BITANGENT SURFACES AND INVOLUTIONS OF QUARTIC SURFACES

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ABSTRACT. We study the congruence of bitangent lines of an irreducible surface in $\mathbb{P}^3$ in arbitrary characteristic, with special attention to quartic surfaces with rational double points and, in particular, Kummer quartic surfaces.

1. INTRODUCTION

Let $G_1(\mathbb{P}^3)$ be the Grassmannian of lines in $\mathbb{P}^3$. For a point in $G_1(\mathbb{P}^3)$, the corresponding line in $\mathbb{P}^3$ is called a *ray*. A surface $F \subset G_1(\mathbb{P}^3)$ is called a *congruence of lines*. Its algebraic class $[F]$ in the Chow ring $A^\bullet(G_1(\mathbb{P}^3))$ is determined by two numbers: the *order* m and the *class* n. The order (resp. class) is the number of lines on F passing through a general point (resp. contained in a general plane) in $\mathbb{P}^3$. We will call the pair (m, n) the *bidegree* of F. The number $m + n$ is equal to the degree of F in the Plücker embedding $G_1(\mathbb{P}^3) \hookrightarrow \mathbb{P}^5$ (cf. [22, Chapter X, §2], [3], [12]).

A surface X in $\mathbb{P}^3$ defines a congruence of lines in $\mathbb{P}^3$ equal to the closure of the set of lines that are tangent to X at two distinct points. It is classically known as the *bitangent surface*, and we denote it by $\mathrm{Bit}(X)$. For a general X, its order and class are also classically known, and we will reproduce the calculation in this paper with the novelty of taking care of the case of characteristic 2.

For example, it is known that, for a general surface of degree d, the bidegree of the surface $\mathrm{Bit}(X)$ is equal to $(\frac{1}{2}d(d-1)(d-2)(d-3), \frac{1}{2}d(d-2)(d^2-9))$ [21, Volume 1, p. 281]. In particular, the bidegree of the bitangent surface of a general quartic surface is equal to $(12, 28)$. Moreover, the surface is a smooth surface of degree 40 in $\mathbb{P}^5$. It is of general type with $p_g = 45$ and $K^2 = 360$ [26], [29].

Although, for a general smooth quartic X, the surface $\mathrm{Bit}(X)$ is irreducible, for special quartic surfaces, even smooth ones, it can be reducible. It is known that, if a quartic surface X is smooth and does not contain lines, then $\mathrm{Bit}(X)$ is smooth [1]. However, the surface $\mathrm{Bit}(X)$ could be reducible even when X is smooth but contains lines. For example, a general Cayley's symmetroid quartic surface admits a smooth quartic model with reducible bitangent surface. One of its irreducible components is a Reye congruence of bidegree $(7, 3)$ [2], [5, 7.4].

Although admitting some mild singular points in X doesn't change the bidegree, it may drastically change the surface $\mathrm{Bit}(X)$, it even may become reducible. For example, when X is realized as the focal surface of a congruence of lines of bidegree $(2, n)$ (the focal surface of a congruence F of lines of order $n > 1$ is a surface in $\mathbb{P}^3$ such that all rays of F are its bitangents, see, for example, [22, Chapter X, §2]), the surface $\mathrm{Bit}(X)$ is always singular although its singular points are

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ordinary double points. In the case $n = 2$, the congruence of lines is a del Pezzo surface of degree 4 and the focal surface is the famous 16-nodal Kummer quartic surface X [16], and the number of irreducible components of $\mathrm{Bit}(X)$ is equal to 22, six of them are of bidegree $(2,2)$ and 16 of them are of bidegree $(0,1)$. They are planes of lines contained in one of the 16 planes intersecting X along a double conic, a *trope plane*.

The paper is addressing the problem of decomposition of $\mathrm{Bit}(X)$ into irreducible components for quartic surfaces over an algebraically closed field of arbitrary characteristic p. The case $p = 2$ and X being a Kummer quartic surface is of special importance to us. Recall that smooth curves C of genus 2 in characteristic two are divided into three types, that is, *ordinary curves*, *curves of* 2*-rank* 1 and *supersingular curves*. The Jacobian $J(C)$ has four, two or one 2-torsion point(s) accordingly. The quotient surface $X = J(C)/(\iota)$ by the negation involution ι can be embedded in $\mathbb{P}^3$ with the image isomorphic to a quartic surface, the *Kummer quartic surface* associated to C. Instead of 16 nodes in the case $p \neq 2$, the Kummer surface X has four rational double points of type D_4, two rational double points of type D_8, or an elliptic singularity of type $④^1_{0,1}$ in the sense of Wagreich [27]. The number of trope-planes also drops; it is equal to $4, 2$ and 1. We will show that $\mathrm{Bit}(X)$ consists of the following irreducible components as in Table 1.

	bidegree	Number
$p \neq 2$	$(2,2)$ $(0,1)$	6 16
$p = 2$, ordinary	$(1,1)$ $(0,1)$	3 4
$p = 2$, 2-rank 1	$(1,1)$ $(0,1)$	2 2
$p = 2$, supersingular	$(1,1)$ $(0,1)$	1 1

TABLE 1. Irreducible components of $\mathrm{Bit}(X)$ for Kummer quartic surfaces X

Observe that $(12, 28)$ is a multiple of the sum of the bidegrees of irreducible components of the bitangent surfaces of ordinary Kummer surfaces in characteristic 2. We do not know any explanation of this fact.

One reason for the dropping of the order of $\mathrm{Bit}(X)$ in characteristic two is explained by the fact that the discriminant polynomial of a binary form is a square, and another is that irreducible components of bidegree $(1,1)$ appear instead of bidegree $(2,2)$. On the other hand, the class, being equal to the number of bitangent lines to a general plane section H of X, also drops. It is equal to $7, 4, 2$, or 1 if the Hasse–Witt invariant of H is equal to $3, 2, 1$, or 0, respectively [20], [24].

On the way, we discuss different kinds of involutions on quartic surfaces and their relationship to Cremona transformations. For example, an irreducible component of the bitangent surface $\mathrm{Bit}(X)$ with non-zero order and non-zero class defines a birational involution of X with the pairs of tangency points of bitangent lines as its general orbits.

The plan of the paper is as follows. In Section 2, we will reproduce Salmon's proof for the formula of the bidegree of the bitangent surface of a general surface X of degree d in characteristic

0 (Theorem 2.2) and show it holds also for smooth surfaces or surfaces with rational double points (Corollary 2.5). In Section 3, we discuss the classical results about the bitangent surface of a Kummer quartic surface in characteristic $p = 0$. They extend without change to the case $p \neq 2$. In Section 4, we discuss birational involutions of quartic surfaces. In Section 5, we give a proof (due to G. Kemper) of the fact that the discriminant of a binary form in the case $p = 2$ is a square (Proposition 5.3). This reduces the order of congruences of bitangent lines to half. Finally, in the last three sections 6, 7 and 8 we determine the bidegrees of congruences of bitangent lines of Kummer quartic surfaces in characteristic two according to ordinary, 2-rank 1 and supersingular, respectively (Theorems 6.5, 7.1, 8.1).

Throughout the paper, we assume that the base field $\Bbbk$ is an algebraically closed field of characteristic $p \geq 0$.

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2. Generalities on the surface of bitangent lines

Let X be a normal surface of degree $d \geq 4$. We keep this assumption during the whole paper. A line ℓ in $\mathbb{P}^3$ is called a *bitangent line* if either it is contained in X or X cuts out in ℓ a divisor $D = 2a + 2b + D'$.

Let $\mathrm{Bit}(X) \subset G_1(\mathbb{P}^3)$ be the variety of bitangent lines. If $\mathrm{char}(\Bbbk) \neq 2$ we will show in this section that each irreducible component of $\mathrm{Bit}(X)$ is a surface. In section 5, we extend this result to the case where $\mathrm{char}(\Bbbk) = 2$. Considered as a surface in $G_1(\mathbb{P}^3)$ it has the bidegree (m, n). Its degree in the Plücker embedding is equal to $m + n$.

Recall that $G_1(\mathbb{P}^3)$ contains two types of planes, one is called an α-*plane* consisting of lines passing through a point of $\mathbb{P}^3$ and other is called a β-*plane* consisting of lines contained in a plane of $\mathbb{P}^3$.

The following fact is, of course, well known, but to be sure that it is true without assumption on the characteristic, we supply a proof.

Proposition 2.1. *An irreducible integral surface S in $G_1(\mathbb{P}^3)$ of bidegree $(m, 0)$ (resp. $(0, n)$) is an α-plane (resp. β-plane). In particular, $m = 1$ (resp. $n = 1$).*

Proof. Passing to the dual congruence of lines in the dual space of $\mathbb{P}^3$, it is enough to prove that any irreducible congruence of bidegree $(0, n)$ is a β-plane. Let $Z_S = \{(x, \ell) \in \mathbb{P}^3 \times S : x \in \ell\}$ be the restriction of the tautological projective bundle over $G_1(\mathbb{P}^3)$ to S. Since S is irreducible, Z_S is an irreducible 3-fold. Since $m = 0$, its image Y under the first projection is an irreducible subvariety of dimension 1 or 2. Suppose $\dim Y = 1$. Then, the fibers of $Z_S \to Y$ are α-planes that sweep at least one plane, and hence S contains an α-plane. So, Y cannot be a curve. Thus, Y is a surface. The image of the fiber of $Z_S \to S$ over a point $y \in Y$ in S is a curve contained in the α-plane $\Omega(y)$ of lines through y. The corresponding rays sweep a cone with vertex at y contained in Y. Thus, Y is a cone at each of its points. Since Y is reduced, this can happen only if Y is a plane. So, all rays of S are contained in a plane, and hence, S is a β-plane. □

Proposition 2.2. *Assume* $\mathrm{char}(\Bbbk) = 0$ *and let* $X = V(F)$ *be a normal surface of degree* $d \geq 4$. *Then,* $\mathrm{Bit}(X)$ *is a congruence of lines of bidegree* (m, n), *with*

$$1 \leq m \leq \frac{1}{2}d(d-1)(d-2)(d-3), \quad n = \frac{1}{2}d(d-2)(d^2-9).$$

Proof. We follow Salmon [21, Volume 2, p. 281]. Let q be a general point in $\mathbb{P}^3$ and let $\ell = \langle q, q' \rangle$ be the line containing q and tangent to X at some point $q' = [x_0, y_0, z_0, w_0]$. Without loss of generality, we may assume that $q = [0,0,0,1]$ and

$$F = w^d + A_1 w^{d-1} + \cdots + wA_{d-1} + A_d,$$

where A_k are homogeneous forms of degree k in x, y, z.

Plugging in the parametric equation $[s,t] \mapsto [sv + tv']$, where $[v] = q, [v'] = q'$, we get

$$f := F(sv + tv') = (s + tw_0)^d + \sum_{i=1}^{d} t^i A_i(x_0, y_0, z_0)(s + tw_0)^{d-i}.$$

By polarizing, we can rewrite it in the form

$$f = \sum_{k+m=d} s^k t^m P_{v^k}(F)(v'),$$

where $P_{v^k}(F)(v')$ is the value of the totally polarized symmetric multilinear form defined by F at $(v, \ldots, v, v', \ldots, v')$. Geometrically, following the notation from [3, Chapter 1], the locus of zeros of $P_{v^k}(F)(v')$ with fixed v is the k-th polar $P_{q^k}(V(F))$ of the hypersurface $V(F)$. Recall from loc. cit. that the first *polar hypersurface* $P_q(V(F))$ is the locus of zeros of $\sum a_i \frac{\partial F}{\partial x_i}$, where $q = [a_0, \ldots, a_n]$. Since $q' \in X$, we get $P_{v^0}(F)(v') = F(v') = 0$. Moreover, because ℓ is tangent to X at q', we obtain $P_v(F)(v') = 0$. Thus, we can rewrite

$$f = s^2 g_{d-2}(s,t).$$

The line ℓ is tangent to X at some other point if and only if the binary form g_{d-2} of degree $d-2$

$$g_{d-2}(s,t) = \sum_{k=0}^{d-2} s^k t^{d-2-k} P_{v^{k+2}}(F)(v')$$

has a multiple root.

Recall that the discriminant polynomial $D(a_0, \ldots, a_n)$ of a binary form $\sum_{i=0}^n a_i s^{n-i} t^i$ of degree n is a homogenous polynomial of degree $2(n-1)$. The polynomial $D(a_0, \ldots, a_n)$ is also a bi-homogeneous polynomial in variables $a_0, \ldots, a_n$ of bidegree $(n(n-1), n(n-1))$ with respect to the action of $\mathbb{G}_m^2$ via

$$(a_0, \ldots, a_n) \mapsto (\lambda^n a_0, \lambda^{n-1}\mu a_1, \ldots, \lambda\mu^{n-1}a_{n-1}, \mu^n a_n).$$

In other terms, it is a weighted homogeneous polynomial of degree $2(n-1)$ with the weights of the a_k's equal to k.

Applying this to the discriminant of the polynomial $g_{d-2}(s,t)$ we obtain that its discriminant $D(P_{v^d}(F)(v'), \ldots, P_{v^2}(F)(v'))$ is a polynomial of degree $\frac{1}{2}(d-2)(d-3)$ in x_0, y_0, z_0 and hence we obtain the locus of points q' such that the line $\langle q, q'\rangle$ is tangent to X at two points including q' is contained in the intersection of hypersurfaces of degrees $d, d-1$, and $(d-2)(d-3)$.

This implies that the expected number of bitangent lines passing through q is equal to $\frac{1}{2}d(d-1)(d-2)(d-3)$.

The class n of $\mathrm{Bit}(X)$ is equal to the number of bitangent lines to a general plane section H of X. Since X is normal, the hyperplane section is a smooth plane curve of degree d. Their number is well-known classically, and it is equal to $\frac{1}{2}d(d-2)(d^2-9)$ (see [3, 5.5.1, formula (5.33)]).

Since $\mathrm{char}(\Bbbk) = 0$, the discriminants of the polynomials g_{d-2} are not equal to zero. This shows that $m \geq 1$.

To see that $\mathrm{Bit}(X)$ is a congruence of lines, i.e., each of its irreducible components is a surface, we consider the incidence variety

$$M = \{(q, \ell) \in \mathbb{P}^3 \times G_1(\mathbb{P}^3) : \ell \in \mathrm{Bit}(X), q \in \ell\}.$$

The fiber of the first projection consists of bitangent lines ℓ passing through q. By the above, it consists of finitely many lines. This implies that each irreducible component of M is of dimension 3. Since the fibers of the second projections are lines, the image of each irreducible component of M is a surface. □

We denote by $\mathrm{Flex}(X) \subset G_1(\mathbb{P}^3)$ the variety of lines in $\mathbb{P}^3$ that are either contained in X or intersect X at some point with multiplicity ≥ 3. It intersects $\mathrm{Bit}(X)$ at the set of lines contained in X and the set of bitangents that intersect X at one point.

Proposition 2.3. *The expected bidegree of the congruence of lines* $\mathrm{Flex}(X)$ *is equal to* (m, n)*, where* $m \leq d(d-1)(d-2)$ *and* $n \leq 3d(d-2)$.

Proof. Following the proof of the previous proposition, we find that the order m of $\mathrm{Flex}(X)$ is equal to the degree of the reduced complete intersection $X \cap P_q(X) \cap P_{q^2}(X)$. We have $m \leq d(d-1)(d-2)$. The class n of $\mathrm{Flex}(X)$ is equal to the number of flex tangents in a general plane section H of X. It is equal to the number of intersection points of a plane section $H \cap X$ with its Hessian curve of degree $3(d-2)$. Clearly, $m \leq 3d(d-2)$. □

Let $\mathrm{pr}_q : X \to \mathbb{P}^2$ be the projection of X to a general plane. The ramification curve $\mathrm{Rm}(q)$ of the projection is equal to $X \cap P_q(X)$. Its degree is equal to $d(d-1)$. Using the well-known formula for the arithmetic genus of a complete intersection of hypersurfaces, we obtain that its arithmetic genus is equal to $1 + \frac{1}{2}d(d-1)(2d-5)$. The branch curve $B(q)$ of pr_q is equal to the projection of $\mathrm{Rm}(X)$, hence it is of degree $d(d-1)$. If $\mathrm{Rm}(q)$ is smooth, then it is isomorphic to the normalization of $B(q)$. A bitangent line containing the center of the projection is a secant line of $\mathrm{Rm}(X)$, hence its projection has a singular point of $B(q)$ with at least two different branches. For a quartic surface, it is an ordinary node. A flex line from $\mathrm{Flex}(X)$ containing q is a tangent line to $P_q(X)$. Its projection is a singular point with at least two equal branches.

We use the following theorem due to Valentine and Viktor Kulikov [15, Theorem 0.1]:

Theorem 2.4. *Assume that* $\mathrm{char}(\Bbbk) = 0$ *and* X *has only rational double points as singularities. Let* $\mathrm{pr}_q : X \to \mathbb{P}^2$ *be a general projection of* X*. Then,* $\mathrm{pr}_q^*(B(q)) = 2\mathrm{Rm}(q) + C$*, where both* $\mathrm{Rm}(q)$ *and* C *are reduced. The projection of a singular point of type* A_n, D_n, E_n *is a simple* *singular point of $B(q)$ of type a_n, d_n, e_n, other singular points of $B(q)$ are ordinary nodes and ordinary cusps. They are the projections of the bitangents lines and flex lines of X.*

Corollary 2.5. *Assume that $\mathrm{char}(\Bbbk) = 0$ and X is smooth or has only ordinary double points. Then, the bidegrees of $\mathrm{Bit}(X)$ and $\mathrm{Flex}(X)$ are equal to $(\frac{1}{2}d(d-1)(d-2)(d-3), \frac{1}{2}d(d-2)(d^2-9))$ and $(d(d-1)(d-2), 3d(d-2))$, respectively.*

Proof. We choose q general enough such that no bitangent or flex line through q passes through a singular point of X. The ramification curve $\mathrm{Rm}(q)$ of the projection pr_q is equal to the intersection $X \cap P_q(X)$ of X with its polar with respect to q. We know that the jacobian ideal of an ordinary double point $x \in X$ is the maximal ideal $\mathfrak{m}_{X,x}$. This easily implies that $P_q(X) = V(\sum_{i=0}^{3} a_i \frac{\partial F_d}{\partial x_i})$ is smooth at x and, under the projection pr_q, the double points of $\mathrm{Rm}(q)$ are mapped bijectively to ordinary double points of $B(q)$ different from the nodes and cusps coming from bitangent and flex lines.

Since $\mathrm{Rm}(q)$ is a complete intersection of degrees $(d, d-1)$, its arithmetic genus p_a is equal to $1 + \frac{1}{2}d(d-1)(2d-5)$. The projection pr_q defines a birational isomorphism between the curves $\mathrm{Rm}(q)$ and $B(q)$. The geometric genus g of $B(q)$ is equal to $\frac{1}{2}(d(d-1)-1)(d(d-1)-2) - \delta - \kappa - \delta_0$, where δ is the number of bitangent lines, κ is the number of flex lines, and δ_0 is the number of nodes of $\mathrm{Rm}(q)$. This gives an inequality

$$p_a = 1 + \frac{1}{2}d(d-1)(2d-5)$$
$$= \frac{1}{2}(d(d-1)-1)(d(d-1)-2) - \frac{1}{2}d(d-1)(d-2)(d-3) - d(d-1)(d-2)$$
$$\geq g \geq \frac{1}{2}(d(d-1)-1)(d(d-1)-2) - \delta - \kappa$$

that implies the inequality

$$\delta + \kappa \geq \frac{1}{2}d(d-1)(d-2)(d-3) + d(d-1)(d-2). \tag{2.1}$$

Since δ is equal to the number of bitangents of X dropped from q, we obtain δ is the order of $\mathrm{Bit}(X)$. Similarly, we get that κ is the order of the $\mathrm{Flex}(X)$. Applying Propositions 2.2 and 2.3, we obtain $\delta \leq \frac{1}{2}d(d-1)(d-2)(d-3)$ and $\kappa \leq d(d-1)(d-2)$. It follows from (2.1) that $\delta = \frac{1}{2}d(d-1)(d-2)(d-3)$ and $\kappa = d(d-1)(d-2)$.

In this proof, we silently assumed that the curves $\mathrm{Rm}(q)$ and $B(q)$ are irreducible. One can treat the reducible case in a similar manner by dividing the bitangents and flex lines into subsets corresponding to irreducible components of the curves. We leave it to the reader to finish the proof. □

3. Quartic surfaces: $p \neq 2$

In the special case $d = 4$, we expect that, for a smooth or nodal quartic surface in characteristic zero, the bidegree (m, n) of $\mathrm{Bit}(X)$ is equal to $(12, 28)$. Let S be an irreducible component of $\mathrm{Bit}(X)$ and let $q_S : Z_S \to S$ be the restriction of the tautological line bundle $q : Z \to G_1(\mathbb{P}^3)$ to the congruence S. The pre-image $\tilde{X}$ of X under the projection $p_S : Z_S \to \mathbb{P}^3$ is a double cover of S defined by the projection q_S:

$$\begin{tikzcd} & Z_S \arrow[dl, "p_S"'] \arrow[dr, "q_S"] & \\ \mathbb{P}^3 & & S. \end{tikzcd}$$

Thus, any irreducible component S of $\mathrm{Bit}(X)$ with non-zero order and non-zero class defines a birational involution of the surface $\tilde{X}$ with the quotient isomorphic to S. In the case when the projection p_S restricts to a birational isomorphism $\tilde{X} \to X$, we obtain a birational involution σ_S of X. The set of fixed points of σ_S is equal to the pre-image of the locus of points on X such that there exists a line intersecting X at this point with multiplicity 4.

Example 3.1. It follows from Kummer's classification of congruences of order 2 without fundamental curves that the moduli space of quartic surfaces with $16 \geq \mu \geq 11$ nodes contains an irreducible component such that a general surface from this component has reducible congruence $\mathrm{Bit}(X)$. Some of the irreducible components are confocal congruences, i.e., congruences of the same bidegree that share the same focal surface (for example, as we will explain below, the first case in the following Table 2 is a quartic del Pezzo surface). The number of irreducible components and their bidegree are given in Table 2 below (see [16], [25, Art. 348]).

μ	bidegree	Number
16	$(2,2)$	6
	$(0,1)$	16
15	$(2,3)$	6
	$(0,1)$	10
14	$(2,4)$	4
	$(0,1)$	6
	$(4,6)$	1
13	$(2,5)$	3
	$(0,1)$	3
	$(6,10)$	1
12_{I}	$(2,6)$	2
	$(0,1)$	1
	$(8,15)$	1
12_{II}	$(2,6)$	3
	$(6,10)$	1
11	$(2,7)$	1
	$(10,21)$	1

TABLE 2. Irreducible components of $\mathrm{Bit}(X)$

The first case is a Kummer quartic surface X. It follows that X admits six involutions with quotient isomorphic to a congruence of bidegree $(2,2)$. This is a quartic del Pezzo surface embedded in $G_1(\mathbb{P}^3)$ via its anti-canonical linear system. Each pair from the six congruences intersects at four points. The branch curve has $28 = 16 + 12$ ordinary nodes and 24 cusps. Its geometric genus is equal to 3. The ramification curve $\mathrm{Rm}(q)$ is a curve on X of degree 12 with 16 nodes, its geometric genus is also equal to 3.

We have already mentioned in the introduction that a Kummer surface admits 16 planes cutting the surface along a smooth conic. Recall that a Kummer quartic surface is the quotient of the Jacobian of a smooth curve of genus two by the inversion automorphism. The sixteen double conics (tropes) are the images of the theta divisor and its translations by sixteen 2-torsion points. This is a classical, well-known fact that can be found in any treatment of Kummer surfaces (see, for example, [3, 10.3], [10]). Let us explain how to see the six irreducible components of bidegree $(2,2)$. Recall that X admits a smooth model $\tilde{X}$ as a surface of degree 8 in $\mathbb{P}^5$ (see [3, 10.3.3]). The surface $\tilde{X}$ is a complete intersection of three quadrics in $\mathbb{P}^5$ which is a $K3$ surface. One can choose projective coordinates such that $\tilde{X}$ is given by the following equations:

$$\sum_{i=1}^{6} z_i^2 = \sum_{i=1}^{6} a_i z_i^2 = \sum_{i=1}^{6} a_i^2 z_i^2 = 0, \tag{3.1}$$

where $a_1, \ldots, a_6$ are distinct constants. The projective transformations

$$[z_1, z_2, z_3, z_4, z_5, z_6] \mapsto [\pm z_1, \pm z_2, \pm z_3, \pm z_4, \pm z_5, \pm z_6]$$

induce automorphisms of $\tilde{X}$ which generate a 2-elementary group of order 2^5. The group of such automorphisms contains a subgroup of index 2 that consists of the identity and 15 projective involutions that restrict to symplectic automorphisms of $\tilde{X}$.[1] They are the involutions that change 2 or 4 signs at the coordinates and non-symplectic involutions are the ones that change 1, 3 or 5 signs. The quotient surface of $\tilde{X}$ by a symplectic automorphism is a K3 surface with rational double points. The remaining sixteen involutions are decomposed into two sets. One set consists of ten involutions that change exactly three signs at the coordinates. They are fixed-point-free with the orbit space isomorphic to an Enriques surface. Another set consists of six involutions g_i that change only one coordinate with the orbit space isomorphic to a del Pezzo surface of degree 4. These are the involutions we are interested in.

The first quadric $V(\sum z_i^2)$ can be identified with the Grassmannian quadric $G_1(\mathbb{P}^3)$ written in Klein coordinates z_i corresponding to six apolar line complexes (see, for example, [3, 10.2]). It is known that any automorphism of $G_1(\mathbb{P}^3)$ comes from a projective collineation or correlation. The involutions g_i preserve the apolar line complexes $V(z_i)$, and, hence, come from a correlation. They transform points to planes, and therefore, act on $G_1(\mathbb{P}^3)$ by transforming the plane of lines through a point (an α-plane) to the plane of lines contained in a given plane (a β-plane).

Lemma 3.2. *Let g_i be considered as a birational involution of X. Then, the closure of lines spanned by the orbits of g_i is an irreducible component of* $\mathrm{Bit}(X)$.

[1] This means that the involution leaves invariant a non-zero regular 2-form on the surface, or, equivalently, the set of its fixed points is non-empty and finite.

Proof. First, we recall the relation between $X, \tilde{X}$ and the quadratic line complex $Y = V(\sum a_i z_i^2) \cap G_1(\mathbb{P}^3)$ (see, e.g., [3, §10.3.3]). For a point $x \in \mathbb{P}^3$, let $\Omega(x) \subset G_1(\mathbb{P}^3)$ be the α-plane consisting of lines passing through x, and for a plane $\Pi \subset \mathbb{P}^3$, denote by $\Omega(\Pi)$ the β-plane consisting of lines lying on Π. The Kummer quartic surface X is the set of $x \in \mathbb{P}^3$ such that $\Omega(x) \cap Y$ is a singular conic in $\Omega(x)$. The surface $\tilde{X}$ sits in $G_1(\mathbb{P}^3)$ as a singular surface of the quadratic line complex, that is, it parameterizes the lines that are singular points of the intersection of Y with α-planes. We can identify each point $x \in X$ with the α-plane $\Omega(x)$. An automorphism g_i acts on X by sending $\Omega(x)$ to $\Omega(\Pi)$. Let $x \in X$ be a general point and let $\Pi = \mathbb{T}_x(X)$ be the embedded tangent space of X at x. As we explained above, g_i interchanges the two families $\{\Omega(x)\}_{x \in \mathbb{P}^3}$, $\{\Omega(\Pi)\}_{\Pi \subset \mathbb{P}^3}$ of planes, and hence,

$$g_i(\Omega(x)) = \Omega(\Pi'), \quad g_i(\Omega(\Pi)) = \Omega(x')$$

for some $x' \in \mathbb{P}^3$ and $\Pi' \subset \mathbb{P}^3$. Since g_i acts on X as a birational automorphism, $x' \in X$ and $\Pi' = \mathbb{T}_{x'}(X)$. Since any point of $H_i = V(z_i)$ is fixed by g_i,

$$\emptyset \neq \Omega(x) \cap H_i = g_i(\Omega(x) \cap H_i) = \Omega(\Pi') \cap H_i,$$

$$\emptyset \neq \Omega(\Pi) \cap H_i = g_i(\Omega(\Pi) \cap H_i) = \Omega(x') \cap H_i.$$

These imply that $x \in \Pi'$ and $x' \in \Pi$. Since the line $\ell = \langle x, x' \rangle \subset \Pi$ and $\ell \subset \Pi'$, ℓ is tangent to X at x and x', that is, it is a bitangent line of X. Thus, ℓ can be identified with the pair $\{x, x' = g_i(x)\}$, that is, a point of the orbit space $\tilde{X}/(g_i)$. □

The quotient surface $\tilde{X}/(g_i)$ is given by

$$\sum_{j \neq i} (a_j - a_i) z_j^2 = \sum_{j \neq i} (a_j^2 - a_i^2) z_j^2 = 0,$$

which is a quartic del Pezzo surface $\mathcal{D}$.

Also, note that the fixed locus of the involution g_i is a canonical curve C of genus 5 given as the intersection of three diagonal quadrics. The curve C has a special group of automorphisms isomorphic to the 2-elementary group 2^4. It is classically known as a *Humbert curve* of genus 5. The image B of C in the quartic del Pezzo surface $\mathcal{D}$ belongs to $|-2K_{\mathcal{D}}|$. The curve B is the branch curve of the double cover $\tilde{X} \to \mathcal{D}$. It has the distinguished property that any of the 16 lines on $\mathcal{D}$ splits into a pair of lines on $\tilde{X}$.

4. Birational involutions of a quartic surface

Let σ be a birational involution of a quartic surface X. The closure of lines spanned by orbits of σ is an irreducible congruence $S(\sigma)$ of lines. Fix a general point $P \in \mathbb{P}^3$, the order of $S(\sigma)$ is equal to the cardinality of the set $\{x \in \mathbb{P}^3 : P \in \langle x, \sigma(x) \rangle\}$.

Suppose that σ is the restriction of a Cremona involution T. Then, the set of points $x \in \mathbb{P}^3$ such that $P \in \langle x, T(x) \rangle$ is classically called an *isologue* of T and P is its center. It is given by the condition that

$$\operatorname{rank} \begin{pmatrix} a & b & c & d \\ x_0 & x_1 & x_2 & x_3 \\ y_0 & y_1 & y_2 & y_3 \end{pmatrix} < 3,$$

where $P = [a,b,c,d], x = [x_0,x_1,x_2,x_3], T(x) = [y_0,y_1,y_2,y_3]$. It is expected to be a curve of degree $2k+1$, where k is the algebraic degree of T [11, p. 175]. However, it also includes the locus of fundamental points of T, as well as the closure of the locus of fixed points of T. The class of $S(\sigma)$ is expected to be equal to the degree of $\Pi \cap T(\Pi) \cap X$ for a general plane Π, which is $4k$.

Let σ be a birational involution of a quartic surface. The rational map

$$\phi_\sigma : X/(\sigma) \dashrightarrow G_1(\mathbb{P}^3), x \mapsto \langle x, \sigma(x)\rangle$$

is of degree 1 or 2, since a general line intersects X with multiplicity 4. There are three possible scenarios:

(i) ϕ_σ is of degree 2;
(ii) ϕ_σ is of degree 1, a general line $\ell_x = \langle x, \sigma(x)\rangle$ intersects X at two points and two fixed points of σ;
(iii) ϕ_σ is of degree 1, a general line ℓ_x is a bitangent line of X.

In the last case, we say that σ is a *bitangent involution*.

Let $S(\sigma) \subset G_1(\mathbb{P}^3)$ be the congruence of lines defined as the closure of the image of the rational map ϕ_σ.

Example 4.1. Here, we give an example of an involution of Type (ii). Let σ be the restriction of a projective involution T in $\mathbb{P}^3$. Suppose that $p \neq 2$ and the fixed locus of T is the union of two skew lines ℓ_1, ℓ_2. It is clear that the lines ℓ_x are invariant lines of T. Hence, each ℓ_x has two fixed points on it, one on ℓ_1 and one on ℓ_2. This shows that the congruence $S(\sigma)$ is of bidegree $(1,1)$ isomorphic to a smooth quadric. It is known that the set X^σ of fixed points of an involution σ of a K3 surface consists of 8 isolated fixed points if σ is a symplectic involution or it does not have isolated fixed points if σ is an anti-symplectic involution. This implies that T intersects X either at ≤ 8 points (one can show that the equality holds since none of the lines can pass through a singular point, however, we will not need this fact) or T is contained in X. In the former case, each line ℓ_x has two fixed points, one on ℓ_1 and another on ℓ_2. Thus, the degree of ϕ_σ is equal to one.

In the second case, assume X is smooth and X^σ consists of two lines ℓ_1 and ℓ_2. Then, $Y = X/(\sigma)$ is also smooth. The cover $X \to Y = X/(\sigma)$ is ramified over $\ell_1 + \ell_2$, and its branch curve is the union of two disjoint smooth rational curves C_1, C_2 with self-intersection -4. By the adjunction formula, $|-K_Y| = \emptyset, |-2K_Y| = \{C_1 + C_2\}$. Thus, $|-K_Y| = \emptyset$ and $|-2K_Y| \neq \emptyset$, i.e., Y is a Coble surface obtained by blowing up 10 points on a smooth quadric Q (see [5, §9.1]), the eight intersection points of two nodal quartic curves $\bar{C}_1, \bar{C}_2$ of bidegree $(2,2)$ and the two nodes. The pre-images of the ten exceptional curves on Y in X are ten invariant lines lying in X that contain infinitely many orbits of σ.

Here is a concrete example: let $X = V(F)$, where

$$F = x_3^3x_0 + x_3^3x_1 + x_3x_0^3 + x_3x_0x_2^2 + x_3x_1^3 + x_0^3x_2 + x_0^2x_1x_2 + x_0x_2^3 + x_1^3x_2 + x_1x_2^3$$

and $\sigma : [x_0,x_1,x_2,x_3] \mapsto [-x_0,-x_1,x_2,x_3]$. The set of fixed points of σ consists of two skew lines $\ell_1 = V(x_0,x_1)$ and $\ell_2 = V(x_2,x_3)$. Plugging in the parametric equation $[(s-t)u_0,(s-t)u_1,(s+t)u_2,(s+t)u_3]$ of a line ℓ_x, where $x = [u_0,u_1,u_2,u_3]$ is not a fixed point of σ, we obtain the expression

$$(s^2-t^2)(F(u_0,u_1,u_2,u_3)s^2+2G(u_0,u_1,u_2,u_3)st+F(u_0,u_1,u_2,u_3)t^2),$$

where

$$G=-u_0^3u_2-u_0^3u_3-u_0^2u_1u_2+u_0u_2^3+u_0u_2^2u_3+u_0u_3^3-u_1^3u_2-u_1^3u_3+u_1u_2^3+u_1u_3^3.$$

Thus, the union of lines contained in X and intersecting the skew lines ℓ_1 and ℓ_2 is equal to $X\cap V(G')$, where G' is obtained from G by replacing u_i with x_i.

We check (using Maple software) that if $\mathrm{char}(\Bbbk)\neq 5,643$ the surfaces $V(F)$ and $V(G')$ intersect transversally at 10 lines and intersect at the two lines of fixed points with multiplicity 3. This shows that $Y=X/(\sigma)\to S(\sigma)=Q$ is the blowing down of ten (-1)-curves on the Coble surface Y.

Example 4.2. Here is an example of a bitangent involution. We refer the reader to [2] or [5, §7.4]. Assume $\mathrm{char}(\Bbbk)\neq 2$. Let W be a general web of quadrics in $\mathbb{P}^3$ and let $\mathcal{D}(W)\subset W$ be the set of singular quadrics in W, classically known as a Cayley quartic symmetric (see [2, Def. 2.1.1]). Choose a basis $\{q_0,q_1,\ldots,q_3\}$ of W and let $q(\lambda)=\sum_i\lambda_iq_i\in W$ for $\lambda=(\lambda_i)$ a coordinate of W. Here, we identify a quadric in $\mathbb{P}^3$ and a symmetric 4×4-matrix. Then, $\mathcal{D}(W)$ is a quartic surface in $W\cong\mathbb{P}^3$ defined by $\det(q(\lambda))=0$. Another attribute of a web of quadrics is the Steinerian (or Jacobian) surface in $\mathbb{P}^3$, the locus of singular points of quadrics from W. This surface is also a quartic surface but it lies in the original $\mathbb{P}^3$ but not in W (see [3, 1.1.6]).

A Reye line is a line in $\mathbb{P}^3$ which is contained in a pencil from W. It is known that the set of Reye lines forms an irreducible congruence $\mathrm{Rey}(W)$ of bidegree $(7,3)$ [2, Prop. 3.4.2], a *Reye congruence*, and each Reye line is bitangent to the Steinerian surface X of W.

The surface X admits a fixed-point-free involution σ such that $S(\sigma)=\mathrm{Rey}(W)$. The involution is defined by $x\mapsto\cap_{Q\in W}P_x(Q)$, where $P_x(Q)$ is the first polar of Q with pole at x.

One can also consider the surface of bitangents $\mathrm{Bit}(\mathcal{D}(W))$ of the quartic symmetroid $\mathcal{D}(W)$. The generality assumption on W implies that $\mathcal{D}(W)$ does not contain lines. So, $\mathrm{char}(\Bbbk)=0$, Corollary 2.5 implies that the bidegree of $\mathrm{Bit}(\mathcal{D}(W))$ is equal to $(12,28)$. It follows from [5, Theorem 7.4.7] that it is an irreducible surface with normalization isomorphic to the Reye congruence $\mathrm{Rey}(W)$. So, $\mathrm{Rey}(W)$ admits two birational models as congruences of bidegree $(7,3)$ and $(12,28)$.

A point of a congruence of lines is called a fundamental point (in classical terminology, a singular point) if the set of rays passing through this point contains a one-dimensional component. In our case of the congruence $\mathrm{Bit}(X)$ a fundamental point must be a singular point of X. For example, if X is a Kummer quartic, the rays through its singular point sweep a trope. It is classically known that all rays of a congruence of lines of bidegree (m,n) with only isolated fundamental points are tangent to the focal surface Φ of the congruence of degree $2m+2g-2$, where g is the genus of a general hyperplane section of the congruence (see [25], [12], or [4]). Since $m=12$, the degree is ≥ 22. However, Φ could be reducible, and X is its irreducible component. So, the existence of a bitangent involution on X implies that the focal surface of $\mathrm{Bit}(X)$ is reducible.

Table 2 gives examples of families of quartic surfaces which admit bitangent involutions with one irreducible component of order 2.

It is clear that, if a quartic surface X admits a bitangent involution σ, then the congruence of lines $S(\sigma)$ is an irreducible component of $\mathrm{Bit}(X)$.

Remark 4.3. As we explained, any birational involution σ of a quartic surface X defines an irreducible congruence of lines $S(\sigma)$. If the order of $S(\sigma)$ is equal to one, the involution σ lifts to a birational involution T of $\mathbb{P}^3$. Indeed, given a general point P in $\mathbb{P}^3$, there is a unique line passing through P and spanned by an orbit of two points $x, \sigma(x)$ on X. We define $T(P)$ to be the fourth point on this line such that the pairs $(x, \sigma(x))$ and $(P, T(P))$ are harmonically conjugate.

5. Quartic surfaces: $p = 2$

In this section, we assume that $\mathrm{char}(\Bbbk) = 2$. We continue to assume that X is a normal quartic surface.

It is obvious that, in characteristic $p \neq 2$, the bitangent surface of a normal quartic surface X does not contain α-planes (otherwise, the projection of X from some point is ramified at every point). This is not anymore true over in characteristic $p = 2$.

A point x in $\mathbb{P}^3$ is called an *inseparable projection center* of a normal surface X if the projection map with the center at x is inseparable. It is clear that the set of lines passing through an inseparable projection center is an α-plane contained in $\mathrm{Bit}(X)$. Conversely, if $\mathrm{Bit}(X)$ contains an α-plane of lines through a point $x \in \mathbb{P}^3$, the points x is an inseparable projection center of X.

Proposition 5.1. *The set of inseparable projection centers of a normal quartic surface in characteristic 2 is a finite set. Any inseparable projection center contained in X is a singular point of X.*

Proof. Since the set of points $x \in X$ such that the line $\langle q, x\rangle$ is tangent at x is equal to the intersection $X \cap P_q(X)$ [3, Theorem 1.1.5], a point $q = [a_0, a_1, a_2, a_3]$ is an inseparable center of a quartic surface $X = V(F)$ if and only if the polar $P_q(X) = V(\sum a_i \frac{\partial F}{\partial x_i}) = \mathbb{P}^3$. Suppose the set of inseparable projection centers contains a curve Z of degree ≥ 2 (not necessarily irreducible). Then, we can choose three non-collinear points on it. Choose projective coordinates such that these points are $[1, 0, 0, 0], [0, 1, 0, 0]$ and $[0, 0, 1, 0]$. Then, the partials $\frac{\partial F}{\partial x_i} = 0$ for $i = 0, 1, 2$. This means that F contains x_0, x_1, x_2 in even power. But, then, it must contain x_3 in even power. Hence, F is a square. So, Z must be a line. In this case, $\mathrm{Bit}(X)$ contains a special hyperplane section of the Grassmannian $G_1(\mathbb{P}^3)$ that consists of lines intersecting Z. A general plane section of X intersects Z and, hence, contains a curve of bitangent lines. However, a smooth quartic curve contains only finitely many bitangents (in fact, less than or equal to 7, see below Proposition 5.8). Thus, a general plane section of X is singular contradicting the Akizuli-Bertini theorem (see [9, Remark 8.18.1]). This proves that the set of inseparable centers is a finite set.

Suppose $q \in X$ is an inseparable projection center. Then, we may choose projective coordinates such that $q = [0, 0, 0, 1]$ and the equation of X can be written in form $\sum A_{4-k}(x, y, z)w^k = 0$, where A_i is a homogeneous polynomial in x, y, z of degree i. Since $q \in X$, $A_0 = 0$ and since $P_q(X) = 0$, $A_1 = 0$. Thus, q is a singular point. □

We can extend the proof of Proposition 2.2 to the case of characteristic 2 to obtain the following Corollary.

Corollary 5.2. $\mathrm{Bit}(X)$ *is a congruence of lines, i.e., each irreducible component of* $\mathrm{Bit}(X)$ *is a surface.*

Recall that a universal binary form of degree d is a homogeneous polynomial of degree d in two variables whose coefficients are algebraically independent over a field k..

Proposition 5.3. *Assume* $\mathrm{char}(\Bbbk) = 2$. *Then, the discriminant* $\mathcal{D}(d)$ *of the universal binary form of degree* d *is a square of a homogeneous polynomial of degree* $d-1$.

Proof. (supplied by G. Kemper) Take a univariate polynomial $f = a_0x^d + \cdots + a_1x + a_d$ whose coefficients are indeterminates over the field $\mathbb{F}_2$. Let $D = D(a_0, \ldots, a_d)$ be the discriminant of f. We know that D, as a function in roots, considered as indeterminants $y_1, \ldots, y_d$ is equal to the square of $P = \prod_{i<j}(y_i - y_j)$. Since $\mathrm{char}(\Bbbk) = 2$, $P = \prod_{i<j}(y_i + y_j)$, and hence, it is invariant under the whole symmetric group $\mathfrak{S}_d$ (but not only under the alternating group $\mathfrak{A}_d$ if $\mathrm{char}(\Bbbk) \neq 2$). Since the Galois group of f permutes the roots r_i, this means that P lies in $\mathbb{F}_2[a_0, \ldots, a_d]$. So, the discriminant of f is a square. □

Corollary 5.4. *Suppose* $\mathrm{Bit}(X)$ *does not contain* α*-planes. Then, the order* m *of* $\mathrm{Bit}(X)$ *satisfies*

$$1 \leq m \leq \frac{1}{4}d(d-1)(d-2)(d-3).$$

Proof. The assumption implies that the order of $\mathrm{Bit}(X)$ is equal to the number of bitangents dropped from a general point in $\mathbb{P}^3$. In the proof of Theorem 2.2, by our assumption, the discriminant of the binary form g_{d-2} is not zero, hence $m \geq 1$. Now apply Proposition 5.3 and obtain the other inequality. □

Definition 5.5. Let $m(d) := \frac{1}{2}d(d-1)(d-2)(d-3)$ (resp. $m(d) := \frac{1}{4}d(d-1)(d-2)(d-3)$) for $p \neq 2$ (resp. $p = 2$). We say that a surface X is a *general projection* surface if the order of $\mathrm{Bit}(X)$ is equal to $m(d)$.

By Corollary 2.5, any surface in characteristic zero with ordinary double points as singularities is a general projection surface.

In the next section, we show that it is not true anymore in characteristic 2.

In the following example of an involution σ of Type (ii) of a quartic surface in characteristic 2 all rays from the congruence of lines $S(\sigma)$ are tangent to the surface.

Example 5.6. Let $X = V(F)$ be a quartic surface, where

$$F = x_0^3x_1 + x_1^3x_2 + x_2^3x_3 + x_3^3x_0.$$

The surface is invariant with respect to the involution

$$\sigma : [x_0, x_1, x_2, x_3] \mapsto [x_2, x_3, x_0, x_1].$$

The set of singular points of X consists of one rational double point $x_0 = [1, 1, 1, 1]$ of type A_2 and four ordinary double points $[1, \zeta, \zeta^3, \zeta^2]$, where $\zeta^5 = 1, \zeta \neq 1$. The fixed locus X^σ of σ is the line $\ell = V(x_0 + x_2, x_1 + x_3)$.

For any point $x = [a, b, c, d] \in X$ not lying on ℓ, the line $\ell_x = \langle x, \sigma(x)\rangle$ contains one extra point $[a+c, b+d, a+c, b+d] \in \ell$. The line ℓ_x is tangent to X at this point. So, σ is an involution of Type (ii).

Any invariant line on X is contained in a plane $x_3 = x_1 + t(x_0 + x_2)$. A straightforward computation shows that there are four invariant lines on X corresponding to the parameters $t = 0, \infty, e, e^2$, where $e^2 + e + 1 = 0$. Under the map $\phi : X/(\sigma) \to S(\sigma)$, these lines are blown down to points. The algebra of invariants $\Bbbk[x_0, x_1, x_2, x_3]^{(\sigma)}$ is generated by the polynomials

$$p_0 = x_0 + x_2,\ p_1 = x_1 + x_3,\ p_2 = x_0x_2,\ p_3 = x_1x_3,\ p_4 = x_0x_1 + x_2x_3.$$

They satisfy the relation

$$p_0^2p_3 + p_0p_1p_4 + p_1^2p_2 + p_4^2 = 0,$$

and embed $\mathbb{P}^3/(\sigma)$ into $\mathbb{P}(1, 1, 2, 2, 2)$ as a weighted homogeneous hypersurface of degree 4 with the double line $V(p_0, p_1, p_4)$. We can write

$$F = p_4(p_3 + p_0^2) + (p_2 + p_1^2)(p_0p_1 + p_4).$$

Therefore, the image of X in $\mathbb{P}^3/(\sigma)$ is the intersection of two hypersurfaces of degree 4, so it has trivial dualizing sheaf. It is singular along the line $V(p_0, p_1, p_4)$. The quotient $X/(\sigma)$ is isomorphic to the normalization of this surface. Note that the polynomials p_i do not generate the algebra of invariants of the projective coordinate ring of X. In fact, $x_0(x_3^3 + x_0x_1)$ and $x_1(x_0^3 + x_1^2x_2)$ are invariant modulo (F).

The map $\phi : X/(\sigma) \to S(\sigma)$ is just the projection map given by (p_0, p_1, p_3). It shows that the congruence of lines $S(\sigma)$ is isomorphic to $\mathbb{P}(1, 1, 2)$. The image $\phi(\ell)$ of the line ℓ is the singular point $[0, 0, 1]$ of $\mathbb{P}(1, 1, 2)$. The images of the four invariant lines are two nonsingular points.

Lemma 5.7. *With no assumption on the characteristic, suppose* $\mathrm{Bit}(X)$ *contains a* β*-plane of lines in a plane* $\Pi = V(L)$*, where* L *is a linear form. Then, the equation of* X *can be written in the form*

$$Q(x, y, z, w)^2 + L(x, y, z, w)F(x, y, z, w) = 0, \tag{5.1}$$

where Q *is a quadratic form and* F *is a cubic form. The singular locus of* X *contains* $V(L, Q, F) \cup (V(Q) \cap \mathit{Sing}(V(F)))$*, where* $\mathit{Sing}(V(F))$ *is the singular locus of* $V(F)$*. In particular,* X *is a singular quartic surface.*

Proof. Let $C = X \cap \Pi$. Any line in Π is a bitangent of X. Thus, it is bitangent to C. This could happen only if C is a double conic, reducible or not. Thus, Π is a trope-conic, and its equation can be written as in (5.1).

Without loss of generality, we may assume that $L = x$ is a coordinate plane. Taking the partials, we find the singular locus contains $V(x, Q, F) \cup (V(Q) \cap \mathrm{Sing}(V(F)))$. □

Recall that the *p-rank* of a smooth curve C of genus $g > 0$ over an algebraically closed field of characteristic $p > 0$ is the p-rank of the elementary abelian group $J(C)[p]$ of p-torsion points of its jacobian variety of C. It takes values in the set $[0, g]$. When $p = 2$ and C is a smooth plane quartic, the p-rank takes values in $\{3, 2, 1, 0\}$ and can be characterized by the number of bitangents of C equal to $7, 4, 2, 1$, respectively [20], [24, §3].

The proof of the following proposition can be found in [28, Proposition 1] (see also [20]).

Proposition 5.8. *Let* C *be a smooth plane quartic curve over an algebraically closed field of characteristic* 2*. Then, it is projectively equivalent to one of the curves*

(1) $Q(x,y,z)^2+xyz(x+y+z)=0, \quad Q(1,0,0),\ Q(0,1,0),\ Q(0,0,1),\ Q(1,1,0),\ Q(1,0,1),$ $Q(0,1,1),\ Q(1,1,1)\neq 0$*;*

(2) $Q(x,y,z)^2+xyz(y+z)=0, \quad Q(1,0,0),\ Q(0,1,0),\ Q(0,0,1),\ Q(0,1,1)\neq 0$*;*

(3) $Q(x,y,z)^2+xy(y^2+xz)=0, \quad Q(1,0,0),\ Q(0,0,1)\neq 0$*;*

(4) $Q(x,y,z)^2+x(y^3+x^2z)=0, \quad Q(0,0,1)\neq 0$*.*

One check that the number of bitangents is indeed equal to $7,4,2,1$, respectively. More precisely, the bitangents are

(1) $V(x), V(y), V(z), V(x+y), V(y+z), V(z+x), V(x+y+z)$.

(2) $V(x), V(y), V(z), V(y+z)$;

(3) $V(x), V(y)$;

(4) $V(x)$.

The loc. cit. paper of Wall also computes the automorphism group of a plane quartic in one of the normal forms (1)-(4). This implies that the codimension of the subspace of plane quartics of the form (1)-(4) is equal to $0,1,2,3$, respectively [28].

Based on Wall's computations, the following proposition is proved in [19, Section 2].

Proposition 5.9. *A smooth plane quartic curve over any field k (not necessary algebraically closed) of characteristic 2 of p-rank less than 3 is isomorphic over K to a curve Q^2+LF, where L is a linear form.*

The next theorem shows, surprisingly for us, that, although the moduli space of non-ordinary (i.e., of p-rank <3) is of dimension 5 in the moduli space of all plane quartics, a general hyperplane section of a nonsingular quartic surface is an ordinary curve.

Theorem 5.10. *Let X be a normal quartic surface over an algebraically closed field of characteristic 2 whose general hyperplane section is not an ordinary plane quartic. Then,* $\mathrm{Bit}(X)$ *contains a β-plane, and hence, X is singular.*

Proof. Let

$$F := \{(x,\Pi)\in X\times\check{\mathbb{P}}^3 : x\in\Pi\}\subset\{(x,\Pi)\in\mathbb{P}^3\times\check{\mathbb{P}}^3 : x\in\Pi\}$$

be the universal family of plane sections of X considered as the closed subset of the universal family of planes. Passing to the generic fiber of the second projection, we obtain a plane quartic curve C_K over the field K of rational functions on $\check{\mathbb{P}}^3$. Applying Proposition 5.9, we find that $C_K = V(Q^2+LF)$, where L is a linear form with coefficients in K. Applying a linear transformation of $\mathbb{P}^3$ over K, we may assume that the equation of $V(L)$ in $\mathbb{P}^3_K$ is $x_0=0$, where (x_0,x_1,x_2,x_3) are coordinates in $\mathbb{P}^3_K$. This implies that any line in the plane $V(x_0)$ is a bitangent line of X. Applying Lemma 5.7, we obtain that X is singular. $\square$

Example 5.11. (suggested by the referee). Consider the surface X with the equation

$$w^2(ax+by+z)^2+x(y^3+x^2z)=0.$$

The surface X has a unique (non-rational) singular point $[0,0,0,1]$ and an ordinary node $[0,0,1,0]$. The point $[0,0,0,1]$ is a unique inseparable projection center of X. In fact, take a general point $P=[x_0,y_0,z_0,w_0]$ in $\mathbb{P}^3$. Substituting the parametric equation $[x,y,z,w]=[sx_0+tu_0, sy_0+tu_1, sz_0+tu_2, sw_0+tu_3]$, we find one bitangent line that connects P with the point $[0,0,0,1]$. Thus, $\mathrm{Bit}(X)$ contains an α-plane. On the other hand, a general hyperplane section $w=\alpha x+\beta y+\gamma z$ is the case of Proposition 5.8, that is, it is a plane quartic curve with 2-rank 0. By Proposition 5.10, it contains a β-plane. Thus, $\mathrm{Bit}(X)$ contains the union of an α-plane and a β-plane. Further computation shows that there is nothing else.

6. Kummer quartic surfaces in characteristic 2, ordinary case

Kummer quartic surfaces in characteristic 2 are divided into three classes according to curves of genus 2 being ordinary, 2-rank 1, or supersingular (see, e.g., [14]). In this section, we discuss the simplest case, an ordinary Kummer quartic surface. The Kummer quartic surface X in characteristic 2 associated with an ordinary genus 2 curve C is given by

$$X=V(a(x^2y^2+z^2w^2)+b(x^2z^2+y^2w^2)+c(x^2w^2+y^2z^2)+xyzw), \tag{6.1}$$

where a,b,c are non-zero constants. They coincide with the coefficients of the Igusa canonical model of C [17], [18], [14]. The Kummer quartic surface X has four singular points

$$p_1=[1,0,0,0],\ p_2=[0,1,0,0],\ p_3=[0,0,1,0],\ p_4=[0,0,0,1]$$

all of which are rational double points of type D_4 [23], [13], and X has four tropes defined by the hyperplane sections $x=0, y=0, z=0, w=0$, respectively.

Applying Proposition 5.8, we find that a general hyperplane section of X is a plane quartic with the 2-rank equal to 3. So, we expect that the class of X is equal to 7. Computing the partial derivatives of the polynomial defining X, we find that there are no inseparable projection centers of X. So, $\mathrm{Bit}(X)$ does not contain α-planes.

Lemma 6.1. *Let S be the congruence of lines of bidegree $(1,1)$ of rays intersecting the skew lines $\ell_1=V(x,y)$ and $\ell_2=V(z,w)$ (or $V(x,z)$ and $V(y,w)$, or $V(x,w)$ and $V(y,z)$). Then, each ray of S is a bitangent line of X.*

Proof. It is enough to consider the first pair of skew lines. A line ℓ passing through a point $q=[x_0,y_0,z_0,w_0]$ not on ℓ_1 or ℓ_2 is the intersection of two planes $\langle \ell_1,q\rangle$ and $\langle \ell_2,q\rangle$. The parametric equation of ℓ is

$$[x,y,z,w]=[sx_0,sy_0,tz_0,tw_0].$$

The line ℓ intersects ℓ_1 at the point $[0,0,z_0,w_0]$ and intersects ℓ_2 at the point $[x_0,y_0,0,0]$. Plugging in these equations into (6.1), we get

$$a(x_0^2y_0^2s^4+z_0^2w_0^2t^4)+(b(x_0^2z_0^2+y_0^2w_0^2)+c(x_0^2w_0^2+y_0^2z_0^2)+x_0y_0z_0w_0)s^2t^2=0. \tag{6.2}$$

This equation is a square of a quadratic equation, and hence, the assertion holds. □

We keep the notation in the proof of Lemma 6.1. If q belongs to X, then it corresponds to $[s,t] = [1,1]$, so $[s,t] = [1,1]$ is one of the solutions of (6.2). The second solution is $[s,t] = [z_0w_0, x_0y_0]$. This defines an explicit bitangent involution σ_1 of X. We see that it is not defined only at the singular points $p_1,\dots,p_4$ of X. The involution is the restriction of the Cremona involution

$$T_1 : [x,y,z,w] \mapsto [xzw, yzw, xyz, xyw]$$

which is equal to the composition of the standard inversion transformation T and the involution

$$g_1 : [x,y,z,w] \mapsto [y,x,w,z].$$

Note that g_1 is induced from a translation of $J(C)$ by a non-zero 2-torsion point. There are three non-zero 2-torsion points on $J(C)$ and all three bitangent involutions are the products of T and the involutions induced from the translations.

Proposition 6.2. *The surface X admits three bitangent involutions $\sigma_1,\sigma_2,\sigma_3$. The congruence of lines $S(\sigma_i)$ is equal to the congruence of lines intersecting two skew lines from the previous lemma. The fixed curve X^{σ_i} is an elliptic curve of degree 4, and it is cut out set-theoretically by the quadric $V(xy+zw)$.*

Proof. It is enough to consider the first involution defined by the two lines $\ell_1 = V(x,y)$ and $\ell_2 = V(z,w)$. We denote it by σ. We have already proved the first assertion. The fixed locus of T_1 is the quadric $Q = V(xy+zw)$. The quadric intersects X along a complete intersection of two surfaces:

$$\begin{aligned}&xy+zw = 0,\\ &b(x^2z^2+y^2w^2)+c(x^2w^2+y^2z^2)+xyzw = 0.\end{aligned}$$

Plugging in $y = zw/x$ in the second equation, we find that the intersection is a quartic curve taken with multiplicity 2. It passes through the singular points, and these points are nonsingular on the quartic. We also check (substituting $x = 1$) that its projection to the plane is a smooth cubic curve. Thus, $C = X^{\sigma}$ is, set-theoretically, a smooth quartic elliptic curve.

Note that it is analogous to the fact that the fixed locus of any of the six bitangent involutions of a Kummer quartic surface in characteristic different from 2 is an octic curve cut out by a quartic surface with multiplicity 2. □

Let us now look at the birational map

$$\phi_\sigma : Y = X/(\sigma) \dashrightarrow S(\sigma) \cong \mathbb{P}^1\times\mathbb{P}^1.$$

Recall that ϕ_σ is not defined at the singular points of X. The pencil of planes $\ell_1^\perp$ (resp. $\ell_2^\perp$) with the base line ℓ_1 (resp. ℓ_2) cuts out in X a pencil of plane quartic curves. The involution σ acts identically on the parameters of the pencil. Let

$$\pi : \tilde{X} \to X$$

be the minimal resolution of X. The involution σ lifts to a biregular involution $\tilde{\sigma}$ of $\tilde{X}$.

Lemma 6.3. *The pencils $\ell_1^\perp$ and $\ell_2^\perp$ define two invariant elliptic pencils $|F_1|$ and $|F_2|$ on $\tilde{X}$. Each pencil has four reducible fibers: two fibers of type $\tilde{D}_6$ and two fibers of type $\tilde{A}_1^*$ (of Kodaira's type* III*).*

Proof. We consider only the pencil defined by $\ell_1^\perp = V(y+tx)$ with parameter $t \in \mathbb{P}^1$. We check that, for a general parameter t, the plane quartic $X \cap V(y+tx)$ has two cusps at p_3 and p_4, hence its geometric genus is equal to 1. The cusps are the base points of the pencil of quartics. There are special values of the parameter t: $0, \infty, \sqrt{c/b}, \sqrt{b/c}$. For $t=0$ (resp. $t=\infty$), the quartic is a double conic, it passes through the third singular point $p_1 \in \ell_2$ (resp. $p_2 \in \ell_2$). The conic is one of the four *trope-conics* on X, i.e. conics cut out set-theoretically by one of the trope planes:

$$\begin{aligned} C_{123} &= V(\sqrt{a}xy+\sqrt{b}xz+\sqrt{c}yz), \\ C_{124} &= V(\sqrt{a}xy+\sqrt{b}yw+\sqrt{c}xw), \\ C_{134} &= V(\sqrt{a}zw+\sqrt{b}xz+\sqrt{c}xw), \\ C_{234} &= V(\sqrt{a}zw+\sqrt{b}yw+\sqrt{c}yz). \end{aligned}$$

Let $E_i = E_0^{(i)}+E_1^{(i)}+E_2^{(i)}+E_3^{(i)}, i=1,2,3,4$, be the exceptional curves over the singular points p_i of X, where $E_0^{(i)}$ is the central component. We check that the proper transform of $V(y)$ intersects one of the components $E_i^{(j)}, i \neq 0, j=1,3,4$. The corresponding fiber of $|F_1|$ is of type $\tilde{D}_6$. If $t=a/b$, the quartic has one cusp at one of the base points and a cusp followed by infinitely near node at another base points. The corresponding fiber of type $\tilde{A}_1^*$. □

Let us look at the orbit space $\tilde{Y} = \tilde{X}/(\tilde{\sigma})$. Recall that each singular point of X is a rational double point of type D_4 with the exceptional curve $E_0^{(i)}+E_1^{(i)}+E_2^{(i)}+E_3^{(i)}$ over p_i, where $E_0^{(i)}$ is the central component. Observe that the transformation T_1 blows down each conic C_{ijk} to a singular point:

$$C_{123} \to p_3,\ C_{124} \to p_4,\ C_{134} \to p_1,\ C_{234} \to p_2.$$

The involution $\tilde{\sigma}$ interchanges the proper transform $\tilde{C}_{123}$ of C_{123} with the central component $E_0^{(3)}$, and similarly for other trope conics. It acts on the corresponding reducible fiber via a symmetry of its dual graph.

Now, we are ready to describe the birational morphism

$$\tilde{\phi}_\sigma : \tilde{Y} \to S(\sigma).$$

Let $|\mathfrak{f}_1|$ and $|\mathfrak{f}_2|$ be the rulings of the quadric $Q = S(\sigma)$ corresponding to the family of planes $\ell_1^\perp$ and $\ell_2^\perp$. Their pre-images under $\tilde{X} \to \tilde{Y} \to Q$ are the two elliptic pencils from Lemma 6.3. Let $L_0^{(1)}, L_\infty^{(1)}$ (resp. $L_0^{(2)}, L_\infty^{(2)}$) be the lines from $|\mathfrak{f}_1|$ (resp. $|\mathfrak{f}_2|$) corresponding to the planes $V(x), V(y) \in \ell_1^\perp$ (resp. $V(z), V(w) \in \ell_2^\perp$). Their pre-images in $\tilde{X}$ are the reducible fibers of type $\tilde{D}_6$ of $|F_1|$ (resp. $|F_2|$). The image of the curve $\tilde{X}^{\tilde{\sigma}}$ in Q is a smooth curve B of bidegree $(2,2)$ that passes through the vertices of the quadrangle of lines $L = L_0^{(1)}+L_\infty^{(1)}+L_0^{(2)}+L_\infty^{(2)}$.

We summarize our discussion above with the following assertion:

Proposition 6.4. *The morphism $\tilde{\phi}_\sigma : \tilde{X}/(\tilde{\sigma}) \to S(\sigma)$ is the blow-up of the following eight points on $S(\sigma)$: four vertices of the quadrangle of lines L and infinitely near points to them corresponding to the tangent direction of the curve B. The pre-images of the two fibers in $|\mathfrak{f}_i|$ corresponding to the ramification points of the projection map $B \to |\mathfrak{f}_i|^* \cong \mathbb{P}^1$ are reducible fibers of $|F_i|$ of type $\tilde{A}_1^*$.*

So far, we have found that the bitangent surface $\mathrm{Bit}(X)$ contains three irreducible components of bidegree $(m, n) = (1, 1)$. They correspond to three bitangent involutions σ_i. The surface also contains four irreducible components of bidegree $(0, 1)$ corresponding to the trope-conics C_{ijk}.

Theorem 6.5. *Let X be the Kummer quartic surface X in characteristic 2 associated with an ordinary curve of genus 2. Then, the surface $\mathrm{Bit}(X)$ is of bidegree $(m, n) = (3, 7)$ in $G_1(\mathbb{P}^3)$ which consists of 7 irreducible components, three of bidegree $(1, 1)$ corresponding to three bitangent involutions and four of bidegree $(0, 1)$ corresponding to four tropes.*

Proof. For any plane Π containing p, the plane quartic curve $X \cap \Pi$ has bitangent lines forming a line in $G_1(\mathbb{P}^3)$, which implies that the plane quartic curve is a double conic. This is impossible for a normal quartic X. As we mentioned before, Lemma 6.1, $\mathrm{Bit}(X)$ does not contain α-planes. Thus it is enough to show that the class n of $\mathrm{Bit}(X)$ is equal to $3 + 4 = 7$. This is the number of bitangent lines contained in a general plane in $\mathbb{P}^3$. We have already shown that $n \geq 7$. On the other hand, it is known that the number of bitangent lines of a smooth plane quartic in characteristic 2 is $7, 4, 2$ or 1 if the Hasse-Witt invariant is equal to $3, 2, 1$ or 0, respectively [24]. The assertion now follows. □

Remark 6.6. Consider the standard inversion transformation

$$T : [x, y, z, w] \mapsto [yzw, xzw, xyw, xyz]$$

of $\mathbb{P}^3$. The restriction of T to $\tilde{X}$ is a fixed-point-free involution σ with $\tilde{X}/(\sigma)$ isomorphic to an Enriques surface. Let us consider the Plücker embedding of the corresponding congruence of lines. The six minors p_{ij} of the matrix

$$\begin{pmatrix} x & y & z & w \\ yzw & xzw & xyw & xyz \end{pmatrix}$$

are given by

$$\begin{aligned} &[p_{12}, p_{13}, p_{14}, p_{23}, p_{24}, p_{34}] \\ = &\left[zw(x^2 + y^2),\ yw(x^2 + z^2),\ yz(x^2 + w^2),\ xw(y^2 + z^2),\ xz(y^2 + w^2),\ xy(z^2 + w^2)\right], \end{aligned}$$

which satisfies

$$p_{12}p_{34} + p_{13}p_{24} + p_{14}p_{23} = xyzw((x + y)(z + w) + (x + z)(y + w) + (x + w)(y + z))^2 = 0$$

and the cubic equation

$$p_{12}p_{13}p_{23} + p_{12}p_{14}p_{24} + p_{13}p_{14}p_{34} + p_{23}p_{24}p_{34} = 0 \tag{6.3}$$

(see [8, (33)]).

Lemma 6.7. *Let $p_0 = [x_0, y_0, z_0, w_0] \in X$ be a general point of the Kummer quartic surface. The line ℓ passing through p_0 and $\sigma(p_0)$ is not a bitangent line of X.*

Proof. The following proof is suggested by a referee. By plugging the parametric equation

$$[x_0 + ty_0z_0w_0, y_0 + tx_0z_0w_0, z_0 + tx_0y_0w_0, w_0 + tx_0y_0z_0]$$

of the line ℓ into the equation (6.1) of X, we can see that the term which is linear in t is equal to $(x_0 + y_0 + z_0 + w_0)t$. Since a general point does not lie in the hyperplane $V(x + y + z + w)$, ℓ is not a bitangent line. □

Let ℓ be a line passing through p_0 and $\sigma(p_0)$. Let $\ell \cap X = \{p_0, \sigma(p_0), p_0', p_0''\}$. The image of ℓ under σ is a cubic curve, and hence, p_0', p_0'' are not conjugate to $p_0, \sigma(p_0)$ under the action of σ. Thus, the map $X \dashrightarrow G_1(\mathbb{P}^3)$ sending p_0 to $\ell = \langle p_0, \sigma(p_0)\rangle$ has degree 2 onto its image. Therefore, the Enriques surface $\tilde{X}/(\sigma)$ can be embedded into $G_1(\mathbb{P}^3)$ satisfying a cubic relation (6.3). This suggests the following question:

Question 6.8. Is the Enriques surface $\tilde{X}/(\sigma)$ a Reye congruence of bidegree $(7,3)$?

7. Kummer quartic surfaces in characteristic 2, 2-rank 1 case

In this section, we discuss the case of Kummer quartic surfaces associated with curves of 2-rank 1. The Kummer quartic surface X_1 associated with a curve of genus two and 2-rank 1 is given by

$$X_1 = V(\beta^2x^4 + \alpha^2x^2z^2 + x^2zw + xyz^2 + y^2w^2 + z^4), \tag{7.1}$$

where α, β are constants with $\beta \neq 0$ [7, §3]. The surface X_1 has exactly two singular points

$$p_1 = [0,0,0,1],\ \ p_2 = [0,1,0,0]$$

of type D_8 [23], [13] and contains two tropes defined by the hyperplane section $x = 0$ and $z = 0$, respectively. The two tropes meet at p_1 and p_2.

Consider the skew lines $\ell = V(x,y)$ and $\ell' = V(z,w)$. In the same way as in the ordinary case, we see that the lines meeting ℓ, ℓ' are bitangent lines of X_1. The corresponding bitangent involution of X_1 is given by the restriction of a Cremona involution

$$\sigma_1 : [x,y,z,w] \mapsto [xz^2, yz^2, \beta x^2z, \beta x^2w]. \tag{7.2}$$

Thus, $\mathrm{Bit}(X_1)$ contains a smooth quadric surface $S(\sigma_1)$.

There exists another bitangent involution

$$\sigma_2 : [x,y,z,w] \mapsto [x^2z, x^2w, xz^2, yz^2], \tag{7.3}$$

which is a composite of σ_1 with a projective linear transformation

$$\tau : [x,y,z,w] \to [z,w,\beta x,\beta y].$$

Note that τ is induced from the translation by the non-zero 2-torsion of the Jacobian of the curve.

Now, let us consider the Plücker embeddings of the congruence of bitangent lines defined by σ_2. The six minors p_{ij} of the matrix

$$\begin{pmatrix} x & y & z & w \\ x^2z & x^2w & xz^2 & yz^2 \end{pmatrix}$$

are given by

$$\begin{aligned} &[p_{12}, p_{13}, p_{14}, p_{23}, p_{24}, p_{34}] \\ &= \big[x^2(xw+yz),\ 0,\ xz(xw+yz),\ xz(xw+yz),\ (xw+yz)^2,\ z^2(xw+yz)\big] \\ &= [x^2, 0, xz, xz, xw+yz, z^2]. \end{aligned}$$

They satisfy

$$p_{13} = 0,\ p_{14} + p_{23} = 0,\ p_{12}p_{34} + p_{14}p_{23} = 0.$$

Thus, the congruence of lines $S(\sigma_2)$ is a quadric cone, a special linear section of the Grassmannian quadric.

We now conclude:

Theorem 7.1. *Let X_1 be the Kummer quartic surface associated with a smooth curve of genus two and of 2-rank 1. Then, $\mathrm{Bit}(X_1)$ is of bidegree $(2,4)$ in $G_1(\mathbb{P}^3)$. It consists of two tropes and two quadric surfaces, one is a smooth quadric $S(\sigma_1)$ and another is a quadric cone $S(\sigma_2)$.*

Proof. We know that $\mathrm{Bit}(X)$ contains two β-planes and two irreducible components of bidegree $(1,1)$. Taking the partial derivatives, we find that they are linearly independent, and hence, there are no inseparable projection centers. So, $\mathrm{Bit}(X)$ does not contain α-planes. The equation of X is of the form

$$Q^2 + x^2zw + xyz^2 = 0.$$

Substituting $w = ax + by + cz$, we obtain the equation of a general plane section $q^2 + x^2z(ax + by) + xyz^2 = 0$. It is immediate to check that it is projectively equivalent to a plane quartic of type (2) from Proposition 5.8. Thus, its 2-rank is equal to 4, and hence, the class of $\mathrm{Bit}(X)$ is equal to 4. So, we have found all irreducible components of $\mathrm{Bit}(X)$: two β-planes and two components of bidegree $(1,1)$. □

Remark 7.2. Let $\ell_0 = V(x,z)$ be the line which is the intersection of the two trope-hyperplanes. Then, any bitangent line defined by σ_2 meets ℓ_0 as follows. Let $p = [x_0, y_0, z_0, w_0] \in \mathbb{P}^3$ be a general point and $q = \sigma_2(p)$. The bitangent line $\langle p, q\rangle$ is given by

$$[x_0(s + tx_0z_0),\ sy_0 + tx_0^2w_0,\ z_0(s + tx_0z_0),\ sw_0 + ty_0z_0^2]$$

where $[s,t] \in \mathbb{P}^1$ is a parameter. The line $\langle p, q\rangle$ meets ℓ_0 at $[0, x_0, 0, z_0]$ when $[s,t] = [x_0z_0, 1]$.

8. Kummer quartic surfaces in in characteristic 2, supersingular case

In this section, we consider the supersingular case. In this case, the Kummer quartic surface X_0 is given by the equation

$$X_0 = V(x^3w + \alpha x^3y + x^2yz + \alpha^2x^2z^2 + xy^3 + y^2w^2 + z^4), \tag{8.1}$$

where α is a constant [6, (5.1)], [7, §3]. The Kummer surface has one singular point $p_0 = [0,0,0,1]$ which is an elliptic singularity of type $④^1_{0,1}$ in the sense of Wagreich [13] and contains a trope defined by the hyperplane $x = 0$.

Let us consider the following Cremona involution:

$$T : [x,y,z,w] \mapsto [x^3, x^2y, \alpha x^3 + x^2z + xy^2, \alpha x^2y + x^2w + y^3]. \tag{8.2}$$

The involution T preserves X_0, and restricts to a bitangent involution σ of X_0. This follows from a direct calculation.

Let us consider the Plücker embeddings of bitangent lines. The six minors p_{ij} of the matrix

$$\begin{pmatrix} x & y & z & w \\ x^3 & x^2y & \alpha x^3 + x^2z + xy^2 & \alpha x^2y + x^2w + y^3 \end{pmatrix}$$

are given by

$$\begin{aligned} &[p_{12}, p_{13}, p_{14}, p_{23}, p_{24}, p_{34}] \\ &= \big[0,\ x^2(\alpha x^2 + y^2),\ xy(\alpha x^2 + y^2),\ xy(\alpha x^2 + y^2),\ y^2(\alpha x^2 + y^2),\ (xw + yz)(\alpha x^2 + y^2)\big] \end{aligned}$$

$$= [0, x^2, xy, xy, y^2, xw + yz],$$

which satisfies

$$p_{12} = 0,\ p_{14} + p_{23} = 0,\ p_{13}p_{24} + p_{14}p_{23} = 0.$$

Thus, the congruence of lines $S(\sigma)$ is a quadric cone.

Theorem 8.1. *Let X_0 be the Kummer quartic surface associated with a supersingular curve. Then, $\mathrm{Bit}(X_0)$ is of bidegree $(1, 2)$ in $G_1(\mathbb{P}^3)$ and it consists of a trope and a quadric cone.*

Proof. Taking the partial derivatives, we find again that there are no inseparable projection centers. So, $\mathrm{Bit}(X)$ does not contain α-planes. Since we have already found irreducible components of $\mathrm{Bit}(X)$ of bidegree $(0, 1)$ and $(1, 1)$, it is enough to check that the 2-rank of a general plane section is equal to 2. This can be seen directly by plugging in $w = ax + by + cz$ in the equation of X and reducing the non-square part of the obtained equation to the form (3) from Proposition 5.8. □

Remark 8.2. Let $\ell_0 = V(x, y)$. Then, any bitangent line meets ℓ_0 as follows. Let $p = [x_0, y_0, z_0, w_0] \in \mathbb{P}^3$ be a general point and $q = \sigma(p)$. The bitangent line $\langle p, q\rangle$ is given by

$$[x_0(s + tx_0^2), y_0(s + tx_0^2), z_0(s + tx_0^2) + tx_0(\alpha x_0^2 + y_0^2), w_0(s + tx_0^2) + ty_0(\alpha x_0^2 + y_0^2)],$$

where $[s, t] \in \mathbb{P}^1$ is a parameter. The line $\langle p, q\rangle$ meets ℓ_0 at $[0, 0, x_0, y_0]$ when $[s, t] = [x_0^2, 1]$.

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