CENTRO INTERNAZIONALE MATEMATICO ESTIVO

(C.I.M.E)

To the memory of
Lucien Godeaux /1887-1975/

ALGEBRAIC SURFACES WITH $q = p_g = 0$.

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CONTENTS

Introduction

1. Notations. Let F be a complex algebraic surface. We will use the following standard notations:

$\mathcal{O}_F$: the structure sheaf of F .

$\mathcal{O}_F(D)$: the invertible sheaf associated with a divisor D on F .

$K_F = -c_1(F)$: minus the first Chern class of F or a canonical divisor on F .

$\omega_F = \mathcal{O}_F(K_F)$: the canonical sheaf of F .

$h^i(D)$: the dimension of the space $H^i(F,\mathcal{O}_F(D))$.

$p_g(F) = h^0(K_F) = h^2(\mathcal{O}_F)$; the geometric genus of F .

$q(F) = h^1(K_F) = h^1(\mathcal{O}_F)$: the irregularity of F .

K_F^2 : the self-intersection index of K_F .

$p^{(1)}(F) = K_{F'}^2 + 1$, where F' is a minimal model of a non-rational surface F ; the linear genus of F .

$c_2(F)$: the topological Euler-Poincare characteristic of F .

$p_a(F) = -q(F) + p_g(F) = 1/12(K_F^2 + c_2(F)) - 1$: the arithmetical genus.

$P_n(F) = h^0(nK_F)$: the n-genus of F .

NS(F) : the Neron-Severi group of F, the quotient of the Picard group $Pic(F)$ by the subgroup of divisors algebraically equivalent to zero (= $Pic(F)$ if $q = 0$).

$Tors(F) = Tors(NS(F)) = Tors(H_1(F,\mathbb{Z}))$.

If not stated otherwise F will be always assumed to be non-singular and projective.

2. Historical. It is easily proved that for a rational surface F (that is birationally equivalent to the projective plane $\mathbb{P}^2$) the invariants $q(F)$ and $p_g(F)$ are zero. The interest to non-rational surfaces with vanishing q and p_g was born in 1896 when Castelnuovo had established the necessary and sufficient conditions for a surface to be rational. Clebsh had proved earlier that a curve of genus 0 is rational. The question whether a surface with $q = p_g = 0$ is rational was a natural problem. In [10] Castelnuovo had shown that the answer is negative in general proving that one must add also the condition $P_2 = 0$ and constructing an example of a non-rational surface with $q = p_g = 0$. In the same paper he also exhibited other examples of such surfaces due to Enriques. The latter were of particular destiny, as it turned out later they play a special role in the general classification of algebraic surfaces representing one of the four classes of surfaces with vanishing Kodaira dimension (see [1], [6]) . Both examples of Enriques and Castelnuovo belong to the class of elliptic surfaces, that is they contain a pencil of elliptic curves. In particular, we have for these

surfaces $p^{(1)} = 1$. Later Enriques gave another construction of his surfaces and also presented other non-rational surfaces with $q = p_g = 0$ [17] . They were also elliptic surfaces.

The first examples of surfaces of general type with $q = p_g = 0$ appeared only in 1931-32 when Godeaux had constructed a surface with $q = p_g = 0$ and $p^{(1)} = 2$ [18] and Campedelli had constructed ([9]) a surface with $p^{(1)} = 3$. Later Godeaux constructed some other examples with $p^{(1)} = 3$ [20].

3. Modern development. The new interest to the surfaces under the title is related to the general problem of the existence of surfaces with given topological invariants which became of the main concern after the period of the reconstruction of Enriques' classification results had happily ended. The particular interest to the surfaces with $p^{(1)} = 2$ and 3 (numerical Godeaux and Campedelli surfaces) is due to Bombieri's paper [4] where for all other surfaces it was settled the question of the birationality of the 3-canonical map Φ_{3K} . Now due to works of Bombieri-Catanese [5,II], Miyaoka [32] and Victor Kulikov (non-published) we know that Φ_{3K} is birational for these surfaces, but I do not include the corresponding proofs in this survey refering to the paper of Catanese in these proceedings.

In Chapter II, I expose in more details the results of my paper [14] which deals with elliptic surfaces with $q = p_g = 0$. The theory of Kodaira-Ogg-Šafarevič allows to classify all such surfaces.

In Chapter III, we study more interesting case of surfaces of the general type. All such surfaces are divided into nine classes corresponding to the possible values of $p^{(1)} = 2, 3, \ldots, 10$. To distinguish the surfaces with the same $p^{(1)}$ one may consider the group Tors(F) or more generally the whole fundamental group $\pi_1(F)$. It can be shown (see Chapter III, §6) that there are only a finite number of possible π_1's for surfaces of the same class, and hence one may ask about some explicit estimate of the order of Tors(F) . Unfortunately, this is known only for the cases $p^{(1)} = 2$ (Bombieri) and 3 (Beauville, Reid) and only in the first case this estimate is the best possible. Moreover, we do not know whether the classes with $p^{(1)} = 8$ and 10 are empty* . The examples of surfaces with $4 \leq p^{(1)} \leq 7$ are due to Burniat [7,8] . We present here a new version of his construction ([7], [37]) which enables us to calculate Tors(F) for such surfaces. The examples of surfaces with $p^{(1)} = 9$ are due to Kuga [29] and Beauville [3] .

4. Acknowledgements. This work owes very much to many people with whom I had a conversation on the subject at different periods of my life. It would be impossible to mention them all. I am especially indepted to Miles Reid and Fabrizio Catanese whose critical remarks were very valuable. It is also a great pleasure to thank C.I.M.E. and M.I.T. for their support during the preparation of this paper.

* see. Epilogue.

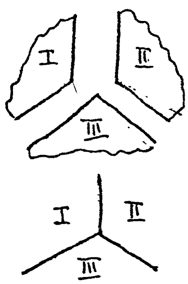
CHAPTER I. CLASSICAL EXAMPLES.

§1. The Enriques surface.

Let $\mathbb{P}^3$ be the projective 3-space with homogeneous coordinates x_i, $i = 0,\dots,3$. Consider the coordinate tetrahedron T : $x_0x_1x_2x_3 = 0$ and let X be a surface in $\mathbb{P}^3$ which passes twicely through the edges E_i $(i = 1,\dots,6)$ of T, that is, has E_i as its ordinary double lines. We also assume that X has no other singular points outside T and other common points with T. Since the section of F by a coordinate plane is the double reducible cubic curve, we see that F must be of order 6. More explicitly we may consider F as given by the equation:

$$(x_0x_1x_2)^2 + (x_0x_1x_3)^2 + (x_0x_2x_3)^2 + (x_1x_2x_3)^2 + x_0x_1x_2x_3(x_0^2+x_1^2+x_2^2+x_3^2) = 0$$

Let F be the normalization of X. Then F is a non-singular surface. To see it one has to look locally at the normalization of the affine coordinate cross : $xyz = 0$ in $\mathbb{A}^3$. Here the normalization will be just the disjoint union of three planes, the inverse image of the singular loci will be the union of six lines lying by pairs in these planes. Two lines in each of the planes correspond to the two axis lying in the same coordinate plane. The inverse image of the origin will be the three points, each of them is the intersection point of the two lines in one of the planes. So, locally the picture is as follows:



Let $p : F \to X$ be the projection. Then the local analysis above shows that for any edge E_i of the tetrahedron T we have

$$p^{-1}(E_i) = C_i = C_i' + C_i''$$

where C_i' and C_i'' are non-singular rational curves meeting each other transversally at two points arising from the two pinch-points of X lying on each of the edges.

C_i and C_j do not meet if E_i and E_j are not incident, otherwise C_i and C_j meet transversally at one point,

$$C_i \cap C_j \cap C_k = \emptyset \text{ for distinct } i, j, k \ .$$

Now we use the classical formula for the canonical sheaf of the normalization of a surface of degree n in $\mathbb{P}^3$:

$$\omega_F = \mathcal{O}_F((n-4)H-\Delta)) \ ,$$

where H is the inverse image of a plane section of X and Δ is the conductor divisor (= the annulator of the sheaf $p_*(\mathcal{O}_F/\mathcal{O}_X)$ (see Mumford's appendix to Chapter III of [43]) . In our case we easily find that

$$\omega_F = \mathcal{O}_F(2H - C) \ ,$$

where $C = C_1 + \ldots + C_6$.

The global sections of ω_F correspond to quadrics in $\mathbb{P}^3$ passing through the edges of the tetrahedron T. Since by trivial reasons such quadrics do not exist we have

$$p_g(F) = h^0(2H - C) = 0 .$$

Next, taking for $2H$ the inverse image of the union of two faces of the tetrahedron, we obtain that

$$K_F \sim 2H - C \sim C + C_i - C_j - C \sim C_i - C_j ,$$

where C_i is the common edge of these faces, and C_j is the opposite edge.

Taking for $4H$ the inverse image of the union of all faces (= the tetrahedron T) we get

$$2K_F \sim 4H - 2C \sim 2C_i - 2C_j \sim 0 .$$

Thus we have

$$P_2(F) = h^0(\mathcal{O}_F) = 1$$

and hence F is non-rational.

Since K_F is numerically equivalent to zero, we have

$$C_i^2 = C_i K_F = C_i(C_i - C_j) = 0 , \quad i + 1,\dots, 6 .$$

By the adjunction formula we get

$$H^0(C_j, \mathcal{O}_{C_j}) = H^0(C_j, \mathcal{O}_{C_j}) = \mathbf{C} .$$

Thus, C_i is a reducible curve of arithmetical genus 1 . Since $2C_i \sim 2C_j$ and C_i does not meet C_j we infer that the linear system $|2C_i|$ contains a pencil of curves of arithmetical genus 1 Since there are no base points of $2C_i$ we obtain by Bertini's theorem that almost all curves form this pencil are non-singular elliptic curves. Note also that this pencil contains two degenerate curves, $2C_i$ and $2C_j$.

Now we may use the formula expressing $c_2(F)$ in terms of the Euler-Poincare characteristic of degenerate curves of the elliptic pencil (see [1], Ch. IV):

$$c_2(F) = \sum_i \chi(B_i)$$

where B_i are all singular curves of the pencil. Since

$$\chi(2C_i) = \chi(2C_j) = \chi(C_i) = 1$$

we deduce that

$$c_2(F) > 0 .$$

Since $K_F^2 = 0$ we get by the Noether formula $12(I - q(F)) = c_2(F) > 0$. This obviously implies that $q(F) = 0$.

§2. The Godeaux surface.

Consider the projective involution σ of $\mathbb{P}^3$ of order 5 given in coordinates by the formula:

$$(x_0,x_1,x_2,x_3) \longmapsto (x_0,\zeta x_1,\zeta^2 x_2,\zeta^3 x_3) ,$$

ζ being a primitive 5-th root of unity. This involution acts freely outside the vertices of the coordinate tetrahedron. Let F' be a non-singular quintic which is invariant under σ and does not pass through these vertices. For example, we may take for F' a quintic with the equation:

$$a_0x_0^5 + a_1x_1^5 + a_2x_2^5 + a_3x_3^5 = 0 .$$

(For a general surface F' with the properties above one has to add to the left side 8 invariant monomials $x_0x_2^2x_3^2,\ldots$) . Let G be the cyclic group of order 5 generated by σ , acting freely on F' . Consider the quotient $F = F'/G$, the projection $\mathbb{p} : F' \to F$ is a finite non-ramified map of non-singular surfaces.

Lemma. Let $p : F' \to F$ be a finite non-ramified map of degree n . Then

$$K_{F'}^2 = n\, K_F^2$$

$$1 + p_a(F') = n\,(1 + p_a(F)) .$$

Proof. The first relation easily follows from the equality $p^*(\omega_F) = \omega_{F'}$, since p is smooth and finite. The second one follows from the Noether formula and the relation $c_2(F') = n\, c_2(F)$, which can be proved either by topological arguments or using the equality

$$p^*(\Omega^1_F) = \Omega^1_{F'} ,$$

Ω^1 being the sheaf of 1-differentials, and standard properties of Chern classes.

Since we have for F', $K^2_{F'} = 5$, $p_a(F') = 4$ we get from the lemma

$$K^2_F = 1 , \quad p_a(F) = 0 .$$

Since, obviously, $q(F) \leq q(F')$, we obtain

$$p_g(F) = 0 .$$

Next, note that F is minimal, that is there are no exceptional curves of the first kind lying on it. Indeed, the inverse image of such curve under p would be the disjoint union of five exceptional curves of the first kind on F'. However, F' is minimal. From the minimality of F and the fact $K^2_F \geq 1$ it follows that F is of general type. Another way to show this is to use the property of ample sheaves: $p^*(\omega_F)$ is ample implies ω_F is ample.

Since F' is simply-connected we obtain that the map p is the universal covering. In particular, $\mathrm{Tors}(F) = \pi_1(F) = \mathbb{Z}/5\mathbb{Z}$.

§3. The Campedelli surface.

This is a double ramified covering of the projective plane $\mathbb{P}^2$ branched along some curve of the IO-th degree (more precisely it is a minimal non-singular model of such covering).

Let W be the following reducible curve of the IO-th degree

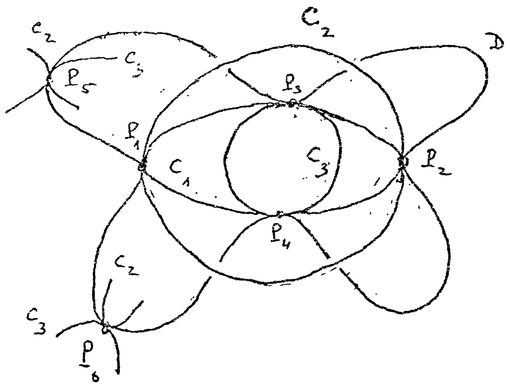
$$W = C_1 \cup C_2 \cup C_3 \cup D \quad ,$$

where C_i are non-singular conics and D is a non-singular quartic with the following properties:

$$C_1 \cap C_2 = 2P_1 + 2P_2 \; ; \quad C_1 \cap C_3 = 2P_3 + 2P_4 \; ; \quad C_2 \cap C_3 = 2P_5 + 2P_6$$

$$D \cap C_1 = 2(P_1 + P_2 + P_3 + P_4) \; ; \quad D \cap C_2 = 2(P_1 + P_2 + P_5 + P_6)$$

$$D \cap C_3 = 2(P_3 + P_4 + P_5 + P_6) \, .$$



To see that such configuration of curves exists one may take for C_2 and C_3 two concentric circles lying in the complement to the line at infinity, the points P_5 and P_6 will be the two cyclic points. The existence of a quartic D touching the conics C_i easily follows from the consideration of the net $\lambda C_1C_2 + \mu C_1C_3 + \nu C_2C_3 = 0$.

Lemma 1. Let X be a non-singular surface and W a reduced curve on it. Suppose that there exists a divisor D on X such that $W \sim 2D$, then there is a double covering

$$f : Y \to X$$

branched exactly along W . Moreover, Y is normal and non-singular over the complement to the singular focus of W .

Proof. Assume firstly that W is non-singular. Let F be the line bundle corresponding to the divisor D and (U_j) a coordinate covering of X such that $F|U_j$ is trivial and W is given by the local equation $\{c_j = 0\}$ on U_j . Let g_{ij} be a system of transition functions for F , then $c_i = g_{ij}^2 c_j$ on $U_i \cap U_j$ and we may consider the subvariety Y of F given by the equations $x_j^2 = c_j$, where x_j is a fibre coordinate of $F|U_j$. It is obviously checked that the projection $Y \to X$ satisfies the properties stated in the lemma.

If W is singular we apply the arguments above to X replaced by $X' = X - S$ and W by $W' = W - S$, where S is the singular locus of W . Then it suffices to take for Y the normalization of X in the double covering $Y' \to X'$ constructed as above.

Remark. The sheaf $L = \mathcal{O}_X(D)$ can be characterized as the subsheaf of antiinvariant sections of the direct image $f_*(\mathcal{O}_Y)$. If $q(X) = 0$ then this sheaf is determined uniquely by W (since they differ by an element of order 2 in $\mathrm{Pic}(X)$) . This shows that in this case any double covering with properties from lemma 1 can be obtained by the construction of the lemma.

Applying this lemma to the plane $\mathbb{P}^2$ and the 10-th degree curve W we may construct a double covering Y of $\mathbb{P}^2$ branched along W . This surface has six singular points lying over the points P_i . The Campedelli surface F will be obtained as the minimal non-singular model of Y .

Let $p : X \to \mathbb{P}^2$ be the minimal resolution of singularities of the curve W . The proper transform of W is given by

$$p^{-1}(W) \sim p^*(10L) - 3\sum_{i=1}^{6} S_i - 6\sum_{i=1}^{6} S_i' ,$$

where L is a line on $\mathbb{P}^2$,

$$p^{-1}(P_i) = S_i + S_i' , \quad i = 1,\dots,6$$

with $S_i^2 = -2$, $S_i'^2 = -1$.

Now we apply the lemma to the surface X and the non-singular curve

$$p^{-1}(W) + \sum_{i=1}^{6} S_i$$

and consider the corresponding double covering $r : F' \to X$. To compute the canonical class $K_{F'}$ we use the following:

Lemma 2. Let $g : V' \to V$ be the double covering of non-singular surfaces branched along the curve W , $g^*(W) \sim 2\bar{D}$ for some divisor $\bar{D}$ on V' . Then

$$K_{V'} \sim g^*(K_V) + \bar{D} .$$

Proof. First, note that our double covering can be obtained by the construction from lemma 1 . In fact, consider the splitting

$$g_*(\mathcal{O}_{V'}) = \mathcal{O}_V \oplus L$$

into invariant and anti-invariant pieces. Then clearly $L^{\otimes 2}$ is contained in the invariant piece that is in $\mathcal{O}_V$. Thus $\mathcal{O}_V$-Algebra $g_*(\mathcal{O}_{V'})$ is the quotient algebra of the symmetric algebra $\mathrm{Symm}(L) = \mathcal{O}_V \oplus L \oplus L^{\otimes 2} \oplus \ldots$ by the Ideal generated by $L^{\otimes 2} - J$, where J is an ideal sheaf in $\mathcal{O}_V$. Taking the spectrums we get that $V' = \mathrm{Spec}(g_*(\mathcal{O}_{V'}))$ is isomorphic to the closed subscheme of the line bundle $F = \mathbb{V}(\check{L}) = \mathrm{Spec}(\mathrm{Symm}(L))$. Looking locally we easily identify J with the sheaf $\mathcal{O}_V(-W)$ and obtain that V' is constructed with the help of a divisor D corresponding to F in the same way as in lemma 1 .

Now, the formula for $K_{V'}$ can be proved very simply. In notations of lemma 1 we consider a 2-form w on V in local coordinates c_j and some other function t_j . Then we use the

relation $dc_j \wedge dt_j = 2x_j dx_j \wedge dt_j$ to obtain that $(g^*(W)) = g^*((W)) +$ $+ \{x_j = 0\}$ (the brackets ( ) denotes the divisor of a 2-form).
This proves the lemma.

Thus, we have

$$K_{F'} \sim r^*(-3p^*(L) + \sum_{i=1}^{6} S_i + 2\sum_{i=1}^{6} S_i') + \frac{1}{2} r^*(10p^*(L) - 2\sum_{i=1}^{6} S_i - 6\sum_{i=1}^{6} S_i'$$

$$\sim r^*(2p^*(L) - \sum_{i=1}^{6} S_i') .$$

Assume that $D \in |K_{F'}|$, then we see from above that $D = r^*(D')$ for some

$$D' \in |2p^*(L) - \sum_{i=1}^{6} S_i'| .$$

The latter linear system is equal to the inverse transform of the system of conics on $\mathbb{P}^2$ passing through all points P_i . To show that the latter does not exist we argue as follows. Taking for P_1 and P_2 the two cyclic infinite points $(1, \pm i, 0)$ we get the equations for C_1 and C_2 in the form:

$$X_1^2 + X_2^2 - a^2 X_0^2 = 0 , \quad X_1^2 + X_2^2 - b^2 X_0^2 = 0$$

and the equation for C_3 on the form:

$$\frac{X_1^2}{a^2} + \frac{X_2^2}{b^2} - X_0^2 = 0 .$$

The points P_3, P_4, P_5, P_6 will have the coordinates $(\pm a, 0, 1)$, $(0, \pm b, 1)$ respectively. Now let C be a conic with an equation

$$a_1x_1^2 + a_2x_2^2 + a_3x_1x_2 + a_4x_1x_0 + a_5x_2x_0 + a_6x_0^2 = 0$$

which passes through the points $P_1, \ldots, P_6$. Since it passes through P_1 and P_2 we may assume that $a_3 = 0$ and $a_1 = a_2 = 1$. Since it passes through P_3 and P_4 we get the equations

$$a^2 \pm a_4 + a_6 = 0$$

which give $a_4 = 0$ and $a_6 = -a^2$. Similarly we get $a_5 = 0$ and $a_6 = -b^2$. This contradiction shows that C does not exist.

Thus $K_{F'} = \emptyset$ and $P_g'(F') = 0$.

Since r is branched along S_i, $i = 1, \ldots, 6$ and $p^{-1}(C_i)$, $i' = 1, 2, 3$, we see that

$$r^*(S_i) = 2\,\bar{S}_i \;, \quad r^*(p^{-1}(C_i)) = 2\,\bar{C}_i$$

for some curves $\bar{S}_i$ and $\bar{C}_i'$ on F' . Also, we notice that

$$\bar{S}_i^2 = \frac{1}{4}(r^*(S_i)^2) = \frac{1}{4}(2S_i^2) = \frac{1}{4}(-4) = -1 \;,$$

$$\bar{C}_i'^2 = \frac{1}{4}(r^*(p^{-1}(C_i))^2) = \frac{1}{4}(2(p^{-1}(C_i))^2) = \frac{1}{4}(-8) = -2 \;.$$

This shows that $\overline{S}_i$ are exceptional curves of the 1st kind. Let $\sigma : F' \to F$ be the blowing down of all $\overline{S}_i$. We will show that F is a minimal model of F' . We have

$$2K_{F'} \sim r^*(4p^*(L) - 2\sum_{i=1}^{6} S_i') \sim$$

$$\sim 2\overline{C}_1' + 2\overline{C}_2' + 2S_1' + 2S_2' + 2(\sum_{i\neq 5,6} \overline{S}_i + \overline{S}_1 + \overline{S}_2)$$

$$\sim 2\overline{C}_1' + 2\overline{C}_3' + 2S_3' + 2S_4' + 2(\sum_{i\neq 3,4} \overline{S}_i + \overline{S}_3 + \overline{S}_4)$$

$$\sim 2\overline{C}_2' + 2\overline{C}_3' + 2S_5' + 2S_6' + 2(\sum_{i\neq 1,2} \overline{S}_i + \overline{S}_5 + \overline{S}_6)$$

since

$$p^{-1}(C_1) \sim p^*(2L) - 2\sum_{i\neq 5,6} S_i' - \sum_{i\neq 5,6} S_i$$

$$p^{-1}(C_2) \sim p^*(2L) - 2\sum_{i\neq 3,4} S_i' - \sum_{i\neq 3,4} S_i$$

$$p^{-1}(C_3) \sim p^*(2L) - 2\sum_{i=1,2} S_i' - \sum_{i\neq 1,2} S_i'$$

This shows that

$$2K_F \sim 2\overline{C}_1 + 2\overline{C}_2 + 2\overline{S}_1 + 2\overline{S}_2 \sim$$

$$\sim 2\overline{C}_2 + 2\overline{C}_3 + 2\overline{S}_3 + 2\overline{S}_4 \sim$$

$$\sim 2\overline{C}_1 + 2\overline{C}_3 + 2\overline{S}_5 + 2\overline{S}_6 ,$$

where $\overline{C}_i = \sigma_*(\overline{C}_i')$, $\overline{S}_i = \sigma_*(S_i')$.

If E is an exceptional curve of the 1st kind on F, then $(E\,K_F) = -1$ and hence E must coincide with one of the curves $\overline{C}_i$ or $\overline{S}_i$. However, neither of them is an exceptional curve, because

$$\overline{C}_i^2 = \overline{C}_i'^2 = -2 \ , \quad (\overline{S}_i K_F) = 1 \quad .$$

To compute K_F^2 we use that

$$K_F^2 = (\overline{C}_1 + \overline{C}_2 + \overline{S}_1 + \overline{S}_2)^2 = -2 - 2 - 1 - 1 + 8 = 2 \ .$$

It remains to notice that F is a surface of general type, since it is minimal and has positive K_F^2. In particular, we have $q(F) \leq p_g(F) = 0$ (see Chap. 3, §1, lemma 3). Also note that $2K_F$ is determined by the net of quartics $\lambda C_1C_2 + \mu C_1C_3 + \nu C_2C_3$ and is of dimension 2 .

We also have the following obvious torsion divisors of order 2 on F :

$$g_1 : K_F - \overline{C}_1 - \overline{C}_2 - \overline{S}_1 - \overline{S}_2 \ ,$$
$$g_2 : K_F - \overline{C}_1 - \overline{C}_3 - \overline{S}_5 - \overline{S}_6 \ ,$$
$$g_3 : K_F - \overline{C}_2 - \overline{C}_3 - \overline{S}_3 - \overline{S}_4 \ ,$$
$$g_4 : K_F - \overline{D} \ , \text{ where } \sigma_*(r^*(p^{-1}(D))) = 2\overline{D} \quad .$$

It is immediately checked that

$$g_1 + g_2 + g_3 = 0$$

and

$$g_1 + g_2 + g_4 = g_3 + g_4 \not\equiv 0 .$$

This shows that

$$\mathrm{Tors}(F) \supseteq (Z/2Z)^3 .$$

It will be shown in Chapter III, §3 that, in fact, we have the equality

$$\mathrm{Tors}(F) = (Z/2Z)^3 .$$

CHAPTER 2. ELLIPTIC SURFACES.

1. Generalities.

A projective non-singulur surface X is called elliptic if there exists a morphism $f : X \to B$ onto a non-singular curve B whose general fibre X is a smooth curve of genus 1 . Such f is called an elliptic fibration on X . From general properties of morphisms of schemes we infer that almost all fibres are non-singular elliptic curves over the ground field k (as everywhere in this paper we assume that $k = \mathbb{C}$ or algebraically closed of characteristic 0) . An elliptic surface X is called minimal if there exist an elliptic fibration without exceptional curves of the 1st kind in its fibres ((such fibration will be called minimal).

Let $f : X \to B$ be an elliptic fibration on an elliptic surface X and X_b a fibre over a point $b \in B$. Consider X_b as a positive divisor on X , then according to Kodaira [26] it is one of the following types:

$m^1 0$: $X_b = mE_0$, $m \geq 1$, where E_0 is a non-singular elliptic curve;

$m^1 1$: $X_b = mE_0$, $m \geq 1$, where E_0 is a rational curve with a node;

$m^1 2$: $X_b = mE_0 + mE_1$, $m \geq 1$, where E_0 and E_1 are non-singular rational curves meeting transversally at two points;

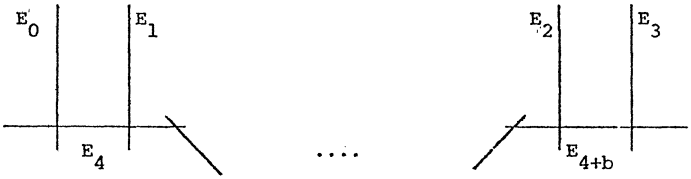
$m^1 b$: $X_b = mE_0 + \ldots + mE_{b-1}$, $m \geq 1$, where E_i are rational non-singular curves with $E_i \cap E_j \cap E_k = \emptyset$ for distinct i, j, and k and $(E_i E_{i+1}) = 1$, $i = 0, \ldots, b-1$, assuming $E_b = E_0$ $(b \geq 3)$.

II : $X_b = E_0$, a rational curve with a cusp;

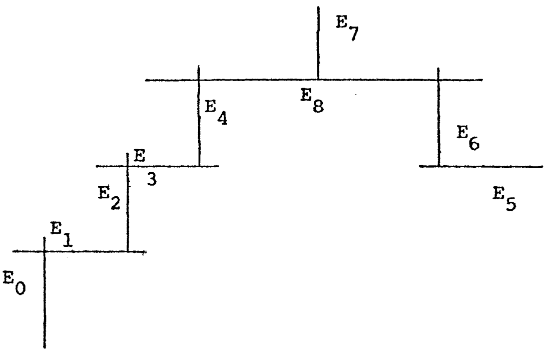
III : $X_b = E_0 + E_1$, where E_0 and E_1 are non-singular rational curves with simple contact at one point;

IV : $X_b = E_0 + E_1 + E_2$, where E_i are non-singular rational curves transversally meeting each other at one point $p = E_0 \cap E_1 \cap E_2$;

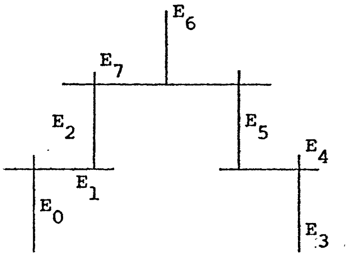
I_b^* : $X_b = E_0 + E_1 + E_2 + E_3 + 2E_{4+b}$, where all E_i are non-singular rational curves transversally interesecting as shown on the picture



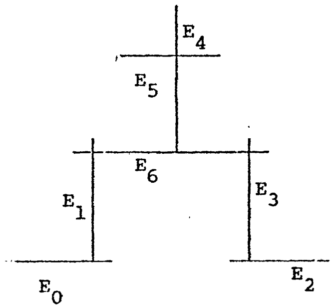
II^* : $X_b = E_0 + 2E_1 + 3E_2 + 4E_3 + 5E_4 + 2E_5 + 4E_6 + 3E_7 + 6E_8$, where E_i are non-singular rational curves interesecting as shown on the picture



III^{*} : $X_b = E_0 + 2E_1 + 3E_2 + E_3 + 2E_4 + 3E_5 + 2E_6 + 4E_7$, where E_i are non-singular rational curves and the picture is



IV^{*} : $X_b = E_0 + 2E_1 + E_2 + 2E_3 + E_4 + 2E_5 + 3E_6$, where E_i are rational non-singular curves and the picture is



A singular fibre of type m^1b, $b \geq 0$, $m \geq 2$, is called <u>multiple</u> of multiplicity m.

Let $f : X \to B$ be an elliptic fibration, then its general fibre X_η is a sooth curve of genus 1 over the field K of rational functions of B and it is an abelian variety over K if and only if it has a K-rational point. In geometric terms the latter is equivalent to the existence of a global section $s : B \to X$ of the morphism f .

Consider the jacobian variety J of X_η , this is again a smooth curve over K of genus 1 with a rational point over K . For any extension K'/K such that X has a K'-rational point there exists a natural isomorphism of K'-curves $X_\eta \otimes_K K' \simeq J \otimes_K K'$. According to general properties of birational transformations of two-dimensional schemes there exists the unique minimal elliptic fibration $j : A \to B$ such that $A_\eta \simeq J$. This surface is called the jacobian surface of the elliptic surface X . Since j has a section, all singular fibres of j are non-multiple.

Proposition 1. For any $b \in B$ such that X_b is a non-multiple fibre the fibrations $f : X \to B$ and $j : A \to B$ are isomorphic over the henselization $\tilde{O}_b$ of the local ring $O_{B,b}$.

Proof. Let $\tilde{f}_b : \tilde{X}(b) \to \mathrm{Spec}\,\tilde{O}_b$ be the restriction of f over $\tilde{O}_b$, and $\tilde{j}_b : \tilde{A}(b) \to \mathrm{Spec}\,\tilde{O}_b$ the same for j . Since $\tilde{f}_b$ is smooth at some point of a component of multiplicity 1, there exists a section of $\tilde{f}_b$. This implies that the general fibre $\tilde{X}(b)_\eta$ is an abelian curve over the fraction field $\tilde{K}_b$ of $\tilde{O}_b$. From this we infer easily that $\tilde{X}(b)_\eta \simeq \tilde{A}(b)_\eta$ and hence in virtue of the uniqueness of the minimal models we get $\tilde{X}(b) \simeq \tilde{A}(b)$.

Proposition 2. Let $b \in B$ such that X_b is a multiple fibre of type ${}_m1_b$. Then the fibre A_b of $j : A \to B$ is of type ${}_1A_b$.

Proof. Let $B' \to B$ be a covering of B ramified at some point $b' \in B'$ over b with the ramification index equal to m .

Let $f'_{b'} : X'(b') \to \mathrm{Spec}\, O_{B',b'}$ be the restriction of the base change map $X \times_B B' \to B'$ over the local ring $O_{B',b'}$. Denote by $\overline{X}'(b')$ the normalization of $X'(b')$ and let $\overline{f}'_{b'} : \overline{X}'(b') \to \mathrm{Spec}\, O_{B',b'}$ be the composite map.

Let $x \in X$ be a double point of the fibre X_b. Then formally at x the map $f : X \to B$ is isomorphic to the map $\mathbb{A}^2 \to \mathbb{A}^1$ given by $t = (xy)^m$. This shows that $X'(b')$ formally at the point x' lying over x is isomorphic to the hypersurface $t^m = (xy)^m$ in $\mathbb{A}^3$. Taking the normalization we observe that there are exactly m points $x'_1, \ldots, x'_m \in \overline{X}'(b')$ lying over x' and formally at each x'_i $\overline{X}'(b')$ is given by the equation $t = xy$. Looking globally we infer that the fibre $\overline{X}(b')_0$ is of type $1^1 mb$.

Performing the same base change for $j : A \to B$ and resolving the singularities of the obtained surface $A'(b')$ we will get the scheme over $O_{B',b'}$ with the closed fibre of type $1^1 mb$ (Proposition 1). Checking case by case we find that it can be only if the fibre of j over b is of type $1^1 b$.

Let $j : A \to B$ be a minimal elliptic fibration with a global section, $W(j)$ be the set of all minimal elliptic fibrations over B for which j serves as the jacobian fibration. For any $f : X \to B$ from $W(j)$ the general fibre X_K is a principal homogeneous space (p.h.s) for A_K over the field K of rational functions on B. As it is well known the set of all p.h.s. for A_K forms the Galois cohomology group $H^1(K, A_K)$. In virtue of the existence and uniqueness of minimal models for A_K the map $W(j) \to H^1(K, A_K)$ is bijective.

To compute $H^1(K, A_K)$ we argue as follows ([34, 38, 41]). Let $i : \eta = \operatorname{Spec} K \hookrightarrow B$ be the inclusion of the general point. Identify A_K with the etale sheaf which it represents and let $\underline{A} = i_* A_K$. The sheaf $\underline{A}$ is representable by the commutative group scheme over B which is obtained by throwing out all points of the surface A where f is non-smooth (the <u>Neron model</u> of A_K). The Leray spectral sequence for i gives the exact cohomology sequence:

$$0 \longrightarrow H^1(B,\underline{A}) \longrightarrow H^1(K,A_K) \xrightarrow{\psi} H^0(B,R^1 i_* A_K) \longrightarrow H^2(B,\underline{A})$$

For any closed point $b \in B$ we have

$$(R^1 i_* A_K)_b = H^1(\tilde{K}_b, A_{\tilde{K}_b}) \quad ,$$

where $\tilde{K}_b$ is the fraction field of the henselization $\tilde{O}_{B,b}$ of the local ring $O_{B,b}$, $A_{\tilde{K}_b} = A_K \times_K \tilde{K}_b$. To compute $H^1(\tilde{K}_b, A_{\tilde{K}_b})$ it suffices to compute for all n the subgroup ${}_nH^1(\tilde{K}_b, A_{\tilde{K}_b})$ of the elements killed by multiplication by n (since H^1 is always periodical). Using the Kummer exact sequence

$$0 \longrightarrow {}_nA_{\tilde{K}_b} \longrightarrow A_{\tilde{K}_b} \xrightarrow{n} A_{\tilde{K}_b} \longrightarrow 0$$

we get

$${}_nH^1(\tilde{K}_b, A_{\tilde{K}_b}) = H^1(\tilde{K}_b, {}_nA_{\tilde{K}_b}) \quad .$$

Now, since $A_{\tilde{K}_b}$ coincides with its Picard variety $\mathrm{Pic}^0(A_{\tilde{K}_b})$, we have

$$_nA_{\tilde{K}_b} = ({}_nR^1\tilde{j}^b_* G_m)_{\tilde{K}_b} = (R^1\tilde{j}^b_*\mu_n)_{\tilde{K}_b}$$

where $\tilde{j}^b : \tilde{A}(b) \to \mathrm{Spec}\,\tilde{O}_{B,b}$ is the strict localization of the morphism j over b. Since $\mathrm{Spec}\,\tilde{O}_{B,b}$ is cohomologically trivial

$$H^1(\tilde{K}_b, {}_nA_{\tilde{K}_b}) = H^1(\tilde{K}_b, i_b^* R^1\tilde{j}^b_*\mu_n) = H^0(\mathrm{Spec}\,\tilde{O}_{B,b}, R^1\tilde{j}^b_*\mu_n) =$$

$$= H^1(\tilde{A}(b), \mu_n) = H^1(A_b, \mu_n) \ ,$$

where $i_b : \mathrm{Spec}\tilde{K}_b \hookrightarrow \mathrm{Spec}\tilde{O}_{B,b}$ is the inclusion of the general point. It remains to add that

$$H^1(A_b, \mu_n) = (Z/nZ)^{d_b} \ ,$$

where $d_b = 2$ in case A_b is of type $_1 1_0$, $d_b = 1$ in case A_b is of type $1^1 1$, $1 \geq 1$, $d_b = 0$ in the remaining cases.

Let $f : X \to B$ be the elliptic fibration representing an element $x \in H^1(K, A_K)$. Then we interprete the composite map

$$\psi_b : H^1(K, A_K) \longrightarrow H^0(B, R^1 i_* A_K) \longrightarrow (R^1 i_* A_K)_b = H^1(\tilde{K}_b, A_{\tilde{K}_b})$$

as follows. The general fibre of the strict localization $\tilde{f}^b : \tilde{X}(b) \to \mathrm{Spec}\,\tilde{O}_{B,b}$ represents a p.h.s. for $A_{\tilde{K}_b}$ over the field $\tilde{K}_b$ and, hence, an element of $H^1(\tilde{K}_b, A_{\tilde{K}_b})$. Also, it can be checked that $\psi_b(X)$ equals the class of the normal sheaf of the reduced fibre X_b^0 in the Picard group $\mathrm{Pic}(X_b^0)$, whose torsion part is identified with $H^1(K_b, A_{K_b}) = \varinjlim H^1(A_b, \mu_n) = \varinjlim H^1(X_b^0, \mu_n)$.

From this observation we immediately obtain the following

Proposition 3. For any $x \in H^1(K, A_K)$ $\psi_b(x) \neq 0$ if and only if the fibre X_b of the corresponding elliptic fibration is multiple. The multiplicity of X_b equals the order of $\psi_b(x)$ in $H^1(\tilde{K}_b, A_{\tilde{K}_b})$.

The last assertion follows from the proof of Proposition 2.

Now we shall compute the kernel $H^1(B, \underline{A})$ of the map (so called the Tate-Shafarevich group of A_K) .. First, we have the following exact sequence:

$$0 \longrightarrow \underline{A} \longrightarrow R^1 j_x \mathbb{G}_m \longrightarrow \mathbb{Z}_B \longrightarrow 0$$

which comes from the identification of A_K with its Picard variety $\mathrm{Pic}^0_{A_K} = \mathrm{Ker}((R^1 j_x \mathbb{G}_m)_K \to \mathbb{Z}_K)$ (see the details in [24]). Since j has a global section, the exact cohomology sequence gives the isomorphism

$$H^1(B, \underline{A}) = H^1(B, R^1 j_x \mathbb{G}_m) .$$

Next, considering the Leray spectral sequence for j and $G_{m,A}$ and using that $R^i j_x G_m = H^1(B, G_m) = 0$, $i > 1$ we get

$$H^1(B, \underline{A}) = \mathrm{Br}(A) = H^2(A, G_m) .$$

In virtue of birational invariance of Br ([24]) we obtain

Proposition 4. Assume that A is a rational surface, then

$$H^1(B,A) = 0 \quad .$$

In particular, any minimal elliptic surface without multiple fibers whose jacobian surface A is rational is isomorphic to A .

The last thing to do is to investigate the group $H^2(B,\underline{A})$. Let $\underline{A}^0$ be the subsheaf of $\underline{A}$ which is representable by the connected component of the unit of the group scheme $\underline{A}$ (equal to the surface A minus all irreducible components of the fibers which do not meet some fixed section of j and also minus singular points of irreducible fibers) . We have the "Kummer exact sequence"

$$0 \longrightarrow {}_n\underline{A}^0 \longrightarrow \underline{A}^0 \xrightarrow{n} \underline{A}^0 \longrightarrow 0 \quad ,$$

which gives the exact sequence

$$H^1(B, \underline{A}^0) \longrightarrow H^2(B, {}_n\underline{A}^0) \longrightarrow {}_nH^2(B, \underline{A}^0) \longrightarrow 0 \ .$$

The quotient sheaf $\underline{A}/\underline{A}^0$ has finite support, hence

$$H^i(B,\underline{A}) = H^i(B,\underline{A}^0) \ , \quad i \geq 1 \ .$$

Applying the global duality theorem [12], we get

$$H^2(B,{}_n\underline{A}^0) = \mathrm{Hom}(H^0(B,{}_n\hat{\underline{A}}^0) \ , Z/nZ) \ .$$

The dual sheaf ${}_n\hat{\underline{A}}$ coincides with ${}_n\underline{A}$ in virtue of the auto-duality of the jacobian variety A_K . Now we use the following.

<u>Lemma</u>. Suppose that $q(A) = 0$. Then

$$H^0(B, {}_n\underline{A}^0) = 0 .$$

<u>Proof</u>. Any element of the group $H^0(B, {}_n\underline{A}^0)$ represents a section of j of order dividing n which meets the same irreducible component of a fiber as the fixed zero section. Moreover, any two such sections do not meet each other, since for any point $b \in B$ the reduction homomorphism $A(\tilde{K}_b) \to A_b(k)$ is an isomorphism on the subgroup of points of finite order. The latter follows, for example, from the equality $H^0(\mathrm{Spec}\ \tilde{O}_{B,b}, {}_n\underline{A}) = H^0(k, {}_n\underline{A}_b)$ which is a particular case of some general property of étale cohomology ([12]). Suppose that $H^0(B, \underline{A}_n^0) \neq 0$, and let S be a section from this group different from the zero section S_0 . Then

$$n(S - S_0) \sim \sum m_i F_i'$$

where F_i' is a divisor supported in some fiber F_i of j . Since S and S_0 meet the same component of fibers we get immediately that $F_i' \theta = 0$ for each component θ of F_i . Applying the main lemma below we get that $F_i' = F_i$ and hence

$$n(S - S_0) \sim kF$$

where F is any fiber (use that since $q(A) = 0$ we have $B = \mathbb{P}^1$ and hence all fibers are linearly equivalent). Now from the computation of K_A (see again below) we get for any section $(S \cdot K_A) = -1 + p_g(A) < 0$. Since $S \simeq \mathbb{P}^1$ we get $S^2 = -2 + 1 - p_g(A) = -1 - p_g(A) < 0$ .. However

$$(n(S - S_0))^2 = n^2 S^2 + n^2 S_0^2 = k^2 (F^2) = 0 \quad .$$

This contradiction proves the lemma.

From this lemma we get the following.

Proposition 5. Suppose that $q(A) = 0$. Then the map

$$\psi : H^1(K, A_K)' \longrightarrow \bigoplus_{b \in B} H^1(K_b, A_{K_b})$$

is surjective. In particular, for any finite set of closed points $b_1, \ldots, b_r \in B$ such that the fiber A_{b_i} is of type $1^1 h_i$ ($i=1,\ldots r$; $h_i > 0$) and any collection of positive numbers $m_1, \ldots, m_r$ there exists a minimal elliptic fibration $f : X \to B$ whose jacobian fibration equals j and whose fibers X_{b_i} are of type $m_i{}^1 h_i$, $i=1,\ldots,r$.

Now we shall compute the canonical class K_X of an elliptic surface X. We restrict ourselves for the simplicity to the case of regular surfaces X (i.e. we assume that $q(X) = 0$). For the general case we refer to [6] or [27]. In particular, we may assume that the base B of any elliptic fibration $f : X \to B$ is the projective line $\mathbb{P}^1$.

Main lemma. ([6]). Let $C = \sum n_i C_i$ be an effective divisor on a surface X with each E_i irreducible. Assume that

$$(C_i, D) \leqq 0 \text{ , all } i$$

and that D is connected.

Then every divisor $Z = \sum m_i C_i$ satisfies $Z^2 \leqq 0$ and equality holds if and only if $D^2 = 0$ and $Z = rD$, $r \in \mathbb{Q}$.

Proof. Write $x_i = m_i/n_i$ and consider the equality

$$\begin{aligned} Z^2 &= \sum x_i x_j n_i n_j (C_i \cdot C_j) \\ &\leqq \sum x_i^2 n_i^2 (C_i C_i) + \sum_{i \neq j} \frac{1}{2}(x_i^2 + x_j^2) n_i n_j (C_i C_j) \\ &= \sum x_i^2 n_i (C_i D) \leqq 0 \text{ .} \end{aligned}$$

If equality holds everywhere, then we have either $x_i = x_j$ or $(C_i C_j) = 0$ for all $i;j$; since D is connected the last possiblity does not occur. Hence x_i is constant, that means that $m_i = rn_i$, $r \in \mathbb{Q}$.

Theorem. Let $f : X \to \mathbb{P}^1$ be an elliptic fibration of an elliptic surface X with $q(X) = 0$. Then

$$K_X \sim (p_g(X) - 1)F + \sum_i (m_i - 1)F_i^0$$

where F is any fibre of f , $F_i = m_i F_i^0$ all multiple fibres of multiplicity m_i .

Proof. For any non-singular fibre X_b we have

$$\mathcal{O}_{X_b} \otimes \omega_X \simeq \omega_{X_b} \simeq \mathcal{O}_{X_b} \quad .$$

Taking a sufficiently large number of distinct "general" points $b_1,\ldots,b_r$ and considering the exact sequence

$$0 \longrightarrow \omega_X \longrightarrow \omega_X \otimes \mathcal{O}(\sum_{i=1}^{r} X_{b_i}) \longrightarrow \bigoplus_{i=1}^{r} \mathcal{O}_{X_{b_i}} \longrightarrow 0$$

we get

$$|K_X + \sum_{i=1}^{r} X_{b_i}| \neq 0 \quad .$$

If D is a divisor in the linear system above, we have

$$(D \cdot F) = 0 \ , \quad \text{for any fibre } F \ .$$

This implies that we can write

$$K_X \sim (\text{sum of fibres}) + \Gamma \quad ,$$

where $\Gamma \geq 0$ is contained in a union of fibres and does not contain fibres of f. Let Γ_0 be a connected component of Γ contained in the fibre X_b. If $X_b = \sum n_i E_i$ then

$$0 = (K_X X_b) = \sum n_i (K_X E_i)$$

and

$$0 = (X_b E_i) = \sum_{j \neq i} n_j (E_j E_i) + (E_i^2) \quad .$$

This shows that $(E_i^2) < 0$ if X_b is reducible, that implies that $(K_X E_i) = 2g(E_i) - 2 - (E_i^2) = 0$, since E_i cannot be an exceptional curve of the 1st kind. Hence, we have $(K_X E_i) = 0$.

Thus, if X_b is reducible, then

$$(\Gamma_0 \cdot E_i) = 0 \text{ , all components } E_i \text{ of } X_b .$$

Applying the main lemma we get that $\Gamma_0 = rX_b$, $r \in \mathbb{Q}$.

So, we have proved that

$$K_X \sim nF + \sum a_i F_i^0 , \quad 0 \leq a_i < m_i$$

and it remains to show that $n = p_g(X) - 1$, and $a_i = m_i - 1$.

For this we note, firstly, that the divisor $\sum a_i F_i^0$ is the fixed part of the linear system K_X . Indeed, any rational function belonging to the space $H^0(X, O_X(K_X))$ must be constant on the general fibre of f , and hence, it is induced by a rational function on $\mathbb{P}^1$. But then it is either regular on the divisor F_i^0 , or has the pole of order multiple to m_i at F_i^0 .

Thus we have

$$p_g = h^0(K_X) = h^0(nF) = n + 1$$

that proves the assertion about n .

Next, by Riemann-Roch

$$h^0(K_X + F_i^0) = 1 + p_g(X) > h^0(K_X)$$

and this shows that

$$K_X = F_i^0 \sim (n+1)F + \sum_i a_i' F_i^0 \ , \quad 0 \leq a_i' < m_i$$

(using again the arguments above) . This, obvioulsy, implies that $a_i + 1 = m_i$.

Corollary 1. For any minimal elliptic surface X

$$K_X^2 = 0 \ .$$

Furthermore, if $q(X) = 0$, then the plurigenus

$$P_n(X) = n(P_g(X) - 1) + \sum_{i=1}^{r} [n(m_i - 1)/m_i] + 1$$

Proof. The first assertion follows easily from the proof of the theorem. Indeed, we have proved without assumption $q(X) = 0$ that K_X is numerically equivalent to a rational linear combination of fibres.

To prove the second assertion, we use that

$$nK_X \sim n(p_g(X) - 1)F + \sum_{i=1}^{r} n(m_i - 1)F_{b_i}^0$$
$$(n(p_g(X) - 1) + \sum_{i=1}^{r} [n(m_i - 1)/m_i]F + \sum_{i=1}^{r} a_i F_{b_i}^0 \ ,$$

where $0 \leq a_i < m_i$. Again, using the arguments of the proof of the theorem, we get that $\sum a_i F_{b_i}^0$ equals the fixed part of $|nK_X|$. This, of course, proves the assertion.

Corollary 2. ([13]). An elliptic surface with $q = p_g = 0$ is rational if and only if its minimal elliptic fibration contains at most one multiple fibre.

In fact, $P_2(X) = 0$ implies that the number of multiple fibres $r \leq 1$. In another direction the assertion follows immediately.

Corollary 3. (Godeaux). Suppose that $q(X) = p_g(X) = 0$. Then

$$K_X \sim (r - 1)F - \sum_{i=1}^{r} F_{b_i}^0 ,$$

where F_{b_i}, $i = 1,\ldots, r$, are all multiple fibres.

Next, we want to compare the numerical invariants of an elliptic surface and its jacobian surface.

Proposition 6. Let $f : X \to B$ be an elliptic fibration. Denote by EP(Z) the topological Euler-Poincare characteristic (in case $k \neq C$, the field of complex numbers, we consider l-adic etale cohomology). Then

$$EP(X) = \sum_{b \in B} EP(F_b) .$$

For the proof we refer to [1], Ch. 4 ($k = \mathbb{C}$) or [12] (arbitrary k). Note that we use here the assumption $\operatorname{char}(k) = 0$. In the general case there is some additional term depending on the wild ramification.

Corollary. Let X be a minimal elliptic surface, A its jacobian surface. Then

$$EP(X) = EP(A) , \quad p_a(X) = p_a(A) .$$

The first equality follows from propositions 1 and 2 , the second one follows from the first and the Noether formula.

Proposition 7. Let $f : X \to B$ be an elliptic fibration. Suppose that for some fiber X_b the reduced curve $X_{b,red}$ is singular. Then

$$q(X) = \text{genus}(B) .$$

Proof. The hypothesis implies that under the Albanese map alb: $X \to \text{Alb}(X)$ the fiber X_b goes to a point (since all of its components are rational curves). This shows that in the canonical commutative diagram

$$\begin{array}{ccc} X & \xrightarrow{\text{alb}} & \text{Alb}(X) \\ \downarrow & & \downarrow \psi \\ B & \xrightarrow{\text{jac}} & J(B) \end{array}$$

ψ is a finite surjective map and hence $\dim \text{Alb}(X) = \dim J(B) = \text{genus}(B)$.

Corollary. Suppose that the jacobian fibration $j : A \to B$ has a singular fiber. Then for any elliptic surface X with the jacobian surface equal to A we have

$$q(X) = q(A) .$$

Corollary. Let X be an elliptic surface with $q = p_g = 0$. Then its jacobian surface is rational. Conversely, any elliptic surface with rational jacobian surface has $q = p_g = 0$.

Thus all elliptic surfaces with $q = p_g = 0$ are obtained from rational jacobian elliptic surfaces by choice of some fibres of type 1^1h and some element of finite order of the Picard group of each of these fibrès.

All rational elliptic surfaces can be described with the help of so called Halphen pencils on the projective plane ([13]) . These are the pencils of curves of degree 3m with 9 multiple points of multiplicity m lying on a cubic. The case $m = 1$ corresponds to jacobian surfaces.

To find a place in the above classification of elliptic surfaces with $q = p_g = o$ for the Enriques surfaces constructed in Chapter 1 we note that for such surfaces $P_2 = 1$. In virtue of the first corollary to the theorem in §1 we get the following relation for the multiplicities m_i of multiple fibres

$$\sum_{i=1}^{r} (1-\frac{1}{m_i}) = 1 .$$

This, of course, can occur only in the case

$$r = 2, m_1 = m_2 = 2 .$$

Applying the formula for the canonical class of elliptic surfaces we see that, on the contrary, for any minimal elliptic surface X with $q = p_g = 0$ and two multiple fibres of multiplicity 2 we have $2K_X = 0$. Notice also that the following result holds:

<u>Theorem</u> (Enriques). Any algebraic surface X with $q = p_g = 0$ and $2K_X = 0$ is an elliptic surface.

The proof is too long to reproduce here (see [1], Ch. 9, and also [6]).

There is also a theorem (again due to Enriques) which states that any surface with $q = p_g = 0$, $2K = 0$ has a sextic surface as its birational model. Again the proof is too long to be reproduced here (see [1], Ch. 9 and also [2]). The particular form of this sextic passing through the edges of a tethraedron corresponds to a particular Enriques surface.

2. Torsiòn.

In this section we shall prove that any finite abelian group can be realized as the torsion group of an elliptic surface with $q = p_g = 0$.

Lemma. Suppose that D is a torsion divisor on an elliptic minimal surface X with $q = 0$. Then D is linearly equivalent to a rational linear combination of fibres of some elliptic fibration on X.

Proof. Since $h^2(K_X + D) = h^0(-D) = 0$, by Riemann-Roch we get

$$h^0(K_X + D) \geq 1 + p_g(X) .$$

Let $D' \in |K_X + D|$. Since D' does not intersect a general fibre of any elliptic fibration (because K_X does not), it equals some linear combination of components of fibres. Moreover, D' does not intersect any component (because K_X does not). Applying the main lemma from §1 we get that D' is a rational linear combination of fibres. Thus, $D' - K_X \sim D$ is also a rational linear combination of fibres.

Theorem. Let $f : X \to B$ be a minimal elliptic fibration with $q(X) = 0$. Let $F_{b_i} = m_i F^0_{b_i}$, $i = 1, \ldots, r$ be all its multiple fibres. Then

$$\mathrm{Tors}(\mathrm{Pic}(X)) = \mathrm{Ker}\Big(\bigoplus_{i=1}^{r} Z/m_i \xrightarrow{\psi} Z/m \Big) ,$$

where $m = m_1 \ldots m_r$, $\psi(a_1, \ldots, a_r) = \sum a_i \bar{m}_i \bmod m$, $\bar{m}_i = m/m_i$.

Proof. Using the lemma we may write any torsion divisor D in the form

$$D \sim \sum_{i=1}^{r} a_i F^0_{b_i} - lF ,$$

where $0 \leqq a_i < m_i$, F any non-multiple fibre.

Intersecting the both sides with some transversal curve C we obtain

$$0 = (D \cdot C) = \sum_{i=1}^{r} a_i (F \cdot C)/m_i - l(F \cdot C) ,$$

and hence

$$l = \sum_{i=1}^{r} a_i/m_i .$$

This shows that l is uniquely determined by a_i and, moreover,

$$(*) \qquad \sum_{i=1}^{r} a_i \bar{m}_i = lm \equiv 0 \mod m .$$

Now we know (see the proof of the theorem in §1) that the divisor $\sum a_i F^0_{b_i}$ is in the fixed part of any linear system containing it. Hence the coefficients a_i are determined uniquely by the divisor class of D . This shows that the map

$$\alpha : \mathrm{Tors}(\mathrm{Pic}(X)) \longrightarrow \mathrm{Ker}(\oplus Z/m_i \xrightarrow{\psi} Z/m)$$

$$D \longmapsto (a_1, \ldots, a_r)$$

is injective.

Now for any $(a_1,\dots,a_r)$ satisfying condition (*) the divisor

$$D = \sum_{i=1}^{r} a_i F_{b_i}^{0} - 1F$$

has zero intersection with any transversal curve and any component of fibres. This shows that D is numerically equivalent to zero, and, hence, D is a torsion divisor. This proves the surjectivity of α .

Corollary 1. In notations above

$$\# \operatorname{Tors}(\operatorname{Pic}(X)) = \text{g.c.d.}(\bar{m}_1 \dots, \bar{m}_r)$$

Corollary 2. For any finite abelian group G there exists an elliptic surface with $q = p_g = 0$ such that

$$\operatorname{Tors}(\operatorname{Pic}(X)) = G .$$

Proof. Applying Proposition 4 we may find such an elliptic surface with multiple fibres of any prescribed multiplicities.

Let

$$G = \bigoplus_{i=1}^{s} (Z/p_i^{n_1^{(i)}} \oplus \dots \oplus Z/p_i^{n_{k(i)}^{(i)}}) , \quad n_1^{(i)} \le \dots \le n_{k(i)}^{(i)}$$

be the primary decomposition of G . Consider a surface X with the following collection of multiplicities:

$$(p_1^{n_1^{(1)}}, \dots, p_1^{n_{k(1)}^{(1)}}, p_1^{n_{k(1)}^{(1)}}; \dots; p_s^{n_1^{(s)}}, \dots, p_s^{n_{k(s)}^{(s)}}, p_s^{n_{k(s)}^{(s)}}) .$$

Then applying the theorem we easily see that

$$\mathrm{Tor}(\mathrm{Pic}(X)) = G .$$

<u>Corollary 3.</u> ([14]) . There exists an elliptic surface with $q = p_g = 0$ which is not a rational surface and has no torsion divisors.

Just take a surface with multiple fibres of coprime multiplicities and apply Corollary 1 and Corollary 2 to the theorem of §1 .

§3. Fundamental group.

Here following to Kodaira [28] and Iithaka [25] we shall compute the fundamental group of an elliptic surface, over the field of complex numbers.

Let $f : X \to B$ be an elliptic fibration.

Lemma 1. Let $U \subset B$ be an open set such that the restriction $f_U : X_U \to U$ of f over U has no multiple fibres. Choose a point $p_0 \in X_U$ lying in a non-singular fibre. Then the following exact sequence holds

$$\pi_1(X_{f(p_0)}, p_0) \longrightarrow \pi_1(X_U, p_0) \longrightarrow \pi_1(U, f(p_0)) \longrightarrow 1$$

Proof. Consider the inclusion map $X_{f(p_0)} \hookrightarrow X_U$ and the projection map $f_U : X_U \to U$ and the correspondent homomorphisms of fundamental groups. Then the image of $\pi_1(X_{f(p_0)}, p_0)$ is clearly contained in the in the kernel of the second homomorphism, and we have to show that it coincides with the kernel and the second homomorphism is surjective. Restricted over sufficiently small U the map f is a differentiable 2-torus fibre bundle, and the corresponding sequence is the exact homotopy sequence. This obviously proves the surjectivity of the second homomorphism.

Let γ be a loop with the origin at p_0. Let X_{u_0} be a singular fibre of f_U, there exists a local section $D_{u_0} \to X_U$, D_{u_0} being a small disc centered at $f(p_0)$ (since X_{u_0} is not a multiple fibre):

Assuming that γ goes to zero under f_* it allows to deform γ to a loop on $X_{f(p_0)}$ keeping the point $p_0 \in \gamma$ fixed. This proves the lemma.

Next, let $D_1,\dots,D_r$ be some open discs around the points $b_1,\dots,b_r$ for which the fibre X_{b_i} is multiple of multiplicity m_i. Assume that over the punctured discs of D_i^* the morphism f is smooth. Let $U = B - D_1 - \dots - D_r$, $X_U = f^{-1}(U)$, $V_i = f^{-1}(D_i)$, $V_i^* = f^{-1}(D_i^*)$.

We shall apply van Kampen's theorem to compute $\pi_1(X)$.

Let

δ, σ be some loops on $X_{f(p_0)}$ originated at p_0 which generates $\pi_1(X_{f(p_0)}; p_0)$

$t_1,\dots,t_2$ be the loops on B starting at $f(p_0)$ and going around the points $b_1,\dots,b_r$;

$a_1,\dots,a_g$; $b_1,\dots,b_g$ another loop originated at $f(p_0)$ which together with t_i generate $\pi_1(U; f(p_0))$;

Denote by t_i', a_i', b_i' some loops on X_U lying over t_i, a_i, b_i with the origin at p_0. Then assuming that a_i, b_i are chosen as the canonical generators of $\pi_1(U; f(p_0))$, we get the following.

<u>Lemma 2</u>. The group $\pi_1(X_U; p_0)$ is generated by $\delta, \sigma, t_1',\dots,t_r'$, $a_1',\dots a_g', b_1',\dots,b_g'$ (g = genus of B) with the following basic relations:

(i) $\delta\sigma = \sigma\delta$;

(ii) the group $\{\delta, \sigma\}$ generated by δ, σ is normal in $\pi_1(X_U ; p_0)$;

(iii) $a_1' b_1' a_1'^{-1} b_1'^{-1} \ldots a_g' b_g' a_g'^{-1} b_g'^{-1} t_1' \ldots t_r' \in \{\delta, \sigma\}$;

(iv) some relation between δ and σ (may be trivial) .

This follows immediately from Lemma 1 and the known structure of $\pi_1(U ; f(p_0))$.

Choose some points p_i, $i = 1, \ldots, r$ lying over D_i^* and some loops δ_i^*, σ_i^* in the fibre $X_{f(p_i)}$ generating $\pi_1(X_{f(p_i)}; p_i)$. Let $\bar{t}_i$ be a loop going around b_i and passing through $f(p_i)$, $\bar{t}_i'$ some loop on V_i^* lying over $\bar{t}_i$ which passes through p_i .

Lemma 3. The group $\pi_1(V_i^*, p_i)$ is generated by $\delta_i^*, \sigma_i^*, \bar{t}_i'$ with the following basic relations:

(i) $\delta_i^* \sigma_i^* = \sigma^* \delta^*$;

(ii) δ_i^* and σ_i^* generate a normal subgroup in $\pi_1(V_i^* ; p_i)$;

(iii) $\bar{t}_i \delta_i^* = \delta_i^* \bar{t}_i$;

(iv) $\bar{t}_i \sigma_i^* = \delta_i^{*h_i} \sigma_i^* \bar{t}_i$, if X_{b_i} is of type $m_i^1 h_i$.

Proof. Applying Lemma 1 we will prove the first assertion and find the first two relations. To obtain another pair of relations we will use the following description of V_i^* which is due to Kodaira [27] . There exists an unramified covering $F \to V_i^*$ whose covering transformation group is a cyclic group of order m_i . The space F is represented in the form

$$F = D_i^* \times \mathbb{C}/\Gamma \qquad , \text{ if } h_i = 0 ,$$

or

$$F = D_i^* \times \mathbb{C}^*/\Gamma \qquad , \text{ if } h_i > 0 ,$$

where in the first case Γ is the discontinuous group of analytic automorphisms

$$(z,\zeta) \longrightarrow (z,\zeta + n_1 j(z^{m_i}) + n_2) \ , \ n_1, n_2 \in \mathbb{Z}$$

$(j(z^{m_i})$ is a holomorphic function of z^{m_i} with $\operatorname{Im} j(z^{m_i}) > 0$) .
In the second case Γ is the infinite cyclic group of analytic automorphisms of $D_i^* \times \mathbb{C}^*$ generated by the automorphism

$$(z, w) \longmapsto (z, wz^{m_i h_i}) .$$

Identifying the universal covering space of D_i^* with the upper half plane $H = \{\tau \mid \operatorname{Im}(\tau) > 0 \}$ and the covering map with the exponential map $\tau \to \exp(2\pi i\tau)$, we get that in the both cases the universal covering space of V_i^* is equal to $H \times \mathbb{C}$ and the covering transformation group $\overline{\Gamma}$ may be described as follows:
If $h_i = 0$, then $\overline{\Gamma}$ consists of analytic automoprhisms

$$(\tau, \zeta) \longmapsto (\tau + \frac{n_1}{m} , \zeta + n_3 j(\exp(2\pi i m_i \tau) + n_2) \ , \ n_1, n_1, n_3 \in \mathbb{Z} .$$

If $h_i > 0$, then $\overline{\Gamma}$ consists of analytic automorphisms

$$(\tau, \zeta) \longrightarrow (\tau + \frac{n_1}{m}, \zeta + n_2 + n_3 m_i h_i (\tau + \frac{n_1}{m})) .$$

Identifying in the usual way the loops originated at p_i with covering transformations, we may assume that

t_i corresponds to the element of $\overline{\Gamma}$ with $(n_1, n_2, n_3) = (1, 0, 0)$,

δ_i corresponds to the element of $\overline{\Gamma}$ with $(n_1, n_2, n_3) = (0, 1, 0)$,

σ_i corresponds to the element of $\overline{\Gamma}$ with $(n_1, n_2, n_3) = (0, 0, 1)$.

The relations (iii) and (iv) are verified now immediately.

To use van Kampen's theorem we consider homomorphisms

$$\pi_1(V_i^*, p_i) \to \pi_1(X_U, p_0) ,$$

which correspond to the natural inclusions $V_i^* \hookrightarrow X$ and to a choice of some paths connecting the points p_i and p_0 , and also the natural surjections

$$r_i : \pi_1(V_i^* ; p_i) \to \pi_1(V_i ; p_i) .$$

Applying the same arguments as in the proof of Proposition 2 from §1 we may assume that the cyclic covering F of V_i^* can be prolonged to an elliptic fibration over D_i , the cycle covering of D_i of degree m , with fibre of type ${}_1 1_{h_i}$ over the origin. This easily implies that

$$r_i(\overline{t}_i)^{m_i} \in \{p(\delta_i), p(\sigma_i)\} .$$

Moreover, if $h_i > 0$ we get that

$$p(\delta_i) = 0 .$$

Collecting everything together we obtain:

Theorem. The fundamental group $\pi_1(X)$ is generated by letters

$$\delta, \sigma, a_1, \dots, a_g, b_1, \dots, b_g, t_1, \dots, t_r .$$

The basic relations are

i) $\delta \cdot \sigma = \sigma \cdot \delta$,

ii) $\{\delta, \sigma\}$ is a normal subgroup ,

iii) $t_i^{m_i} \in \{\delta, \sigma\}$; $i = 1, \dots, r$,

iv) $a_1 b_1 a_1^{-1} b_1^{-1} \dots a_g b_g a_g^{-1} b_g^{-1} t_1 \dots t_r \in \{\delta, \sigma\}$.

v) some relation between δ and σ (may be trivial) .

Corollary. Let $f : X \to \mathbb{P}^1$ be an elliptic fibration. Then $\pi_1(X)$ is abelian if and only if it has at most 2 multiple fibres.

In fact, $\pi_1(X)$ has as its quotient the group $G(m_1, \dots, m_r)$ given by generators $t_1, \dots, t_r$ and relations

$$t_1^{m_1} = \dots = t_r^{m_r} = t_1 \dots t_r = 1 .$$

These groups are well known in the theory of automorphic functions. Namely, there exist natural representations of these groups as a discrete subgroup of the automorphism group of one of the three standard planes: the Riemannian $\mathbb{P}^1(\mathbb{C})$, the Euclidean $\mathbb{C}$, and the Lobachevsky $H = \{z \in \mathbb{C} \mid \mathrm{Im}(z) > 0\}$. We have (see [30]) that each of these cases corresponds to the sign of the number

$$e = \sum_{i=1}^{r} \frac{1}{m_i} - r + 2 .$$

We have the case

$\mathbb{P}^1(\mathbb{C})$	iff $e > 0$ and iff $G(m_1,\dots,m_r)$	is finite;
$\mathbb{C}$	iff $e = 0$ and iff $G(m_1,\dots,m_r)$	is non-commutative nilpotent;
H	iff $e < 0$ and iff $G(m_1,\dots,m_r)$	is infinite non-nilpotent .

Thus, $\pi_1(X)$ can be abelian only in the case $e > 0$. In this case, $G(m_1, m_2, m_3)$ is a finite subgroup of $SL(2,\mathbb{C})/\{\pm 1\}$, that is the rotation group of some regular polyhedron $(r = 3)$ or a cyclic group $(r = 2)$. This, of course, proves the corollary.

Corollary. ([14]) . Let X be an elliptic surface with $q = p_g = 0$, which admits an elliptic fibration $f : X \to \mathbb{P}^1$ with exactly two multiple fibres of coprime multiplicity. Then X is a simply connected non-rational surface.

In fact, its fundamental group being abelian has to coincide with the homology group $H_1(X,\mathbb{Z})$. Since $q(X) = 0$, $H_1(X,\mathbb{Z}) = \mathrm{Tors}(H_1(X,\mathbb{Z}))$. It remains to apply Corollary 1 of §2 .

Remark. In [14] the argument that $\pi_1(X)$ is abelian was not correct. So, in fact, it was proven there only that there exist non-rational surfaces with $q = p_g = 0$ with no torsion divisors. This was the original question of F. Severi.

CHAPTER III. SURFACES OF GENERAL TYPE

§1. Some useful lemmas.

Lemma 1. Let X be a scheme and T is a finite subgroup of the Picard group $\mathrm{Pic}(X)$. Then there exists a finite etale Galois covering $f : X_T \to X$ uniquely determined by the properties $T = \mathrm{Ker}(\mathrm{Pic}(X) \xrightarrow{f^*} \mathrm{Pic}(X_T))$ and the Galois group of f is isomorphic to the character group $\mathrm{Char}(T)$.

Proof. For any $\varepsilon \in T$ let $O_X(\varepsilon)$ be the corresponding invertible sheaf. The locally free sheaf $L = \bigoplus_{\varepsilon\in T} O_X(\varepsilon)$ has a natural structure of an O_X-Algebra corresponding to the isomorphisms $O_X(\varepsilon) \otimes O_X(\varepsilon') \to O_X(\varepsilon+\varepsilon')$. Put $X_T = \mathrm{Spec}(L)$. Then the projection $f : X_T \to X$ is finite and flat. It is also etale, since $\det(L) = \prod_{\varepsilon\in T} O_X(\varepsilon) = O_X$. The group $G = \mathrm{Char}(T)$ acts naturally on X_T multiplying each summand $O_X(\varepsilon)$ by $\chi(\varepsilon)$, $\chi \in G$. Clearly, the invariant subalgebra $L^G = O_X$, hence $X_T/G = X$ and f is a Galois covering. Assume that $\mathbb{L} \in \mathrm{Ker}(\mathrm{Pic}(X) \xrightarrow{f^*} \mathrm{Pic}(X_T))$. Then $f^*(\mathbb{L}) = O_{X_T}$ and $f_* f^*(\mathbb{L}) = f_*(O_{X_T}) = \bigoplus_{\varepsilon\in T} \mathbb{L} \otimes O_X(\varepsilon) = f_*(O_{X_T}) = \bigoplus_{\varepsilon\in T} O_X(\varepsilon)$. This implies that $\mathbb{L} \otimes O_X(\varepsilon) = O_X(\varepsilon')$ for some $\varepsilon' \in T$ and hence $\mathbb{L} = O_X(\varepsilon - \varepsilon') \in T$. The inclusion $T \subset \mathrm{Ker}$ is obvious.

To prove the uniqueness note that for any finite Galois covering $f : X' \to X$ with the Galois group G we have

$$\mathrm{Char}(G) \cong \mathrm{Ker}(\mathrm{Pic}(X) \xrightarrow{f^*} \mathrm{Pic}(X')) .$$

This immediately follows from the Hochshild-Serre spectral sequence or from direct considerations.

Now $f_*(O_{X'})$ must split into eigen subsheaves corresponding to characters of G

$$f_*(O_{X'}) = \bigoplus_{\chi\in \mathrm{Char}(G)} f_*(O_{X'})_\chi$$

Let L_χ be the invertible sheaf corresponding to a character in virtue of the above identification of Char(G) with the subgroup of Pic(X) . Then L_χ being lifted onto X' is trivial, thus it is embedded into $f_*(O_{X'})$ and is isomorphic to one of its summands (namely, $f_*(O_{X'})_\chi$) . This shows that $X' = \mathrm{Spec}(f_*(O_{X'}))$ is isomorphic to X_T constructed above.

<u>Corollary</u>. In the above notations

$$H^i(X_T, O_{X_T}) = \bigoplus_{\varepsilon\in T} H^i(X, O_X(\varepsilon))$$

More generally, for any locally free sheaf L on X we have

$$H^i(X_T, f^*(L)) = \bigoplus_{\varepsilon\in T} H^i(X, L\otimes O_X(\varepsilon)) .$$

<u>Proof</u>. We have

$$f_*(O_{X_T}) = \bigoplus_{\varepsilon\in T} O_X(\varepsilon) ,$$

hence for any locally free L on X

$$f_*(f^*(L)) = f_*(O_{X_T})\otimes L = \bigoplus_{\varepsilon\in T} L\otimes O_X(\varepsilon) .$$

It remains to apply the Leray spectral sequence which degenerates because f is finite.

Lemma 2. (Bombieri [4]). Let F be a surface of general type with $q(F) = 0$, $m =$ Tors(F) the order of the torsion group. Then

$$p_g(F) \leq \frac{1}{2} K_F^2 + \frac{3}{m} - 1$$

and

$$p_g(F) \leq \frac{1}{2} K_F^2 - 1$$

if there exist a finite abelian unramified covering of F of irregularity at least one.

Proof. Let $f : \overline{F} \to F$ be the covering corresponding to the torsion group Tors(F) in virtue of Lemma 1. By the lemma of §2, Chapter 1 we know that

$$K_{\overline{F}}^2 = m\, K_F^2 \quad ,$$

$$1 + p_a(\overline{F}) = m\,(1 + p_a(F)) \quad .$$

Now apply the following classic Noether theorem (see [4], Th. 9):

$$p_g(\overline{F}) \leq \frac{1}{2} K_{\overline{F}}^2 + 2$$

and consider separately the two possible cases:

a) $q(\overline{F}) > 0$: Then $Pic(\overline{F})$ contains a finite subgroup of any order n . Let $\overline{F}(n) \to \overline{F}$ be the corresponding etale covering.

We have

$$p_g(\bar{F}(n)) = n\ (1 + p_a(\bar{F})) + q(\bar{F}(n)) - 1 \leq \frac{1}{2} n\ K_{\bar{F}}^2 + 2 ,$$

dividing by n and letting $n \to \infty$ we get

$$1 + p_a(\bar{F}) \leq \frac{1}{2} K_{\bar{F}}^2 .$$

Now dividing by m we obtain

$$1 + p_a(F) = 1 + p_g(F) \leq \frac{1}{2} K_F^2 .$$

b) $q(\bar{F}) = 0$: Then

$$m(1 + p_g(F)) = 1 + p_g(\bar{F}) \leq \frac{m}{2} K_F^2 + 3$$

and it suffices to divide both sides by m .

<u>Lemma 3</u>. Let F be a surface of general type. Then

$$q(F) \leq p_g(F) .$$

<u>Proof</u>. By Noether's formula

$$1 - q(F) + p_g(F) = \frac{1}{12}(K_F^2 + c_2(F)) .$$

Since $K_F^2 > 0$ and $c_2(F) > 0$ (otherwise, F would be ruled, [4], Th. 13) we get the inequality.

Lemma 4. Let F be a surface of general type and D be a divisor numerically equivalent to mK_F , $m \geq 1$. Then

$$H^1(F, O_F(D + K_F)) = 0 .$$

Proof. This immediately follows from the following Ramanujam's form of Kodaira's Vanishing theorem (C. Ramanujam, J. Indian Math. Soc., 38 (1974), 121-124) : Let X be a complete non-singular surface, L and invertible sheaf on X such that $(c_1(L)^2) > 0$ and $(c_1(L) \cdot C) \geq 0$ for any curve on X . Then $H^i(X, L^{-1}) = 0$ for $i = 0, 1$.

Corollary. The m-th plurigenus P_m of a surface of general type F is given by

$$P_m = \frac{1}{2} m(m - 1) K_F^2 + 1 + p_a(F) ,$$

in particular

$$P_2 = p^{(1)}(F) + p_a(F) .$$

Use Reimann-Roch and Lemma 4 applied to $D = (m - 1)K_F$.

Lemma 5. Let $f : X \to Y$ be a double covering of non-singular surfaces branched along a reduced curve $W \subset Y$. Then

(i) $f_*(O_X) = O_Y \oplus L$, $L^{\otimes 2} \simeq O_Y(-W)$

(ii) $\omega_X = f^*(\omega_Y \otimes L^{-1})$.

<u>Proof</u>. The subsheaf O_Y is naturally identified with the subsheaf of $f_*(O_X)$ invariant under sheet-interchange. Since the characteristic is assumed to be zero (or at least prime to 2), this sheaf is a direct summand of $f_*(O_X)$, the complement being a sheaf L of anti-invariant sections. The sheaf $L^{\otimes 2}$ is obviously a subsheaf of the invariant subsheaf, that is O_Y , thus $L^{\otimes 2} \simeq J$ for some Ideal sheaf $J \subset O_Y$. This shows that X is isomorphic to the subscheme of the vector bundle $\mathbb{V}(L) = \mathrm{Spec}(\bigoplus_{n=0}^{\infty} L^{\otimes n})$ defined by the ideal $(L^{\otimes 2} - J)$. Now, the local arguments of the proof of Lemma 2, Ch. 1, §3 show that $J = O_Y(-W)$ and $\omega_X = f^*(\omega_Y \otimes L^{-1})$.

<u>Corollary</u>. Let F be an invertible sheaf on Y.. Then

$$H^0(X, f^*F) \simeq H^0(Y, F) \oplus H^0(Y, F \otimes L) .$$

In particular

$$H^0(X,\omega_X^{\otimes n}) \simeq H^0(Y, \omega_Y^{\otimes n} \otimes L^{-n}) \oplus H^0(Y,\omega_Y^{\otimes n} \otimes L^{-n+1}) .$$

§2. Numerical Godeaux surfaces.

By this we mean any surface of general type F with

$$p_g(F) = 0 \quad \text{and} \quad p^{(1)}(F) = 2 \ .$$

In virtue of Lemma 3 and corollary to Lemma 4 of §1 we get moreover that

$$q(F) = 0 \quad \text{and} \quad P_m(F) = \frac{1}{2} m(m-1) + 1 \ .$$

We will distinguish these surfaces by the value of its torsion group $\mathrm{Tors}(F)$. First of all, by Lemma 2 of 1 we have the following.

Proposition 1. If $m = \mathrm{Tors}(F)$ then

$$m \leq 6 \ .$$

For any abelian unramified covering $F' \to F$ we have

$$q(F') = 0 \ .$$

Proposition 2. (Bombieri). There are no numerical Godeaux surfaces with $\mathrm{Tors}(F) = 6$.

Proof. Assume that $\mathrm{Tors}(F) = \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/3\mathbb{Z}$. Then there exists an unramified covering $F' \to F$ of order 2 with $\mathrm{Tors}(F') = \mathbb{Z}/3\mathbb{Z}$.

By the lemma of Chapter 1, §2 we have

$$p^{(1)}(F') = 3 \quad \text{and} \quad -q(F') + p_g(F') = 1 .$$

By proposition 1 $q(F') = 0$ and hence we obtain a surface with $p^{(1)} = 3$, $p_g = 1$, $q = 0$ and the torsion group $\mathbb{Z}/3\mathbb{Z}$. However this contradicts Theorem 15 of [4].

Remark. Since the previous proof is a simple application of Theorem 15 of [4], which in its turn is proved using other non-trivial results of [4], it is better to give an independent proof. As suggested by Miles Reid we can argue as follows.

Let Y be the covering of X corresponding to the group of torsion of order 6. Then $p_g(Y) = 5$, $K_Y^2 = 6 = 2p_g(Y) - 4$. Now we will use

Lemma (E. Horikawa). Let Y be a surface of general type with $(K_Y^2) = 2p_g(Y) - 4$. There $|K_Y|$ is an irreducible linear system whose general member is a hyperelliptic curve.

Proof. Suppose that

$$|K_Y| = |C| + F$$

where F is a fixed part. Assuem that $|C|$ is composed of a pencil, say $C \sim a[C_0]$, where $a > 1$ and $[C_0]$ is an irreducible pencil. Then $p_g(Y) \leq a + 1$ and the equality holds if $[C_0]$ is linear (i.e. $\dim H^0(Y, O(C_0)) = 2$). We have $K_Y \cdot F \geq 0$, therefore $K_Y^2 \geq aK_Y \cdot F$ and since $C_0^2 \geq 0$ we get $K_Y \cdot C_0 \geq 2$, because $K_Y^2 \geq 2$.

Hence

$$K_Y^2 \geq 2a \geq 2p_g - 2$$

and we have a contradiction. Thus we may assume that $|C|$ is not composed of a pencil.

Now the analysis of the proof of Noether's inequality $p_g(Y) \leq \frac{1}{2} K_Y^2 + 2$ (see [4], p. 209) shows that in the case of the equality $|K_Y|$ is an irreducible non-singular curve C of genus $g = (K_Y^2) + 1$.

Now the exact sequence

$$0 \longrightarrow O_Y \longrightarrow O_Y(K_Y) \longrightarrow O_C(K_Y \cdot C) \longrightarrow 0$$

shows that $\dim H^0(C, O_C(K_Y \cdot C)) = p_g(Y) - 1$. Let D denotes the restriction of $|K_Y|$ on C . Then $2D \sim K_C$ and $2 \dim H^0(C, O_C(D)) = 2p_g(Y) - 2 = K_Y^2 + 2 = \deg D + 2$. Now by a classical Clifford's theorem on special divisors it follows that C is hyperelliptic (see, for example, H. Martens. J. Reine Angen. Math. 233, (1968), 89–100).

After we have proven the lemma the argument is very simple. If σ is an automorphism of the covering $Y \to X$ then σ acts freely on Y and hence on a general member C of $|K_Y|$. But this is obviously impossible (any automorphism of a hyperelliptic curve has a fixed point)..

Lemma. (Reid [39]). Let F be a minimal numerical Godeaux surface. Then

(i) For any non-zero $g \in \mathrm{Tors}(F)$ there exists a unique positive divisor $D_g \in |K_F + g|$;

(ii) if $g \neq g'$ then D_g and $D_{g'}$ have no common components;

(iii) if g, g' and g'' are distinct non-zero elements of $\mathrm{Tors}(F)$ then $D_g, D_{g'}$ and $D_{g''}$ do not meet.

Proof. (i) By Riemann-Roch

$$h^0(K_F + g) = 1 + h^1(K_F + g) - h^2(K_F + g) .$$

By Serre's duality, $h^2(K_F + g) = h^0(-g) = 0$, since $g \neq 0$. By the same reason, $h^1(K_F + g) = h^1(-g) = 0$ in virtue of the corollary to Lemma 1, §1 and Proposition 1 .

(ii) If one of D_g or $D_{g'}$ is irreducible the result is obvious. Suppose that

$$D = D_g = C + \sum n_i C_i , \quad D' = D_{g'} = C' + \sum n_i' C_i'$$

is the decomposition into irreducible components with C and C' chosen so that $(D \cdot C) = (K_F \cdot C') = 1$ (recall that $(D \cdot K_F) = (D' \cdot K_F) = K_F^2 = 1)$.

If $C = C'$ then $D = D'$, because there are no relations between fundamental curves (that is, curves with no intersection with K_F) other than equality ([4], Prop. 1) .

Let. E be the common part of D and D', then $E^2 < 0$ and even, since it is a positive combination of fundamental curves. Thus $(D - E)^2 = D^2 - 2(D \cdot E) + E^2 = 1 + E^2 \leq -1$. But

$$(D - E)^2 = (D - E)(D' - E)$$

must be non-negative, since $D - E$ and $D' - E$ have no common components.

(iii) Since $K_F^2 = 1$ each two D_g and $D_{g'}$, $g \neq g'$ meet transversally at a non-singular point for both curves. The fact that three distinct D_g, $D_{g'}$ and $D_{g''}$ meet at a point is equivalent to the fact that $O_F(D_g - D_{g'})$ being restricted on $D_{g''}$ is isomorphic to the structure sheaf of $D_{g''}$. Write the exact sequence

$$0 \to O_F(D_g - D_{g'} - D_{g''}) \to O_F(D_g - D_{g'}) \to O_{D_{g''}} \to 0$$

and the corresponding cohomology sequence

$$H^0(F,O_F(D_g - D_{g'})) \to H^0(D_{g''},O_{D_{g''}}) \to H^1(F,O_F(D_g - D_{g'} - D_{g''})) \quad .$$

Since $D_g - D_{g'}$ is a non-zero torsion divisor, the first term is zero. By duality, the third term is equal to $h^1(\varepsilon)$ for some torsion divisor. That is also zero (see the proof of (i)) . This contradicts the non-triviality of the middle term.

Proposition 3. (Bombieri-Catanese, Reid). There are no numerical Godeaux surfaces with $\mathrm{Tors}(F) = \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$.

Proof. Let F be such a surface. Then we have the three distinct non-zero torsion divisors of order 2. Let D, D' and D'' be the three divisors constructed in Reid's lemma. Then the divisors $2D$, $2D'$ and $2D''$ belong to the linear system $|2K_F|$ and by the property (iii) they cannot be members of a pencil. Thus, $\dim |2K_F| \geq 2$. However, we know that $P_2(F) = \dim |2K_F| + 1 = 2$. This contradiction proves the assertion.

Remark. The proof of Bombieri-Catanese [5] uses other more elaborate arguments. The proof from [32] is not complete. Thus, we have the following possible cases:

$$\mathrm{Tors}(F) = \{0\} \ ; \ \mathbb{Z}/2\mathbb{Z} \ , \ \mathbb{Z}/3\mathbb{Z} \ , \ \mathbb{Z}/4\mathbb{Z} \ \text{and} \ \mathbb{Z}/5\mathbb{Z} \ .$$

We know examples of surfaces with $\mathbb{Z}/5\mathbb{Z}$ (the Godeaux surfaces of §2, Chapter I). Let us show that these are essentially all examples of such surfaces. The proof below is due to Miles Reid [39].

Let $\bar{F}$ be the unramified covering of order 5 corresponding to the torsion group $\mathrm{Tors}(F)$. Then by the corollary to Lemma 1 of §1 we have

$$H^0(\bar{F}, O_{\bar{F}}(mK_{\bar{F}})) = \bigoplus_{g \in \mathrm{Tors}} H^0(F, O_F(mK_F + g)) \ .$$

We know form Reid's lemma (i) that $h^0(K + g) = 1$, $g \neq 0$. Let x_1, x_2, x_3, x_4 be non-zero elements corresponding to the four non-zero elements of $\mathrm{Tors}(F)$. We may consider them as elements of $H^0(\bar{F}, O_{\bar{F}}(K_{\bar{F}}))$ generating this space. Since by Reid's lemma the x_i's have no common zero on F, therefore on $\bar{F}$ they define a morphism $f : F \to \mathbb{P}^3$.

Since $K_{\overline{F}}^2 = 5$ and the degree of f must divide 5 we get that f is birational onto a surface F' of degree 5 . This quintic F' must be a normal surface, since the arithmetic genus of its hyperplane sections coincides with the genus of its inverse images (=canonical divisors) on $\overline{F}$. Thus F' coincides with the canonical model of $\overline{F}$ and as such has only double rational points as singularities.

The group $G = \mathrm{Char}(\mathrm{Tors}(F)) = \mathbb{Z}/5\mathbb{Z}$ acting on $\overline{F}$ acts by functoriality on the canonical model $F' = \mathrm{Proj}(\bigoplus_{m=0}^{\infty} H^0(\overline{F}, O_{\overline{F}}(mK_{\overline{F}}))$ multiplying x_i by some ζ^i (ζ a 5-th root of unity). Thus F is "almost" the quotient of a quintic by $\mathbb{Z}/5\mathbb{Z}$. More exactly, the canonical model of F is isomorphic to such quotient.

We refer to [11] and [32] for the study of pluricanonical maps of numerical Godeaux surfaces. Also in [32] it can be found the facts concerning the moduli space of surfaces with $\mathrm{Tors} = \mathbb{Z}/5\mathbb{Z}$.

<u>Surfaces with $\mathrm{Tors}(F) = \mathbb{Z}/4\mathbb{Z}$</u> (Reid-Miyaoke).

To construct such surfaces we will pull ourselves by shoe-strings. Assume that such surface F exists. As for the Godeaux surfaces we consider the elements $x_i \in H^0(F, O_F(K_F + g_i))$, where g_1 , $g_2 = g_1^2$, $g_3 = g_1^3$ are non-zero elements of $\mathrm{Tors}(F)$. Then x_1x_3 and x_2^2 form a basis for $H^0(F, O_F(2K_F + g_2))$ (their linear independence follows from Reid's lemma) . Let y_1 and y_3 be sections of $H^0(F, O_F(2K_F + g_1))$ and $H^0(F, O_F(2K_F + g_3))$ respectively such that (x_2x_3, y_1) and (x_1x_2, y_3) form bases.

<u>Proposition.</u> (Reid). The above elements x_i, y_i generate the pluricanonical ring $A(\overline{F}) = \bigoplus_{m=0}^{\infty} H^0(\overline{F}, O_{\overline{F}}(mK_{\overline{F}})) = \bigoplus_{\substack{m=0 \\ g \in \mathrm{Tors}}}^{\infty} H^0(F, O_F(mK_F + g))$

of the surface $\overline{F}$ which is the unramified covering of F corresponding to the torsion group $\mathrm{Tors}(F)$. There are two basic relations of degree 8 between these generators.

Proof. The monomials

$$x_1^4, x_2^4, x_3^4, x_1^2x_3^2, x_1x_2^2x_3, y_1y_3, y_1x_1x_2, y_3x_3x_2 \in H^0(F, O_F(4K_F)) .$$

However, by the corollary to Lemma 4, §1 we find that

$$h^0(4K_F) = 7 .$$

Thus there is a linear dependence between these 8 monomials, which we will write

$$f_0(x_1, x_2, x_3, y_1, y_3) = 0 .$$

In the same way the 8 monomials ,

$$x_1^2x_2^2, x_3^2x_2^2, x_1^3x_3, x_1x_3^3, y_1^2, y_3^2, x_1x_2y_3, x_3x_2y_1 \in H^0(F, O_F(4K_F + g_2))$$

and $h^0(4K_F + g_2) = 7$. Hence we have the second relation

$$f_1(x_1, x_2, x_3, y_1, y_3) = 0 .$$

Both these relations of degree 4 considering x_i, y_i as elements of the graded canonical ring $A(\overline{F}) = \bigoplus_{m=0}^{\infty} H^0(\overline{F}, O_{\overline{F}}(mK_{\overline{F}}))$.

Next, let

$$B = \mathbb{C}[X_1, X_2, X_3, Y_1, Y_3]/(f_0, f_1)$$

be the quotient polynomial ring. Grade B by the condition $\deg(X_i) = 1$, $\deg(Y_i) = 2$, then we have the morphism of graded algebras

$$\psi : B \to A(\overline{F}) \ , \quad X_i \mapsto x_i \ , \ Y_i \mapsto y_i \quad .$$

The proposition is equivalent to the assertion that ψ is an isomorphism.

Now, the Poincaré function (compare [15])

$$P_B(t) = \sum \dim B_i \ t^i = \frac{(1-t^4)^2}{(1-t)^3(1-t^2)^2} = \frac{(1+t^2)^2}{(1-t)^3} =$$

$$= \sum \left(\frac{(1+2)(i+1)}{2} + i(i-1) + \frac{(i-2)(i-3)}{2}\right) t^i =$$

$$= \sum (2i(i-1)+4)t^i \quad .$$

In virtue of the formula for $P_i(\overline{F})$ this coincides with

$$P_{A(\overline{F})}(t) = \sum P_i(\overline{F}) \ t^i \quad .$$

Thus, it suffices to check that ψ is injective.

If ψ is not injective then the image of the rational map

$$\psi : M = \mathrm{Proj}(A(\overline{F})) \longrightarrow V = \mathrm{Proj}(B)$$

will be a proper closed subscheme of V.

Let j be the embedding $V \hookrightarrow \mathbb{P}^7$ corresponding to the surjection $\mathbb{C}[B_2] \to B^{(2)} = \bigoplus_{i=0}^{\infty} B_{2i}$, $a : \overline{F} \to M$ the canonical map of $\overline{F}$ onto its canonical model M. The composition

$$\overline{F} \longrightarrow M \longrightarrow V \longrightarrow \mathbb{P}^7$$

is easily to be seen coincides with the 2-canonical map

$$\Phi_{2K_{\overline{F}}} : \overline{F} \to \mathbb{P}^7 \quad .$$

In virtue of Reid's lemma $\Phi_{K_{\overline{F}}}$ is regular (see the analogous argument in the previous case of the Godeaux surfaces), thus $\Phi_{2K_{\overline{F}}}$ is also regular. This shows that $\tilde{\psi}$ is in fact a morphism. Let $\overline{V} = \Phi_{2K_{\overline{F}}}(\overline{F})$. By our assumption, $\overline{V}$ is a proper closed subscheme of V. Since $\overline{V}$ spans $\mathbb{P}^7$ its degree is at least 6. Since $(2K_{\overline{F}})^2 = 16$ and $|K_{\overline{F}}|$ has no fixed part it implies that $\overline{V}$ is a surface and $\deg \overline{V} = 8$ or 16. Moreover, in the first case, $\Phi_{2K_{\overline{F}}}$ defines a 2-sheet covering

$$g : \overline{F} \longrightarrow \overline{V} ,$$

and in the second case g is a birational morphism. Since $\deg j(V) = 16$

(this follows from the equality of the Poincare functions for $A(\overline{F})$ and B) we get that in the second case $\overline{V} = V$. So, we may assume that $\varphi_{2K_{\overline{F}}}$ is a 2-sheeted covering onto its image $\overline{V}$. Let $C \in |K_{\overline{F}}|$ be a non-singular curve, the map $g|_C$ equals the canonical map of C and since it is 2-sheeted C must be a hyperelliptic curve and $g|_C$ its hyperelliptic involution. Now, notice that the canonical map $\varphi_{K_{\overline{F}}}$ also factors through $\tilde{\psi}$ and hence through g . Then $K_{\overline{F}}$ cuts out on C a g_4^1 which is composed with hyperelliptic g_2^1 . This implies that $K_{\overline{F}}|_C$ is not a complete linear system. But the latter contradicts the vanishing of $H^1(\overline{F}, O_{\overline{F}})$.

Corollary. Let F be a numerical Godeaux surface with $\mathrm{Tors}(F) = \mathbb{Z}/4\mathbb{Z}$, $\overline{F}$ its unramified covering corresponding to the torsion group. Then the canonical model $\overline{M}$ of $\overline{F}$ is isomorphic to a weighted complete intersection $V_{4,4}(1,1,1,2,2)$. The action of the group $\mathrm{Char}(\mathbb{Z}/4\mathbb{Z}) = \mu_4$ on $\overline{M}$ is induced by the action of this group on the weighted projective space $\mathbb{P}(1,1,1,2,2)$ which multiplies the first three coordinates by ζ, ζ^2, ζ^3 accordingly and the fourth and the fifth coordinate by ζ, ζ^3 accordingly (ζ a primitive 4-th root of 1). The canonical model M of F is obtained by dividing $\overline{M}$ by this action.

This corollary prompts to us the way to construct F . For this one may take a non-singular $F = V_{4,4}(1,1,1,2,2)$ invariant under the above action on $\mathbb{P}(1,1,1,2,2)$ and not containing the fixed point of this action. Using the general properties of weighted complete intersection (which are quite analogous to the ones of usual non-singular complete intersections) we find (see, for example, [15]):

$$q(\overline{F}) = 0 \ , \quad \omega_{\overline{F}} = \mathcal{O}_{\overline{F}}(4+4-1-1-1-2-2) = \mathcal{O}_{\overline{F}}(1) \ ,$$

$$p_g(\overline{F}) = \dim H^0(\overline{F}, \mathcal{O}_{\overline{F}}(1)) = 3, \ P_2(\overline{F}) = \dim H^0(F, \mathcal{O}_{\overline{F}}(2)) = 8 \ ,$$

$$p^{(1)}(\overline{F}) = P_2(\overline{F}) - P_a(\overline{F}) = 5 \ .$$

Dividing F by the free action of μ_4 we get the surface F with

$$q(F) = p_a(F) = p_g(F) = 0 \ , \quad p^{(1)} = 2 \ .$$

Notice also that we have $\pi_1(\overline{F}) = 0$ and thus

$$\pi_1(F) = \mathrm{Tors}(F) = \mathbb{Z}/4\mathbb{Z} \ .$$

An explicit example of $V_{4,4}(1,1,1,2,2)$ with the properties above:

$$x_0^4 + x_1^4 + x_2^4 + x_3x_4 = 0$$

$$x_0^2x_1^2 + x_2^2x_1^2 + x_3^2 + x_4^2 = 0 \ .$$

For a more general example see [32] .

Surfaces with $\mathrm{Tors}(F) = \mathbb{Z}/3\mathbb{Z}$.

Here the same method of Miles Reid shows that the covering $\overline{F}$ of such surface F is embedable into the weighted projective space $\mathbb{P}(1,1,2,2,2,3,3)$, unfortunately, not as a complete intersection. There are not any explicit constructions of $\overline{F}$ (the example in [39] does not work) and, thus, the question of the existence of such surfaces F is still open* .

* see Epilogue.

Surfaces with $\mathrm{Tors}(F) = \mathbb{Z}/2\mathbb{Z}$ (Campedelli-Kulikov-Oort).

The main idea here belongs to Campedelli, who proposed to construct a surface with $p^{(1)} = 2$ as a double plane branched along a 10th order curve with 5 triple points of type $x^3 + y^6 = 0$ and an ordinary 4-ple point. Unfortunately, his construction of such a curve is false (see below). Victor Kulikov (non-published) proposed to modify the Campedelli curve, taking the union of two conics and two cubics such that one of the cubics has a double point, both conics pass through this point and touch both the cubics at other points. Oort gave an explicit construction of this configuration ([35]): Let $W = C_1 \cup C_2 \cup D_1 \cup D_2$, where

$$C_1 : y^2 + (x-t)(2x-2y-3t) = 0$$

$$C_2 : y^2 + (x-t)(2x+2y-3t) = 0$$

$$D_1 : y^2 t + x(x-t)(x-3t) = 0$$

$$D_2 : [(y^2 t + x(x-t)(x-3t))(2t-x)+(x^2-3xt+3t^2)^2]/t = 0$$

It is easily checked that

$$C_1 \cap D_1 = 2P_1 + 2P_2 + 2P_5 ,$$

$$C_2 \cap D_1 = 2P_3 + 2P_4 + 2P_5 ,$$

$$C_1 \cap D_2 = 2P_1 + 2P_2 + 2P_6 ,$$

$$C_2 \cap D_2 = 2P_3 + 2P_4 + 2P_6 ,$$

$$C_1 \cap C_2 = 3P_5 + P_6 ,$$

$$D_1 \cap D_2 = 2P_1 + 2P_2 + 2P_3 + 2P_4 + P_7 ,$$

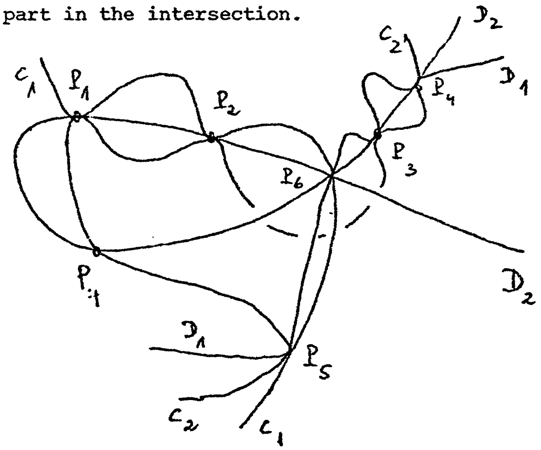
where

$$P_1 = (x,y,t) = (\frac{3+\sqrt{-3}}{2}, \frac{3+\sqrt{-3}}{2}, 1); \; P_2 = (\frac{3-\sqrt{-3}}{2}, \frac{3-\sqrt{-3}}{2}, 1);$$

$$P_3 = (\frac{3+\sqrt{-3}}{2}, \frac{3+\sqrt{-3}}{2}, 1), \quad P_4 = (\frac{3-\sqrt{-3}}{2}, -\frac{3-\sqrt{-3}}{2}, 1),$$

$$P_5 = (1,0,1), \; P_6 = (\frac{3}{2},0,1), \; P_7 = (0,1,0),$$

the point P_6 is an ordinary double point of D_2, and the combination of the points above is considered as a divisor on any non-singular curve taking part in the intersection.



Let F be the minimal non-singular model of the double plane branched along the curve W .

<u>Assertion 1:</u>

$$p_g(F) = 0, \quad p^{(1)}(F) = 2.$$

Proof. This is similar to the proof used at the construction of the classical Campedelli surface from Chapter 1, §3 .

Let $p : X \to \mathbb{P}^2$ be the minimal resolution of singular points of the branch curve W . Then the strict inverse transform of W

$$p^{-1}(W) \sim 10p^*(L) - 3\sum_{i=1}^{5} S_i - 6\sum_{i=1}^{5} S_i' - 8S_5'' - 4S_6 - 2S_7 ,$$

where L is a line on $\mathbb{P}^2$,

$$p^{-1}(P_i) = S_i + S_i' , \ i = 1,\ldots, 4; \ p^{-1}(P_5) = S_5 + S_5' + S_5'' ;$$

$$p^{-1}(P_i) = S_i , \ i = 6, 7 ;$$

with $S_i^2 = -2$, $1 \leq i \leq 5$; $S_i^2 = -1$, $i = 6, 7$; $S_i'^2 = -2$, $S_5''^2 = -1$.

Let $r : F' \to X$ be the double covering of X branched along the divisor $p^{-1}(W) + \sum_{i=1}^{5} S_i$, then

$$\begin{aligned} K_{F'} &\sim r^*(K_X) + \tfrac{1}{2}r^*(p^{-1}(W) + \sum_{i=1}^{5} S_i) \\ &\sim r^*(p^*(-3L) + \sum_{i=1}^{7} S_i + 2\sum_{i=1}^{5} S_i' + 3S_5'') + \\ &+ \tfrac{1}{2}r^*(10p^*(L) - 2\sum_{i=1}^{5} S_i - 6\sum_{i=1}^{5} S_i' - 8S_5'' - 4S_6 - 2S_7) \\ &\sim r^*(p^*(2L) - \sum_{i=1}^{5} S_i' - S_5'' - S_6) . \end{aligned}$$

Assume that $D \in |K_{F'}|$, then we see from above and corollary to Lemma 5, §1 that $D = r^*(D')$, where $D' \in |2p^*(L) - \sum_{i=1}^{5} S_i' - S_5'' - S_6|$ and hence equals the proper inverse image under p of a conic passing through the points $P_1,\dots,P_6$. However, obviously these points are not situated on a conic. This shows that $|K_{F'}| = \emptyset$ and thus

$$p_g(F) = 0 .$$

Since r is branched along S_i, i=1,...,5 and $p^{-1}(C_i)$, i=1,2, we see that

$$r^*(S_i) = 2\overline{S}_i) \ , \ r^*(p^{-1}(C_i)) = 2\overline{C}_i'$$

for some curves S_i and $\overline{C}_i'$ on F' . Also, we have

$$\overline{S}_i^2 = \frac{1}{4}(r^*(S_i)^2) = \frac{1}{4}(2S_i^2) = \frac{1}{4}(-4) = -1$$

$$\overline{C}_i'^2 = \frac{1}{4}(r^*(p^{-1}(C_i))^2) = \frac{1}{4}(2(p^{-1}(C_i))^2) = \frac{1}{4}(-8) = -2 .$$

This shows that $\overline{S}_i$ are exceptional curves of the 1st kind. Let $\sigma : F' \to F$ be the blowing down of all $\overline{S}_i$. We will show that F is the minimal model of F' . We have

$$\begin{aligned} 2K_F &\sim \sigma_*(r^*(4p^*(L) - 2\sum_{i=1}^{5} S_i' - 2S_5'' - 2S_6)) \\ &\sim \sigma_*(r^*(p^{-1})(C_1)) + r^*(p^{-1}(C_2)) + 2r^*(S_5') + 4r^*(S_5'')) \\ &\sim \sigma_*(2\overline{C}_1' + 2\overline{C}_2' + 2r^*(S_5') + 4r^*(S_5'')) \end{aligned}$$

and hence

$$2K_F \sim 2\overline{C}_1 + 2\overline{C}_2 + 2\overline{S}_5' + 4\overline{S}_5'' ,$$

where we put

$$\sigma_*(\overline{C}_1' = \overline{C}_i, \ \ \sigma_*(r^*(S_5')) = \overline{S}_5', \ \sigma_*(r^*(S_5'')) = \overline{S}_5'' .$$

Assuming that E is an exceptional curve of the 1st kind on F, we get that $(E \cdot 2K_F) = -2$ and hence E coincides with one of the curves $\bar{C}_i$, $\bar{S}'_5$ or $\bar{S}''_5$. However, we saw above that $\bar{C}_i^2 = \bar{C}'^2_i = -2$ and also $\bar{S}'^2_5 = r^*(S'_5)^2 + 1 = 2\, S'^2_5 + 1 = -4 + 1 = -3$, $\bar{S}''^2_5 = r^*(S''_5)^2 = -2$.

Now

$$p^{(1)}(F) = K_F^2 + 1 = \frac{1}{4}(2K_F)^2 + 1 = \frac{1}{4}(-8-8-12-32+32+16+16\,) + 1 = 2$$

and the assertion is proven.

<u>Assertion 2.</u>

$$\mathrm{Tors}(F) = \mathbb{Z}/2\mathbb{Z} \quad .$$

<u>Proof.</u> In the proof of Assertion 1 we have found already a torsion divisor of order 2, this is

$$K_F - \bar{C}_1 - \bar{C}_2 - \bar{S}'_5 - 2\bar{S}''_5 \quad .$$

In virtue of the analysis of the torsion of numerical Godeaux surfaces we know that $\mathrm{Tors}(F) = \mathbb{Z}/2\mathbb{Z}$ or $\mathbb{Z}/4\mathbb{Z}$. Let us exclude the second possiblilty.

Assume that g is a torsion divisor of order 4. Consider the involution δ of F corresponding to its rational projection onto $\mathbb{P}^2$. If $\delta^*(g) \sim g$, then $2g \sim 0$, since there are no torsion divisors on $\mathbb{P}^2$. Thus, $\delta^*(g) \sim -g$, because δ defines an automorphism of the torsion group $\mathbb{Z}/4\mathbb{Z}$. Let D_g be the unique curve from $|K_F + g|$. Then

$$D_g + \delta^*(D_g) = D_g + D_{-g} \in |2K_F| \quad .$$

The bicanonical system $|2K_F|$ is a pencil generated by the two curves

$$2\bar{C}_1 + 2\bar{C}_2 + 2\bar{S}'_5 + 4\bar{S}''_5$$

and

$$\bar{D}_2 + \bar{H} + \bar{S}''_5 + 2\sigma_*(2^*(S_7)) \sim \sigma_*(2^*(2p^*(L) - 2\sum_{i=1}^{4} S'_i - 2S_6 - S_7)) +$$
$$+ \sigma_*(2^*(p^*(L) - 2S'_5 - 3S''_5 - S_7)) +$$
$$+ \bar{S}''_5 + 2\sigma_*(2^*(S_7)) \sim \sigma_*(2^*(4p^*(L) -$$
$$- 2\sum_{i=1}^{5} S'_i - 2S''_5 - 2S_6)) \sim 2K_F \quad .$$

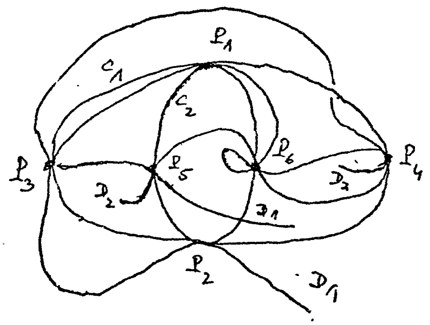
We see that $|2K_F|$ has the fixed component, namely $\bar{S}''_5$, which has to be contained in both D_g and D_{-g}. However, by Reid's lemma the curves D_g and D_{-g} has no common components. This contradiction proves the assertion.

Remark. Campedelli proposed to construct the branch curve W as the union of 3 conics C_1, C_2, C_3 and a quartic D such that C_1 and C_2 are bitangent to C_3, touch each other at a point, D has a node at one of the two ordinary intersection points of C_1 and C_2, passes through the five contact points of the conics with the same tangent direction (see [9]) .

The arguments similar to the one used above show that the bicanonical system of the corresponding double plane is equal to the inverse image of the pencil of quartics on $\mathbb{P}^2$ touching D at the points of contact with $C_1 \cup C_2 \cup C_3$ and having a node at the node of D. Considering the two curves from this pencil $C_1 + C_2$ and D we will find two torsion divisors of order 2. This contradicts Proposition 3. Thus the Campedelli construction does not exist.

Surfaces with Tors(F) = 0 .

There are no examples of such surfaces. Maybe it is worth to consider a version of the example above with the branch curve W equal to the union of two conics and two cubics forming the following configuration (Kulikov):



where C_1 and C_2 are conics, and D_1, D_2-cubics.

Arguing as above we would show that the bicanonical system is equal to the inverse image of the pencil of quartics passing through $P_1, \ldots, P_5$ with the same tangent direction as W and having a node at P_6 . It is seen that there are no members of this pencil composed of components of W . This easily proves that there are no torsion elements of order 2 .

Of course, the existence of this configuration is not easy to justify.

3. Numerical Campedelli surfaces.

These are surfaces with $p_g = 0$ and $p^{(1)} = 3$. They are distinguished by the order m of its torsion gorup. It waś proved by Beauville [3] and Reid that $m \leq 10$. Here we exhibit examples of numerical Campedelli surfaces with $m = 2, 4, 7$ and 8 . There are no examples of such surfaces with other possible value of m*, moreover there are no examples of numerical Campedelli surfaces with $\mathrm{Tors}(F) = \mathbb{Z}/4\mathbb{Z}$.

a) Classical Campedelli surfaces. For them we already know (Chapter 1, §3) that $\mathrm{Tors}(F) \supset (\mathbb{Z}/2\mathbb{Z})^3$. We will prove now that we have the equality.

Proposition (Miyaoke [32], Reid [39]). Let $r : \bar{F} \to F$ be the unramified covering of the classical Campedelli surface corresponding to the subgroup $T = (\mathbb{Z}/2\mathbb{Z})^3$ of the torsion group $\mathrm{Tors}(F)$. Then the canonical system $K_{\bar{F}}$ defines the birational morphism of $\bar{F}$ onto the intersection of 4 quadrics in $\mathbb{P}^6$.

Proof. We know (Chpater III, §1) that

$$H^0(\bar{F}, O_{\bar{F}}(nK_{\bar{F}})) = \bigoplus_{g \in T} H^0(F, O_F(nK_F + g)) .$$

Let us show that

$$h^0(K_F+g) = \dim H^0(F, O_F(K_F+g)) = 1 \text{ , for all non-zero } g \in T .$$

Since $h^0(2K_F) = 3$, we get that $h^0(K_F+g) \leq 2$. If we have the equality, then $|2K_F|$ is composed of the pencil $+g|$.

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* see Epilogue.

Considering the restriction of $|K_F + g|$ onto $\overline{S}_1$, we see that this pencil has a base point on $\overline{S}_1$. This shows that $|2K_F|$ has also this point as its base point. However, the curves $2\overline{C}_3 + 2\overline{C}_2 + 2\overline{S}_3 + 2\overline{S}_4$ and $2\overline{C}_1 + 2\overline{C}_3 + 2\overline{S}_5 + 2\overline{S}_6$ from $|2K_F|$ intersect $\overline{S}_1$ at two distinct points. This contradiction proves the needed assertion.

Denote the lements of T by 000, 100, 110, 010, 001, 011, 101, and 111. Let

$$x_0 \in H^0(K_F + 100),\ x_1 \in H^0(K_F + 010),\ x_2 \in H^0(K_F + 001)\ ,$$
$$x_3 \in H^0(K_F + 011)\ \ x_4 \in H^0(K_F + 101),\ x_5 \in H^0(K_F + 110)\ ,$$
$$x_6 \in H^0(K_F + 111)$$

be non-zero sections.

Clearly, $r^*(x_i) = y_i$, $i=0,\ldots,6$, generate $H^0(\overline{F},O_{\overline{F}}(K_{\overline{F}}))$. All squares x_i^2 belong to $H^0(F,O_F(2K_F))$ and, since $h^0(2K_F) = 3$, there must be 4 relations among them. This shows that there are 4 relations between y_i^2 in $H^0(\overline{F},O_{\overline{F}}(K_{\overline{F}}))^2$. Now we can find explicitly these relations. We know that the bicanonical system $|2K_F|$ is represented by the net of quartics

$$\lambda_1 C_1 C_2 + \lambda_2 C_1 C_3 + \lambda_3 C_2 C_3$$

(in notation of Ch. I, §3). Up to a permutation we easily find that

x_1^2 corresponds to C_1C_2

x_2^2 " " C_2C_3

x_3^2 " " C_1C_3

$$
\begin{array}{lll}
x_4^2 & \text{corresponds to} & C_1\ell_2^2 \\
x_5^2 & \text{"} \quad\quad \text{"} & C_2\ell_2^2 \\
x_6^2 & \text{"} \quad\quad \text{"} & C_3\ell_3^2 \\
x_7^2 & \text{"} \quad\quad \text{"} & D
\end{array}
$$

where ℓ_1 (resp. ℓ_2, resp. ℓ_3) is the line through the points P_5 and P_6 (resp. P_3 and P_4, resp. P_1 and P_2).

This gives the following relations among y_i

$$y_1^2 = ay_3^2 + by_4^2$$

$$y_2^2 = cy_3^2 + dy_6^2$$

$$y_3^2 = cy_1^2 + ey_5^2$$

$$y_7^2 = fy_1^2 + gy_2^2 + hy_3^2$$

for some non-zero constants $a, b, \ldots, g, h$.

Thus we obtain that the canonical image $\Phi_{\bar{K}}(\bar{F})$ is contained in the complete intersection V of the four quadrics given above. It is easily checked that V has only isolated singular points (in fact 24 double ordinary points) and hence being a complete intersection is an irreducible surface. This implies that $\Phi_{\bar{K}}(\bar{F}) = V$ if only $\dim \Phi_{\bar{K}}(\bar{F}) = 2$. Assume that $\Phi_{\bar{K}}(\bar{F})$ is a curve. Then its normalization X is isomorphic to the projective line $\mathbb{P}^1$ (since $g(\bar{F}) = 0$ in view of the corollary to Lemma 1, Ch. III, §1 and the remark above concluding that $h^0(K_F + \varepsilon) = 1$ for any $\varepsilon \in T$). Clearly the group $T = (Z/2)^3$ acts faithfully on $\mathbb{P}^7 = \mathbb{P}(H^0(\bar{F}, O_{\bar{F}}(K_{\bar{F}}))$ and hence on the image $\Phi_{\bar{K}}(\bar{F})$. this shows that T is isomorphic to a subgroup of $\mathrm{Aut}(\mathbb{P}^1)$, but this is impossible.

Thus we obtain that

$$V = \operatorname{Proj}\left(\bigoplus_{m=0}^{\infty} H^0(\overline{F}, O_{\overline{F}}(K_{\overline{F}}))^m\right) = \cdot(\overline{F})$$

is a complete intersection of four quadrics.

Remark. Computing the Poincare function of the canonical ring $A(\overline{F}) = \bigoplus_{m=0}^{\infty} H^0(\overline{F}, O_{\overline{F}}(mK_{\overline{F}}))$ we see that it coincides with the Poincare function of its subring $\bigoplus_{m=0}^{\infty} H^0(\overline{F}, O_{\overline{F}}(K_{\overline{F}}))^m$. This shows that these rings are isomorphic and V is the canonical model of $\overline{F}$. In particular V has exactly 24 double ordinary points corresponding to the inverse images of the three (-2)-curves on F : $\overline{C}_1$, $\overline{C}_2$ and $\overline{C}_3$. Also we get that the canonical model of F is the quotient of V by the group $(\mathbb{Z}/2)^3$. In this way it is easily to get the moduli space of the classical Campedelli surfaces. It is a unirational variety of dimension 6 (look at the coefficients of the four equations of V above). See the details in [32].

Corollary. Let F be a classical Campedelli surface. Then

$$\operatorname{Tors}(F) = \pi_1(F) = (\mathbb{Z}/2\mathbb{Z})^3 .$$

In fact, the surface F obtained as the unramified covering of F corresponding to the subgroup $(\mathbb{Z}/2\mathbb{Z})^3 \subset \operatorname{Tors}(F)$ is simply-connected (because it is isomoprhic to a minimal resolution of double rational points of a complete intersection).

b) Godeaux' surfaces. These surfaces were constructed by Godeaux as the quotients of suitable intersections of four quadrics in $\mathbb{P}^6$ by cyclic group of order 8 acting freely ([20]).

Consider four quadrics given by the equations:

$$a_1x_0x_6 + a_2x_1x_5 + a_3x_2x_4 + a_4x_3^2 = 0$$
$$b_1x_0^2 + b_2x_4^2 + b_3x_2x_6 + b_4x_3x_5 = 0$$
$$c_1x_1^2 + c_2x_5^2 + c_3x_0x_2 + c_4x_4x_6 = 0$$
$$d_1x_2^2 + d_2x_6^2 + d_3x_0x_4 + d_4x_1x_3 = 0$$

where a generator of $G = \mathbb{Z}/8\mathbb{Z}$ acts on the intersection X of these quadrics by the formulas:

$$(x_0,x_1,x_2,x_3,x_4,x_5x_6) \longrightarrow (x_0,\zeta x_1,\zeta^2x_2,\zeta^3x_3,\zeta^4x_4,\zeta^5x_5,\zeta^6x_6)$$

where $\zeta = \exp(2\ i/8)$.

The same argument as in the case of classical Godeaux surfaces shows that the quotient X/G is a numerical Campedelli surface with

$$\mathrm{Tors(Pic}(X/G) = \pi_1(X/G) = \mathbb{Z}/8\mathbb{Z} .$$

c) Godeaux-Reid surfaces. These are also quotients of the intersection of four quadrics by other groups of order 8 ([39]). First, consider the group $G = (\mathbb{Z}/2\mathbb{Z})^3$. Define the action of G on $\mathbb{P}^6$ by the formulas:

$$g_1 : (x_0,x_1,x_2,x_3,x_4,x_5,x_6) \longrightarrow (-x_0,x_1,x_2,x_3,-x_4,-x_5,-x_6)$$
$$g_2 : (x_0,x_1,x_2,x_3,x_4,x_5,x_6) \longrightarrow (x_0,-x_1,x_2,-x_3,x_4,-x_5,-x_6)$$
$$g_3 : (x_0,x_1,x_2,x_3,x_4,x_5,x_6) \longrightarrow (x_0,x_1,-x_2,-x_3,-x_4,x_5,-x_6)$$

It is clear that for any fixed point (i.e. a point with non-trivial isotropy subgroup) at least three of its coordinates must be zero. This shows that G acts freely on the surface given by the equations

$$\sum a_i x_i^2 = \sum b_i x_i^2 = \sum c_i x_i^2 = \sum d_i x_i^2 = 0 \quad ,$$

where all minors of maximal order of the matrix

$$\begin{pmatrix} a_0 & \cdots & a_6 \\ b_0 & \cdots & b_6 \\ c_0 & \cdots & c_6 \\ d_0 & \cdots & d_6 \end{pmatrix}$$

are non-zero.

Second, consider the group $G = \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/4\mathbb{Z}$. Let $g_1 = (1,0)$, $g_2 = (0,1)$ be its generators. Define the action of G on $\mathbb{P}^6$ by the formulas ($\zeta = e^{\pi i/2}$) :

$$g_1 : (x_0,x_1,x_2,x_3,x_4,x_5,x_6) \longrightarrow (-x_0,x_1,x_2,x_3,-x_4,-x_5,-x_6)$$

$$g_2 : (x_0,x_1,x_2,x_3,x_4,x_5,x_6) \longrightarrow (x_0, x_1,-x_2, \zeta^3 x_3,\zeta x_4,-x_5,\zeta^3 x_6)$$

Now notice that any fixed point is fixed either under g_1 or under g_2^2 . Thus, the set of the fixed point in $\mathbb{P}^6$ with respect to the action of G is the set

$$F = \{x_1=x_2=x_3=0\} \cup \{x_0=x_4=x_5=x_6=0\} \cup \{x_1=x_3=x_4=x_6=0\} \cup \{x_0=x_2=x_5=0\} \ .$$

This shows that the surface X given by the equations

$$a_0x_0^2 + a_1x_2^2 + a_2x_5^2 + a_3x_1x_3 + a_4x_4x_6 = 0$$

$$b_0x_0^2 + b_1x_2^2 + b_2x_5^2 + b_3x_1x_3 + b_4x_4x_6 = 0$$

$$c_0x_1^2 + c_1x_3^2 + c_2x_6^2 + c_3x_0x_5 + c_4x_4^2 = 0$$

$$d_0x_1^2 + d_1x_3^2 + d_2x_6^2 + d_3x_0x_5 + d_4x_4^2 = 0$$

is easily can be chosen not passing through F . Since it is obviously G-invariant we may consider the quotient X/G , which is a numerical Campedelli surface with

$$\pi_1(X/G) = \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/4\mathbb{Z} \quad .$$

The last example is more interesting [40] . Let $Q = \{\pm 1, \pm i, \pm j, \pm k\}$ be the quaternion group. Consider tis action on $\mathbb{P}^6$ by the formulas:

$$-1 : (x_0,x_1,x_2,x_3,x_4,x_5,x_6) \longrightarrow (x_0,x_1,x_2,-x_3,-x_4,-x_5,-x_6)$$

$$i : (x_0,x_1,x_2,x_3,x_4,x_5,x_6) \longrightarrow (-x_0,x_1,x_2,x_4,-x_3,x_6,-x_5)$$

$$j : (x_0,x_1,x_2,x_3,x_4,x_5,x_6) \longrightarrow (x_0,-x_1,x_2,x_5,-x_6,-x_3,x_4)$$

$$j : (s_0,x_1,x_2,x_3,x_4,x_5,x_6) \longrightarrow (-x_0,-x_1 + x_2,x_6,x_5,-x_4,-x_3)$$

Since $g^2 = -1$ for all $g \neq 1$, any fixed point is fixed by -1 . This shows that the set of fixed points

$$F = \{x_0=x_1=x_2=0\} \cup \{x_3=x_4=x_5=x_6=0\} \quad .$$

Now, the surface X given by the equations:

$$a_0x_0x_1 + a_1x_3x_4 + a_2x_5x_6 = 0$$

$$b_0x_1x_2 + b_1x_3x_5 + b_2x_4x_6 = 0$$

$$c_0x_0x_2 + c_1x_3x_6 + c_2x_4x_5 = 0$$

$$d_0x_0^2 + d_1x_1^2 + d_2x_2^2 + d_3(x_3^2+x_4^2+x_5^2+x_6^2) = 0$$

is G-invariant and obviously can be chosen to be non-singular and not passing through F. Taking the quotient $V = X/G$ we obtain a numerical Campedelli surface with

$$\pi_1(V) = Q_8, \quad \mathrm{Tors}(V) = \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} .$$

d) Surfaces with $\mathrm{Tors} = \mathbb{Z}/7\mathbb{Z}$. It is proven by Godeaux [21] and Reid [39] that if such surface F exists then the canonical model $\bar{F}$ of its covering corresponding to the torsion group is given by seven cubical equations in $\mathbb{P}^5$. More precisely, it is shown by Reid that the surface $X \subset P^5$ given by the equations

$$x_2^2x_0+x_4^2x_3+x_5^2x_1+x_0x_1x_3-x_0^2x_4-x_1^2x_2-x_3^2x_5-x_2x_4x_5 = 0$$

$$-x_4^3+x_5^2x_2+x_0^2x_5+x_0x_2x_3-x_1^2x_3-x_2^2x_1 = 0$$

$$-x_2^3+x_4^2x_5+x_2^2x_5-x_1x_5x_0-x_3^2x_0-x_5^2x_3 = 0$$

$$-x_0^3+x_3^2x_1+x_2^2x_3-x_2x_1x_4+x_5^2x_4-x_1^2x_5 = 0$$

$$-x_5^3+x_2^2x_4-x_3^2x_2+x_3x_4x_1-x_0^2x_1+x_4^2x_0 = 0$$

$$-x_3^3+x_1^2x_0-x_4^2x_1+x_0x_4x_5+x_2^2x_5+x_0^2x_2 = 0$$

$$-x_1^3+x_0^2x_3-x_5^2x_0-x_5x_3x_2-x_4^2x_2+x_3^2x_4 = 0$$

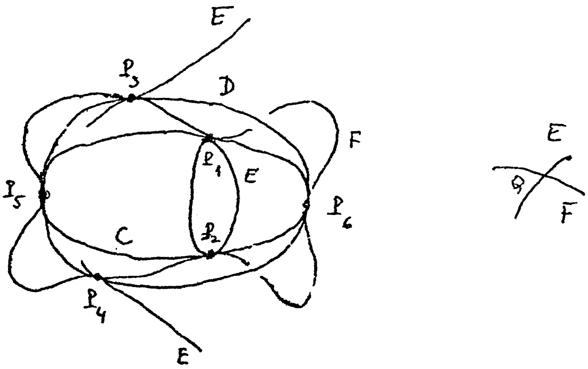
is a very good candidate to be such surface $\bar{F}$. It is certainly invariant with respect to the involution δ of $\mathbb{P}^5$

$$(x_0,x_1,x_2,x_3,x_4,x_5) \longrightarrow (x_0,\zeta^2x_1,\zeta^3x_2,\zeta^4x_3,\zeta^5x_4,\zeta^6x_5) ,$$

where $\zeta = \exp(2\pi i/7)$. Also, this involution acts freely on X.

It has the same Hilbert polynomial as $\bar{F}$. The only thing that has to be proven is that X is non-singular and canonically embeded.

e) Campedelli-Oort-Kulikov surfaces. The history here is the same as in the case of similar surfaces with $p^{(1)} = 2$. Kulikov proposed to modify the classical Campedelli surface replacing the branch curve W by another curve also of the 10th order. More precisely, the new W is constructed as the union $W = E \cup F \cup C \cup D$, where E and F are non-singular cubics, C and D are conics, which intersect each other according to the following picture:



Oort gave the explicit equations (in affine coordinates):

$$E : y^2 + x(x^2 + x + 2) = 0$$

$$F : (x + 1)^2(x - 3)^2 - (x + 3)(y^2 + x(x^2 + x + 2)) = 0$$

$$C : y^2 - x^2 + x = 0$$

$$D : y^2 + 7x^2 - 7x = 0$$

The same arguments as in the case of all other double planes considered above show that the bicanonical system of the surface equals the inverse image of the linear system of quartics passing through P_i with the same tangent direction as W. Also, in the same manner it can be shown that the minimal non-singular model of the corresponding double plane is a numerical Campedelli surface. The curves $C \cup D$, $C \cup 2L$, $D \cup 2L'$, where L (resp. L') is the line given by the equation $x + 1 = 0$ (resp. $x - 3 = 0$) determine the bicanonical divisors effectively divisible by 2. Thus, they define three torsion divisors of order 2, whose sum is, in fact, linearly equivalent to zero. This shows that

$$\mathrm{Tors}(F) \supset (\mathbb{Z}/2\mathbb{Z})^2 .$$

It is easy to see that there are no more torsion divisors of order 2. Applying Beauville's estimate of #Tors we get that

$$\mathrm{Tors}(F) = (\mathbb{Z}/2\mathbb{Z})^2 \quad \text{or} \quad (\mathbb{Z}/2) \oplus \mathbb{Z}/4\mathbb{Z} .$$

Unfortunately, I cannot see how to exclude the second possibility. But it is conjectured that it can be done.

Remark. We have two different constructions of surfaces with $\mathrm{Tors} = (\mathbb{Z}/2\mathbb{Z})^3$, these are the classicla Campedelli surfaces and the Godeaux-Reid surfaces. It is easy to see (using the proposition from this section) that the Godeaux-Reid surface is a deformation of the classical Campedelli surface (see the details in [36]).

4. Burniat's surfaces

These surfaces were constructed in [7,8] as certain (2,2)-covers of the projective plane. The linear genus $p^{(1)}$ takes value 3, 4, 5, 6, and 7 for them. Later this construction was reproduced in a modern way by C. Peters [37] . Here I give some other version of this construction which allows to compute the torsion group.*

First, we consider a minimal rational elliptic surface $V \to \mathbb{P}^1$ with two exceptional fibres $F_0 = 2E_0 + E_1 + E_2 + E_3 + E_4$ and $F_0' = 2E_0' + E_1' + E_2' + E_3' + E_4'$ of type 1_0^* (see Ch. II, §1). We also suppose that there exist 4 sections S_1, S_2, S_3, S_4 nonintersecting each other with the properties:

$$(S_i E_i) = (S_i E_i') = 1 ,$$

$$2S_i + E_i + E_i' \sim 2S_j + E_j + E_j' .$$

To construct such a surface V one may consider the ruled surface $\mathbb{F}_2$, that is a $\mathbb{P}^1$-bundle over $\mathbb{P}^1$ with a section s_0 for which $(s_0^2) = -2$, an elliptic pencil on it generated by the curves $2s_0 + \ell_1 + \ell_2 + \ell_3 + \ell_4$ and $2s$, s being any section nonintersecting s_0 and ℓ_i any four distinct fibres of $\mathbb{F}_2$. The minimal resolution of the base points of this pencil $s \cap \ell_i$ provides the needed elliptic surface V .

Next, let F_1 and F_2 be any two distinct non-singular fibres of V , consider the pencil P generated by the divisors $F_1+2S_3+E_3+E_3'$ and $F_2+2S_4+E_4+E_4'$. It is easily seen that P has 2 base points

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* See Epilogue

of multiplicity 2, namely, $Q_1 = F_1 \cap S_4$ $Q_2 = F_2 \cap S_3$. Moreover, F_1 (resp. F_2) touches non-singular curves of the pencil at Q_1 (resp. Q_2) .

Let D_1 and D_2 be two curves of $\mathcal{P}$ without common components. Consider the following five possible cases (it will be shown later that all of them can be realized):

A) D_i are both non-singular;

B) $D_1 = E_1 + D_1'$, where D_1' is non-singular, D_2 as in A);

C) D_1 as in B), $D_2 = E_1' + D_2'$, where D_2' is non-singular;

D) $D_1 = E_1 + E_2' + D_1'$, where D_1' is non-singular, D_2 as in C);

E) D_1 as in D), $D_2 = E_2 + E_1' + D_2'$, where D_2' is non-singular.

The following properties are easily checked:

$$(D_i^2) = 4 \ , \quad (D_i K_V) = -(D_i F) = -2 \quad (F \text{ any fibre}) \ ,$$

D_1' touches D_2' at Q_1 and Q_2 , $(D_1' \cdot D_2') = 4$,

D_i does not meet any of E_j or E_j' ,

$(D_i' \cdot E) = 2$, where E denotes any other irreducible component of D_i .

The Burniat surfaces will be constructed as minimal non-singular models of the double covering of V branched along the curve W , where in each of the cases A)-E) the curve W is as follows:

A). $$W = D_1+D_2+F_1+F_2+\sum_{i=1}^{4} E_i + \sum_{i=1}^{4} E_i' \sim 6F+4S_1+2E_1+2E_1'-2E_0-2E_0' \ ,$$

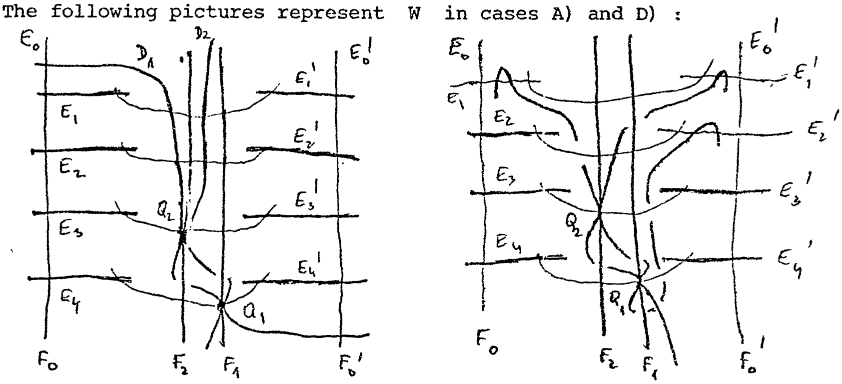
B) $$W = D_1'+D_2+F_1+F_2+\sum_{i=2}^{4} E_i + \sum_{i=1}^{4} E_i' \sim 6F+4S_1+2E_1'-2E_0-2E_0' \ ,$$

$$\text{C)}\quad W = D'_1+D'_2+F_1+F_2+\sum_{i=2}^{4} E_i+\sum_{i=2}^{4} E'_i \sim 6F+4S_1-2E_0-2E'_0-2E'_2 \ ,$$

$$\text{D)}\quad W = D'_1+D'_2+F_1+F_2+\sum_{i=3}^{4} E_i+\sum_{i=2}^{4} E'_i \sim 6F+4S_1-2E_0-2E'_0-2E_2-2E'_2 \ ,$$

$$\text{E)}\quad W= D'_1+D'_2+F_1+F_2+\sum_{i=3}^{4} E_i+\sum_{i=3}^{4} E'_i \sim 6F+4S_1-2E_0-2E'_0-2E_2-2E'_2 \ :$$

The following pictures represent W in cases A) and D) :



(the thick curves denote the components of W) .

To get a minimal non-singular model of this double covering we proceed as in the case of the classical Campedelli surfaces. Let $\sigma : V' \to V$ be the birational morphism which blows up the curves R_i and R'_i at the points Q_i $(i=1,2)$, where we assume that

$$(R_i^2) = -1 \ , \quad (R_i'^2) = -2 \ .$$

Then the divisor

$$p^{-1}(W) + R'_1 + R'_2 \sim p^*(W) - 2R'_1 - 2R'_2 - 6R_1 - 6R_2$$

is 2-divisible and non-singular. Thus we may form a double covering $r : X' \to B'$ branched along this divisor which will be a non-singular model of X .

To compute $K_{X'}$ we use the formula of Ch. I, §2:

$$K_{X'} = r^*(K_{V'}) + \frac{1}{2} r^*(p^{-1}(W) + R_1' + R_2') =$$

$$= r^*(p^*(-F) + R_1' + R_2' + 2R_1 + 2R_2) +$$

$$+ \frac{1}{2} r^*(p^*(W) - 2R_1' - 2R_2' - 6R_1 - 6R_2) \sim$$

$$\sim r^*(-p^*(F) + R_1' + R_2' + 2R_1 + 2R_2 + 3p^*(F) +$$

$$+ 2p^*(S_1) - p^*(E_0') + B - R_1' - R_2' - 3R_1 - 3R_2) \sim$$

$$\sim r^*(p^*(2F + 2S_1 - E_0 - E_0') + B - R_1 - R_2) ,$$

where

$$B = r^*(p^*(E_1 + E_1') , \quad \text{in case A)} ,$$

$$= r^*(p^*(E_1')) , \quad \text{in case B)} ,$$

$$= 0 , \quad \text{in case C)} ,$$

$$= -r^*(p^*(E_2')) , \quad \text{in case D)} ,$$

$$= -r^*(p^*(E_2 + E_2')) , \quad \text{in case E)} .$$

Now notice that $p^*(R_i')$ are exceptional curves of the 1st kind taken with multiplicity 2 . The same is true also for $r^*(p^*(E_i))$ or $r^*(p^*(E_1'))$ if r is branched along $p^*(E_i)$ or $p^*(E_i')$. Let $\sigma : X' \to X$ be the blowing down these exceptional curves. Put $\bar{D} = \sigma_*(r^*(p^*(D)))$ for any divisor D on V , and also $\bar{R}_i = \sigma_*(r^*(R_i))$. Then, we get

$$K_X \sim 2\bar{F} + 2\bar{S}_1 - \bar{E}_0 - \bar{E}_0' - \bar{R}_1 - \bar{R}_2 + B ,$$

where

$$\hat{B} = \begin{cases} 0 & , \text{ in cases A), B), C)} , \\ -\overline{E}_2' & , \text{ in case D)} , \\ -\overline{E}_2 - \overline{E}_2' & , \text{ in case E)} . \end{cases}$$

Since

$$\begin{aligned} \overline{F} \sim 2\overline{E}_0 \sim 2\overline{E}_0' & , \text{ in case A)} , \\ 2\overline{E}_0 + \overline{E}_1 \sim 2\overline{E}_0' & , \text{ in case B)} , \\ 2\overline{E} + \overline{E}_1 \sim 2\overline{E}_0' + \overline{E}_1' & , \text{ in case C)} , \\ 2\overline{E}_0 + \overline{E}_1 \sim 2\overline{E}_0' + \overline{E}_1' + \overline{E}_2' & , \text{ in case D)} , \\ 2\overline{E}_0 + \overline{E}_1 + \overline{E}_2 \sim 2\overline{E}_0' + \overline{E}_1' + \overline{E}_2' & , \text{ in case E)} , \end{aligned}$$

and

$$\overline{F}' \sim 2\hat{F}_0 + 2\overline{R}_1 \sim 2\hat{F}_0' + 2\overline{R}_2 ,$$

where $\overline{F}_0 = 2\hat{F}_0$, $\overline{F}_0' = 2\hat{F}_0'$, we get

$$\begin{aligned} 2K_X & \sim 4\overline{F} + 4\overline{S}_1 - 2\overline{E}_0 - 2\overline{E}_0' - 2\overline{R}_1 - 2\overline{R}_2 + 2\hat{B} \\ & \sim 2\hat{F}_0 + 2\hat{F}_0' + 4\overline{S}_1 , \text{ in case A)} , \\ & \sim 2\hat{F}_0 + 2\hat{F}_0' + 4\overline{S}_1 + \overline{E}_1 , \text{ in case B)} , \\ & \sim 2\hat{F}_0 + 2\hat{F}_0' + 4\overline{S}_1 + \overline{E}_1 + \overline{E}_1' , \text{ in case C)} , \\ & \sim 2\hat{F}_0 + 2\hat{F}_0' + 4\overline{S}_1 + \overline{E}_1 + \overline{E}_1' - \overline{E}_2' \sim 2\hat{F}_0 + 2\hat{F}_0' + 2\overline{S}_1 + 2\overline{S}_2 + \overline{E}_2, \text{ in case D)} , \\ & \sim 2\hat{F}_0 + 2\hat{F}_0' + 4\overline{S}_1 + \overline{E}_1 + \overline{E}_1' - \overline{E}_2 - \overline{E}_2' \sim 2\hat{F}_0 + 2\hat{F}_0' + 2\overline{S}_1 + 2\overline{S}_2 , \text{ in case E)} . \end{aligned}$$

This implies

a) $K_X^2 = \frac{1}{4}((2K_X)^2) = 6$, in case A) ,

$= 5$, in case B) ,

$= 4$, in case C) ,

$= 3$, in case D) ,

$= 2$, in case E) .

b) X is non-rational (since $2K_X$ is positive) .

c) X is a minimal model (since for any exceptional curve of the 1st kind C $(2K_X C) < 0$ and this implies that C is one of the curves $\hat{F}_0$, $\hat{F}'_0$, $\bar{E}_i$ or $\bar{E}'_i$, but it is easily checked that neither of them is an exceptional curve of the 1st kind).

It remains to show that

$$p_g(X) = 0 .$$

For simplicity we will prove it only in the case A) . In other cases the proof is similar.

Suppose that $|K_X| \neq \emptyset$. Then taking its inverse transform on X' we get

$$r^*(p^*(2F-E_0-E'_0+ \sum E_i+ \sum E'_i+2S_1) \geq r^*(R_1) + r^*(R_2) \quad .$$

This implies that

$$p^*(2F-E_0-E'_0+ \sum E_i+ \sum E'_i+2S_1) \geq R_1 + R_2 .$$

This means that there exists a positive divisor

$$D \in |2F - E_0 - E_0' + \sum E_i + \sum E_i' + 2S_1| = |E_0 + E_0' + \sum E_i + \sum E_i' + 2S_1|$$

which passes through the points Q_1 and Q_2.

Now notice that

$$|D| \supset |E_1 + E_1' + 2S_1| + |E_0 + E_0' + \sum_{i=2}^{4} E_i + \sum_{i=2}^{4} E_i'|$$

moreover, $D^2 = 0$, and $(D \cdot K_V) = -2$. If $\dim |D| > 1$ then for the moving part $|D'|$ of $|D|$ we must have $(D'^2) > 0$. Thus $|D|$ has some fixed part which clearly consists of components of $E_0 + E'_0 + \sum_{i=2}^{4} E_i + \sum_{i=2}^{4} E_i'$ (since $|E_1 + E_1' + 2S_1|$ is an urreducible pencil of rational curves). However, it can be seen that adding any of these compoents to $E_1 + E_1' + 2S_1$ does not increase the self-intersection index. This shows that $|E_1 + E_1' + 2S_1|$ is, in fact, equal to the moving part of $|D|$. Thus, since the fixed part of D does not contain the points Q_1 and Q_2, we have to show that there are no curves in $|E_1 + E_1' + 2S_1|$ passing through Q_1 and Q_2. But this is easy, because the only curve linearly equivalent to $E_1 + E_1' + 2S_1$ passing thrugh Q_1 is the curve $E_3 + E_3' + 2S_3$ which does not pass through Q_2.

The only thing hanging on us is the proof of the existence of the cases A)-E). Of course, for A) it is easy, since the general member of the pencil P is non-singular. To construct other cases we use a representation of V as a double plane which comes from the inversion involution of the general elliptic fibre of V. Dividing V by this involution we get the surface Z obtained from the quardic

$\mathbb{P}^1 \times \mathbb{P}^1$ by blowing up 8 points, the four of them P_1, P_2, P_3, P_4 are situated on a fibre F of the first projection, and other 4 , P'_1, P'_2, P'_3, P'_4 on a fibre $F' \neq F$ of the same projection. The branch locus of the projection $V \to Z$ equals the union of the proper inverse transforms onto Z of the curves F , F' , and four fibres N_1, N_2, N_3, N_4 of the second projection, each of them N_i passing through P_i and P'_i . These N_i correspond to the sections S_i on V , L_0, L'_0 correspond to the curves E_0, E'_0 , and the lines blown up from the points P_i , P'_i correspond to the curves E_i , E'_i . Consider the rational map $Z \to \mathbb{P}^2$ which is the composition of the blowing down $Z \to \mathbb{P}^1 \times \mathbb{P}^1$ and the linear projection of the quadric onto $\mathbb{P}^2$ with center at some point lying outside the branch locus of $V \to Z \to \mathbb{P}^1 \times \mathbb{P}^1$. Then the image of the branch locus will be equal to the union of six lines, two of them passing through some point A_1 , say ℓ_0, ℓ'_0 , and four of them passing through other point $A_2 \neq A_1$, say n_1, n_2, n_3, n_4 . The pencil of elliptic curves on V is obtained from the pencil of lines through A_1 , the curves F_1 and F_2 correspond to some lines m_1 and m_2 through A_1 . Let $B_1 = m_1 \cap \ell_3$, $B_2 = m_2 \cap \ell_4$. The pencil $\mathcal{P}$ on V corresponds to the pencil of conics passing through A_1, A_2, B_1 and B_2 . To get the case B) we just take for D_1 a conic from this pencil passing through the point $\ell_0 \cap n_1$; in the case C) we take D_1 as in B) , and for D_2 take a conic from this pencil passing through the point $\ell'_0 \cap n_1$. To get the case D) we take for D_1 a conic from the pencil passing through the points $\ell_0 \cap n_1$ and $\ell'_0 \cap n_2$ (that can be done only for some special choice of the lines), and D_2 as in C). Finally, to get the case E) we take for D_1 the same conic as in D), and for D_2 the conic passing through the points $\ell_0 \cap n_2$ and $\ell'_0 \cap n_1$ (also take some special choice of the lines) .

Now we will compute the torsion of Burniat's surfaces. Obviously, we have the following torsion divisors of order 2:

Case A): $g_1 = \bar{E}_0 - \bar{E}'_0$, $g_2 = \bar{E}_0 - \hat{F}_1 - \bar{R}_1$, $g_3 = \bar{E}_0 - \hat{F}_2 - \bar{R}_2$,

$g_4 = \bar{S}_2 - \bar{S}_4 - \bar{R}_1 g_5 = \bar{S}_2 - \bar{S}_3 - \bar{R}_2$, $g_6 = \hat{D}_1 - \hat{F}_1 - \bar{S}_3$,

Case B): $g_1 = \bar{E}'_0 - \hat{F}_1 - \bar{R}_1$, $g_2 = \bar{E}'_0 - \hat{F}_2 - \bar{R}_2$, $g_3 = \bar{S}_2 - \bar{S}_4 - R_1$,

$g_4 = \bar{S}_2 - \bar{S}_3 - \bar{R}_2$, $g_5 = \hat{D}_2 - \hat{F}_1 - \bar{S}_3$

Case C): $g_1 = \hat{F}_1 + \bar{R}_1 - \hat{F}_2 - \bar{R}_2$, $g_2 = \bar{S}_2 - \bar{S}_3 - R_2$,

$g_3 = \bar{S}_2 - \bar{S}_4 - \bar{R}_1$, $g_4 = \hat{D}'_2 + R_1 - \bar{E}'_0 - \bar{S}_3$,

Case D): $g_1 = \hat{F}_1 + \bar{R}_1 - \hat{F}_2 - \bar{R}_2$, $g_2 = S_3 + R_2 - S_4 - R_1$,

$g_3 = \hat{D}'_1 + R_1 + R_2 - \bar{S}_2 - \bar{E}_0$.

Case E): $g_1 = \hat{F}_1 + \bar{R}_1 - \hat{F}_2 - \bar{R}_2$, $g_2 = \bar{S}_3 + \bar{R}_2 - \bar{S}_4 - \bar{R}_1$.

(where $\sigma_*(r^*(p^{-1}(D'_2))) = 2\hat{D}'_2$ and $\bar{S}_i = \sigma_*(r^*(p^{-1}(S_i)))$, i=3, 4) .

We will show that, in fact, these divisors generate the whole torsion group.

<u>Lemma</u>. Let ${}_2\mathrm{Tors}(X)$ denote the subgroup of elements of order 2 in Tors(X) . Then

$$\mathrm{Tors}(X) = {}_2\mathrm{Tors}(X) .$$

Proof. Let $\delta: X \to X$ be the involution of the second order induced by the rational double projection of X onto V. Then δ induces an automorphism of Tors(X) of order 2 $\delta^*: \text{Tors}(X) \to \text{Tors}(X)$.

For any $g \in \text{Tors}(X)$ the divisor $g+\delta^*(g)$ is invariant with respect to δ and hence being taken twicely comes from a torsion divisor on V. Since V is rational, we get that the latter is linearly equivalently to zero. Thus

$$2(g + \delta^*(g)) = 2g + \delta^*(2g) \sim 0 .$$

Replacing g by 2g we get that $\text{Tors}(X) \neq {}_2\text{Tors}(X)$ implies the existence of a non-trivial torsion divisor g such that $g + \delta^*(g) \sim 0$.

Let D_g be an effective divisor from the linear system $|K_X+g|$, where g as above. Then

$$D_g + \delta^*(D_g) \sim D_g + D_{\delta^*(g)} \sim D_g + D_{-g} \in |2K_X|$$

Using the computation of $2K_X$ on the page 90 we get that there exists a curve

$$C \in | F_o + F_o' + 4S_1 + 2E_1 + 2E_1' + E |$$

(E is a linear combination of other E_i, E_i') such that

$$D_g + \delta^*(D_g) = \sigma_*(r^*(p^*(C))) .$$

Since $p^*(C)$ splits under the covering $r : X' \to V'$, it must touch the branch curve $W' = p^{-1}(W) + R'_1 + R'_2$. Counting the intersection indices we easily find that $p^*(C)$ touches the curves $p^{-1}(F_1)$ and $p^{-1}(F_2)$ at one point P_1 and P_2 respectively, and touches the curves $p^{-1}(D_1)$ (or $p^{-1}(D'_1)$ and $p^{-1}(D_2)$ (or $p^{-1}(D'_2)$) at two points P_3, P'_3 and P_4, P'_4 respectively. Also, it does not touch the components E_i or E'_i of W'.

Now notice that both W' and $p^*(C)$ are invariant with respect to the automorphism h of V' induced by the inversion automorphism of the elliptic pencil. This shows that the points P_1 and P_2 are fixed under h (and hence are situated on one of the sections $p^{-1}(S_i)$), and the points P_3 and P'_3 (resp. P_4, P'_4) are conjugate with respect to h. Using this we observe that any curve $C' \sim p^*(C)$ which passes through P_1 and P_2 and touches $p^*(C)$ at P_3 and P_4 will necessarily touch $p^*(C)$ at all 6 points $P_1, P_2, P_3, P'_3, P_4, P'_4$. Since $\dim |p^*(C)| = \dim |2K_X| = 6$ we always can choose such C'. Considering $r^*(C')$ we get the contradiction in view of the following:

Sublemma. Let F be a non-singular projective surface with $q(F) = 0$, D_1 and D_2 effective divisors such that $D_1 - D_2$ is a non-trivial torsion divisor. Then for any D $|D_1+D_2|$ with no common component with $D_1 + D_2$ there exists a point $P \in F$ such that $(D \cdot D_1)_P \neq (D \cdot D_2)_P$.

Proof. Assume the contrary, let $D \in |D_1 + D_2|$ which does not satisfy the assertion of the lemma. Consider the linear pencil generated by the divisors D and $D_1 + D_2$. Resolving its base points we get a morphism $f : F' \to \mathbb{P}^1$ of a surface F' birationally equivalent to F onto $\mathbb{P}^1$ with a fibre containing two numerically equivalent components.

The main lemma of Chapter 2, §1 shows that it is possible only in the case when the general fibre of f is disconnected. Moreover, in this case f has to factor through $f' : F' \to B$, where B is a non-rational curve. This of course, contradicts the assumption $q(F) = 0$.

Theorem. Let X be a Burniat surface of linear genus $p^{(1)}$. Then

$$\mathrm{Tors}(X) = (\mathbb{Z}/2)^{p^{(1)}-1} .$$

Proof. We already know that $\mathrm{Tors}(X) = {}_2\mathrm{Tors}(X)$ and, even more, that any torsion divisor class is invariant with respect to the involution induced by the projection $r : X' \to V'$. Consider the morphism $f : X \to \mathbb{P}^1$ which is defined by the inverse image of the elliptic pencil on V'. We have the following multiple fibres of this morphism:

Case A): $2\bar{E}_0,\ 2\bar{E}_0',\ 2\hat{F}_1 + 2R_1,\ 2\hat{F}_2 + 2R_2$;

Case B): $2\bar{E}',\ 2\hat{F}_1 + 2R_1,\ 2\hat{F}_2 + 2R_2$;

Case C), D), E): $2\hat{F}_1 + 2R_1,\ 2\hat{F}_2 + 2R_2$.

Let $\mathrm{Tors}_f(X)$ be the subgroup of $\mathrm{Tors}(F)$ generated by components of fibres of f. Using the main lemma from Chapter 2, §1 we see that

$$\mathrm{Tors}_f(X) = \begin{cases} (\mathbb{Z}/2\mathbb{Z})^3 & \text{in case A)} \\ (\mathbb{Z}/2\mathbb{Z})^2 & \text{in case B)} \\ \mathbb{Z}/2\mathbb{Z} & \text{in cases C), D), E)} \end{cases}$$

and can be generated by the first three (resp. two, resp. one) divisors g_i indicated on page 94.

Let X_η be the general fibre of f. The restriction homomorphism $Pic(X) \to Pic(X_\eta)$ induces the imbedding

$$Tors(X)/Tors_f(X) \hookrightarrow {}_2Pic(X_\eta)^r ,$$

where $Pic(X_\eta)^r$ denotes the subgroup of divisors on X which are invariant with respect to the automorphism induced by the projection $r_\eta : X_\eta \to V_\eta$, V_η being the general elliptic fibre on V. The covering r_η is ramified along the two points defined by the curves D_1 (or D_1') and D_2 (or D_2').

This shows that each $D \in Pic(X_\eta)^r$ can be represented by a linear combination of the curves $\hat{D}_1, \hat{D}_2, \bar{S}_1, \bar{S}_2, \bar{S}_3, \bar{S}_4$ (the latter four generates $Pic(V_\eta)$). Using the relations on V

$$2S_i \sim 2S_j \quad \text{modulo} \quad E_i, E_i'$$

$$D_i \sim 2S_j \quad \text{modulo} \quad E_i, E_i'$$

we find that each divisor $\hat{D}_i - \bar{S}_j$, $\bar{S}_i - \bar{S}_j$ defines an element of ${}_2Pic(X_\eta)^r$,

Now we notice that the covering $r : X' \to V'$ is defined by the line bundle corresponding to the divisor

$$p^*(3F+2S_1) - R_1' - R_2' - 3R_1 - 3R_2 \quad \text{mod.} \quad E_i, E_j'$$

(see p. 88). This implies that

$$\hat{D}_1 + \hat{D}_2 \sim 2\bar{S}_2 \quad \text{modulo components of fibres of} \quad f .$$

There is also a relation between $\bar{S}_i$

$$\bar{S}_1 + \bar{S}_2 \sim \bar{S}_3 + \bar{S}_4 \text{ modulo components of fibres of } f$$

because S_i defines the 4 points of order 2 on V.

Summarizing we get that ${}_2\mathrm{Pic}(X_\eta)^r$ is generated by the three divisors

$$\bar{S}_3 - \bar{S}_2 \ , \quad \bar{S}_4 - \bar{S}_2 \ , \text{ and } \hat{D}_1 - \bar{S}_3$$

which as it is easily checked are independent.

The arguments above show that any element of $\mathrm{Tors}(X)/\mathrm{Tors}_f(X)$ can be represented by a sum of the above divisors plus a combination of components of fibres of f. It is easy to find in each of the cases A)-E) the corresponding torsion divisors. In fact, we obtain that these divisors are combinations of divisors g_i (i=4, 5, 6 in case A), i=3, 4, 5 in case B), i=2, 3, 4 in case C), i=2, 3 in Case D), i=2 in case E)) indicated on p. 73. This proves the theorem.

Remark. As we observed above the morphism $f : X \to \mathbb{P}^1$ has 4 multiple fibres of multiplicity 2 in case A). Let $B \to \mathbb{P}^1$ be the 2-sheeted covering of $\mathbb{P}^1$ by an elliptic curve B branched at the four points corresponding to the multiple fibres. The normalization X' of the surface $X \times_{\mathbb{P}^1} B$ is a double covering of X non-ramified outside the two points $Q_1 = \hat{F}_1 \cap R_1$ and $Q_2 = \hat{F}_2 \cap R_2$. Also, $\bar{X}$ being mapped onto B has the infinite fundamental group, the points $\bar{Q}_1$ and $\bar{Q}_2$ lying over Q_1 and Q_2 are ordinary double points. This shows that the complement $X - \{Q_1, Q_2\}$ has a non-ramified covering with infinite fundamental group, hence X itself has infinite fundamental group.

Another way to prove that the fundamental group of the Burniat surface with $p^{(1)} = 7$ is infinite is based on the corollary to Lemma 1 of Chapter III, §1 . Consider the surface X_T corresponding to the torsion group T of X . Then we have

$$g(X_T) = \sum_{g\in T} h^1(g) = \sum_{\substack{g\in T\\ g\neq 0}} (h^0(K_X+g)-1)$$

Consider the inverse image of the pencil $\mathcal{P}$ onto X . The divisor $2\hat{D}_1$ belongs to this pencil and $h^0(2\hat{D}_1+\bar{R}_1+\bar{R}_2) = 2$. Now

$$2(\bar{R}_1 + \bar{R}_2 + 2\hat{D}_1 - K_X) \sim 2(2\hat{D}_1 - 2\bar{F} - 2\bar{S}_1 + \bar{E}_0 + \bar{E}_0' + 2\bar{R}_1 + 2\bar{R}_2)$$

$$2(\bar{F} + 2\bar{S}_1 - 2\bar{R}_1 - 2\bar{R}_2 - 2\bar{F} - 2\bar{S}_1 + \bar{E}_0 + \bar{E}_0' + 2\bar{R}_1 + 2\bar{R}_2) \sim 2(\bar{E}_0' - \bar{E}_0) \sim 0 .$$

This shows that $2\hat{D}_1 + \bar{R}_1 + \bar{R}_2 \sim K_X + g$ and hence

$$q(X_T) > 0 .$$

This, of course, implies that X_T and thus X has infinite fundamental group. *)

*) See Epilogue

§5. Surfaces with $p^{(1)} = 9$.

Such surfaces were constructed by M. Kuga [29] and A. Beauville [3].

Kuga's construction:

Let $H = \{z \in \mathbb{C} : \mathrm{Im}(z) > 0\}$ be the upper half plane. The Lie group $\mathbb{P}GL(2,\mathbb{R}) = S\mathbb{L}(2,\mathbb{R})/\pm 1$ is identified in a natural way with its group of analytic automorphisms.

Let Γ be a discrete subgroup of $\mathbb{P}GL(2,\mathbb{R}) \times \mathbb{P}GL(2,\mathbb{R})$ acting freely on $H \times H$ with compact quotient $V = H \times H/\Gamma$. By Matsushima-Shimura [31] we have

$$h^{1,0}(V) = h^{0,1}(V) = q(V) = 0 ;$$

$$h^{1,1}(V) = 2p_g(V) + 2 .$$

Therefore,

$$c_2(V) = 4p_g(V) + 4 , \quad K_V^2 = 8p_g(V) + 8 .$$

Next, notice that V has no exceptional curves of the first kind (and more generally, no rational curves), because the projection $H \times H \to H \times H/\Gamma = V$ splits over such curve, but $H \times H$ does not contain any complete curves.

Thus, to find the needed surfaces with $p_g(V) = 0$ and $p^{(1)} = 9$ it suffices to choose such Γ that

$$c_2(H \times H/\Gamma) = 4 .$$

By the Gauss-Bonnet formula

$$c_2(V) = \frac{1}{4\pi^2} \operatorname{vol}(V) \quad ,$$

where the volume $\operatorname{vol}(V)$ is computed by integration of the invariant volume element

$$dv = \frac{dx_1 \wedge dy_1}{y_1^2} \wedge \frac{dx_2 \wedge dy_2}{y_2^2} \quad ,$$

($(z_1, z_2) = (x_1+iy_1,\ x_2+iy_2)$ being the coordinates on $H \times H$).

Now, let

$k = \mathbb{Q}(\sqrt{d})$ be a real quadratic field, d the discriminant;

$A = A(k,\theta)$ be the division quaternion algebra with the center k and with the discriminant $\theta = p_1 p_2 \cdots p_{2r}$ assumed to be totally indefinite (that is, $A \otimes_{\mathbb{Q}} \mathbb{R} = M_2(R) \oplus M_2(R)$).

$N : A \to k$ be the reduced norm of A;

$\underline{O}$ be the maximal order of A (unique up to conjugation if the class number of k equals 1);

$E(\underline{O})$ be the group of all units of $\underline{O}$;

$\overline{\Gamma} = \{g \in E(\underline{O}) : N(g) = 1\}$

Consider the natural injection $i : A \to A \otimes_{\mathbb{Q}} \mathbb{R} = M_2(\mathbb{R}) \oplus M_2(\mathbb{R})$ and the projection $j : GL_2(\mathbb{R}) \times GL_2(\mathbb{R}) \to \mathbb{P}GL(2,\mathbb{R}) \times \mathbb{P}GL(2,\mathbb{R})$. Let $\Gamma = j(i(\overline{\Gamma}))$ be a discrete subgroup of $\mathbb{P}GL(2,\mathbb{R}) \times \mathbb{P}GL(2,\mathbb{R})$ with compact quotient $V = H \times H/\Gamma$; we note that Γ is isomorphic to the image of $\overline{\Gamma}$ into A^*/k^*.

According to Simizu ([42]) the volume $\operatorname{vol}(H \times H/\Gamma)$ can be expressed through the zeta function $\zeta_k(s)$ of k by the formula:

$$\mathrm{vol}(H \times H/\Gamma) = \frac{2}{\pi^2} d^{3/2} \zeta_k(2) \Pi(|p_i| - 1)$$

($|p|$ denotes the norm of prime ideal p of k) .

Now

$$\zeta_k(s) = \zeta(s)\, L(s,\chi) ,$$

where $\zeta(s)$ is the Riemann zeta function and $L(s,\chi)$ is the Dirichlet L-function associated with the character χ mod d

$$\chi(n) = \begin{cases} (\frac{n}{d}) & ; \text{ if } d \equiv 1 \mod 4 \\ (\frac{n}{m})(-1)^{(n-1)/2} & , \text{ if } d = 4m,\ m \equiv 3 \mod 4 \\ (\frac{n}{m'})(-1)^{(n^2-1)/8} & , \text{ if } d = 8m',\ m \equiv 1 \mod 4 \\ (\frac{n}{m'})(-1)^{(n^2-1)/8+(n-1)/2} & , \text{ if } d = 8m',\ m' \equiv 3 \mod 4 \end{cases}$$

The value of the Riemann zeta at 2 equals $\pi^2/6$. The value $L(s,\chi)$ at 2 equals

$$L(2,\chi) = \frac{1}{2}(\frac{2}{d})^2 \tau(\chi) B_d ,$$

where

$$\tau(\chi) = \sum_{n=1}^{d-1} \chi(n) e^{2\pi i n/d} \quad , \text{ the Gauss' sum}$$

$$B_d = \frac{1}{2d}(\sum_{m=1}^{d-1} m^2 \chi(m)) .$$

Thus, we have

$$c_2(H \times H/\Gamma) = \frac{2d^{3/2}}{2^2\pi^4} (\frac{\pi^2}{6})\frac{1}{2}(\frac{2\pi}{d})^2 \tau(\chi) B_d \Pi_{p/\theta}(|p|-1) =$$

$$= \frac{1}{6} \frac{1}{d^{1/2}} \tau(\chi) B_d \Pi(|p|-1) .$$

Since the Gauss' sum $\tau(\chi)$ has absolute value $|\tau(\chi)| = d^{1/2}$ and $c_2 = \frac{1}{4\pi^2}$ vol is positive, we get

$$c_2(H\times H/\Gamma) = \frac{1}{6}|B_d| \prod_{p|\theta} (|p| - 1)$$

Next, we have to be assured that the group Γ acts freely on HxH, and hence $H\times H/\Gamma$ is smooth. Since the stabilizator group of any point is a finite subgroup of Γ, that can be if and only if Γ has no elements of finite order.

Let $g \in \Gamma$ be an element of order N, $\bar{g} \in \bar{\Gamma}$ some of its preimages in $\bar{\Gamma}$. We have $g^N = \pm 1$, and thus $\bar{g}^{2N} = 1$. Then the quaternion algebra A has to contain a subfield isomorphic to the field

$$\mathbb{Q}(e^{\pi i/N}) = \mathbb{Q}(\bar{g}) .$$

Conversely, if the class number $h(k) = 1$, then $A \supset \mathbb{Q}(e^{2\pi i/N})$ implies that $\bar{\Gamma}$ has an element of order N.

Since the maximal subfield of A has degree 2 over k, we have

$$\phi(N) = [\mathbb{Q}(e^{2\pi i}) : \mathbb{Q}] \qquad \text{divides } 4 .$$

Thus the only possible orders for N are

$$N = 2,3,4,5,6,8,10,12 .$$

Obviously, an element of order 2 in $\bar{\Gamma}$ defines the unit element of Γ

Now, if $\phi(N) = 2$ (N=3,4,6) then the maximal subfield K of A coincides with $k(e^{2\pi i/N})$, if $\phi(N)=4$ then $K = \mathbb{Q}(e^{2\pi i/N})$ and k is the real quadratic subfield of $\mathbb{Q}(e^{2\pi i/N})$.

Let K be a quadratic extension field of k; then the local arguments show that K is embedable into $A = A(k,\theta)$ if and only if $p|\theta$ does not decompose in K.

Now we are ready to give an explicit example.

Example. $k = \mathbb{Q}(\sqrt{2})$, $d = 8$, $\theta = p_2 p_5$, where p_2 and p_5 lie over 2 and 5 accordingly.

We compute

$$B_8 = 1 \ , \ c_2(H\times H/\Gamma) = \frac{1}{6} B_8(2-1)(25-1) = 4 \ .$$

To check the smoothness of $H\times H/\Gamma$ we observe that the only cyclotomic field containig k is $\mathbb{Q}(e^{2\pi i/8})$, and in this case p_2 and p_5 do not decompose. Thus it suffices to consider the cases N = 3,4,and 6 . In the second case $K = \mathbb{Q}(\sqrt{2},i)$, and in the first and the third, $K = \mathbb{Q}(\sqrt{2},\sqrt{-3})$. In the both cases we easily verify that p_2 and p_5 do not decompose.

Notice that other examples can be also obtained by taking instead of some other discrete subgroups in $\underline{O}$, for example,

$$\bar{\Gamma}' = \{g \in E(\underline{O}) : N(g) \text{ is a totally positive unit of } k\} \ .$$

We refer to [29] for the examples of the corresponding surfaces $H\times H/\Gamma$.

To compute the torsion group $\mathrm{Tors}(H\times H/\Gamma)$ we note that

$$\mathrm{Tors}(H^2(V,Z)) = H_1(V,Z) = \Gamma/[\Gamma,\Gamma] = \bar{\Gamma}/(\pm 1)[\bar{\Gamma},\bar{\Gamma}] \ .$$

For any maximal two-sided ideal $p\underline{O}$ in $\underline{O}$ we may consider the image $\phi(\bar{\Gamma})$ in $\underline{O}/p$ ($= M_2(\mathbb{F}_q)$ or $\mathbb{F}_{q^2}$, $q=\mathrm{Norm}_{k/\mathbb{Q}}(p)$, depending on whether $p\nmid\theta$ or $p|\theta$).

Moreover, by the Eichler approximation theorem we have

$$\phi(\overline{\Gamma}) = \begin{cases} SL_2(\mathbb{F}_q) & , \ p \nmid \theta \\ U & , \ p \mid \theta \end{cases}$$

where $U = \{ a \in \mathbb{F}_{q^2} : N_{\mathbb{F}_{q^2}/\mathbb{F}_q}(a) = 1 \}$ is a cyclic group of order q+1 .

This immediately shows that it is always

$$\mathrm{Tors}(V) = \overline{\Gamma} / (\underline{+}1)[\overline{\Gamma},\overline{\Gamma}] \neq 1 \ .$$

The more detailed analysis gives the following result:

Theorem([29]). There exists a subgroup M of Γ containing $[\Gamma,\Gamma]$ such that

$$\widetilde{\Gamma}/M = \bigoplus_{i=1}^{r} \mathbb{Z}/(q_i+1) \oplus (\mathbb{Z}/2)^a \oplus (\mathbb{Z}/3)^b \ ,$$

where

$$q_i = N_{k/\mathbb{Q}}(p_i) \ , \ \theta = p_1 \cdots p_r \ ;$$

$$a = \begin{cases} 2 & \text{if } (2) = p_2 p_2' \ , \ p_2 \neq p_2' \ \text{and} \ p_2 \nmid \theta \ , p_2' \nmid \theta \\ 1 & \text{if } p_2 \mid 2 \ , \ |p_2| = 2 \ , \ p_2 \nmid \theta \quad \text{but other divisor of 2 divides} \ \theta\theta \\ 0 & \text{otherwise} \end{cases}$$

$$b = \begin{cases} 2 & \text{if } (3) = p_3 p_3' \ , \ p_3 \neq p_3' \ \text{and} \ p_3 \nmid \theta \ , \ p_3' \nmid \theta \\ 1 & \text{if } (3) = p_3^2 \ , \ p_3 \nmid \theta \quad \text{or} \quad p_3 \mid 3 \ \text{and other divisor of 3 divides} \ \theta \\ 0 & \text{otherwise} \end{cases}$$

Moreover, $M = [\overline{\Gamma},\overline{\Gamma}]$ if the congruence subgroup conjecture of Bass-Serre is true for $\overline{\Gamma}$. Also, $-1 \in M$ if and only if one of $q_i \equiv 1 \mod 4$.

In the above example we have

$$\overline{\Gamma}/M = \mathbb{Z}/3 \oplus \mathbb{Z}/6 .$$

Beauville's examples ([3]).These surfaces are constructed as the quotients $V = C\times D/G$, where C and D are complete non-singular algebraic curves of genus g at least 2, G is a finite group acting freely on the product.

To construct the quotient with the needed properties Beauville proposes to take for G a finite group of order $(g(C)-1)(g(D)-1)$ acting on the both C and D with the rational quotients. In order to get a free action on $C\times D$ he puts

$$g(x,y) = (g(x),\sigma(g)(y)) \quad , \quad g\in G \quad , \quad (x,y)\in C\times D \quad ,$$

where σ is an automorphism of G such that for all $g\in G$ acting non-freely on C $\sigma(g)$ acts freely on D .

In virtue of the lemma of Chap.I,§2 we have

$$p_a(V) = 0 \quad , \quad K_V^2 = 8 \quad .$$

Moreover, V does not contain any rational curves, since the projection $C\times D \to V$ has to split over such curve and there are no rational curves on $C\times D$. This implies that V is a minimal model.

It remains to prove that the irregularity $q(V) = 0$. We have

$$H^1(V,\mathcal{O}_V) = H^1(C\times D,\mathcal{O}_{C\times D})^G = H^1(C,\mathcal{O}_C)^G \oplus H^1(D,\mathcal{O}_D)^{\sigma(G)}$$

but, since C/G and $D/\sigma(G)$ are rational curves, the both summands are zeros.

Example 1. $C = D$ is the plane curve with the equation:

$$x^5 + y^5 + z^5 = 0 ,$$

$G = (Z/5)^2$ acts on C by the formulas:

$$(p,q)(x,y,z) = (\xi^p x, \xi^q y, z) , \quad \xi = e^{2\pi i/5} ,$$

σ is the automorphism of G given by $(1,0) \to (1,1), (0,1) \to (1,2)$.

The set of elements of G which act freely is $A = \{(p,q), p \neq q\}$ and $G = \{1\} \cup A \cup \sigma(A)$.

Example 2. $C = D$ is the curve of genus 4 given by the equation in $\mathbb{P}^3$:

$$x^3 + y^3 + z^3 + t^3 = 0 , \quad xy + zt = 0 .$$

$G = (Z/3)^2$ acts on C by the formulas:

$$(p,q)(x,y,z,t) = (\xi^p x, \xi^{-p} y, \xi^q z, \xi^{-q} t) , \quad \xi = e^{2\pi i/3} ,$$

σ is the automorphism given by $(1,0) \to (1,1)$, $(0,1) \to (1,2)$.

The set of elements of G acting freely on C is the set $A = \{(p,q), p \pm q \neq 0\}$ and $G = \{1\} \cup A \cup \sigma(A)$

Applying the well known Hochshild-Serre exact sequence:

$$0 \to \mathrm{Hom}(G,\mathbb{C}^*) \to \mathrm{Pic}(C\times D/G) \to \mathrm{Pic}(C\times D)^G \to H^2(G,\mathbb{C}^*)$$

we see that

$$\mathrm{Tors}(C\times D/G) \supset G/[G,G] .$$

In particular, in the above examples the torsion group is non-trivial.

6. Concluding remarks.

It would be very optimistic to expect the complete classification of all surfaces of general type with $p_g=0$. However, there are still many problems to answer in the visible future.

One of the most interesting from my point of view is the following:

Problem 1. Is there a simply connected surface of general type with $p_g=0$?

Or more weak

Problem 1'. Is there a surface of general type with $p_g=0$ and trivial torsion group?

Consider the class of all surfaces of general type with $p_g=0$ and fixed $P_2= p^{(1)}$. Then there exists a number N such that the N-canonicla system defines a birational morphism for all such surfaces([4]). Thus the set of its N-canonicla models can be parametrized by an open subset of the Hilbert scheme corresponding to some Hilbert polynomial. Since the latter is of finite type, this open subset consists of finite number of connected components. The surfaces parametrized by a connected variety are diffeomorphic, and,in particular,have the same fundamental group. This argument shows that there are only finite number of possibilities for the fundamental group of a surface. In particular, the order of the torsion group is bounded by a constant depending only on $p^{(1)}$.

Problem 2. Find a bound for the order of the torsion group of surfaces with the fixed $p^{(1)}$ (as always of general type and with $p_g=0$).

We remind that it is done in the cases of numerical Godeaux and Campedelli surfaces.

Consider the class of all surfaces with the fixed value the torsion group T. Denote it by $\mathcal{M}$ (a,T).

Problem 3. Can $\mathcal{M}$(a,T) be parametrized by a connected variety? In particular, are the elements of $\mathcal{M}$(a,T) diffeomorphic to each other?

For the start it would be very interesting to know the answer at least in the cases $\mathcal{M}(2,Z/2)$, $\mathcal{M}(2,Z/3)$ and $\mathcal{M}(3,Z/2\oplus Z/2)$. Recall that in the last case we know two (and possibly even three) different constructions of surfaces from this class. In some cases the answer is positive (e.g. $\mathcal{M}(2,Z/4)$, $\mathcal{M}(2,Z/5)$, $\mathcal{M}$(3,abelian of order 8)).

We still do not know if all possible values of $p^{(1)}$ are realized*).

Problem 4. Are there surfaces with $p^{(1)} = 8$ and 10 ?

There is much hope to solve the following

Problem 5. Find all possible torsion groups of numerical Godeaux and Campedelli surfaces.*)

The validity of the following assertion is observed in all known examples:

Problem 6. Prove that the fundamental group is infinite in the case $p^{(1)} \geq 7$ and finite otherwise.

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*) See Epilogue

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EPILOGUE

After this work has been almost done the author was informed in many new results.

1. Numerical Godeaux surfaces with Tors = $\mathbb{Z}/3$ have been constructed by Miles Reid [45]. The construction is very delicate.

2. The final version of Peters' preprint [37] has been published [44].It can be found there the result about the torsion of Burniat's surfaces(the proof is not complete). Also it is proven there that the fundamental group is infinite in case $p^{(1)}=7$. This result is also refered to M.Reid.

3. F.Oort and C.Peters also have proven that the torsion of Campedelli--Oort-Kulikov surfaces with $p^{(1)}=2$ is equal to $\mathbb{Z}/2$ ([51]).

4. M.Inoue has constructed surfaces with $p^{(1)}=8$ and also calculated the fundamental group for Burniat's surfaces ([46]).

5. M.Reid has computed the canonical ring of numerical Godeaux surfaces with Tors=$\mathbb{Z}/2$ ([46]).

6. M.Reid has proven that $\#$ Tors ≤ 9 for numerical Campedelli surfaces. He conjectures that 9 can be replaced by 8 and the surfaces with the torsion group of order 8 are the Godeaix-Reid surfaces .Another conjecture: $\#$ Tors < 30 for surfaces with $p^{(1)}=4$ ([47]).

7. Using the nonarchimedean uniformization theory D.Mumford has construced a surface with $p^{(1)}=10$ ([48]).

8. Many people have discovered independently a surface with Tors = $\mathbb{Z}/5$ and $p^{(1)}=3$ ([46]). As it was explained to me by Fabrizio Catanese it can be

constructed in the foloowing way. Let F be a quintic surface in P^3 which is invariant under an involution of order 5 and posseses 20 ordinary double points.Also assume that there exists a quartic surface B tangent to F along a curve C which passes through these double points and smooth at them. The existence of such surfaces F and B is proven in [49]. Blow up F at these 20 double points to the surface $\bar{F}$, then the sum of the twenty exceptional -2-curves on $\bar{F}$ is linearly equivalent to the strict inverse transform of C taken twicely. Let $\bar{V}$ be the double covering of $\bar{F}$ branched at those curves, V the blowing down of the strict trnasforms of the branch locus. Then it can be easily shown that $K_V^2 = 10$, $p_g(V) = 4$. The $\mathbb{Z}/5$-action on F extends to a free action on V and the quotient defines the needed surface X. By Reid's result (see 6.) we get $Tors(X) = \mathbb{Z}/5$. Moreover, the surface $\bar{V}$ can be realized as a non-singular compactification of a quotient of the upper half planes by a discrete group of Hilbert's type ([50]), this implies that $\bar{V}$ is simply connected, and hence the fundamental group of X is $\mathbb{Z}/5$.

9. C.Peters conjectures that for any double plane of general type with $p_g=o$ the torsion group consists of elements of order 2 ([44]').

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