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A complex ball uniformization of the moduli space of cubic surfaces via periods of $K3$ surfaces

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Abstract. In this paper we show that the moduli space of nodal cubic surfaces is isomorphic to a quotient of a 4-dimensional complex ball by an arithmetic subgroup of the unitary group. This complex ball uniformization uses the periods of certain $K3$ surfaces which are naturally associated to cubic surfaces. A similar uniformization is given for different covers of the moduli space corresponding to geometric markings of the Picard group or a choice of a line on the surface. We also give a detailed description of the boundary components corresponding to singular surfaces.

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1. Introduction

There are two main approaches to the construction of moduli spaces in algebraic geometry. One uses geometric invariant theory which allows one to construct the moduli space as a quotient of an open subset of an appropriate Hilbert scheme, the other one uses

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period maps to construct the moduli space as a quotient of an open subset of a Hermitian symmetric homogeneous domain by a discrete subgroup of its group of holomorphic automorphisms. Both approaches suggest a way to compactify the moduli space. In the algebraic approach one adds the equivalence classes of semi-stable points. In the transcendental approach one considers the whole domain together with its boundary.

There are several remarkable cases where both approaches work. Comparing the constructions gives a beautiful interplay between the algebraic theory of invariants and the theory of automorphic functions. The historically first example of such an interplay is of course the moduli space of elliptic curves which, on one hand, is the quotient of the space of binary forms of degree 4 by the group SL(2) and, on the other hand, is a natural quotient of the upper half-plane by the modular group. Similarly, binary forms of degree 5, 6, 8 and 12 give the moduli spaces of Del Pezzo surfaces of degree 4, and hyperelliptic curves of genus 2, 3 and 5, respectively. Using the theory of hypergeometric functions one can show that the corresponding domains are complex balls of dimension 2, 3, 5 and 9, respectively. Increasing the number of variables by one, one finds the ternary cubic forms which leads again to the moduli space of elliptic curves, the forms of degree 4 corresponding to the moduli space of non-hyperelliptic curves of genus 3 (in this case the domain is the Siegel upper half space of degree 3) and the forms of degree 6 corresponding to $K3$ surfaces with degree 2 polarization (the domain is of type IV in Cartan's classification).

Using domains of type IV one can also give a uniformization of the moduli space of cubic and quartic forms in 4 variables. The case of forms of degree 3 (cubic surfaces) was treated in the work of K. Matsumoto, T. Sasaki and M. Yoshida [MSY], and degree 4 ($K3$ surfaces with degree 4 polarization) much earlier by J. Shah [Sha]. Although cubic surfaces do not admit non-zero holomorphic 2-forms, so that the periods are not defined, there are identifications of this moduli space with other moduli spaces for which the period map is defined. In [MSY] one uses the moduli space of $K3$ surfaces which have a certain primitive sublattice of rank 16 in the Picard group. Such a surface can be realized as a double cover of $\mathbb{P}^2$ branched along the union of 6 lines in a general position. The blow-up of the dual set of 6 points in $\mathbb{P}^2$ is a nonsingular cubic surface. Recent work of D. Allcock, J. Carlson and D. Toledo [ACT] gives a different uniformization of the moduli space of cubic surfaces where the domain of type IV is replaced by a complex ball. This ball quotient is the moduli space of principally polarized abelian varieties of dimension 5 with complex multiplication in the Eisenstein ring $\mathbb{Z}[\zeta_3]$. Each such variety can be realized as the intermediate Jacobian of the triple cyclic cover of $\mathbb{P}^3$ branched over a nonsingular cubic surface. Independently this construction was found by the second author and B. Hunt. Subsequently, Allcock and Freitag [AF] found modular forms on the ball quotient which embed it into a nine dimensional projective space. Freitag [F] later proved that the ideal of the image is defined by cubic polynomials and that the quotient ring is the full ring of modular forms. The image variety turns out to be isomorphic to a compactification of the moduli space of marked cubic surfaces.

A similar approach works for Del Pezzo surfaces of degree 2 and 1 which can be realized as surfaces of degree 4 and 6 in weighted projective spaces $\mathbb{P}(1,1,1,3)$ and $\mathbb{P}(1,1,2,3)$, respectively (see also [HL] for another approach to a complex ball uniformization of the moduli space of Del Pezzo surfaces of degree 1). All of this is based on the existence of an embedding of a complex ball into a Siegel domain. It is also known that a complex ball can be embedded into a type IV domain. For example a moduli space of lattice polarized $K3$

surfaces admitting an automorphism of order 3 or 4 which acts non-trivially on the lattice of transcendental cycles is parametrized by an arithmetical quotient of an open subset of a complex ball. This observation was used by the third author [Ko1] and independently by the second author (unpublished) to construct a complex ball uniformization of the moduli space of Del Pezzo surfaces of degree 2. This moduli space is isomorphic to the moduli space of non-hyperelliptic curves of genus 3 via the map which associates to a Del Pezzo surface the fixed curve of the Geizer involution. The $K3$ surface associated to such a surface is its double cover branched along this fixed curve. In [Ko2] a similar description of the moduli spaces of curves of genus 4 and of Del Pezzo surfaces of degree 1 is given.

In this paper we give a similar construction for the moduli space of cubic surfaces. To each stable cubic surface S we associate a $K3$ surface X_S with an automorphism of order 3. Its periods are parametrized by a complex 4-ball and we do in fact recover most of the results from [ACT]. Our construction is also closely related to the work of K. Matsumoto and T. Terasoma [MT] who associate to a line on a cubic surface a certain curve C of genus 10 which admits an involution σ with two fixed points such that the $\mathrm{Prym}(C,\sigma)$ is isomorphic to the intermediate Jacobian of the triple cover of $\mathbb{P}^3$ branched along the cubic surface. The curve C also admits an automorphism τ of order 6 such that $\sigma = \tau^3$. The $K3$ surface associated to the cubic is the minimal nonsingular model of the quotient $(C \times E)/\langle\tau\rangle$, where E is an elliptic curve with an automorphism of order 6. The branching of the map $C \to C/\langle\tau\rangle \cong \mathbb{P}^1$ is very special, we have 7 branch points, 5 of which have ramification index $(3,3)$ and two have index (6). According to Deligne-Mostow [DM] the moduli space of such covers is isomorphic to an open subset of a complex ball quotient $\mathscr{B}/\Gamma$. We identify this moduli space with the moduli space of $K3$ surfaces X_S and interprete the monodromy group Γ in terms of the orthogonal group of the lattice of transcendental cycles on the $K3$ surfaces. We also give an interpretation of a compactification of the ball quotient in terms of $K3$ surfaces.

Here is the review of the contents of the paper. In section 2 we study stable cubic surfaces. Since these have at most nodes as singularities we refer to them as nodal cubic surfaces. We define markings of nodal cubic surfaces and we introduce the moduli space of marked nodal cubic surfaces $\mathscr{M}^m_{\mathrm{ncub}}$. The Weyl group $\mathrm{W}(E_6)$ acts on $\mathscr{M}^m_{\mathrm{ncub}}$ (the action can be described by planar Cremona transformations) and the quotient variety is $\mathscr{M}_{\mathrm{ncub}}$, the moduli space of stable cubic surfaces. It has a natural compactification $\bar{\mathscr{M}}_{\mathrm{ncub}}$, the moduli space of semi-stable cubic surfaces, which is obtained by adding one point. The moduli space $\mathscr{M}^m_{\mathrm{ncub}}$ admits also a natural compactification $\bar{\mathscr{M}}^m_{\mathrm{ncub}}$ which is obtained by adding 40 points. It admits a $\mathrm{W}(E_6)$-equivariant embedding into $\mathbb{P}^9$. We discuss different constructions of the moduli space $\bar{\mathscr{M}}^m_{\mathrm{ncub}}$.

For a nodal cubic surface and a line on it we define in section 3 a pair of binary forms, of degree 2 and 5, modulo the action of $\mathrm{SL}(2)$. Using this, we prove that the moduli space of cubic surfaces together with a choice of a line on it is a rational variety.

In section 4 we define a $K3$ surface $X_{S,l}$ associated to a nodal cubic surface S together with the choice of a line l on S. The surface $X_{S,l}$ admits a natural elliptic fibration as well as an automorphism of order three. We show that this $K3$ surface depends only on S (and not on the choice of l) by defining a $K3$ surface $X_{S,l,m}$, where l and m are skew lines on S, which can be seen to be isomorphic to both $X_{S,l}$ and $X_{S,m}$. We write X_S for the (isomorphism class of such a) $K3$ surface associated to (S,l). We relate X_S to the $K3$ surface associated to a cubic fourfold with a plane, to the cubic threefold V associated to S by Allcock, Carlson and Toledo and to the 'Matsumoto-Terasoma curve' C.

In section 5 we show that the Picard lattice of a generic X_S is isomorphic to the lattice $M = U \oplus A_2^{\oplus 5}$. The lattice of transcendental cycles is isomorphic to the lattice $T = A_2(-1) \oplus A_2^{\oplus 4}$. This follows from the fact that the elliptic fibration on the generic X_S has 5 singular fibres of type IV and 2 fibres of type II and some lattice theoretic considerations. We also compute the Picard lattices of the $K3$ surfaces associated to general nodal cubic surfaces.

In section 6 we study the moduli space of M-polarized $K3$ surfaces $(X, \phi : M \to \mathrm{Pic}(X))$. If $\phi(M) = \mathrm{Pic}(X)$, an M-polarization ϕ is equivalent to the data which consists of an elliptic fibration with a unique section, an order on the 5 reducible fibres of type IV or I_3, and an order on the set of irreducible components of each fibre which do not meet the section. An M-polarization on the $K3$ surface X_S associated to a smooth cubic surface S is equivalent to a marking on S, that is, an order on the set of 27 lines (or, equivalently, a choice of an ordered set of six skew lines). The M-polarized $K3$ surfaces (X_S, ϕ) are distinguished from general M-polarized $K3$ surfaces by the property that there exists an automorphism σ of order 3 which is the identity on $\phi(M)$ and, for smooth S, coincides with some explicitly described isometry ρ on the orthogonal complement of $\phi(M)$ in $H^2(X_S, \mathbb{Z})$. The isometry ρ fixes the period $H^{2,0}(X_S)$ of X_S so that the image of the period map of the surfaces X_S lies in the fixed locus of a certain automorphism of order 3 on the period space of M-polarized $K3$ surfaces. This fixed locus turns out to be isomorphic to a 4-dimensional complex ball $\mathscr{B}$. In this way we construct the moduli space $\mathscr{K}3^m_{M,\rho}$ of (M, ρ)-polarized $K3$ surfaces as a quotient of $\mathscr{B}$. The Weyl group $\mathrm{W}(E_6)$ acts naturally on $\mathscr{K}3^m_{M,\rho}$ by changing the polarizations.

In section 7 we establish a natural $\mathrm{W}(E_6)$-equivariant isomorphism from the moduli space of marked nonsingular cubic surfaces $\mathscr{M}^m_{\mathrm{cub}}$ onto an open subset $\mathscr{K}3^m_{M,\rho} \backslash \Delta^m$ of $\mathscr{K}3^m_{M,\rho}$. The moduli space of isomorphism classes of pairs (S, l) of cubic surfaces together with a choice of a line is isomorphic to the quotient of $\mathscr{K}3^m_{M,\rho} \backslash \Delta^m$ by a subgroup of $\mathrm{W}(E_6)$ isomorphic to $\mathrm{W}(D_5)$. In this way we obtain an interpretation of a line on a general cubic surface S as a choice, up to automorphisms of X_S, of an elliptic pencil with 5 fibres of type IV on the associated $K3$ surface X_S.

In section 8 we study in detail the geometry of the discriminant locus Δ^m. We show that each point $[(X, \phi)] \in \Delta^m$ admits an automorphism σ of order 3 such that $H^2(X, \mathbb{Z})^{\sigma^*}$ contains $\phi(M) \oplus R$, where R is spanned by all (-2)-vectors in $\phi(M)^\perp \cap \mathrm{Pic}(X)$. The lattice R is isomorphic to r $(\leqq 4)$ copies of the root lattice A_2. The polarization ϕ defines an elliptic fibration on X and we describe its possible singular fibres. We also prove that Δ^m consists of 36 irreducible components on which $\mathrm{W}(E_6)$ acts transitively. The cubic surfaces with Eckardt points define another divisor in $\mathscr{K}3^m_{M,\rho}$ and we prove that it consists of 45 irreducible components permuted transitively by $\mathrm{W}(E_6)$. Finally we show that the Satake-Baily-Borel compactification of $\mathscr{K}3^m_{M,\rho}$ contains 40 cusps, again transitively permuted by $\mathrm{W}(E_6)$. This agrees with the results obtained in [ACT].

In section 9 we show that the $\mathrm{W}(E_6)$-equivariant isomorphism from $\mathscr{M}^m_{\mathrm{cub}}$ onto $\mathscr{K}3^m_{M,\rho} \backslash \Delta^m$ can be extended to an equivariant isomorphism from the moduli space of marked nodal cubics $\mathscr{M}^m_{\mathrm{ncub}}$ to $\mathscr{K}3^m_{M,\rho}$. We also show that the quotient $\mathscr{K}3^m_{M,\rho}/\mathrm{W}(D_5)$

and the moduli space of nodal cubic surfaces together with a choice of a line $\mathscr{M}^1_{\mathrm{ncub}} = \mathscr{M}^m_{\mathrm{ncub}}/\mathrm{W}(D_5)$ are isomorphic. Moreover, the latter space is naturally isomorphic to the GIT-quotient $P_1(2^5, 1, 1)/S_5 \times S_2 = (\mathbb{P}^1)^7//\mathrm{SL}(2) \times (g)$, where the linearization of SL(2) is defined by weighting the first five factors with weight 2 and the last two factors with weight 1. Here S_5 acts by permutation of the first five factors and S_2 acts by permutations of the last two factors.

The configuration space $P_1(2^5, 1, 1)/(g) = (\mathbb{P}^1)^7//\mathrm{SL}(2) \times (g)$ occurs in the work of Deligne and Mostow [DM] and we show that the group Γ is isogenous to the reflection group Π acting on $\mathscr{B}$ which is generated by the reflection group Π' of the hypergeometric function defined by the multi-valued form

$$\omega = z^{-1/6}[(z-1)(z-a_1)(z-a_2)(z-a_3)(z-a_4)]^{-1/3}\,dz$$

and an involution g. Moreover, we match the types of degeneration of the elliptic fibration corresponding to the polarization and the type of degeneration of a stable point set through this morphism.

Finally, in section 10, we compare the Hodge structure on the $K3$ surface X_S with the principally polarized Hodge structure on $H^1(P, \mathbb{Z})$, where P is the intermediate Jacobian of a cubic threefold associated to the cubic surface S.

2. Nodal cubic surfaces

2.1. Nodal cubics and points in $\mathbb{P}^2$. A *nodal cubic surface* is a surface of degree 3 in $\mathbb{P}^3$ which has at most ordinary double points as singularities. Let $S \subset \mathbb{P}^3$ be a nodal cubic surface with a node $P = (0,0,0,1)$. Then its equation is of the form:

$$F_2(x_0, x_1, x_2)x_3 + F_3(x_0, x_1, x_2) = 0, \tag{2.1}$$

where the F_i are homogeneous of degree i and $F_2 = 0$ defines a smooth conic. Projection from P is a birational isomorphism $S \dashrightarrow \mathbb{P}^2$ with inverse given by:

$$\mathbb{P}^2 \dashrightarrow S, \quad x = (x_0, x_1, x_2) \mapsto \big(F_2(x)x_0, F_2(x)x_1, F_2(x)x_2, -F_3(x)\big).$$

It is a rational map given by the linear system of cubics through $B = (F_2 = 0) \cap (F_3 = 0)$. The inverse image of a point in B is a line on S. There are at most two nodes on a line in S which implies that each point in B has multiplicity at most 2. In particular, S has at most 4 nodes. It also follows easily from considering equation (2.1) that other nodes of S appear only when the cubic defined by F_3 is simply tangent to the conic defined by F_2. Equivalently, S can be obtained as the blow-up 6 points on a conic, where among the points there could be infinitely near points of order at most 2.

Let S be a nodal cubic surface and let $\tilde{S} \to S$ be the desingularization of S. The fibre over a node is a (-2)-curve, i.e. a smooth rational curve with selfintersection -2. The rational map $S \dashrightarrow \mathbb{P}^2$ defines a morphism $\pi : \tilde{S} \to \mathbb{P}^2$ which is the composition of birational morphisms

$$\pi : \tilde{S} = \tilde{S}_0 \to \tilde{S}_1 \to \cdots \to \tilde{S}_6 = \mathbb{P}^2,$$

where each $\pi_i : \tilde{S}_{i-1} \to \tilde{S}_i$, $i = 1, \ldots, 6$, is the blow-down of an exceptional curve of the first kind (a (-1)-curve for short).

Let $E_i' \subset \tilde{S}_i$ be the exceptional curve of π_i and put $E_i = (\pi_{i-1} \circ \cdots \circ \pi_1)^*(E_i')$. Let e_i be the divisor class of E_i and let e_0 be the divisor class of the pre-image of a line $l \subset \mathbb{P}^2$ under π. The classes $e_0, e_1, \ldots, e_6$ form an orthonormal basis in

$$H^2(\tilde{S}, \mathbb{Z}) = \mathrm{Pic}(\tilde{S}) = \mathbb{Z}e_0 \oplus \mathbb{Z}e_1 \oplus \cdots \oplus \mathbb{Z}e_6$$

in the sense that $e_0^2 = 1$, $e_i^2 = -1$, $i \neq 0$, $(e_i, e_j) = 0$, $i \neq j$. The canonical class $K_{\tilde{S}}$ of $\tilde{S}$ is equal to $-3e_0 + e_1 + \cdots + e_6$.

The anti-canonical map $\tilde{S} \to \mathbb{P}^3$ maps $\tilde{S}$ onto S and contracts the (-2)-curves to nodes. In particular, $K_{\tilde{S}}$ is orthogonal to the class of each (-2)-curve. Such a class is, up to sign, one of the following 36 classes:

$$(2.2) \qquad e_i - e_j, \quad e_0 - e_i - e_j - e_k, \quad 2e_0 - e_1 - e_2 - \cdots - e_6,$$

with $1 \leqq i < j < k \leqq 6$. Let $p_i = \pi(E_i) \in \mathbb{P}^2$. Then $e_i - e_j$, $i > j$, is effective if and only if p_i and p_j coincide, $e_0 - e_i - e_j - e_k$ is effective if and only if the points p_i, p_j and p_k are on a line and $2e_0 - e_1 - e_2 - \cdots - e_6$ is effective if and only if the six points $p_1, \ldots, p_6$ are on a conic.

2.2. Geometric markings. A minimal resolution of a nodal cubic surface is a Del Pezzo surface of degree 3. In this paper a Del Pezzo surface of degree d is a smooth surface X with $-K_X$ nef and $K_X^2 = d > 0$. For $d \geqq 3$, the anti-canonical linear system $|-K_X|$ maps X birationally to a surface of degree d in $\mathbb{P}^d$ with at most rational double points as singularities. Notice that we do not assume that $-K_X$ is ample, in that case one should call X a Fano surface. It is known that a Del Pezzo surface admits a birational morphism $\pi : X \to \mathbb{P}^2$ as in 2.1. A choice of such π and its decomposition $\pi = \pi_{9-d} \circ \cdots \circ \pi_1$ is called a *geometric marking* of X. Two geometric markings $X = X_0 \to X_1 \to \cdots \to X_{9-d} = \mathbb{P}^2$ and $X' = X_0' \to X_1' \to \cdots \to X_{9-d}' = \mathbb{P}^2$ are called *isomorphic* if there exist isomorphisms $\phi_i : X_i \to X_i'$, $i = 0, \ldots, 9-d$, such that $\pi_{i+1}' \circ \phi_i = \phi_{i+1} \circ \pi_{i+1}$, $i = 0, \ldots, 9-d-1$.

2.3. Lattice markings. The Picard lattice of a Del Pezzo surface X of degree d is isomorphic to

$$I_{1,9-d} = \langle 1 \rangle \oplus \langle -1 \rangle^{9-d},$$

the standard odd unimodular hyperbolic lattice with the standard orthonormal basis $(\boldsymbol{e}_0, \ldots, \boldsymbol{e}_{9-d})$. Let $k = -3\boldsymbol{e}_0 + \boldsymbol{e}_1 + \cdots + \boldsymbol{e}_{9-d}$. Let $k^\perp$ be the orthogonal complement of $\mathbb{Z}k$ in $I_{1,9-d}$. Assume $d \leqq 6$. Then the sublattice $k^\perp$ is isomorphic to E_{9-d}, where E_{9-d} is the root lattice E_{9-d} if $d = 1, 2, 3$, the root lattice D_5 if $d = 4$, the root lattice A_4 if $d = 5$, and the root lattice $A_2 + A_1$ if $d = 6$, spanned by vectors $\boldsymbol{e}_0 - \boldsymbol{e}_1 - \boldsymbol{e}_2 - \boldsymbol{e}_3$, $\boldsymbol{e}_1 - \boldsymbol{e}_2, \ldots, \boldsymbol{e}_{9-d+1} - \boldsymbol{e}_{9-d}$. A *lattice marking* of a Del Pezzo surface X is an isometry

$$\phi : I_{1,9-d} \to \mathrm{Pic}(X), \quad \text{such that } \phi(k) = K_X.$$

In particular, the restriction of ϕ to $k^\perp$ is an isometry $k^\perp \to K_X^\perp$.

A geometric marking defines a lattice marking by $\phi(\boldsymbol{e}_i) = e_i$ with e_i as in 2.1.

Let $\mathrm{W}(X)$ be the subgroup of the orthogonal group of $\mathrm{Pic}(X)$ generated by reflections in the classes of the (-2)-curves on X. Two lattice markings $\phi, \phi' : I_{1,9-d} \to \mathrm{Pic}(X)$ are called *equivalent* if there exists an element $\sigma \in \mathrm{W}(X)$ such that $\phi = \sigma \circ \phi'$.

The proof of the following result can be found in [Lo].

2.4. Proposition. *Let X be a Del Pezzo surface. Then there is a natural bijection between the isomorphism classes of geometric markings and equivalence classes of lattice markings on X.*

2.5. The moduli space of marked smooth cubics. We denote by $\mathcal{M}^m_{\mathrm{cub}}$ the moduli space of marked smooth cubic surfaces. Its points correspond to isomorphism classes of pairs (S, ϕ), where S is a smooth cubic surface and ϕ is a lattice marking of S. There is an isomorphism:

$$\mathcal{M}^m_{\mathrm{cub}} \cong \big((\mathbb{P}^2)^6 - \Delta\big)/\mathrm{SL}(3), \quad (S, \phi) \mapsto (p_1, \dots, p_6),$$

where the $p_i \in \mathbb{P}^2$ are the images of the lines with classes $\phi(\boldsymbol{e}_i) \in \mathrm{Pic}(S)$ in the blow-down $\mathbb{P}^2$ of S and Δ is the set of 6-tuples of points where either two points coincide, or three are on a line or all six are on a conic. The inverse image of a 6-tuple consists of the surface S obtained by blowing up the points p_i and the marking is defined by putting $\phi(\boldsymbol{e}_i)$ equal to the class of the exceptional divisor over p_i.

2.6. The Cremona action on $\mathcal{M}^m_{\mathrm{cub}}$. The Weyl group $\mathrm{W}(E_6)$ is the subgroup of $\mathrm{O}(I_{1,6})$ which fixes the element $k \in I_{1,6}$. It acts naturally on $\mathcal{M}^m_{\mathrm{cub}}$ by composing a lattice marking with (the inverse of) an isometry in $\mathrm{W}(E_6)$:

$$\mathrm{W}(E_6) \to \mathrm{Aut}(\mathcal{M}^m_{\mathrm{cub}}), \quad \sigma \mapsto [(S, \phi) \mapsto (S, \phi \circ \sigma^{-1})].$$

Equivalently, $\mathrm{W}(E_6)$ acts via the Cremona action on 6 ordered points in $\mathbb{P}^2$ (see [DO]). From now on we will simply identify $\mathrm{W}(E_6)$ with its image in $\mathrm{Aut}(\mathcal{M}^m_{\mathrm{cub}})$.

The quotient of $\mathcal{M}^m_{\mathrm{cub}}$ by $\mathrm{W}(E_6)$ is the moduli space of smooth cubic surfaces $\mathcal{M}_{\mathrm{cub}}$. Let p_{cub} be this quotient map:

$$p_{\mathrm{cub}} : \mathcal{M}^m_{\mathrm{cub}} \to \mathcal{M}^m_{\mathrm{cub}}/\mathrm{W}(E_6) \cong \mathcal{M}_{\mathrm{cub}}.$$

2.7. The GIT compactification. Geometric Invariant Theory provides a natural compactification of the moduli space of cubic surfaces $\mathcal{M}_{\mathrm{cub}}$:

$$\bar{\mathcal{M}}_{\mathrm{cub}} = \mathbb{P}\big(H^0\big(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(3)\big)\big)^{\mathrm{ss}}//\mathrm{SL}(4).$$

The stable points in $\mathbb{P}\big(H^0\big(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(3)\big)\big)$ are the nodal cubic surfaces. Points in

$$\mathcal{M}_{\mathrm{ncub}} = \mathbb{P}\big(H^0\big(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(3)\big)\big)^{\mathrm{s}}//\mathrm{SL}(4)$$

are thus isomorphism classes of nodal cubic surfaces. The strictly semi-stable points all map to one point in $\bar{\mathcal{M}}_{\mathrm{cub}}$. The corresponding minimal closed orbit is the orbit of the cubic surface with equation

(2.3) $$w^3 + xyz = 0.$$

The complement of this point in $\bar{\mathscr{M}}_{\text{cub}}$ is denoted by $\mathscr{M}_{\text{ncub}}$, the moduli space of nodal cubic surfaces.

The explicit computation of invariants of cubic quaternary forms, due to A. Cayley and G. Salmon [Sa1] (see a modern account in [Be]), gives an isomorphism

(2.4) $$\bar{\mathscr{M}}_{\text{cub}} \cong \mathbb{P}(1, 2, 3, 4, 5).$$

The moduli space of nonsingular surfaces is isomorphic to the complement of a hypersurface of degree 4 defined by the discriminant. In particular, $\mathscr{M}_{\text{cub}}$ is affine.

2.8. Moduli of marked nodal cubics. We can construct the moduli space of marked nodal cubic surfaces as follows. Let $\mathbb{C}(\mathscr{M}^m_{\text{cub}})$ be the field of rational functions of $\mathscr{M}^m_{\text{cub}}$. It is an extension, with Galois group $\mathrm{W}(E_6)$, of $\mathbb{C}(\mathscr{M}_{\text{cub}}) = \mathbb{C}(\bar{\mathscr{M}}_{\text{cub}})$. Now we define $\bar{\mathscr{M}}^m_{\text{cub}}$ to be the normalisation of $\bar{\mathscr{M}}_{\text{cub}}$ in the field $\mathbb{C}(\mathscr{M}^m_{\text{cub}})$.

By its definition, $\bar{\mathscr{M}}^m_{\text{cub}}$ is a normal projective variety and, since $\mathscr{M}^m_{\text{cub}}$ is smooth (see sections 2.9 and 2.10), we have

$$\mathscr{M}^m_{\text{cub}} \hookrightarrow \bar{\mathscr{M}}^m_{\text{cub}},$$

the complement of $\mathscr{M}^m_{\text{cub}}$ will be called the boundary of $\bar{\mathscr{M}}^m_{\text{cub}}$. By construction, the Weyl group $\mathrm{W}(E_6)$ acts on $\bar{\mathscr{M}}^m_{\text{cub}}$ with quotient $\bar{\mathscr{M}}_{\text{cub}}$:

$$\bar{p}_{\text{cub}} : \bar{\mathscr{M}}^m_{\text{cub}} \to \bar{\mathscr{M}}_{\text{cub}} = \bar{\mathscr{M}}^m_{\text{cub}}/\mathrm{W}(E_6)$$

and $\bar{p}_{\text{cub}} = p_{\text{cub}}$ on the subvariety $\mathscr{M}^m_{\text{cub}}$. Finally we define the moduli space of marked nodal cubic surfaces to be:

$$\mathscr{M}^m_{\text{ncub}} := \bar{p}^{-1}(\mathscr{M}_{\text{ncub}}).$$

This moduli space is the complement of a finite set of points, called the cusps, in $\bar{\mathscr{M}}^m_{\text{cub}}$ and the cusps are all in one $\mathrm{W}(E_6)$-orbit.

Despite its abstract definition, the variety $\bar{\mathscr{M}}^m_{\text{cub}}$ is rather well-known. Below we present some other constructions of it, and we show that the points in $\mathscr{M}^m_{\text{ncub}}$ correspond to isomorphism classes of marked nodal cubic surfaces. We do not know whether $\mathscr{M}^m_{\text{ncub}}$ is the (coarse) moduli space of some functor.

2.9. Naruki's model. In [Nar], Naruki constructs a smooth, projective compactification of the moduli space $\mathscr{M}^m_{\text{cub}}$ which he calls the cross-ratio variety. Its boundary contains 40 disjoint divisors which can be blown down to 40 singular points of a normal variety $\mathscr{N}$. At each singular point $\mathscr{N}$ is locally isomorphic to a cone over a Segre embedding of $(\mathbb{P}^1)^3$. Naruki also shows that the action of $\mathrm{W}(E_6)$ on $\mathscr{M}^m_{\text{cub}}$ $(\subset \mathscr{N})$ extends to a biregular action on $\mathscr{N}$ with quotient $\mathscr{N}/\mathrm{W}(E_6) = \bar{\mathscr{M}}_{\text{cub}}$. Using the universal property of the normalization in a field extension and Zariski's Main Theorem (see [Mu], III.9, Proposition 1) we obtain an isomorphism

$$\phi_{\mathcal{N}} : \mathcal{N} \xrightarrow{\sim} \bar{\mathcal{M}}^m_{\mathrm{cub}}.$$

From Naruki's description of $\mathcal{N}$, see also [CvG], one obtains that the forty singular points of $\mathcal{N}$ map to the cusps of $\bar{\mathcal{M}}^m_{\mathrm{cub}}$. Moreover, the boundary of $\bar{\mathcal{M}}^m_{\mathrm{cub}}$ consists of 36 divisors, each of which is isomorphic to the Segre cubic threefold $\mathcal{S}_3$, best seen as a subvariety of $\mathbb{P}^5$:

$$\mathcal{S}_3\colon \quad \sum_{i=1}^{6} x_i = 0, \quad \sum_{i=1}^{6} x_i^3 = 0. \tag{2.5}$$

The group $\mathrm{W}(E_6)$ acts transitively on the set of 36 boundary divisors. The stabilizer of each of the 36 divisors is isomorphic to the permutation group S_6 which acts on $\mathcal{S}_3$ by permuting the coordinates. Also notice that there is an isomorphism ([DO])

$$\mathcal{S}_3 \cong (\mathbb{P}^1)^6//\mathrm{SL}(2). \tag{2.6}$$

Again, the action of S_6 on $\mathcal{S}_3$ is the natural one.

2.10. A GIT model. Since the interpretation of $\mathcal{N}\backslash\mathrm{Sing}(\mathcal{N})$ as the moduli space of marked nodal cubic surfaces is not obvious in Naruki's construction we sketch another model of $\bar{\mathcal{M}}^m_{\mathrm{ncub}}$, where this interpretation is more apparent. First we recall the explicit construction of the GIT-quotient $X = (\mathbb{P}^2)^6//\mathrm{SL}(3)$ given in [DO]. The graded ring of invariants

$$R = \bigoplus_{n=0}^{\infty} \left(H^0\left((\mathbb{P}^2)^6, \bigotimes_{i=1}^{6} \pi_i^* \mathcal{O}_{\mathbb{P}^2}(n) \right) \right)^{\mathrm{SL}(3)}$$

is generated by elements t_0, t_1, t_2, t_3, t_4 of degree 1 and one element t_5 of degree 2. Here π_i is the i-th projection from $(\mathbb{P}^2)^6$. The relation between the generators is $t_5^2 + F_4(t_0, t_1, t_2, t_3, t_4) = 0$, where F_4 is a homogeneous polynomial of degree 4. Thus X is isomorphic to a hypersurface of degree 4 in the weighted projective space $\mathbb{P} = \mathbb{P}(1,1,1,1,1,2)$. Note that the involution $t_5 \mapsto -t_5$ corresponds to the association (or the Gale transform) of the point sets (see [DO]). Its locus of fixed points is isomorphic to the quartic hypersurface $V(F_4)$ in $\mathbb{P}^4$ and parametrizes the self-associated point sets, i.e. point sets lying on a conic.

The quartic 3-fold $V(F_4)$ in $\mathbb{P}^4$ has 15 double lines l_{ij} corresponding to minimal semi-stable orbits of point sets $(p_1, \dots, p_6)$ where $p_i = p_j$. Three lines l_{ij}, l_{kl}, l_{mn}, where $\{1,2,3,4,5,6\} = \{i,j\} \cup \{k,l\} \cup \{m,n\}$, intersect at one point $P_{ij,kl,mn}$. It represents the orbit of the point set $p_i = p_j$, $p_k = p_l$, $p_m = p_n$. It follows from the explicit equation of F_4 that its local equation at $P_{ij,kl,mn}$ is given by $w^2 + z_1 z_2 z_3 = 0$, where $w = z_i = z_j = 0$ is the local equation of one of the 3 double lines meeting at the point. This implies that X is given locally at the point $P'_{ij,kl,mn} = (P_{ij,kl,mn}, 0)$ by the equation $uv + xyz = 0$.

Let Z be the singular locus of X and $\mathcal{I}_Z$ its sheaf of ideals. One considers the linear system $|\mathcal{I}_Z(3)| \subset R_3$. A. Coble [Co] gives explicitly 40 elements of $|\mathcal{I}_Z(3)|$ which span a $\mathbb{P}V \cong \mathbb{P}^9$ and shows that the birational action of $\mathrm{W}(E_6)$ on X induces a linear action on V.

We construct the moduli space of marked cubic surfaces as the image Y of X under the rational map given by the linear system $\mathbb{P}V$.

First we blow up the ambient space $\mathbb{P}$ at the points $P'_{ij,kl,mn}$. Let $E_{ij,kl,mn} \cong \mathbb{P}^4$ be the exceptional divisor at the point $P'_{ij,kl,mn}$. The proper inverse transform X_1 of X intersects each $E_{ij,kl,mn}$ along the union of two hyperplanes $H_{ij,kl,mn}$, $H'_{ij,kl,mn}$ corresponding to the tangent cone of the singular point. The proper inverse transforms of the lines l_{ij} are double curves C_{ij} on X_1. Each of the curves C_{ij}, C_{kl}, C_{mn} intersects $E_{ij,kl,mn}$ at a point. The three points span the plane $\Pi_{ij,kl,mn} = H_{ij,kl,mn} \cap H'_{ij,kl,mn}$. Next we blow up the 15 singular curves C_{ab} to get a variety X_2. The proper inverse transform of the linear system $\mathbb{P}V$ in X_2 has base locus equal to the union of the proper transforms $\bar{\Pi}_{ij,kl,mn}$ of the planes $\Pi_{ij,kl,mn}$. Each surface $\bar{\Pi}_{ij,kl,mn}$ is isomorphic to the blow-up of 3 points on the plane. The proper transforms of the lines joining three pairs of points are double curves of X_2. Next we blow up the surfaces $\bar{\Pi}_{ij,kl,mn}$ to get a nonsingular variety X_3. Now the proper inverse transforms of the hyperplanes $H_{ij,kl,mn}$, $H'_{ij,kl,mn}$ become separated and the proper inverse transform of the linear system $\mathbb{P}V$ has no base points.

Let $Y \subset \mathbb{P}^9$ be the image of X_3 under this linear system. Observe first that Y is a compactification of the geometric quotient $\mathscr{M}^m_{\mathrm{cub}} = U/\mathrm{SL}(3)$, where $U = (\mathbb{P}^2)^6 - \Delta$ as in 2.5.

Next we shall see its complement. First of all we have 20 divisors D'_{ijk} in X representing 6-tuples of points where p_i, p_j, p_k are collinear. The sum of the two divisors D'_{ijk} and D'_{lmn}, where $\{i,j,k\} \cup \{l,m,n\} = \{1,\dots,6\}$, is defined by a linear function $L_{ijk} = L_{lmn} \in R_1$ (see [DO]). The corresponding hyperplane $V(L_{ijk})$ cuts out the quartic $V(F_4)$ along a nonsingular quadric $Q_{ijk} = Q_{lmn}$. The quadric contains 6 double lines l_{ij}, l_{ik}, l_{jk}, l_{lm}, l_{ln}, l_{mn}. Let D_{ijk} be the proper inverse transforms of D'_{ijk} in Y. Let D_{ij} be the proper inverse transforms in X_3 of the pre-images of the curves C_{ij} in X_2. We have 15 such divisors. Finally, let D_0 be the proper inverse transform of $V(t_5) \cong V(F_4)$ in Y. It is easy to see that under the map $X_3 \to Y$ the proper inverse transforms of the quadrics Q_{ijk} are blown down to points $c_{ijk} = c_{lmn}$. Also let $c_{ij,kl,mn}$, $c'_{ij,kl,mn}$ be the images in Y of the hyperplanes $H_{ij,kl,mn}$, $H'_{ij,kl,mn}$. Altogether we have 40 points which we call the cusps. The forty cusps is the set of singular points of the variety Y. So, we see that the complement of the image of $U/\mathrm{SL}(3)$ in Y is equal to the union of 36 divisors D_{ijk}, D_{ij}, D_0.

The Weyl group $\mathrm{W}(E_6)$ acts on Y interchanging the boundary divisors. This makes them all isomorphic to each other. This is easy to check. The restriction of the linear system $\mathbb{P}V$ to the quartic $V(F_4)$ is the map given by the partials of F_4. It maps $V(F_4)$ to the dual variety known to be isomorphic to the Segre cubic $\mathscr{S}_3 \subset \mathbb{P}^4$. This shows that $D_0 \cong \mathscr{S}_3$.

One can check that the variety Y is a normal proper $\mathrm{W}(E_6)$-variety containing the $\mathrm{W}(E_6)$-variety $\mathscr{M}^m_{\mathrm{cub}}$ as an open subset. Thus there is a birational morphism $f : Y \to \bar{\mathscr{M}}^m_{\mathrm{cub}}$. We claim that f is an isomorphism. Let E be an irreducible component of the exceptional locus of f. It is contained in one of the 36 boundary divisors D. However $D \cong \mathscr{S}_3$ has $\mathrm{Pic}(D) \cong \mathbb{Z}$. Nothing can be blown down on D. Thus we obtain that

$$Y \cong \bar{\mathscr{M}}^m_{\mathrm{cub}}. \tag{2.7}$$

2.11. Remark. In [ACT], $\mathcal{M}^m_{\text{cub}}$ is identified with an open subset of a smooth ball quotient. In [AF] Allcock and Freitag show, using modular forms constructed via a Borcherds lift, that this ball quotient embeds into a $\mathbb{P}^9$ and that the closure of its image is isomorphic to the Satake compactification of the ball quotient, the boundary consists of 40 singular points. Freitag [F] proved that ideal of the image of the ball quotient is generated by explicitly given cubics and that it is a normal variety.

Coble, in [Co], defines a rational map $(\mathbb{P}^2)^6 \to \mathbb{P}^9$ which is SL(3)-invariant and hence factors over $\mathcal{M}_{\text{cub}}$. It is easily seen to be a birational isomorphism between $\mathcal{M}^m_{\text{cub}}$ and its image. This map is moreover equivariant with respect to the Cremona action of $W(E_6)$. See also [Y] where in particular the restriction to a boundary divisor is worked out. It is easy to verify that the image of $\mathcal{M}^m_{\text{cub}}$ lies in the subvariety defined by the cubics.

In [vG2] the corresponding rational functions on Naruki's variety $\mathcal{N} \cong \bar{\mathcal{M}}^m_{\text{cub}}$ are explicitly identified, and also the 40 functions used by Coble are given.

Matsumoto and Terasoma [MT] showed how to get this embedding via an embedding of the complex ball into the Siegel space (of genus 5) followed by a map to $\mathbb{P}^9$ given by explicitly determined theta constants.

2.12. Boundary divisors. Since a node of S corresponds to a (-2)-curve in $K^{\perp}_{\tilde{S}}$, the 36 boundary divisors are parametrized by the 36 positive simple roots of E_6. Let α be one of the 36 positive roots (see (2.2)). To each α we assign the divisor D_α in $\mathcal{M}^m_{\text{ncub}}$, we write:

$$D_\alpha = \begin{cases} D_{ij} & \text{if } \alpha = e_i - e_j, \\ D_{ijk} & \text{if } \alpha = e_0 - e_i - e_j - e_k, \\ D_0 & \text{if } \alpha = 2e_0 - e_1 - e_2 - e_3 - e_4 - e_5 - e_6. \end{cases}$$

Each D_α parametrizes marked nodal cubic surfaces (S, ϕ) for which $\phi(\alpha)$ is effective. The isomorphism between D_α and the Segre cubic $\mathcal{S}_3$ becomes apparent and the isomorphism (2.6) is the natural isomorphism between D_0 and $(\mathbb{P}^1)^6/\!/\mathrm{SL}(2)$. Of course all divisors D_α are mutually isomorphic, being permuted by the action of $W(E_6)$.

If $\phi(\alpha)$ is effective and r_α denotes the reflection in $W(E_6)$ defined by the root α, then the lattice marked nodal cubic surfaces (S, ϕ) and $(S, \phi \circ r_\alpha)$ are equivalent. This suggests that in the Cremona action of $W(E_6)$ on $\mathcal{M}^m_{\text{ncub}}$ the reflection r_α acts identically on D_α. This is in fact the case ([Nar], p. 22).

The Segre cubic has 10 nodes p_{1ij}, for example, $p_{125} = (1 : 1 : -1 : -1 : 1 : -1)$, corresponding to the minimal orbit of sixtuples $(p_1, \ldots, p_6)$ of points on $\mathbb{P}^1$ such that $p_1 = p_i = p_j$, $p_l = p_m = p_n$. Identifying $\mathcal{S}_3$ with D_0, the nodes of $\mathcal{S}_3$ are the cusps of $\bar{\mathcal{M}}^m_{\text{cub}}$ lying on D_0.

The image $\bar{p}(D)$ of a boundary divisor in $\bar{\mathcal{M}}_{\text{cub}}$ is the locus of singular cubic surfaces. It is defined by the vanishing of the discriminant invariant on the space of cubic surfaces, which is of degree 32 in the coefficients of the cubic form. In the isomorphism

(2.4) it corresponds to the hyperplane defined by the unknown with weight 4. Thus $\bar{p}(D)$ is isomorphic to $\mathbb{P}(1,2,3,5)$. On the other hand, if we identify D with the Segre cubic $\mathscr{S}_3$, and the stabilizer of D in $\mathrm{W}(E_6)$ with the permutation group S_6 (see the next subsection), we see that $\mathbb{P}(1,2,3,5)$ must be isomorphic to $\mathscr{S}_3/S_6$. This is easy to see: the group S_6 acts on $\mathscr{S}_3$ given by equations (2.5) by permuting the coordinates. This easily implies that the subring of invariants of the homogeneous coordinate ring of $\mathscr{S}_3$ is generated by elementary symmetric polynomials of degree 2, 4, 5, 6 and hence $\mathscr{S}_3/S_6 \cong \mathbb{P}(2,4,5,6) \cong \mathbb{P}(1,2,3,5)$.

2.13. Moduli of r-nodal cubics. The irreducible components of the locus of marked nodal cubics with r nodes are parametrized by unordered subsets of r orthogonal roots (up to sign) in E_6. We denote by $D_{\alpha_1,\dots,\alpha_r}$ the intersection of the divisors $D_{\alpha_1},\dots,D_{\alpha_r}$ corresponding to r orthogonal roots $\alpha_1,\dots,\alpha_r$.

The stabilizer in $\mathrm{W}(E_6)$ of such a locus $D_{\alpha_1,\dots,\alpha_r}$ is the product of the subgroup of order 2^r, generated by the corresponding r roots (this subgroup acts trivially on the component), the permutations on r roots $\alpha_1,\dots,\alpha_r$ ($\cong S_r$) and the subgroup generated by reflections in the roots orthogonal to the r simple roots. The stabilizer modulo the subgroup of order 2^r is the group of permutations of geometric markings on S.

In case $r=1$, the 30 roots $e_i - e_j$ are all orthogonal to the root $\alpha = 2e_0 - e_1 - \cdots - e_6$, so we see that $\mathbb{Z}/2\mathbb{Z} \times \mathrm{W}(A_5) \cong \mathbb{Z}/2\mathbb{Z} \times S_6$ acts on D_α. Thus we recover the fact that $\mathrm{W}(A_5) \cong S_6$ acts on a boundary divisor.

In case $r=2$, there are 12 roots $e_i - e_j$ $(3 \leqq i, j \leqq 6)$ orthogonal to the two roots $\alpha_1 = 2e_0 - e_1 - \cdots - e_6$ and $\alpha_2 = e_1 - e_2$. Together with α_1, α_2 they generate the root sublattice $A_1^2 \oplus A_3$ of E_6. So the subgroup of $\mathrm{W}(E_6)$ leaving this sublattice invariant is isomorphic to $(\mathbb{Z}/2\mathbb{Z})^2 \cdot S_2 \times \mathrm{W}(A_3) \simeq (\mathbb{Z}/2\mathbb{Z})^2 \cdot S_2 \times S_4$ and it acts on D_{α_1,α_2}.

In case $r=3$, there are two roots $\pm(e_5 - e_6)$ orthogonal to the three roots $\alpha_1 = 2e_0 - e_1 - \cdots - e_6$, $\alpha_2 = e_1 - e_2$ and $\alpha_3 = e_3 - e_4$. Together with α_1, α_2, α_3 they generate a root system of type A_1^4. So $(\mathbb{Z}/2\mathbb{Z})^3 \cdot S_3 \times \mathbb{Z}/2\mathbb{Z}$ acts on $D_{\alpha_1,\alpha_2,\alpha_3}$.

In case $r=4$, there are no roots orthogonal to the four roots $\alpha_1 = 2e_0 - e_1 - \cdots - e_6$, $\alpha_2 = e_1 - e_2$, $\alpha_3 = e_3 - e_4$ and $\alpha_4 = e_5 - e_6$. So $(\mathbb{Z}/2\mathbb{Z})^4 \cdot S_4$ acts on $D_{\alpha_1,\dots,\alpha_4}$.

2.14. Lines on a nodal cubic surface. A nonsingular cubic surface contains 27 lines. They represent the classes $e_0 - e_i - e_j$, $1 \leqq i < j \leqq 6$, e_i, $2e_0 - e_1 - \cdots - e_6 + e_i$, $i = 1,\dots,6$.

Assume now that S has a node s_0. Projecting from s_0, we see that $\tilde{S}$ admits a geometric marking $\pi : \tilde{S} \to \mathbb{P}^2$ such that the images p_i of the E_i (as in 2.1) lie on an irreducible conic C. If S has no more nodes, the six points p_i are distinct. If there is one more node, we may assume without loss of generality that p_2 is infinitely near to p_1 (i.e. $E_2 = E_1 + C$, where C is a (-2)-curve and the point p_2 corresponds to the tangent direction of C at p_1). If S has three nodes we can further assume that p_4 is infinitely near to p_3 with the similar tangency condition. Finally if S has 4 nodes we can further assume that p_6 is infinitely near to p_5. From this we easily deduce the following facts.

If S has one node, there are 21 lines on S. Six of them contain the node, and are represented by the exceptional curves $E_i = \phi(\boldsymbol{e}_i)$, where ϕ is the lattice marking corresponding to the geometric marking. We will simply omit ϕ in what follows. The remaining 15 lines have the classes $e_0 - e_i - e_j$. The (-2)-curve C has class $\alpha_1 = 2e_0 - (e_1 + \cdots + e_6)$ and the classes $e_i + \alpha_1 = s_{\alpha_1}(e_i)$ also represent the lines on the node. So the lines on the nodes are limits of pairs of lines on a smooth cubic surface.

If S has 2 nodes, there are 16 lines on S. The (-2)-curves are $\alpha_1 = 2e_0 - (e_1 + \cdots + e_6)$ and $e_2 - e_1$, the orbits on the set of classes of 27 lines of the group generated by s_{α_1} and s_{α_2} correspond to the lines on S. One line connects the two nodes and represents the orbit $\{e_1, e_2 = e_1 + \alpha_2, e_1 + \alpha_1, e_2 + \alpha_1\}$. There are 4 lines passing through the node s_0 which represent the orbits $\{e_i, e_i + \alpha_1\}$, $i = 3, 4, 5, 6$. Another 4 lines pass through the second node. They represent the orbits $\{e_0 - e_2 - e_i, e_0 - e_1 - e_i\}$, $i = 3, 4, 5, 6$. The remaining 7 lines do not contain nodes. They represent orbits with one element, given by the classes $e_0 - e_i - e_j$, $3 \leqq i < j \leqq 6$ and $e_0 - e_1 - e_2$.

If S has 3 nodes, there are 12 lines. There are 3 lines connecting pairs of nodes. They represent the classes e_1, e_3, $e_0 - e_1 - e_3$. There are 6 lines each containing one node. They represent the classes e_5, e_6, $e_0 - e_1 - e_i$, $e_0 - e_3 - e_i$, $i = 5, 6$. The remaining 3 lines do not contain nodes. They represent the classes $e_0 - e_1 - e_2$, $e_0 - e_3 - e_4$, $e_0 - e_5 - e_6$.

If S has 4 nodes there are 9 lines. Six of them connect the six pairs of nodes. They represent the classes e_1, e_3, e_5, $e_0 - e_1 - e_3$, $e_0 - e_1 - e_5$, $e_0 - e_3 - e_5$. The remaining three lines do not contain nodes and represent the classes $e_0 - e_1 - e_2$, $e_0 - e_3 - e_4$, $e_0 - e_5 - e_6$.

2.15. Pencils of conics. A conic on a nodal cubic surface S is cut out by a plane. The residual component of the plane section is a line. The pencil of planes through this line defines a pencil of conics. Thus the number of pencils of conics is equal to the number of lines. The preimage of the pencil on $\tilde{S}$ is a conic bundle, i.e. a morphism $f : \tilde{S} \to \mathbb{P}^1$ with general fibre isomorphic to $\mathbb{P}^1$. A standard computation shows that singular fibres of f are of the following three types:

Type I: $F = E_1 + E_2$, where E_1, E_2 are two (-1)-curves and $E_1 \cdot E_2 = 1$.

Type II: $F = E_1 + E_2 + R$, where E_1, E_2 are (-1)-curves, R is a (-2)-curve, $E_1 \cdot E_2 = 0$, $E_1 \cdot R = E_2 \cdot R = 1$.

Type III: $F = R_1 + R_2 + 2E$, where R_1, R_2 are (-2)-curves, E is a (-1)-curve, $R_1 \cdot R_2 = 0$, $R_1 \cdot E = R_2 \cdot E = 1$.

The number of singular fibres is equal to 5 if we count the fibres of type II and III with multiplicity 2.

The pre-image of the line l corresponding to the pencil defines a bisection B of f. There are three possible cases:

No nodes on l: B is irreducible.

One node on l: $B = B_0 + R$, where B_0 is a (-1)-curve, R is a (-2)-curve, $B_0 \cdot R = 1$. Each component of B is a section of f.

Two nodes on l: $B = B_0 + R_1 + R_2$, where B_0 is a (-1)-curve, R_1, R_2 are (-2)-curves, $B_0 \cdot R_1 = B_0 \cdot R_2 = 1$. The components R_1 and R_2 are sections of f. The component B_0 is contained in a fibre.

Let $p_1, \dots, p_s \in \mathbb{P}^1$ be the points such that the fibre $f^{-1}(p_i)$ is singular. We assign to each point p_i the multiplicity m_i equal to 2 if the fibre is of type I and equal to 4 otherwise. The divisor $D = \sum_{s=1}^{s} m_i p_i$ will be called the *discriminant* of the conic pencil. Let $p_{s+1}, p_{s+2} \in \mathbb{P}^1$ be the points such that the bisection B ramifies over these points. If B is reducible, we assume that $p_{s+1} = p_{s+2} = q$, where B has a singular point over q. The divisor $T = p_{s+1} + p_{s+2}$ will be called the *bisection branch divisor*. Let us write the divisor $D + T = \sum_{i=1}^{s} m_i p_i + p_{s+1} + p_{s+2}$ as $\sum_{i=1}^{s'} n_i p_i$, where $s' \leqq s + 2$. We order the points in such a way that $n_1 \geqq n_2 \geqq \cdots \geqq n_{s'}$. The vector $\boldsymbol{t} = (n_1, \dots, n_{s'})$ will be called the *type vector* of the conic pencil.

Table 1 below lists all possible type vectors. Also we indicate the total number r of nodes on S, the number e of Eckardt points on l (i.e. points where three lines meet).

The column "Kodaira fibres" will be explained later in section 4.3.

2.16. Types of lines. Let l be a line defining the pencil of conics.

Case 1), 2), 3) in Table 1: l is any line.

Case 4): l is one of 6 lines containing the node.

Case 5), 6), 7): l is one of 15 lines not passing through the node.

Case 8): l is one of 8 lines through exactly one node.

Case 8*): l is the unique line containing two nodes.

Case 9), 11): l is one of 6 lines not containing a node and not meeting the line of type 8*).

Case 10), 12): l is the unique line not containing a node and meeting the line of type 8*).

Case 13): l is one of 6 lines passing exactly through one node.

Case 13*): l is one of 3 lines passing through two nodes.

Case 14, 15): l is one of 3 lines not containing a node.

Case 16): l is one of 6 lines passing through two nodes.

Case 17): l is one of 3 lines not containing a node.

	t	singular fibres	Kodaira fibres	r	e
1)	(2222211)	5I	5IV, 2II	0	0
2)	(322221)	5I	I_0^*, 4IV, II	0	1
3)	(33222)	5I	$2I_0^*$, 3IV	0	2
4)	(222222)	5I	6IV	1	0
5)	(422211)	II, 3I	IV^*, 3IV, 2II	1	0
6)	(43221)	II, 3I	IV^*, I_0^*, 2IV, II	1	1
7)	(4332)	II, 3I	IV^*, $2I_0^*$, IV	1	2
8)	(42222)	II, 3I	IV^*, 4IV	2	0
8*)	(42222)	5I	IV^*, 4IV	2	0
9)	(44211)	2II, I	$2IV^*$, IV, 2II	2	0
10)	(52221)	III, 3I	II^*, 3IV, II	2	0
11)	(4431)	2II, I	$2IV^*$, I_0^*, II	2	1
12)	(5322)	III, 3I	II^*, I_0^*, 2IV	2	1
13)	(4422)	2II, I	$2IV^*$, 2IV	3	0
13*)	(4422)	II, 3I	$2IV^*$, 2IV	3	0
14)	(5421)	III, II, I	II^*, IV^*, IV, II	3	0
15)	(543)	III, II, I	II^*, IV^*, I_0^*	3	1
16)	(444)	2II, I	$3IV^*$	4	0
17)	(552)	2III, I	$2II^*$, IV	4	0

Table 1. Pencils of conics

3. Cubic surfaces and 2+5 points on the line

3.1. The forms (F_2, F_5). Let S be a nodal cubic surface and let l be a line on S. Consider the pencil of conics through the line l, cf. section 2.15. Let $D = \sum_{i=1}^{s} m_i p_i$ be its discriminant divisor and let $T = p_{s+1} + p_{s+2}$ be the bisection branch divisor. Let $F_5(x_0, x_1)$ be a homogeneous form of degree 5 defining D and let $F_2(x_0, x_1)$ be a homogeneous form of degree 2 defining T.

It follows from section 2.15 that the following properties are satisfied:

(i) $F_2 \neq 0$.

(ii) F_5 has at most double roots.

(iii) F_2 and F_5 do not have common multiple roots.

A pair of binary forms (F_5, F_2) satisfying properties (i)–(iii) will be called a *stable pair*. Let $V(d)$ be the space of binary forms of degree d. A pair of nonzero binary forms (F_5, F_2) defines a point in $\mathbb{P}(V(5)) \times \mathbb{P}(V(2))$.

3.2. Proposition. *A pair of nonzero binary forms* (F_5, F_2) *is stable if and only if it is a stable point with respect to the diagonal action of* SL(2) *and the linearization defined by the invertible sheaf* $\mathcal{O}_{\mathbb{P}(V(5))}(2) \boxtimes \mathcal{O}_{\mathbb{P}(V(2))}(1)$. *The strictly semistable points all map to one point in the quotient, the corresponding unique minimal closed orbit is the one of a pair* $(L_1^3 L_2^2, L_2^2)$ *with* L_1, L_2 *nonproportional linear forms.*

Proof. This easily follows from the Hilbert-Mumford numerical criterion of stability and is left to the reader. □

3.3. Line marked cubic surfaces. Let (S, ϕ) be a nodal cubic surface with a geometric marking ϕ on its minimal resolution and let l be a line on S with divisor class $\phi(e_6)$. The stabilizer of e_6 in $\mathrm{W}(E_6)$ is isomorphic to the Weyl group $\mathrm{W}(D_5)$. The quotient space

$$\mathcal{M}^1_{\text{ncub}} = \mathcal{M}^m_{\text{ncub}}/\mathrm{W}(D_5)$$

is the moduli space of isomorphism classes of pairs (S, l), where S is a nodal cubic surface and l is a line on it.

To a pair (S, l) we associate the binary forms F_2, F_5 as in 3.1. It is easy to see that this can be defined for families of (S, l) and therefore we have a morphism

$$\mathcal{M}^1_{\text{ncub}} \to \big(\mathbb{P}\big(V(2)\big) \times \mathbb{P}\big(V(5)\big)\big)^{\mathrm{s}}/\mathrm{SL}(2), \quad (S, l) \mapsto [(F_2, F_5)], \tag{3.1}$$

where $\big(\mathbb{P}\big(V(2)\big) \times \mathbb{P}\big(V(5)\big)\big)^{\mathrm{s}}$ is the open subset corresponding to stable pairs of binary forms.

3.4. Lemma. *Let* $f : X \to Y$ *be a birational surjective morphism with finite fibres. Assume that* X *and* Y *admit normal projective completions* $\overline{X}$ *and* $\overline{Y}$ *with zero-dimensional complements. Then* f *extends to an isomorphism* $\bar{f} : \overline{X} \to \overline{Y}$.

Proof. Let $\partial X = \overline{X} \backslash X$, $\partial Y = \overline{Y} \backslash Y$, these are finite sets. Let $\Gamma \subset X \times Y$ be the graph of f and let $\bar{\Gamma}$ be its closure in $\overline{X} \times \overline{Y}$. Obviously

$$\bar{\Gamma} \backslash \Gamma \subset X \times \partial Y \cup \partial X \times Y.$$

Moreover, since $\overline{X}$ and $\overline{Y}$ are normal and hence irreducible, $\bar{\Gamma}$ does not contain $X \times \{y_0\}$, for any $y_0 \in \partial Y$, nor $\{x_0\} \times Y$, for any $x_0 \in \partial X$. In particular, the first projection $p : \bar{\Gamma} \to \overline{X}$ is an isomorphism over an open subset of X and has finite, non-empty, fibres over X. By Zariski's Main Theorem ([Mu], III.9, Proposition 1), p is an isomorphism over X. Thus $p^{-1}(X) = \Gamma \subset \bar{\Gamma}$ is the graph of the composition $X \to Y \to \overline{Y}$.

Now we show that the projection $q : \bar{\Gamma} \to \overline{Y}$ is birational, surjective with finite fibres. The map q is a birational isomorphism since Γ is the graph of the birational isomorphism f and the complement of the set $\{y \in Y : (x_0, y) \in \bar{\Gamma}$ for some $x_0 \in \partial X\}$ contains a non-empty open subset of Y. The surjectivity is trivial since $q(\Gamma) = Y$ and $q(\bar{\Gamma})$ is closed in $\overline{Y}$. Let $y \in Y$ and let $(x, y) \in \bar{\Gamma}$. If $x \in X$, then $(x, y) \in p^{-1}(X)$, which is the graph of f so $y = f(x)$. Else $(x, y) \in \partial X \times \{y\}$ which is a finite set. Thus for $y \in Y$ the fiber $q^{-1}(y)$ is finite. As $p^{-1}(X)$ is the graph of f, a point $(x, y_0) \in \bar{\Gamma}$ with $y_0 \in \partial Y$ has $x \in \partial X$, hence also $q^{-1}(y_0)$ is finite.

We conclude, again by Zariski's Main Theorem, that q is an isomorphism. Thus $\bar{\Gamma}$ is the graph of a morphism $g = p \circ q^{-1} : \overline{Y} \to \overline{X}$ such that $g \circ f$ is the inclusion $X \subset \overline{X}$. Since $f(X) = Y$ we get $g(Y) = X$. By interchanging the role of f and g, we find that p is also an isomorphism. Hence g is an isomorphism. $\square$

3.5. Let

$$\bar{\mathcal{M}}^1_{\text{ncub}} = \bar{\mathcal{M}}^m_{\text{ncub}}/\mathrm{W}(D_5).$$

It is easy to see that $\mathrm{W}(D_5)$ acts transitively on the set of 40 cusps. For example, it follows easily from the well-known description of maximal subgroups of $\mathrm{W}(E_6)$ of index 40. Thus $\bar{\mathcal{M}}^1_{\text{ncub}}$ is a normal one-point compactification of $\mathcal{M}^1_{\text{ncub}}$. The corresponding point in $\bar{\mathcal{M}}_{\text{ncub}}$ is represented by the cubic surface (2.3). It has three lines permuted by the automorphism group of the cubic.

We also know from Proposition 3.2 that the target space in (3.1) admits a one-point normal compactification isomorphic to the GIT-quotient $\mathbb{P}(V(5))^{\text{ss}}//\mathrm{SL}(2)$.

3.6. Theorem. *The morphism* (3.1) *extends to an isomorphism*

$$\bar{\mathcal{M}}^1_{\text{ncub}} \to \big(\mathbb{P}(V(2)) \times \mathbb{P}(V(5))\big)^{\text{ss}}//\mathrm{SL}(2). \tag{3.2}$$

Proof. Applying Lemma 3.4 it is enough to check that the map (3.1) satisfies the assumption of the lemma. Assume that (S, l) is a nonsingular surface. Let us show how to reconstruct (S, l) from the $\mathrm{SL}(2)$-orbit of a pair (F_5, F_2). We view the zeroes of the binary forms as the tangent directions at a fixed point $p \in \mathbb{P}^2$ and identify them with the pencil of lines through p. Given (F_2, F_5), fix a conic Q not containing p such that the lines through p defined by F_2 are tangents of Q. Then a choice of 5 points $p_1, \dots, p_5$ on the intersection of the lines defined by F_5 with the conic, no two lying on the same line, defines uniquely (up to isomorphism) a cubic surface S with a line l corresponding to the conic. It is isomorphic to the blow-up of $\mathbb{P}^2$ at the points $p_1, \dots, p_5$, p. Let p_i' be the point on Q such that p_i, p_i', p are collinear. Let us show that replacing p_i with p_i' leads to an isomorphic pair (S', l').

Note that replacing $(p_1, \dots, p_5)$ with $(p_1', \dots, p_5')$ leads to the same surface because the points $(p_1, \dots, p_5, p)$ and $(p_1', \dots, p_5', p)$ are projectively equivalent. This can be easily seen by choosing projective coordinates such that $p = (0, 0, 1)$ and $Q = V(x_0x_1 - x_2^2)$. Then $p_i = (1, a_i^2, a_i)$ and $p_i' = (1, a_i^2, -a_i)$.

Now it is enough to show that fixing a pair $\{p_i, p_j\}$ and interchanging $p_k \mapsto p_k'$ for $k \neq i, j$ defines an isomorphic surface. Choose coordinates so that $p = (0, 0, 1)$, $p_i = (1, 0, 0)$, $p_j = (0, 1, 0)$. The equation of the conic Q through the points $p_1, \dots, p_5$ is, after scaling the coordinates,

$$z^2 + xy + a(x + y)z = 0, \quad a \neq 0$$

(use that the lines $\langle p, p_i \rangle$ and $\langle p, p_j \rangle$ are not tangent to Q). The Cremona transformation $T : (x, y, z) \mapsto (xz, yz, xy)$ with base points at p, p_i, p_j maps the conic Q to itself. A general line l through p is mapped to itself. As T is a non-trivial involution on such a line, it maps p_k to p_k' if $k \neq i, j$. The cubic surface obtained by blowing up the three points which are the images under T of the lines $\langle p_i, p_j \rangle$, $\langle p, p_j \rangle$ and $\langle p, p_i \rangle$ and the images of the three other

p_k is a cubic surface isomorphic to S. The images of the lines are $(0,0,1) = p$, $(0,1,0) = p_j$ and $(1,0,0) = p_i$ respectively, the images of the other three p_k are the p_k', so we get the result.

Thus we know that (3.1) is one-to-one on the open subset U equal to the pre-image of $\mathcal{M}_{\mathrm{cub}}$ under the projection $\mathcal{M}^1_{\mathrm{ncub}} \to \mathcal{M}_{\mathrm{cub}}$. The complement is the quotient of the union of the 36 boundary divisors in $\mathcal{M}^m_{\mathrm{cub}}$ by the action of $\mathrm{W}(D_5)$. It is easy to see that $\mathrm{W}(D_5)$ has two orbits on the set of 36 positive roots in E_6 of cardinality 16 and 20. Thus the complement is the union of two irreducible divisors D_1 and D_2 each isomorphic to a finite quotient of the Segre cubic S_3 minus its set of singular points (belonging to the boundary of $\mathcal{M}^m_{\mathrm{cub}}$ in $\bar{\mathcal{M}}^m_{\mathrm{cub}}$). It is immediately checked that the map (3.1) is not constant on D_1 and D_2. On the other hand, being a finite quotient of a hypersurface in $\mathbb{P}^4$ (minus a finite set of points), the varieties D_1 and D_2 have Picard group of rank 1, hence no curves blow down on these varieties. This shows that no positive-dimensional subvariety on the source space of the map (3.1) is mapped to a point. Hence the map has finite fibres.

It remains to show the surjectivity of (3.1). Any stable pair of binary forms (F_5, F_2) defines the divisor $D + T$ as in section 2.15 by reading off the zeroes of the forms. The type vector of this divisor can be found in Table 1. It corresponds to a pencil of conics defined by a line on a cubic surface of type listed in section 2.16. The image of the corresponding pair (S, l) is the orbit of (F_5, F_2). $\square$

3.7. Since the variety $\big(\mathbb{P}(V(2)) \times \mathbb{P}(V(5))\big)^{\mathrm{s}}/\mathrm{SL}(2)$ is obviously birationally isomorphic to the quotient $\mathbb{P}(V(5))^{\mathrm{s}}/\mathbb{C}^*$ (by fixing first a binary form of degree 2), we obtain the following:

3.8. Corollary. *The moduli space $\mathcal{M}^1_{\mathrm{cub}}$ is isomorphic to an open subset of a toric variety. In particular, it is rational.*

3.9. Remark. It follows from the isomorphism (2.4) that the moduli space of cubic surfaces is rational. However, as far as we know, the rationality of the space $\mathcal{M}^1_{\mathrm{cub}}$ was not known. Note also that the moduli space $\mathcal{M}^1_{\mathrm{cub}}$ is birationally isomorphic to the universal surface over the moduli space of Del Pezzo surfaces of degree 4.

4. The $K3$ surface associated to a cubic surface

4.1. In the previous section we associated a pair of binary forms (F_2, F_5) to a nodal cubic surface S with a line l. We now use these binary forms to define a $K3$ surface $X_{S,l}$.

We will show that $X_{S,l}$ depends only on the nodal cubic S and that the lines on a generic S correspond to certain 'standard' elliptic fibrations (cf. section 6.20, Corollary 7.6). Finally we relate $X_{S,l}$ to S using a cubic fourfold.

4.2. Definition. Let S be a nodal cubic surface and let l be a line on S. Let $F_2(x_0, x_1)$ be a homogeneous form of degree 2 and let $F_5(x_0, x_1)$ be a homogeneous form of degree 5 associated to (S, l) as in 3.1.

To the pair (S, l) we associate a surface $X_{S,l}$ which is a nonsingular minimal model of the double plane with the branch divisor

$$W:\quad x_2\big(F_2(x_0,x_1)x_2^3+F_5(x_0,x_1)\big)=0. \tag{4.1}$$

It is easy to check that the properties (i)–(iii) in 3.1 are equivalent to the property that any singular point of the curve W is analytically equivalent to a singularity $f(x,y)=0$ such that the surface singularity $z^2+f(x,y)=0$ is a double rational point. This implies that $X_{S,l}$ is a $K3$ surface. The multiplication of x_2 by a primitive cube root of unity induces an automorphism of $X_{S,l}$ of order 3.

4.3. The elliptic fibration. Consider the pencil of lines

$$L(t_0,t_1):\quad t_1x_0-t_0x_1=0$$

in $\mathbb{P}^2$ passing through the point $(0,0,1)$. Since a general line $L(\lambda,\mu)$ intersects W at four nonsingular points, we obtain that the pre-image of the pencil of lines on $X_{S,l}$ is an elliptic pencil. Thus we have an elliptic fibration

$$f=f_l:X_{S,l}\to\mathbb{P}^1.$$

The singular fibres correspond to lines $L(t_0,t_1)$ such that $F_5(t_0,t_1)=0$ or $F_2(t_0,t_1)=0$. The proper transform of W in the blow-up $V\cong\mathbf{F}_1$ of the point $(0,0,1)$ is a curve $\overline{W}$ in the linear system $|6f+4e|$, where e is the exceptional section and f is a fibre. The pre-image s of the line $x_2=0$ is a component of $\overline{W}$. It is a section with the divisor class $f+e$. The pre-image of a line corresponding to a zero (x_0,x_1) of F_5 is a fibre of $V\to\mathbb{P}^1$ over (x_0,x_1) which intersects $B=\overline{W}-s$ with multiplicity 3 at a point where B intersects s. A line corresponding to a zero of F_2 is a fibre which intersects B with multiplicity 3 at a point where B intersects e. The surface $X_{S,l}$ is isomorphic to a minimal resolution of the double cover of V branched along $\overline{W}$.

Now it is easy to describe the singular fibres of the elliptic fibration $f:X_{S,l}\to\mathbb{P}^1$. For example, in the case when F_5 and F_2 have no multiple roots and have no common roots, the fibres over the zeroes of F_2 are cuspidal cubics. The fibres over the zeroes of F_5 are reducible of type IV in Kodaira's notation. If F_2 has a common zero with F_5, the fibre of $V\to\mathbb{P}^1$ becomes an irreducible component of B. The corresponding fibre of f is of type I_0^*. If F_2 has a double root which is not a root of F_5, then B acquires a cusp. Instead of two irreducible fibres of f we obtain one reducible fibre of type IV. If F_5 has a double root which is not a root of F_2, then B acquires a cusp at the curve s. The corresponding fibre of f is of type IV^*. It is not difficult to describe the fibres in all possible cases. Their Kodaira types are given in Table 1. Note that the irreducible singular fibres correspond to zeroes of F_2 which are not zeroes of F_5. Observe also that the pre-image of s in $X_{S,l}$ is a section s of the elliptic fibration. The pre-image of e is a bisection b. If B acquires a cusp at the exceptional section e or has a fibre component, then b splits in two disjoint sections.

4.4. Let l be a line on a nodal cubic surface S, and let m be another line disjoint from l. Consider the rational map $\Phi:l\times m\dashrightarrow S$ defined by taking the third intersection point of the line through the points $(p,q)\in l\times m$ with S. We denote by L and M the irreducible curves in $l\times m$ which map onto the lines l and m in S respectively under Φ.

4.5. Lemma. *The rational map Φ extends to an isomorphism from the blow-up Z of $l\times m$ along $L\cap M$, which is a set of 5 points (including infinitely near points) to a minimal resolution $\tilde{S}$ of S. The curves L and M have bi-degree $(2,1)$ and $(1,2)$ respectively.*

Proof. This is just a straightforward computation. Choose coordinates on $\mathbb{P}^3$ such that $m : x_0 = x_1 = 0$ and $l : x_2 = x_3 = 0$ so that the equation of S is given by

$$\sum_{i,j=0}^{1} A_{ij}(x_2, x_3)x_i x_j + 2\sum_{i=0}^{1} B_i(x_2, x_3)x_i = 0, \tag{4.2}$$

where A_{ij}, B_i are homogeneous forms of degree 1 and 2, respectively. Let $p = (a_0, a_1, 0, 0) \in l$, $q = (0, 0, a_2, a_3) \in m$. The line l' spanned by p, q has parametric equation $(x_0, x_1, x_2, x_3) = (sa_0, sa_1, ta_2, ta_3)$. Plugging it in equation (4.2), we obtain

$$st\left(s\sum_{i,j=0}^{1} A_{ij}(a_2, a_3)a_i a_j + 2t\sum_{i=0}^{1} B_i(a_2, a_3)a_i\right) = 0.$$

Thus the rational map Φ is given by the formula

$$\Phi(p, q) = (Ma_0, Ma_1, La_2, La_3), \tag{4.3}$$

where

$$M(p, q) = -2\sum_{i=0}^{1} B_i(a_2, a_3)a_i, \quad L(p, q) = \sum_{i,j=0}^{1} A_{ij}(a_2, a_3)a_i a_j. \tag{4.4}$$

It is easy to see that the base locus Z of the linear system of divisors of bi-degree $(3, 3)$ defining Φ is the complete intersection of the divisor $M = 0$ of bi-degree $(1, 2)$ and $L = 0$ of bi-degree $(2, 1)$. Local computations show that Z is reduced and consists of 5 points if and only if S is smooth. The rational map Φ is obviously birational, and defines a birational morphism $\Phi' : Z \to S$ of the blow-up Z of $l \times m$ along $L \cap M$. It is clear that the proper images under Φ of the divisors $L = 0$ and $M = 0$ are the lines l and m, respectively. Comparing the Betti numbers of Z and $\tilde{S}$, we see that they are equal. Thus Φ' defines an isomorphism from Z to $\tilde{S}$. □

4.6. Remark. Assume S is nonsingular. Then we obtain that S is isomorphic to the blow-up of 5 distinct points in $\mathbb{P}^1 \times \mathbb{P}^1$. The map $S \to \mathbb{P}^1 \times \mathbb{P}^1$ is the blowing down of 5 disjoint lines intersecting the lines l and m. This is of course well-known. Take any two skew lines on S. It is known that there are exactly five skew lines on S which intersect l, m. The easiest way to see it is to complete l, m to a set of six skew lines $n_1 = l$, $n_2 = m$, $n_3, \ldots, n_6$, then consider the blow-down $\pi : S \to \mathbb{P}^2$ of these lines to points $p_1, \ldots, p_6$ in the plane. The five skew lines are the proper inverse transforms of the line spanned by p_1, p_2 and the four conics C_i passing through all p_j's except p_i with $3 \leqq i \leqq 6$. Blowing down the five lines, we obtain $\mathbb{P}^1 \times \mathbb{P}^1$. The images of the lines l, m are the curves of bi-degree $(2, 1)$ and $(1, 2)$. The blowing down morphism $S \to \mathbb{P}^1 \times \mathbb{P}^1$ which inverts Φ is the Cartesian product of the linear projections from the lines l and m.

4.7. The surface $X_{S,l,m}$. The divisor $W' = L + M$ on $l \times m = \mathbb{P}^1 \times \mathbb{P}^1$ is of bi-degree $(3, 3)$. Let us consider the cyclic triple cover $Y \to l \times m$ branched along W'. It has singular points over the singular locus of W'. If L intersects M transversally, Y has 5 double rational points of type A_2. Let $X_{S,l,m}$ be a nonsingular minimal model of Y.

4.8. Lemma. *Let*

$$f = f_{l,m} : X_{S,l,m} \to m \cong \mathbb{P}^1$$

be the composition of the blow down map $X_{S,l,m} \to Y$, the triple covering $Y \to l \times m$ and the second projection $l \times m \to m$. Then f is an elliptic fibration with a section whose Weierstrass form is given by

$$y^2 + x^3 + F_5(t_0, t_1)^2 F_2(t_0, t_1) = 0, \tag{4.5}$$

where the binary forms $F_2(t_0, t_1)$ and $F_5(t_0, t_1)$ coincide with the binary forms F_2 and F_5 associated to (S, l) in section 3.1.

Proof. For any general point $(t_0, t_1) \in \mathbb{P}^1$, the fibre of f over this point is isomorphic to a plane cubic curve with the equation

$$x_2^3 + \big(B_0(t_0, t_1)x_0 + B_1(t_0, t_1)x_1\big)\big(A_{00}(t_0, t_1)x_0^2 + 2A_{01}(t_0, t_1)x_0x_1 + A_{11}(t_0, t_1)x_1^2\big) = 0. \tag{4.6}$$

The cubic curve has an obvious automorphism of order 3 defined by multiplying x_2 by the third roots of unity. As is well-known such a cubic can be reduced by a projective transformation to the Weierstrass form

$$y^2t + x^3 + bt^3 = 0.$$

The coefficient b is the value of a certain SL(3)-invariant T on the space of homogeneous polynomials of degree 3 in 3 variables. Using the explicit formula for T (see [Sa2], p. 192), a direct computation shows that

$$b = F_5(t_0, t_1)^2 F_2(t_0, t_1), \tag{4.7}$$

where

$$F_5 = B_0(t_0, t_1)^2 A_{11}(t_0, t_1) + B_1(t_0, t_1)^2 A_{00}(t_0, t_1) - 2A_{01}(t_0, t_1)B_0(t_0, t_1)B_1(t_0, t_1),$$

$$F_2 = A_{00}(t_0, t_1)A_{11}(t_0, t_1) - A_{01}(t_0, t_1)^2.$$

Let $t_1x_2 - t_0x_3 = 0$ be the pencil of planes through the line $l : x_2 = x_3 = 0$. Using the equation (4.2) of S we find that the pencil of conics defined by the line l has the equation

$$A_{00}(t_0, t_1)x_0^2 + 2A_{01}(t_0, t_1)x_0x_1 + A_{11}(t_0, t_1)x_1^2 + 2B_0(t_0, t_1)x_2x_0 + 2B_1(t_0, t_1)x_2x_1 = 0. \tag{4.8}$$

Its discriminant is equal to

$$\det\begin{pmatrix} A_{00} & A_{01} & B_0 \\ A_{01} & A_{11} & B_1 \\ B_0 & B_1 & 0 \end{pmatrix} = -F_5(t_0, t_1). \tag{4.9}$$

The restriction of the member of the pencil corresponding to the parameters (t_0, t_1) to the line l is given by the binary form

$$A_{00}(t_0,t_1)x_0^2 + 2A_{01}(t_0,t_1)x_0x_1 + A_{11}(t_0,t_1)x_1^2 = 0. \tag{4.10}$$

The discriminant of this binary form is equal to

$$\det\begin{pmatrix} A_{00} & A_{01} \\ A_{01} & A_{11} \end{pmatrix} = F_2(t_0,t_1). \tag{4.11}$$

If l does not contain nodes, the equation (4.10) defines a base-point free pencil of divisors of degree 2 on l, and we see that $F_2 = 0$ describes the locus of points in the parameter space of the pencil of conics where the bisection l ramifies. If l contains a node, we may assume that its coordinates are $(1,0,0,0)$. Then $A_{11} = 0$ and we get a pencil of divisors of degree 1 on l with one base point. The discriminant is equal to $-A_{01}^2$ and describes one point with multiplicity 1 corresponding to the singular point of the bisection B defined by l. Finally, if l contains two nodes, we may assume that $A_{11} = A_{00} = 0$. Then the pencil (4.10) cuts out the fixed divisor with equation $A_{01}(t_0,t_1)x_0x_1 = 0$. It is equal to zero when $A_{01}(t_0,t_1) = 0$. These points correspond to fibre components of the bisection B of the conic bundle. The discriminant is again $-A_{01}(t_0,t_1)^2$. □

4.9. Theorem. *Let S be a nodal cubic surface and let l be a line on S. Then the isomorphism class of the $K3$ surface $X_{S,l}$ associated to a pair (S,l) is independent on the choice of the line l.*

Proof. We compare the elliptic fibration f_l on $X_{S,l}$ obtained from the pencil of lines through the singular point $(0,0,1)$ of the branch curve W and the elliptic fibration $f_{l,m}$ on the triple cover $X_{S,l,m}$, where m is a line disjoint from l. The fibre of f_l corresponding to a general line $t_1x_0 - t_0x_1 = 0$, with $t_0 = 1$, passing through the point $(0,0,1)$ is birationally isomorphic to the curve

$$z^2 + x_2x_0^2\big(F_2(1,t_1)x_2^3 + F_5(1,t_1)x_0^3\big) = 0.$$

After the change of variables $y = F_5z/x_0x_2^2$, $x = F_5x_0/x_2$ we reduce this equation to the Weierstrass form (4.5) from Lemma 4.8. This shows that the surfaces $X_{S,l}$ and $X_{S,l,m}$ have isomorphic elliptic pencils. Hence $X_{S,l} \cong X_{S,l,m}$. Switching the roles of l and m, we see that $X_{S,l} \cong X_{S,m}$. It is easy to see that if two lines l, m on S are not skew, then there exists a third line n which is disjoint from l and m, so again $X_{S,l} \cong X_{S,n} \cong X_{S,m}$. We conclude that $X_{S,l}$ does not depend on a choice of a line l. □

4.10. Definition. Let S be a nodal cubic surface. A $K3$ surface associated to S is a $K3$ surface X_S isomorphic to the surface $X_{S,l}$ associated to a pair (S,l), where l is a line on S defined in section 4.2 or the surface $X_{S,l,m}$ associated to a triple (S,l,m), where l, m is a pair of skew lines on S defined in 4.7.

As a corollary of the results above and those of the previous section we have:

4.11. Corollary. *The moduli space $\mathcal{M}^1_{\mathrm{ncub}}$ is isomorphic to the moduli space of elliptic $K3$ surfaces with the Weierstrass form*

$$y^2 + x^3 + F_5(t_0,t_1)^2F_2(t_0,t_1) = 0, \tag{4.12}$$

where (F_5,F_2) is a stable pair of binary forms of degrees 5 and 2.

4.12. Cubic fourfolds. Let us give another proof of the independence of the $K3$ surface $X_{S,l}$ on the choice of the line l. Although it is more geometric, it requires to go beyond the theory of algebraic surfaces. We assume that $S = V(F)$ is a nonsingular surface. Consider the cubic fourfold V defined by the equation

$$F(x_0, x_1, x_2, x_3) + x_4 x_5 (x_4 + x_5) = 0. \tag{4.13}$$

It is well-known (see [Voi]) that the projection from a plane Π contained in a nonsingular cubic fourfold defines a structure of a quadric bundle on the blow-up V' along the plane. The discriminant curve of the quadric bundle is a plane sextic, and the double cover of the plane branched over this sextic is a $K3$ surface $X(V,\Pi)$. It parametrizes the pairs (Q,r), where Q is a fibre of the quadric bundle and r is a ruling of lines on it. Suppose we have another plane Π' in V disjoint from Π. It intersects each fibre Q of the quadric bundle at a point x, and the choice of the ruling r on Q picks up a line on V intersecting both planes Π and Π'. This gives an isomorphism from the $K3$-surface $X(V,\Pi)$ and the surface $X(V,\Pi,\Pi')$ parametrizing lines in V intersecting Π and Π'. Reversing the roles of Π and Π' we see that

$$X(V,\Pi) \cong X(V,\Pi') \cong X(V,\Pi,\Pi'). \tag{4.14}$$

4.13. Proposition. *Let $l : L_1 = L_2 = 0$, $m : M_1 = M_2 = 0$ be disjoint lines on a nonsingular cubic surface $S = V(F) \subset \mathbb{P}^3$ and Π, Π' be two disjoint planes on the cubic fourfold* (4.13) *given by the equations $L_1 = L_2 = x_4 = 0$ and $M_1 = M_2 = x_5 = 0$. Then the $K3$ surface $X(V,\Pi)$ is isomorphic to the $K3$ surface $X_{S,l}$.*

Proof. We may assume that $l : x_2 = x_3 = 0$ and $m : x_0 = x_1 = 0$. Write the equation (4.13) in the form similar to (4.2)

$$\sum_{i,j=0}^{1} A_{ij}(x_2,x_3)x_i x_j + 2\sum_{i=0}^{1} B_i(x_2,x_3)x_i + x_4 x_5^2 + x_5 x_4^2 = 0. \tag{4.15}$$

Let $t_1 x_2 - t_0 x_3 = t_2 x_3 - t_0 x_4 = 0$ be the net of 3-planes through the plane $\Pi : x_2 = x_3 = x_4 = 0$. The corresponding quadric bundle is given by

$$\sum_{i,j=0}^{1} A_{ij}(t_0,t_1)x_i x_j + 2\sum_{i=0}^{1} B_i(t_0,t_1)x_i x_2 + t_2 x_5^2 + t_2^2 x_2 x_5 = 0.$$

Computing the discriminant of the quadric $Q(t_0,t_1,t_2)$ we find, using (4.9) and (4.11), that the discriminant curve of the quadric bundle is given by the equation

$$t_2\big(t_2^3 F_2(t_0,t_1) + 4F_5(t_0,t_1)\big) = 0.$$

After scaling the unknowns we obtain the equation of the branch curve of the $K3$ surface $X_{S,l}$ from (4.1). Thus the $K3$ surfaces $X(V,\Pi)$ and $X_{S,l}$ are isomorphic. □

Since for any two lines l_1, l_2 on a nonsingular cubic surface there exists a line m disjoint from l_1 and l_2, Proposition 4.13 and the isomorphism (4.14) show that the surfaces X_{S,l_1} and X_{S,l_2} are isomorphic. This gives another proof of Theorem 4.9 in the case when S is a nonsingular surface. Similar proof can be given in the case of a nodal cubic.

4.14. Remark. The lattice of transcendental cycles of X_S and that of the cubic fourfold Y are isomorphic. In fact, the blow-up Y' of Y along the union of two disjoint planes is isomorphic to the blow-up of $\mathbb{P}^2 \times \mathbb{P}^2$ along the $K3$ surface $X \cong X_{l,m}$. This gives an isomorphism of Hodge structures

$$H^4(Y', \mathbb{Z}) \cong H^4(\mathbb{P}^2 \times \mathbb{P}^2, \mathbb{Z}) \oplus H^2(X, \mathbb{Z})(-1).$$

This isomorphism is compatible with the cup-product such that the two summands become orthogonal. Here $H^2(X, \mathbb{Z})(-1)$ is identified with $\xi \cdot \pi^*\big(H^2(X, \mathbb{Z})\big)$, where $\pi : Y' \to \mathbb{P}^2 \times \mathbb{P}^2$ is the natural morphism of the blow-up and ξ is a cohomology class from $H^2(Y', \mathbb{Z})$ which cuts out the tautological class of the exceptional divisor isomorphic to the projectivization of the normal bundle of X. This implies that the sublattice consisting of algebraic cycles in $H^4(Y', \mathbb{Z})$ is isomorphic to $H^4(\mathbb{P}^2 \times \mathbb{P}^2, \mathbb{Z}) \oplus \mathrm{Pic}(X)[-1]$. Passing to the orthogonal complements we get the result.

4.15. Cubic threefolds. We relate the $K3$ surface X_S to the Matsumoto-Terasoma curve associated to (S, l). Given a smooth cubic surface S in $\mathbb{P}^3$, we define, following [ACT], the cubic threefold $V \subset \mathbb{P}^4$ to be the triple cover of $\mathbb{P}^3$ branched along S. So if

$$S\colon \quad F(x_0, x_1, x_2, x_3) = 0,$$

then

$$V\colon \quad F(x_0, x_1, x_2, x_3) + x_4^3 = 0.$$

Note that $S \subset V$ (the points of V with $x_4 = 0$), hence a line $l \subset S$ defines a line, also denoted by l, in V. The projection of a cubic threefold away from a line in $\mathbb{P}^4$ defines the structure of a conic bundle on the blow-up of V along the line. The associated discriminant curve in $\mathbb{P}^2$ is a plane quintic. A straightforward computation shows that the discriminant curve is a plane quintic with the equation

$$W'\colon \quad F_5(t_0, t_1) + t_2^3 F_2(t_0, t_1) = 0,$$

where the F_i are as in 3.1, so W' is a component of W.

4.16. Remark. Each smooth point t of the plane quintic W' defines two lines (the components of the singular conic in the fibre of $V \to \mathbb{P}^2$ over t). Thus there is a natural double cover $C' \to W'$. This double cover was studied by Matsumoto and Terasoma in [MT], the corresponding double cover $C \to \overline{W}'$ of the normalizations of these curves is ramified in two points, which are identified in C'. The curve C is isomorphic to the affine curve ([MT], (3.1)):

$$v^3 - xf(x^2) = 0,$$

where f is a polynomial of degree 5. The Prym variety of the double cover $C \to \overline{W}'$ is a 5-dimensional principally polarized abelian variety which is isomorphic to the intermediate Jacobian variety P of the cubic threefold V (cf. [MT]). The Matsumoto-Terasoma curve C has the following property.

4.17. Proposition. *Let $f : X_{S,l} \to \mathbb{P}^1$ be the elliptic fibration as in the subsection* 4.3. *The pull-back of $X_{S,l}$ along the base change $C \to \mathbb{P}^1$, $(v, x) \mapsto x$, is birationally equivalent to the product $C \times E$ where E is the elliptic curve with $j = 0$: $E \cong \mathbb{C}/(\mathbb{Z} + \mathbb{Z}\zeta_3)$.*

Proof. In [MT] it is proved that $W = C/\iota$ where ι is the (Clemens-Griffiths) involution $\iota : (v, x) \mapsto (-v, -x)$. Therefore the quotient curve is given by $y^3 = u^2 f(u)$ where $u = x^2$ and $y = xv$. This curve is birationally equivalent to W'. In fact, choosing coordinates such that $F_2(y_0, y_1) = y_0 y_1$ the equation of W' is $y_2^3 y_0 y_1 + F_5(y_0, y_1)$, hence $y_2^3 y_1 + F_5(1, y_1)$ is an affine equation. Putting $v = -y_1 y_2$, $u = y_1$ we find the birational isomorphism with $f(u) = F_5(1, u)$.

The function field of $X_{S,l}$ is defined by $s^2 = y_0 y_1 + F_5(y_0, y_1)$. The elliptic fibration is given by the rational function $t = y_1/y_0$. Rewriting the equation we get: $(s/y_0)^2 = t + y_0^3 F_5(1, t)$, equivalently, since $F_5(1, t) = f(t)$:

$$Y^2 = X^3 + tf(t)^2 \quad (X = y_0 f(t), Y = sf(t)/y_0).$$

Since on C we have $v^6 = tf(t)^2$ we can write this as $\left(sf(t)/y_0 v^3\right)^2 = \left(y_0 f(t)/v^2\right)^3 + 1$, which is the equation $Y^2 = X^3 + 1$ of the curve E. □

4.18. Remark. According to Donagi and Smith [DS], the Prym map $\mathscr{R}_6 \to \mathscr{A}_5$ has degree 27 with the Galois group $\mathrm{W}(E_6)$. Identifying the branch points on W and the ramification points on C, we obtain the admissible double cover $C' \to W'$ in $\mathscr{R}_6$. Thus we get 27 'natural' pre-images of P under the Prym map. However, the Prym map has 2-dimensional fibre over the intermediate Jacobian of a cubic threefold, in fact any line in the threefold defines an admissible double cover in $\mathscr{R}_6$.

5. The Picard lattice

In this section we compute the Picard lattice $\mathrm{Pic}(X_S) \subset H^2(X_S, \mathbb{Z})$ of the $K3$ surface X_S associated to a nodal cubic surface and its orthogonal complement, the lattice of transcendental cycles $T_{X_S} := \mathrm{Pic}(X_S)^\perp$.

5.1. Lattices. Recall the following two lattices:

$$U = \left(\mathbb{Z}^2, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\right), \quad A_2 = \left(\mathbb{Z}^2, \begin{pmatrix} -2 & 1 \\ 1 & -2 \end{pmatrix}\right).$$

The second cohomology group $H^2(X, \mathbb{Z})$ equipped with the quadratic form defined by the cup-product is an even unimodular lattice of signature $(3, 19)$. It is isomorphic to the $K3$ lattice

$$L = U^{\oplus 3} \oplus E_8^{\oplus 2},$$

where $E_8 = \mathbb{Z}^8$ with the quadratic form defined by the opposite of the Cartan matrix of the root system of type E_8. In general, A_m, D_n, E_k denote the root lattices of the simple root systems of the corresponding symbol (with the Cartan matrix multiplied by -1).

For any lattice M we denote by $M(n)$ the lattice M with the quadratic form multiplied by n. Let M be a nondegenerate even lattice. The dual abelian group M^* contains M as a subgroup of finite index, the quotient group $D(M) = M^*/M$ is called the *discriminant group* of M. It is equipped with a quadratic form

$$q : D(M) \to \mathbb{Q}/2\mathbb{Z}, \quad q(m^* + M) = t^{-2}(tm^*, tm^*) + 2\mathbb{Z},$$

where $t \in \mathbb{Z}$ is such that $tm^* \in M$. We use the notation $\mathrm{O}(M)$ (resp. $\mathrm{O}(D)$) to denote the group of automorphisms of M (resp. $\mathrm{D}(M)$) preserving the quadratic form. If M is a primitive sublattice of a unimodular lattice there is a natural isomorphism $\mathrm{D}(M) \cong \mathrm{D}(M^\perp)$.

5.2. Lattices $M(t)$ and $T(t)$. Recall that a choice of a line on a nodal cubic surface S defines an elliptic pencil $f : X_S \to \mathbb{P}^1$. Its type is determined by the type vector $\boldsymbol{t}$ of the conic bundle on S corresponding to l, cf. 2.15. We call it the type vector of (S, l) and the type vector of the elliptic fibration. We will explain later that for any possible type vector $\boldsymbol{t}$ there exists a pair (S, l) of type $\boldsymbol{t}$ such that the Picard lattice of the $K3$ surface X_S is of rank $12 + 2r + 2e$, where r is the number of nodes on S and e is the number of Eckardt points on l. We denote by $M(\boldsymbol{t})$ the smallest primitive sublattice of $H^2(X_S, \mathbb{Z})$ containing the sections and components of fibres of the elliptic fibration defined by the line l. Note that $\mathrm{Pic}(X_S) \cong M(\boldsymbol{t})$. We will compute the lattice $M(\boldsymbol{t})$ and its orthogonal complement $T(\boldsymbol{t})$ in $H^2(X_S, \mathbb{Z})$.

5.3. Proposition. *Assume that the Mordell-Weil group* $\mathrm{MW}(f)$ *is finite. Then the lattices* $M(\boldsymbol{t})$ *and* $T(\boldsymbol{t})$ *are as in Table* 2.

	$\boldsymbol{t}$	$M(\boldsymbol{t})$	$T(\boldsymbol{t})$
1)	(2222211)	$U \oplus A_2^{\oplus 5}$	$A_2(-1) \oplus A_2^{\oplus 4}$
2)	(322221)	$U \oplus D_4 \oplus A_2^{\oplus 4}$	$A_2(-2) \oplus A_2^{\oplus 3}$
3)	(33222)	$U \oplus D_4^{\oplus 2} \oplus A_2^{\oplus 3}$	$A_2(-1) \oplus A_2(2)^{\oplus 2}$
4)	(222222)	$U \oplus E_6 \oplus A_2^{\oplus 3}$	$A_2(-1) \oplus A_2^{\oplus 3}$
5)	(422211)	$U \oplus E_6 \oplus A_2^{\oplus 3}$	$A_2(-1) \oplus A_2^{\oplus 3}$
6)	(43221)	$U \oplus D_4 \oplus E_6 \oplus A_2^{\oplus 2}$	$A_2(-2) \oplus A_2^{\oplus 2}$
7)	(4332)	$U \oplus D_4^{\oplus 2} \oplus E_6 \oplus A_2$	$A_2(-2) \oplus A_2(2)$
8)	(42222)	$U \oplus E_6^{\oplus 2} \oplus A_2$	$A_2(-1) \oplus A_2^{\oplus 2}$
9)	(44211)	$U \oplus E_6^{\oplus 2} \oplus A_2$	$A_2(-1) \oplus A_2^{\oplus 2}$
10)	(52221)	$U \oplus E_8 \oplus A_2^{\oplus 3}$	$A_2(-1) \oplus A_2^{\oplus 2}$
11)	(4431)	$U \oplus E_6^{\oplus 2} \oplus D_4$	$A_2(-2) \oplus A_2$
12)	(5322)	$U \oplus E_8 \oplus D_4 \oplus A_2^{\oplus 2}$	$A_2(-2) \oplus A_2$
13)	(4422)	$U \oplus E_8 \oplus E_6 \oplus A_2$	$A_2(-1) \oplus A_2$
14)	(5421)	$U \oplus E_8 \oplus E_6 \oplus A_2$	$A_2(-1) \oplus A_2$
15)	(543)	$U \oplus E_8 \oplus E_6 \oplus D_4$	$A_2(-2)$
16)	(444)	$U \oplus E_8^{\oplus 2} \oplus A_2$	$A_2(-1)$
17)	(552)	$U \oplus E_8^{\oplus 2} \oplus A_2$	$A_2(-1)$

Table 2. The Picard lattices

Proof. We will consider only the first two cases. Let $f : X_S \to \mathbb{P}^1$ be the elliptic fibration of type $\boldsymbol{t} = (2222211)$ with Picard lattice $\mathrm{Pic}(X_S) \cong M(\boldsymbol{t})$. It follows from 4.3 that it has 5 reducible fibres of type IV and a section s defined by the line $x_2 = 0$. It also has 2 irreducible cuspidal fibres. We will use the Shioda-Tate formula [Shi]:

$$(\#\mathrm{MW})^2 \cdot \mathrm{D}(M(\boldsymbol{t})) = d_1 \ldots d_k, \tag{5.1}$$

where MW is the Mordell-Weil group and $d_1, \dots, d_k$ are the discriminants of the lattices generated by components of reducible fibres not intersecting the zero section. It follows from (5.1) that the Mordell-Weil group MW is a torsion group of order 3^l. Since the fibration has a cuspidal fibre, which has trivial torsion group, MW is trivial. Thus f has a unique section s. Now we use (5.1) again and find that the discriminant of M is equal to 3^5. Since $M = M(\boldsymbol{t})$ obviously contains the sublattice $U \oplus A_2^{\oplus 5}$ of the same rank and discriminant (it is spanned by the class of a fibre, the section, and irreducible components of reducible fibres), it must coincide with it. The discriminant group is then easy to compute. Let q_T be the discriminant form of T, then $q_T = -q_M$ ([N1], Prop. 1.6.1). We can easily see that T and $A_2(-1) \oplus A_2^4$ have the same discriminant form. It now follows from Nikulin [N1], Cor. 1.13.3 that $T \cong A_2(-1) \oplus A_2^4$.

Assume that the fibration is of type (322221). The product $d_1 \dots d_k$ is equal to $2^2 3^4$. The Shioda-Tate formula gives that either $\#\mathrm{MW} = 1, 3$, or 3^2, or 6. Since this fibration also has a cuspidal fibre (i.e. of type II), which has trivial torsion group, MW is trivial. So, the Shioda-Tate formula tells us that $\mathrm{D}\big(M(\boldsymbol{t})\big)$ is of order $2^2 3^4$. The remaining arguments are similar to the previous case. □

5.4. The lattices M, T. We set

$$M := U \oplus A_2^{\oplus 5}, \quad T := A_2(-1) \oplus A_2^{\oplus 4}.$$

Since their discriminant groups are isomorphic and the quadratic forms are the negative of each other, they are orthogonal complements of each other in the unimodular lattice L (see [N1]). We set

$$D = \mathrm{D}(M) \cong \mathrm{D}(T).$$

These lattices correspond to the type $\boldsymbol{t} = (2222211)$.

5.5. An automorphism σ of order 3. As in section 4.7, we choose two skew lines on a nodal cubic surface S and consider the associated $K3$ surface $X = X_S \cong X_{S,l,m}$. Recall that it is obtained as a minimal resolution of the triple cyclic cover Y of $\mathbb{P}^1 \times \mathbb{P}^1$ branched along the union of two divisors L and M of bidegree $(1,2)$ and $(2,1)$. It is easy to describe the set of fixed points of the automorphism σ of X defined by the triple cover. We do it only in the case when S is a nonsingular surface. Let $q_1, \dots, q_5$ be the intersection points of L and M. The cubic surface S is obtained by blowing up the points q_i. The surface S is nonsingular if and and only if no two points lie on a ruling, and no four points lie on a plane section. An Eckardt point on the line l corresponds to a ruling which is tangent to L at some point q_i.

Assume that there are no Eckardt points on l. Consider the elliptic fibration on $f : X \to \mathbb{P}^1$ corresponding to the projection $\mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^1$ such that L is a section. Its reducible singular fibres correspond to the ruling passing through the points q_i. Each fibre is of type IV. Two components are the exceptional curves of the resolution $X \to Y$ of a singular point of type A_2. The third component is the proper transform of the ruling passing through the corresponding point q_i. The bisection b intersects the latter component and one of the first two components. The section s intersects the other component coming from the resolution of singularities. The set of fixed points of σ is equal to the union of the section s, the bisection b and the singular points of the reducible fibres.

In the case when l contains one Eckardt point, the elliptic fibration acquires one reducible fibre of type I_0^*. Other reducible fibres are of type IV. The bisection b intersects the multiple component E_0 of this fibre. The section s intersects a reduced component E_1. The fixed points of the involution σ is the union of the section s, the bisection b, the point $E_0 \cap E_1$, and the singular points of fibres of type IV. If l has two Eckardt points, we have two reducible fibres of type IV and the set of fixed points is described similarly to the previous case.

5.6. The involution τ. Let $f : X \to \mathbb{P}^1$ be the elliptic fibration with a section s as in section 5.5. Let τ be the involution of X defined by the inversion $x \mapsto -x$ of each fibre. Then τ switches the two components of each singular fibre of type IV which do not meet s and preserves each component of any singular fibre of type I_0^*.

If f has five singular fibres of type IV and two singular fibres of type II, then the fixed locus of τ is the union of s and a smooth curve C of genus 5 which passes through the singular point of each singular fibre. If f has four singular fibres of type IV, one of type I_0^* and one of type II, then the fixed locus of τ is the union of s, the multiple component of the fibre of type I_0^* and a smooth curve of genus 3. If f has three singular fibres of type IV and two fibres of type I_0^*, then the fixed locus of τ is the union of s, two multiple components of singular fibres of type I_0^* and a smooth elliptic curve.

5.7. Remark. The automorphism group of the $K3$ surface X is infinite. For example, consider the divisor consisting of the 2-section and the two components of a reducible singular fibre of f not meeting the section. It defines an elliptic fibration on X with a section which has two reducible singular fibres, one is of type I_3 and another of type I_0^*. This elliptic fibration has a Mordell-Weil group of rank 4. Considering translations by the sections of infinite order we see that $\mathrm{Aut}(X)$ is an infinite group.

5.8. Lemma. *Assume S is nonsingular. Then*

$$H^2(X,\mathbb{Z})^{\sigma^*} \subset \mathrm{Pic}(X), \quad H^2(X,\mathbb{Z})^{\sigma^*} \cong M.$$

The automorphism σ acts trivially on the discriminant lattice $\mathrm{D}\big(H^2(X,\mathbb{Z})^{\sigma^*}\big) \cong \mathrm{D}(M)$.

Proof. Consider the elliptic fibration on X defined in 4.3. From 5.5 we know the description of fixed points of σ. Assume first that all reducible fibres are of type IV. Let P be the sublattice of $\mathrm{Pic}(X)$ spanned by the divisor classes of a fibre, of the section s and of the irreducible components of fibres which do not intersect s. It is immediate that $P \cong M$ and σ acts identically on P. The fixed locus X^σ of the automorphism σ consists of 5 isolated fixed points (the singular points of the reducible fibres) and two smooth rational curves (the section s and the bisection b). Applying the Lefschetz fixed point formula we obtain that the trace of σ^* on $H^2(X,\mathbb{Z})$ is equal to 7. Thus the trace of σ^* on $P^\perp$ is equal to $7 - 12 = -5$. This easily implies that the characteristic polynomial of σ^* on $P^\perp \otimes \mathbb{C}$ is equal to $(t^2+t+1)^5$. Therefore $P^\perp \otimes \mathbb{C}$ does not contain non-zero σ^*-invariant elements, so $H^2(X,\mathbb{Z})^{\sigma^*} = P \cong M$. Since σ^* acts trivially on $P \cong M$, it also acts trivially on $\mathrm{D}(P) \cong \mathrm{D}(M)$.

Suppose now that f contains a fibre $F = 2E_0 + E_1 + E_2 + E_3 + E_4$ of type I_0^*. Assume that E_1 intersects the section s. Then the divisor classes $E_0 + E_2 + E_3 + E_4$ and E_0 are σ-

invariant and span a lattice of type A_2. We define the lattice P similar to the above by using this contribution from a fibre of type I_0^*. The remaining arguments are the same. □

6. The moduli space of $K3$ surfaces associated to a cubic surface

6.1. We first recall the basic facts about moduli of $K3$ surfaces. In the subsections before 6.5, M will be any even non-degenerate sublattice of signature $(1, t)$.

6.2. Markings. We recall the definition of an M-polarization of a projective $K3$ surface X (see [Do]). Fix a connected component $V(M)^+$ of the cone $V(M) = \{x \in M \otimes \mathbb{R} : (x, x) > 0\}$. Let

$$\Delta(M) = \{\delta \in M : (\delta, \delta) = -2\}.$$

For any $\delta \in \Delta(M)$ let $H_\delta = \{x \in V(M)^+ : (x, \delta) = 0\}$. Choose a connected component $C(M)^+$ of the complement of the union of hyperplanes H_δ, $\delta \in \Delta(M)$, in $V(M)^+$. Let

$$\Delta(M)^\pm = \{\delta \in \Delta(M) : \pm(x, \delta) > 0, \forall x \in C(M)^+\}.$$

We have $\Delta(M) = \Delta^+(M) \amalg \Delta^-(M)$.

Now we define an M-polarization of X as a primitive lattice embedding $\phi : M \to \mathrm{Pic}(X)$ such that $C(X)^+ \cap \phi(M \otimes \mathbb{R}) \subset \phi(C(M)^+)$, where $C(X)^+$ is the cone in $\mathrm{Pic}(X) \otimes \mathbb{R}$ spanned by the pseudo-ample (i.e. nef and big) divisor classes of X.

Note that the closure of $C(X)^+$ is the nef cone $C(X)$. The closure $C(M)$ of $C(M)^+$ is the subset of the closure of $V(M)^+$ which consists of vectors v such that $(v, \delta) \geqq 0$ for any $\delta \in \Delta(M)^+$. The polarization ϕ embeds $C(X) \cap \phi(M \otimes \mathbb{R})$ in $\phi(C(M))$. For any $\delta \in \Delta(M)^+$ the image $\phi(\delta)$ is a divisor class R with $R^2 = -2$. For any $v \in C(M)$ the image $\phi(v)$ is a pseudo-ample divisor D with $D^2 \geqq 0$. Since $R \cdot D = (\delta, v) > 0$, it follows from Riemann-Roch that R is effective. Note that R is not necessarily the divisor class of an irreducible curve (a (-2)-curve).

The polarization is called *ample* if $\phi(C(M)^+) \cap \mathrm{Pic}(X)^+ \neq \emptyset$, where $\mathrm{Pic}(X)^+$ is the ample cone of X. It is easy to see that a polarization ϕ is ample if and only if the orthogonal complement of $\phi(M)$ in $\mathrm{Pic}(X)$ does not contain the divisor classes of (-2)-curves. In particular, any polarization with $\phi(M) = \mathrm{Pic}(X)$ is ample.

A pair (X, ϕ), where ϕ is an M-polarization (resp. an ample M-polarization), is called an *M-polarized $K3$ surface* (resp. *ample M-polarized $K3$ surface*). Two M-polarized $K3$ surfaces (X, ϕ) and (X', ϕ') are called isomorphic if there exists an isomorphism $f : X \to X'$ such that $\phi = f^* \circ \phi'$.

6.3. Moduli of M-polarized $K3$ surfaces. It is known (see [Do]) that there exists a coarse moduli space $\mathcal{M}_{K3,M}$ of isomorphism classes of M-polarized $K3$ surfaces. Let us assume that M admits an embedding into the $K3$ lattice $L = U^{\oplus 3} \oplus E_8^{\oplus 2}$ which is unique up to isometry. Fix such an embedding. Let T be the orthogonal complement of M in L.

Any M-polarization ϕ of a $K3$ surface X extends to an isometry $\tilde{\phi}: L \to H^2(X, \mathbb{Z})$ (a cohomology marking of X). Extending $\tilde{\phi}$ $\mathbb{C}$-linearly, we get a one dimensional subspace $\tilde{\phi}^{-1}(H^{2,0}(X)) \subset T \otimes \mathbb{C}$ which is called the *period* of $(X, \tilde{\phi})$.

$$\begin{array}{ccc} M & \subset & L \\ \phi \downarrow & & \downarrow \tilde{\phi} \\ \mathrm{Pic}(X) & \hookrightarrow & H^2(X, \mathbb{Z}) \end{array}$$

The moduli space $\mathscr{M}_{K3,M}$ is isomorphic to the quotient $\mathscr{D}_M/\Gamma_M$, where $\mathscr{D}_M$ is the union of two copies of a Hermitian symmetric domain of type IV corresponding to the inner product vector space $T \otimes \mathbb{R}$ of signature $(2, 20 - t)$, $\mathscr{D}_M$ is a subset of the projective space $\mathbb{P}(T \otimes \mathbb{C})$. The group Γ_M is the subgroup of the orthogonal group $\mathrm{O}(L)$ of L which leaves M pointwise fixed. It is also isomorphic to the subgroup of $\mathrm{O}(T)$ which acts identically on the discriminant group $D(T) = T^*/T$.

The isomorphism classes of ample M-polarized $K3$ surfaces are parametrized by an open subset of $\mathscr{M}_{K3,M}$ whose complement is the image in $\mathscr{M}_{K3,M}$ of the union of hypersurfaces in $\mathscr{D}_M$ defined by lines in $T \otimes \mathbb{C}$ orthogonal to vectors $r \in T$ with $r^2 = -2$.

6.4. The group W(M). For any $\delta \in \Delta(M)$ we can define a reflection $s_\delta \in \mathrm{O}(M)$ associated to δ by $s_\delta : v \mapsto v + (v, \delta)\delta$. Let $\mathrm{W}(M)$ be the subgroup of $\mathrm{O}(M)$ generated by all s_δ's. The set $C(M)$ is a fundamental domain for $\mathrm{W}(M)$ in the closure of $V(M)^+$. Thus for any $v \in M$ with $v^2 \geqq 0$ there exists a $w \in \mathrm{W}(M)$ such that $(w(v), \delta) \geqq 0$, for any $\delta \in \Delta(M)^+$.

Let (X, ϕ) be an M-polarized $K3$ surface. Then for any $v \in M$ with $v^2 \geqq 0$ there is a $w \in \mathrm{W}(M)$ such that $\phi(w(v)) \in C(M)$. In particular, for any given embedding $\phi : M \to \mathrm{Pic}(X)$, there is a $w \in \mathrm{W}(M)$ such that $C(X)^+ \cap \phi(M \otimes \mathbb{R}) \subset (\phi \circ w)(C(M)^+)$, i.e., $\phi \circ w$ is an M-polarization.

6.5. Fixing $V(M)^+$ and $\Delta(M)^+$. The lattice M from 5.4 has a unique (up to an isometry) primitive embedding in the $K3$ lattice L [N1] and we identify M with a primitive sublattice of L from now on. We fix a basis in U formed by two isotropic vectors f_1, f_2 with $(f_1, f_2) = 1$ and a simple root basis r_1, r_2 in A_2, i.e., $(r_1)^2 = (r_2)^2 = -2$ with $(r_1, r_2) = 1$. We define a basis of M by taking f_1, f_2 in U and r_1, r_2 in each copy of A_2.

We define $V(M)^+$ by requiring that $f_1 + f_2 \in V(M)^+$. We define $\Delta(M)^+$ as follows. Firstly, (-2)-vectors v with $(f_1 + f_2, v) > 0$ belong to it. Secondly, if $(f_1 + f_2, v) = 0$, then $v \in \Delta(M)^+$ if and only if it is a nonnegative combination of $f_2 - f_1$ and the r_i's in each copy of A_2.

6.6. Automorphisms of L. Let ρ_o be the isometry of A_2 defined by

$$\rho_o(r_1) = r_2, \quad \rho_o(r_2) = -r_1 - r_2.$$

Obviously ρ_o is of order 3, has no non-zero fixed vectors and acts trivially on $\mathrm{D}(A_2) = (A_2)^*/A_2$. Let ρ be the isometry of $T = A_2(-1) \oplus A_2^{\oplus 4}$ defined by $\rho = (\rho_o)^{\oplus 5}$. Then ρ is of order 3, has no non-zero fixed vectors and acts trivially on $\mathrm{D}(T)$. Thus the isometry $(1_M, \rho)$ of $M \oplus T$ can be extended to the one of the $K3$ lattice L (Nikulin [N1], Corollary 1.5.2). For simplicity we denote this isometry of L by the same letter ρ.

6.7. Period domains. The period domain for M-polarized $K3$ surfaces is

$$\mathscr{D}_M = \{\omega \in \mathbb{P}(T \otimes_{\mathbb{Z}} \mathbb{C}) : (\omega, \omega) = 0, (\omega, \bar{\omega}) > 0\}.$$

Note that $\mathscr{D}_M$ has two connected components so it is not a domain in the strict meaning of this notion. Let ρ be the isometry of T defined in 6.6. Let

$$T \otimes \mathbb{C} = V_+ \oplus V_-$$

be the decomposition of $T \otimes \mathbb{C}$ into the two 5-dimensional eigenspaces of ρ with eigenvalues $\zeta_3 = e^{2\pi i/3}$ and ζ_3^{-1}, respectively. Since

$$(\omega, \omega) = \big(\rho(\omega), \rho(\omega)\big) = \zeta^2(\omega, \omega),$$

we see that $(\omega, \omega) = 0$ for all $\omega \in V_+$, and similarly for V_-. Let

$$\mathscr{B} = \{\omega \in \mathbb{P}(V_+) : (\omega, \bar{\omega}) > 0\} = \mathscr{D}_M \cap \mathbb{P}(V_+).$$

In a suitable basis of V_+ we have $(\omega, \bar{\omega}) = x_0\bar{x}_0 - (x_1\bar{x}_1 + \cdots + x_4\bar{x}_4)$. Thus, if $(\omega, \bar{\omega}) > 0$, then $x_0 \neq 0$ and we can normalize $x_0 = 1$, hence $\mathscr{B}$ is a 4-dimensional complex ball:

$$\mathscr{B} \cong \Big\{x = (x_1, \ldots, x_4) \in \mathbb{C}^4 : \sum_i x_i\bar{x}_i < 1\Big\}.$$

The 4-ball is a bounded symmetric domain of type $\mathrm{I}_{1,4}$.

6.8. Discrete groups. We define the following four groups using the notation from 6.6:

$$\Gamma_M = \{g \in \mathrm{O}(L) : g(m) = m, \forall m \in M\},$$

$$\tilde{\Gamma}_\rho = \{g \in \mathrm{O}(L) : g \circ \rho = \rho \circ g\},$$

$$\Gamma_\rho = \{g \in \mathrm{O}(T) : g \circ \rho = \rho \circ g\},$$

$$\Gamma_{M,\rho} = \mathrm{Ker}\big(\Gamma_\rho \to \mathrm{O}(D)\big).$$

6.9. The Hermitian module. The isometry ρ of T gives T the structure of a free module Λ of rank 5 over the ring of Eisenstein integers $\mathbb{Z}[\zeta_3]$: for any $a + b\zeta_3 \in \mathbb{Z}[\zeta_3]$ and any $x \in T$ we have

$$(a + b\zeta_3) \cdot x = (a1_T + b\rho)(x).$$

If r_i, r_i' is the simple root basis of the i-th copy of A_2 with $\rho(r_i) = r_i'$, then $\zeta_3 r_i = r_i'$ and any element in this A_2 can be written as $r = zr_i$ with $z = a + b\zeta_3 \in \mathbb{Z}[\zeta_3]$. Note that

$$z\bar{z} = (a + b\zeta_3)(a + b\zeta_3^{-1}) = a^2 - ab + b^2 = -(r, r)/2.$$

Therefore the quadratic form on T is twice the real part of the $\mathbb{Z}[\zeta_3]$-valued Hermitian form H, of signature $(1, 4)$, on the Eisenstein lattice T with

$$H(z, w) = z_0\overline{w}_0 - (z_1\overline{w}_1 + \cdots + z_4\overline{w}_4).$$

The group Γ_ρ is the unitary group $\mathrm{U}(T)$ of T considered as a Hermitian lattice over the ring of Eisenstein integers (see [ACT], [AF]).

6.10. The discriminant group. The residue field $\mathbb{Z}[\zeta_3]/\sqrt{-3}\mathbb{Z}[\zeta_3]$ is isomorphic to $\mathbb{F}_3$ and ζ_3 maps to $1 \bmod 3$. Thus $V = \Lambda/\sqrt{-3}\Lambda$ acquires a natural structure of a 5-dimensional vector space over $\mathbb{F}_3$ equipped with a non-degenerate quadratic form. We show that the discriminant group $\mathrm{D}(T)$ is isomorphic to V. Define a $\mathbb{Z}$-linear homomorphism

$$h : \Lambda \to T^*, \quad h(x) = \bigl(x + 2\rho(x)\bigr)/3, \tag{6.1}$$

where we identify Λ with T as a $\mathbb{Z}$-module. Then

$$h(\sqrt{-3}x) = h\bigl((1 + 2\zeta_3)x\bigr) = (1 + 2\rho)^2 x/3 = -x \in T.$$

This shows that h factors through an isomorphism

$$V = \Lambda/\sqrt{-3}\Lambda \to \mathrm{D}(T) = T^*/T.$$

The basis $(r_1, \ldots, r_5)$ of Λ (as $\mathbb{Z}[\zeta_3]$-module) is an orthonormal basis with respect to H. Since $h(r_i)^2 = (r_i + 2r_i')^2/9 = -2/3$, $\bigl(h(r_i), h(r_j)\bigr) = 0$, $i \neq j$, we obtain that

$$h(x)^2 = -\frac{2}{3}x^2.$$

In particular, if we identify $\mathrm{D}(T)$ with V, then the quadratic form on $\mathrm{D}(T)$ is obtained from the quadratic form on V by multiplying it by $-2/3$.

If Q is the root lattice of type E_6, then $Q/3Q$ inherits a non-degenerate quadratic form such that $Q/3Q$ is isomorphic to V as quadratic spaces over $\mathbb{F}_3$. This defines an isomorphism of groups

$$(6.2) \qquad \begin{aligned} \mathrm{W}(E_6) &\cong \mathrm{SO}(V), \\ \mathrm{O}\bigl(D(T)\bigr) &\cong \mathrm{O}(V) \cong \{1, -1\} \times \mathrm{SO}(V). \end{aligned}$$

All of this is well-known and can be found, for example, in [Bo], Chapter 6, §4, exercise 2.

6.11. Proposition. *Each of the natural maps*

$$\tilde{\Gamma}_\rho \to \Gamma_\rho \to \mathrm{O}\bigl(\mathrm{D}(T)\bigr)$$

is surjective. In particular,

$$\Gamma_\rho/\Gamma_{M,\rho} \cong \mathrm{O}\bigl(\mathrm{D}(T)\bigr) \cong \{\pm 1\} \times \mathrm{W}(E_6).$$

Moreover, any isometry in $\Gamma_{M,\rho}$ can be extended to an isometry of L which acts trivially on M defining an injective homomorphism of groups

$$\Gamma_{M,\rho} \hookrightarrow \Gamma_M.$$

Proof. For the surjectivity of the map $\Gamma_\rho \to \mathrm{O}(\mathrm{D}(T))$ see [ACT], Lemma 4.5. It is proven in Nikulin [N1], Theorem 1.14.2 that the natural map $\mathrm{O}(M) \to \mathrm{O}(\mathrm{D}(M))$ is surjective. By Corollary 1.5.2 of loc. cit. this implies that the map $\tilde{\Gamma}_\rho \to \Gamma_\rho$ is surjective. The inclusion $\Gamma_{M,\rho} \to \Gamma_M$ follows from (Nikulin [N1], Corollary 1.5.2). □

6.12. Definition. An (ample) (M,ρ)-polarized $K3$ surface is an (ample) M-polarized $K3$ surface (X,ϕ) such that there is an extension $\tilde{\phi} : L \to H^2(X,\mathbb{Z})$ of ϕ which satisfies

$$\tilde{\phi}^{-1}\big(H^{2,0}(X)\big) \in \mathscr{B} \quad (\subset \mathbb{P}(T \otimes \mathbb{C})).$$

Two (M,ρ)-polarized $K3$ surfaces (X,ϕ) and (X',ϕ') are said to be isomorphic if there is an isomorphism $f : X \to X'$ such that $\phi = f^* \circ \phi'$ and $\tilde{\phi}^{-1} \circ f^* \circ \tilde{\phi}' \in \mathrm{O}(L)$ commutes with $\rho \in \mathrm{O}(L)$.

6.13. Lemma. *Let (X,ϕ) be an ample (M,ρ)-polarized $K3$ surface. Then X has an automorphism σ of order 3 such that $\sigma^* = \tilde{\phi} \circ \rho \circ \tilde{\phi}^{-1}$ for an extension $\tilde{\phi} : L \to H^2(X,\mathbb{Z})$ of ϕ. In particular, σ acts trivially on $\phi(M)$ $(\subset \mathrm{Pic}(X))$.*

Proof. Choosing $\tilde{\phi}$ as in the definition of (M,ρ)-polarization, the period of X is fixed by ρ. Since (X,ϕ) is amply polarized, $\mathrm{Pic}(X) \cap M^\perp$ contains no (-2)-vectors. Moreover, the M-polarization of X is ample and ρ acts trivially on M. Therefore X has an automorphism σ with $\sigma^* = \tilde{\phi} \circ \rho \circ \tilde{\phi}^{-1}$ (cf. [Na], Theorem 3.10). □

6.14. The moduli spaces $\mathscr{K}3^m_{M,\rho}$ and $\mathscr{K}3_{M,\rho}$. We know from section 6.3 that the moduli space of M-polarized $K3$ surfaces is isomorphic to $\mathscr{D}/\Gamma_M$. The isometry ρ acts naturally on $T_{\mathbb{C}}$ as is described in 6.7 and induces an automorphism of order 3 of the domain $\mathscr{D}_M \subset \mathbb{P}(T_{\mathbb{C}})$. It defines the union of two balls $\mathscr{B}_\pm = \mathscr{D}_M \cap \mathbb{P}(V_\pm)$. Complex conjugation switches the two balls $\mathscr{B}_\pm$. Obviously the group Γ_ρ is the stabilizer subgroup of $\mathscr{B} = \mathscr{B}_+$ in Γ_M. We set

$$\mathscr{K}3^m_{M,\rho} = \mathscr{B}/\Gamma_{M,\rho}, \quad \mathscr{K}3_{M,\rho} = \mathscr{B}/\Gamma_\rho.$$

The element $-I \in \Gamma_\rho$ acts trivially on $\mathbb{P}(T \otimes \mathbb{C})$ and thus on $\mathscr{B}$, and $-I$ maps to $-1 \in \mathrm{O}(D)$. Thus $\mathrm{O}(D)/\{\pm 1\} \cong \mathrm{W}(E_6)$ acts on $\mathscr{K}3^m_{M,\rho}$ and there is a natural map:

$$\pi_M : \mathscr{K}3^m_{M,\rho} \to \mathscr{K}3_{M,\rho} \cong \mathscr{K}3^m_{M,\rho}/\mathrm{W}(E_6).$$

For $r \in L$, let $r^\perp$ be the hyperplane in $\mathbb{P}(V_+)$ of lines orthogonal to r, and let $H(r)$ be its intersection with $\mathscr{B}$. The *discriminant locus* is the subset $\mathscr{H} \subset \mathscr{B}$ defined by:

$$\mathscr{H} = \bigcup_r H(r),$$

where r varies over the set of all (-2)-vectors in $T = M^\perp$. The image of $\mathscr{H}$ in $\mathscr{K}3^m_{M,\rho}$ (resp. $\mathscr{K}3_{M,\rho}$) will be denoted by Δ^m (resp. Δ).

It follows from Lemma 6.13 that the quasi-projective variety $\mathscr{K}3^m_{M,\rho} \backslash \Delta^m$ is the coarse moduli space of ample (M,ρ)-polarized $K3$ surfaces. We will refer to $\mathscr{K}3^m_{M,\rho}$ as the moduli space of (M,ρ)-polarized $K3$ surfaces.

6.15. Remark. If $[(X,\phi)], [(X',\phi')] \in \mathscr{K}3^m_{M,\rho}$ are in the same fibre of π_M, then the $K3$ surfaces X and X' are isomorphic. This follows from the surjectivity of the map $\tilde{\Gamma}_\rho \to \Gamma_\rho$ and the Torelli Theorem for $K3$ surfaces. Let $\alpha \in \mathrm{O}(\mathrm{D}(M))$. As we already noticed in the proof of Proposition 6.11, we can lift α to an isometry $\tilde{\alpha}$ of M. Composing it with some element of $\mathrm{W}(M)$ which acts identically on $\mathrm{D}(M)$, we may assume that $\tilde{\alpha}$ leaves $\Delta(M)^+$ invariant. Now α acts on $[(X,\phi)] \in \mathscr{K}3^m_{M,\rho}$ by $[(X,\phi)] \mapsto [(X,\phi\circ\tilde{\alpha}^{-1})]$. This describes the action of $\mathrm{O}(\mathrm{D}(M))$ on $\mathscr{K}3^m_{M,\rho}$. If $\phi(M) = \mathrm{Pic}(X)$, then $\mathrm{O}(\mathrm{D}(M))$ acts transitively on the polarizations of X. Thus we can interpret a general point of $\mathscr{K}3_{M,\rho}$ as the isomorphism class of a $K3$ surface which admits an ample (M,ρ)-polarization.

6.16. Recall that the subspaces V_+ and V_- (see 6.7) are defined over $\mathbb{Q}(\zeta_3)$ where ζ_3 is a primitive cube root of unity. Let K be the extension field of $\mathbb{Q}(\zeta)$ obtained by adjoining all primitive $6l$-th roots of unity for which the value of the Euler function satisfies $\varphi(6l) \leqq 10 = \mathrm{rank}(T)$. The only possible values of l are as follows: $l = 1,2,3,4,5$. We consider the union $\mathscr{W}$ of hyperplanes of $\mathbb{P}(V_+)$ defined over K. A non-singular cubic surface S is called *generic* if the period of the associated $K3$ surface X_S is contained in the complement of $\mathscr{W}$. For example, a cubic surface with an Eckardt point is not generic (we shall show in 8.9 that the period of X_S is contained in the hyperplane orthogonal to some vector $r \in T$).

6.17. Lemma. *Assume that S is a generic cubic surface and let X_S be the associated $K3$ surface. Then the image of the natural map*

$$\mathrm{Aut}(X_S) \to \mathrm{O}(T)$$

is a cyclic group of order 6 *generated by τ and σ (for τ, σ, see* 5.5, 5.6*). In particular the image of the natural map*

$$\mathrm{Aut}(X_S) \to \mathrm{O}(D(T))$$

is $\{\pm 1\}$.

Proof. The proof is similar to the one given in [BP], Lemma 2.9. It is well-known that the image G in $\mathrm{O}(T)$ is a cyclic group (cf. [N3], Theorem 3.1). Let m be the order of G. If $g \in \mathrm{Aut}(X_S)$ is a generator of G, then $g^*\omega_X = \zeta_m \cdot \omega_X$ where ω_X is a nowhere vanishing holomorphic 2-form on $X = X_S$ and ζ_m is a primitive m-th root of unity. Since $\tau^*\omega_X = -\omega_X$ and $\sigma^*\omega_X = \zeta_3\omega_X$, m is divisible by 6. Since g^* is defined over $\mathbb{Q}$, the eigenspaces of g^* are defined over $\mathbb{Q}(\zeta_m)$. If $m > 6$, then an eigenspace is a non-trivial subspace of V_+. This contradicts the assumption of genericity of S. σ^* acts trivially on $\mathrm{D}(T)$ and τ^* acts as -1. Hence the second assertion follows. □

6.18. Corollary. *The map $\pi_M : \mathscr{K}3^m_{M,\rho} \to \mathscr{K}3_{M,\rho}$ is a Galois cover with the Galois group isomorphic to* $\mathrm{W}(E_6)$.

Proof. As we explained in 6.14 the group $\mathrm{O}(\mathrm{D}(T))/\{\pm 1\} \cong \mathrm{W}(E_6)$ acts on $\mathscr{K}3^m_{M,\rho}$ with quotient isomorphic to $\mathscr{K}3_{M,\rho}$. The isotropy subgroup of $[(X,\phi)]$ is isomorphic to the image of $\mathrm{Aut}(X)$ in $\mathrm{D}(\phi(M)^\perp)/\{\pm 1\}$. By the previous lemma it is trivial for a generic surface X. □

6.19. Nef divisors. Let (X,ϕ) be an ample M-polarized $K3$ surface. Then X has an automorphism σ of order 3 (6.13). For any $v \in M$ with $v^2 \geqq 0$ there is a $w \in \mathrm{W}(M)$ such that $\phi(w(v)) \in C(M)$. If $\phi(w(v))$ is not nef, then there is a smooth rational curve R with

$\big(R, \phi\big(w(v)\big)\big) < 0$. Since $\phi(M)^{\perp} \cap \mathrm{Pic}(X)$ does not contain (-2)-vectors, $R = r + r'$ where $r \in M^*$, $r' \in T^*$ and $r^2 < 0$, $(r')^2 < 0$. Since $r^2 + (r')^2 = R^2 = -2$, $r^2 = -2/3$ or $-4/3$. Since σ is an automorphism, $\big(R, \sigma(R)\big) \geqq 0$. Hence $(3r)^2 = \big(R + \sigma(R) + \sigma^2(R)\big)^2 \geqq -6$. Thus $r^2 = -2/3$. Then r defines a reflection

$$s_r : x \mapsto x + 3(x, r)r$$

which acts trivially on T. Obviously $\big(R, \phi\big(s_r\big(w(v)\big)\big)\big) > 0$. If necessary, by using these reflections successively, we may assume that $\phi\big(w(v)\big) \in C(X)$, i.e., $\phi\big(w(v)\big)$ is nef. In particular, any primitive isotropic vector f in M defines, uniquely, a nef divisor in $\mathrm{Pic}(X)$. As is well-known a primitive nef divisor F with $F^2 = 0$ defines an elliptic fibration with the cohomology class of a fibre equal to F ([PS], §3, Cor. 3).

6.20. Elliptic fibrations. Let (X, ϕ) be an ample M-polarized $K3$ surface. With the definitions from 6.5, we have $f_1 \in C(M)$ and f_1 is obviously isotropic and primitive. Therefore, $\phi(f_1) \in \mathrm{Pic}(X)$ defines an elliptic fibration on V (cf. 6.19) which we denote by

$$\Phi_\phi : X \to \mathbb{P}^1$$

and we call it the *standard elliptic fibration*. Since $\phi(f_2 - f_1) \cdot \phi(f_1) = (f_2 - f_1, f_1) = 1$, the divisor class $\phi(f_2 - f_1)$ is an effective class with $D^2 = -2$. Let D be the effective representative of this class written as a sum $\sum n_i R_i$, where R_i are irreducible curves. Since D intersects any fibre F with multiplicity 1, we see that one of the components R_i, say R_1, is a section of the fibration. We also have $n_1 = 1$ and $R_i \cdot F = 0$ for $i > 1$. By the Hodge Index Theorem, $R_i^2 < 0$ for $i > 1$. By the adjunction formula, all R_i's are (-2)-curves and the R_i's, $i \neq 1$, are contained in fibres of the fibration. This easily implies that R_1 is determined uniquely by $\phi(f_2 - f_1)$. We shall denote the section corresponding to R_1 by s. We remark that R_1 is obtained from D by applying suitable reflections corresponding to R_i $(i > 1)$. Thus, up to isometries, we may assume that the classes f_1 and $f_2 - f_1$ define an elliptic fibration Φ_ϕ with a section s.

The images under ϕ of the simple root bases $\{r_i, r_i'\}$, $i = 1, \dots, 5$, of each copy of A_2 are effective divisor classes R_i, R_i' on X which are orthogonal to F and to the section s. As above we can show that each such divisor class is a sum of (-2)-curves contained in a fibre. Thus X has at least 10 smooth rational curves contained in fibres of Φ_ϕ.

6.21. Lemma. *Let (X, ϕ) be an ample (M, ρ)-polarized $K3$ surface, let σ be an automorphism of order three as in* 6.13 *and let Φ_ϕ be the standard elliptic fibration on X.*

Then σ preserves Φ_ϕ and fixes pointwisely its section s and a smooth bisection b. Moreover, the types of singular fibres of Φ_ϕ are one of the following:

$$(\mathrm{II}, \mathrm{II}, \mathrm{IV}, \mathrm{IV}, \mathrm{IV}, \mathrm{IV}, \mathrm{IV}), \quad (\mathrm{II}, \mathrm{IV}, \mathrm{IV}, \mathrm{IV}, \mathrm{IV}, \mathrm{I}_0^*), \quad (\mathrm{IV}, \mathrm{IV}, \mathrm{IV}, \mathrm{I}_0^*, \mathrm{I}_0^*).$$

In each case the fibration has exactly 5 *reducible fibres.*

Proof. Let X^σ be the fixed locus of the automorphism σ. Since σ can be locally linearized, X^σ is a smooth closed subset of X. It is easy to see that the trace of ρ in its action on $L \cong H^2(X, \mathbb{Z})$ is equal to 7. Applying the Lefschetz fixed point formula, we obtain that the Euler characteristic of X^σ is equal to 9. Since σ acts identically on $\phi(M)$, it preserves the section s and the divisor class of a fibre of Φ_ϕ. Let us show that σ fixes the section s pointwisely, or, equivalently, leaves invariant each fibre of Φ_ϕ. Assuming otherwise, we obtain that X^σ is contained in fibres of Φ_ϕ. Thus any irreducible one-dimensional component of X^σ has the Euler characteristic equal to 0 (if it is nonsingular fibre) or 2 (if it is a component of a reducible fibre), the smoothness of the fixed point set excludes nodal cubics. Let l be the number of irreducible one-dimensional components of X^σ different from a fibre, and let k be the number of isolated fixed points. Then $2l + k = \chi(X^\sigma) = 9$. Since σ has exactly two fixed points on s, it leaves invariant the two fibres F_1, F_2 passing through these points. Obviously the curves R_i, R_i' (see 6.20) are contained in the union $F_1 \cup F_2$. In particular, the number of irreducible components of the divisor $F_1 + F_2$ is greater than or equal to 12. Since a Dynkin diagram of type ADE admits a non-trivial automorphism of order 3 only in the case D_4, the automorphism σ acts identically on the set of irreducible components of a fibre F_i unless it is of type I_0^*. Note that either F_1 or F_2 is not of type I_0^* because $F_1 + F_2$ has at least 12 components. Assume that both of the F_i's are not of this type. We apply the Lefschetz fixed point formula to the cell complex F_i. Let n_i be the number of irreducible components of F_i. The Lefschetz number of $\sigma|F_i$ is equal to n_i if F_i is of type I_n and to $n_i + 1$ otherwise. Let l_i be the number of one-dimensional rational components of X^σ contained in F_i and let k_i be the number of isolated fixed points of σ contained in F_i. We have $2l_i + k_i \geqq n_i$, hence $9 = 2l + k \geqq 2l_1 + k_1 + 2l_2 + k_2 \geqq n_1 + n_2 \geqq 12$, a contradiction. Assume that one of the fibres, say F_1 is of type I_0^*. Then $2l_2 + k_2 \geqq n_2 \geqq 12 - 5 = 7$. The automorphism σ has a fixed point on the non-multiple component E of F_1 which is intersected by s. The multiple component E_0 of F_1 is σ-invariant. If σ is the identity on E_0, then $l_1, k_1 \geqq 1$, and $2l_1 + k_1 \geqq 3$. If σ does not act identically on E_0, it has 2 fixed points on it. In both cases it is easy to see that $2l_1 + k_1 \geqq 3$ again. Thus we get $2l_1 + k_1 + 2l_2 + k_2 \geqq 3 + n_2 \geqq 3 + 7 = 10$, again a contradiction.

Now we know that σ preserves every fibre of Φ_ϕ, so that the general fibre has a non-trivial automorphism of order 3 over the function field of the base. This implies that the j-function of the fibration is constant 0. In particular, the singular fibres must be of type II, IV, IV^*, II^*, I_0^*. Each nonsingular fibre has exactly 3 fixed points of σ, one lies on the section s, and the pairs of others lie on a bisection b (which could be the union of two sections). The bisection b is a part of X^σ and hence smooth.

Let $\pi : X' \to X$ be the blow-up of the 0-dimensional part of X^σ. We know that σ is not symplectic (i.e. does not leave invariant a non-zero holomorphic 2-form on X). This easily shows that it lifts to an automorphism σ' of X' with $X'^{\sigma'}$ purely one-dimensional. Let $\overline{X}'$ be the quotient surface $X'/(\sigma')$. It is a smooth surface. Let C be a smooth rational curve on X such that $\sigma(C) = C$ but $\sigma|C$ is not the identity. Then σ has two fixed points p, q on C. If p, q are isolated fixed points of σ on X, then the proper inverse transform C' on X' has self-intersection -4. Since C' is equal to the pre-image of some curve on $\overline{X}'$ and -4 is not divisible by 3, we get a contradiction. Similarly, if p, q belong to the one-dimensional part of X^σ, we get $C'^2 = -2$ and again a contradiction. Thus, one fixed point is an isolated fixed point of σ and another one belongs to the one-dimensional part of X^σ.

As we have already observed before, σ acts identically on the set of irreducible components of any fibre, unless it is of type I_0^*. In the case of I_0^*, σ preserves the multiple component E and permutes the three simple components E_1, E_2, E_3 not meeting the section. Notice that any σ-invariant irreducible component of a fibre not intersecting the section s

must belong to $\phi(M) \cap \phi(U)^{\perp} = \phi(A_2^5)$. The fixed part of $D_4 = \langle E, E_1, E_2, E_3 \rangle$ under σ^* is $\langle E, E + E_1 + E_2 + E_3 \rangle \cong A_2$. Since E_6 and E_8 can not be embedded into A_2^5, singular fibres of type IV*, II* do not appear.

Using that the Euler characteristics of the fibres add up to 24, it remains to show that we have exactly 5 reducible fibres. Since a fibre of type I_0^* or IV contributes one copy of A_2 in $A_2^5 \cong \phi(M) \cap \phi(U)^{\perp}$, there must be five of them. The lemma is now proven. □

7. A complex ball uniformization

7.1. From $K3$'s to cubics. We are going to construct a map

$$G : \mathscr{K}3^m_{M,\rho} \backslash \Delta^m \to \mathscr{M}^m_{\mathrm{cub}},$$

where $\mathscr{M}^m_{\mathrm{cub}}$ is the moduli space of marked smooth cubic surfaces, i.e., smooth cubic surfaces with an ordered set of six skew lines $L_1, \dots, L_6$.

Let $[(X, \phi)] \in \mathscr{K}3^m_{M,\rho} \backslash \Delta^m$ be an ample (M, ρ)-polarized $K3$ surface. We use the notation of Lemma 6.21 and its proof. For simplicity we consider the case where Φ_ϕ has two singular fibres of type II and five singular fibres of type IV. The construction for the other two cases is similar. It follows from the proof of Lemma 6.21 that on each reducible fibre σ has one fixed point, the point of intersection of the three components. The bisection b intersects two components, and the section s intersects the third one. Let X' be the blow-up of the five isolated fixed points of σ as in the proof of the lemma. The quotient $\overline{X}'$ of X' by the action of σ is a smooth rational surface and the images of the components of the fibers of type IV are (-1)-curves in $\overline{X}'$. The polarization ϕ gives an ordering of the 2 components in each fibre which meet the bisection b, and we blow down the first one in each of the 5 fibres as well as the component in the fibre which meets the section. The result is a smooth rational surface S which has (-1)-curves $L_1, \dots, L_5$ the images of the remaining components in the type IV fibres (these are numbered by the polarization ϕ) as well as the (-1)-curve m which is the image of the section s. These six curves do not intersect and thus can be blown down to get a smooth rational surface with $b_2 = 1$, hence this surface must be $\mathbb{P}^2$. Therefore S is a cubic surface and the six (-1)-curves define a marking on S. It is easy to see that this marked cubic surface S depends only on the isomorphism class of (X, ϕ). We may now define:

$$G : [(X, \phi)] \mapsto (S, L_1, \dots, L_5, L_6 = m).$$

Note that the 2-section C maps to a line l in S which is skew with m and does meet $L_1, \dots, L_5$. By the uniqueness of the triple cover (Theorem 4.9) we have that $X \cong X_{S,l,m}$ and, by construction (see 6.13) $\sigma^* = \tilde{\phi} \circ \rho \circ \tilde{\phi}^{-1}$ for some extension $\tilde{\phi} : L \to H^2(X, \mathbb{Z})$ of ϕ.

7.2. Theorem. *The map G defines a* $\mathrm{W}(E_6)$*-equivariant isomorphism*

$$G : \mathscr{K}3^m_{M,\rho} \backslash \Delta^m \xrightarrow{\cong} \mathscr{M}^m_{\mathrm{cub}}.$$

Proof. We first construct the inverse map

$$G^{-1} : \mathscr{M}^m_{\mathrm{cub}} \to \mathscr{K}3^m_{M,\rho} \backslash \Delta^m.$$

Given $(S, L_1, \dots, L_6) \in \mathscr{M}^m_{\mathrm{cub}}$, let $m = L_6$ and let l be the (unique) line which meets $L_1, \dots, L_5$ but not m (if we blow down the L_i to points $x_i \in \mathbb{P}^2$, l maps to the conic on $x_1, \dots, x_5$).

Let $X_{l,m}$ be the $K3$ surface associated to (S, l, m) and let $f : X_{l,m} \to \mathbb{P}^1$ be the elliptic fibration from subsection 4.3. We define a polarization $\phi_{l,m} : M \to \mathrm{Pic}(X_{l,m})$ as in the proof of Lemma 5.8 by fixing an order on the set of reducible fibres and the order on the set of components of fibres of type IV which do not intersect the section s. Thus $\phi(f_1)$ is the class of a fibre of f and $\phi(f_2)$ is the sum of the class of a fibre and the class of the section (see 6.20). The image of r_1 in the i-th copy of $A_2 \subset M$ is the first component of the i-th fibre if it is of type IV, and it is the divisor class $E + E_1 + E_2 + E_3$ if the i-th fibre is of type I_0^* (see the notation in the proof of Lemma 6.21).

The $K3$ surface $X_{l,m}$ is a triple cyclic covering of S with an automorphism σ. We proved in Lemma 5.8 that σ^* acts identically on $\phi(M)$ and has the trace -5 on $\phi(M)^\perp$. This implies that σ^* has no eigenvectors in $\phi(M)^\perp \otimes \mathbb{Q}$, and hence $\phi(M)^\perp$ is a free module of rank 5 over the ring of Eisenstein integers $\mathbb{Z}[\zeta_3]$. In particular, the maps σ^* glue to a locally constant map on the local system with fibers $H^2(X_{l,m}, \mathbb{Z})$. The construction of the map G is such that if $(S', L'_1, \dots, L'_6) = G(X, \phi)$ for some (X, ϕ), then $\rho = \tilde{\phi}^{-1} \circ \sigma^*_{S'} \circ \tilde{\phi}$ where $\tilde{\phi} : L \to H^2(X_{l,m}, \mathbb{Z})$ is a cohomology marking of X such that $\tilde{\phi}|M = \phi$ and $\tilde{\phi}(T) = \phi(M)^\perp$. As σ^* is locally constant we conclude that there is an extension $\tilde{\phi}_{l,m}$ of the polarization $\phi_{l,m}$ such that $\rho = \tilde{\phi}^{-1}_{l,m} \circ \sigma^* \circ \tilde{\phi}_{l,m}$. This shows that $G^{-1}[(S, L_1, \dots, L_6)] := [(X_{l,m}, \phi)]$ belongs to $\mathscr{K}3^m_{M,\rho} \backslash \Delta^m$. It is obvious that G^{-1} is the inverse of G.

We remark that the above construction of $X_{l,m}$ can be done as a family, and hence G^{-1} is analytic. Let $(\mathscr{S}, \mathscr{L}_1, \dots, \mathscr{L}_6)$ be an analytic family of marked smooth cubic surfaces over the base Y. Then by taking the triple cover and taking the resolution of singularities, we have an analytic family of $K3$ surfaces $\mathscr{X}$ over Y. The covering transformation of $\mathscr{X} \to \mathscr{S}$ induces an automorphism σ_y of each member $X_{l,m,y}$ $(y \in Y)$ of the family $\mathscr{X}$ and defines an isometry $\bar{\phi}_y : L \to H^2(X_{l,m,y}, \mathbb{Z})$ with $\bar{\phi}_y^{-1} \circ \sigma_y^* \circ \bar{\phi}_y = \rho$ which depends analytically on y. Thus we have an analytic family of ample (M, ρ)-polarized $K3$ surfaces over Y.

We show that G^{-1} is $\mathrm{W}(E_6)$-equivariant, then $G = (G^{-1})^{-1}$ is obviously equivariant as well. The group $\mathrm{W}(E_6)$ acts on $\mathscr{M}^m_{\mathrm{cub}}$ in the standard way via symmetries of the set of lines and $\mathrm{W}(E_6) = \mathrm{Gal}(\mathscr{M}^m_{\mathrm{cub}}/\mathscr{M}_{\mathrm{cub}})$. Let $\mu : \mathrm{Gal}(\mathscr{M}^m_{\mathrm{cub}}/\mathscr{M}_{\mathrm{cub}}) \to \mathrm{Aut}(\mathscr{K}3^m_{M,\rho} \backslash \Delta^m)$ be the action defined via the isomorphism G^{-1}, obviously μ is injective. Let $S \in \mathscr{M}_{\mathrm{cub}}$, the main result of the section 3 (Theorem 4.9) was that $X_{l,m}$ is independent of the choice of the lines l, m in S, hence $\mu(g)$ is a covering transformation of $\mathscr{K}3^m_{M,\rho} \backslash \Delta^m \to \mathscr{K}3_{M,\rho} \backslash \Delta$ for any $g \in \mathrm{W}(E_6)$. Thus we have an injection:

$$\mu : \mathrm{W}(E_6) \cong \mathrm{Gal}(\mathscr{M}^m_{\mathrm{cub}}/\mathscr{M}_{\mathrm{cub}}) \to \mathrm{Gal}(\mathscr{K}3^m_{M,\rho}/\mathscr{K}3_{M,\rho}).$$

Since $\mathrm{Gal}(\mathscr{K}3^m_{M,\rho}/\mathscr{K}3_{M,\rho}) \cong \mathrm{W}(E_6)$ (see 6.18), μ is an isomorphism. □

7.3. The moduli space of cubic surfaces $\mathscr{M}_{\mathrm{cub}}$ is the quotient of $\mathscr{M}^m_{\mathrm{cub}}$ by $\mathrm{W}(E_6)$. Let $\mathrm{W}(E_6)_l \subset \mathrm{W}(E_6) \subset \mathrm{Aut}(\mathrm{Pic}(S))$ be the subgroup which fixes the class of a line l on S. It is well-known that $\mathrm{W}(E_6)_l \cong \mathrm{W}(D_5)$, which is the semi-direct product of $(\mathbb{Z}/2)^4$ and S_5.

The action of $S_5 \subset \mathrm{W}(D_5)$ on a marking $(L_1, \dots, L_6 = l)$ of a cubic surface is by permuting the first 5 lines. The group $\mathrm{W}(D_5)$ is generated by these permutations and an element c_{123} of order two which acts as the standard Cremona transformation on $\mathbb{P}^2$ defined by the points p_1, p_2 and p_3 where $\pi : S \to \mathbb{P}^2$ is the blow down of the L_i and $p_i = \pi(L_i)$. Thus c_{123} maps L_1 to L_1', the strict transform of the line on p_2 and p_3, and it fixes L_4, L_5 and L_6. It also permutes the $2 \cdot 5$ lines on S which meet l. Let l_i be the line which maps to the line through p_i and p_6 and let m_i be the conic through all 6 points except p_i. Then c_{123} fixes the l_i and m_i except for permuting $l_4 \leftrightarrow m_5$ and $l_5 \leftrightarrow m_4$. This implies that an element in $\mathrm{W}(D_5)$ permutes the indices and exchanges an even number of l_i with an even number of m_i.

7.4. Recall from Proposition 6.11 that

$$\Gamma_\rho/\Gamma_{M,\rho} \cong \mathrm{O}(D) \cong \mathrm{W}(E_6) \times \{\pm 1\}$$

acts on the discriminant lattice $D = \mathrm{D}(T) \cong \mathbb{F}_3^5$. The subgroup of $\mathrm{O}(D)$ which consists of isometries preserving an unordered basis (up to signs) of $\mathrm{D}(T)$ is isomorphic to $\mathrm{W}(D_5) \times \{\pm 1\}$. This provides us with a natural copy of $\mathrm{W}(D_5)$ in $\Gamma_\rho/\Gamma_{M,\rho}$. Let $\Gamma'_{M,\rho}$ be the inverse image in Γ_ρ of this subgroup. The group $\Gamma'_{M,\rho}$ acts on $\mathscr{K}3^m_{M,\rho}$ by changing the polarizations without changing the standard elliptic fibration defined by the polarization. Since $\mathrm{W}(D_5)$ is a maximal subgroup of $\mathrm{W}(E_6)$ we see that any $w \in \mathrm{W}(E_6) \backslash \mathrm{W}(D_5)$ does not preserve the isomorphism class of the standard elliptic fibration. This implies the following corollaries:

7.5. Corollary. *Let $\mathscr{M}_{\mathrm{cub}}$ be the moduli space of cubic surfaces. There are isomorphisms*

$$(\mathscr{B} \backslash \mathscr{H})/\Gamma_{M,\rho} \cong \mathscr{K}3_{M,\rho} \backslash \Delta \cong \mathscr{M}_{\mathrm{cub}}.$$

Let $\mathscr{M}^1_{\mathrm{cub}}$ be the moduli space of cubic surfaces with a line. There are isomorphisms

$$(\mathscr{B} \backslash \mathscr{H})/\Gamma'_{M,\rho} \cong (\mathscr{K}3^m_{M,\rho} \backslash \Delta^m)/\mathrm{W}(D_5) \cong \mathscr{M}^1_{\mathrm{cub}}$$

as well as a birational isomorphism

$$\mathscr{B}/\Gamma'_{M,\rho} \simeq \mathscr{M}^1_{\mathrm{cub}}$$

where $\Gamma'_{M,\rho}$ is the inverse image of $\mathrm{W}(E_6)_l \times \{\pm 1\} \subset \mathrm{W}(E_6) \times \{\pm 1\} \cong \Gamma_\rho/\Gamma_{M,\rho}$ in Γ_ρ.

7.6. Corollary. *Assume that S is a generic cubic surface. Then X_S has exactly 27 ($=$ the index of $\mathrm{W}(D_5)$ in $\mathrm{W}(E_6)$) non-isomorphic standard elliptic fibrations.*

8. The geometry of the discriminant locus

8.1. Here we will give a geometric interpretation of the points in $\mathscr{K}3^m_{M,\rho}$ belonging to the discriminant locus Δ^m. We know that each such point represents the isomorphism class of a non-amply M-polarized $K3$ surface (X, ϕ). For such a surface there is a (-2)-vector r in $\phi(M)^\perp \cap \mathrm{Pic}(X)$. This implies that ρ (cf. 6.6) can not be represented by an automorphism of X. Let R be the sublattice of $\mathrm{Pic}(X)$ generated by all (-2)-vectors in

$\phi(M)^{\perp} \cap \mathrm{Pic}(X)$. Then R is a negative definite lattice generated by (-2)-vectors, i.e., a root lattice. Hence R is an orthogonal direct sum

$$R = R_1 \oplus \cdots \oplus R_r,$$

where R_i is an indecomposable root lattices of type A_m, D_n, E_k. Obviously ρ preserves R. Since ρ has no non-zero fixed vectors in R, ρ preserves each R_i. Thus R_i is an indecomposable root lattice with an isometry of order 3 without non-zero fixed vectors. In the following we shall show that $R_i \cong A_2$ and $r \leqq 4$ (see 8.7).

8.2. Lemma. *$R_i \cong A_2$ for any i.*

Proof. First of all, note that the rank of R_i is even because it has an isometry of order 3 without non-zero fixed vectors. Since the rank of $\mathrm{Pic}(X) \leqq 20$, R_i is isometric to A_{2n}, D_{2n}, E_6 or E_8 ($n \leqq 4$). Let K be a primitive sublattice of $H^2(X, \mathbb{Z})$ generated by M and R. Let $l(K)$ be the minimal number of generators of the 3-elementary subgroup of K^*/K. Then $K^*/K \cong (K^{\perp})^*/K^{\perp}$ and $l(K) = l(K^{\perp}) \leqq \mathrm{rank}(K^{\perp})$. Using this observation and the fact $l(M) = 5$, we can easily see that R is isometric to D_4, $A_2^{\oplus n}$ ($1 \leqq n \leqq 4$) or E_6. (For example if $R = E_8$, then $K = M \oplus E_8$ and $l(K) = 5$. This contradicts the fact $l(K^{\perp}) \leqq \mathrm{rank}(K^{\perp}) = 2$.) Next we shall show that R is not isometric to D_4. In this case $K = M \oplus D_4$ and the elliptic fibration defined by an M-polarization has five singular fibres of type IV and one of type I_0^*. This contradicts the fact that the Euler number of $K3$ surface is 24. By the same argument, the case $R = E_6$ does not occur. □

8.3. We remark that all R_i are 3-elementary, i.e., $R_i^*/R_i \cong (\mathbb{Z}/3\mathbb{Z})^l$ for some non-negative integer l and ρ acts trivially on R_i^*/R_i.

Let

$$T' = (\phi(M) \oplus R)^{\perp}, \quad S = (T')^{\perp} \quad (\subset H^2(X, \mathbb{Z})).$$

Thus S is the smallest primitive sublattice of $H^2(X, \mathbb{Z})$ containing $\phi(M) \oplus R$. By definition, the lattice $T' \cap \mathrm{Pic}(X)$ contains no (-2)-vectors.

8.4. Lemma. *Let (X, ϕ) be an (M, ρ)-polarized $K3$ surface. Let S, R, T' be as above. Then S, T' are 3-elementary lattices, and ρ acts trivially on $(T')^*/T'$. Moreover X has an automorphism σ' of order three such that $S = H^2(X, \mathbb{Z})^{(\sigma')^*}$.*

Proof. We have a chain of lattices:

$$\phi(M) \oplus R \subset S \subset S^* \subset (\phi(M) \oplus R)^*$$

and $S^*/S \cong (S^*/(\phi(M) \oplus R))/(S/(\phi(M) \oplus R))$. Since M and R are 3-elementary, S is a 3-elementary lattice, i.e., $S^*/S \cong (\mathbb{Z}/3\mathbb{Z})^l$. Since ρ acts trivially on $(\phi(M) \oplus R)^*/(\phi(M) \oplus R) \cong \phi(M)^*/\phi(M) \oplus R^*/R$, ρ acts trivially on S^*/S. Since T' is the orthogonal complement of S in unimodular lattice $H^2(X, \mathbb{Z})$, T' is 3-elementary and ρ acts trivially on $(T')^*/T'$ (see Nikulin [N1], Proposition 1.6.1). Hence the isometry $(1_S, \rho|T')$ can be extended to an isometry ρ' of $H^2(X, \mathbb{Z})$ (Nikulin [N1], Corollary 1.5.2). Then ρ' is represented by an automorphism σ' of X (see [Na], Theorem 3.1). □

The following fact was first observed by Vorontsov [Vor].

8.5. Lemma. *We keep the same assumption as in Lemma* 8.4. *Define a non-negative integer* $l(T')$ *by*: $(T')^*/T' \cong (\mathbb{Z}/3\mathbb{Z})^{l(T')}$. *Then*

$$\operatorname{rank}(T') \geqq 2l(T').$$

Proof. Let $x \in T'$. Since

$$\bigl(x, \rho'(x)\bigr) = \bigl(\rho'(x), (\rho')^2(x)\bigr) = \bigl(\rho'(x), -x - \rho'(x)\bigr),$$

we get $2\bigl(x, \rho(x)\bigr) = -(x,x)$. Hence x and $\rho'(x)$ generate a sublattice $A_2(m)$, where $m = (x,x)$. From this we can find a sublattice $K = A_2(m_1) \oplus \cdots \oplus A_2(m_k)$ of T' of finite index. Moreover we have $(T')^*/T' \cong \bigl((T')^*/K\bigr)/(T'/K)$. If m_i is not divisible by 3, the contribution from $A_2(m_i)$ to $l(T')$ is at most 1. In case m_i is divisible by 3, the fixed part under ρ' in $A_2(m_i)^*/A_2(m_i)$ is $\mathbb{Z}/3\mathbb{Z}$. Since ρ acts trivially on $(T')^*/T'$, the contribution from $A_2(m_i)$ is at most 1. This implies the assertion. □

8.6. Lemma. *We keep the same notation as in Lemma* 8.4. *Then* $R \cong A_2^{\oplus r}$ *and* $l(S) = 5 - r$.

Proof. Let

$$R = R_1 \oplus \cdots \oplus R_r$$

be the orthogonal decomposition of R into indecomposable root lattices R_i. We know that R_i is isomorphic to A_2 (Lemma 8.2). Obviously R_i^*/R_i is $\mathbb{Z}/3\mathbb{Z}$. Since

$$S^*/S \cong \bigl(S^*/\bigl(\phi(M) \oplus R\bigr)\bigr)/\bigl(S/\bigl(\phi(M) \oplus R\bigr)\bigr),$$

we have $l(T') = l(S) \geqq \bigl(l(M) + r\bigr) - 2r = 5 - r$. On the other hand, it follows from Lemma 8.5 that $10 - 2r \geqq \operatorname{rank}(T') \geqq 2l(T')$. Hence $l(S) = 5 - r$. □

Let us summarize the previous lemmas by stating the following:

8.7. Theorem. *Let* $(X, \phi) \in \mathscr{K}3^m_{M,\rho}$. *Then* X *admits an automorphism* σ' *of order* 3 *such that* $H^2(X,\mathbb{Z})^{(\sigma')^*} = S$, *the smallest primitive sublattice of* $\operatorname{Pic}(X)$ *which contains* $\phi(M)$ *and the sublattice* R *generated by all* (-2)*-vectors in* $\phi(M)^\perp \cap \operatorname{Pic}(X)$. *The sublattices* $\phi(M)$ *and* R *are orthogonal to each other and the lattice* R *is isomorphic to* r $(\leqq 4)$ *copies of the lattice* A_2. *The number* r *will be called the* d e g e n e r a c y r a n k *of* (X, ϕ).

The degeneracy rank of (X, ϕ) is equal to the number of nodes of the associated nodal cubic surface (see 2.15). This is easy to see from Table 2 by computing the quotient of $M(\boldsymbol{t})$ by $M = U \oplus A_2^{\oplus 5}$ and comparing the result with the value of r in Table 1. The next theorem generalizes Lemma 6.21.

8.8. Theorem. *Let* $[(X, \phi)] \in \mathscr{K}3^m_{M,\rho}$. *Then the* M*-polarization* ϕ *of* X *defines an elliptic fibration. Its singular fibres are given in the column Kodaira fibres of Table* 1 *from above. The Picard lattice* S_X *and its lattice of transcendental cycles* T_X *can be found in the corresponding rows of Table* 2 (*under the assumption in Proposition* 5.3). *The degeneracy rank is given in the column* r *in Table* 1.

Proof. By the same arguments as in 6.19, 6.20, the M-polarization on X defines an elliptic fibration with a section. The proof of the assertion about possible combinations of singular fibres is very similar to the proof of Lemma 6.21 and is omitted. The description of the transcendental lattice follows from the following easy facts:

$$q_{E_6} = -q_{A_2}, \quad q_{A_2(-1)} = -q_{A_2}, \quad q_{A_2} \oplus q_{A_2} = q_{A_2(-1)} \oplus q_{A_2(-1)}, \quad q_{A_2(-2)} = q_{D_4} \oplus q_{A_2}$$

and Theorem 1.14.2 from [N1]. □

8.9. The Eckardt locus. Let $[(X,\phi)] \in \mathscr{K}3^m_{M,\rho} \backslash \Delta^m$. We know that the corresponding marked cubic surface $(S, L_1, \dots, L_6)$ has an Eckardt point on the unique line l intersecting $L_1, \dots, L_5$ if and only if the standard elliptic fibration Φ_ϕ on (X,ϕ) has a fibre of type I_0^*. In that case $\phi(M) \neq \mathrm{Pic}(X)$, but for general S with such property, the orthogonal complement $\phi(M)^\perp_{\mathrm{Pic}(X)}$ of $\phi(M)$ in $\mathrm{Pic}(X)$ is isomorphic to $A_2(2)$. In fact if $F = 2E_0 + E_1 + \cdots + E_4$ is the fibre of type I_0^* and E_4 meets the section, then $\phi(M)^\perp_{\mathrm{Pic}(X)}$ is spanned by $E_1 - E_2$ and $E_2 - E_3$.

The involution τ (cf. 5.6) defined by the elliptic fibration also acts on $\phi(M)$, via $\iota = \tau^*$, in a different way. If all fibres are of type IV, then the action of ι on $\phi(M) \cong U \oplus A_2^5$ permutes the simple root basis in each copy of A_2. Let $N = \phi(M)^\iota$ be the sublattice of the invariant elements, then

$$N \cong U \oplus A_1^5.$$

However, if one of the fibres is of type I_0^*, then $\phi(M)^\iota \cong U \oplus A_2 \oplus A_1^4$. The orthogonal complement of $\phi(N)$ in $\phi(M)^\iota$ is spanned by the class of the divisor $E_1 + E_2 + E_3$. Also $r = [E_1] \in \phi(N)^\perp_L$ but not in $\phi(M)$.

For any (-2)-vector $r \in N^\perp \backslash T \subset L$ consider the hyperplane $r^\perp$ in $\mathbb{P}(V_+)$ of lines orthogonal to r. Let $H(r)_\iota$ be the intersection of this hyperplane with the ball $\mathscr{B} \subset \mathbb{P}(V_+)$. Let $\mathscr{H}_\iota$ be the union of the hyperplanes $H(r)_\iota$. If an ample (M,ρ)-marked surface (V,ϕ) has a fibre of type I_0^* in its standard elliptic fibration Φ_ϕ, then its period belongs to $\mathscr{H}_\iota$. Let Δ_ι^m (resp. in Δ_ι) be the image of $\mathscr{H}_\iota$ in $\mathscr{K}3^m_{M,\rho}$ (resp. in $\mathscr{K}3_{M,\rho}$). In this notation we have

8.10. Theorem. *Under the isomorphism $\mathscr{M}_{\mathrm{cub}} \cong \mathscr{K}3_{M,\rho} \backslash \Delta$, the image of the locus of smooth cubic surfaces with Eckardt points* (*the Eckardt locus*) *is mapped to $\Delta_\iota \backslash (\Delta \cap \Delta_\iota)$.*

8.11. It is well-known that any nonsingular cubic surface admits 45 tritangent planes, i.e. planes which intersect the surface along the union of three lines. A marking of a cubic surface defines an order on the set of tritangent planes. Let $\mathscr{E}_i$ be the locus of points in $\mathscr{M}^m_{\mathrm{cub}}$ corresponding to marked cubic surfaces which contain an Eckardt point in the i-th tritangent plane. The Weyl group $\mathrm{W}(E_6)$ acts on $\mathscr{M}^m_{\mathrm{cub}}$ and permutes the loci $\mathscr{E}_i$'s transitively. Let $(S, L_1, \dots, L_6)$ be a marked cubic surface and let M_i be the line on S which meets L_i and L_{i+3} for $i = 1,2,3$ but none of the other L_j. The M_i lie in a tritangent plane and they meet in a point if and only if the points $p_1, \dots, p_6 \in \mathbb{P}^2$ obtained by blowing down the L_i are such that the three lines $\langle p_i, p_{i+3} \rangle$ (the images of the M_i), intersect at some point q. Let $\mathscr{E}_j$ be the corresponding component of the Eckardt locus in $\mathscr{M}^m_{\mathrm{cub}}$. Its pre-image Z in $(\mathbb{P}^2)^6$ consists of 6-tuples of points $(p_1, \dots, p_6)$ such that the lines $\langle p_i, p_{i+3} \rangle$, $i = 1,2,3$ intersect. Assigning the intersection point q to the 6-tuple defines a surjective map from Z to $\mathbb{P}^2$

whose fibres, as is easy to see, are irreducible and of the same dimension. This shows that Z, and hence $\mathscr{E}_j$ is irreducible. The image of each $\mathscr{E}_i$ in $\mathscr{M}_{\mathrm{cub}}$ is then an irreducible hypersurface.

The irreducibility of the Eckardt locus in 8.15 follows also from our ball uniformization of $\mathscr{M}_{\mathrm{cub}}$. We follow the proof given in [AF].

8.12. Lemma. *Let $D = T^*/T$ be the discriminant group of T as in* 5.1 *and let $N = M^\iota$. The group $\mathrm{W}(E_6) = \mathrm{O}(D)/\{\pm 1\}$ acts transitively on the subsets of $(D - \{0\})/\{\pm 1\}$ of vectors of the same norm. There are three such subsets.*

(i) *The set of vectors of norm* 0 *has* 40 *elements. Each non-zero isotropic vector is represented by $\big(e + 2\rho(e)\big)/3$, where $e \in T$ is a primitive isotropic vector.*

(ii) *The set of vectors of norm $-2/3$ has* 36 *elements. Each $(-2/3)$-vector is represented by a vector $\big(r + 2\rho(r)\big)/3$ in T^* with $r \in T$, $r^2 = -2$ and $\big(r, \rho(r)\big) = 1$.*

(iii) *The set of vectors of norm $-4/3$ has* 45 *elements. Each $(-4/3)$-vector in $\mathrm{D}(T)$ is represented by r'' where $r = r' + r'' \in N^\perp \backslash T$ is a (-2)-vector and r', r'' is the projection of r into $(N^\perp \cap M)^*$, T^* respectively.*

Proof. If we consider T as a free Hermitian module Λ over $\mathbb{Z}[\zeta_3]$ (see 6.9), then [ACT], [AF] define an isotropic vector, a short vector and a long vector as a vector with Hermitian square equal to 0, -1, -2, respectively. The images of these vectors in T^* with respect to the isomorphism $h : \Lambda \to T^*$ (6.1) are vectors with square 0, $-2/3$, $-4/3$, respectively. It is proven in [AF], Proposition 2.1 that there are exactly three Γ_ρ-orbits of the images of these vectors in $\mathrm{D}(T)$. Their cardinality is 40, 36 and 45, respectively. This gives three orbits of $\mathrm{O}\big(\mathrm{D}(T)\big)$ in $\mathrm{D}(T)$ of the same cardinality. The assertions (i) and (ii) follow from the explicit formula for the isomorphism h (6.1). To prove (iii), we consider an ample (M, ρ)-polarized $K3$ surface X whose standard elliptic fibration acquires fibres of type I_0^*. Let $\tilde{\phi} : L \to H^2(X, \mathbb{Z})$ be a cohomology marking with $\tilde{\phi}|M = \phi$. In the notation of 8.9, we may assume that the image of the first copy of A_2 of M in $\mathrm{Pic}(X)$ is spanned by E_0 and $E_0 + E_1 + E_2 + E_3$. Let $r = \tilde{\phi}^{-1}([E_1])$. Then $r \in N^\perp \backslash T$ and $r' = \dfrac{1}{3}\big(r + \rho(r) + \rho^2(r)\big) = \dfrac{1}{3}\tilde{\phi}^{-1}(E_1 + E_2 + E_3) \in (M \cap N^\perp)^*$. We easily check that $r'^2 = -2/3$. Then $r'' = r - r' \in T^*$ and $(r'')^2 = -4/3$. □

8.13. Moduli interpretation. Consider the three Γ_ρ-orbits of vectors from T^*:

(1) $\dfrac{1}{3}\big(e + 2\rho(e)\big)$, where e is a primitive isotropic vector in T;

(2) $\dfrac{1}{3}\big(r + 2\rho(r)\big)$, where r is a (-2)-vector in T (this corresponds to a short root in Λ);

(3) r'' equal to the projection of a (-2)-vector $r \in N^\perp \backslash T$ (this corresponds to a long root in Λ).

Each vector $v \in T^*$ defines a hyperplane $v^\perp$ in $\mathbb{P}(V_+)$ of lines orthogonal to v. So, we have three Γ_ρ-orbits of such hyperplanes corresponding to vectors from the above list. It is shown in [AF] that there is a bijective correspondence between the $\Gamma_{M,\rho}$-orbits of these vectors and their images in $\mathrm{D}(T)$. Thus each Γ_ρ-orbit consists of 40, 36, 45 $\Gamma_{M,\rho}$-orbits, respectively.

8.14. The boundary divisors. We know that the discriminant $\mathscr{H}$ is equal to the union of hyperplanes $H(r) = r^\perp \cap \mathscr{B}$, where r is a (-2)-vector from T. For any $x \in V_+$, we can easily see that $(r, x) = 0$ if and only if $\big(r + 2\rho(r), x\big) = 0$. This shows that the hyperplane corresponding to a vector of type (2) in 8.13 is one of the hyperplanes $H(r)$. Thus the discriminant locus Δ^m in $\mathscr{K}3^m_{M,\rho}$ consists of 36 hypersurfaces Δ^m_α $(\alpha \in D/\{\pm 1\}$ with norm $-2/3)$ which are permuted transitively by $\mathrm{W}(E_6)$. The discriminant locus Δ in $\mathscr{K}3_{M,\rho}$ is irreducible. It is well-known that the stabilizer of each Δ^m_α in $\mathrm{W}(E_6)$ is $G_1 = S_6 \times \mathbb{Z}/2\mathbb{Z}$ (see 2.12).

Take a generic point in $H(r)$. Then the corresponding $K3$ surface has $A_2(-1) \oplus A_2^{\oplus 3}$ as its transcendental lattice (see the cases 4), 5) in Table 2). The automorphism σ' in Theorem 8.7 defines a hermitian lattice structure on $A_2(-1) \oplus A_2^{\oplus 3}$ of signature $(1,3)$ over the Eisenstein integers as in 6.9. Then Δ^m is the quotient of $H(r)$ by the stabilizer subgroup of r in $\Gamma_{M,\rho}$. It is known that Δ^m is isomorphic to the smooth locus of the Segre cubic $\mathscr{S}_3$ (cf. [Hu], Chap. 3, 3.2.3). Its Satake-Baily-Borel compactification is obtained by adding 10 cusps and isomorphic to $\mathscr{S}_3$.

Now we fix an orthogonal basis $\{\alpha_i\}$ of D such that $q_T(\alpha_i) = -4/3$. This defines an isomorphism of quadratic forms

$$D \simeq \mathbb{F}_3^5$$

where the quadratic form q on $\mathbb{F}_3^5$ is given by

$$q(0, \ldots, 0, 1, 0, \ldots, 0) = -\frac{4}{3}.$$

Recall that the stabilizer of a basis of D in $\mathrm{W}(E_6)$ is $\mathrm{W}(D_5) \simeq (\mathbb{Z}/2\mathbb{Z})^4 \cdot S_5$.

Then there are 36 $(-2/3)$-vectors in D which are divided into two orbits of $\mathrm{W}(D_5)$. One consists of 16 vectors containing $(1,1,1,1,1)$ and another consists of 20 vectors containing $(1,1,0,0,0)$. The stabilizer in $\mathrm{W}(D_5)$ of $(1,1,1,1,1)$ is S_5, and that of $(1,1,0,0,0)$ is $(\mathbb{Z}/2\mathbb{Z})^3 \cdot (S_2 \times S_3)$. Note that the sum of indices of these groups in G_1 is $12 + 15 = 27$. The orbit of cardinality 20 corresponds to markings such that the marked line does not contain the node. For example, if the line corresponds to e_6 under a geometric marking defined by $(e_1, \ldots, e_6)$, then the effective class corresponding to the node could be either of type $e_i - e_j$, $1 \leqq i < j < 6$ or $e_0 - e_i - e_j - e_k$, $1 \leqq i < j < k < 6$.

8.15. Eckardt loci. If v is of type (3) in 8.13 the hyperplane $v^\perp \cap \mathscr{B}$ is equal to the hyperplane $\mathscr{H}(r)_\iota$ defined in 8.9. Thus we obtain that the image of the Eckardt locus Δ^m_ι in $\mathscr{K}3^m_{M,\rho}$ consists of 45 irreducible hypersurfaces. The Eckardt locus Δ_ι in $\mathscr{K}3_{M,\rho}$ is irreducible. This shows that the Eckardt locus in $\mathscr{M}_{\mathrm{cub}}$ is irreducible (as promised).

8.16. Cusps. For a non-zero isotropic vector e in T we define a totally isotropic sublattice

$$I(e) := \langle e, \rho(e) \rangle \quad (\subset T).$$

Then $\bar{\mathscr{B}} \cap (\mathbb{P}(I(e) \otimes \mathbb{C}))$ is a cusp of $\mathscr{B}$ (i.e. a rational boundary component), and any cusp of $\mathscr{B}$ corresponding to a parabolic subgroup of Γ_ρ is obtained in this manner. Thus we obtain that the Satake-Baily-Borel compactification of $\mathscr{K}3^m_{M,\rho} = \mathscr{B}/\Gamma_{M,\rho}$ (resp. $\mathscr{K}3_{M,\rho} = \mathscr{B}/\Gamma_\rho$) is obtained by adding 40 cusps (resp. one cusp). As in the case of $(-2/3)$-vectors, we can see that $\mathrm{W}(D_5)$ acts on 40 cusps transitively, and hence the Satake-Baily-Borel compactification of $\mathscr{K}3^m_{M,\rho}/\mathrm{W}(D_5)$ is obtained by adding one cusp.

9. Extension of the isomorphism to the boundary

In this section we will extend the $\mathrm{W}(E_6)$-equivariant isomorphism

$$G : \mathscr{K}3^m_{M,\rho} \backslash \Delta^m \to \mathscr{M}^m_{\mathrm{cub}}$$

from Theorem 7.2 to a $\mathrm{W}(E_6)$-equivariant isomorphism

$$\mathscr{K}3^m_{M,\rho} \cong \mathscr{M}^m_{\mathrm{ncub}}.$$

It follows from Lemma 6.21 that for any $[(X, \phi)] \in \mathscr{K}3^m_{M,\rho} \backslash \Delta^m$ the standard elliptic fibration defined by the polarization ϕ has the Weierstrass model as in Corollary 4.11. Let $[(S, l)]$ be the isomorphism class of a nonsingular cubic surface together with a line corresponding to the pair (F_5, F_2) under isomorphism (3.2). It follows from the construction of the map G that the image of $G([(X, \phi)])$ under the canonical projection $\mathscr{M}^m_{\mathrm{cub}} \to \mathscr{M}^m_{\mathrm{cub}}/\mathrm{W}(D_5) = \mathscr{M}^1_{\mathrm{cub}}$ is equal to $[(S, l)]$. Applying Theorem 8.8 and using Table 1 we see that the standard elliptic fibration on any (X, ϕ) defined by a point in $\mathscr{K}3^m_{M,\rho}$ has Weierstrass model (4.12), where (F_5, F_2) is a stable pair of binary forms. Using the isomorphism (3.2), the pair (F_5, F_2) defines a point $[(S, l)] \in \mathscr{M}^1_{\mathrm{ncub}}$. Obviously this can be done in families, so this gives a morphism $\mathscr{K}3^m_{M,\rho} \to \mathscr{M}^1_{\mathrm{ncub}}$ which obviously factors through the map

$$f : \mathscr{K}3^m_{M,\rho}/\mathrm{W}(D_5) \to \mathscr{M}^1_{\mathrm{ncub}}. \tag{9.1}$$

By the above this map extends the isomorphism G modulo $\mathrm{W}(D_5)$.

9.1. Theorem. *The map* (9.1) *extends to an isomorphism of compactifications*:

$$\bar{f} : \overline{\mathscr{K}3}^m_{M,\rho}/\mathrm{W}(D_5) \to \bar{\mathscr{M}}^1_{\mathrm{ncub}}.$$

Here the compactification of the target space is the Satake-Baily-Borel compactification of $\mathscr{K}3^m_{M,\rho}/\mathrm{W}(D_5)$ (*see* 8.16) *and the compactification of the source space is from the proof of Theorem* 3.6.

Proof. We will apply Lemma 3.4. By 8.16 both compactifications are one-point compactifications. Since $\bar{f}$ extends an isomorphism f, it is a birational morphism. The map is obviously surjective since we can always choose a structure of an M-polarization on the elliptic surface defined by the Weierstrass model from (4.12). It remains to check the last assumption from Lemma 3.4, i.e. the finiteness of fibres. For this we argue as in the proof of Theorem 3.6. It follows from 8.14 that the complement $\overline{\mathscr{K}3}^m_{M,\rho} \backslash \mathscr{K}3^m_{M,\rho}$ consists of 36 divisors isomorphic to the Segre cubic hypersurface. Thus the complement $\overline{\mathscr{K}3}^m_{M,\rho}/\mathrm{W}(D_5)\backslash\mathscr{K}3^m_{M,\rho}/\mathrm{W}(D_5)$ consists of two irreducible divisors isomorphic to a finite quotient of the Segre cubic (minus a finite set of points). Now we can finish as in the proof of Theorem 3.6. □

9.2. Theorem. *The isomorphism* $\mathscr{K}3^m_{M,\rho}\backslash\Delta^m \cong \mathscr{M}^m_{\mathrm{cub}}$ *extends to a* $\mathrm{W}(E_6)$*-equivariant isomorphism*

$$\mathscr{K}3^m_{M,\rho} \cong \mathscr{M}^m_{\mathrm{ncub}}.$$

Passing to the quotients it defines an isomorphism

$$\mathscr{K}3_{M,\rho} \cong \mathscr{M}_{\mathrm{ncub}}.$$

Proof. The isomorphism $\mathscr{K}3^m_{M,\rho}/\mathrm{W}(D_5) \cong \mathscr{M}^m_{\mathrm{ncub}}/\mathrm{W}(D_5)$ constructed in Theorem 9.1 lifts to a $\mathrm{W}(E_6)$-equivariant isomorphism $\mathscr{K}3^m_{M,\rho} \cong \mathscr{M}^m_{\mathrm{ncub}}$. In fact, this is true for open Zariski subsets defined by nonsingular cubic surfaces, hence each of the varieties is the normalization of the quotient in the field of rational functions $\mathbb{C}(\mathscr{K}3^m_{M,\rho}) = \mathbb{C}(\mathscr{M}^m_{\mathrm{ncub}})$. Now we have an isomorphism α of varieties which defines a birational isomorphism of $\mathrm{W}(E_6)$-varieties. Obviously, it is an isomorphism of $\mathrm{W}(E_6)$-varieties (for each $g \in \mathrm{W}(E_6)$ the maps $g \circ \alpha$ and $\alpha \circ g$ coincide on an open Zariski subset, hence coincide everywhere). □

9.3. Corollary. *The isomorphism*

$$(\mathscr{B}\backslash\mathscr{H})/\Gamma_{M,\rho} \cong \mathscr{M}_{\mathrm{cub}}$$

from Corollary 7.5 *extends to an isomorphism*

$$\mathscr{B}/\Gamma_{M,\rho} \cong \mathscr{M}_{\mathrm{ncub}}.$$

9.4. Remark. As in the proof of Theorem 9.1 (also see (3.2)) , the isomorphism

$$\mathscr{K}3^m_{M,\rho}/\mathrm{W}(D_5) \xrightarrow{\cong} \mathscr{M}^1_{\mathrm{ncub}}$$

extends to the isomorphism of their compactifications. The geometric meaning is as follows.

The strictly semistable cubic surface defined by

$$X_3^3 - X_0X_1X_2 = 0 \tag{9.2}$$

(cf. [ACT], (4.6)) has three double rational points of type A_2 and has only three lines which lie in one $\mathrm{Aut}(S)$-orbit. This defines three planes in the cubic fourfold X defined by $X_5^3 + X_4^3 + X_3^3 - X_0X_1X_2 = 0$ (one such plane is Π: $X_2 = X_3 = X_4 + X_5 = 0$) and projection away from such a plane defines a quadric bundle structure on X. The discriminant curve is easily computed and is a sextic given by

$$t_2\big(L_1(t_0,t_1)^3L_2(t_0,t_1)^2 + t_2^3L_2(t_0,t_1)^2\big) = 0, \tag{9.3}$$

where L_1, L_2 are independent linear forms.

It follows from Proposition 3.2 that the pair $(F_5, F_2) = (L_1^3 L_2^2, L_2^2)$ represents a semi-stable but not stable point in $\mathbb{P}(V(5)) \times \mathbb{P}(V(2))$ whose orbit is closed in the set of semi-stable points. The corresponding point in $(\mathbb{P}(V(5)) \times \mathbb{P}(V(2)))^{\mathrm{ss}}//\mathrm{SL}(2)$ compactifies $(\mathbb{P}(V(5)) \times \mathbb{P}(V(2)))^{\mathrm{s}}/\mathrm{SL}(2)$. Thus we see that $\mathscr{M}^1_{\mathrm{ncub}}$ admits a one-point compactification corresponding to the surface (9.2) together with its unique (up to automorphism) line.

The sextic curve (9.3) appears as a semistable sextic in Shah [Sha], Theorem 2.4, Group II, (2). The double cover X of $\mathbb{P}^2$ branched along this sextic is a Type II degeneration of $K3$ surfaces, i.e. corresponding to a point on an 1-dimensional rational boundary component of the period domain of polarized $K3$ surfaces of degree 2 (= a bounded symmetric domain of type IV and of dimension 19). The 1-dimensional rational boundary components of a bounded symmetric domain of type IV bijectively correspond to the set of totally isotropic primitive sublattices of rank 2 of its underlying lattice of signature $(2, r)$. In our situation, ρ-invariant totally isotropic primitive sublattices of rank 2 of T correspond to the set of cusps of $\mathscr{B}$. Thus X corresponds to the boundary of the Satake-Baily-Borel compactification of $\mathscr{K}3^m_{M,\rho}/\mathrm{W}(D_5)$.

9.5. Configurations of 7 points in $\mathbb{P}^1$. Recall from Theorem 3.6 that we have a natural isomorphism

$$\mathscr{M}^1_{\mathrm{ncub}} \cong (\mathbb{P}(V(5)) \times \mathbb{P}(V(2)))^{\mathrm{s}}/\mathrm{SL}(2),$$

where $(\mathbb{P}(V(5)) \times \mathbb{P}(V(2)))'$ is the open subset corresponding to stable pairs of binary forms (F_5, F_2). Consider the product $(\mathbb{P}_1)^7$ as the product $(\mathbb{P}^1)^5 \times (\mathbb{P}^1)^2$. We have an isomorphism

$$\psi : (\mathbb{P}^1)^7/S_5 \times S_2 \to \mathbb{P}(V(5)) \times \mathbb{P}(V(2)).$$

Let $p : (\mathbb{P}^1)^7 \to \mathbb{P}(V(5)) \times \mathbb{P}(V(2))$ be the composition of the quotient map and ψ and

$$\mathscr{L} = p^*(\mathscr{O}_{\mathbb{P}(V(5))}(2) \boxtimes \mathscr{O}_{\mathbb{P}(V(2))}(1)) \cong \mathop{\boxtimes}_{i=1}^{5} \mathscr{O}_{\mathbb{P}^1}(2) \otimes (\mathscr{O}_{\mathbb{P}^1}(1) \boxtimes \mathscr{O}_{\mathbb{P}^1}(1)).$$

Since the stability is preserved under the action of finite groups, we see that semi-stable (stable) points in $\mathbb{P}(V(5)) \times \mathbb{P}(V(2))$ with respect to the action of SL(2) and the linearization defined by the invertible sheaf $\mathscr{O}_{\mathbb{P}(V(5))}(2) \boxtimes \mathscr{O}_{\mathbb{P}(V(1))}(1)$ correspond to semi-stable (stable) points in $(\mathbb{P}_1)^7$ with respect to the diagonal action of SL(2) and the linearization defined by the line bundle $\mathscr{L}$. Let

$$P_1(2^5, 1, 1) = ((\mathbb{P}_1)^7)^s/\mathrm{SL}(2).$$

We have

$$(\mathbb{P}(V(5)) \times \mathbb{P}(V(2)))^{\mathrm{s}}/\mathrm{SL}(2) \cong P_1(2^5, 1, 1)/S_5 \times S_2.$$

We know that $\mathscr{M}^1_{\mathrm{ncub}} = \mathscr{M}^m_{\mathrm{ncub}}/\mathrm{W}(D_5)$. The group $\mathrm{W}(D_5)$ is equal to the semi-direct product $(\mathbb{Z}/2\mathbb{Z})^4 \ltimes S_5$. Here S_5 is the subgroup of $\mathrm{W}(D_5)$ which acts on markings on nonsingular surfaces by permuting the divisor classes $e_1, \dots, e_5$. It stabilizes the divisor class $2e_0 - e_1 - \cdots - e_5$ of a line l. The subgroup $H = (\mathbb{Z}/2\mathbb{Z})^4$ is generated by the conjugates of the product of two commuting reflections $s_{e_0-e_1-e_2-e_6} \circ s_{e_1-e_2}$. Let l_i' be the lines representing the classes $e_0 - e_i - e_6$. Then H acts by switching even numbers of l_i's with l_i''s. The proof of Theorem 3.6 shows that the map $\mathcal{M}^1_{\mathrm{ncub}} \to \big(\mathbb{P}(V_5) \times \mathbb{P}(V_2)\big)'/\mathrm{SL}(2)$ induces an S_5-equivariant isomorphism

$$\mathcal{M}^m_{\mathrm{ncub}}/H \cong P_1(2^5, 1, 1)/S_2.$$

9.6. Monodromy groups. According to Deligne and Mostow [DM], the variety $P_1(2^5, 1, 1)$ is isomorphic to the quotient of a complex 4 ball by a reflection subgroup Π' corresponding to hypergeometric function defined by the multi-valued form

$$\omega = z^{-1/6}[(z-1)(z-a_1)(z-a_2)(z-a_3)(z-a_4)]^{-1/3}\,dz.$$

They also show that Π' and S_2 generate a reflection subgroup Π such that the ball quotient is isomorphic to $P_1(2^5, 1, 1)/S_2$. As shown in 4.17, X is the minimal model of a quotient $(C \times E)/(\mathbb{Z}/6\mathbb{Z})$. This correspondence gives us an isogeny between our group Γ_ρ and Π.

10. Half twists

10.1. To a smooth cubic surface S one can associate a principally polarized Hodge structure of rank 10 and weight 1, it is $H^1(P, \mathbb{Z})$ where P is the intermediate Jacobian of the cubic threefold V (cf. 4.15) associated to S. In [ACT], see also [MT], it is shown that this Hodge structure, with its automorphism of order three, determines S.

The automorphism of order three defines the structure of a free $\mathbb{Z}[\zeta]$-module on $H^1(P, \mathbb{Z})$. It defines eigenspaces $H^{1,0}(P)_\chi$ and $H^{1,0}(P)_{\bar{\chi}}$ of dimension 4 and 1 respectively. This allows one to define a weight two Hodge structure W, with Hodge numbers $(1, 8, 1)$, and with the same underlying lattice $W = H^1(P, \mathbb{Z})$ as follows:

$$W^{2,0} = H^{1,0}(P)_{\bar{\chi}}, \quad W^{1,1} = H^{1,0}(P)_\chi \oplus H^{0,1}(P)_{\bar{\chi}}, \quad W^{0,2} = H^{0,1}(P)_\chi,$$

in fact it is easy to check that $W^{p,q} = \overline{W^{q,p}}$. The automorphism of order three of $H^1(P, \mathbb{Z})$ preserves this decomposition, hence also W has an automorphism of order three. The polarization E on $H^1(P, \mathbb{Z})$ defines a $\mathbb{Q}[\zeta_3]$-valued Hermitian form H on $H^1(P, \mathbb{Z}) \cong \mathbb{Z}[\zeta_3]^5$ (cf. [ACT]) with imaginary part E. The real part Q of H is a polarization of W. The lattice (W, Q) is of type $A_2^4 \oplus A_2(-1)$. The polarized Hodge structure (W, Q) is the (negative) half twist of $\big(H^1(P, \mathbb{Z}), E\big)$ ([vG1]).

10.2. The lattice $(W, Q) \cong A_2^{\oplus 4} \oplus A_2(-1)$ has a unique (up to an isometry) embedding in the $K3$ lattice L and the automorphism of order three on W extends to an automorphism of order three on the $K3$ lattice. The polarized Hodge structure (W, Q) is invariant under this automorphism and defines a $K3$ surface with an automorphism of order three. So the half twist of $H^1(P, \mathbb{Z})$ provides a purely Hodge theoretic approach to the $K3$ surfaces which were constructed as triple covers of cubic surfaces in this paper.

References

[ACT] *D. Allcock*, *J. A. Carlson*, *D. Toledo*, The complex hyperbolic geometry of the moduli space of cubic surfaces, J. Alg. Geom. **11** (2002), 659–724.

[AF] *D. Allcock*, *E. Freitag*, Cubic surfaces and Borcherds products, Comm. Math. Helv. **77** (2002), 270–296.

[BP] *W. Barth*, *C. Peters*, Automorphisms of Enriques surfaces, Invent. Math. **73** (1983), 383–411.

[B1] *A. Beauville*, Variétés de Prym et jacobiennes intermédiaires, Ann. Sci. Éc. Norm. Sup. **10** (1977), 309–391.

[B2] *A. Beauville*, Les singularités du diviseur Θ de la jacobienne intermédiaire de l'hypersurface cubique dans P^4, in: Algebraic threefolds (Varenna 1981), Springer Lect. Notes Math. **947** (1982), 190–208.

[Be] *N. D. Beklemishev*, Invariants of cubic forms of four variables, Vestnik Mosk. Univ, Ser. 1 Mat. Mekh. **1982**, No. 2, 42–49; Engl. Transl.: Moscow Univ Math. Bull. **37** (1982), 54–62.

[Bo] *N. Bourbaki*, Lie algebras and Lie groups, vol. 2, Springer, 2002.

[Co] *A. B. Coble*, Point sets and allied cremona groups (Part III), Trans. Amer. Math. Soc. **18** (1917), 331–372.

[CvG] *E. Colombo*, *B. van Geemen*, The Chow group of the moduli space of marked cubic surfaces, math.AG/0210465.

[DM] *P. Deligne*, *G. W. Mostow*, Monodromy of hypergeometric functions and non-lattice integral monodromy, Publ. Math. IHES **63** (1986), 5–89.

[Do] *I. Dolgachev*, Mirror symmetry for lattice polarized $K3$-surfaces, J. Math. Sci. **81** (1996), 2599–2630.

[DO] *I. Dolgachev*, *D. Ortland*, Point sets in projective spaces and theta functions, Astérisque **165** (1988).

[DS] *R. Donagi*, *R. C. Smith*, The structure of the Prym map, Acta Math. **146** (1981), 25–102.

[F] *E. Freitag*, A graded algebra related to cubic surfaces, Kyushu J. Math. **56** (2002), 299–312.

[vG1] *B. van Geemen*, Half twists of Hodge structures of CM-type, J. Math. Soc. Japan **53** (2001), 813–833.

[vG2] *B. van Geemen*, A linear system on Naruki's moduli space of marked cubic surfaces, Internat. J. Math. **13** (2002), no. 2, 183–208.

[HL] *G. Heckman*, *E. Looijenga*, The moduli space of rational elliptic surfaces, Algebraic Geometry 2000, Azumino, Adv. Stud. Pure Math. **36** (2002), 185–248.

[Hu] *B. Hunt*, The geometry of some special arithmetic quotients, Springer Lect. Notes Math. **1637** (1996).

[Ko1] *S. Kondō*, A complex hyperbolic structure of the moduli space of curves of genus three, J. reine angew. Math. **525** (2000), 219–232.

[Ko2] *S. Kondō*, The moduli space of curves of genus 4 and Deligne-Mostow's complex reflection groups, Algebraic Geometry 2000, Azumino, Adv. Stud. Pure Math. **36** (2002), 383–400.

[Lo] *E. Looijenga*, Rational surfaces with an anticanonical cycle, Ann. Math. **114** (1981), 267–322.

[MSY] *K. Matsumoto*, *T. Sasaki*, *M. Yoshida*, The monodromy of the period map of a 4-parameter family of $K3$-surfaces and the hypergeometric function of type $(3,6)$, Internat. J. Math. **3** (1992), 1–164.

[MT] *K. Matsumoto*, *T. Terasoma*, Theta constants associated to cubic threefolds, J. Alg. Geom. **3** (2003), 741–775.

[Mos] *G. W. Mostow*, On discontinuous actions of monodromy groups on the complex n-ball, J. A. M. S. **1** (1988), 555–586.

[Mu] *D. Mumford*, The red book of varieties and schemes, Springer-Verlag, 1988.

[Na] *Y. Namikawa*, Periods of Enriques surfaces, Math. Ann. **270** (1985), 201–222.

[Nar] *I. Naruki*, Cross ratio variety as a moduli space of cubic surfaces (Appendix by E. Looijenga), Proc. London Math. Soc. **45** (1982), 1–30.

[N1] *V. V. Nikulin*, Integral symmetric bilinear forms and its applications, Math. USSR Izv. **14** (1980), 103–167.

[N2] *V. V. Nikulin*, Factor groups of groups of automorphisms of hyperbolic forms with respect to subgroups generated by 2-reflections, J. Sov. Math. **22** (1983), 1401–1475.

[N3] *V. V. Nikulin*, Finite groups of automorphisms of Kählerian surfaces of type $K3$, Moscow Math. Soc. **38** (1980), 71–137.

[PS] *I. Piatetski-Shapiro*, *I. R. Shafarevich*, A Torelli theorem for algebraic surfaces of type $K3$, Math. USSR Izv. **5** (1971), 547–587.

[Sa1] *G. Salmon*, A Treatise on the Analytic Geometry of Three Dimensions, Longmans and Green, 1912–1915; reprinted by Chelsea Publ. Co., 1965.

[Sa2] *G. Salmon*, A Treatise on the Higher Plane Curves, Hodges, Foster and Figgis, Dublin 1879; reprinted by Chelsea Publ. Co., 1960.

[Sha] *J. Shah*, A complete moduli space for $K3$ surfaces of degree 2, Ann. Math. **112** (1980), 485–510.

[Shi] *T. Shioda*, On elliptic modular surfaces, J. Math. Soc. Japan **24** (1972), 20–59.

[Y] *M. Yoshida*, A W(E_6)-equivariant projective embedding of the moduli space of cubic surfaces, Preprint Kyushu University, 1999-26.

[Voi] *C. Voisin*, Théorème de Torelli pour les cubiques de $\mathbb{P}^5$, Invent. Math. **86** (1986), 577–601.

[Vor] *S. P. Vorontsov*, Automorphisms of even lattices that arise in connection with automorphisms of algebraic $K3$ surfaces, Vest. Mosk. Univ. Math. **38** (1983), 19–21.

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