension of fields of characteristic 0 of degree n and D be a central simple algebra over K. Recall that isomorphism classes of central simple algebras over a field F form a group, the *Brauer group* of this field. It is isomorphic to $H^2(\mathrm{Gal}(\bar{F}/F),\bar{F}^*)$. The extension L/K gives rise to the natural homomorphism of group cohomology $\mathrm{Cor} : \mathrm{Br}(K) \to \mathrm{Br}(K_0)$ that assigns to D the isomorphism class of the algebra $\mathrm{Cor}_{K/K_0}(D)$ constructed as follows. Let $\sigma_1,\ldots,\sigma_n : L \to \bar{K}$ be the set of distinct K_0-embeddings of K into its algebraic closure, then the Galois group of K acts naturally on the tensor product $E = \otimes(D\otimes_{\sigma_i}\bar{K})$ and $\mathrm{Cor}_{K/K_0}(D)$ is the subalgebra of invariants for this action. An element d of the tensor product can be written in the form

$$d = \sum a_{i_1\ldots i_n}\sigma_1(e_{i_1})\otimes\cdots\sigma_n(e_{i_n}),$$

where $e_1,\ldots,e_{r^2}$ is a basis of D over K and $a_{i_1\ldots i_n} \in K$. An element τ of the Galois group acts by sending d to

$$\tau(d) = \sum \tau(a_{i_1\ldots i_n})\sigma_{\tau(1)}(e_{i_1})\otimes\cdots\sigma_{\tau(n)}(e_{i_n}),$$

where $\sigma_{\tau(i)} := \tau\circ\sigma_i$. By choosing a normal basis of L over K, one sees that Corr_{K/K_0} is a central simple algebra of degree r^{2n} over K and $\mathrm{Corr}_{K/K_0}\otimes_K\bar{K} \cong E$. It comes equipped with the norm homomorphism

$$\mathrm{Nm} : D^* \to \mathrm{Cor}_{K/K_0}(D)^* \tag{17.3}$$

that sends an invertible element $d \in D^*$ to the tensor product $(d \otimes 1) \otimes \cdots (d \otimes 1) \in E$ which is obviously invariant with respect to the action of the Galois group.

For example, if $D = K$, we obtain $\mathrm{Cor}_{K/K_0}(K) = K_0$ and the norm homomorphism is the usual norm map for field extensions.

In Mumford's example one takes K to be a totally real cubic extension of $K_0 = \mathbb{Q}$ and D be a quaternion division algebra over K. One chooses the extension and D in such a way that

$$\mathrm{Cor}_{K/K_0}(D) \cong \mathrm{Mat}_8(\mathbb{Q}), \quad D \otimes_{\mathbb{Q}} \mathbb{R} \cong \mathbb{K} \oplus \mathbb{K} \oplus \mathrm{Mat}_2(\mathbb{R}),$$

where $\mathbb{K} = H((\frac{-1,-1}{\mathbb{Q}}))$ is the standard quaternion algebra over $\mathbb{Q}$. The norm map becomes a natural homomorphism $D^* \to \mathrm{GL}(8, \mathbb{Q})$.

Let $G_{\mathbb{Q}}$ be an algebraic group over $\mathbb{Q}$ such that its set of F-points is equal to $\{x \in D \otimes_{\mathbb{Q}} F)^* : xx' = 1\}$, where $x \mapsto x'$ is the standard involution of D. For example, $G_{\mathbb{Q}}(\mathbb{Q}) = D_1^* = \{x \in D^* : xx' = 1\}$. and

$$G_{\mathbb{Q}}(\mathbb{R}) = \mathbb{K}_1^* \times \mathbb{K}_1^* \times \mathrm{SL}(2, \mathbb{R}) \cong \mathrm{SU}(2) \times \mathrm{SU}(2) \times \mathrm{SL}(2, \mathbb{R}).$$

The group $\mathrm{SU}(2) \times \mathrm{SU}(2)$ embeds naturally in $\mathrm{SU}(4)$ and hence acts on $\mathbb{C}^4$ preserving the standard Hermitian form on $\mathbb{C}^4$. Thus it preserves its real part that gives an embedding $\mathrm{SU}(2) \times \mathrm{SU}(2) \hookrightarrow \mathrm{SO}(4)$. The group $\mathrm{SO}(4) \times \mathrm{SL}(2, \mathbb{R})$ acts naturally on the tensor product $W = \mathbb{R}^4 \otimes \mathbb{R}^2 \cong \mathbb{R}^8$ preserving the skew-symmetric form A, the tensor product of the standard symmetric bilinear form on $\mathbb{R}^4$ and the standard symplectic form on $\mathbb{R}^2$. This gives rise to a real linear representation $\rho : G_{\mathbb{Q}}(\mathbb{R}) \to \mathrm{Sp}(W, A) \cong \mathrm{Sp}(8, \mathbb{R})$ that can be shown to correspond to the norm homomorphism (17.3).

Let Λ be a lattice in $\mathbb{R}^8$ and Γ be an arithmetic subgroup of $G_{\mathbb{Q}}$ that preserves this lattice. Also, let

$$\phi_0 : U(1) \to \mathrm{SU}(2) \times \mathrm{SU}(2) \times \mathrm{SL}(2, \mathbb{R}) \subset \mathrm{Sp}(W, A), \quad e^{i\theta} \mapsto (I_2, I_2, \left(\begin{smallmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{smallmatrix}\right)).$$

It is clear that $\rho(G(\mathbb{R}))$ is normalized by $\phi_0(U(1))$ and hence we obtain the data $(G_{\mathbb{Q}}, \Gamma, \rho, \phi_0)$ from (17.1) satisfying (17.2). This allows us to construct a Kuga family

$$\mathcal{X}(G_{\mathbb{Q}}, \Gamma, \rho, \phi_0) \to \Gamma \backslash G_{\mathbb{R}}/K_{\mathbb{R}}^0$$

of abelian 4-folds, where the base is a compact Shimura curve. The abelian varieties in the family correspond to $\rho(g)\phi_0\rho(g)^{-1} : U(1) \to \mathrm{Sp}(V, A)$. Obviously, the Hodge group H containing the image of $\rho(g)\phi_0\rho(g)^{-1}$ cannot be a proper subgroup of G for all g. Hence, the Hodge group of a general member coincides with $G_{\mathbb{Q}}$. On the other hand, since the representation ρ is irreducible over $\mathbb{C}$, we obtain, for any point where the Hodge group coincides with $G_{\mathbb{Q}}$, the corresponding abelian variety does not have non-trivial endomorphisms.

The *André-Oort Conjecture* asserts that any special family is a pull-back of some family of Hodge type. We refer to [78] for a more precise and a general definition of this conjecture.

Note that a marked family $\tilde{\mathcal{X}} \to \tilde{T}$ as above defines a family of the Mumford-Tate groups $\mathrm{MT}_{\tilde{t}}$ of the Mumford-Tate groups of fibers. This gives a stratification of T by the type of the Mumford-Tate group of fibers. Since the Mumford-Tate group is determined by the set of Hodge tensors that it fixes, the loci of points with fixed Mumford-Tate group are called the *Hodge loci*. Among them are of course the loci of abelian varieties with some special algebra of endomorphisms (since any endomorphism give rise to a Hodge class on the self-product of the variety).

Let $f : \mathcal{C} \to T$ be a smooth family of projective curves of genus $g \geq 2$. It defines a smooth family of Jacobian varieties $\mathcal{J} \to T$ of fibers of f. One asks whether such a family can be a special family of principally polarized abelian varieties. The current conjecture is that it is possible only if $g \leq 7$ (the modified *Oort Conjecture*). In fact, a modified *Coleman Conjecture* asserts that, for $g > 7$, the locus of Jacobians in $\mathcal{A}_g$ contains only a finitely many CM-points.

We refer for the discussion of these conjectures to an excellent surveys [77] and [78]. We only discuss one example.

Example 17.2. Fix a finite group G and consider families $f : \mathcal{C} \to T$ of curves together with a faithful homomorphism $\rho : G \to \mathrm{Aut}(\mathcal{C}/T)$. We call such a G-family of curves. For example a family of curves (16.2) can be considered as such a family where the base T is the open subset of $(\mathbb{P}^1 \setminus \{\infty\}^r$ that consists of distinct points. Under the map $T \to \mathcal{M}_g$, the image is a subvariety of $\mathcal{M}_g$ of dimension $r-2$. Similarly, one defines a G-family of polarized abelian varieties.

In general, the local deformation theory of the pair (C, G) tells us that the local dimension of the moduli space of pairs (C, G) is a smooth variety of dimension $\dim H^1(C, T_C)^G = H^0(C, K_C^{\otimes 2})^G$. Note that the linear space $H^1(C, T_C)$ can be naturally identified with the tangent space of the local deformation space of C. The tangent space of the local deformation space for a polarized abelian variety $A = V/\Lambda$ is naturally isomorphic to the tangent space of the corresponding point V in $zeros\mathsf{G}(g, \Lambda_\mathbb{C})_E$. The tangent space of the Grassmannian $\mathsf{G}(g, \Lambda_\mathbb{C})$ at the point V is naturally isomorphic to $\mathrm{Hom}(V, \Lambda_\mathbb{C}/V)$. If V happens to be a Lagrangian subspace with respect to a symplectic form E, then we can identify $\Lambda_\mathbb{C}/V$ with V^* and one can show that the tangent space of $\mathsf{G}(g, \Lambda_\mathbb{C})_E$ at V is isomorphic to the symmetric square of $S^2(V^*) \subset \mathrm{Hom}(V, V^*) = V^* \otimes V^*$. In our case $V = H^{-1,0}(A) \cong \Omega^1(A)^*$, and the tangent space becomes naturally isomorphic to the space of quadratic forms on $\Omega^1)(A)$. In this way one proves that the moduli space of abelian varieties with a fixed action of a finite group G has local dimension equal to $\dim(S^2(\Omega^1(A))^*)^G$.

A G-family of principally polarized varieties is a special case of a family of Hodge type. So, it is a special family of principally polarized varieties. Thus, a G-family of Jacobian varieties is special if its dimension is equal to $\dim(S^2(\Omega^1(A))^*)^G$, where $A = \mathrm{J}(C)$ is a general member of the family. These dimensions can be computed by using a formula of Hurwitz and Chevalley-Weil (16.4). We have

$$\dim S^2(\Omega^1(\mathrm{J}(C))^*)^G = \dim S^2(\Omega^1(C)^*)^G = \dim S^2(H^{0,1}(C))^G$$

Since we know the characters of G in its representation on $H^{0,1}(C)$, we easily find

$$\dim S^2(\Omega^1(\mathrm{J}(C))^*)^G = \sum_{n=1}^{m_1} d_n d_{m-n} + \begin{cases} d_k(d_k+1)/2 & \text{if , } m = 2k \text{ is even} \\ 0 & \text{otherwise.} \end{cases}$$

The following is the Table from [77] that gives a list of 20 triples $(m, r, (a_1, \ldots, a_r)$ defining families of cyclic covers such that its image S in $\mathcal{A}_g$ under the Torelli map coincides with the locus of abelian varieties with a cyclic group action that contains S. It is proven by J. Rohde in [89] that the list is complete.

	g	m	$(a_1,\dots,a_{r+1})$			g	m	$(a_1,\dots,a_{r+1})$
(1)	1	2	$(1,1,1,1)$		(11)	4	5	$(1,3,3,3)$
(2)	2	2	$(1,1,1,1,1,1)$		(12)	4	6	$(1,1,1,3)$
(3)	2	3	$(1,1,2,2)$		(13)	4	6	$(1,1,2,2)$
(4)	2	4	$(1,2,2,3)$		(14)	4	6	$(2,2,2,3,3)$
(5)	2	6	$(2,3,3,4)$		(15)	5	8	$(2,4,5,5)$
(6)	3	3	$(1,1,1,1,2)$		(16)	6	5	$(2,2,2,2,2)$
(7)	3	4	$(1,1,1,1)$		(17)	6	7	$(2,4,4,4)$
(8)	3	4	$(1,1,2,2,2)$		(18)	6	10	$(3,5,6,6)$
(9)	3	6	$(1,3,4,4)$		(19)	7	9	$(3,5,5,5)$
(10)	4	3	$(1,1,1,1,1,1)$		(20)	7	12	$(4,6,7,7)$

Remark 17.3. The case (16) is especially nice. Consider a general curve C from the family as a plane quintic

$$t_2^5 = (t_1 - x_1 t_0)(t_1 - x_2 t_0)(t_1 - x_3 t_0)(t_1 - x_4 t_0)(t_1 - x_5 t_0).$$

Let L be the line $t_2 = 0$ that intersects it at 5 distinct points. Now let X' be the double cover of $\mathbb{P}^2$ branched along the union $C \cup L$:

$$X : t_3^2 + t_2^5 + f_5(t_0, t_1) = 0.$$

After we blow-up its 5 singular points coming from $L \cap C$, we obtain a K3 surface X whose group of automorphisms contains a non-symplectic automorphism g of order 5. The moduli space of such K3 surfaces was studied in [61]. It is isomorphic to the moduli space of cyclic covers of type (16). Both spaces are naturally isomorphic to the quotient of an open subset of a 2-dimensional ball by an arithmetic hypergeometric reflection group of type $(\frac{2}{5}, \frac{2}{5}, \frac{2}{5}, \frac{2}{5}, \frac{2}{5})$. Kondo shows that a surface X as above is a quotient of the product $D \times C$ by a cyclic group of automorphisms of order 5. The curve D is isomorphic to the genus 2 curve with an automorphism of order 5. Under the rational projection $D \times C \dashrightarrow X$, the transcendental lattice $T_X \otimes \mathbb{Q}$, considered as a 3-dimensional vector space over $\mathbb{Q}(\zeta_5)$ becomes isomorphic to a direct summand of the rational Hodge structure on $H^1(D, \mathbb{Q}) \otimes H^1(C, \mathbb{Q}) \cong \mathbb{Q}(\zeta_5)^{12}$. So, our family of K3 surfaces is a special family.

Another example of appearance of an isomorphic moduli space of K3 surfaces is case (6) from the list. Here we consider a family of K3 surfaces birationally isomorphic to the double cover of $\mathbb{P}^2$ branched along the union of a nonsingular plane quartic curve C with equation

$$z^3 x + f_4(x, y) = 0$$

and the lines $L : z = 0$ and $M : x = 0$. The double cover has 5 ordinary singular points at points in $L \cap C$ and $L \cap M$, and a singular point of type E_6 over the point $[0, 0, 1]$. The pencil of lines through the point $[0, 0, 1]$ lifted to the cover gives a pencil of elliptic curves with 4 reducible fibers of type IV and one reducible fiber of type $\tilde{E}_6 = IV^*$. There are no more singular fibers. After base change by using the cover $C \to \mathbb{P}^1$ ramified over the 5 points corresponding to the singular fibers, we elliptic fibration becomes isomorphic to the constant fibration $E \times C \to C$. Here E is an elliptic curve with complex multiplication by $\mathbb{Q}(\zeta_3)$. The transcendental lattice is isomorphic to

$U(3) \oplus U(3) \oplus A_2$. It is a free module over $\mathbb{Z}[\zeta_3]$. The surface is birationally isomorphic to the quotient $C \times E/(\sigma)$, where σ is an automorphism of order 3. It has 15 fixed points over the points $x_1, \ldots, x_4, \infty \in C$. The action at a point over x_i is locally given by $(z_1, z_2) \mapsto (\zeta_3 z_1, \zeta_3 z_2)$ that gives rise a quotient singularity of type $\frac{1}{3}(1,1)$. The action at a point over ∞ is locally given by $(z_1, z_2) \mapsto (\zeta_3 z_1, \zeta_3^2 z_2)$ that gives rise a singular point of type A_2.

Note that, under transcendental lattice of $C \times E \to X$ is isomorphic as a $\mathbb{Z}[\zeta_3]$-module to the tensor product $H^1(C, \mathbb{Z}) \otimes H^1(E, \mathbb{Z}) \cong \mathbb{Z}[\zeta_3]^6$. Under the cover $C \times E \to X$, the transcendental lattice of X become a direct summand of $\mathbb{Z}[\zeta_3]^6$. This implies that the family of K3's is special, since for all CM point in the family of curves C, the Hodge structure on the transcendental part of the corresponding K3 is also of CM type. An example of a CM point in the family is the curve $C: z^3 x = x^4 + y^4$ which is isomorphic to the curve

$$z^4 + x^4 - 2\sqrt{-3}x^2y^2 + y^4 = 0$$

with automorphism group of order 48 isomorphic to $4.\mathfrak{A}_4$ (type III from Table 6.1. in [28]). The corresponding K3 surface is isomorphic to the surface

$$w^2 = (x^2 - (i+1)x^2y^2 - iy^2)(z^4 + x^4 - 2\sqrt{-3}x^2y^2 + y^4).$$

Note that the moduli space of K3 surfaces birationally isomorphic to the double plane

$$w^2 + xz(z^3x + f_4(x, y)) = 0$$

is a closed subvariety of one of the three irreducible components of the moduli space of K3 surfaces with a non-symplectic automorphism of order 3. Our component is of dimension 9 and its general member has the lattice of invariant algebraic cycles isomorphic to U. It is a quotient of a 9-dimensional ball and our family is the quotient of a 2-dimensional subball.

Also we may consider the case (17). The curve is isomorphic to the plane curve of degree 7

$$z^7 = x^4y(x - y)(x - ay).$$

Applying the Cremona transformation $[x, y, z] \mapsto [z^2, xy, xz]$ we transform this curve to a curve of degree 6

$$x^6 = yz(z^2 - xy)(z^2 - axy).$$

The curve has one triple point $[0, 1, 0]$ and one double point infinitely near to the triple point. So, its genus is equal to 6, as it should be. Now we can consider the double cover of the plane branched along this sextic curve

$$w^2 + x^6 + yz(z^2 - xy)(z^2 - axy) = 0.$$

The cyclic group (ζ_7) acts by $[x, y, z, w] \mapsto [\zeta_7^2 x, y, \zeta_7 z, \zeta_7^6 w]$. Its minimal resolution is a K3 surface with a non-symplectic automorphism of order 7. The pencil of lines in the plane through the point $[0, 1, 0]$ defines an elliptic pencil on X with two reducible fibers of type $I_0^* = \tilde{D}_4$ and IV, and 12 singular fibers of type I_1. The transcendental lattice is of rank 14, so that the Hodge structure $(T_X)_{\mathbb{Q}}$ is a 2-dimensional linear space over the field $\mathbb{Q}(\zeta_7)$. This shows that the moduli space of such surfaces is a modular curve $\mathbb{H}/\Gamma$ embedded in the moduli space of lattice polarized K3 surfaces.

Finally, we refer to [69] for some recent advance on the existence of special families of Jacobian varieties. This is based on the characterizations of families $f : X \to T$ of abelian varieties achieving the *Arakelov's bound* for the slope of the sheaf $f_*\Omega^1_{X/T}$. In particular, the authors prove the non-existence of special families of hyperelliptic Jacobians of genus $g \geq 8$. Special families of hyperelliptic curves of genus 3 were constructed in [35] and in [69].

Special family of abelian varieties must define a geodesic subvariety in $\mathcal{A}_g$. A recent paper [21] studies totally geodesic submanifolds of $\mathcal{A}_g$ that are contained in the Jacobian locus.

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