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Finite groups of symplectic automorphisms of K3 surfaces in positive characteristic

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Abstract

We show that Mukai's classification of finite groups which may act symplectically on a complex K3 surface extends to positive characteristic p under assumptions that (i) the order of the group is coprime to p and (ii) either the surface or its quotient is not birationally isomorphic to a supersingular K3 surface with Artin invariant 1. In the case without assumption (ii) we classify all possible new groups which may appear. We prove that assumption (i) on the order of the group is always satisfied if $p > 11$. For $p = 2, 3, 5, 11$, we give examples of K3 surfaces with finite symplectic automorphism groups of order divisible by p which are not contained in Mukai's list.

1. Introduction

A remarkable work of S. Mukai [Mu] gives a classification of finite groups which can act on a complex algebraic K3 surface X leaving invariant its holomorphic 2-form (symplectic automorphism groups). Any such group turns out to be isomorphic to a subgroup of the Mathieu group M_{23} which has at least five orbits in its natural action on a set of 24 elements. A list of maximal subgroups with this property consists of 11 groups, each of which can be realized on an explicitly given K3 surface. A different proof of Mukai's result was given later by S. Kondō [Ko]. G. Xiao [Xiao] classified all possible topological types of a symplectic action. Neither Mukai's nor Kondō's proof extends to the case of K3 surfaces over algebraically closed fields of positive characteristic p. In fact there are known examples of surfaces over a field of positive characteristic whose automorphism group contains a finite symplectic subgroup which is not realized as a subgroup of M_{23} (e.g. the Fermat quartic over a field of characteristic 3, or the surface from [DKo] over a field of characteristic 2).

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The main tool used in Mukai's proof is the characterization of the representation of a symplectic group $G \subset \mathrm{Aut}(X)$ on the 24-dimensional cohomology space

$$H^*(X,\mathbb{Q}) = H^0(X,\mathbb{Q}) \oplus H^2(X,\mathbb{Q}) \oplus H^4(X,\mathbb{Q}).$$

Using the Lefschetz fixed-point formula and the description of possible finite cyclic subgroups of $\mathrm{Aut}(X)$ and their fixed-point sets due to Nikulin [Ni1], one can compute the value $\chi(g)$ of the character of this representation at any element of finite order n. It turns out that

$$\chi(g) = \varepsilon(n)$$

for some function $\varepsilon(n)$ and the same function describes the character of the 24-permutation representation of M_{23}. A representation of a finite group G in a finite-dimensional vector space of dimension 24 over a field of characteristic 0 is called a Mathieu representation if its character is given by

$$\chi(g) = \varepsilon(\mathrm{ord}(g)).$$

The obvious fact that G leaves invariant an ample class, $H^0(X,\mathbb{Q})$, $H^4(X,\mathbb{Q})$, $H^2(X,\mathcal{O}_X)$ and $H^0(X,\Omega^2_X)$, shows that

$$\dim H^*(X,\mathbb{Q})^G \geq 5.$$

These properties and the known classification of finite subgroups of $\mathrm{SL}(2,\mathbb{C})$ which could be realized as stabilizer subgroups of G in its action on X show that the 2-Sylow subgroups of G can be embedded in M_{23}. By clever and nontrivial group theory arguments Mukai proves that subgroups of M_{23} with at least five orbits are characterized by the properties, that they admit a rational Mathieu representation V with $\dim V^G \geq 5$, and that their 2-Sylow subgroups are embeddable in M_{23}.

The main difficulties in the study of K3 surfaces over algebraically closed fields of positive characteristic p arise from the absence of the Torelli theorem, the absence of a natural unimodular integral lattice containing the Neron-Severi lattice, the presence of supersingular K3 surfaces, and the presence of wild automorphisms.

A group of automorphisms is called *wild* if it contains a wild automorphism, an automorphism of order equal to a power of the characteristic p. Otherwise, it is called *tame*.

In this paper we first improve the results of our earlier paper [DK1] by showing that a finite symplectic group of automorphisms G is always tame if $p > 11$. Next, we show that Mukai's proof can be extended to finite tame symplectic groups in any positive characteristic p, unless both the surface and its quotient are birationally isomorphic to a supersingular K3 surface with Artin invariant equal to 1 (the exceptional case). To do this, we first prove that Nikulin's classification of finite order elements and their sets of fixed points extends to positive characteristic p, as long as the order is coprime to p. Next we consider the 24-dimensional representations of G on the l-adic cohomology, $l \neq p$,

$$H^*_{\text{et}}(X, \mathbb{Q}_l) = H^0_{\text{et}}(X, \mathbb{Q}_l) \oplus H^2_{\text{et}}(X, \mathbb{Q}_l) \oplus H^4_{\text{et}}(X, \mathbb{Q}_l)$$

and on the crystalline cohomology

$$H^*_{\text{crys}}(X/W) = H^0_{\text{crys}}(X/W) \oplus H^2_{\text{crys}}(X/W) \oplus H^4_{\text{crys}}(X/W).$$

It is known that the characteristic polynomial of any automorphism g has integer coefficients which do not depend on the choice of the cohomology theory. Comparing $H^2_{\text{crys}}(X/W)$ with the algebraic de Rham cohomology $H^2_{\text{DR}}(X)$ allows one to find a free submodule of $H^*_{\text{crys}}(X/W)^G$ of rank 5 except when both X and X/G are birationally isomorphic to a supersingular K3 surface with the Artin invariant equal to 1 (the exceptional case). This shows that in a nonexceptional case, for all prime $l \neq p$, the vector spaces

$$V_l = H^*_{\text{et}}(X, \mathbb{Q}_l)$$

are Mathieu representations of G with

$$\dim V_l^G \geq 5.$$

A careful analysis of Mukai's proof shows that this is enough to extend his proof.

In the exceptional case, it is known that a supersingular K3 surface with the Artin invariant equal to 1 is unique up to isomorphism. It is isomorphic to the Kummer surface of the product of two supersingular elliptic curves if $p > 2$, and to the surface from [DKo] if $p = 2$. We call a tame group G *exceptional* if it acts on such a surface with $\dim V_l^G = 4$. We use arguments from Mukai and some additional geometric arguments to classify all exceptional groups. All exceptional groups turn out to be subgroups of the Mathieu group M_{23} with 4 orbits. The problem of realizing exceptional groups will be discussed in another publication.

In the last section we give examples of K3 surfaces in characteristic

$$p = 2, 3, 5, 11$$

with wild finite symplectic automorphism groups which are not contained in Mukai's list. We do not know a similar example in characteristic $p = 7$, however we exhibit a K3 surface with a symplectic group of automorphisms of order 168 which does not lift to a surface from Mukai's list.

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2. Automorphisms of order equal to the characteristic p

Let X be a K3 surface over an algebraically closed field k of positive characteristic p.

The automorphism group $\mathrm{Aut}(X)$ acts on the 1-dimensional space of regular 2-forms $H^0(X,\Omega_X^2)$. Let

$$\chi_{2,0} : \mathrm{Aut}(X) \to k^*$$

be the corresponding character. An automorphism g is called *symplectic* if $\chi_{2,0}(g) = 1$.

Obviously any automorphism of order equal to the characteristic is symplectic. The main result of this section is the following.

THEOREM 2.1. *Let g be an automorphism of X of order $p = \mathrm{char}(k)$. Then $p \le 11$.*

First, we recall the following result from our previous paper [DK1].

THEOREM 2.2. *Let g be an automorphism of order p and let X^g be the set of fixed points of g. Then one of the following cases occurs.*

(1) *X^g is finite and consists of $0, 1,$ or 2 points; X^g may be empty only if $p = 2$, and may consist of 2 points only if $p \le 5$.*

(2) *X^g is a divisor such that the Kodaira dimension of the pair (X, X^g) is equal to 0. In this case X^g is a connected nodal cycle, i.e. a connected union of smooth rational curves.*

(3) *X^g is a divisor such that the Kodaira dimension of the pair (X, X^g) is equal to 1. In this case $p \le 11$ and there exists a divisor D with support X^g such that the linear system $|D|$ defines an elliptic or quasi-elliptic fibration $\phi : X \to \mathbb{P}^1$.*

(4) *X^g is a divisor such that the Kodaira dimension of the pair (X, X^g) is equal to 2. In this case X^g is equal to the support of some nef and big divisor D. Take D minimal with this property. Let*

$$d := \dim H^0(X, \mathcal{O}_X(D - X^g)) \quad \text{and} \quad N := \frac{1}{2}D^2 + 1.$$

Then

$$p(N-d-1) \leq 2N-2.$$

Theorem 2.2 does not give the bound for p in cases (1), (2) and (4). First we take care of case (4).

PROPOSITION 2.3. *Let g be an automorphism of order $p = \mathrm{char}(k)$ and $F = X^g$ (with reduced structure). Assume $\kappa(X,F) = 2$. Then $p \leq 5$. Moreover, if $p = 5$, then X^g contains a curve C of arithmetic genus 2 and the linear system $|C|$ defines a double cover $X \to \mathbb{P}^2$. The surface is birationally g-isomorphic to the affine surface*

$$z^2 = (y^5 - yx^4)P_1(x) + P_6(x),\ \ g(x,y,z) = (x, y+x, z),$$

where $P_i(x)$ is a polynomial of degree i.

Proof. We improve an argument from our paper [DK1].

Case 1: *F is* nef *and $|F|$ has nonempty fixed part.* By a well-known result due to B. Saint-Donat [SD], $F \sim aE + \Gamma$, where $a \geq 2$ and E is an irreducible curve of arithmetic genus 1 and Γ is the fixed part which is a (-2)-curve with $E \cdot \Gamma = 1$. In this case g leaves invariant a genus 1 pencil defined by the linear system $|E|$. Note that Γ is a section of the fibration and is fixed pointwisely by g. It is well known that no elliptic curve admits an automorphism of order ≥ 5 fixing the origin. Thus $p \leq 3$.

Case 2: *F is* nef *and $|F|$ has no fixed part.* Assume $F^2 = 0$. Then $F \sim aE$, where E is an irreducible curve of arithmetic genus 1; hence $\kappa(X,F) = 1$.

Assume $F^2 = 2$. Then $|F|$ has no base points, and $|F|$ defines a map of degree 2 onto $\mathbb{P}^2$. Since we can assume that $p \neq 2$, the map is separable and the branch curve is a curve of degree 6 with simple singularities which is invariant under a projective transformation of order p. Since F is the pre-image of a line, g fixes pointwisely a line in $\mathbb{P}^2$. Thus it is conjugate to a transformation $(x_0, x_1, x_2) \mapsto (x_0, x_1, x_2 + x_1)$. It is known that the ring of invariants $k[x_o, x_1, x_2]^g$ is generated by the three polynomials $x_0, x_1, x_2^p - x_2 x_1^{p-1}$. Thus $p \leq 5$.

Assume $F^2 \geq 4$. Then F is nef and big, hence Theorem 2.2 (4) applied to $D = F$ gives $d = 1$ and $p(N-2) \leq 2N-2$. Since $N = D^2/2 + 1 \geq 3$, this gives $p \leq 3$.

Case 3: *F is not* nef. Since F is reduced and connected, being non-nef means that

$$F = F_1 + C_1 + \ldots + C_k,$$

where C_i's are chains of (-2)-curves with no common components and $C_i \cdot F_1 = 1$ and F_1 is nef. In fact, let E be a (-2)-curve such that $F \cdot E < 0$.

Then $F = E + F'$, and $E \cdot F = -2 + E \cdot F' > -2$. Hence $E \cdot F' = 1$. This implies that E is an end-component of F and F' is connected. If F' is not nef, repeat the same process. Continuing in this way, we prove the claim.

If $F_1^2 = 0$, then g fixes F_1 and C_1; hence g fixes pointwisely a fibre of an elliptic fibration and a section. We argue as in Case 1 to get $p \leq 3$. If $F_1^2 = 2$, we use Case 2 and get $p \leq 5$. Let us assume that $F_1^2 \geq 4$; thus F_1 is big and nef.

Take $D = F + F_1 = 2F_1 + C_1 + \ldots + C_k$. It is nef, and indeed minimal-nef. Then

$$N = h^0(D) - 1 = \frac{1}{2}D^2 + 1 = \frac{1}{2}(4F_1^2 + 4k - 2k) + 1 = 2F_1^2 + k + 1,$$

$$d = h^0(D - F) = h^0(F_1) = \frac{1}{2}F_1^2 + 2 \geq 4.$$

Hence $N = 4d + k - 7$ and the inequality of Theorem 2.2 (4) gives

$$p(N - d - 1) = p(3d + k - 8) \leq 2(N - 1) = 8d + 2k - 16.$$

Thus,

$$p \leq \frac{(8d + 2k - 16)}{(3d + k - 8)} = 2 + \frac{2d}{3d + k - 8} < 4. \qquad \square$$

Now, consider cases (1) and (2) of Theorem 2.2, where X^g is either one point or a connected nodal cycle. We first recall the following information from [DK1, Lemma 2.1 and Th. 2.4].

PROPOSITION 2.4. *The following is true.*

(1) *If X^g consists of a point, then $X/(g)$ is either a rational surface with trivial canonical divisor and one isolated elliptic Gorenstein singularity, or a* K3 *surface with one rational double point. The latter case occurs only if $p \leq 5$.*
(2) *If X^g is a nodal cycle, then $X'/(g)$ is a rational surface with trivial canonical divisor with one isolated elliptic Gorenstein singularity, where X' is the surface obtained from X by blowing down the nodal cycle X^g.*

In the following proposition, we prove that case (2) of Theorem 2.2 occurs only if $p = 2$.

PROPOSITION 2.5. *Suppose X^g is a nodal cycle. Then $p = 2$.*

Proof. The quotient $Z = X/(g)$ is known to be a rational surface with at most rational singularities and $-K_Z = (p-1)B$, where $B_{\rm red}$ is the image of the nodal cycle in Z (see [DK1, §3]). Let $\pi : Z' \to Z$ be a minimal resolution of singularities. Then $-K_{Z'} = -\pi^*(K_Z) + \Delta' = (p-1)\pi^*(B) + \Delta'$, where Δ' is a positive divisor supported on the exceptional locus. Obviously Z' is a resolution of singularities of the surface $X'/(g)$, where X' is obtained from X by blowing down the nodal cycle. Let $a : Z' \to V$ be the blowing down to a minimal resolution of $X'/(g)$. Then $-K_V = a_*((p-1)\pi^*(B) + \Delta')$ is the fundamental cycle of V. By Corollary 3.6 of [DK1], we have $H^1((p-1)B, \mathcal{O}_{(p-1)B}) \cong k$. Since Z has only rational singularities this implies that $H^1((p-1)\pi^*(B), \mathcal{O}_{(p-1)\pi^*(B)}) \cong k$, hence $H^1(a_*(p-1)B, \mathcal{O}_{a_*(p-1)B}) \neq \{0\}$. It is known that for any proper part A of the fundamental cycle of a minimal elliptic singularity, we have $H^1(A, \mathcal{O}_A) = 0$. This implies that

$$-K_V = (p-1)a_*(\pi^*(B)).$$

Since V is a nonminimal rational surface, it contains a (-1)-curve E. Intersecting both sides of the previous equality with E, we get $1 = (p-1)\pi^*(B) \cdot E$; hence $p = 2$. □

LEMMA 2.6 (P. Samuel). *Let A be a normal noetherian local k-algebra of dimension ≥ 2 with maximal ideal $\mathfrak{m}$. Let G be a finite group of automorphisms of A with local ring of invariants A^G. Assume that G acts freely on the punctured local scheme $V = \operatorname{Spec}(A) \setminus \{\mathfrak{m}\}$. Then the class group $\operatorname{Cl}(A^G)$ fits in the following exact sequence of groups*:

$$0 \to H^1(G, A^*) \to \operatorname{Cl}(A^G) \to H^0(G, H^1(V, \mathcal{O}_V^*)).$$

In particular, if in addition A is factorial, $\operatorname{Cl}(A^G) \cong H^1(G, A^*)$.

Proof. Let $U = \operatorname{Spec}(A^G) \setminus \{\mathfrak{m}^G\}$. It is known that $\operatorname{depth}(A^G) \geq 2$ [Fo]. Thus the class group $\operatorname{Cl}(A^G)$ is isomorphic to the Picard group $\operatorname{Pic}(U)$. We apply the two spectral sequences which we used in [DK1] to the free action of G on V with quotient U and the G-linearized sheaf $\mathcal{O}_V^*$:

$$E_2^{i,j} = H^i(G, H^j(V, \mathcal{O}_V^*)) \Rightarrow \mathbb{H}^n,$$

$${}'E_2^{i,j} = H^i(U, \mathcal{H}^j(G, \mathcal{O}_V^*)) \Rightarrow \mathbb{H}^n.$$

Since G acts freely,

$$\mathcal{H}^j(G, \mathcal{O}_V^*) = 0 \quad \text{for} \quad j > 0$$

and the second spectral sequence gives an isomorphism

$$\operatorname{Cl}(A^G) = \operatorname{Pic}(U) = H^1(U, \mathcal{O}_U^*) \cong \mathbb{H}^1.$$

Now the first assertion follows from the first spectral sequence.

If A is factorial, then

$$\operatorname{Cl}(A) = H^1(V, \mathcal{O}_V^*) = 0,$$

which proves the last assertion. □

The well-known property of cohomology groups ([CE, Ch. XII, Prop. 2.5] gives the following.

COROLLARY 2.7. *In the situation of the above lemma, if in addition* $|\mathrm{Cl}(A)|$ *is finite, then* $\mathrm{Cl}(A^G)$ *is killed by multiplication by the product* $|\mathrm{Cl}(A)| \cdot |G|$.

We will also use the following formula from [KS, Lemma 4.1.8] or [Sa, Th. 7.4].

LEMMA 2.8. *Let* X *be a smooth surface over an algebraically closed field* k *of characteristic* $p \geq 0$ *and* $x \in X$ *be a closed point of* X. *Let* G *be a finite group of automorphisms of* X *such that* $X^G = \{x\}$. *Let* $U = X \setminus \{x\}$ *and* $V = U/G$. *Then*

$$e_c(U) = (\#G - 1)(e_c(V) + 1) + e_c(V) - \sum_{g \in G \setminus \{1\}} l(g), \tag{2.1}$$

where $e_c(Z)$ *denotes the* l*-adic Euler-Poincaré characteristic with compact support for any* $l \neq p$ *and* $l(g)$ *is the intersection index of the graph of* g *with the diagonal at the point* (x, x).

PROPOSITION 2.9. *Let* g *be a wild automorphism of order* p *acting on a* K3 *surface such that* $|X^g| = 1$. *Assume that the quotient surface is rational. Then there exists a* g*-invariant elliptic or quasi-elliptic fibration on* X.

Proof. Note that the quotient surface $X/(g)$ is a rational surface with trivial canonical divisor with one isolated elliptic Gorenstein singularity Q. Let

$$\sigma : Y \to X/(g)$$

be a minimal resolution. We have

$$K_Y = -\Delta,$$

where Δ is an effective divisor whose support is equal to the exceptional set of σ and satisfies

$$\Delta \cdot R_i \leq 0$$

for any irreducible component R_i of Δ (see, for example, [Re], 4.21). Let N be the sublattice of $\mathrm{Pic}(Y)$ spanned by the divisor classes of the curves R_i and N^* be its dual lattice. It is known that the class group of the local ring of the singular point Q is mapped surjectively onto the group N^*/N isomorphic to the group of connected components of the local Picard group ([Gr], pp. 189-191). By Corollary 2.7, the group N^*/N is a p-elementary finite abelian group. Since Y is a rational surface, the Picard lattice $\mathrm{Pic}(Y)$ is a unimodular hyperbolic lattice; hence the orthogonal complement $N^{\perp}$ is a p-elementary lattice of signature $(1,t)$, $t \geq 0$. Since N contains K_Y, it is also an even lattice. In particular, $\operatorname{rank} N^\perp \geq 2$ if p is odd.

Case 1: $p > 2$ *and* $\operatorname{rank} N^\perp \geq 3$, *or* $p = 2$ *and* $\operatorname{rank} N^\perp \geq 2$. It follows from [RS] that, if $p > 2$, an even p-elementary hyperbolic lattice of rank ≥ 3 is determined uniquely by its rank and discriminant. An explicit construction of such a lattice shows that it always contains an isotropic vector. Also it is known that an even 2-elementary hyperbolic lattice of rank ≥ 2 always contains an isotropic vector ([Ni2, Th. 4.3.3]). Thus $N^\perp$ contains an isotropic vector v. By Riemann-Roch,

$$h^0(\mathcal{O}_Y(v)) + h^0(\mathcal{O}_Y(K_Y - v)) \geq 1.$$

Since K_Y is anti-effective, it follows that v or $-v$ is effective. We may assume that v is effective. Since the homomorphism

$$\sigma^* : \operatorname{Pic}(X/\langle g\rangle) \to N^\perp$$

has finite cokernel, some multiple of v is linearly equivalent to $\sigma^*(D)$, where D is a Cartier effective divisor on $X/\langle g\rangle$ with $D^2 = 0$. (Here we use the intersection theory for $\mathbb{Q}$-divisors on $\mathbb{Q}$-factorial surfaces.) The pre-image A of D in $\operatorname{Pic}(X)$ is a g-invariant effective divisor with $A^2 = 0$. Dividing it by a positive integer, we may assume that the divisor class $[A]$ is primitive. Applying to $[A]$ a sequence of reflections with respect to (-2)-curves we obtain a nef divisor class $[F]$ uniquely determined by $[A]$ (see [RS, §3, Prop. 3]). Since a reflection preserves primitivity, $[F]$ is primitive. It is easy to see that g acts on the orbit of $[A]$ with respect to the reflection group. Since $[F]$ is the unique nef member in the orbit and any automorphism preserves nefness, we have $g([F]) = [F]$. The linear system $|F|$ is a g-invariant pencil of elliptic curves ([SD] or [Re, Th. 3.8]). This proves the assertion.

Case 2: $p > 2$ *and* $\operatorname{rank} N^\perp = 2$. We apply Lemma 2.8 to our situation. Since $e_c(X \setminus \{x\}) = 23$, formula (2.1) gives

$$23 = (p-1)(e_c(V) + 1 - l(g)) + e_c(V). \tag{2.2}$$

Keeping the notation of the lemma, $V \cong Y \setminus \Delta_{\mathrm{red}}$. By the additivity of e_c,

$$e_c(V) = e_c(Y) - e_c(\Delta_{\mathrm{red}}) = 2 + \operatorname{rank}\operatorname{Pic}(Y) - (1 + \operatorname{rank} N) = \operatorname{rank} N^\perp + 1 = 3.$$

Here we use that the graph of components of Δ is a tree of smooth rational curves or an irreducible cuspidal curve of arithmetic genus 1. Otherwise, the local Picard group of $X/(g)$ contains a connected algebraic group isomorphic to an elliptic curve or the multiplicative group; hence it contains elements of finite order not killed by multiplication by p. The latter contradicts Corollary 2.7. Now formula (2.2) gives $20 = (p-1)a$, where $a < 4$. The only possibility is $p = 11$.

Assume that $p = 11$. We will first show that $N^\perp$ contains an isotropic vector. In this case the intersection matrix of $N^\perp$ is of the form

$$\begin{pmatrix} 2a & c \\ c & 2b \end{pmatrix}.$$

Since $N^\perp$ is indefinite and 11-elementary,

$$\det N^\perp = 4ab - c^2 = -1, \quad -11 \quad \text{or} \quad -121.$$

In the first case, $N^\perp \cong U$, where U is an even indefinite unimodular lattice. The second case cannot occur, since no square of an integer is congruent to 3 modulo 4. Assume the third case. Since $N^\perp$ is 11-elementary, all of the coefficients of the matrix must be divisible by 11, and hence $N^\perp \cong U(11)$. Therefore, $N^\perp$ contains an isotropic vector in any case. By the same proof as in Case 1, X admits a g-invariant elliptic pencil.

Case 3: $p = 2$ *and rank* $N^\perp = 1$. In this case $e_c(V) = 2$; hence formula (2.2) gives

$$21 = (2-1)(2+1-l(g)),$$

which is absurd. □

Remark 2.10. If $\operatorname{rank} N^\perp > 2$, formula (2.2) gives

$$23 - k = (p-1)(k+1-l(g)),$$

where $k = e_c(V) > 3$. If $k \neq 23$, this gives a weaker bound $p \leq 19$ with possible cases $(p, k, l(g)) = (19, 5, 5), (17, 7, 7), (13, 11, 11)$, etc.

PROPOSITION 2.11. *Let g be an automorphism of X of order $p = \mathrm{char}(k)$. If X admits a g-invariant arithmetic genus* 1 *fibration, then $p \leq 11$.*

Proof. Since quasi-elliptic fibrations occur only in characteristic $p = 2$ or 3, we may assume that our fibration is an elliptic fibration. Assume first that g^* acts as the identity on the base curve $\mathbb{P}^1$ of the elliptic fibration. Then g becomes an automorphism of the elliptic curve $X/\mathbb{P}^1$ over the function field of $\mathbb{P}^1$. On the Jacobian of this elliptic curve, g induces an automorphism g' of order p. Let

$$j : J \to \mathbb{P}^1$$

be the Jacobian elliptic fibration. Note that J is a $K3$ surface (cf. [CD, Th. 5.7.2]). Since the order of g' is $p > 3$, g' must be a translation by a p-torsion section. By Corollary 5.9 from [DK1], $p \leq 11$.

Next assume that g^* acts non-identically on the base $\mathbf{P}^1$ of the elliptic fibration. Let s_0 be the unique fixed point of g^* on the base. The fibre X_{s_0} contains X^g. The remaining singular fibres form orbits of fibres of the same type. By the same argument as in the proof of [DK1, Cor. 5.6], we see that $p \leq 11$. □

Now, Propositions 2.3, 2.5, 2.9 and 2.11 together with Theorem 2.2 prove our Theorem 2.1.

Remark 2.12. One should compare the result of Theorem 2.1 with the known result that an abelian surface A over a field of characteristic $p > 0$ does not admit an automorphism g of order $p > 5$ which fixes a point. This result can be proved by considering the action of g on the Tate module $H^1_{\mathrm{et}}(A, \mathbb{Q}_l)$, $l \neq p$ and applying the Weyl theorem. A similar proof for K3 surfaces only shows that $p > 23$ is impossible. We thank Yuri Zarkhin for this remark.

If g preserves an elliptic or a quasi-elliptic pencil, then it either acts non-identically on the base or is realized by an automorphism of its Jacobian fibration. In the latter case, if $p > 3$, then it is realized by a translation by a p-torsion section. In [DK1, Cor. 5.9] we have proved that no non-trivial p-torsion section exists if $p > 11$. One can improve this bound to 7 by the same proof. In the case $p = 11$, if an 11-torsion section exists, then the proof (see (i)–(iii) in [DK1, p. 126]) gives only one possible combination of types of singular fibres: three singular fibres of type I_{11}, I_{11}, II. If this happens, formula (5.1) in [DK1] gives a contradiction; in this case the left-hand side of the formula cannot be an integer.

We state this result for future reference.

THEOREM 2.13. *Let $f : X \to \mathbb{P}^1$ be an elliptic fibration with a section on a* K3*-surface over an algebraically closed field of characteristic $p > 7$. Then the group of p-torsion sections is trivial.*

A different proof of the above result is given by A. Schweizer [Sc].

3. Tame symplectic automorphisms

In this section we consider symplectic automorphisms of X of finite order prime to the characteristic p. We will show that they behave as in the complex case.

LEMMA 3.1. *Let Γ be a finite subgroup of* $\mathrm{SL}(2,k)$ *of order prime to p. Then Γ is isomorphic to a finite subgroup of* $\mathrm{SL}(2,\mathbb{C})$, *i.e. one of the following groups*: *a cyclic*, *binary dihedral*, *binary tetrahedral*, *binary octahedral*, *or binary icosahedral group.*

Proof. This is of course well-known. For completeness sake let us recall the usual proof (going back to Felix Klein). Let $\Gamma' \subset \mathrm{PSL}(2,k)$ be the image of Γ in $\mathrm{PSL}(2,k)$. Any non-trivial element $g \in \Gamma'$ has exactly two fixed points in the natural action on $S = \mathbb{P}^1(k)$. Let $\mathfrak{P}$ be the union of the sets of fixed points $S^g, g \in \Gamma' \setminus \{1\}$. Let $O_1, \ldots, O_r$ be the orbits of Γ' in $\mathfrak{P}$ and $n_1, \ldots, n_r$ be the orders of the corresponding stabilizers. An easy argument, using the Burnside counting formula, gives the equation

$$\sum_{i=1}^{r}(1-\frac{1}{n_i}) = 2 - \frac{2}{\#\Gamma'}.$$

This immediately implies that either $r = 2$ and Γ' is a cyclic group, or $r = 3$, and

$$(n_1, n_2, n_3; \#\Gamma') = (2,2,n;2n), (2,3,3;12), (2,3,4;24), \text{ or } (2,3,5;60).$$

An easy exercise in group theory shows that the groups Γ are isomorphic to a cyclic or a binary polyhedral group. $\square$

For a nondegenerate lattice L, we denote by $\mathrm{disc}(L)$ the discriminant group L^*/L of L. We define

$$d_L := \#\mathrm{disc}(L),$$

as the order of group L^*/L.

Let $\phi : X \to C$ be an elliptic surface, with or without a section. For any reducible fibre X_c, let S_c be the sublattice of the Picard lattice S_X generated by all irreducible components of the fibre. The Gram matrix with respect to the basis formed by the irreducible components is described by a Dynkin diagram of affine type $\tilde{A}_n, \tilde{D}_n, \tilde{E}_n$, where $n+1$ is the number of irreducible components. The radical of S_c is spanned by the (scheme-theoretical) fibre X_c considered as a divisor on X. The quotient $\bar{S}_c$ by the radical is isomorphic to the corresponding negative definite root lattice of types A_n, D_n, E_n. (We say that the fibre is of the corresponding type.) If $M_c \subset S_c$ is a negative definite sublattice of maximal rank, then the composition with the projection to $\bar{S}_c$ defines an embedding of lattices

$$M_c \hookrightarrow \bar{S}_c.$$

The orthogonal sum $\oplus_{c\in C}\bar{S}_c$ is the quotient of the sublattice S_X^{vert} of S_X generated by components of fibres by the rank 1 sublattice generated by the divisor class of any fibre. We denote the orthogonal sum by $\mathcal{R}(\phi)$ and call it the *root lattice* of the elliptic surface ϕ.

A negative definite sublattice $M = \oplus_{c\in C} M_c \subset S_X^{\mathrm{vert}}$ is called *maximal* if each M_c is a sublattice of S_c of maximal possible rank. Its image in $\mathcal{R}(\phi)$ is a sublattice of finite index, say a. In particular,

$$d_M = a^2 d_{\mathcal{R}(\phi)}. \tag{3.1}$$

LEMMA 3.2. *Let G be a finite group of symplectic automorphisms of a* K3 *surface of order prime to p. Let $Y \to X/G$ be a minimal resolution of the* *quotient X/G and let $\mathcal{R}_G$ be the sublattice of* $\mathrm{Pic}(Y)$ *generated by the irreducible components of the exceptional divisor. Then the discriminant of $\mathcal{R}_G$ is coprime to p. Moreover, if rank $\mathcal{R}_G = 20$, then the discriminant is not a square.*

Proof. The lattice $\mathcal{R}_G$ is a direct sum of irreducible root lattices $\mathcal{R}_i$ of type A, D, E generated by irreducible components of a minimal resolution of quotient singularities corresponding to stabilizer subgroups G_i of G. Via the action of G_i on the tangent space of X at one of its fixed points, the group G_i becomes isomorphic to a finite subgroup H of $\mathrm{SL}(2, k)$. Since $\#G_i$ is prime to p, the quotient singularity is formally isomorphic to the singularity $\mathbb{A}_k^2/H$. Now we apply Lemma 3.1 and use the well-known resolution of the quotient singularity $\mathbb{A}^2/H$. If $\mathcal{R}_i$ is of type A_k, then $\#G_i = d_{\mathcal{R}_i} = k+1$. If $\mathcal{R}_i$ is of type D_n, then $\#G_i = 4(n-2), d_{\mathcal{R}_i} = 4$. If $\mathcal{R}_i$ is of type E_6, then $\#G_i = 24, d_{\mathcal{R}_i} = 3$. If $\mathcal{R}_i$ is of type E_7, then $\#G_i = 48, d_{\mathcal{R}_i} = 2$. If $\mathcal{R}_i$ is of type E_8, then $\#G_i = 120, d_{\mathcal{R}_i} = 1$. In all cases we see that if $p | d_{\mathcal{R}_i}$, then $p | \#G_i$, a contradiction to the assumption that $\#G$ is prime to p. This proves the first assertion.

Assume that rank $\mathcal{R}_G = 20$. Obviously the Picard number ρ of Y satisfies $\rho \geq 21$. Thus Y is a supersingular K3 surface. It is known ([Ar1]) that the discriminant group of the Picard lattice of a supersingular K3 surface is a p-elementary abelian group $(\mathbb{Z}/p)^{2\sigma}$, where σ is the Artin invariant of the surface. Let N be the orthogonal complement of $\mathcal{R}_G$ in S_Y. We have

$$d_{\mathcal{R}_G} \cdot d_N = i^2 p^{2\sigma},$$

where i is the index of the sublattice

$$\mathcal{R}_G \oplus N \subset S_Y.$$

Assume that $d_{\mathcal{R}_G}$ is a square. Then d_N is a square. The lattice N is an indefinite lattice of rank 2 whose discriminant is the negative of a square. It must contain a primitive isotropic vector. By Riemann-Roch, we can represent it by an effective divisor A with self-intersection 0. It is known that a suitable composition of reflections with respect to the divisor classes of (-2)-curves sends A to a nef divisor E with $E^2 = 0$. Let

$$\psi : S_Y \to S_Y, \ \ \psi(A) = E$$

be the composition. Let $R_i, i = 1, \ldots, 20$, be the irreducible components of the exceptional divisor. Since ψ is an isometry, the images $\psi(R_i)$ generate a sublattice M of S_Y isomorphic to $\mathcal{R}_G$. Since $A \cdot R_i = 0$ for all i, we have

$$E \cdot \psi(R_i) = 0, \ \ i = 1, \ldots, 20.$$

Since, by Riemann-Roch, each $\psi(R_i)$ is effective or anti-effective, this implies that

$$E \cdot C_{ij} = 0$$

for all irreducible components C_{ij} of $\psi(R_i)$ or $-\psi(R_i)$; that is, C_{ij}'s are irreducible components of divisors from the pencil $|E|$ of genus 1 curves. Since M is a negative definite lattice of rank 20, it is a maximal sublattice of S_X^{vert} with respect to the elliptic fibration ϕ given by the pencil. By (3.1),

$$a^2 d_{\mathcal{R}(\phi)} = d_M = d_{\mathcal{R}_G}. \tag{3.2}$$

Let

$$f : J \to \mathbb{P}^1$$

be the Jacobian fibration of ϕ. The surface J is a K3 surface with the same type of singular fibres ([CD, Th. 5.3.1]). In particular,

$$\mathcal{R}(\phi) \cong \mathcal{R}(f).$$

The surface J is a supersingular K3 surface, since S_J^{vert} is of rank 21. The orthogonal complement of the root lattice $\mathcal{R}(f)$ in S_J is generated by the divisor classes of the zero-section and a fibre, and hence is unimodular. Thus we obtain

$$d_{S_J} m^2 = d_{\mathcal{R}(f)}$$

for some number m (equal to the order of the Mordell-Weil group of sections of f). Since $d_{S_J} = p^{2\sigma'}$, we obtain that $p | d_{\mathcal{R}(f)}$, and, by (3.2), $p | d_{\mathcal{R}_G}$. This contradicts the first assertion. □

THEOREM 3.3. *Let g be a symplectic automorphism of finite order n. If $(n,p) = 1$, then g has only finitely many fixed points $\#X^g$ and the possible pairs $(n, \#X^g)$ are as follows*:

$$(2,8), (3,6), (4,4), (5,4), (6,2), (7,3), (8,2).$$

Proof. At a fixed point, g is linearizable because $(n,p) = 1$. This implies that the quotient surface $Y = X/(g)$ has at worst cyclic quotient Gorenstein singularities and its minimal resolution is a $K3$ surface. Therefore, Nikulin's argument [Ni1] for the complex case works (when we replace the rational cohomology with the l-adic cohomology) except in the following two cases.

Case 1. $n = 11$ and $X/(g)$ has two A_{10}-singularities.

Case 2. $n = 15$ and $X/(g)$ has three singularities of type A_{14}, A_4 and A_2.

In any of these cases the discriminant of the lattice $\mathcal{R}_G$ defined in the previous lemma is a square. Thus, these cases cannot happen. □

4. The main theorem

A Mathieu representation of a finite group G is a 24-dimensional representation on a vector space V over a field of characteristic zero with character

$$\chi(g) = \varepsilon(\mathrm{ord}(g)),$$

where

$$\varepsilon(n) = 24\left(n\prod_{p|n}\left(1+\frac{1}{p}\right)\right)^{-1}. \tag{4.1}$$

The number

$$\mu(G) = \frac{1}{\#G}\sum_{g\in G}\varepsilon(\mathrm{ord}(g)) \tag{4.2}$$

is equal to the dimension of the subspace V^G of V. The natural action of a finite group G of symplectic automorphisms of a complex K3 surface on the singular cohomology

$$H^*(X,\mathbb{Q}) = \bigoplus_{i=0}^{4} H^i(X,\mathbb{Q}) \cong \mathbb{Q}^{24}$$

is a Mathieu representation with

$$\mu(G) = \dim H^*(X,\mathbb{Q})^G \geq 5.$$

From this, Mukai deduces that G is isomorphic to a subgroup of M_{23} with at least five orbits. In positive characteristic the formula for the number of fixed points is no longer true and the representation of G on the l-adic cohomology, $l \neq p$,

$$H^*_{\mathrm{et}}(X,\mathbb{Q}_l) = \bigoplus_{i=0}^{4} H^i_{\mathrm{et}}(X,\mathbb{Q}_l) \cong \mathbb{Q}_l^{24}$$

is not Mathieu in general. In this section, using Theorem 3.3, we will show that if G is tame; i.e., the order of G is coprime to p, then the natural representation of G on $H^*_{\mathrm{et}}(X,\mathbb{Q}_l) \cong \mathbb{Q}_l^{24}$ is Mathieu. We will also show that $\dim H^*_{\mathrm{et}}(X,\mathbb{Q}_l)^G \geq 5$ under the additional assumption that either X or a minimal model of X/G is not supersingular with Artin invariant $\sigma = 1$.

First let us recall that the analog of the lattice of transcendental cycles on a surface X in characteristic $p > 0$ is the group $T_l\mathrm{Br}(X)$ equal to the projective limit of groups $\mathrm{Br}(X)[l^n]$, where $\mathrm{Br}(X) = H^2_{\mathrm{et}}(X,\mathbb{G}_m)$ is the cohomological Brauer group. Recall that the Kummer sequence in étale cohomology [Mi] gives the exact sequence

$$0 \to \mathrm{Pic}(X)/l^n\mathrm{Pic}(X) \to H^2_{\mathrm{et}}(X,\mu_{l^n}) \to \mathrm{Br}(X)[l^n] \to 0. \tag{4.3}$$

Passing to the projective limit we have the exact sequence of $\mathbb{Z}_l$-modules

$$0 \to \mathrm{Pic}(X) \otimes \mathbb{Z}_l \to H^2_{\mathrm{et}}(X, \mathbb{Z}_l) \to T_l\mathrm{Br}(X) \to 0. \tag{4.4}$$

Tensoring with $\mathbb{Q}_l$, we get an exact sequence of $\mathbb{Q}_l$-vector spaces

$$0 \to \mathrm{Pic}(X) \otimes \mathbb{Q}_l \to H^2_{\mathrm{et}}(X, \mathbb{Q}_l) \to V_l\mathrm{Br}(X) \to 0. \tag{4.5}$$

It gives the analog of the usual formula for the second Betti number of a surface

$$b_2(X) = \rho(X) + \lambda(X),$$

where $\rho(X)$ is the Picard number of X and $\lambda(X) = \dim_{\mathbb{Q}_l} V_l\mathrm{Br}(X)$ is the Lefschetz number of X. Since $b_2(X)$ and $\rho(X)$ do not depend on $l \neq p$, the Lefschetz number $\lambda(X)$ does not depend on l either.

PROPOSITION 4.1. *Let G be a finite group of symplectic automorphisms of a* K3 *surface X defined in characteristic $p > 0$. Assume that G is tame, i.e. the order of G is coprime to p. Then for any prime $l \neq p$, the natural representation of G on $H^*_{\mathrm{et}}(X, \mathbb{Q}_l) \cong \mathbb{Q}_l^{24}$ is Mathieu.*

Proof. By Theorem 3.3,

$$\mathrm{ord}(g) \in \{1, \ldots, 8\}$$

for all $g \in G$. By the Lefschetz fixed point formula, the character $\chi(g)$ of the representation on the l-adic cohomology is equal to the number of fixed points of g, which is equal to $\varepsilon(\mathrm{ord}(g))$. This proves the assertion. This also shows that the action of G on $H^2_{\mathrm{et}}(X, \mathbb{Q}_l)$ is faithful. $\square$

LEMMA 4.2. *Let G be a finite tame group of symplectic automorphisms of a* K3 *surface X. If a nonsingular minimal model Y of X/G is not supersingular, then*

$$\dim H^*_{\mathrm{et}}(X, \mathbb{Q}_l)^G \geq 5. \tag{4.6}$$

Proof. Since Y is not supersingular, $\lambda(Y) > 0$. If $\lambda(Y) = 1$, the Picard number of Y is equal to 21, and hence Y admits an elliptic fibration. By Artin [Ar1], the height h of the formal Brauer group of an elliptic non-supersingular surface is finite and

$$\lambda(Y) = 22 - \rho(Y) \geq 2h \geq 2.$$

Choose l coprime to the order of G. It is known that $\dim_{\mathbb{Q}_l}(V_l\mathrm{Br}(X))^G = \lambda(Y)$ ([Shio1, Prop. 5]). Taking a G-invariant ample divisor class defining a nonzero element in $(\mathrm{Pic}(X) \otimes \mathbb{Q}_l)^G$, we see that $\dim H^2_{\mathrm{et}}(X, \mathbb{Q}_l)^G \geq 3$. Since the characteristic polynomial does not depend on $l \neq p$, this is true for all $l \neq p$. Together with $H^0_{\mathrm{et}}(X, \mathbb{Q}_l)$ and $H^4_{\mathrm{et}}(X, \mathbb{Q}_l)$ we get (4.6). $\square$

It remains to consider the case when X/G is birationally isomorphic to a supersingular K3 surface.

LEMMA 4.3. *Assume $p \neq 2$. Assume that a* K3 *surface X admits a symplectic automorphism g of order* 2. *Then X admits an elliptic fibration.*

Proof. As is well-known, it suffices to show that $\rho(X) \geq 5$, or equivalently, $\lambda(X) \leq 17$. Let U be the open set where g acts freely. We know that $X \setminus U$ consists of 8 fixed points of g. Let $G = (g), V = U/G$. We shall use the two spectral sequences employed in the proof of Lemma 2.6. It is easy to see that they give the following exact sequence:

$$0 \to H^1(G, \mathrm{Pic}(U)) \to \mathrm{Br}(V) \to \mathrm{Br}(U)^G \to H^2(G, \mathrm{Pic}(U)). \tag{4.7}$$

Let t_+ (resp. t_-) be the rank of the g-invariant (resp. g-anti-invariant) part of $\mathrm{Pic}(U) \cong \mathrm{Pic}(X)$. We have

$$H^1(G, \mathrm{Pic}(U)) = H^1(G, \mathrm{Pic}(X)) = \mathrm{Ker}(1+g^*)/\mathrm{Im}(1-g^*) \cong (\mathbb{Z}/2\mathbb{Z})^{t_-},$$
$$H^2(G, \mathrm{Pic}(U)) = H^2(G, \mathrm{Pic}(X)) = \mathrm{Ker}(1-g^*)/\mathrm{Im}(1+g^*) \cong (\mathbb{Z}/2\mathbb{Z})^{t_+}.$$

Splitting (4.7) into two short exact sequences and passing to the 2-torsion subgroups we get the following exact sequences of 2-elementary groups:

$$0 \to (\mathbb{Z}/2\mathbb{Z})^{t_-} \to \mathrm{Br}(V)[2] \to A \to (\mathbb{Z}/2\mathbb{Z})^{t_-},$$
$$0 \to A \to \mathrm{Br}(U)^G[2] \to (\mathbb{Z}/2\mathbb{Z})^{t} \to 0,$$

where $t \leq t_+$. This gives

$$\begin{aligned} \dim_{\mathbb{F}_2} \mathrm{Br}(U)^G[2] \leq \dim_{\mathbb{F}_2} A + t &\leq (\dim_{\mathbb{F}_2} \mathrm{Br}(V)[2] - t_-) + t_- + t \\ &\leq \dim_{\mathbb{F}_2} \mathrm{Br}(V)[2] + t_+. \end{aligned} \tag{4.8}$$

Let Y be a minimal resolution of singularities of X/G and $\mathcal{E}$ be the exceptional divisor. According to [DeMF], the exact sequence of local cohomology for the pair $(Y, \mathcal{E})$ and the sheaf $\mathbb{G}_m$ defines an exact sequence (modulo p-groups)

$$0 \to \mathrm{Br}(Y) \to \mathrm{Br}(Y \setminus \mathcal{E}) \to H^1(\mathcal{E}, \mathbb{Q}/\mathbb{Z}).$$

Since $\mathcal{E}$ is the disjoint union of eight smooth rational curves, we obtain

$$\mathrm{Br}(Y) \cong \mathrm{Br}(Y \setminus \mathcal{E}) \cong \mathrm{Br}(V).$$

Similarly, we obtain

$$\mathrm{Br}(U) \cong \mathrm{Br}(X).$$

It follows from (4.4) that, up to a finite group,

$$\mathrm{Br}(Y)[2] = (\mathbb{Z}/2\mathbb{Z})^{\lambda(Y)}.$$

Applying (4.8), we obtain

$$\dim_{\mathbb{F}_2} \mathrm{Br}(X)^G[2] \leq \lambda(Y) + t_+. \tag{4.9}$$

If $\rho(X) \geq 5$, we are done. Otherwise, $\rho(X) \leq 4$; hence $t_+ \leq 4$. Since Y contains eight disjoint smooth rational curves and also the pre-image of a class of an ample divisor on X/G, we have $\rho(Y) \geq 9$, and therefore $\lambda(Y) \leq 22 - 9 = 13$. Now (4.9) implies

$$\dim_{\mathbb{F}_2} \mathrm{Br}(X)^G[2] \leq 17. \tag{4.10}$$

The exact sequence of sheaves in étale topology

$$0 \to \mu_{2^n} \xrightarrow{[2]} \mu_{2^{n+1}} \to \mu_2 \to 0$$

gives, after passing to cohomology and taking the projective limits, the exact sequence

$$H^2(X, \mathbb{Z}_2) \xrightarrow{[2]} H^2(X, \mathbb{Z}_2) \to H^2(X, \mu_2) \to 0.$$

Since an automorphism of order ≤ 2 of a free $\mathbb{Z}_l$-module acts trivially modulo 2, we obtain that

$$H^2(X, \mu_2)^G = H^2(X, \mu_2).$$

Applying the Kummer exact sequence (4.3), we see that

$$\mathrm{Br}(X)^G[2] = \mathrm{Br}(X)[2].$$

It remains to apply (4.10). □

Lemma 4.4. *Let G be a finite tame group of symplectic automorphisms of a* K3*-surface X of order $\neq 7, 21$. Assume that a minimal resolution Y of X/G is a supersingular* K3 *surface. Then X is supersingular.*

Proof. Recall from [Ar1] that the formal Brauer group $\hat{\mathrm{Br}}(S)$ of a supersingular K3-surface S is isomorphic to the formal additive group $\hat{\mathbb{G}}_a$. It is conjectured that the converse is true, and it has been verified if S is an elliptic surface (loc.cit. Theorem (1.7)). Since the Brauer group is a birational invariant, the projection $\pi : X \to X/G$ defines a natural homomorphism of the formal Brauer groups $\pi^* : \hat{\mathrm{Br}}(Y) \to \hat{\mathrm{Br}}(X)$. The corresponding map of the tangent spaces is $\pi^* : H^2(Y, \mathcal{O}_Y) \to H^2(X, \mathcal{O}_X)$. Since the order of G is prime to the characteristic, the trace map shows that this homomorphism is nonzero. Since there are no nontrivial maps between a formal group of finite height and $\hat{\mathbb{G}}_a$ we obtain that $\hat{\mathrm{Br}}(X) \cong \hat{\mathbb{G}}_a$. If G contains an element of order 2, we are done by Lemma 4.3. Assume that G has no elements of order 2. By Proposition 4.1, the representation of G in $H^*(X, \mathbb{Q}_l)$ is a Mathieu representation. Following Mukai's arguments from [Mu], we obtain that the order of G must divide $3^2.5.7$.

Suppose that three distinct prime numbers divide $\#G$. It is known that no simple non-abelian group of order dividing $3^2.5.7$ exists. Thus G is solvable and hence contains a subgroup of order 35 ([Ha, Th. 9.3.1]). It follows easily from Sylow's Theorem that such a group is cyclic and hence is not realized as a group of symplectic automorphisms of X.

Suppose $\#G = 3^a.5$ or $3^a.7$ with $a \neq 0$. Again, by Sylow's theorem we obtain that a Sylow 5-subgroup (or 7-subgroup) is normal. Since no element of order 3 can commute with an element of order 5 or 7, we obtain that G is a non-abelian group of order 21. This case is excluded by the assumption.

The remaining possible cases are $\#G = 3, 5, 9$. This gives that X/G has either six singular points of type A_2, or four singular points of type A_4, or eight singular points of type A_2. Since $\rho(Y) = 22$, this immediately implies that $\dim \mathrm{Pic}(X/G)\otimes\mathbb{Q} = 22-12 = 10$ or $22-16 = 6$. Hence $\mathrm{rank}\,\mathrm{Pic}(X)^G \geq 6$. Thus X is an elliptic surface, and, by Artin's result cited above, we obtain that X is supersingular. □

Proposition 4.5. *Let G be a finite tame group of symplectic automorphisms of a* K3 *surface X. Assume that either X or a minimal model of X/G is not a supersingular* K3 *surface with Artin invariant $\sigma = 1$. Then*

$$\dim H^*_{\mathrm{et}}(X, \mathbb{Q}_l)^G \geq 5.$$

Proof. By Lemma 4.2 we may assume that a minimal nonsingular model Y of X/G is supersingular. A symplectic group of order 7 or 21 is uniquely determined and is in Mukai's list [Xiao] and satisfies the assertion of the proposition. By Lemma 4.4 we obtain that X is supersingular.

Assume that the Artin invariant σ of X is greater than 1.

Let us consider the representation of G on the crystalline cohomology $H^*_{\mathrm{crys}}(X/W)$. We refer to [Il] for the main properties of crystalline cohomology and to [RS], [Og] for particular properties of crystalline cohomology of K3 surfaces. The cohomology $H^*_{\mathrm{crys}}(X/W)$, where X is a K3 surface, is a free module of rank 24 over the ring of Witt vectors $W = W(k)$. The vector space $H^*_{\mathrm{crys}}(X/W)_K$, where K is the field of fractions of W, is of dimension 24. The ring W is a complete noetherian local ring of characteristic 0 with maximal ideal (p) and the residue field isomorphic to k. The quotient module $H^*_{\mathrm{crys}}(X/W)/pH^*_{\mathrm{crys}}(X/W)$ is a k-vector space of dimension 24 isomorphic to the algebraic de Rham cohomology $H^*_{\mathrm{DR}}(X)$. Let

$$H = H^2_{\mathrm{DR}}(X).$$

It is known that the Hodge spectral sequence

$$E_1^{p,q} = H^q(X, \Omega^p_X) \Rightarrow H^n_{\mathrm{DR}}(X)$$

degenerates and we have the following canonical exact sequences:

$$0 \to F^1H \to H \to H^2(X, \mathcal{O}_X) \to 0, \tag{4.11}$$

$$0 \to H^0(X, \Omega^2_X) \to F^1H \to H^1(X, \Omega^1_X) \to 0. \tag{4.12}$$

Here

$$F^iH = \sum_{p\geq i} H^q(X, \Omega_X^p) \cap H$$

is the Hodge filtration of the de Rham cohomology. Obviously, the subspace F^1H is G-invariant. Since the order of G is prime to the characteristic, the representation of G is semi-simple and hence the G-module H is isomorphic to the direct sum of G-modules

$$H \cong H^0(X, \Omega_X^2) \oplus H^2(X, \mathcal{O}_X) \oplus H^1(X, \Omega_X^1). \tag{4.13}$$

By definition,

$$H^0(X, \Omega_X^2) \subset H^G.$$

By Serre's duality,

$$H^2(X, \mathcal{O}_X) \subset H^G.$$

This shows that $\dim H^G \geq 2$.

Let

$$V = H^2_{\text{crys}}(X/W).$$

The multiplication by p map $[p]$ defines the exact sequence

$$0 \to V \xrightarrow{[p]} V \to H \to 0.$$

Taking G-invariants we obtain the exact sequence

$$0 \to V^G \xrightarrow{[p]} V^G \to H^G \to H^1(G, V). \tag{4.14}$$

Since the ring W has characteristic 0, the multiplication by $|G|$ defines an injective map $V \to V$. On the other hand, it induces the zero map on the cohomology $H^1(G, V)$. This implies that

$$H^1(G, V) = 0.$$

This shows that V^G is a free submodule of V of rank equal to $\dim H^G$.

It is known that the Chern class map

$$c_1 : S_X \to H^2_{\text{crys}}(X/W)$$

is injective and its composition with the reduction mod p map

$$H^2_{\text{crys}}(X/W) \to H^2_{\text{DR}}(X)$$

defines an injective map

$$c : S_X/pS_X \to H^2_{\text{DR}}(X) \tag{4.15}$$

with image contained in $F^1H^2_{\text{DR}}(X)$ (see [Og]).

If X is supersingular with Artin invariant $\sigma > 1$, the composition of c with the projection $F^1H^2_{\text{DR}}(X) \to H^1(X, \Omega_X^1)$ is injective. This result is implicitly contained in [Og] (use Remark 2.7 together with the fact that a supersingular surface with $\sigma > 1$ admits a non-trivial deformation to a supersingular surface with Artin invariant equal to 1). Let L be a G-invariant ample line bundle on X. We may assume that its isomorphism class defines a non-zero element in S_X/pS_X. Thus its image in $H^1(X, \Omega^1_X)$ is a nonzero G-invariant element and we get three linearly independent elements in H^G, each from one of the three direct summands of H. Applying (4.14), we find three linearly independent elements in $H^2_{\mathrm{crys}}(X/W)^G$.

Since $H^0_{\mathrm{crys}}(X/W)$, $H^4_{\mathrm{crys}}(X/W)$ are trivial G-modules, we obtain

$$\dim H^*_{\mathrm{crys}}(X/W)^G_K \geq 5. \tag{4.16}$$

It remains to use the fact that the characteristic polynomials of $g \in G$ on $H^*_{\mathrm{crys}}(X/W)_K$ and on $H^*_{\mathrm{et}}(X, \mathbb{Q}_l)$, $l \neq p$, have integer coefficients and coincide with each other ([Il, 3.7.3]). Thus if the assertion is not true, X must be supersingular of Artin invariant $\sigma = 1$.

Assume that Y is supersingular with Artin invariant $\sigma > 1$. Let X' be the open subset of X where G acts freely and let $Y' = X'/G$. The standard Hochshild-Serre spectral sequence implies that the pull-back under the projection $f : X' \to Y'$ defines an isomorphism

$$\mathrm{Pic}(Y')/\mathrm{Hom}(G, k^*) \cong \mathrm{Pic}(X')^G \cong \mathrm{Pic}(X)^G.$$

Let R be the sublattice of S_Y spanned by the irreducible components of exceptional curves of the resolution $\pi : Y \to X/G$. It is isomorphic to the orthogonal sum of root lattices of discriminants prime to p. The restriction map $\mathrm{Pic}(Y) \to \mathrm{Pic}(Y')$ is surjective and its kernel is R. The torsion subgroup of $\mathrm{Pic}(Y')$ is isomorphic to $\mathrm{Hom}(G, k^*)$. Let R' be the saturation of R in $\mathrm{Pic}(Y)$. We have

$$N := \mathrm{Pic}(Y)/R' \cong \mathrm{Pic}(Y')/\mathrm{Hom}(G, k^*) \cong \mathrm{Pic}(X)^G. \tag{4.17}$$

Since $\#G$ is coprime to p, the discriminant of the sublattice R' is coprime to p. The discriminant group D_Y of $\mathrm{Pic}(Y)$ is an elementary p-group of rank $2\sigma \geq 4$ and is a subquotient of the discriminant group of $R' \oplus R^\perp$. This implies that $\mathrm{rank}\, N = \mathrm{rank}\, R^\perp > 2$, it follows from (4.17) that $\mathrm{rank}\,\mathrm{Pic}(X)^G > 2$, and by the above arguments we will find 5 linearly independent elements in $H^*_{\mathrm{crys}}(X/W)^G$. □

Remark 4.6. If p divides $|G|$, the exact sequences (4.11) and (4.12) may not split as G-modules. In fact, there are examples where

$$\dim H^*_{\mathrm{crys}}(X/W)^G_K = 4,$$

so that (4.16) does not hold.

THEOREM 4.7. *Let G be a finite group of symplectic automorphisms of a* K3 *surface X. Assume that G is tame and that either X or a minimal model of X/G is not a supersingular* K3 *surface with Artin invariant $\sigma = 1$. Then G is a subgroup of the Mathieu group M_{23} which has ≥ 5 orbits in its natural permutation action on the set of* 24 *elements. All such groups are subgroups of the* 11 *groups listed in* [Mu].

Proof. Let us consider the linear representation ρ of G on $H^*_{\text{et}}(X, \mathbb{Q}_l)$, $l \neq p$. Applying Proposition 4.1 and 4.5, we find that ρ is a Mathieu representation over the field $\mathbb{Q}_l$ with $\dim H^*_{\text{et}}(X, \mathbb{Q}_l)^G \geq 5$. Replacing $\mathbb{Q}$ with $\mathbb{Q}_l$ we repeat the arguments of Mukai. At several places, he uses the fact that the representation is over $\mathbb{Q}$. The only essential place where he uses the fact that the representation is over $\mathbb{Q}$ is in Proposition (3.21), where G is assumed to be a 2-group containing a maximal normal abelian subgroup A and the case of $A = (\mathbb{Z}/4)^2$ with $\#(G/A) \geq 2^4$ is excluded by using the fact that a certain 2-dimensional representation of the quaternion group Q_8 cannot be defined over $\mathbb{Q}$. We use the fact that G also admits a Mathieu representation on the 2-adic cohomology, and it is easy to see that the representation of Q_8 cannot be defined over $\mathbb{Q}_2$. To show that the 2-Sylow subgroup of G can be embedded in M_{23}, he uses the fact that the stabilizer of any point on X is isomorphic to a finite subgroup of $\mathrm{SL}(2, \mathbb{C})$, and the classification of such subgroups allows him to exclude some groups of order 2^n. By Lemma 3.1, we have the same classification, so we can do the same. □

Applying Theorem 2.1, we obtain the following.

COROLLARY 4.8. *Assume that $p > 11$ and either X or a minimal model of X/G is not a supersingular* K3 *surface with Artin invariant $\sigma = 1$. Then G is a subgroup of M_{23} with ≥ 5 orbits and hence belongs to Mukai's list.*

5. The exceptional case

Here we investigate the case when the order of G is prime to p and

$$\dim H^*_{\text{crys}}(X/W)^G_K = 4. \tag{5.1}$$

By Theorem 4.7 this may happen only if both X and a minimal nonsingular model Y of X/G is a supersingular K3 surface with Artin invariant $\sigma = 1$. We refer to this as the exceptional case and the group G will be called an *exceptional group.*

It is known that a supersingular surface with Artin invariant $\sigma = 1$ is unique up to isomorphism. More precisely, we have the following (see [Og, Cor. 7.14]).

PROPOSITION 5.1. *Let X be a supersingular surface with Artin invariant $\sigma = 1$. Assume that $p \neq 2$. Then X is birationally isomorphic to the Kummer surface of the abelian surface $E \times E$, where E is a supersingular elliptic curve.*

If $p = 2$, the surface is explicitly described in [DKo]. Note that the Kummer surface does not depend on E. If $p \equiv 3 \mod 4$ (resp. $p \equiv 2 \mod 3$) we can take for E an elliptic curve with Weierstrass equation $y^2 = x^3 - x$ (resp. $y^2 = x^3 + 1$).

It follows from the proof of Proposition 4.5 and Lemma 3.2 that an exceptional group satisfies the following properties:

(EG1) G admits a Mathieu representation V_l over any $\mathbb{Q}_l, l \neq p$;

(EG2) $\mu(G) = \dim V_l^G = 4$;

(EG3) The root lattice $\mathcal{R}_G$ spanned by irreducible components of the exceptional locus of the resolution $Y \to X/G$ is of rank 20 (this is equivalent to (EG2));

(EG4) $d_{\mathcal{R}_G}$ is coprime to p and is not a square.

We use the following notation of groups from [Mu] and the ATLAS [CN]:

C_n the cyclic group of order n, sometimes denoted by n;

D_{2n} the dihedral group of order $2n$;

Q_{4n} the binary dihedral group of order $4n$;

T_{24} the binary tetrahedral group;

O_{48} the binary octahedral group;

$\mathfrak{S}_n$ the symmetric group of degree n;

$\mathfrak{A}_n$ the alternating group of degree n;

$\mathfrak{S}_{n_1,\ldots,n_k}$ a subgroup of of $\mathfrak{S}_{n_1+\ldots+n_k}$ which preserves the decomposition of a set of $n_1+\ldots+n_k$ elements as a disjoint union of subsets of cardinalities $n_1, \ldots, n_k$;

$\mathfrak{A}_{n_1,\ldots,n_k} = \mathfrak{S}_{n_1,\ldots,n_k} \cap \mathfrak{A}_{n_1+\ldots+n_k}$;

M_k the Mathieu group of degree k;

$L_n(q) = \mathrm{PSL}_n(\mathbb{F}_q)$;

$A_\bullet B$ a group which has a normal subgroup isomorphic to A with quotient isomorphic to B;

$A^\bullet B$ as above but the extension does not split;

$A : B$ a semidirect product with normal subgroup A;

$A \circ B$ the central product of two groups.

We will also use the notation for 2-groups from [HS] and [Mu].

The goal of this section is to prove the following.

THEOREM 5.2. *An exceptional group G is isomorphic to one of the following groups* (*the corresponding root lattice $\mathcal{R}_G$ is given in the parenthesis*):

(I) *Non-solvable groups*:

(i) $\mathfrak{S}_6$ *of order* $2^4.3^2.5$ $(A_4+2A_3+2A_5)$;

(ii) M_{10} *of order* $2^4.3^2.5$ $(A_4+A_3+A_2+A_7+D_4)$;

(iii) $2^4:\mathfrak{A}_6$ *of order* $2^7.3^2.5$ $(E_7+A_2+A_3+2A_4)$;

(iv) M_{21} *of order* $2^6.3^2.5.7$ $(A_2+2A_4+A_6+D_4)$;

(v) $\mathfrak{A}_7$ *of order* $2^3.3^2.5.7$ $(A_2+A_3+A_4+A_6+D_5)$;

(vi) $M_{20}'=2^4:\mathfrak{A}_5(\not\cong M_{20})$ *of order* $2^6.3.5$ $(A_1+2A_4+D_5+E_6)$;

(vii) $M_{20}:2\cong 2^4:\mathfrak{S}_5$ *of order* $2^7.3.5$ $(A_2+A_3+A_4+A_5+D_6)$;

(viii) $2^3:L_2(7)$ *of order* $2^6.3.7$ $(A_2+2A_3+A_6+E_6)$.

(II) *Solvable groups*:

(ix) $3^2:C_8$ *of order* $2^3.3^2$ $(2A_7+A_3+A_2+A_1)$;

(x) $3^2:SD_{16}$ *of order* $2^4.3^2$ $(A_7+A_5+D_4+A_3+A_1)$;

(xi) $2^3:7$ *of order* $2^3.7$ $(3A_6+2A_1)$;

(xii) $2^4:(5:4)$ *of order* $2^6.5$ $(A_7+A_4+3A_3)$;

(xiii) $2^2{}_{\bullet}(\mathfrak{A}_4\times\mathfrak{A}_4)=\Gamma_{13}a_1:3^2$ *of order* $2^6.3^2$ $(E_6+2A_2+2A_5)$;

(xiv) $2^2{}_{\bullet}\mathfrak{A}_{4,4}=\Gamma_{13}a_1:\mathfrak{A}_{3,3}=2^4:\mathfrak{A}_{3,4}$ *of order* $2^7.3^2$ $(E_7+2D_5+A_2+A_1)$;

(xv) $2^4:(3\times D_6)$ *of order* $2^5.3^2$ $(3A_5+A_3+A_2)$;

(xvi) $2^4:(3^2:4)$ *of order* $2^6.3^2$ $(A_7+D_5+2A_3+A_2)$;

(xvii) $2^4:\mathfrak{S}_{3,3}=\mathfrak{S}_{4,4}=2^4:\mathfrak{A}_{2,3,3}$ *of order* $2^6.3^2$ $(D_5+D_4+2A_5+A_1)$;

(xviii) O_{48} *of order* $2^4.3$ $(E_7+D_6+D_5+A_2$ *or* $2E_7+D_4+A_2)$;

(xix) $T_{24}\times 2$ *of order* $2^4.3$ $(E_6+D_4+2A_5)$;

(xx) $O_{48}:2$ *of order* $2^5.3$ $(E_7+D_6+A_5+2A_1)$;

(xxi) $(Q_8\circ Q_8)^{\bullet}\mathfrak{S}_3=\Gamma_5a_1{}^{\bullet}\mathfrak{S}_3$ *of order* $2^6.3$ $(A_1+A_2+A_3+2E_7)$;

(xxii) $(2^4:2)^{\bullet}\mathfrak{S}_3=2^4:Q_{12}$ *of order* $2^6.3$ $(A_1+A_2+A_7+2D_5)$;

(xxiii) $\Gamma_{13}a_1:3=2^4:\mathfrak{A}_4$ *of order* $2^6.3$ $(A_1+3A_5+D_4$ *or* $2A_1+3E_6)$;

(xxiv) $\Gamma_{13}a_1:\mathfrak{S}_3=2^4:\mathfrak{S}_4$ *of order* $2^7.3$ $(A_1+A_3+A_5+D_5+D_6)$;

(xxiv′) $\Gamma_{13}a_1:\mathfrak{S}_3=2^4:\mathfrak{S}_4$ *of order* $2^7.3$ $(A_1+2A_3+E_6+E_7)$;

(xxv) $\Gamma_{25}a_1:3=2^4:\mathfrak{A}_4$ *of order* $2^6.3$ $(A_1+A_3+2A_5+E_6)$;

(xxvi) $\Gamma_{25}a_1:\mathfrak{S}_3=2^4:\mathfrak{S}_4=\Gamma_5a_1{}^{\bullet}D_{12}$ *of order* $2^7.3$ $(A_1+A_3+A_5+D_4+E_7)$;

(xxvii) $(2^3:7):3$ *of order* $2^3.3.7$ $(A_6+2A_5+2A_2)$.

Remark 5.3. (1) All these groups are subgroups of the Mathieu group M_{23} with number of orbits equal to 4 (see Proposition 5.14). Note that not all subgroups of M_{23} with 4 orbits belong to our list. For example, a group containing elements of order > 8 must be wild by Theorem 3.3, hence is not contained in our list. Examples of such groups are $L_2(11), 2 \times L_2(7), \mathfrak{A}_{3,5}$. We thank D. Allcock and S. Kondō for confirming this. Also, as we will see in the proof of Lemma 5.10, Case 3, there is a degree-2 extension of M_{20}' isomorphic to $2^4 : S_5$ which cannot act symplectically. According to Allcock, this group can be realized as a subgroup of M_{23} preserving the partition $(5, 2, 1, 16)$ with $(5, 2, 1)$ forming an octad of the Steiner system. Our double extension $M_{20} : 2 \cong 2^4 : S_5$ preserves the partition $(1, 1, 2, 20)$.

(2) Most exceptional groups seem to realize in infinitely many different characteristics. A full account on the realization problem will be given in another publication.

(3) Among the 27 groups, the following ten groups are maximal: (i)–(v), (vii), (viii), (x), (xiv), (xvii).

We will prove the theorem by analyzing and extending Mukai's arguments from [Mu]. We can use the arguments only based on property (EG1) of G and do not use the assumption that $\mu(G) \geq 5$.

Recall that

$$\mu(G) = \frac{1}{\#G} \sum_{g \in G} \varepsilon(\mathrm{ord}(g)),$$

where $\varepsilon(n)$ is given in (4.1). Let g be an element of order n prime to p acting symplectically on a K3 surface. Using Theorem 3.3 and computation of $\#X^g$ from [Ni1], one checks that

$$\varepsilon(n) = \#X^g. \tag{5.2}$$

The possible values of $\varepsilon(n)$ are given in Theorem 3.3.

LEMMA 5.4.

$$\sum_{Gx \in X/G} \frac{1}{\#G_x} = \frac{24}{\#G} + k - \mu(G),$$

where k is the number of singularities on X/G.

Proof. Let

$$S = \{(x, g) \in X \times G \setminus \{1\} : g(x) = x\}.$$

By projecting to X we get

$$\#S = \#G \sum_{Gx \in \mathrm{Sing}(X/G)} \left(1 - \frac{1}{\#G_x}\right).$$

By projecting to G, and using (5.2), we get

$$\#S = \sum_{g\in G\setminus\{1\}} \#X^g = \sum_{g\in G\setminus\{1\}} \varepsilon(\mathrm{ord}(g)) = \#G\mu(G) - 24.$$

This gives

$$\sum_{Gx\in \mathrm{Sing}(X/G)} \frac{1}{\#G_x} = \frac{24}{\#G} + k - \mu(G). \qquad \square$$

Remark 5.5. We also have the following formula from [Xiao]:

$$\sum_{Gx\in X/G} \frac{1}{\#G_x} = \frac{24}{\#G} + k + \mathrm{rank}\, R_G - 24.$$

Comparing this with the previous formula, we get

$$\mu(G) = 24 - \mathrm{rank}\, R_G. \tag{5.3}$$

The classification of finite subgroups of $\mathrm{SL}(2,k)$ which admit a Mathieu representation is given in [Mu], Proposition (3.12). The groups Q_{4n}, $n\geq 5$, and the binary icosahedral group I_{120} are not realized since they contain an element of order > 8. The following table gives the information about possible stabilizer groups G_x, their orders o_x, the number c_x of irreducible components in a minimal resolution, types of singular points and the structure of the discriminant group D_x of the corresponding root lattice.

G_x	C_2	C_3	C_4	C_5	C_6	C_7	C_8	Q_8	Q_{12}	Q_{16}	T_{24}	O_{48}
o_x	2	3	4	5	6	7	8	8	12	16	24	48
c_x	1	2	3	4	5	6	7	4	5	6	6	7
Type	A_1	A_2	A_3	A_4	A_5	A_6	A_7	D_4	D_5	D_6	E_6	E_7
D_x	C_2	C_3	C_4	C_5	C_6	C_7	C_8	C_2^2	C_4	C_2^2	C_3	C_2

Table 1

Note that $\mu(O_{48}) = 4$ so this case may occur only in the exceptional case. For an exceptional group G, using the above table and the formula from Lemma 5.4 it is easy to show that

(EG5) the number k of singularities on X/G is 4 or 5.

Let G be an exceptional group acting on X. It defines a set of k numbers $o_1,\ldots,o_k$ and $c_1,\ldots,c_k$ from Table 1. We have

(i) $c_1+\ldots+c_k = 20$;

(ii) $\frac{1}{o_1}+\ldots+\frac{1}{o_k} = k-4+\frac{24}{N}$, where N is a positive integer (equal to $\#G$);

(iii) $o_i|N$ for all $i = 1, \dots, k$;

(iv) $d_1 \dots d_k$ is not a square, where d_i is the order of the discriminant group D_i;

(v) $k = 4$ or 5.

We are grateful to Daniel Allcock who ran a computer program which enumerates all collections of numbers $o_1, \dots, o_k$ and the corresponding numbers c_i, d_i, N satisfying properties (i)–(v). This gives all possible orders N of possible exceptional groups G as well as all possible root lattices describing a minimal resolution of singularities of X/G. We refer to this as the *List*. It is reproduced in Table 2.

Order	Root lattices
$2^6.3^2.5.7$	$A_2A_4A_4A_6D_4$
$2^3.3^2.5.7$	$A_2A_3A_4A_6D_5$
$2^4.5.7$	$A_3A_3A_4A_4A_6$
$3^2.5.7$	$A_2A_4A_4A_4A_6$
$2^7.3^2.5$	$A_2A_3A_4A_4E_7$
$2^5.3^2.5$	$A_3A_4A_4A_4A_5$
$2^4.3^2.5$	$A_1A_3A_4E_6E_6,\ A_1A_4A_5D_4E_6,\ A_1A_4A_5D_5D_5$ $A_2A_3A_4A_7D_4,\ A_3A_3A_4A_5A_5$
$2^3.3^2.5$	$A_2A_4A_4A_5A_5$
$2^6.3.7$	$A_2A_3A_3A_6E_6,\ A_2A_3A_5A_6D_4$
$2^3.3.7$	$A_1A_3A_5A_6D_5,\ A_2A_2A_5A_5A_6$
$2^7.3.5$	$A_2A_3A_4A_5D_6$
$2^6.3.5$	$A_1A_4A_4D_5E_6$
$2^3.3.5$	$A_1A_2A_4A_7E_6,\ A_1A_4A_5A_5A_5$
$2^7.3^2$	$A_1A_2D_5D_5E_7,\ A_1A_2D_5D_6E_6$
$2^6.3^2$	$A_1A_2D_4D_6E_7,\ A_1A_2D_5D_6D_6,\ A_1A_5A_5D_4D_5,$ $A_2A_2A_2E_7E_7,\ A_2A_2A_5A_5E_6,\ A_2A_3A_3A_7D_5$
$2^5.3^2$	$A_2A_2A_5A_7D_4,\ A_2A_3A_5A_5A_5,\ A_2A_2A_3A_7E_6$
$2^4.3^2$	$A_1A_1D_5D_6E_7,\ A_1A_1D_6D_6E_6,\ A_1A_3A_5A_7D_4,\ A_1A_3A_3A_7E_6$
$2^3.3^2$	$D_4D_5D_5E_6,\ A_1A_2A_3A_7A_7$
$2^3.7$	$A_1A_1A_6A_6A_6$
$2^6.5$	$A_1A_3A_4D_6D_6,\ A_1A_4A_7D_4D_4,\ A_3A_3A_3A_4A_7$
$2^7.3$	$A_1A_3A_3E_6E_7,\ A_1A_3A_5D_4E_7,\ A_1A_3A_5D_5D_6$
$2^6.3$	$A_1A_1E_6E_6E_6,\ A_1A_2A_3E_7E_7,\ A_1A_2A_7D_5D_5,$ $A_1A_3A_5A_5E_6,\ A_1A_5A_5A_5D_4$
$2^5.3$	$A_1A_1A_5D_6E_7,\ A_1A_1A_7D_5E_6,\ A_2A_2A_2A_7A_7$
$2^4.3$	$A_2D_4E_7E_7,\ A_2D_5D_6E_7,\ A_3A_5E_6E_6,\ A_5A_5D_4E_6$

Table 2

LEMMA 5.6. *Let $l \neq p$ be a prime number. Assume that the minimal number of generators of the l-torsion part of the discriminant group of the lattice $\mathcal{R}_G$ is greater than 2. Then the l-part of the abelianized group $G/[G,G]$ is nontrivial.*

Proof. Let M be the saturation of the lattice $\mathcal{R}_G$ in $\mathrm{Pic}(Y)$. It follows from the proof of Proposition 4.5 that $G/[G,G] \cong M/\mathcal{R}_G$. The lattice $M \oplus M^\perp$ is a sublattice of the finite index of the lattice S_Y with discriminant group $(\mathbb{Z}/p\mathbb{Z})^2$. Tensoring with the localized ring $\mathbb{Z}_{(l)}$ at the prime ideal $(l) \subset \mathbb{Z}$, we obtain

$$M_l \oplus (M^\perp)_l \subset (S_Y)_l.$$

Since $(S_Y)_l$ is unimodular over $\mathbb{Z}_{(l)}$, the discriminant groups of M_l and $(M^\perp)_l$ are isomorphic to each other. Since $\mathrm{rank}_{\mathbb{Z}_{(l)}} (M^\perp)_l = 2$, we obtain that the discriminant group of M_l is generated by ≤ 2 elements. If l does not divide $\#G/[G,G]$, then $M_l \cong (\mathcal{R}_G)_l$. The discriminant group of $(\mathcal{R}_G)_l$ is the l-torsion part of the discriminant group of $\mathcal{R}_G$, and the assertion follows. □

LEMMA 5.7. *If G is of order divisible by 35, then G is isomorphic to the simple group $M_{21} \cong L_3(4)$ of order $2^6.3^2.5.7$ or the alternating group $\mathfrak{A}_7$ of order $2^3.3^2.5.7$.*

Proof.

Case 1: $G = [G,G]$. By Lemma 5.6, for any prime $l \neq p$, the l-part of the discriminant group of $\mathcal{R}_G$ is generated by ≤ 2 elements.

The List shows that there are 4 possible orders N divisible by 35. We will eliminate the orders $N = 2^4.5.7$ and $3^2.5.7$. In the first case, considering a 7-Sylow subgroup, we see that no elements of order 5 or 2 can normalize it, because G does not contain elements of order 35 or 14. Furthermore, G does not contain D_{14}, which does not admit a Mathieu representation. By Sylow's theorem, $\#\mathrm{Syl}_7(G) = 2^4.5 \equiv 1 \mod 7$, a contradiction. In the second case, no elements of order 7 or 3 can normalize a 5-Sylow subgroup, and hence $\#\mathrm{Syl}_5(G) = 3^2.7 \equiv 1 \mod 5$, again a contradiction.

Assume $\#G = 2^3.3^2.5.7$. This is the order of $\mathfrak{A}_7$. Let us show that G is simple. Assume G is not simple and let H be a normal subgroup such that G/H is simple (non-abelian because $G = [G,G]$). It follows from [Mu, Prop. (3.3)] that $\mu(G/H) > \mu(G) = 4$. The group G/H acts symplectically on a minimal resolution of X/H and belongs to Mukai's list. It follows from Proposition (4.4) of *loc. cit.* that $G/H \cong \mathfrak{A}_5, \mathfrak{A}_6$, or $L_2(7)$. In the first case, $\#H = 42$. There is no such group in Mukai's list as well as in our List. In the second case, $\#H = 7$. It is known that $\mathfrak{A}_6$ does not admit such a nontrivial extension (necessarily central). In the last case $\#H = 15$ and again we use the fact that G does not contain an element of order 15. Thus G is simple. It is known that there is only one simple group of order equal to the order of $\mathfrak{A}_7$. The List gives only one possible root lattice $\mathcal{R}_G = A_2 + A_3 + A_4 + A_6 + D_5$.

Assume $\#G = 2^6.3^2.5.7$. This is the order of the Mathieu group M_{21}. As in the previous case we show that G is simple by analyzing the kernel H of a homomorphism onto a simple quotient of G. As before, $G/H \cong \mathfrak{A}_5, \mathfrak{A}_6$, or $L_2(7)$. In the first case, $\#H = 2^4.3.7$. One checks that there are no such groups in Mukai's list (Theorem (5.5) and Proposition (4.4)) and in our List. In the second case, $\#H = 2^3.7$ and a group of this order with $\mu(G) = 4$ is possible. We will show later that H must be isomorphic to $C_2^3 : C_7$. The Sylow subgroup C_2^3 of H is a characteristic subgroup; hence G acts on it via conjugation. This defines a nontrivial homomorphism $f : G \to \mathrm{Aut}(C_2^3) \cong L_2(7)$. Let us treat this case which also covers the third case for a possible quotient of G. Since $G = [G, G]$, the image of f is not a solvable subgroup of $L_2(7)$. The known classification of subgroups of $L_2(7)$ shows that it is equal to $L_2(7)$. Thus the kernel of f is a subgroup of order $2^3.3.5$. This order is in the List, but as we will see in the next lemma it cannot be realized as the order of an exceptional group. There is only one group with this order in Mukai's list, which is $\mathfrak{S}_5$. This group contains $\mathfrak{A}_5$ as a unique subgroup of index 2. The group G acts on it by conjugation. Since the group $\mathrm{Out}(\mathfrak{A}_5)$ of outer automorphisms (modulo inner automorphisms) of $\mathfrak{A}_5$ is abelian, we get a nontrivial homomorphism $G \to \mathfrak{A}_5 = \mathrm{Inn}(\mathfrak{A}_5)$. As before we infer that it is surjective. But we have seen above that this is impossible. This proves that G is simple. It is known that there are two simple groups of order $2^6.3^2.5.7$; one is M_{21} and another is $\mathfrak{A}_8$. The latter group contains an element of order 15 and must be excluded. The List gives only one possibility $\mathcal{R}_G = A_2 + 2A_4 + A_6 + D_4$.

Case 2: $G \neq [G, G]$. Since 5 and 7 divide $\#G$, X/G has singularities of type A_4 and A_6. This follows from Table 1. Assume G has a normal subgroup H with an abelian quotient of order 5 (resp. 7). Then $X/H \to X/G$ is a cyclic cover of degree 5 (resp. 7). The pre-image of a singularity of type A_6 (resp. A_4) consists of five singularities of type A_6 (or seven singularities of type A_4). This gives $\mathrm{rank}\,\mathcal{R}_H > 20$, a contradiction. This implies that $35|\#[G, G]$. If $\mu([G, G]) \geq 5$, this contradicts [Mu, Prop. (4.2)]. So $\mu([G, G]) \leq 4$. On the other hand, by [Mu, Cor. (3.5)], $\mu([G, G]) \geq \mu(G) = 4$. Thus $\mu([G, G]) = 4$. Replacing G with $[G, G]$ and repeating the argument, we see that G is obtained by taking extensions of a proper subgroup K with $35|\#K$, $K = [K, K]$ and $\mu(K) = 4$. By the result of Case 1, $K \cong M_{21}$ or $\mathfrak{A}_7$. The first case can be excluded, since M_{21} has order maximal in the List. The second case can also be excluded, since $\mathfrak{A}_7$, though its order is not maximal in the List, admits no extensions in the set of finite symplectic groups. This easily follows from the fact that $X/\mathfrak{A}_7$ has only one singularity of type A_4, and $5k,\ k > 1$, does not divide the order of a stabilizer subgroup. $\square$

It follows from the List that a 2-group is not exceptional and hence is in Mukai's list. We need the following description of symplectic groups of order 2^6.

LEMMA 5.8 ([Mu], [Xiao]). *There are five symplectic groups of order* 2^6: $\Gamma_{13}a_1$, $\Gamma_{22}a_1$, $\Gamma_{25}a_1$, $\Gamma_{23}a_2$, *and* $\Gamma_{26}a_2$.

(i) $\Gamma_{23}a_2$ *and* $\Gamma_{26}a_2$ *do not contain a subgroup* $\cong 2^4$.

(ii) $\Gamma_{13}a_1 \cong 2^4 : 2^2 \cong 4^2 : 2^2 \cong 2^2.\, 2^4$.

(iii) $\Gamma_{22}a_1 \cong 2^4 : 4 \cong 2^3.\, (2 \times 4)$.

(iv) $\Gamma_{25}a_1 \cong 2^4 : 2^2 \cong 2^3.\, 2^3$.

(v) $\Gamma_{22}a_1$ *and* $\Gamma_{25}a_1$ *have only one normal subgroup* $\cong 2^4$.

(vi) $\Gamma_{13}a_1$, $\Gamma_{22}a_1$ *and* $\Gamma_{25}a_1$ *always split over every normal subgroup* $\cong 2^4$.

In (ii)–(iv), *the last isomorphism is given by* (commutator).(quotient).

Proof. We give a proof of the last assertion (vi), which is not given explicitly in [Mu] or [Xiao]. By (ii)–(iv), each of these three groups has a normal subgroup $\cong 2^4$ over which it splits. By (v) the assertion follows for $\Gamma_{22}a_1$ and $\Gamma_{25}a_1$.

Let $G = \Gamma_{13}a_1$, and let $H_1,\ H_2$ be two distinct normal subgroups $\cong 2^4$ of G. If $\#(H_1 \cap H_2) = 8$, then the join H_1H_2 is of order 2^5. There is only one symplectic group of this order containing a subgroup $\cong 2^4$ [Mu]. It is Γ_4a_1, but this group does not contain two subgroups $\cong 2^4$. This proves that $\#(H_1 \cap H_2) \le 4$. In this case H_i contains a subgroup $K_i \cong 2^2$ with $K_i \cap H_j = \{1\}$ ($\{i,j\} = \{1,2\}$). Thus $H_j < G$ splits for $j = 1, 2$. □

COROLLARY 5.9. *Let G be an exceptional group or in Mukai's list. Let H be a normal subgroup of G. Assume either $H \cong 2^4$ or $\#H = 2^6$. Then G splits over H.*

Proof. Let P be a 2-Sylow subgroup of G. Then P contains H as a normal subgroup. By Gaschütz's theorem cited in [Mu, p. 204 in the proof of Prop. (4.8)], it suffices to show that $H < P$ splits. We may assume $P \neq H$.

Assume $H \cong 2^4$. If $\#P = 2^5$, then $P \cong \Gamma_4a_1$ and $H < P$ splits. If $\#P = 2^6$, then by Lemma 5.8, $H < P$ splits. If $\#P = 2^7$, then $P \cong F_{128}$, a unique symplectic group of order 128. It follows from [Mu, Prop. (3.18)] that $H < P$ splits.

Assume $\#H = 2^6$. Then $P \cong F_{128}$, which splits over every subgroup of index 2. □

LEMMA 5.10. *Let G be an exceptional group of order $2^a.3^b.5$ or $2^a.3.7$ with $a, b \ge 1$. Then G is isomorphic to one of the following seven groups*:

$$(2^3 : 7) : 3,\ \mathfrak{S}_6,\ M_{10},\ 2^4 : \mathfrak{A}_6,\ 2^4 : \mathfrak{A}_5 \not\cong M_{20},\ M_{20} : 2 = 2^4 : \mathfrak{S}_5,\ 2^3 : L_2(7).$$

Proof. Assume that G is solvable. A solvable group of order mn, where $(m,n)=1$ contains a subgroup of order m (see [Ha, Th. 9.3.1]). If $\#G=2^a.3^b.5$, G contains a subgroup of order $3^b.5$. Such an order cannot be found in Mukai's list or in our List. The case $\#G=2^6.3.7$ can also be excluded, because there is no group of order $2^6.7$ in both lists. Assume $\#G=2^3.3.7$. If G is simple, then $G\cong L_2(7)$ with $\mu(L_2(7))=5$, hence, not exceptional. Thus G is not simple. Let H be a nontrivial normal subgroup of G. By inspecting possible orders of H and G/H, and using that an order 7 element does not normalize an order 3 element, we infer that $\#H=2^3$ or $2^3.7$. In the first case, the quotient G/H has a normal subgroup of order 7, hence we may assume the second case. A group of order $2^3.7$ does not appear in Mukai's list, so must be exceptional. Later, we will see that such a group must be isomorphic to $2^3:7$ and its lattice is $3A_6+2A_1$. This implies that $G\cong(2^3:7):3$ and $\mathcal{R}_G=A_6+2A_5+2A_2$. This order with the type of lattice can be found in the List.

Now we assume that G is not solvable. Therefore G contains a normal subgroup H and a normal subgroup T of H such that H/T is a non-commutative simple group.

Case 1: $T=\{1\}$; *i.e., G contains a simple nonabelian normal subgroup H.* Assume $5|\#G$. The known classification of simple groups of order $2^a.3^b.5$ (see [Br]) gives that H is isomorphic to $\mathfrak{A}_5$ or $\mathfrak{A}_6$. Let $\bar{G}=G/H$. Since the groups $\mathfrak{A}_5,\mathfrak{A}_6$ are in Mukai's list and hence satisfy $\mu(G)\geq 5$, we may assume that $\bar{G}$ is a nontrivial group. It is known (see [Xiao]) that in both cases X/H has two singular points of type A_4. Since $5k, k>1$ does not divide the order of a stabilizer subgroup of G, we see that $\bar{G}$ acts simply transitively on the set of two singularities, and hence is a cyclic group of order 2. If $H\cong\mathfrak{A}_5$, the group G is isomorphic to $\mathfrak{S}_5$ or the direct product $\mathfrak{A}_5\times C_2$. The first group is in Mukai's list and $\mu(G)=5$. The second group contains a cyclic group of order 10 which cannot act symplectically on X.

Thus, we may assume that $H\cong\mathfrak{A}_6$. Again, G cannot be the direct product $\mathfrak{A}_6\times C_2$. The ATLAS [CN] shows that $G\cong\mathfrak{S}_6$ or $G\cong M_{10}$. The orders of these groups are in the List and there are five possible root lattices for groups of order 720. Three of them contain sublattices of type E_6 or D_5. It is easy to see that the corresponding stabilizer subgroups isomorphic to T_{24} or Q_{12} are not subgroups neither of S_6 nor M_{10}. This leaves only two possibilities for groups of this order: $\mathcal{R}_G=2A_3+A_4+2A_5$ or $\mathcal{R}_G=A_2+A_3+A_4+A_7+D_4$. Since $\mathfrak{S}_6$ has no elements of order 8 but M_{10} has, we see that the first case could be realized for $\mathfrak{S}_6$ and the second for M_{10}.

Assume $7|\#G$. The known classification of simple groups of order $2^a.3.7$ [Br] gives that $H\cong L_2(7)$. This group is from Mukai's list and $\mu(L_2(7))=5$. Thus, the quotient $\bar{G}=G/H$ is a group of order 2^{a-3}. The orbit space X/H has one singular point of type A_6 which must be fixed under the action of $\bar{G}$ on

X/H giving a singularity Gx on X/G with stabilizer group G_x of order $7.\#\bar{G}$. Table 1 shows that $\bar{G}$ must be trivial. Therefore, $\mu(G) = 5$ and this case is not realized.

Case 2: $T \neq \{1\}$ *and* $H = G$. Assume $5|\#G$. Again $\bar{G} = G/T \cong \mathfrak{A}_5$ or $\mathfrak{A}_6$. Thus T is a normal subgroup of order $2^{a-2}.3^{b-1}$ if $\bar{G} \cong \mathfrak{A}_5$, or 2^{a-3} if $\bar{G} \cong \mathfrak{A}_6$. Assume $b = 2$ and $G/T = \mathfrak{A}_5$. Since an element g_5 of order 5 from G does not normalize a subgroup of order 3, we see that the number of Sylow 3-subgroups in T must be divisible by 5. This is a contradiction. Assume T is of order 2^{a-2} and $G/T = \mathfrak{A}_5$. Since an element of order 5 does not commute with an element of order 2, it cannot act identically on the center of T. It is easy to check that an abelian group A of order 2^n admits an automorphism of order 5 only if $n > 4$ or $A = C_2^4$. A group which contains a central subgroup of index 2 is abelian. Thus T is either a 2-elementary abelian group of order 2^4 (if $a = 6$), or an abelian group of order 2^5 (if $a = 7$). It follows from Nikulin ([Mu, Prop. (3.20)]) that an abelian group of order 2^5 does not admit the Mathieu representation. Therefore, $T \cong C_2^4$. The order $2^6.3.5$ is in our List. There is only one possible root lattice in this case: $\mathcal{R}_G = A_1 + 2A_4 + D_5 + E_6$. This group is not isomorphic to M_{20}, since $\mu(M_{20}) = 5$. A possible scenario is that X/T has 15 singular points of type A_1 and $G/H \cong \mathfrak{A}_5$ has two orbits of 5 and 10 points on this set. One orbit gives a singularity of type E_6, another one of type D_5. By Corollary 5.9, $T < G$ splits; i.e., $G \cong 2^4 : \mathfrak{A}_5$. There are two nonisomorphic actions of $\mathfrak{A}_5$ on $\mathbb{F}_2^4$ (see §2.8 in [Mu]). One gives the group M_{20} from Mukai's list. Another one gives a group from (vii). We denote this group by M_{20}'. It is realized as a subgroup of M_{23}.

Assume $G/T = \mathfrak{A}_6$. A similar argument shows that $T \cong C_2^4$. A direct computation shows that $\mu(\mathfrak{A}_6) = 5$. For any nontrivial normal subgroup T of G we have $\mu(G/T) > \mu(G)$ ([Mu, Prop. (3.3)]). Thus $\mu(G) \leq 4$ and such group may appear. The order $2^4.\#\mathfrak{A}_6$ appears in the List with $\mathcal{R}_G = A_2 + A_3 + 2A_4 + E_7$. This case cannot be excluded. By Corollary 5.9, $T < G$ splits and G is isomorphic to a semi-direct product $2^4 : \mathfrak{A}_6$.

Assume $7|\#G$. We have $\bar{G} \cong L_2(7)$ and T is of order 2^{a-3}. It follows from the List that $a = 6$ and $\mathcal{R}_G = A_2 + A_3 + A_5 + A_6 + D_4$ or $A_2 + 2A_3 + A_6 + E_6$. Let us exclude the first possibility for $\mathcal{R}_G$. Since the 2-part of the discriminant group is generated by > 2 elements, by Lemma 5.6, it suffices to show that $G = [G, G]$. Since $L_2(7)$ is simple, the image of $[G, G]$ in $\bar{G}$ is either trivial or the whole $\bar{G}$. In the first case $[G, G] \subset T$ and the abelian group $G/[G, G]$ maps surjectively to a noncommutative group $L_2(7)$. In the second case, $\#L_2(7) \leq \#[G, G] \leq 2^3\#L_2(7)$. The List shows that one of the inequalities is the equality. If it is the first one, G contains a normal subgroup isomorphic to $L_2(7)$ and hence is isomorphic to the product $T \times L_2(7)$ which contains elements of order larger than allowed. If it is the second one, then $G = [G, G]$. This proves that $\mathcal{R}_G = A_2 + 2A_3 + A_6 + E_6$. Next we prove that the extension $G = 2^3.L_2(7)$

splits. From the type of $\mathcal{R}_G$ we see that G contains no elements of order 8. Let P be a 2-Sylow subgroup of G. Since the 2-Sylow subgroups of $L_2(7)$ are isomorphic to the dihedral group D_8, we have the extension $P = 2^3.D_8$. By Gaschütz's theorem cited in [Mu, p. 204 in the proof of Prop. (4.8)], it suffices to show that $2^3 < P$ splits. Since P is isomorphic to one of the five groups from Lemma 5.8 and contains no elements of order 8, we see that P must be isomorphic to $\Gamma_{25}a_1$, $\Gamma_{23}a_2$, or $\Gamma_{13}a_1$. The last group can further be excluded as it does not have a normal subgroup $\cong 2^3$ with quotient $\cong D_8$ [HS]. Consider a subgroup $K \cong 2^3.\mathfrak{S}_4$ of G containing P. The Mukai's list contains three groups of the order $\#K$, T_{192}, H_{192}, and $\Gamma_{13}a_1 : 3$ [Xiao]. The first two are split extensions $2^3 : \mathfrak{S}_4$ ([Mu, Remark, p. 192]), and the last one cannot contain P, since $P \not\cong \Gamma_{13}a_1$. If K is exceptional, then as we will see later, K is isomorphic to (xi) $(Q_8 \circ Q_8)^{\bullet}\mathfrak{S}_3$, (xxii) $(2^4 : 2)^{\bullet}\mathfrak{S}_3$, (xxiii) $\Gamma_{13}a_1 : 3$, or (xxv) $\Gamma_{25}a_1 : 3$. The first two contain elements of order 8, as their $\mathcal{R}_K$ show, and the third can also be excluded for the same reason as above. Thus $P \cong \Gamma_{25}a_1$. Since $\Gamma_{25}a_1$ is a 2-Sylow subgroup of $\mathrm{Hol}(2^3) = 2^3 : L_2(7)$ [HS], $2^3 < P$ splits.

Case 3: $T \neq \{1\}, G \neq H$. It is shown in [Mu, Th. (4.9)] that a non-solvable group G with $\mu(G) \geq 5$ is isomorphic to $\mathfrak{A}_5, \mathfrak{A}_6, \mathfrak{S}_5, L_2(7)$ or $M_{20} \cong 2^4 : \mathfrak{A}_5$. If $\mu(H) \geq 5$, we must have $H \cong M_{20}$ and $T = C_2^4$. Since X/M_{20} has two singularities of type A_4, G/H must be a cyclic group of order 2, and hence $G \cong M_{20\bullet}2$. The order $2\#M_{20}$ appears in the List with $\mathcal{R}_G = A_2+A_3+A_4+A_5+D_6$. From this lattice, it is easy to compute the order breakdown for G; in particular, G has more elements of order 2 than M_{20}. Thus, the extension splits and $G = M_{20} : 2 \cong 2^4 : \mathfrak{S}_5$.

Assume $\mu(H) = 4$. It follows from Case 2 that H is isomorphic to one of the two groups M_{20}', or $2^4 : \mathfrak{A}_6$. There are no orders in the List of orders strictly divisible by the order of the latter group. In the first case, as we saw in Case 2, X/M_{20}' has singularities of type $A_1 + 2A_4 + D_5 + E_6$. It has two singular points of type A_4, so $G/M_{20}' \cong C_2$ and its action on X/M_{20}' fixes the unique singularity of type D_5. However a degree 2 extenson of Q_{12} cannot be a stabilizer subgroup. This proves that the group M_{20}' has no extension in the set of finite symplectic groups. □

It remains to consider the cases where $\#G$ is divisible by at most two primes. These are all solvable groups.

It follows from the List that an exceptional group G cannot be of order $q^a, q = 2, 3, 5, 7$ or $3^b5,\ 3^b7, b = 1, 2$.

LEMMA 5.11. *Let G be an exceptional group. Assume G is solvable of order $2^6 \cdot 5$, or $2^3 \cdot 7$, or $2^a \cdot 3^2, (3 \leq a \leq 7)$. Then G is one of the groups from* (ix)–(xvii) *in Theorem* 5.2.

Proof. First of all Proposition (5.1) from [Mu] gives that a nilpotent G is either abelian with no elements of order 4 or a 2-group. By the order condition

G is not a 2-group. Assume that G is abelian with no elements of order 4. Then $G \cong C_2^a \times C_3^2, C_2^6 \times C_5$, or $C_2^3 \times C_7$. The latter two cases have no Mathieu representations. In the first case, $\mu(G) = 4$ implies that $a = 2$. But such an order is not in the List. Thus we may assume that G is not nilpotent and hence its Fitting subgroup F (maximal nilpotent normal subgroup) is a proper nontrivial subgroup. We use Mukai's classification by analyzing only the cases where the assumption $\mu(G) \geq 5$ was used.

The first case is when $F \cong C_3^2$. The quotient G/F must be a 2-subgroup H of the group $\mathrm{Aut}(C_3^2) \cong \mathrm{GL}_2(\mathbb{F}_3)$ of order 48. The List shows that $\#H = 8$ or 16. Assume that $\#H = 8$. There are two possible root lattices for exceptional groups of order 72: $\mathcal{R}_G = A_1 + A_2 + A_3 + 2A_7$ or $D_4 + 2D_5 + E_6$. It is known that X/F has eight singular points of type A_2. To get a singular point of type E_6 in X/G the group H must fix exactly one of the singular points on X/F. This is obviously impossible. Thus only the first case is realized. It corresponds to the case $G/F \cong C_8$ which leads to a group with $\mu(G) = 4$ (see [Mu, p. 206]). If $\#H = 16$, the group H is a 2-Sylow subgroup of $\mathrm{GL}_2(\mathbb{F}_3)$, known to be isomorphic to the semi-dihedral group SD_{16}. The order of $3^2 : SD_{16}$ is 144 and it is in the List with $\mathcal{R}_G = A_7 + D_4 + A_5 + A_3 + A_1$ or three other possibilities, all containing one copy of the root lattice E_6 or E_7. These cases can be easily excluded. This gives the groups from (ix) and (x).

The second possible case is when G/F is of order divisible by 7 and G has a group K isomorphic to $C_2^3 : C_7$ as a subquotient group. The only possible order from the List of the form $2^a.7$ is $2^3.7$. The corresponding root lattice is $2A_1 + 3A_6$. This leads to our group in (xi).

The third case is when $\#G = 2^a.5$ and G admits a quotient $\bar{G}$ containing a subgroup G_0 with $\mu(G_0) = 5$ isomorphic to $C_2^4 : C_5$. The inspection of the List gives that $\#G = 2^6.5$. If $\bar{G} \neq G$, then $\mu(G) < \mu(G_0)$. Assume this is the case. The kernel H of the projection $G \to \bar{G}$ is of order 2^s with $s \leq 2$. Since it is a normal subgroup and an element of order 5 does not commute with an element of order 2, the order of $\mathrm{Aut}(H)$ must be divisible by 5. This implies that $H = \{1\}$. Thus the Fitting subgroup F of G is a normal subgroup isomorphic to C_2^4, and G must contain $C_2^4 : C_5$ as a proper subgroup. It is known that G/F is mapped injectively in $\mathrm{Out}(F)$. The quotient G/F is of order $2^2.5$. Since it cannot contain an element of order 10, the quotient is isomorphic to $C_5 : C_4$. There are three possible root lattices $\mathcal{R}_G$ in the List. Two of them contain a sublattice of type D_4 or D_6. It is easy to see that the corresponding stabilizer subgroups Q_8, Q_{16} are not subgroups of G. The remaining case $3A_3 + A_4 + A_7$ cannot be excluded. By Corollary 5.9, the extension $G = 2^4.\,(5:4)$ splits, and gives case (xii).

The fourth case is when G/F is of order divisible by 9. In this case G is of order $2^a.3^2$ by the order assumption. Mukai considers the Frattini subgroup Φ of F (the intersection of maximal subgroups) and shows that $F/\Phi \cong C_2^4$ and

G/Φ contains a subgroup G_0 with $G_0 \cong C_2^4 : C_3^2 \cong \mathfrak{A}_4 \times \mathfrak{A}_4$ and $\mu(G_0) = 5$. As in the previous case, to get $\mu(G) = 4$ we must have either $\Phi \neq \{1\}$ or $\Phi = \{1\}$ and G_0 is a proper subgroup of G.

Assume $\Phi \neq \{1\}$ and let $\#\Phi = 2^s, s \leq 3$. Assume $s = 1$. The quotient X/Φ has eight singular points of type A_1. The group G_0 acts on X/Φ and permutes these points with at least one stabilizer subgroup of order divisible by 3^2. The known structure of the stabilizers shows that this is impossible. No stabilizer G_x is of order divisible by 9. Assume $s = 2$ thus $\Phi \cong C_2^2$ or C_4. In the first case the quotient X/Φ has 12 singular points of type A_1 and the stabilizer subgroups of G_0 of these points are groups of order $2^2.3$ (one orbit), or $2^3.3$ (two orbits of size 6 each). The first case gives one singularity of type E_6, and the second two singularities of type E_7. Both appear in the List. But, since G_0 has singularities of type $2A_5 + 4A_2 + A_1$ (no A_3) on a minimal resolution Y of X/Φ (see [Xiao]) we easily exclude the second case, obtaining that the singularities of Y/G_0 are of types $E_6 + 2A_2 + 2A_5$. This case can be found in the List. This gives a possible case $G \cong 2^2.\ (\mathfrak{A}_4 \times \mathfrak{A}_4) \cong \Gamma_{13}a_1 : 3^2$ from (xiii). If $\Phi = C_4$, X/Φ has singularities of type $4A_3 + 2A_1$. The stabilizer of G_0 of a point of type A_1 must be a group of order divisible by $2^3.3^2$. No stabilizer G_x is of order divisible by 9. Thus this case does not occur. A degree 2 extension of the group from (xiii) may also appear. In this case G/Φ is a degree 2 extension of $G_0 = \mathfrak{A}_4 \times \mathfrak{A}_4$ and $\mu(G/\Phi) > \mu(G) = 4$, therefore G/Φ appears on Mukai's list. There appears only one such group in [Xiao], and it is $\mathfrak{A}_{4,4}$. Thus $G \cong 2^2.\ \mathfrak{A}_{4,4} \cong \Gamma_{13}a_1 : \mathfrak{A}_{3,3}$, and we obtain that $\mathcal{R}_G = E_7 + 2D_5 + A_2 + A_1$. It occurs in the List. This is the case (xiv).

Assume $s = 3$. Then F is a Sylow 2-subgroup of G of order 2^7 with normal subgroup Φ of order 8. Thus $\Phi \cong Q_8, D_8, 2^3, 2 \times 4$ or C_8. Assume that $\Phi \cong Q_8$. The quotient X/Φ has singular points of type $2D_4 + 3A_3$ or $4D_4 + A_1$. In any case the group G_0 of order $2^4.3^2$ permutes points of type D_4 with stabilizer of order divisible by 9. No stabilizers of this order could occur. Similar argument rules out the remaining possibilities $\Phi \cong D_8, 2^3, 2 \times 4$ or C_8. This shows that the case $\#G = 2^7.3^2$ and $\#\Phi = 8$ does not occur.

Assume $\Phi = \{1\}$. Thus G contains $2^4 : 3^2$ as a proper subgroup and $F = 2^4$. If $a = 5$, the quotient group G/F is of order 18. There are two possible groups of order 18 which can act symplectically on X: a group isomorphic to $\mathfrak{A}_{3,3}$, or $C_3 \times D_6$. The first case leads to a group $G \cong \mathfrak{A}_{4,4}$ with $\mu(G) = 5$. The second case gives singularities of X/G of types $3A_5 + A_3 + A_2$. A case with a group of this order and the same $\mathcal{R}_G$ can be found in the List. By Corollary 5.9, the extension $F < G$ splits, and gives the case (xv). If $a = 6$, we have three possible groups G/F isomorphic to $3^2 : 4$, $3 \times \mathfrak{A}_4$, or $\mathfrak{S}_{3,3}$. In the second case, G/F contains a normal subgroup of order 4, and its pullback is a nilpotent normal subgroup of G contaning F. This contradicts the maximality of the Fitting subgroup F. This excludes the second group. The first and the third groups have singularities of the quotient of types $4A_3 + 2A_2 + 2A_1$ and $2A_5 + A_2 + 6A_1$, resp.. The groups act on the set of 15 singular points of X/F. It is easy to see that in these two cases X/G has singularities of types $A_7 + D_5 + 2A_3 + A_2$, $D_5 + D_4 + 2A_5 + A_1$. Both cases can be found in the List. Both extensions split by Corollary 5.9, giving the groups from (xvi) and (xvii). If $a = 7$, we have three possible groups G/F isomorphic to one of the following groups $\mathfrak{A}_{4,3}$, $N_{72} \cong 3^2.D_8$, $M_9 \cong 3^2.Q_8$ of order 72. The first case can be excluded by the maximality of F, as the group contains a normal subgroup of order 4. The order $2^7.3^2$ appears in the List with possible root lattices $\mathcal{R}_G$ of types $A_1 + A_2 + 2D_5 + E_7$ or $A_1 + A_2 + D_5 + D_6 + E_6$. It is easy to see that the stabilizer subgroups T_{24} and O_{48} of singularities of type E_6 and E_7 are not subgroups of $2^4_{\bullet}N_{72}$ or $2^4_{\bullet}M_9$. Thus these two groups are also excluded. □

Lemma 5.12 *Let G be an exceptional group. Assume G is solvable of order $2^a \cdot 3$, $(4 \leq a \leq 7)$. Then G is one of the groups from* (xviii)–(xxvi) *in Theorem* 5.2.

Proof. Again we follow the arguments from [Mu]. It follows from the List that a 2-group is not exceptional and hence is in Mukai's list. Hence his assumptions (7.1) are satisfied except the last one where we have to replace the condition $\mu(G) \geq 5$ with the condition $\mu(G) \geq 4$.

First Mukai considers the case when the Fitting subgroup F of G is of order divisible by 3, or equivalently, G has a unique 3-Sylow subgroup T. Let S be a 2-Sylow subgroup and $\phi : S \to \mathrm{Aut}(T) = C_2$ be the natural homomorphism. The classification of abelian nilpotent symplectic groups shows that $\#\mathrm{Ker}(\phi)$ is of order ≤ 4, thus $\#G \leq 24$. There are no exceptional groups of order ≤ 24.

Thus, we may assume that the Fitting subgroup F is a 2-group. In this case F is the intersection of all 2-Sylow subgroups of G. If F is equal to a unique 2-Sylow subgroup, then $G/F \cong C_3$. Otherwise G contains three 2-Sylow subgroups and $G/F \cong \mathfrak{S}_3$.

Using the classification of 2-groups, Mukai lists all possible groups F. In our case they can be only of order 2^c, $c = 3, 4, 5, 6, 7$ ($c = a$, or $a - 1$).

Case $c = 3$. In this case $F \cong C_2^3$ or Q_8 and $G/F \cong \mathfrak{S}_3$. There are four possible root lattices for exceptional groups of order 48. If $F \cong C_2^3$, the quotient X/F has 14 singular points of type A_1 so the largest stabilizer for the action of G on X is of order 12. Since all possible root lattices contain a subdiagram E_6 or E_7 this case is not realized. The quotient X/Q_8 has two singular points of type D_4 and three singular points of type A_3, or other possibility is four points of type D_4 and one of type A_1. It is easy to see that the first possibility leads to the root lattice of type $A_2 + D_4 + 2E_7$ and the second one to $A_2 + D_5 + D_6 + E_7$. In both cases the group G must coincide with the stabilizer of a singular point of type E_7 and hence is isomorphic to O_{48}. This is our case (xviii).

Case $c = 4$. In this case $F \cong C_2^4, C_4^2$, or $Q_8 \times C_2$, or $Q_8 \circ C_4$. In the last case, Mukai shows that $G \cong T_{24} \circ C_4$, or $(T_{24} \circ C_4)_\bullet 2$, and excludes them because $T_{24} \circ C_4$ contains an element of order 12.

Assume $F \cong Q_8 \times C_2$. This case is also excluded by Mukai because of the assumption $\mu(G) \geq 5$. It cannot be excluded in our case and leads to two groups $G = T_{24} \times C_2$, if $G/F \cong C_3$, and $G = (T_{24} \times C_2)_\bullet 2$, if $G/F \cong \mathfrak{S}_3$. Let us determine the root lattice of $G = T_{24} \times C_2$. The singular points of X/T_{24} are of type $E_6 + D_4 + A_5 + 2A_2$, or $2E_6 + A_3 + 2A_2$. In the first case, the group C_2 fixes the unique singularity of type E_6. Since $T_{24} \times C_2 \neq O_{48}$, we get a contradiction. In the second case X/G has singularities of type $E_6 + D_4 + 2A_5$. A group of order 48 with this root lattice is in the List and gives case (xix). It is easy to see that $G = (T_{24} \times C_2)_\bullet 2$ has root lattice $2A_1 + A_5 + D_6 + E_7$. Computing order breakdown, we see that G has more elements of order 2 than $T_{24} \times C_2$ or O_{48}. Therefore the extension splits and $G = (T_{24} \times C_2) : 2 \cong O_{48} : 2$. This is also in the List and gives case (xx).

Assume $F \cong C_2^4$ or C_4^2. If F is a 2-Sylow subgroup, then $\#G = 48$. The quotient G/F has either 15 singular points of type A_1 or six singular points of type A_3. Thus the largest possible order of a stabilizer subgroup of G is 12. The List shows that the root lattice $\mathcal{R}_G$ always contains a copy of E_6 or E_7. This shows that this case does not occur. So, F is not a 2-Sylow subgroup and $G/F \cong \mathfrak{S}_3$. There are three possible root lattices for groups of order 96 in our List. They are of types $2A_1 + A_5 + D_6 + E_7$, $2A_1 + A_7 + D_5 + E_6$, $3A_2 + 2A_7$. If $F \cong C_2^4$, X/F has 15 singular points of type A_1 and hence the largest possible stabilizer subgroup of G is of order 12 and no stabilizers of order 8. This shows that this case is not realized. If $F \cong C_4^2$, X/F has six singular points of type A_3 and hence the largest possible order of a stabilizer subgroup of G is 24. Since $T_{24} \not\cong 4_\bullet \mathfrak{S}_3$, the root lattice of G cannot contain E_6. This rules out the first two root lattices, and the remaining root lattice is $2A_7 + 3A_2$. The 3-part of its discriminant group is isomorphic to 3^3. But $G \cong 4^2_\bullet \mathfrak{S}_3$ does not admit a non-trivial homomorphism to C_3. This contradicts Lemma 5.6. Thus, this case must be excluded.

Case $c = 5$. In this case F is isomorphic to $Q_8 \circ Q_8$ or $2^4 : 2 = \Gamma_4 a_1$. Assume $F = Q_8 \circ Q_8$. In this case X/F has nine singular points of type A_1 and 2 singular points of type D_4. If $G/F \cong C_3$, the group C_3 fixes the points of type D_4 and define two points of type E_6 on the quotient X/G. No root lattices with such sublattices are realized for groups of order 96. Thus $G/F \cong \mathfrak{S}_3$. There are five possible root lattices for groups of order 192. If $\mathfrak{S}_3$ leaves points of type D_4 invariant, then X/G has two singular points of type E_7. If it permutes these point, we have one singular point of type E_6. By inspection of the diagrams, we see only one lattice $2E_7 + A_3 + A_2 + A_1$ fits. From the lattice, it is easy to see that G has exactly 19 elements of order 2.

Since the subgroup $Q_8 \circ Q_8$ has the same number of elements of order 2, the extension does not split. This is the group from (xxi).

Assume that $F \cong 2^4 : 2$. Then X/F has three singular points of type A_3 and eight singular points of type A_1. Using the list of possible root lattices for exceptional groups of order 96, we see that $G/F \cong \mathfrak{S}_3$. Since $T_{24} \ncong 4_\bullet\mathfrak{S}_3$, the group $\mathfrak{S}_3$ does not leave a point of type A_3 invariant, and hence permutes three singular points of type A_3, giving a singular point on X/G of type D_4 or A_7. Moreover, this argument shows that X/G does not have a singular point of type E_7 or E_6. By inspecting the five possible root lattices for groups of order 192, we see only two lattices $A_1+A_2+A_7+2D_5$ and $A_1+3A_5+D_4$ survive. But the latter can also be eliminated by considering the orbit decomposition of $\mathfrak{S}_3$ on the eight singular points of type A_1. This gives $\mathcal{R}_G = A_1 + A_2 + A_7 + 2D_5$. By a similar computation of the number of elements of order 2, we see that the extension does not split. This is case (xxii).

Case $c = 6$. In this case F is a 2-group of type $\Gamma_{13}a_1$ or $\Gamma_{25}a_1$ from Proposition (6.12) of [Mu]. Assume $F \cong \Gamma_{13}a_1$. The quotient X/F has three singular points of type D_4 and six singular points of type A_1. If $G/F \cong C_3$, this leads to the two possible root lattices $A_1 + D_4 + 3A_5$ and $2A_1 + 3E_6$. This is case (xxiii). If $G/F \cong \mathfrak{S}_3$, then we have a group of order $2^7.3$. The corresponding root lattices in our List are of the following types: $A_1 + A_3 + A_5 + D_5 + D_6, A_1 + A_3 + A_5 + D_4 + E_7, A_1 + 2A_3 + E_6 + E_7$. The first and the third correspond to the two root lattices of the index 2 subgroup. By Corollary 5.9, the extension $G = F.\ \mathfrak{S}_3$ splits, and gives case (xxiv) and (xxiv′).

Assume $F \cong \Gamma_{25}a_1$. Then X/F has one singular point of type D_4, three singular points of type A_3 and five singular points of type A_1. If $G/F \cong C_3$, $\mathcal{R}_G$ must contains only one copy of E_6. There is only one such lattice $A_1+A_3+2A_5+E_6$ for groups of order 192. This gives case (xxv). If $G/F \cong \mathfrak{S}_3$, the root lattice $A_1 + A_3 + A_5 + D_4 + E_7$ may occur. Again, by Corollary 5.9, the extension splits, and gives case (xxvi).

Case $c = 7$. In this case $G/F \cong C_3$ and Mukai leads this to contradiction. It is still true in our situation. The group $F \cong F_{128}$ of order 2^7 has singularities on X/F_{128} of types $D_6 + D_4 + 2A_3 + 3A_1$. Since F is normal, the group C_3 acts on X/F_{128}. It must fix the singular points of type A_3 and gives on X/G singular points of type D_5. But $Q_{12} \ncong C_4.C_3$. □

Remark 5.13. Here we explain the difference between the groups $2^4 : \mathfrak{S}_4$. Mukai's list contains a unique group of order $384 = 2^7.3$ which he denotes by F_{384}. Our list contains three groups of this order non-isomorphic to F_{384}. The difference between these groups and F_{384} is very subtle. The group F_{384} is isomorphic to $2^4 : \mathfrak{S}_4$ and the pre-image of the normal subgroup 2^2 of $\mathfrak{S}_4$ is isomorphic to $\Gamma_{13}a_1$ (see [Mu, p. 212]). Thus $F_{384} \cong \Gamma_{13}a_1 : \mathfrak{S}_3$ is an extension of the same type as our groups (xxiv) and (xxiv′). The difference is of course

G	action of $\mathfrak{S}_4$ on $2^4 \setminus \{0\}$	action of $\mathfrak{S}_4$ on Ξ	composition series
F_{384}	$3+12$	$1+1+2+4$	$(2,2)$
(xxiv)	$3+4+8$	$1+3+4$	$(2,1,1)$
(xxiv′)	$1+2+12$	$1+3+4$	$(1,1,2)$
(xxvi)	$1+6+8$	$1+1+6$	$(1,2,1)$

Table 3

in the action of $\mathfrak{S}_4$ on $2^4 \setminus 0$, whose orbit decomposition can be read off from the corresponding root lattice $\mathcal{R}_G$ and is given in the first column of Table 3.

Recall that in the action of M_{24} on a set Ω of 24 elements, the complement of an octad Ξ is identified with the affine space $\mathbb{F}_2^4$, and each element $g \in \mathfrak{A}_8$ embeds in M_{24} by acting on Ξ as an even permutation and acting on the complement $\Omega \setminus \Xi$ as a linear map $i(g)$, where $i : \mathfrak{A}_8 \to L_4(2) = \mathrm{GL}_4(\mathbb{F}_2)$ is the exceptional isomorphism of simple groups. For F_{384} Mukai shows that the image of $\mathfrak{S}_4$ in $\mathfrak{A}_8$ is a subgroup with orbit decomposition $1+1+2+4$ ([Mu, p. 218 and Cor. (3.17)]), dependent on the assumption $\mu(G) \geq 5$. In our case, $\mu(G) = 4$ and hence the number of orbits must be equal to 3. It is easy to see that $1+3+4$ and $1+1+6$ are the only possible such decompositions for a subgroup of $\mathfrak{A}_8$ isomorphic to $\mathfrak{S}_4$. On the other hand, it is known that $\mathfrak{S}_4$ has only one non-trivial irreducible linear representation over a field of characteristic 2. It is isomorphic to the representation $\mathfrak{S}_4 \to \mathfrak{S}_3 \cong L_2(2)$. This easily implies that any faithful linear representation of $\mathfrak{S}_4 \to L_4(2)$ is either decomposable as a sum of the trivial representation and an indecomposable 3-dimensional representation, or is an indecomposable representation with composition series with factors of dimensions $(1,1,2), (2,1,1), (1,2,1), (2,2)$.

In the reducible case, the representation is the direct sum of one-dimensional representation and an indecomposable 3-dimensional representation with composition series of type $(1,2)$, or $(2,1)$. In the first case $\mathfrak{S}_4$ has three fixed points in $2^4 \setminus \{0\}$. Assume that the corresponding extension $G = 2^4 : \mathfrak{S}_4$ realizes. Then in the quotient $X/2^4$ we have 15 ordinary double points permuted by $\mathfrak{S}_4$ according to its action on $2^4 \setminus \{0\}$. Three of them are fixed. This implies that X/G has three singular points of type E_7; this is too many. In the second case $\mathfrak{S}_4$ has orbit decomposition of type $1+3+3+4+4$. Again it is easy to see that this leads to a contradiction. This argument can be also used to determine the second column of Table 3 using the known information about the root lattice $\mathcal{R}_G$ (from [Xiao] for a non-exceptional group or otherwise from the List.)

Thus we may assume that the representation of $\mathfrak{S}_4$ in $\mathbb{F}_2^4$ is indecomposable. One can show that, up to isomorphism, the representation is determined by its composition series, hence we have four different cases. They correspond to the cases in Table 3, where the last column indicates the type of the composition series.

Note that the linear representations $\mathfrak{S}_4 \to L_4(2)$ from (xxiv) and (xxiv′) differ by an outer automorphism of $L_4(2)$ defined by a correlation. The images of $\mathfrak{S}_4$ under the representations defined by F_{384} and (xxvi) are conjugate to subgroups of $\mathfrak{S}_6$ embedded in $\mathfrak{A}_8 \cong L_4(2)$ as the subgroup $\mathfrak{A}_{2,4}$. They differ by an outer automorphism of $\mathfrak{S}_6$.

The distinction between M_{20} and M_{20}' also can be given similarly; the image of $\mathfrak{A}_5$ in $\mathfrak{A}_8$ has orbit decomposition 1+1+1+5 and 1+1+6, respectively.

Proposition 5.14. *All exceptional groups are contained in M_{23}.*

Proof. We will indicate the chain of maximal subgroups starting from a maximal subgroup of M_{23} and ending at the subgroup containing the given group. If no comments are given, the verification is straightforward using the list of maximal subgroups of M_{23} which can be obtained from ATLAS. A useful fact is that in all extensions of type $2^4 : K$, the group K acts faithfully in 2^4 ([Mu, Prop. (3.16)]) and hence defines an injective homomorphism $K \to L_4(2) \cong \mathfrak{A}_8$.

The subgroup $G(\Xi)$ of M_{24} which preserves an octad Ξ is isomorphic to the affine group $\mathrm{AGL}_4(2) = 2^4 : \mathfrak{A}_8$. Here 2^4 acts identically on the octad, and the quotient $\mathfrak{A}_8$ acts on the octad by even permutations. A section S of the semi-direct product is a stabilizer subgroup of a point outside the octad. Taking this point as the origin, we have an isomorpohism $i : \mathfrak{A}_8 \to S \cong L_4(2)$ which we used in Remark 5.13. We will write any element from $G(\Xi)$ as (g_1, g_2), where g_1 is a permutation of Ξ and g_2 is a permutation of $\Omega \setminus \Xi$. The group $\mathrm{AGL}_4(2)$ is generated by elements $(g, i(g)), g \in \mathfrak{A}_8$, and translations $(1, t_a), a \in 2^4$. The image of $\mathfrak{A}_8$ consists of elements $(g, i(g))$, where $i(g) \in L_4(2)$. In particular, all elements of $\mathfrak{A}_8$ fix the origin in $\Omega \setminus \Xi$, and hence $\mathfrak{A}_8$ is isomorphic to a subgroup of M_{23}. Recall that the latter is defined as the stabilizer of an element of Ω.

(i) $M_{23} \supset \mathfrak{A}_8$. We use that $\mathfrak{S}_6$ embeds in $\mathfrak{A}_8$ as the subgroup $\mathfrak{A}_{2,6}$.

(ii), (iii), (v), (vii) $M_{23} \supset M_{22}$.

(iv) $M_{23} \supset M_{21}{\bullet}2$.

(vi) $\subset$ (iii). We use the embedding of $\mathfrak{A}_5$ in $\mathfrak{A}_6$ as the composition of the natural inclusion $\mathfrak{A}_5 \subset \mathfrak{A}_6$ and the outer automorphism of $\mathfrak{S}_6$.

(viii) $M_{23} \supset M_{22}$. There is only one split extension $2^3 : L_2(7)$.

(ix) $\subset$ (x).

(x) $M_{23} \supset M_{21}{\bullet}2 \supset (3^2 : Q_8){\bullet}2$.

(xi) $\subset$ (xxvii).

(xii) $\subset$ (vii). The group $5 : 4$ is the normailzer of a 5-Sylow subgroup of $\mathfrak{S}_5$. The group $\mathfrak{S}_5$ embeds in $\mathfrak{A}_8 = L_4(2)$ as a subgroup $\mathfrak{A}_{1,2,5}$ and its conjugacy class is unique. The conjugacy class of the subgroup $5 : 4$ is also unique, and hence its action on 2^4 is defined uniquely up to isomorphism.

(xiii) $\subset$ (xiv).

(xiv) $M_{23} \supset 2^4 : \mathfrak{A}_7 \supset 2^4 : \mathfrak{A}_{3,4} = \Gamma_{13}a_1 : \mathfrak{A}_{3,3}$.

(xv) $\subset$ (xvii).

(xvii) $M_{23} \supset \mathfrak{A}_8 \supset 2^4 : \mathfrak{A}_{2,3,3} \cong 2^4 : \mathfrak{S}_{3,3}$.

(xvi) $\subset$ (iii), we use that $3^2 : 4$ is the normalizer of a 3-Sylow subgroup of $\mathfrak{A}_6$ and it is a unique (up to conjugation) maximal subgroup of $\mathfrak{A}_6$.

(xviii) $\subset$ (xxvi). First we use that O_{48} is a central extension of $\mathfrak{S}_4$. The group (xxvi) is an extension $2^4 : \mathfrak{S}_4$ and $\mathfrak{S}_4$ fixes a unique point $a \in 2^4 \setminus 0$ (see Table 3). We embed $\mathfrak{S}_4$ as a subgroup of $L_4(2)$ generated by the matrices x, y satisfying $x^4 = y^2 = (xy)^3 = 1$:

$$\begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Let $a = (0,0,0,1), b = (1,1,1,1), c = (0,1,0,1)$. Then we immediately check that $t_b i(x) = i(x) t_c$, $(t_b i(x))^4 = (t_c i(y))^2 = t_a$. The subgroup generated by the elements $(x, t_b i(x))$ and $(y, t_c i(y))$ is isomorphic to O_{48}.

(xviii), (xix) $\subset$ (xx). In particular, it gives another embedding of O_{48} in M_{23}.

(xx) $\subset$ (xxiv$'$). The group (xxiv$'$) is an extension $2^4 : \mathfrak{S}_4$ and $\mathfrak{S}_4$ fixes a unique point $a \in 2^4 \setminus 0$ and leaves invariant a 2-dimensional subspace $< a, a' > \subset 2^4$ (see Table 3). In this case, we realize $\mathfrak{S}_4$ as a subgroup of $L_4(2)$ generated by the matrices

$$\begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix}.$$

Let $a = (1,0,0,0), a' = (0,1,0,0), b = (1,1,1,1), c = (0,1,0,1)$. It is checked that $t_b i(x) = i(x) t_c$, $(t_b i(x))^4 = (t_c i(y))^2 = t_a$, and the subgroup generated by the elements $(x, t_b i(x))$, $(y, t_c i(y)), (1, t_{a'})$ is isomorphic to $O_{48} : 2$.

(xxi) $\subset$ (xxvi). Note $Q_8 \circ Q_8 = \Gamma_5 a_1$; thus the group (xxi) has a 2-Sylow subgroup $\cong \Gamma_{26} a_2 = \Gamma_5 a_1{}^\bullet 2$. If the group (xxi) admits a degree 2 extension in our list of exceptional groups or in the Mukai's list, it must be the group (xxvi). This follows from the types of singularities. On the other hand, the group (xxvi) contains a non-split extension of the from $(Q_8 \circ Q_8)^\bullet \mathfrak{S}_3$. To show this, denote the groups (xxi) and (xxvi) by G and K, respectively. The normal subgroup $\Gamma_{25} a_1$ of K contains only one subgroup $\cong \Gamma_5 a_1$. Denote this subgroup by H. Then H is normal in K, and its quotient K/H is of order 12 and contains a subgroup $\cong \mathfrak{S}_3$; hence $K/H \cong D_{12}$. Since all symplectic groups of order 2^6 are contained in the unique symplectic group of order 2^7 which is isomorphic to a 2-Sylow subgroup of K, there is a chain of subgroups $H \subset \Gamma_{26}a_2 \subset K$. This implies that there is an order 2 subgroup A of K/H such that $\phi^{-1}(A)$ is a non-split extension $\Gamma_{26}a_2 = H^\bullet 2$, where $\phi : K \to K/H$ is the projection. Since $K/H \cong D_{12}$, one can always find a subgroup $B \subset K/H$ such that $A \subset B \cong \mathfrak{S}_3$. Now $\phi^{-1}(B)$ gives a non-split extension $(Q_8 \circ Q_8)^\bullet \mathfrak{S}_3$.

(xxii) $\subset$ (xiv). Since $G = \Gamma_4 a_1{}^\bullet \mathfrak{S}_3$, it has a 2-Sylow subgroup $\cong \Gamma_{22}a_1 = \Gamma_4 a_1{}^\bullet 2 = 2^4 : 4$. The normal subgroup $\Gamma_4 a_1$ of G contains only one subgroup $\cong 2^4$. Denote this subgroup by H. Then H is normal in G, and its quotient G/H is of order 12 and contains a cyclic subgroup of order 4; hence $G/H \cong Q_{12}$. Thus $G \cong 2^4 : Q_{12}$, a split extension by Corollary 5.9. The group $\mathfrak{A}_{3,4}$ contains $\langle (12)(4567), (123)\rangle \cong Q_{12}$.

(xxiii) $\subset$ (xxiv) or (xxiv$'$).

(xxiv), (xxiv$'$) $\subset$ (xiv). Here and in the next inclusion use Table 3.

(xxv) $\subset$ (xxvi) $\subset$ (iii).

(xxvii) $\subset$ (viii). We use that $L_3(2) \cong L_2(7)$ contains $7 : 3$ as the normalizer of a 7-Sylow subgroup. □

Example 5.15. The group $O_{48} \cong 2_\bullet \mathfrak{S}_4$ is contained in a maximal subgroup of $U_3(5) = \mathrm{PSU}_3(\mathbb{F}_{25})$ isomorphic to $2_\bullet \mathfrak{S}_5$. Thus it acts on a K3 surface from the example in the next section in the case $p = 5$, and its order is prime to p. Unfortunately, we do not know how to realize explicitly other exceptional groups. It is known that the group $\mathfrak{A}_5$ admits a symplectic action on the Kummer surface of the product of two supersingular curves in characteristic $p \equiv 2, 3 \mod 5$ [Ibu]. Together with the group 2^4 defined by the translations, we have a symplectic group isomorphic to an extension $2^4 : \mathfrak{A}_5$. Unfortunately, this group is M_{20} not M_{20}' as we naively hoped.

6. Examples of finite groups of symplectic automorphisms in positive characteristic p

Non-exceptional tame groups of symplectic automorphisms. A glance at Mukai's list of examples of K3 surfaces with maximal finite symplectic group of automorphisms shows that all of them can be realized over a field of positive characteristic $p > 7$. A complete Mukai's list consists of 80 groups (81 topological types) [Xiao]. All of them realize over a field of positive characteristic p, as long as the order is not divisible by p.

Wild groups of symplectic automorphisms. Here we give a list of examples of K3 surfaces over a field of characteristic $p = 2, 3, 5, 11$ with symplectic finite automorphism groups of order divisible by p not from the Mukai list.

$(p = 2)$ $\quad X = V(x^4 + y^4 + z^4 + w^4 + x^2y^2 + x^2z^2 + y^2z^2 + xyz(x + y + z)) \subset \mathbb{P}^3$

It is a supersingular K3 surface with Artin invariant $\sigma = 1$ with symplectic action of the group $\mathrm{PSL}(3, \mathbb{F}_4)_\bullet 2 \cong M_{21\bullet}2$, whose order is $(20, 160).2 = (2^6.3^2.5.7).2$ (see [DKo]). Although the order of this group divides the order of M_{23}, it is not a subgroup of M_{23}.

$$(p = 3) \quad X = V(x^4 + y^4 + z^4 + w^4) \subset \mathbb{P}^3.$$

Fermat quartic surface is supersingular with Artin invariant $\sigma = 1$ in characteristic $p = 3 \bmod 4$. The general unitary group $\mathrm{GU}(4, \mathbb{F}_9)$ acts on the Hermitian form $x^4 + y^4 + z^4 + w^4$ over $\mathbb{F}_9$, and so $\mathrm{PSU}(4, \mathbb{F}_9) = U_4(3)$ acts on X, which is simple of order $3, 265, 920 = 2^7.3^6.5.7$. This example was known to several people (A. Beauville, S. Mukai, T. Shioda, J. Tate). The order of this group does not divide the order of M_{23}.

$$(p = 5) \quad X = V(x^6 + y^6 + z^6 - w^2) \subset \mathbb{P}(1, 1, 1, 3).$$

It is supersingular with Artin invariant $\sigma = 1$. The general unitary group $\mathrm{GU}(3, \mathbb{F}_{5^2})$ acts on the Hermitian form $x^6 + y^6 + z^6$ over $\mathbb{F}_{5^2}$, so $\mathrm{PSU}(3, \mathbb{F}_{5^2}) = U_3(5)$, a simple group, acts symplectically on X. The order of this group is equal to $126, 000 = 2^4.3^2.5^3.7$ and does not divide the order of M_{23}.

$$(p = 7) \quad X = V(x^3 + (y^8 + z^8)x - w^2) \subset \mathbb{P}(4, 1, 1, 6).$$

It is supersingular with Artin invariant $\sigma = 1$. The general unitary group $\mathrm{GU}(2, \mathbb{F}_{7^2})$ acts on the Hermitian form $y^8 + z^8$ over $\mathbb{F}_{7^2}$, and so $\mathrm{PSU}(2, \mathbb{F}_{7^2}) \cong L_2(7)$, a simple group of order 168, acts symplectically on X. Although this group can be found in Mukai's list, the group action on the surface in his example degenerates in characteristic 7. This surface is birationally isomorphic to the affine surface $y^2 = x^3 + (t^7 - t)x$ ([DK1, Exs. 5.8]). This surface is also isomorphic to the Fermat quartic surface in characteristic $p = 7$, and hence admits a symplectic action of the group F_{384} of order 384.

$$(p = 11) \quad X = V(x^3 + y^{12} + z^{12} - w^2) \subset \mathbb{P}(4, 1, 1, 6).$$

It is supersingular with Artin invariant $\sigma = 1$. The general unitary group $\mathrm{GU}(2, \mathbb{F}_{11^2})$ acts on the Hermitian form $y^{12}+z^{12}$ over $\mathbb{F}_{11^2}$, and so $\mathrm{PSU}(2, \mathbb{F}_{11^2}) \cong L_2(11)$, a simple group of order 660, acts symplectically on X. This is a subgroup of M_{23} but has four orbits, so it is not realized in characteristic 0. This surface is birationally isomorphic to the affine surface $y^2 = x^3 + t^{11} - t$ ([DK1, Exs. 5.8]). This surface is also isomorphic to the Fermat quartic surface in characteristic $p = 11$, and hence admits a symplectic action of the group F_{384} of order 384 (see Remark 5.13).

The last three examples were obtained through a discussion with S. Kondō. One can show using an algorithm from [Shio2] and its generalization [Go] that all the previous examples are supersingular K3 surfaces with the Artin invariant equal to 1.

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