LINEAR SYSTEMS OF QUADRICS. MÜNICH.
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Contents

Lecture 1

Quadrics

1.1 Quadratic forms

Here we recall some standard facts and notations from the theory of quadratic forms that can be found in a good text-book in linear algebra. We let $\Bbbk$ to be a field of characteristic $\neq 2$ and refer to the Appendix for the case when the characteristic is equal to 2.

Let E be a linear space of dimension $n+1$ over $\Bbbk$. A *quadratic form* q on E is an element of the second symmetric square $S^2E^\vee$ of the dual linear space $E^\vee$. If we choose a basis $e_1, \ldots, e_{n+1}$ in E, then q is given by an expression

$$q = \sum_{1 \leq i \leq j \leq n+1} a_{ij} t_i t_j, \tag{1.1.1}$$

where $t_1, \ldots, t_{n+1}$ is the dual basis in $E^\vee$ (it serves as coordinates in E) and $a_{ij} \in \Bbbk$. A quadratic form can be considered as a function $q : E \to \Bbbk$ characterized by the properties

(a) $q(\lambda v) = \lambda^2 q(v)$, for any $\lambda \in \mathbf{k}$ and $v \in E$;

(b) the function $b_q : E \times E \to \Bbbk : (v, w) \mapsto q(v+w) - q(v) - q(w)$, is a symmetric bilinear form on E, i.e. an element of the linear space $(S^2E)^\vee \subset E^\vee \otimes E^\vee$.

The symmetric bilinear form b_q is called the *polar bilinear form* associated with q. Its expression in the basis $(t_i \otimes t_j)_{1 \leq i \leq j \leq n+1}$ of $E^\vee \otimes E^\vee$ is given by

$$b_q = 2\sum_{i=1}^{n+1} a_{ii} t_i \otimes t_i + \sum_{1 \leq i < j \leq n+1} a_{ij}(t_i \otimes t_j + t_j \otimes t_i).$$

The symmetric matrix A with diagonal elements $2a_{ii}$ and off-diagonal elements $a_{ij} = a_{ji}$ is called the matrix of b_q in the basis $(e_1, \ldots, e_{n+1})$. We have $b_q(e_i, e_j) = b_q(e_j, e_i) = a_{ij}$ if $i < j$ and $b_q(e_i, e_i) = 2a_{ii}$ otherwise. Since the characteristic is different from two, $q(x) = \frac{1}{2}b_q(x, x)$, so a quadratic form is completely determined by the polar bilinear form.

As any bilinear form, the polar bilinear form b_q can be identified with a linear map $E \to E^\vee$ that assigns to a vector v the linear function $w \mapsto b_q(v, w)$. If we identify $(E^\vee)^\vee$ with E, then the symmetry condition is equivalent to that the linear map coincides with its transpose map. The matrix of this map in a basis of E and its dual basis of $E^\vee$ is the matrix A of b_q defined in above. Its determinant is defined uniquely up to squares in $\Bbbk$ and is called the *discriminant* of q.

The kernel of the linear function $b_q : E \to E^\vee$ is called the *radical* of b_q and will be denoted by $\mathrm{rad}(b_q)$. Its dimension is called the *corank* or the *defect* of q and of b_q and the dimension of its image is called the *rank* of q or of b_q.

A quadratic form q is called *degenerate* if its defect $\delta(q)$ is not equal to zero. A non-zero vector v in the radical $\mathrm{rad}(b_q)$ of b_q is called a *singular vector*.

For any subset L of E, we denote by $L^\perp$ the orthogonal complement of L with respect to b_q, i.e

$$L^\perp := \{v \in E : b_q(v, w) = 0, \text{ for any } w \in L\}. \tag{1.1.2}$$

This is obviously a linear subspace of E. For example, $\mathrm{rad}(b_q) = E^\perp$ and for any L, we have $\mathrm{rad}(b_q) \subset L^\perp$. Note that, if q is non-degenerate, and L is a subspace, then

$$\dim L^\perp = \dim E - \dim L.$$

Let q be a totally degenerate quadratic form, i.e. $\mathrm{rad}(b_q) = 0$. If $p \neq 2$, then $q = 0$. Otherwise $q = \sum a_i x_i^2$ in some basis in E. In some radical extension $K/\Bbbk$ and $\mathrm{rad}(b_q)$ be the radical of b_q. Choose a basis $e_1, \ldots, e_c$ of $\mathrm{rad}(b_q)$ and let L be the complementary subspace of $\mathrm{rad}(b_q)$. Then there exists a finite extension $K/\Bbbk$ such that q vanishes on some nonzero vector $v_1 \in L_K$. (**Prove that one may choose a separable extension even when** $\mathrm{char}(\Bbbk) = 2$). Let $w_1 \in L_K$ be an element from $v_1^\perp$. Since the radical of b_q restricted to L is trivial, we can choose w_1 with $b_q(v_1, w_1) = 1$. Let H_1 be the plane spanned by v_1 and w_1, and $E' = H^\perp$ be its orthogonal complement in E_K. Arguing, by induction on n, we find a basis $(v_1, w_1, \ldots, v_m, w_m)$ if $n+1 = 2m$ and $(v_1, w_1, \ldots, v_m, w_m, e)$ if $n = 2m$ such that in coordinates defined by this basis the expression of q is given by

$$q = \sum_{i=1}^{m} t_{2i}t_{2i-1} \text{ if } n = 2m-1, \tag{1.1.3}$$

$$q = \sum_{i=1}^{m} t_{2i}t_{2i-1} + t_{2m+1}^2 \text{ if } n = 2m. \tag{1.1.4}$$

A vector $v \in E$ is called an *isotropic vector* of a quadratic form q if $q(v) = 0$. A linear subspace F in E such that the restriction of b_q to F is identical zero is called a *totally isotropic subspace*. Since $p = \mathrm{char}(\Bbbk) \neq 0$, then F is totally isotropic if and only if the restriction of q to F is zero. An example of a totally isotropic subspace of q is its radical $\mathrm{rad}(b_q)$. If L is isotropic, we have $L \subset L^\perp$. If q is non-degenerate, applying (**??**), we find that $\dim L \leq [\frac{n+1}{2}]$. It follows from the formulas (**??**) that there exists an isotropic subspace of dimension $[\frac{n+1}{2}]$. It is defined by the equations $t_2 = \ldots = t_{2m} = 0$ and t_{2m+1} if n is even. A totally isotropic subspace of this dimension is called a *maximal isotropic subspace*.

By a theorem of Witt, any totally isotropic subspace is contained in a maximal totally isotropic subspace of dimension $m = [\frac{1}{2}(n+1)]$ (if q is non-degenerate). Two totally isotropic subspaces of the same dimension can be transformed to each other by an element of the *orthogonal group* $\mathrm{O}(E; q)$ of linear transformations of E that preserve the quadratic form q.

Recall that the orthogonal group $\mathrm{O}(E, q) \cong \mathrm{O}_\Bbbk(n+1)$ is an algebraic group of dimension $n(n+1)/2$. It consists of two connected components, the connected component of the identity is the group $\mathrm{SO}_\Bbbk(n+1)$. The latter is not simply-connected if n is odd, its universal cover is the spin group $\mathrm{Spin}_\Bbbk(n+1)$. If n is even, the group $\mathrm{SL}_\Bbbk(n+1)$ acts transitively on the variety of $\mathrm{Isot}_k(Q)$ of totally isotropic subspaces of fixed dimension k. The latter is considered as a subvariety of the Grassmann variety $G(k, n+1)$ of k-dimensional subspaces of E.[1] If n is odd, the latter is true if $k < m$. The variety $\mathrm{Isot}_m(q)$ of maximal isotropic subspaces has two orbits with respect to $\mathrm{SO}_\Bbbk(n+1)$. It consists of two connected components.

Assume $n+1 = 2m$ and q is non-degenerate. Let U be a maximal totally isotropic subspace of q. By Witt's Theorem, we may choose coordinates as in (**??**) such that U is given by equations $t_{2i-1} = 0, i = 1, \ldots, m$. Let V be the complementary isotropic subspace given by equations $t_{2i} = 0, i = 1, \ldots, m$. Then $E = U \oplus V$. Let W be any other maximal totally isotropic

[1] Other common notations are $G(r, n+1)$ if $E = \Bbbk^{n+1}$ or $G_{r-1}(n)$ as the set of $r-1$ projective subspaces in $\mathbb{P}^n$.

subspace with $U \cap W = \{0\}$ and p_U, p_V be the projection maps from W to the subspaces U and V. Then the projection map p_V is bijective, and $\phi = p_U \circ p_V^{-1}$ is a linear map from $V \to U$ and the map $(p_U, p_V) : W \to U \oplus V$ has the image equal to the graph of f. Choose a basis $(e_1, \ldots, e_m)$ of U and a basis $f_1, \ldots, f_m$ of V such that they correspond to the coordinates $(t_1, \ldots, t_{2m-1})$ in U and coordinates $(t_2, \ldots, t_{2m})$ in V. Then $b_q(e_i, f_j) = \delta_{ij}$ and $\phi(f_i) = \sum a_{ij} e_j, i = 1, \ldots, m$, and W, being the graph, has a basis $(f_1 - \sum a_{1j} e_j, \ldots, f_m - \sum a_{mj} e_j)$. Since W is isotropic, we get $a_{ii} = 0$ and $a_{ij} = -a_{ji}$. Thus, W is defined by an alternating matrix $A = (a_{ij})$. The subspace V corresponds to the zero matrix. It is clear that $(\lambda_1, \ldots, \lambda_m)$ belongs to the left nullspace of A if and only if the linear combination $\sum \lambda_i e^i$ belongs to V. This shows that $\dim W \cap V = \operatorname{corank} A$. Since the corank of an alternating matrix is congruent to m modulo 2, we see that $\dim W \cap V \equiv m \mod 2$.

Let W_1, W_2 be two maximal totally isotropic subspaces, suppose there exists a third maximal totally isotropic subspace W_3 such that $W_3 \cap W_1 = \{0\}, W_3 \cap W_2 = \{0\}$, then, taking $U = W_3, V = W_1$, we obtain that $\dim W_1 \cap W_2 \equiv m \mod 2$. Conversely, suppose that this condition is satisfied. Let us show that there exists W_3 complementary to W_1 and W_2. The bilinear form b_q defines an isomorphism from $W_1 / W_1 \cap W_2$ to $(W_2 / W_1 \cap W_2)^\vee$. The condition on $\dim W_1 \cap W_2$ implies that the dimension r of these spaces is even. Let us choose a basis $e_{r+1}, \ldots, e_m$ of $W_1 \cap W_2$, extend it to a basis $e_1, \ldots, e_r, e_{r+1}, \ldots, e_m$ of W_1 and a basis $f_1, \ldots, f_r, e_{r+1}, e_m$ of W_2 satisfying $b_q(e_i, f_j) = \delta_{ij}$ for $1 \le i, j \le r$. Then define W_3 to be the span of m vectors

$$f_1, \ldots, f_r, e_1 + f_2, e_2 - f_1, \ldots, e_{r-1} + f_r, e_r - f_{r-1}. \tag{1.1.5}$$

It is immediately checked that W_3 a totally isotropic and is complementary to W_1 and W_2.

We say that two maximal totally isotropic subspaces W_1, W_2 belong to the same *ruling* if there is a maximal totally isotropic subspace W_3 complementary to W_1 and W_2. We have seen already that if n is odd, two subspaces belong to the same ruling if and only if $\dim W_1 / W_1 \cap W_2$ is even.

Recall that the *Grassmann variety* $G(r, E)$ is a projective algebraic variety. It embeds in $|\bigwedge^r E|$ by the *Plücker map* $L \mapsto \bigwedge^r(L) \subset \bigwedge^r(E)$. In coordinates it is given by maximal minors of the matrix Z that expresses elements of a basis of L as linear combinations of a basis in L (*Plücker coordinates*). Since the rank of this matrix is equal to r, we can change a basis in L to assume that one of the maximal minors is different from zero. Thus the matrix M has m unit vectors among its columns. The rest of entries are arbitrary. This defines an open subset of the Grassmann variety isomorphic to the affine space of dimension $r(n+1-r)$. The whole variety is covered by $\binom{n+1}{r}$ such open subsets. This is similar to the case when $r=1$ in which $G(1,E)=|E|$ is the projective space. It is covered by $n+1$ affine subsets with a nonzero ith projective coordinate.

Returning to our case, where $r=m$ and $n+1=2m$, we obtain that we may choose a basis in W such that the matrix M looks like $[I_rA]$. This defines an affine open subset of $G(r,E)$ isomorphic to the affine space of dimension $\binom{n+1}{r}$.

Proposition 1.1. *Let q be an non-degenerate quadratic form.*

(i) If n is even, then there is only one ruling.

(ii) If n is odd, there are two rulings.

(iii) Each ruling is an irreducible closed subvariety of the Grassmann variety $G(m,E)$ of m-dimensional subspaces in E that admits an open cover by affine spaces of dimension $\frac{m(m-1)}{2}$.

Proof. (i) Let W_1 and W_2 be two maximal totally isotropic subspace. We choose W_3 to be defined by the same basis as in (**??**) if $r=\dim W_1/W_1\cap W_2$ is even. Otherwise we replace the index r with $[2\frac{r}{2}]$ and add an additional vector $2e_r-f_r+2e_0$, where $e_0\in(W_1+W_2)^{\perp}$ and $q(e_0)=-\frac{1}{2}$ if $p\neq 2$ and $q(e_0)=1$ otherwise.

(ii) We have seen already that two subspaces W_1 and W_2 are in the same ruling if and only if $\dim W_1/W_1\cap W_2$ is even. Two maximal totally isotropic subspaces extending the same $(m-1)$-dimensional totally isotropic subspace are obviously belong to different rulings. Thus there are two rulings.

(iii) We have seen this already in above.

□

We can derive the formula for the dimension of a family of maximal isotropic subspaces in one ruling by looking at the stabilizer of one of such space in $\mathrm{O}(E,q)$. We choose a standard basis defined by two complimentary maximal totally isotropic subspaces $U=\langle e_1,\ldots,e_m\rangle, V=\langle f_1,\ldots,f_m\rangle$ with $b_q(e_i,f_j)=\delta_{ij}$. Then the matrix of b_q can be written as a block-matrix $J_{2m}=\left(\begin{smallmatrix}0 & I_m\\ I_m & 0\end{smallmatrix}\right)$. A transformation $\sigma\in\mathrm{O}(E_K,q)$ is defined by a matrix X with entries in $K/\Bbbk$ satisfying

$$ {}^tX\cdot J_{2m}\cdot X=J_{2m}. $$

If we write it in the block form $X = \begin{pmatrix} X_1 & X_2 \\ X_3 & X_4 \end{pmatrix}$, then the previous equation is equivalent to matrix equations

$$ {}^tX_3 \cdot X_1 + {}^tX_1 \cdot X_3 = 0, \quad {}^tX_3 \cdot X_2 + {}^tX_1 \cdot X_4 = I_m, \quad {}^tX_4 \cdot X_2 + {}^2X_2 \cdot X_4 = 0. $$

This shows that the functor $K \mapsto \mathrm{O}(E_K, q)$ is representable by an affine algebraic variety given by the equations in entries of X (taken as unknowns) expressing the matrix equations from above.

Also if X represents a linear operator that fixes U, then $X_3 = 0$, and the equations give ${}^tX_1 \cdot X_4 = I_m$ and ${}^tX_3X_1 + {}^tX_1 \cdot X_3 = 0$. The second equation says that X_3 is invertible, and then X_4 is its inverse, and the third equation gives $m(m+1)/2$ linearly independent conditions on the entries of X_2. It is known that the dimension of the orthogonal group $\mathrm{O}_{\Bbbk}(2m)$ is equal to $m(2m-1)$ (its Lie algebra is the Lie algebra of skew-symmetric matrices of size $2m$). Thus the dimension of the orbit is equal to $m(2m-1) - (m^2 + \frac{1}{2}m(m-1)) = \frac{1}{2}m^(m-1)$ that agrees with the previous computation.

Remark 1.2. Recall that a non-degenerate quadratic form q on E defines the *Clifford algebra* $\mathsf{C}(q)$, the quotient of the tensor algebra $T(E)$ by the two-sided ideal generated by tensors $v \otimes v - q(v)$. The Clifford algebra has a natural $\mathbb{Z}/2\mathbb{Z}$-grading $\mathsf{C}^+(q)^+ \oplus C^-(q)$ induced by the grading of $T(E)$. Let $\rho : E = T^1(E) \to \mathsf{C}(q)$ be the natural projection map. Since it is an injective map, we can identify E with a linear subspace of $\mathsf{C}(q)$. Fix a decomposition $E = U \oplus V$ into the direct sum of two totally isotropic subspaces. There is a natural isomorphism Φ from $\mathsf{C}(q)^+$ to the algebra of endomorphisms of the exterior algebra $\bigwedge^* V$ uniquely determined by the properties that $\Phi(u)(\omega) = u \llcorner \omega$ and $\Phi(v)(\omega) = v \wedge \omega$, where $u \in U, v \in V$ and $\llcorner$ denote operation of contraction $U \otimes \bigwedge^k V \to \bigwedge^{k-1} V$, the subspace U is considered here as the dual subspace of V via the bilinear form b_q.

Let $e(U, V)$ be the unique element of $\mathsf{C}(q)$ defined by the property that $\Phi(e(U, V))$ is the identity on $\bigwedge^{\mathrm{ev}} V$ and zero on $\bigwedge^{\mathrm{odd}} V$. Obviously, $e(U, V)$ satisfies $e(U, V)^2 = e(U, V)$ (i.e. it is an *idempotent*). The center $Z(q)$ of $\mathsf{C}^+(q)$ is a 2-dimensional subspace of $\mathsf{C}^+(q)$ spanned by the idempotents $e(U, v)$ and $1 - e(V, U)$. Fixing U and changing V to any $W \in \mathrm{Isot}_U(q)$, we obtain idempotents $e(U, W)$. Thus each W defines a decomposition of the center (as a $\Bbbk$-algebra) into the direct sum of two one-dimensional subalgebras generated by $e(U, W)$ and $1 - e(W, U)$.

One can show that two maximal totally isotropic subspaces W, W' belong to the same ruling if and only $e(U, W) = e(U, W')$ (see [**?**], Proposition 1.12).

1.2 Quadrics

Let $|E|$ denote the projective space of lines in E, it is equal to the projective space $\mathbb{P}(E^\vee)$ of one dimensional quotients of the dual space $E^\vee$. Here we follow the notation for a projective space used, for example, in [?]. For any extension of fields $K/\Bbbk$, the set of K-points of $|E|$ are lines in the linear space $E_K = E \otimes_{\Bbbk} K$ obtained by extension of scalars. For any vector $v \in E_K$ we denote by $[v]$ the corresponding point in $|E|(K)$. When $\Bbbk$ is algebraically closed, we will identify $|E|$ with $|E|(\Bbbk)$.

Recall that $|E|$ comes with a natural line bundle denoted by $\mathcal{O}_{|E|}(1)$. Its sections are linear forms on E. Sections of $\mathcal{O}_{|E|}(k)$ are elements of the symmetric power $S^k(E^\vee)$. In particular, a quadratic form can be considered as a section of $\mathcal{O}_{|E|}(2)$. Its zero subscheme Q is denoted by $V(q)$. It is called a *quadric hypersurface* in $|E|$, or just a *quadric* defined by equation $q = 0$. We have $Q(K) = \{[v] \in |E_K| : q(v) = 0\}$.

Proposition 1.3. *A quadric $Q = V(q)$ is smooth if and only if $\mathrm{rad}(b_q) = \{0\}$. The locus of singular points $\mathrm{Sing}(Q)$ is the projective subspace $|\mathrm{rad}(b_q)|$*

Proof. Fix projective coordinates so that q is given by expression (??). Recall that the tangent space T_xX of a subvariety X of $\mathbb{P}^n$ at a point $x \in X$ given by homogeneous equations $f_1 = \ldots = f_N$ in unknowns $t_0, \ldots, t_{n+1}$ is a linear projective subspace of $\mathbb{P}^n$ given by equations

$$\frac{\partial f_i}{\partial t_1}(x)t_1 + \cdots + \frac{\partial f_i}{\partial t_{n+1}}(x)t_{n+1} = 0.$$

The point x is *nonsingular* or *smooth* if and only if $\dim T_xX = \dim_x X$. The set of singular points $\mathrm{Sing}(X)$ is equal to the set of points where $\dim T_xX > \dim_x X$.

In our casem $\dim_x Q = n-1$, so a point is singular if and only if $T_x(Q) = |E|$ is the whole projective space. Computing the partials, we see that the tangent space at a point $x_0 = [v_0]$ is defined by the linear equation $b_q(v_0, x) = 0$. It is the whole space if and only if $v_0 \in \mathrm{rad}(b_q)$.

□

Note that, if $v_0 \notin \mathrm{rad}(b_q)$, then the restriction of the quadric to this subspace has radical equal to $\langle v_0 \rangle$, so the corresponding quadric $q|_{T_{x_0}(Q)}$ has x_0 as its singular point. Let $\bar{E} = E/\langle v_0 \rangle$. Then the bilinear form b_q induces a bilinear form on $\bar{E}$ and defines a quadric $\bar{Q}$ in $|\bar{E}|$. Its singular locus is $|\mathrm{rad}(b_q) + \langle v_0 \rangle / \langle v_0 \rangle|$.

A geometric interpretation of a totally isotropic subspace of q of dimension $r+1$ is of course a projective subspace of dimension r (an r-plane, for short) contained in the quadric $Q = V(q)$. We see from the previous section that the maximal dimension of such a subspace is equal to $m = [\frac{n-1}{2}]$. A subspace of this dimension is called a *generator*. Also we infer that all generators are parameterized by an algebraic variety of dimension $\frac{1}{2}m(m-1)$ if $n+1 = 2m$ and consists of two connected components if $n = 2m$. It is irreducible of dimension $m(m+1)/2$ if $n = 2m$ is even. The connected component is a closed subvariety of the Grassmann variety $G(m, n+1)$. In affine subsets parameterized by square matrices A of size m, the subvariety is given by the condition that the matrix is skew-symmetric. This variety is called the *spinor variety* and denoted by S_m or *orthogonal Grassmannian variety* and denoted by $OG(m, 2m)$. This variety is a homogeneous variety with respect to the *spinor group* $\mathrm{Spin}_{\mathbb{k}}(n+1)$, a double cover of the orthogonal group $\mathrm{O}_{\mathbb{k}}(n+1)$.

By a theorem of Witt, any r-plane is contained in a ruling and all ruling are equivalent with respect to the orthogonal group $\mathrm{O}(E, q)$.

Remark 1.4. It is known that the spinor group $\mathrm{Spin}(2m)$ admits two irreducible half-spinor representations $V^{\pm}$ of dimension 2^{m-1}. One can show that the spinor variety S_m can be embedded in $|V^{\pm}|$. Also, one can show that in the case $n = 2m$, the variety $\mathrm{Gen}(Q)$ of generators of a smooth quadric in $\mathbb{P}^n$ is isomorphic to any irreducible component of $\mathrm{Gen}(Q')$, where Q' is a smooth quadric in $\mathbb{P}^{n+1}$.

We can compute the dimension of the variety of generators in another way. In fact, we compute the dimension of the variety $F_k(Q)$ of k-dimensional projective subspaces (for short they are called *k-planes*)contained in a nonsingular quadric Q in $\mathbb{P}^n$ for any $k \le [\frac{n+1}{2}]$. Since this variety does not depend on Q we denote it by $F_{k,n}$.

Consider the incidence variety

$$I_{k,n} = \{(x, \Lambda) \in Q \times F_k(Q) : x \in \Lambda\}.$$

It comes with two projections $p_1 : I_{k,n} \to Q$ and $p_2 : I_{k,n} \to F_k(Q)$. The fibers of the second projection are isomorphic to a k-plane, hence $\dim I_k = \dim F_k(Q) + k$. Since a k-plane containing a point x must be contained in the tangent space $\mathbb{T}_x Q$, the fiber $p_1^{-1}(x)$ is isomorphic to the variety of k-planes on the quadric $Q_0 = \mathbb{T}_x Q \cap Q$. We know that it is the cone with vertex at x over a nonsingular quadric of dimension $n-3$. Any k-plane containing x is a cone over a $(k-1)$-plane contained in Q_0. Thus $p_1^{-1}(x)$ is isomorphic to $F_{k-1,n-2}$. This gives an inductive formula

$$\dim F_{k,n} = \dim I_{k,n} - k = \dim Q + \dim F_{k-1,n-2} - k = n-1-k+\dim F_{k-1,n-2}.$$

It also shows that whenever $F_{k-1,n-2}$ is irreducible, then $F_{k,n}$ is irreducible. Easy computations show that

$$\dim F_{k,n} = (k+1)(n-k) - \frac{1}{2}(k+1)(k+2), \tag{1.2.1}$$

and $F_{k,n}$ is irreducible if $k < m = [\frac{n}{2}]$ or $k = m$ when n is even.

Remark 1.5. Let $\ell \in F_1(Q)$ be a line on a smooth quadric. Recall the Lie algebra of the orthogonal group $\mathrm{O}(E,q)$ can be identified with the exterior power $\bigwedge^2 E$, or, after a choice of a basis in E, with the Lie algebra of skew-symmetric matrices of size $n+1$. Thus a line ℓ can be identified with a point in the projective space $|\bigwedge^2 E|$. Of course, this corresponds to the Plücker embedding of the Grassmannian $G(2,E)$. The group $G = \mathrm{O}(E,q)$ has a natural linear representation in its Lie algebra, the *adjoint representation.* If we identify $\mathrm{Lie}(G)$ with the tangent space at G at the unity elements e, the action is the action $g \mapsto (dg_e)^{-1}$. Let us consider the corresponding projective representation of $\mathrm{O}(E,q)$ in $|\bigwedge^2 E|$. The set $F_1(Q)$ is an orbit of this action, in fact, the only one which is closed. It is an example of an *adjoint variety* which can be considered for other semi-simple algebraic groups and their adjoint projective representations. It is also an example of a *contact Fano variety.*

1.3 Birational geometry of quadrics

Choose a basis in E such that $t_{r+1} = \ldots = t_{n+1} = 0$ are the equations of the linear projective subspace $\mathrm{Sing}(Q)$. Then the equation of Q is

$$\sum_{1\le i\le j\le r} a_{ij}t_it_j = 0.$$

For any point $x_1 \in \mathrm{Sing}(Q)$ with coordinates $[a_1,\ldots,a_r,0,\ldots,0]$ and a point $x_2 = [0,\ldots,0,b_{r+1},\ldots,b_{n+1}]$ in the subspace $x_1 = \ldots = x_r = 0$ the line $\overline{x_1,x_2}$ consists of points with coordinates $[\alpha a_1,\ldots,\alpha a_r,\beta b_{r+1},\ldots,b_{n+1}b_{n+1}]$. Obviously it is contained in Q. This shows that Q is the *join* of the subspace $\mathrm{Sing}(Q)$ with a nonsingular quadric Q_0 in the projective subspace $t_1 = \ldots = t_r = 0$, i.e. it is equal to the union of lines joining point in $\mathrm{Sing}(Q)$ with points in Q_0. In particular, when $\dim \mathrm{Sing}(Q) = 0$, the quadric Q is a *cone* over a quadric in $\mathbb{P}^{n-1}$.

Recall that the product of projective subspaces $|E_1| \times |E_2|$ is a projective variety isomorphic to the subvariety of $|E_1 \otimes E_2|$ whose points are represented by tensors $v_1 \otimes v_2, v_i \in E_i$. A closed subvariety of $|E_1| \times |E_2|$ is given by equations representing tensors $S^m E_1^\vee \otimes S^n E_2^\vee$. In coordinates, they are bihomogeneous forms of bidegree (m, n).

For example, if we take $E_1 = E_2 = E$, the diagonal Δ in $|E| \times |E|$ is given by the bilinear form $v \wedge w = 0$ expressing the condition that two vectors are proportional.

Let $\phi : E_1 \to E_2$ be a nonzero linear map with kernel K, it defines a rational map $|\phi| : |E_1| \to |E_2|$ which is not defined on the subspace $|K|$. Its graph is defined by

$$\Gamma_{|\phi|} := \{([v_1], [v_2]) \in |E_1| \times |E_2| : \phi(v_1) \wedge v_2 = 0\}.$$

It is clear that the subset of Γ_ϕ of pairs $(x, y), x \notin |F|$, is the graph of the restriction of $|\phi|$ to $|E_1| \setminus |F|$. We have two projections

$$\begin{array}{ccc} & \Gamma_{|\phi|} & \\ \swarrow^{\pi_1} & & \searrow^{\pi_2} \\ |E_1| & & |E_2|. \end{array}$$

The image of the second projection is the linear subspace $|\phi(E_1)|$ of $|E_2|$. Its fiber over a point $[\phi(v_1)]$ is isomorphic to the projective subspace $|K + \Bbbk v_1|$ of dimension $\dim K$.

The image of the first projection is equal to $|E_1|$. The projection is an isomorphism over $|E_1| \setminus |K|$. Its fiber over a point $[v_1] \in |K|$ is isomorphic to the projective space $|E_2|$. If we choose a complementary subset F of $\phi(E_1)$ in E_2, then

$$\Gamma_{|\phi|} \cong |F| \times \mathrm{Bl}_{|K|} |E_1|,$$

where $\mathrm{Bl}_Z X$ denotes the blow-up of a variety X along the closed subvariety Z.

We apply this construction to the case when ϕ is a surjective map $\pi : E \to E/K$. The map $|\phi| : |E| \dashrightarrow |E/F|$ is the *projection map* from $|E|$ to $|E/K|$ from $|K|$. Choosing a complementary subspace E' of K in E, the projection map is the composition of the projection map to the subspace $|E'|$ and an isomorphism $|E'| \to |E/K|$. The graph $\Gamma_{|\pi|}$ becomes isomorphic to the blow-up $\mathrm{Bl}_{|K|} |E|$ of $|E|$ along the subspace $|K|$. The second projection

$p_1 : \mathrm{Bl}_{|K|}|E| \to |E'|$ is a projective bundle with each fiber isomorphic to $\mathbb{P}^{\dim K}$. The pre-image of $|K|$ under the first projection is the *exceptional divisor* $\mathrm{Ex}(\pi_1)$ of the blow-up map π_1. It is isomorphic to $|K| \times |E'|$, and the second projection restricted to the exceptional divisor coincides with the projection $|K| \times |E'| \to |E'|$.

Let X be a closed irreducible subvariety of $|E|$ and $p_{|K|} : X \dashrightarrow |E'|$ be the restriction of $|\pi|$ to X. If $|K| \cap X = \emptyset$, then $p_{|K|}$ is a regular map. Its image is a closed subvariety of $|E'|$. The fiber of the projection over a point $y \in |E'|$ is equal to the intersection of X with the subspace spanned by $|K|$ and y. If $Z = X \cap |K| \neq \emptyset$, the projection is only a rational map not defined on Y. One can *regularize* it as follows.

We consider the pre-image of $X \setminus Z$ in $\mathrm{Bl}_{|K|}|E|$ and then take its Zariski closure $\bar{X}$ there (it is called the *proper inverse transform* of X in $\mathrm{Bl}_{|K|}|E|$). The restriction of the first projection $\mathrm{Bl}_{|K|}|E| \to |E|$ to $\bar{X}$ is a regular map $\sigma : \bar{X} \to X$. It is an isomorphism over $X \setminus Z$. The pre-image $\sigma^{-1}(Z)$ is the exceptional divisor $\mathrm{Ex}(\sigma)$ of σ. It is a closed subvariety of the exception divisor $\mathrm{Ex}(\pi_1)$ of the blow-up. If Z is a nonsingular subvariety of X, then $\bar{X} \cong \mathrm{Bl}_Z X$ is the blow-up of X along Z. The exceptional divisor $\mathrm{Ex}(\sigma)$ of $\sigma : \bar{X} \to X$ is a projective bundle over Z with fibers isomorphic to $\mathbb{P}^s$, where $s = \dim X - \dim Y - 1$. For experts, it is the projective bundle associated to the normal sheaf $\mathcal{N}_{Z/X}$. The second projection $\pi_2 : \mathrm{Bl}_{|K|}|E| \to |E'|$ defines a regular map $\tau : \bar{X} \to |E'|$. Since $\bar{X}$ is closed in $\mathrm{Bl}_{|K|}|E|$, the image of τ is a closed subvariety of $|E'|$. It is called the *image of the projection map* $p_{|K|} : X \dashrightarrow |E'|$.

Let us apply all of this to the case when X is a quadric Q in $|E|$. Choose a projective subspace $|K|$ of $|E|$ and consider the projection map $\pi_{|K|} : |E| \dashrightarrow |E/K|$ with center $|K|$. Assume first that $\dim |K| = 0$, i.e. $|K| = x_0$ is a point. This is the only case when Q may not intersect the center of the projection. Assume it does not, i.e. $x_0 \notin Q$. For any $x \in Q$, the line $\overline{x_0, x}$ spanned by x_0 and x intersects Q at two points counting with multiplicity. This defines a degree 2 map $Q \to |E/K| \cong \mathbb{P}^{n-1}$. Its fiber consists of two points x_1, x_2 where the line $\overline{x_0, x}$ intersects Q or one point x or one point x if the line is tangent to Q at the point x. Let $x_0 = [v_1]$ and $x = [v]$. We have $-q(v) + q(v + \lambda v_0) - \lambda^2 q(v_0) = \lambda b_q(v_0, v)$ for any $\lambda \in \Bbbk$. Assume that $q(v) = 0$ and $b_q(v_0, v) = 0$, then any point $y \neq x$ on the line $\overline{x_0, x}$ can be written in the form $y = [v + \lambda v_0]$ for some $\lambda \neq 0$. Since we assume that $q(v_0) \neq 0$, we obtain that $q(v + \lambda v_0) \neq 0$, i.e. $y \notin Q$. This shows that the line $\overline{x_0, x}$ is tangent to Q. It is easy to see that the converse is also true. Thus we obtain that the set Q^0 of points $x \in Q$ where the line $\overline{x_0, x}$ is tangent

Q is equal to the intersection of Q with the hyperplane $V(b_q(v_0))$, where b_q is considered as a linear map $E \to E^\vee$. This hyperplane is called the *first polar* of Q with respect to the point $x_0 = [v_0]$ and is denoted by $P_{x_0}Q$.

In coordinates, it is given by the equation

$$\sum_{i=1}^{n+1} a_i \frac{\partial q}{\partial t_i} = 0,$$

where $x_0 = [a_1, \ldots, a_{n+1}]$.

We have $b_q(v_0, v_0) = 2q(v_0) \neq 0$, so $P_{x_0}Q \neq |E|$ and Q^0 is a quadric in $P_{x_0}Q \neq |E|$. The map $E \to E/\Bbbk$ is an isomorphism on $P_{x_0}Q$ and the image of Q_0 is a quadric $\bar{Q}^0$ in $E/\Bbbk v_0$. Thus the projection defines a double cover

$$p_{x_0} : Q \to |E/\Bbbk v_0| \cong \mathbb{P}^{n-1}$$

ramified over $Q^0 = Q \cap P_{x_0}Q$ and branched over the quadric $\bar{Q}^0$ in $|E/\Bbbk v_0|$.

For example, assume Q has the equation

$$t_{n+1}^2 + l(t_1, \ldots, t_n)t_{n+1} + q(t_1, \ldots, t_n) = 0,$$

where l isa linear form and q' is a quadratic form. Then the projection from the point $[0, \ldots, 0, 1]$ is defined by the map $[t_1, \ldots, t_{n+1}] \to [t_1, \ldots, t_n]$. If $l = 0$ and $p \neq 2$, then the projection is a double cover branched over the quadric $V(q')$. If $p = 2$, then it is an inseparable cover if $l = 0$ and a separable double cover otherwise ramified over the hyperplane $V(l)$.

Next we consider the case when the center of the projection is a point x_0 lying on the quadric not contained in $\mathrm{Sing}(Q)$. In elementary geometry, when Q is a quadric over $\mathbb{R}$ and $Q(\mathbb{R})$ is sphere, this is known as the *stereographic projection*.

The polar hyperplane $P_{x_0}Q$ in this case is the tangent hyperplane of Q at the point $x_0 = [v_0]$. It intersects Q along a quadric Q^0 with $x_0 \in \mathrm{Sing}(Q^0)$. As we observed before Q^0 is a cone over a quadric Q_0^0 in a projective subspace of dimension $n-2$ in $|E/\Bbbk v_0|$. Any line joining x_0 with a point $x \neq x_0 \in Q^0$ is contained in Q. Thus the projection blows down Q^0 to the quadric Q_0^0 lying in a hyperplane $H \subset |E/\Bbbk v_0|$. On other other hand, outside Q_0, the projection is an isomorphism onto the complement of Q_0^0 in $|E/\Bbbk v_0|$. The projection map is an example of a birational map $Q \dashrightarrow \mathbb{P}^{n-1}$. The inverse rational map $\mathbb{P}^{n-1} \dashrightarrow Q$ is given as follows. Consider the linear space of quadratic forms q such that $V(q)$ contains the quadric Q_0^0. Choose a basis $q_1, \ldots, q_{n+1}$ of this linear space and consider the map given by the formulas

$y \mapsto [q_1(y), \ldots, q_{n+1}(y)]$. Its image is projectively isomorphic to the quadric Q. Changing a basis we adjust it to make the image equal to Q.

For example, assume $|E| = \mathbb{P}^2$ with coordinates t_0, t_1, t_2. Take Q_0 given by the equations $t_0 = 0, t_1^2 + t_2^2 = 0$. Then the linear space of quadratic forms vanishing on Q_0 is spanned by $t_0^2, t_0t_1, t_0t_2, t_1^2 + t_2^2$ Consider the rational map $\mathbb{P}^2 \dashrightarrow \mathbb{P}^3$ defined by the formula

$$[t_0, t_1, t_2] \mapsto [y_0, y_1, y_2, y_3] = [t_0^2, t_0t_1, t_0t_2, t_1^2 + t_2].$$

The image is a smooth quadric given by equation

$$y_0y_3 - y_1^2 - y_2^2 = 0.$$

Note that the pre-images of hyperplanes in $\mathbb{P}^3$ are conics with equations $at_0^2 + bt_0t_1 + ct_0t_2 + dt_1^2 + t_2^2 = 0$. Over reals they are circles.

sectionCohomology of quadrics Since a quadric $Q \subset |E|$ is a hypersurface in $\mathbb{P}^n$, we can apply computations of cohomology from [Hartshorne]. We have

$$\begin{aligned} H^i(Q, \mathcal{O}_Q(m)) &= 0,\ i \neq 0, n,\ m \in \mathbb{Z}; \\ H^0(Q, \mathcal{O}_Q(1)) &= n+1,\ H^0(Q, \mathcal{O}_Q(m)) \cong S^m(E^\vee)/QS^{m-2}(E^\vee),\ m \geq 2; \\ H^n(Q, \mathcal{O}_Q(m)) &\cong H^0(Q, \mathcal{O}_Q(-n+1-m), \end{aligned}$$

The last isomorphism uses that the canonical sheaf ω_Q of a quadric is isomorphic to $\mathcal{O}_Q(-n+1)$.

We can also compute topological cohomology. We use the l-adic cohomology or usual cohomology if $\mathbb{k} = \mathbb{C}$.

Proposition 1.6. *Let Q be a smooth quadric in $\mathbb{P}^n$ and $\eta_Q = c_1(\mathcal{O}_Q(1)$ be the divisor class of its hyperplane section. Then*

(i) $\mathrm{Pic}(Q) \cong \mathbb{Z}$ *if* $n \neq 3$ *and* $\mathrm{Pic}(Q) \cong \mathbb{Z}^2$ *if* $n = 3$.

(ii) The Chern class homomorphism $c_1 : \mathrm{Pic}(X) \to H^2(X, \mathbb{Z}_l)$ is an isomorphism.

(iii) $H^{2i+1}(Q, \mathbb{Z}_l) = 0$;

(iv) $H^{2k}(Q, \mathbb{Z}_l)$ is freely generated by η_Q^k if n is even, or n is odd and $n = 2k + 1$.

(v) $H^{2k}(Q, \mathbb{Z}_l) \cong \mathbb{Z}_\ell[r_1] + \mathbb{Z}_\ell[r_2]$ if $n = 2k + 1$, where $\eta_Q^k = [r_1] + [r_2]$ and r_1, r_2 are the cohomology classes of generators from two different rulings of Q.

Proof. We use induction by n. Let $X \to Q$ be the blow-up of a point $x_0 \in Q$. We know that X is isomorphic to the blow-up of a smooth quadric Q_0 in a hyperplane $H \subset \mathbb{P}^{n-1}$. The assertion will follow from the known behaviour of cohomology under a blow-up $f : X \to Y$ of a smooth closed subvariety Z in Y. We have

$$H^*(X, \mathbb{Z}_l) \cong f^*H^*(X, \mathbb{Z}_l) \oplus H^*(E, \mathbb{Z}_l)/g^*H^*(Z, \mathbb{Z}_l),$$

where is the exceptional divisor and $g : E \to Z$ is the restriction of f to E (see [Griffiths-Harris], p. 605 when $\Bbbk = \mathbb{C}$). Also, we have to use that $E = \mathbb{P}(\mathcal{N}_{Z/Y})$ is the projective bundle over Z associated to the normal sheaf $\mathcal{N}_{Z/Y} \cong (\mathcal{I}_Z/\mathcal{I}_Z^2)^\vee$, where $\mathcal{I}_Z$ is the ideal sheaf of Z in Y. It is known that $H^*(E, \mathbb{Z}_l)$ is a free module over $H^*(Z, \mathbb{Z}_l)$, where the structure of a module is defined by the homomorphism g^*. Its generators are $\eta^i_{E/Z}$, where $\eta_{E/Z} = c_1(\mathcal{O}_E(1))$. they satisfy a relation

$$\eta^r_{E/Z} + c_1\eta^{r-1}_{E/Z} + \cdots + c_r = 0,$$

where $c_i = c_i(\mathcal{N}_{Z/Y})$ are the Chern classes of $\mathcal{N}_{Z/Y}$ and $r = \operatorname{codim}(Z, Y) = \operatorname{rank}(\mathcal{N}_{Z/E})$. We leave to the reader to finish the proof by computing cohomology of X in two ways, as the blow-up of Q_0 in $\mathbb{P}^{n-1}$ and as the blow-up of a point on Q. □

1.5. Examples.

Example 1.7. A one-dimensional quadric $Q = V(q)$ is called a *conic*. It is singular if and only if the rank of q is less than 3, hence if and only if it is irreducible over an extension of $\Bbbk$. A conic with a $\Bbbk$-point is isomorphic to the projective line via the projection from this point. A conic of rank 1 is a double line.

The polar line of a conic with respect to a point p_0 outside of the conic intersects the conic at two different points x, x'. The tangent lines at these points intersect at p_0.

All conics in $|E|$ are parameterized by the 5-dimensional projective space $|S^2E^\vee| \cong \mathbb{P}^5$. Singular conics are parameterized by the *discriminant hypersurface* D_2. It is a cubic hypersurface in $\mathbb{P}^5$. In coordinates, its equation is given by the determinant of a symmetric 3×3-matrix $\det(A_{ij})$. Its singular locus parametrize double lines. It is isomorphic to the *Veronese surface* in $\mathbb{P}^5$, the image of the Veronese map $\nu_2 : \mathbb{P}^2 = \mathbb{P}(E^\vee) \to \mathbb{P}^5 = \mathbb{P}(S^2E^\vee)$ given by $[l] \to [l^2]$.

Example 1.8. A quadric in $\mathbb{P}^3$ is a *quadric surface*. A smooth quadric surface over an algebraically closed field $\Bbbk$ can be given by equation

$$t_1t_2 + t_3t_4 = 0.$$

It has two families of lines given by equations

$$\begin{aligned} \lambda t_1 - \mu t_3 &= \mu t_2 + \lambda t_4 = 0, \\ \lambda t_2 - \mu t_4 &= \mu t_1 + \lambda t_3 = 0. \end{aligned}$$

The lines of each family do not intersect, each line from family intersects all lines in another family. The *Segre map*

$$\mathbb{P}^1 \times \mathbb{P}^1 \to Q, \quad ([u_0, u_1], [v_0, v_1]) \mapsto [u_0v_0, u_0v_1, u_1v_0, u_1v_1]$$

is an isomorphism. The images of fibers of each projection $\mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^1$ is a family of lines on Q.

The group of automorphism of a smooth quadric contains a subgroup of index 2 isomorphic to $\mathrm{Aut}(\mathbb{P}^1) \times \mathrm{Aut}(\mathbb{P}^1)$. It is also isomorphic to $\mathrm{PSL}_{\Bbbk}(4)$. The quotient by this subgroup is generated by the involution that switches the factors. Thus, we have

$$\mathrm{Aut}(Q) \cong (\mathrm{PGL}_{\Bbbk}(2) \times \mathrm{PGL}_{\Bbbk}(2)) \rtimes \mathbb{Z}/2\mathbb{Z}.$$

The projection from a point $x_0 \in Q$ to $\mathbb{P}^2$ extends to a regular map from the blow-up $\mathrm{Bl}_{x_0} Q$ to $\mathbb{P}^2$.

$$\begin{array}{ccc} & \mathrm{Bl}_{x_0} Q & \\ {}^{\sigma}\swarrow & & \searrow^{\pi} \\ Q & \longrightarrow & \mathbb{P}^2. \end{array}$$

The map σ is an isomorphism outside two points y_1, y_2 in $\mathbb{P}^2$. The fibers over these points are the proper transforms of two generators of Q containing the point x_0. The inverse map $\mathbb{P}^2 \dashrightarrow Q$ given by conics through the points y_1, y_2. It decomposes as in the following diagram:

$$\begin{array}{ccc} & \mathrm{Bl}_{y_1, y_2} \mathbb{P}^2 & \\ {}^{\sigma}\swarrow & & \searrow^{\pi} \\ \mathbb{P}^2 & \longrightarrow & Q. \end{array}$$

The two families of lines are the images of pencils of lines through the points y_1, y_2 under the inverse rational map $\mathbb{P}^2 \dashrightarrow Q$. The image of the line $\overline{y_1, y_2}$ is the point x_0.

A quadric of corank 1 has an isolated singular point. It is called a *quadratic cone* because it is isomorphic to the cone over a conic. A quadric of corank 2 is the union of two planes. Its singular locus is the intersection of the planes. A quadric of corank 1 is the double plane. It is singular everywhere.

The space of quadrics $|S^2E^\vee|$ is isomorphic to $\mathbb{P}^9$. Singular quadrics form the *discriminant hypersurface* D_3. It is a quartic hypersurface in $\mathbb{P}^9$ given by the discriminant equation. Its singular locus parameterizes quadrics of corank ≥ 2. It is isomorphic to the quotient of $\mathbb{P}^2 \times \mathbb{P}^2$ by the involution that switches the factors. It is the image of the map $\mathbb{P}^2 \times \mathbb{P}^2 \to \mathbb{P}^9$ given by the linear system of $|S^2E^\vee| \subset |E^\vee \otimes E^\vee|$. Its singular locus is the image of the diagonal, and isomorphic to $\mathbb{P}^2$.

Example 1.9. Let $n = 4$, a nonsingular quadric over an algebraically closed field $\Bbbk$ can be given by equation

$$t_1t_2 + t_3t_4 + t_5^2 = 0.$$

The projection map $\pi_{x_0} : Q \dashrightarrow \mathbb{P}^3$ decomposes according to the following commutative diagram

$$\begin{array}{ccc} & \mathrm{Bl}_{x_0} Q & \\ {}^{\sigma}\swarrow & & \searrow^{\pi} \\ Q & & \mathbb{P}^3. \end{array}$$

We know that the inverse map of the projection map $Q \dashrightarrow \mathbb{P}^3$ from a point $x_0 \in Q$ is given by quadrics containing some nonsingular conic C in a hyperplane $H \subset \mathbb{P}^3$, the image of the tangent hyperplane $\mathbb{T}_{x_0}Q$. We have an irreducible 2-dimensional family of lines on Q. Each line containing x_0 is blown down to a point in $\mathbb{P}^3$. All the images of such lines lie in the conic C. Any other line intersects $\mathbb{T}_{x_0}Q$, and hence intersects some line through x_0. Its projection is a line in $\mathbb{P}^3$ that intersects a conic C at one point. Conversely, the inverse map $\mathbb{P}^3 \dashrightarrow Q$ is given by quadrics containing the conic C. It decomposes as in the following diagram:

$$\begin{array}{ccc} & \mathrm{Bl}_C \mathbb{P}^3 & \\ {}^{\sigma}\swarrow & & \searrow^{\pi} \\ \mathbb{P}^3 & \longrightarrow & Q. \end{array}$$

Each line intersecting C at two points is blown down to the point x_0. The family of lines on Q is the image of the family of lines in $\mathbb{P}^3$ that intersect C at one point.

The set of lines $F_1(Q)$ is an irreducible 3-dimensional variety isomorphic to $\mathbb{P}^3$.

Example 1.10. Assume $n = 5$. a nonsingular quadric in $\mathbb{P}^5$ over an algebraically closed field $\Bbbk$ can be given by equation

$$t_1t_2 + t_3t_4 + t_5t_6 = 0.$$

Consider $\mathbb{P}^5$ as the projective space $|\bigwedge^2 U|$, where U is a 4-dimensional linear space. Let e_1, e_2, e_3, e_4 be a basis in U, then $e_i \wedge e_j, 1 \le i < j \le 4$ is a basis in $\bigwedge^2$. For any two distinct points $[v], [w]$ in $|U|$, the line joining these two points defines a decomposable 2-vector $v \wedge w$ in $\bigwedge^2 U$. Obviously, each decomposable 2-vectors is obtained from a unique line in $\mathbb{P}^3$. It is easy to check that a 2-vector λ is decomposable if and only if $\lambda \wedge \lambda = 0$. Thus, the Grassmannian of lines $G(2, U)$ is isomorphic to the subvariety of $|| \bigwedge^2 U|$ defined by the condition that $\lambda \wedge \lambda = 0$. In coordinates p_{ij} dual to the basis $(e_i \wedge e_j)$ (they are the Plücker coordinates), the condition is translated into the condition

$$p_{12}p_{34} + p_{13}p_{24} - p_{14}p_{23} = 0.$$

Thus we see that any nonsingular quadric Q in $\mathbb{P}^5$ is isomorphic the Grassmannian $G(2, 4)$ of lines in $\mathbb{P}^3$. A For this reason, the Grassmannian $G(2, 4)$ is called sometimes the *Klein quadric*. Now, we easily find two irreducible 3-dimensional families planes in Q. One family consists of planes σ_y formed by lines in $\mathbb{P}^3$ passing through a point $y \in \mathbb{P}^3$. Another family consists of planes σ_Λ formed by lines in $\mathbb{P}^3$ contained in a plane $\Lambda \subset \mathbb{P}^3$. A line in Q is formed by lines in $\mathbb{P}^3$ that are contained in a plane Λ and contains a fixed point $y \in \Lambda$. Obviously, the family of lines is of dimension 5. It agrees with formula (**??**).

We have $\mathrm{PO}(Q)^0 \cong \mathrm{Aut}(\mathbb{P}^3) \cong \mathrm{PGL}(4)$. The group $\mathrm{Aut}(\mathbb{P}^3)$ acts on Q via its identification with the Grassmannian $G(2, U)$. An extra automorphism comes from a *polarity* which is an isomorphism $|U| \to |U^\vee|$ defined by a linear map $c : E \to E^\vee$ such that $c \circ {}^t c^{-1}$ is the identity (see [Classical Algebraic Geometry], p. 10).

1.4 Appendix 1:Quadrics in characteristic 2

The first different feature of this case is that a quadratic form is not determined by its polar bilinear form. In fact we see that b_q is an alternating form in this case, and, for example, the quadratic form $\sum t_i^2$ has zero polar bilinear form.

A vector in the radical of q may not be in the zero set of q, so a *singular vector* in this case means that it is an element $v \in \mathrm{Rad}(b_q)$ such that $q(v) = 0$. The jacobian criterion of smoothness implies that a singular point of the associated quadric $Q = V(q)$ is equal to $[v]$, where v is a singular vector. Since $Q(\bar{\Bbbk})$ either contains $|\mathrm{Rad}(b_q)|(\Bbbk)$ or intersects it along the locus $V(l^2)$, where l is a linear form, we see that the singular locus $\mathrm{Sing}(Q)$ of Q is a linear subspace of dimension equal to $\delta - 1$, or $\delta - 2$, where $\delta = \dim \mathrm{Rad}(b_q)$ is the defect of q. In particular, if n is odd, the defect of b_q must be even, and hence Q is singular if and only if b_q is degenerate. On the other hand, if n is even, the defect is odd, and Q could be smooth but b_q is always degenerate.

If b_q is degenerate the determinant of its matrix is zero, but this does not imply that $V(q)$ is singular. The notion of the discriminant has to be replaced with the notion of the $\frac{1}{2}$-*discriminant*.

Exercise 1.11. *Let B be a skew-symmetric matrix of odd size $n+1$. Let $\mathrm{Pf}_1, \ldots, \mathrm{Pf}_{n+1}$ are the pfaffians of the principal submatrices of B obtained by deleting the ith row and the ith column of B.*

(i) Show that $V(q)$ is smooth if and only

$$q(\mathrm{Pf}_1, \ldots, \mathrm{Pf}_{n+1}) = 0$$

has no solutions in any extension of $\Bbbk$.

(b) Show that the left-hand side is a polynomial of degree $n+1$ in the coefficients a_{ij} of q. It is called the $\frac{1}{2}$-discriminant *of q (Of course the discriminant of any q is the determinant* $\det(B)$ *which is zero if $n+1$ is odd and* $\mathrm{char}(\Bbbk) = 2$*).*

(c) Compute the $\frac{1}{2}$-discriminant for small n and guess the general form of it (it must be analogous of the general form of the discriminant).

(d) Let $q = \sum A_{ij} t_i t_j$ be a quadratic form whose coefficients are independent variables. The determinant of the matrix of the polar bilinear form is a polynomial D in $\mathbb{Z}[(A_{ij})$. Show that $D = 2D'$, for some polynomial $D' \in \mathbb{Z}[(A_{ij}])$.

(e) Show that, for any homomorphism $\phi : \mathbb{Z}[(A_{ij}) \to \Bbbk$, the image $\phi(D')$ of D' is the $\frac{1}{2}$-discriminant of $\phi(q) = \sum a_{ij} t_i t_j$, where $a_{ij} = \phi(A_{ij}$.

(f) Show that when $n = 2$ and $n = 4$, the equations of $\frac{1}{2}$-discriminant for conics and quadrics of dimension 3 are

$$\begin{aligned}\mathrm{Disc}_2' &= A_{11}A_{23}^2 + A_{22}A_{13}^2 + A_{33}A_{12}^2 + A_{12}A_{23}A_{13},\\ \mathrm{Disc}_4' &= (A_{11}A_{23}^2A_{45}^2 + \cdots) + (A_{12}^2A_{34}A_{45}A_{35} + \cdots)\\ &\quad +(A_{12}A_{23}A_{34}A_{45}A_{15} + \cdots)\end{aligned}$$

We say that a subspace F is *totally singular* if the restriction of q to F is zero. In geometric terms this means that $|F|$ is a linear projective subspace contained on Q. A totally singular subspace is totally isotropic (i.e the restriction of b_q to F is zero) but the converse is not true. In coordinates, the restriction of q to a totally isotropic subspace F of b_q is equal to $\sum a_i t_i^2$, where t_i is a basis in $F^\vee$. Thus, over the algebraic closure $\bar{\mathbb{k}}$ of $\mathbb{k}$, it is equal to the square of a linear function $l : F \to \mathbb{k}$. In particular, the restriction of q to F is either the whole F or a hyperplane F_0 in F. In geometric terms, $|F|$ or $F_0|$ is a linear projective subspace contained in $Q(\bar{\mathbb{k}})$. It follows from the equations (**??**) that a maximal isotropic subspace of Q coincides with a maximal singular subspace if $n + 1 = 2m$ and its dimension is larger by one if $n = 2m$. So, we have the same properties of families of generators as in characteristic $p \neq 2$.

The restriction of q to $|\operatorname{rad}(b_q)|$ is either the whole space or a hyperplane. This shows that the set of singular vectors of q is a linear subspace in $E_{\mathbb{k}^{\mathrm{perf}}}$ of dimension $\delta(q)$ or $\delta(q) - 1$. The set of the corresponding points in the quadric $Q = V(q)$ is the locus $\mathrm{Sing}(Q)$ of singular points in Q.

Assume $p = 2$. If $v_0 \in \operatorname{rad}(b_q)$, then $P_{x_0}Q = |E|$ and any line $\overline{x_0, x}, x \in Q$ is tangent to Q at x. The projection p_{x_0} is an inseparable map of degree 2 onto $\mathbb{P}^{n-1}$. The point x_0 is called the *strange point* of the quadric Q. For example, when n is even, there is always such point. If n is odd such point exists only if b_q is degenerate. If $v_0 \neq \operatorname{rad}(b_q)$, then $P_{x_0}Q \neq |E|$ but $x_0 \in P_{x_0}Q$. The first polar is tangent to Q at any intersection point with Q. Thus Q^0 becomes the double hyperplane of dimension $n - 2$ contained in Q. The projection is a separable double cover ramified along Q^0. The image of Q_0 in $|E/\mathbb{k}v_0|$ is a hyperplane. This is the branch divisor of the double cover.

1.5 Appendix 2:Orthogonal Grassmannians

Let $Q = V(q)$ be a non-degenerate quadric in an odd-dimensional projective space $\mathbb{P}^{2k+1} = |E|$. In this Lecture we have introduced the orthogonal

Grassmannian $\mathrm{OG}(k+1, 2k+2)$, a connected component of the variety of linear subspaces of dimension k contained in Q. Its dimension is equal to $\frac{1}{2}k(k+1)$. Via the Plücker embedding, the orthogonal Grassmannian embeds in $|\bigwedge^2 E| \cong \mathbb{P}^{\binom{2k+2}{k+1}-1}$. However, the image is contained in a proper projective subspace $|S|$ of dimension 2^{k-1}. The linear subspace S of E is one of two irreducible representations of the orthogonal group $\mathrm{O}(E,q)$ of dimension 2^k, the *half-spinor representations*.

Let us recall its definition. Let $C(q)$ be the Clifford algebra of (E,q). It is the quotient of the tensor algebra $T^*(E)$ by the two-sided ideal generated by elements $v \otimes v - q(v)$. The natural $\mathbb{Z}/2\mathbb{Z}$-grading of the tensor algebra induces a natural grading $C(q) = C(q)^+ \oplus C(q)^-$ of the Clifford algebra. Suppose q admits a decomposition $E = F \oplus F'$ into the direct sum of to totally isotropic subspaces. It always happens if the ground field is algebraically closed, and, we make this assumption. Using the restriction of the bilinear form $b_q : E \to E^\vee$ to $F' \times F$, we can naturally identify F' with The dual linear space $F^\vee$. The subalgebra of $C(q)$ generated by vectors $f \in F \subset E = T^1(E)$ is naturally isomorphic to the exterior algebra $\bigwedge^\bullet(F)$ of dimension 2^{k+1}. It has a natural $\mathbb{Z}/2\mathbb{Z}$-grading and decomposes accordingly. into the direct sum $\bigwedge^\bullet(F) = \bigwedge^\bullet_+(F) \oplus \bigwedge^\bullet_-(F)$. Each summand is linear space of dimension 2^k. Let us define a homomorphism of algebras

$$\rho : C(q) \to \mathrm{End}_{\Bbbk}(\bigwedge_-^\bullet(F)).$$

Let $f \in F$(resp. $f^* \in F^\vee$). It acts on $\bigwedge^\bullet(F)$ via the wedge-product (resp. the contraction). This action first extends, by additivity to the whole E, and then by the derivation rule to the tensor algebra. It is checked that the elements of the two-sided ideal act as the zero operators. Each summand is obviously invariant. In this way one identifies $C(q)$ with the algebra of endomorphisms of $\mathrm{End}_{\Bbbk}(\bigwedge^\bullet(F))$ isomorphic to the algebra of matrices of size 2^{k+1}. The subalgebra $C(q)^+$ is isomorphic to the sum of two left ideals in the matrix algebra, each is isomorphic to the algebra of matrices of size 2^k. These are two *half-spinor representations* $S_\pm$ of $C^+(q)$.

Let $G(q)$ be the group of invertible elements x in $C^+(q)$ such that, for any $v \in E$, one has $x \cdot v \cdot x^{-1} \in E$. The image of the homomorphism $\phi : G(q) \to \mathrm{GL}(E)$ is obviously contained in the orthogonal group $\mathrm{O}(E,q)$. Suppose $\mathrm{char}(\Bbbk) \neq 2$. It is known that the group $\mathrm{O}(E,q)$ is generated by reflections $s_r : v \to v - (v,r)r$, where $r \in E, q(e) = 1$. Since $r \otimes r = 1$ in $T(E)$, we obtain that $r^2 = 1$ is $C(q)$, hence $r \in G(q)$. It is immediately checked that $\phi(r) = -s_r$ and, conversely, the image of any element r in

$G(q)\cap E$ is equal to $-s_r$. Using this one can show that homomorphism ϕ is surjective (we have to use that the element $-\mathrm{id}_E$ in $\mathrm{O}(E,q)$ is the product of even number of reflections). The kernel of the homomorphism ϕ is equal to the group Z^* of invertible elements in the center of the algebra $C^+(q)$. The extension

$$1 \to Z^* \to G(q) \to \mathrm{O}(E,q) \to 1$$

does not split, however, there is a unique subgroup $\mathrm{Spin}(q)$ in $G(q)$ that is mapped surjectively onto $\mathrm{O}(E,q)$ with kernel {
*pm*1}. The group $\mathrm{Spin}(q)$ is called the *spinor group* of (E,q). Being the subgroup of $G(q)$ it admits two linear representations $S_\pm$, called the half-spinor representations of $\mathrm{Spin}(q)$. They are isomorphic (via the exterior automorphism of the group) irreducible representations of dimension 2^k.

Being a closed subvariety of the usual Grassmannian $G(k+1,2k+2)$, the orthogonal Grassmannian embeds in the Plücker space $|\bigwedge^{k+1}(E)|$. However, it spans a proper projective subspace of the Plücker space. let us explain which one.

Let $b_q : E \to E^\vee$ be the linear isomorphism define by the non-degenerate quadratic form q. By passing to the exterior powers, it defines, for any $m \geq 0$, a linear isomorphism $\wedge^{k+1}(b_q) : \bigwedge^m(E) \to \bigwedge^m(E^\vee) = \bigwedge^m(E)^\vee$. Taking $m = \dim E$, we can identify the one-dimensional linear spaces $\bigwedge^{2k+2}(E^\vee)$ and $\bigwedge^{2k+2}(E^\vee)$ with the field of scalars $\Bbbk$ (by assuming that the discriminant of q is equal to 1). The wedge-product defines a non-degenerate pairing

$$\bigwedge^{k+1}(E) \times \bigwedge^{k+1}(E) \to \bigwedge^{2k+2}(E) = \Bbbk$$

and, hence defines a linear isomorphism $\epsilon : \bigwedge^{k+1}(E^\vee) \to \bigwedge^{k+1}(E^\vee)$. The automorphism $\wedge^{k+1}(b_q) \circ \epsilon^{-1} : \bigwedge^{k+1}(E) \to \bigwedge^{k+1}(E)$ is an involution with two eigenvalues ± 1 that decomposes $\bigwedge^{k+1}(E)$ into the direct sum of two eigensubspaces $\bigwedge^{k+1}(E)_\pm$ of the same dimension. Under the Plücker space, the image of each connected component $F_k(Q)_\pm$ of the variety $F_k(Q)$ is equal to the intersection of $G(k+1,2k+2)$ with $|\bigwedge^{k+1}(E)_\pm|$. Thus we see that the orthogonal Grassmannian is a linear section of the usual Grassmannian.

Example 1.12. Let Q be a non-degenerate quadric in $|E| = \mathbb{P}^3$, so that $k = 1$ and $F_1(Q)_\pm$ parameterizes lines in one of the two rulings of lines in Q. We know that $G(2,4)$ is the Klein quadric in $\mathbb{P}^5 = |\bigwedge^2 E|$. The image of $F_1(Q)_\pm$ is a conic in $G(2,4)$ cut our by a plane in $\mathbb{P}^5$.

If we take Q to be a quadric in $|E| = \mathbb{P}^5$, for example the Klein quadric $G(2,4) = G_1(\mathbb{P}^3)$ in $\mathbb{P}^5$, then we know that $F_2(Q)_\pm$ are the two families of planes (of lines passing through a point in $\mathbb{P}^3$ or lines contained in a plane in $\mathbb{P}^3$). They are naturally isomorphic to $\mathbb{P}^3$ (or the dual $\mathbb{P}^3$). In the Plücker space $|\bigwedge^3 E| \cong \mathbb{P}^{19}$ they are equal to the intersection $G(3,6) = G_2(\mathbb{P}^5)$ with $\mathbb{P}^9$. One can show that the embedding $\mathbb{P}^3 \hookrightarrow \mathbb{P}^9$ obtained in this way is the second Veronese embedding $\nu_2 : \mathbb{P}^3 \to \mathbb{P}^9$.

It follows from the previous example that the embedding $\mathrm{O}(k+1,E)$ in $|\bigwedge^{k+1}(E)_\pm|$ is not a minimal projective embedding. The minimal embedding is the *spinor embedding*. which we proceed to describe. Fix a maximal totally isotropic subspace F of E and consider the associated half-spinor representation $S_\pm$ based on $\bigwedge^\bullet(E)_\pm$. The line $\bigwedge^{k+1}(F)$ is spanned by a vector belonging to the subspace S_+ if k is odd and to the subspace S_- otherwise. Consider the orbit of $\mathrm{Spin}(E,q)$ of this line in its natural action in the projective space $|S_\pm|$. Of course, the projective representation of $\mathrm{Spin}(E,q)$ in $|S_\pm|$ defines a projective representation of the orthogonal group $\mathrm{O}(E,q)$ but it does not leave to a linear representation of this group (this why do we need the spinor group!). Since $\mathrm{O}(E,q)$ acts transitively on totally isotropic subspaces of the same dimension, the orbit coincides with $F_k(Q)_|pm|$. One can show that the orbit spans $S_\pm$. In fact, the orbit coincides with a homogeneous space of the form $\underline{\mathrm{Spin}}(E,q)/P$, where P is a maximal parabolic subgroup of an algebraic group $\underline{\mathrm{Spin}}(E,q)$ with $\underline{\mathrm{Spin}}(E,q)(\Bbbk) = \mathrm{Spin}(E,q)$. Its Picard group is generated by a line bundle that defines the embedding $F_k(Q)$ into $|S_\pm$.

Example 1.13. In the previous example, we have seen the spinor embeddings of $\mathrm{O}(2,4)$ and $\mathrm{O}(3,6)$ into $\mathbb{P}^1$ and $\mathbb{P}^3$, respectively. The 6-dimensional variety $\mathrm{O}(4,8)$ representing one family of 3-spaces in a nonsingular quadric $Q \subset \mathbb{P}^7$ is minimally embedded in $\mathbb{P}^7$. It is a nonsingular quadric in $\mathbb{P}^7$. In fact, it must a hypersurface in $\mathbb{P}^7$, the only homogeneous hypersurface in a projective space is either a nonsingular quadric or a projective subspace.

Example 1.14. The 10-dimensional variety $\mathrm{O}(5,10)$ embeds into $\mathbb{P}^{15} = |\bigwedge^\bullet_\pm(F)$. One can show see [Kapustka, Projections of Mukai varieties] that it is given as the zero locus of the quadratic map

$$\Phi : \bigwedge_{\pm}^{\bullet}(F) = \Bbbk \oplus \bigwedge^2 F \oplus \bigwedge^4 F \to \bigwedge^4 F \oplus F$$

defines as follows. We first identify $\Bbbk$ with $\bigwedge^5 F$ and $\bigwedge^4 F$ with $F^\vee$. Then we set

$$\Phi(v,\omega,\phi) = (v \llcorner \phi + \frac{1}{2}\omega \wedge \omega, v \llcorner \omega).$$

The zero locus is given by 10 quadratic equations.

The homogeneous variety $\mathrm{O}(5, 10)$ is an example of a *Muklai variety*, a non-degenerate subvariety of a projective space whose general linear sections of dimension one are canonically embedded algebraic curves of some genus $g > 1$. In particular, the degree of such a variety is equal to $2g - 2$. In our case, $g = 7$ and the curve is cut out by a general projective subspace in $\mathbb{P}^{15}$ of codimension 9. A linear section by a subspace of codimension 8 (resp. 7) is a Fano 3-fold (resp. a K3 surface) of degree 12.

Lecture 2

Normal elliptic curves

2.1 First example

We start with the intersection of two irreducible conics $C_1 \cap C_2$. For simplicity, let us assume that the field is algebraically closed. By Bezout's theorem, $C_1 \cap C_2$ consists of $k \leq 4$ points. Let $C_1 = V(q_1), C_2 = V(q_2)$ and $C(t_1, t_2) = V(t_1 q_1 + t_2 q_2)$. When $[t_1, t_2]$ runs $\mathbb{P}^1$, we have a *pencil of conics*. Let D be the discriminant of the conic $t_1 q_1 + t_2 q_2$. It is a homogeneous polynomial $a_1 t_1^3 + a_2 t_1^2 t_2 + a_3 t_1 t_2^2 + a_4 t_2^3$ of degree 3 in t_1, t_2. Plugging in $(t_1, t_2) = (1, 0)$ and $(0, 1)$, and using that C_1, C_2 are smooth, we get that $a_1, a_4 \neq 0$. Clearly, $V(D)$ consists of $r \leq 3$ points.

The number r is equal to the number of singular conics in the pencil. If two conics intersect at a point P with multiplicity m, then there exists a member in the pencil that has a point of multiplicity $\geq m$ at this point. If $k = 4$, and $C_1 \cap C_2 = \{p_1, p_2, p_3, p_4\}$, then we have three singular members formed by three pairs of lines $\langle p_i, p_j \rangle + \langle p_k, p_l \rangle$. In fact, no three points are collinear, because otherwise all conics in the pencil contain a line as their irreducible component. So, we may choose projective coordinates such that the points are $[1, 0, 0], [0, 1, 0], [0, 0, 1], [1, 1, 1]$. Then the conics in the pencil have equations

$$at_0 t_1 + bt_0 t_2 + ct_1 t_2 = 0, \quad a + b = c = 0.$$

The three singular conics are $V(t_0(t_1 - t_2)), V(t_1(t_2 - t_3))$ and $V(t_2(t_0 - t_2))$.

If $k = 3$ and $C_1 \cap C_2 = 2p_1 + p_2 + p_3$, then we have two singular members. One is the union of the line $\langle p_2, p_3 \rangle$ and the common tangent line to the conics at the point p_1. The second one is the union of the lines $\langle p_1, p_2 \rangle$ and $\langle p_1, p_3 \rangle$.

If $C_1 \cap C_2 = 2p_1 + 2p_2$, then we have a singular member equal to the union of tangent lines at p_1 and p_2 and the double line $\langle p_1, p_2\rangle$.

If $C_1 \cap C_2 = 3p_1 + p_2$, we have one singular member, the union of the tangent line at p_1 and the line $\langle p_1, p_2\rangle$.

Finally, if $C_1 \cap C_2 = 4p$, we have one singular member, the double tangent line at p.

So, we see that the number r depends on k in a rather complicated way. If $p \neq 2$, the different cases are determined by considering *elementary divisors* of the λ-matrix $\det(\lambda A + B)$, where A, B are matrices of the bilinear forms b_{q_1}, b_{q_2}. Possible Jordan forms are the following: one with three distinct eigenvalues ($r = 3$), two with two distinct eigenvalues ($r = 2$), and two with one eigenvalue ($r = 1$). We will discuss this later.

Let $f : \mathbb{P}^2 \dashrightarrow \mathbb{P}^1$ be the rational map defined by

$$x = [x_0, x_1, x_2] \mapsto [q_1(x), q_2(x)]. \tag{2.1.1}$$

Let X be the Zariski closure of the graph of the map $f' : \mathbb{P}^2 \setminus C_1 \cap C_2 \to \mathbb{P}^1$. If $k = 4$, X is a nonsingular surface, and the projection to $\mathbb{P}^2$ blows up the intersection points. It is a *del Pezzo surface* of degree 5.

Recall from [Dolgachev,Classical Algebraic Geometry] that a del Pezzo surface is a smooth algebraic surface S such that $-K_S$ is ample. Each such surface is a rational surface. It is isomorphic to $\mathbb{P}^2$ or $\mathbb{P}^1 \times \mathbb{P}^1$, or admits a birational morphism $f : S \to \mathbb{P}^2$ whose inverse is the blow-up of k points $p_1, \ldots, p_k$ in the plane. Since $K_S^2 > 0$, the number k is at most 8. We have $K_S^2 = 9 - k$ (under a blow-up the self-intersection of the canonical class decreases by one). The number $d = K_S^2$ is called the *degree* of the del Pezzo surface S. For $d \geq 3$, the linear system $|-K_S|$ defines an embedding of S into $\mathbb{P}^d$ as a surface S^{ac} of degree d. It is called an *anti-canonical model* of a del Pezzo surface. Conversely, every smooth surface of degree d in $\mathbb{P}^d$ not lying in a proper projective linear subspace is isomorphic to a del Pezzo surface of degree d. The linear system $|-K_S|$ defines a birational map $\mathbb{P}^2 \dashrightarrow S^{\mathrm{ac}}$ onto an anti-canonical model. By adjunction formula, for any smooth curve C, we have $C^2 + C \cdot K_S = 2g(C) - 2$, where g is the genus of C. In particular, if C is a (-1)-*curve*, i.e. a smooth rational curve C with self-intersection $C^2 = -1$, we have $(-K_S) \cdot C = 1$. Thus its image on S^{ac} is a line. A (-1)-curve is characterized by the property that it can be realized as the exceptional curve of the blow-up of a nonsingular point on a surface. Or, equivalently, it can be blown down to such a point. Under the birational morphism $S \to \mathbb{P}^2$, the image of a (-1)-curve R is an irreducible curve of some degree m. Since under the blow-up the self-intersection of the proper inverse transform of an irreducible curve $\bar{R}$ decreases by one, we see that $m^2 - a = -1$, where $a = \sum m_i$ with m_i equal to the multiplicity of $\bar{R}$ at the point p_i. This implies that for $d \geq 3$, the curve R is either blown down to one of the points p_i, or $m = 1$ and $a = 2$, or $m = 2$ and $a = 5$. The linear system $|-K_S|$ is defined by plane cubic curves passing through the points $p_1, \ldots, p_5$. If $S = \mathbb{P}^2$, then $|-K_S|$ is the complete linear system of plane cubics and it defines a *Veronese map* $\nu_3 : \mathbb{P}^2 \to \mathbb{P}^9$ onto a *Veronese surface* of degree 9 in $\mathbb{P}^9$. An anti-canonical del Pezzo surface S^{ac} of degree $d \geq 3$ is isomorphic to a projection of a Veronese surface of degree 9 into a linear subspace of dimension $d \geq 3$.

Let us go back to our case when $d = 5$. We denote a del Pezzo surface of degree 5 by $\mathcal{D}_5$. It is unique, up to isomorphism. By above a (-1)-curve on $\mathcal{D}_5$ is either one of four exceptional curves or equal to the proper inverse transform of one of the six lines $\langle p_i, p_j \rangle$. Altogether with have ten (-1)-curves on $\mathcal{D}_5$. Their images in the anti-canonical model S^{ac} are lines.

It is known that the canonical sheaf of the Grassmannian $G = G(k, m)$ is isomorphic to $\mathcal{O}_G(-m)$. In particular, for the Grassmannian $G(2, 5)$, we have $\omega_G \cong \mathcal{O}_G(-5)$. By the adjunction formula, if H is a hyperplane section of G, we have $\omega_H \cong \mathcal{O}_F(-4)$. Continuing in this way, we see that the intersection of $G)2, 5)$ by a linear subspace of codimension 4 is a surface X in $\mathbb{P}^5$ with canonical sheaf isomorphic to $\mathcal{O}_X(-1)$. It is known that the degree of $G(2, 5)$ in its Plücker embedding is equal to 5, so X is of degree 5 and, hence must be isomorphic to an anti-canonical model S^{ac} of a del Pezzo surface of degree 5.

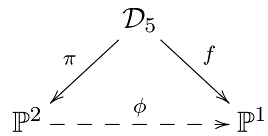
We know that a del Pezzo surface of degree 5 is unique up to isomorphism. So that the freedom of choosing a linear subspace of codimension 4 is illusory. Different choices lead to projectively isomorphic surfaces. To see this we count parameters. Codimension 4 linear subspaces in $\mathbb{P}^9$, where $G(2, 5)$ is embedded, depend on $\dim G(6, 10) = 24$ parameters. On the other hand, the projective linear group $\mathrm{PGL}(5)$ acts in $\mathbb{P}^9$ via its natural action in $\mathbb{P}^4$. Its dimension is equal to $25 - 1 = 24$. Thus we expect that all codimension 4 sections of $G(2, 5)$ form one orbit with respect $\mathrm{PGL}(5)$.

One more remark is in order. A Grassmann variety $G(m, n+1)$ embedded in $|\bigwedge^m(\mathbb{k}^{n+1})|$ via the P"ucker embedding is either isomorphic to $\mathbb{P}^n$ (when $m = n$ or $m = 1$) or it is equal to the intersection of quadrics. The quadrics are given by the Plücker quadratic relations between the Plücker coordinates.

$$\sum_{k=1}^{m+1} (-1)^k p_{i_1,\ldots,i_{m-1},j_k} p_{j_1,\ldots,j_{k-1},j_{k+1},\ldots,j_{m+1}} = 0, \tag{2.1.2}$$

where $(i_1,\dots,i_{m-1})$ and $(j_1,\dots,j_{m+1})$ are two strictly increasing subsets of $[1,n+1]$. These relations are easily obtained by considering the left-hand-side expression as an alternating $(m+1)$-multilinear function on $\mathbb{C}^m$. It is known that these equations define $G(m,n+1)$ scheme-theoretically in $\mathbb{P}^{\binom{n+1}{m}-1}$ (see, for example, [Hodge-Pedoe, vol. 2].

Consider the rational map $\phi:\mathbb{P}^2 \dashrightarrow \mathbb{P}^1$ (**??**) and let



be its resolution of indeterminacy points. The map π is the blow-up of the points p_1,p_2,p_3,p_4 and the map f is a *conic bundle*. The latter means that its general fiber is isomorphic via the projection π to an irreducible conic. Its singular fibers are isomorphic to reducible conics. In our case, we have three such singular fibers. They are the reducible conics in the pencil $\mathcal{P}=V(\lambda q_1+\mu q_2)$ of conics. The irreducible components of the fibers are the proper transforms of lines $\langle p_i,p_j\rangle$. Let us denote them by U_{ij}. The exceptional curves over the points p_i we denote by U_{i5}. Thus we have 10 (-1)-curves $U_{ab}, 1\le a<b\le 5$. In the anti-canonical model S^{ac} of S they are 10 lines. It is easy to see that each curve U_{ab} intersects three other curves. If we define the graph with vertices U_{ab} and draw the edges when two curves intersect, we obtain the famous *Petersen graph*:

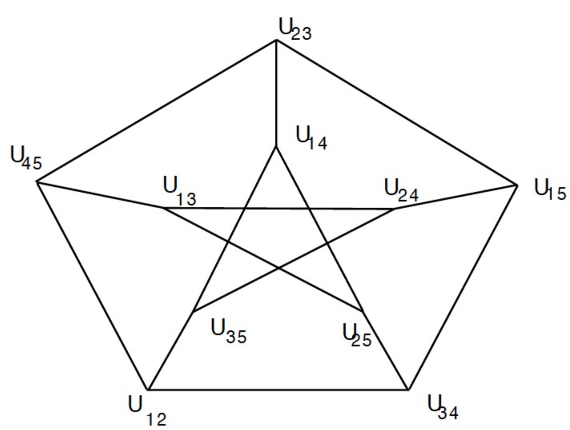


Figure 2.1: Petersen graph

The pair $(\mathcal{D}_5,f)$ has a nice moduli-theoretical interpretation. Take a point $x \in S$ which does not lie on the union $\mathbf{U}$ of the curves U_{ab}. Then the fiber $C(x)$ over $f(x)$ is an irreducible conic on which four ordered points are fixed, they are the intersections of the curves R_{i5} with this fiber. Or, in other words, they are the points which are mapped to the points $p_1, \ldots, p_4$ under the morphism π. Since $C(x)$ is isomorphic to $\mathbb{P}^1$, we get an ordered set of 4 points on $\mathbb{P}^1$. Choosing coordinates in $\mathbb{P}^1$, we may assume that the reducible fibers correspond to the points $\{0, 1, \infty\}$. In this way, $\mathbb{P}^1 \setminus \{0, 1\infty\}$ becomes isomorphic to the moduli space $\mathcal{M}_{0,4}$ of ordered sets of 4-points in $\mathbb{P}^1$ modulo projective equivalence. The pre-image of this open subset of $\mathbb{P}^1$ is the open set $\mathcal{D}_5^o = \mathcal{D}_5 \setminus \mathbf{U}$. A point x in this open set defines a fiber through this point identified with $\mathbb{P}^1$ and an ordered set of 5 points (p_1, p_2, p_3, p_4, x). In this way S^o becomes isomorphic to the moduli space $\mathcal{M}_{0,5}$ of rational curves together with an ordered set of 5 points on it. The whole curve $\mathbb{P}^1$ (resp. the surface $\mathcal{D}_5$) becomes isomorphic to a compactification $\overline{\mathcal{M}}_{0,4}$ (resp. $\overline{\mathcal{M}}_{0,5}$).

2.2 Modular surfaces

If $p \neq 2$, the conic bundle $f : S \to \mathbb{P}^1$ has also a natural moduli interpretation as the *Kummer universal family* for the fine moduli space of elliptic curves with a *2-level structure*.

Let us explain what these words mean. First, a *N-level structure* on an elliptic curve C is a choice of a basis in the group $E[N]$ of its N-torsion points. We assume that $(N, p) = 1$. If $N = 2$, the negation involution $\tau : x \mapsto -x$ has the quotient $C/(\tau)$ isomorphic to $\mathbb{P}^1$. The double cover $\pi : C \to \mathbb{P}^1$ ramifies at 4 points, the 2-torsion points. If we use the coordinates on $\mathbb{P}^1$ such that the zero point p_0 goes to ∞, the complement of p_0 has the *Weierstrass affine equation*

$$f(x, y) = y^2 + x^3 + a_2 x^2 + a_4 x + a_6 = 0$$

The point $(x, y) = (x_i, 0), i = 1, 2, 3$ on the curve are the non-trivial 2-torsion points. An order on the set of these points defines a basis in the group $E[2]$. So, we see that an elliptic curve with level 2 structure is defined by an ordered set of 4 points on $\mathbb{P}^1$. Thus, we can identify $\mathcal{M}_{0,4}$ with the moduli space of elliptic curves with level 2-structure. It is isomorphic to $\mathbb{P}^1 \setminus \{0, 1, \infty\}$.

The fine moduli space of elliptic curves with N-level structure exists and is isomorphic to an affine curve $X(N)^o$. Over $\mathbb{C}$ this curve is isomorphic to the quotient of the upper half-plane $\mathcal{H} = \{z = a + bi \in \mathbb{C} : b > 0\}$ by the modular group $\Gamma(N)$ of matrices from $\mathrm{SL}(2,\mathbb{Z})$ congruent to the identity matrix mod N. It admits a compactification $X(N)$, called the *modular curve of level* N. The complement consists of

$$c_N := \frac{1}{2}N^2\prod_{p|N}(1-\frac{1}{p^2})$$

points, called *cusps.*. For $N \geq 3$, the moduli space is a fine moduli space and the universal curve

$$\pi : S(N) \to X(N)$$

over the moduli space exists. It is a *modular elliptic surface* of level N. For $N = 2$, the moduli space exists as a stack only.

Recall that an *elliptic surface* is a smooth projective algebraic surface S together with a morphism $f : S \to B$ (called an *elliptic fibration*) to a smooth projective curve B such that all fibers over an open non-empty subset are curves of genus one. An elliptic surface is called *relatively minimal* if any birational morphism $S \to S'$ over B is an isomorphism. It follows from the theory of algebraic surfaces that this is equivalent to that no fiber of f contains a (-1)-curve. All possible singular fibers of relatively minimal elliptic surfaces, considered as positive divisors, were described by Kodaira. They are distinguished by *Kodaira's types* $II, III, IV, I_n, I_n^*, II^*, III^*, IV^*$. Irreducible fibers have one ordinary double point (type I_1) or one ordinary cusps (type II). Other fibers consist of n components if type is I_n, $n+6$ components of type is I_n^*, two components if type III, three components if type IV, $n = 7, 8, 9$ if the type is II^*, III^*, IV^*. The intersection of these components is given by the following graphs:

Here we use another notation that comes from the theory of root systems of simple Lie algebras. The graphs which you see are Dynkin diagrams of affine root systems of types $\tilde{A}_n, \tilde{D}_n, \tilde{E}_6, \tilde{E}_7, \tilde{E}_8$.

Let E_η be the fiber of $f : S \to B$ over the generic point η of B (in the scheme-theoretical sense). This is a curve over the field of rational functions $\Bbbk(\eta)$ of the curve B. It has a rational point over this field if and only if the fibration f admits a section $s : B \to S$. We identify this section with its image, a smooth curve isomorphic to B under the morphism f. A curve E of genus one over a field K tsuch that $E(K) \neq \emptyset$ has a structure of an elliptic curve, a complete one-dimensional algebraic group. The structure of an algebraic group is determined by a choice of a rational point that serves as the unit element. With respect to this group structure the set of sections aquires a structure of a group, the *Mordell-Weil group* of the

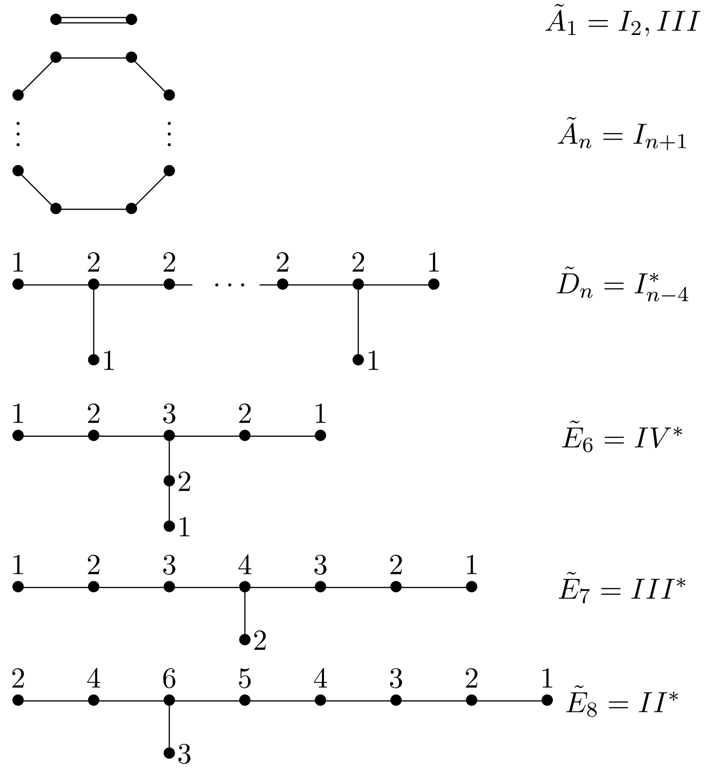


Figure 2.2: Reducible fibers of genus one fibration

elliptic surface. If the surface admits at least one singular fiber, the group is a finitely generated abelian group.

Let us return to modular elliptic surfaces. It follows from the definition of a universal family for a fine moduli space that that the Mordell-Weil group of such a surface contains a subgroup isomorphic to $(\mathbb{Z}/N\mathbb{Z})^2$. In fact, one can show that there is nothing else. A fiber over a point $x \in X(N)^o$ is an elliptic curve with level N whose isomorphism class is the point x. The level structure on all such curves is defined by N^2 disjoint sections which form a group isomorphic to $(\mathbb{Z}/N)^2$. If $N > 2$, a fiber over a cusp is a singular fiber of Kodaira's type I_N. The genus of a section C is equal to the genus of $X(N)$. A section C_0 of $S(N) \to X(N)$ is a curve isomorphic to the base $X(N)$. Its genus is equal to

$$g(X(N)) = 1 + \frac{c_N(N-6)}{12}.$$

Th self-intersection of a section C is equal to

$$C^2 = -c_N N/12.$$

The geometric genus $p_g := h^0(\mathcal{O}_{S(N)})$ of the surface $S(N)$ is equal

$$p_g(S_N) = \frac{c_N(N-3)}{6}.$$

Although for $N = 2$ a fine moduli space does not exist, one can define a close approximation to it, an elliptic surface $S(2)^* \to \mathbb{P}^1$ with Mordell-Weil isomorphic to $(\mathbb{Z}/2\mathbb{Z})^2$. The surface $S(2)^*$ is obtained from the del Pezzo surface $\mathcal{D}_5$ of degree 5 by the following construction. Let $\mathbf{U}$ be the union of the 10 lines on $\mathcal{D}_5$. Its divisor classe in the basis e_0, e_1, e_2, e_3, e_4 of $\mathrm{Pic}(S)$ defined by the class of $e_0 = \pi^*(\mathcal{O}_{\mathbb{P}^2}(1))$ and the classes of the exceptional curves $R_1, \ldots R_4$ is equal to

$$\sum_{i=1}^{4} e_i + \sum_{1 \le i < j \le 4} (e_0 - e_i - e_j) = 6e_0 - 2(e_1 + e_2 + e_3 + e_4). \tag{2.2.1}$$

The double cover of S branched along the divisor $\mathbf{U}$ is defined locally by $z_V^2 - \phi_V = 0$, where $\phi_V = 0$ are local equations of $\mathbf{U}$ in an affine open set V. It can be glued to a global cover if the divisor class of $\mathbf{U}$ is divisible by 2 in the divisor class group $\mathrm{Pic}(\mathcal{D}_5)$. As formula (**??**) shows, this is satisfied in our case. Let $X \to \mathcal{D}_5$ be such a cover. If $x \in U$ is a nonsingular point of $\mathbf{U}$, the local equation of $\mathbf{U}$ can be chosen to be $u = 0$, where u is a local parameter. Thus over this point the equation of X is given by $z^2 - u = 0$, it is a nonsingular open subset of X. On the other hand, if x is a singular point of $\mathbf{U}$, i.e. equal to one of the intersection points of two components of $\mathbf{U}$, the local equation can be chosen in the form $uv = 0$, thus $z^2 - uv = 0$ defined a surface that is singular of the point x. The singularity is as simple as possible. It is called an *ordinary double point*, an *ordinary node*, or a *rational double point* of type A_1. The elliptic surface $S(2)^*$ is obtained as a minimal resolution $\sigma : S(2)^* \to X$ of singularities of X. Its elliptic fibration is the pre-image of one of five pencils of conics on $\mathcal{D}_5$. They are the pre-images on $\mathcal{D}_5$ of the pencils of of lines through one of the points $p_1, \ldots, p_4$ or the pencil of conics through all of these points. The pre-images of singular points of X on $S(2)^*$ are smooth rational curves C_i with self-intersection -2 (a (-2)-*curve*, for short). A fiber of $S \to \mathbb{P}^1$ over cusps $0, 1, \infty$ consists of two (-1)-curves intersecting at one point. It is also intersected by four curves R_1, R_2, R_3, R_4. So, we see that X has 15 singular points, five over each fiber. The pre-image of the fiber of $X \to S \to \mathbb{P}^1$ over a cusp is a fiber of Kodaira's type I_3^*. The elliptic surface $S(2)^*$ is a *K3 surface*. i.e. an algebraic surface X satisfying $K_X = 0$ and $H^1(X, \mathcal{O}_X) = \{0\}$. It is characterized (among elliptic surfaces) by the property that $B \cong \mathbb{P}^1$ and

$$\sum_{b \in B} e(f^{-1}(b)) = 24, \tag{2.2.2}$$

where $e()$ denotes the topological Euler-Poincaré characteristic (computed in étale topology if $\Bbbk \neq \mathbb{C}$).

The choice of a zero section defines a group law on each nonsingular fiber. The involutions $y \mapsto -y$ on each fiber are glued together to define a global involution $T : S(N) \to S(N)$, the quotient $\mathrm{Kum}(N) = S(N)/(T)$by this involution is called the *universal Kummer family.* It comes with a projection map $\mathrm{Kum}(N) \to X(N)$ whose fibers over points in $X(N)^o$ are isomorphic to $\mathbb{P}^1$. For $N > 2$, the fibers over cusps are chains of $k = [N + 2/2]$ smooth rational curves $R_1 + R_2 + \cdots + R_k$ with $R_1^2 = R_k^2 = -1, R_j^2 = -2, j \neq i, k$ and $R_i \cdot R_{i+1} = 1, i = 1, \ldots, k-1$, all other intersections are zeros. Starting from R_1 we blow down the curves in each fiber until we get a minimal ruled surface.

2.3 Normal elliptic curves

Again we assume here that $\Bbbk$ is algebraically closed. Let Q_1, Q_2 be two quadrics in $\mathbb{P}^3$ such that $C = Q_1 \cap Q_2$ is a smooth curve. Projecting from a point $x_0 \in C$ to $\mathbb{P}^2$, we find that the image of C is a plane cubic. It is a curve of genus 1, an *elliptic curve.* The same fact follows from the *adjunction formula.* If we fix a smooth quadric Q in the pencil spanned by Q_1, Q_2 (we will see later that this is always possible if the intersection is smooth), then C is a divisor on Q given by a section of $\mathcal{O}_Q(2)$. The canonical sheaf of $\mathbb{P}^3$ is $\mathcal{O}_{\mathbb{P}^3}(-4)$, and the adjunction formula for Q gives $\omega_Q = \mathcal{O}_Q(-2)$, then the canonical sheaf of C is $\mathcal{O}_C$. This property is the definition of an elliptic curve.

Let E be an elliptic curve and p_0 be the point on E which we choose to be the zero point in the group law. By Riemann-Roch,

$$h^0(np_0) := \dim H^0(E, \mathcal{O}_E(np_0)) = n.$$

Let $(s_0, \ldots, s_{n-1})$ be a basis of the space $H^0(E, \mathcal{O}_E(np_0))$. Its elements have a pole of order n at p_0 and vanish at some set $(s_i)_0$ of n points in E. Assume

$n \geq 3$. The map

$$\phi : E \to \mathbb{P}^{n-1}, \quad x \mapsto [s_0(x), \dots, s_{n-1}(x)]$$

is a projective embedding of E in $\mathbb{P}^{n-1}$.

Note that the linear system $|np_0|$ of effective divisors linearly equivalent to np contains n^2 divisors $n(p_0 \oplus e)$, where $\oplus$ denotes the addition in the group law and $e \in E[n]$ is a n-torsion point. They must be cut out by hyperplanes in $\mathbb{P}^{n-1}$. They are called *osculating hyperplanes*. They intersect the image of E in $\mathbb{P}^{n-1}$ at one point with multiplicity n.

Assume that $(n, \mathrm{char}(\Bbbk)) = 1$. An elliptic curve E has n^2 points x such that $nx = 0$ in the group law. The group $E[n]$ of such points is isomorphic to $(\mathbb{Z}/n\mathbb{Z})^2$. If $\Bbbk = \mathbb{C}$, then $E = \mathbb{C}/\mathbb{Z} + \mathbb{Z}\tau$, where $\tau = a + bi, b > 0$. The n-torsion points are the cosets of the point $\frac{1}{n}(k + m\tau), 0 \leq k, m \leq n - 1$.

Let $G_n = (\mathbb{Z}/n\mathbb{Z})^2$. Consider its projective representation in a vector space V with a basis $(e_0, \dots, e_{n-1})$ that is given on generators $\sigma = (1, 0)$ and $\tau = (0, 1)$ by the formual

$$\sigma : e_i \to e_{i-1}$$
$$\tau : e_i \mapsto \epsilon_n^i e_i,$$

where $i \in \mathbb{Z}/n\mathbb{Z}$ and ϵ_n is a generator of the group $\mu_n(\Bbbk) = \{a \in \Bbbk : a^n = 1\}$. It is an irreducible projective representation that originates from the *Schrödinger representation* of the *Heisenberg group* H_n that fits in the exact sequence

$$1 \to \mu_n \to H_n \to (\mathbb{Z}/n\mathbb{Z})^2 \to 1.$$

Here $[\sigma, \tau] = \epsilon_n \,\mathrm{id}$, so group H_n is not commutative but rather nilpotent.

Proposition 2.1. *There exists a unique basis in $H^0(E, \mathcal{O}_E(np_0))$ such that the embedding of E in $\mathbb{P}^{n-1}$ defined by this basis is invariant with respect to the Schrödinger representation.*

Proof. Since under the addition map $\tau_\alpha : p \mapsto p \oplus \alpha$, where $\alpha \in E[n]$, the divisor class np does not change, we see that invertible sheaf $\mathcal{L} = \mathcal{O}_E(np)$ satisfies $\tau_\alpha^*(\mathcal{L}) \cong \mathcal{L}$. Let ϕ_α be an isomorphism $\tau_\alpha^*(\mathcal{L}) \to \mathcal{L}$. It is defined uniquely up to an automorphism of $\mathcal{L}$ which is given by a scalar automorphism $c : \mathcal{O}_E \to \mathcal{O}_E$. It is immediately checked that the set of automorphisms (ϕ_α) satisfies

$$\phi_{\alpha+\beta} = c_{\alpha,\beta}\phi_\beta^*(\tau_\alpha) \circ \phi_\alpha,$$

where $c_{g,g'}$ is a nonzero constant. The set $(c_{\alpha,\beta})_{\alpha,\beta\in G}$ is a 2-cocycle whose cohomology class belongs to $H^2(G, \Bbbk^*)$. Computing the cohomology group, we find that this cocycle can be chosen to be equal to the cocycle defined by $(\alpha,\beta)\mapsto[\alpha,\beta]$. By a general yoga of group cohomology, this implies that the projective representation $\rho:G\to \mathrm{PGL}(H^0(E,\mathcal{L}))$ defined by $\rho(\alpha)(s)=\phi_\alpha(\tau_\alpha^*(s))$ lifts to a linear representation of H_n in $H^0(E,\mathcal{L})$. It follows from the definitions of the sheaves $\tau_\alpha^*(\mathcal{L})$, that it is compatible with the action of its quotient G on E. The latter means that, for all $s\in H^0(E,\mathcal{L})$ and $x\in E$,

$$h^*(s)(x)=s(\bar{h}^{-1}(x)),$$

where $\bar{h}$ is the image of h in G.

Now, we see that there exits a central extension $\tilde{E}[n]$ of $E[n]$ isomorphic to H_n that admits a linear representation in $H^0(E,\mathcal{L})$. By means of the dual representation, it acts in the projective space $\mathbb{P}(H^0(E,\mathcal{L}))\cong\mathbb{P}^{n-1}$ in which E is embedded by means of the linear system $|np_0|$. The action factors through the group $E[n]$ and defines a projective representation of $E[n]$ in $\mathbb{P}^{n-1}$.

One can show that the Schrödinger V representation of H_n is a unique (up to isomorphism) irreducible linear representation on which the subgroup μ_n acts in a natural way by scalar multiplication. It also can be characterized by the property that, for any cyclic subgroup of order n of H_n that projects isomorphically to G, the linear subspace V^K of invariant elements is one-dimensional. Our representation of $E[n]$ on $H^0(E,\mathcal{O}_E(np_0))$ satisfied these properties. The last property is shown as follows. We choose a cyclic subgroup K of $E[n]$ that can be isomorphically lifted to a subgroup of $\tilde{E}[n]$. Under an isomorphism $\tilde{E}[n]\to H_n$, it can be taken to be the subgroup $\langle\sigma\rangle$ or $\langle\tau\rangle$. Let $f:E\to E/K$ be the quotient map. The curve $E'=E/K$ is an elliptic curve isogenous to E. Since $\mathcal{L}$ admits a linearization with respect to K, we see that $\mathcal{L}=f^*\mathcal{M}$, for some invertible sheaf $\mathcal{M}$. Since $\deg\mathcal{L}=n$, we have $\deg\mathcal{M}=1$. By Riemann-Roch, $\dim H^0(E',\mathcal{M})=1$. Also, we have $H^0(E',\mathcal{M})=H^0(E,\mathcal{L})^K$.

Thus we have proved that one can choose an isomorphisms $\phi:H_n\to\tilde{E}[n]$ that descends to a level n-structure $G_n=(\mathbb{Z}/n\mathbb{Z})^{\oplus 2}\to E[n]$ such that the linear action of $\tilde{E}[n]$ on $H^0(E,\mathcal{O}_E(np_0))$ is isomorphic to the Schrödinger representation of H_n in V.

□

Remark 2.2. What we have glimpsed in during this proof is the general theory of *linearization* of an action of an algebraic group G on an algebraic variety X (not necessary faithful). It is a choice of a sheaf $\mathcal{F}$ such that there exist isomorphisms $\phi_g:g^*(\mathcal{F})\to\mathcal{F}, g\in G$ satisfying $\phi_{g'\circ g}=\phi_{g'}^*(\phi_g)\circ\phi_g$.

This allows to define a linear representation of G in the space of sections of $\mathcal{F}$. We refer for the details to [Mumford, Geometric Invariant Theory].

Let $[-1] : E \to E$ be the negation involution of an elliptic curve with respect to its group law. It satisfies

$$[-1] \circ t_a = t_{-a} \circ [-1], \tag{2.3.1}$$

where $t_a : E \to E$ is the translation map $x \mapsto x \oplus a$. Let V be the vector space $H^0(E, \mathcal{O}_E(np_0))$ with a chosen basis $(e_0, e_1, \ldots, e_{n-1})$ as above. Since $[-1](p_0) = p_0$, the sheaf $\mathcal{O}_E(np_0)$ admits a canonical linearization with respect to $[-1]$ (in fact, one can show that an invertible sheaf $\mathcal{O}_C(D))$ on an algebraic curve C on which a finite group G acts admits a linearization with respect to G if and only if $g(D) = D$ for all $g \in G$). Let $\iota : V \to V$ be the linear action of $[-1]$ on V. The identity (**??**) gives, for all $g \in G$.

$$\iota \circ g = g^{-1} \circ \iota.$$

It is easy to write the possible matrix of ι satisfying these identity. We get

$$\iota : t_i \mapsto t_{-i}.$$

The linear space V decomposes into two eigensubspaces $V^{\pm}$ with respect to ι with eigenvalues ± 1. We have

$$\begin{aligned} V^+ &= \langle e_0, e_1 + e_{n-1}, e_2 + e_{n-2}, \ldots, e_{[\frac{n}{2}]} + e_{[\frac{n+1}{2}]}\rangle, \\ V^- &= \langle e_1 - e_{n-1}, e_2 - e_{n-2}, \ldots, e_{[\frac{n}{2}]} - e_{[\frac{n+1}{2}]}\rangle, \end{aligned}$$

where the last vector in V^- should not be ignored if n is even. In particularly, we find that

$$\dim V^+ = 1 + [\frac{n}{2}], \quad \dim V^- = n - 1 - [\frac{n}{2}].$$

If $n = 2k+1$ is odd, we get $\dim V^{\pm} = \frac{1}{2}(n \pm 1)$. If $n = 2k$ is even, we get $\dim V^+ = 1 + k, \dim V^+ = k - 1$. Let $V_g^- = g(V^-), g \in G$. We have n^2 such subspaces, and hence n^2 corresponding linear projective subspaces in $\mathbb{P}(V)$ of dimension equal to $\dim V^- - 1$. Let C be a coset of some cyclic subgroup of $E[n]$. The image of C in $\mathbb{P}(V)$ consists of n points that span a hyperplane H_C.

Assume that n is an odd number. There are $n+1$ cyclic subgroups and $n(n+1)$ cosets. Thus we have a configuration of $n(n+1)$ hyperplanes H_C and

p^2 subspaces $\mathbb{P}(V_g^-)$ of dimension $\frac{1}{2}(n-3)$. Each H_C contains n subspaces $\mathbb{P}(V_g^-)$ and each $\mathbb{P}(V_g^-)$ is contained in $n+1$ hyperplanes H_C. This realizes an *abstract configuration* $(n(n+1)_n, n^2_{n+1})$. An abstract configuration (a_c, b_d) is a relation on two sets A, B of cardinalities a, b such that each element of A is related to c elements of B, and each element of B is related to d elements in A. The subspaces $\mathbb{P}(V^-)$ intersects $\phi(E)$ at the unique fixed point, the origin (here we use that n is odd). The other subspaces $\mathbb{P}(V_g^-)$ intersects $\phi(E)$ at other n-torsion points.

When $\Bbbk = \mathbb{C}$, one constructs the basis in $H^0(E, \mathcal{O}_E(np))$ explicitly by using the theory of Riemann theta functions or Weierstrass σ-functions on $\mathbb{C}$. Let

$$\sigma(z) := z \prod_{r\in\Lambda\setminus\{0\}} (1-\frac{z}{r})e^{\frac{z}{r}+\frac{z^2}{2r^2}},$$

where $\Lambda = \mathbb{Z} + \mathbb{Z}\tau$. It satisfies

$$\begin{aligned} \sigma(z+1) &= -e^{\eta_1(z+\frac{1}{2})}\sigma(z), \\ \sigma(z+\tau) &= -e^{\eta_2(z+\frac{\tau}{2})}\sigma(z). \end{aligned}$$

for some $\eta_1, \eta_2 \in \mathbb{C}$. Let

$$\sigma_{pq}(z) := \sigma(z - \frac{p+q\tau}{n}), \quad p, q \in \mathbb{Z},$$

$$\omega_1 := -e^{-\frac{n-1}{2}\frac{\eta_2\omega_1}{n}}, \quad \omega_2 := e^{-\frac{\eta_1}{2n}}.$$

Then one defines the sections $s_m, m = 0, \ldots, n-1$ by

$$s_m = \omega_1^m \omega_2^{m^2} e^{m\eta_1 z}\sigma_{m,0}(z)\cdots\sigma_{m,n-1}(z).$$

Example 2.3. Assume $n = 3$. Then the ring of invariant polynomial $P(t_0, t_1, t_2)$ with respect to the Schrödinger representation is generated by $t_0^3, t_1^3, t_2^3, t_0t_1t_2$ and the equation of $\phi(E)$ in $\mathbb{P}^2$ (after scaling the coordinates) is a *Hesse equation*

$$t_0^3 + t_1^3 + t_2^3 + \lambda t_0 t_1 t_2 = 0,$$

where $8\lambda^3 + 1 \neq 0$ if the cubic is nonsingular. The osculating hyperplanes in this case are nine *flex tangent lines* that are tangent to the cubic curve at one point. The corresponding points are *inflection points* of the cubic. Their coordinates do not depend on the parameter and they lie on the coordinate lines $t_i = 0$. We may assume that the points on the line $t_0 = 0$ form a subgroup H of $E[3]$. Then points on the other two lines form two nontrivial cosets with respect to this subgroup. There are 4 proper subgroups of G, and there are 12 cosets altogether. Each contain three collinear inflection points. The corresponding lines are given by equations

$$x = 0,\ y = 0,\ z = 0,\ H_{i,j} := t_0 + \epsilon_3^i t_1 + \epsilon_3^j t_2 = 0, \quad 0 \leq i, j \leq 2.$$

The two sets of 9 points and 12 lines form the famous abstract *Hesse configuration* $(12_3, 9_4)$. There are 4 singular members in the Hesse pencil. One is given by equation $t_0t_1t_2 = 0$ corresponding to the parameter $\lambda = \infty$. Other fibers are also triangles of lines, they correspond to the parameters λ satisfying $8\lambda^3 + 1 = 0$. The blow-up of 9 base points of the pencil is the modular elliptic surface $S(3)$. The elliptic fibration is given by the resolving the indeterminacy points of the rational map

$$\mathbb{P}^2 \dashrightarrow \mathbb{P}^1, \quad [t_0, t_1, t_2] \mapsto [t_0^3 + t_1^3 + t_2^3, t_0t_1t_2].$$

Its singular members are of Kodaira's type I_3

Example 2.4. Assume $n = 4$. Let $\phi(E) = C \subset \mathbb{P}^4 = |V^\vee|$, where $V = H^0(E, \mathcal{O}_E(4p))$, be invariant quartic curve with respect to the Schrödinger representation. Let Q be a quadric in $\mathbb{P}^4$, it either contains C or cuts out a divisor of degree 8 on C equal to the zero scheme of a section of $\mathcal{O}_C(2)$. By Riemann-Roch, $h^0(\mathcal{O}_C(2)) = 8$. Since there are 10 linear independent quadratic forms in 4 variables, we see that the restriction homomorphism

$$r : H^0(C, \mathcal{O}_{\mathbb{P}^3}(2)) \to H^0(C, \mathcal{O}_C(2))$$

has two-dimensional kernel. Thus there exists two linear independent quadrics $Q_1 = V(q_1), Q_2 = V(q_2)$ that contain C. Since the degree of $Q_1 \cap Q_2$ is equal to 4, we obtain that $C = Q_1 \cap Q_2$.

Since C is invariant with respect to the Schrödinger representation, the pencil $\langle q_1, q_2 \rangle$ must be invariant two with respect to the representation of the Heisenberg group in the symmetric square of $V = H^0(C, \mathcal{O}_C(2)) \cong \Bbbk^4$. It is easy to see that

$$S^2V = \oplus_{i=1}^5 V_i,$$

where V_i are irreducible sub-representations given explicitly by

$$V_1 = \langle t_0^2 + t_2^2, t_1^2 + t_3^2 \rangle, \quad V_2 = \langle t_0^2 - t_2^2, t_1^2 - t_3^2 \rangle, \quad V_3 = \langle t_0t_3, t_1t_2 \rangle,$$

$$V_4 = \langle t_0t_1 + t_2t_3, t_0t_2 + t_1t_3 \rangle, \quad V_5 = \langle t_0t_1 - t_2t_3, t_0t_2 - t_1t_3 \rangle.$$

Note that V_1 and V_3 are isomorphic representations, the generators σ and τ act similarly on their bases. So, our quartic curve C must be the intersection of two quadrics either spanning V_4 or V_5 or to be contained in $V_1 \oplus V_3$. The two quadrics can not spend V_4 or V_5 since we find, looking at the matrix of partials, that the intersection point $[1,0,1,0]$ will be a singular point. It cannot be equal neither to V_2 nor to V_1 or V_3. It is easy to see that the only possibility is that

$$C(\lambda) : t_0^2 + t_2^2 + \lambda t_1 t_3 = t_1^2 + t_3^2 + \lambda t_0 t_2 = 0.$$

The parameter λ here satisfies

$$\lambda(\lambda^4 = 1) \neq 0, \lambda \neq \infty.$$

Thus there are six singular members in the family. Each is isomorphic to the union of four lines forming a quadrangle. In the case $n = 4$, we have $n+1$ obvious cyclic subgroups generated by $(1,0),(1,1),(1,2),(1,3),(0,1)$ of $G_4 = (\mathbb{Z}/4\mathbb{Z})^{\oplus 2}$ as well as a new subgroup generated by $(2,1)$. So, the configuration of cosets H_C and subspaces $\mathbb{P}(V_g^-)$ realizes an abstract configuration $(24_4, 16_6)$. Note that, since n is even in this case, the point $\mathbb{P}(V^-)$ is not on the embedded curve $\phi(E)$. The fixed points of the involution $[-1]$ are four 2-torsion points, and they all lie in the hyperplane $\mathbb{P}(V^+)$ which intersects the quartic curve at four points.

Let S be the union of all curves $C(\lambda), \lambda \in \mathbb{P}^1$. Fix a line ℓ in $\mathbb{P}^3$. The quadrics $Q_1(\lambda)$ intersect ℓ at two points, and then λ moves in $\mathbb{P}^1$ and form a linear series of degree 2. Another degree 2 series of degree 2 is formed by intersecting $Q_2(\lambda)$ with ℓ. The graphs of the corresponding maps of degree 2 $\ell \to \mathbb{P}^1$ are curves of bi-degree $(2,1)$ on $\mathbb{P}^1 \times \mathbb{P}^1$. They intersect at 4 points. This implies that ℓ intersects four curves in the family, and hence S is a quartic surface in $\mathbb{P}^3$. It is a modular elliptic surface $S(4)$ of level 4. It contains 6 singular fibers of Kodaira's type I_4, the modular cuve $X(4)$ is of genus 0, hence, applying formula (**??**) we obtain that $S(4)$ is a K3 surface.

Example 2.5. Finally, assume that $n = 5$. Here we skip the details referring to [Hulek, Asterisques]. The curves C are given by intersection of 5 quadrics in $\mathbb{P}^4$.

$$
\begin{aligned}
Q_0 &= t_0^2 + \lambda t_2 t_3 - \lambda^{-1} t_1 t_4 = 0, \\
Q_1 &= t_1^2 + \lambda t_3 t_4 - \lambda^{-1} t_0 t_2 = 0, \\
Q_2 &= t_2^2 + \lambda t_0 t_4 - \lambda^{-1} t_1 t_3 = 0, \\
Q_3 &= t_3^2 + \lambda t_0 t_1 - \lambda^{-1} t_2 t_3 = 0, \\
Q_4 &= t_4^2 + \lambda t_1 t_2 - \lambda^{-1} t_0 t_3 = 0,
\end{aligned}
$$

These are normal quintic elliptic curves. One can show that $C(\lambda)$ is equal to the intersection of the Grassmann variety $G(2,5) \subset \mathbb{P}^9$ with a linear subspace of dimension 4. The linear subspaces of dimension 4 in $\mathbb{P}^9$ are parameterized by the Grassmann variety $G(5,10)$ of dimension 25. The group of projective automorphisms of $\mathbb{P}^5$ is of dimension 24. Thus the number of projective moduli of elliptic quintics is equal to 1, as expected. There are 12 singular curves among $C(\lambda)$. Each consists of a pentagon of lines. The union of the curves $C(\lambda)$ is an elliptic modular surface $S(5)$. The parameter a belongs to the modular curve $X(5)$ of genus 0. The surface $S(5)$ has $p_g := h^0(\omega_{S(5)}) = 4$.

One can prove that, for any $n \geq 4$, a normal elliptic curve of degree n is given by a linear system of quadratic equations.

Lecture 3

Pencils of quadrics

3.1 Discriminant hypersurface

Let $|S^2E^\vee|$ be the projective space of quadrics in $|E| \cong \mathbb{P}^n$. The reduced subvariety D_n of $|S^2E^\vee|$ that consists of singular quadrics is called the *discriminant hypersurface*. Choose a basis $e_1, \ldots, e_{n+1}$ in E with the dual basis $(t_1, \ldots, t_{n+1})$ in $E^\vee$. The basis in $S^2E^\vee$ consists of monomials t_it_j and the coordinates in $S^2E^\vee$ are generic coefficients A_{ij} of quadratic forms. If $\mathrm{char}(\Bbbk) \neq 2$, the equation of the discriminant hypersurface is

$$\det(B_{ij}) = 0,$$

where $B_{ii} = 2A_{ii}$ and $B_{ij} = B_{ji} = A_{ij}$. It is a hypersurface of degree $n+1$. If n is even, and $\mathrm{char}(\Bbbk) = 2$, then the equation of D_n is

$$\sum_{1 \le i \le j \le n+1} A_{ij} \,\mathrm{Pf}(A)_i \,\mathrm{Pf}(A_j) = 0,$$

where $\mathrm{Pf}(A)_i$ is the Pfaffian of the matrix obtained from the matrix (B_{ij}) from above by deleting the ith row and the ith column. Again, this is a hypersurface of degree $n+1$. If n is odd and $\mathrm{char}(\Bbbk) = 2$, we take for the equation of D_n the Pfaffian of the matrix $B = (B_{ij})$. This is a hypersurface of degree $\frac{1}{2}(n+1)$.

Let us look at the singularities of D_n. We assume here that $\mathrm{char}(\Bbbk) \neq 2$. Let $\tilde{\mathsf{D}}_n$ be the subscheme of $|S^2E^\vee| \times |E|$ given by $n+1$ bi-homogeneous equations

$$B \cdot \mathbf{t} = 0,$$

where $\mathbf{t}$ is the column-matrix with entries $(t_1, \ldots, t_{n+1})$. It is clear that its $\Bbbk$-points are pairs (Q, x), where Q is a quadric and x is its singular point.

There are two projections

$$\begin{array}{ccc} & \tilde{\mathsf{D}}_n & \\ {}^{p_1}\swarrow & & \searrow^{p_2} \\ \mathsf{D}_n & & |E| \end{array}.$$

The fiber of the first projection over a quadric Q is its singular locus, a linear subspace of $|E|$. The fiber of the second projection over a point $x \in |E|$ is a linear subspace of $|S^2E^\vee|$ of quadrics that contain x in its singular locus. It is a linear projective subspace of dimension equal to $\dim |S^2E^\vee| - n - 1 = \binom{n+1}{2} - n - 1$. It follows that the scheme $\tilde{\mathsf{D}}_n$ is smooth of dimension equal to $\dim |S^2E^\vee| - 1 = \frac{1}{2}(n^2 + 3n - 2)$. Thus the projection p_1 is a *resolution of singularities* of the discriminant variety, i.e. a proper morphism of a smooth scheme which is an isomorphism over an open subset of smooth points.

The following proposition follows from the theory of *determinantal varieties*, we omit the proof.

Proposition 3.1. *Let* $\mathrm{SM}_m(k)$ *be the affine variety of symmetric matrices of size* m *and corank* $\geq k$. *Then*

- $\mathrm{SM}_m(k)$ *is an irreducible Cohen-Macaulay subvariety of codimension* $\frac{1}{2}k(k+1)$;
- $\mathrm{Sing}(\mathrm{SM}_m(k)) = \mathrm{SM}_m(k+1)$;
- $\deg \mathrm{SM}_m(k) = \prod_{0 \leq i \leq k-1} \frac{\binom{m+i}{k-i}}{\binom{2i+1}{i}}$.

This gives the stratification of the singular locus $\mathrm{Sing}(\mathsf{D}_n)$ of the discriminant hypersurface:

$$\mathrm{Sing}(\mathsf{D}_n) = \mathrm{Sing}(\mathsf{D}_n)_0 \supset \mathrm{Sing}(\mathsf{D}_n)_1 \supset \ldots \supset \mathrm{Sing}(\mathsf{D}_n)_{n-1} \supset \emptyset,$$

where $\mathrm{Sing}(\mathsf{D}_n)_t$ is the closed subvariety parameterizing quadrics Q with singular locus of dimension t. Here each $\mathrm{Sing}(\mathsf{D}_n)_t$ is the singular locus of $\mathrm{Sing}(\mathsf{D}_n)_{t-1}$. We also have

$$\mathrm{codim}(\mathrm{Sing}(\mathsf{D}_n)_t, \mathsf{D}_n) = \frac{1}{2}(t+1)(t+2) - 1 = \frac{1}{2}t(t+3).$$

We can also describe the tangent space of D_n at its smooth point Q_0. Let $\mathrm{Sing}(Q) = \{x_0\}$. Since p_1 is an isomorphism over Q_0, the tangent space is isomorphic to the tangent space of $\tilde{\mathsf{D}}_n$ at the point (Q_0, x_0). The description of the second projection shows that it is isomorphic to the tangent space of the fiber of p_2 over the point x_0 (it is certainly the subspace of the tangent space and its dimensions agree, so it must be the whole space). Thus we obtain

Proposition 3.2. *Let Q_0 be a smooth point of D_n. Then the tangent space of D_n at the point Q_0 is naturally isomorphic to the space of quadrics that contain the singular point of Q_0.*

The projective space $|S^2E^\vee|$ is the complete linear system of quadrics $|\mathcal{O}_{|E|}(2)|$ in the projective space $|E| = \mathbb{P}(E^\vee)$. Let $|L| \subset |S^2E^\vee|$ is a linear system of quadrics in $|E|$ of dimension r. If $r = 1, 2, 3, > 3$, we say that $|L|$ is a *pencil*, a *net*, a *web*, a *hyperweb*.

Let $\mathsf{D}(|L|)$ be the intersection of $|L|$ with the discriminant hypersurface D_n. We say that L is a *regular* if $|L|$ intersects D_n *transversally*. This means that, for all $t \geq 0$, the intersection $|L| \cap (\mathrm{Sing}(\mathsf{D}_n)_t \setminus \mathrm{Sing}(\mathsf{D}_n)_{t+1})$ is smooth of codimension in $|L|$ equal to the codimension of $\mathrm{Sing}(\mathsf{D}_n)_t$ in $|S^2E^\vee|$. This is true when $|L|$ is chosen to be generic, i.e. belongs to some open subset in the Grassmannian $G(r+1, |S^2E^\vee|)$.

Example 3.3. Let $n = 2$. Assume $\mathrm{char}(\Bbbk) \neq 2$. The discriminant hypersurface D_2 is a cubic hypersurface in $\mathbb{P}^5$ given by equation

$$\det \begin{pmatrix} 2A_{11} & A_{12} & A_{13} \\ A_{12} & 2A_{22} & A_{23} \\ A_{13} & A_{23} & 2A_{33} \end{pmatrix} = 0.$$

Its singular locus is the Veronese surface of degree 4, the image of the map $\mathbb{P}^2 \to \mathbb{P}^5$ given by

$$(x, y, z) \mapsto [A_{11}, A_{22}, A_{33}, A_{12}, A_{13}, A_{23}] = [x^2, y^2, 2z^2, 2xy, 2xz, 2yz].$$

Assume $p = 2$. Then the discriminant hypersurface is again a cubic hypersurface. Its equation is

$$A_{11}A_{23}^2 + A_{22}A_{13}^2 + A_{33}A_{12}^2 + A_{12}A_{23}A_{13} = 0.$$

Its singular locus is given by equations $A_{12} = A_{13} = A_{23} = 0$. It is a linear subspace of dimension 2.

3.2 Pencils of quadrics

Assume $r = 1$, i.e. $|L|$ is a pencil of quadrics. Let $|L|$ be generated by quadrics Q_1, Q_2 and $X = Q_1 \cap Q_2$ be the base locus of $|L|$. The line $|L|$ intersects the hypersurface D_n at $\deg \mathsf{D}_n = n+1$ points counting with multiplicities.

Theorem 3.4. *Assume* $\mathrm{char}(\Bbbk) \neq 2$. *The following properties are equivalent:*

(i) *The variety X is nonsingular;*

(ii) $\mathsf{D}_n(|L|)$ *consists of $n+1$ distinct points.*

(iii) *X is projectively isomorphic over $\bar{\Bbbk}$ to the variety defined by the equations*

$$\sum_{i=1}^{n+1} t_i^2 = \sum_{i=1}^{n+1} a_i t_i^2, \tag{3.2.1}$$

where $a_i \neq a_j, i \neq j$.

Proof. (i) $\Rightarrow$ (ii). Suppose $|L|$ consists of less than $n+1$ points. This means that $|L|$ either intersects D_n at its singular point, or belongs to the tangent space of D_n at its nonsingular point. In the first case $|L|$ contains a quadric Q with $\dim \mathrm{Sing}(Q) \geq 1$. This implies that any other quadric Q' in the pencil intersects $\mathrm{Sing}(Q)$ at some point x. Let $\mathbb{T}_x(X)$ denote the tangent space of a projective subvariety $X \subset \mathbb{P}^n$ at its point x. It is a projective subspace of $\mathbb{P}^n$ of dimension equal to $\dim_x(X)$ if x is a nonsingular point of X. We have

$$\mathbb{T}_x(X) = \mathbb{T}_x(Q) \cap \mathbb{T}_x(Q') = \mathbb{T}_x(Q'), \tag{3.2.2}$$

hence $\dim \mathbb{T}_x(X)$ is larger than the dimension of X (equal to $n-2$). Thus x is a singular point of X. This contradiction proves the implication.

(ii) $\Rightarrow$ (i) Suppose X is singular at some point x. Then, it follows from (**??**) that any two smooth quadrics in $|L|$ are tangent at some point x (i.e. their tangent spaces at this point coincide). If we write $Q = V(q), Q' = V(q'), x = [v]$, then this means that the linear forms $b_q(v)$ and $b_{q'}(v)$ are proportional (recall, that we view a bilinear form as a linear map $E \to E^\vee$). Thus, a linear combination of q, q' defines a quadric Q that is singular at the point x. So, we see that all quadrics in $|L|$ contain a singular point of Q. Applying Proposition **??**, we find that $|L|$ is contained in the tangent space of D_n at the point Q, and hence it intersects D_n at Q with multiplicity larger than 1. This contradiction proves the implication.

(ii) $\Rightarrow$ (iii) We know that a nonsingular quadric can be reduced to the form given by one of the equations (**??**). Since $\mathrm{char}(\Bbbk) \neq 2$, we can reduce any of these two equations to the sum of squares $\sum_{i=1}^{n+1} t_i^2$. Since $|L|$ contains a nonsingular quadric Q_1, we can reduce its equation to this form. Let A be the matrix of the bilinear form b_q for some other nonsingular quadric $Q_2 = V(q)$ different from Q_1. Then $\mathsf{D}(|L|)$ consists of points $[1, \lambda] \in \mathbb{P}^1$ such that $\det(A - \lambda I_{n+1}) = 0$. Since it consists of $n+1$ different points, we see that the matrix A has $n+1$ distinct eigenvalues. Since A is symmetric, there exists a matrix C such that $C^t \cdot A \cdot C$ is a diagonal matrix Λ with distinct entries (belonging to $\bar{\Bbbk}$) at the diagonal. To do this, we first find S such that $S^{-1} \cdot A \cdot S = \Lambda$, then use that A is symmetric to obtain that $({}^tS \cdot S) \cdot \Lambda \cdot ({}^tS \cdot S)^{-1} = \Lambda$, hence ${}^tS \cdot S$ is diagonal, and then scale the coordinates to obtain that $({}^tS \cdot S) = I_{n+1}$. Thus we can find a basis such that the quadrics Q_1 and Q_2 are given by the equation in the assertion (iii).

(iii) $\Rightarrow$ (i) Obviously the quadrics $V(a_i q_1 - q_2)$ are singular with isolated singular point, and the number of them is equal to $n+1$. □

3.3 Segre's symbol

The classification of pencils of quadrics with singular base locus is given by the *Segre symbol*

$$[(e_1^{(1)} \dots e_1^{(s_1)})(e_2^{(1)} \dots e_2^{(s_2)}) \dots (e_r^{(1)} \dots e_r^{(s_r)})].$$

For any $e \geq 1$ and $\alpha \in \Bbbk$ consider the following quadratic forms:

$$\begin{aligned} p(\alpha, e) &= \alpha \sum_{i=1}^{e} t_i t_{e+1-i} + \sum_{i=1}^{e-1} t_{i+1} t_{e+1-i}, \\ q(e) &= \sum_{i=1}^{e} t_i t_{e+1-i}. \end{aligned}$$

Theorem 3.5. *Assume* $\mathrm{char}(\Bbbk) \neq 2$ *and* $\Bbbk$ *is algebraically closed. Assume that one of the quadrics in the pencil is nonsingular. Any pair of non-proportional quadratic forms* q_1, q_2 *can be written in some coordinates in the form*

$$\begin{aligned} q_1 &= \sum_{i=1}^{r} \sum_{j=1}^{s_i} p(\alpha_i, e_i^{(j)}), \\ q_2 &= \sum_{i=1}^{r} \sum_{j=1}^{s_i} q(e_i^{(j)}), \end{aligned}$$

Proof. A proof can be found, for example, in [Hodge-Pedoe, vol. 2] or [Gantmacher]. Let us give it, for completeness sake.

Let $R = \Bbbk[t]$ (or any principal ideal ring) and M be matrix with entries in R. Recall that the definition of *elementary divisors* of M (see, for example, [Gelfand, Linear Algebra]). Let p_k be monic polynomials equal to the greatest common divisors of minors of M of size k. We have $P_1|P_2|\cdots P_n$ where n is the smallest of the numbers of rows or columns of M. Each P_k is the product of some powers of irreducible polynomials. These powers collected from each P_k are called the elementary divisors of M. Let $T : E \to E$ be an endomorphism of a linear space of dimension n over $\Bbbk$. Define a structure of a R-module on E by setting $t \cdot v = T(v), v \in E$. A fundamental theorem about modules over a principal ring (that can be found in many books on abstract algebra) gives that E is isomorphic to the direct sum of modules of the form $R/(p_i^{e_j})$, where $p_i^{e_j}$ are elementary divisors of the characteristic matrix $t - A$, where A is the matrix of T in some basis of E. Note that the product of elementary divisors of a square matrix is equal to the determinant of the characteristic matrix.

Let $E[t] = \Bbbk[t] \otimes_{\Bbbk} E$ be the R-module obtained from E by extension of scalars. Let We have an exact sequence of R-modules

$$0 \to E[t] \xrightarrow{\psi} E[t] \xrightarrow{\phi} E \to 0,$$

where

$$\psi(f(t) \otimes v) = tf(t) \otimes v - f(t) \otimes T(v), \quad \psi(f(t) \otimes v) = f(T)(v).$$

It is immediately checked that the sequence is exact (or, see [Bourbaki, Algebra, Chapter 3]). So, one can interpret E as the cokernel of the map of free modules over R defined by the endomorphism $t - T$. Choosing a basis in E, we identify T with a matrix A, and the endomorphism is given by the characteristic matrix $t - A$. The decomposition of E into the sum of primary modules of the form $k[t]/(p_i^{k_j})$, where $p_i^{k_j}$ are the elementary divisors of the characteristic matrix $t - A$.

Let us apply this to our situation. We choose a basis (q_1, q_2) of L and a basis in E such that b_{q_1}, b_{q_2} are defined by symmetric matrices A, B such that A is invertible and consider the endomorphism of E given by the matrix $A^{-1} \cdot B$. The characteristic matrix $t - A^{-1} \cdot B$ obtained from the matrix $At - B$ by multiplication by a scalar matrix. It is easy to see the elementary divisors of the matrix $At - B$ doe not depend on a choice of a basis in L nor do they depend on a choice of a basis in E. Thus they are invariants of the pencil.

We shall show that two pencils are projectively equivalent if their elementary divisors are the same. The assertion of the theorem will follow, since the canonical forms represent all possible collections of elementary divisors.

Suppose two pencils defined by two pairs of symmetric matrices (A, B) and (A', B') as above have the same elementary divisors. Then they define isomorphic structures of $\Bbbk[t]$-module on E. Let S be the matrix of this transformation. It satisfies $A^{-1} \cdot B = S^{-1} \cdot A'^{-1} \cdot B' \cdot S$. Let $P = A \cdot S^{-1} \cdot A'^{-1}$. Then, we have

$$B = P \cdot B' \cdot S, \quad A = P \cdot A' \cdot S.$$

It is not what we want, we need a matrix Q such that ${}^tQ \cdot A \cdot Q = A'$ and ${}^tQ \cdot B \cdot Q = B'$. It will define the needed projective transformation. Let us find such a matrix.

Since A, B, A', B' are symmetric, we get

$${}^tS \cdot B' \cdot {}^tP = B, \quad {}^tS\dot{A}' \cdot {}^tP = A.$$

This implies that

$$({}^tS \cdot P^{-1}) \cdot A \cdot (S^{-1} \cdot {}^tP) = A$$

and similar equality for B. Let $M = {}^tS \cdot P^{-1}$. We have

$$M \cdot A = A \cdot {}^tM, \quad M \cdot B = B \cdot {}^tM.$$

Obviously, we have $f(M) \cdot A = A \cdot {}^tf(M), f(M) \cdot B = B \cdot {}^tf(M)$ for any polynomial $f(t)$. Choose $f(t)$ such that $f(M)^2 = M$. It is always possible since the matrices $I_{n+1}, \ldots, M^{(n+1)^2}$ are linearly dependent. Let $N = f(M)$. Then, we have $N \cdot A = A \cdot {}^tN, N \cdot B = B \cdot {}^tN$, hence

$$A' = P^{-1} \cdot A \cdot S^{-1} = {}^tS^{-1} \cdot M \cdot A \cdot S^{-1}$$

$$= {}^tS^{-1} \cdot N^2 \cdot A \cdot S^{-1} = ({}^tS^{-1} \cdot N) \cdot A \cdot ({}^tN \cdot S^{-1}).$$

Now the matrix $S' = {}^tS^{-1} \cdot N$ is the matrix that does the job. This proves the assertion. □

Note that if $\mathsf{D}(|L|)$ has $n+1$ distinct roots, the elementary divisors are all linear polynomials, and the canonical form coincides with the form given in Theorem **??**. This gives another proof of the implication (i)⇒ (iii).

3.4 The case when the characteristic is equal to 2

Here we only state the results. First assume that n is odd. In this case the discriminant of the pencil is given by the pfaffian of the matrix of even size $n+1$ with entries homogeneous linear forms in two variables. It is expected to have $\frac{1}{2}(n+1)$ zeroes.

The following result belongs to Usha Bhosle (Crelle Journal, vol. 407).

Theorem 3.6. *Assume that the discriminant of the pencil is a reduced polynomial of degree $k = \frac{1}{2}(n+1)$. Then, on some coordinates, the pencil is generated by quadrics $V(q_1)$ and $V(q_2)$, where*

$$\begin{aligned} q_1 &= \sum_{i=1}^{k} t_{2i-1}t_{2i}, \\ q_2 &= \sum_{i=1}^{k} a_i t_{2i-1}t_{2i} + c_i t_{2i-1}^2 + d_i t_{2i}, \end{aligned}$$

where $a_1, \ldots, a_k$ are the roots of the discriminant. The base locus $X = V(q_1) \cap V(q_2)$ is nonsingular if and only if $\prod_{i=1}^{k} c_i d_i \neq 0$.

Suppose n is even. Then a recent result of myself and Alex Duncan is the following:

Theorem 3.7. *Let (q_1, q_2) be a pair of quadratic forms on a vector space E of dimension $n = 2m+1 \geq 3$ over a field $\Bbbk$ of characteristic 2. Suppose that the intersection of quadrics $V(q_1) \cap V(q_2)$ is smooth. Then there exists a basis $(x_0, \ldots, x_m, y_0, \ldots, y_{m-1})$ in $E^\vee$ such that*

$$\begin{aligned} q_1 &= \sum_{i=0}^{m} a_{2i}x_i^2 + \sum_{i=0}^{m-1} x_{i+1}y_i + \sum_{i=0}^{m-1} r_{2i+1}y_i^2, \\ q_2 &= \sum_{i=0}^{m} a_{2i+1}x_i^2 + \sum_{i=0}^{m-1} x_i y_i + \sum_{i=0}^{m-1} r_{2i}y_i^2, \end{aligned} \tag{3.4.1}$$

where the coefficients $a_0, \ldots, a_n$ are equal to those of the half-discriminant polynomial, and $r_0, \ldots, r_{n-2}$ are in $\Bbbk$. If $\Bbbk$ is algebraically closed, one can choose r_i's to be zero.

3.5 Intersection of two quadrics in $\mathbb{P}^{2n}$

Let $X = Q_1 \cap Q_2$ be a smooth intersection of two quadrics in $\mathbb{P}^{2k}$. It is a subvariety of $\mathbb{P}^{2k}$ of dimension $2k-2$ and degree 4. We have already considered the case $k = 1$, where X consisted of four distinct points in the plane. The next case is $k = 2$, where X is an algebraic surface in $\mathbb{P}^4$. By the adjunction formula, $\omega_X \cong \mathcal{O}_X(-1)$, it is a *del Pezzo surface of degree* 4. As we explained in the previous Lecture, X is isomorphic to the blow-up of 5 points $p_1, \ldots, p_5$ in the plane. The anti-canonical class is the divisor class $3e_0 - e_1 - \cdots - e_5$. Since it is ample, it has positive intersection with any curve on X. In particular, no three points p_i lie on a line since otherwise the proper inverse image of this line belongs to the divisor class $e_0 - e_i - e_j - e_k$, and hence does not intersect $-K_X$. We also explained how to see lines on X. Since $X = X^{\mathrm{ac}}$, each line comes from a (-1)-curve on X, and the latter are obtained as proper inverse transforms of a line $\langle p_i, p_j \rangle$, or of the conic through $p_1, \ldots, p_5$, or of the exceptional curve over one of the points p_i. Altogether we have 16 lines. They intersect each other according to the following graph.

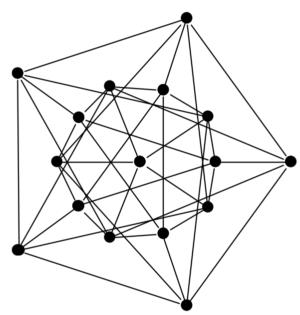


Figure 3.1: Lines on a del Pezzo quartic surface

Note that a birational morphism $\sigma : X \to \mathbb{P}^2$ is defined by a choice of five skew lines in X. Inspecting the graph, we find that there are 5! of such choices. Fix a line ℓ on X. Then there will be exactly five disjoint lines $\ell_1, \ldots, \ell_5$ intersecting ℓ (you can easily verify it by taking ℓ to be the proper inverse image of the conic through the five points). Let $\pi : X \to \mathbb{P}^2$ be the blow-down of these lines. The image of ℓ will be a conic K. Fix a point $x_0 \in \ell$ not belonging to the set S of the intersection points of ℓ with other five lines. For any other point $x \in \ell \setminus S$, consider the line $\ell = \langle \pi(x_0), \pi(x) \rangle$ in the plane. The union of this line and the conic is a cubic curve through the five points p_i, the images of the lines ℓ_i. It corresponds to a hyperplane section of the anti-canonical model X. Since it is singular at the point x_0, it contains the tangent space $\mathbb{T}_{x_0}X$. There will be a unique nonsingular quadric Q that contains this hyperplane as its tangent space at x_0. Also, if $x \in S$ and equal to the intersection point $\ell \cap \ell_i$, then the restriction of any quadrics to the plane spanned by ℓ and ℓ_i contains the same conic $\ell \cup \ell_i$, hence there will be a quadric Q_i that contains the whole plane. Since a nonsingular quadric does not contain planes, Q_i is one of the five singular quadrics in the pencil. This shows that we can identify ℓ with $|L|$. Under this identification, the set $\{\ell \cap \ell_1, \ldots, \ell \cap \ell_5\}$ is identified with the $\mathsf{D}(|L|)$. Thus, via the Veronese map whose image is the conic through $p_1, \ldots, p_5$, the image of $\mathsf{D}(|L|)$ is equal to the set of points $\{p_1, \ldots, p_5\}$.

Let $\pi : X \to \mathbb{P}^2$ be the blowing down of 5 disjoint lines $\ell_1, \ldots, \ell_5$ and let $e_0, e_1, \ldots, e_5$ be the corresponding basis in $\mathrm{Pic}(X)$. Since $K_X = 3e_0 - (e_1 + \cdots + e_5)$, we see that the orthogonal complement of K_X has a basis $(e_0 - e_1 - e_2 - e_3, e_1 - e_2, e_2 - e_3, e_3 - e_4, e_4 - e_5)$ (it is called a *geometric basis*). Computing the intersection matrix, we find that it coincides with the Cartan matrix of the root system of type D_5. Fixing an order on the set of $\ell_1, \ldots, \ell_5$, defines a basis in $\mathrm{Pic}(X)$. A del Pezzo surface with a fixed ordered set of five skew lines is called a *marked quartic del Pezzo surface.* We see from the previous discussion that a marking of a quartic del Pezzo surface is equivalent to any of the following:

- a choice of an ordered set of five skew lines on X;
- a choice of an order on the set of singular quadrics in the pencil of quadrics containing X;
- a choice of a geometric basis in $\mathrm{Pic}(X)$;
- a choice of an isomorphism between the subgroup $K_X^{\perp} \subset \mathrm{Pic}(X)$ equipped with the intersection product quadratic form and the root lattice of a simple Lie algebra of type D_5;
- a choice of a bijection of the graph of 16 lines on X together with the incidence relation defined by the intersection of lines and the graph given in Figure **??**.

Thus, the set of isomorphism classes of marked quartic del Pezzo surfaces of degree 4 is equal to the set of PGL(3)-orbits of 5 ordered points in $\mathbb{P}^2$ nor three of which are collinear. Via the Veronese map it is the same as the set of PGL(2)-orbits of 5 ordered points in $\mathbb{P}^1$. Since we can always fix the first four points by a projective transformation, we obtain that the moduli space of marked quartic del Pezzo surfaces can be identified with the open subset of a del Pezzo surface of degree 5 with the complemet equato the set of 10 lines on it. It coincides with the moduli space $\mathcal{M}_{0,5}$. Thus, we see that the same space could serve as the moduli space for different moduli problems.

Now let X be a smooth complete intersection of two quadrics in $\mathbb{P}^{2k}$ with $k > 2$. Most of the previous discussion extends to this case. We refer for details to Miles thesis "Intersection of two and more quadrics" which can be found online or to my joint preprint (AGArchive) with A. Duncan that deals with the case when $\mathrm{char}(\Bbbk) = 2$.

Let Q_1, Q_2 be two smooth quadrics in the pencil $|L|$ of quadrics containing X. We know that X contains one ruling of subspaces of dimension $k-1$. Its dimension is equal to $\frac{1}{2}k(k+1)$. Since $\dim G(k+1, 2k+1) = k(k+1)$, we expect that X contains finitely many linear subspaces of dimension k. In fact, this is always true, and the number of such subspaces (called *generators*) is equal to 2^{2k}. By the adjunction formula $-K_X = (2k-3)\eta$, where η is the class of a hyperplane section. The variety X is a Fano variety of dimension $2k-2$. Let $A^{k-1}(X)$ be the group $H^{k-1}(X, \mathbb{Z})$ if $\Bbbk = \mathbb{C}$, or the group $H^{k-1}(X, \mathbb{Z}_\ell)$ otherwise (or the group of algebraic cycles of dimension $k-1$ modulo rational equivalence, see [Fulton]). The classes of η^{k-1} and the classes of generators span $A^{k-1}(X)$. The group $A^{k-1}(X)$ is equipped with the intersection symmetric bilinear form

$$A^{k-1}(X) \times A^{k-1} \to A^{2k-2}(X) = \mathbb{Z}\eta^{2k-2} \cong \mathbb{Z}.$$

The orthogonal complement of η^{k-1} is a free abelian group equipped with a integral values symmetric bilinear form. In some basis, it coincides with the Cartan matrix of the root system of type D_{2k+1}. Every generator Λ is intersected by $2k+1$ generators, and a choice of an order on this set of $2k+1$ generators defines a basis in $A^{k-1}(X)$. Such a choice is called a marking of X, and as in the two-dimensional case, we obtain that the moduli space of marked complete intersections of two quadrics in $\mathbb{P}^{2k}$ is isomorphic to the moduli space of ordered $2k+1$ points in $\mathbb{P}^1$ modulo projective equivalence. Its dimension is equal to $2k-2$.

3.6 Intersection of two quadrics in an odd-dimensional projective space

We have already studied the intersection of two quadrics in $\mathbb{P}^3$. This is a quartic elliptic curve. Let us study such intersections in a higher-dimensional projective space of odd dimension. We assume that $X = Q_1 \cap Q_2 \subset \mathbb{P}^{2g+1}$ is smooth and $\mathrm{char}(\Bbbk) \neq 2$. By Theorem **??**, the pencil $|L|$ generated by Q_1, Q_2 contains exactly $2g+2$ singular quadrics. Their singular locus is a point. We identify $|L|$ with $\mathbb{P}^1$ and let $C \to \mathbb{P}^1$ be the double cover ramified over $\mathsf{D}(|L|)$. We can choose the equations of two quadrics $V(q_1), V(q_2)$ in

$|L|$ as in Theorem **??** and choose a basis in L formed by q_1, q_2. Then the singular quadrics have coordinates $[a_1, -1]$, and the double cover has the equation

$$x_2^2 - (x_1 + a_1 x_0) \cdots (x_1 + a_{2g+2} x_0) = 0 \qquad (3.6.1)$$

(understood to be the equation of a hypersurface in the weighted projective space $\mathbb{P}(1, 1, g+1)$). Or, if we change the basis of L to $(q_1, a_{2g+2}q_1 - q_2)$, we may assume that the singular quadric $V(a_{2g+2}q_1 - q_2)$ corresponds to the point at infinity $[1, 0]$, and we can write C in affine equation

$$y^2 - (x + a_1) \cdots (x + a_{2g+1}) = 0.$$

This is a familiar equation of a *hyperelliptic curve* of genus g (a rational curve if $g = 0$, an elliptic curve if $g = 1$).

Thus a smooth intersection of two quadrics in $\mathbb{P}^{2g+1}$ defines a hyperelliptic curve of genus g. Conversely, by taking such curve as in equation (**??**), we can define two quadrics with equations (**??**). Note that the sets of isomorphism classes of both sets are the same and coincide with the orbit of PGL(2) on the set of distinct ordered $2g + 2$ points. It is a quasi-projective algebraic variety of dimension $2g - 1$. Thus we see that the geometry of smooth complete intersections of two quadrics in $\mathbb{P}^{2g+1}$ must be related to the geometry of hyperelliptic curves of genus g. Of course, when $g = 1$, it is not surprising.

Recall that the *Jacobian variety* of a nonsingular projective curve C of genus g is an abelian variety $\mathrm{Jac}(C)$ whose set of $\Bbbk$-rational points is the group of divisor classes of degree 0 modulo linear equivalence. Over $\mathbb{C}$, it is the torus $\mathbb{C}^g/\Lambda$, where Λ is spanned by $2g$-vectors

$$v_i = (\int_{\gamma_i} \omega_1, \ldots, \int_{\gamma_i} \omega_g), \quad i = 1, \ldots, 2g,$$

where $(\omega_1, \ldots, \omega_g)$ is a basis of holomorphic differential 1-forms and $(\gamma_1, \ldots, \gamma_{2g})$ is a basis of homology classes of 1-cycles. The map of the group $\mathrm{Div}(C)^0$ of divisor classes of degree 0 to $\mathrm{Jac}(C)$ is given by the Abel-Jacobi map that assigns to a point $c \in C$, the vector $(\int_{c_0}^{c} \omega_1, \ldots, \int_{c_0}^{c} \omega_g)$ modulo the lattice. It is defined by fixing a point $c_0 \in C$. By Abel's Theorem, the kernel of this map is the group of principal divisors.

The following is a fundamental theorem, attributed to A. Weil.

Theorem 3.8. *Let X be a smooth intersection of two quadrics in $\mathbb{P}^{2g+1}$ and C be the associated hyperelliptic curve of genus g. Then the variety $F_{g-1}(X)$ of $(g-1)$-planes contained in X is naturally isomorphic to the Jacobian variety of C.*

Proof. We will restrict ourselves only to the case $g = 2$, leaving the general case to the reader. For each $\ell \in F(X) := F_1(X)$ consider the projection map $p_\ell : X' = X \setminus \ell \to \mathbb{P}^3$. For any point $x \in X$ not on ℓ, the fiber over $p_\ell(x)$ is equal to the intersection of the plane $\ell_x = \langle \ell, x \rangle$ with X'. The intersection of this plane with a quadric Q from the pencil $|L|$ is a conic containing ℓ and another line ℓ'. If we take two nonsingular generators of $|L|$, we find that the fiber is the intersection of two lines or the whole $\ell' \in F(X)$ intersecting ℓ. In the latter case, all points on $\ell' \setminus \ell$ belong to the same fibre. Since all quadrics from the pencil intersect the plane $\langle \ell, \ell' \rangle$ along the same conic $\ell \cup \ell'$, there exists a unique quadric $Q_{\ell'}$ from the pencil which contains $\langle \ell, \ell' \rangle$. The plane belongs to one of the two rulings of planes on $Q_{\ell'}$ (or a unique family if the quadric is singular). Note that each quadric from the pencil contains at most one plane in each ruling which contains ℓ (two members of the same ruling intersect along a subspace of even codimension). Thus we can identify the following sets:

- pairs (Q, r), where $Q \in |L|$, r is a ruling of planes in Q,
- $B = \{\ell' \in F(X) : \ell \cap \ell' \neq \emptyset\}$.

If we identify $\mathbb{P}^3$ with the set of planes in $\mathbb{P}^5$ containing ℓ, then the latter set is a subset of $\mathbb{P}^3$. Let D be the union of ℓ''s from B. The projection map p_ℓ maps D to B with fibres isomorphic to $\ell' \setminus \ell \cap \ell'$.

Extending p_ℓ to a morphism $f : \bar{X} \to \mathbb{P}^3$, where $\bar{X}$ is the blow-up of X with center at ℓ, we obtain that f is an isomorphism outside B and that the fibres over points in B are isomorphic to $\mathbb{P}^1$. Observe that $\bar{X}$ is contained in the blow-up $\bar{\mathbb{P}}^5$ of $\mathbb{P}^5$ along ℓ. The projection f is the restriction of the projection $\bar{\mathbb{P}}^5 \to \mathbb{P}^3$ which is a projective bundle of relative dimension 2. The crucial observation now is that B is isomorphic to our hyperelliptic curve C. In fact, consider the incidence variety

$$\mathcal{X} = \{(Q, \pi) \in |L| \times G_2(\mathbb{P}^5) : \pi \subset Q\}.$$

Its projection to $|L|$ has fiber over Q isomorphic to the rulings of planes in Q. It consists of two connected components outside of the set of singular quadrics and one connected component over the set of singular quadrics. Taking the Stein factorization, we get a double cover of $|L| = \mathbb{P}^1$ branched along the discriminant. It is isomorphic to C.

A general plane in $\mathbb{P}^3$ intersects B at $d = \deg B$ points. The preimage of the plane under the projection $p_\ell : X \dashrightarrow \mathbb{P}^3$ is isomorphic to the complete intersection of two quadrics in $\mathbb{P}^4$. Taking a general hyperplane, we may assume that the intersection of the two quadrics is nonsingular. Thus it is a del Pezzo surface of degree 4, hence it is obtained by blowing up five points in $\mathbb{P}^2$. We know that any line in such a surface intersect five other lines. Thus $d = 5$ and B is isomorphic to a genus 2 curve of degree 5 in $\mathbb{P}^3$. The exact sequence

$$0 \to \mathcal{I}_B(2) \to \mathcal{O}_{\mathbb{P}^3}(2) \to \mathcal{O}_B(2) \to 0,$$

together with the fact that $\dim H^0(B, \mathcal{O}_B(2)) = 9$ that follows from the Riemann-Roch formula, shows that B is contained in a unique quadric $\mathcal{Q}$ in $\mathbb{P}^3$. As we explained in Lecture 1, the exceptional divisor E of the blowing-up $\tilde{X} \to X$ is isomorphic to the product $\ell \times \mathbb{P}^1 = \mathbb{P}^1 \times \mathbb{P}^1$. It is also isomorphic to a nonsingular quadric. The image of E under the map $\tilde{X} \to \mathbb{P}^3$ is a quadric containing B. Thus B is contained in a unique nonsingular quadric $\mathcal{Q}$. It is easy to see that B must be a curve of bi-degree $(2,3)$. Recall that it means that the divisor classes f_1 and f_2 of lines in each ruling intersect B with degree 2 and 3. The lines from the first ruling cut out in B the linear series of degree 2 that coincides with the canonical linear system $|K_B|$.

We see now that the projection $p_\ell : \bar{X} \to \mathbb{P}^3$ is a birational isomorphism which blows down the union of lines intersecting ℓ to the curve B. The inverse rational map $\mathbb{P}^3 \dashrightarrow \mathbb{P}^5$ whose image is X is obtained by the linear system of cubics containing B. In fact, cosnider the exact sequence

$$0 \to \mathcal{I}_B(3) \to \mathcal{O}_{\mathbb{P}^3}(3) \to \mathcal{O}_B(3) \to 0.$$

Since $\deg B = 5$, we get $\deg \mathcal{O}_B(3)) = 15$. Applying Riemann-Roch, we get $h^0(\mathcal{O}_B(3)) = 14$. Since $h^0(\mathcal{O}_{\mathbb{P}^3}(3)) = 20$, we obtain that $h^0(\mathcal{I}_B(3)) = 6$. Thus $\dim |\mathcal{I}_B(3)| = 5$, and the map ϕ_B given by the linear system $|\mathcal{I}_B(3)|$ of cubics through B maps $\mathbb{P}^3$ to $\mathbb{P}^5$. Every honest bisecant of B intersects a member of the linear system at one point outside of B. Thus it is mapped to a line contained in the image X' of the map. A tri-secant of B is blown down to a point. The set of trisecants coincides with the linear series g^1_3 on B cut out by a ruling of the quadric containing B. So, the set of trisecants is mapped to a line $\ell \subset X'$. The degree of X' is easy to compute. A general line in $\mathbb{P}^5$ corresponds to the intersection iof two cubic surfaces from $|\mathcal{I}_B(3)|$. They intersect along a curve of degree 9 that contains the curve B. So, the residual curve is of degree 4. This shows that a general line in $\mathbb{P}^5$ intersects X' at four points, hence $\deg X' = 4$. Thus the composition of rational maps $\Phi = \phi_B \circ p_\ell$ is a map $X \dashrightarrow X'$. Since $p + l$ sends a general line in X to a secant of B and ϕ_B sends a bisecant to a line in X', we see that Φ sends a general line X to a general line in X'. It is easy to see that this implies that Φ is a projective isomorphism $X \cong X'$.

It follows from the previous discussion that X is birationally isomorphic to the blow-up of $\mathbb{P}^3$ along the curve B. Since X is assumed to be smooth, this easily implies that B is a smooth hyperelliptic curve of genus 2.

Let us construct an isomorphism between $\mathrm{Jac}(C)$ and $F_1(X)$. Recall that $\mathrm{Jac}(C)$ is birationally isomorphic to the symmetric square $C^{(2)}$ of the curve C. The canonical map $C^{(2)} \to \mathrm{Pic}^2(C)$ defined by $x+y \mapsto [x+y]$ is an isomorphism over the complement of one point represented by the canonical class of C. Its fiber over K_C is the linear system $|K_C|$ isomorphic to $\mathbb{P}^1$. Also note that $\mathrm{Pic}^2(C)$ is canonically identified with $\mathrm{Jac}(C)$ by sending a divisor class ξ of degree 2 to the class $\xi - K_C$.

Each line ℓ' skew to ℓ is projected to a secant line of B. In fact, $\langle \ell, \ell' \rangle \cap X$ is a quartic curve in the plane $\langle \ell, \ell' \rangle \cong \mathbb{P}^3$ that contains two skew line components. The residual part is the union of two skew lines m, m' intersecting both ℓ and ℓ' (use that the hyperplane section if a curve of arithmetic genus one, this forces the residual part to be the union of two lines. Thus ℓ' is projected to the secant line joining two points on C which are the projections of the lines m, m'. If $m = m'$, then ℓ' is projected to a tangent line of B. Thus the open subset of lines in X skew to ℓ is mapped bijectively to an open subset of $C^{(2)}$ represented by "honest" secants of C, i.e. secants which are not 3-secants. Each line $\ell' \in F_1(X) \setminus \{\ell\}$ intersecting ℓ is projected to a point b of B. The line f of the ruling of $\mathcal{Q}$ intersecting B with multiplicity 3 and passing through a point $b \in B$ defines a positive divisor D of degree 2 such that $f \cap B = b + D$. The divisor class $[D] \in \mathrm{Pic}^2(C)$ is assigned to ℓ'. Finally, the line ℓ itself corresponds to K_C. This establishes an isomorphism between $\mathrm{Pic}^2(C)$ and $F(X)$. □

Remark 3.9. Note that we have shown during the proof, that X admits a birational morphism to $\mathbb{P}^3$, hence it is is a rational variety. It is one of an examples of Fano 3-dimensional variety that happens to be rational. Not all of them are rational. For example, a cubic hypersurface in $\mathbb{P}^4$ is known to be non-rational.

Remark 3.10. Note that the proof works in any characteristic, even in characteristic 2. In the latter case, the curve B in this case admits a separable double cover $B \to \mathbb{P}^1$ ramified at the discriminant of the pencil. Its equation can be given by

$$y^2 + a_3(u,v)y + a_6(u,v) = 0,$$

where a_3, a_6 are binary forms of degrees 3 and 6. The zeros of a_3 are the zeros of the discriminant of the pencil. Thus a_3 is given by the pfaffian of the matrix of the bilinear form $uq_1 + vq_2$, where $V(q_1), V(q_2)$ generate the pencil. The binary form is not uniquely defined, we can replace it by $a_6' = a_6 + b_3^2 + a_3b_3$, where b_3 is any binary form of degree 3. One can show that a_6 can be chosen in a such a way that it vanishes at zeros of $[1, a_3]$. In this case the curve is nonsingular if and only if it has only simple zeros at the zeros of a_3 (see cited paper of U. Bhosle, Proposition 1.5. Theorem **??** gives a canonical equations of a smooth X when a_3 is a reduced polynomial of degree 3. In this case B has 3 distinct non-trivial 2-torsion divisor classes equal to $[p_2 - p_1], [p_3 - p_1], [p_2 + p_3 - 2p_1]$, where $p_i = [1, a_i, 0]$. It is known that $\mathrm{Jac}(C)[2]$ is isomorphic to eitehr $(\mathbb{Z}/2\mathbb{Z})^2$, or $\mathbb{Z}/2\mathbb{Z}$, or trivial. A theorem of David Leep [Journal of algebra and its Appl. vol. 1 (2002)] states that, without assumption on a_3, X is smooth if and only if the curve C is nonsingular. The polynomial a_6 in the equation of C is expressed in terms of the *Arf-invariant* of the pencil defined by $Arf = a_6/a_3^2$.

3.7 Quadratic line complex

Recall that a smooth quadric in $\mathbb{P}^5$ could be identified with the Klein quadric, the Grassmannian $G = G(2,4) = G_1(\mathbb{P}^3)$ of lines in $\mathbb{P}^3$ embedded via the Plücker embedding. Thus the intersection of G with another quadric Q can be viewed as a hypersurface in $G(2,4)$. It is called a *quadratic line complex*. We will assume that $X = G \cap Q$ is smooth. We already know that X defines a genus 2 curve, the double cover of the pencil $|L|$ generated by quadrics G, Q branched along the discriminant variety $\mathsf{D}(|L|)$. There is much more fascinating geometry related to this correspondence.

Let

$$\begin{array}{ccc} & Z_G = \{(x,\ell) \in \mathbb{P}^3 \times G\} & \\ {}^{p_1}\swarrow & & \searrow^{p_2} \\ \mathbb{P}^3 & & G \end{array}$$

be the incidence variety with its two natural projections. The fiber of the first projection over a point $x \in \mathbb{P}^3$ is the plane π_x of lines containing this point. As we know from Lecture 1, its image under the second projection is a plane in G. The fiber of the second projection over a line $\ell \in G$, is mapped to ℓ itself under the first projection. It is easy to see that these maps are bijections, so we may identify the first fiber with a plane π_x, and the second fiber with the line ℓ. In fact, the second projection is projective line bundle over G, and the first projection is a projective bundle of relative dimension

2. Now, let us restrict the incidence variety over $X = G \cap Q$

$$
\begin{array}{ccc}
 & Z_X = \{(x,\ell) \in \mathbb{P}^3 \times X\} & \\
 {}^{p_1}\swarrow & & \searrow^{p_2} \\
\mathbb{P}^3 & & X
\end{array}
$$

The image of the second projection is the variety $F_1(X)$ of lines on X. We know that it is isomorphic to $\mathrm{Jac}(C)$. The fiber of the first projection over $x \in \mathbb{P}^3$ is the intersection of $\pi_x \cap Q$. It could be the whole plane, a nonsingular conic, or a singular conic. We know that the first case does not occur since it would imply that X contains a plane Λ. Let us see that the smoothness assumption on X implies that this is impossible. For any point $x \in L$ the tangent hyperplanes $\mathbb{T}_x(G)$ and $\mathbb{T}_x(Q)$ are hyperplanes in $\mathbb{P}^5$. By varying x inside Λ, we define two (projective) linear maps $\phi_G : \Lambda \to \check{\mathbb{P}}^5$ and $\phi_Q : \Lambda \to \check{\mathbb{P}}^5$. The image of each map is the plane of hyperplanes containing Λ. The self-map $\phi_G^{-1} \circ \phi_Q : \Lambda \to \Lambda$ is a projective automorphism of the plane, hence has a fixed point. The fixed point $x_0 \in \Lambda$ corresponds to a hyperplane which is tangent to both Q and G at the point x_0. This would imply that X is singular at x_0. An alternative proof is to use the Lefschetz Theorem on a hyperplane section that tells that $\mathrm{Pic}(\mathbb{P}^5) \to \mathrm{Pic}(X)$ is an isomorphism. Thus all surfaces in X have even dimension, so X does not contains planes.

Definition 3.11. The *Kummer surface* associated with a quadratic line complex is the locus of points $x \in \mathbb{P}^3$ such that the fiber $p_1^{-1}(x)$ is a reducible conic.

Theorem 3.12. *The Kummer surface is an irreducible surface K of degree 4 with 16 ordinary double points. The fibers of p_1 over these points are double lines.*

Proof. We know that the variety of lines $F_1(X)$ is isomorphic to the Jacobian variety $\mathrm{Jac}(C)$. Each line in $F_1(X)$ is a line in G, and hence it is equal to the pencil $\pi_{x,\Lambda} = \pi_x \cap \pi_\Lambda$ of lines in some plane Λ passing through a point $x \in \Lambda$. The plane $\pi_x$intersects Q along a reducible conic that contains this line as its irreducible component. Thus all lines in X come in pairs, each pair makes a reducible fiber of p_1 over a point in the Kummer surface. This shows that $\mathrm{Jac}(C)$ admits a degree 2 map onto K.

Recall that an abelian variety A of dimension g has the involution $[-1]_A$ that sends a point x to $-x$. The fixed points of this involution are 2-torsion points. There are 2^{2g} of them if $\mathrm{char}(\Bbbk) \neq 2$. The quotient space $A/([-1]_A)$ is an algebraic variety with 2^{2g} singular points. It is called the *Kummer variety* of A and is denoted by $\mathrm{Kum}(A)$. When $g = 2$, as in our case, we obtain a surface with 16 ordinary double points. Its minimal resolution is a K3 surface. So, in the following we are going to give a geometric construction of $\mathrm{Kum}(\mathrm{Jac}(C))$ as a surface in $\mathbb{P}^3$ parameterizing pairs of lines in $X = G \cap Q$ that lie in a plane contained in G. By definition, the pairs of these lines form an orbit in $F_1(X) = \mathrm{Jac}(C)$ of some involution τ of $\mathrm{Jac}(C)$. Note that any involution on an abelian variety looks like $x \mapsto -x + t_a$, where t_a is a translation map $x \mapsto x + a$. Its fixed points satisfy $2x = a$, and form a principal homogeneous space over $A[2]$. In particular, their number is 2^{2g}. By changing the origin in A, we may assume that the involution is the negation involution. Thus, we see that our Kummer surface must have 16 singular points. They correspond to lines ℓ in X such that the plane π_x is tangent to Q along ℓ. So, we see that our Kummer surface is isomorphic to $\mathrm{Kum}(\mathrm{Jac}(C))$. Its 16 nodes should correspond to the points over which the fiber is a double line. In fact, by definition of the involution on $F_1 * X) = \mathrm{Jac}(C)$, the fixed points of this involutiion correspond to pairs of coinciding lines.

It remains to show that our Kummer surface K is a quartic surface. To show this we have to check that a general line ℓ in $\mathbb{P}^3$ intersects k at four points. Consider the union $\cup_{x\in\ell}\pi_x$ of all lines that intersect ℓ. It is known that they form the intersection of $G(2,4)$ with a hyperplane Hin the Plĕker space $\mathbb{P}^5$ that is tangent to $G(2,4)$ at the point represented by ℓ. The intersection $S = H \cap X = H \cap G \cap Q$ is a del Pezzo surface S of degree 4. The restriction of p_1 to $p_2^{-1}(S)$ defines a fibration $S \to \ell$ whose general fiber is a conic and singular fibers are the unions of two lines. To see that the number of sich lines is exactly four, we use the theory of quartic del Pezzo surface. If we exhibit S as the blow-up of 5 points, a conic bundle on such a surface is equal to the pre-image of lines through one point. There are 4 lines which pass through the remaining 4 points which define singular members of the pencil.

□

Remark 3.13. There are many other facts about Kummer quartic surfaces which are discussed in [DolgachevCAG] or [Griffiths-Harris]. For example, projection from a node, gives its birational model isomorphic to the double cover of $\mathbb{P}^2$ branched along the union of six lines. The six lines are all tangent to the same conic, and the six tangency points are projectively equivalent to to the six points in the discriminant equation of the intersection of two quadrics. Another remarkable fact is that, by considering the other ruling of planes in the Grassmannian, one finds in $\mathbb{P}^3$ sixteen planes that tangent the Kummer surface along a conic. The sixteen conics and sixteen nodes form an abstract configuration $(16_6, 16_6)$, the *Kummer configuration*. One more fact which I cannot resist not to mention is that the set of intersection points of pairs of lines over points in the Kummer surface is a smooth surface Y in $\mathbb{P}^5$ of degree 8 birationally isomorphic to the Kummer surface. If we write equation of X as the intersection of two quadrics with equations $\sum_{i=1}^6 t_i^2 = \sum_{i=1}^6 a_i t_i^2$, then the equation of Y is given by adding one more equation $\sum_{i=1}^6 a_i^2 t_i^2 = 0$. The six points $[1, a_i, a_i^2]$ the plane are the images of the points in the discriminant variety of X under the Veronese map. They correspond to the Weierstrass points of the genus 2 curve C.

Remark 3.14. A *conic bundle* is a flat morphism of smooth varieties $f : X \to Y$ that whose general fiber is isomorphic to $\mathbb{P}^2$ and singular fibers are isomorphic to the union of two $\mathbb{P}^1$'s intersecting transversally at one point. The subvariety of Y parameterizing singular fibers is called the *discriminant* of a conic bundle. . In our previous discussion we found an example of a conic bundle $Z_X \to \mathbb{P}^3$ whose discriminant is a Kummer surface. the variety Z_X is isomorphic to a $\mathbb{P}^1$-bundle over $X = G(2,4) \cap Q$. Since the latter is rational, the total space Z_X of the conic bundle is rational. In general, it is avery difficult problem to decide whether the total space of a conic bundle is rational even when its base is rational. We will see in a future lecture, and example of Artin and Mumford when this is not the case.

sectionPrincipally polarized abelian varieties Here we give a more abstract construction of the Kummer surface of the jacobian variety as a 16-nodal quartic surface in $\mathbb{P}^3$. Let A be an abelian variety of dimension g. A principal polarization on A is an ample divisor class Δ such that $h^0(\Delta) := \dim H^0(A, \mathcal{O}_A(\Delta)) = 1$. By Riemann-Roch on an abelian variety, we have $h^0(m\Delta) = \frac{m^g}{g!}\Delta^g$. Thus Δ is a principal polarization if and only if $\Delta^g = g!$. Taking $m = 2$, we obtain a linear system $|2\Delta|$ with $\dim|2\Delta| = 2^g - 1$. Let t_a denote the translation automorphism of A by a point $a \in A$. One can find a such that $[-1]_A^*(t_a^*(\Delta)) = t_a^*(\Delta)$. Replacing Δ with $t_a^*(\Delta)$, we find a principal polarization such that $[-1]^*(\Delta) = \Delta$. It is called a symmetric principal polarization. The map given $\Phi_\Delta : A \to |\Delta|^* \cong \mathbb{P}^{2^g-1}$ given by the linear system $|2\Delta|$ commutes with the involution $[1]_A$ and hence factors through the $\mathrm{Kum}(A) = A/([-1]_A)$. One can show that it defines an embedding of $\mathrm{Kum}(A)$ in $\mathbb{P}^{2^g-1}$ unless (A, Δ) is a reducible polarization, i.e. $(A, \Delta) \cong (A_1, \Delta_1) \times \cdots (A_1, \Delta_1)$. The latter means that $A \cong A_1 \times \cdots A_k$ and $\Delta = p_1^*(\Delta_1) + \cdots p_k^*(\Delta_k)$. It follows that, if this is not the case, $\mathrm{Kum}(A)' = \phi_{2\Delta}(\mathrm{Kum}(A))$ is a g-dimensional subvariety of $\mathbb{P}^{2^g-1}$ of degree

$$\deg(\mathrm{Kum}(A)) = (2\Delta)^g/2 = 2^{g-1}g!,$$

In particular, we seee that the image is quartic surface if $g = 2$. In this case the polarization is irreducible if and only $A \cong \mathrm{Jac}(C)$ fotr some smooth curve C of genus 2. The images of the set of 2-torsion points is a set of singular points on $\phi(\mathrm{Kum}(A))$. The number of them is equal to 2^{2g} if $\mathrm{char}(\Bbbk) \neq 2$. Each singular point is formally isomorphic to the singular point of the cone over a Veronese variety $\nu_g(\mathbb{P}^{g-1})$ in $\mathbb{P}^g$.

It follows from the definition of the map $\phi_{2\Delta}$ defined by the linear system $|2\Delta$ that if we choose a representative $2D$ of 2Δ with $[-1]_A^*(D) = D$, then, the image of $2D$ is cut out in $\mathrm{Kum}(A)'$ by a hyperplane that is tangent to $\mathrm{Kum}(A)'$ along the image of D. There are 2^g such divisors, each is obtained from one of them by translation t_a, where $a \in A[2]$ is a 2-torsion point. Thus we have 2^{2g} hyperplanes that are tangent to $\mathrm{Kum}(A)'$ along the divisor D', the image of D with respect to $\phi_{2\Delta}$. These hyperplanes are called *tropes*. So, there are 2^{2g} tropes. One can show that each trope contains $2^{g-1}(2^g-1)$ singular points, and each singular point is contained in so many tropes. This realizes an abstract configuration $(2^{2g}_{2^{g-1}(2^g-1)}, 2^{2g}_{2^{g-1}(2^g-1)})$ which is called the *Kummer configuration*.

In our case when $A = Jac(C)$ and $g = 2$, we see that the configuration is $(16_6, 16_6)$. In this case the principal polarization is defined by the image of the Abel-Jacobi map $C \to \mathrm{Jac}(C), c \mapsto [c - c_0]$. The divisors D are isomorphic to C, the image of C is a conic D'in the Kummer quartic surface that contains 6 singular points. They are the branch points of the double cover $D \to D'$.

Note that the construction works also if the characteristic is equal to 2. In this case, we have $A[2] \cong (\mathbb{Z}/2\mathbb{Z})^{g-k}$, where $0 \leq k \leq g$. For example, in our situation, we still have a quartic Kummer surface, however, it has $4, 2$ or 1 singular points. The singular points are more complicated than ordinary nodes. For example, in the last case, the Kummer surface is a rational surface with one (elliptic) singularity.

Lecture 4

Conic bundles

4.1 The Brauer group

Recall the Brauer group of a field $\Bbbk$ is the group of equivalence classes of *central simple algebras* over $\Bbbk$. This means that the algebra has center equal to $\Bbbk$ and it has no two-sided ideals. Each algebra is isomorphic to the matrix algebra $M_n(K)$ over some separable extension K of $\Bbbk$. This implies that $\dim_k A = n^2$. Another way to see this is the fact that any central simple algebra is isomorphic to the matrix algebra over a division ring (a non-commutative associative algebra over a field where each non-zero element is invertible). The equivalence for simple algebras is defined by $A_1 \equiv A_2$ if $A \otimes M_k(\Bbbk) \cong A_2 \otimes M_n(\Bbbk)$ for some k, n. A simple central algebra is a $\Bbbk$-form of the matrix algebra $M_n(\Bbbk)$, hence it defines an element of the Galois cohomology $H^1(\Bbbk, \mathrm{PGL}_n)$ (we use that, by Skolem-Noether theorem, the group of automorphisms of $M_n(\Bbbk)$ is isomorphic to PGL_n). The exact sequence of algebraic groups

$$1 \to \mathbb{G}_m \to \mathrm{GL}_n \to \mathrm{PGL}_n \to 1$$

defines, via the coboundary map, the class in $\mathrm{Br}(\Bbbk) := H^2(\Bbbk, \mathbb{G}_m)$, the *Brauer class*. By Hilbert 90 Theorem, $H^1(\Bbbk, \mathrm{GL}_n) = \{1\}$, hence $H^1(\Bbbk, \mathrm{PGL}_n)$ becomes a subgroup of the group $\mathrm{Br}(\Bbbk)$. One can show that, in fact, it coincides with this group. The group law is defined by the tensor product of algebras.

Example 4.1. Let A be the *quaternion algebra* $(a, b)_{\Bbbk}$ over $\Bbbk$. It is generated over $\Bbbk$ by elements i, j such that $i^2 = a, j^2 = b, ij = -ji$, where $a, b \in \Bbbk^*$. When $\Bbbk = \mathbb{R}$ and $a = b = -1$, we get the usual definition of the quaternion algebra (where $k = ij$). It is easy to see that $(au^2, b)_{\mathbb{k}} \cong (a,b)_{\mathbb{k}}$, thus if a or b is a square in $\mathbb{k}$, we have $(a,b)_{\mathbb{k}} \cong (1,1)_{\mathbb{k}}$ (by sending

$$1 \mapsto \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right), i \to \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right), j \mapsto \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right), k \mapsto \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right).)$$

Let $K = \mathbb{k}(\sqrt{a}$ or $K = \sqrt{b})$. Then $A_K \cong M_2(K)$. It is clear that $A \otimes_{\mathbb{k}} A$ is generated by $x \otimes y, x, y = 1, i, j, k$ and $(x \otimes y)^2$ is a square in $\mathbb{k}$. Thus it becomes isomorphic to the matrix algebra over $\mathbb{k}$. This shows that the Brauer class of a quaternion algebra is of order 2. It is a still an open problem where any element of order 2 in $\mathrm{Br}(\mathbb{k})$ is equivalent to a quaternion algebra.

Now let us move from 0-dimensional scheme Spec $\mathbb{k}$ to any scheme S. We use étale topology of S that replaces the category of separable extensions in the Galois theory of fields. An immediate generalization of a central simple algebra is the notion of an *Azumaya algebra*. It is a sheaf of Algebras over S locally isomorphic in étale topology to the algebra $\mathcal{E}\backslash\lceil(\mathcal{E})$ of some locally free sheaf $\mathcal{E}$ over S. Two Azymaya algebras $\mathcal{A}, \mathcal{B}$ are called equivalent if $\mathcal{A} \otimes \mathcal{E}\backslash\lceil(\mathcal{E}_1) \cong \mathcal{A}_2 \otimes \mathcal{E}\backslash\lceil(\mathcal{E}_2)$ for some locally free sheaves $\mathcal{E}_1, \mathcal{E}_2$. We have an exact sequence of sheaves on S

$$1 \to \mathbb{G}_{m,S} \to \mathrm{GL}_{n,S} \to \mathrm{PGL}_{n,S} \to 1$$

and the exact sequence of cohomology gives a homomorphism

$$\phi : H^1(S, \mathrm{PGL}_{n,S}) \to H^2(S, \mathbb{G}_m).$$

In general, this homomorphism is neither injective nor surjective. The group $H^1(S, \mathrm{PGL}_{n,S})$ is the *Brauer group* of S, and the group $H^2(S, \mathbb{G}_m)$ is called the *cohomological Brauer group*. However, under some conditions on S, for example, when S is a smooth algebraic variety, the two groups are isomorphic and denoted by $\mathrm{Br}(X)$.

Another geometric interpretation of a cohomology class from $H^1(S, \mathrm{PGL}_{n,S})$ is an isomorphism class of a *Severi-Brauer* variety. It is a variety X over S such that the fibers of the structure morphism $X \to S$ are isomorphic to a projective space $\mathbb{P}^r$. Locally, in étale topology, it is isomorphic to a projective bundle over S, i.e Proj $\mathrm{S}(\mathcal{E})$ for some locally free sheaf of rank $r+1$ over S.

For example, the Severi-Brauer variety whose cohomology class coincides with the class of a quaternion algebra $Q(a,b)$ is a conic over $\mathbb{k}$ with equation $C : ax^2 + by^2 - z^2 = 0$. If we take the intersection of this conic with a line $x = 0$, we obtain a point in $C(K)$, where K is a quadratic extension $\mathbb{k}(\sqrt{a})$.

One can show that the group of endomorphisms of this quadratic algebra over $\mathbb{k}$ is isomorphic to the quaternion algebra $(a,b)_{\mathbb{k}}$.

Globalizing this example, we obtain that a Severi-Brauer variety over S of relative dimension 1 is given by a smooth conic bundle over S.

The latter is given by a locally free sheaf of rank 3 over S and a section of q of $S^2\mathcal{E}$. It corresponds to a section of $\mathcal{O}_{\mathbb{P}(\mathcal{E})}(2)$ under the isomorphism $\pi_*\mathcal{O}_{\mathbb{P}(\mathcal{E})}(2) \cong S^2\mathcal{E}$. If we view q as a linear map $\mathcal{E}^\vee \to \mathcal{E}$, its rank at each point is equal to 3, so in each fiber of $\mathbb{P}(\mathcal{E}) \to S$, its defines a nonsingular conic.

Consider the *Kummer exact sequence* of sheaves in étale topology:

$$1 \to \mu_n \to \mathbb{G}_m \xrightarrow{[n]} \mathbb{G}_m \to 0 \to 0.$$

We use that $\mathrm{Pic}(S) \cong H^1(S, \mathbb{G}_m)$. The exact sequence gives an exact sequence of group cohomology:

$$0 \to \mathrm{Pic}(S) \otimes \mathbb{Q}/\mathbb{Z} \to H^2(S, \mu_n) \to \mathrm{Br}(S)[n] \to 0,$$

where $\mathrm{Br}(S)[n]$ is the group of n-torsion elements. This also gives an exact sequence

$$0 \to \mathrm{Pic}(S) \otimes \mathbb{Q}/\mathbb{Z} \to H^2(S, \mathbb{Q}/\mathbb{Z}) \to \mathrm{Tors}(\mathrm{Br}(S)) \to 0, \qquad (4.1.1)$$

Now we use the exact sequence

$$0 \to \mathbb{Z} \to \mathbb{Q} \to \mathbb{Q}/\mathbb{Z}$$

to get an exact sequence

$$0 \to H^2(S, \mathbb{Z}) \otimes \mathbb{Q}/\mathbb{Z} \to H^2(S, \mathbb{Q}/\mathbb{Z}) \to \mathrm{Tors}(H^3(S, \mathbb{Z}) \to 0.$$

Another way to see this exact sequence is to consider the exact sequence

$$0 \to \mathbb{Z} \to \mathbb{Z} \to \mathbb{Z}/n\mathbb{Z} \to 0$$

that gives an exact sequence

$$H^2(S, \mathbb{Z})/nH^2(S, \mathbb{Z}) \to H^2(S, \mathbb{Z}/n\mathbb{Z}) \to H^2(3, \mathbb{Z})[n] \to 0,$$

and then take the inductive limit. Together with (**??**), we get a commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathrm{Pic}(S) \otimes \mathbb{Q}/\mathbb{Z} & \longrightarrow & H^2(S, \mathbb{Q}/\mathbb{Z}) & \longrightarrow & \mathrm{Br}(S) & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle c_1} & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & H^2(S, \mathbb{Z}) \otimes \mathbb{Q}/\mathbb{Z} & \longrightarrow & H^2(S, \mathbb{Q}/\mathbb{Z}) & \longrightarrow & \mathrm{Tors}(H^3(S, \mathbb{Z}) & \longrightarrow & 0
\end{array}$$

This shows that the homomorphism $\mathrm{Br}(S) \to \mathrm{Tors}(H^3(S,\mathbb{Z})$ is always surjective. Also, if the first Chern class map $c_1 : \mathrm{Pic}(S) \to H^2(S,\mathbb{Z})$ is an isomorphism (e.g. S is a rational variety), we have an isomorphism

$$\mathrm{Br}(S) \cong \mathrm{Tors}(H^3(S,\mathbb{Z}).$$

Here we used cohomology with integer coefficients, this requires the assumption that $\Bbbk = \mathbb{C}$. However, the same argument, replacing $\mathbb{Z}$ with $\mathbb{Z}_l$, $l \neq p = \mathrm{char}(\Bbbk)$. Then we obtain isomorphism

$$\mathrm{Br}(S)(l)_{\neq p} \cong \mathrm{Tors}(H^3(S,\mathbb{Z}_l),$$

where $\mathrm{Br}(s)_{\neq p}$ is the Brauer group modulo p-torsion.

Proposition 4.2. *Let $f : X' \dashrightarrow X$ be a birational isomorphism of smooth varieties. Then*

$$Tors(H^3(X,\mathbb{Z}) \cong Tors(H^3(X',\mathbb{Z}).$$

Proof. Since any birational morphism of nonsingular varieties is decomposed into blow-ups and blow-downs with non-singular centers, it is enough to assume that $f : X' \to X$ is a blow-up with a nonsingular center Y. Let $f^* : H^k(X,\mathbb{Z}) \to H^k(X',\mathbb{Z})$ be the usual pull-back homomorphism of cohomology and $f_* : H^k(X',\mathbb{Z}) \to H^k(X,\mathbb{Z})$ be the homomorphism obtained by identifying, via the Poincaré Duality, $H^k(X',\mathbb{Z})$ with $H_{\dim X'-k}(X',\mathbb{Z})^\vee$. Since f is birational, $f_*f^*[X] = [X]$, hence, by the projection formula, $f_*(x \cdot f^*(y)) = f_*(x) \cdot y$ shows that f_*f^* is the identity. Thus $H^k(X,\mathbb{Z})$ injects in $H^k(X',\mathbb{Z})$. In particular, $\mathrm{Tors}(H^3(X,\mathbb{Z}))$ injects in $\mathrm{Tors}(H^3(X',\mathbb{Z})$. Next we use the Leray spectral sequence for f. Since the fibers over points in the center Y of the blow up are projective space, we obtain $R^kf_*\mathbb{Z} = 0$ for odd k. Consider the Leray spectral sequence

$$E^{p,q} := H^p(X, R^qf_*\mathbb{Z}) \Rightarrow H^*(X',\mathbb{Z})$$

The boundary homomorphism $d_2 : E^{p,q} \to E^{p+2,q-1}$ all zeroes. Thus the spectral sequence degenerates and gives an exact sequence

$$0 \to E^{3,0} \to H^3(X',\mathbb{Z}) \to E^{1,2} \to 0.$$

Since $E^{1,2} = H^1(Y,\mathbb{Z})$ is torsion-free (this follows from the formula for universal coefficients) and we know that from above that $E^{3,0} = H^3(X,\mathbb{Z})$ is a direct summand of $H^3(X',\mathbb{Z})$, we get that the torsion of two groups coincide. $\square$

Note that more generally, Grothendieck proves that $\mathrm{Br}(X)$ is a birational invariant.

Corollary 4.3. *Let X be a nonsingular rational variety, then*

$$Br(X) = 0.$$

We use that $c_1 : \mathrm{Pic}(X) \to H^2(X, \mathbb{Z})$ is always an isomorphism for a rational variety. Thus $\mathrm{Br}(X) \cong \mathrm{Tors}(H^3(X, \mathbb{Z}))$.

4.2 Cayley quartic symmetroid

Let L be a web of quadrics in $\mathbb{P}^3$. We assume that L is general in the sense that it intersects D_3 transversally. In particular, $S = D(|L|)$ is a quartic surface with 10 ordinary double points. It is called a *Cayley quartic symmetroid*. Note that it is not true that any quartic surface with 10 nodes is isomorphic to a Cayley quartic symmetroid, although the number of parameters for projective equivalence classes for quartics with 10 nodes and Cayley symmetroids is the same and is equal to 9. A Cayley quartic symmetroid is characterized by the property that the projection from a node exhibits it as a double cover branched along the union of two cubic curves such that there is a conic that is tangent to all of them at three points. Recall that, if we choose the node to have coordinates $[1, 0, 0, 0]$, the equation of a quartic surface with this node becomes

$$x_0^2 Q_2(x_1, x_2, x_3) + 2x_0 F_3(x_1, x_2, x_3) + G_4(x_1, x_2, x_3) = 0.$$

The branch curve of this cover is given by the equation

$$B : F_3^2 - Q_2 G_4 = 0.$$

It is a curve of degree 6 such that the conic $V(Q_2)$ is tangent to it at six points $V(F_3) \cap V(Q_2)$. Since a quartic with 10 nodes has 9 other nodes besides the center of the projection, we see that B has 9 singular points. In our case B is the union of two cubics intersecting at 9 points. We will prove it later. Also, we will explain the relationship between the Cayley quartic symmetroid and Enriques surfaces. For a non-believer of this strange fact, we can recompute the count of constants by other method. As double covers, quartic symmetroids are parameterized by webs of quadrics modulo projective linear group PGL(4). The webs are parameterized by the Grassmannian $G_3(|\mathcal{O}_{\mathbb{P}^3}(2)|)$ of dimension $4(10-4) = 24$. The number of parameters is again $24 - 15 = 9$. Also, we can recompute the number of parameters of 10-nodal quartics by counting the number of parameters of 9-nodal plane sextics that admit an everywhere tangent conic. The number of them is equal $\dim |\mathcal{O}_{\mathbb{P}^2}(6)| - 1 - 9 - 8 = 27 - 18 = 9$.

Note that the moduli space of Cayley quartic symmetroids is of dimension 9. In fact, pairs of cubics depend on 18 parameters (since $\dim |\mathcal{O}_{\mathbb{P}^2}(3)| = 9$) and the preojective linear group PGL(3) is of dimension 9. On the other hand 10-nodal quartic surfaces in $\mathbb{P}^3$ also depend on 9 parameters. In fact, $\dim |\mathcal{O}_{\mathbb{P}^3}(4)| = 34$, each node gives one condition and $\dim \mathrm{PGL}(4) = 15$, all of this gives again $9 = 34 - 10 - 15$ parameters. This shows that quartic symmetroids defines an irreducible component in the space of all 10-nodal quartic surfaces.

4.3 Artin-Mumford counter-example to the Lüroth Problem

Recall that a variety X is *unirational* (resp. *rational*) if there exists a dominant rational map $\mathbb{P}^N \dashrightarrow X$ (resp. a birational map). The Lüroth Problem asked whether a unirational algebraic variety over an algebraically closed field of characteristic 0 is always rational. This is always true when $\dim X \leq 2$ but not always true in higher dimension. The first (rogorous) counter-examples were forund in 1972 by three different teams M. Artin and d. Mumford, H. Clemems and Ph. Griffiths, V. Iskovskilh and Yu. Manin. Here, following A. Beauville, we give an application of our study of linear systems of quadrics to discuss Artin-Mumford example.

Proposition 4.4. *Let $X \to \mathbb{P}^3$ be the double cover of $\mathbb{P}^3$ branched along the Cayley quartic symmetroid $S = |L|$. Then X is unirational.*

Proof. Let $\mathcal{X} = \{(Q, \ell) \in |L| \times G_1(\mathbb{P}^3)\}$. Since a nonsingular quadric in $\mathbb{P}^3$ contains two ruling of lines, a general fiber of the first projection to $|L|$ has two connected components isomorphci to $\mathbb{P}^1$. By Stein factorization, it factors through the double cover of $|L|$ branched along $D(|L|)$. This doubel cover can be identified with X. Let ℓ be a general line in $\mathbb{P}^3$. Then a quadric from $|L|$ contains ℓ if and only if it contains 3 distinct points on it. This gives three conditions in order that Q contains ℓ. Since $\dim |L| = 3$, we obtain that there is a unique quadric containing ℓ. Thus the second projection is of degree 1, hence $\mathcal{X}$ is birationally isomorphic to $G_1(\mathbb{P}^3)$, and, hence, it is rational. Since X is the image of $\mathcal{X}$ under a regular map, is unirational. □

Remark 4.5. In fact, one can show that a double cover of $\mathbb{P}^n$ branched along an irreducible quartic hypersurface is unirational (see [Beauville, Lüroth Problem]).

The next theorem belongs to M. Artin and D. Mumford. We follow Beauville's proof.

Theorem 4.6. *X is not rational.*

Proof. It follows from the proof of the previous theorem that the projection $p_1 : \mathcal{X} \to X$ has fibers isomorphic to $\mathbb{P}^1$. Under the Plücker embedding, they are conics. Let X^o be the smooth locus of X. Its complement consists of 10 isolated points. The restriction $\mathcal{X}^o \to X^o$ of of p_1 over X^o is a Severi-Brauer variety that defines, if it is not trivial, an element in $\mathrm{Br}(X^o)$ of order 2. Suppose it is trivial. Then it is a vector bundle over X^o, and hence admits a rational section. So, it suffices to show that it does not admit such a section.

Suppose $\sigma : X^o \dashrightarrow \mathcal{X}^o$ is such a section. Let $(Q, r), (Q, r_1)$ be two points in X lying in $\pi^{-1}(Q), Q \in |L|$. Here r_1, r_2 are two rulings of lines in Q. Then σ map $(Q, r_1), (Q, r_2)$ to two points to two lines $\ell_1 \subset r_1$ and $\ell_2 \subset r_2$ in Q. These lines intersect at one point $x = \ell_1 \cap \ell_2$. Thus, σ defines a rational section $|L| \to \mathcal{Q}$, where $\mathcal{Q} = \{(x, Q) \in \mathbb{P}^3 \times |L| : x \in Q\}$ is the universal family of the web of quadrics.

Let us show that the universal family $\mathcal{F}$ of any base-point-free r-dimensional linear system $L|$ of hypersurfaces of degree d in $\mathbb{P}^n$ has no rational sections. Suppose it has a rational section $\sigma : |L| \to \mathcal{Q}$. Then its closure is a subvariety of $\mathcal{F}$ birationally isomorphic to $|L|$. Let $\eta \in H^{2r}(\mathcal{Q}, \mathbb{Z})$ be its cohomology class. The subvariety $\mathcal{F}$ is a hypersurface in $|L| \times \mathbb{P}^n$ given by an equation of bi-degree $(1, d)$. Since $|L|$ is base-point-free, $r \geq n$ and $\dim \mathcal{F} = r + n - 1 > 2n - 2$. By Lefschetz's Theorem on a hyperplane section, $H^{2n-2}(|L| \times \mathbb{P}^n) \to H^{2n-2}(\mathcal{F}, \mathbb{Z})$ is an isomorphism. Thus, $H^{2n-2}(\mathcal{F}, \mathbb{Z})$ is spanned by the images of the classes $h_1^i \cdot h_2^{n-1-i}$, where h_1, h_2 are the pre-images of the classes of hyperplanes in $|L|$ and $\mathbb{P}^n$. Under the fist projection $q : \mathcal{F} \to |L|$, the image of $h_1^i \cdot h_2^{n-1-i}$ is equal to $h_1^i \cdot p_*(h_2^{n-1-i})$. Since $\dim |L| > n - 1$, they are all zero, except when $i = 0$, in which case $q : h_2^{n-1} \to |L|$ is of degree d. hence, the image of q_* to $H^0(|L|, \mathbb{Z}) \cong \mathbb{Z}$ is equal to $d\mathbb{Z}$. But the image of $q_*(\eta)$ must be a generator of $H^0(|L|, \mathbb{Z})$. This contradiction proves the theorem.

Thus we have seen that the $\mathbb{P}^1$-fibration over X^o is not isomorphic to a $\mathbb{P}^1$-bundle, hence gives a non-zero 2-torsion element in $\mathrm{Br}(X^o)$. Let $\tilde{X}$ be the blow-up of $\mathbb{P}^3$ at the nodes of X. Its exceptional divisors over the singular points p_i are isomorphic to quadrics E_i. In the commutative diagram

$$\begin{array}{ccc} \mathrm{Pic}(\tilde{X}) & \xrightarrow{c_1} & H^2(\tilde{X},\mathbb{Z}) \\ \downarrow & & \downarrow r \\ \mathrm{Pic}(X^0) & \xrightarrow{c_1} & H^2(X^o,\mathbb{Z}) \end{array}$$

the top horizontal arrow is surjective. To see this use that $H^2(\tilde{X},\mathcal{O}_{\tilde{X}})=0$ since $\tilde{X}$ is a unirational and apply the exponential exact sequence

$$0\to\to\mathbb{Z}\to\mathcal{O}_{\tilde{X}}\to\mathcal{O}^*_{\tilde{X}}\to 0.$$

Since $\tilde{X}\setminus X$ is the disjoint union Y of 10 quadrics, the Gysin exact sequence $H^2(\tilde{X},\mathbb{Z})\to H^2(X^o,\mathbb{Z})\to H^1(Y,\mathbb{Z})=0$ shows that the restriction homomorphism r is surjective. Thus, by Proposition **??**, we get that $\mathrm{Tors}(H^3(X^o,\mathbb{Z}))\neq 0$. Using again the Gysin exact sequence

$$0\to H^3(\tilde{X},\mathbb{Z})\to H^3(X^0,\mathbb{Z})\to H^2(Y,\mathbb{Z})$$

we find that $\mathrm{Tors}(H^3(X^o,\mathbb{Z}))\cong\mathrm{Tors}(H^3(\tilde{X},\mathbb{Z}))$. This proves that $\tilde{X}$ and hence X is not rational. □

Remark 4.7. Suppose $|L|$ has a base point. Then, the universal family $\mathcal{F}\to|L|$ has a section. It assigns to the quadric $Q\in|L|$ the one of the base points x_0 of $|L|$. Thus, the conic bundle $\mathcal{X}\to X$ acquires a section too. It assigns to (Q,r) the line $\ell\in r$ that contains the point x_0. Thus, the Severi-Brauer variety is trivial and $\mathcal{X}$ becomes *stably rational* (i.e. the product with $\mathbb{P}^1$ becomes a rational variety). Note that the non-vanishing of the Brauer group implies that the variety is not stably rational.

An example of the case when the web $|L|$ acquires a base point is the web of quadrics passing through 6 points in a general position in $\mathbb{P}^3$. The discriminant hypersurface in this case is the Kummer quartic surface $\mathrm{Kum}(\mathrm{Jac}(C))$, where the curve C is the double cover of the unique rational normal curve passing through the six points ramified over these points. The six base points are additional nodes of the quartic surface, so all together we have sixteen of them. The double cover X of $\mathbb{P}^3$ branched over the Kummer surface is a rational variety. To see this we consider the *Segre 10-nodal cubic hypersurface* in $\mathbb{P}^4$ It is projectively isomorphic to the cubic hypersurface given in $\mathbb{P}^5$ by two equations (one of them is linear)

$$\sum_{i=1}^{6}x_i^3=\sum_{i=1}^{6}x_i=0.$$

The group $\mathfrak{S}_6$ acts on it by permutation of coordinates. Tt contains 10 singular points forming the $\mathfrak{S}_6$-orbit of the point $[1, 1, 1, -1, -1, -1]$. It also contains 15 planes forming the $\mathfrak{S}_6$-orbit of the plane $x_1 + x_2 = x_3 + x_4 = 0$. It is known that the Segre cubic is isomorphic to the image of $\mathbb{P}^3$ under a birational map given by the linear system of quadrics through 5 points (see [Dolgachev]). In particular, it is a rational variety. Now, the projection from a nonsingular point defined a birational isomorphism to the double cover of $\mathbb{P}^3$ branched along a Kummer quartic surface. Ten of the 16 nodes of the Kummer surface are the projections of the ten nodes of the Segre cubci. The other 6 nodes are the images of the six lines that pass through any nonsingular point on the Segre cubic.

Remark 4.8. Let $f : X \subset \mathbb{P}^4$ be a nonsingular cubic hypersurface and ℓ is a line on it (lines on X are parameterized by a surface, called the *Fano surface* of lines on X). Let $X' \to X$ be the blow-up of ℓ. Let $f : X' \to \mathbb{P}^2$ be the projection morphism with center at ℓ. For any point $x \not\ell$, the plane spanned by x and ℓ intersects X along the union of ℓ and a conic. The conic is isomorphic to the fiber $f^{-1}(f(x))$. Let C be the plane curve parametrizing reducible fibers. One can show that C is a plane curve of degree 5 and the fibers of its points are isomorphic to reduced reducible conics. This defines a non-ramified double cover $\tilde{C} \to C$ whose fibers are components of these reducible conics. One of the proofs of non-rationality of X (it is always is unirational) consists of proving that the *Prym variety*

$$\mathrm{Prym}(\tilde{C}/C) : \mathrm{Ker}(Nm : \mathrm{Jac}(\tilde{C}) \to \mathrm{Jac}(C))$$

is isomorphic to the intermediate Jacobian variety $J(X)$ and it is not isomorphic, as a principally polarized abelian variety to a Jacobian variety. It would be interesting to find the proof of non-rationality by showing that the conic bundle over $\mathbb{P}^2 \setminus C$ is a non-trivial Severi-Brauer variety.

Lecture 5

Quartic symmetroid and Enriques surfaces.

Let us discuss the Cayley quartic symmetroids in more details. Let $L \subset E \cong \Bbbk^4$ be generated by four quadrics $Q_i = V(q_i), i = 1,2,3,4$. If L is general, the base scheme $\mathrm{Bs}(|L|) = Q_1 \cap Q_2 \cap Q_3 \cap Q_4$ is empty (too many homogeneous equations in 4 variables). Let $b_i = b_{q_i}$. Being a symmetric bilinear form in E, it defines a section s_i of an invertible sheaf

$$\mathcal{O}_{\mathbb{P}^3 \times \mathbb{P}^3}(1,1) := p_1^* \mathcal{O}_{\mathbb{P}^3}(1) \otimes p_2^* \mathcal{O}_{\mathbb{P}^3}(1) \cong \mathcal{O}_{\mathbb{P}^3 \times \mathbb{P}^3}(1),$$

where $\mathbb{P}^3 \times \mathbb{P}^3$ is considered to the Severi variety $S_{3,3}$ be in $\mathbb{P}^{15}$ by the Segre embedding $\mathbb{P}^3 \times \mathbb{P}^3 \hookrightarrow \mathbb{P}15$. Let $PB(|L|)$ be the intersection $Z(s_1) \cap \cdots \cap Z(s_4)$ of the schemes of zeros of these sections. This is a complete intersection of four divisors of type $(1,1)$ in $\mathbb{P}^3 \times \mathbb{P}^3$, or as four hyperplane sections of the Severi variety. Since $\omega_{\mathbb{P}^3 \times \mathbb{P}^3} \cong \mathcal{O}_{S_{3,3}}(-4)$, by the adjunction formula,

$$\omega_{\mathrm{PB}(|L|} \cong \mathcal{O}_{\mathrm{PB}(|L|)}.$$

Thus, $\mathrm{PB}(|L|)$ is a K3 surface, assuming that it is nonsingular. Since the bilinear forms b_i are symmetric, the subvariety $\mathrm{PB}(|L|)$ is invariant with respect to the involution σ defined by switching the factors. Its set of fixed point is the diagonal. A fixed point $(x,x) = ([v],[v])$ satisfies $q_i(v) = b_i(v,v) = 0, i = 1,2,3,4$, hence belongs to the base scheme. By assumption, it is empty, hence σ is fixed-point-free. The quotient $S = \mathrm{PB}(|L|)/(\sigma)$ is an *Enriques surface* and the surface $\mathrm{PB}(|L|)$ is its K3-cover.

Consider a morphism $\mathrm{PB}(|L|) \to G_1(\mathbb{P}^3)$ that assigns to a point $(x,y) \in \mathrm{PB}(|L|)$ the line $\langle x,y \rangle$ spanned by the points x,y. Since $\mathrm{PB}(|L|)$ does not intersect the diagonal, the map is well-defined and it factors through S.

Suppose $\langle x, y\rangle = \langle x', y'\rangle$. Let $x = [v], y = [u], x' = [v'], y' = [u']$, we can write $v' = \alpha u + \beta v, u' = \alpha' u + \beta' v$ to obtain

$$0 = b_i(u', v') = \alpha\alpha' b_i(u, u) + \beta\beta' b_i(v, v) + (\alpha\beta' + \alpha'\beta)b_i(u, v)$$

$$= \alpha\alpha' b_i(u, u) + \beta\beta' b_i(v, v) = q_i(\lambda u) + q_i(\mu v),$$

where $\lambda^2 = \alpha\alpha', \mu^2 = \beta\beta'$. This gives

$$0 = b_i(\lambda u, \mu v) = q_i(\lambda u) + q_i(\mu v) - q_i(\lambda u + \mu v) = -q_i(\lambda u + \mu v) = 0.$$

This shows that the point $[\lambda u + \mu v]$ belongs to $\mathrm{Bs}(|L|)$ contradicting the assumption

Let $\ell = \langle [v], [u]\rangle$, where $([v], [u]) \in \mathrm{PB}(|L|)$. A quadric $Q = V(q) \in |L|$ contains ℓ if and only if it contains 3 points on ℓ. Since $0 = b_q(u, v) = q(u) + q(v) - q(u + v)$, if Q contains the points $[v], [u]$, it automatically contains the third point $[u+v]$ on ℓ. This implies that there will be a pencil of quadrics in $|L|$ that contains ℓ but just a unique quadric containing ℓ if it were a general line. Conversely, if ℓ is contained in a pencil of quadrics from $|L|$ the previous equality implies that $b_q(u, v) = 0$ for all $q \in L$, and hence $\ell = \langle [v], [u]\rangle$, where $([v], [u]) \in \mathrm{PB}(|L|)$

A line in a web $|L|$ that is contained in a pencil of quadrics form $|L|$ is called a *Reye line*. Thus we see that the image of the map $\mathrm{PB}(|L|) \to G_1(|E|)$ is a surface parameterizing Reye lines of $|L|$.

A surface in $G_1(\mathbb{P}^3)$ is called a *congruence of lines*. We know that

$$\begin{aligned} H^2(G_1(\mathbb{P}^3), \mathbb{Z}) &= \mathbb{Z}\sigma, \\ H^4(G_1(\mathbb{P}^3), \mathbb{Z}) &= \mathbb{Z}\sigma_1 + \mathbb{Z}\sigma_2, \end{aligned}$$

where σ is the class of a hyperplane section in the Plücker embedding $G_1(\mathbb{P}^3) \hookrightarrow \mathbb{P}^5$, and σ_1, σ_2 are the classes of planes in of two rulings of planes in a 4-dimensional quadric $G_1(\mathbb{P}^3)$. We may assume that $\sigma_1 = [\pi_x], \sigma_2 = [\pi_\Lambda]$. We use the intersection theory on the quadric $G_1(\mathbb{P}^3)$. We have $\sigma_1^2 = [\pi_x \cap \pi_y] = [\text{point}]$, and $\sigma_2^2 == [\pi_\Lambda \cap \pi_\Lambda] = [\text{point}]$. Thus $\sigma_i^2 = 1$. Also, we have $\sigma_1 \cdot \sigma_2 = [\pi_x \cap \pi_\Lambda] = [\emptyset]$, so $\sigma_1 \cdot \sigma_2 = 0$. We can represent σ by the variety π_ℓ of lines intersecting a fixed line ℓ. We have $\pi_\ell \cap \pi_{\ell'} \cap \pi_\Lambda$ is the unique line in Λ intersecting the points $\ell \cap \Lambda$ and $\ell' \cap \Lambda$. This gives $\sigma^2 \cdot \sigma_1 = 1$. Similarly we get $\sigma^2 \cdot \sigma_2 = 1$. Thus $\sigma^2 = \sigma_1 + \sigma_2$. All of this is a special case of the *Schubert calculus* that describes the intersection theory of Grassmannians (see [Fulton]).

Let $Z \subset G_1(\mathbb{P}^3)$ be a congruence of lines in $\mathbb{P}^3$ and let $[Z] \in H^4(G_1(\mathbb{P}^3), \mathbb{Z})$ be its cohomology class. Then $[Z] = m\sigma_1 + n\sigma_2$ and intersecting with σ_1, σ_2, and using the previous computations, we find that

$$\begin{aligned} m &= [Z] \cdot \sigma_1 = \#\{\text{lines in } Z \text{ that contain a general point in } \mathbb{P}^3\}, \\ n &= [Z] \cdot \sigma_2 = \#\{\text{lines in general plane in } \mathbb{P}^3 \text{ that are contained in } Z\}. \end{aligned}$$

The number m (resp. n) is called the *order* (resp. *class*) of the congruence Z, The pair (m, n) is called the *bidegree* of Z. Note that the degree of Z in the Plücker embedding is equal to

$$\deg Z = \sigma^2 \cdot [Z] = \sigma^2 \cdot (m\sigma_1 + n\sigma_2) = m + n.$$

Let Z be the congruence of Reye lines of $|L|$, called the *Reye congruence* of $|L|$.

Proposition 5.1. *The bidegree of the Reye congruence of $|L|$ is equal to $(7, 3)$.*

Proof. Fix two general planes $|N_1|$ and $|N_2$ in $|L|$. Suppose ℓ is contained in a pencil $\mathcal{P}$ of quadrics in $|L|$ which is not contained in $|N_1|$ or $|N_2|$. Since $\mathcal{P}$ intersects $|M_1|$ and $|M_2|$ at one point, the line ℓ is contained in unique quadric in $|N_1$ and in a unique quadric in $|N_2|$. Let X be the variety of lines contained in some quadric from a net of quadrics in $\mathbb{P}^3$. It is a hypersurface in $G_1(\mathbb{P}^3)$, called the *Montesano cubic complex*. Let us see that its class in $H^2(G_1(\mathbb{P}^3), \mathbb{Z})$ is equal to 3σ, i.e. it is equal to the intersection of the quadric $G_1(\mathbb{P}^3)$ and a cubic hypersurface. It is an example of a Fano variety of degree 6 in $\mathbb{P}^5$. To see this, it is enough to compute the number of lines in X that are contained in $\pi_x \cap \pi_\Lambda$ for some general plane Λ and a general point $x \in \Lambda$. When we restrict the net $|M|$ to the plane Λ we get a net $\mathcal{N}(\Lambda)$ of conics in $\Lambda \cong \mathbb{P}^2$. Its discriminant curve is a cubic curve. It parameterizes reducible conics in the net. Let $\mathcal{P}(\Lambda, x)$ be the pencil of conics in $\mathcal{N}(\Lambda)$ that pass through a general point $x \in \Lambda$. Then its discriminant consists of 3 points, and thus there are three singular conics passing through x. When x is general enough we may assume that x is not the singular point of the conic. Thus there will three line components passing through x, and hence $\deg X = 3$.

Thus, we see that our congruence is contained in the intersection of two Montesano complexes. The latter is a surface of class $(3\sigma)^2 = 9(\sigma_1 + \sigma_2)$. So, its bidegree is equal to $(9, 9)$. It remains to compute the bidegree of the residual surface and to show that it is equal to $(2, 6)$. The residual surface is the surface of lines that is contained in some quadric from the pencil $|M_1| \cap |M_2|$. The base curve of this pencil is a quartic elliptic curve. As a curve on a nonsingular quadric, it is a curve of bidegree $(2,2)$. Thus any line contained in some quadric will intersect this curve at 2 points. This we have to compute the bidegree of the congruence of bisecant lines of a quartic elliptic curve. The number of secants passing through a general point is equal to 2 because, projecting from this point we get a plane curve of degree 4 of arithmetic genbus 3 and geometric genus 1, so it must have two nodes. Thus the order of the congruence is equal to 2. Now intersecting the quartic curve with a general plane we get four points. Thus the number of bisecants is equal to $\binom{4}{2} = 6$. So, the class of the residual congruence is equal to $(2,6)$, as asserted. □

Theorem 5.2. *Assume that $D(|L|)$ does not contain lines. The K3-cover* $\mathrm{PB}(|L|)$ *of the Reye congruence Enriques surface S is a minimal resolution of the Cayley quartic symmetroid. It is also isomorphic to a nonsingular quartic surface in $\mathbb{P}^3$ with 10 lines equal to the singular loci of quadrics of corank 2 in the web.*

Proof. Let $D(|L|)$ be the quartic symmetroid of a general web of quadrics $|L|$ in $\mathbb{P}^3$. Let $\tilde{D}(|L|) = \{(Q,x) \in |L| \times \mathbb{P}^3 : x \in \mathrm{Sing}(Q)\}$ be the incidence variety which we had already used in discussing the discriminant variety of a linear system of quadrics. For the complete linear system of quadrics, the first projection defines a resolution of singularities of the discriminant variety D_3 of quadrics in $\mathbb{P}^3$. One can show that, when $|L|$ intersects transversally D_3, the projection $p_1 : \tilde{D}(|L|) \to D(|L|)$ is still a resolution of singularities. Its fibers over each of the ten singular points is isomorphic to the singular line of the corresponding quadric of corank 2. This is (-2)-curve on the K3 surface $\tilde{D}(|L|)$. Now let us consider a rational map $\mathrm{st} : D(|L|) \dashrightarrow \mathbb{P}^3$ (the *Steiner map*) that assigns to a singular quadric $Q \in D(|L|)$ its singular point. The map is not defined at the singular points of $D(|L|)$ but extends to a regular map of $\tilde{D}(|L|)$. Let us find the image $\mathrm{St}(|L|)$ of the map. Choose a basis (q_1,q_2,q_3,q_4) of L and let (A_1,A_2,A_3,A_4) be the corresponding matrices of the bilinear forms b_{q_i}'s. Let $x = [v] = [x_1,x_2,x_3,x_4]$ be the singular point of a quadric $V(\sum \lambda_i q_i)$ of corank 1. It spans the null space of the matrix $\sum \lambda_i A_i$. Let us write A_i in terms of its columns $A_i = [A_i^{(1)}, \ldots, A_i^{(4)}]$. Then

$$(\sum_{i=1}^{4} \lambda_i A_i) \cdot v = \sum_{i=1}^{4} \lambda_i (\sum_{j=1}^{4} x_j A_i^{(j)}).$$

Considered as a system of linear equations in unknowns $\lambda_1, \ldots, \lambda_4$, it has a non-trivial solution if and only if

$$\det[\sum_{j=1}^{4} x_j A_1^{(j)}, \sum_{j=1}^{4} x_j A_2^{(j)}, \sum_{j=1}^{4} x_j A_3^{(j)}, \sum_{j=1}^{4} x_j A_4^{(j)}] = 0.$$

Obviously, it is a homogeneous polynomial of degree 4 in the coordinates $x_1, \ldots, x_4$. Thus, we see that $D(|L|)$ is birationally isomorphic to the quartic surface $\mathrm{St}(|L|)$ and the Steiner map extends to an isomorphism $\tilde{D}(|L|) \to \mathrm{St}(|L|)$. Thus $\mathrm{St}(|L|)$ is isomorphic to a minimal resolution of the Cayley quartic symmetroid.

It remains to prove that $\mathrm{PB}(|L|)$ is also isomorphic to the surface $\mathrm{St}(|L|)$. For this we consider the first projection $\mathrm{PB}(|L|) \to \mathbb{P}^3$ and show that its image coincides with $\mathrm{St}(|L|)$. Let $([v],[u]) \in \mathrm{PB}(|L|)$, then the hyperplane $b_i(v) = 0$ is the polar of Q_i at the point $[v]$. The intersection of these four polar hyperplanes contains the point $y = [u]$. Thus the linear forms $b_i(v)$ are linear dependent, hence there is a non-trivial linear relation $\sum \lambda_i b_i(v) = 0$. This shows that the quadric $Q = V(\sum_{i=1}^{4} \lambda_i q_i)$ is singular at $x = [v]$. So the point $[v]$ belongs to $\mathrm{St}(|L|)$. It is clear that $y \in p_1^{-1}(x)$. Suppose that $y' \in p_1^{-1}(x)$ then the previous argument shows that the intersection of polar hyperplanes $b_i(v) = 0$ contains the line $\langle y, y' \rangle$. Hence there is at least one-dimensional linear space of linear relations between the hyperplanes, i.e. there is a pencil of quadrics with a singular point at $x = [v]$. This implies that $\mathsf{D}(|L|)$ contains a line. Thus, under our assumption, the map $\mathrm{PB}(|L|) \to \mathrm{St}(|L|)$ is bijective. Now the fiber of the $\tilde{\mathrm{st}} : \tilde{D}(|L|) \to \mathrm{St}(|L|)$ over the point x is the same space of quadrics singular at x. Thus, under our assumption, the surfaces $\tilde{D}(|L|), \mathrm{St}(|L|), \mathrm{PB}(|L|)$ are isomorphic. □

Remark 5.3. Let $\mathrm{PB}(|L|) \subset |E| \times |E| \hookrightarrow |E \otimes E| \cong \mathbb{P}^{14}$ be the Segre embedding. Since $\mathrm{PB}(|L|)$ is equal to the intersection of 4 hyperplanes, it is contained in $|E \otimes E/W|$, where $|W|$ is spanned by hyperplanes containing $\mathrm{PB}(|L|)$. Under the involution σ the space $(E \otimes E)^\vee \cong E^\vee \otimes E$ decomposes into the direct product $S^2 E^\vee \oplus \bigwedge^2 E^\vee$. The linear system $|\bigwedge^2 E^\vee|$ defines the Plücker embedding of $S = \mathrm{PB}(|L|)/(\sigma)$ in $|\bigwedge^2 E|$ with the image equal to the Reye congruence. The linear system $|S^2 E^|$ restricts on $\mathrm{PB}(|L|)$ to a 5-dimensional linear system $|W^\perp| \subset |S^2 E| = |(S^2 E^\vee)^\vee|$. It embeds S into another $\mathbb{P}^5$ with the image not lying on a quadrics. The image is contained in the determinantal quartic hypersurface $D(W^\perp)$ of the five-dimensional linear system of quadrics $|W^\perp|$. It is equal to the singular locus of $D(W^\perp)$ parameterizing quadrics of corank 2. It is a surface of degree 10 given by partial derivatives of the equation defining $D(W^\perp)$.

Remark 5.4. We constructed three different nonsingular birational models of the Cayley quartic symmetroid $D(|L|)$. They are $\tilde{D}(|L|), \mathrm{St}(|L), \mathrm{PB}(|L|)$. They are isomorphic to the K3-cover of an Enriques surface S which is isomorphic to a Reye congruence of lines of bi-degree (7,3) in $G_1(\mathbb{P}^3)$. We know that the double cover of $\mathbb{P}^3$ branched along $D(|L|)$ is a unirational but non-rational 3-fold. A natural question: is the same true for the double cover of $\mathbb{P}^3$ branched along the nonsingular quadric $\mathrm{St}(|L|)$?

Finally, we have to prove Cayley's theorem characterizing quartic symmetroids among quartics with 10 nodes. Let η_H and η_S be the divisor classes in $\tilde{D}(|L|)$ equal to the pre-image of a hyperplane under the morphisms $\pi : \tilde{D}(|L|) \to D(|L|) \subset |L|$ and $\tilde{st} : \tilde{D}(|L|) \to \mathrm{St}(|L|) \subset |E|$. One can show that

$$2\eta_S = 3\eta_H - r_1 - \cdots - r_{10}, \tag{5.0.1}$$

where r_i are the classes of the exceptional curves of the projection π. The relation (**??**) characterize a minimal resolution of a quartic symmetroid from a minimal resolution of any 10-nodal quartics (see [Dolgachev, Classical AG]).[1] The Steinerian map is given by a column of minors of the adjugate matrix defining $D(|L|)$. They coincide with polar cubics of $D(|L|)$ and define a linear subsystem of $|3\eta_H - r_1 - \cdots - r_{10}|$. Let Q_i be the quadric of corank 2 corresponding to a singular point p_i of $D(|L|)$. It is the union of two planes, and the pre-image of $Q_i \cap \mathrm{St}(|L|)$ under the Steinerian map is the union of two cubics, the residual cubics of the intersection of each of the planes with $\mathrm{St}(|L|)$. These cubics are projected from p_i to the union of two components of the branch curve of the double cover.

[1] For experts: the Picard lattice of the former is $U \perp E_8(2) \perp \langle -4 \rangle$ and the Picard lattice of the latter is $\langle 4 \rangle \perp \langle -2 \rangle^{\oplus 10}$.

Index