On rank 2 vector bundles with $c_1^2 = 10$ and $c_2 = 3$ on Enriques surfaces.

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1. Introduction. Let S be an Enriques surface over an algebraically closed field k and Δ be a numerically effective divisor on S with $\Delta^2 = 10$. It is known that one can always find such Δ with the property that the linear system $|\Delta|$ gives a birational map onto a surface of degree 10 in $\mathbb{P}^5$ with at most double rational points as singularities. We will assume that there exists an ample Δ. Then S can be embedded by $|\Delta|$ into $\mathbb{P}^5$. Δ is said to be a Reye polarization if the image of S lies on a quadric. Δ is a said to be a Cayley polarization if $\Delta+K_S$ is a Reye polarization. It can be proven **[CD2]** that S admits a Reye polarization if and only if it is nodal, i.e. contains smooth rational curves (nodal curves). Note that every Enriques surface lying on a quadric is isomorphic to a Reye congruence of lines in $\mathbb{P}^3$ (char(k) $\neq 2$). The Cayley polarization maps S onto a surface in $\mathbb{P}^5$ isomorphic to the surface of reducible quadrics in a 5-dimensional linear system of quadrics in $\mathbb{P}^5$.

In this note we will study rank 2 vector bundles E on S with

$$c_1(E) = \Delta \text{ and } c_2(E) = 3,$$

where Δ is an ample divisor on S with $\Delta^2 = 10$. If Δ is a Reye polarization, we may assume that S lies in the Grassmann variety G(2,4) in its Plücker embedding. Then an example of such a bundle is the restriction of the universal quotient bundle on G(2,4). One of the motivation for this work was to verify whether this bundle is stable. The formula for the dimension of the moduli space of stable vector bundles shows that one expects the existence of at most finitely many isomorphisms classes of stable E's as above. We will see that the existence of at least one such E depends very much on the property of the polarization Δ. More precisely, we show that, if it exists, then it is unique and Δ is a Reye polarization.

As an application we give a characterization of the Cayley polarizations Δ by the condition that the variety of trisecants of (S,Δ) is three-dimensional. By other means this result was obtained by A. Conte and A. Verra **[CV]**.

1. Fano polarizations of Enriques surfaces. A numerically effective (nef) divisor Δ on an Enriques surface S is called a Fano polarization if $\Delta^2 = 10$ and $\Delta \cdot F \geq 3$ for every nef divisor F with $F^2 = 0$.

Proposition 1. *Every Enriques surface S admits a Fano polarization Δ. The complete linear system $|\Delta|$ defines a birational map $S \to \mathbb{P}^5$ whose image is a surface with at most double rational points as its singularities.*

PROOF. This is proven in **[CD1]** under the assumption that char(k) = 0. The description of all vectors x in Pic(S) with $x^2 = 10$ is given in Corollary 2.5.7 in **[CD1]**. From this it follows that there exists a vector x with $x^2 = 10$ and $x \cdot f \geq 3$ for all vectors f with $f^2 = 0$. Applying Theorem 3.2.1 from loc. cit. we find a nef divisor Δ with $\Delta^2 = 10$ and $\Delta \cdot f \geq 3$ for all f with $f^2 \geq 3$. This is a Fano polarization. The property of the map given by the linear system $|\Delta|$ follows from Corollary 2 of appendix to Chapter 4 of **[CD1]**. The assumption of characteristic 0 can be avoided by applying a recent result from **[S–B]** where it was shown that the Bogomolov's criterion of

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unstability of rank 2 vector bundles on algebraic surfaces ($c_1^2 > 4c_2$) is valid in arbitrary characteristic unless the surface is of general type or a quasi-elliptic of Kodaira dimension 1.

It follows from the previous Proposition that an ample Fano polarization defines an embedding $S \hookrightarrow \mathbb{P}^5$ onto a surface of degree 10 (a Fano model of an Enriques surface). For example, if S does not contain smooth rational curves (it is called unnodal in this case), every Fano polarization is ample. Another example is a Reye congruence which is defined as the surface of lines in $\mathbb{P}^3$ which are contained in a subpencil of a fixed web of quadrics satisfying a certain condition of regularity. This surface is a nodal Enriques surface and its Plücker embedding is defined by an ample Fano polarization. One can show that, in general, an Enriqures surface admits an ample Fano polarization if and only if its non-degeneracy invariant d(S) (see **[CD1]**, p. 182) is maximal (= 10). For every Fano polarization Δ the divisor $\Delta+K_S$ is a Fano polarization which is ample if and only if Δ is ample. Recall from the introduction that an ample Fano polarization Δ is said to be a Reye polarization if $|\Delta|$ maps S onto a quadric. It can be shown **[CD2]** that S admits a Reye polarization if and only if S is nodal and d(S) = 10. If char(k) $\neq 2$ the quadric must be non-singular, and under its identification with the Grassmann variety G(2,4), the image of S is equal to the Reye congruence of some web W of quadrics. The adjoint polarization $|\Delta+K_S|$ of a Reye polarization Δ is called a Cayley polarization. The corresponding linear system maps S onto the variety of reducible quadrics from the 5-dimensional linear system of quadrics $W^{\perp}$ which is apolar to the web W (see **[CD2]**, **[Co]**).

In the following we assume that

$$K_S \neq 0,$$

i.e., S is a classical Enriques surface (e.g., char(k) $\neq 2$).

Lemma 1. *Let S be embedded in $\mathbb{P}^5$ by $|\Delta|$, where Δ is an ample Fano polarization. Then S contains 20 plane cubic curves F_i, $i = 1,\dots,20$, such that*

$$F_i \bullet F_j = \begin{cases} 1 & \text{if } |i-j| \neq 10, \\ 0 & \text{otherwise.} \end{cases}$$

No three of these curves have a common point. Moreover $|2F_i| = |2F_{i+10}|$ is a pencil of curves of arithmetic genus 1 for each i = 1,...,10, and every plane *cubic curve on S is equal to some F_i, $i = 1,\dots,20$.*

PROOF. The first assertion is proven in **[CD1]**, Thm. 3.3.1 (cf. also **[BP]**). It follows from the fact that in the Picard lattice Num(S) one has $3\Delta \equiv f_1+\dots+f_{10}$ for some isotropic vectors f_i with $f_i \bullet f_j = 1$ for $i \neq j$.

Suppose three curves, say F_1, F_2 and F_3 intersect at one point. Then we have an exact sequence:

$$0 \to \mathcal{O}_S(F_1-F_2-F_3) \to \mathcal{O}_S(F_1-F_2) \to \mathcal{O}_{F_3} \to 0.$$

Since $(F_1-F_2-F_3)^2 = -2$, and neither $F_1-F_2-F_3$, nor F_1-F_2 is effective, we have

$$h^1(F_1-F_2-F_3) = h^0(F_1-F_2) = 0.$$

Considering the exact cohomology sequence, this immediately leads to contradiction.

It is known that every nef divisor F with $F^2 = 0$ is effective and can be written as a sum of indecomposable divisors with the same property. For each indecomposable F, which we call a genus 1 curve, |2F| or |F| is a pencil. Also it is known that every pencil of genus 1 curves contains two double fibres 2F'. In particular, $|F_i| = \{F_i\}$ (since $\Delta \bullet F_i$ is odd) but $|2F_i|$ is a pencil. Let F be a genus 1 curve, for instance a plane cubic curve. Intersecting both sides of the equality $3\Delta \equiv F_1 + \dots + F_{10}$ with F, we obtain $\Delta \bullet F > 3$ unless $F \bullet F_i = 0$ for some $i \in \{1,\dots,10\}$. In the latter case $|2F_i|$ must contain F or 2F as its fibre. In the first case $F = F_i$ or F_{i+10}, in the second case $F \sim 2F_i$, and $\Delta \bullet F = 6$.

Corollary. *For every curve C of arithmetic genus 1 moving in a pencil*

$$\Delta \cdot C \geq 6.$$

The equality takes place if and only if $C \in |2F_i|$ for some i.

Note that, in the above notation,

$$F_{i+10} \in |F_i + K_S|.$$

Sometimes we will denote F_{i+10} by F_i', where $i = 1,\dots,10$. We denote by π_i the plane containing the curve F_i. It is easy to see, by using the previous Corollary, that $\dim|\Delta - F_i - F_j| = 0$ if $i \neq j$, hence the planes π_i and π_j span a hyperplane and therefore intersect at one point.

Lemma 2.

$$\pi_i \cap S = F_i.$$

PROOF. The linear system $|\Delta - F_i|$ is cut out by hyperplanes passing through the plane π_i. Therefore our assertion follows from the fact that $|\Delta - F_i|$ has no base points. Obviously each base point must lie in the plane π_i. Assume C is a fixed component of $|\Delta - F_i|$. Then $C \cdot F_i \geq 3$, hence $(\Delta - F_i - C) \cdot F_i = 3 - C \cdot F_i$ shows that $C \cdot F_i = 3$, i.e. C is a line. So $C^2 = -2$, $\Delta \cdot C = 1$ and $(\Delta - F_i - C)^2 = 6$. By Riemann–Roch, $\dim |\Delta - F_i - C| \geq 3$ which is absurd. To show that $|\Delta - F_i|$ has no isolated base points, it is enough to verify that for every nef divisor F with $F^2 = 0$ one has

$$(\Delta - F_i) \cdot F \geq 2.$$

(**[CD1]**, Thm. 4.4.1). By Riemann–Roch, $\Delta - F_i - F_j$ is effective if $i \neq j$. Thus

$$(\Delta - F_i) \cdot F = (\Delta - F_i - F_j) \cdot F + F_j \cdot F \geq F_j \cdot F.$$

If $F \cdot F_j > 1$ for some $j \neq i$, we are done. If $F \cdot F_j = 1$ for all $j \neq i$, then $3\Delta \cdot F = 9 + F \cdot F_i$, and

$$(\Delta - F_i) \cdot F = 3 - \tfrac{2}{3} F \cdot F_i.$$

Thus, if we are wrong, $F \cdot F_i = 3$, $\Delta \cdot F = 4$. But then $\Delta \cdot (F + F_i) = 7$, $(F + F_i)^2 = 6$, and $(\Delta^2 \cdot (F + F_i)^2 - (\Delta \cdot (F + F_i))^2) = 60 - 49 > 0$. The latter contradicts the Hodge Index theorem.

Lemma 3. *Let D be an effective divisor on S with $D \cdot \Delta \leq 5$. Then $D^2 \leq 0$.*

PROOF. By Hodge's Index theorem:

$$\Delta^2 D^2 = 10 D^2 \leq (\Delta \cdot D)^2 \leq 25,$$

that yields $D^2 \leq 2$. Assume $D^2 = 2$. Let $|D| = |M| + A$, where A is the fixed part of $|D|$. Since $D \cdot \Delta \leq 5$, $A = \emptyset$ or $\Delta \cdot M \leq 4$. In the first case, $D \sim F + F'$ or $D \sim 2F + R$, where F and F' are irreducible curves of arithmetic genus 1 (**[CD1]**, Proposition 3.6.2). By Lemma 1, $\Delta \cdot F \geq 3$, hence $\Delta \cdot D \geq 6$. In the second case, $M \cdot \Delta \leq 4$, and by Hodge's Index theorem, $10 M^2 \leq 16$. This implies that $M^2 \leq 0$, hence $h^0(M) = h^0(D) = 1$ (Corollary to Lemma 1).

2. **A characterization of Reye polarizations.** By using vector bundles, we reprove the following result of F. Cossec **[Co]**:

Theorem 1. *In the notation above*

$$|\Delta - F_i - F_{i+10}| \neq \emptyset$$

if and only if Δ is a Reye polarization.

PROOF. Note first that non-empty $|\Delta-F_i-F_{i+10}|$ is represented by a curve C with $C\bullet F_i = 3$ and $C^2 = -2$. Assume Δ is a Reye polarization. Then S lies on a non-singular quadric. Let π_i be the plane containing F_i. Since a nonsingular quadric has two families of rulings, and $F_i\bullet F_j = F_{i+10}\bullet F_j = 1$, the planes π_i, π_{i+10} and π_j lie in the same family, hence there exists a hyperplane containing any pair of them. This hyperplane cuts out a unique curve in $|-F_i-F_{i+10}+\Delta|$. Let us prove the converse. Assume $h^0(-F_i-F_{i+10}+\Delta) \neq 0$. By Riemann-Roch, $h^1(-F_i-F_{i+10}+\Delta) = 1$. Let E be the non-trivial extension:

$$(*) \qquad 0 \to \mathcal{O}_S(F_i) \to E \to \mathcal{O}_S(\Delta-F_i) \to 0$$

which corresponds to a non-trivial element in the group

$$\mathrm{Ext}^1(\mathcal{O}_S(\Delta-F_i),\mathcal{O}_S(F_i)) \cong H^1(\mathcal{O}_S(2F_i-\Delta) \cong H^1(\mathcal{O}_S(\Delta-F_i-F_{i+10}) \cong k.$$

Taking cohomology and using Riemann-Roch, we obtain $h^0(E) = 4$. Let us show that E is spanned by its global sections. Let s_{F_i} be a non-zero section of $\mathcal{O}_S(F_i)$. For every $s\in H^0(E)$ the section $s_{F_i}\wedge s$ is either zero, or vanishes on a curve $F_i+D(s)\in|\mathcal{O}_S(\Delta)|$ for some $D(s)\in|\Delta-F_i|$. Since the map $H^0(E) \to H^0(\mathcal{O}_S(\Delta-F_i))$ is surjective and $|\Delta-F_i|$ has no base points, we find that E is generated by its global sections outside the curve F_i. Now let us show that the same E can be also represented as an extension

$$(**) \qquad 0 \to \mathcal{O}_S(F_j) \to E \to \mathcal{O}_S(\Delta-F_j) \to 0$$

for any $j \neq i$, $|j-i| \neq 10$. Then, repeating the argument from above we obtain that E is generated by global sections outside F_j. Since no three F_i's have a common point, we deduce that E is generated by its global sections everywhere. Tensoring (*) by $\mathcal{O}_S(-F_j)$ we obtain an exact sequence

$$0 \to \mathcal{O}_S(F_i-F_j) \to E(-F_j) \to \mathcal{O}_S(\Delta-F_i-F_j) \to 0$$

Since F_i-F_j is not effective and $(F_i-F_j)^2 = -2$, we have $h^1(F_i-F_j) = 0$. Since $(\Delta-F_i-F_j)^2 = 0$, we have $h^0(\Delta-F_i-F_j) \neq 0$. This shows tha $h^0(E(-F_j)) \neq 0$, and there is a non-trivial homomorphism of sheaves $\mathcal{O}_S(F_j) \to E$. Let L be a saturated line subbundle of E containing the image of $\mathcal{O}_S(F_j)$. Assume $L = \mathcal{O}_S(F_j)$. Then the quotient sheaf E/L is torsion free, and we have an exact sequence:

$$0 \to \mathcal{O}_S(F_j) \to E \to \mathcal{I}_Z(D) \to 0$$

for some 0-dimensional subscheme Z and a divisor D. Counting the Chern classes of E we find that $D \sim \Delta-F_j$, and $Z = \emptyset$. This gives (**). Assume now that $L \neq \mathcal{O}_S(F_j)$. Let $\varphi: L \to \mathcal{O}_S(\Delta-F_i)$ be the composition of the inclusion $L \to E$ and the projection $E \to \mathcal{O}_S(\Delta-F_i)$. If φ is trivial, L is a subsheaf of $\mathcal{O}_S(F_i)$ hence $h^0(F_i-D)$ and $h^0(D-F_j) \geq 0$ which is obviously impossible. Thus φ is non-trivial, hence $L \cong \mathcal{O}_S(D)$, where $|\Delta-F_i-D| \neq \emptyset$. Intersecting $\Delta-F_i-D$ with Δ, we obtain $\Delta\bullet D \leq 7$. If the equality holds we have $\Delta-F_i-D \sim 0$, hence $L \cong \mathcal{O}_S(\Delta-F_i)$, and E splits. Thus we may assume that $\Delta\bullet D \leq 6$. If $\Delta\bullet D = 6$, then $(\Delta-D)\bullet\Delta \leq 5$. By Lemma 3 and Lemma 2 we have $h^0(\Delta-D) = 1$. Using the exact sequence

$$0 \to \mathcal{O}_S(D) \to E \to \mathcal{I}_Z(\Delta-D) \to 0$$

for some 0-dimensional subscheme Z, we obtain $h^0(D) \geq 3$. But $h^0(D) \leq h^0(\Delta-F_i) = 3$, so $h^0(D) = h^0(\Delta-F_i)$, and $\Delta-F_i-D$ is the fixed part of $|\Delta-F_i|$. Since the latter is base-point-free , we have $D \sim \Delta-F_i$, hence E splits. So we have $\Delta\bullet D \leq 5$, hence $D^2 \leq 0$. Counting the Chern classes of E, we get

$$D\bullet(\Delta-D) \leq c_2(E) = 3.$$

that implies $\Delta\bullet D \leq 3$. On the other hand $|D-F_j| \neq \emptyset$ yields $\Delta\bullet D \geq \Delta\bullet F_j = 3$. The equality $\Delta\bullet D = 3$ gives $D = F_j$ contrary the assumption.

3. The main result

Definition. The vector bundle E constructed in the proof of the previouis theorem is called the Reye bundle. Obviously, $c_1(E) = \Delta$ is the Reye polarization, and $c_2(E) = 3$. Since E is the restriction of the universal quotient bundle of G(2,4) to S, its isomorphism class is independent of the choice of F_i.

Proposition 1. *Let E be a rank 2 vector bundle on S with $c_1(E) = \Delta$ and $c_2(E) = 3$. Then*

$$h^0(E) \geq 4.$$

PROOF. By Riemann-Roch:

$$h^0(E)+h^0(E^*(K)) = 4+h^1(E).$$

If $h^0(E^*(K)) = 0$, the assertion is obvious. Assume $h^0(E^*(K)) \neq 0$. Let $\mathcal{O}_S(D)$ be an effective saturated line subbundle of $E^*(K)$. It defines an exact sequence

$$0 \to \mathcal{O}_S(D) \to E^*(K) \to \mathcal{I}_Z(D') \to 0$$

for some divisor D' and an effective 0-dimensional cycle Z. We have

$$c_1(E^*(K)) = -\Delta = D+D' < 0.$$

Dualizing the above exact sequence and twisting it by $\mathcal{O}_S(K)$, we get an exact sequence

$$0 \to \mathcal{O}_S(-D'+K) \to E \to \mathcal{I}_Z(-D) \to 0.$$

It yields

$$h^0(E) \geq h^0(-D'+K) = h^0(D+\Delta+K) \geq h^0(\Delta) = 6.$$

Definition. A vector bundle is called regular if it has a section with only isolated zeroes.

Theorem 2. *Let E be a regular rank 2 vector bundle on S with $c_1(E) = \Delta$ and $c_2(E) = 3$.*
(i) If Δ is not Cayley or Reye, then:

$$E \cong \mathcal{O}_S(F_i) \oplus \mathcal{O}_S(\Delta-F_i)$$

for some $i = 1,\dots,20$.
(ii) If Δ is Cayley, then E is either as in (i), or is isomorphic to one of the 20 non-split extensions:

$$0 \to \mathcal{O}_S(\Delta-F_i) \to E \to \mathcal{O}_S(F_i) \to 0.$$

(iii) If Δ is Reye, then E is either as in (i), or is isomorphic to the Reye bundle given by a non-split extension:

$$0 \to \mathcal{O}_S(F_i) \to E \to \mathcal{O}_S(\Delta-F_i) \to 0.$$

Moreover, in (iii) the isomorphism class of E does not depend on the choice of F_i.

PROOF. Let E be a regular rank 2 vector bundle as in the statement of the theorem. By assumption, there exists a section of E with only isolated zeroes. Let

$$0 \to \mathcal{O}_S \xrightarrow{s} E \to \mathcal{I}_Z(\Delta) \to 0.$$

be the corresponding exact sequence. Since

$$\mathrm{Ext}^1(\mathcal{I}_Z(\Delta),\mathcal{O}_S) \cong H^1(\mathcal{I}_Z(\Delta+K)) \neq 0,$$

the cycle Z is special with respect to $|\Delta+K|$, i.e. the canonical restriction map:

$$H^0(\mathcal{O}_S(\Delta+K)) \to H^0(\mathcal{O}_Z(\Delta+K)) = k^3$$

is not surjective. This implies that

$$h^0(\mathcal{I}_Z(\Delta+K)) = 4$$

(instead of expected 3), and Z lies on the the base line $\ell(Z)$ of $|\mathcal{I}_Z(\Delta+K)| \subset |\Delta+K|$. In particular, $\ell(Z)$ is a trisecant of S in the embedding $S \subset |\Delta+K|^* = \mathbb{P}^5$. Conversely, if ℓ is a trisecant of S which cuts out a cycle Z of length 3 on S, we can reverse the argument and construct a rank 2 vector bundle E as above.

Returning to our E, let Z be a cycle of length 3 corresponding to E and ℓ be the trisecant which contains it. Since $3\Delta \sim F_1'+\ldots+F_{10}'$, the line ℓ can intersect at most three F_i's. Choose F_i' such that $Z\cap F_i' = \emptyset$.

We claim that there exists a non-trivial morphism

$$\mathcal{O}_S(F_i) \to E.$$

For every j there is a hyperplane in $|\Delta+K_S|^* = \mathbb{P}^5$ which contains ℓ and one of the planes π_j' containing F_j (F_i and F_i' are plane curves in both embeddings $S \hookrightarrow |\Delta|$ and $S \hookrightarrow |\Delta+K_S|$). Since $F_i'\cap Z = \emptyset$, there exists a curve in $|\Delta+K_S-F_i'|$ which contains Z. Thus

$$h^0(\mathcal{I}_Z(\Delta-F_i)) \neq 0.$$

Consider the exact sequence

$$0 \to \mathcal{O}_S(-F_i) \to E(-F_i) \to \mathcal{I}_Z(\Delta-F_i) \to 0.$$

Since $h^0(F_i') = 1$, By Riemann-Roch, $h^1(\mathcal{O}_S(-F_i)) = h^1(F_i') = 0$. This implies that the map

$$H^0(S,E(-F_i)) \to H^0(S,\mathcal{I}_Z(\Delta-F_i)$$

is bijective. Hence $H^0(S,E(-F_i)) \neq 0$ proving our claim.

Let $L = \mathcal{O}_S(D)$ be an invertible subsheaf of E with the maximal degree $\Delta\bullet D$. It yields the following exact sequence:

$$0 \to \mathcal{O}_S(D) \to E \to \mathcal{I}_{Z'}(\Delta-D) \to 0$$

By the choice of L, $D\bullet\Delta \geq F_i\bullet\Delta = 3$. On the other hand, counting $c_2(E)$ from the exact sequence, we obtain

$$(*)\ D\bullet(\Delta-D) + \deg(Z') = c_2(E) = 3,$$

hence

$$-D^2+\deg(Z') = (3-D\bullet\Delta) \leq 0.$$

From this it follows

$$D^2 \geq 0$$

with equality holding if and only if $D\bullet\Delta = 3$, $Z' = \emptyset$.

If $D\bullet\Delta = 3$. $D = F_j$ for some j (Lemma 1), and we obtain the following exact sequence:

$$0 \to \mathcal{O}_S(F_j) \to E \to \mathcal{O}_S(\Delta-F_j) \to 0$$

Thus E is either isomorphic to $\mathcal{O}_S(F_j)\oplus\mathcal{O}_S(\Delta-F_j)$ or E is a non-trivial extension. In the latter case

$$\mathrm{Ext}^1(\mathcal{O}_S(\Delta-F_j),\mathcal{O}_S(F_j)) \cong H^1(S,\mathcal{O}_S(2F_j-\Delta)) \cong H^1(S,\mathcal{O}_S(\Delta-F_j-F_j')) \neq 0.$$

By Riemann-Roch, $h^0((\Delta-F_j-F_j')) \neq 0$. By the proof of Theorem 1, we obtain that Δ is a Reye polarization, and E is the Reye bundle.

Now we turn our attention to the case $D\bullet\Delta > 3$, $D^2 > 0$. Twisting the Koszul sequence for Z by $\mathcal{O}_S(-D)$, we get

$$h^0(\mathcal{O}_Z(\Delta-D)) \neq 0,$$

hence

$$(\Delta-D)\bullet\Delta \geq \deg(Z) = 3,$$

and, by (*)

$$(\Delta-D)^2 = (\Delta-D)\bullet\Delta-(\Delta-D)\bullet D \geq 3-3 +\deg(Z') = \deg(Z') \geq 0.$$

By Riemann–Roch, $h^0(D) \geq 2$. Lemma 3 implies $D\bullet\Delta \geq 6$. This yields

$$(\Delta-D)\bullet\Delta \leq 4,$$

and, again by Lemma 3,

$$(\Delta-D)^2 \leq 0.$$

Thus we obtain $(\Delta-D)^2 = 0$, hence $(\Delta-D)\bullet\Delta = 3$, $\deg(Z') = 0$. Thus $\Delta-D = F_j$ for some j, and we get the exact sequence

$$0 \to \mathcal{O}_S(\Delta-F_j) \to E \to \mathcal{O}_S(F_j) \to 0.$$

Again as above, either E spits or

$$\mathrm{Ext}(\mathcal{O}_S(F_j),\mathcal{O}_S(\Delta-F_j)) \cong H^0(S,\mathcal{O}_S(\Delta-2F_j)) \cong H^0(S,\mathcal{O}_S(\Delta+K_S-F_j-F_j') \neq 0.$$

By Theorem 1, we find that Δ is a Cayley polarization. Since $|F_j-(\Delta-F_i)| = \emptyset$ for all i and j, all the sheaves from case (ii) are non–isomorphic.

It remains to prove the uniqueness statement for E from case (iii). In fact, the corresponding trisecant cannot lie in any plane cubic F. If it does, then F passes through the cycle Z corresponding to E, and there exists a divisor in $|\Delta-(\Delta-F)|$ containing Z. Thus there exists a non–trivial map $\mathcal{O}_S(\Delta-F_i) \to E$, which splits the extension. The proof of the theorem shows that for every F the subsheaf $\mathcal{O}_S(F)$ is saturated in E, and hence E is independent of the choice of F.

The next corollary follows immediately from the previous theorem by using Riemann–Roch and the vanishing theorem (see **[CD1]**).

Corollary 1. *Let E be as in the previous theorem. Then*

$$h^0(E) = 4,\ h^1(E) = h^2(E) = 0.$$

4. Stability. Recall that a vector bundle E is H–stable (resp. H–semi–stable), where H is a divisor, if for every line subbundle L in E

$$L\bullet H < \tfrac{1}{2}c_1(E)\bullet H \text{ (resp. } L\bullet H \leq \tfrac{1}{2}c_1(E)\bullet H).$$

Theorem 3. *Let E be a rank 2 vector bundle on an Enriques surface S with $c_1(E) = \Delta$ and $c_2(E) = 3$. The following assertions are equivalent:*

(i) E is Δ–semi–stable;

(ii) E is isomorphic to the Reye bundle.

(iii) E is Δ–stable.

PROOF. (i) $\Rightarrow$ (ii) By Proposition 1, $h^0(E) \neq 0$. We may assume that the zero set of any non–zero section has a 1-dimensional part D. Otherwise E is regular, and the semi-stability implies that case (iii) of Theorem 2 must hold. Consider the exact sequence

$$0 \to \mathcal{O}_S(D) \to E \to \mathcal{J}_Z(\Delta-D) \to 0.$$

By semi-stability we have

$$D\bullet\Delta \le \tfrac{1}{2}\Delta^2 = 5.$$

From Lemma 3 it follows

$$D^2 \le 0.$$

Counting the Chern classes of E by using the above exact sequence, we obtain

$$D\bullet(\Delta-D) + \deg(Z') = 3$$

which yields

$$D\bullet\Delta \le 3+D^2.$$

Since $D\bullet\Delta \ge 1$ this leaves us with the cases $D^2 = -2$ or $D^2 = 0$.

Case 1. $D^2 = -2$. Then $D\bullet\Delta = 1$, $Z' = \emptyset$, and D is a line in the embedding defined by $|\Delta|$. We have the exact sequence:

$$0 \to \mathcal{O}_S(D) \to E \to \mathcal{O}_S(\Delta-D) \to 0.$$

By Lemma 2, D does not lie in a any plane π_i unless it is a component of F_i. This shows that $D\bullet F_i \le 1$ for every i. Since $\Delta\bullet D = 1$, there are exactly three F's with $F_i\bullet D = 1$. Choose one of them. Then $(\Delta-D-F_i)^2 = 2$, and, by Riemann-Roch, $h^0(\Delta-D-F_i) \ge 2$. Twisting the above sequence by $\mathcal{O}_S(-F_i)$, we obtain the cohomology exact sequence:

$$0 \to H^0(S,\mathcal{O}_S(D-F_i)) \to H^0(S,E(-F_i)) \to H^0(S,\mathcal{O}_S(\Delta-D-F_i) \to H^1(S,\mathcal{O}_S(D-F_i)).$$

Since $(D-F_i)\bullet\Delta < 0$, the first space is zero. Since $(D-F_i)^2 = -4$, by Riemann-Roch, the last space is one-dimensional. This implies that $H^0(S,E(-F_i)) \ne 0$, hence E contains $\mathcal{O}_S(F_i)$ as a subsheaf, and therefore is represented as an extension

$$0 \to \mathcal{O}_S(F_i) \to E \to \mathcal{O}_S(\Delta-F_i) \to 0.$$

If the extension splits, $\mathcal{O}_S(\Delta-F_i)$ will be the subbundle of E with $(\Delta-F_i)\bullet\Delta = 7 > 5$. This contradicts the semi-stability of E. By Theorem 2, E must be the Reye bundle.

Case 2. $D^2 = 0$. Then $D\bullet\Delta = 3$, hence $D = F_i$ for some i. Also $\deg(Z')' = \emptyset$, and we get the exact sequence as above. Applying Theorem 2, we obtain that E is the Reye bundle.

(ii) $\Rightarrow$ (iii). This follows from the proof of Theorem 1 (take a destabilizing subbundle L and argue as in the proof of this theorem).

(iii) $\Rightarrow$ (ii) Obvious.

Examples. Here we give examples of non-regular rank 2 vector bundles on S with $c_1(E) = \Delta$ and $c_2(E) = 3$. The first example is a decomposable bundle

$$E = \mathcal{O}_S(D)\oplus\mathcal{O}_S(\Delta-D),$$

where $D = F_1+F_2+F_3$. Then

$$D^2 = 6,\ D\bullet(\Delta-D) = 3,\ (\Delta-D)^2 = -2,\ \Delta\bullet(\Delta-D) = 1.$$

If $|\Delta-D| = \emptyset$, i.e. S does not contain lines (with respect to Δ), then

$$H^0(E) = H^0(\mathcal{O}_S(D)),$$

hence all sections of E vanish on a curve from $|D|$.

The second example is an indecomposable extension:

$$0 \to \mathcal{O}_S(D) \to E \to \mathcal{O}_S(\Delta-D) \to 0,$$

where D is as above. We assume again that S does not contain lines, hence $|\Delta-D| = \emptyset$. But this time we assume that Δ is Cayley. Then

$$h^1(2D-\Delta) = \dim \mathrm{Ext}^1(\mathcal{O}_S(\Delta-D),\mathcal{O}_S(D)) \neq 0$$

if and only if $|2D-\Delta| \neq \emptyset$ (note that $(2D-\Delta)^2 = -2$ and apply Riemann–Roch). Since

$$2D-\Delta \sim 2F_1+2F_2+2F_3-\Delta = (\Delta-2F_1)+2(F_2+F_3+2F_1-\Delta) \equiv (\Delta-2F_1) \bmod 2\mathrm{Pic}(S)$$

and $\Delta-2F_1$ is effective with $(\Delta-2F_1)^2 = -2$, $|2D-\Delta| \neq \emptyset$ (Looijenga's lemma, see **[CD2]**). Thus, if Δ is Cayley, we can construct an indecomposable extension. Since $h^0(\Delta-D) = 0$, $h^0(E) = h^0(\mathcal{O}_S(D))$, and E is non-regular.

Remark 1. It is easy to see that the Reye bundle is extremal, i.e. satisfies:

$$E \cong E(K_S),\ \mathrm{Ext}^0(E,E) \cong \mathrm{Ext}^2(E,E) \cong k,\ \mathrm{Ext}^1(E,E) = 0.$$

It is interesting to find other vector bundles on an Enriques surface satisfyng these conditions*. We refer to **[Ku]** for the study of extremal vector bundles on K3-surfaces.

5. An application. In this section we give another proof of the following result of A. Conte and A. Verra **[CV]**:

Theorem 4. *Let X be the subvariety of G(2,6) parametrizing trisecants of an Enriques surface S of degree 10 in $\mathbb{P}^5$. Then dim X = 3 if $\Delta = \mathcal{O}_S(1)$ is Cayley and dim X = 2 (and consists of 20 planes) otherwise.*

PROOF. Let ℓ be a trisecant of S and Z be the corresponding cycle of length 3. As in the proof of Theorem 2, we construct a vector bundle E given by an extension:

$$0 \to \mathcal{O}_S \to E \to \mathcal{I}_Z(\Delta) \to 0.$$

If S is unnodal, $E \cong \mathcal{O}_S(\Delta-F)\oplus\mathcal{O}_S(F)$ for some plane cubic, and $Z = (\Delta-F)\cap F$ lies in the plane containing F. Conversely, every line in such a plane is a trisecant. This yields that the variety X of trisecants is equal to the union of 20 Schubert planes (of lines in each plane of F).

Assume S is nodal. If $|\Delta|$ does not map S into a quadric, every E as in the theorem splits and X is the same as in the previous case. Assume that $|\Delta|$ maps S into a quadric (then $|\Delta+K|$ does not map S into any

*All stable exceptional rank 2 bundles E on an Enriques surface have been described in a recent thesis of Hoil Kim **[Ki]**. They satisfy $c_2(E) = t$, $c_1(E)^2 = 4t-2$ and exist only for nodal Enriques surfaces (for any $t \geq 3$). Each such a bundle is uniquely determined by its Chern classes and can be obtained from an extension

$$0 \to \mathcal{O}_F \to E \to \mathcal{O}_F(R) \to 0,$$

where R is a nodal cycle, by tensoring by an invertible sheaf. In **[CV]** it is shown that each generic nodal Enriques surface can be embedded into the Grassmannian G(2,t+1) as a congruence of bidegree (3t−2,t). This allows one to define a Reye bundle E with $c_2(E) = t$, $c_1(E)^2 = 4t-2$. It is stable and extremal. We do not know whether any extremal stable rank 2 vector bundle on an Enriques surface is isomorphic to a Reye bundle.

quadric). Let E be a non-trivial bundle E as in the statement of the theorem. We know that $\mathbb{P}(\Gamma(E)) \cong \mathbb{P}^3$. For general section s of E its zero cycle lies on a trisecant in the embedding by $|\Delta+K|$. This defines a rational map $\mathbb{P}^3 \to X$. Since E is unique, this map is dominant.

Remark 2 ([CV]). Assume that Δ is a Cayley polarization. Then the union of trisecants of S is isomorphic to the quartic hypersurface of singular quadrics in the 5-dimensional linear system of quadrics parametrized by $|\Delta|^*$.

Corollary 3. *Let S be an Enriques surface of degree 10 in $\mathbb{P}^5$ and C be its smooth hyperplane section. If $\Delta = \mathcal{O}_S(1)$ is not Reye, then C is a non-trigonal curve of genus 6. If Δ is Reye, then C is a trigonal curve of genus 6 if and only if the hyperplane is tangent to the quadric containing S.*

PROOF. It is clear that any nonsingular curve $C\in|\Delta|$ is of genus 6. It is easy to see that C is not hyperelliptic (see **[CD1]**). Assume C is trigonal. Then its canonical image lies on a scroll, hence C has infinitely many (∞^1) trisecants. Note that the canonical map of C is given by $|\Delta+K|$. It is known that every smooth curve with finitely many trisecants in its Prym-canonical embedding has at most 20 trisecants (see **[Ve]**). Thus in the embedding $S \to |\Delta+K|^*$, we can find a trisecant not lying on any of the 20 planes plane cubic curves of S. By Corollary 1 this happens if and only if Δ is Reye. Let S lie on a non-singular quadric Q and $C = S\cap H$ be its smooth hyperplane section, where H is a tangent hyperplane to Q at some point $p\in Q$. The intersection $H\cap Q$ is a cone over a quadric in $\mathbb{P}^3$ with the vertex at p. One of the projection to $\mathbb{P}^1$ from the point p has fibres equal to the planes belonging to the family of planes on Q defining $c_2(E)$. This shows that the induced projection of $C = H\cap S$ to $\mathbb{P}^1$ is defined by a trigonal linear series on C. Thus we have an irreducible 4-dimensional family of trigonal hyperplane sections curves on S. The family of 0-cycles $Z\in \mathrm{Sym}^3(S)$ which define a trisecant on the Cayley embedding of S is 3-dimensional. Each such cycle is contained in a 2-dimensional family of hyperplane sections C of Q. Since each trigonal curve has infinitely many "trisecant" cycles Z, the variety of trigonal hyperplane sections of Q is an irreducible variety of dimension 4. Hence it coincides with the variety of hyperplane sections $H\cap Q$, where H is a tangent hyperplane to Q.

Remark 3. If Z is a zero cycle of section of E, and $c_1(E) = \Delta$ is Cayley or S is unnodal, then $H\in|\Delta|$ containing Z is reducible, and equals the union of a plane cubic F and a curve from $|\Delta-F|$.

6. Congruences of lines. The stability of the Reye bundle implies the next Corollary. We give another version of its proof.

Corollary 4. *Let E be the tautological quotient bundle on G(2,4) and E be its restriction to a Reye congruence S of bidegree (7,3). Then E is stable.*

PROOF. It is known that S is a nodal Enriques surface embedded into $\mathbb{P}^5$ by $|\Delta|$. Clearly $c_1(E) = [\Delta]$, $c_2(E) = 3$. It is easy to see that E is regular (the zero set of a generic section of E is equal to the set of rays lying in a plane of $\mathbb{P}^3$, which cuts out 3 points on S). By Theorem 2 (ii), E is either stable or isomorphic to the direct sum $\mathcal{O}_S(F)\oplus\mathcal{O}_S(\Delta-F)$. The projection $E \to \mathcal{O}_S(F)$ defines a section $i:S \to \mathbb{P} = \mathbb{P}(E)$ such that $i^*(\mathcal{O}_{\mathbb{P}}(1)) = \mathcal{O}_S(F)$. On the other hand, the linear system $|\mathcal{O}_{\mathbb{P}}(1)|$ defines a map $\mathbb{P} \to \mathbb{P}^3$ with the property that the image of the fibre of $\mathbb{P} \to S$ over a point $s\in S$ is equal to the ray in $\mathbb{P}^3$ corresponding to $s\in G(2,4)$. This shows that the composition $S \to \mathbb{P} \to \mathbb{P}^3$, given by $|\mathcal{O}_S(F)|$ is constant, i.e. all rays of s pass through one point. But then S is a Schubert plane of lines passing through a point. Absurd.

Conjecture. *Let $S \subset G(2,4)$ be a nonsingular congruence of lines. F and E be the tautological subbundle and quotient bundle, respectively. Assume that S is non-degenerate, i.e. does not lie in a hyperplane section of G(2,4) (with respect to the Plucker embedding). Then the restrictions of E and F to S are both semi-stable.*

If this is true, applying Bogomolov's theorem, we obtain that

$$m+n = c_1(E|S)^2 \leq 4c_2(E|S) = 4n,$$

$$m+n = c_1(F|S)^2 \leq 4c_2(F|S) = 4m,$$

that is,

$$m \leq 3n,\ n \leq 3m.$$

This has been observed for all known smooth non-degenerate congruences in G(2,4)*

References

[BP] W. Barth, C. Peters, Automorphisms of Enriques surfaces, Inv. Math. 73 (1983), 383–411.

[CD1] F. Cossec, I. Dolgachev, Enriques surfaces I. Birkhauser. 1989

[CD2] F. Cossec, I. Dolgachev, Enriques surfaces II (in preparation).

[Co] F. Cossec, Reye congruences, Trans. Amer. Math. Soc. 280 (1983), 737–751.

[CV] A. Conte, A. Verra, Reye constructions for nodal Enriques surfaces.(to appear in Trans. Amer. Math. Soc.)

[Ki] H. Kim, Vector bundles on Enriques surfaces. Ph. D. thesis. Univ. of Mich. 1990.

[Ku] S. Kuleshov, A theorem on existence of exceptional bundles on surfaces of type K3, Izv. Akad. Nauk SSSR, Ser. Math., 53, (1989), 363–378.

[S-B] N. Shepherd-Barron, Unstable vector bundles and linear systems on surfaces in characteristic p (preprint).

[Ve] A. Verra, A short proof of unirationality of $\mathfrak{a}_5$, Indagationes Mathematicae, 46 (1984), 339–355.

* Many interesting results related to this conjecture were obtained by Mark Gross in his 1990 Berkeley Ph.D. thesis. In particular he verified this conjecture for all non-degenerate congruences of Kodaira dimension $\neq 2$.