

Chilean configuration of conics, lines and points

IGOR DOLGACHEV, ANTONIO LAFACE, ULF PERSSON
AND GIANCARLO URZÚA

Abstract. Using the theory of rational elliptic fibrations, we construct and discuss a one parameter family of configurations of 12 conics and 9 points in the projective plane that realizes an abstract configuration $(12_6, 9_8)$. This is analogous to the famous Hesse configuration of 12 lines and 9 points forming an abstract configuration $(12_3, 9_4)$. We also show that any Halphen elliptic fibration of index 2 with four triangular singular fibers arises from such configuration of conics.

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1. Introduction

The famous Hesse configuration $(12_3, 9_4)$ of 12 lines and 9 points, with 3 points on each line and 4 lines through each point, can be geometrically realized by the Hesse pencil of cubic curves

$$x^3 + y^3 + z^3 + txyz = 0$$

with 9 base points and 4 reducible members each consisting of three lines. Each line contains three base points, and each base point is on four reducible members. By blowing up the base points, we obtain a rational elliptic surface with four reducible fibers of type I_3 in Kodaira's notation for singular fibers of elliptic fibrations on algebraic surfaces. A pencil of cubic curves is the first in the series of Halphen pencils of plane curves whose general member is of degree $3m$ with nine

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m -multiple points (including infinitely near points). The number m is called the index of the Halphen pencil.

In this paper we give an explicit construction and a complete discussion of an Halphen pencil of index 2 that contains exactly four reducible members, each is the union of three smooth conics. Each of these 12 conics contains 6 base points and each base point lies on 8 of these conics, forming an abstract configuration $(12_6, 9_8)$ which is analogous to the Hesse configuration. Even more surprising is that one can find the dual Hesse configuration $(9_4, 12_3)$ of 9 lines and 12 points embedded in our configuration of conics. It consists of the 12 singular points of the reducible members of the Halphen pencil and 12 lines that are analogous to the 9 harmonic polar lines in the Hesse configuration.

The paper originates from a question of Piotr Pokora who asked one of the authors about an interesting configuration of conics in the plane. In [22] Pokora and T. Szemberg proved that our configuration of conics gives an example of a free arrangement of conics. Independently, and about the same time, the construction of the configuration of conics $(12_6, 9_8)$ was given by D. Koher, X. Roulleau and A. Sarti [18] who used it for constructing a configuration of 12 disjoint smooth rational curves on a generalized Kummer surface obtained as the double cover of the projective plane ramified over the dual of a smooth cubic curve. The 12 conics are conics through a set of six cusps among the nine cusps of the dual curve (see Proposition 5.1). In [23], motivated by computation of the Harbourne constant of a configuration of curves on a rational surface (see section 8 in our paper), Roulleau studied the conic configuration $(12_6, 9_8)$ as an example of a conic configurations related to the duals of a plane algebraic curve.

The present construction arose from discussions during the fourth Latin American School on Algebraic Geometry and its Applications (ELGA IV) held on December 2019 in Talca, Chile. This explains the name for our configuration of conics.

We work over any algebraically closed field of characteristic different from 2 and 3.

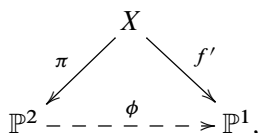
The paper is organized as follows. In Section 2, we discuss Halphen pencils and, in particular, provide structure theorems for (-1) -curves on rational elliptic surfaces coming from such pencils. Starting from this point we focus on rational elliptic surfaces whose Jacobian surface is the Hesse surface. The definition of the latter surface is given in Section 3. Section 4 is devoted to the explicit construction of Chilean surfaces: a one-parameter family of rational elliptic surfaces of index two whose Jacobian is the Hesse surface. In Section 5, we show that any such rational elliptic surface is a Chilean surface. Section 6 introduces a double plane model for elliptic surfaces of index two which allows one to describe their (-1) -curves. Such description is given in the same section for the Chilean surfaces. Section 7 deals with rational elliptic surfaces of higher index whose Jacobian surface is the Hesse surface. Finally, in Section 8, we discuss some few properties and applications such as log Chern numbers for configurations of lines and conics which are naturally associated to our Halphen pencils.

2. Halphen pencils

Let $\lambda F + \mu G = 0$ be any pencil of plane curves whose general fiber is birationally isomorphic to an elliptic curve, a smooth projective curve of genus 1. It defines a rational map

$$\phi: \mathbb{P}^2 \dashrightarrow \mathbb{P}^1, \quad (x_0 : x_1 : x_2) \mapsto (F(x_0, x_1, x_2) : G(x_0, x_1, x_2)). \tag{2.1}$$

Let $\pi: X \rightarrow \mathbb{P}^2$ be a resolution of base points of the pencil. We have the following commutative diagram



where f' is an elliptic fibration. The birational morphism π admits a factorization

$$\pi: X = X_N \xrightarrow{\pi_N} X_{N-1} \xrightarrow{\pi_{N-1}} \dots \xrightarrow{\pi_2} X_1 \xrightarrow{\pi_1} X_0 = \mathbb{P}^2, \tag{2.2}$$

where each morphism $\pi_i: X_i \rightarrow X_{i-1}$ is the blow-up of one point $x_i \in X_{i-1}$. For any $i \geq j$, let $\pi_{i,j} := \pi_j \circ \dots \circ \pi_i: X_i \rightarrow X_{j-1}$. A point $x_i \in X_i$ with $\pi(x_i) = x_j \in X_{j-1}$ is called *infinitely near* to x_j (of order $i - j$). The points $\pi_{i,1}(x_i) \in \mathbb{P}^2, i = 1, \dots, N$, are the intersection points of two general members of the pencil. Their number could be less than N .

The points $\pi_{i,2}(x_i) \in X_1, i = 2, \dots, N$, are the intersection points of the proper transforms of two general members of the pencil in X_1 , and so on, until the proper transforms of the members of the pencil on X has no intersection points and hence we obtain a morphism $f': X \rightarrow \mathbb{P}^1$ whose general fiber is birationally isomorphic to the general member of the pencil. Since X is smooth, a general fiber F of f being birationally isomorphic to an elliptic curve must be smooth, and hence F is an elliptic curve. Let $\sigma: X \rightarrow S$ be a birational morphism to a relatively minimal model of the fibration $f': X \rightarrow \mathbb{P}^1$. By definition, f' is a composition of σ with a morphism $f: S \rightarrow \mathbb{P}^1$ whose fibers do not contain (-1) -curves, smooth rational curves with self-intersection -1 . Let F be a general fiber of f . Let D be a divisor on S with $\mathcal{O}_S(D) \cong \mathcal{O}_S(K_S)$. Restricting D to the general fiber S_η we obtain a divisor linearly equivalent to zero on S_η . Replacing it by a linearly equivalent divisor we obtain that the restriction of D to S_η is the zero divisor, *i.e.* the support of D is contained in fibers of f .

We use the well-known fact that the restriction of the intersection form of divisor classes on S to the subgroup generated by irreducible components of a fiber is semi-negative definite with the radical generated over \mathbb{Q} by F itself [3, Proposition VIII.3]. This implies that any irreducible component Θ of a reducible fiber has negative self-intersection, hence we obtain $\Theta^2 \leq -2$, and, by adjunction formula, $\Theta \cdot K_S \geq 0$. Since Θ is a part of F , we obtain $\Theta \cdot K_S = 0$. Since K_S has a representative contained in fibers, we obtain that

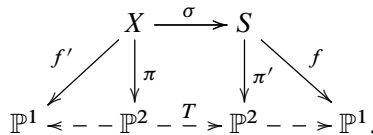
$$F = -mK_S \tag{2.3}$$

for some rational number m . Intersecting both sides with the divisor class of a (-1) -curve E on S , we get that $m = F \cdot E$ is a positive integer. Let $\pi' : S \rightarrow V$ be a birational morphism to a minimal rational surface V . It follows from (2.3) that S has no smooth rational curves with self-intersection less than -2 . This easily implies that we can choose V to be \mathbb{P}^2 . Let us take X in (2.2) to be our surface S .

Let $\mathcal{E}_i = \pi_{N,i}^*(\pi_i^{-1}(x_i)), i = 1, \dots, 9$. Then $e_0 := \pi^*(\text{line}), e_i := \mathcal{E}_i$ form a basis in the Picard group $\text{Pic}(X)$ of divisor classes on X satisfying $e_0^2 = 1, e_i^2 = -1, e_i \cdot e_j = 0, i \neq j$. It is called a *geometric basis*. The well-known behavior of the canonical class under the blow-up at a point gives

$$K_S = -3e_0 + e_1 + \dots + e_N. \tag{2.4}$$

Since $K_S^2 = 0$, formula (2.4) shows that $N = 9$. Thus we obtain a commutative diagram of rational maps



The birational map T transforms our original pencil to a pencil of elliptic curves with 9 base points y_1, \dots, y_9 with some of them may be infinitely near points. The existence of such transformation T was first proven by E. Bertini in 1877 [4] (a modern proof following the arguments from above was first given in [7]). Its proper transform on S belongs to the linear system $| -mK_S |$. Applying formulas (2.4) and (2.3), we obtain that a general member F of the elliptic pencil on S satisfies $F \sim mF_0$. Thus its image in the plane is a curve $F_{3m} = 0$ of degree $3m$ with m -multiple points at y_1, \dots, y_9 . The pencil can be written in the form

$$\lambda F_{3m} + \mu G_3^m = 0. \tag{2.5}$$

It is called an *Halphen pencil* (of index m) in honor of G. Halphen who was the first to give a detailed discussion of the properties of such pencils [15]. Of course, if $m = 1$, this is a pencil of plane cubics. Note that, if $m > 1$, one of the fibers of f is of the form $F = mF_0$ (because $F \sim -mK_S$). It is called the m -multiple fiber of the elliptic fibration. Obviously there is only one multiple fiber since otherwise we can take F_{3m} to be equal to $F_{3s}^{m/s}$ for some s dividing m and obtain that a general member of the pencil is reducible.

Lemma 2.1. *Let $f : S \rightarrow \mathbb{P}^1$ be a relatively minimal elliptic fibration on a rational surface S defined by an Halphen pencil of index m with a fiber $F = mF_0$. Let (e_0, e_1, \dots, e_9) be the geometric basis on S defined by a birational morphism $\pi : S \rightarrow \mathbb{P}^2$. Then $\mathcal{O}_{F_0}(-K_S) = \mathcal{O}_{F_0}(3e_0 - e_1 - \dots - e_9)$ has order m in the Picard group $\text{Pic}(F_0)$. Moreover, if mF_0 is a multiple fiber with m prime to the characteristic $p = \text{char}(\mathbb{k})$ of \mathbb{k} (respectively divisible by p), then F_0 is of Kodaira's type I_n (respectively of other types and $m = p$).*

Proof. We have already used that $\mathcal{O}_F(K_S) \cong \mathcal{O}_F$ for a general fiber F of the fibration. Since we can find a fiber F disjoint from F_0 , we obtain

$$\mathcal{O}_{F_0}(F) \cong \mathcal{O}_{F_0}(-mK_S) \cong \mathcal{O}_{F_0}(-K_S)^{\otimes m} \cong \mathcal{O}_{F_0}.$$

This shows that the order of $\mathcal{O}_{F_0}(-K_S)$ in $\text{Pic}(F_0)$ divides m . Suppose it is equal to $k < m$. Since $|lF_0| + (m-l)F_0 \subset |mF_0|$ for any $l < m$ and $|mF_0|$ is an irreducible pencil, we have $h^0(lF_0) = 1$. Applying Riemann-Roch, we deduce from this that $h^1(lF_0) = 0$. Then the exact sequence

$$0 \rightarrow \mathcal{O}_S((k-1)F_0) \rightarrow \mathcal{O}_S(kF_0) \rightarrow \mathcal{O}_{F_0}(kF_0) \cong \mathcal{O}_{F_0} \rightarrow 0$$

shows that there exists a section s of $\mathcal{O}_S(kF_0)$ whose divisor of zeros D is an element of the linear system $|kF_0|$ that is disjoint from kF_0 . This shows that kF_0 moves in a pencil generated by D and kF_0 , contradicting equality $h^0(kF_0) = 1$ from above.

The last assertion follows from the fact that the m -torsion subgroup $\text{Pic}(F_0)$ is non-trivial only in the cases from the assertion from the lemma [6, Chapter 4, Section 1]. □

Remark 2.2. We used here that S is a rational elliptic surface. In general, an elliptic surface may have multiple fibers mF_0 such that $\mathcal{O}_{F_0}(F_0)$ is of order strictly dividing m and even could be equal to 1. Such fibers are called wild and occur only if $\text{char}(\mathbb{k})$ divides m . Any elliptic surface X with $H^1(X, \mathcal{O}_X) = 0$, as in our case, has no wild multiple fibers.

Assume that F_0 is a smooth elliptic curve, and let $\{G_3 = 0\}$ be its image in the plane. Then one can interpret the assertion of the previous lemma by saying that choosing a group law on the plane cubic $G_3 = 0$, the base points y_1, \dots, y_9 add up to a m -torsion point. This follows from the linear equivalence $m(y_1 + \dots + y_9) \sim 3mh$, where h is the divisor class of the intersection of the cubic with a general line in the plane. This makes sense even when some of the points are infinitely near.

Conversely, choose 9 distinct points y_1, \dots, y_9 on a smooth plane cubic C such that the divisor class of $y_1 + \dots + y_9 - 3h$ is of order m in the Picard group. Choose an inflection point q with the tangent line $\{L = 0\}$ to be the zero point in the group law on $C : \{G_3 = 0\}$. Then the points y_i taken with multiplicity m add up to zero. Let F_{3m} be a homogeneous polynomial such that the restriction of the rational function F_{3m}/L^{3m} to C is a rational function ϕ with $\text{div}(\phi) = m(y_1 + \dots + y_9) - 3mq$. The choice of F_{3m} is not unique and one can choose P_{3m} such that the points y_1, \dots, y_9 are points on the curve $F_{3m} = 0$ of multiplicities m [8, Lemma 4.4]. The pencil $\{\lambda F_{3m} + \mu G_3^m = 0\}$ is an Halphen pencil of index m .

Let F_η be the generic fiber of an elliptic fibration considered as an elliptic curve over the field $K = \mathbb{k}(t)$ of rational functions on \mathbb{P}^1 . A rational point of $F_\eta(L)$ over a finite extension L/K defines, by passing to its Zariski closure on S , an irreducible curve such that the restriction of f to it is a finite cover of degree equal to $d = [L : K]$. We call such a curve a d -section. In particular, a rational

point in $F_\eta(K)$ defines a section of the fibration, and conversely any section arises in this way from a rational point on F_η .

Suppose $f: S \rightarrow \mathbb{P}^1$ is a rational elliptic surface arising from an Halphen pencil of index m . Since one of the fibers is of the form $F = mF_0$, intersecting a d -section with F , we obtain that m divides d . On other hand, any (-1) -curve E on S satisfies $E \cdot F = -mE \cdot K_S = m$, hence it defines an m -section.

Suppose $m > 1$, then $F_\eta(K) = \emptyset$. The Jacobian variety $\text{Jac}(F_\eta)$ parametrizes the divisor classes of degree 0 on F_η . It is an elliptic curve with the group law defined by the addition of the divisor classes. It becomes isomorphic to F_η over a field extension L of K such that $F_\eta(L) \neq \emptyset$. The curve F_η is a torsor (= principal homogenous space) over $\text{Jac}(F_\eta)$. This means that there is an action morphism $a: \text{Jac}(F_\eta) \times F_\eta \rightarrow F_\eta$ over K such that the morphism $(a, p_2): \text{Jac}(F_\eta) \times F_\eta \rightarrow F_\eta \times F_\eta$, where p_2 is the second projection map, is an isomorphism. The theory of minimal models of algebraic surfaces allows one to find a relatively minimal elliptic fibration $j: J \rightarrow \mathbb{P}^1$ with the generic fiber J_η isomorphic to $\text{Jac}(F_\eta)$. It is called the *Jacobian fibration* of f . One can show that the isomorphism class of a non-trivial torsor F_η of J_η realized as the generic fiber of a rational elliptic surface is uniquely determined by a choice of the data (x_0, τ) that consists of a point $x_0 \in \mathbb{P}^1$ and a m -torsion class τ in the connected component of the identity of the Picard scheme of the fiber J_{x_0} of j over the point x_0 [6, Chapter 4, Section 8]. The corresponding fiber of $f: S \rightarrow \mathbb{P}^1$ over x_0 is the m -multiple fiber mF_0 . All other fibers of f are isomorphic to fibers of j . We refer to Remark 5.5 where this construction is made very explicit for Halphen surfaces of index 2.

Thus we obtain that to any Halphen pencil of index $m > 1$ given by (2.5) one can associate an Halphen pencil of index 1 where one of the members is isogenous of degree m to the cubic curve $\{G_3 = 0\}$. All other members are birationally isomorphic to the members of the Halphen pencil.

Remark 2.3. Over the complex numbers Kodaira introduced a highly transcendental transformation (a so called logarithmic transformation, see [2]) that for any (positive) integer m makes any smooth fiber (or more generally any so called semi-stable fiber I_n) into a fiber with multiplicity m while leaving the complements biholomorphic (but not birational in general). In particular, other fibers are left unscathed, however it causes havoc among transversal divisors. As the transform has an inverse, this is the ‘only’ way multiple fibers occur. There is also the inverse transformation that coincides, in the category of algebraic surfaces, with taking the Jacobian fibration. One can prove, using the formula for the canonical class of an elliptic surfaces that Halphen surface are the only ones that are remain of the same birational type as their Jacobian surface. This is a unique situation, which accounts for its particular allure. In particular we note that any configuration of degenerate fibers that occur in a cubic pencil ($m = 1$) also occur for any $m > 1$ and conversely (with the obvious caveat); but their projective realizations are of course very different. To make those explicit has provided an intriguing challenge for some of the authors, the elaboration of a specific instance resulted in the present paper.

We begin with the following well-known lemma. We provide a proof for the sake of completeness.

Lemma 2.4. *Let S be a smooth rational surface with nef anticanonical divisor $-K_S$. Then any divisor E such that $E^2 = E \cdot K_S = -1$ which has non-negative intersection with all the (-2) -curves of S is linearly equivalent to a (-1) -curve.*

Proof. Assume E satisfies the hypothesis of the theorem. By Riemann-Roch E is linearly equivalent to an effective divisor. Without loss of generality, we can assume E itself to be effective. Since $-K_S$ is nef it has non-negative intersection with any irreducible curve and, by adjunction, the only curves which have intersection 0 with $-K_S$ are the (-2) -curves. It follows that $E = E' + R$, where E' is a (-1) -curve and R is a non-negative sum of (-2) -curves. From $E^2 = -1$ we deduce $E' \cdot R + (E' + R) \cdot R = 0$. Since $E' \cdot R \geq 0$, and E' being irreducible and not contained in R , we deduce that $0 \leq E \cdot R = (E' + R) \cdot R \leq 0$. Thus $E' \cdot R = R^2 = 0$. Since R is in K_S^\perp , which is a negative-semidefinite lattice, we conclude that $R \sim nK_S$ for some $n \in \mathbb{Z}$. The equation $(E' + nK_S)^2 = -1$ forces $n = 0$, so that $E = E'$. \square

Proposition 2.5. *Let S be a rational elliptic surface which is relatively minimal, that is, there are no (-1) -curves in the fibers of the elliptic fibration $S \rightarrow \mathbb{P}^1$. Assume $F_0 \in |-K_S|$ is smooth. Then the kernel of the restriction map*

$$\text{res}: \text{Pic}(S) \rightarrow \text{Pic}(F_0)$$

contains the subgroup Λ generated by the classes of the (-2) -curves of S . Moreover if π is extremal, that is Λ has rank 9, then $\Lambda = \ker(\text{res})$.

Proof. The first statement is clear because any (-2) -curve is disjoint from F_0 . For the second statement see [17, Theorem IV.5]. \square

We now delve into the description of the set $\text{Exc}(S)$ of (-1) -curves on S . For a similar approach see [17]. Recall that the normal bundle $\mathcal{O}_{F_0}(F_0) \cong \mathcal{O}_{F_0}(-K_S)$ is of order m in the Picard group of F_0 . We denote it by $\text{res}(F_0)$. Let pts_0 be the subset of F_0 which consists of points cut out by all the (-1) -curves of S as well as their translates by any element of the subgroup of $\text{Pic}(F_0)$ generated by $\text{res}(F_0)$.

Proposition 2.6. *The action of K_S^\perp/Λ on F_0 by translations induces a transitive action of the same group on pts_0 . If $\Lambda = \ker(\text{res})$, then the action is free.*

Proof. Let $q \in \text{pts}_0$. By the definition of pts_0 , there is a translate p of q by a multiple of $\text{res}(F_0)$ which is cut out by a (-1) -curve E . Given $[D] \in K_S^\perp$ the divisor $E + D$ has intersection -1 with K_S , so that $(E + D)^2$ is an odd integer by the genus formula. Then there exists an integer r such that $(E + D + rF_0)^2 = -1$. By Riemann-Roch the divisor $E + D + rF_0$ is linearly equivalent to an effective divisor L . Write $L = H + R$, where H contains all the irreducible components which are not contracted by π , while R contains the contracted ones, so that $[R] \in$

$\Lambda + \langle F_0 \rangle$. By the definition of H it follows that it has non-negative intersection with any (-2) -curve of S . Since $H \cdot F_0 = 1$, it follows that H^2 is an odd integer by the genus formula. Then there exists an integer s such that $(H + sF_0)^2 = -1$. By Lemma 2.4 the divisor $H + sF_0$ is linearly equivalent to a (-1) -curve E' . The class of the difference $E' - (E + D) \sim H + sF_0 - (H + R - rF_0)$ is in $\Lambda + \langle F_0 \rangle$. So, if we denote by p' the intersection point of E' with F_0 , the class of the difference $p + \text{res}(D) - p'$ is a multiple of $\text{res}(F_0)$. As a consequence $p'' \sim p + \text{res}(D)$ is in pts_0 . This shows that K_S^\perp acts on pts_0 . Since K_S^\perp contains the classes of differences of (-1) -curves and the class of F_0 , the action is transitive. This proves the first statement. The second statement is clear. \square

Let M be a negative definite irreducible root lattice of type A_n, D_n, E_n , and $\alpha_1, \dots, \alpha_n$ be its basis of simple roots. Let \tilde{M} be its extension to an affine root lattice by adding a root α_0 such that the radical \mathfrak{f} of \tilde{M} is generated by $\mathfrak{f} = \alpha_0 + \alpha_{\max}$, where $\alpha_{\max} = \sum_{i=1}^n m_i \alpha_i$ is the maximal root with respect to the basis of simple roots.

We extend this definition to the case when the lattice M is the orthogonal sum $M_1 \oplus \dots \oplus M_k$ of irreducible root lattices by taking the lattice \tilde{M} with the radical of rank one such the quotient by the radical is M .

Fix an integer valued linear function l on \tilde{M} and consider the convex subset in $\tilde{M}_{\mathbb{R}}$ defined by

$$\Pi_M(l) := \{x \in \tilde{M}_{\mathbb{R}} : x \cdot \alpha_i \leq l(\alpha_i), i = 0, \dots, n\}.$$

For any α_i , we have $\mathfrak{f} - \alpha_i$ is a positive root (*i.e.* a positive integer linear combination of simple roots). This implies that, for any $x \in \Pi_M(l)$, $-x \cdot \alpha_i = x \cdot (\mathfrak{f} - \alpha_i) \leq l(\mathfrak{f} - \alpha_i)$, hence $x \cdot \alpha_i \geq l(\mathfrak{f} - \alpha_i)$ is bounded from below by $a_i = l(\mathfrak{f} - \alpha_i)$. On the other hand, by definition, $x \cdot \alpha_i$ is bounded from above. Let $\alpha_1^*, \dots, \alpha_n^*$ be the dual basis of $(\alpha_1, \dots, \alpha_n)$ in M^\vee (the fundamental weights). Then we can write any vector $x \in \Pi_M(l)$ as $x = x_0 \mathfrak{f} + \sum_{i=1}^n x_i \alpha_i^*$ and obtain, by taking the intersection with α_i , that $a_i \leq x_i \leq l(\alpha_i)$ and $\sum_{i=1}^n m_i x_i \leq l(\alpha_0)$. Obviously, $\Pi_M(l)$ is preserved under translations by the radical spanned by \mathfrak{f} . So, its image $\bar{\Pi}_M(l) = \Pi_M(l) / \mathbb{R}\mathfrak{f}$ in the quotient $M_{\mathbb{R}}$ of $\tilde{M}_{\mathbb{R}}$ by this subspace is a bounded compact rational polyhedron in $M_{\mathbb{R}}$. Its intersection with the lattice M consists of all roots α satisfying $\alpha \cdot \alpha_i \leq l(\alpha_i), i = 1, \dots, n$. Since the polyhedron is compact, this set is finite. Note that in the case $l \equiv 0$, $\bar{\Pi}_M(l)$ is the image of the fundamental chamber corresponding to the root basis $(\alpha_0, \dots, \alpha_n)$ in $M_{\mathbb{R}}$.

We apply this to our situation when the lattice \tilde{M} arises as the lattice Λ generated by (-2) -curves on S and $[F_0]$ (the last vector is not needed if F_0 is reducible). Its radical is the vector $[F_0]$ and the quotient by $\langle F_0 \rangle$ is a root lattice of finite type. Any divisor class D in $\text{Pic}(S)$, via the intersection product, defines a function l on \tilde{M} as above. Copying the definition of the convex set $\Pi_M(l)$ we introduce the following definition.

Definition 2.7. Given $D \in \text{Pic}(S)$ define the following *tropical Riemann-Roch space*:

$$L^{\text{trop}}(D) := \{R \in \Lambda : (D + R) \cdot C \geq 0 \text{ for any } (-2)\text{-curve } C \text{ of } S\}.$$

The above definition is due to the fact that $L^{\text{trop}}(D)$ behaves like a Riemann-Roch space with respect to the following tropical sum: if $R = \sum_i a_i R_i$ and $R' = \sum_i b_i R_i$ are in $L^{\text{trop}}(D)$ then their tropical sum $R \oplus R' := \sum_i \max\{a_i, b_i\} R_i$ is in $L^{\text{trop}}(D)$.

The next proposition follows from the above discussion.

Proposition 2.8. *The set $L^{\text{trop}}(D)/\langle mK_S \rangle$ is finite for any $D \in \text{Pic}(S)$.*

Proposition 2.9. *Let $E \in \text{Exc}(S)$ and let $p := \text{res}(E)$ be its intersection point with F_0 . Then the assignment $R \mapsto E + R - \frac{1}{2}(2E \cdot R + R^2)F_0$ induces an injection*

$$\phi: L^{\text{trop}}(E)/\langle mK_S \rangle \rightarrow \{E' \in \text{Exc}(S) : \text{res}(E' - E) \in \langle \text{res}(F_0) \rangle\}.$$

If $\Lambda = \ker(\text{res})$, then ϕ is a bijection.

Proof. First of all observe that if $R \in L^{\text{trop}}(E)$ then $D := E + R - \frac{1}{2}(2E \cdot R + R^2)F_0$ has non-negative intersection with all the (-2) -curves of S and $D^2 = D \cdot K_S = -1$. Then D is linearly equivalent to a (-1) -curve E' of S , by Lemma 2.4. Observe that E' intersects F_0 at a point p' such that $p' - p \in \langle \text{res}(F_0) \rangle$. Then ϕ is well defined. We now show that ϕ is injective. Let $R, R' \in L^{\text{trop}}(D)$ be such that $\phi(R) = \phi(R')$. Then $R - R'$ is linearly equivalent to an integer multiple of $F_0 \sim -K_S$. This must be a multiple of mK_S because it lies in Λ , which proves the statement. Assume now that $\Lambda = \ker(\text{res})$. To prove the surjectivity of ϕ , let E' be a (-1) -curve of S such that $\text{res}(E' - E) = n \text{res}(F_0)$ for some integer n . Then $E' - E \sim nF_0 + R$ with $R \in \ker(\text{res}) = \Lambda$. In particular, $E + R + nF_0 \sim E'$ has non-negative intersection with all the (-2) -curves of S and the same holds for $E + R$, so that $R \in L^{\text{trop}}(E)$. □

Remark 2.10. The surface S is acted by the group of η -rational points of $\text{Pic}^0(F_\eta)$, where F_η is the generic fiber. This group is the homomorphic image of K_S^\perp via the pullback of the restriction map to the generic fiber. The kernel consists of the classes of vertical divisors $\Lambda + \langle F_0 \rangle$. Thus $K_S^\perp/\Lambda + \langle F_0 \rangle$ acts on S . The map

$$\text{res}: \text{Exc}(S) \rightarrow \text{pt}_{\mathfrak{s}_0}$$

is equivariant with respect to the above action. If $\Lambda = \ker(\text{res})$ then the action is free on the codomain and the latter is subdivided into m orbits where m is as usual the index of the rational elliptic surface. In particular, if Λ has rank 9, then $\text{Exc}(S)$ is finite. To prove this observe first that $\text{pt}_{\mathfrak{s}_0}$ is finite, being the union of a finite number of orbits for the action of a finite group. By Proposition 2.5 the equality $\Lambda = \ker(\text{res})$ holds. Thus, by Proposition 2.9, the number of elements of $\text{Exc}(S)$ which contain a given point $q \in \text{pt}_{\mathfrak{s}_0}$ is in bijection with $L^{\text{trop}}(E)/\langle mK_S \rangle$, where E is a (-1) -curve which contains q . The latter set is finite by Proposition 2.8.

Let X be a Jacobian elliptic surface. A choice of a section E defines a group law on the generic fiber F_η and hence the negation involution $a \mapsto -a$. This involution extends to a biregular involution of the surface which is the lift of the Bertini involution on the weak del Pezzo surface of degree 1 obtained by blowing down the section E . Its fixed locus consists of the union of E and a 3-section of the fibration (which might be reducible).

There is an analogue of this involution on any Halphen surface X of index 2. Any (-1) -curve defines a 2-section of the elliptic fibration, and as such it defines a point x_η of degree 2 on its generic fiber F_η . The linear system $|x_\eta|$ defines a separable degree 2 map $F_\eta \rightarrow \mathbb{P}_\eta^1$ over the generic point η of the base. Its deck transformation defines an involution on F_η defined over η . The minimality of the fibration allows us to extend it to a biregular involution of the surface X , which we call the *Bertini involution* and denote it by β_E . The scheme of the fixed points of this involution on F_η is a reduced effective divisor of degree 4. Its closure B_E in X is a curve that intersects each fiber with multiplicity 4 (the curve E is invariant but not pointwise fixed!). It intersects transversally each smooth fiber.

Proposition 2.11. *The action of β_E^* is trivial on $L^{\text{trop}}(E)$.*

Proof. Observe that any (-1) -curve E' which has the same restriction of E to the generic fiber is preserved by β_E . Indeed the divisors cut out by E and E' on a general fiber are linearly equivalent and so each of them is mapped to itself by β_E . Now, for any $R \in L^{\text{trop}}(E)$, adding to the divisor $E + R$ some multiple of F_0 , we obtain a divisor $D \in L^{\text{trop}}(E)$ with $D \cdot K_S = D^2 = -1$. Applying Lemma 2.4, we obtain that D is linearly equivalent to a (-1) -curve E' as above. The statement follows. □

3. The Hesse pencil

The Hesse pencil is the pencil of plane cubic curves of the form

$$H_{\lambda;\mu} : \{\lambda(x^3 + y^3 + z^3) + \mu xyz = 0\}.$$

It has 9 base points x_1, \dots, x_9 with coordinates $(0 : 1 : -\epsilon), (1 : 0 : -\epsilon), (1 : -\epsilon : 0)$, where $\epsilon^3 = 1$. It has 4 singular fibers, whose equations are given by (see [1], [11, 3.1.3] for this and other information about the Hesse pencil):

$$\begin{aligned} H_{0;1} &: \{xyz = 0\}, \\ H_{1;-3} &: \{(x + y + z)(x + \epsilon y + \epsilon^2 z)(x + \epsilon^2 y + \epsilon z) = 0\}, \\ H_{1;-3\epsilon} &: \{(x + \epsilon y + z)(x + \epsilon^2 y + \epsilon^2 z)(x + y + \epsilon z) = 0\}, \\ H_{1;-3\epsilon^2} &: \{(x + \epsilon^2 y + z)(x + \epsilon y + \epsilon z)(x + y + \epsilon z) = 0\}. \end{aligned}$$

Each base point is an inflection point of a smooth member of the pencil. Each line contains 3 base points, and each base point lies on 4 lines. The nine points and 12

lines form the abstract Hesse configuration $(9_4, 12_3)$. All nonsingular members of the pencil have an inflection point at each base point. The polar conic of a general member of the pencil with pole at a base point x_i is equal to the union of the tangent line and another line ℓ_i that does not depend on the choice of a nonsingular member of the pencil. It is called a *harmonic polar line* of the pencil. Each line ℓ_i intersects a general member with multiplicity 3. It passes through one of the singular points of a singular member and intersects the opposite side of the triangle at one point. The 12 singular points of the four triangles and 9 harmonic polar lines form an abstract configuration $(12_3, 9_4)$ dual to the Hesse configuration.

The blow-up $\pi: S \rightarrow \mathbb{P}^2$ of the base points x_1, \dots, x_9 is a rational elliptic surface $f: S \rightarrow \mathbb{P}^1$ of index 1. The exceptional curves $\mathcal{E}_i = \pi^{-1}(x_i)$ are sections of the fibration. As we remarked earlier, they correspond to rational points of the elliptic curve F_η over the field $K = \mathbb{k}(t)$ of rational functions on the base \mathbb{P}^1 of the fibration. Fixing one base point, say x_1 , and the corresponding section E_1 , and hence a rational point on F_η , we equip F_η with a group law. It defines a group law on any nonsingular fiber F with the zero point $F \cap E_1$. It also defines a group law of a one-dimensional commutative algebraic group on the set of nonsingular points of each fiber. The group of rational points on F_η is a finitely generated Abelian group, called the *Mordell-Weil group* of the elliptic surface. Any element of this group defines a translation automorphism of F_η that extends to a biregular automorphism of S . In our case the 9 sections generate the Mordell-Weil group isomorphic to $(\mathbb{Z}/3\mathbb{Z})^{\oplus 2}$.

Definition 3.1. An *Halphen pencil of Hesse type* is an Halphen pencil defining a rational elliptic surface (*Halphen surface of Hesse type*) with the Jacobian fibration isomorphic to the elliptic fibration defined by the Hesse pencil.

Note that the correspondence between relatively minimal rational elliptic surfaces and Halphen pencils of index m is far from being bijective since it depends on a choice of a birational morphism $\pi: S \rightarrow \mathbb{P}^2$. Two Halphen pencils corresponding to the same elliptic surface are the images of its elliptic fibration with respect to different birational morphisms from the surface to the plane. It follows that they differ by a Cremona transformation of the plane.

4. The Chilean configuration of conics

Our goal in this section is to give an explicit construction of an Halphen pencil of Hesse type and index 2 such that its four singular members are unions of three conics.

Although we know that there exists a rational elliptic surface of index 2 with the Jacobian surface defined by the Hesse pencil, it is not obvious that we can find a morphism $\pi: S \rightarrow \mathbb{P}^2$ such that the images of the singular fibers are the unions of three conics (other possibility is the union of two lines and a plane quartic).

To do so we first fix a nonsingular plane cubic $C: \{G_3 = 0\}$. Then we choose 9 distinct points such that they can be partitioned in 4 different ways in three

subsets (x_i, x_j, x_k) of 3 points such that $(x_i + x_j + x_k - h) \sim 0$. In this way we have 12 triples. Note that the base points on each line component $x + \epsilon y + \epsilon^2 z = 0$ of a singular member of the Hesse pencil add up to 0 in the group law on a fixed nonsingular member of the pencil with the zero point defined by a choice of one of the base points x_i . We can parameterize the set of line components of each reducible fiber by \mathbb{F}_3 in such a way that each section is represented by a vector (a_1, a_2, a_3, a_4) indicating which component of a fiber it intersects. This identifies the set of sections with the plane \mathbb{F}_3^2 embedded in \mathbb{F}_3^4 in such a way that the difference of two vectors has only one zero coordinate [20]. For example, we may assume that

x_1	$(0,0,0,0)$	x_4	$(1,1,0,2)$	x_7	$(2,2,0,1)$
x_2	$(0,1,2,1)$	x_5	$(1,2,2,0)$	x_8	$(2,1,1,0)$
x_3	$(0,2,1,2)$	x_6	$(1,0,1,1)$	x_9	$(2,0,2,2)$

Now we identify the 12 line components of reducible fibers with lines in the finite affine plane \mathbb{F}_3^2 . Three points lie on a line if and only if the sum of the corresponding 4-vectors is equal to 0. This gives a finite projective geometry realization of the Hesse configuration. Note that 3 points in a line add up to zero in the group law on a nonsingular member $F = F_t$ of the pencil with the zero point equal to x_1 . The four lines containing the origin x_1 are given by the first column of the previous table, the first row, the diagonal, and x_1, x_6, x_8 .

We can order the coordinates of the nine base points of the Hesse pencil as follows:

$$\begin{aligned}
 x_1 &= (0 : 1 : -1), & x_2 &= (0 : 1 : -\epsilon), & x_3 &= (0 : 1 : -\epsilon^2), \\
 x_4 &= (1 : 0 : -1), & x_5 &= (1 : 0 : -\epsilon^2), & x_6 &= (1 : 0 : -\epsilon), \\
 x_7 &= (1 : -1 : 0), & x_8 &= (1 : -\epsilon : 0), & x_9 &= (1 : -\epsilon^2 : 0),
 \end{aligned}
 \tag{4.1}$$

where ϵ is a primitive 3rd root of unity.

Let $F[3]$ be the group of 3-torsion points on the curve F with the point x_1 as the zero element. Then the lines of the Hesse configuration that contain x_1 contain 3 points which form a cyclic subgroup of $F[3]$. Each line in the same member of the pencil is identified with a coset with respect to this subgroup.

Now fix a non-trivial 2-torsion τ point on F . Consider 9 points y_i equal to $x_i + \tau$ in the group law. Then each of the 12 cosets define a set of 3 points on F added up to τ in the group law. We have partitioned the 9 points in 12 different ways as the union of three triples of points added up to τ in the group law. Take two cosets H_i, H_j with no common elements. Then their union defines a set of 6 points whose sum in the group law is equal to zero. Take 3 cosets H_i, H_j, H_k defining a singular fiber of the Hesse pencil. Adding up τ , we obtain 3 disjoint sets $H_i + \tau, H_j + \tau, H_k + \tau$ of triples on F that add up to τ . It follows that there exists a conic that intersects F at 6 points from two of these sets. The sum of these three conics pass through 9 points of the Halphen pencil with multiplicity 2 at each point. This defines a singular fiber of the Halphen pencil equal to the union of 3 conics.

We have 4 singular fibers, hence 12 conics. Each base point defines a 2-section of the elliptic fibration that intersects 2 components of a singular fiber.

This means that each base point lies on 8 conics, two from each fiber. Moreover, each conic passes through 6 points, hence we have constructed the Chilean configuration $(12_8, 9_6)$ of 12 conics and 9 points in the plane.

Remark 4.1. $F[3]$ is clearly isomorphic to the affine plane over the prime field \mathbf{F}_3 which can be completed to the projective plane, with the line at infinity distinguished. Components of reducible fibers corresponding to 'finite' lines in the projective plane, two components belong to the same fiber if and only if they are parallel (meaning cosets of the same cyclic subgroup), *i.e.* meeting at the line at infinity, whose points hence correspond to the reducible fibers, each given by the pencil of lines through the points. Points will correspond to sections, and we see how two components can only share one section and each section will intersect exactly one component of each fiber. Note that with a proper choice of zero, lines in the finite projective planes correspond to bona fide lines in the complex setting, but for the combinatorial codification this is not important. In fact, by a slight abuse of notation, we can refer to the same finite plane when we are talking about the translated points in the index two setting. Then three points on a line, does no longer mean that they are literally on a line. This extended use will turn out to be useful further down. Thus any two parallel lines in the new interpretation can be associated to a conic component of the Halphen pencil.

We conclude the section with some observation of the automorphism group of S . First of all, any birational automorphism that preserves the fibration is a biregular automorphism. This follows from the relative minimality assumption since neither horizontal nor vertical curve can be blow down. An automorphism of S preserves the pencil but may act non-trivially on it. The subgroup of automorphisms that act trivially on the base (which could be called *vertical*) is easy to describe. It is mapped to the group of automorphisms of the Jacobian fibration with the kernel equal to the group of translations of S by the elements of the Mordell-Weil group of the Jacobian fibration. The group of automorphisms of the Jacobian fibration is the group of automorphisms of its generic fiber defined over the field of rational functions of the base. It is the semi-direct product of the group of Mordell-Weil group (defined when we fix a section) and the group of automorphisms fixing a section. The latter depends on the j -invariant of the elliptic curve but always contains the Bertini involution. When $m = 2$, as we explained in Section 2, we have similar involutions, the Bertini involutions β_E . We refer for the study of automorphism groups of Halphen surfaces to [12].

Recall that the Hesse group G_{216} is one of maximal finite groups of projective transformations of the plane. It can be realized as the group of automorphisms of the Hesse pencil of cubic curves, or equivalently, as the group of automorphisms of a rational Halphen surface of Hesse type and of index 1 (see [1] or [11]). The group G_{216} has a normal subgroup H isomorphic to $(\mathbb{Z}/3\mathbb{Z})^2$ and the quotient is isomorphic to the binary octahedral group $2.\mathfrak{A}_4$ which is isomorphic to $\mathrm{SL}(2, \mathbb{F}_3)$. The

pre-image of the central subgroup of $\mathrm{SL}(2, \mathbb{F}_3)$ under the quotient map is the subgroup G_{18} isomorphic to the semi-direct product $(\mathbb{Z}/3\mathbb{Z})^2 \rtimes (\mathbb{Z}/2\mathbb{Z}) \cong \mathbb{Z}/3\mathbb{Z} \rtimes \mathfrak{S}_3$. It consists of automorphisms generated by translation and any of the nine Bertini involutions defined by the negation automorphism of a general fiber when we fix the structure of group on it by fixing one of the sections. The projection to $G_{168} \rightarrow \mathfrak{A}_4$ corresponds to the action on the base of the elliptic fibration as the octahedral group of automorphisms of \mathbb{P}^1 .

Corollary 4.2. *The automorphism group of an Halphen surface of index 2 of Hesse type is isomorphic to the subgroup G_{18} of the Hesse group G_{216} of projective transformations of \mathbb{P}^2 . Its normal subgroup of order 9 consists of translation automorphisms by elements of the Mordell-Weil group of the Jacobian fibration. Its nine involutions are the Bertini transformations.*

Proof. An ordered set \mathcal{E} of (-1) -curves that are blown down to the set of base points of an Halphen pencil that gives rise to a Chilean configuration of conics defines a geometric basis (e_0, e_1, \dots, e_9) of $\mathrm{Pic}(S)$. By Corollary 6.7, $e_1 + \dots + e_9$ is invariant with respect to the action of $\mathrm{Aut}(S)$ on $\mathrm{Pic}(S)$, and since $K_S = 3e_0 - (e_1 + \dots + e_9)$ is obviously invariant too, we obtain that the divisor class e_0 is invariant. This shows that any $g \in \mathrm{Aut}(S)$ descends to an automorphism T of \mathbb{P}^2 under the blowing down morphism $\pi : S \rightarrow \mathbb{P}^2$ defined by the linear system $|e_0|$. Thus we can identify $\mathrm{Aut}(S)$ with a group of projective transformations of \mathbb{P}^2 . Composing any $g \in \mathrm{Aut}(S)$ with the translation automorphism t_a for some element of the Mordell-Weil group, we may assume that g fixes a (-1) -curve $E_1 \in \mathcal{E}$. Then it commutes with the Bertini involution β_{E_1} , descends to the del Pezzo surface \mathcal{D} obtained by blowing down E_1 , and hence fixes the point s' on the image of F_0 on \mathcal{D} equal to the unique base point of the anti-canonical linear system $| -K_{\mathcal{D}} |$. We identify this point with a point on F_0 . Thus T fixes 4 points on the image \bar{F}_0 of F_0 in the plane, the images of s' , $s = E_1 \cap F_0$, and the images of the fixed points of β_{E_1} on F_0 .

Since a projective transformation cannot pointwise fix a cubic curve, g^2 has four fixed points on F_0 and hence is either identity or coincides with an involution of F_0 with four fixed points. Since β_{E_1} fixes the same points on F_0 , $g^2 = \beta_{E_1}$. It follows that g has two fixed points on \bar{E}_1 , one is s and another is $s_0 = F_1 \cap E_1$, where F_1 is some other fiber. Since E_1 intersects F_1 with multiplicity 2, it must be tangent to F_1 at the fixed point s_0 . This is obviously impossible since the blow-up of s_0 would have a fixed point with three different tangent directions defined by F_1, E_1 and the exceptional curve, hence acts identity at the tangent space of the surface at this point is hence identity (because we assume that characteristic is different from 2). Thus g coincides with β_{E_1} and this proves the assertion. \square

Remark 4.3. The same proof works for degenerate Chilean configuration when $2F_0$ is a double fiber. In fact, it is even easier. The group of automorphisms contains G_{18} and leaves invariant a triangle of lines in the plane. This means that the group G of automorphisms is an imprimitive linear group, *i.e.* it contains a normal

subgroup that arises from a reducible linear representation. It follows from Blichfeldt’s classification of finite groups of automorphisms of the projective plane (the complete list can be found in [9, 4.2]) that this group coincides with G_{18} .

5. Explicit formulas and uniqueness

We take the harmonic polar line ℓ_1 with respect to the point $x_7 = (1 : -1 : 0)$. It has the equation $x - y = 0$. It intersects the nonsingular cubics $H_{1;t}$ at three 2-torsion points $(1 : 1 : a)$, where a is a root of the cubic equation $X^3 + tX + 2 = 0$. This gives us 9 base points of the Halphen pencil of Hesse type and index 2:

$$\begin{aligned} p_1 &= (a : 1 : 1), & p_2 &= (\epsilon a : \epsilon^2 : 1), & p_3 &= (\epsilon^2 a : \epsilon : 1), \\ p_4 &= (1 : a : 1), & p_5 &= (\epsilon : \epsilon^2 a : 1), & p_6 &= (\epsilon^2 : \epsilon a : 1), \\ p_7 &= (1 : 1 : a), & p_8 &= (\epsilon : \epsilon^2 : a), & p_9 &= (\epsilon^2 : \epsilon : a). \end{aligned}$$

Given the nine base points it is easy to compute the classes of the (-2) -curves of S . These are all the conics through six of the nine points whose sum is zero. The strict transforms of these conics are the irreducible components of the four reducible fibers of type I_3 of the following pencil of plane sextics:

$$\begin{aligned} &x^3y^3 + x^3z^3 + y^3z^3 - \frac{1}{a}(x^4yz + xy^4z + xyz^4) \\ &+ \frac{1 - a^3}{a^2}x^2y^2z^2 + \lambda \left(x^3 + y^3 + z^3 - \frac{a^3 + 2}{a}xyz \right)^2 = 0. \end{aligned} \tag{5.1}$$

Recall that for any smooth plane cubic curve $C \subset \mathbb{P}^2$, the dual curve C^* is a plane sextic in the dual plane $\check{\mathbb{P}}^2$ with 9 cusps. If one considers a plane cubic $F_3 \subset \check{\mathbb{P}}^2$ passing through these cusps, we can define a pencil of plane sextic curves spanned by C^* and F_3^2 . It is an example of an Halphen pencil of index 2. We say that this pencil is generated by the dual plane cubic C^* .

Recall from [11, 3.2] that the *Caylean* of a plane cubic is the locus of lines $\langle p, q \rangle$ that are line components of reducible polar conics.

Proposition 5.1. *The Halphen pencil of Hesse type in equation (5.1) is generated by C^* , where $C = \{u^3 + v^3 + w^3 - 3auvw = 0\}$, and the multiple fiber is the Caylean of C .*

Proof. We can rewrite equation (5.1) as:

$$(xy - az^2)(xz - ay^2)(yz - ax^2) + \lambda (a(x^3 + y^3 + z^3) - (a^3 + 2)xyz)^2 = 0.$$

The pencil contains a unique sextic with nine cusps. Its equation is the following:

$$x^6 + y^6 + z^6 + (4a^3 - 2)(x^3y^3 + x^3z^3 + y^3z^3) - 6a^2(x^4yz + xy^4z + xyz^4) - 3a(a^3 - 4)x^2y^2z^2 = 0.$$

Its dual is the plane cubic C , whose equation in dual coordinates u, v, w is

$$u^3 + v^3 + w^3 - 3auvw = 0$$

(see [11, 3.2.3]). The equation of the Caylean of this curve is [11, (3.27)]

$$x^3 + y^3 + z^3 - \frac{2 + a^3}{a}xyz = 0.$$

Since $t = -\frac{2+a^3}{a}$, we see that the cubic

$$x^3 + y^3 + z^3 + txyz = 0$$

coincides with the Caylean of the cubic C and our pencil coincides with the pencil generated by the dual C^* of C . □

The following is a nice corollary previously observed by D. Testa

Corollary 5.2. *A smooth plane cubic can be uniquely reconstructed from the nine cusps of its dual sextic.*

In fact, the Halphen pencil with the nine cusps as its set of its base points contains a unique irreducible member with 9 cusps, its dual is the plane cubic.

Remark 5.3. We are not writing here the values of $\lambda \neq 0, \infty$ which correspond to the special fibers because these values λ_1, λ_2 are nasty. One can check that their cross ratio R satisfies the equation $R^2 - R + 1 = 0$ that means that the double cover of \mathbb{P}^1 branched over $0, \infty, \lambda_1, \lambda_2$ is an elliptic curve with the absolute invariant j equal to zero as it happens in the case of the Jacobian fibration. It is known that the invariant of binary quartics $a_0u^4 + a_1u^3v + a_2u^2v^2 + a_3uv^3 + a_4v^4$ that vanishes on the set of quartics whose zeroes are four points on \mathbb{P}^1 with such cross-ratio (*equianharmonic quartics*) is equal to $S = a_0a_4 - 3a_1a_3 + 4a_2^2$. When we fix the zeroes 0 and ∞ corresponding to vanishing of the coefficients a_0, a_4 , we see that the set of unordered pairs $\{\lambda_1, \lambda_2\} \in (\mathbb{P}^1)^{(2)} \cong \mathbb{P}^2$ is a conic $-3a_1a_3 - 4a_2^2 = 0$. We have to throw away four points from this conic corresponding to $\lambda_1, \lambda_2 \in \{0, \infty\}$ or $\lambda_1 = \lambda_2$. This should be compared with Remark 5.5 below where we show that the parameter t of our Halphen pencils of Hesse type take value in $\mathbb{P}^1 \setminus \{4 \text{ points}\}$. This shows that the map that assigns to a parameter t the unordered pair $\{\lambda_1, \lambda_2\}$ is bijective.

The classes of the (-2) -curves of the surface S expressed in a geometric basis by the columns of the following matrix

$$\begin{pmatrix} 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 \\ -1 & -1 & 0 & -1 & -1 & 0 & -1 & -1 & 0 & -1 & 0 & -1 \\ -1 & -1 & 0 & -1 & 0 & -1 & -1 & 0 & -1 & -1 & -1 & 0 \\ -1 & -1 & 0 & 0 & -1 & -1 & 0 & -1 & -1 & 0 & -1 & -1 \\ -1 & 0 & -1 & -1 & -1 & 0 & -1 & 0 & -1 & 0 & -1 & -1 \\ -1 & 0 & -1 & -1 & 0 & -1 & 0 & -1 & -1 & -1 & 0 & -1 \\ -1 & 0 & -1 & 0 & -1 & -1 & -1 & -1 & 0 & -1 & -1 & 0 \\ 0 & -1 & -1 & -1 & -1 & 0 & 0 & -1 & -1 & -1 & -1 & 0 \\ 0 & -1 & -1 & -1 & 0 & -1 & -1 & -1 & 0 & 0 & -1 & -1 \\ 0 & -1 & -1 & 0 & -1 & -1 & -1 & 0 & -1 & -1 & 0 & -1 \end{pmatrix}. \tag{5.2}$$

For example, the first column exhibits a conic passing through the first six points p_1, \dots, p_6 . The reducible fibers are the sums of columns $(1, 2, 3)$, $(4, 5, 6)$, $(7, 8, 9)$ and $(10, 11, 12)$. The defining polynomials for the above 12 conics are the following:

$$\begin{aligned} &xy - az^2, \\ &xz - ay^2, \\ &yz - ax^2, \\ &x^2 + (\epsilon a + \epsilon)xy + \epsilon^2 y^2 + (\epsilon^2 a + \epsilon^2)xz + (a + 1)yz + \epsilon z^2, \\ &x^2 + (\epsilon^2 a + \epsilon^2)xy + \epsilon y^2 + (\epsilon a + \epsilon)xz + (a + 1)yz + \epsilon^2 z^2, \\ &x^2 + (a + 1)xy + y^2 + (a + 1)xz + (a + 1)yz + z^2, \\ &x^2 + (\epsilon a + 1)xy + y^2 + (\epsilon^2 a + \epsilon)xz + (\epsilon^2 a + \epsilon)yz + \epsilon^2 z^2, \\ &x^2 + (\epsilon^2 a + \epsilon)xy + \epsilon^2 y^2 + (\epsilon a + 1)xz + (\epsilon^2 a + \epsilon)yz + z^2, \\ &x^2 + (a + \epsilon^2)xy + \epsilon y^2 + (a + \epsilon^2)xz + (\epsilon^2 a + \epsilon)yz + \epsilon z^2, \\ &x^2 + (\epsilon a + \epsilon^2)xy + \epsilon y^2 + (\epsilon^2 a + 1)xz + (\epsilon a + \epsilon^2)yz + z^2, \\ &x^2 + (a + \epsilon)xy + \epsilon^2 y^2 + (a + \epsilon)xz + (\epsilon a + \epsilon^2)yz + \epsilon^2 z^2, \\ &x^2 + (\epsilon^2 a + 1)xy + y^2 + (\epsilon a + \epsilon^2)xz + (\epsilon a + \epsilon^2)yz + \epsilon z^2. \end{aligned}$$

Remark 5.4. When we choose the parameter t to be ∞ or satisfy $t^3 + 27 = 0$, the cubic curve $\{x^3 + y^3 + z^3 + txyz = 0\}$ becomes the union of three lines. The corresponding cubic equation $X^3 + tX + 2 = 0$ has two roots, one of them is a double root. In the case of the root of multiplicity 1, we get a pencil of sextics with the corresponding 9 distinct base points p_1, \dots, p_9 as above, which lie on the triangle $H_{\lambda;\mu}$. We call the corresponding configuration of 9 conics and 3 lines the *degenerated Chilean configuration*. In the construction of the pencil as a torsor of the Jacobian fibration, the first parameter corresponds to one choice of a non-trivial

2-torsion element of the connected component of the Picard scheme of the singular fiber isomorphic to the multiplicative group \mathbb{G}_m .

The pencil corresponding to the double root a of $X^3 + tX + 2 = 0$ is a pencil of elliptic curves with 3 triple points x_1, x_2, x_3 (specialization of the nine p_i 's above) and 9 simple infinitely near points $x_i^{(1)} \succ x_i, x_i^{(2)} \succ x_i, x_i^{(3)} \succ x_i, i = 1, 2, 3$, where \succ indicates an infinitely near point of order 1. For example, if we take $t = -3$, then the multiple root a is equal to 1 and the pencil is given by equation

$$(x^2 - yz)(z^2 - xy)(y^2 - xz) + \lambda(x^3 + y^3 + z^3 - 3xyz)^2 = 0.$$

Each of the conics in the fiber over $\lambda = 0$ passes through three base points x_1, x_2, x_3 , and they intersect each other transversally at one of points $(1 : 0 : 0), (0 : 1 : 0), (0 : 0 : 1)$. The base points are also the singular points of the cubic that enter with multiplicity 2. If we apply a quadratic Cremona transformation with fundamental points at the base points, the pencil is transformed to an Halphen pencil of Hesse type of index 1. In fact, one can prove that the Halphen surfaces of Hesse type from the pencil in (5.1) have four degenerations (one for each triangle and corresponding double root a) into a singular surface with three singularities, which has an elliptic fibration with four I_3 fibers such that at the 3 vertices of one of them we have cyclic quotient singularities of type $1/4(1, 1)$ (*i.e.* germ at $(0, 0)$ of the quotient of \mathbb{C}^2 by $(x, y) \mapsto (ix, iy)$). The construction of the singular surface is the following. Take the Jacobian Hesse fibration, and consider one of the four I_3 fibers (this is chosen by the degeneration), blow-up the 3 nodes and contract the 3 proper transforms from this I_3 fiber. Hence we have that Halphen surfaces of Hesse type degenerate into a singular model of their Jacobian fibration. This degeneration is in fact a \mathbb{Q} -Gorenstein smoothing, this is, the canonical class of the corresponding 3-fold is \mathbb{Q} -Cartier.

Roughly speaking, the way one constructs the degeneration is by going backwards. We take the Hesse surface, and construct the singular surface described above. One can prove that this surface has no local-to-global obstructions to deform (*e.g.*, as in [21, Section 4]). In fact, this is also true after contracting three pairs of (-2) -curves from the rest of the I_3 fibers (one can use [21, Theorem 4.4]). So we do contract them as well, and consider a \mathbb{Q} -Gorenstein degeneration keeping these three A_2 singularities and producing a new A_2 singularity from the singular fiber of type I_3 . The later is possible because we can reduce that deformation to a deformation of $1/12(1, 5)$ which has a 1 parameter deformation into an A_2 singularity (see, *e.g.*, [14, Proposition 2.3]). It can be proved that the general fiber is an Halphen surface of index 2 keeping an elliptic fibration from the one in the Hesse surface, and since we have four A_2 singularities with no two in one fiber, we have no more room for other Halphen surface than the Halphen surface of Hesse type. Now since we know that there is just one family of them, this corresponds to the Chilean elliptic pencil. This phenomenon may be general, it could happen for any Jacobian fibration with a fiber of type I_n : Any Halphen surface of index m can \mathbb{Q} -Gorenstein degenerate to a singular model of its Jacobian.

We call the Halphen pencil corresponding to the choice of a reducible fiber F_t and the non-multiple root of $X^3 + tX - 2 = 0$ the *degenerate Halphen pencil of*

Hesse type. It can be obtained from an Halphen pencil that defines the degenerated Chilean configuration.

Remark 5.5. We already know from explicit formulas that a Chilean configurations of conics depends on one parameter. We will prove later in Proposition 6.3 that there is only one Chilean pencil (up to projective equivalence) that gives rise to an Halphen surface of index 2 of Hesse type. Thus the moduli space of Chilean configurations (including degenerate ones) is the same as the moduli space of Halphen surfaces of index 2 of Hesse type. To describe it more precisely, we have to invoke the theory of torsors of Jacobian elliptic surfaces (see [6, Chapter 4]). In the case when the Jacobian elliptic surface $j : J \rightarrow \mathbb{P}^1$ is rational it gives that the isomorphism class of a torsor of index m is determined uniquely by the data that consists of points x_1, \dots, x_k on the base and non-trivial m_i -torsion points in the connected group of the Picard variety $\mathbf{Pic}(J_{x_i})$ of the fibers J_{x_i} . The corresponding torsor is an elliptic fibration with m_i -multiple fibers over the points x_i . In the special case when all m_i are equal to 2, so that the index of the torsor is equal to 2, one can make it more explicit by considering the Weierstrass model of the Jacobian fibration. This defines a degree 2 map from J to the minimal ruled surface \mathbf{F}_2 whose branch divisor is the union of the special section E with $E^2 = -2$ and a curve B disjoint from E from the linear system $|6f + 3e|$, where f is the divisor class of the ruling and e is the divisor class of E . A choice of a nonsingular point b_i on B defines a point x_i on the base and a non-trivial 2-torsion element in $\mathbf{Pic}_{J_{x_i}}$. Thus the variety of torsors of index 2 with fixed number of double fibers is isomorphic to the symmetric product $(B^\#)^{(k)}$, where $B^\#$ is the set of smooth points of B . Applying this to our situation we find that the pre-image of B on J is a 3-section of the elliptic fibration that intersects each singular fiber at one of its singular points. Its image in the plane is a harmonic polar line. Thus the moduli space can be identified with a harmonic polar line minus four singular points of the singular fibers lying on it.

One can also apply the quadratic twist construction of the torsor. Fix a section and consider the Weierstrass model as above. The image of the harmonic polar line on it is a curve of arithmetic genus four with four cusps. Choose a twisted cubic K on \mathbf{F}_2 (i.e an irreducible curve from the linear system $|3f + e|$) that passes through the four cusps and passes through some simple point b of B . Since $\dim |3f + e| = 5$, we can do it. The curve K intersects the exceptional section E at one point b' .

The pre-image of K on J is a smooth rational 2-section R that is tangent to a fiber at a point \tilde{b} and passes through the point \tilde{b}' on the proper transform of E , the zero section of $j : J \rightarrow \mathbb{P}^1$. Let $X \rightarrow J$ be the double cover of J ramified over the two fibers at $t = j(\tilde{b})$ and $t' = j(\tilde{b}')$. It follows from the formula for the canonical class of a double cover that its canonical class is trivial. For simplicity, we assume that J_t and $J_{t'}$ are smooth. This implies that X is a smooth K3 surface (otherwise we have to replace X by its minimal resolution). The pre-image of R in X splits into two sections R_+ and R_- intersecting at two points transversally, one point over \tilde{b}' where it intersects the pre-image \mathbf{O} of the zero section. Now we define an involution τ on X that is the composition of the deck transformation of

the double cover $X \rightarrow J$ and the translation $\iota_{\mathcal{O}}$ by \mathcal{O} , where we take R_+ as the zero section. The involution τ acts identically on $X_{t'}$ and acts as a translation by a 2-torsion point on X_t . The quotient is our surface S .

Lemma 5.6. *Let S be an Halphen surface of Hesse type of index 2. Assume that the half-fiber is not one of the four triangles. Then S contains a (-1) -curve which intersects eight (-2) -curves.*

Proof. Let E be a (-1) -curve of S . If E intersects all the four triangles in two sides there is nothing to prove. Assume this is not the case and let T_1, \dots, T_k , with $1 \leq k \leq 4$, be the triangles which are intersected by E at just one side and let $R_i^{(1)}, R_i^{(2)}$ be the two curves in T_i which are disjoint from E . Let

$$E' := E - \sum_{i=1}^k (R_i^{(1)} + R_i^{(2)}) + kF_0.$$

Observe that $E'^2 = E' \cdot K_S = -1$, so that by Riemann-Roch E' is linearly equivalent to an effective divisor which, with abuse of notation, we will denote by the same letter. Moreover, E' has non-negative intersection with any of the twelve (-2) -curves of S . By Lemma 2.4, E' is a (-1) -curve in S . Finally E' intersects each triangle along two sided as desired. □

The next theorem is a uniqueness result. It shows that, up to projective equivalence, the family of Chilean configurations from Equation 5.1 represents all Halphen pencils of Hesse type of index 2.

Theorem 5.7. *Let S be an Halphen surface of index 2 whose Jacobian is the Hesse surface. Assume that the half-fiber is not one of the four triangles. Then, S is a Chilean surface, this is, it comes from a Chilean configuration via the pencil in Equation (5.1).*

Proof. Let E be a (-1) -curve as in Lemma 5.6, let $R_i^{(1)}, R_i^{(2)}$ be the two sides of the triangle T_i which are intersected by E and let $R_i^{(0)}$ the remaining side. Observe that we are free to (re)label the four singular fibers as well as the components in each such fiber as we wish. Let $\Lambda \subseteq K_S^{\frac{1}{2}}$ be the sublattice spanned by the classes of the (-2) -curves, as before. Let $\sigma \in \text{Aut}(S)$ be such that $\sigma(E) \neq E$. Observe that $\sigma(E)$ and E do not have the same intersection product with any (-2) -curve of S . Indeed, if this were the case, the class of $\sigma(E) - E$ would be proportional to F_0 , but the only divisor of the form $E + nF_0$ with self-intersection -1 is E , a contradiction. The (-2) -classes generate $K_S^{\frac{1}{2}}$ over the rationals and $\sigma(E) - E$ belongs to this space, so that there exists rational numbers a_{ij} such that

$$\sigma(E) - E \sim \sum_{i=1}^4 (a_{i0}R_i^{(0)} + a_{i1}R_i^{(1)} + a_{i2}R_i^{(2)}).$$

Let $i \in \{1, 2, 3, 4\}$ be the index of a singular fiber, where E and $\sigma(E)$ have different intersection numbers with the three irreducible components. After possibly switching the labels of $R_i^{(1)}$ and $R_i^{(2)}$, we can assume $\sigma(E) \cdot R_i^{(0)} = 1$, $\sigma(E) \cdot R_i^{(1)} = 0$, and $\sigma(E) \cdot R_i^{(2)} = 1$. These equations imply $a_{i0} = -\frac{1}{3} + n$, $a_{i1} = \frac{1}{3} + n$, and $a_{i2} = n$, where $n \in \mathbb{Z}$. In other words, the class of the \mathbb{Q} -divisor

$$\frac{1}{3} \left(R_i^{(1)} - R_i^{(0)} \right)$$

must appear in $\sigma(E) - E$. Since the above divisor has self-intersection $-2/3$, the difference $\sigma(E) - E$ must contain exactly the sum of three such divisors, whose self-intersection is -2 . Moreover, we can assume without loss of generality that the sum is supported at the last three singular fibers, so we can write

$$D_{0111} := \frac{1}{3} \left(R_2^{(1)} - R_2^{(0)} + R_3^{(1)} - R_3^{(0)} + R_4^{(1)} - R_4^{(0)} \right).$$

Here the notation D_{0111} means that the divisor is supported at the first three singular fibers, and in each such fiber we are taking $R_i^{(1)} - R_i^{(0)}$ instead of $R_i^{(2)} - R_i^{(0)}$. We showed that the class of D_{0111} is in $\text{Pic}(S)$, but the sublattice $\Lambda + \langle D_{0111}, F_0 \rangle / \langle F_0 \rangle \subseteq K_S / \langle F_0 \rangle$ is still not unimodular, so that another class has to be added to get the full lattice K_S . Acting with another automorphism τ which comes from the Jacobian action, we get a (-1) -curve $\tau(E)$ distinct from E and $\sigma(E)$. Reasoning as above with $\tau(E) - E$ one concludes that the new class to be added is again the sum of three divisors of type $\frac{1}{3}(R_i^{(a)} - R_i^{(0)})$, where $a \in \{1, 2\}$. Moreover the new class must have integer intersection product with D_{0111} . Recalling that $R_i^{(0)}, R_i^{(1)}, R_i^{(2)}$ are the three components of a fiber of type I_3 , we get $\frac{1}{3}(R_i^{(1)} - R_i^{(0)}) \cdot \frac{1}{3}(R_i^{(2)} - R_i^{(0)}) = \frac{1}{9}(1 - 1 - 1 - 2) = -\frac{1}{3}$. Thus the only possibilities for the new class are $D_{0222}, D_{a012}, D_{a021}, D_{a210}, D_{a120}, D_{a201}, D_{a102}$, where $a \in \{1, 2\}$. The first possibility cannot occur because $D_{0222} \sim 2F_0 - R_2^{(0)} - R_3^{(0)} - R_4^{(0)} - D_{0111}$ is already in $\Lambda + \langle F_0, D_{0111} \rangle$. Without loss of generality, we can relabel the last three fibers and relabel the components in the first fiber, because these operations leave D_{0111} unaltered. In this way, we can assume the second divisor to be

$$D_{1012} := \frac{1}{3} \left(R_1^{(2)} - R_1^{(0)} + R_3^{(1)} - R_3^{(0)} + R_4^2 - R_4^{(0)} \right).$$

Observe that $D_{0111} \cdot D_{1012} = -1$ and each of the two divisors has intersection 1 with E . The class of each of the following divisors is in the Picard group, because it is a linear combination with integer coefficients of (-2) -curves, F_0 , E , D_{0111} and D_{1201}

$$\begin{array}{lll} E_1 := E & E_2 \sim E + D_{0222} & E_3 \sim E + D_{0111} \\ E_4 \sim E + D_{2021} & E_5 \sim E + D_{2210} & E_6 \sim E + D_{2102} \\ E_7 \sim E + D_{1012} & E_8 \sim E + D_{1201} & E_9 \sim E + D_{1120}. \end{array}$$

Each such divisor has non negative intersection with any (-2) -curve, it has self-intersection -1 and intersection -1 with K_S , so it is linearly equivalent to a (-1) -curve by Lemma 2.4. Moreover, $E_i \cdot E_j = 0$ for any $i \neq j$. The orthogonal complement of the classes of these nine (-1) -curves is generated by the class of

$$H := 3E + 3F_0 - \sum_{i=1}^4 R_i^{(0)},$$

which has self-intersection 1 and intersection product 2 with any (-2) -curve of the surface. Thus all the (-2) -curves are conics in the plane model defined by H . To conclude we recall that, according to Proposition 2.6, the subset $\text{pts}_0 \subseteq F_0$ of points cut out by the (-1) -curves of S is acted freely and transitively by $K_S^\perp/\Lambda \simeq \mathbb{Z}/3\mathbb{Z} \oplus \mathbb{Z}/6\mathbb{Z}$. Of these 18 points, we already know the nine base points p_1, \dots, p_9 cut out by the above disjoint exceptional curves of S . Moreover $\text{res}(H) = 3 \text{res}(E) + \text{res}(F_0)$, because $\text{res}(F_0)$ has order two, so that $x_i \sim p_i + \text{res}(F_0)$ is an inflection point of the plane cubic F_0 in the model defined by H . This is exactly how the Chilean configuration is constructed in Section 4. \square

We will give another proof of the uniqueness of the Chilean pencil in Proposition 6.6.

Remark 5.8. The set of nine (-1) -curves constructed in Theorem 5.7 is invariant for any automorphism of the surface. Indeed let E be the (-1) -curve in the proof of the theorem. Any automorphism maps E to a (-1) -curve E' which intersects any singular fiber at two curves. Then, by the same argument in the proof of the theorem, the difference $E' - E$ must be linearly equivalent to one of the divisors $D_{0222}, D_{0111}, D_{2021}, D_{2210}, D_{2102}, D_{1012}, D_{1201}, D_{1120}$.

Remark 5.9. It follows from the Borel-de Siebenthal-Dynkin algorithm for embedding of root lattices that, in our case, the embedding $\Lambda_0 := \Lambda/\langle F_0 \rangle \hookrightarrow K_S^\perp/\langle K_S \rangle \cong E_8$ is unique modulo the action of the Weyl group $W(E_8)$. By [19, Lemma 2.5] the embeddings of Λ into $K_S^\perp \simeq \tilde{E}_8$ which are compatible with the embedding $\Lambda_0 \rightarrow E_8$ are in bijection with the elements of the group $\text{Ext}_{\mathbb{Z}}^1(E_8/\Lambda_0, \mathbb{Z}/2\mathbb{Z}) \simeq \text{Ext}_{\mathbb{Z}}^1((\mathbb{Z}/3\mathbb{Z})^2, \mathbb{Z}/2\mathbb{Z})$, which is trivial. This provides another proof of the fact that the classes of D_{0111} and D_{1012} are in $\text{Pic}(S)$.

The base points of the Halphen pencil in (5.1) define nine disjoint (-1) -curves on S . However, there are more (-1) -curves on Halphen surfaces of higher index.

Proposition 5.10. *The number of (-1) -curves on the rational elliptic surface corresponding to the Halphen pencil of Hesse type of index 2 in (5.1) is equal to 144. The set of (-1) -curves consists of the following subsets:*

- 9 exceptional curves;
- 36 proper transforms of lines through two base points;
- 54 proper transforms of conics through five base points;

- 36 proper transforms of cubics with a double point at one of the base points and six simple base points;
- Proper transforms of 9 quartics with one triple point and passing through other 8 points.

The set of these 2-sections is freely acted by the Mordell-Weil group of the Jacobian fibrations isomorphic to $(\mathbb{Z}/3\mathbb{Z})^{\oplus 2}$ with 16 orbits represented by the columns of the following matrices

$$\begin{pmatrix} 0 & 2 & 2 & 2 & 2 & 2 & 2 & 4 \\ 0 & -1 & -1 & -1 & 0 & 0 & 0 & -1 \\ 0 & -1 & 0 & 0 & -1 & -1 & 0 & -1 \\ 0 & 0 & -1 & 0 & -1 & 0 & -1 & -1 \\ 0 & -1 & 0 & 0 & -1 & -1 & 0 & -1 \\ 0 & -1 & -1 & -1 & 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 & -1 & 0 & -1 & -1 \\ 0 & 0 & 0 & -1 & 0 & -1 & -1 & -1 \\ 0 & 0 & 0 & -1 & 0 & -1 & -1 & -1 \\ 1 & -1 & -1 & -1 & -1 & -1 & -1 & -3 \end{pmatrix} \quad \begin{pmatrix} 1 & 1 & 1 & 1 & 3 & 3 & 3 & 3 \\ -1 & 0 & 0 & 0 & -1 & -1 & -1 & 0 \\ 0 & -1 & 0 & 0 & -1 & -1 & 0 & -1 \\ 0 & 0 & -1 & 0 & -1 & 0 & -1 & -1 \\ 0 & -1 & 0 & 0 & -1 & -1 & 0 & -1 \\ -1 & 0 & 0 & 0 & -1 & -1 & -1 & 0 \\ 0 & 0 & -1 & 0 & -1 & 0 & -1 & -1 \\ 0 & 0 & 0 & -1 & 0 & -1 & -1 & -1 \\ 0 & 0 & 0 & -1 & 0 & -1 & -1 & -1 \\ 0 & 0 & 0 & 0 & -2 & -2 & -2 & -2 \end{pmatrix}.$$

The curves in the first matrix restrict to the point $p_9 \in F_0$ while the curves in the second matrix restrict to the inflection point $x_9 \in F_0$.¹

Proof. Recall that by Proposition 2.9 the subset of $\text{Exc}(S)$ which intersects F_0 at p or $p + \text{res}(F_0)$ is in bijection with $L^{\text{trop}}(E)/\langle 2K_S \rangle$, where E is any such (-1) -curve. We know by Section 2 that this set is the set of integer points of a polytope in $\Lambda_{\mathbb{Q}}/\langle 2K_S \rangle$ which we are now going to describe. According to Proposition 2.6 the restriction map induces a surjection

$$\text{Exc}(S) \rightarrow \text{pts}_0$$

which is equivariant with respect to the free action of $K_S^{\perp}/\Lambda + \langle F_0 \rangle \simeq (\mathbb{Z}/3\mathbb{Z})^2$. The set pts_0 consists of two orbits under this action: the set of flexes $\{x_1, \dots, x_9\}$ and their translates with respect to the 2-torsion class $\{p_1, \dots, p_9\}$ which are cut out by the exceptional divisors of the Chilean model. Denote by E one such exceptional divisor, by $R_i^{(1)}, R_i^{(2)}$ the two sides of the i -triangle which are intersected by E and by $R_i^{(0)}$ the side which is not intersected. Then, up to multiples of $2F_0 \sim R_i^{(0)} + R_i^{(1)} + R_i^{(2)}$, the only linear combination $D := E + a_0R_i^{(0)} + a_1R_i^{(1)} + a_2R_i^{(2)}$ which has non-negative intersection with all the three (-2) -curves is $E - R_i^{(0)}$. To prove this observe that, after possibly adding an integer multiple of $2F_0$ we can assume $a_2 = 0$. Then the three inequalities $D \cdot R_i^{(0)} \geq 0, D \cdot R_i^{(1)} \geq -1, D \cdot R_i^{(2)} \geq -1$ become $-2a_0 + a_1 \geq 0, a_0 - 2a_1 \geq -1, a_0 + a_1 \geq -1$, whose solution set is the triangle in the below picture.

¹ The first row gives the degrees (a_i) of the curves $a_i H + \sum_{j=1}^9 a_{ij} e_j$, and the remaining nine the multiplicity at the points p_j .

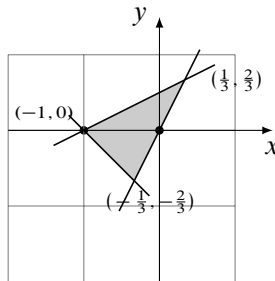


Figure 5.1.

The only integer points in the triangle are $(0, 0)$ and $(-1, 0)$, which correspond to the divisors E and $E - R_i^{(0)}$ respectively. Thus a set of representatives for $L^{\text{trop}}(E)/\langle 2F_0 \rangle$ consists of all the sums $-\sum_I R_i^{(0)}$, where I is any subset of $\{1, \dots, 4\}$. As a consequence we can write down explicitly all the curves which intersect F_0 at x_9 or at p_9 , these are the following 16 curves:

$$E - \sum_{i \in I} R_i^{(0)} + |I|F_0,$$

where I varies along the subsets of $\{1, 2, 3, 4\}$ and $|I|$ is its cardinality. Observe that in our case we can take $E = E_9$, so that $E \cap F_0 = \{p_9\}$. When I has even cardinality we get the first set of eight curves which intersect F_0 at p_9 . When I has odd cardinality we get the second set of eight curves which intersect F_0 at the flex x_9 . By Proposition 2.6 we deduce that the number of (-1) -curves of S is $8 \cdot 18 = 144$. To describe the remaining (-1) -curves one can use the action of the group of η -rational points of $\text{Pic}^0(F_\eta)$. This group, isomorphic to $K_S^\perp/\Lambda + \langle 2F_0 \rangle \simeq (\mathbb{Z}/3\mathbb{Z})^2$ acts by translations by 3-torsion classes on F_0 . Computing the action on $\{p_1, \dots, p_9\}$ the nine points are permuted by the elements of the following group

$$\langle (195)(276)(384), (186)(294)(375) \rangle.$$

The statement follows. □

Remark 5.11. The set $\text{Exc}(S)$ is equipped with the following non-geometric involution which preserves each fiber of the map $\text{res}: \text{Exc}(S) \rightarrow \text{pts}_0$. The involution is

$$E - \sum_{i \in I} R_i^{(0)} + |I|F_0 \mapsto E - \sum_{i \in I^c} R_i^{(0)} + |I^c|F_0,$$

where I^c is the complement of I in $\{1, 2, 3, 4\}$. Observe that the two curves have intersection number $-1 + |I| + |I^c| = 3$ and their sum is the divisor $2E - \sum_{i=1}^4 R_i^{(0)} + 4F_0$.

Remark 5.12. To determine the classes of the (-1) -curves in X one can proceed as in [19] (see also [17]). The classes of the (-2) -curves span a sublattice $\Lambda \subseteq K_S^\perp$

of rank 9 of $\text{Pic}(S)$. Thus using the perfect pairing given by the intersection form on the Picard group we deduce that the polyhedra

$$\Delta_S := \text{convex hull}(E \in \text{Pic}(S) : E \cdot K_S = -1 \text{ and } E \cdot C \geq 0 \text{ for any } (-2)\text{-curve } C)$$

has dimension 9 and it decomposes as the following Minkowski sum $\Delta_S = \Delta + \mathbb{Q} \cdot K_S$, where Δ is a polytope of dimension 8. It turns out that the (-1) -curves of S are in bijection with the integer points of Δ . Indeed clearly a (-1) -curve of S has integer intersection with all the (-2) -curves and thus it gives an integer point of Δ . On the other hand an integer point of Δ can be lifted to an affine line contained in Δ_S . This line has the form $E + nK_S$ and there is only one integer value of n such that the self-intersection of $E + nK_S$ is -1 . By Riemann-Roch the class is effective and since it has non-negative intersection with all the (-2) -curves of S one concludes that it is irreducible. One checks with a computer that Δ has 144 integer points.

In the next section we will give a proof of Proposition 5.10 that does not rely on any computer computation.

Remark 5.13. Since all nine exceptional curves of $\pi : S \rightarrow \mathbb{P}^2$ form one orbit with respect to the Mordell-Weyl group of the Jacobian fibration, the restriction of $f : S \rightarrow \mathbb{P}^1$ to each of them is a double cover of the base \mathbb{P}^1 branched over the same pair of points. The fiber of f over one of the points is the double fiber and the image under π of the fiber over the other point is a 9-cuspidal sextic which we discussed in Proposition 5.1.

6. A double plane model of S

Let S be an Halphen surface of index 2 of Hesse type. We do not assume that it comes from a Chilean configuration. We denote the half-fiber by F_0 and the general fiber by F . It has also four reducible fibers F_1, \dots, F_4 of type I_3 . Fix a (-1) -curve E_0 on S and let $\sigma_{E_0} : S \rightarrow S'$ be its blow-down morphism. The surface S' is a *weak del Pezzo surface* of degree 1. For example, if E_0 is chosen to be the exceptional curve over one of the base points of the Halphen pencil, then S' is the blow-up of the remaining base points. It is well known and well documented [11, Section 8.8.2] that the anti-canonical linear system $| -K_{S'} |$ is a pencil with one base point s' and the anti-bicanonical linear system $| -2K_{S'} |$ defines a degree 2 map

$$\phi' : S' \dashrightarrow Q \subset \mathbb{P}^3,$$

where Q is a singular quadric with vertex $v_0 = \phi'(s')$. The deck transformation of ϕ' defines a biregular involution β , known as the *Bertini involution* of S' .

The branch curve B' of ϕ' is cut out by a cubic surface in \mathbb{P}^3 . If we project Q to \mathbb{P}^2 from a point $q \notin B'$, we obtain a *double plane model* of S' with branch curve a plane sextic W with a triple point q_1 and a triple infinitely near point $q_1^{(1)} \succ q_1$.

The image of q is a line L that intersects W at q_1 with multiplicity 6. However, if we project from a point $q \in B'$, then the branch curve W of the double plane model is the union of a quintic curve W_0 with double points at $q_1, q_1^{(1)}$ and the line L that intersects W_0 at q_1 with multiplicity 4 and intersects W_0 with multiplicity 1 at some point q_2 .

We denote the composition $S \rightarrow S' \dashrightarrow Q \dashrightarrow \mathbb{P}^2$ by $\phi: S \dashrightarrow \mathbb{P}^2$. It defines a *double plane model* of S .

Since $K_S = \sigma_{E_0}^*(K_{S'}) + E_0$ and $| -K_S | = \{F_0\}$, we see that the point s' lies on the half-fiber F_0 . Let $s = E_0 \cap F_0$. Since the birational lift of the Bertini involution β leaves invariant K_S and $\sigma_{E_0}^*(K_{S'})$, it leaves invariant E_0 and hence fixes the point s . Thus the image $q \in Q$ of s belongs to B' . Then we project from q to obtain that the branch curve W of ϕ is the union of a plane quintic W_0 and the line L as above.

The map ϕ is given by the linear system $| -2K_S + E_0 |$. It blows down E_0 to the point q_2 and the image of s' is the point q_1 . It also blows down the irreducible components of the fibers F_1, \dots, F_4 to singular points of W_0 . If E_0 intersects two components of F_i , it blows down the remaining component to an ordinary node of W_0 . If E_0 intersects one component with multiplicity 2, then it blows down the remaining two components to an ordinary cusp. Thus W_0 has four additional double points y_1, \dots, y_4 .

The map ϕ is a rational map of degree 2 with indeterminacy point s . Let $\sigma_{s,s'}: \tilde{S} \rightarrow S$ be the blow-up of the points s, s' and $\tau: X \rightarrow \mathbb{P}^2$ be the blow-up of the points $q_1, q_1^{(1)}, q_2, y_1, \dots, y_4$. We have a commutative diagram

$$\begin{CD} \tilde{S} @>\tilde{\phi}>> X \\ @V\sigma_{s,s'}VV @VV\tau V \\ S @>\phi>> \mathbb{P}^2, \end{CD}$$

where the top arrow is a finite map of degree 2. The lift β_{E_0} of the Bertini involution generates the Galois group G of the map $\tilde{\phi}$. The locus of fixed point $\tilde{S}^{\beta_{E_0}}$ consists of the union of the exceptional curves $\mathcal{E}_{s'}, \mathcal{E}_s$ and a smooth curve B . The map $\tilde{\phi} \circ \tau: \tilde{S} \rightarrow \mathbb{P}^2$ is given by the linear system $|F + \bar{E}_0|$, where we identify the divisor class of a general fiber F of f with its pull-back on X and use a bar to denote the proper transform.

Let $\mathcal{E}_{q_1} = \tau^*(q_1)$ be the exceptional configuration over q_1 . It is equal to the sum of the exceptional curve $\mathcal{E}_{q_1^{(1)}}$ over $q_1^{(1)}$ and a (-2) -curve R_{q_1} . We have

$$\begin{aligned} \tilde{\phi}^*(R_{q_1}) &= 2\mathcal{E}_{s'}, \\ \tilde{\phi}^*(\bar{L}) &= 2\mathcal{E}_s, \\ \tilde{\phi}^*(\mathcal{E}_{q_1^{(1)}}) &= \bar{F}_0, \\ \tilde{\phi}^*(\bar{W}_0) &= 2B. \end{aligned}$$

The elliptic pencil on S is equal to the pre-image of the pencil of lines through q_2 . The double fiber is clearly mapped to the line L . The following picture summarizes the construction of the double plane model of S .

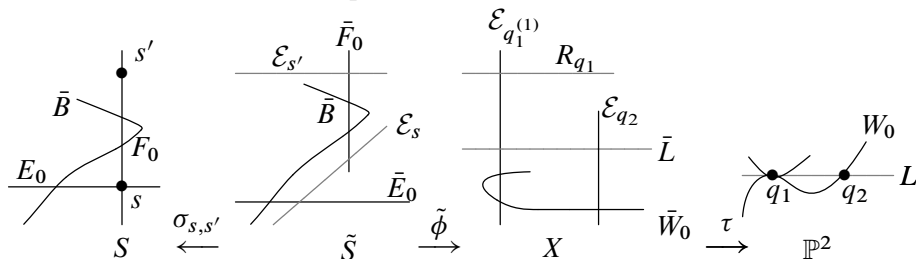


Figure 6.1.

Remark 6.1. If we blow-up the point s' on S' we obtain a Jacobian rational elliptic surface \tilde{S}' with the section equal to the exceptional curve over s' . Following [16], the birational transformation $h : S \dashrightarrow \tilde{S}'$ is called an *Halphen transform*. It replaces an index 2 elliptic fibration $f : S \rightarrow \mathbb{P}^1$ with a Jacobian elliptic fibration $f' : \tilde{S}' \rightarrow \mathbb{P}^1$. Note that the elliptic fibration f' is not, in general, isomorphic to the Jacobian fibration of $f : S \rightarrow \mathbb{P}^1$ although it is defined by the same data that consists of a choice of a fiber on f' and a non-trivial 2-torsion element in its Picard group (in our case defined by $\mathcal{O}_{h(F_0)}(s - s')$). Note that the elliptic pencil on \tilde{S}' is the pre-image of the pencil of lines through the point q_1 .

Remark 6.2. In the case of a degenerate configuration where $2F_0$ is a reducible fiber, we obtain a similar double plane model. The difference only is that the singular point q_1 of the quintic is not a tacnode anymore but a double cusp and the line L intersects the quintic at this point with multiplicity 2. Its proper transform on S is a component of the double fiber which intersects E_1 . The other two components are mapped to the curve $\mathcal{E}_{q_1^{(1)}}$ that splits in the cover.

Recall that we have proved in Lemma 5.6 that there exists a (-1) -curve E_0 which intersects two irreducible components of each reducible fiber of S . In Theorem 5.7 we proved the existence of eight (-1) -curves in S which together with E_0 form a set of nine disjoint (-1) -curves which can be blow-down to \mathbb{P}^2 so that the images of the reducible fibers is a Chilean configuration. Let us see this from the double cover.

Let C_i be a conic through $q_1, q_1^{(1)}$ and three points $y_j, j \neq i$. It does not intersect the branch curve and its proper transform \tilde{C}_i on X is a (-1) -curve. So it splits in the cover $\tilde{\phi} : \tilde{S} \rightarrow X$ into two disjoint (-1) -curves that are disjoint from the exceptional divisor of $\tilde{S} \rightarrow S$. Thus, the proper transform of C_i is the union of two disjoint (-1) -curves $E_i + E_{-i}$. Since the curves \tilde{C}_i are disjoint and do not pass through q_2 , we obtain a set of nine disjoint (-1) -curves E_0, E_i, E_{-i} that define a birational morphism $\pi : S \rightarrow \mathbb{P}^2$.

Proposition 6.3. *Every (-1) -curve E_0 can be included into a set $\mathcal{E}(E_0)$ of nine disjoint (-2) -curves $E_0, E_1, E_{-1}, \dots, E_4, E_{-4}$ such that the Bertini involution β_{E_0}*

sends E_i to E_{-i} and leaves invariant E_0 . Moreover, if y_i is an ordinary node, the image of the fiber F_i is the union of three conics. If y_i is a cusp, then the image of F_i is the union of a quartic and two lines.

Proof. Only the last assertion has not been proven yet. For simplicity of notation we assume that $y_i = y_1$. Suppose y_1 is a node. The curve E_1 intersects $R_1^{(1)}$ and $R_1^{(2)}$. Then each conic $C_i, i \neq 1$, passes through y_1 and hence $E_{\pm i}$ intersects $R_1^{(0)}$ and $R_1^{(1)}$ (or $R_1^{(2)}$). The curve $E_{\pm 1}$ intersects either $R_0^{(1)}$ or $R_0^{(2)}$ with multiplicity 1 or intersects one of these curves with multiplicity 2. In the first case each component $R_0^{(k)}$ intersects six curves from $\mathcal{E}(E_0)$ and hence its image under the blowing down morphism is a conic. In the second case, $R_0^{(1)}$ intersects four curves E_0, E_2, E_3, E_4 with multiplicity 1 and one curve E_1 with multiplicity 2. The self-intersection of its image under π is equal to $-2 + 4 + 4 = 6$, a contradiction. This proves the assertion in the case when y_i is a node.

Suppose y_1 is a cusp. The curve E_1 intersects $R_1^{(0)}$ with multiplicity 2. Each conic $C_i, i \neq 1$, passes through y_1 and hence $E_{\pm i}$ intersects both $R_1^{(1)}$ and $R_1^{(2)}$. Each curve $E_{\pm 1}$ intersects $R_1^{(0)}$ with multiplicity 2. Thus $R_1^{(\pm 1)}$ intersects 3 curves from $\mathcal{E}(E_0)$ and the curve $R_1^{(0)}$ intersects 6 curves from $\mathcal{E}(E_1)$ with multiplicity 1 and three curves from this set with multiplicity 2. This implies that the image of $R_1^{(\pm 1)}$ is a line and the image of $R_1^{(\pm 1)}$ has self-intersection $-2 + 6 + 12 = 16$, i.e. it is a quartic. This proves the assertion in the case when y_i is cusp. \square

Let $\mathcal{E}(E_0)$ denote the set of nine disjoint (-1) -curves including E_0 obtained by the construction from Proposition 6.3.

Definition 6.4. A set of nine disjoint (-1) -curves on S is called a *Chilean set of exceptional curves* if the image of the elliptic pencil under a birational map that blows down these curves is an Halphen pencil whose reducible members consist of three conics.

Corollary 6.5. *Suppose E_0 intersects eight irreducible components of reducible fibers (the existence of such curve follows from Lemma 5.6). Then $\mathcal{E}(E_0)$ is a Chilean set of exceptional curves and the branch curve of the double plane model does not have cusps.*

Proof. Since E_0 does not intersect any component $R_i^{(k)}$ with multiplicity 2, it follows from the proof of the previous proposition that all singular points y_i are nodes. \square

Proposition 6.6. *There is only one Chilean set of exceptional curves on S .*

Proof. Let \mathcal{E}_1 and \mathcal{E}_2 be two Chilean sets of exceptional curves. Let $\Sigma_i = \sum_{E \in \mathcal{E}_i} E$. Since each $R_i^{(k)}$ intersects six curves from \mathcal{E}_1 and \mathcal{E}_2 , we obtain $\Sigma_i \cdot R_i^{(k)} = 6$. Thus $\Sigma_1 - \Sigma_2$ belongs to the orthogonal complement of the sublattice $\text{Pic}_{\text{fib}}(S)$ of

$\text{Pic}(S)$ spanned by the components of fibers. Since $\text{Pic}_{\text{fib}}(S)$ is a lattice of rank 9 with radical spanned by $[F_0]$, its orthogonal complement is equal to $\mathbb{Z}[F_0]$. This shows that $\Sigma_1 - \Sigma_2 \sim nF_0$. Then $(\Sigma_1 - \Sigma_2)^2 = -9 - 2(9n - 9) - 9 = 0$ and so $n = 0$. Therefore $\Sigma_1 \sim \Sigma_2$, but now we intersect with each $E \in \mathcal{E}_1$ to get that $\Sigma_1 = \Sigma_2$. \square

Corollary 6.7. *The Chilean set \mathcal{E} of exceptional curves forms one orbit with respect to the action of $\text{Aut}(S)$.*

Proof. By the uniqueness of \mathcal{E} , it is invariant with respect to any automorphism of S . The group $\text{Aut}(S)$ contains as a subgroup H the Mordell-Weil group of $j(f)$ isomorphic to $(\mathbb{Z}/3\mathbb{Z})^2$. It acts simply transitively on the set, because it acts transitively on the set of nine sections of $j(f)$. This can be seen by using that H permutes cyclically components of each singular fiber and hence cannot fix any of the curves from \mathcal{E} . \square

Let us now fix a Chilean set $\{E_1, \dots, E_9\}$ of (-1) -curves. We would like to find the set $\text{Exc}(S)$ of (-1) -curves on S in terms the proper transforms under $\phi: S \dashrightarrow \mathbb{P}^2$ of some curves C in the plane. We already know that the curves E_2, \dots, E_9 come from conics passing through three of the double points of the branch quintic curve and passing through q_1 and its infinitely near point $q_1^{(1)}$. We fix E_1 that intersects F_0 at a nonsingular point s . We also have another point s' such that $\mathcal{O}_{F_0}(s - s') \cong \text{res}(F_0)$. Let $\text{Exc}_0(S)$ be the subset of $\text{Exc}(S)$ of (-1) -curves passing either through s or s' . It coincides with the set $\text{res}^{-1}(\text{pt}_{\mathfrak{s}_0})$ defined in Section 2. By Propositions 2.5 and 2.9, it coincides with $L^{\text{trop}}(E_1)/(mK_S)$. By Proposition 2.11, the Bertini involution β_{E_1} acts trivially on it. Thus all curves from the set $\text{Exc}_0(S)$ are invariant with respect to β_{E_1} . In other words, their images in the plane under the map ϕ do not split in the cover.

We write $a_s = 1$ if $s \in E$ and $a_{s'} = 1$ if $s' \in E$.

Let E be any (-1) -curve invariant with respect to β_{E_1} . Let $R_i^{(0)}, R_i^{(1)}, R_i^{(2)}, i = 1, 2, 3, 4$, be the components of singular fibers indexed as in the proof of Lemma 5.6, i.e. we assume that E_1 does not intersect $R_i^{(0)}$. Since the group $\text{Pic}(S)_{\mathbb{Q}}^{\beta_{E_1}^*}$ of rational divisor classes invariant with respect to $\beta_{E_1}^*$ is freely generated by the divisor classes of E_1, F_0 and $R_1^{(0)}, \dots, R_4^{(i)}$ we can write

$$E \sim k_0 E_1 - \sum_{i=1}^4 k_i R_i^{(0)} + k F_0$$

for some rational coefficients. Let

$$n := E \cdot E_1.$$

Intersecting both sides with F_0 , we find $k_0 = 1$. Intersecting both sides with E_1 we find $k = n + 1$. Since E is invariant, it either intersects both $R_i^{(1)}$ and $R_i^{(2)}$, or

intersects $R_i^{(0)}$ with multiplicity 2. Intersecting with $R_i^{(0)}$, we find that $k_i = 0$ in the former case and $k_i = 1$ in the latter case.

Let $v_C = \sum_{i=1}^4 k_i$ be the number of nodes y_i lying on C . The equality $-1 = E^2 = E_1^2 + 2(n + 1) - 2 \sum_{i=1}^4 k_i = -1 + 2(n + 1) - 2v_C$ gives

$$n + 1 = v_C.$$

The curve C passes through v_C nodes, passes through q_1 if $a_{s'} = 1$ and passes through q_2 with multiplicity $n - a_s$. Since the proper transform \tilde{E} of E on \tilde{S} is the pre-image of a curve on X , \tilde{E}^2 must be even (and this confirms that $a_s + a_{s'} = 1$), and since blowing only one of points on E , we have $\tilde{E}^2 = -2$, and hence the proper transform \tilde{C} on X has self-intersection equal to -1 . Since \tilde{C} intersects the exceptional curve \mathcal{E}_s over q_2 with multiplicity $\frac{1}{2}(n - a_s)$, this gives

$$\text{deg}(C)^2 = v_C + a_{s'} + \frac{1}{4}(n - a_s)^2 - 1.$$

Now we can list all possible C :

- (i) $n = 0, v_C = 1, n = 1, q_1 \in C, q_2 \notin C, \text{deg}(C) = 1$;
- (ii) $n = 1, v_C = 2, a_s = 1, q_1, q_2 \notin C, \text{deg}(C) = 1$;
- (iii) $n = 2, v_C = 3, a_{s'} = 1, q_1, q_2 \in C, \text{deg}(C) = 2$;
- (iv) $n = 3, v_C = 4, a_s = 1, q_1 \in C, q_2 \notin C, \text{deg}(C) = 2$.

In another words, C is either a line passing through q_2 and one node (case (i)), or a line passing through two nodes (case (ii)), or a conic passing through q_1, q_2 and three points y_i (case (iii)), or a conic passing through q_2 and all points y_i (case (iv)).

Let (e_0, e_1, \dots, e_9) be the geometric basis corresponding to the base points of the Chilean Halphen pencil. We take the double model corresponding to E_1 . Then the divisor class of a (-1) -curve E is $de_0 - \sum_{i=1}^9 m_i e_i$. We have $m_1 = E \cdot E_1 = n$. Since E_2, \dots, E_9 are the split pre-images of the conics through three nodes and $q_1, q_1^{(1)}$, we can group them in pairs and redenoting them by $E_2, E_{-2}, E_3, E_{-3}, E_4, E_{-4}$. We assume that $\phi(E_i + E_{-i})$ is the conic passing through all points y_i except y_i . Intersecting $C = \phi(E)$ with one of the curves, we find the divisor class of $[E]$ to be equal to

$$[E] = \begin{cases} e_0 - e_i - e_{-i} & \text{in case (i)} \\ 2e_0 - e_1 - e_i - e_{-i} - e_j - e_{-j} & \text{in case (ii)} \\ 3e_0 - 2e_1 - \sum_{j \neq i} (e_j + e_{-j}) & \text{in case (iii)} \\ 4e_0 - 3e_1 - \sum (e_i + e_{-i}) & \text{in case (iv)}. \end{cases}$$

As we see that, together with E_1 there are 16 such curves as expected from our general theory.

Since each $E_i \cap \beta_{E_1}(E_i) = \emptyset$ for $i \neq 1$, the branch curve B does not intersect such E_i . It intersects E_1 with multiplicity 1. This gives

$$[B] = e_0 - e_1. \tag{6.1}$$

In other words, the image of B in the plane is a line passing through x_1 .

To find the rest of 144–16 curves, we use that the translation group acts simply transitively on the set of curves E_1, E_i, E_{-i} . Thus it leaves the class $\sum_{i=1}^9 e_i$ invariant, and hence leaves invariant the class e_0 that defines the blowing down $\pi : S \rightarrow \mathbb{P}^2$. This means that the group of translations descend to a group of projective transformations of the plane. Using the explicit formulas for the base points of the Chilean pencil from Section 5, we find that the group is generated by transformations $(x : y : z) \mapsto (z : y : x)$ and $(x : y : z) \mapsto (x : \epsilon y : \epsilon^2 z)$. Now the rest of the curves are obtained by applying these transformations to the 16 that has been already found. The following table gives the result, and it also gives the images of (-1) -curves under ϕ .

Table 6.1. The classes of the 144 (-1) -curves on S .

$\text{deg}(C)$	n	v_C	u_C	a_s	$a_{s'}$	$\text{deg}(\pi(E))$	split	class	#
0	0	0	0	0	0	0	no	e_1	1
1	0	1	0	0	1	1	no	$e_0 - e_i - e_{-i}$	4
1	1	2	0	1	0	2	no	$2e_0 - e_1 - e_i - e_{-i} - e_j - e_{-j}$	6
2	2	3	0	0	1	3	no	$3e_0 - 2e_1 - \sum_{j \neq i} (e_j + e_{-j})$	4
2	3	4	0	1	0	4	no	$4e_0 - 3e_1 - \sum (e_i + e_{-i})$	1
2	0	3	0	0	0	0	yes	$e_{\pm i}$	8
2	0	2	0	0	0	1	yes	$e_0 - e_{\pm i} - e_{\pm j}$	24
2	0	1	0	0	0	2	yes	$2e_0 - e_i - e_{-i} - e_{\pm j} - e_{\pm k} - e_{\mp l}$	24
2	0	0	0	0	0	3	yes	$3e_0 - 2e_{\pm i} - \sum_{j \neq i} (e_j + e_{-j})$	8
3	1	3	1	0	0	1	yes	$e_0 - e_1 - e_{\pm i}$	8
3	1	2	1	0	0	2	yes	$2e_0 - e_1 - e_i - e_{-i} - e_{\pm j} - e_{\mp k}$	24
3	1	1	1	0	0	3	yes	$3e_0 - e_1 - e_i - e_{-i} - 2e_{\pm j} - e_{\mp j} - e_{\pm k} - e_{\pm l}$	24
3	1	0	1	0	0	4	yes	$4e_0 - e_1 - 3e_{\pm i} - e_{\mp i} - \sum_{j \neq i} (e_j + e_{-j})$	8

Here u_C denotes the number of double points of C among y_1, \dots, y_4 . We see that all 144 exceptional curves are accounted for.

Remark 6.8. An explicit computer computation gives the equation of the branch quintic curve W_0 :

$$\begin{aligned} &x^3y^2 + 2\epsilon x^2y^3 + \epsilon^2xy^4 + 2\epsilon^2x^3yz + (2a^3 + 4)x^2y^2z + 2\epsilon a^3xy^3z \\ &+ 2\epsilon^2(2a^3 - 1)y^4z + \epsilon(-4a^3 + 1)x^3z^2 + \epsilon^2(-10a^3 + 4)x^2yz^2 \\ &+ (a^6 - 12a^3 - 4)xy^2z^2 + 4\epsilon(a^3 - 2)y^3z^2 + (-4\epsilon a^6 - 16\epsilon a^3 + 2\epsilon)x^2z^3 \\ &- 8\epsilon^2(5a^3 + 1)xyz^3 + (2a^6 - 32a^3 - 16)y^2z^3 \\ &+ (-16\epsilon a^6 - 16\epsilon a^3 - 4\epsilon)xz^4 - 8\epsilon^2(5a^3 + 2)yz^4 + (-16\epsilon a^6 - 8\epsilon)z^5 = 0. \end{aligned}$$

Now consider the blow-up of the base points of the Halphen pencil defining a Chilean configuration. Fix one of the exceptional curves E_i , and consider the

double plane model $\phi_i : S \dashrightarrow \mathbb{P}^2$ defined by E_i . Using (6.1), we find that the branch curve B_i of ϕ_i is represented in the geometric basis by $e_0 - e_i$. So the image of B_i under the blowing down map $\pi : S \rightarrow \mathbb{P}^2$ is a line ℓ_i passing through the base point x_i . Since B_i intersects each reducible fiber at one of its singular point and intersects the component opposite to this point with multiplicity 2, we see that ℓ_i passes through the intersection points of four conics in the Chilean configuration that contain the base point x_i . This defines a configuration of 9 lines and 12 points isomorphic to the dual Hesse configuration $(9_4, 12_3)$.

We have proved the following.

Theorem 6.9. *A Chilean configuration of 12 conics defines 9 lines ℓ_i and 12 intersection points of conics isomorphic to the dual Hesse configuration $(9_4, 12_3)$.*

Remark 6.10. Note that the Hesse pencil is defined over \mathbb{F}_4 if we assume that the characteristic is equal to 2. Its set of 12 line components of fibers, 9 harmonic polars and the set of 9 base points and 12 singular points of fibers is equal to the set of 21 lines and 21 points in $\mathbb{P}^2(\mathbb{F}_4)$. The specific of characteristic 2 is that each harmonic polar line passes through the corresponding base point. This is similar to what we have in the case of the Chilean pencil. The union of 21 lines and 21 points form a symmetric configuration (21_5) . Let X be the blow-up of all 21 points. It admits an inseparable finite map of degree 2 isomorphic to a supersingular K3 surface with Artin invariant 1 [10]. It is a minimal resolution of the double plane

$$w^2 + xyz(x^3 + y^3 + z^3) = 0.$$

The Chilean pencil is defined over any non-trivial extension $K = \mathbb{F}_4(a)$ of the finite field \mathbb{F}_4 and defines a configuration of 21 points, 12 conics and 9 lines over K . There is no degenerate configuration in this case because the multiplicative group of \bar{K} does not have elements of order 2. We again blow-up the 21 points, and consider the double plane

$$w^2 + F(x, y, z) = 0,$$

where $F(x, y, z)$ is any smooth irreducible non-multiple member of the pencil. The double cover has 21 ordinary double points defined over K , and its minimal resolution is a supersingular K3 surface. Our 1-dimensional family of such surfaces is one of the three irreducible components of the moduli space of supersingular K3 surfaces with Artin invariant σ equal to 2 studied by I. Shimada [24]. Each family is isomorphic to the affine line with one point deleted. The closures of these families contain one common point that corresponds to the supersingular K3 surface with the Artin invariant 1. The three families are distinguished by a certain 9-dimensional linear code in \mathbb{F}_2^{21} . Our family has the weight polynomial

$$W(t) = 1 + 9t^5 + 102t^8 + 144t^9 + 144t^{12} + 102t^{13} + 9z^{16} + z^{21}.$$

The code is generated by 9 words of weight 5 representing our harmonic polar lines B_i and 66 words of weight 8 represented by our 54 conics and 12 conic components of reducible fibers.

Remark 6.11. An inspection of the list $\text{Exc}(S)$ of 144 (-1) -curves from Proposition 5.10 suggests a certain involution on this set that interchanges the exceptional curves E_i with quartics, lines with cubics and pairs conics. An explanation of this duality is the following. Given a (-1) -curve E we consider the corresponding double plane and its ramification curve B on S . We have $E \cdot B = 1$, so that $(F + B - E)^2 = -1$ and $K_S \cdot (F + B - E) = -1$. It is easy to see that, for any (-2) -curve R , we have either $R \cdot (F + B - E) \geq 0$ (if R intersects E , then it also intersects B). Applying Lemma 2.4, we see that $F + B - E \sim E'$, where E' is a (-1) -curve. This defines an involution on the set $\text{Exc}(S)$. The inspection of Table 6.1 confirms that it matches the curves as above.

Remark 6.12. Any five points determine a conic, but we want to exclude the 12 conics which are part of reducible fibers. Such forbidden sets of fives corresponds to removing one point of the six of such on each component conic, thus the number of allowed sets of fives are given by $\binom{9}{5} - 6 \cdot 12 = 54$. Examples of such sets are those given by the union of two non-parallel lines in our affine space of nine points. The number of such sets can easily be determined to be $54 (12 \cdot (12 - 3) / 2$ or $9 \cdot \binom{4}{2})$ thus all our allowable sets are of this type. This makes the duality $I \rightarrow I'$ explicit. Given I with special point p (the intersection of the lines) then choose I' as the two residual line through p (there are 4 lines through each point).

Remark 6.13. It is known that two (-1) -curves on a weak del Pezzo surface of degree d intersect with multiplicity ≤ 3 (2 if $d = 2$ and 1 if $d \geq 3$ because in the latter case they become lines in the anti-canonical model). An Halphen surface is obtained by blowing up one point from a weak del Pezzo surface of degree 1. We observed that in our case two (-1) -curves intersect with multiplicity ≤ 3 . This raises the question: *What is the maximal intersection number of two (-1) -curves on an Halphen surface with extremal Jacobian fibration?*

7. Higher indices

Let m be a positive integer which is not a multiple of 3 and let S_m be an elliptic Halphen surface of index m whose Jacobian is the Hesse surface. Such surface can be constructed as in the case $m = 2$ by blowing-up the plane at the nine points p_1, \dots, p_9 , where $p_i := x_i + \eta$ and η is an m -torsion point of the curve of equation $x^3 + y^3 + z^3 + txyz = 0$. We say that the points p_i, p_j, p_k are *Hesse-collinear* if x_i, x_j, x_k are collinear. We consider two cases.

- (1) If $m = 3k + 1$, then there are 12 irreducible plane curves, each of degree m with three Hesse-collinear points of multiplicity $k + 1$ and the remaining six points of multiplicity k ;
- (2) If $m = 3k + 2$, then there are 12 irreducible plane curves, each of degree m with three Hesse-collinear points of multiplicity k and the remaining six points of multiplicity $k + 1$.

In both cases one can verify the existence of the curves by checking that the points with the given multiplicities sum to 0 in the group law of the cubic. The irreducibility is a consequence of the fact that a curve of self-intersection -2 on a relatively minimal elliptic surface can only be the sum of irreducible (-2) -curves, but we cannot have more than 12 of them.

Here are some m -torsion loci with respect to the point $x_7 = (1 : -1 : 0)$.² When $m = 4$ the locus is cut out by

$$xy^3 - y^4 - txy^2z - xz^3 - 2yz^3 = 0.$$

When $m = 5$ the locus is cut out by

$$\begin{aligned} &2x^2y^6 - xy^7 + 2y^8 - x^2y^3z^3 - xy^4z^3 + 5y^5z^3 - x^2z^6 + 2xyz^6 + 2y^2z^6 \\ &+ t(-x^2y^5z + 3xy^6z - y^7z + x^2y^2z^4 + 3xy^3z^4 + yz^7) \\ &+ t^2(x^2y^4z^2 - xy^5z^2 + xy^2z^5) = 0. \end{aligned}$$

To construct elliptic surfaces of index 3 whose Jacobian is the Hesse surface we proceed as follows. Let $\pi : \mathbb{Z}/9\mathbb{Z} \oplus \mathbb{Z}/3\mathbb{Z} \rightarrow \mathbb{Z}/3\mathbb{Z} \oplus \mathbb{Z}/3\mathbb{Z}$ be the homomorphism induced by the projection on the first coordinate. Let β_{E_0} be a set-theoretic section of π such that the sum of all the elements in its image generates the kernel of π . An example of such a β_{E_0} is given by the columns of the following matrix

$$\begin{pmatrix} 0 & 0 & 0 & 1 & 4 & 1 & 5 & 5 & 5 \\ 0 & 1 & 2 & 0 & 1 & 2 & 0 & 1 & 2 \end{pmatrix},$$

where the first coordinate of each vector is in $\mathbb{Z}/9\mathbb{Z}$ while the second is in $\mathbb{Z}/3\mathbb{Z}$. Choose nine points p_1, \dots, p_9 on a smooth plane cubic C in such a way that p_1 is a flex of C and the class of $p_i - p_1$ is given by the i -th column of the above matrix. The blow-up of \mathbb{P}^2 at the nine points is an elliptic surface of index 3 whose Jacobian is the Hesse surface and whose (-2) -classes are the columns of the following matrix

$$\begin{pmatrix} 1 & 4 & 4 & 1 & 4 & 4 & 1 & 4 & 4 & 1 & 4 & 4 \\ -1 & -1 & -1 & -1 & -1 & -1 & 0 & -2 & -1 & 0 & -2 & -1 \\ -1 & -1 & -1 & 0 & -2 & -1 & -1 & -1 & -1 & 0 & -1 & -2 \\ -1 & -1 & -1 & 0 & -1 & -2 & 0 & -1 & -2 & -1 & -1 & -1 \\ 0 & -2 & -1 & 0 & -1 & -2 & 0 & -2 & -1 & 0 & -1 & -2 \\ 0 & -2 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 \\ 0 & -2 & -1 & 0 & -2 & -1 & 0 & -1 & -2 & 0 & -2 & -1 \\ 0 & -1 & -2 & 0 & -2 & -1 & 0 & -2 & -1 & -1 & -1 & -1 \\ 0 & -1 & -2 & 0 & -1 & -2 & -1 & -1 & -1 & 0 & -2 & -1 \\ 0 & -1 & -2 & -1 & -1 & -1 & 0 & -1 & -2 & 0 & -1 & -2 \end{pmatrix}.$$

² We have computed these loci with Magma moving to a Weierstraß model of the elliptic curve and calculating the m -torsion subscheme there.

The 9-torsion points with respect to the point $x_7 = (1 : -1 : 0)$ are cut out by the following cubics:

$$\begin{array}{ll}
 xy^2 + \epsilon x^2z + \epsilon^2 yz^2, & xy^2 + \epsilon^2 x^2z + \epsilon yz^2, \\
 xy^2 + x^2z + yz^2, & x^2y + \epsilon^2 y^2z + \epsilon xz^2, \\
 x^2y + \epsilon y^2z + \epsilon^2 xz^2, & x^2y + y^2z + xz^2, \\
 3x^3 + (\epsilon + 2)txyz - 3\epsilon^2z^3, & 3x^3 + (-\epsilon + 1)txyz - 3\epsilon z^3.
 \end{array}$$

8. Applications

In this section, we exhibit some nice properties of the Chilean configuration.

A simple crossings configuration of curves on a surface Z is a collection of nonsingular irreducible curves $\mathbf{A} = \{C_1, \dots, C_d\}$ such that C_i, C_j are transversal for all $i \neq j$. For $n \geq 2$, an n -point of \mathbf{A} is a point which belongs to exactly n curves in \mathbf{A} . Let $\sigma : Y \rightarrow Z$ be the blow-up of all n -points with $n \geq 3$. Then $D := \sigma^*(\mathbf{A})_{\text{red}}$ is a simple normal crossings divisor, and we define the log Chern numbers \bar{c}_1^2, \bar{c}_2 of (Z, \mathbf{A}) as the Chern numbers of $\Omega_Y^1 \log(D)$. We have (see [25, Section 4])

$$\begin{aligned}
 \bar{c}_1^2 &= c_1^2(Z) - \sum_{i=1}^d C_i^2 + \sum_{n \geq 2} (3n - 4)t_n + 4 \sum_{i=1}^d (g(C_i) - 1) \\
 \bar{c}_2 &= c_2(Z) + \sum_{n \geq 2} (n - 1)t_n + 2 \sum_{i=1}^d (g(C_i) - 1),
 \end{aligned}$$

where t_n is the number of n -points. The highest the log Chern slope \bar{c}_1^2/\bar{c}_2 is, the more special the configuration is. As the Chilean configuration of 12 conics is simple crossings with $t_2 = 12, t_8 = 9$, and $t_n = 0$ else, we have $\bar{c}_1^2 = 117, \bar{c}_2 = 54$, and so $\bar{c}_1^2/\bar{c}_2 = 13/6 = 2.1\bar{6}$. The Hesse configuration of 12 lines has $\bar{c}_1^2/\bar{c}_2 = 2.5$.

We recall that log Chern slopes do have constraints. As a general example, let us consider a simple crossings arrangement of plane curves $\{C_1, \dots, C_d\}$ with $\bigcap_{i=1}^d C_i = \emptyset$. It is a divisible arrangement as defined in [25, Section 4] (see [25, Example 4.3]), and so by [25, Theorem 6.1], if $\bar{c}_2 \neq 0$, we can compare \bar{c}_1^2/\bar{c}_2 with the Chern slope of a smooth complex projective surface. Typically these surfaces are not ruled, and so we can use the Bogomolov-Miyaoka-Yau inequality for complex algebraic surfaces to show that $\bar{c}_1^2 \leq 3\bar{c}_2$. See [13] for a systematic study of log Chern slopes for configurations of lines. In that case, we have the combinatorial (independent of the field of definition) inequalities $\bar{c}_2 \leq \bar{c}_1^2 \leq 3\bar{c}_2$ (up to trivial configurations). For complex configurations of lines, the inequalities improve to the sharp $\bar{c}_2 \leq \bar{c}_1^2 \leq 8/3\bar{c}_2$, where the upper bound is achieved if and only if the configuration is the dual Hesse configuration. It is easy to prove (combinatorially) that $[1, 2[$ has no limit points for log Chern slopes over any fixed

field. In [13] it is proved that log Chern slopes are dense in $[2, 2.5]$ over \mathbb{C} , and it is conjectured that there are no limit points in $]2.5, 2.\bar{6}]$. In positive characteristic, it is proved that they are dense in $[2, 3]$.

There are at least four special configurations after we add the degenerated Chilean configuration, and the dual Hesse configuration into the picture.

- (A₀) The degenerated Chilean configuration is a simple crossings configuration of 9 conics and 3 lines. We have $t_2 = 12$ and $t_7 = 9$, and so $\bar{c}_1^2 = 99$, $\bar{c}_2 = 45$, and $\bar{c}_1^2/\bar{c}_2 = 11/5 = 2.2$. Thus when we go from the Chilean configuration to the degenerated Chilean configuration, the log Chern slope jumps up;
- (A₁) Consider the configuration A_1 consisting of the 12 irreducible conics of the Chilean configuration, and the 9 lines of the dual Hesse configuration. Then A_1 has $t_2 = 6 \times 3 \times 4 = 72$, $t_5 = 12$, $t_8 = 9$, $t_n = 0$ else. Therefore $\bar{c}_1^2 = 324$, $\bar{c}_2 = 144$, and $\bar{c}_1^2/\bar{c}_2 = 9/4 = 2.25$;
- (A₂) Consider instead the configuration A_2 formed by one of the four degenerated Chilean configurations of 9 conics and 3 lines together with the 9 lines in the dual Hesse configuration. Then A_2 has $t_2 = 6 \times 3 \times 3 = 54$, $t_5 = 12$, $t_7 = 9$, $t_n = 0$ else. Therefore $\bar{c}_1^2 = 270$, $\bar{c}_2 = 117$, and $\bar{c}_1^2/\bar{c}_2 = 30/13 = 2.307692$;
- (A₃) This is just putting together the Hesse configuration of 12 lines with the dual Hesse configuration of 9 lines as appear in the Chilean configurations. Then we have $t_2 = 9 \times 4 = 36$, $t_4 = 9$, $t_5 = 12$, $t_n = 0$ else. A nice observation is that the 9×4 2-points are the 9×4 base points of the degenerated Chilean arrangements. We have $\bar{c}_1^2 = 180$, $\bar{c}_2 = 72$, and $\bar{c}_1^2/\bar{c}_2 = 5/2 = 2.5$.

Another application is suggested by Anatoly Libgober. Let \mathcal{C} be the union of 12 conics from the Chilean configuration.

Proposition 8.1. *The fundamental group $\pi_1(U)$ of the complement $U = \mathbb{P}^2 \setminus \mathcal{C}$ fits in the exact sequence*

$$F_{19} \rightarrow \pi_1(U) \rightarrow F_3 \rightarrow 1,$$

where F_k denote the free group with k generators.

Proof. The restriction of the Halphen pencil to U defines a morphism $U \rightarrow \mathbb{P}^1 \setminus \{4 \text{ points}\}$. Its fibers are isomorphic to plane sextic curves with 9 nodes deleted. Passing to the normalization we obtain that the fiber is isomorphic to an elliptic curve with 18 points deleted. Its fundamental group is isomorphic to the free group F_{19} . The exact sequence from the assertion is the exact sequence for fundamental groups of smooth fibrations. □

We refer to [5] for the discussion of the problem of classification of configuration of curves on a simply connected algebraic surface that admits a surjection to a free non-commutative group.

Another application of our configuration can be found in [22]. Recall that a plane curve C is called *free* if the logarithmic tangent sheaf $\Theta^1(\log C)$ is locally free, or, equivalently, the Jacobian sheaf of C has projective dimension 2.

It is well-known that the union of 12 lines from the Hesse configuration, being a special case of an arrangement of reflection hyperplanes of a complex reflection group, is a free divisor.

The following proposition is proved in [22, Proposition 1 and 2];

Proposition 8.2. *The union of 12 conics from the Chilean configuration as well as the union of 12 conics and 9 lines from the embedded dual Hesse configuration are free divisors in the projective plane.*

Another interesting attribute of a reducible curve C on a rational surface is the *Harbourne constant* $H(C)$ defined by

$$H(C) = \frac{C^2 - \sum_{x \in X} \text{mult}_x(C)^2}{s},$$

where $\text{mult}_x(C)$ is the multiplicity of a singular point $x \in C$ and s is the number of such points. It is conjectured that $H(C) \geq -4.5$ for curves over \mathbb{C} .

In our case we may consider the union C of nine 2-sections E_i and four reducible fibers of the elliptic fibration on S . We have $C^2 = -9 + 9 \cdot 16 = 15.9 = 135$ and we have $84 = 8.9 + 12$ points of multiplicity 2, so $H(C) = \sum_{x \in X} \text{mult}_x(C)^2 = -3 + \frac{17}{28}$. On the other hand, if we take the degenerate Chilean configuration, we obtain $C^2 = -9 + 14.9 = 117$ and $\sum_{x \in X} \text{mult}_x(C)^2 = 7.9 + 12 = 75$ that gives $H(C) = -3 + \frac{14}{25}$, that is slightly less.

References

- [1] M. ARTEBANI and I. DOLGACHEV, *The Hesse pencil of plane cubic curves*, Enseign. Math. (2) **55** (2009), 235–273.
- [2] W. P. BARTH, K. HULEK, C. A. M. PETERS and A. VAN DE VEN, “Compact Complex Surfaces”, Ergebnisse der Mathematik und ihrer Grenzgebiete, 2nd ed., 3 Folge. A Series of Modern Surveys in Mathematics, Vol. 4. Springer-Verlag, Berlin, 2004.
- [3] A. BEAUVILLE, “Surfaces Algébriques Complexes”, Société Mathématique de France, Paris, 1978. Avec une sommaire en anglais; Astérisque, n. 54.
- [4] E. BERTINI, *Ricerche sulle trasformazioni univoche involutorie del piano*, Ann. Mat. Pura Appl. (2) **8** (1877), 244–282.
- [5] J. I. COGOLLUDO and A. LIBGOBER, *Free quotients of fundamental groups of smooth quasi-projective varieties*, Proc. Edinb. Math. Soc. **64** (2021), 924–946.
- [6] F. R. COSSEC and I. V. DOLGACHEV, “Enriques Surfaces, I”, Progress in Mathematics, Vol. 76, Birkhäuser Boston, Inc., Boston, MA, 1989.
- [7] I. V. DOLGACHEV, *Rational surfaces with a pencil of elliptic curves*, Izv. Akad. Nauk SSSR Ser. Mat. **30** (1966), 1073–1100.
- [8] I. V. DOLGACHEV, “Lectures on Invariant Theory”, London Mathematical Society Lecture Note Series, Vol. 296, Cambridge University Press, Cambridge, 2003.
- [9] I. V. DOLGACHEV and V. A. ISKOVSKIKH, *Finite subgroups of the plane Cremona group*, J. Algebraic Geom. **269** (2009), 443–548.

- [10] I. V. DOLGACHEV and S. KONDO, *A supersingular K3 surface in characteristic 2 and the Leech lattice*, Int. Math. Res. Not. **1** (2003), 1–23.
- [11] I. V. DOLGACHEV, “Classical Algebraic Geometry: A Modern View”, Cambridge University Press, Cambridge, 2012.
- [12] I. V. DOLGACHEV and G. MARTIN, *Automorphism groups of rational elliptic and quasi-elliptic surfaces in all characteristics*, Adv. Math. **400** (2022), paper n. 108274.
- [13] S. ETEROVIĆ, F. FIGUEROA and G. URZÚA, *On the geography of line arrangements*, to appear in Adv. Geom., <https://doi.org/10.1515/advgeom-2021-0025>.
- [14] P. HACKING and Y. PROKHOROV, *Smoothable del Pezzo surfaces with quotient singularities*, Compos. Math. **146** (2010), 169–192.
- [15] G. H. HALPHÉN, *Sur les courbes planes du sixième degré à neuf points doubles*, Bull. Soc. Math. France **10** (1882), 162–172.
- [16] B. HARBOURNE and W. E. LANG, *Multiple fibers on rational elliptic surfaces*, Trans. Amer. Math. Soc. **307** (1988), 205–223.
- [17] B. HARBOURNE and R. MIRANDA, *Exceptional curves on rational numerically elliptic surfaces*, J. Algebra **128** (1990), 405–433.
- [18] D. KOHEL, X. ROULLEAU and A. SARTI, *A special configuration of 12 conics and generalized Kummer surfaces*, Manuscripta Math. (2021), <https://doi.org/10.1007/s00229-021-01334-2>.
- [19] A. LAFACE and D. TESTA, *On minimal rational elliptic surfaces*, arXiv:1502.00275.
- [20] R. MIRANDA and U. PERSSON, *On extremal rational elliptic surfaces*, Math. Z. **193** (1986), 537–558.
- [21] H. PARK, D. SHIN and G. URZÚA, *A simply connected numerical Campedelli surface with an involution*, Math. Ann. **357** (2013), 31–49.
- [22] P. POKORA and T. SZEMBERG, *Conic-line arrangements in the complex projective plane*, to appear in Discrete and Computational Geometry.
- [23] X. ROULLEAU, *Conic configurations via dual of quartic curves*, Rocky Mountain J. Math. **51** (2021), 721–732.
- [24] I. SHIMADA, *Moduli curves of supersingular K3 surfaces in characteristic 2 with Artin invariant 2*, Proc. Edinb. Math. Soc. (2) **49** (2006), 435–503.
- [25] G. URZÚA, *Arrangements of curves and algebraic surfaces*, J. Algebraic Geom. **19** (2010), 335–365.

Department of Mathematics
 University of Michigan
 Ann Arbor, USA
 idolga@umich.edu

Departamento de Matemática
 Facultad de Ciencias Físicas y Matemáticas
 Universidad de Concepción
 Concepción, Chile
 alaface@udec.cl

University of Gothenburg Chalmers
 Mathematical Sciences, Algebra and Geometry
 Gothenburg, Sweden
 ulfp@chalmers.se

Facultad de Matemáticas
 Pontificia Universidad Católica de Chile
 Santiago, Chile
 urzua@mat.uc.cl