AUTOMORPHISMS OF
CLUSTER ALGEBRAS OF RANK 2

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To the memory of Andrei Zelevinsky (1953–2013)

Abstract. We compute the automorphism group of the affine surfaces with the coordinate ring isomorphic to a cluster algebra of rank 2.

1. Introduction

1.1. Definition of $\mathcal{C}(a,b)$

A cluster algebra $\mathcal{C}(a,b)$ of rank 2 is a subring of the field of rational functions $\mathbb{Q}(y_1, y_2)$ generated by elements $y_n, n \in \mathbb{Z}$, defined inductively by the relations

$$y_{n-1}y_{n+1} = \begin{cases} y_n^a + 1 & \text{if } n \text{ is even,} \\ y_n^b + 1 & \text{otherwise} \end{cases} \tag{1}$$

(see [5]). Here a, b are fixed positive integers. The elements y_n are called *cluster variables* and the pairs y_n, y_{n+1} are called *clusters*. It follows from [2, Cor. 1.21] that any four consecutive cluster variables, say y_1, y_2, y_3, y_4, generate $\mathcal{C}(a,b)$ as a $\mathbb{Z}$-algebra and the relations are defining relations. Thus

$$\mathcal{C}(a,b) \simeq \mathbb{Z}[y_1, y_2, y_3, y_4]/(y_1y_3 - y_2^a - 1, y_2y_4 - y_3^b - 1).$$

In the general context of cluster algebras, the algebra $\mathcal{C}(a,b)$ corresponds to a skew-symmetrizable seed matrix

$$\begin{pmatrix} 0 & a \\ -b & 0 \end{pmatrix}.$$

If $a = b$, it can be also defined by the quiver with two vertices and a arrows from the first one to the second one. Note that one can also consider the case $a = b = 0$ when the algebra $\mathcal{C}(0,0)$ is the algebra of Laurent polynomials in two variables. We omit this well-known case.

The cluster algebra $\mathcal{C}(a,b)$ is called of *finite type* if the number of cluster variables is finite. This happens if and only if $ab \leq 3$ (this follows from [4, Thm. 6.1]). In this case, we have the periodicity

$$y_n = y_m \iff n \equiv m \mod h + 2,$$

where $h = 3, 4, 6$ if $ab = 1, 2, 3$, respectively. The algebra $\mathcal{C}(a,b)$ is called of type $\mathsf{A}_2, \mathsf{B}_2, \mathsf{G}_2$ if $ab = 1, 2, 3$, respectively (the remaining type $\mathsf{A}_1 \times \mathsf{A}_1$ of a finite root system of rank 2 corresponds to the case $(a,b) = (0,0)$). In this paper we compute the group of automorphisms of the cluster algebra $\mathcal{C}(a,b)$.

1.2. The cluster automorphisms σ_p

As observed in [5], the transformations $\{\sigma_p\}_{p \in \mathbb{Z}}$ of $\mathcal{C}(a,b)$ defined by

$$\sigma_p : y_n \mapsto y_{2p-n}$$

preserve the relations (1) and define automorphisms of $\mathcal{C}(a,b)$ for arbitrary parameters a, b.

For example, σ_2, σ_3 send y_1, y_2, y_3, y_4 respectively to y_0, y_1, y_2, y_3 and y_2, y_3, y_4, y_5, confirming that $\{y_0, y_1, y_2, y_3\}$ and $\{y_2, y_3, y_4, y_5\}$ are also sets of generators.

Using the identities $y_1^b + 1 = y_1^b(y_3^b + 1) - (y_1 y_3 - 1) \sum_{i=0}^{b-1}(y_1 y_3)^i$ and $y_4^a + 1 = y_4^a(y_2^a + 1) - (y_2 y_4 - 1) \sum_{i=0}^{a-1}(y_2 y_4)^i$, we obtain the equalities

$$y_0 = \frac{y_1^b + 1}{y_2} = y_1^b y_4 - y_2^{a-1} \sum_{i=0}^{b-1}(y_1 y_3)^i,$$

$$y_5 = \frac{y_4^a + 1}{y_3} = y_4^a y_1 - y_3^{b-1} \sum_{i=0}^{b-1}(y_2 y_4)^i.$$

Hence, $\sigma_2, \sigma_3 \in \mathrm{Aut}(\mathcal{C}(a,b))$ correspond respectively to the following automorphisms of $\mathrm{Spec}\,\mathcal{C}(a,b)$ (we again write σ_i the dual geometric action induced by σ_i, and will in fact only work with this latter):

$$\sigma_2 \colon (y_1, y_2, y_3, y_4) \mapsto \left(y_3, y_2, y_1, y_1^b y_4 - y_2^{a-1} \sum_{i=0}^{b-1}(y_1 y_3)^i \right),$$

$$\sigma_3 \colon (y_1, y_2, y_3, y_4) \mapsto \left(y_4^a y_1 - y_3^{b-1} \sum_{i=0}^{b-1}(y_2 y_4)^i, y_4, y_3, y_2 \right).$$

It is immediately checked that σ_2, σ_3 satisfy $\sigma_2^2 = \sigma_3^2 = 1$ and generate a finite dihedral group D_{2n} of order $2n$ or infinite dihedral group $D_\infty \simeq \mathbb{Z}/2\mathbb{Z} \star \mathbb{Z}/2\mathbb{Z} \simeq \mathbb{Z} \rtimes \mathbb{Z}/2\mathbb{Z}$. The periodicity of the set of cluster variables easily implies that

$$n = \begin{cases} 10, & ab = 1, \\ 6, & ab = 2, \\ 8, & ab = 3, \\ \infty, & ab > 3. \end{cases}$$

1.3. The results

Our description of the group $\operatorname{Aut}(\mathcal{C}(a,b)) \simeq \operatorname{Aut}(\operatorname{Spec} \mathcal{C}(a,b))$ is given by the description of $\operatorname{Aut}(\operatorname{Spec} \mathcal{C}(a,b) \otimes \mathbb{Q})$ with geometric tools, and by observing that the generators of the automorphism groups are defined over $\mathbb{Z}$, hence both groups are equal.

In fact, we will precisely describe the group structure of the group

$$\operatorname{Aut}(\operatorname{Spec} \mathcal{C}(a,b) \otimes \mathrm{k})$$

for any field k of characteristic 0 or characteristic prime to ab. This is the group of automorphisms of the affine surface over k

$$X(a,b) = \operatorname{Spec} \mathcal{C}(a,b) \otimes \mathrm{k}.$$

Theorem 1. *The group*

$$\mu_{a,b} := \{(\mu,\nu) \in \mathrm{k}^* \mid \mu^a = \nu^b = 1\} \subset \mathrm{k}^{*2}$$

acts on $X(a,b)$ as

$$(y_1, y_2, y_3, y_4) \mapsto (\nu^{-1}y_1, \mu y_2, \nu y_3, \mu^{-1}y_4).$$

If $a = b$, then there is a group $H_{a,a} \subset \operatorname{Aut}(X(a,b))$ of order 2, acting on $X(a,b)$ via

$$\sigma_{5/2} \colon (y_1, y_2, y_3, y_4) \mapsto (y_4, y_3, y_2, y_1).$$

The group $\operatorname{Aut}(X(a,b))$ is isomorphic to

$$\begin{cases} \langle \sigma_2, \sigma_3 \rangle & \simeq D_{10} & \text{if} \ (a,b) = (1,1), \\ \langle \sigma_2, \sigma_3 \rangle \times \mu_{2,1} & \simeq D_6 \times \mu_{2,1} \simeq D_{12} & \text{if} \ (a,b) = (2,1), \\ \langle \sigma_2, \sigma_3 \rangle \ltimes \mu_{3,1} & \simeq D_8 \ltimes \mu_{3,1} & \text{if} \ (a,b) = (3,1), \\ (\langle \sigma_2, \sigma_3 \rangle \ltimes \mu_{a,a}) \rtimes H_{a,a} \simeq (D_\infty \ltimes \mu_{a,a}) \rtimes \mathbb{Z}/2\mathbb{Z} & \text{if} \ a = b \geq 2, \\ \langle \sigma_2, \sigma_3 \rangle \ltimes \mu_{a,b} & \simeq D_\infty \ltimes \mu_{a,b} & \text{if} \ a \neq b, \ ab \geq 4. \end{cases}$$

Note that $\mu_{a,b}$ is isomorphic to $\mathbb{Z}/a\mathbb{Z} \times \mathbb{Z}/b\mathbb{Z}$ if k is algebraically closed, but is smaller in general. The group $\mu_{a,b}$ is the diagonalizable commutative algebraic group with the group of characters isomorphic to the abelian group N corresponding to the seed skew-symmetrizable matrix defining a cluster algebra. It is always a part of its automorphism group and corresponds to its grading by the group N.[2]

The proof of Theorem 1 is given by five propositions. More precisely, Proposition 4.5 gives the cases $X(a,1)$, $a \geq 4$, Propositions 5.2, 5.5, 5.7 give respectively the cases A_2, B_2, G_2, which correspond to $X(1,1), X(2,1), X(3,1)$, and the general case $X(a,b)$ with $a,b \geq 2$ is done in Proposition 6.4.

An automorphism of the $\mathbb{Z}$-algebra $\mathcal{C}(a,b)$ is called a *cluster automorphism* if it sends each cluster to a cluster (see [1]). Examples of such automorphisms are the automorphisms σ_p. It follows from Theorem 1 that the group of cluster automor-

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[2]We owe this remark to Greg Muller.

phisms of $\mathcal{C}(a,b)$ is generated by σ_2 and σ_3, except when $a = b \neq 1$, in which case $\sigma_{5/2} \in H_{a,a}$ is a cluster automorphism not generated by σ_2, σ_3. In the case $(a, b) = (1, 1)$, we have $\sigma_{5/2} = \sigma_2\sigma_3\sigma_2\sigma_3\sigma_2$ (see Remark 5.3).

We thank Sergey Fomin and Greg Muller for coaching the second author in the rudiments of the theory of cluster algebras. Thanks also to the referees for their constructive remarks that helped to improve the text.

2. Compactifications with a pentagon

In the sequel, all algebraic varieties are defined over a field k of characteristic zero or of characteristic p with $ab \not\equiv 0$ (mod p).

Proposition 2.1. *The surface $X(a,b)$ admits a smooth compactification $\overline{X}(a,b)$ that is isomorphic to the blow-up of $\mathbb{P}^2$ at the points*

$$\{[1 : 0 : \xi] \mid \xi^b + 1 = 0\} \text{ and } \{[1 : \lambda : 0] \mid \lambda^a + 1 = 0\}.$$

The boundary is the strict transform of the union of the coordinate lines described by the following picture (where the numbers indicate the self-intersections of the irreducible components of the boundary):
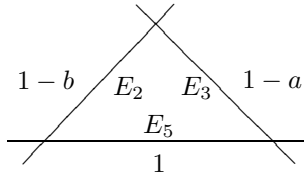
Moreover, the boundary divisor is an anti-canonical divisor.

Remark 2.2. In the above proposition, the points are taken in a finite Galois extension K of k (this works because $ab \neq 0$ in k). The blow-up map is defined over k, because the Galois group preserves the set of blown-up points. Also, each of the irreducible components of the boundary is defined over k. Note that the choice of the names of the curves here could seem strange to the reader, but it was motivated by the sequel (see Corollary 2.3).

Proof. Consider the projection map

$$\pi : X(a,b) \to \mathbb{A}^2, \quad (y_1, y_2, y_3, y_4) \mapsto (y_2, y_3).$$

The preimage of a point (y_2, y_3) corresponds to points (y_1, y_2, y_3, y_4) where $y_1y_3 = y_2^a + 1$ and $y_2y_4 = y_3^b + 1$. In particular π restricts to an isomorphism $U \to V$, where U and V are the open subsets of $X(a,b)$ and $\mathbb{A}^2 = \text{Spec}\,(\text{k}[y_2, y_3])$ given by $y_2y_3 \neq 0$. However, each point of the set

$$\Delta = \{(0, \xi) \mid \xi^b + 1 = 0\} \cup \{(\lambda, 0) \mid \lambda^a + 1 = 0\} \subset \mathbb{A}^2$$

has a preimage which is isomorphic to an affine line. Let $\eta \colon Z \to \mathbb{A}^2$ be the blow-up of $\mathbb{A}^2$ at points from Δ. It remains to show that $\varphi = \eta^{-1} \circ \pi$ is an open embedding

of $X(a, b)$ into Z, whose restriction is an isomorphism $X(a, b) \to Z \setminus (E_2 \cup E_3)$, where E_2, E_3 are respectively the strict transforms of the lines of equation $y_2 = 0$ and $y_3 = 0$.

We first restrict ourselves to the open subsets $U_2 \subset X(a, b)$ and $V_2 \subset \mathbb{A}^2$ where $y_2 \neq 0$. The restriction of η is then the blow-up of the ideal $(y_2^a + 1, y_3)$, that we can write as

$$\eta^{-1}(V_2) = \{((y_2, y_3), [u : v]) \in V_2 \times \mathbb{P}^1 \mid y_3 v = (y_2^a + 1)u\},$$

where η corresponds to the projection on the first factor. The map $\eta^{-1} \circ \pi$ sends thus (y_1, y_2, y_3, y_4) to $((y_2, y_3), [y_1 : 1])$. As the curve $v = 0$ corresponds to E_3, we obtain an isomorphism $U_2 \to \eta^{-1}(V_2) \setminus E_3$. Exchanging coordinates y_2, y_3, we obtain the same result when $y_3 \neq 0$. Since $(y_2, y_3) \neq (0, 0)$ on $X(a, b)$, this gives the result.

Since the anti-canonical class of $\mathbb{P}^2$ is represented by the union of the coordinate lines, the strict transform of this divisor at its simple points is the anti-canonical divisor of the blow-up.   □

Corollary 2.3. *The surface $X(a, b)$ admits a smooth compactification $Z(a, b)$ with an anticanonical boundary described by the following picture:*
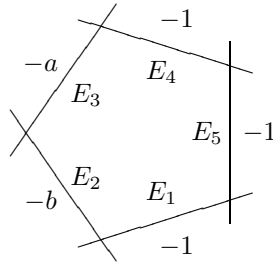
Moreover, $Z(a, b)$ is obtained by blowing-up the points

$$\{[1 : 0 : \xi] \mid \xi^b + 1 = 0\}, \{[1 : \lambda : 0] \mid \lambda^a + 1 = 0\}, [0 : 1 : 0], [0 : 0 : 1]$$

of $\mathbb{P}^2$.

Proof. It suffices to blow-up the two points $E_5 \cap E_3$ and $E_5 \cap E_2$ from the compactification of Proposition 2.1, which correspond to $[0 : 1 : 0]$ and $[0 : 0 : 1]$.   □

Remark 2.4. The boundary $B = Z(a, b) \setminus X(a, b)$ being anti-canonical, every curve $C \subset Z(a, b)$ that is not contained in B intersects the anti-canonical divisor $-K$ non-negatively. If C is an irreducible curve of B, then $C^2 + CK = -2$, which implies that $C(-K) = C^2 + 2$. This shows that $Z(a, b)$ is a weak Del Pezzo surface ($-K_X$ is big and nef) if and only if $a, b \leq 2$.

Remark 2.5. If $a \geq 2$ and $b \geq 2$, another natural normal compactification of the surface $X(a, b)$ is a complete intersection $Y(a, b)$ of two surfaces

$$x_1 x_3 - x_2^a - x_0^a = x_2 x_4 - x_3^b - x_0^b = 0$$

of degrees a and b in the weighted projective space $\mathbb{P}(1, a-1, 1, 1, b-1)$. The surface is singular at the point $[0, 1, 0, 0, 0]$ and $[0, 0, 0, 0, 1]$. Via the projection to the coordinates x_0, x_2, x_3, it admits a birational map onto $\mathbb{P}^2$. In general, the compactification $Y(a, b)$ is related to the compactification $\overline{X}(a, b)$ in a rather complicated way by a sequence of blow-ups at singular points and then blow-downs. For example, when $a = b$, it is enough to resolve the singular points which are quotient singularities of type $(1/a)(1, 1)$, and then blow down the strict transform of the line $x_0 = x_2 = x_3 = 0$.

3. Birational maps between n-gons pairs

Definition 3.1. Let Y be a smooth projective surface, and let $n \geq 2$ be an integer. An n-*gon* on Y is a divisor $B = E_1 + \cdots + E_n$, where the E_i are curves of Y isomorphic to $\mathbb{P}^1$, such that

$$E_i \cdot E_j = \begin{cases} 1 & \text{if } |i - j| \in \{1, n-1\}, \\ 0 & \text{if } |i - j| \in \{2, \ldots, n-2\}. \end{cases}$$

The pair (Y, B) will be called an n-*gon pair.* The type of B (or of the pair (Y, B)) is the sequence $(E_1^2, \ldots, E_n^2)$, which is an n-uple of integers, defined up to cyclic permutation and reversion.

We say that the n-gon B (or the pair (Y, B)) is *standard* if $n \geq 3$, and if there is an ordering of the E_i such that $(E_1)^2 = (E_2)^2 = 0$, $E_1 \cdot E_2 = 1$ and $(E_i)^2 \leq -2$ for $i \geq 3$.

Example 3.2. Corollary 2.3 gives examples of pentagons (n-gons with $n = 5$) which are not standard. In the sequel, we will use this example to provide either quadrangles or triangles, in a standard form.

Definition 3.3. Let (Y, B) be an n-gon pair and (Y', B') be an m-gon pair. A birational map $f: Y \dashrightarrow Y'$ is called a *birational map of pairs* if it restricts to an isomorphism $Y \setminus B \to Y' \setminus B'$. If the map f is regular (resp. biregular), we will moreover say that f is a *birational morphism of pairs* (respectively an *isomorphism of pairs*).

Example 3.4. Let (Y, B) be an n-gon pair of type $(0, 0, -a, -b_1, \ldots, -b_{n-3})$.

Blowing-up the intersection point of the first two curves and contracting the strict transform of the second curve, we obtain an n-gon of type $(-1, 0, -a + 1, -b_1, \ldots, -b_{n-3})$.

Blowing-up again the intersection of the first two curves and contracting the strict transform of the second curve, we obtain an n-gon of type $(-2, 0, -a + 2, -b_1, \ldots, -b_{n-3})$.

After $a - 1$ steps, we obtain a birational map of pairs $(Y, B) \dashrightarrow (Y', B')$, where (Y', B') is an n-gon pair of type $(-a, 0, 0, -b_1, \ldots, -b_{n-3})$:
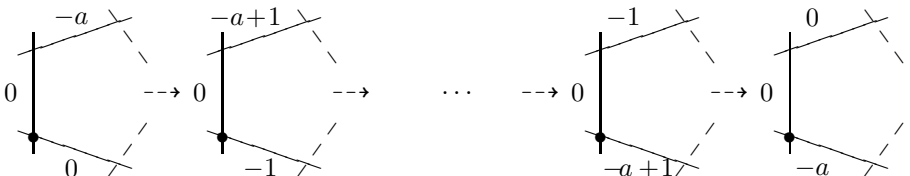
Definition 3.5. The birational maps $f \colon Y \dashrightarrow Y'$ as in Example 3.4 will be called *fibered modifications*.

Remark 3.6. In Example 3.4, if Y is a rational surface, then the linear system associated to the second (0)-curves[3] of B and B' induce morphisms $\pi \colon Y \to \mathbb{P}^1$ and $\pi' \colon Y' \to \mathbb{P}^1$ with general fibre isomorphic to $\mathbb{P}^1$. There exists then an automorphism θ of $\mathbb{P}^1$ such that the following diagram commutes:

$$
\begin{array}{ccc}
Y & \overset{f}{\dashrightarrow} & Y' \\
{\scriptstyle \pi}\downarrow & & \downarrow{\scriptstyle \pi'} \\
\mathbb{P}^1 & \overset{\theta}{\longrightarrow} & \mathbb{P}^1.
\end{array}
$$

Note that the restriction of π and π' on the surfaces $Y \setminus B$ and $Y' \setminus B'$ yield fibrations with general fibres isomorphic to $\mathbb{A}^1 \setminus \{0\}$. Hence, f restricts to an isomorphism of fibered surfaces $Y \setminus B \to Y' \setminus B'$. This explains why we call f a fibered modification.

Proposition 3.7. *Let (Y, B) be a standard n-gon pair, and let (Y', B') be a standard m-gon pair.*

Any birational map of pairs $f \colon (Y, B) \dashrightarrow (Y', B')$ decomposes into

$$f = f_k \circ \cdots \circ f_1,$$

where each $f_i \colon (Y_i, B_i) \dashrightarrow (Y_{i+1}, B_{i+1})$ is either an isomorphism of pairs or a fibered modification (and where $Y_0 = Y, Y_k = Y', B_0 = B, B_k = B'$).

In particular, $m = n$.

Proof. We can assume that f is not an isomorphism. Let us take a minimal resolution

$$
\begin{array}{ccc}
 & Z & \\
{\scriptstyle \pi}\swarrow & & \searrow{\scriptstyle \eta} \\
Y & \overset{f}{\dashrightarrow} & Y'
\end{array}
$$

of f. The fact that f is a birational map of pairs $(X, B) \dashrightarrow (X', B')$ implies that the base-points of f and f^{-1} are in B and B' and that the restrictions of π and η yield isomorphisms $Z \setminus B_Z \to Y \setminus B$ and $Z \setminus B_Z \to X' \setminus B'$, for some divisor B_Z on Z. Since B and B' are cycles, there is one cycle in B_Z, plus a priori some branches, which are then contracted by η and π. By the minimality condition, this implies that B_Z is in fact a cycle. In particular, the indeterminacy points of f and f^{-1} are singular points of B and B', respectively.

Let us observe that η contracts exactly one (-1)-curve. Firstly, the map η is

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[3]For any integer m, an (m)-curve is a smooth rational curve with self-intersection equal to m.

not an isomorphism because f is not an isomorphism and because there is no (-1)-curve on B'. Secondly, if η contracts at least two (-1)-curves, these are the strict transforms of the two (0)-curves of B. Hence the intersection point of these curves is blown-up by π and the exceptional divisor of the point is sent by f onto a curve of self-intersection ≥ 1 of B'. However, B' does not contain such curves.

The same argument for f^{-1} implies that π also contracts exactly one (-1)-curve. Hence, f has a unique proper indeterminacy point q, and π is a *tower-resolution*, namely a sequence of blow-ups such that each point blown-up belongs to the exceptional curve of the previous point. In other words, π is the blow-up of a chain of infinitely near points $x_r \succ x_{r-1} \succ \cdots \succ x_1 = q$. The (-1)-curve contracted by η is the strict transform of one (0)-curve E of B. Hence, q is a singular point of B, lying on E. Note that q is the intersection point of E with the other (0)-curve F, since otherwise F would be sent by f onto a curve of self-intersection ≥ 1.

We denote by $(0, 0, -a, -b_1, \ldots, -b_{n-3})$ the type of B, where F and E correspond to the first and second curve, respectively. Blowing-up q and contracting the strict transform of E, we obtain a birational map $\theta_1 \colon (Y, B) \dashrightarrow (Y_1, B_1)$, where B_1 is an n-gon of type $(-1, 0, -a + 1, -b_1, \ldots, -b_{n-3})$. The map $\varphi_1 = f \circ \theta_1^{-1} \colon (Y_1, B_1) \dashrightarrow (Y', B')$ has one indeterminacy point less than f (including in the counting all infinitely near points). Moreover, the fact that the minimal resolution π of f is a tower-resolution implies that φ_1 has a unique proper (i.e., not infinitely near) indeterminacy point q_1, which is the intersection of the first two curves. We write $\pi_1 \colon Z_1 \to Y_1$ the minimal resolution of φ_1, which is again a tower-resolution, and denote by $\eta_1 \colon Z_1 \to Y'$ the birational morphism $\varphi_1 \circ \pi_1$.

Since B_1 contains exactly one curve of self-intersection ≥ -1, η_1 contracts only one (-1)-curve, which is the strict transform of the (0)-curve of B_1. Blowing-up q_1 and contracting the strict transform of the (0)-curve, we obtain a birational map $\theta_2 \colon (Y_1, B_1) \dashrightarrow (Y_2, B_2)$, where B_2 is an n-gon of type $(-2, 0, -a + 2, -b_1, \ldots, -b_{n-3})$. After $a - 1$ steps, we obtain a pair (Y_{a-1}, B_{a-1}), where B_{a-1} is an n-gon of type $(-a + 1, 0, -1, -b_1, \ldots, -b_{n-3})$ on a smooth projective surface Y_{a-1}, and the birational map $\varphi_{a-1} = f \circ (\theta_{a-1} \circ \cdots \circ \theta_1)^{-1}$ has $a - 1$ indeterminacy points less than f. The unique indeterminacy point of φ_{a-1} is the intersection point q_{a-1} of the first two curves, but now the unique (-1)-curve contracted by η_{a-1} is the strict transform of either the second or the third curve. Blowing-up q_{a-1} and contracting anyway the strict transform of the (0)-curve, we obtain a birational map $\theta_a \colon Y_{a-1} \dashrightarrow Y_a$ such that $\varphi_a = \varphi_{a-1} \circ (\theta_a)^{-1} = f \circ (\theta_a \circ \cdots \circ \theta_1)^{-1} \colon (Y_a, B_a) \dashrightarrow (Y', B')$ has either $a - 1$ or a indeterminacy points less than φ. Since $\theta_a \circ \cdots \circ \theta_1$ is a fibered modification, the result follows by induction on the number of indeterminacy points. $\square$

4. Compactifications of $X(a, 1)$ with a triangle

The case of $X(a, 1)$ is a bit different from the general case of $X(a, b)$ with $a, b \geq 2$, since the curve $E_2 \subset Z(a, 1)$ has self-intersection -1. Denote by $\eta \colon (Z(a, 1), B_Z) \to (T(a, 1), B_T)$ the birational morphism of pairs which contracts the curves E_2 and E_4. The boundary B_T of $X(a, 1)$ in $T(a, 1)$ consists of a triangle $\eta(E_1) + \eta(E_3) + \eta(E_5) = \mathcal{E}_1 + \mathcal{E}_3 + \mathcal{E}_5$ of type $(0, -(a - 2), 0)$.
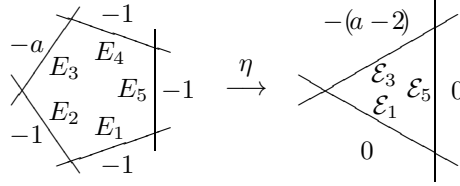
If $a \geq 4$, the triangle is a standard triangle, so the automorphisms of $X(a, 1)$ can be described with the help of Proposition 3.7. The special cases where $a \leq 3$ will then be described separately. The following lemma allows us to view $T(a, 1)$ as a blow-up of points in $\mathbb{P}^1 \times \mathbb{P}^1$.

Lemma 4.1. *The smooth projective surface $T(a, 1)$ is the blow-up $\pi' \colon T(a, 1) \to \mathbb{P}^1 \times \mathbb{P}^1$ of the following a points*

$$\{([\xi : 1], [\xi : 1]) \mid \xi^a + 1 = 0\}$$

and the boundary $T(a, 1) \setminus X(a, 1)$ consists of the strict transform of the curves $\pi'(\mathcal{E}_1) = \mathbb{P}^1 \times [0 : 1]$, $\pi'(\mathcal{E}_5) = [1 : 0] \times \mathbb{P}^1$ and of the diagonal $\pi'(\mathcal{E}_3)$.
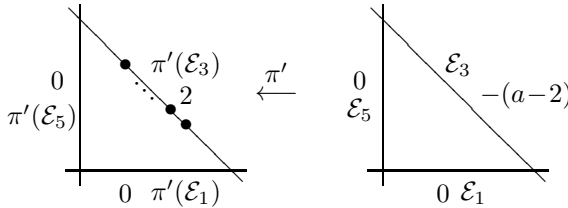
Moreover, the restriction of π' to the affine surface $X(a, 1)$ is given by

$$(y_1, y_2, y_3, y_4) \mapsto ([y_2 : 1], [1 : y_4]).$$

Remark 4.2. As in Proposition 2.1, the points blown-up belong to a finite Galois extension of k so not necessarily to k, but the morphism π' is in any case defined over k.

Proof. Recall that the birational morphism $\pi \colon Z(a, 1) \to \mathbb{P}^2$ of Corollary 2.3 is the blow-up of the $a + 3$ points

$$[1 : 0 : -1], \{[1 : \lambda : 0] \mid \lambda^a + 1 = 0\}, [0 : 1 : 0], [0 : 0 : 1].$$

The birational morphism $\eta \colon Z(a, 1) \to T(a, 1)$ contracts the curves E_2, E_4, which are respectively the strict transform of the second coordinate line $\pi(E_2)$ and the exceptional divisor of $[0 : 1 : 0]$. Hence, denoting by $\kappa \colon \mathbb{P}^2 \dashrightarrow \mathbb{P}^1 \times \mathbb{P}^1$ the blow-up of $[0 : 0 : 1], [1 : 0 : -1]$ followed by the contraction of the strict transform of $\pi(E_2)$, the map $\pi' = \kappa \pi \eta^{-1} \colon T(a, 1) \to \mathbb{P}^1 \times \mathbb{P}^1$ is the blow-up of the a points $\{\kappa([1 : \lambda : 0]) \mid \lambda^a + 1 = 0\}$. Explicitely, we can choose κ to be given by

$$[x_0 : x_1 : x_2] \dashrightarrow ([x_1 : x_0], [x_1 : x_0 + x_2]),$$

which implies that $\{\kappa([1 : \xi : 0]) \mid \xi^a + 1 = 0\} = \{([\xi : 1], [\xi : 1]) \mid \xi^a + 1 = 0\}$. The restriction of $\pi \colon Z(a, 1) \to \mathbb{P}^2$ being $(y_1, y_2, y_3, y_4) \mapsto (1 : y_2 : y_3)$ (see the proof of Proposition 2.1), the restriction of π' to $X(a, 1)$ is

$$(y_1, y_2, y_3, y_4) \mapsto \kappa([1 : y_2 : y_3]) = ([y_2 : 1], [y_2 : y_3 + 1]) = ([y_2 : 1], [1 : y_4]). \qquad \square$$

Lemma 4.3. *The action of the group* $\mathrm{Aut}(T(a,1), B_T)$ *of automorphisms of the pair* $(T(a,1), B_T)$ *on the set* $\{\mathcal{E}_1, \mathcal{E}_3, \mathcal{E}_5\}$ *induces a split exact sequence*

$$1 \to \mu_{a,1} \to \mathrm{Aut}(T(a,1), B_T) \to R_a \to 1,$$

where $\mu_{a,1} \simeq \{\mu \in \mathrm{k}^* \mid \mu^a = 1\}$ *acts on* $X(a,1)$ *via*

$$(y_1, y_2, y_3, y_4) \mapsto (y_1, \mu y_2, y_3, \mu^{-1} y_4),$$

and where

$$R_a = \begin{cases} \langle \sigma_2, \sigma_3 \rangle \simeq \mathfrak{S}_3 & \text{if } a = 2, \\ \langle \sigma_2 \rangle \simeq \mathbb{Z}/2\mathbb{Z} & \text{if } a \neq 2. \end{cases}$$

Proof. Denote by $\mu_{a,1}$ the kernel of the action of $\mathrm{Aut}(T(a,1), B_T)$ on the set $\{\mathcal{E}_1, \mathcal{E}_3, \mathcal{E}_5\}$. Let us observe that the set of a curves contracted by π' is invariant by $\mu_{a,1}$. Indeed, the image by $\mu_{a,1}$ of one of the curves is an irreducible curve, not intersecting $\mathcal{E}_1$ and $\mathcal{E}_5$. The image of this curve by π' does not intersect the two fibres $\pi'(\mathcal{E}_1) = \mathbb{P}^1 \times [0 : 1]$, $\pi'(\mathcal{E}_5) = [1 : 0] \times \mathbb{P}^1$, so it is a point.

The group $\mu_{a,1}$ is then the lift of automorphisms of $\mathbb{P}^1 \times \mathbb{P}^1$ which leave invariant the three curves $\pi'(\mathcal{E}_1)$, $\pi'(\mathcal{E}_3)$, $\pi'(\mathcal{E}_5)$ and which preserve the set $\{([\xi : 1], [\xi : 1]) \mid \xi^a + 1 = 0\}$. This group is isomorphic to $\{\mu \in \mathrm{k}^* \mid \mu^a = 1\}$, acts on $\mathbb{P}^1 \times \mathbb{P}^1$ via

$$([u_1 : u_2], [v_1 : v_2]) \mapsto ([\mu u_1 : u_2], [\mu v_1 : v_2]),$$

and then on $X(a,1) = \mathrm{Spec}\, \mathrm{k}[y_1, y_2, y_3, y_4]/(y_1 y_3 - y_2^a - 1, y_2 y_4 - y_3 - 1)$ via

$$(y_1, y_2, y_3, y_4) \mapsto (y_1, \mu y_2, y_3, \mu^{-1} y_4).$$

It coincides then with the group $\mu_{a,1}$ already defined in the introduction.

The explicit description of $\pi' \colon T(a,1) \to \mathbb{P}^1 \times \mathbb{P}^1$ given in Lemma 4.1 shows that the automorphism

$$([u_1 : u_2], [v_1 : v_2]) \mapsto ([v_2 : v_1], [u_2 : u_1])$$

of $\mathbb{P}^1 \times \mathbb{P}^1$ lifts to an automorphism of $T(a,1)$ which preserves the boundary, exchanging the two (0)-curves and preserving the $(-a)$-curve. In affine coordinates, this gives the following automorphism of $X(a,1) \subset \mathbb{A}^4$

$$(y_1, y_2, y_3, y_4) \mapsto \left(\frac{y_4^a + 1}{y_2 y_4 - 1}, y_4, y_3, y_2 \right) = \left(y_1 y_4^a - \sum_{i=0}^{a-1} (y_2 y_4)^i, y_4, y_3, y_2 \right),$$

which corresponds to the automorphism σ_3. If $a \neq 2$, then this element generates the image of the action, since the three curves $\mathcal{E}_1, \mathcal{E}_3, \mathcal{E}_5$ have self-intersection $0, 2-a, 0$, respectively.

If $a = 2$, then π' blows-up the two points $([\xi : 1], [\xi : 1])$, where $\xi^2 + 1 = 0$, which are the two indeterminacy points of the birational involution

$$([u_1 : u_2], [v_1 : v_2]) \mapsto ([u_1 : u_2], [u_1 v_2 - u_2 v_1 : u_1 v_1 + u_2 v_2])$$

of $\mathbb{P}^1 \times \mathbb{P}^1$. The lift of the involution gives an automorphism of $B(a,1)$, which fixes $\mathcal{E}_5$ and exchanges the two curves $\mathcal{E}_1$ and $\mathcal{E}_3$. This involution yields the automorphism of order 2 of $X(2,1)$, given by

$$\sigma_2 \colon (y_1, y_2, y_3, y_4) \mapsto (y_3, y_2, y_1, y_1 y_4 - y_2). \qquad \square$$

If $a \geq 4$, we can decompose any automorphism of $X(a,1)$ into a sequence of isomorphisms of pairs and fibered modifications (Proposition 3.7). A priori, the fibered modification could go from one pair to a different one, but we will show that in the case of $X(a,1)$, we can only consider fibered modifications $T(a,1) \dashrightarrow T(a,1)$, hence each of them can be seen as a unique automorphism of $X(a,1)$, up to automorphisms of the pair $X(a,1) \subset T(a,1)$ (and these latter automorphisms have been described in Lemma 4.3).

Example 4.4. The following birational involution of $\mathbb{P}^1 \times \mathbb{P}^1$

$$f \colon ([u_1 : u_2], [v_1 : v_2]) \dashrightarrow ([u_1 : u_2], [u_2^{a-2}(u_1 v_2 - u_2 v_1) : u_1^{a-1} v_1 + u_2^{a-1} v_2])$$

is not defined only at $([1 : 0], [0 : 1])$ and at $\{([\xi : 1], [\xi : 1]) \mid \xi^a + 1 = 0\}$. It follows from the explicit description that the lift of f is a birational map $\widehat{f} = (\pi')^{-1} f \pi'$ of $T(a,1)$ which restricts to an automorphism of $T(a,1) \setminus \mathcal{E}_5$, and which exchanges $\mathcal{E}_1$ and $\mathcal{E}_3$.

The map $\widehat{f}$ is therefore a fibered modification if $a \neq 2$ and an isomorphism if $a = 2$. Moreover, $\widehat{f}$ restricts to an automorphism of $X(a,1)$, that we will show to be equal to σ_2.

To compute this, we use the map $X(a,1) \to \mathbb{P}^1 \times \mathbb{P}^1$ given by $(y_1, y_2, y_3, y_4) \mapsto ([y_2 : 1], [1 : y_4])$. The composition with f yields

$$([y_2 : 1], [1 : y_4]) \dashrightarrow ([y_2 : 1], [(y_2 y_4 - 1) : y_2^{a-1} + y_4]) = ([y_2 : 1], [1 : y_1 y_4 - y_2^{a-1}]).$$

Hence, $y_3 = y_2 y_4 - 1$ is exchanged with $y_2(y_1 y_4 - y_2^{a-1}) - 1 = y_1$. The involutive automorphism of $X(a,1)$ is thus given by

$$\sigma_2 \colon (y_1, y_2, y_3, y_4) \mapsto (y_3, y_2, y_1, y_1 y_4 - y_2^{a-1}).$$

Proposition 4.5. *If $a \geq 4$, then $\mathrm{Aut}(X(a,1)) = \mu_{a,1} \rtimes \langle \sigma_2, \sigma_3 \rangle$. Moreover, $\langle \sigma_2, \sigma_3 \rangle = \langle \sigma_2 \rangle \star \langle \sigma_3 \rangle \simeq \mathbb{Z}/2\mathbb{Z} \star \mathbb{Z}/2\mathbb{Z} \simeq \mathbb{Z} \rtimes \mathbb{Z}/2\mathbb{Z}$ is an infinite dihedral group and $\mu_{a,1}$ is a finite cyclic group.*

Proof. Because $a \geq 4$, the pair $(T(a,1), B_T)$ is standard. According to Proposition 3.7, every automorphism of $X(a,1)$ decomposes into fibered modification and isomorphisms of pairs. Each fibered modification is equal to σ_2, up to isomorphism of pairs (Example 4.4), and each automorphism of the pair $(T(a,1), B_T)$ is generated by σ_3 and $\mu_{a,1}$ (Lemma 4.3). Hence, $\mathrm{Aut}(X(a,1))$ is generated by $\mu_{a,1}$, σ_2 and σ_3. To achieve the proof, it remains to observe that $\sigma_2 \sigma_3$ is of infinite order. The map $\sigma_2 \sigma_3$ and its inverse have both a unique proper indeterminacy point, and these two points are different. Proceeding by induction, we obtain that $(\sigma_2 \sigma_3)^n$ has again a unique proper indeterminacy point for any $n \geq 1$, always being the proper indeterminacy point of $\sigma_2 \sigma_3$. $\square$
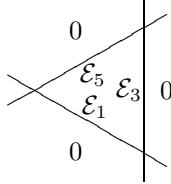
5. Cluster algebras of types $\mathsf{A_2, B_2, G_2}$

By contrast to the case $a \geq 4$, we will see that the group of automorphisms of $X(a,1)$ with $a = 1, 2, 3$ is finite, and is in fact contained in the group of automorphisms of a symmetric n-gon $(Y(a,1), B_Y)$ that we define now.
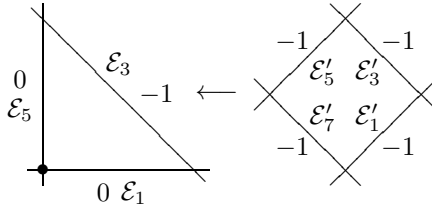
(1) The pair $(Y(1,1), B_Y) = (Z(1,1), B_Z)$ is a pentagon of five (-1)-curves:
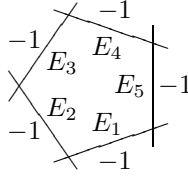
(2) The pair $(Y(2,1), B_Y) = (T(2,1), B_T)$ is a triangle of three (0)-curves:

(3) The pair $(Y(3,1), B_Y)$ is obtained by blowing-up the point $\mathcal{E}_1 \cap \mathcal{E}_5$ in $(T(3,1), B_T)$, and is a square of four (-1)-curves. We denote by $\mathcal{E}_i'$ the strict transform of $\mathcal{E}_i$ and by $\mathcal{E}_7'$ the exceptional curve produced, and obtain the following diagram:

Lemma 5.1. *For $a = 1, 2, 3$, every automorphism of $X(a,1)$ extends to an automorphism of the pair $(Y(a,1), B_Y)$.*
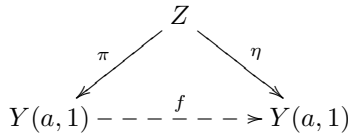
Proof. Suppose for contradiction the existence of a birational map

$$f \colon (Y(a,1), B_Y) \dashrightarrow (Y(a,1), B_Y)$$

which is not an automorphism of pairs. Recall that the type of B_Y is

$$(-1, -1, -1, -1, -1), \quad (0,0,0) \quad \text{or} \quad (-1, -1, -1, -1).$$

Let us take a minimal resolution

of f. As observed in Proposition 3.7, it follows from the minimality condition that the preimage B_Z of B under π is equal to the preimage of B under η and consists of a cycle of smooth rational curves. In particular, the indeterminacy points of f and f^{-1} are singular points of B.

Since f is not an isomorphism, π and η contract at least one (-1)-curve.

If η contracts at least two (-1)-curves, these are sent by π onto two curves of $E_1, E_2 \subset B$ of self-intersection ≥ -1. If E_1, E_2 are (-1)-curves of B, π does not blow-up any point of these two disjoint curves. There is one irreducible curve of B touching these two curves, which is thus sent by f onto a curve of self-intersection ≥ 0; this is impossible. If E_1 and E_2 are (0)-curves, then π blows-up the point of intersection, but no other point. This is impossible since the boundary obtained would have only two curves.

The only remaining case is when η contracts exactly one (-1)-curve, and by symmetry we can also assume that π also contracts one (-1)-curve, which implies that f has exactly one proper indeterminacy point. If B is a triangle, we observe that the image of the (0)-curve not touching the indeterminacy point is a curve of self-intersection ≥ 1, which is impossible. The remaining case is when B only contains (-1)-curves, and so the (-1)-curve contracted by η is the strict transform by π of a (-1)-curve E of B. No point of E is then blown-up by π. Since f has only one proper indeterminacy point, there is a (-1)-curve E' of B which touches E and which does not contain any indeterminacy point. Its image by f is a curve of self-intersection ≥ 0, which is impossible. $\square$

Proposition 5.2 (Case A_2). *The group* $\mathrm{Aut}(X(1,1))$ *is a dihedral group* D_{10} *of order* 10 *generated by the cluster transformations* σ_2, σ_3, *which act on the pentagon* $(E_1, \ldots, E_5)$ *via the following actions:*
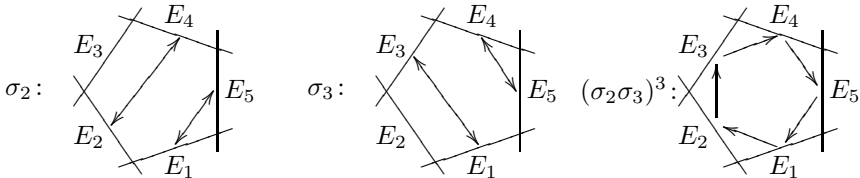

Remark 5.3. The natural automorphism $\sigma_{5/2} \colon (y_1, y_2, y_3, y_4) \mapsto (y_4, y_3, y_2, y_1)$ of $X(1,1)$ corresponds to the permutation $E_1 \leftrightarrow E_4$, $E_2 \leftrightarrow E_3$ and is thus equal to $\sigma_2 \sigma_3 \sigma_2 \sigma_3 \sigma_2$.

Proof. According to Lemma 5.1, we have $\mathrm{Aut}(X(1,1)) = \mathrm{Aut}(Y(1,1), B_Y)$. Recall that $Y(1,1) = Z(1,1) \to T(1,1)$ contracts E_2 and E_4. By Lemma 4.1, σ_2 extends to an automorphism of $(T(1,1), B_T)$ which exchanges $\mathcal{E}_1$ and $\mathcal{E}_5$ and which fixes $\mathcal{E}_3$. Hence, the corresponding automorphism of $Y(1,1)$ corresponds to the permutation $E_1 \leftrightarrow E_5, E_2 \leftrightarrow E_4$. The action of σ_3 is given in Example 4.4. Since it exchanges $\mathcal{E}_1$ and $\mathcal{E}_3$, it corresponds to the permutation $E_1 \leftrightarrow E_3, E_4 \leftrightarrow E_5$.

Observe that $\sigma_2 \sigma_3$ acts as $E_1 \to E_3 \to E_5 \to E_2 \to E_4$, which implies that $(\sigma_2 \sigma_3)^3$ is the permutation $E_1 \to E_2 \to E_3 \to E_4 \to E_5$. Since both σ_2 and σ_3 conjugate $(\sigma_2 \sigma_3)^3$ to its inverse, the group generated by σ_2, σ_3 admits a surjective homomorphism to D_{10}.

Let us observe that $Y(1,1)$ is a del Pezzo surface of degree 5, a fact which directly follows from the description of $\pi \colon Y(1,1) = Z(1,1) \to \mathbb{P}^1 \times \mathbb{P}^1$ (see also Remark 5.4 for another argument).

It follows from the classification of automorphism groups of del Pezzo surfaces that $\mathrm{Aut}(Y(1,1)) \subset \mathfrak{S}_5$, and equality holds if k is algebraically closed. The group $\mathfrak{S}_5$ is the group of symmetries of the union of ten (-1)-curves on $Y(1,1)$ whose intersection graph is the Petersen graph (see [3]):
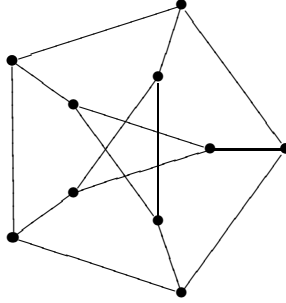

In the anti-canonical model, these (-1) curves are the 10 lines on the surface. The boundary B consists of five (-1)-curves forming a subgraph of the Petersen graph isomorphic to a pentagon. There are 12 such subgraphs, and the stabilizer group of each one is isomorphic to D_{10}. This shows that $\mathrm{Aut}(X(1,1))$ is contained in D_{10}, and hence coincides with it. $\qquad\square$

Remark 5.4. Note that, expressing y_2 in terms of y_1 and y_3, we obtain that $X(1,1)$ admits a natural compactification in $\mathbb{P}^3$ isomorphic to a cubic surface X with equation
$$x_1 x_2 x_3 - x_0^2 x_4 - x_2 x_0^2 - x_0^3 = 0.$$
The boundary $x_0 = 0$ consists of three coplanar lines and the surface has two of the intersection points as ordinary double points. The intersection points of these lines are singular points of the cubic surface. The points $[0:1:0:0]$ and $[0:0:1:0]$ are ordinary double points and the point $[0:0:1:0]$ is a double rational point of type A_2. Let $X' \to X$ be a minimal resolution of singularities. The pre-image of the boundary is a 7-gon of type $(-1,-2,-2,-1,-2,-1,-2)$. By blowing down the first and fourth curve, we obtain a smooth compactification $Y(1,1)$ with the boundary equal to a pentagon of (-1)-curves. Since X' is a weak del Pezzo surface of degree 3, the surface $Y(1,1)$ is a del Pezzo surface of degree 5.

Proposition 5.5 (Case B_2). *The group* $\mathrm{Aut}(X(2,1))$ *is isomorphic to* $\mathfrak{S}_3 \times \mu_{2,1} \simeq D_{12}$. *The group* $\mathfrak{S}_3$ *is generated by* σ_2 *and* σ_3, *and* $\mu_{2,1}$ *by the automorphism* $(y_1, y_2, y_3, y_4) \mapsto (y_1, -y_2, y_3, -y_4)$, *which fixes the three curves* $\mathcal{E}_1, \mathcal{E}_3, \mathcal{E}_5$. *The actions of* σ_2 *and* σ_3 *on the triangle are the following:*
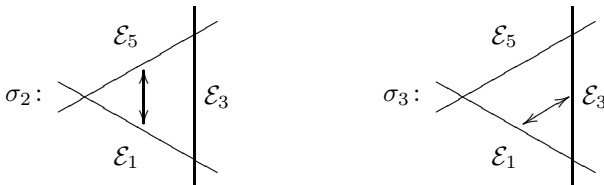
Proof. According to Lemma 5.1,

$$\mathrm{Aut}(X(2,1)) = \mathrm{Aut}(Y(2,1), B_Y) = \mathrm{Aut}(T(2,1), B_T).$$

The action of σ_2 and σ_3 on the triangle are given in Lemma 4.3 and Example 4.4. One can moreover check that σ_2 and σ_3 generate a group isomorphic to $\mathfrak{S}_3$. By Lemma 4.3, we have a split exact sequence

$$1 \to \mu_{2,1} \to \mathrm{Aut}(X(1,2)) \to \mathfrak{S}_3 \to 1.$$

We can then easily check that σ_2, σ_3 commute with $\mu_{2,1}$. Hence, $\mathrm{Aut}(X(1,2))$ is isomorphic to $\mathfrak{S}_3 \times \mu_{2,1}$. In particular, $\mu_{2,1} \simeq \mathbb{Z}/2\mathbb{Z}$ and $\mathrm{Aut}(X(1,2)) \simeq \mathfrak{S}_3 \times \mathbb{Z}/2\mathbb{Z} = D_{12}$. $\square$

Remark 5.6. Similarly to the previous case, the surface $X(2,1)$ admits a compactification X isomorphic to a cubic surface

$$x_1 x_2 x_3 - x_0^2 x_4 - x_2^2 x_0 - x_0^3 = 0.$$

The boundary $x_0 = 0$ consists of three coplanar lines. The surface has 2 singular points of types A_2 and A_1. We leave to the reader to find a birational isomorphism from X to our compactification $Y(2,1)$. The surface $Y(2,1)$ is a del Pezzo surface of degree 6. This latter observation also follows from the description of the morphism $Y(2,1) = T(2,1) \to \mathbb{P}^1 \times \mathbb{P}^1$, which is the blow-up of two general points.

Proposition 5.7 (Case G_2). *The group $\mathrm{Aut}(X(3,1))$ is isomorphic to $\mu_{3,1} \rtimes D_8$. It is generated by the group of cluster automorphisms D_8 generated by σ_2 and σ_3, and by $\mu_{3,1} \simeq \{\mu \in \mathrm{k}^* \mid \mu^3 = 1\}$, acting on $X(3,1)$ via*

$$(y_1, y_2, y_3, y_4) \mapsto (y_1, \mu y_2, y_3, \mu^{-1} y_4),$$

and fixing the four curves $\mathcal{E}_1', \mathcal{E}_3', \mathcal{E}_5', \mathcal{E}_7'$. The actions of σ_2 and σ_3 on the square are the following:
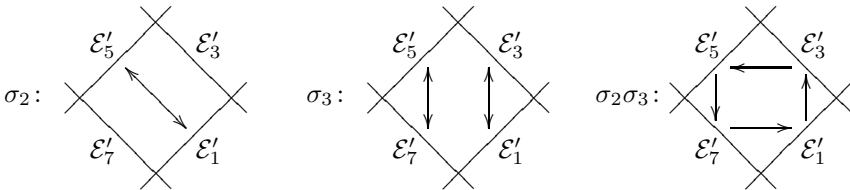
Proof. According to Lemma 5.1, we have $\mathrm{Aut}(X(3,1)) = \mathrm{Aut}(Y(3,1), B_Y)$. There is thus an action of $X(3,1)$ on the set $\{\mathcal{E}_1', \mathcal{E}_3', \mathcal{E}_5', \mathcal{E}_7'\}$. The kernel corresponds to automorphisms of $(T(3,1), B_T)$ acting trivially on the triangle and is thus equal to $\mu_{a,3}$ by Lemma 4.3. We obtain an exact sequence

$$1 \to \mu_{a,3} \to \mathrm{Aut}(X(3,1)) \to G \to 1,$$

where G is a subgroup of the dihedral group D_8. By Lemma 4.3, σ_2 exchanges $\mathcal{E}_1$ and $\mathcal{E}_5$, so corresponds to the transposition $\mathcal{E}_1' \leftrightarrow \mathcal{E}_5'$. The map σ_3 exchanges $\mathcal{E}_1$ and $\mathcal{E}_3$ (see Example 4.4), so corresponds to the permutation $\mathcal{E}_1' \leftrightarrow \mathcal{E}_3', \mathcal{E}_5' \leftrightarrow \mathcal{E}_7'$.

The map $\sigma_2\sigma_3$ corresponds thus to the permutation $\mathcal{E}_1' \to \mathcal{E}_3' \to \mathcal{E}_5' \to \mathcal{E}_7'$, so the map $\mathrm{Aut}(X(3,1))$ induces a surjective morphism $\langle \sigma_2, \sigma_3 \rangle \to D_8$. The explicit formulas for σ_2, σ_3 imply that it is injective. $\square$

Remark 5.8. As in the previous two cases, $X(3,1)$ admits a compactification X isomorphic to a cubic surface

$$x_1 x_2 x_3 - x_0^2 x_4 - x_3^2 - x_0^3 = 0.$$

The boundary consists of the union of the line $\ell : x_0 = x_2 = 0$ and the conic $C : x_0 = x_1 x_3 - x_2^2 = 0$. The points $[0:1:0:0]$ and $[0:0:1:0]$ are singular. The first point is an ordinary node, the second one is of type A_2.

Let $X' \to X$ be a minimal resolution of singularities. The preimage of the boundary is a pentagon of type $(-1,-2,0,-2,-2)$. After we blow up the point $E_2 \cap E_3$, and then blow down the curves E_1, E_5, we obtain a del Pezzo surface of degree 4 containing $X(3,1)$ with the boundary B equal to a quadrangle of four (-1)-curves. This is our compactification $Y(3,1)$. It is known that a del Pezzo surface of degree 4 contains 16 lines in its anti-canonical embedding in $\mathbb{P}^4$. There are 40 quadrangles among them, and the Weyl group of type $W(\mathsf{D}_5)$ of order $2^4.5!$ acts transitively on this set with the stabilizer isomorphic to the group $\mathfrak{S}_3 \rtimes D_8$ of order 48, the normalizer of the subgroup $\mathfrak{S}_3$ of $\mathfrak{S}_5$. Our group of automorphisms of $X(3,1)$ is a subgroup of this group of index 2.

6. Compactifications of $X(a,b)$ with $a, b \geq 2$, with a square

Let us now study the general case $X(a,b)$ with $a, b \geq 2$ (other cases were treated in Sections 4 and 5). The only (-1)-curves of the pair $Z(a,b)$ are then E_4, E_5, E_1. Denote by $\eta \colon (Z(a,b), B_Z) \to (S(a,b), B_S)$ the birational morphism of pairs which contracts the curve E_5. The boundary B_S of $X(a,b)$ in $S(a,b)$ consists of a square $\eta(E_1) + \eta(E_2) + \eta(E_3) + \eta(E_4) = \mathcal{E}'_1 + \mathcal{E}'_2 + \mathcal{E}'_3 + \mathcal{E}'_4$ of type $(0, -(b-1), -(a-1), 0)$.
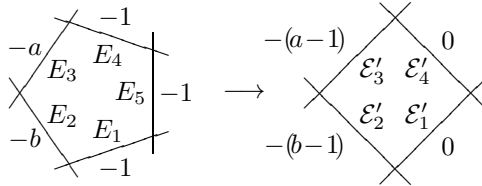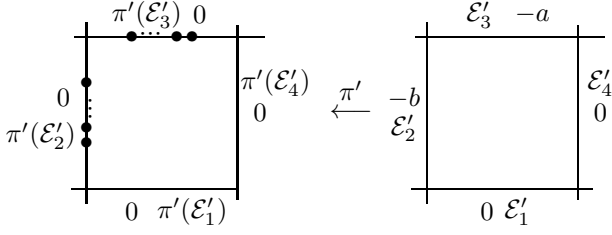

Since the square is standard (because $a, b \geq 2$), we can apply Proposition 3.7 to describe the automorphism group of $X(a,b)$. The description of $S(a,b)$ is given by Corollary 2.3:

Lemma 6.1. *The smooth projective surface $S(a,b)$ is the blow-up $\pi' \colon S(a,b) \to \mathbb{P}^1 \times \mathbb{P}^1$ of the $a + b$ points*

$$\{([\xi:1],[0:1]) \mid \xi^a + 1 = 0\}, \quad \{([0:1],[\xi:1]) \mid \xi^b + 1 = 0\},$$

and the boundary $S(a,b) \setminus X(a,b)$ consists of the strict transform of the curves $\pi'(\mathcal{E}'_1) = \mathbb{P}^1 \times [0:1]$, $\pi'(\mathcal{E}'_2) = [0:1] \times \mathbb{P}^1$, $\pi'(\mathcal{E}'_3) = \mathbb{P}^1 \times [0:1]$ and $\pi'(\mathcal{E}'_4) = [1:0] \times \mathbb{P}^1$.

Moreover, the restriction of π' to the affine surface $X(a,b)$ is given by

$$(y_1, y_2, y_3, y_4) \mapsto ([y_2 : 1], [y_3 : 1]).$$

Proof. Recall that the birational morphism $\pi \colon Z(a,1) \to \mathbb{P}^2$ of Corollary 2.3 is the blow-up of the $a + b + 2$ points

$$\{[1 : 0 : \xi] \mid \xi^b + 1 = 0\}, \{[1 : \lambda : 0] \mid \lambda^a + 1 = 0\}, [0 : 1 : 0], [0 : 0 : 1].$$

Composing π with the birational map $\kappa \colon \mathbb{P}^2 \dashrightarrow \mathbb{P}^1 \times \mathbb{P}^1$ given by

$$[x_0 : x_1 : x_2] \dashrightarrow ([x_1 : x_0], [x_2 : x_0]),$$

which blows-up $[0 : 1 : 0], [0 : 0 : 1]$ and contracts $\pi(E_5)$, we obtain π', which is the blow-up of

$$\{\kappa([1 : 0 : \xi]) \mid \xi^b + 1 = 0\}, \{\kappa([1 : \lambda : 0]) \mid \lambda^a + 1 = 0\}.$$

The restriction of $\pi \colon Z(a,b) \to \mathbb{P}^2$ being $(y_1, y_2, y_3, y_4) \mapsto (1 : y_2 : y_3)$ (see the proof of Proposition 2.1), the restriction of π' to $X(a,b)$ is

$$(y_1, y_2, y_3, y_4) \mapsto \kappa([1 : y_2 : y_3]) = ([y_2 : 1], [y_3 : 1]). \quad \square$$

Lemma 6.2. *The action of the group* $\mathrm{Aut}(S(a,b), B_S)$ *of automorphisms of the pair* $(S(a,b), B_S)$ *on the set* $\{\mathcal{E}_1', \mathcal{E}_2', \mathcal{E}_3', \mathcal{E}_4'\}$ *gives a split exact sequence*

$$1 \to \mu_{a,b} \to \mathrm{Aut}(S(a,b), B_S) \to H_{a,b} \to 1,$$

where $\mu_{a,b} \simeq \{(\mu, \nu) \in (\mathrm{k}^*)^2 \mid \mu^a = \nu^b = 1\}$, *and* $H_{a,b}$ *is trivial if* $a \neq b$ *and isomorphic to* $\mathbb{Z}/2\mathbb{Z}$ *if* $a = b$.
   The group $\mu_{a,b}$ *acts on* $X(a,b)$ *via*

$$(y_1, y_2, y_3, y_4) \mapsto (\nu^{-1}y_1, \mu y_2, \nu y_3, \mu^{-1}y_4).$$

The group $H_{a,a}$ *corresponds to the subgroup of* $\mathrm{Aut}(X(a,a))$ *generated by*

$$(y_1, y_2, y_3, y_4) \mapsto (y_4, y_3, y_2, y_1).$$

Proof. Denote by $\mu_{a,b}$ the kernel of the action of $\mathrm{Aut}(S(a,b), B_S)$ on the set $\{\mathcal{E}'_1, \mathcal{E}'_2, \mathcal{E}'_3, \mathcal{E}'_4\}$. Let us observe that the set of a curves contracted by π' and touching $\mathcal{E}_3$ is invariant by $\mu_{a,b}$. Indeed, the image by $\mu_{a,b}$ of one of the curves is an irreducible curve, not intersecting $\mathcal{E}'_1$ and $\mathcal{E}'_2$. The image of this curve by π' does not intersect the two fibres $\pi'(\mathcal{E}'_1) = \mathbb{P}^1 \times [0 : 1]$, $\pi'(\mathcal{E}'_2) = [0 : 1] \times \mathbb{P}^1$, so is a point. The same argument works for the b curves contracted by π' and touching $\mathcal{E}'_2$.

The group $\mu_{a,b}$ is then the lift of automorphisms of $\mathbb{P}^1 \times \mathbb{P}^1$ which leave invariant the four curves $\pi'(\mathcal{E}'_i)$, $i = 1, \ldots, 4$ and which preserve the sets $\{([\xi : 1], [0 : 1]) \mid \xi^a + 1 = 0\}, \{([0 : 1], [\xi : 1]) \mid \xi^b + 1 = 0\}$. This group is isomorphic to $\{(\mu, \nu) \in (\mathrm{k}^*)^2 \mid \mu^a = \nu^b = 1\}$, it acts on $\mathbb{P}^1 \times \mathbb{P}^1$ via

$$([u_1 : u_2], [v_1 : v_2]) \mapsto ([\mu u_1 : u_2], [\nu v_1 : v_2]),$$

and then on $X(a,b) = \mathrm{Spec}\,\mathrm{k}[y_1, y_2, y_3, y_4]/(y_1 y_3 - y_2^a - 1, y_2 y_4 - y_3^b - 1)$ via

$$(y_1, y_2, y_3, y_4) \mapsto (\nu^{-1} y_1, \mu y_2, \nu y_3, \mu^{-1} y_4).$$

If $a \neq b$, the action on the set of four curves of B_S is trivial, because the self-intersections have to be preserved.

If $a = b$, the explicit description of $\pi' : S(a,1) \to \mathbb{P}^1 \times \mathbb{P}^1$ given in Lemma 6.1 shows that the automorphism

$$([u_1 : u_2], [v_1 : v_2]) \mapsto ([v_1 : v_2], [u_1 : u_2])$$

of $\mathbb{P}^1 \times \mathbb{P}^1$ lifts to an automorphism of $T(a,1)$ which preserves the boundary, exchanging the two (0)-curves and the two $(-a)$-curves. In affine coordinates, this gives the following automorphism of $X(a,1) \subset \mathbb{A}^4$:

$$(y_1, y_2, y_3, y_4) \mapsto (y_4, y_3, y_2, y_1). \quad \square$$

Example 6.3. The following birational involution of $\mathbb{P}^1 \times \mathbb{P}^1$

$$f \colon ([u_1 : u_2], [v_1 : v_2]) \dashrightarrow ([u_1 : u_2], [(u_1^a + u_2^a)v_2 : v_1 u_2^a])$$

is not defined only at $([1 : 0], [1 : 0])$ and at $\{([\xi : 1], [0 : 1]) \mid \xi^a + 1 = 0\}$. On the open subset $U \subset \mathbb{P}^1 \times \mathbb{P}^1$ where $u_2 = 1$, we obtain the birational map

$$([x : 1], [v_1 : v_2]) \dashrightarrow ([x : 1], [(x^a + 1)v_2 : v_1])$$

whose base-points are $\{([\xi : 1], [0 : 1]) \mid \xi^a + 1 = 0\}$. Hence, the blow-up $\widehat{U} \to U$ of these points conjugates f to an automorphism of $\widehat{U}$. Since f preserves the set $\{([0 : 1], [\xi : 1]) \mid \xi^b + 1 = 0\}$, which is the set of remaining points blown-up by π', the map π' conjugates f to a birational map $\widehat{f} = (\pi')^{-1} f \pi'$ of $S(a,b)$ which restricts to an automorphism of $S(a,b) \setminus \mathcal{E}'_4 = (\pi')^{-1}(U)$, and which exchanges $\mathcal{E}'_1$ and $\mathcal{E}'_3$.

Since $a \neq 0$, the map $\widehat{f}$ is not an isomorphism, and is thus a fibered modification $(S(a,b), B_S) \dashrightarrow (S(a,b), B_S)$. Moreover, $\widehat{f}$ restricts to an automorphism of $X(a,b)$, that we will show to be equal to σ_2.

To compute this, we use the map $X(a,b) \to \mathbb{P}^1 \times \mathbb{P}^1$ given by $(y_1, y_2, y_3, y_4) \mapsto ([y_2 : 1], [y_3 : 1])$. The composition with f yields

$$([y_2 : 1], [y_3 : 1]) \dashrightarrow ([y_2 : 1], [(y_2^a + 1) : y_3]) = ([y_2 : 1], [y_1 : 1]).$$

Hence, y_3 is exchanged with y_1. The involutive automorphism of $X(a,1)$ is thus given by

$$\sigma_2 \colon (y_1, y_2, y_3, y_4) \mapsto \left(y_3, y_2, y_1, y_1^b y_4 - y_2^{a-1} \sum_{i=0}^{b-1} (y_1 y_3)^i \right).$$

Similarly, the birational involution of $\mathbb{P}^1 \times \mathbb{P}^1$

$$([u_1 : u_2], [v_1 : v_2]) \dashrightarrow ([(v_1^b + v_2^b)u_2 : u_1 v_2^b], [v_1 : v_2])$$

yields a fibered modification $(S(a,b), B_S) \dashrightarrow (S(a,b), B_S)$ which restricts to an automorphism of $S(a,b) \setminus \mathcal{E}_1'$ and to the automorphism σ_3 of $X(a,b)$.

In particular, if $\psi \colon (S(a,b), B_S) \dashrightarrow (Y', B')$ is a fibered modification, there is an isomorphism $\tau \colon (Y', B') \to (S(a,b), B_S)$ such that $\tau\psi$ restricts to σ_2 or σ_3 on $X(a,b) = S(a,b) \setminus B_S$.

Proposition 6.4. *If $a, b \geq 2$, then $\mathrm{Aut}(X(a,b)) = (\mu_{a,b} \rtimes \langle \sigma_2, \sigma_3 \rangle) \rtimes H_{a,b}$, where $\mu_{a,b}$ and $H_{a,b}$ are as in Lemma 6.2.*

Moreover, $\langle \sigma_2, \sigma_3 \rangle = \langle \sigma_2 \rangle \star \langle \sigma_3 \rangle \simeq \mathbb{Z}/2\mathbb{Z} \star \mathbb{Z}/2\mathbb{Z} \simeq \mathbb{Z} \rtimes \mathbb{Z}/2\mathbb{Z}$ is an infinite dihedral group, $\mu_{a,b}$ is a finite abelian group, $H_{a,b}$ is trivial if $a \neq b$ and/or order 2 if $a = b$.

Proof. Because $a, b \geq 2$, the pair $(S(a,b), B_S)$ is standard. According to Proposition 3.7, every automorphism of $X(a,b)$ decomposes into fibered modification and isomorphisms of pairs. Each fibered modification is equal to σ_2 or σ_3, up to isomorphism of pairs (Example 6.3), and each automorphism of the pair $(S(a,b), B_S)$ is generated by $\mu_{a,b}$ and $H_{a,b}$ (Lemma 6.2). Hence, $\mathrm{Aut}(X(a,b))$ is generated by $\mu_{a,b}$, $H_{a,b}$, σ_2 and σ_3.

In the case where $H_{a,b}$ is not trivial, i.e., when $a = b$, we observe that the involution normalises $\mu_{a,b}$ (sending (μ, ν) onto (ν, μ)) and also $\langle \sigma_2, \sigma_3 \rangle$ (exchanging σ_2 and σ_3). To achieve the proof, it remains to observe that $\sigma_2 \sigma_3$ is of infinite order. This of course follows from characterizations of cluster algebras $\mathcal{C}(a,b)$ with finitely many clusters: they must be of types $\mathsf{A}_2, \mathsf{B}_2,$ or G_2 [5]. However, we can give an independent proof. It is exactly the same as the proof of Proposition 4.5: the map $\sigma_2 \sigma_3$ and its inverse have both a unique proper indeterminacy point, and these two points are different. Proceeding by induction, we obtain that $(\sigma_2 \sigma_3)^n$ has again a unique proper indeterminacy point for any $n \geq 1$, always being the proper indeterminacy point of $\sigma_2 \circ \sigma_3$. $\square$

7. Isomorphisms between two surfaces

We finish this note with the following result.

Proposition 7.1. *Let $a, b, c, d \geq 1$. The surfaces $X(a, b)$ and $X(c, d)$ are isomorphic if and only if $(a, b) = (c, d)$ or $(a, b) = (d, c)$.*

Proof. If $(a, b) = (d, c)$, the isomorphism is given by $(y_1, y_2, y_3, y_4) \mapsto (y_4, y_3, y_2, y_1)$.

Suppose now that $X(a, b)$ is isomorphic to $X(c, d)$. The automorphism groups of $X(a, b)$ and $Y(c, d)$ being isomorphic, the only cases to consider are when $ab \geq 4$ and $cd \geq 4$ (by Theorem 1).

We take a compactification (Y_1, B_1) of $X(a, b)$ by a standard square of type $(0, 0, -a, -b)$ (see Section 6 and in particular Lemma 6.1), and a compactification (Y_2, B_2) of $X(c, d)$ by a standard square of type $(0, 0, -c, -d)$. The isomorphism $X(a, b) \to X(c, d)$ decomposes into fibered modification and isomorphisms of pairs (Proposition 3.7). These maps do not affect the type of the boundary (which is defined up to permutations), so $(a, b) = (c, d)$ or $(a, b) = (d, c)$.   □

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