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Finite subgroups of the plane Cremona group

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To the memory of Vasily Iskovskikh

Abstract.

We survey some old and new results about finite subgroups of the Cremona group $\mathrm{Cr}_2(k)$ of birational automorphisms of the projective plane over a field k.

Contents

§1. Introduction

The Cremona group $\mathrm{Cr}_n(k)$ of degree n over a field k is the group of birational automorphisms of $\mathbb{P}^n_k$. In algebraic terms,

$$\mathrm{Cr}_n(k) = \mathrm{Aut}_k(k(t_1, \ldots, t_n)).$$

In this article I will survey some old and new results on classification of conjugacy classes of finite subgroups of $\mathrm{Cr}_2(k)$. Recall that in the case $n = 1$, we have

$$\mathrm{Cr}_1(k) \cong \mathrm{Aut}(\mathbb{P}^1_k) \cong \mathrm{PGL}_2(k).$$

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The classification of finite subgroups of $\mathrm{PGL}_2(k)$ is well-known. If k is algebraically closed of characteristic zero, then each such group is isomorphic to either a cyclic group C_n, or a dihedral group D_n of order $2n$, or the tetrahedron group T, or the octahedron group O, or the icosahedron group I. There is only one conjugacy class for each group in $\mathrm{Cr}_1(k)$. If $\mathrm{char}(k) = p > 0$, then G is isomorphic to a subgroup of $\mathrm{PGL}_2(\mathbb{F}_q)$ for some $q = p^s$.

In this survey we will be concerned with the case $n = 2$. We will consider three essentially different cases:

- k is the field of complex numbers $\mathbb{C}$;
- k is an arbitrary field of characteristic prime to the order $|G|$ of G;
- k is algebraically closed of characteristic p dividing the order of G.

Although in the first case the classification is 'almost' complete, in the remaining cases it is very far from being complete.

This work arises from collaboration with my old, now deceased, friend and colleague Vasya Iskovskikh. His help and guidance is hard to overestimate.

§2. General facts

2.1. G-varieties

Let G be a finite subgroup of $\mathrm{Cr}_n(k)$. We say that a rational variety X *regularizes* G if there exists a birational isomorphism $\phi : X - \rightarrow \mathbb{P}^n$ such that $\phi^{-1} \circ G \circ \phi$ is a subgroup of automorphisms of X.

Lemma 1. *Each finite subgroup of* $\mathrm{Cr}_n(k)$ *can be regularized.*

Proof. First we find an open subset U of $\mathbb{P}^n$ on which G acts biregularly. For example, we may take $U = \cap_{g\in G}\mathrm{dom}(g)$, where $\mathrm{dom}(g)$ denotes the largest open subset on which $g : \mathbb{P}^n - \rightarrow \mathbb{P}^n$ is defined. Then we consider the orbit space $V = U/G$, take some compactification $\bar{V}$ of V and let X' to be the normalization of $\bar{V}$ in the field of rational functions of U. The group G is the Galois group of the cover $X' \rightarrow \bar{V}$ and acts biregularly on X'. Q.E.D.

Assume that k is of characteristic zero (resp. $n = 2$). Then a G-variety X admits a G-equivariant resolution of singularities [1] (resp. [23]). Thus we can regularize the action on a nonsingular variety X.

Definition 2. Let G be a finite group. A G-variety X is a pair (X, ρ), where X is a projective algebraic variety over k and $\rho : G \rightarrow$

$\mathrm{Aut}(X)$ is an injective homomorphism of groups. A rational map of G-varieties $f : (X, \rho) \to (Y, \rho')$ is a rational map $f : X - \to Y$ together with a group automorphism $\phi : G \to G$ such that, for any $g \in G$,

$$f \circ \rho(g) \circ f^{-1} = \rho'(\phi(g)).$$

A birational isomorphism of G-varieties is an invertible rational map of G-varieties.

For any G-variety (X, ρ), we choose a birational isomorphism $\phi : X \to \mathbb{P}^n$ and, for any $g \in G$, we let $\iota(g) = \phi \circ \rho(g) \circ \phi^{-1}$. This defines an injective homomorphism from G to $\mathrm{Cr}_n(k)$. The previous lemma easily implies the following.

Theorem 3. *There is a natural bijective correspondence between birational isomorphism classes of rational G-varieties and conjugacy classes of subgroups of $\mathrm{Cr}_n(k)$ isomorphic to G.*

Definition 4. A *minimal* G-variety is a G-variety (X, ρ) such that any birational morphism of G-surfaces $(X, \rho) \to (X', \rho')$ is an isomorphism. A group G of automorphisms of a rational variety X is called a *minimal group of automorphisms* if the pair (X, ρ) is minimal.

So our goal is to classify minimal G-varieties (X, ρ) up to birational isomorphism of G-varieties. For this we need an analog of the theory of minimal models in a G-equivariant setting. If k is algebraically closed of characteristic zero we can equivariantly resolve singularities of (X, G) and then run the equivariant version of Mori's program in dimension 3 [22] (and arbitrary dimension when such program will be fully established). We obtain that G regularizes on a minimal G-variety with $G\mathbb{Q}$-factorial terminal singularities (nonsingular if $n = 2$). Since X is rational, we obtain that X is a G-equivariant minimal Mori's fibration $f : X \to Z$, where $\dim Z < \dim X$, the Weil divisor $-K_X$ is relatively ample and the relative G-invariant Picard number $\rho^G_{X/Z}$ is equal to 1.

If $n = 2$, then X is a nonsingular Fano variety (a Del Pezzo surface) with $\mathrm{Pic}(X)^G \cong \mathbb{Z}$ or $Z = \mathbb{P}^1$ and $f : X \to Z$ is a conic bundle with $\mathrm{Pic}(X)^G \cong \mathbb{Z}^2$. One can also run the Mori program not assuming that the field k is algebraically closed (see [11]).

From now on we restrict ourselves with the case $n = 2$. There are a few classification results in the cases $n > 2$. However, recently all simple and p-elementary subgroups of $\mathrm{Cr}_3(\mathbb{C})$ have been classified by Yuri Prokhorov [28], [29].

2.2. Lift to characteristic 0

A finite subgroup G of $\mathrm{Cr}_n(k)$ is called *wild* if its order is divisible by the characteristic of k. It is called *tame* otherwise. The following result belongs to J.-P. Serre [33]

Theorem 5. *Let G be a finite tame subgroup of $\mathrm{Cr}_2(k)$. Then there exists a field K of characteristic 0 such that G is isomorphic to a subgroup of $\mathrm{Cr}_2(K)$.*

The proof is based on the following result.

Proposition 6. *Let G be a finite group of automorphisms of a projective smooth geometrically connected variety X over a field k. Suppose the following conditions are satisfied.*

- *$|G|$ is prime to $\mathrm{char}(k)$;*
- *$H^2(X, \mathcal{O}_X) = 0$;*
- *$H^2(X, \Theta_X) = 0$, where Θ_X is the tangent sheaf of X.*

Let A be a complete noetherian ring with residue field k. Then there exists a smooth projective scheme X_A over A on which G acts over A and the special fibre is G-isomorphic to X.

We apply this proposition by taking X to be a rational surface over k and A to be the ring of Witt vectors with residue field k. It is easy to see that all conditions of the Proposition are satisfied. Let K be the field of fractions of A. It follows from the rationality criterion for surfaces that the general fibre X_K is a geometrically rational surface, i.e. becomes rational when we replace K by its algebraic closure $\bar{K}$. This proves the assertion of the theorem.

Note that, even if k is algebraically closed, the lifts of two non-conjugate subgroups of $\mathrm{Cr}_2(k)$, may be conjugate in $\mathrm{Cr}_2(\bar{K})$.

§3. The case $k = \mathbb{C}$

Let k be an algebraically closed field of characteristic 0. Without loss of generality, we may assume that $k = \mathbb{C}$. In this section we survey results obtained in [11] and [3], [4]. We refer for the very old history of the problem to [11].

3.1. Conic bundles

We start with minimal groups acting on a conic bundle. Let $\pi : S \to \mathbb{P}^1$ be a conic bundle with t singular fibres over points in a finite set $\Sigma \subset \mathbb{P}^1$. Each singular fibre $F_x, x \in \Sigma$, is the bouquet of two $\mathbb{P}^1$'s.

Assume first that $t = 0$, i.e. $S = \mathbf{F}_n$ is a minimal ruled surface. Since G acts minimally, $n \neq 1$. We identify $\mathbf{F}_n$ with the weighted projective plane $\mathbb{P}(1,1,n)$. If $n \neq 0$, an automorphism is given by the formula

$$(t_0, t_1, t_2) \mapsto (at_0 + bt_1, ct_0 + dt_1, et_2 + f_n(t_0, t_1)),$$

where f_n is a homogeneous polynomial of degree n. The following proposition is easy to prove.

Proposition 7. *Let $S = \mathbf{F}_n, n \neq 0$. We have*

$$\mathrm{Aut}(\mathbf{F}_n) \cong \mathbb{C}^{n+1} \rtimes (\mathrm{GL}_2(\mathbb{C})/\mu_n),$$

where $\mathrm{GL}_2(\mathbb{C})/\mu_n$ acts on $\mathbb{C}^{n+1}$ by means of its natural linear representation in the space of binary forms of degree n. Moreover,

$$\mathrm{GL}(2)/\mu_n \cong \begin{cases} \mathbb{C}^* \rtimes \mathrm{PSL}(2), & \text{if } n \text{ is even,} \\ \mathbb{C}^* \rtimes \mathrm{SL}(2), & \text{if } n \text{ is odd.} \end{cases}$$

Using this proposition, it is not hard to list all finite subgroups which may act on $\mathbf{F}_n$.

Next we assume that $\pi : S \to \mathbb{P}^1$ is a conic bundle with $t > 0$ of singular fibres. The Picard group of S is freely generated by the divisor classes of a section E of π, the class F of a fibre, and the classes of t components of singular fibres, no two in the same fibre. The next lemma follows easily from the intersection theory on S.

Lemma 8. *Let E and E' be two sections with negative self-intersection $-n$. Let r be the number of components of singular fibres which intersect both E and E'. Then $t - r$ is even and*

$$2E \cdot E' = t - 2n - r.$$

In particular,

$$t \geq 2n + r.$$

Since a conic bundle S is isomorphic to a blowup of a minimal ruled surface, it always contains a section E with negative self-intersection $-n$. If $n \geq 2$, we obviously get $t \geq 4$. If $n = 1$, since (S, G) is minimal, there exists $g \in G$ such that $g(E) \neq E$ and $E \cap g(E) \neq \emptyset$. Applying the previous lemma we get

$$t \geq 4.$$

Let $G \to \mathrm{Aut}(\mathbb{P}^1)$ be the natural action of G on the base of the conic fibration. Let $\bar{G}$ be the image of G in $\mathrm{Aut}(\mathbb{P}^1)$ and K be the kernel. The group K is isomorphic to a subgroup of $\mathrm{Aut}(S_\eta)$, where S_η

is a general fibre of π isomorphic to the projective line over the field of rational functions of the base of the fibration.

Suppose G acts faithfully on the Picard group $\mathrm{Pic}(S)$. Then the subgroup K acts non-trivially on the subgroup of $\mathrm{Pic}(S)$ generated by the components of fibres of the conic fibration. This implies that K is isomorphic to a subgroup of the group $2^t := (\mathbb{Z}/2\mathbb{Z})^t$. Since $\mathrm{PGL}_2(\mathbb{C})$ does not contain subgroups isomorphic to 2^t for $t > 2$ we obtain that $K \cong 2$ or 2^2.

Theorem 9. *Assume that G acts faithfully on $\mathrm{Pic}(S)$. Then the subgroup K is isomorphic to either $\mathbb{Z}/2\mathbb{Z}$ or $(\mathbb{Z}/2\mathbb{Z})^2$. In the first case a generator of K fixes pointwise an irreducible smooth bisection C of ϕ and switches the components in $m \leq t$ fibres over the branch points of the g_2^1 on C defined by the projection π. The curve C is a curve of genus $g = (m-2)/2$. In the second case, each nontrivial element g_i of K fixes pointwise an irreducible smooth bisection C_i. The set Σ is partitioned in 3 subsets $\Sigma_1, \Sigma_2, \Sigma_3$ such that the set of ramification points of the projection $\phi : C_i \to \mathbb{P}^1$ is equal to $\Sigma_j + \Sigma_k, i \neq j \neq k$.*

In [11] we investigate possible extensions $1 \to K \to G \to \bar{G} \to 1$.

Next we assume that G acts on $\mathrm{Pic}(S)$ with a non-trivial kernel G_0. A conic bundle that admits such an action is called an *exceptional conic bundle*. All such conic bundles can be explicitly described. Here we give only one possible construction of an exceptional conic bundle. Other constructions can be found in [11].

Let us consider a quasi-smooth hypersurface Y of degree $2g+2$ in weighted projective space $\mathbb{P} = \mathbb{P}(1, 1, g+1, g+1)$ given by an equation

$$p_{2g+2}(t_0, t_1) + t_2 t_3 = 0, \tag{1}$$

where $p_{2g+2}(t_0, t_1)$ is a homogeneous polynomial of degree $2g+2$ without multiple roots. The surface is a double cover of $\mathbb{P}(1, 1, g+1)$ (the cone over a Veronese curve of degree $g+1$) branched over the curve isomorphic to the curve $p_{2g+2}(t_0, t_1) + t_2^2 = 0$. The preimages of the singular point of $\mathbb{P}(1, 1, g+1)$ with coordinates $[0, 0, 1]$ is a pair of singular points of Y with coordinates $[0, 0, 1, 0]$ and $[0, 0, 0, 1]$. The singularities are locally isomorphic to the singular points of a cone of the Veronese surface of degree $g+1$. Let S be a minimal resolution of Y. The preimages of the singular points are disjoint smooth rational curves E and E' with self-intersection $-(g+1)$. The projection $\mathbb{P}(1, 1, g+1, g+1) \to \mathbb{P}^1, [t_0, t_1, t_2, t_3] \mapsto [t_0, t_1]$ lifts to a conic bundle on S with sections E, E'. The pencil $\lambda t_2 + \mu t_3 = 0$ cuts out a pencil of curves on Y which lifts to a pencil of bisections of the conic bundle S with $2g+2$ base points $[t_0, t_1, 0, 0]$, where $p_{2g+2}(t_0, t_1) = 0$.

The following proposition describes the automorphism group of an exceptional conic bundle. We denote by Y_g an exceptional conic bundle given by equation (1). Since we are interested only in minimal groups we assume that $g \geq 1$.

Proposition 10. *The group of automorphisms of an exceptional conic bundle* (1) *is isomorphic to an extension $N.P$, where P is the subgroup of* $\mathrm{PGL}_2(\mathbb{C})$ *leaving the set of zeroes of* $p_{2g+2}(t_0, t_1)$ *invariant and $N \cong \mathbb{C}^* \rtimes 2$ is a group of matrices with determinant ± 1 leaving $t_2 t_3$ invariant. Moreover, the extension splits and defines an isomorphism*

$$\mathrm{Aut}(Y_g) \cong N \times P$$

if and only if g is odd, or g is even and P is either a cyclic group or a dihedral group D_{4k+2}.

3.2. De Jonquières transformations

A Cremona transformation T of the plane which is defined by a linear system L of plane curves of degree d which pass through a point q with multiplicity $d-1$ and points $p_1, \ldots, p_{2d-2}$ with multiplicity 1 is called a *De Jonquières transformation*. One can show that there exists a curve Γ of degree $d-1$ with singular point at q of multiplicity $d-2$ and passing through the points $p_1, \ldots, p_{2d-2}$. If we choose $q = [0, 0, 1]$, then Γ can be given by the equation

$$a(t_0, t_1, t_2) = t_2 a_{d-2}(t_0, t_1) + a_{d-1}(t_0, t_1) = 0,$$

where $a_s(t_0, t_1)$ denotes a binary form of degree s. Let

$$b(t_0, t_1, t_2) = t_2 b_{d-1}(t_0, t_1) + b_d(t_0, t_1) = 0$$

define a curve from L which does not belong to the pencil formed by $\Gamma + \ell$, where ℓ is a line through q. Then the transformation T with homaloidal net L is equal to the composition $\phi \circ T_0$, where ϕ is a projective automorphism and T_0 is given by the formula

$$T_0 : [t_0, t_1, t_2] \to [t_0 a(t_0, t_1, t_2), t_1 a(t_0, t_1, t_2), b(t_0, t_1, t_2)]. \tag{2}$$

It easy to see that T transforms the pencil of lines through the point q to the pencil of lines through the point $\phi(q)$. Assume $\phi(q) = q$, for example, if T is of finite order, then in affine coordinates $x = t_1/t_0, y = t_2/t_0$, the transformation T can be given by the formula

$$T : (x, y) \mapsto \Big(\frac{\alpha_1 x + \alpha_2}{\alpha_3 x + \alpha_4}, \frac{r_1(x) y + r_2(x)}{r_3(x) y + r_4(x)}\Big), \tag{3}$$

where $r_i(x)$ are certain rational functions in x. All such transformations form a subgroup of $\mathrm{Cr}_2(k)$, called a De Jonquières subgroup. It depends on the choice of generators, x, y of the field of rational functions $k(\mathbb{P}^2)$ of $\mathbb{P}^2$. Any Cremona transformation leaving a pencil of rational curves invariant belongs to a De Jonquières subgroup.

Consider the transformation T_0, where $b_{d-1}(t_0,t_1) = -a_{d-1}(t_0,t_1)$. Then one checks that T_0^2 is the identity. Its set of fixed points is a plane curve of degree d

$$H_d : t_2^2 a_{d-2}(t_0,t_1) + 2t_2 a_{d-1}(t_0,t_1) + b_d(t_0,t_1) = 0.$$

It has a singular point of multiplicity $d-2$ at q and passes through the points $p_1,\dots,p_{2d-2}$. Its normalization is a curve of genus $g = d-2$. Let $S \to \mathbb{P}^2$ be the blow-up of the points $q, p_1, \dots, p_{2d-2}$. The pencil of lines through the point q defines a conic bundle structure on S. Its singular fibres are the full pre-images of lines $\langle q, p_i\rangle$. The transformation T_0 lifts to S and interchanges the exceptional curve E at the point q with the proper inverse transform of the curve Γ. Its fixed locus on S is the proper inverse transform of the curve H_d.

Theorem 11. *Let G be a finite subgroup of $\mathrm{Cr}_2(k)$. The following properties are equivalent:*

- (i) *G leaves invariant a pencil of rational curves;*
- (ii) *G belongs to a De Jonquières subgroup of $\mathrm{Cr}_2(k)$;*
- (iii) *G can be regularized by a group of automorphisms of a conic bundle.*

More can be said about cyclic groups. A De Jonquières transformation of order 2 is called a *De Jonquières involution.* The transformation T_0 from (2) with $b_{d-1}(t_0,t_1) = -a_{d-1}(t_0,t_1)$ is an example of a De Jonquières involution. In affine coordinates it can be given by

$$T_0 : (x,y) \mapsto (x, \frac{f(x)}{y}),$$

where $f(x)$ is a polynomial of degree $2g+1$ with no multiple roots. The fixed locus of T_0 is birationally isomorphic to the curve $y^2 - f(x) = 0$.

The proof of the following result can be found in [4].

Theorem 12. *Let g be an element of a De Jonquières group of finite order. Assume that g is not conjugate to a projective automorphism. Then g is of even order $2n$ and, up to a conjugate,*

$$g^n : (x,y) \mapsto (x, \frac{F(x^n)}{y})$$

for some polynomial $F(x)$ with no multiple roots.

3.3. Automorphism groups of Del Pezzo surfaces

Let S be a Del Pezzo surface of degree $d = K_S^2$. We start with the case $d = 9$, i.e. $S = \mathbb{P}^2$. The classification of conjugacy classes of finite subgroups of $\mathrm{Aut}(\mathbb{P}^2) \cong \mathrm{PGL}_3(k)$ is known since the beginning of the 20th century.

Recall some standard terminology from the theory of linear groups. Let G be a subgroup of the general linear group $\mathrm{GL}(V)$ of a vector space V over a field k. The group G is called *intransitive* if the representation of G in V contains an invariant non-zero subspace. Otherwise it is called *transitive*. A transitive group G is called *imprimitive* if it contains a proper intransitive normal subgroup G'. In this case, if G is tame, V decomposes into a direct sum of G'-invariant proper subspaces, and elements from G permute them. A group is *primitive* if it is neither intransitive, nor imprimitive. We reserve this terminology for finite subgroups of $\mathrm{PGL}(V)$ keeping in mind that each such group can be represented by a subgroup of $\mathrm{GL}(V)$.

We restrict ourselves with transitive subgroups, since intransitive groups are easy to classify.

Theorem 13. *Let G be a transitive imprimitive finite subgroup of $\mathrm{PGL}_3(\mathbb{C})$. Then G is conjugate to one of the following groups*

- $G \cong n^2 \rtimes 3$ *generated by transformations*

$$[\epsilon_n t_0, t_1, t_2],\ [t_0, \epsilon_n t_1, t_2],\ [t_2, t_0, t_1];$$

- $G \cong n^2 \rtimes S_3$ *generated by transformations*

$$[\epsilon_n t_0, t_1, t_2],\ [t_0, \epsilon_n t_1, t_2],\ [t_0, t_2, t_1],\ [t_2, t_0, t_1];$$

- $G = G_{n,k,s} \cong (n \times \frac{n}{k}) \rtimes 3$, *where* $k > 1, k|n$ *and* $s^2 - s + 1 = 0 \mod k$. *It is generated by transformations*

$$[\epsilon_{n/k} t_0, t_1, t_2],\ [\epsilon_n^s t_0, \epsilon_n t_1, t_2],\ [t_2, t_0, t_1].$$

- $G \simeq (n \times \frac{n}{3}) \rtimes S_3$ *generated by transformations*

$$[\epsilon_{n/3} t_0, t_1, t_2],\ [\epsilon_n^2 t_0, \epsilon_n t_1, t_2],\ [t_0, t_2, t_1],\ [t_1, t_0, t_2].$$

Here we denote by $[l_1(t), \ldots, l_n(t)]$, where $l_i(t)$ are linear forms in variables $t_0, \ldots, t_n$, the projective transformation $[t_0, \ldots, t_n] \mapsto [l_1(t), \ldots, l_n(t)]$. The next theorem is a well-known result of Blichfeldt [5].

Theorem 14. *Any primitive finite subgroup G of $\mathrm{PGL}_3(\mathbb{C})$ is conjugate to one of the following groups.*

(1) *The icosahedral group* $I \cong A_5$. *It leaves invariant a nonsingular conic.*

(2) *The* Hessian group *of order 216 isomorphic to* $3^2 \rtimes \mathrm{SL}_2(\mathbb{F}_3)$. *It is realized as the group of automorphisms of the Hesse pencil of cubics*

$$x^3 + y^3 + z^3 + txyz = 0.$$

(3) *The* Klein group *of order 168 isomorphic to* $L_2(7)$ *(realized as the full group of automorphisms of the Klein quartic* $x^3y + y^3z + z^3x = 0$*).*

(4) *The* Valentiner group *of order 360 isomorphic to* $\mathfrak{A}_6$. *It can be realized as the full group of automorphisms of the nonsingular plane sextic*

$$10x^3y^3 + 9zx^5 + y^5 - 45x^2y^2z^2 - 135xyz^4 + 27z^6 = 0.$$

(5) *Subgroups of order 36 and 72 of the Hessian group.*

A Del Pezzo surface of degree $d = 8$ is isomorphic to either $\mathbf{F}_0$ or $\mathbf{F}_1$. The second surface is not G-minimal. We have $\mathrm{Aut}(\mathbf{F}_0) \cong \mathrm{PGL}_2(k) \wr 2$. All finite subgroups of this group are easy to find using the following Goursat Lemma.

Lemma 15. *Let G be a finite subgroup of the product $A \times B$ of two groups A and B. Let $p_1 : A \times B \to A, p_2 : A \times B \to B$ be the projection homomorphisms. Let $G_i = p_i(G), H_i = \mathrm{Ker}(p_j|G), i \neq j = 1, 2$. Then H_i is a normal subgroup in G_i. The map $\phi : G_1/H_1 \to G_2/H_2$ defined by $\phi(aH_1) = p_2(a)H_2$ is an isomorphism, and*

$$G = G_1 \times_D G_2,$$

where $D = G_1/H_1, \alpha : G_1 \to D$ is the projection map to the quotient, and β is the composition of the projection $G_2 \to G_2/H_2$ and ϕ^{-1}.

We refer to [11] for a complete list.

A Del Pezzo surface of degree $d \leq 7$ is isomorphic to the blow-up of $9 - d$ distinct points in $\mathbb{P}^2$ such that no three are on a line, no six are on a conic, and, if $d = 1$, no plane cubic passes through the points with one of them being a singular point. A Del Pezzo surface of degree $d = 7$ is not G-minimal, since the proper inverse image of the line through the two points is a G-invariant exceptional curve.

From now on we assume that $d \leq 6$. Recall that the orthogonal complement $\mathcal{R}_S$ of the canonical class K_S in $\mathrm{Pic}(S)$ equipped with the intersection form is isomorphic to a root quadratic lattice $\mathcal{Q}_d$ of type $A_2 \oplus A_1$ $(d = 6)$, A_4 $(d = 5)$, D_5 $(d = 4)$, E_6 $(d = 3)$, E_7 $(d = 2)$, E_8 $(d =$

1). The subgroup of isometries of $\mathcal{R}_S$ generated by reflections in divisor classes R with $R^2 = -2$ is denoted by $W(S)$. It is isomorphic to the Weyl group $W(\mathcal{Q}_d)$ of the lattice $\mathcal{Q}_d$. It coincides with the whole group of isometries for $d = 1, 2$ and its index is equal to 2 in other cases. The natural representation of $\mathrm{Aut}(S)$ in $\mathrm{O}(\mathrm{Pic}(S))$ defines a homomorphism

$$\rho : \mathrm{Aut}(S) \to W(\mathcal{R}_S).$$

Let $\pi : S \to \mathbb{P}^2$ be the blow-up morphism of $N = 9 - d$ points $p_1, \ldots, p_N$. Let $E_i = \pi^{-1}(p_i)$ and $e_i = [E_i] \in \mathrm{Pic}(S)$ be the divisor class of E_i. Let e_0 be the divisor class of the pre-image of a line in $\mathbb{P}^2$. The classes $(e_0, e_1, \ldots, e_N)$ form a basis of $\mathrm{Pic}(S)$ which we call a *geometric basis*. It is an orthonormal basis in the sense that it defines an isomorphism of lattices $\mathrm{Pic}(S) \to I^{1,N}$, where $I^{1,N}$ is the quadratic lattice defined by the diagonal matrix $[1, -1, \ldots, -1]$. The vectors

$$\alpha_1 = e_0 - e_1 - e_2 - e_3, \alpha_i = e_{i-1} - e_i, i = 2, \ldots, N$$

form a basis of the lattice $\mathcal{R}_S$. Its intersection matrix is equal to the negative of the Cartan matrix with the Dynkin diagram of the corresponding type. The Weyl group $W(\mathcal{R}_S)$ is generated by reflections in the vectors α_i.

$$s_i : x \mapsto x + (x, \alpha_i)\alpha_i.$$

Proposition 16. *The homomorphism ρ is injective if $d < 6$. If $d = 6$, the homomorphism ρ is surjective and its kernel is a connected algebraic group isomorphic to the two-dimensional torus $\mathbb{C}^{*2}$. It acts on S via its natural action on the projective plane.*

If $d = 6$, we have

$$W(\mathcal{Q}_6) = W(A_2 \oplus A_1) \cong S_3 \times 2 \cong D_{12},$$

where D_{12} is the dihedral group of order 12.

The reflection s_1 is realized by the lift of the standard quadratic transformation $\tau : [t_0, t_1, t_2] \mapsto [t_1t_2, t_0t_2, t_0t_1]$ of $\mathbb{P}^2$. The reflection s_2(resp. s_3) is realized by the projective transformations $[t_0, t_1, t_2] \mapsto [t_1, t_0, t_2]$ (resp. $[t_0, t_1, t_2] \mapsto [t_0, t_2, t_1]$). This shows that $W(\mathcal{Q}_6) = \langle s_1 \rangle \times \langle s_2, s_3 \rangle = 2 \times S_3$. The normalizer $N(T)$ of $T = \mathrm{Ker}(\rho)$ in $\mathrm{PGL}_3(k)$ is generated by T and s_2, s_3.

Let G be a minimal finite subgroup of $\mathrm{Aut}(S)$. Obviously, $\rho(G)$ contains s_1 and s_2s_3 since otherwise G leaves invariant α_1 or one of the vectors $2\alpha_1 + \alpha_2$, or $\alpha_1 + 2\alpha_2$. This shows that $G \cap N(T)$ is an imprimitive subgroup of $\mathrm{PGL}_3(\mathbb{C})$. This gives

Theorem 17. *Let G be a minimal subgroup of a Del Pezzo surface of degree* 6*. Then*

$$G = H.\langle s_1\rangle,$$

where H is an imprimitive finite subgroup of $\mathrm{PGL}_3(\mathbb{C})$.

Note that one of the groups from the theorem is the group $2^2 \rtimes S_3 \cong S_4$. Its action on S given by the equation

$$x_0y_0z_0 - x_1y_1z_1 = 0$$

in $(\mathbb{P}^1)^3$ is given in [2].

Next we assume that S is a Del Pezzo surface of degree $d = 5$.

In this case S is isomorphic to the blowup of the reference points $p_1 = [1,0,0],\ p_2 = [0,1,0],\ p_3 = [0,0,1], p_4 = [1,1,1]$. The lattice $\mathcal{Q}_5$ is of type A_4 and $W_S \cong \mathfrak{S}_5$ is the permutation group of degree 5. The homomorphism

$$\rho : \mathrm{Aut}(S) \to \mathfrak{S}_5$$

is an isomorphism.

One of the ways to see the isomorphism $\mathrm{Aut}(S) \cong \mathfrak{S}_5$ is to use a well-known isomorphism between S and the moduli space $\overline{\mathcal{M}_{0,5}} \cong (\mathbb{P}^1)^5//\mathrm{SL}_2(\mathbb{C})$. The group $\mathfrak{S}_5$ acts by permuting the factors.

Theorem 18. *Let (S,G) be a minimal Del Pezzo surface of degree $d = 5$. Then $G = \mathfrak{S}_5, \mathfrak{A}_5, 5 \rtimes 4, 5 \rtimes 2$, or* 5.

Proof. The group $\mathfrak{S}_5$ acts on $\mathcal{Q}_5 \cong \mathbb{Z}^4$ by means of its standard irreducible 4-dimensional representation (view $\mathbb{Z}^4$ as a subgroup of $\mathbb{Z}^5$ of vectors with coordinates added up to zero and consider the representation of $\mathfrak{S}_5$ by switching the coordinates). It is known that a maximal proper subgroup of $\mathfrak{S}_5$ is equal (up to a conjugation) to one of three subgroups $\mathfrak{S}_4, D_{12}, \mathfrak{A}_5, 5 \rtimes 4$. A maximal subgroup of $\mathfrak{A}_5$ is either 5×2 or $\mathfrak{S}_3$ or D_{10}. It is easy to see that the groups $\mathfrak{S}_4$ and D_{12} have invariant elements in the lattice $\mathcal{Q}_5$. It is known that an element of order 5 in $\mathfrak{S}_5$ is a cyclic permutation, and hence has no invariant vectors. Thus any subgroup G of $\mathfrak{S}_5$ containing an element of order 5 defines a minimal surface (S,G). So, if (S,G) is minimal, G must be equal to one of the groups from the assertion of the theorem. Q.E.D.

Let S be a Del Pezzo surface of degree 4. It is well-known that S is isomorphic to a nonsingular surface of degree 4 in $\mathbb{P}^4$ given by equations

$$F_1 = \sum_{i=0}^{4} t_i^2 = 0, \quad F_2 = \sum_{i=0}^{4} a_i t_i^2 = 0, \tag{4}$$

where all a_i's are distinct.

The Weyl group $W(\mathcal{Q}_4) \cong W(D_5)$ is isomorphic to the group $2^4 \rtimes \mathfrak{S}_5$. The normal subgroup 2^4 is generated as a normal subgroup by the element $s_1 s_5$. It is realized by automorphisms of S which act in $\mathbb{P}^4$ by multiplying the coordinates by ± 1. The subgroup of the Weyl group isomorphic to $\mathfrak{S}_5$ acts on $\mathrm{Pic}(S)$ by permuting the divisor classes $e_1, \ldots, e_5$ of 5 skew lines and leaving the class e_0 fixed.

The group $\mathrm{Aut}(S)$ could be larger than 2^4. Its image H in $\mathfrak{S}_5$ is isomorphic to a subgroup of $\mathrm{PGL}_2(\mathbb{C})$ leaving the set of 5 points $p_1, \ldots, p_k$ invariant. Since there is a unique conic through these points, H is a subgroup of $\mathrm{PGL}_2(\mathbb{C})$ leaving an effective divisor of degree 5 invariant. It follows that H is one of the following groups $2, 3, 4, 5, \mathfrak{S}_3, D_{10}$. The corresponding surfaces are projectively equivalent to the following surfaces

$$
\begin{aligned}
2 &: \quad x_0^2 + x_1^2 + x_2^2 + x_3^2 + x_4^2 = x_0^2 + a x_1^2 - x_2^2 - a x_3^2 = 0, \ a \neq -1, 0, 1, \\
4 &: \quad x_0^2 + x_1^2 + x_2^2 + x_3^2 + x_4^2 = x_0^2 + i x_1^2 - x_2^2 - i x_3^2 = 0, \\
\mathfrak{S}_3 &: \quad x_0^2 + \epsilon_3 x_1^2 + \epsilon_3^2 x_2^2 + x_3^2 = x_0^2 + \epsilon_3^2 x_1^2 + \epsilon_3 x_2^2 + x_4^2 = 0, \\
D_{10} &: \quad \sum_{i=0}^{4} \epsilon_5^i x_i^2 = \sum_{i=0}^{4} \epsilon_5^{4-i} x_i^2 = 0
\end{aligned}
$$

The analysis of all minimal finite subgroup of $\mathrm{Aut}(S)$ is rather tedious and non-trivial. We only give the final result (see [11]).

$\mathrm{Aut}(S)$	Subgroups
2^4	$2^4,\ 2^3,\ 2^2$
$2^4 \rtimes 2$	$2 \times 4,\ D_8,\ L_{16},\ 2^4 \rtimes 2$
$2^4 \rtimes 4$	$8,\ 2^2 \rtimes 8,\ 2^4 \rtimes 4,$
$2^4 \rtimes \mathfrak{S}_3$	$2^2 \times 3,\ 2 \times \mathfrak{A}_4,\quad 2^4 \rtimes 3,\ 2^4 \rtimes \mathfrak{S}_3,$
$2^4 \rtimes D_{10}$	$2^4 \rtimes D_{10},\ 2^4 \rtimes 5$

Table 1. Minimal subgroups of automorphisms of a Del Pezzo surface of degree 4.

Here L_{16} is a solvable group of order 16 with generators a, b, c and defining relations $a^4 = b^2 = c^2 = [c, a]b = [a, b] = [b, c] = 1$. Note that we did not include subgroups occurring in the previous rows.

Now let us consider the case of cubic surfaces. The groups of automorphisms of nonsingular cubic surfaces were essentially known in the 19th century ([20], [38]). A general cubic surface does not admit non-trivial automorphisms. There are 11 classes of cubic surfaces with non-trivial automorphisms. They are reproduced in the following table.

Type	Order	Structure	$F(t_0,t_1,t_2,t_3)$	Parameters
I	648	$3^3:\mathfrak{S}_4$	$t_0^3+t_1^3+t_2^3+t_3^3$	
II	120	$\mathfrak{S}_5$	$t_0^2t_1+t_0t_2^2+t_2t_3^2+t_3t_1^2$	
III	108	$H_3(3):4$	$t_0^3+t_1^3+t_2^3+t_3^3+6at_1t_2t_3$	$20a^3+8a^6=1$
IV	54	$H_3(3):2$	$t_0^3+t_1^3+t_2^3+t_3^3+6at_1t_2t_3$	$a-a^4\neq 0$, $8a^3\neq -1$, $20a^3+8a^6\neq 1$
V	24	$\mathfrak{S}_4$	$t_0^3+t_0(t_1^2+t_2^2+t_3^2)$ $+at_1t_2t_3$	$9a^3\neq 8a$ $8a^3\neq -1$,
VI	12	$\mathfrak{S}_3\times 2$	$t_2^3+t_3^3+at_2t_3(t_0+t_1)+t_0^3+t_1^3$	$a\neq 0$
VII	8	8	$t_3^2t_2+t_2^2t_1+t_0^3+t_0t_1^2$	
VIII	6	$\mathfrak{S}_3$	$t_2^3+t_3^3+at_2t_3(t_0+bt_1)+t_0^3+t_1^3$	$a\neq 0, b\neq 0,1$
IX	4	4	$t_3^2t_2+t_2^2t_1+t_0^3+t_0t_1^2+at_1^3$	$a\neq 0$
X	4	2^2	$t_0^2(t_1+t_2+at_3)+t_1^3+t_2^3$ $+t_3^3+6bt_1t_2t_3$	$8b^3\neq -1$
XI	2	2	$t_1^3+t_2^3+t_3^3+6at_1t_2t_3$ $+t_0^2(t_1+bt_2+ct_3)$	$b^3,c^3\neq 1$ $b^3\neq c^3$ $8a^3\neq -1$,

Table 2. Groups of automorphisms of cubic surfaces.

A proof can be found in [14] and [11].

The important tool is the classification of conjugacy classes of elements of finite order in the Weyl groups. According to [7] they are indexed by certain graphs. We call them *Carter graphs*. One writes each element $w\in W$ as the product of two involutions w_1w_2, where each involution is the product of reflections with respect to orthogonal roots. Let $\mathcal{R}_1,\mathcal{R}_2$ be the corresponding sets of such roots. Then the graph has vertices identified with elements of the set $\mathcal{R}_1\cup\mathcal{R}_2$ and two vertices α,β are joined by an edge if and only if $(\alpha,\beta)\neq 0$. A Carter graph with no cycles is a Dynkin diagram. The subscript in the notation of a Carter graph indicates the number of vertices. It is also equal to the difference between the rank of the root lattice Q and the rank of its fixed sublattice $Q^{(w)}$.

Note that the same conjugacy classes may correspond to different graphs (e.g. D_3 and A_3, or $2A_3+A_1$ and $D_4(a_1)+3A_1$).

The Carter graph determines the characteristic polynomial of w. In particular, it gives the trace $\mathrm{Tr}_2(g)$ of g^* on the cohomology space

Graph	Order	Characteristic polynomial
A_k	$k+1$	$t^k + t^{k-1} + \cdots + 1$
D_k	$2k-2$	$(t^{k-1}+1)(t+1)$
$D_k(a_1)$	l.c.m$(2k-4,4)$	$(t^{k-2}+1)(t^2+1)$
$D_k(a_2)$	l.c.m$(2k-6,6)$	$(t^{k-3}+1)(t^3+1)$
$\vdots$	$\vdots$	$\vdots$
$D_k(a_{\frac{k}{2}-1})$	even k	$(t^{\frac{k}{2}}+1)^2$
E_6	12	$(t^4-t^2+1)(t^2+t+1)$
$E_6(a_1)$	9	t^6+t^3+1
$E_6(a_2)$	6	$(t^2-t+1)^2(t^2+t+1)$
E_7	18	$(t^6-t^3+1)(t+1)$
$E_7(a_1)$	14	t^7+1
$E_7(a_2)$	12	$(t^4-t^2+1)(t^3+1)$
$E_7(a_3)$	30	$(t^5+1)(t^2-t+1)$
$E_7(a_4)$	6	$(t^2-t+1)^2(t^3+1)$
E_8	30	$t^8+t^7-t^5-t^4-t^3+t+1$
$E_8(a_1)$	24	t^8-t^4+1
$E_8(a_2)$	20	$t^8-t^6+t^4-t^2+1$
$E_8(a_3)$	12	$(t^4-t^2+1)^2$
$E_8(a_4)$	18	$(t^6-t^3+1)(t^2-t+1)$
$E_8(a_5)$	15	$t^8-t^7+t^5-t^4+t^3-t+1$
$E_8(a_6)$	10	$(t^4-t^3+t^2-t+1)^2$
$E_8(a_7)$	12	$(t^4-t^2+1)(t^2-t+1)^2$
$E_8(a_8)$	6	$(t^2-t+1)^4$

Table 3. Carter graphs and characteristic polynomials.

$H^2(S,\mathbb{C}) \cong \mathrm{Pic}(S)\otimes\mathbb{C}$. The latter should be compared with the Euler–Poincaré characteristic of the fixed locus S^g of g by applying the Lefschetz fixed-point formula.

$$\mathrm{Tr}_2(g) = s - 2 + \sum_{i\in I}(2-2g_i), \tag{5}$$

where S^g the disjoint union of smooth curves $R_i, i\in I$, of genus g_i and s isolated fixed points.

To determine whether a finite subgroup G of $\mathrm{Aut}(S)$ is minimal, we use the well-known formula from the character theory of finite groups

$$\mathrm{rank}\ \mathrm{Pic}(S)^G = \frac{1}{\#G}\sum_{g\in G}\mathrm{Tr}_2(g).$$

The tables for conjugacy classes of elements from the Weyl group W_S give the values of the trace on the lattice $\mathcal{R}_S = K_S^{\perp}$. Thus the group is minimal if and only if the sum of the traces add up to 0.

We first give the list of minimal cyclic groups of automorphisms.

Proposition 19. *The following conjugacy classes define minimal cyclic groups of automorphisms of a cubic surface S.*

- $3A_2$ *of order 3,*
- $E_6(a_2)$ *of order 6,*
- $A_5 + A_1$ *of order 6,*
- $E_6(a_1)$ *of order 9,*
- E_6 *of order 12.*

A very tedious computation gives the final classification of minimal finite subgroups of automorphisms of cubic surfaces.

Surface Type	Subgroups
I	$\mathfrak{S}_4,\ \mathfrak{S}_3,\ \mathfrak{S}_3\times 2,\ \mathfrak{S}_3\times 3,\ 3^2\rtimes 2,\ 3^2\rtimes 2^2,$
II	$H_3(3)\rtimes 2,\ H_3(3),\ 3^3\rtimes 2,\ 3^3\rtimes 2^2,\ 3^3\rtimes 3,$ $3^3\rtimes\mathfrak{S}_3,\ 3^3\rtimes D_8,\ 3^3\rtimes\mathfrak{S}_4,$ $3^3\rtimes 4, 3^3,\ 3^2,\ 3^2\times 2,\ 9,\ 6\ (2),\ 3.$
II	$\mathfrak{S}_5,\quad \mathfrak{S}_4.$
III	$H_3(3)\rtimes 4,\ H_3(3)\rtimes 2,\ H_3(3),$ $\mathfrak{S}_3\times 3,\ \mathfrak{S}_3,\ 3^2,\ 12,\ 6,\ 3.$
IV	$H_3(3)\rtimes 2,\ H_3(3),\ \mathfrak{S}_3,\ 3\times\mathfrak{S}_3,\ 3^2,\ 6,\ 3.$
V	$\mathfrak{S}_4,\ \mathfrak{S}_3.$
VI	$6,\quad \mathfrak{S}_3\times 2,\quad \mathfrak{S}_3.$
VIII	$\mathfrak{S}_3.$

Table 4. Minimal subgroups of automorphisms of cubic surfaces.

Here $H_3(3)$ is the Heisenberg group of order 27 isomorphic to the group of unipotent upper-triangular matrices of size 3×3 over the finite field $\mathbb{F}_3$. Note that there could be more than one conjugacy class of isomorphic groups. The number of these classes can be found in [11].

Next we consider the case of Del Pezzo surfaces of degree 2. It is known that the linear system $|-K_S|$ defines a double cover $f : S \to \mathbb{P}^2$ branched along a nonsingular plane curve B of degree 4. This implies that the group of automorphisms of S is mapped isomorphically onto the group $\mathrm{Aut}(B)$ with kernel of order 2 generated by the deck transformation γ of the cover. The automorphism γ of the rational surface S is conjugate in the Cremona group to the *Geiser birational involution* of the plane. The automorphism group of a plane quartic curve have been also determined in the 19th century. A modern proof can be found in [13].

It is known that the center of the Weyl group $W(\mathcal{Q}_2) = W(E_7)$ is generated by an element w_0 which acts on $\mathcal{Q}_2$ as the negative of the identity. Its conjugacy class is of type A_1^7. The quotient group $W(E_7)' = W(E_7)/\langle w_0\rangle$ is isomorphic to the simple group $\mathrm{Sp}(6, \mathbb{F}_2)$. The extension $2.\mathrm{Sp}(6, \mathbb{F}_2)$ splits by the subgroup $W(E_7)^+$ equal to the kernel of the determinant homomorphism $\det : W(E_7) \to \{\pm 1\}$. Thus we have

$$W(E_7) = W(E_7)^+ \times \langle w_0\rangle.$$

Let H be a subgroup of $W(E_7)'$. Denote by H^+ its lift to an isomorphic subgroup of W^+. Any other isomorphic lift of H is defined by a nontrivial homomorphism $\alpha : H \to \langle w_0\rangle \cong 2$. Its elements are the products $h\alpha(h), h \in H^+$. We denote such a lift by H_α. Thus all lifts are parameterized by the group $\mathrm{Hom}(H, \langle w_0\rangle)$ and H^+ corresponds to the trivial homomorphism. Note that $wH_\alpha w^{-1} = (w' H w'^{-1})_\alpha$, where w' is the image of w in $W(E_7)'$. In particular, two lifts of the same group are never conjugate.

It is convenient to view a Del Pezzo surface of degree 2 as a hypersurface in the weighted projective space $\mathbb{P}(1, 1, 1, 2)$ given by an equation of degree 4

$$t_3^2 + F_4(t_0, t_1, t_2) = 0. \tag{6}$$

The automorphism of the cover is the Geiser involution $\gamma = [t_0, t_1, t_2, -t_3]$. For any divisor class D on S we have $D + \gamma_0^*(D) \in |-mK_S|$ for some integer m. This easily implies that γ^* acts as the minus identity in $\mathcal{Q}_2$. Its image in the Weyl group $W(E_7)$ is the generator w_0 of its center. Thus the Geiser involution is the geometric realization of w_0.

Let $\rho : \mathrm{Aut}(S) \to W(E_7)$ be the natural injective homomorphism corresponding to a choice of a geometric basis in $\mathrm{Pic}(S)$. Denote by $\mathrm{Aut}(S)^+$ the full preimage of $W(E_7)^+$. Since $W(E_7)^+$ is a normal subgroup, this definition is independent of a choice of a geometric basis. Under the restriction homomorphism $\mathrm{Aut}(S) \to \mathrm{Aut}(B)$, the group

$\mathrm{Aut}(S)^+$ is mapped isomorphically to $\mathrm{Aut}(B)$ and we obtain

$$\mathrm{Aut}(S)^+ \cong \mathrm{Aut}(S)/\langle\gamma\rangle \cong \mathrm{Aut}(B).$$

From now on we will identify any subgroup G of $\mathrm{Aut}(B)$ with a subgroup of $\mathrm{Aut}(S)$ which we call the *even lift* of G. Under the homomorphism $\rho : \mathrm{Aut}(S) \to W(E_7)$ all elements of G define *even conjugacy classes*, i.e. the conjugacy classes of elements from $W(E_7)^+$. It is immediate to see that a conjugacy class is even if and only if the sum of the subscripts in its Carter graph is even. An isomorphic lift of a subgroup G to a subgroup of $\mathrm{Aut}(S)$ corresponding to some nontrivial homomorphism $G \to \langle\gamma\rangle$ (or, equivalently to a subgroup of index 2 of G) will be called an *odd lift* of G.

The odd and even lifts of the same group are never conjugate, two minimal lifts are conjugate in $\mathrm{Aut}(S)$ if and only if the groups are conjugate in $\mathrm{Aut}(B)$. Two odd lifts of G are conjugate if and only if they correspond to conjugate subgroups of index 2 (inside of the normalizer of G in $\mathrm{Aut}(B)$).

Lemma 20. *Let G be a subgroup of* $\mathrm{Aut}(B)$ *and H be its subgroup of index 2. Assume H is a minimal subgroup of* $\mathrm{Aut}(S)$ *(i.e. its even lift is such a subgroup). Then G is minimal in its even lift and its odd lift corresponding to H. Conversely, if G is minimal in both lifts, then H is a minimal subgroup.*

Since γ generates a minimal subgroup of automorphisms of S, any group containing γ is minimal. So, we classify first subgroups of $\mathrm{Aut}(B)$ which admit minimal lifts. These will be all minimal subgroups of $\mathrm{Aut}(S)$ which do not contain the Geiser involution γ. The remaining minimal groups will be of the form $\langle\gamma\rangle \times \widetilde{G}$, where $\widetilde{G}$ is any lift of a subgroup G of $\mathrm{Aut}(B)$. Obviously, the product does not depend on the parity of the lift.

We first give the list of minimal cyclic groups.

(1) Order 2 (A_1^7) (The Geiser involution) $g = [t_0, t_1, t_2, -t_3]$

$$F = t_3^2 + F_4(t_0, t_1, t_2).$$

(2) Order 4 ($2A_3 + A_1$) $g = [t_0, t_1, it_2, t_3]$

$$F = t_3^2 + t_2^4 + L_4(t_0, t_1).$$

(3) Order 6 ($E_7(a_4)$) $g = [t_0, t_1, \epsilon_3 t_2, -t_3]$

$$F = t_3^2 + t_2^3 L_1(t_0, t_1) + L_4(t_0, t_1).$$

(4) Order 6 ($A_5 + A_2$) $g = [t_0, -t_1, \epsilon_3 t_2, -t_3]$

$$F = t_3^2 + t_0^4 + t_1^4 + t_0 t_2^3 + a t_0^2 t_1^2.$$

(5) Order 6 ($D_6(a_2) + A_1$) $g = [t_0, \epsilon_3 x_1, \epsilon_3^2 x_2, -x_3]$

$$F = t_3^2 + t_0(t_0^3 + t_1^3 + t_2^3) + t_1 t_2(\alpha t_0^2 + \beta t_1 t_2).$$

(6) Order 12 ($E_7(a_2)$) $g = [t_0, \epsilon_4 t_1, \epsilon_3 t_2, t_3]$

$$F = t_3^2 + t_0^4 + t_1^4 + t_0 t_2^3,\ (t_0, t_1, t_2, t_3).$$

(7) Order 14 ($E_7(a_1)$) $g = [t_0, \epsilon_4 t_1, \epsilon_3 t_2, t_3]$

$$F = t_3^2 + t_0^3 t_1 + t_1^3 t_2 + t_2^3 t_0.$$

(8) Order 18 (E_7) $g = [t_0, \epsilon_3 t_1, \epsilon_9^2 t_2, -t_3]$

$$F = t_3^2 + t_0^4 + t_0 t_1^3 + t_2^3 t_1.$$

Using the information about cyclic groups of automorphisms of plane quartics, it is not hard to get the classification of possible automorphism groups (see [13]). It is given in Table 5.

Here AS_{16} is a solvable group of order 16 with generators $a.b, c$ and defining relations $a^4 = b^2 = c^2 = [a, b] = [c, b]a^{-2} = [c, a] = 1$.

Let us describe minimal subgroups of automorphisms of a Del Pezzo surface of degree 2.

To summarize our investigation we give two lists. In the first one we list all groups which do not contain the Geiser involution γ. We indicate by $+$ or $-$ the types of their lifts.

All other minimal groups are of the form $\langle\gamma\rangle \times G$, where G is one of the lifts of a subgroup of $\mathrm{Aut}(B)$. In the second list we give only groups $2 \times G$, where G does not admit a minimal lift. All other groups are of the form $2 \times G$, where G is given in Table 6.

Here M_{16} is a group of order 16 defined by generators a, b with relations $a^8 = b^2 = [a, b]a^4 = 1$. Note that some isomorphic groups may not be conjugate in $\mathrm{Cr}_2(k)$. We compute the number of conjugacy classes in [11].

Theorem 21. *Let G be a minimal group of automorphisms of a Del Pezzo surface of degree 2. Then G is either equal to a minimal lift of a subgroup from Table 6 or equal to $\gamma \times G'$, where G' is either from the table or one of the following groups of automorphisms of the branch quartic curve B*

(1) *Type I:* $7 : 3,\ \mathfrak{A}_4,\ \mathfrak{S}_3,\ 7,\ 4,\ 3,\ 2.$

Type	Order	Structure	Equation	Parameters
I	336	$2 \times \mathrm{PSL}_2(\mathbb{F}_7)$	$t_3^2 + t_0^3 t_1 + t_1^3 t_2 + t_2^3 t_0$	
II	192	$2 \times (4^2 : \mathfrak{S}_3)$	$t_3^2 + t_0^4 + t_1^4 + t_2^4$	
III	96	$2 \times 4.A_4$	$t_3^2 + t_2^4 + t_0^4 + at_0^2t_1^2 + t_1^4$	$a^2 = -12$
IV	48	$2 \times \mathfrak{S}_4$	$t_3^2 + t_2^4 + t_1^4 + t_0^4$ $+at_0^2t_1^2 + t_0^2t_2^2 + t_1^2t_2^2$	$a \neq \frac{-1\pm\sqrt{-7}}{2}$
V	32	$2 \times AS_{16}$	$t_3^2 + t_2^4 + t_0^4 + at_0^2t_1^2 + t_1^4$	$a^2 \neq 0, -12, 4, 36$
VI	18	18	$t_3^2 + t_0^4 + t_0t_1^3 + t_1t_2^3$	
VII	16	$2 \times D_8$	$t_3^2 + t_2^4 + t_0^4 + t_1^4$ $+at_0^2t_1^2 + bt_2^2t_0t_1$	$a, b \neq 0$
VIII	12	2×6	$t_3^2 + t_2^3t_0 + t_0^4 + t_1^4 + at_0^2t_1^2$	
IX	12	$2 \times \mathfrak{S}_3$	$t_3^2 + t_2^4 + at_2^2t_0t_1$ $+t_0(t_2^3 + t_0^3) + bt_0^2t_1^2$	
X	8	2^3	$t_3^2 + t_2^4 + t_1^4 + t_0^4$ $+at_2^2t_0^2 + bt_1^2t_2^2 + ct_0^2t_1^2$	distinct $a, b, c \neq 0$
XI	6	6	$t_3^2 + t_2^3t_0 + L_4(t_0, t_1)$	
XII	4	2^2	$t_3^2 + t_2^4$ $+t_2^2L_2(t_0, t_1) + L_4(t_0, t_1)$	
XIII	2	2	$t_3^2 + F_4(t_0, t_1, t_2)$	

Table 5. Groups of automorphisms of Del Pezzo surfaces of degree 2.

(2) *Type II:* $2^2,\ \mathfrak{S}_3,\ 8,\ 4,\ 3,\ 2.$
(3) *Type III:* $2^2,\ 4,\ 2.$
(4) *Type IV:* $\mathfrak{S}_3,\ 2^2,\ 3,\ 2.$
(5) *Type V:* $2^2,\ 2.$
(6) *Type VI:* $9,\ 3.$
(7) *Type VII:* $2^2,\ 4,\ 2$
(8) *Type VIII:* $3.$
(9) *Type IX:* $\mathfrak{S}_3,\ 3,\ 2.$
(10) *Type X:* $2^2,\ 2.$
(11) *Type XI:* $3.$
(12) *Type XII:* $\{1\}.$

Let S be a Del Pezzo surface of degree 1. The linear system $|-2K_S|$ defines a finite map of degree 2 onto a quadric cone Q in $\mathbb{P}^3$. Its branch locus is a nonsingular curve B of genus 4 cut out by a cubic surface. Recall that a singular quadric is isomorphic to the weighted projective space $\mathbb{P}(1,1,2)$. A curve of genus 4 of degree 6 cut out in Q by a cubic surface is given by equation $F(t_0, t_1, t_2)$ of degree 6. After change of

Type of S	Group	Lift
I	$L_2(7), \mathfrak{S}_4, D_8$	+
II	$4^2 \rtimes \mathfrak{S}_3, \mathfrak{S}_4, D_8 \times 4, AS_{16}$	+,-
	$4^2 \rtimes 3, \mathfrak{A}_4, M_{16}, D_8$	+
	$4^2, 2 \times 4, 4$	-
III	$4\mathfrak{A}_4, AS_{16}$	+,-
	$D_8, D_8 \rtimes 3$	+
	$12, 6, 2 \times 4, 4$	-
IV	$\mathfrak{S}_4, D_8$	+
V	AS_{16}	+,-
	D_8	+
	$2 \times 4, 4$	-
VII	D_8	+
VIII	6	-

Table 6. Minimal groups of automorphisms not containing γ.

coordinates it can be given by an equation $t_2^3 + a(t_0,t_1)t_2 + b(t_0,t_1) = 0$, where $a(t_0,t_1)$ and $b(t_0,t_1)$ are binary forms of degree 4 and 6 (or identically zero). The double cover of Q branched along such curve is isomorphic to a hypersurface of degree 6 in $\mathbb{P}(1,1,2,3)$

$$t_3^2 + t_2^3 + a(t_0,t_1)t_2 + b(t_0,t_1) = 0. \tag{7}$$

The vertex of Q has coordinates $[0,0,1]$ and its preimage in the cover consist of one point $[0,0,1,a]$, where $a^2+1=0$ (note that $[0,0,1,a]$ and $[0,0,1,-a]$ represent the same point on $\mathbb{P}(1,1,2,3)$). This is the base-point of $|-K_S|$. The members of $|-K_S|$ are isomorphic to genus 1 curves with equations $y^2 + x^3 + a(t_0,t_1)x + b(t_0,t_1) = 0$. The locus of zeros of $\Delta = 4a^3 + 27b^2$ is the set of points in $\mathbb{P}^1$ such that the corresponding genus 1 curve is singular. It consists of a simple roots and b double roots. The zeros of a are either common zeros with b and Δ, or represent nonsingular elliptic curves isomorphic to an anharmonic plane cubic curve. The zeros of b are either common zeros with a and Δ, or represent nonsingular elliptic curves isomorphic to a harmonic plane cubic curve.

Observe that no common root of a and b is a multiple root of b since otherwise the surface is singular.

Since the ramification curve of the cover $S \to Q$ (identified with the branch curve B) is obviously invariant with respect to $\mathrm{Aut}(S)$, we have a natural surjective homomorphism

$$\mathrm{Aut}(S) \to \mathrm{Aut}(B). \tag{8}$$

Its kernel is generated by the deck involution β which is called the *Bertini involution*. It defines the Bertini involution in $\mathrm{Cr}(2)$. The Bertini involution is the analog of the Geiser involution for Del Pezzo surfaces of degree 2. The same argument as above shows that β acts in $\mathcal{R}_S$ as the minus identity map. Under the homomorphism $\mathrm{Aut}(S) \to W(E_8)$ defined by a choice of a geometric basis, the image of β is the elements w_0 generating the center of $W(E_8)$. This time w_0 is an even element, i.e. belongs to $W(E_8)^+$. The quotient group $W(E_8)^+/\langle w_0 \rangle$ is isomorphic to the simple group $\mathrm{O}^+(8, \mathbb{F}_2)$.

Since Q is a unique quadric cone containing B, the group $\mathrm{Aut}(B)$ is a subgroup of $\mathrm{Aut}(Q)$. Consider the natural homomorphism $r : \mathrm{Aut}(S) \to \mathrm{Aut}(\mathbb{P}^1)$ which is the composition of (8) and the natural homomorphism $\mathrm{Aut}(B) \to \mathrm{Aut}(\mathbb{P}^1)$. Let G be a subgroup of $\mathrm{Aut}(S)$ and P be its image in $\mathrm{Aut}(\mathbb{P}^1)$. We assume that elements from G act on the variables t_0, t_1 by linear transformations with determinant 1. The polynomials $a(t_0, t_1)$ and $b(t_0, t_1)$ are the relative invariants of the binary group $\bar{P} = 2.P$. They are polynomials in the known basic relative invariants (Gründformen). Each relative invariant $p(t_0, t_1)$ defines a character $\chi : \bar{P} \to \mathbb{C}^*$ via $g^*(p) = \chi(g)p$. Let χ_4, χ_6 be the corresponding characters of $\bar{P}$ defined by the binary forms a, b. Let χ_2, χ_3 be the characters of G defined by the action on the variables t_2, t_3. Assume that $a \neq 0$. Then

$$\chi_4\chi_2 = \chi_6 = \chi_3^3 = \chi_3^2.$$

If $g \in G \cap \mathrm{Ker}(r) \setminus \{1\}$, then g acts on the variables t_0, t_1 by either the identity or the minus identity. Thus $\chi_4(g) = \chi_6(g) = 1$ and we must have $\chi_2(g) = \chi_3(g)^2 = 1$. This shows that $g = [t_0, t_1, t_3, -t_3] = [-t_0, -t_1, t_2, -t_3] = \beta$. If $a = 0$, then we must have only $\chi_2(g)^3 = \chi_3(g)^2 = 1$.

Using these arguments it is easy to list all possible automorphism groups of the curve B, and then describe their lifts to $\mathrm{Aut}(S)$ similarly to the case of Del Pezzo surfaces of degree 2. We state the results.

Table 7 gives the list of the full automorphism groups of Del Pezzo surfaces of degree 1.

Here Q_8 denotes the quaternion group of order 8.

The following is the list of cyclic minimal groups $\langle g \rangle$ of automorphisms of Del Pezzo surfaces $V(F)$ of degree 1.

(1) Order 2

Type	Order	Structure	F_4	F_6	Parameters
I	144	$3 \times (\overline{T} : 2)$	0	$t_0t_1(t_0^4 - t_1^4)$	
II	72	$3 \times 2D_{12}$	0	$t_0^6 + t_1^6$	
III	36	$6 \times D_6$	0	$t_0^6 + at_0^3t_1^3 + t_1^6$	$a \neq 0$
IV	30	30	0	$t_0(t_0^5 + t_1^5)$	
V	24	$\overline{T}$	$a(t_0^4 + \alpha t_0^2t_1^2 + t_1^4)$	$t_0t_1(t_0^4 - t_1^4)$	$\alpha = 2\sqrt{-3}$
VI	24	$2D_{12}$	$at_0^2t_1^2$	$t_0^6 + t_1^6$	$a \neq 0$
VII	24	2×12	t_0^4	t_1^6	
VIII	20	20	t_0^4	$t_0t_1^5$	
IX	16	D_{16}	$at_0^2t_1^2$	$t_0t_1(t_0^4 + t_1^4)$	$a \neq 0$
X	12	D_{12}	$t_0^2t_1^2$	$t_0^6 + at_0^3t_1^3 + t_1^6$	$a \neq 0$
XI	12	2×6	0	$G_3(t_0^2, t_1^2)$	
XII	12	2×6	t_0^4	$at_0^6 + t_1^6$	$a \neq 0$
XIII	10	10	t_0^4	$t_0(at_0^5 + t_1^5)$	$a \neq 0$
XIV	8	Q_8	$t_0^4 + t_1^4 + at_0^2t_1^2$	$bt_0t_1(t_0^4 - t_1^4)$	$a \neq 2\sqrt{-3}$
XV	8	2×4	$at_0^4 + t_1^4$	$t_0^2(bt_0^4 + ct_1^4)$	
XVI	8	D_8	$t_0^4 + t_1^4 + at_0^2t_1^2$	$t_0t_1(b(t_0^4 + t_1^4)$ $+ct_0^2t_1^2)$	$b \neq 0$
XVII	6	6	0	$F_6(t_0, t_1)$	
XVIII	6	6	$t_0(at_0^3 + bt_1^3)$	$ct_0^6 + dt_0^3t_1^3 + t_1^6$	
XIX	4	4	$G_2(t_0^2, t_1^2$	$t_0t_1F_2(t_0^2, t_1^2)$	
XX	4	2^2	$G_2(t_0^2, t_1^2$	$G_3(t_0^2, t_1^2)$	
XXI	2	2	$F_4(t_0, t_1)$	$F_6(t_0, t_1)$	

Table 7. Groups of automorphisms of Del Pezzo surfaces of degree 1.

- A_1^8 (the Bertini involution) $g = [t_0, t_1, t_2, -t_3]$

$$F = t_3^2 + t_2^3 + a(t_0, t_1)t_2 + b(t_0, t_1),$$

(2) Order 3
- $4A_2$ $g = [t_0, t_1, \epsilon_3 t_2, t_3]$

$$F = t_3^2 + t_2^3 + b(t_0, t_1),$$

(3) Order 4
- $2D_4(a_1)$ $g = [t_0, -t_1, -t_2, \pm it_3]$

$$F = t_3^2 + t_2^3 + (at_0^4 + bt_0^2t_1^2 + ct_1^4)t_2 + t_0t_1(dt_0^4 + et_1^4),$$

(4) Order 5

- $2A_4$ $g = [t_0, \epsilon_5 t_1, t_2, t_3]$

$$F = t_3^2 + t_2^3 + at_0^4 t_2 + t_0(bt_0^5 + t_1^5),$$

(5) Order 6
- $E_6(a_2) + A_2$ $g = [t_0, -t_1, \epsilon_3 t_2, t_3]$

$$F = t_3^2 + t_2^3 + G_3(t_0^2, t_1^2),$$

- $E_7(a_4) + A_1$ $g = [t_0, \epsilon_3 t_1, t_2, -t_3]$

$$F = t_3^2 + t_2^3 + (t_0^4 + at_0 t_1^3)t_2 + bt_0^6 + ct_0^3 t_1^3 + dt_1^6,$$

- $2D_4$ $g = [\epsilon_6 t_0, \epsilon_6^{-1} t_1, t_2, t_3]$

$$F = t_3^2 + t_2^3 + at_0^2 t_1^2 t_2 + bt_0^6 + ct_0^3 t_1^3 + et_1^6,$$

- $E_8(a_8)$ $g = [t_0, t_1, \epsilon_3 t_2, -t_3]$

$$F = t_3^2 + t_2^3 + F_6(t_0, t_1),$$

- $A_5 + A_2 + A_1$ $g = [t_0, \epsilon_6 t_1, t_2, t_3]$

$$F = t_3^2 + t_2^3 + at_0^4 t_2 + t_0^6 + bt_1^6,$$

(6) Order 8
- $D_8(a_3)$ $g = [it_0, t_1, -it_2, \pm\epsilon_8 t_3]$

$$F = t_3^2 + t_2^3 + at_0^2 t_1^2 t_2 + t_0 t_1(t_0^4 + t_1^4),$$

(7) Order 10
- $E_8(a_6)$ $g = [t_0, \epsilon_5 t_1, t_2, -t_3]$

$$F = t_3^2 + t_2^3 + at_0^4 t_2 + t_0(bt_0^5 + t_1^5),$$

(8) Order 12
- $E_8(a_3)$ $g = [-t_0, t_1, \epsilon_6 t_2, it_3]$

$$F = t_3^2 + t_2^3 + t_0 t_1(t_0^4 + at_0^2 t_1^2 + t_1^4),$$

(9) Order 15
- $E_8(a_5)$ $g = [t_0, \epsilon_5 t_1, \epsilon_3 t_2, t_3]$

$$F = t_3^2 + t_2^3 + t_0(t_0^5 + t_1^5),$$

(10) Order 20

- $E_8(a_2)$ $g = [t_0, \epsilon_{10}t_1, -t_2, it_3]$

$$F = t_3^2 + t_2^3 + at_0^4t_2 + t_0t_1^5,$$

(11) Order 24
- $E_8(a_1)$ $g = [it_0, t_1, \epsilon_{12}t_2, \epsilon_8t_3]$

$$F = t_3^2 + t_2^3 + t_0t_1(t_0^4 + t_1^4),$$

(12) Order 30
- E_8 $g = [t_0, \epsilon_5t_1, \epsilon_3t_2, -t_3]$

$$F = t_3^2 + t_2^3 + t_0(t_0^5 + t_1^5).$$

To list all minimal subgroups of $\mathrm{Aut}(S)$ is very easy. We know that any subgroup in $\mathrm{Ker}(r)$ contains one of the elements $\alpha, \beta, \alpha\beta$ which are all minimal of types $8A_1, 4A_2, E_8(a_8)$. So, a subgroup is not minimal only if its image P in $\mathrm{Aut}(B)$ can be lifted isomorphically to $\mathrm{Aut}(S)$.

We use the following lemma.

Lemma 22. *Let $P \subset \mathrm{Aut}(\mathbb{P}^1)$ and $G \subset \mathrm{Aut}(S)$ be contained in $r^{-1}(P)$. Then G is a minimal group unless $G = \tilde{P} \cong P$ and G is a non-minimal cyclic group or non-minimal dihedral group D_6.*

Here is the list of minimal groups of automorphisms of a Del Pezzo surface of degree 1.

Type I. $P \cong S_4$.

- $P = \{1\}$: $\langle\beta\alpha\rangle \cong 6, \langle\alpha\rangle \cong 3, \langle\beta\rangle \cong 2$;
- $P = 2$: 4, 12;
- $P = 2$: 4, 12;
- $P = 3$: 3^2, 3×6;
- $P = 2^2$: Q_8, $Q_8 \times 3$;
- $P = 2^2$: D_8, $D_8 \times 3$;
- $P = 4$: 8, 8×3;
- $P = D_8$: D_{16}, $D_8 \times 3$;
- $P = D_6$: $D_6 \times 2$, $D_6 \times 3, D_6 \times 6$;
- $P = \mathfrak{A}_4$: $\overline{T}$, $\overline{T} \times 3$;
- $P = \mathfrak{S}_4$: $\overline{T} : 2$, $3 \times (\overline{T} : 2)$.

Type II: $P = D_{12}$.

- $P = \{1\}$: $\langle\beta\alpha\rangle \cong 6, \langle\alpha\rangle \cong 3, \langle\beta\rangle \cong 2$;
- $P = 2$: 4, 12;
- $P = 2$: 2^2, $2^2 \times 3$, 6;
- $P = 3$: 3^2, $3^2 \times 2$;

- $P = 2^2$: Q_8, $Q_8 \times 3$;
- $P = 6$: 2×6,
- $P = D_6$: $2 \times D_6$, $D_6 \times 3$, $D_6 \times 6$;
- $P = D_{12}$: $2D_{12}$, $3 \times 2D_{12}$.

Type IV: $P = 5$

- $P = \{1\}$: $\langle \beta\alpha \rangle \cong 6, \langle \alpha \rangle \cong 3, \langle \beta \rangle \cong 2$;
- $P = 5$: $5,\ 10, 15, 30$;

Type VII: $P \cong 12$.

- $P = 2$: 2^2.
- $P = 3$: 6;
- $P = 4$: 2×4;
- $P = 6$: 2×6;
- $P = 12$: 2×12.

Type VIII: $P \cong 10$.

- $P = \{1\}$: $\langle \beta\alpha \rangle \cong 6, \langle \alpha \rangle \cong 3, \langle \beta \rangle \cong 2$;
- $P = 2$: 2^2.
- $P = 5$: 10;
- $P = 10$: 20.

Type XV: $P \cong 4$.

- $P = \{1\}$: $\langle \beta\alpha \rangle \cong 6, \langle \alpha \rangle \cong 3, \langle \beta \rangle \cong 2$;
- $P = 2$: 2^2.
- $P = 4$: 2×4.

3.4. Elementary links

We will be dealing with minimal Del Pezzo G-surfaces or minimal conic bundles G-surfaces. In the G-equivariant version of the Mori theory they are interpreted as extremal contractions $\phi : S \to C$, where $C = \mathrm{pt}$ is a point in the first case and $C \cong \mathbb{P}^1$ in the second case. They are also two-dimensional analogs of rational Mori G-fibrations.

A birational G-map between Mori fibrations are diagrams

(9)
$$\begin{tikzcd} S \arrow[r, dashed, "\chi"] \arrow[d, "\phi"] & S' \arrow[d, "\phi'"] \\ C & C' \end{tikzcd}$$

which in general do not commute with the fibrations. These maps are decomposed into *elementary links*. These links are divided into the four following types.

- Links of type I:

They are commutative diagrams of the form

(10)
$$\begin{tikzcd} S \arrow[d, "\phi"'] & Z = S' \arrow[l, "\sigma"'] \arrow[d, "\phi'"'] \\ C = \mathrm{pt} & C' = \mathbb{P}^1. \arrow[l, "\alpha"'] \end{tikzcd}$$

Here $\sigma : Z \to S$ is the blowup of a G-orbit, S is a minimal Del Pezzo surface, $\phi' : S' \to \mathbb{P}^1$ is a minimal conic bundle G-fibration, α is the constant map. For example, the blowup of a G-fixed point on $\mathbb{P}^2$ defines a minimal conic G-bundle $\phi' : \mathbf{F}_1 \to \mathbb{P}^1$ with a G-invariant exceptional section.

- Links of type II:

They are commutative diagrams of the form

(11)
$$\begin{tikzcd} S \arrow[d, "\phi"'] & Z \arrow[l, "\sigma"'] \arrow[r, "\tau"] & S' \arrow[d, "\phi'"'] \\ C & = & C'. \end{tikzcd}$$

Here $\sigma : Z \to S, \tau : Z \to S'$ are the blowups of G-orbits such that rank $\mathrm{Pic}(Z)^G =$ rank $\mathrm{Pic}(S)^G + 1 =$ rank $\mathrm{Pic}(S')^G + 1$, $C = C'$ is either a point or $\mathbb{P}^1$. An example of a link of type II is the Geiser (or Bertini) involution of $\mathbb{P}^2$, where one blows up 7 (or 8) points in general position which form one G-orbit. Another frequently used link of type II is an elementary transformation of minimal ruled surfaces and conic bundles.

- Links of type III:

These are the birational maps which are the inverses of links of type I.

- Links of type IV:

They exist when S has two different structures of G-equivariant conic bundles. The link is the exchange of the two conic bundle structures

(12)
$$\begin{tikzcd} S \arrow[d, "\phi"'] & = & S' \arrow[d, "\phi'"'] \\ C & & C'. \end{tikzcd}$$

One uses these links to relate elementary links with respect to one conic fibration to elementary links with respect to another conic fibration. Often the change of the conic bundle structures is realized via an involution in $\mathrm{Aut}(S)$, for example, the switch of the factors of $S = \mathbb{P}^1 \times \mathbb{P}^1$ (see the following classification of elementary links).

Theorem 23. *Let $f : S - \to S'$ be a birational map of minimal G-surfaces. Then χ is equal to a composition of elementary links.*

The proof of this theorem is the same as in the arithmetic case considered in [17], Theorem 2.5.

To start a link, one has to blow-up base points of maximal multiplicity of a linear system defining the birational map. To do it equivariantly, we blow up the orbits of points of maximal multiplicity. One uses the following

Lemma 24. *Let S a G-minimal Del Pezzo surface of degree d and $\mathcal{H} = |-aK_S - \sum m_i \kappa_i|$ be a linear system defining a birational G-equivariant map $\sigma : S - \to S'$. Here κ_i are the G-orbits of base points of $\mathcal{H}$ and $a \in \frac{1}{2}\mathbb{Z}$. Then*

$$\sum \#\kappa_i < d.$$

It follows from this lemma and Theorem 23 that σ is an isomorphism if G has no orbits of points of cardinality less than d. For example, this obviously happens if $d = 1$. So any minimal group of automorphisms of a Del Pezzo surface of degree 1 cannot be conjugate to a minimal group of automorphisms of another Del Pezzo surface or a conic bundle. It is superrigid in the sense of the following definition.

Definition 25. A minimal Del Pezzo G-surface is called *superrigid* (resp. *rigid*) if any birational G-map $\chi : S - \to S'$ is a G-isomorphism (resp. there exists a birational G-automorphism $\alpha : S - \to S$ such that $\chi \circ \alpha$ is a G-isomorphism).

A minimal conic bundle $\phi : S \to \mathbb{P}^1$ is called *superrigid* (resp. *rigid*) if for any birational G-map $\chi : S - \to S'$, where $\phi' : S' \to \mathbb{P}^1$ is a minimal conic bundle, there exists an isomorphism $\delta : \mathbb{P}^1 \to \mathbb{P}^1$ such that the following diagram is commutative

(13)
$$\begin{tikzcd} S \arrow[r, dashed, "\chi"] \arrow[d, "\phi"'] & S' \arrow[d, "\phi'"'] \\ \mathbb{P}^1 \arrow[r, "\delta"] & \mathbb{P}^1 \end{tikzcd}$$

(resp. there exists a birational G-automorphism $\alpha : S - \to S'$ such that the diagram is commutative after we replace χ with $\chi \circ \alpha$).

3.5. Final classification

Let S be a minimal G-surface S and $d = K_S^2$. We will classify all birational isomorphism classes of (S, G) according to the increasing parameter d. Since the number of singular fibres of a minimal conic bundle is at least 4, we have $d \leq 4$ for conic bundles.

- $d \leq 0$.

In this case, Iskovskikh's classification of elementary links shows that S is a superrigid conic bundle with $k = 8 - d$ singular fibres. The number k is a birational invariant.

Also observe that if $\phi : S \to \mathbb{P}^1$ is an exceptional conic bundle and $G_0 = \mathrm{Ker}(G \to \mathrm{O}(\mathrm{Pic}(S))$ is non-trivial, then no links of type II is possible. Thus the conjugacy class of G is uniquely determined by the isomorphism class of S.

- $d = 1$, S is a Del Pezzo surface.

The surface S is superrigid. Hence the conjugacy class of G is determined uniquely by its conjugacy class in $\mathrm{Aut}(S)$. All such conjugacy classes have been listed.

- $d = 1$, S is a conic bundle.

Let $\phi : S \to \mathbb{P}^1$ be a minimal conic bundle on S. It has $t = 7$ singular fibres. If $-K_S$ is ample, i.e. S is a (non-minimal) Del Pezzo surface, then the center of $\mathrm{Aut}(S)$ contains the Bertini involution β. We know that β acts as -1 on $\mathcal{R}_S$, thus β does not act identically on $\mathrm{Pic}(S)^G$, hence $\beta \notin G$. Since t is odd, the conic bundle is not exceptional, so we cam apply Theorem 9. It follows that G must contain a subgroup isomorphic to 2^2, adding β we get that S is a minimal Del Pezzo 2^3-surface of degree 1. However, the classification shows that there are no such surfaces.

Thus $-K_S$ is not ample. It follows from classification of elementary links that the structure of a conic bundle on S is unique. Any other conic bundle birationally G-isomorphic to S is obtained from S by elementary transformations with G-invariant set of centers.

- $d = 2$, S is a Del Pezzo surface.

By Lemma 24, S is superrigid unless G has a fixed point on S. If $\sigma : S - \to S'$ is a birational G-map, then the unique maximal base point of the linear system defining σ does not lie on a (-1)-curve. We can apply an elementary link $S \leftarrow Z \to S'$ of type II which together with the projections $S \to \mathbb{P}^2$ resolves the Bertini involution. These links together with the G-automorphisms (including the Geiser involution) generate the group of birational G-automorphisms of S (see [17], Theorem 4.6). Thus the surface is rigid. The conjugacy class of G in $\mathrm{Cr}(2)$ is determined uniquely by the conjugacy class of G in $\mathrm{Aut}(S)$. All such conjugacy classes have been listed.

- $d = 2$, $\phi : S \to \mathbb{P}^1$ is a conic bundle.

If $-K_S$ is ample, then ϕ is not an exceptional conic bundle. The center of $\mathrm{Aut}(S)$ contains the Geiser involution γ. Since γ acts non-trivially on $\mathrm{Pic}(S)^G = \mathbb{Z}^2$, we see that $\gamma \notin G$. Applying γ we obtain another conic bundle structure. In other words, γ defines an elementary link of type IV. Using the factorization theorem we show that the group of birational G-automorphisms of S is generated by links of type II, the Geiser involution, and G-automorphisms (see [17], [18], Theorem 4.9). Thus $\phi : S \to \mathbb{P}^1$ is a rigid conic bundle.

If S is not a Del Pezzo surface, ϕ could be an exceptional conic bundle with $g = 2$. In any case the group G is determined in Proposition 10. It is not known whether S can be mapped to a conic bundle with $-K_S$ ample.

One can show that any conic bundle with $d \geq 3$ is a non-minimal Del Pezzo surface, unless $d = 4$ and S is an exceptional conic bundle. In the latter case, the group G can be found in Proposition 10. It is not known whether it is birationally G-isomorphic to a Del Pezzo surface.

- $d = 3$, S is a minimal Del Pezzo surface.

The classification of elementary links shows that S is rigid. The conjugacy class of G in $\mathrm{Cr}(2)$ is determined by the conjugacy class of G in $\mathrm{Aut}(S)$.

- $d = 3$, S is a minimal conic bundle.

Since $k = 5$ is odd, G has 3 commuting involutions, the fixed-point locus of one of them must be a rational 2-section of the conic bundle. It is easy to see that it is a (-1)-curve C from the divisor class $-K_S - f$. The other two fixed-point curves are of genus 2. The group G leaves the curve C invariant. Thus blowing it down, we obtain a minimal Del Pezzo G-surface S' of degree 4. The group G contains a subgroup isomorphic to 2^2. Thus G can be found in the list of minimal groups of degree 4 Del Pezzo surfaces with a fixed point. For example, the group 2^2 has 4 fixed points.

- $d = 4$, S is a minimal Del Pezzo surface.

If $S^G = \emptyset$, then S admits only self-links of type II, so it is rigid or superrigid. The conjugacy class of G in $\mathrm{Cr}(2)$ is determined by the conjugacy class of G in $\mathrm{Aut}(S)$. If x is a fixed point of G, then we can apply a link of type I, to get a minimal conic bundle with $d = 3$. So, all groups with $S^G \neq \emptyset$ are conjugate to groups of de Jonquières type realized on a conic bundle $S \in \mathcal{C}_5$.

- $d = 4$, S is a minimal conic bundle.

Since $k = 4$, then either S is an exceptional conic bundle with $g = 1$, or S is a Del Pezzo surface with two sections with self-intersection -1 intersecting at one point. In the latter case, S is obtained by regularizing a de Jonquéres involution. They are minimal if and only if the kernel of the map $G \to \mathrm{PGL}_2(\mathbb{C})$ contains an involution not contained in $G_0 = \mathrm{Ker}(G) \to \mathrm{O}(\mathrm{Pic}(S))$. If G_0 is not trivial, then no elementary transformation is possible. So, S is not birationally isomorphic to a Del Pezzo surface.

- $d = 5$, S is a Del Pezzo surface, $G \cong 5$.

Let us show that (S, G) is birationally isomorphic to $(\mathbb{P}^2, G)$. Since rational surfaces are simply-connected, G has a fixed point x on S. The anti-canonical model of S is a surface of degree 5 in $\mathbb{P}^5$. Let P be the tangent plane of S at x. The projection from P defines a birational G-equivariant map $S - \to \mathbb{P}^2$ given by the linear system of anti-canonical curves with double point at x. It is an elementary link of type II.

- $d = 5$, S is a Del Pezzo surface, $G \cong 5 \rtimes 2, 5 \rtimes 4$.

The cyclic subgroup of order 5 of G has two fixed points on S. This immediately follows from the Lefschetz fixed-point formula. Since it is normal in G, the groups G has an orbit κ with $\#\kappa = 2$. Using an elementary link of type II with $S' = \mathbf{F}_0$, we obtain that G is conjugate to a group acting on $\mathbf{F}_0$.

- $d = 5$, S is a Del Pezzo surface, $G \cong \mathfrak{A}_5,\ \mathfrak{S}_5$.

It is clear that $S^G = \emptyset$ since otherwise G admits a faithful 2-dimensional linear representation. It is known that it does not exist. Since $\mathfrak{A}_5$ has no index 2 subgroups G does not admit orbits κ with $\#\kappa = 2$. The same is obviously true for $G = \mathfrak{S}_5$. It follows from the classification of links that (S, G) is superrigid.

- $d = 6$.

One of the groups from this case, namely $G \cong 2 \times \mathfrak{S}_3$ was considered in [18], [19]. It is proved there that (S, G) is not birationally isomorphic to $(\mathbb{P}^2, G)$ but birationally isomorphic to minimal $(\mathbf{F}_0, G)$.

- $d = 8$.

In this case $S = \mathbf{F}_0$ or $\mathbf{F}_n, n > 1$. In the first case (S, G) is birationally isomorphic to $(\mathbb{P}^2, G)$ if $S^G \neq \emptyset$ (via the projection from the fixed point). This implies that the subgroup G' of G belonging to the connected component of the identity of $\mathrm{Aut}(\mathbf{F}_0)$ is an extension of cyclic groups. One can show that this implies that G' is an abelian group of transformations $(x, y) \mapsto (\epsilon_{nk}^a x, \epsilon_{mk}^b y)$, where $a = sb \mod k$ for some s coprime to k. If $G \neq G'$, then we must have $m = n = 1$ and $s = \pm 1 \mod k$.

If $\mathbf{F}_0^G = \emptyset$ and $\mathrm{Pic}(\mathbf{F}_0)^G \cong \mathbb{Z}$, then the classification of links shows that links of type II with $d = d' = 7, 6, 5, d = 3, d' = 1$ map $\mathbf{F}_0$ to $\mathbf{F}_0$ or to minimal Del Pezzo surfaces of degrees 5 or 6. These cases have been already considered. If rank $\mathrm{Pic}(S)^G = 2$, then any birational G-map $S - \to S'$ is composed of elementary transformations with respect to one of the conic bundle fibrations. They do not change K_S^2 and do not give rise a fixed points. So, G is not conjugate to any subgroup of $\mathrm{Aut}(\mathbb{P}^2)$.

Assume $n > 1$. Then $G = A.B$, where $A \cong n$ acts identically on the base of the fibration and $B \subset \mathrm{PGL}_2(\mathbb{C})$. The subgroup B fixes pointwise two disjoint sections, one of them is the exceptional one. Let us consider different cases corresponding to possible groups B.

$B \cong n$. In this case B has two fixed points on the base, hence G has 2 fixed points on the non-exceptional section. Performing an elementary transformation with center at one of these points we descend G to a subgroup of $\mathbf{F}_{n-1}$. Proceeding in this way, we arrive to the case $n = 1$, and then obtain that G is a group of automorphisms of $\mathbb{P}^2$.

$B \cong D_n$. In this case B has an orbit of cardinality 2 in $\mathbb{P}^1$. A similar argument shows that G has an orbit of cardinality 2 on the non-exceptional section. Applying the product of the elementary transformations at these points we descend G to a subgroup of automorphisms of $\mathbf{F}_{n-2}$. Proceeding in this way we obtain that G is a conjugate to a subgroup of $\mathrm{Aut}(\mathbb{P}^2)$ or of $\mathrm{Aut}(\mathbf{F}_0)$. In the latter case it does not have fixed points, and hence is not conjugate to a linear subgroup of $\mathrm{Cr}(2)$.

$B \cong T$. The group B has an orbit of cardinality 4 on the non-exceptional section. A similar argument shows that G is conjugate to a group of automorphisms of $\mathbf{F}_2, \mathbf{F}_0$, or $\mathbb{P}^2$. Now suppose we arrive at $\mathbf{F}_2$. The group T has an orbit O_1 of length 6 on the exceptional section and an orbit O_2 of length 6 on a non-exceptional section. Make the elementary operations at O_1 to get a surface that has a section C with $C^2 = -8$ and a disjoint section Z with $Z^2 = 8$. Now we take two orbits of cardinality 4 on Z and make the corresponding elementary operations to arrive at $\mathbf{F}_0$.

$B \cong O$. Using orbits of cardinality 6 we first educe to the case $S = \mathbb{P}^2, \mathbf{F}_n, n = 0, 2, 3$. Suppose $S = \mathbf{F}_3$. Using an orbit of cardinality 8 on the exceptional section we get a surface admitting a section C with $C^2 = -11$ and a disjoint section Z with $Z^2 = 11$. Now using two orbits of cardinality 6 on Z, we arrive at $\mathbf{F}_1$, and then at $\mathbb{P}^2$. If $S = \mathbf{F}_2$, we do the same by using first an orbit of cardinality 6 on the exceptional section, and then an orbit of cardinality 8 to arrive at $\mathbf{F}_0$.

$B \cong I$. A similar argument shows that G is conjugate to a subgroup of group of automorphisms of $\mathrm{Aut}(\mathbb{P}^2)$, or $\mathrm{Aut}(\mathbf{F}_0)$, or $\mathrm{Aut}(\mathbf{F}_2)$.[1]

- $d = 9$.

[1] The argument in the last three cases is due to I. Cheltsov.

In this case $S = \mathbb{P}^2$ and G is a finite subgroup of $\mathrm{PGL}_3(\mathbb{C})$. The methods of the representation theory allows us to classify them up to conjugacy in the group $\mathrm{PGL}_3(\mathbb{C})$. However, some of non-conjugate groups can be still conjugate inside the Cremona group.

For example all cyclic subgroups of $\mathrm{PGL}_3(\mathbb{C})$ of the same order n are conjugate in $\mathrm{Cr}_2(\mathbb{C})$. Any element g of order n in $\mathrm{PGL}_3(\mathbb{C})$ is conjugate to a transformation g given in affine coordinates by the formula $(x, y) \mapsto (\epsilon_n x, \epsilon_n^a y)$. Let T be given by the formula $(x, y) \mapsto (x, x^a/y)$. Let $g' : (x, y) \mapsto (\epsilon_n^{-1} x, y)$. We have

$$g' \circ T \circ g : (x, y) \mapsto (\epsilon_n x, \epsilon_n^a y) \mapsto (\epsilon_n x, x^a/y) \mapsto (x, x^a/y) = T.$$

This shows that g' and g are conjugate.

We do not know whether any two isomorphic non-conjugate subgroups of $\mathrm{PGL}_3(\mathbb{C})$ are conjugate in $\mathrm{Cr}_2(\mathbb{C})$.

§4. Cyclic tame subgroups of $\mathrm{Cr}_2(k)$, where k is a perfect field

In this section we survey the results from [12], [32] and [31].

4.1. Elements of finite order in reductive algebraic groups

If the base field k is algebraically closed, then any cyclic tame group can be realized as a group of projective transformations. Also, it follows from the classification that any cyclic group of prime order $\ell > 5$ is conjugate to a group of projective transformations. Both of these statements are not true anymore if k is not algebraically closed.

For any integer N and a prime number ℓ prime to $\mathrm{char}(k)$ we denote by $\nu_\ell(N)$ the largest n such that ℓ^n divides N. For any finite group A we set $\nu_\ell(A)$ to be equal to $\nu_\ell(|A|)$. Let $t_\ell = [k(\zeta_\ell) : k], m_\ell = \sup\{d \geq 1 : \zeta_{\ell^d} \in k(\zeta_\ell)\}$, where ζ_ℓ generales $\nu_\ell(\bar{k})$.

For example, when $k = \mathbb{Q}$, we have $t_\ell = \ell - 1$ and $m_\ell = 1$. If $k = \mathbb{F}_q$, then t_ℓ is equal to the order of q in $\mathbb{F}_\ell^*$ and $m_\ell = \nu_\ell(q^{\ell-1} - 1)$.

The following is a special case of Theorem 6 from [31].

Theorem 26. *Let A be a finite subgroup of $\mathrm{PGL}_{n+1}(k)$. For any $\ell > 2$,*

$$\nu_\ell(A) \leq \sum_{2 \leq s \leq n+1, t_\ell | s} (m_\ell + \nu_\ell(s)),$$

(if the index set is empty, $\nu_\ell(A) = 0$).

Corollary 27. *Assume $t_\ell \geq n + 2$. Then $\mathrm{PGL}_{n+1}(k)$ does not contain elements of prime order ℓ.*

For example, if k is of characteristic zero and $m_\ell(k) = \{1\}$ (e.g. $k = \mathbb{Q}$), then $t_\ell = \ell - 1$ and we get

$$n \geq \ell - 2,$$

if $\mathrm{PGL}_{n+1}(k)$ contains an element of order ℓ. In particular, $\mathrm{PGL}_{n+1}(k)$ contains an element of order 7 only if $n \geq 5$.

On the other hand, if $k = \mathbb{F}_2$ and $\ell = 7$, then $t_\ell = 3$ and it is known that $\mathrm{PGL}_3(k)$ is isomorphic to a simple group of order 168 and it contains an element of order 7.

The next result of Serre [31], Theorems 4 and 4', concerns elements of finite order in an algebraic k-torus. Note that any 2-dimensional k-torus is known to be rational over k [37], hence any its element defines an element in $\mathrm{Cr}_2(k)$.

Theorem 28. *Let T be an algebraic k-torus and A be a finite subgroup of $T(k)$. Then*

$$\nu_\ell(A) \leq m_\ell \Big[\frac{\dim T}{\phi(t_\ell)}\Big],$$

where ϕ is the Euler function. Assume $m_\ell < \infty$ (e.g. k is finitely generated over its prime subfield). For any $n \geq 1$ there exists an n-dimensional k-torus T and a finite subgroup A of $T(k)$ such that $\nu_\ell(A) = m_\ell\big[\frac{\dim T}{\phi(t_\ell)}\big]$.

Corollary 29. *A two-dimensional k-torus T with $T(k)$ containing an element of prime order $\ell > 2$ exists if and only if t_ℓ takes values in the set $\{1, 2, 3, 4, 6\}$.*

Proof. In fact, the set $\{1, 2, 3, 4, 6\}$ is the set of positive integers t_ℓ such that $\phi(t_\ell) \leq 2$. If $\phi(t_\ell) > 2$, Serre's bound implies that no such torus exists. If $\phi(t_\ell) = 2$, Serre's construction from above exhibits such a torus. If $\phi(t_\ell) = 1$, i.e. $t_\ell = 1$ or 2, we can take $T = \mathbb{G}_{m,k}^2$ in the first case and $T = R_{k(\zeta_\ell)/k}(\mathbb{G}_m)$ in the second case. Q.E.D.

4.2. Elements of order ≥ 7

Looking at the table of conjugacy classes of elements in the Weyl groups, we notice that an element of order > 7 does not exist in these groups, and an element g of order 7 exists only in the Weyl groups $W(E_7)$ and $W(E_8)$. When k is algebraically closed, we checked that no such element is minimal. If $d = 2$ this can be shown directly as follows. It is known that, over the algebraic closure $\bar{k}$, S contains 576 sets of 7 disjoint (-1)-curves. An element g of order 7 acts on this set and has a fixed element because $566 \equiv 2 \mod 7$. Blowing this invariant set down, we see that g arises from a projective automorphism. If $k \neq \bar{k}$

this argument does not work since one may not be able to blow down the seven disjoint curves over the ground field. Nevertheless, we prove that if this happens, the surface S must be a k-minimal rational surface. Then we use the following fundamental result from arithmetic of rational surfaces.

Theorem 30. *A minimal geometrically rational surface X over a perfect field k is k-rational if and only if the following two conditions are satisfied:*

(i) $X(k) \neq \emptyset$;

(ii) $d = K_X^2 \geq 5$.

This result is a culmination of several results due to V. Iskovskikh and Yu. Manin. Its modern proof based on the theory of elementary links can be found in [17], §4, p. 642.

A similar argument works in the case $d = 1$. Thus, an element of order $\ell \geq 7$ may act minimally either on a Del Pezzo surface of degree $d \geq 6$ or on a conic bundle. In the latter case, by using Corollary 27, we obtain that $t_\ell \leq 2$. More precisely, we prove the following.

Proposition 31. *Assume $\ell \geq 5$ and σ acts minimally as an automorphism of a k-rational conic bundle X. Then $t_\ell \leq 2$ and σ is conjugate in $\mathrm{Cr}_2(k)$ to an element defined by a rational point on a 2-dimensional algebraic k-torus.*

Recall that a Del Pezzo surface S of degree 6 over an algebraically closed field is obtained by blowing up 3 non-collinear points in the plane. It contains an open subset T isomorphic to a 2-dimensional torus which acts on S extending the action on itself by translations. In other word, S has a structure of a toric surface. The complement of T is the hexagon of (-1)-curves. There is a unique toric surface $\mathcal{D}$ with Picard number 4 defined over $\mathbb{Z}$, and $S_{\bar{k}} \cong \mathcal{D}_{\bar{k}}$. Since the set of all (-1)-curves on $S_{\bar{k}}$ is defined over k, its complement U in S becomes isomorphic to a torus over $\bar{k}$. This implies that U is a torsor (= principally homogeneous space) over a two-dimensional k-torus T (see [24], Chapter IV, Theorem 8.6). Since S is rational, $S(k) \neq \emptyset$ and hence $U(k) \neq \emptyset$ ([21], Proposition 4). This shows that U is an algebraic k-torus. Thus S is a toric surface over k.

Proposition 32. *Assume a cyclic group $G = \langle \sigma \rangle$ of prime order $\ell \geq 5$ acts minimally on a k-rational Del Pezzo surface S of degree 6. Let T be the complement of the union of (-1)-curves on S that acts on X via its structure of a toric surface over k. Then σ is defined via the action by an element $\tilde{\sigma} \in T(k)$. The torus T splits over $k(\zeta_\ell)$ with cyclic*

Galois group $\langle\gamma\rangle$ of order 6 and $t_\ell = 6$. The G-surface (S, G) is unique up to k-isomorphism.

Similarly, we deal with Del Pezzo surfaces of degree $d > 6$,

Proposition 33. *Assume that σ of prime order $\ell \geq 7$ acts minimally on a k-rational Del Pezzo surface X of degree d. Then one of the following cases occurs:*

(i) $d = 6, t_\ell = 6$;
(ii) $d = 8, t_\ell = 4$ *or* $t_\ell = 2$;
(iii) $d = 9, t_\ell \leq 3$.

In all cases X has a structure of a toric surface and σ belongs to $T(k)$, where T is an open subset of X isomorphic to a k-torus.

Summing up, we get the following main result of [12].

Theorem 34. *Let k be a perfect field of characteristic $p \geq 0$. Then* $\mathrm{Cr}_2(k)$ *contains an element of prime order $\ell > 5$ not equal to p if and only if there exists a 2-dimensional algebraic k-torus T such that $T(k)$ contains an element of order ℓ.*

Assume char $k = 0$. We assume also that

(*) $k \cap \mathbb{Q}(\zeta_\ell) = \mathbb{Q}$.

Thus

$$t_\ell = \ell - 1.$$

Assume $\ell \geq 7$. By Proposition 31, σ cannot act minimally on a conic bundle. By Proposition 33, σ can only act minimally on a Del Pezzo surface X of degree 6 in which case $\ell = 7$. By Proposition 32, X is a unique (up to isomorphism) toric surface T over k split over $E = k(\zeta_7)$. The Galois group Γ acts on T_E via the subgroup $H \subset \mathrm{Aut}(\Sigma)$ isomorphic to the cyclic group of order 6. The action of H on T_E is Γ-equivariant, and hence admits a descent to an action of H on X. The 7-torsion subgroup $T(k)[7]$ of $T(k)$ is H-invariant. Hence H acts on the cyclic group $\langle\sigma\rangle$ of order 7 by automorphisms. This shows that all non-trivial powers of σ are conjugate in $\mathrm{Cr}_2(k)$.

This proves the following.

Theorem 35. *Assume (*) is satisfied. Then* $\mathrm{Cr}_2(k)$ *does not contain elements of prime order > 7 and all elements of order 7 are conjugate.*

It follows from the proof of Proposition 33 that an element of order $\ell = 5$ can be realized as an automorphism of a Del Pezzo surface of degree 8 defined over $\mathbb{Q}$. One can also show that it can be realized as a minimal automorphism of a Del Pezzo surface of degree 5 over $\mathbb{Q}$.

In the case when k is finitely generated over its prime field, J.-P. Serre gives a bound for the order of any finite group in $\mathrm{Cr}_2(k)$ [32].

Theorem 36. *Assume k is finitely generated over its prime subfield. Then finite tame subgroups of Cr(k) of order prime to* $\mathrm{char}(k)$ *have bounded order. Let $M(k)$ be the least common multiple of their orders. Then*

(i) *If $k = \mathbb{Q}$, we have $M(k) = 120960 = 2^7.3^3.5.7$.*
(ii) *If k is finite with q elements, we have:*

$$M(k) = \begin{cases} 3(q^4-1)(q^6-1) & \textit{if } q \equiv 4 \text{ or } 7 \mod 9 \\ (q^4-1)(q^6-1) & \textit{otherwise.} \end{cases}$$

§5. Wild cyclic groups

Here we assume that the ground field k is of characteristic $p > 0$ and study wild subgroups of $\mathrm{Cr}_2(k)$.

5.1. Conic bundles

Let $G = \langle\sigma\rangle$ be a cyclic subgroup of $\mathrm{Cr}_n(k)$ of order $p^s m$, where $(p, m) = 1$. Assuming that we know the classification of tame cyclic subgroups (for example when k is algebraically closed and $n = 2$), we are interested only in wild cyclic p-groups. An example of a wild cyclic group of order p^n in $\mathrm{Cr}_n(k)$ is easy to give (see [33]). We consider the additive group $W_n(k)$ of Witt vectors of length n. It is an affine space $\mathbb{A}^n_k$ with a group law containing a cyclic subgroup of order p^n. The latter acts on $W_n(k)$ by translation, and hence embeds in $\mathrm{Aut}(\mathbb{A}^n_k) \subset \mathrm{Cr}_n(k)$.

Conjecture 37. $\mathrm{Cr}_n(k)$ does not contains elements of order p^s with $s > n$.

In [14] we prove this conjecture for $n = 2$.

Let us sketch a proof. It is enough to assume that k is algebraically closed. First we consider the case when G regularizes on a conic bundle.

Lemma 38. *Let σ be an element of order p^s in* $\mathrm{Aut}(\mathbb{P}^r_k)$. *Then* $s < 1 + \log_p(r+1)$.

Proof. Let $A \in \mathrm{GL}_{r+1}(k)$ represent σ and $A^{p^s} = cI_{r+1}$ for some constant c. Multiplying A by $c^{\frac{1}{p^s}}$ we may assume that $A^{p^s} = I_{r+1}$ but $A^{p^{s-1}} \neq I_{r+1}$. Since k^* does not contain non-trivial p-th roots of unity, we can reduce A to the Jordan form with 1 at the diagonal. Obviously $A^{p^{s-1}} = I_{r+1} + (A - I_{r+1})^{p^{s-1}}$. Since, for any Jordan block-matrix J with zeros at the diagonal, we have $J^{r+1} = 0$, we get $p^{s-1} < r+1$. The assertion follows. Q.E.D.

Corollary 39. *Let $f : X \to \mathbb{P}^1_k$ be a conic bundle and σ be an automorphism of X of order p^s preserving the conic bundle. Then $s \leq 2$.*

Proof. Let $\bar{g}$ be the image of σ in the automorphism group of the base of the fibration. By the previous lemma $\bar{\sigma}^p = 1$. Thus σ^p acts identically on the base and hence acts on the general fibre of f. By Tsen's Theorem, the latter is isomorphic to the projective line over the function field of the base. Applying the lemma again we obtain that $\sigma^{p^2} = 1$. Q.E.D.

This checks the theorem in the case of a conic bundle. A closer look at elements of order p^2 gives the following.

Theorem 40. *Let σ be a minimal automorphism of order p^2 of a conic bundle $X \to \mathbb{P}^1_k$. Then $p = 2$.*

5.2. Del Pezzo surfaces

Next we consider the case when G regularizes on a Del Pezzo surface.

Theorem 41. *A Del Pezzo surface of degree $d \geq 4$ does not contain elements of order p^3. An automorphism of order p^2 not conjugate to a projective automorphism in $\mathrm{Cr}_2(k)$ exists only if $p = 2$. It is minimally realized on $X = \mathbb{P}^1_k \times \mathbb{P}^1_k$ or on a Del Pezzo surface of degree 4.*

Here the cases $d \geq 6$ are easy and the cases $d = 4, 5$ are treated by using the representation of automorphisms in the Weyl group.

Consider the case $d = 3$. Again looking at the Weyl group, we immediately check the conjecture in this case. Moreover, we obtain that a minimal element of order p^2 exists only if $p = 3$. The following nice argument due to J.-P. Serre excludes this case.

It follows from the classification of conjugacy classes of elements of $W(E_6)$ that the trace of σ in its action in $K_X^{\perp}$ is equal to 0. Thus the Lefschetz number of σ in in the ℓ-adic cohomology of X is equal to 3. This implies that σ has a fixed point x_0. Since σ acts trivially on $|-K_X - x_0| \cong \mathbb{P}^2_k$, we obtain that it acts trivially on $|-K_X| \cong \mathbb{P}^3_k$.

We have proved the following.

Theorem 42. *A cubic surface does not admit minimal automorphisms of order p^s with $s > 1$.*

In the case of Del Pezzo surfaces of degree 2, the structure of the Weyl group $W(E_7)$ shows that there are no elements of order p^3 unless $p = 2$ and $s = 3$. The following argument of Serre excludes this case (our original proof is a little more complicated). We use that $W(E_7) = W(E_7)^+ \times \langle w_0 \rangle$, where w_0 generates the center of $W(E_7)$. In the faithful representation $\rho : \mathrm{Aut}(X) \to W(E_7)$, the image of the Geiser involution

γ is equal to w_0. This implies that a subgroup G of order 8 of $\mathrm{Aut}(X)$ is isomorphic to a subgroup of $A\times\langle\gamma\rangle$, where A is isomorphic to a subgroup of $\mathrm{Aut}(\mathbb{P}^2_k)$. Since the latter has no elements of order 8, we are done.

The case of Del Pezzo surfaces of degree 1 is the most difficult and interesting case.

Theorem 43. *A Del Pezzo surface of degree 1 may admit an automorphism of order p^s only if $p = 2$ and $s = 2$.*

We refer for the conceptual proof to [14]. A computational proof of this result was also given by J.-P. Serre.

As we have seen in the previous sections, an element of order p^2 not conjugate to a projective transformation exists only for $p = 2$. It can be realized as a minimal automorphism of a conic bundle, or a Del Pezzo surfaces of degree 1 or 4. Del Pezzo surfaces of degree 1 are super-rigid, i.e. a minimal automorphism of such a surface could be conjugate only to a minimal automorphism of the same surface. A minimal automorphism of a Del Pezzo surface of degree 4 is conjugate to a minimal automorphism of a conic bundle with 5 singular fibres (see [11], §8).

Thus we have proved the following.

Theorem 44. *An element of order p^2 not conjugate to a projective transformation exists only if $p = 2$. An element of order 4 is conjugate to either a projective transformation, or a transformation realized by a minimal automorphism of a conic bundle or of a Del Pezzo surface of degree 1.*

For the completeness sake let us add that elements of order p not conjugate to a projective transformations occur for any p. They can be realized as automorphisms of conic bundles, and if $p = 2, 3, 5$ as automorphisms of Del Pezzo surfaces.

§6. Wild simple groups

We use the notation from [9]. Thus $L_n(q)$ denotes $\mathrm{PSL}_n(\mathbb{F}_q)$, $U_n(q)$ denotes $\mathrm{PSU}_n(\mathbb{F}_{q^2})$, $\mathrm{PGU}_n(q)$ denotes $\mathrm{PU}_n(q^2)$. The group $L_n(q)$ is a subgroup of index $(q-1, n)$ of $\mathrm{PGL}_n(\mathbb{F}_q)$ and the group $U_n(q)$ is a subgroup of index $(q+1, n)$ of $\mathrm{PU}_n(q)$. We will assume that k is algebraically closed of characteristic $p > 0$.

6.1. Projective linear groups

Let us first recall the classification of finite subgroups of $\mathrm{Cr}_1(k) = \mathrm{PGL}_2(k)$ and $\mathrm{PGL}_3(k)$.

First let us make some remarks. In the wild case, a primitive group may arise from a reducible linear representation, i.e. a representation that admits an invariant proper subspace which does not split as a summand. So the right analog of a primitive group is an irreducible group, a group arising from an irreducible linear representation. If a group G is simple, then it is either an irreducible group or is isomorphic to an irreducible subgroup in lower dimension. We will also use that an irreducible subgroup of $\mathrm{PGL}_n(k)$ is conjugate to a subgroup of $\mathrm{PGL}_n(\mathbb{F}_q)$ for some finite subfield $\mathbb{F}_q$ of k [39].

A proof of the following result can be found in [36], Chapter 3, Theorem 6.17.

Theorem 45. *Let G be a proper wild subgroup of $\mathrm{PGL}_2(k)$. Then G is isomorphic to one of the following groups*

(i) *the group $G_{\xi,A}$ of affine transformations $x \mapsto \xi^t x + a$, where a belongs to a finite subgroup A of the additive group of k containing 1 and ξ is a root of unity such that $\xi A = A$.*
(ii) *$p = 2$ and G is a dihedral group of order $2n$, where n is odd.*
(iii) *$p = 3$ and $G \cong L_2(5) \subset L_2(9)$.*
(iv) *$L_2(q)$ or $\mathrm{PGL}_2(\mathbb{F}_q)$ for some $q = p^r$.*

The proof of the next result can be found in [16].

Theorem 46. *Assume $p = 2$. Let G be a finite irreducible subgroup of $\mathrm{PGL}_3(k)$. Then G is conjugate to one of the following groups.*

(i) *$L_3(2^s)$ or $\mathrm{PGL}_3(2^s)$ for some s (the groups are equal if s is odd);*
(ii) *$U_3(2^s)$ or $\mathrm{PGU}_3(2^s)$ for some s (the groups are equal if s is even);*
(iii) *$\mathfrak{A}_6 \subset L_3(4)$;*
(iv) *$3^2 \rtimes 4 \subset \mathrm{PGU}_3(2)$.*

Note that some of these groups are familiar from the case when $k = \mathbb{C}$. We have $L_3(2) \cong L_2(7)$ is the Klein group of order 168, $\mathrm{PU}_3(\mathbb{F}_4)$ is the Hessian group of order 216 and $U_3(2) = 3^2 : Q_8$ is its subgroup of index 3. Note that in characteristic 2, $\mathfrak{A}_5$ is realized as primitive group but leaves a point invariant (the intersection of tangents to an invariant conic).

The proof of the next theorem can be found in [6], [25].

Theorem 47. *Assume $p > 2$. Let G be a finite irreducible subgroup of $\mathrm{PGL}_3(k)$ which does not leave a point or a line invariant. Then G is conjugate to one of the following groups.*

(i) *$L_3(p^s)$ or $\mathrm{PGL}_3(p^s)$ for some s (the groups are equal if $3|p^s-1$);*

(ii) *$U_3(p^s)$ or $\mathrm{PGU}_3(p^s)$ for some s (the groups are equal if $3|p^s+1$);*
(iii) *The Hessian group of order 216 and its subgroups of order 72 and 36;*
(iv) *$\mathrm{SO}_3(p^s) \cong L_2(p^s)$;*
(iv) *$L_2(7)$ (isomorphic to $O_3(7)$ if $p = 7$);*
(v) *$\mathfrak{A}_6$;*
(vi) *$\mathfrak{S}_6$ if $p = 5$;*
(vii) *$\mathfrak{A}_7$ if $p = 5$.*

Note that $\mathfrak{A}_5$ is realized in characteristic 5 as the group $\mathrm{SO}_3(5)$.

6.2. Conic bundles

Let G be a wild subgroup of $\mathrm{Cr}_2(k)$ minimally realized as a group of automorphisms of a conic bundle. We assume that it has $t > 0$ singular fibres. By Lemma 8 which applies in our case, we get $t \geq 4$.

Lemma 48. *Let G_K be the kernel of the action of G on the base of the conic bundle. Then G_K contains an element of order 2. If $p \neq 2$ it switches the irreducible components of some singular fibre.*

Proof. In the complex case this is Lemma 5.6 from [11]. The proof extends to the wild case if $p \neq 2$. Assume $p = 2$. Since G acts minimally, there exists $g \in G$ which switches two components R and R' of some singular fibre. If the order of g is an odd number $2m + 1$, then g^{2m} and g^{2m+1} fix the components, hence g fixes the components. Thus the order g is even. Replacing g by some odd power, we may assume that the order of g is equal to 2^a for some $a > 0$. Assume $a > 1$. Since $\mathrm{PGL}_2(k)$ does not contain elements of order 2^a, the element $g^{a-1} \in G_K$ and satisfies the assertion of the theorem. If $a = 1$, applying Theorem 2.4 from [27], we obtain that the point q is not an isolated fixed point. This implies that there exists a curve of fixed points passing through q. Since g switches the components of the fibre, this curve is mapped surjectively to the base of the fibration. This immediately implies that $g \in G_K$. Q.E.D.

Assume $p \neq 2$. Applying Theorem 40, we find that either $\bar{G}$ or K is a tame group. If K is tame, then we have a complete analog of the description of G in the tame case, except the group $\bar{G}$ is given by Theorem 45. Assume K is wild. If K is in case (iii) or (iv) from Theorem 45, then it is simple, and hence the natural homomorphism $K \to 2^t$, where t is the number of singular fibres, is the identity. This shows that K acts trivially on the Picard group. Let g_0 be a nontrivial element from K of order divisible by p and E be a section of the conic bundle with negative self-intersection. Then g_0 fixes E and hence has a fixed point on each component of singular fibre. Since a wild element has only one fixed point on $\mathbb{P}^1$ and the singular point of the fibre is obviously fixed, we get a contradiction.

Assume K is as in case (i) of Theorem 45. Then, the kernel of the homomorphism $K \to 2^t$ is a wild group. By the above, this leads to contradiction.

Assume $p = 2$. By Lemma 48, K is a wild group. By the above, the kernel of the homomorphism $K \to \mathrm{Pic}(S)$ is a tame group. This implies that either it is trivial, or S is an exceptional conic bundle. The automorphism group of an exceptional conic bundle can be described similarly to the tame case. If S is not exceptional, then K must be a subgroup of 2^t isomorphic to a subgroup of k. Note that this case includes the example of a De Jonquières involution $(x, y) \to (x, F(x)/y)$ which makes sense over any field.

Next we consider the case when the conic bundle is a minimal ruled surface $\mathbf{F}_n$. Proposition 7 has a similar statement where $\mathbb{C}$ is replaced by k. However, a wild group may have nontrivial intersection with the kernel of the homomorphism $\mathrm{Aut}(\mathbf{F}_n) \to \mathrm{Aut}(\mathbb{P}^1)$. A simple wild group must be isomorphic to a subgroup of $\mathrm{Aut}(\mathbb{P}^1)$.

6.3. Del Pezzo surfaces

Next we assume that G is a wild group of automorphisms of a Del Pezzo surface. Consider first the following two examples.

Example 49. Assume $p = 2$ and let S be the Fermat cubic

$$t_0^3 + t_1^3 + t_2^3 + t_3^3 = 0.$$

We consider the left-hand-side F as a hermitian form over the finite field $\mathbb{F}_4$. Then the unitary group $\mathrm{PGU}_4(2) \cong U_4(2)$ acts faithfully on the surface. It is known that $\mathrm{PU}_4(2)$ is isomorphic to the simple subgroup $W(E_6)'$ of index 2 of the Weyl group $W(E_6)$. I claim that $\mathrm{Aut}(S) \cong \mathrm{PU}_4(2)$. Suppose that $\mathrm{Aut}(S) \cong W(E_6)$. Choose a double-six $(\ell_1, \ldots, \ell_6), (\ell'_1, \ldots, \ell'_6)$ of lines on S. Let $\sigma \in W(S) \cong W(E_6)$ which acts by sending $[\ell_i]$ to $[\ell'_i]$. If we choose the first six lines to define a geometric marking $(e_0, e_1, \ldots, e_6)$ on S, then σ is represented in $W(E_6)$ by the reflection in the vector $2e_0 - e_1 - \ldots - e_6$. Any of the 15 lines with the divisor class $e_0 - e_i - e_j$ is invariant with respect to σ. Suppose there exists $g \in \mathrm{Aut}(S)$ such that $g^* = \sigma$. Then g has a fixed point on each of the 15 lines. Since no more than three lines pass through one point, we have ≥ 9 fixed points of g on S. Looking at the Jordan form of a matrix representing g acting in $\mathbb{P}^3$ we see that g has a plane section of fixed points in S. This plane intersects each line ℓ_i. Since $g(\ell_i) = \ell'_i$ and ℓ and ℓ'_i are skew, we get a contradiction. Obviously, $\mathrm{Aut}(S) \cong W(E_6)'$ is a minimal subgroup of automorphisms.

Example 50. Let $p = 3$ and S be a Del Pezzo surface of degree 2 with equation

$$t_0^4 + t_1^4 + t_2^4 + t_3^2 = 0.$$

Then the polynomial $H : t_0^4 + t_1^4 + t_2^4$ can be considered as a hermitian form over $\mathbb{F}_9$. The group of its automorphisms is the simple group $U_3(3) \cong \mathrm{PGU}_3(3)$ of order $2^5.3^3.7$. It is isomorphic to a subgroup of index 4 of a maximal subgroup of $W(E_7)$ of index 240.

Together with the Geiser involution $t_3 \mapsto -t_3$ we obtain a subgroup of $\mathrm{Aut}(S)$ isomorphic to $U_3(3) \times 2$. Let us show that this is the whole group. It suffices to show that the automorphism group of the quartic curve $H = 0$ is isomorphic to $U_3(3)$. We have an upper bound $B(g,p)$ for the automorphism group of a curve of genus $g \geq 2$ over an algebraically closed field of characteristic $p > 0$. If $2g+1 > p > g+1$, or $p > 2g+1$ we have the Hurwitz bound $B(g,p) = 84(g-1)$ (see [30]). If $p \leq g+1$ we have the bound (see [35])

$$B(g,p) = \frac{16p^2g^4}{(p-1)^3 + p - 1}\big(\frac{2g}{p-1} + 1\big). \tag{14}$$

This gives $B(3,3) = 6048 = \#\mathrm{U}_3(3)$. The subgroup $U_3(3)$ admits a unique lift to a subgroup of $\mathrm{Aut}(S)$. It is known to contain an element of order 12. It follows from Table 3.3 that its conjugacy class is minimal of type $E_7(a_2)$. Thus $U_3(3)$ is realized as a minimal subgroup of a Del Pezzo surface of degree 2. Since it is rigid (note that the theory of elementary links applies without change to the case of positive characteristic), it is not conjugate in $\mathrm{Cr}_2(k)$ to a group of linear transformations.

The previous examples show that there are new conjugacy classes of finite subgroups in $\mathrm{Cr}_2(k)$ which are not realized in the case of characteristic 0.

Theorem 51. *Let G be a simple non-abelian wild subgroup of $\mathrm{Cr}_2(k)$.*

(i) *If G is conjugate to a group of projective transformations, then it is isomorphic to one of the following groups*

$$L_2(p^m)(p^m \neq 2,3),\ U_3(p^m), \mathfrak{A}_6, \mathfrak{S}_6(p=5),\ \mathfrak{A}_7(p=5).$$

(ii) *If G is not conjugate to a linear group, then it is isomorphic to one of the following groups:*

$$L_2(p^m)(p^m \neq 2,3), \mathfrak{A}_5,\ L_2(7)(p \neq 2),\ U(3,3), U_4(2).$$

Moreover, the groups $L_2(p^m)(p^m \neq 2,3)$ *are realized as minimal subgroups of a minimal ruled surface, the groups* $U_3(3), L_2(7)$ *are realized as minimal subgroups of a Del Pezzo surface of degree 2 in characteristic* $p = 3$ *and the latter group also when* $p = 7$*, the group* $\mathfrak{A}_5$ *is realized as a minimal subgroup of Del Pezzo surface of degree 4 (*$p = 2$*) and a Del Pezzo surface of degree 1 (*$p = 5$*), the group* $U_4(2)$ *is realized as a minimal subgroup of a cubic surface in characteristic* $p = 2$.

Proof. Suppose G is minimally regularized on a conic bundle $\pi : S \to \mathbb{P}^1$ with $t > 0$ singular fibres. Then the homomorphism $G \to \mathrm{Aut}(\mathbb{P}^1)$ is either injective or trivial. Assume that it is injective. Since G is minimal, the G-orbit of any component of a singular fibre does not consist of disjoint curves. Thus there exists an element $g \in G$ of necessarily even order switching two components R and R' of some singular fibre. Replacing g by some power we may assume that its order is $2^r > 1$. Obviously it fixes the intersection point $R \cap R'$. If $p = 2$, an element of order 2^r acts identically on the tangent space at a fixed point (because $\mathrm{GL}_2(k)$ has no elements of order $2^r, r > 1$). Thus it cannot switch the components. This shows that G cannot be minimal in the case $p = 2$. If $p > 2$, the proof of Lemma 5.6 from [11] applies in our case and shows that K cannot be trivial. So, we see that $G = K$ and hence G admits a non-trivial homomorphism $G \to 2^t$, contradicting the simplicity assumption.

If $t = 0$, i.e. S is a minimal ruled surface, then G embeds in the group of automorphisms of the base, hence is isomorphic to $L_2(p^m), p^m \neq 2,3$. Since it has no fixed point on $\mathbb{P}^1$, it cannot be conjugate to a linear group.

Next we assume that S is a Del Pezzo surface of degree d. If $S = \mathbb{P}^2$, applying Theorems 46 and 47, we obtain the groups from our list. If $S = \mathbb{P}^1 \times \mathbb{P}^1$, then G, being simple, does not switch the factors, and hence leaves invariant $\mathrm{Pic}(S)$. This contradicts the minimality of (S, G).

If $d = 6$, G is mapped isomorphically to a subgroup of $W(A_2 + A_1) \cong D_6$. So this case does not occur.

If $d = 5$, G is a subgroup of $W(A_4) \cong \mathfrak{S}_5$. Thus $G \cong \mathfrak{A}_5$. This case is realized in all characteristics. The group contains a minimal element of order 5, so it is minimal. We know that from the theory of elementary links this group is not conjugate to a linear group.

If $d = 4$, G is mapped isomorphically to a subgroup $\mathfrak{S}_5$ of $W(D_5)$. The only simple subgroup is $\mathfrak{A}_5$. Comparing the characters of elements of $W(D_5)$ acting on the root lattice of type D_5 isomorphic to $K_S^\perp$ we learn that the representation of $\mathfrak{A}_5$ on $K_S^\perp$ is not irreducible. Thus $\mathfrak{A}_5$

cannot act minimally on S. It acts leaving a set of 5 skew lines invariant, and hence is conjugate to a subgroup of $\mathrm{Aut}(\mathbb{P}^2)$.

If $d = 3$, G is isomorphic to a simple subgroup of $W(E_6)$, hence, it must be isomorphic to $U_4(2)$ or one of its subgroups $\mathfrak{A}_6$ or $\mathfrak{A}_5$ (see [9]). The first group is conjugate to a subgroup of a maximal subgroup of $W(E_6)$ isomorphic to $\mathfrak{S}_6$. The latter group is not minimal, it leaves a set of 6 skew lines invariant. The group $\mathfrak{A}_5$ is also non-minimal since it has no irreducible representations of dimension $6 = \dim K_S^{\perp}$. As we have seen in Example 49 the group $U_4(2)$ is realized in the case $p = 2$.

If $d = 2$, G is isomorphic to a simple subgroup of $W(E_7)$. Using [9] we find that it must be isomorphic to one of the following groups

$$\mathrm{U}_3(3), L_2(7), L_2(8), \mathrm{U}_4(2), \mathfrak{A}_m,\ 5 \le m \le 8.$$

It is known that $|-K_S|$ defines a degree 2 separable finite map $S \to \mathbb{P}^2$ ramified over a nonsingular plane quartic curve if $p > 2$ (see [10]). This implies that the group G is isomorphic to a subgroup of $\mathrm{PGL}_3(k)$. Comparing our list with the list from Theorem 47, we find that the groups $\mathfrak{A}_8$ and $\mathrm{U}_4(2)$ are not realized. Any of the groups $\mathfrak{A}_5, \mathfrak{A}_6, \mathfrak{A}_7$ has no irreducible 7-dimensional representation, so it can not be minimally realized. As we saw in Example 50 the group $\mathrm{U}_3(3)$ is realized. The group $L_2(7)$ is realized as a minimal wild group in characteristic 3 and 7. In characteristic 3 it is realized as a subgroup of $U_3(3)$. In characteristic 7 we use that $L_2(7) \cong O_3(7)$ and occurs as the automorphism group of the Klein plane quartic [15]. It can be lifted to a minimal automorphism group of a Del Pezzo surface of degree 2.

Assume $p = 2$. The equation of S in $\mathbb{P}(1, 1, 1, 2)$ is

$$t_3^2 + a_2(t_0, t_1, t_2)t_3 + a_4(t_0, t_1, t_2) = 0.$$

The conic $B : a_2(t_0, t_1, t_2) = 0$ is the branch curve of the cover. The group G is isomorphic to a subgroup of $\mathrm{Aut}(\mathbb{P}^2)$ leaving the conic B invariant. If B is not smooth, then G leaves each component of B invariant and becomes isomorphic to a subgroup of a solvable group, hence it is not simple. Thus B is smooth and G is isomorphic to a subgroup of the projective orthogonal group $\mathrm{PO}(3)$. The group has a fixed point q in the plane where all tangents to the conic meet. Thus, acting on the pencil of lines through this point, it becomes isomorphic to a subgroup of $\mathrm{PGL}_2(k)$. Applying Theorem 45 and the classification of subgroups of $W(E_7)$, we find that $G \cong L_2(8)$ or $L_2(4) \cong \mathfrak{A}_5$. The latter group does not admit an irreducible 7-dimensional representation so cannot act minimally on S. Let us show that $L_2(8)$ also cannot occur. The pre-image of the pencil of lines through q on S is a pencil of elliptic curves with two base points q' and q'' which are mapped to q under the map $S \to \mathbb{P}^2$. Blowing them up we obtain an elliptic surface $f : X \to \mathbb{P}^1$. The group G acts on the base of the fibration leaving invariant a set of ≤ 12 points corresponding to singular fibres. The set of these points is the locus of zeros of a binary homogeneous form of degree $d \leq 12$. Now we use that the algebra of invariants $k[T_1, \ldots, tT_n]^{\mathrm{SL}_n(\mathbb{F}_q)}$ is freely generated by polynomials of degrees $\frac{q^n-1}{q-1}$ and $q^n - q^i, i = 1, \ldots, n-1$ (see [26]).[2] Taking $q = 8$ and $n = 2$ we see that the degrees of these polynomials are larger than 12. This contradiction proves the assertion.

Finally let S be a Del Pezzo surface of degree 1. As we have explained in the previous section G is mapped to the automorphism group of the base of the corresponding elliptic fibration leaving invariant the set of points corresponding to singular fibres. Since the general fibre does not admit a simple non-abelian group of automorphisms we see that G is isomorphic to a subgroup of $\mathrm{PGL}_2(k)$ in its action on the base. Applying Theorem 45 we obtain that $G \cong L_2(q)$ for some $q = p^s$ or $G \cong \mathfrak{A}_5$. The latter case is excluded for the same reason as in the case $d = 2$. We also know from the previous case that the algebra of $L_2(q)$-invariant binary forms is generated by two polynomials of degrees $q+1$ and $q(q-1)$.

Assume $p > 3$. Then equation (15) shows that G leaves invariant polynomials of degree 6. This implies that $q = p = 5$. Also coefficient a_4 in the equation must be equal to zero (since it is also invariant). The locus of zeros of the invariant of degree $q+1$ is of course the set of points in $\mathbb{P}^1(\mathbb{F}_q)$. So the equation of a Del Pezzo surface S of degree 1 in characteristic 5 with $\mathrm{Aut}(S) \cong L_2(5) \cong \mathfrak{A}_5$ is

$$z^2 + x^3 + t_0t_1(t_0+t_1)(t_0+2t_1)(t_0+3t_1)(t_0+4t_1) = 0.$$

Assume $p = 3$. Then $q = 3$ and the group $L_2(3)$ is not simple. For completeness sake, or for the future use, we mention that the surface must be given by a unique equation of the form

$$y^2 + x^3 + a_4(t_0,t_1)x + a_6(t_0,t_1) = 0, \ a_4 \neq 0, \tag{15}$$

where a_4 and a_6 are the Dickson invariants.

Assume $p = 2$. Then the Weierstrass equation of an rational elliptic surface in characteristic 2 must be of the form

$$y^2 + xy + x^3 + a_2(t_0,t_1)x^2 + a_6(t_0,t_1) = 0, \ a_6 \neq 0, \tag{16}$$

[2] Usually this theorem is stated in the case $k = \mathbb{F}_q$, however the polynomials define a system of parameters over k and the product of their degrees is equal to the order of $\mathrm{SL}_n(q)$, so they freely generate the algebra of invariants over k.

or

$$y^2 + a_3(t_0,t_1)y + x^3 + a_4(t_0,t_1)x + a_6(t_0,t_1) = 0,\ a_3 \neq 0. \tag{17}$$

This implies that $q = 2$ and hence the group is $L_2(2) \cong \mathfrak{S}_3$ is not simple. Q.E.D.

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