Derived categories. Winter 2008/09

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Lecture 1

Derived categories

1.1 Abelian categories

We assume that the reader is familiar with the concepts of categories and functors. We will assume that all categories are *small*, i.e. the class of objects $\mathrm{Ob}(\mathsf{C})$ in a category C is a set. A small category can be defined by two sets $\mathrm{Mor}(\mathsf{C})$ and $\mathrm{Ob}(C)$ together with two maps $s, t : \mathrm{Mor}(\mathsf{C}) \to \mathrm{Ob}(\mathsf{C})$ defined by the source and the target of a morphism. There is a section $e : \mathrm{Ob}(\mathsf{C}) \to \mathrm{Mor}(\mathsf{C})$ for both maps defined by the identity morphism. We identify $\mathrm{Ob}(\mathsf{C})$ with its image under e. The composition of morphisms is a map $c : \mathrm{Mor}(\mathsf{C}) \times_{s,t} \mathrm{Mor}(\mathsf{C}) \to \mathrm{Mor}(\mathsf{C})$. There are obvious properties of the maps (s, t, e, c) expressing the axioms of associativity and the identity of a category. For any $A, B \in \mathrm{Ob}(\mathsf{C})$ we denote by $\mathrm{Mor}_{\mathsf{C}}(A, B)$ the subset $s^{-1}(A) \cap t^{-1}(B)$ and we denote by $\mathbf{id}_A$ the element $e(A) \in \mathrm{Mor}_{\mathsf{C}}(A, A)$.

A *functor* from a category C defined by $(\mathrm{Ob}(\mathsf{C}), \mathrm{Mor}(\mathsf{C}), s, t, c, e)$ to a category C' defined by $(\mathrm{Ob}(\mathsf{C}'), \mathrm{Mor}(\mathsf{C}'), s', t', c', e')$ is a map of sets $F : \mathrm{Mor}(\mathsf{C}) \to \mathrm{Mor}(\mathsf{C}')$ which is compatible with the maps (s, t, c, e) and (s', t', c', e') in the obvious way. In particular, it defines a map $\mathrm{Ob}(\mathsf{C}) \to \mathrm{Ob}(C')$ which we also denote by F.

For any category C we denote by C^{op} the *dual category*, i.e. the category $(\mathrm{Mor}(\mathsf{C}), \mathrm{Ob}(\mathsf{C}), s', t', c, e)$, where $s' = t, t' = s$. A *contravariant functor* from C to C' is a functor from C^{op} to C. For any two categories C and $\mathcal{D}$ we denote by D^{C} (or by $\mathrm{Funct}(\mathsf{C}, \mathsf{D})$) the category of functors from C to D. Its set of objects are functors $F : \mathsf{C} \to \mathsf{D}$. Its set of morphisms with source F_1 and target F_2 are natural transformation of functors, i.e. maps $\Phi = (\Phi_1, \Phi_2) : \mathrm{Mor}(\mathsf{C}) \to \mathrm{Mor}(\mathsf{D}) \times \mathrm{Mor}(\mathsf{D})$ such that $(s \times t) \circ \Phi = (s \times t) \circ (F_1, F_2)$ and $(t \times s) \circ \Phi = (t \times s) \circ (F_1, F_2)$, and $\Phi_2 \circ F_1 = F_2 \circ \Phi_1$. It is also required that $\Phi_i(u \circ v) = \Phi_i(u) \circ \Phi_i(v)$ and $\Phi_i(\mathbf{id}_A) = \mathbf{id}_{F_i(A)}$, for any $A \in \mathrm{Ob}(\mathsf{C})$.

Let Sets be the category of small sets (i.e. subset of some set which we can always enlarge if needed). We denote by $\widehat{\mathsf{C}}$ the category $\mathsf{Sets}^{\mathsf{C}^{\mathrm{op}}}$. A typical example is when we take $\mathsf{C} = \mathrm{Open}(X)$ to be the category of open subsets of a topological space X with inclusions as morphisms, a contravariant functor $F : \mathrm{Open}(X) \to \mathsf{Sets}$ is a presheaf on X. For this reason, the objects of $\widehat{\mathsf{C}}$ are called *presheaves* on C.

Any $A \in \mathrm{Ob}(\mathsf{C})$ defines the presheaf

$$h_A : (\phi : X \to Y) \to (\mathrm{Mor}_{\mathsf{C}}(Y, A) \xrightarrow{\phi\circ} \mathrm{Mor}_{\mathsf{C}}(X, A))$$

For any morphism $u : A \to A'$ in C and any $A \in \mathrm{Ob}(\mathsf{C})$, composing on the left defines a map $h_A(A) \to h_{A'}(A)$, and the set of such maps makes a morphism of functors $h_A \to h_{A'}$. This defines a functor

$$h : \mathsf{C} \to \widehat{\mathsf{C}},\ \mathrm{Mor}_{\mathsf{C}}(A, A') \to \mathrm{Mor}_{\widehat{\mathsf{C}}}(h_A, h_{A'})$$

called the *Yoneda functor* or the *representation functor*. According to the *Yoneda lemma* this functor is *fully faithful*, i.e. defines a bijection

$$\mathrm{Mor}_{\mathsf{C}}(A, A') \to \mathrm{Mor}_{\widehat{\mathsf{C}}}(h_A, h_{A'}).$$

Via the Yoneda functor, the category C becomes equivalent to a full subcategory of the category $\widehat{\mathsf{C}}$ (a subcategory C' is *full* if $\mathrm{Mor}_{\mathsf{C}'}(A, B) = \mathrm{Mor}_{\mathsf{C}}(A, B)$). Also recall that a category C' is *equivalent* to a category C if there exist functors $F : \mathsf{C}' \to \mathsf{C}, G : \mathsf{C} \to \mathsf{C}'$ such that the compositions $F \circ G, G \circ F$ are isomorphic (in the category of functors) to the identity functors. A presheaf $F \in \widehat{\mathsf{C}}$ is called *representable* if it is isomorphic to a functor of the form h_S for some $S \in \mathsf{C}$. We say that F and G are *quasi-inverse functors*. The object S is called the *representing object* of F. It is defined uniquely, up to isomorphism. Dually, one defines the functor $h^A : \mathsf{C} \to \mathsf{Sets}$ whose value at an object X is equal to $\mathrm{Mor}_{\mathsf{C}}(A, X)$. A functor $\mathsf{C} \to \mathsf{Sets}$ isomorphic to a functor h^A is called *corepresentable*.

Let S be a subcategory of the category Sets. A category C is called an S-category if for any $X \in \mathrm{Ob}(\mathsf{C})$ the presheaf h_X takes values in S and for any $(A \to B) \in \mathrm{Mor}(\mathsf{C})$, the map $h_X(A) \to h_X B)$ is a morphism in S. We will be interested in the case when $\mathsf{S} = \mathsf{Ab}$ is the category of abelian groups. In this case an Ab-category is called a *$\mathbb{Z}$-category*. It follows from the definition that the sets of morphisms is equipped with a structure of an abelian group, moreover, the left composition map $\mathrm{Mor}_{\mathsf{C}}(A, B) \to \mathrm{Mor}_{\mathsf{C}}(A, C)$ and the right composition map $\mathrm{Mor}_{\mathsf{C}}(B, C) \to \mathrm{Mor}_{\mathsf{C}}(A, C)$ are homomorphisms of groups.

Another useful example is when $\mathcal{A}$ is the category of linear spaces over a field $\mathbb{K}$. This allows to equip the sets of morphism with compatible structures of linear spaces. In this case a category is called $\mathbb{K}$-*linear*.

From now on, whenever we deal with a $\mathbb{Z}$-category category we set $\mathrm{Hom}_{\mathsf{C}}(A, B) := \mathrm{Mor}_{\mathsf{C}}(A, B)$.

Let $\widehat{\mathsf{C}}^{\mathrm{ab}}$ be the category of abelian presheaves on C, i.e. the category of contravariant functors from C to Ab. If C is a $\mathbb{Z}$-category, the Yoneda functor is the functor from C to $\widehat{\mathsf{C}}^{\mathrm{ab}}$. For any abelian group A one defines the *constant presheaf* $\underline{A}_{\mathsf{C}}$ (or just A if no confusion arises) by

$$\underline{A}_{\mathsf{C}}(S) = A,\ (S \to S') \to \mathbf{id}_A.$$

In particular, we have the *zero presheaf* $\underline{0}$. For any $F \in \widehat{\mathsf{C}}^{\mathrm{ab}}$, we have $\mathrm{Hom}_{\widehat{\mathsf{C}}^{\mathrm{ab}}}(\underline{0}, F) = \{0\}$ and the zero represents the unique morphism $\underline{0} \to F$.

A zero object of a $\mathbb{Z}$-category is an object representing the functor $\underline{0}$. It may not exist, but when they exist all of them are isomorphic and can be identified with one object denoted by 0. The zero object 0 is characterized by the property that $\mathrm{Hom}_{\mathsf{C}}(0,0) = \{0\}$. This immediately implies that 0 is the *initial* and the *final* object in C, i.e., for any $X \in \mathrm{Ob}(\mathsf{C})$ there exists a unique morphism $0 \to X$ (resp. $X \to 0$). It represents the zero element in $\mathrm{Hom}_{\mathsf{C}}(0, X)$ (resp. in $\mathrm{Hom}_{\mathsf{C}}(X, 0)$).

For any morphism $u : F \to G$ of abelian presheaves we can define the kernel $\ker(u)$ by

$$\ker(u)(A) = \ker(F(A) \to G(A)).$$

It can be characterized by the following universal property: there is a morphism $\ker(u) \to F$ such that the composition $\ker(u) \to F \to G$ is the zero morphism, and any morphism $K \to F$ with this property factors through the morphism $\ker(u) \to F$. We say that C has kernels if, for any $X \to Y$ the kernel of $h_X \to h_Y$ in $\widehat{\mathsf{C}}^{ab}$ is representable. By the Yoneda Lemma, the kernel $\ker(X \to Y)$ satisfies the universal property from above, and this can be taken as the equivalent definition.

The definition of the cokernel $\mathrm{coker}(u)$ of a morphism $X \to Y$ in a pre-additive category is obtained by reversing the arrows. It comes with a unique morphism $Y \to \mathrm{coker}(u)$ such that any morphism $Y \to K$ with the zero composition $F \to G \to K$ factors through $G \to \mathrm{coker}(u) \to K$. In other words,

$$\mathrm{coker}(u) = \ker(u^{\mathrm{op}})^{\mathrm{op}},$$

where $u^{\mathrm{op}} : G \to F$ is a morphism in the dual category C^{op} and $\ker(u^{\mathrm{op}})^{\mathrm{op}}$ is the morphism $\ker(u^0) \to G$ in C^{op} considered as a morphism $G \to \ker(u^{\mathrm{op}})$ in C.

Note that $\widehat{\mathsf{C}}^{\mathrm{ab}}$ has cokernels defined by

$$\mathrm{coker}(F \to G)(S) = \mathrm{coker}(F(S) \to G(S)).$$

However, even when $\mathsf{C} = \mathsf{Ab}$ where kernels and cokernels are the usual ones, $\mathrm{coker}(h_A \to h_B)$ may not be representable. For example, if we take $[n] : \mathbb{Z} \to \mathbb{Z}$ the multiplication by $n > 1$ in Ab and $u = h([n]) : h_{\mathbb{Z}} \to h_{\mathbb{Z}}$, then we get

$$\mathrm{coker}(u)(\mathbb{Z}/n\mathbb{Z}) = \mathrm{coker}(h_{\mathbb{Z}}(\mathbb{Z}/n\mathbb{Z}) \to h_{\mathbb{Z}}(\mathbb{Z}/n\mathbb{Z})) = \mathrm{coker}(0 \to 0) = \{0\},$$

because $\mathrm{Hom}(\mathbb{Z}/n\mathbb{Z}, \mathbb{Z}) = \{0\}$. On the other hand,

$$h_{\mathrm{coker}([n])}(\mathbb{Z}/n\mathbb{Z}) = h_{\mathbb{Z}/n\mathbb{Z}}(\mathbb{Z}/n\mathbb{Z}) = \mathrm{Hom}(\mathbb{Z}/n\mathbb{Z}, \mathbb{Z}/n\mathbb{Z}) \neq \{0\}.$$

It follows from the definition that there is a canonical morphism

$$\mathrm{coker}(\ker(u) \to s(u)) \xrightarrow{\alpha} \ker(t(u) \to \mathrm{coker}(u)), \tag{1.1}$$

where $u : s(u) \to t(u)$ is a morphism in C (provided that all the objects in above exist). The morphism u factors as

$$s(u) \to \operatorname{coker}(\ker(u) \to s(u)) \to \ker(t(u) \to \operatorname{coker}(u)) \to t(u).$$

The category $\widehat{\mathsf{C}}^{\mathrm{ab}}$ contains finite direct products over the final object $\underline{0}$ defined by the standard universality property. We have

$$(\prod_{i \in I} F_i)(A) \cong \prod_{i \in I} F_i(A).$$

It also contains finite direct sums $\oplus_{i \in I} F_i$ over the initial object 0_{C}, defined as the direct product in the dual category. Moreover,

$$\prod_{i \in I} F_i \cong \bigoplus_{i \in I} F_i.$$

It follows from the Yoneda Lemma that the object representing the direct products (sums) of representable presheaves is the direct product (sum) of the representing objects.

Definition 1.1.1. An *additive category* is a $\mathbb{Z}$-category such that

(i) the zero presheaf $\underline{0}$ is representable;

(ii) finite direct sums and direct products in $\widehat{\mathsf{C}}^{\mathrm{ab}}$ are representable;

An additive category is called *abelian* if additionally

(iii) kernels and cokernels exist;

(iv) for any morphism u,

$$\operatorname{coker}(\ker(u) \to s(u)) = \ker(t(u) \to \operatorname{coker}(u)),$$

and the canonical morphism α in (1.1) is an isomorphism. The object $\operatorname{coker}(\ker(u) \to s(u))$ is denoted by $\operatorname{im}(u)$ and is called the *image* of the morphism u.

Recall that a morphism $u : s(u) \to t(u)$ is called a *monomorphism* if the corresponding morphism $h(u) : h_{s(u)} \to h_{t(u)}$ in $\widehat{\mathsf{C}}$ is injective on its values. In other words, for any $S \in \operatorname{Ob}(\mathsf{C})$ the map of sets $h_{s(u)}(S) = \operatorname{Mor}_{\mathsf{C}}(S, s(u)) \xrightarrow{\circ u} \operatorname{Mor}_{\mathsf{C}}(S, t(u))$ is injective. By reversing the arrows one defines the notion of an *epimorphism*. If C is an abelian category, then u is a monomorphism (resp. epimorphism) if and only if $\ker(u) = 0$ (resp. $\operatorname{coker}(u) = 0$). Also u is an isomorphism (i.e. the left and the right inverses exist) if and only if it is a monomorphism and an epimorphism.

In an abelian category short exact sequences make sense. Also one can define the *cochain complexes* as sequences $K^\bullet = (u^n : K^n \to K^{n+1})_{n \in I}$ of morphisms with $u^{n+1} \circ u^n = 0 \in \mathrm{Hom}_{\mathsf{C}}(K^n, K^{n+2})$. Here n runs an interval I (finite or infinite) in the set of integers. Also one can define the *cohomology* of complexes as the complex

$$(H^n(K^\bullet), 0) = \mathrm{coker}(\mathrm{im}(u^n) \to \ker(u^{n+1}))$$

with the zero morphisms $H^n \to H^{n+1}$. By reversing the arrows one defines *chain complexes* and the notion of *homology* of a complex $K^\bullet = (u_n : K_n \to K_{n-1})_{n \in I}$.

Finally, let us give some examples of abelian categories. Of course the first example must be the category Ab of abelian groups, it is the motiviating example of the whole theory. If A is an abelian category, then the category $\mathsf{A}^{\mathsf{C}^{\mathrm{op}}}$ of A-valued presheaves on C is an abelian category. In particular, the category $\widehat{\mathsf{C}}^{\mathrm{ab}}$ is an abelian category.

Let $\mathcal{S}h_X^{\mathrm{ab}}$ be the category of abelian sheaves on a topological space (or on a Grothendieck topology) considered as a full subcategory of the category of abelian presheaves $\mathcal{PS}h_X^{\mathrm{ab}} = \widehat{\mathsf{C}}^{\mathrm{ab}}$. It is not a trivial fact that it is an abelian category. One uses the fact that the forgetting functor $\iota : \mathcal{S}h_X^{\mathrm{ab}} \to \mathcal{PS}h_X^{\mathrm{ab}}$ admits the left inverse functor $\mathrm{as} : \mathcal{PS}h_X^{\mathrm{ab}} \to \mathcal{S}h_X^{\mathrm{ab}}$ (i.e. $\mathrm{as} \circ \iota$ is isomorphic to the identity functor in $\mathcal{S}h_X^{\mathrm{ab}}$). The functor as is defined by the standard construction of the sheafication $\mathcal{F}^\#$ of a presheaf $\mathcal{F}$. For any morphism $u : \mathcal{F} \to \mathcal{G}$ of abelian sheaves one has

$$\ker(u) = \ker(\iota(u)))^\#,\ \mathrm{coker}(u) = \mathrm{coker}(\iota(u))^\#$$

(in fact, $\iota(\ker(u)) = \ker(\iota(u)))$.

Another example is the category $\mathrm{Mod}(R)$ of (left) modules over a ring R (all rings will be assumed to be associative and contain 1). It is considered as a subcategory of Ab. The kernels and cokernels in Ab of a morphism in $\mathrm{Mod}(R)$ can be equipped naturally with a stricture of R-modules and represent the kernels and cokernels in $\mathrm{Mod}(R)$.

A generalization of this example is the category of sheaves of modules over a ringed space $(X, \mathcal{O}_X)$. The most important for us example will be the category $\mathrm{Qcoh}(X)$ of qusi-coherent sheaves and its full subcategory Coh_X of coherent sheaves on a scheme X.

Another frequently used example of an abelian category is the *category of quivers* with values in an abelian category A (or the *category of diagrams* in A as defined by Grothendieck). We fix any oriented graph Γ and assign to each its vertex v an object $X(v)$ from A. To each arrow a with tail $t(a)$ and head $h(a)$ we assign a morphism from $u(a) : X(t(a)) \to X(h(a))$ in A.

A *morphism of diagrams* $(X(v), u(a)) \to (X'(v), u'(a))$ is a set of morphisms $\phi_v : X(v) \to X'(v)$ such that, for any arrow a the diagram

$$\begin{array}{ccc} X(t(a)) & \xrightarrow{\phi_{t(a)}} & X'(t(a)) \\ \downarrow{\scriptstyle u(a)} & & \downarrow{\scriptstyle u'(a)} \\ X(h(a)) & \xrightarrow{\phi_{h(a)}} & X'(h(a)) \end{array}$$

is commutative.

One can consider different natural subcategories of the category $\mathrm{Diag}(\Gamma, \mathsf{A})$ of the category of diagrams in A with fixed graph Γ. A diagram is called *commutative* if for any path $p = (a_1, \ldots, a_k)$ the composition $u(p) = u(a_1) \circ \cdots \circ u(a_k)$ depends only on $t(p) := t(a_k)$ and $h(p) := h(a_1)$. It is easy to check that commutative diagrams form an abelian subcategory of the category of diagrams. A commutative diagram is called a *diagram complex* if for any path different from an arrow, the morphism $u(p)$ is the zero morphism. A complex in A defined in above corresponds to a linear graph. All diagram complexes form an abelian subcategory of the category $\mathrm{Diag}(\Gamma, \mathsf{A})$. In particular, the category of cochain complexes in an abelian category is an abelian category.

Here are some examples of additive but not abelian categories. Obviously an additive subcategory of an abelian category is not abelian in general. For example, the category of projective modules over a ring R is additive but has no kernels or cokernels for some homomorphisms of projective R-modules. An additive subcategory may have kernels or cokernels but (1.1) may not be an isomorphism. The most notorious example is the category of filtered abelian groups $A = \cup_{i\in\mathbb{N}} A^i, A^i \subset A^{i+1}$ with morphisms compatible with filtrations. Changing the filtration by ${}'A^i := A^{i+1}$ and considering the identity morphism we obtain a non-invertible morphism that is an monomorphism and an epimorphism.

Let us consider the category $\mathrm{Mod}(R)$ of left modules over a ring R. Note that the category of right R-modules is the category $\mathrm{Mod}(R^{\mathrm{op}})$, where R^{op} is the opposite ring. The category $\mathrm{Mod}(R)$ is obviously an abelian category. It satisfies the following properties

(i) For each set $(M_i)_{i\in I}$ of objects in $\mathrm{Mod}(R)$ indexed by any set of I there exists the direct sum $\oplus_{i\in I} M_i$. It corepresents the functor $\prod_{i\in I} h^{M_i}$.

(ii) The ring R considered as a module over itself is a *generator* P of $\mathrm{Mod}(R)$ (i.e. any object M admits an epimorphism $R^I \to M$ for some set I, where R^I denotes the direct sum $\oplus_{i\in I} R_i$ of objects R_i isomorphic to R).

(iii) the generator P is a *projective object* (i.e. the functor $M \to \mathrm{Hom}_{\mathsf{A}}(R, M)$ sends epimorphisms to epimorphisms;

(iv) the functor h^P commutes with arbitrary set-indexed direct sums.

Theorem 1.1.1. *Let A be an abelian category satisfying the conditions* $(i)-(iv)$ *from above where* $\mathrm{Mod}(R)$ *is replaced with A. Then A is equivalent to the category* $\mathrm{Mod}(R)$*, where* $R = \mathrm{End}(P) := \mathit{Hom}_{\mathsf{A}}(P, P)$

Proof. For any ring R we define the new category $\mathrm{Mod}(R, \mathsf{A})$. Its objects are homomorphisms of rings $\phi : R \to \mathrm{End}_{\mathsf{A}}(X), X \in \mathrm{Ob}(\mathsf{A})$. The set of morphisms from $\phi : R \to \mathrm{End}(X)$ to $\psi : R \to \mathrm{End}(Y)$ is the subset of morphisms $f \in \mathrm{Hom}_{\mathsf{A}}(X, Y)$ such that $\psi(r) \circ f = f \circ \phi(r)$ for any $r \in R$. It is easy to check that $\mathrm{Mod}(R, \mathsf{A})$ is an abelian category. For example, if $f \in \mathrm{Hom}_{\mathsf{A}}(X, Y)$ is a morphism in $\mathrm{Mod}(R, \mathsf{A})$, then its kernel consists of homomorphisms from R to $\mathrm{End}(\ker(f))$. Obviously $\mathrm{Mod}(R, \mathsf{Ab}) \approx \mathrm{Mod}(R)$.

Let $R = \mathrm{End}_{\mathsf{A}}(U)^{\mathrm{op}}$, where U is a fixed object of A. The composition of morphisms defines a map

$$R \times \mathrm{Hom}_{\mathsf{A}}(U, X) \to \mathrm{Hom}_{\mathsf{A}}(U, X).$$

It is easy to see that it equips the abelian group $h^U(X) = \mathrm{Hom}_{\mathsf{A}}(U, X)$ with a structure of a R-module. Also it is easy to check that $h^U : \mathsf{A} \to \mathsf{Sets}$ factors through the subcategory $\mathrm{Mod}(R)$ of Sets, i.e. defines a functor

$$\mathcal{S} = h^U : \mathsf{A} \to \mathrm{Mod}(R).$$

For any R^{op}-module M, any $X \in \mathrm{Ob}(\mathrm{Mod}(R, \mathsf{A}))$ and any $Y \in \mathrm{Ob}(\mathsf{A})$ we can consider the set $\mathrm{Hom}_{R^{\mathrm{op}}}(M, h^X(Y))$. The assignment $Y \to \mathrm{Hom}_{R^{\mathrm{op}}}(M, h^X(Y))$ is a presheaf on A^{op} with values in the category of R^{op}-modules. When it is representable, a representing object is denoted by $M \otimes_R X$. Replacing R by R^{op} we have the notation $X \otimes_R M$, where M is a R-module, and $X \in \mathrm{Mod}(R^{\mathrm{op}}, \mathsf{A})$. This agrees with the usual notation of the tensor product $A \otimes_R B$ of a right R-module A and a left R-module B.

Now we assume that U is a projective generator of A. Let us consider U as an object of $\mathrm{Mod}(R^{\mathrm{op}})$ defined by the identity homomorphism $R^{\mathrm{op}} \to \mathrm{End}_{\mathsf{A}}(U)$. One proves that under the conditions of the theorem, $U \otimes_R M$ exists. To do so we first consider the case $M = R$, where we get a natural bijection $\mathrm{Hom}_{R^{\mathrm{op}}}(R, h^U(Y)) \to \mathrm{Hom}_{\mathsf{A}}(U, Y)$. Then we consider the case when $M = R^I$ by considering the bijection $\mathrm{Hom}_{R^{\mathrm{op}}}(R^I, h^U(Y)) \to \mathrm{Hom}_{\mathsf{A}}(U^I, Y)$. Finally, if $M = \mathrm{coker}(R^J \to R^I)$ we take $U \otimes M$ to be the cokernel of $U^J \to U^I$.

We choose a representing object and denote it by $U \otimes_R M$. Now we can define a functor

$$\mathcal{T} : \mathrm{Mod}(R) \to \mathsf{A},\ M \to U \otimes_R M.$$

By definition, for any $A \in \mathrm{Ob}(\mathsf{A})$ and any $M \in \mathrm{Mod}(R)$, there is a natural bijection

$$\mathrm{Hom}_R(M, \mathcal{S}(A)) = \mathrm{Hom}_R(M, \mathrm{Hom}_{\mathsf{A}}(U, N)) \to \mathrm{Hom}_{\mathsf{A}}(\mathcal{T}(M), A).$$

In particular, the functor $\mathcal{T}$ is the *left adjoint functor* to the functor $\mathcal{S}$. Recall that a functor $G : \mathsf{C}' \to \mathsf{C}$ is called a left adjoint functor to the functor $F : \mathsf{C} \to \mathsf{C}'$ (and F is the *right adjoint* to G) if there is an isomorphism of bifunctors on $\mathsf{C} \times \mathsf{C}'^{\mathrm{op}}$ defined by

$$(X, Y) \to \mathrm{Mor}_{\mathsf{C}}(G(X), Y),\ (X, Y) \to \mathrm{Mor}_{\mathsf{C}'}(X, F(Y)).$$

I leave to the reader to define the product of two categories and a bifunctor defined on the product. By taking $Y = G(X)$, the image of the identity $\mathbf{id}_Y$ defines a morphism $X \to F \circ G(X)$. All such morphisms define a morphism of functors $\mathbf{id}_{\mathsf{C}'} \to F \circ G$. Similarly we define a morphism of functors $G \circ F \to \mathbf{id}_{\mathsf{C}}$. Conversely, a pair of morphisms of functors

$$\alpha : \mathbf{id}_{\mathsf{C}'} \to F \circ G,\ \beta : G \circ F \to \mathbf{id}_{\mathsf{C}}$$

satisfying the property that the compositions

$$G \xrightarrow{G\circ\alpha} G\circ F\circ G \xrightarrow{\beta\circ G} G,\ F \xrightarrow{\alpha\circ F} F\circ G\circ F \xrightarrow{F\circ\beta} F$$

are the identity morphisms of the functors make G left adjoint to F. In particular, an equivalence of categories is defined by a pair of adjoint functors.

Note that a pair (G, F) of adjoint functors, in general, do not define an equivalence of categories, i.e. neither $\alpha : G \circ F \to \mathbf{id}_{\mathsf{C}}$ nor $\beta : \mathbf{id}_{\mathsf{C}'} \to F \circ G$ is an equivalence of categories. However, if F is an equivalence of categories then any quasi-inverse functor to F is a left and a right adjoint functor.

An example of a pair of adjoint functors is $(G, F) = (f^*, f_*)$, where $f : X \to Y$ is a morphism of schemes and $(\mathsf{C}, \mathsf{C}') = (\mathrm{Qcoh}(Y), \mathrm{Qcoh}(X))$, the categories of quasicoherent sheaves. Another example is the functor

$$\mathrm{Hom}_R(B, ?) : \mathrm{Mod}(S) \to \mathrm{Mod}(R),\ N \to \mathrm{Hom}_S(B, N)$$

and its left adjoint functor

$$? \otimes_R B : \mathrm{Mod}(R) \to \mathrm{Mod}(S),\ M \to M \otimes_R B,$$

where R, S are rings and B is a $(R - S)$-bimodule (i.e. a left R-module and a right S-module).

Recall that a functor $F : \mathsf{C} \to \mathsf{C}'$ is called *left exact* (resp. *right exact*) if it transforms monomorphisms (resp. epimorphisms) to monomorphisms (resp. epimorphimsm). It is called *exact* if is left and right exact. Suppose F admits a left adjoint functor G. Then F is left exact and G is right exact. Let us see it in the case of functors of abelian categories. It is enough to show that F transforms kernels to kernels. Let $K = \ker(u : X \to Y)$ and $Y \to \ker(F(X) \to F(Y)$ be a morphism in A'. Applying G we get a morphism $G(Y) \to \ker(G(F(X)), G(F(Y)))$. Composing $G(Y) \to G(F(X))$ with $G(F(X)) \to X$ we get a morphism $G(Y) \to X$ which as is easy to see factors through $G(Y) \to \ker(X \to Y)$. This shows that there is a morphism $G(Y) \to K$ and applying F we get a morphism $Y \to F(G(Y)) \to F(K)$. This shows that $F(K)$ is the kernel of $F(u)$.

Since $\mathcal{T}$ is the left adjoint of $\mathcal{S}$, we obtain that $\mathcal{S}$ is left exact. Since U is a projective object in A, we obtain that $\mathcal{S}$ is right exact, hence *exact*, i.e. left and right exact. Condition (iv) implies that $\mathcal{S}(U^I) = \mathcal{S}(U)^I$. It is easy to see that

$$\mathrm{Hom}_{\mathsf{A}}(U^I, U^j) = \prod_i \oplus_{j\in I}\mathrm{Hom}_{\mathsf{C}}(U_i, U_j) = \prod_{i\in I} \oplus_{j\in I}\mathrm{Hom}_R(\mathcal{S}(U_i), \mathcal{S}(U_j))$$

$$= \mathrm{Hom}_R(\oplus_{i\in I}\mathcal{S}(U_i), \oplus_{j\in I}\mathcal{S}(U_j)) = \mathrm{Hom}(R, \mathcal{S}(\oplus_{i\in I}U_i), \oplus_{j\in I}F(U_j))).$$

This shows that $\mathcal{S}$ defines an equivalence from the subcategory of $\mathcal{A}$ formed by objects isomorphic to direct sums U^I and the subcategory of $\mathrm{Mod}(R)$ formed by free modules. Since U and R are projective generators this easily allows to extend the equivalence to an equivalence between the categories $\mathcal{A}$ and $\mathrm{Mod}(R)$. $\square$

Example 1.1.2. Let $\mathsf{A} = \mathcal{PS}h^{\mathrm{ab}}(X)$ be the category of abelian presheaves on a topological space X. For any open subset V denote by $\mathbb{Z}_V$ the sheaf equal to the constant sheaf $\mathbb{Z}$ on V and zero outside U. Then $\oplus_{V \in \mathrm{Ob}(\mathrm{Open}(X))}\mathbb{Z}_V$ is a generator of A. The generator U is not projective, however if we pass to the category of sheaves and replace U with the associated sheaf $U^{\#}$ we obtain a projective generator.

More generally, consider the category of abelian presheaves $\widehat{\mathsf{C}}^{\mathrm{ab}}$ on a category C. Then $U = \oplus_{X \in \mathrm{Ob}(\mathsf{C})}\mathbb{Z}^{h_X}$ is a generator of this category.

Definition 1.1.2. An abelian category satisfying properties from the assertion of the previous theorem is called a *Grothendieck category.*

Corollary 1.1.3. *For any abelian category* A *there exists a fully faithful additive exact functor to a a category* $\mathrm{Mod}(R)$ *(an* additive functor *of abelian categories is a functor which is a functor of the corresponding abelian presheaves).*

Proof. It is enough to construct such a functor with values in a Grothendieck category. We first embed A in the category of abelian presheaves $\widehat{\mathsf{A}}^{\mathrm{ab}}$. Then we consider a canonical Grothendieck topology on A where all representable presheaves become automatically sheaves. We take B be the category of sheaves on A with respect to the canonical topology. The product of representable sheaves will serve as a projective generator. The Yoneda functor $h : \mathsf{A} \to \mathsf{B}$ will be an additive fully faithful exact functor. Then we apply the previous theorem. □

Suppose $F : \mathrm{Mod}(R) \to \mathcal{A}$ is an equivalence of abelian categories. Then $U = F(R)$ is a projective generator of A and $R \cong \mathrm{End}(U)$. For example, we may take $\mathsf{A} = \mathrm{Mod}(S)$ for some other ring S. Then $\mathrm{Mod}(R) \approx \mathrm{Mod}(S)$ implies that there exists a projective S-module U generating $\mathrm{Mod}(S)$ such that $\mathrm{End}(S) \cong R$. In fact, the proof of the previous theorem shows that U has a structure of $S-R$-bimodule and the equivalence of categories is defined by the functor $M \to M \otimes_S U$. Two rings are called *Morita equivalent* if the categories $\mathrm{Mod}(R)$ and $\mathrm{Mod}(S)$ are equivalent. A good example is the equivalence between the category $\mathrm{Mod}(R)$ and the category $\mathrm{Mod}(S)$, where $S = \mathrm{End}(R^n)$.

1.2 Derived categories

Let $\mathrm{Cp}(\mathsf{A})$ denote the category of cochain complexes $K^\bullet = (K^n, d_K^n)_{n \in \mathbb{Z}}$ in an abelian category A. We can always assume that the interval parametrizing K^n is the whole $\mathbb{Z}$ by adding the zero objects and the zero differentials.

We shall denote by $\mathrm{Cp}(\mathsf{A})^+$ (resp. $\mathrm{Cp}(\mathsf{A})^-$) the full subcategory formed by complexes such that there exists N such that $K^i = 0$ for $i < N$ (resp. $K^i = 0$ for $i > N$). They are called *bounded* from below (resp. from above). A *bounded complex* is a complex bounded from below and from above. The category of those is denoted by $\mathrm{Cp}^b(\mathsf{A})$. The category $\mathrm{Cp}(\mathsf{A})$ comes with the *shift functor* $T : \mathrm{Cp}(\mathsf{A}) \to \mathrm{Cp}(\mathsf{A})$, defined by $T(K^\bullet) = K^\bullet[1]$, where

$$K^\bullet[1]^n = K^{n+1}, d^n_{T(K^\bullet)} = -d^{n+1}_K.$$

We set $K^\bullet[n] = T^n(K^\bullet)$, where T^n denotes the composition of functors. Note that T has the obvious inverse, so we may define $K[n]$ for any integer n. By definition,

$$d^i_{K^\bullet[n]} = (-1)^n d^{i+n}_{K^\bullet}.$$

The differentials (d^n_K) define a morphism $d : K \to K[1]$ such that the composition $T(d) \circ d : K \to K[1] \to K[2]$ is the zero morphism.

Recall that a *simplicial complex* is a presheaf on the category whose objects are natural numbers and morphisms are non-decreasing maps of intervals $f : [0, n] \to [0, m]$. A simplicial complex $X = (X_m \xrightarrow{X(f)} X_n)_{n \le m}$ defines a topological space $|X|$ called the *geometric realization*. One considers the sets

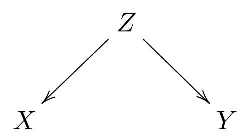
$$\Delta_n = \{(x_0, \ldots, x_n) \in \mathbb{R}^{n+1} : x_0 + \ldots + x_n = 1, x_i \ge 0\},$$

and let

$$|X| = \coprod_{n=0}^{\infty} \Delta_n \times X_n/R,$$

where R is the minimal equivalence relation which identifies the points $(s, x) \in \Delta_n \times X_n$ and $(t, y) \in \Delta_m \times X_m$ if $y = X(f)(x), s = \Delta(f)(t)$ for some morphism $f : m \to n$. Here $\Delta(f)$ is the unique affine map that sends the vertex e_i to the vertex $e_{f(i)}$. The topology is the factor-topology. For example, Δ_n is homeomorphic to the topological realization of the simplicial complex $h_{[n]}$ whose value on $[m]$ is equal to the set of non-decreasing maps $[m] \to [n]$.

A topological space is called *triangulazable* if it is homeomorphic to $|X|$ for some simplicial set X. A chosen homeomorphism is called a *triangulation*. Let $C_n(X) = \mathbb{Z}^{X_n}$ and let $\iota^i_n : [n-1] \to [n]$ whose image omits $\{i\}$. The map $X(\iota^i_n) : X_n \to X_{n-1}$ defines a homomorphism of abelian groups $\delta^i_n : C_n(X) \to C_{n-1}(X)$ and we set $d_n : C_n(X) \to C_{n-1}(X)$ to be equal to the alternating sum $\sum_{i=0}^n (-1)^n \delta^i_n$ of the homomorphisms δ^i_n. One checks that $(C_n(X), d_n)$ is a chain complex. One defines the *homology group* of a triangulizable topological space $\mathcal{X}$ by choosing a triangulation $|X| \to \mathcal{X}$ and setting $H_n(\mathcal{X}, \mathbb{Z}) = H_n(C_\bullet(X))$. Dually one defines the *cohomology*, as the cohomology of the dual complex $C^\bullet(X)$, where $C^n(X) = \mathrm{Hom}(C_n(X), \mathbb{Z})$ and d^n is the transpose of d_n. It does not depend on a choice of a triangulation. Passing to homology or cohomology losses some information about the topology of $|X|$. For example, two simplicial sets may have the same homology but their topological realizations may not be homotopy equivalent. However, one has an important theorem of Whitehead that states that $|X|$ is homotopy equivalent to $|Y|$ if and only if there is a diagram of simplicial maps (i.e. morphisms of functors)



inducing the isomorphism of the homologies of the corresponding chain complexes. Note that a morphism of simplicial complexes $f : X \to Y$ defines a morphism $f_* = (f_n)$ of the chain complexes. One defines a *homotopy* between two morphisms of simplicial complexes $f, g : X \to Y$ as a morphism of simplicial complexes $h_{[1]} \times X \to Y$ such that the composition of this map with $h_{[0]} \to h_{[1]}$ defined by sending 0 to 0 (resp. 0 to 1) is equal to f (resp. g). It induces the maps $h_i : C_i(X) \to C_{i+1}(Y)$ such that $f_i - g_i = d_{i+1}^X \circ h_i + h_{i-1} \circ d_i^Y$ for all $i \in \mathbb{Z}$. This easily implies that f_* and g_* induce the same maps on homology. This allows one to introduce the category of hopotopy types of simplicial complexes, define a functor to the category of complexes of abelian groups modulo homotopy of complexes, and use the derived category of complexes to interpret Whitehead's Theorem by stating that two triangulizable spaces are homotopy equivalent if and only if their derived categories of chain complexes are equivalent.

After this brief motivation let us proceed with the categorical generalizations of the previous discussion.

Let $\mathrm{Cp}(\mathsf{A})$ be the category of complexes in an abelian category A.

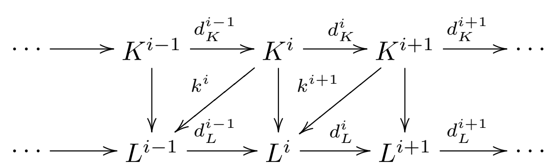
Definition 1.2.1. Two morphisms $f, g : (K^\bullet, d_K^\bullet) \to (L^\bullet, d_L^\bullet)$ are called *homotopy equivalent* if there exist morphisms $h : K^\bullet \to L^\bullet[-1]$ such that

$$f - g = h \circ d_{K^\bullet} + d_{L^\bullet} \circ h.$$

Here we view the differential $d_{X^\bullet}$ of a complex $X^\bullet$ as a morphism $d_{X^\bullet} : X^\bullet \to X^\bullet[1]$. In components, $h = (h^i : K^i \to L^{i-1})$ and

$$f^i - g^i = h^{i+1} d_{K^\bullet}^i + d_{L^\bullet}^{i-1} h^i.$$

In pictures



It is clear that the homotopy to zero morphisms form a subgroup in the group $\mathrm{Hom}_{\mathrm{Cp}(\mathsf{A})}(K^\bullet, L^\bullet)$. Thus the homotopy equivalence is an equivalence relation. Let $\mathrm{Hom}_{\mathbf{K}(\mathsf{A})}(K^\bullet, L^\bullet)$ be the quotient group by the subgroup of morphisms which are homotopy to zero.

Let $f : K^\bullet \to L^\bullet, g : L^\bullet \to M^\bullet$ be two morphisms of complexes. Assume that f is homotopy to 0. Then $g \circ f$ is homotopy to zero. To see this we check that, if (k^i) define a homotopy for f to the zero morphism, then $g(k^i)$ will define the homotopy for $g \circ f$.

This allows one to introduce the category $\mathbf{K}(\mathsf{A})$ whose objects are compexes in A and morphisms are equivalence classes of morphisms in $\mathrm{Cp}(\mathsf{A})$ modulo the homotopy equivalence relation.

It is easy to verify that the *cohomology functor*

$$H : \mathrm{Cp}(\mathsf{A}) \to \mathrm{Cp}(\mathsf{A}),\ K^\bullet \to (H^n(K^\bullet))$$

factors through the category $\mathbf{K}(\mathsf{A})$.

To imitate the Whitehead Theorem we need to convert morphisms in $\mathbf{K}(\mathsf{A})$ that induce the isomorphism of the cohomology into isomorphisms. This can be done using the general notion of localization in a category.

Theorem 1.2.1. *Let C be a small category and S be a set of its morphisms. There exists a category C_S and a functor $Q : \mathsf{C} \to \mathsf{C}_S$ satisfying the following properties:*

(L1) for any $f \in S$, $Q(f)$ is an isomorphism;

(L2) if $F : \mathsf{C} \to \mathsf{C}'$ is a functor satisfying property (i), then there exists a unique functor $G : \mathsf{C}_S \to \mathsf{C}'$ such that $F = G \circ Q$.

Proof. The idea is simple, we have to formally add the inverses of all $s \in S$. Let us consider an oriented graph Γ' whose vertices are objects of C and whose arrows from A to B are morphisms from A to B. For each $s \in S$ from $t(s)$ to $h(s)$ that has no inverse, we add an arrow from $t(s)$ to $h(s)$. Let Γ be the new graph. Now define the category C_S as follows. Its objects are vertices of Γ. Its morphisms correspond to paths in Γ modulo the following equivalence relation: two loops are equivalent if they obtained from each other by the following elementary operations:

(a) two edges u, v with $h(u) = t(v)$ can be replaced by the edge corresponding to the composition $u \circ v$;

(b) the loop $(t(s), h(s), t(s))$ (resp. $(h(s), t(s), h(s))$) corresponding to an edge $s \in S$ are equivalent to the loop $(t(s), t(s))$ (resp. $(h(s), h(s))$) corresponding to the identity morphisms $\mathbf{id}_{t(s)}$ (resp. $\mathbf{id}_{t(s)}$).

The functor $Q : \mathsf{C} \to \mathsf{C}_S$ is defined by considering the inclusion of the graphs $\Gamma' \subset \Gamma$. The properties (L1) and (L2) are checked immediately. □

Definition 1.2.2. The category C_S is called the *localization* of C with respect to the set of morphisms S.

Now we take $\mathsf{C} = \mathrm{Cp}(\mathsf{A})$ and let S be a set of morphisms $f : K^\bullet \to L^\bullet$ such that $H^\bullet(f)$ is an isomorphism (such morphisms are called *quasi-isomorphisms*.

Definition 1.2.3. The derived category is the category $D(\mathsf{A}) = \mathrm{Cp}(\mathsf{A})[S^{-1}]$, where S is the set of quasi-isomorphims. Similarly one defines the categories $D^+(\mathsf{A}), D^-(\mathsf{A})$ and $D^b(\mathsf{A})$.

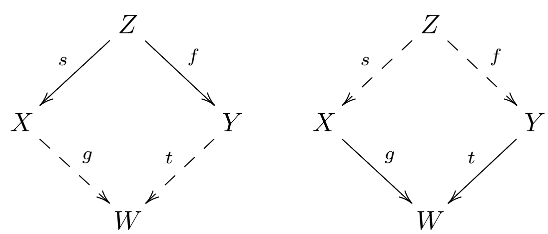
Unfortunately, after localizing an additive category, we will get in general a non-additive category. In order that this does not happen we need impose some additional properties of the set of localizing morphisms S.

Definition 1.2.4. A set S of morphisms in a category C is called a *localizable* set if it satisfies the following properties:

(L1) S is closed under compositions and contains the identity morphisms;

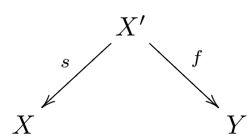
(L1)' If $s \in S$ and $f \circ s$ or $s \circ f \in S$, then $f \in S$;

(L2) for any $s : Z \to X$ from S and a morphism $f : Z \to Y$ there exists $g : X \to W$ from C and $t : Y \to W$ from S such that $g \circ s = t \circ f$. Also the similar property holds when we reverse the arrows

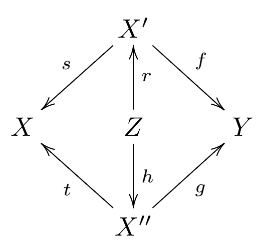


(L3) for any $f, g : A \rightrightarrows B$ the existence of $s \in S$ such that $s \circ f = s \circ g$ is equivalent to the existence of $t \in S$ such that $f \circ t = g \circ t$.

Condition (L2) means that we can write each $s^{-1}f$ in C_S in the form gt^{-1} or can write each fs^{-1} in the form $t^{-1}g$. Let $f : X' \to Y$ be a morphism in C and $s : X' \to X$ belongs to S. We say that the morphism fs^{-1} in C_S by a *roof*



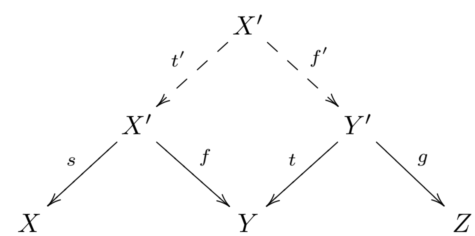
Two roofs define the same morphism if they can be extended to a common roof



where $r \in S$. This is an equivalence relation on the set of roofs and a morphism in C_S is the equivalence class of roofs. To check that this is indeed an equivalence relation one has to use the third property of localizing sets. If $(X', s, f) : X \to Y$

is equivalent to (X'', t, g) by means of (Z', r, h) and (X'', t, g) is equivalent to (X''', u, e) by means of (Z'', p, i), then we first define $sr : Z' \to X, tp : Z'' \to X$, then take $v : W \to Z', k : W \to Z''$ such that $srv = tpk$. Then $f_1 = hv, f_2 = pk$ satisfy $tf_1 = tf_2$. Thus we find $w : Z''' \to W$ such that $f_1 w = f_2 w$. If we take $q = rvw : Z''' \to X'$ and $j = ikw : Z''' \to X'''$ we get the equivalence $(X', s, t) \sim (X''', u, e)$.

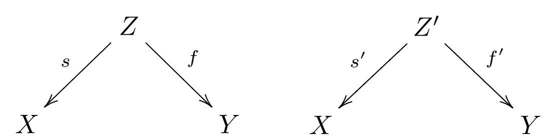
The composition is defined by composing the roofs:



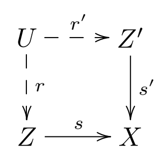
where the top square exists by property (L2). One has to check here that this definition does not depend on the choice of representatives in the equivalence class of the corresponding roofs. We refer for this verification to Milicoc's lectures. Note that here we must use property (L4).

Proposition 1.2.2. *Let S be a localizing set of morphisms in an additive category C. Then C_S is an additive category.*

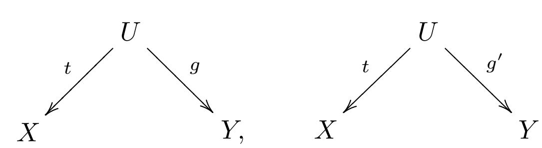
Proof. The idea is to reduce to common denominator. Suppose we have two morphisms $\phi, \phi' : X \to Y$ represented by two roofs $fs^{-1} = (s, f)$ and $f's'^{-1} = (s', f')$



Let $U = Z \times_X Z'$ with respect to $s : Z \to X, s' : Z \to X$. By property (L2) we can find a commutative square



where r' belongs to S. Since $s, s' \in S$, property (L1a) implies that $r \in S$. Thus our two morphisms can be represented by



where $t = s \circ r = s' \circ r', g = f \circ r, g' = f' \circ r'$. It remains to define the sum $\phi + \phi'$ as the morphism represented by the roof $X \xleftarrow{t} U \xrightarrow{g+g'} Y$.

Next observe that the zero morphism $0 : X \to Y$ is equivalent to any roof of the form $X \xleftarrow{s} X' \xrightarrow{0} Y$. They are related by the morphism $X' \xleftarrow{1} X' \xrightarrow{f} X$. In particular, the zero object 0 in C is the zero object in C_S since $\mathrm{End}_{\mathsf{C}_S}(0) = \{0\}$. Finally, the direct sum in C_S is represented by the direct sum in C and the canonical injections and the projections are defined by the corresponding morphisms in C. □

Note that one can prove the proposition without using property (F1a) (see Milicic's Lectures, www.math.utah.edu/ milicic/dercat.pdf). However it greatly simplifies the proof and it is checked in the case of derived categories.

Since replacing a morphism of complexes by a homotopy equivalent morphism does not change the map on the cohomology, the definition of a quasi-isomorphism extends to the category $\mathbf{K}(\mathsf{A})$.

It turns out that the set of quasi-isomorphisms in $\mathbf{K}(\mathsf{A})$ is a localizing set (but not in $\mathrm{Cp}(\mathsf{A})$). We will see later that the corresponding localization of $\mathbf{K}(\mathsf{A})$ is equivalent to the derived category $D(\mathsf{A})$.

Before we show that the set of quasi-isomorphisms in $\mathbf{K}(\mathsf{A})$ is a localizing set we have to introduce some constructions familiar from homotopy theory.

Recall the *cone construction* from algebraic topology. Let $f : X \to Y$ be a continuous map of topological spaces. We define the cone $C(f)$ of f as the topological space

$$C(f) = Y \coprod X \times [0,1]/\sim,$$

where $(x, 1) \sim f(x)$ and $(x, 0) \sim (x_0, 0)$ for some fixed $x_0 \in X$. In the case Y is a point, $C(f) = \Sigma X$ is the *suspension* of X (ΣX is a 'double cone', it is obtained from $X \times [0, 1]$ with $X \times \{0\}$ and $X \times \{1\}$ identified with a point). In the case when $X \to Y$ is an inclusion, $C(f)$ is homotopy equivalent to Y/X (the space obtained by contracting Y to a point). So in this case $C(f)$ is an analog of a cokernel. Also note that there is an inclusion $Y \to C(f)$ and if we apply the cone construction to this we get that $C(f)/Y$ is homotopy equivalent to ΣX. Thus we have a sequence of morphisms in the homotopy category:

$$X \to Y \to C(f) \to \Sigma X.$$

Recall that we have the suspension isomorphism:

$$H_{i+1}(\Sigma X) \cong H_i(X),$$

and

$$H_i(C(f)) = H_i(Y, X).$$

This gives a long exact sequence

$$\ldots \to H_i(X) \to H_i(Y) \to H_i(Y, X) \to H_{i-1}(X) \to H_{i-1}(Y) \to \ldots$$

There is an analogous construction of the cone $C(f)$ of a morphism $f : X \to Y$ of simplicial complexes which is I am not going to remind. The cochain complex of $C(f)$ is

$$C^\bullet(C(f)) = C^\bullet(X)[1] \oplus C^\bullet(Y), \quad d = \begin{pmatrix} d_X[1] & 0 \\ f[1] & d_Y \end{pmatrix}$$

in the sense that $(a, b) \in C^{i+1}(X) \oplus C^i(Y)$ goes to $(d_X^{i+1}(a), f^{i+1}(a) + d_Y^i(b)) \in C^{i+2}(X) \oplus C^{i+1}(Y)$. This can be taken for the definition of the cone $C(f)$ for any morphism of cochain complexes $f : X^\bullet \to Y^\bullet$ in $\mathrm{Cp}(\mathsf{A})$.

Definition 1.2.5. Let $f : X^\bullet \to Y^\bullet$ be a morphism in $\mathrm{Cp}(\mathsf{A})$. Define the *cone* of f as the complex

$$C(f) = X^\bullet[1] \oplus Y^\bullet, \quad d = \begin{pmatrix} d_{X^\bullet[1]} & 0 \\ f[1] & d_{Y^\bullet} \end{pmatrix}.$$

Define the *cylinder* $\mathrm{Cyl}(f)$ as the complex

$$\mathrm{Cyl}(f) = X^\bullet \oplus X^\bullet[1] \oplus Y^\bullet, \quad d = \begin{pmatrix} d_{X^\bullet} & -1 & 0 \\ 0 & d_{X^\bullet[1]} & 0 \\ 0 & f[1] & d_{Y^\bullet} \end{pmatrix}.$$

Example 1.2.3. Let $f : X \to Y$ be a morphism in A considered as a morphism of complexes with support at $\{0\}$. Then $(C(f))^{-1} = X, C(f)^0 = Y$ and $d = f$. Thus $H^0(C(f)) = \ker(f)$ and $H^1(C(f)) = \mathrm{coker}(f)$.

Example 1.2.4. Take $f = \mathbf{id}_{K^\bullet} : K^\bullet \to K^\bullet$. Let $k^i : X^{i+1} \oplus X^i \to X^i \oplus X^{i-1}$ defined by $(x^{i+1}, x^i) \to (x^i, 0)$ This defines the homotopy between $\mathbf{id}_{C(f)}$ and $0_{C(f)}$. In particular, all cohomology of the complex $C(f)$ vanish.

Lemma 1.2.5. *Let $f : X^\bullet \to Y^\bullet$ be a morphism of complexes. There is a commutative diagram in $\mathrm{Cp}(A)$:*

$$\begin{array}{cccccccccccc}
 & & 0 & \longrightarrow & Y^\bullet & \xrightarrow{i_f} & C(f) & \xrightarrow{p_f} & X^\bullet[1] & \longrightarrow & 0 \\
 & & & & \downarrow{\scriptstyle \alpha} & & \| & & & & \\
0 & \longrightarrow & X^\bullet & \longrightarrow & \mathrm{Cyl}(f) & \longrightarrow & C(f) & \longrightarrow & 0 & & \\
 & & \|{\scriptstyle \bar{f}} & & \downarrow{\scriptstyle \beta} & & & & & & \\
 & & X^\bullet & \xrightarrow{f} & Y^\bullet & & & & & &
\end{array}$$

Its rows are exact and the vertical arrows are quasi-isomorphisms.

Proof. Let us describe the morphisms in this diagram. The morphism $i_f : Y^\bullet \to C(f) = X^\bullet[1] \oplus Y^\bullet$ is the direct sum inclusion. The morphism $p_f : C(f) \to X^\bullet[1]$ is the projection to the first summand. The morphism $\mathrm{Cyl}(f) = X^\bullet \oplus X^\bullet[1] \oplus Y^\bullet \to X^\bullet[1] \oplus Y^\bullet$ is the projection to the last two summands. We take α to be the direct sum inclusion and $\beta = (f, \oplus\mathbf{id}_{Y^\bullet})$. The morphism $\bar{f} : X^\bullet \to \mathrm{Cyl}(f)$ is the direct sum inclusion. We leave to the reader to check that this defines morphisms of complexes, the diagram is commutative and its rows are exact.

Obviously, $\beta\circ\alpha = \mathbf{id}_{Y^\bullet}$ Let us check that $\alpha\circ\beta$ is homotopy equivalent to the identity. We define the homotopy $h^i : \mathrm{Cyl}(f)^i i \to \mathrm{Cyl}(f)^{i-1}$ by $(x^i, x^{i+1}, l^i) = (0, x^i, 0)$. We have

$$\alpha\circ\beta(x^i, x^{i+1}, l^i) = (0, 0, f(x^{i+1}) + l^i),$$

$$(d^{i-1}_{\mathrm{Cyl}(f)}h^i + h^i d^i_{\mathrm{Cyl}(f)})(x^i, x^{i+1}, l^i) = (-x^i, -d^i_{X^\bullet}(x^i), f(x^i)) + (0, d^i_K(x^i) - x^{i+1}, 0) =$$

$$(-x^i, -x^{i+1}, f(x^{i+1})) = (\alpha\circ\beta - \mathbf{id}_{\mathrm{Cyl}(f)})(x^i, x^{i+1}, l^i).$$

This checks that $\alpha\circ\beta \sim \mathbf{id}_{Y^\bullet}$. □

Corollary 1.2.6. *Let $f : X^\bullet \to Y^\bullet$ be a morphism in $\mathbf{K}(\mathsf{A})$. Then it can be extended to a sequence*

$$X^\bullet \xrightarrow{f} Y^\bullet \xrightarrow{g} C(f) \xrightarrow{h} X^\bullet[1],$$

where the composition of any two morphisms is zero.

Proof. We define $g : Y^\bullet \to C(f) = X^\bullet[1] \oplus Y^\bullet$ and $h : C(f) \to X^\bullet[1]$ as in the first row of the diagram from the lemma. By the proof of the previous lemma, the composition $g\circ f : X^\bullet \to Y^\bullet \to C(f)$ is homotopy equivalent to the composition $X^\bullet \to \mathrm{Cyl}(f) \to C(f)$ which is zero. The composition $h\circ g : Y^\bullet \to C(f) \to X^\bullet[1]$ is zero because the top row in the lemma is an exact sequence. □

Definition 1.2.6. A *triangle* in $\mathrm{Cp}(\mathsf{A})$ is a diagram of the form

$$X^\bullet \to Y^\bullet \to Z^\bullet \to X^\bullet[1].$$

A *distinguished triangle* is a triangle which is quasi-isomorphic to the triangle

$$X^\bullet \xrightarrow{f} Y^\bullet \xrightarrow{i_f} C(f) \xrightarrow{p_f} X^\bullet[1].$$

It follows from Lemma 1.2.5 that a distinguished triangle is quasi-isomorphic to the triangle

$$X^\bullet \to \mathrm{Cyl}(f) \to C(f) \to X^\bullet[1]$$

with morphisms defined in the lemma.

Lemma 1.2.7. *Any short exact sequence of complexes is quasi-isomorphic to the middle row of the diagram from Lemma 1.2.5.*

Proof. Let

$$0 \to X^\bullet \xrightarrow{f} Y^\bullet \xrightarrow{g} Z^\bullet \to 0$$

be an exact sequence in $\mathrm{Cp}(\mathsf{A})$. We define $\beta : \mathrm{Cyl}(f) \to Y^\bullet$ to be equal to β from Lemma 1.2.5 and $\gamma : C(f) \to Z^\bullet$ by composing the natural projection $C(f) \to Y^\bullet$ with g. We have $\ker(\gamma) = X^\bullet[1] \oplus \mathrm{im}(f) = X^\bullet[1] \oplus X^\bullet = C(\mathbf{id}_{X^\bullet})$. By Example 1.2.4, the latter complex has trivial cohomology. Thus, using the exact sequence $0 \to \ker(\gamma) \to C(f) \to Z^\bullet \to 0$ we obtain that γ is a quasi-isomorphism. □

Theorem 1.2.8. *Any distinguished triangle*

$$X^\bullet \to Y^\bullet \to Z^\bullet \to X^\bullet[1]$$

defines an exact sequence of cohomology:

$$\ldots \to H^i(X^\bullet) \to H^i(Y^\bullet) \to H^i(Z^\bullet) \to H^i(X^\bullet[1]) = H^{i+1}(X^\bullet) \to \ldots. \quad (1.2)$$

Proof. It is enough to prove it for the distinguished triangle:

$$X^\bullet \to \mathrm{Cyl}(f) \to C(f) \xrightarrow{g} X^\bullet[1].$$

We have the exact sequence

$$0 \to X^\bullet \to \mathrm{Cyl}(f) \to C(f) \to 0.$$

It gives the exact sequence of cohomology

$$\ldots \to H^i(X^\bullet) \to H^i(\mathrm{Cyl}(f)) \to H^i(C(f)) \xrightarrow{\delta} H^{i+1}(X^\bullet) \to \ldots.$$

It remains to identify the coboundary morphism δ with $H^i(g)$. We use that $\mathrm{Cyl}(f)^i = X^i \oplus C(f)^i$ and check the definitions. □

Theorem 1.2.9. *In the category $\mathbf{K}(A)$ quasi-isomorphims form a localizing set of morphisms.*

Proof. Properties (L1) and (L1') are obvious.

Let $t : Y^\bullet \to W^\bullet$ and $g : X^\bullet \to W^\bullet$, where t is a quasi-isomorphism. We have to find a quasi-isomorphism $s : Z^\bullet \to X^\bullet$ and a morphism of complexes $f : Z^\bullet \to Y^\bullet$ such that $f \circ t = s \circ g$. Let $X^\bullet \to C(t)$ be the composition $i_t \circ g$, where $i_t : W \to C(t)$ is the canonical inclusion. Let $s : C(i_t \circ g)[-1] \to X^\bullet$ be the morphism $p_{i_t \circ g}[-1]$, where $p_{i_t\circ g} : C(i_t \circ g) = X^\bullet[1] \oplus C(f) \to X^\bullet[1]$ is the canonical projection. We have the following diagram

$$\begin{array}{ccccccc}
C(i_t \circ g)[-1] & \xrightarrow{s} & X^\bullet & \xrightarrow{i_t\circ g} & C(t) & \longrightarrow & C(i_t \circ g) \\
\downarrow{\scriptstyle f} & & \downarrow{\scriptstyle g} & & \| & & \downarrow{\scriptstyle f[1]} \\
Y^\bullet & \xrightarrow{t} & W^\bullet & \xrightarrow{i_t} & C(t) & \longrightarrow & Y^\bullet[1]
\end{array}$$

where the morphism f is constructed as follows. An element from $C(i_t \circ g)[-1]$ is a triple $(x^i, y^i, w^{i-1}) \in X^i \oplus Y^i \oplus W^{i-1}$. We set $f(x^i, y^i, w^{i-1}) = -y^i$. I claim that this diagram is commutative in the category $\mathbf{K}(\mathsf{A})$ (but not in $\mathrm{Cp}(\mathsf{A})$!). We have

$$g \circ s - t \circ f(x^i, y^i, w^{i-1}) = g(x^i) - t(y^i),$$

Let $\chi = \{\chi^i\}$, where $\chi^i : C(i_t \circ g)[-1]^i = C(i_t \circ g)^{i-1} \to W^{i-1}$ is given by $\chi^i(x^i, y^i, w^{i-1}) = -w^{i-1}$. We have

$$(\chi \circ d_{C(i_t \circ g)}[-1] + d_{W^\bullet} \circ \chi)((x^i, y^i, w^{i-1})$$

$$= \chi((d_{X^\bullet}(x^i), d_{Y^\bullet}(y^i), -d_{W^\bullet}(w^{i-1}) - t(x^i) - g(x^i)) + d_{W^\bullet}(-w^{i-1}) =$$

$$d_{W^\bullet}(w^{i-1}) - t(x^i) + g(x^i) - d_{W^\bullet}(w^{i-1}) = g(x^i) - t(y^i).$$

This shows that $g \circ s - t \circ f$ is homotopy to zero. It remains to show that s is a quasi-isomorphism. Since t is a quasi-isomorphism, exact sequence of cohomology (1.2) implies that all cohomology of $C(f)$ are equal to zero. Applying the same exact sequence to the top row of the diagram, we obtain that s is a quasi-isomorphism.

To finish the verification of property (L2) we have also verify that a pair $s : Z^\bullet \to X^\bullet, f : Z^\bullet \to Y^\bullet$, where s is a quasi-isomorphism, defines a pair $g : X^\bullet \to W^\bullet, t : Y^\bullet \to W^\bullet$, where t is a quasi-isomorphism, and $g \circ s = t \circ f$. This follows from a similar argument using the following commutative diagram in $\mathbf{K}(\mathsf{A})$:

$$\begin{array}{ccccccc}
C(s)[-1] & \xrightarrow{\tau} & Z^\bullet & \xrightarrow{s} & X^\bullet & \longrightarrow & C(s) \\
\| & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle g} & & \| \\
C(s)[-1] & \xrightarrow{f\circ\tau} & Y^\bullet & \xrightarrow{t} & C(f\circ\tau) & \longrightarrow & C(s)
\end{array}$$

We leave it to the reader.

Finally we have to check property (L3). Let $f : X^\bullet \to Y^\bullet$ be a morphism in $\mathbf{K}(\mathsf{A})$. Assume $s \circ f = 0$ in $\mathbf{K}(\mathsf{A})$ for some quasi-isomorphism $s : Y^\bullet \to Z^\bullet$. Let $h^i : X^\bullet \to Z^\bullet[-1]$ defines a homotopy between $s \circ f$ and the zero morphism. We have to find a quasi-isomorphism $t : W^\bullet \to X^\bullet$ and a homotopy between $f \circ t$ and the zero morphism. Consider the following commutative diagram:

$$\begin{array}{ccccc}
C(s)[-1] & \longrightarrow & Y^\bullet & \xrightarrow{s} & Z^\bullet \\
\| & & \uparrow{\scriptstyle f} & & \\
C(s)[-1] & \xleftarrow{g} & X^\bullet & \xleftarrow{t} & C(g)[-1]
\end{array}$$

where the morphism $g : X^\bullet \to C(s)[-1]$ is defined by $g^i(x^i) = (f^i(x^i), -h^i(x^i))$ and t is the natural projection. Then $t \circ f = 0$ because $t \circ g = 0$. Also t is a quasi-isomorphism because s is a quasi-isomorphism and hence $C(s)$ has trivial homology. A similar assertion with the roles of s and t reversed can be proven in analogous manner. □

It remains to show that the localization of $\mathbf{K}(\mathsf{A})$ with respect to the localizing set of quasi-isomorphisms is equivalent to the derived category $D(\mathsf{A})$. By the universality property we have a functor from $D(\mathsf{A}) \to \mathbf{K}(\mathsf{A}_S)$, where S is the set of quasi-isomorphisms. This functor is bijective on the sets of objects and surjective on the set of morphisms. It remains to show that it is injective on the set of morphisms. This follows from the following lemma.

Lemma 1.2.10. *Let $f, g : X^\bullet \to Y^\bullet$ two homotopy equivalent morphisms. Then their images in $D(\mathsf{A})$ are equal.*

Proof. Let $h : X^\bullet \to Y^\bullet[-1]$ be a homotopy between two morphisms $f, g : X^\bullet \to Y^\bullet$. First we extend h to a morphism $c(h) : C(f) \to C(g)$ of the cones by setting

$$c(h)(x^{i+1}, y^i) = (x^{i+1}, y^i + h(x^{i+1}).$$

It is a morphism of complexes. In fact, we have

$$d_{C(g)}(c(h)((x^{i+1}, y^i))) = d_{C(g)}(x^{i+1}, y^i + h(x^{i+1}))$$

$$= (-d_{X^\bullet}(x^{i+1}), g(x^{i+1}) + d_{Y^\bullet}(y^i) + d_{Y^\bullet}(h(x^{i+1})),$$

$$c(h)(d_{C(f)}((x^{i+1}, y^i))) = c(h)(-d_{X^\bullet}(x^{i+1}), f(x^{i+1}) + d_{Y^\bullet}(y^i))$$

$$= (-d_{X^\bullet}(x^{i+1}), f(x^{i+1}) + d_{Y^\bullet}(y^i) - h(d_{X^\bullet}(x^{i+1}))).$$

Since $f(x^{i+1}) - g(x^{i+1}) = d_{Y^\bullet} h(x^{i+1})) + h(d_{X^\bullet}(x^{i+1}))$, this checks the claim. Similarly, we extend h to a morphism $\mathrm{cyl}(h) : \mathrm{Cyl}(f) \to \mathrm{Cyl}(g)$ of the cylinders by setting

$$\mathrm{cyl}(h)(x^i, x^{i+1}, y^i) = (x^i, x^{i+1}, y^i + h(x^{i+1})).$$

Consider the following commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & Y^\bullet & \xrightarrow{i_f} & C(f) & \xrightarrow{p_f} & X^\bullet[1] & \longrightarrow & 0 \\
 & & \| & & \downarrow{\scriptstyle c(h)} & & \| & & \\
0 & \longrightarrow & Y^\bullet & \xrightarrow{i_g} & C(g) & \xrightarrow{p_g} & X^\bullet[1] & \longrightarrow & 0
\end{array}$$

where the arrows in the horizontal rows are the natural inclusions and the projections. The commutativity is easy to check. Applying the exact cohomology sequences and the five-homomorphism lemma (extended easily to abelian categories), we obtain that $c(h)$ is a quasi-isomorphism. Similarly, one proves that $\mathrm{cyl}(h)$ is a quasi-isomorphism.

Finally we consider the diagram

$$\begin{array}{ccccccc}
 & & X^\bullet & = & X^\bullet & & \\
 & \swarrow^{g} & \downarrow{\scriptstyle \bar{g}} & & \downarrow{\scriptstyle \bar{f}} & \searrow^{f} & \\
Y^\bullet & \xrightarrow{\alpha(g)} & \mathrm{Cyl}(g) & \xrightarrow{\mathrm{cyl}(h)} & \mathrm{Cyl}(f) & \xrightarrow{\beta(f)} & Y^\bullet
\end{array}$$

Here we employ the notation $\alpha(f), \beta(f)$ and $\alpha(g), \beta(g)\bar{f}, \bar{g}$ from Lemma 1.2.5. One easily checks that the square and the right triangle are commutative. The left triangle becomes commutative in $D(\mathsf{A})$. In fact we know from Lemma 1.2.5 that $\alpha(g)$ has the inverse $\beta(g)$. Since $g = \beta(g) \circ \bar{g}$, we have $\alpha(g) \circ g = \bar{g}$ in $D(\mathsf{A})$. This implies that left triangle is commutative in $D(\mathsf{A})$. Finally one checks that $\beta(f) \circ \mathrm{cyl}(h) \circ \alpha(g) = \mathbf{id}_{Y^\bullet}$, hence the images of f and g in $D(\mathsf{A})$ are equal. □

A generalization of the notion of the derived category of an abelian category is the notion of a *triangulated category*.

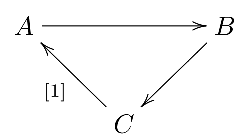
Definition 1.2.7. An additive category C is called *triangulated* if it is equipped with the following data:

(i) An additive auto-equivalence functor $T : \mathsf{C} \to \mathsf{C}$ (the *shift functor*).

(ii) A class of distinguished triangles (closed under a naturally defined isomorphism of triangles)

$$A \to B \to C \to T(A)$$

(one writes them as



to justify the name).

The following axioms must be satisfied:

(TR1) $A \stackrel{\mathbf{id}_A}{\to} A \to 0 \to T(A)$ is distinguished;

(TR2) any morphism $f : A \to B$ can be completed to a distinguished triangle;

(TR3) a triangle $A \stackrel{u}{\to} B \stackrel{v}{\to} C \stackrel{w}{\to} T(A)$ is distinguished if and only if

$$B \stackrel{v}{\longrightarrow} C \stackrel{w}{\longrightarrow} T(A) \stackrel{-T(u)}{\longrightarrow} T(B)$$

is a distinguished triangle;

(TR4) Any commutative diagram

$$\begin{array}{ccc} A & \stackrel{u}{\longrightarrow} & B \\ \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle g} \\ A' & \stackrel{u'}{\longrightarrow} & B' \end{array}$$

extends to a morphism of triangles (i.e. a commutative diagram whose rows are distinguished triangles)

$$\begin{array}{ccccccc} A & \xrightarrow{u} & B & \xrightarrow{v} & C & \xrightarrow{w} & T(A) \\ \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle h} & & \downarrow{\scriptstyle T(f)} \\ A' & \xrightarrow{u'} & B & \xrightarrow{v'} & C & \xrightarrow{w'} & T(A') \end{array}$$

(TR5) Given three distinguished triangles

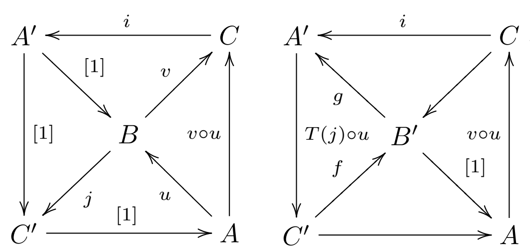
$$A \xrightarrow{u} B \xrightarrow{j} C' \to T(A), \quad B \xrightarrow{v} C \xrightarrow{i} A' \to T(B), \quad A \xrightarrow{v\circ u} C \to B' \to T(A)$$

there exist two morphisms $f : C' \to B', g : B' \to A'$ such that $(\mathbf{id}_A, v, f), (u, \mathbf{id}_C, g)$ define morphisms of the triangles and

$$C' \xrightarrow{f} B' \xrightarrow{g} A' \xrightarrow{T(j)\circ i} C'[1]$$

is a distinguished triangle.

One can illustrate it by using the following diagrams:



Here the upper and the bottom triangle in the left diagram are distinguished triangles and the other two triangles are commutative. In the right diagram, the upper and the bottom triangles are commutative and the other two are distinguished triangles. It is also required that the two possible morphisms $B \to B'$ (factored through C and C') are equal. The axiom says that the left diagram can be completed to the right diagram.

Intuitively, if one expresses a distinguished triangle $A \to B \to C \to T(A)$ by saying that $C = B/A$ is the cokernel of $A \to B$, then axiom (TR5) says that f defines a morphism $B/A \to C/A$ with cokernel C/B.

Proposition 1.2.11. *Let A be an abelian category. Each of the categories $\mathbf{K}(\mathsf{A})$ and $D(\mathcal{A})$ has a structure of a triangulated category with distinguished triangles defined by the cone construction and the shift functor defined by the shift of complexes.*

Proof. Let us first check that $\mathbf{K}(\mathsf{A})$ is a triangulated category. Axiom (TR1) follows from Example 1.2.4 since the cone $C(\mathbf{id}_A)$ of the identity morphism is homotopy to zero morphism, hence $C(\mathbf{id}_A) \cong 0$ in $\mathbf{K}(\mathsf{A})$.

Axiom (TR2) follows from Corollary 1.2.6.

Axiom (TR3) follows from Lermma 1.2.5 where we replace the morphism f with the morphism $f[-1] : X^\bullet[-1] \to Y^\bullet[-1]$ and apply Corollary 1.2.6.

Axiom (TR4) is immediate. We may assume that $C = C(u)$, $C' = C(u')$ and take $h = f[1] \oplus g$.

We skip the verification of axiom (TR5) since we are not going to use this property (see [Gelfand-Manin] or [Kashiwara]).

To check that $D(\mathcal{A})$ is a triangulated category, we use a more general assertion. Suppose C is a triangulated category and S is a localizing set of morphisms in C satisfying the following additional properties

(L4) $s \in S$ if and only if $s[1]$ belongs to S;

(L5) if in axiom (TR4) the morphisms f, g belong to S, the morphism h belongs to S.

Then we claim that C_S inherits the structure of a triangulated category. For any morphism u in C_S represented by a roof $A \xleftarrow{s} A' \xrightarrow{f} B$ we can define the shift $T(u)$ as the morphism represented by the roof $A[1] \xleftarrow{s[1]} A'[1] \xrightarrow{f[1]} B[1]$. It is easy to check that it does not depend on the choice of a representative roof. This defines the shift functor in C_S. We define distinguished triangles in C_S as triangles $A \to B \to C \to A[1]$ isomorphic (in C_S) to distinguished triangles in C.

Now, axiom $(TR1)$ becomes obvious. Suppose $u : A \to B$ is represented by a roof (s, f) as above. Let $A' \xrightarrow{f} B \xrightarrow{v} C \to A'[1]$ be a distinguished triangle in C. Then it is isomorphic to the triangle $A \xrightarrow{u} B \xrightarrow{v} C \to A'[1]$ in C_S. This checks axion (TR2). Axiom (TR3) follows immediately from Axiom (TR3) in C.

To check Axiom (TR4) we may assume that the distinguished triangles extending the morphisms $A \to B$ and $A' \to B'$ are distinguished triangles in C. Consider the following diagram

$$\begin{array}{ccccccc}
A & \xrightarrow{u} & B & \xrightarrow{v} & C & \xrightarrow{w} & A[1] \\
\uparrow{\scriptstyle s} & & \uparrow{\scriptstyle t} & & \uparrow{\scriptstyle r} & & \uparrow{\scriptstyle T(s)} \\
X & & Y & & Z & & X[1] \\
\downarrow{\scriptstyle \tilde{f}} & & \downarrow{\scriptstyle \tilde{g}} & & \downarrow{\scriptstyle \tilde{h}} & & \downarrow{\scriptstyle T(f)} \\
A' & \xrightarrow{u'} & B' & \xrightarrow{v'} & C' & \xrightarrow{w'} & A'[1],
\end{array}$$

where the arrows (s, t, r) belong to S. Here the morphism $f : A \to A'$ in the localized category is represented by a roof $(s, \tilde{f})$ and similar for the morphism

$g : B \to B'$. We are looking for a morphism $h : C \to C'$ represented by some roof $(r, \tilde{h})$.

First, by property (L2) of localizing sets, we find a morphism $\tilde{t} : X' \to X$ in S and a morphism $\tilde{u} : X' \to Y$ in C such that $\tilde{t} \circ u \circ s = t \circ \tilde{u}$. Replacing X with X', s with $s \circ \tilde{t}$m and $\tilde{f}$ with $\tilde{f} \circ \tilde{t}$, we may assume that there exists a morphism $u'' : X \to Y$ such that $u \circ s = t \circ u''$. Since $u' \circ \tilde{f} \circ s^{-1} = \tilde{g} \circ t^{-1} \circ u = \tilde{g} \circ u'' \circ s^{-1}$ in C_S, we can apply property (L2) of a localizing set to obtain that there exists $a : X' \to X$ in S such that $\tilde{g} \circ u'' \circ a = u' \circ \tilde{f} \circ a$. Replacing X with X' we obtain a commutative diagram of morphisms in C

$$\begin{array}{ccccccc}
A & \xrightarrow{u} & B & \xrightarrow{v} & C & \xrightarrow{w} & A[1] \\
\uparrow{\scriptstyle s} & & \uparrow{\scriptstyle t} & & \uparrow{\scriptstyle r} & & \uparrow{\scriptstyle T(s)} \\
X & \xrightarrow{u''} & Y & \xrightarrow{v''} & Z & \xrightarrow{w'} & X[1] \\
\downarrow{\scriptstyle \tilde{f}} & & \downarrow{\scriptstyle \tilde{g}} & & \downarrow{\scriptstyle \tilde{h}} & & \downarrow{\scriptstyle T(f)} \\
A' & \xrightarrow{u'} & B' & \xrightarrow{v'} & C' & \xrightarrow{w'} & A'[1],
\end{array}$$

where the middle row is a distinguished triangle in C and r is defined by Axiom (TR3) in C. By property (L5), r must belong to S. Now the roof $(r, \tilde{h})$ together with morphisms f, g define a morphism of distinguished triangles in C_S satisfying Axiom (TR3) in C_S.

We skip the verification of Axiom (TR5).

Since the set of quasi-isomorphisms in $\mathbf{K}(\mathsf{A})$ obviously satisfies properties (L4) and (L5), we obtain the assertion of the proposition. □

Definition 1.2.8. A subcategory of a triangulated category is called *triangulated subcategory* if each its morphism $A \to B$ can be extended to a distinguished triangle with morphisms in the subcategory. A functor of triangulated categories is a functor which commutes with the shift functors and sends distinguished triangles to distinguished triangles (it is also called a *δ-functor*). An equivalence of triangulated categories is a functor of triangulated categories such that its quasi-inverse functor is also a functor of triangulated categories.

1.3 Derived functors

Let $F : \mathsf{A} \to \mathsf{B}$ be an additive functor of abelian categories. We would like to extend it to a functor of the corresponding derived categories. Of course, we can immediately extend it, componentwise, to a functor $\mathrm{Cp}(\mathsf{A}) \to \mathrm{Cp}(\mathsf{B})$. Also it does extend to a functor $\mathbf{K}(F) : \mathbf{K}(\mathsf{A}) \to \mathbf{K}(\mathsf{B})$ since F commutes with homotopy morphisms. It is easy to see that it transforms cones to cones, and hence defines a δ-functor. To extend it further we need to check that $\mathbf{K}(F)(u)$ is a quasi-isomorphism for any quasi-isomorphism u in $\mathbf{K}(\mathsf{A})$. Then, by the universal property of localizations, we obtain a functor $D(F) : D(\mathsf{A}) \to D(\mathsf{B})$ such that $D(F) \circ Q_{\mathsf{A}} = Q_{\mathsf{B}} \circ \mathbf{K}(F)$. It is easy to see that an exact functor F

transforms quasi-isomorphism to quasi-isomorphisms, but this is a very special case.

Let $\mathbf{F} : \mathbf{K}(\mathsf{A}) \to \mathbf{K}(\mathsf{A})$ be a functor of triangulated categories in the sense of Definition 1.2.8. Note that we do not assume that $\mathbf{F}$ is of the form $\mathbf{K}(F)$. Obviously, $\mathbf{F}$ extends to derived categories if it transforms quasi-isomorphisms to quasi-isomorphisms, and, in particularly, acyclic complexes (i.e. with zero cohomology) to acyclic complexes. In this case it becomes a functor of triangulated categories. Also by considering

Conversely, suppose $\mathbf{F}$ is a functor of triangulated categories. Let $f : X^\bullet \to Y^\bullet$ be a quasi-isomorphism of complexes from $\mathbf{K}(\mathsf{A})$. Extending it to a distinguished triangle $X^\bullet \to Y^\bullet \to C(f) \to X^\bullet[1]$ we obtain an acyclic complex $C(f)$ (apply the exact cohomology sequence). Consider the distinguished triangle $\mathbf{F}(X)^\bullet \to \mathbf{F}(Y)^\bullet \to \mathbf{F}(C(f)) \to \mathbf{F}(X)^\bullet[1]$. If moreover we know that $\mathbf{F}$ transforms acyclic complexes to acyclic complexes, then $\mathbf{F}(C(f))$ is acyclic, and $\mathbf{F}(X)^\bullet \to \mathbf{F}(Y)^\bullet$ is a quasi-isomorphism. The idea of defining the derived functor is to find a sufficiently large subcategory of $\mathbf{K}(\mathsf{A})$ such that the restriction of $\mathbf{F}$ to it transforms acyclic complexes to acyclic complexes.

In the following $\mathrm{Cp}^*(\mathsf{A})$ denotes either $\mathrm{Cp}(\mathsf{A})$, or $\mathrm{Cp}^\pm(\mathsf{A})$, or $\mathrm{Cp}^b(\mathsf{A})$ and similar definitions for $\mathbf{K}^*(\mathsf{A}), D^*(\mathsf{A})$.

Definition 1.3.1. A full triangulated subcategory $\mathbf{K}^*(\mathsf{A})'$ of $\mathbf{K}^*(\mathsf{A})$ is called *left (right) adapted* for a left (right) exact functor $\mathbf{F}$ if the following properties are satisfied

(A1) $\mathbf{F}(X^\bullet)$ is acyclic for any acyclic complex $X^\bullet$ in $\mathbf{K}^*(\mathsf{A})'$;

(A2) for any object in $X^\bullet$ in $\mathbf{K}^*(\mathsf{A})$ there is a quasi-isomorphism $X^\bullet \to R^\bullet$ ($R^\bullet \to X^\bullet$), where $R^\bullet$ is an object in $\mathbf{K}^*(\mathsf{A})'$;

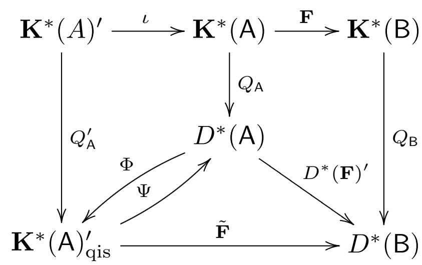
(A3) the inclusion of categories $\iota : \mathbf{K}^*(\mathsf{A})' \to \mathbf{K}^*(\mathsf{A})$ defines an equivalence of triangulated categories $\Psi : \mathbf{K}^*(\mathsf{A})'_{\mathrm{qis}} \to D^*(\mathsf{A})$, where qis is the set of quasi-isomorphisms.

By property (A1), $\mathbf{F} \circ \iota$ transforms acyclic complexes to acyclic complexes. By the universality property of localizations this defines a functor $\tilde{\mathbf{F}} : \mathbf{K}^*(\mathsf{A})'_{\mathrm{qis}} \to D^*(\mathsf{A})$ such that $Q_{\mathsf{B}} \circ \mathbf{F} \circ \iota = \tilde{\mathbf{F}} \circ Q'_{\mathsf{A}}$. Let $\Phi : D^*(\mathsf{A}) \to \mathbf{K}^*(\mathsf{A})'_{\mathrm{qis}}$ be a quasi-inverse functor. We set

$$D^*(\mathbf{F})' = \tilde{\mathbf{F}} \circ \Phi.$$

By property (A3), the functor $D^*(\mathbf{F})'$ is a functor of triangulated categories.

We have the following diagram:



Take $X^\bullet \in \text{Ob}(\mathbf{K}^*(\mathsf{A}))$ and consider it as an object of $D^*(\mathsf{A})$. Using the isomorphism of functors $\mathbf{id}_{D^*(\mathsf{A})} \to \Psi \circ \Phi$ we can find a (functorial) isomorphism in $D^*(\mathsf{A})$ from $X^\bullet$ to $\Psi(Y^\bullet)$, where $Y^\bullet = \Phi(X^\bullet) \in \text{Ob}(\mathbf{K}^*(\mathsf{A})')$. The isomorphism can be given by $X^\bullet \xrightarrow{s} Z^\bullet \xleftarrow{t} Y^\bullet$, where s, t are quasi-isomorphisms. By property (A2) we can find a quasi-isomorphism $a : Z^\bullet \to W^\bullet$, where $W^\bullet \in \mathbf{K}^*(\mathsf{A})'$. Replacing s with $a \circ s$ and t with $a \circ t$, we may assume that $Z^\bullet \in \mathbf{K}^*(\mathsf{A})'$. Applying $\mathbf{F}$ we get a morphisms in $\mathbf{K}^*(\mathsf{B})$

$$\mathbf{F}(X^\bullet) \xrightarrow{\mathbf{F}(s)} \mathbf{F}(Z^\bullet) \xleftarrow{\mathbf{F}(t)} \mathbf{F}(Y^\bullet) .$$

Since $t : Y^\bullet \to Z^\bullet$ is a quasi-isomorphism in $\mathbf{K}(\mathsf{A})'$, $\mathbf{F}(t)$ is a quasi-isomorphism. Applying Q_{B}, we get a morphism in $D^*(\mathsf{B})$

$$Q_{\mathsf{B}} \circ \mathbf{F}(X^\bullet) \to Q_{\mathsf{B}} \circ \mathbf{F}(\iota(Y^\bullet)) = \tilde{\mathbf{F}} \circ Q'_{\mathsf{A}}(Y^\bullet) = \tilde{\mathbf{F}} \circ \Phi(Q_{\mathsf{A}}(X^\bullet)) = D^*(\mathbf{F})' \circ Q_{\mathsf{A}}(X^\bullet).$$

This defines a morphism of functors

$$\epsilon_{\mathbf{F}} : Q_{\mathsf{B}} \circ \mathbf{F} \to D^*(\mathbf{F})' \circ Q_{\mathsf{A}}.$$

Note that, by definition, $\tilde{\mathbf{F}} \circ \Phi \circ \Psi = D^*(\mathbf{F})' \circ \Psi$. Thus we have an isomorphism of functors $\tilde{\mathbf{F}} \to D^*(\mathbf{F})' \circ \Psi$. Composing with Q'_{A} we get an isomorphism of functors

$$Q_{\mathsf{B}} \circ \mathbf{F} \circ \iota = \tilde{\mathbf{F}} \circ Q_{\mathcal{R}} \to D^*(\mathbf{F})' \circ \Psi \circ Q_{\mathcal{R}} = D^*(\mathbf{F})' \circ Q_{\mathsf{A}} \circ \iota. \tag{1.3}$$

This shows that $\epsilon_{\mathbf{F}}$ defines an isomorphism of functors after we restrict them to $\mathbf{K}^*(\mathsf{A})'$.

One can show that the pair $(D^*(\mathbf{F})', \epsilon_{\mathbf{F}})$ satisfies the following definition of the derived functor.

Definition 1.3.2. Let $\mathbf{F} : \mathbf{K}(\mathsf{A}) \to \mathbf{K}(\mathsf{B})$ be a left exact additive functor of abelian categories. A right *derived functor* of $\mathbf{F}$ is a pair consisting of an exact additive functor $D^+(\mathbf{F}) : \mathcal{D}^+(A) \to D^+(\mathsf{B})$ and an morphism of functors $\epsilon_{\mathbf{F}} : Q_{\mathsf{B}} \circ \mathbf{K}^+(\mathbf{F}) \to D^+(\mathbf{F}) \circ Q_{\mathsf{A}}$, where $Q_{\mathsf{A}} : \mathbf{K}^+(\mathsf{A}) \to D^+(\mathsf{A})$ and Q_{B} :

$\mathbf{K}^+(\mathsf{B}) \to D^+(\mathsf{B})$ are the natural morphisms of the localizations.

$$\begin{array}{ccc} \mathbf{K}^+(\mathsf{A}) & \xrightarrow{Q_{\mathsf{A}}} & D^+(\mathsf{A}) \\ \downarrow{\scriptstyle \mathbf{K}^+(F)} & & \downarrow{\scriptstyle D^+(F)} \\ \mathbf{K}^+(\mathsf{B}) & \xrightarrow{Q_{\mathsf{B}}} & D^+(\mathsf{B}) \end{array}$$

This pair must satisfy the following universality property: for any exact functor $G : D^+(\mathsf{A}) \to D^+(\mathsf{B})$ and a morphism of functors $\epsilon : Q_{\mathsf{B}} \circ \mathbf{K}^+(\mathbf{F}) \to G \circ Q_{\mathsf{A}}$ there exists a unique morphism of functors $\eta : D^+(\mathbf{F}) \to G$ such that the following diagram is commutative

$$\begin{array}{ccc} & Q_{\mathsf{B}} \circ \mathbf{K}^+(\mathbf{F}) & \\ {\scriptstyle \epsilon_{\mathbf{F}}} \swarrow & & \searrow {\scriptstyle \epsilon} \\ G \circ Q_{\mathsf{A}} & \xleftarrow[\eta \circ Q_{\mathsf{A}}]{} & \mathrm{D}^+(\mathbf{F}) \circ Q_{\mathsf{A}} \end{array}$$

Similarly we give a definition of the derived functor of a right exact additive functor $\mathbf{F}$. In this case there exists a morphism of functors $D^*(\mathbf{F})' : D^*(\mathsf{A}) \to D^*(\mathsf{B})$ and a morphism of functors

$$\epsilon_{\mathbf{F}} : D^*(\mathbf{F})' \circ Q_A \circ \mathbf{F}.$$

satisfying the previous definition with the appropriate change of the universality property.

The next theorem will be left without proof.

Theorem 1.3.1. *Assume that* $\mathbf{F}$ *has an adapted subcategory* $\mathbf{K}(A)'$. *The functor* $D^+_{\mathcal{R}}(\mathbf{F})$ *is a derived functor of* $\mathbf{F}$. *In particular, it does not depend, up to isomorphism of functors, on a choice of an adapted subcategory.*

To define a left (right) adapted subcategory $\mathbf{K}^{\pm}(\mathsf{A})'$ for left (right) additive functor $\mathbf{F} : \mathbf{K}^{\pm}(\mathsf{A}) \to \mathbf{K}^{\pm}(\mathsf{B})$ we choose it to be the full subcategory $\mathbf{K}^{\pm}(\mathcal{R})$ of $\mathbf{K}^{\pm}(\mathsf{A})$ formed by complexes of objects belonging to a left (right) adapted subset $\mathcal{R} \subset \mathrm{Ob}(\mathsf{A})$ in the sense of the following definition.

Definition 1.3.3. A set $\mathcal{R}$ of objects in $\mathcal{A}$ is called *left (right) adapted* for $\mathbf{F}$ if it satisfies the following properties

(i) For any acyclic complex $X^\bullet$ in $\mathrm{Cp}^+(\mathcal{R})$ the complex $\mathbf{F}(X^\bullet)$ is acyclic.

(ii) Any object A in $\mathcal{A}$ admits a monomorphism $A \to R$ (epimorphism $R \to A$), where $R \in \mathrm{Ob}(\mathcal{R})$;

(iii) $\mathcal{R}$ is closed under taking finite direct sums.

We will show that $\mathbf{K}^{\pm}(\mathcal{R})$ is an adapted subcategory for $\mathbf{F} : \mathbf{K}^{+}(\mathsf{A}) \to \mathbf{K}^{+}(\mathsf{B})$. This will allow us to define the derived functor

$$D^{+}_{\mathcal{R}}(\mathbf{F}) : D^{+}(\mathcal{A}) \to D^{+}(\mathsf{B}).$$

By above we obtain a morphism of functors

$$\epsilon_{\mathbf{F}} : Q_{\mathsf{B}} \circ \mathbf{K}^{+}(\mathbf{F}) \to D^{+}_{\mathcal{R}}(\mathbf{F}) \circ Q_{\mathsf{A}}.$$

Lemma 1.3.2. *Any $X^{\bullet} \in \mathbf{K}^{+}(A)$ admits a quasi-isomorphism to an object in $\mathbf{K}^{+}(\mathcal{R})$.*

Proof. Without loss of generality we may assume that $X^{p} = 0, p < 0$. Let $i_0 : X^0 \to R^0$ be a monomorphism of X^0 to an object from $\mathcal{R}$. Let $R^0 \amalg_{X^0} X^1$ be the direct product over X^0 (the cokernel of $X^0 \to R^0 \oplus X^1$). Composing $X^1 \to R^0 \amalg_{X^0} X^1$ with a monomorphism $a : R^0 \amalg_{X^0} X^1 \to R^1$ for some R^1 from $\mathcal{R}$, we obtain a morphism $i_1 : X^1 \to R^1$. We define a complex $d^0_R : R^0 \to R^1$ by taking d^0_R equal to the composition of the morphism $R^0 \to R^0 \amalg_{X^0} X^1$ with a. This defines a morphism of complexes $(X^0 \xrightarrow{d^0_{X^\bullet}} X^1) \to (R^0 \to R^1)$. Next we consider the diagram

$$\begin{array}{ccccccc}
 & & X^1 & \xrightarrow{d^1_{X^\bullet}} & X^2 & & \\
 & \swarrow^{i_1} & \downarrow{\scriptstyle p\circ i_1} & & \downarrow & \searrow^{i_2} & \\
R^1 & \xrightarrow{p} & \operatorname{coker}(d^0_R) & \xrightarrow{b} & \operatorname{coker}(d^0_R) \amalg_{X^1} X^2 & \longrightarrow & R^2
\end{array}$$

and define i_2 as the composition of $X^2 \to \operatorname{coker}(d^0_R) \amalg_{X^1} X^2$ and a monomorphism $\operatorname{coker}(d^0_R) \amalg_{X^1} X^2 \to R^2$. We define $d^1_R : R^1 \to R^2$ as the composition of the morphisms in the second row. It follows from the definition that $d^1_R \circ d^0_R = 0$. Continuing in this way we use the diagrams

$$\begin{array}{ccccccc}
 & & X^n & \xrightarrow{d^n_{X^\bullet}} & X^{n+1} & & \\
 & \swarrow^{i_n} & \downarrow{\scriptstyle p\circ i_n} & & \downarrow & \searrow^{i_{n+1}} & \\
R^n & \xrightarrow{p} & \operatorname{coker}(d^{n-1}_R) & \xrightarrow{b} & \operatorname{coker}(d^{n-1}_R) \amalg_{X^n} X^{n+1} & \longrightarrow & R^{n+1}
\end{array}$$

to define a bounded from below a complex $R^{\bullet}$ of objects from $\mathcal{R}$ and a morphism of complexes $i : X^{\bullet} \to R^{\bullet}$.

To check that $i : X^{\bullet} \to R^{\bullet}$ is a quasi-isomorphism, we fully embed $\mathcal{A}$ in the category of modules over some ring to assume that all our complexes are complexes of modules so we can use set-theoretical definitions of monomorphims and epimorphisms. Consider the morphism of cohomology

$$H(i_n) : H^n(X^{\bullet}) = \ker(d^n_{X^\bullet})/\operatorname{im}(d^{n-1}_{X^\bullet}) \to H^n(R^{\bullet}) = \ker(d^n_{R^\bullet})/\operatorname{im}(d^{n-1}_{R^\bullet}).$$

An element $\bar{r}^n \in H^n(R^{\bullet})$ can be represented by an element in $r^n \in \operatorname{coker}(d^{n-1}_R)$ which is sent to zero under the map $b : \operatorname{coker}(d^{n-1}_R) \to \operatorname{coker}(d^{n-1}_R) \amalg_{X^n} X^{n+1}$.

Since $\operatorname{coker}(d_R^{n-1}) \amalg_{X^n} X^{n+1} = \operatorname{coker}(X^n \to \operatorname{coker}(d_R^{n-1}) \oplus X^{n+1})$, we obtain that $(r^n, 0)$ must be the image of some element from X^n, in particular $r^n = p(i_n(x^n))$ for some $x^n \in X^n$. Obviously, $x^n \in \ker(d^n_{X^\bullet})$. This checks that $H(i_n)$ is surjective. We leave to the reader to check that $H(i_n)$ is injective. □

Proposition 1.3.3. *Let $\mathcal{R}$ be an adapted set of objects for a left exact functor* $\mathbf{F}$. *Then the subcategory $\mathbf{K}^+(\mathcal{R})$ of $\mathbf{K}^+(\mathsf{A})$ is an adapted subcategory.*

Proof. Since the cone of a morphism of complexes in $\mathbf{K}^+(\mathcal{R})$ is an object from $\mathbf{K}^+(\mathcal{R})$, the subcategory $\mathbf{K}^+(\mathcal{R})$ is a triangulated subcategory of $\mathbf{K}^+(\mathsf{A})$.

Property (A1) follows from the definition. Property (A2) follows from Lemma 1.3.2.

It remains to verify (A3). Let us first show that the functor $\mathbf{K}^+(\mathcal{R})_{\text{qis}} \to D^+(\mathsf{A})$ is an equivalence of categories. Applying Lemma 1.3.2, it suffices to prove that this functor is fully faithful. Any morphism $u : X^\bullet \to Y^\bullet$ in $D^+(\mathsf{A})$ of objects from $\mathbf{K}^+(\mathcal{R})$ is represented by a roof $g : X^\bullet \to Z^\bullet, t : Y^\bullet \to Z^\bullet$, where $Z^\bullet$ is an object of $\mathbf{K}(\mathsf{A})$ and t is a quasi-isomorphism. Applying Lemma 1.3.2, we find a quasi-isomorphism $s : Z^\bullet \to W^\bullet$, where $W \in \mathbf{K}^+(\mathcal{R})$. The roof $(s \circ t, s \circ g)$ is a morphism $u' : X^\bullet \to Y^\bullet$ in $\mathbf{K}^+_{\text{qis}}$ such that $\Psi(u') = u$. We leave to the reader to check the injectivity of the map on Hom's defined by Ψ. This proves the assertion.

Obviously, the set of quasi-isomorhisms in $\mathbf{K}^+(\mathsf{A})$ is a localizing set satisfying the additional properties (L4) and (L5). Thus $\mathbf{K}^+(\mathcal{R})_{\text{qis}}$ is a triangulated category and the inclusion functor $\mathbf{K}^+(\mathcal{R}) \to \mathbf{K}^+(\mathsf{A})$ defines a functor of triangulated categories. To show that it is an equivalence of triangulated categories, we have to verify that its quasi-inverse functor is a functor of triangulated categories. This follows from the lemma below. □

Lemma 1.3.4. *A triangle in $\mathbf{K}^+(\mathcal{R})_{\text{qis}}$ isomorphic to a distinguished triangle in $D^+(\mathsf{A})$ is isomorphic to a distinguished triangle with objects in $\mathcal{R}$.*

Proof. Without loss of generality we may assume that a triangle $X^\bullet \xrightarrow{f} Y^\bullet \to Z^\bullet \to X^\bullet[1]$ in $\mathbf{K}(\mathcal{R})$ is isomorphic in $D^+(\mathsf{A})$ to a distinguished triangle of the form $\tilde{X}^\bullet \xrightarrow{\tilde{f}} \tilde{Y}^\bullet \to C(\tilde{f}) \to \tilde{X}^\bullet[1]$.

$$\begin{array}{ccccccc} X^\bullet & \xrightarrow{f} & Y^\bullet & \longrightarrow & Z^\bullet & \longrightarrow & X^\bullet[1] \\ \downarrow{\scriptstyle \phi} & & \downarrow{\scriptstyle \psi} & & \downarrow{\scriptstyle \gamma} & & \downarrow{\scriptstyle \phi[1]} \\ \tilde{X}^\bullet & \xrightarrow{\tilde{f}} & \tilde{Y}^\bullet & \longrightarrow & C(\tilde{f})^\bullet & \longrightarrow & X^\bullet[1] \end{array}$$

The morphism f here is a morphism in the derived category, it is represented by a roof $X^\bullet \xleftarrow{s} W^\bullet \xrightarrow{g} Y^\bullet$, where s is a quasi-isomorphism and $W^\bullet \in \operatorname{Ob}(\mathrm{Cp}^+(\mathcal{R}))$. Let

$$W^\bullet \xrightarrow{g} Y^\bullet \to C(g) \to W^\bullet[1]$$

be a standard distinguished triangle in $\mathbf{K}^+(\mathcal{R})_{\text{qis}}$. We have a morphism $\delta : C(g) = W^\bullet[1] \oplus Y^\bullet \to C(\tilde{f}) = \tilde{X}^\bullet[1] \oplus \tilde{Y}^\bullet$ given as the direct sum of the morphisms $\phi \circ s[1] : W^\bullet[1] \to \tilde{X}^\bullet[1]$ and ψ. The composition $r = \gamma^{-1} \circ \delta : C(g) \to Z^\bullet$ defines a commutative diagram

$$\begin{array}{ccccccc} W^\bullet & \xrightarrow{g} & Y^\bullet & \longrightarrow & C(g) & \longrightarrow & W^\bullet[1] \\ \downarrow{\scriptstyle s} & & \downarrow & & \downarrow{\scriptstyle r} & & \downarrow{\scriptstyle g[1]} \\ X^\bullet & \xrightarrow{f} & Y^\bullet & \longrightarrow & Z^\bullet & \longrightarrow & X^\bullet[1] \end{array}$$

Since s is an isomorphism in $D^+(\mathsf{A})$, we obtain an isomorphism of triangles in $D^+(\mathsf{A})$. Since the upper row is a distinguished triangle in $\mathbf{K}^+(\mathcal{R})_{\text{qis}}$, we are done.

□

From now on we will identify any $A \in \mathsf{Ob}(\mathsf{A})$ with the complex $A^\bullet$ such that $A^0 = A$ and $A^i = 0$ if $i \neq 0$. We will call such a complex an *object-complex*. Note that now $A[n]$ makes sense for any object A. This identification of objects with object-complexes defines a canonical fully faithful functor $\mathsf{A} \to \mathrm{Cp}^b(\mathsf{A})$. Composing it with the functor $\mathrm{Cp}^b(\mathsf{A}) \to \mathbf{K}^b(\mathsf{A})$ we obtain a functor $\mathsf{A} \to \mathbf{K}^b(\mathsf{A})$. It is clear that a morphism $A \to B$ in A is homotopy equivalent to the zero morphism only if it is the zero morphism. This shows that the functor $\mathsf{A} \to \mathbf{K}^b(\mathsf{A})$ is fully faithful. Finally, since $H^\bullet(A) = A$ any quasi-isomorphism of object-complexes is an isomorphism. Thus composing the functor $\mathcal{A} \to \mathbf{K}^b(\mathsf{A})$ with the localization functor $\mathbf{K}^b(\mathsf{A}) \to D^b(\mathsf{A})$ we obtain a fully faithful functor

$$\mathsf{A} \to D^b(\mathsf{A}).$$

It can be used to identify $\mathcal{A}$ with a full subcategory of $D^b(\mathsf{A})$. Let $F : \mathsf{A} \to \mathsf{B}$ be a left exact additive functor and $\mathbf{F}$ be its extension to a functor of triangulated categories $\mathbf{K}(\mathsf{A}) \to \mathbf{K}(\mathsf{B})$. We denote by $\mathbf{R}\mathrm{F}$ the right derived functor $D^+(\mathbf{F}) : \mathcal{D}^+(\mathsf{A}) \to \mathcal{D}^+(\mathsf{B})$ defined by some choice of a an adapted set objects. This defines a functor

$$\mathrm{R}^n F = H^n \circ \mathbf{R}\mathrm{F} : \mathsf{A} \to \mathsf{B}$$

which is called the *n-th right derived functor* of F. If F is right exact we can similarly define the left derived functor $\mathbf{L}F : D^-(\mathsf{A}) \to D^-(\mathsf{B})$ and the *n-th left derived functor*

$$L_n\mathbf{F} = H^{-n} \circ \mathbf{L}\mathrm{F} : \mathsf{A} \to \mathsf{B}.$$

It follows from the construction of the derived functor that

$$\mathrm{R}^n F(A) = D^+(\mathbf{K}^+(F))(A[n]),\ L_n F(A) = D^-(\mathbf{K}^-(F))(A[-n]),$$

where $\mathbf{K}^\pm(F)$ is the canonical extension of F to a functor of triangulated categories $\mathbf{K}^\pm(\mathsf{A}) \to \mathbf{K}^\pm(\mathsf{B})$.

For any distinguished triangle,

$$X^\bullet \to Y^\bullet \to Z^\bullet \to X^\bullet[1]$$

we have the distinguished triangle

$$\mathbf{R}\mathrm{F}(X^\bullet) \to \mathbf{R}\mathrm{F}(X^\bullet) \to \mathbf{R}\mathrm{F}(X^\bullet) \to \mathbf{R}\mathrm{F}(X^\bullet)[1]$$

that defines a long exact sequence of cohomology

$$\cdots \to H^n(\mathbf{R}\mathrm{F}(X^\bullet)) \to H^n(\mathbf{R}\mathrm{F}(Y^\bullet)) \to H^n(\mathbf{R}\mathrm{F}(Z^\bullet)) \to H^{n+1}(\mathbf{R}\mathrm{F}(X^\bullet)) \to \cdots .$$

In particular, a short exact sequence of objects in $\mathcal{A}$

$$0 \to A \xrightarrow{f} B \to C \to 0$$

considered as the distinguished triangle (see Lemma 1.2.7)

$$A \to \mathrm{Cyl}(f) \to C(f) \to A[1]$$

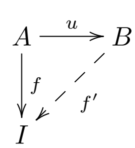
defines a long exact sequence

$$0 \to F(A) \to F(B) \to F(C) \to \mathrm{R}^1F(A) \to \mathrm{R}^1F(B) \to \mathrm{R}^1F(C) \to \cdots .$$

Similarly, a right exact functor defines a long exact sequence

$$\cdots \to L_1F(A) \to L_1F(B) \to L_1F(C) \to F(A) \to F(B) \to F(C) \to 0.$$

Very often we will choose $\mathcal{R}$ to be the set of injective objects in $\mathcal{A}$. Recall that an *injective object* in a category C is an object I which is a projective object in the dual category, i.e. the functor $h_I : \mathsf{C} \to \mathsf{Sets}$ transforms monomorphisms to surjective maps of sets. In other words for any monomorphism $u : A \to B$ in C and a morphism $f : A \to I$ there exists a morphism $f' : B \to I$ such that the diagram



is commutative. If C is an additive category, an injective object is characterized by the property that the functor h_I is exact.

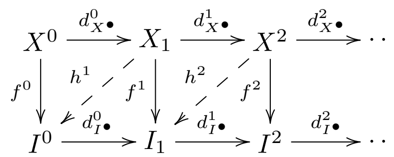
We say that an additive category has *enough injective objects* if any object admits a monomorphism to an injective object.

Theorem 1.3.5. *Suppose that an abelian category $\mathcal{A}$ has enough injective objects. Then the set $\mathcal{I}$ of injective objects is an adapted set for any left exact additive functor F.*

We need to check that for any acyclic complex $I^\bullet$ of injective objects its image under $\mathbf{K}(F)$ is an acyclic complex. Let us show this.

Lemma 1.3.6. *A morphism $f : X^\bullet \to I^\bullet$ of an acyclic bounded from below complex to a complex from $\mathrm{Cp}^+(\mathcal{I})$ is homotopic to zero.*

Proof. We may assume that $I^n = 0, n < 0$. We have to construct a morphism $h : X^\bullet \to I^\bullet[-1]$ such that $d_{I^\bullet}^{n-1} \circ h^n + h^{n+1} \circ d_{X^\bullet}^n = 0$.



By definition of an injective object, the identity morphism $I^0 \to I^0$ extends to a morphism $h^1 : I^1 \to I^0$ such that $h^1 \circ d_{I^\bullet}^0 = f^0$. Because $I^\bullet$ is acyclic, the natural morphism $\mathrm{coker}(d_{I^\bullet}^0) \to I^2$ is a monomorphism. Consider the morphism $d_{I^\bullet}^0 \circ h^1 - f^1$. Since $h^1 \circ d_{X^\bullet}^0 = f^0$, we get

$$(d_{I^\bullet}^0 \circ h^1 - f^1) \circ d_{X^\bullet}^0 = d_{I^\bullet}^0 \circ f^0 - f^1 \circ d_{X^\bullet}^0 = 0.$$

Thus $d_{I^\bullet}^0 \circ h^1 - f^1 : X^1 \to I^1$ defines a morphism $\mathrm{coker}(d_{I^\bullet}^0) \to I^1$ which can be extended to a morphism $h^2 : I^2 \to I^1$. Continuing in this way, we construct a set of morphisms $h = (h^n : I^n \to I^{n-1})$ which as is easy to see define a homotopy between f and 0. $\square$

Applying the lemma to the identity morphism of an acyclic complex in $\mathrm{Cp}^+(\mathcal{I})$, we obtain the following.

Corollary 1.3.7. *An acyclic complex in* $\mathrm{Cp}^+(\mathcal{I})$ *is isomorphic to the zero object in* $\mathbf{K}^+(\mathsf{A})$.

Corollary 1.3.8. *Let* $f : I^\bullet \to X^\bullet$ *be a quasi-isomorphism from an object of* $\mathbf{K}^+(\mathcal{I})$ *to an object from* $\mathbf{K}^+(\mathsf{A})$. *Then* f *admits a left inverse. If* $X^\bullet \in \mathbf{K}(\mathcal{I})$, *then* f *is an isomorphism.*

Proof. We have to find a morphism of complexes $g : X^\bullet \to I^\bullet$ such that $g \circ f$ is homotopic to $\mathbf{id}_{I^\bullet}$. Consider the distinguished triangle

$$I^\bullet \to X^\bullet \to C(f) \to I^\bullet[1].$$

Since f is a quasi-isomorphism, the complex $C(f)$ is acyclic. By Lemma 1.3.6, the image of the morphism $\delta : C(f) \to I^\bullet[1]$ in $\mathbf{K}(\mathsf{A})$ is the zero morphism. Thus there exists a homotopy map $h : C(f) = I^\bullet[1] \oplus X^\bullet \to I^\bullet$ between δ and the zero morphism. On the other hand, we know that δ is the projection to the first summand of $C(f)$. The homotopy h is defined by two morphisms $h_1 : I^\bullet[1] \to I^\bullet$, and $h_2 : J \to I^\bullet$ satisfying

$$(\mathbf{id}_{I^\bullet[1]}, 0) = (h_1[1], h_2[1]) \circ d_{C(f)} + d_{I^\bullet} \circ (h_1, h_2).$$

This gives

$$\mathbf{id}_{I^\bullet[1]}, = h_1 \circ d_{I^\bullet[1]} + d_{I^\bullet} \circ h_1 + h_2 \circ f[1] = h_2 \circ f[1],$$

$$h_2 \circ d_{C(f)} + d_{I^\bullet[1]} \circ h_2.$$

This implies that h_2 is a morphism of complexes and becomes the left inverse of f in $\mathbf{K}(\mathsf{A})$.

Suppose that $X^\bullet \in \mathbf{K}(\mathcal{I})$. Since f has the left inverse, it must be a monomorphism and $h_2 : X^\bullet \to I^\bullet$ must be an epimorphism. Now we replace f with h_2. Since f is a quasi-morphism, we get that h_2 is a quasi-isomorphism. The previous argument shows that h^2 admits a left inverse, hence h_2 is a monomorphism and, since it was an epimorphism, it must be an isomorphism. Therefore f is an isomorphism. □

Remark 1.3.9. The same argument shows that all epimorphisms in $\mathbf{K}(\mathsf{A})$, and in particular in $\mathcal{A}$, split if the category $\mathbf{K}(\mathsf{A})$ is abelian. In fact, assume $f : X^\bullet \to Y^\bullet$ is an epimorphism. Since $\mathbf{K}(\mathsf{A})$ is abelian, and the exact sequence $0 \to X \to \mathrm{Cyl}(f) \to C(f) \to 0$ is isomorphic to the sequence $X \to Y \to C(f) \to 0$ (see Lemma 1.2.5), we obtain that $C(f) = 0$ in $\mathbf{K}(\mathsf{A})$. This implies that $C(f) \to X^\bullet[1]$ is the zero morphism in $\mathbf{K}(\mathsf{A})$. Now we us the homotopy and the previous argument to construct the left inverse of u.

The previous corollary shows that the localization morphism $\mathbf{K}(\mathcal{I}) \to \mathbf{K}^+(\mathcal{I})_{\mathrm{qis}}$ is an equivalence of categories. Thus we obtain

Theorem 1.3.10. *Assume that $\mathcal{A}$ has enough injective objects. Then*

$$\mathbf{K}^+(\mathcal{I}) \approx D^+(\mathsf{A}).$$

An object-complex A is a special case of a complex $X^\bullet \in \mathrm{Cp}(\mathsf{A})^+$ such that $H^i(X^\bullet) = 0, i \neq 0$. A complex of this sort with $X^i = 0$ for $i < 0$ and $H^0(X^\bullet) \cong X^0$ is called a *resolution* of X^0. If $\mathcal{R}$ is a set of objects and all $X^i, i \neq 0$, belong to $\mathcal{R}$ we call it an $\mathcal{R}$-resolution. For example, we can define *injective resolutions*.

Let $X^\bullet$ be a resolution of A. A choice of an isomorphism $A \to H^0(X^\bullet)$ defines an acyclic complex

$$0 \to A \to X^0 \to X^1 \to \cdots \to X^n \to \cdots .$$

Proposition 1.3.11. *Let $\mathcal{A}$ be an abelian category with enough injective objects. For each object A in $\mathcal{A}$ there exists an injective resolution $I^\bullet$ of A. Any morphism $f : A \to B$ in $\mathcal{A}$ can be extended to a morphism of injective resolutions, and this extension is unique up to homotopy.*

Proof. The existence of an injective resolution is obvious. We first find a monomorphism $d^0 : A \to I^1$, then find a monomorphism $\mathrm{coker}(d^0) \to I_2$ and so on. We search for a commutative diagram

$$\begin{array}{ccccccccccccc}
A & \xrightarrow{e_A} & I^0 & \longrightarrow & I^1 & \longrightarrow & \cdots & \longrightarrow & I^{n-1} & \longrightarrow & I^n & \longrightarrow & \cdots . \\
\downarrow{\scriptstyle f} & & \downarrow{\scriptstyle f^0} & & \downarrow{\scriptstyle f^1} & & & & \downarrow{\scriptstyle f^{n-1}} & & \downarrow{\scriptstyle f^n} & & \\
B & \xrightarrow{e_B} & J^0 & \longrightarrow & J^1 & \longrightarrow & \cdots & \longrightarrow & J^{n-1} & \longrightarrow & J^n & \longrightarrow & \cdots
\end{array}$$

Since $e_A : A \to I^0$ is a monomorphism, and J^0 is injective, the composition $e_B \circ f : A \to J^0$ extends to a morphism $f^0 : I^0 \to J^0$ such that $f^0 \circ e_A = e_B \circ f$. Assume that we can define f^n in this way. Since $d_J^n \circ f^n \circ d_I^{n-1} = d_J^n \circ d_J^{n-1} \circ f^{n-1} = 0$ we see that f^n defines a morphism from $\mathrm{im}(d_I^n)$ to J^{n+1}. Since I^{n+1} is injective we can extend it to a morphism $f^{n+1} : I^{n+1} \to J^{n+1}$. This extends f to a morphism f^{n+1}. This proves the existence of an extension.

Let us prove the second assertion. Let $\tilde{f}, \tilde{g}$ be two extensions of $f : A \to B$ to morphisms of the resolutions $I^\bullet \to J^\bullet$. Obviously, $\tilde{f} - \tilde{g}$ induce the zero morphism on the cohomology. Then $\tilde{f} - \tilde{g}$ is an extension of the zero morphism $A \to B$. So, we may assume that $f : A \to B$ is the zero morphism. We need to show that the extension $\tilde{f}$ is homotopy to zero. Since $A \to B \to J^0$ is zero, we have a morphism $\mathrm{coker}(d_I^0) \to J^0$. Since J^0 is injective, it extends to a morphism $h^2 : I^1 \to J^0$. Clearly, $f^0 = h^2 \circ d_I^1 + h^1 \circ d_J^0$, where $h^1 = 0$. Assume we can construct homotopy morphisms $(h^i), i \leq n : I^n \to J^{n-1}$. Thus $f^n = h^{n+1} \circ d_I^n + d_J^{n-1} \circ h^n$. Let $\alpha = f^{n+1} - h^{n+1} \circ d_I^n : I^{n+1} \to J^{n+1}$. It is easy to see that $d_J^n \circ \alpha = 0$. Thus α factors through $\mathrm{im}(d_I^{n+1})$ and then extends to $h^{n+1} : I^{n+1} \to J^n$. We have $f^{n+1} = k^{n+1} \circ d_I^{n+1} + d_J^n \circ t^{n+1}$ and, by induction, we are done.

□

Corollary 1.3.12. *Suppose A has enough injective objects. Then A is equivalent to the full subcategory of $\mathbf{K}^+(\mathcal{I})$ that consists of injective resolutions.*

Proposition 1.3.13. *Let $F : \mathsf{A} \to \mathsf{B}$ be a left exact additive functor of abelian categories. Suppose $\mathcal{A}$ has enough injective objects. There is an isomorphism of functors*

$$F \cong \mathrm{R}^0 F.$$

Proof. We take the set of injective objects as an adapted set of F and define the derived functor accordingly. Let $I^\bullet$ be an injective resolution of A. It follows from the definitions that $\mathrm{R}^0F(A) = H^0(F(I^\bullet))$. Since F is left exact, $F(I^0) = F(A) \to F(I^1)$ is a monomorphism. This shows that $H^0(F(I^\bullet)) \cong F(A)$. To make this isomorphism functorial, we use that any morphism $A \to A'$ in A defines a unique morphism in $\mathbf{K}(\mathsf{A})$ of their injective resolutions. Since taking cohomology H^0 is a functor $\mathbf{K}(\mathsf{A}) \to \mathsf{B}$ we see that R^0F is isomorphic to F. □

Note that, if the derived functor is defined by using an adapted set of objects, we always have an isomorphism $\mathrm{R}^0F(A) \cong F(A)$ but we do not have an isomorphism of functors.

Example 1.3.14. We assume that A has enough injective objects. Consider the additive functor $F = \mathrm{Hom}_{\mathsf{A}}(A, ?) : B \to \mathrm{Hom}_{\mathsf{A}}(A, B)$ from A to Ab. By definition of a monomorphism, F is a left exact functor. We denote by

$$\mathbf{R}\mathrm{Hom}(A, ?) : D^+(\mathsf{A}) \to D^+(\mathsf{Ab})$$

its right derived functor. The functors $\mathrm{Ext}^i(A, ?) : \mathsf{A} \to \mathsf{Ab}$ are defined by

$$\mathrm{Ext}^i_{\mathsf{A}}(A, ?) := \mathrm{R}^i\mathrm{Hom}_{\mathsf{A}}(A, ?).$$

Let us recall the definition. First we extend the functor $\mathrm{Hom}_{\mathsf{A}}(A, ?)$ to a functor $\mathbf{K}\mathrm{Hom}_{\mathsf{A}}(A, ?) : \mathbf{K}^+(\mathsf{A}) \to \mathbf{K}^+(\mathsf{Ab})$. By definition,

$$\mathbf{K}\mathrm{Hom}_{\mathsf{A}}(A, X^\bullet) = \mathrm{Hom}^\bullet(A, X^\bullet),$$

where $\mathrm{Hom}^i(A, X^\bullet) = \mathrm{Hom}_{\mathsf{A}}(A, X^i) = \mathrm{Hom}_{\mathbf{K}(\mathsf{A})}(A, X^\bullet[i])$. To extend it to a derived functor, we replace $X^\bullet$ with a quasi-isomorphic complex of injective objects $I^\bullet$ and apply the extended functor to $I^\bullet$ to get

$$\mathrm{Hom}^i_{\mathsf{A}}(A, X^\bullet) := H^i(\mathbf{R}\mathrm{Hom}(A, I^\bullet)) \cong \mathrm{Hom}_{\mathbf{K}(\mathsf{A})}(A, I^\bullet[n]))$$
$$\cong \mathrm{Hom}_{D^+(\mathsf{A})}(A, X^\bullet[n])).$$

If $X^\bullet = B$ is an object-complex, then $I^\bullet$ is an injective resolution of B, and $\mathrm{Hom}^i_{\mathsf{A}}(A, B)$ coincides with the familiar definition of $\mathrm{Ext}^i_{\mathsf{A}}(A, B)$ from homological algebra.

More generally, let $A^\bullet$ and $B^\bullet$ be two complexes in A, we define the complex of abelian groups $\mathrm{Hom}^\bullet(A^\bullet, B^\bullet) = (\mathrm{Hom}^\bullet(A^\bullet, B^\bullet)^n, d^n)$ in A by setting

$$\mathrm{Hom}^\bullet(A^\bullet, B^\bullet)^n = \prod_{i\in\mathbb{Z}} \mathrm{Hom}_{\mathsf{A}}(A^i, B^{i+n}),$$

$$d^n(f_i) = d_{B^\bullet} \circ f_i - (-1)^n f_i \circ d_{A^\bullet},\ \ f_i : A^i \to B^{i+n}.$$

Note that the kernel of d^n consists of morphisms $A^\bullet \to B^\bullet[n]$ in $\mathrm{Cp}(\mathsf{A})$ and the image of d^{n-1} consists of morphisms of complexes homotopic to zero. Thus

$$H^n(\mathrm{Hom}^\bullet(A^\bullet, B^\bullet)) \cong \mathrm{Hom}_{\mathbf{K}(\mathsf{A})}(A^\bullet, B^\bullet[n]) \cong \mathrm{Hom}_{\mathbf{K}(\mathsf{A})}(A^\bullet[-n], B^\bullet).$$

Via the composition of morphisms in $\mathrm{Cp}(\mathsf{A})$ we get a bi-functor

$$\mathrm{Hom}^\bullet(?, ?) : \mathrm{Cp}(\mathsf{A})^{\mathrm{op}} \times \mathrm{Cp}(\mathsf{A}) \to \mathrm{Cp}(\mathsf{A})$$

that can be extended to a bifunctor

$$\mathrm{Hom}^\bullet(?, ?) : \mathbf{K}(\mathsf{A})^{\mathrm{op}} \times \mathbf{K}(\mathsf{A}) \to \mathbf{K}(\mathsf{Ab}).$$

It follows easily from the definition that both partial functors are δ-functors. If $\mathcal{A}$ has a set of right adapted objects for the first partial functor (e.g. $\mathcal{A}$ has enough projective objects), then we can extend $\mathrm{Hom}^\bullet(?, ?)$ to a bi-functor

$$\mathbf{R}\mathrm{Hom}^\bullet(?, ?) : D^-(\mathsf{A})^{\mathrm{op}} \times D^+(\mathsf{A}) \to D^b(\mathsf{Ab}).$$

The composition of both partial functors with the cohomology functor $H^i : D^b(\mathsf{Ab}) \to \mathsf{Ab}$ are isomorphic functors (so we may choose one, if only one partial functor is defined) and we set

$$\mathrm{Hom}^i_{\mathsf{A}}(A^\bullet, B^\bullet) := H^i(\mathrm{Hom}^\bullet(A^\bullet, B^\bullet)).$$

If the second partial derived functor exists, we have

$$\mathrm{Hom}^i_{\mathsf{A}}(A^\bullet, B^\bullet) = \mathrm{Hom}_{D^+(\mathsf{A})}(A^\bullet, B^\bullet[i]).$$

If the second partial derived functor exists, we have

$$\mathrm{Hom}^i_{\mathsf{A}}(A^\bullet, B^\bullet) = \mathrm{Hom}_{D^-(\mathsf{A})}(A^\bullet[-i], B).$$

The restriction of the bifunctors $\mathrm{Hom}^i_{\mathsf{A}}(?,?)$ to $\mathsf{A}^{\mathrm{op}} \times \mathsf{A}$ and taking the cohomology, we get the familiar bifunctors $\mathrm{Ext}^i_{\mathsf{A}}(?,?)$.

Example 1.3.15. Let $(X, \mathcal{O}_X)$ be a ringed space. For any sheaf of right $\mathcal{O}_X$-modules $\mathcal{M}$ and a sheaf of left $\mathcal{O}_X$-modules $\mathcal{N}$ one can define its *tensor product* $\mathcal{M} \otimes_{\mathcal{O}_X} \mathcal{N}$. This is a just an abelian sheaf on X. If furthermore, $\mathcal{M}$ (resp. $\mathcal{N}$) has a structure of a $\mathcal{O}_X$-bimodule, then the tensor product is a sheaf of left (resp. right) $\mathcal{O}_X$-modules. In particular, this is true if $\mathcal{O}_X$ is a sheaf of commutative rings. Fix $\mathcal{M}$ and consider the functor

$$\mathcal{M}\otimes_{\mathcal{O}_X} : \mathrm{Mod}(\mathcal{O}_X) \to \mathcal{S}h_X^{\mathrm{ab}},\ \mathcal{N} \to \mathcal{M} \otimes_{\mathcal{O}_X} \mathcal{N}.$$

This functor is right exact and, after passing to the homotopy category of complexes defines a δ-functor. Similarly, we define the right exact functor

$$\otimes_{\mathcal{O}_X}\mathcal{N} : \mathrm{Mod}(\mathcal{O}_X^{\mathrm{op}}) \to \mathcal{S}h_X^{\mathrm{ab}},\ \mathcal{M} \to \mathcal{M} \otimes_{\mathcal{O}_X} \mathcal{N}.$$

A sheaf of $\mathcal{O}_X$-modules is called *flat* if the functor $\mathcal{M}\otimes_{\mathcal{O}_X}$ is exact. An example of a flat sheaf is a locally free sheaf (maybe of infinite rank). Any sheaf of $\mathcal{O}_X$-modules admits a flat resolution. One uses the sheaves $\mathcal{O}_U$ defined by

$$(\mathcal{O}_U)_x = \begin{cases} \mathcal{O}_{X,x} & \text{if } x \in U\ , \\ 0 & \text{otherwise,} \end{cases}$$

where U is an open subset. Since we have a natural bijection $\mathrm{Hom}_{\mathcal{O}_X}(\mathcal{O}_U, \mathcal{F}) \to \mathcal{F}(U)$, it is easy to see that the sheaf $\oplus_U \mathcal{O}_U$ is a projective generators in $\mathrm{Mod}(\mathcal{O}_X)$.

The set of flat sheaves is a right adapted set for the functor $\mathcal{M}\otimes_{\mathcal{O}_X}$ and one defines its left derived functor

$$\mathcal{M}\overset{\mathbf{L}}{\otimes}_{\mathcal{O}_X} : D^-(\mathrm{Mod}(\mathcal{O}_X)) \to D^-(\mathcal{S}h_X^{\mathrm{ab}}).$$

By definition,

$$\mathrm{Tor}_n^{\mathcal{O}_X}(\mathcal{M}, ?) = H^{-n}(\mathcal{M}\overset{\mathbf{L}}{\otimes}_{\mathcal{O}_X}).$$

Replacing $\mathcal{N}$ by its flat resolution

$$\cdots \to \mathcal{P}_2 \to \mathcal{P}_1 \to \mathcal{P}_0 \to \mathcal{N} \to 0,$$

we obtain

$$\mathrm{Tor}_n^{\mathcal{O}_X}(\mathcal{M}, \mathcal{N}) = H^{-n}(\cdots \to \mathcal{M} \otimes \mathcal{P}_2 \to \mathcal{M} \otimes \mathcal{P}_1 \to \mathcal{M} \otimes \mathcal{P}_0 \to \mathcal{M} \otimes \mathcal{N}).$$

If X is a point and $\mathcal{O}_X = R$ is a commutative ring, this is a familiar definition from commutative algebra.

For any two bounded complexes $\mathcal{M}^\bullet, \mathcal{N}^\bullet$ in $\mathrm{Cp}^*(\mathrm{Mod}(\mathcal{O}_X))$, one defines their *tensor product* by

$$\mathcal{M}^\bullet \otimes_{\mathsf{A}} \mathcal{N}^\bullet = ((\mathcal{M}^\bullet \otimes_{\mathsf{A}} \mathcal{N}^\bullet)^n, d^n), \tag{1.4}$$

where $(\mathcal{M}^\bullet \otimes_{\mathsf{A}} \mathcal{N}^\bullet)^n = \oplus_{i+j=n} \mathcal{M}^i \otimes \mathcal{N}^j$, and $d^n(x^i \otimes y^j) = d_{\mathcal{M}^\bullet}(x^i) \otimes y^j + (-1)^n x^i \otimes d_{\mathcal{N}^\bullet}(y^j)$. If $\mathcal{A}$ has enough flat objects, we can extend this definition to the derived category to define

$$\mathcal{M}^\bullet \overset{\mathbf{L}}{\otimes} \mathcal{N}^\bullet \in D^-(\mathcal{S}h_X^{\mathrm{ab}}).$$

If $\mathcal{M}^\bullet = \mathcal{M}, \mathcal{N}^\bullet = \mathcal{N}$ are object-complexes, we define

$$\mathrm{Tor}_i^{\mathcal{A}}(\mathcal{M}, \mathcal{N}) := H^{-i}(\mathcal{M} \overset{\mathbf{L}}{\otimes} \mathcal{N}) = H^{-i}(\mathcal{M} \overset{\mathbf{L}}{\otimes} \mathcal{N}).$$

It follows from the definitions that the left derived functor

$$\mathcal{M} \overset{\mathbf{L}}{\otimes}_{\mathcal{O}_X} : D^-(\mathrm{Mod}(\mathcal{O}_X)) \to D(\mathcal{S}h_X^{\mathrm{ab}})$$

of the functor $\mathcal{M} \otimes_{\mathcal{O}_X}$ coincides with the functor $\mathcal{N}^\bullet \to \mathcal{M} \overset{\mathbf{L}}{\otimes} \mathcal{N}^\bullet$. We also get that

$$\mathrm{Tor}_i^{\mathcal{A}}(\mathcal{M}, \mathcal{N}) = H^{-i}(\mathcal{M} \overset{\mathbf{L}}{\otimes}_{\mathcal{O}_X} \mathcal{N}).$$

Note that one can compute $\mathrm{Tor}_i^{\mathcal{O}_X}(\mathcal{M}, \mathcal{N})$ by using either flat resolutions of $\mathcal{M}$ or $\mathcal{N}$, the result is the same.

Example 1.3.16. Let $f : X \to Y$ be a morphism of ringed spaces and

$$f_* : \mathrm{Mod}(\mathcal{O}_X) \to \mathrm{Mod}(\mathcal{O}_Y)$$

be the direct image functor, where $f_*(\mathcal{M})$ is the sheaf $U \to \mathcal{M}(f^{-1}(U))$. This functor is left exact. An injective object in $\mathrm{Mod}(\mathcal{O}_X)$ is a sheaf $\mathcal{I}$ whose stalks $\mathcal{I}_x$ are injective $\mathcal{O}_{X,x}$-modules. The category $\mathrm{Mod}(\mathcal{O}_X)$ admits enough injective objects (see [Hartshorne], Chap. III, Prop.2.2). Also an injective $\mathcal{O}_X$-module is *flabby*, i.e. the restriction maps $\mathcal{F}(U) \to \mathcal{F}(V)$ are surjective. It follows from the definition of $f_*\mathcal{F}$ that f_* is exact on the subcategory of flabby sheaves. Thus we can define the right derived functor

$$\mathbf{R}f_* : D(\mathrm{Mod}(\mathcal{O}_X) \to D(\mathrm{Mod}(\mathcal{O}_Y)).$$

Specializing to object-complexes we define the right derived functors

$$\mathrm{R}^i f_* : \mathrm{Mod}(\mathcal{O}_X) \to \mathrm{Mod}(\mathcal{O}_X),\ \mathcal{F} \to H^i(\mathbf{R}f_*(\mathcal{F})).$$

In particular, taking Y to be a point so that $\mathcal{O}_Y$ is defined by a ring R, we obtain that $f_*(\mathcal{F}) = \Gamma(X, \mathcal{F})$ is the R-module of global sections. Also $\mathrm{R}^i f_*(\mathcal{F}) = H^i(X, \mathcal{F})$, is the *i-th cohomology* R-module. One can show that $\mathrm{R}^i f_*(\mathcal{M})$ is the sheaf associated with the presheaf $U \to H^i(f^{-1}(U), \mathcal{M})$.

We also have the inverse image functor

$$f^*: \mathrm{Mod}(\mathcal{O}_Y) \to \mathrm{Mod}(\mathcal{O}_X),\ \mathcal{M} \to \mathcal{M} \otimes_{\mathcal{O}_Y} \mathcal{O}_X.$$

Since we can compute Tor's by using either of its two arguments, we check that flat sheaves is an adapted set of objects for f^*. This allows one to define the left derived functor

$$\mathbf{L}f^*: D^-(\mathrm{Mod}(\mathcal{O}_X)) \to D^-(\mathrm{Mod}(\mathcal{O}_X)).$$

1.4 Spectral sequences

Suppose we have two left exact additive functors $F : \mathsf{A} \to \mathsf{B}, G : \mathsf{B} \to \mathsf{C}$ of abelian categories. We would like to compare the functors $\mathbf{R}(G \circ F)$ with $\mathbf{R}\mathrm{G} \circ \mathbf{R}\mathrm{F}$, provided that both of them are defined.

Theorem 1.4.1. *Suppose A has a left adapted set of objects $\mathcal{R}_A$ for a left exact functor F, and $F(\mathcal{R}_A)$ is contained in an adapted set of objects $\mathcal{R}_B$ with respect to G. In this case there is natural isomorphism of functors*

$$\mathbf{R}(\mathrm{G} \circ \mathrm{F}) \to \mathbf{R}\mathrm{G} \circ \mathbf{R}\mathrm{F}.$$

Proof. It follows from the definition that $\mathcal{R}_\mathsf{A}$ is adapted for $G \circ F$. Thus the derived functor of the composition exists. We have canonical morphism of functors

$$Q_\mathsf{A} \circ \mathbf{K}(G \circ F) = Q_\mathsf{A} \circ \mathbf{K}(G) \circ \mathbf{K}(F) \to \mathbf{R}\mathrm{G} \circ Q_\mathsf{B} \circ \mathbf{K}(F) \to \mathbf{R}\mathrm{G} \circ \mathbf{R}\mathrm{F} \circ Q_\mathsf{A}.$$

On the other hand, we have a canonical morphism of functors $Q_\mathsf{A} \circ \mathbf{K}(G \circ F) \to \mathbf{R}(\mathrm{G} \circ \mathrm{F}) \circ Q_\mathsf{A}$. By the universality property we obtain a canonical morphism of functors $\mathbf{R}(\mathrm{G} \circ \mathrm{F}) \to \mathbf{R}\mathrm{G} \circ \mathbf{R}\mathrm{F}$. By construction of the derived functor, we replace any complex $X^\bullet$ in $\mathbf{K}(\mathsf{A})$ by quasi-isomorphic complex from $\mathbf{K}(\mathcal{R}_\mathsf{A})$ and send it to $\mathbf{K}(\mathsf{B})$ via F. Since $F(X^\bullet)$ belongs to $\mathbf{K}(\mathcal{R}_\mathsf{B})$, the value of $\mathbf{R}\mathrm{G}$ on $F(X^\bullet)$ is equal to $G(F(X^\bullet))$. Also it coincides with $(G \circ F)(K^\bullet F)$ since $\mathcal{R}_\mathsf{A}$ is adapted for $G \circ F$. This defines an isomorphism of the functors. □

To compute explicitly $\mathrm{R}^n(G \circ F)$ in terms of $\mathrm{R}^q G \circ \mathrm{R}^p F$ one uses spectral sequences.

Recall the definition of a spectral sequence. Let A be an abelian category. A *spectral sequence* in A is a collection $(E_r^\bullet, H^n), r, n \in \mathbb{Z},\ r \geq 1$ of complexes $(E_r^\bullet, d_r)$ and a collection of objects H^n, called the *limit* of the spectral sequence, with filtration of subobjects $F^\bullet = (\cdots \to F^i(H^n) \xrightarrow{u_i} F^{i-1}(H^n) \to \cdots)$, where u_i are monomorphisms. The following properties must be satisfied:

(SS1) each $E_r^n \cong \oplus_{p,q \in \mathbb{Z}, p+q=n} E_r^{p,q}$;

(SS2) the composition of $E_r^{p,q} \to E_r$ with d_r defines a morphism

$$d_r^{p,q}: E_r^{p,q} \to E_r^{p+r,q-r+1};$$

(SS3) there are isomorphisms

$$\alpha_r^{p,q} : \ker(\operatorname{coker}(d_r^{p-r,q+r-1}) \to E_r^{p+r,q-r+1}) \cong E_{r+1}^{p,q}.$$

(SS4) there exists r_0 such that $d_r^{p,q}$ and $d_r^{p-r,q+r-1}$ are equal to zero for $r \geq r_0$, thus $E_r^{p,q} \cong E_{r_0}^{p,q}$ for $r \geq r_0$ (we say that the spectral sequence *degenerates* at E_{r_0}). The isomorphic objects $E_r, t \geq r_0$, are denoted by $E_\infty^{p,q}$

(SS5) for each $p, q \in \mathbb{Z}$, there is an isomorphism $\beta_{p,q} : E_\infty^{p,q} \to \operatorname{Gr}^p(H^{p+q}) := \operatorname{coker}(F^{p+1}(H^{p+q}) \to F^p(H^{p+q}))$.

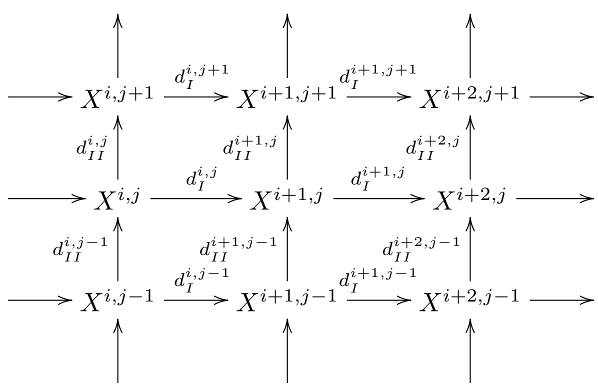
One often uses the following notation for a spectral sequence

$$E_r^{p,q} \Longrightarrow H^n.$$

We leave to the reader to define *morphisms of spectral sequences*.

Example 1.4.2. A *double complex* or a *bicomplex* in an abelian category A is a diagram in A on the graph with set of vertices $\mathbb{Z} \times \mathbb{Z}$ and the set of arrows from $(i, j) \to (i, j+1)$ and $(i, j) \to (i+1, j)$. Each arrow defines the differential $d_I^{i,j} : X^{i,j} \to X^{i+1,j}$ with $d_I^{i+1,j} \circ d_I^{i,j} = 0$ and $d_{II}^{i,j} : X^{i,j} \to X^{i,j+1}$ with $d_{II}^{i,j+1} \circ d_{II}^{i,j} = 0$. We impose the commuting relation

$$d_{II}^{i+1,j} \circ d_I^{i,j} = d_I^{i,j+1} \circ d_{II}^{i,j}. \tag{1.5}$$



Restricting the diagram to the set $\mathbb{Z} \times \{q\}$ we obtain a complex $X^{\bullet,q}$ with (horizontal) differentials $d_I^q : X^{\bullet,q} \to X^{\bullet,q}[1]$ formed by $d_I^{p,q} : X^{p,q} \to X^{p+1,q}$. Restricting the diagram to the set $\{p\} \times \mathbb{Z}$ we obtain a complex $X^{p,\bullet}$ with (vertical) differentials $d_{II}^p : X^{p,\bullet} \to X^{p,\bullet}[1]$ formed by $d_{II}^{p,q} : X^{p,q} \to X^{p,q+1}$.

The relation (1.5) allows one to consider a double complex as a two-way complex in the category $\operatorname{Cp}(\mathsf{A})$. The first complex $X_I^{\bullet,\bullet}$ is formed by the "column" complexes $X^{p,\bullet}$ with differentials $d_I^q : X^{p,\bullet} \to X^{p+1,\bullet}$. The second complex $X_{II}^{\bullet,\bullet}$ if formed by the "row" complexes $X^{\bullet,q}$ with differentials $d_{II}^p : X^{\bullet,q} \to X^{\bullet,q+1}$.

A double complex defines the associated *diagonal complex* or *total complex*

$$\operatorname{tot}(X^{\bullet,\bullet}) = (X^n, d^n),\ X^n = \oplus_{p+q=n} X^{p,q},$$

$$d^n(x^{p,q}) = d_I^{p,q}(x^{p,q}) + (-1)^p d_{II}^{p,q}(x^{p,q}).$$

Note that we need the sign change in order to get

$$d^2 = (d_I + \bar{d}_{II})(d_I + \bar{d}_{II}) = 0,$$

where $\bar{d}_{II}^{p,q} = (-1)^p d_{II}^{p,q}$.

Let

$$H_I(X^{\bullet,\bullet}) = (H^p(X_I^{\bullet,\bullet})),\ H_{II}(X^{\bullet,\bullet}) = (H^q(X_{II}^{\bullet,\bullet})),$$

where the cohomology are taken in the category of complexes $\mathrm{Cp}(\mathsf{A})$. By definition, $H_I^p(X^{\bullet,\bullet}) = (H_I^{p,q}, q \in \mathbb{Z})$, where

$$H_I^{p,q} = \ker(\mathrm{coker}(d_I^{p,q}) \to X^{p+1,q}).$$

Similarly, $H_{II}^p(X^{\bullet,\bullet}) = (H_{II}^{p,q}(X^{\bullet,\bullet}), p \in \mathbb{Z})$, where

$$H_{II}^{p,q}(X^{\bullet,\bullet}) = \ker(\mathrm{coker}(d_{II}^{p,q}) \to X^{p,q+1}).$$

The differential morphisms $d_{II}^{p,q}$ induce a differential $H_I^{p,q} \to H_I^{p,q+1}$ in $H_I^p(X^{\bullet,\bullet})$ and we can take the cohomology of this complex to define the objects $H_{II}^q(H_I^p(X^{\bullet,\bullet}))$. Similarly we define the cohomology objects $H_I^p(H_{II}^q(X^{\bullet,\bullet})$.

Next we define a decreasing filtration in the total complex $\mathrm{tot}(X^{\bullet,\bullet})$. First set

$$F_I^p(\mathrm{tot}(X^{\bullet,\bullet})^n) = \bigoplus_{i+j=n, i\geq p} X^{i,j},\ F_{II}^q(\mathrm{tot}(X^{\bullet,\bullet})^n) = \bigoplus_{i+j=n, j\geq q} X^{i,j},$$

and let $F_I^p(E^n)$ be the image of $F_I^p(X^\bullet)^n$ under the morphism $H^n(F^p(\mathrm{tot}(X^{\bullet,\bullet})^n) \to \mathrm{tot}(X^{\bullet,\bullet})^n)$. This defines the complexes $F_I^p(\mathrm{tot}(X^{\bullet,\bullet}))^\bullet$ and $F_{II}^q(\mathrm{tot}(X^{\bullet,\bullet})^\bullet$. Their differentials are induced by the differential of the total complex. Also these differentials define the complexes

$${}^I\mathrm{Gr}_r^p = \mathrm{coker}(F_I^{p+r}(\mathrm{tot}(X^{\bullet,\bullet})^\bullet) \to F_I^p(\mathrm{tot}(X^{\bullet,\bullet})^\bullet),$$

$${}^{II}\mathrm{Gr}_r^q = \mathrm{coker}(F_{II}^{q+r}(\mathrm{tot}(X^{\bullet,\bullet})^\bullet) \to F_{II}^q(\mathrm{tot}(X^{\bullet,\bullet})^\bullet)$$

For example, ${}^I\mathrm{Gr}_1^p \cong X^{p,\bullet}, {}^{II}\mathrm{Gr}_1^q \cong X^{\bullet,q}$. Let

$$\begin{aligned} {}^IZ_r^{p,q} &= \mathrm{im}(H^{p+q}({}^I\mathrm{Gr}_r^p) \to H^{p+q}({}^I\mathrm{Gr}_1^p)), \\ {}^IB_r^{p,q} &= \mathrm{im}(H^{p+q-1}({}^I\mathrm{Gr}_{r-1}^{p-r+1}) \to H^{p+q}({}^I\mathrm{Gr}_1^p)), \\ {}^IE_r^{p,q} &= \mathrm{coker}(B_r^{p,q} \to Z_r^{p,q}). \end{aligned}$$

Similarly, we define ${}^{II}E_r^{p,q}$. The projections $F_I^p(\mathrm{tot}(X^{\bullet,\bullet})^\bullet \to {}^I\mathrm{Gr}_r^p$ induce the morphisms of the cohomology ${}^IZ_r^{p,q} \to {}^IZ_r^{p+r,q-r+1}$ and define the differentials

$${}^Id_r^{p,q} : {}^IE_r^{p,q} \to {}^IE_r^{p+r,q-r+1}.$$

Assume now that there exist $p_+(n)$ and $p_-(n)$ such that

$$F_I^{p_+(n)}(\mathrm{tot}(X^{\bullet,\bullet})^n) = \mathrm{tot}(X^{\bullet,\bullet})^n, \quad F_I^{p_-(n)}(\mathrm{tot}(X^{\bullet,\bullet})^n) = 0.$$

Then for any (p,q) and $r_0 = \max\{p_+(p+q+1) - p_-(p+q)+1, p_+(p+q) - p_-(p+q-1)+1\}$

$$d_r^{p,q} = d_r^{p-r,q+r-1} = 0, r \geq r_0.$$

Thus we can define the groups ${}^I E_\infty^{p,q}$. We have

$${}^I E_\infty^{p,q} \cong \mathrm{coker}(F_I^{p+1}H^{p+q} \to F_I^p H^{p+q}).$$

This defines the *spectral sequences of the double complex*

$${}^I E_1^{p,q} \Longrightarrow E^n, \ {}^{II} E_1^{q,p} \Longrightarrow E^n.$$

It follows from the definition that

$${}^I E_1^{p,q} = H_{II}(X^{\bullet,\bullet})^{p,q}, \ {}^{II} E_1^{q,p} = H_I(X^{\bullet,\bullet})^{p,q}. \tag{1.6}$$

Also we have

$${}^I E_2^{p,q} = H_I^p(H_{II}^q(X^{\bullet,\bullet})), \ {}^{II} E_2^{q,p} = H_{II}^q(H_I^p(X^{\bullet,\bullet})).$$

Example 1.4.3. Let V be a compact complex manifold, and $\mathcal{E}_V^n$ be the sheaf of smooth complex differential n-forms on V. By writing the coordinate functions z_i as $z_i = x_i + \sqrt{-1}y_i$ we can express each local section of $\mathcal{E}_V^n$ as as a sum of forms of type (p,q) of type $\sum a_{I,J}(z,\bar{z})dz_I \wedge d\bar{z}_J$, where $dz_I = dz_{i_1} \wedge \cdots \wedge dz_{i_p}, d\bar{z}_J = d\bar{z}_{i_1} \wedge \cdots \wedge d\bar{z}_{j_q}$. This gives a direct sum decomposition of sheaves

$$\mathcal{E}_V^n = \bigoplus_{p+q+n} \mathcal{E}_V^{p,q}.$$

The differential $d : \mathcal{E}_V^n \to \mathcal{E}_V^{n+1}$ can be written in the form $d = d' + d''$, where d' is the composition of d and the projection to the $(p+1,q)$-summand. Similarly, d'' is the composition of d and the projection to the $(p,q+1)$-summand. Since $d^2 = 0$, we get $d' \circ d'' + d'' \circ d' = 0$. This shows that $(\mathcal{E}_V^{p,q}, d', (-1)^p d'')$ is a double complex in the category of abelian sheaves with total complex $(\mathcal{E}_V^n, d)$. By *Dolbeault Theorem*, each column complex $\mathcal{E}_V^{p,\bullet}$ represents a resolution of the sheaf Ω_V^p of holomorphic p-forms on V. Each row complex $\mathcal{E}_V^{\bullet,q}$ represents a resolution of the sheaf $\bar{\Omega}_V^q$ of anti-holomorphic q-forms on V.

The total complex $\mathrm{tot}(\Omega_X^{\bullet,\bullet})$ defines the *De Rham complex*, a resolution of the constant sheaf

$$0 \to (\mathbb{C})_V \xrightarrow{d} \mathcal{E}^1 \xrightarrow{d} \mathcal{E}^2 \xrightarrow{d} \cdots.$$

Applying the functor of global sections, we obtain a double complex $\Gamma^{p,q} = \Gamma(\Omega_V^{p,q})$ and its total complex

$$0 \to C \xrightarrow{d} \Gamma(\mathcal{E}^1) \xrightarrow{d} \Gamma(\mathcal{E}^2) \xrightarrow{d} \cdots.$$

By *De Rham's Theorem*, its nth cohomology are the cohomology $H^n(M,\mathbb{C})$. By Dolbeault Theorem, we have

$$H_I^{p,q}(\Gamma^{\bullet,\bullet}) \cong H^q(V,\Omega_V^p), \ H_{II}^{p,q}(\Gamma^{\bullet,\bullet}) \cong H^p(V,\bar{\Omega}_V^q).$$

This gives the spectral sequence (the *Frölicher spectral sequence*)

$$E_1^{p,q} = H^q(V, \Omega_V^p) \Longrightarrow H^n(V, \mathbb{C}).$$

If V is a Kähler manifold (e.g. a smooth projective algebraic variety), the spectral sequence degenerates in the first term (i.e. $E_\infty^{p,q} \cong E_1^{p,q}$) and gives the *Hodge decomposition*

$$H^n(V, \mathbb{C}) \cong \oplus_{p+q=n} H^q(V, \Omega_V^p).$$

Since $\overline{\Omega}_V^{p,q} \cong \Omega_V^{q,p}$, we also get isomorphisms $H^q(V, \Omega_V^p) \cong \overline{H^q(V, \Omega_V^p)}$. The numbers

$$h^{p,q}(V) = \dim_{\mathbb{C}} H^q(V, \Omega_V^p)$$

are called the *Hodge numbers* of V. They satisfy

$$b_n(V) = \dim H^n(V, \mathbb{C}) = \sum_{p+q=n} h^{p,q}, \; h^{p,q} = h^{q,p}.$$

For examples it implies that $b_{odd}(V)$ are even numbers. so a complex manifold with odd b_1 is not Kähler.

Theorem 1.4.4. *Under the assumption of Theorem 1.4.1, for any object $A^\bullet$ in $D^+(\mathsf{A})$, there exists a functorial in $A^\bullet$ spectral sequence*

$$E_2^{p,q} = \mathrm{R}^p G(\mathrm{R}^q F(A^\bullet)) \Longrightarrow \mathrm{R}^n(G \circ F)(A^\bullet).$$

Here is a sketch of a proof. To compute $\mathrm{R}^n(\mathrm{G} \circ \mathrm{F})(A^\bullet)$ we apply $G \circ F$ to an complex $R^\bullet$ of adapted objects quasi-isomorphic to $A^\bullet$ and take the n-th cohomology of $G(F(R^\bullet))$. Here we use that $\mathcal{R}_A$ is adapted for $G \circ F$ so you don't need to replace it by a quasi-isomorphic complex with objects from $\mathcal{R}$. On the other hand, to compute $\mathrm{R}^{\mathbf{p}}\mathbf{G}(\mathrm{R}^{\mathbf{q}}\mathbf{F}(\mathbf{A}^\bullet))$ we need to find a resolution of $\mathrm{R}^q F(A^\bullet)$ in $\mathrm{Cp}(\mathcal{R}_{\mathsf{B}})$, send it to $\mathbf{K}(\mathsf{C})$ and apply p-th cohomology.

We will consider the case when $\mathcal{R}$ are injective objects and $\mathcal{A}$ has enough of them. We will consider double complexes $X^{\bullet,\bullet}$ with $X^{\bullet,q} = 0$ for $q < 0$.

Let

$$K^\bullet \to L^{\bullet,0} \to L^{\bullet,0} \to \cdots \tag{1.7}$$

be a resolution of $K^\bullet$ in the category $\mathrm{Cp}(\mathsf{A})$. This means that $H_I^0(L^{\bullet,\bullet}) \cong K^\bullet$ and $H_I^n(L^{\bullet,\bullet}) = 0, n > 0$. For any complex $X^\bullet$ let

$$\begin{aligned} 0 \to B(X^\bullet) &= \mathrm{im}(d_{X^\bullet} : X^\bullet \to X^\bullet[1]), \\ 0 \to Z(X^\bullet) &= \ker(d_{X^\bullet} : X^\bullet \to X^\bullet[1]), \\ 0 \to H(X^\bullet) &= \mathrm{coker}(B(X^\bullet) \to Z(X^\bullet)). \end{aligned} \tag{1.8}$$

The resolution (1.7) defines the following complexes

$$\begin{aligned} B(K^\bullet) &\to B_{II}(L^{0,\bullet}) \to B_{II}(L^{1,\bullet}) \to \cdots, \\ Z(K^\bullet) &\to Z_{II}(L^{0,\bullet}) \to Z_{II}(L^{1,\bullet}) \to \cdot, \\ H(K^\bullet) &\to H_{II}(L^{0,\bullet}) \to H_{II}(L^{1,\bullet}) \to \cdots. \end{aligned} \tag{1.9}$$

Definition 1.4.1. A resolution (1.7) is called a *A Cartan-Eilenberg resolution* if the following properties are satisfied:

(CE1) All complexes $L^{\bullet,i}, p \geq 0$ are injective resolutions of K^i.

(CE2) The complexes (1.9) are resolutions.

(CE3) The exact sequences

$$0 \rightarrow B_{II}(L^{\bullet,q}) \rightarrow Z_{II}(L^{\bullet,q}) \rightarrow H_{II}(L^{\bullet,q}) \rightarrow 0, \tag{1.10}$$

$$0 \rightarrow Z_{II}^{\bullet,q} \rightarrow L^{\bullet,q} \rightarrow B_{II}^{\bullet,q+1} \rightarrow 0 \tag{1.11}$$

split.

Since a direct summand of an injective object is injective, property (CE3) implies that all complexes (1.9) are injective resolutions. In particular, for any left exact functor G for which injective objects are adapted, one can compute $\mathrm{R}^pG(H^q(K^\bullet))$ by using the injective resolution

$$H^q(K^\bullet) \rightarrow H^q(L^{0,\bullet}) \rightarrow H^q(L^{1,\bullet}) \rightarrow \cdots .$$

Let us apply the second spectral sequence of the double complex $L^{\bullet,\bullet}$ (1.6). Since $H_I^n(L^{\bullet,\bullet}) = 0, n > 0, H_I^0(L^{\bullet,\bullet}) = K^\bullet$, we obtain that $E_2^{p,q} = 0, p > 0, E_2^{0,n} = K^n$. By Exercise 3.4, we have an isomorphism $E_2^{0,n} \cong G^n(H^n) = H^n$. This shows that the canonical monomorphism $K^\bullet \rightarrow \mathrm{tot}(L^{\bullet,\bullet})$ is a quasi-isomorphism.

Let us apply this to our situation. We take $K^\bullet = F(I^\bullet)$, where $I^\bullet$ is an injective resolution of an object A. We consider its Cartan-Eilenberg resolution defining a double complex $L^{\bullet,\bullet}$. All its objects $L^{p,q}$ are injective and all cohomology $H_{II}^{p,q}(L^{\bullet,\bullet})$ are injective. In particular, we have an injective resolution

$$R^qF(A) = H^q(K^\bullet) \rightarrow H_{II}^{0,q}(L^{\bullet,\bullet}) \rightarrow H_{II}^{1,q}(L^{\bullet,\bullet}) \rightarrow \cdots .$$

Applying the functor G we obtain $G^p(R^qF(A)) = H^p(G(H_{II}^{\bullet,q}))$. By condition (CE3), we have $H^p(G(H_{II}^{\bullet,q})) = H_I^pH_{II}^q(G(L^{\bullet,\bullet}))$. Using the first spectral sequence of the double complex $G(L^{\bullet,\bullet})$ we obtain that it converges to

$$H^n(G(\mathrm{tot}(L^{\bullet,\bullet})) = H^n(G(F(I^\bullet)) = R^n(F \circ G).$$

Remark 1.4.5. For any additive functor $F : \mathsf{A} \rightarrow \mathsf{B}$ and a double complex $L^{\bullet,\bullet}$ in A, the cohomology of $\mathrm{tot}(F(L^{\bullet,\bullet}))$ are called the *hypercohomology* of $L^{\bullet,\bullet}$ with respect to the functor F. For example, the hypercohomology of the double complex $(\mathcal{E}_V^{p,q}, d', (-1)^pd'')$ from Example 1.4.3 with respect to the functor of global sections are isomorphic to the cohomology $H^\bullet(X, \mathbb{C})$.

Remark 1.4.6. Similarly, one shows that there exists a spectral sequence for left derived factors. Let $F^\pm : \mathsf{A} \rightarrow \mathsf{B}, G^\pm : \mathsf{B} \rightarrow \mathsf{C}$ be derived factors, where $+$ means right and $-$ means left derived factor. Then there is a spectral sequence

$$E_2^{p,q} = R^{\pm p}G \circ R^{\pm q}F(A^\bullet) \Longrightarrow R^{p+q}(G \circ F)(A^\bullet) \tag{1.12}$$

functorial in $A^\bullet \in \mathrm{Ob}(D^\pm(\mathsf{A}))$.

Example 1.4.7. Let $(X, \mathcal{O}_X)$ be a ringed space. Let $R = \mathcal{O}_X(X)$. Consider the global section functor

$$\Gamma_X : \mathcal{S}h^{\mathrm{ab}} \to \mathrm{Mod}(R),\ \mathcal{M} \to \mathcal{M}(X) = \mathrm{Hom}_{\mathrm{Mod}(\mathcal{O}_X)}(\mathcal{O}_X, \mathcal{M}).$$

The set of injective $\mathcal{O}_X$-modules is an adapted set for Γ_X. This defines the right derived functor

$$\mathbf{R}\Gamma_X : D^+(\mathrm{Mod}(\mathcal{O}_X) \to D^+(\mathrm{Mod}(R)).$$

By definition, for any sheaf of $\mathcal{O}_X$-modules $\mathcal{M}$, we have

$$H^n(X, \mathcal{M}) := R^n\Gamma_X(\mathcal{M}),$$

the *nth cohomology* of X with coefficients in $\mathcal{M}$. Of course, this is the special case of the direct image functor from Example 1.3.16, where we take Y to be a point equipped with the sheaf defined by the ring R. Let $f : X \to Y$ be a morphism of ringed spaces as in this example. We have

$$\Gamma_Y \circ f_* = \Gamma_X,\ \mathbf{R}\Gamma_X = \mathbf{R}f_* \circ \mathbf{R}\Gamma_Y. \tag{1.13}$$

The spectral sequence of the composition of functors

$$E_2^{p,q} = H^p(Y, R^q f_*\mathcal{M}) \Longrightarrow H^n(X, \mathcal{M})$$

is called the *Leray spectral sequence*.

Example 1.4.8. Let $f : X \to Y$ be a smooth projective morphism of complex varieties. Instead of $\mathrm{Qcoh}(X)$ we can take the category of *local coefficient systems* on X, i.e. sheaves of complex vector spaces locally isomorphic to the constant sheaf $\mathbb{C}_X^r$. We have the spectral sequence

$$E_2^{p,q} = H^p(Y, \mathrm{R}^q f_*\mathcal{L}) \Longrightarrow H^n(X, \mathcal{L}). \tag{1.14}$$

According to a fundamental result of P. Deligne this spectral sequence degenerates at E_2.

1.5 Exercises

1.1 Consider the functor from the category of R-modules $\mathrm{Mod}(R)$ to the category of R-algebras which assigns to M its exterior algebra. Find its right adjoint. Does it have a left adjoint?

1.2 Find the left and the right adjoints for the forgetful functor $\mathrm{Mod}(R) \to \mathsf{Ab}$.

1.3 Give a direct proof that the categories $\mathrm{Mod}(R)$ and $\mathrm{Mod}(\mathrm{End}_R(R^n))$ are equivalent.

1.4 An abelian category A is called *semi-simple* if each exact sequence $0 \to A \to B \to C \to 0$ splits, i.e. there exists a section $C \to B$ of the morphism $B \to C$. Prove that the category $D(\mathsf{A})$ is equivalent to the full subcategory of $\mathrm{Cp}(\mathsf{A})$ consisting of acyclic complexes.

1.5 Show that for any abelian category A, the category $\mathbf{K}(\mathsf{A})$ is abelian.

1.6 Let $f : X^\bullet \to Y^\bullet$ be a morphism of complexes that induces the zero morphism of the cohomology complexes. Is it isomorphjic to the zero morphism in $D(\mathsf{A})$?

1.7 A *differential graded* additive category is an additive category A such that any $\mathrm{Hom}_{\mathsf{A}}(A, B)$ has a grading $\mathrm{Hom}_{\mathsf{A}}(A, B) = \oplus_{n\in\mathbb{Z}}\mathrm{Hom}^n_{\mathsf{A}}(A, B)$, the composition defines a pairing

$$\mathrm{Hom}^i_{\mathsf{A}}(A, B) \times \mathrm{Hom}^j_{\mathsf{A}}(B, C) \to \mathrm{Hom}^{i+j}_{\mathsf{A}}(A, C).$$

and there is a differential $d : \mathrm{Hom}_{\mathsf{A}}(A, B) \to \mathrm{Hom}_{\mathsf{A}}(A, B)$ of degree 1 satisfying $d^2 = 0$ and $d(f \circ g) = df \circ g + (-1)^{\deg f} f \circ dg$. The homotopy category $\mathrm{Ho}(\mathsf{A})$ of A is the category with the same set of objects and morphisms $\mathrm{Hom}_{\mathrm{Ho}(\mathsf{A})}(A, B) = H^0(\mathrm{Hom}_{\mathsf{A}}(A, B))$. Show that the category of complexes $\mathrm{Cp}(\mathsf{A})$ has a structure of a differential graded category if we take $\mathrm{Hom}^n(X^\bullet, Y^\bullet) = \mathrm{Hom}_{\mathrm{Cp}(\mathsf{A})}(X^\bullet, Y^\bullet[n])$ and d defined by $df = d_{Y^\bullet} \circ f + (-1)^{\deg f} f \circ d_{X^\bullet}$. Show that the corresponding homotopy category coincides with the category $\mathbf{K}(\mathsf{A})$.

1.8 Let $(X, \mathcal{O}_X)$ be a ringed space with the sheaf of commutative rings $\mathcal{O}_X$. Consider the functor

$$\mathcal{H}om(\mathcal{M}, ?) : \mathrm{Mod}(\mathcal{O}_X) \to \mathrm{Mod}(\mathcal{O}_X),\ \mathcal{N} \to \mathcal{H}om_{\mathcal{O}_X}(\mathcal{M}, \mathcal{N}),$$

where $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{M}, \mathcal{N})$ is the sheaf of $\mathcal{O}_X$-modules defined by

$$U \to \mathrm{Hom}_{\mathcal{O}_X(U)}(\mathcal{M}(U), \mathcal{N}(U)).$$

Show that the functor $\mathcal{H}om(\mathcal{M}, ?)$ admits the right derived functor $\mathbf{R}\mathcal{H}om_{\mathcal{O}_X}(\mathcal{M}^\bullet, ?) : D^+(\mathrm{Mod}(\mathcal{O}_X)) \to D^+(\mathcal{S}h^{\mathrm{ab}})$. For any sheaves of $\mathcal{O}_X$-modules $\mathcal{M}, \mathcal{N}$ we define

$$\mathcal{E}xt^n_{\mathcal{O}_X}(\mathcal{M}, \mathcal{N}) = \mathrm{R}^n\mathcal{H}om(\mathcal{M}, \mathcal{N}).$$

Consider the global section functor Γ_X from Example 2.2.3. Show that there exists a spectral sequence

$$E_2^{p,q} = H^p(X, \mathcal{E}xt^q_{\mathcal{O}_X}(\mathcal{M}, \mathcal{N})) \Longrightarrow \mathrm{Ext}^n_{\mathcal{O}_X}(\mathcal{M}, \mathcal{N}).$$

1.9 In the notation of the previous example, show that, for any $\mathcal{M}^\bullet, \mathcal{N}^\bullet$ in $D^b(\mathrm{Mod}(\mathcal{O}_X)$ there is an isomorphism in $D^b(\mathrm{Mod}(\mathcal{O}_X))$

$$\mathbf{R}\mathcal{H}om_{\mathcal{O}_X}(\mathcal{M}^\bullet, \mathcal{N}^\bullet) \cong \mathbf{R}\mathrm{Hom}_{\mathcal{O}_X}(\mathcal{M}^\bullet, \mathcal{O}_X) \overset{\mathbf{L}}{\otimes} \mathcal{N}^\bullet$$

functorial in $\mathcal{M}^\bullet$ and $\mathcal{N}$. Specializing to object-complexes, show that there is a spectral sequence

$$E_2^{p,q} = \mathrm{Tor}_p^{\mathcal{O}_X}(\mathcal{E}xt^q_{\mathcal{O}_X}(\mathcal{M}, \mathcal{O}_X)) \Longrightarrow \mathcal{E}xt^n_{\mathcal{O}_X}(\mathcal{M}, \mathcal{N}).$$

1.10 Let $F : \mathsf{A} \to \mathsf{B}, G : \mathsf{B} \to \mathsf{A}$ be a pair of additive functors of abelian categories such that G is left adjoint to F. Assume that $\mathbf{R}\mathrm{F} : D^+(\mathsf{A}) \to D^+(\mathsf{B})$

and $\mathbf{R}G : D^-(\mathsf{B}) \to D^-(\mathsf{A})$ exist. Show that the restriction of these functors to $D^b(\mathsf{A}), D^b(\mathsf{B})$ are adjoint to each other.

1.11 Let $\chi : Ob(\mathsf{A}) \to \mathbb{Z}$ be a function satisfying the following properties: for any short exact sequence $0 \to A \to B \to C \to 0$ in $\mathcal{A}$ we have

$$\chi(B) = \chi(A) + \chi(C).$$

Prove that

$$\sum_{p,q} (-1)^{p+q} \chi(E_r^{p,q}) = \sum_n \chi(E^n).$$

provided all the sums contain only finitely many non-zero terms.

1.12 Assume that $E_2^{p,q} = 0$ unless $p \geq 0, q = 0$ (or $E_2^{p,q} = 0$ unless $q \geq 0, p = 0$). Prove that $E_2^{p,0} \cong G^p(H^p)$ (resp. $E_2^{0,q} \cong G^q(H^q)$.).

1.13 Assume that $E_2^{p,q} = 0$ when $p, q < 0$ and $F^i(H^n) = 0$ for $i > n$, $H^n = F^0(H^n)$. Show that there exists the following *five-term exact sequence*:

$$0 \to E_2^{1,0} \to H^1 \to E_2^{0,1} \to E_2^{2,0} \to H^2.$$

1.14 Prove that a Cartan-Eilenberg resolution of a complex $K^\bullet$ is an injective resolution of $K^\bullet$ in the category $\mathrm{Cp}(\mathsf{A})$ and converse is also true.

Lecture 2

Derived McKay correspondence

2.1 Derived category of coherent sheaves

Let X be a noetherian scheme of finite Krull dimension. We will consider it as a scheme over Spec$R$, where R is any subring of $\mathcal{O}_X(X)$. We denote by Qcoh(X) the category of quasi-coherent $\mathcal{O}_X$-modules and by Coh(X) its full subcategory of coherent sheaves. We make the following rather mild assumption on X

- Each coherent sheaf on X is a quotient of a locally free $\mathcal{O}_X$-module.

This happens, for example, when X is a quasi-projective over an affine scheme. We use that such schemes carry ample locally free sheaves of rank 1. By definition (see [Hartshorne], Chap. II, §7), for any coherent sheaf $\mathcal{F}$ and an ample sheaf $\mathcal{L}$, the tensor product $\mathcal{F} \otimes \mathcal{L}^n$ is generated by global sections if n is sufficiently large. This gives a surjection $\mathcal{O}_X^N \to \mathcal{F} \otimes \mathcal{L}^{\otimes n}$, and tensoring by the dual sheaf of $\mathcal{L}^{\otimes n}$, we get that $\mathcal{F}$ is the quotient of a locally free sheaf.

Proposition 2.1.1. *The natural functor*

$$D^b(\mathrm{Coh}(X)) \to D^b(\mathrm{Qcoh}(X)) \tag{2.1}$$

is a fully faithful functor of triangulated categories. It defines an equivalence between $D^b(\mathrm{Coh}(X))$ and the full subcategory of $D^b(\mathrm{Qcoh}(X))$ of bounded complexes with coherent cohomology sheaves.

Proof. The first assertion is obvious. To prove the second one, we use that for any surjection $\mathcal{G} \to \mathcal{F}$ in Qcoh(X), where $\mathcal{F}$ is coherent, there is a coherent subsheaf $\mathcal{G}'$ of $\mathcal{G}$ that is mapped surjectively onto $\mathcal{F}$. Let $\mathcal{G}^\bullet$ be a bounded complex of quasi-coherent sheaves with coherent cohomology $\mathcal{H}^i = \ker(d^i)/\mathrm{im}(d^{i-1})$. We may assume that $\mathcal{G}^i = 0, i > a$. Suppose $\mathcal{G}^i$ are coherent for $i > r$ for some r (for example, we take $r = a+1$). Then $\mathrm{im}(d^r) \subset \mathcal{G}^{r+1}$ is coherent. Thus there exists a coherent subsheaf $\mathcal{F}_1^r$ of $\mathcal{G}^r$ with $d^r(\mathcal{F}_1^r) = \mathrm{im}(d^r)$. We also have a surjection $\ker(d^r) \to \mathcal{H}^r$ which gives a surjection of some coherent subsheaf $\mathcal{F}_2^r$ of $\ker(d^r)$ onto $\mathcal{H}^r$. Now replace $\mathcal{G}^r$ with the coherent sheaf $\mathcal{F}^r \subset \mathcal{G}^r$ generated by $\mathcal{F}_1^r$ and $\mathcal{F}_1^r$ and replace $\mathcal{G}^{r-1}$ with $(d^{r-1})^{-1}(\mathcal{F}^r)$. Since $\mathrm{im}(d^r) = d^r(\mathcal{F}^r)$ we have not changed the cohomology $\mathcal{H}^{r+1}$. It is also clear that we have not changed the cohomology $\mathcal{H}^r$. By induction on r, we have been able to replace the complex $\mathcal{G}^\bullet$ with a quasi-isomorphic complex of coherent sheaves. □

Note that one can prove a similar assertion where D^b is replaced with D^- (see [SGA 6]).

A bounded complex of locally free sheaves is called *perfect*.

Proposition 2.1.2. *Let X be a regular scheme. Then any bounded complex of coherent sheaves is quasi-isomorphic to a perfect complex.*

Proof. An argument dual to one used in the proof of Lemma 1.3.2 shows that any bounded from above complex $F^\bullet$ of coherent sheaves is quasi-isomorphic to a bounded from above complex $L^\bullet$ of locally free sheaves. Now assume $\mathcal{F}^\bullet$ is a bounded complex. Without loss of generality we may assume that $H^i(\mathcal{E}^\bullet) = 0, i < 0$. Since X is regular, $\mathrm{im}(\mathcal{E}^{-1} \to \mathcal{E}^0)$ admits a finite locally free resolution $0 \to \mathcal{M}^{-n} \to \ldots \to \mathcal{M}^{-1}$ (see [Hartshorne], Chapter III, Exercise 6.9). Replacing $\mathcal{E}^\bullet$ with the qusi-isomorphic complex of locally free sheaves

$$0 \to \mathcal{M}^{-n} \to \ldots \to \mathcal{M}^{-1} \to \mathcal{M}^{-1} \to \mathcal{E}^0 \to \mathcal{E}^1 \to \ldots$$

we get the assertion. □

Let $\mathbf{F} : D^*(\mathrm{Qcoh}(X)) \to D^*(\mathsf{A})$ be any functor of triangulated categories, We can compose it with the functor (2.1) to get a functor of triangulated categories

$$\mathbf{F} : D^*(\mathrm{Coh}(X)) \to D^*(\mathsf{A}).$$

Let us consider some examples. From now on we set

$$D^*(X) = D^*(\mathrm{Coh}(X)),\ D^*_{\mathrm{qc}}(X) = D^*(\mathrm{Qcoh}(X)).$$

Note that the category $\mathrm{Qcoh}(X)$ is a R-linear category, i.e. its each Hom is equipped with a natural structure of a R-module. In other words the Yoneda functor factors through the subcategory of R-modules. Extending this to derived categories, we obtain that $D(\mathrm{Qcoh}(X))$ is an additive R-linear category.

Example 2.1.3. Consider the functor

$$\Gamma_X : \mathrm{Qcoh}(X) \to \mathrm{Mod}(R),\ \mathcal{F} \to \mathcal{F}(X).$$

Since injective sheaves are flabby, they form an adapted set for Γ_X. This defines a functor

$$\mathbf{R}\Gamma_X : D^+_{\mathrm{qc}}(X) \to D^+(\mathrm{Mod}(R)).$$

For any complex $\mathcal{F}^\bullet \in \mathrm{Ob}(D^+_{\mathrm{qc}}(X))$

$$\mathbf{R}\Gamma_X(\mathcal{F}^\bullet) = (\Gamma_X(\mathcal{I}^\bullet)),$$

where $\mathcal{I}^\bullet$ is an object of injective sheaves quasi-isomorphic to $\mathcal{F}^\bullet$. Applying, cohomology we can define

$$H^i(X, \mathcal{F}^\bullet) = H^i(\mathbf{R}\Gamma(\mathcal{F}^\bullet)).$$

Let

$$\mathcal{H}^i(\mathcal{F}^\bullet) = H^i(\mathcal{F}^\bullet).$$

For any complex $\mathcal{F}^\bullet$ quas-isomorphic to a complex of coherent sheaves the sheaves $\mathcal{H}^i(\mathcal{F}^\bullet)$ are coherent. There is a spectral sequence

$$E_2^{p,q} = H^p(X, \mathcal{H}^q(\mathcal{F}^\bullet)) \Longrightarrow H^n(X, \mathcal{F}^\bullet). \tag{2.2}$$

It is the first spectral sequence for the double complex $\Gamma_X(\mathcal{L}^{\bullet,\bullet}) = (\Gamma_X(\mathcal{L}^{p,q}))$, where $\mathcal{L}^{\bullet,\bullet}$ is a Cartan-Eilenberg resolution of the complex of the injective complex quasi-isomorphic to $\mathcal{F}^\bullet$. For this reason, the cohomology groups $H^n(X, \mathcal{F}^\bullet)$ are called sometimes the *hyper-cohomology* of $\mathcal{F}^\bullet$ and denoted by $\mathbb{H}^i(X, \mathcal{F}^\bullet)$ (see Remark 1.4.5).

Suppose $\mathcal{F}^\bullet = \mathcal{F}[m]$, where $\mathcal{F}$ is an object-complex. Then $\mathcal{H}^n(\mathcal{F}^\bullet) = \mathcal{F}$ if $n = -m$ and zero otherwise. The spectral sequence degenerates, and we obtain

$$H^p(X, \mathcal{F}[m]) \cong H^{p+m}(X, \mathcal{F}).$$

Assume X is proper over Spec R, for example X is a projective variety over a field k. Then $H^p(X, \mathcal{H}^q(\mathcal{F}^\bullet))$ are finitely generated R-modules ([EGA III], 3.2.1). It follows from the construction of the spectral sequence (2.2) that $H^n(X, \mathcal{F}^\bullet)$ are finitely generated R-modules. Also we know that for any coherent sheaf $\mathcal{F}$ the cohomology $H^i(X, \mathcal{F})$ vanish for $i > \dim X$ ([Hartshorne], Chap. III, Theorem 2.7). Thus we obtain a functor

$$\mathbf{R}\Gamma : D^b(X) \to D^b(\mathrm{Mod}(R)^{\mathrm{fg}})).$$

More generally, let $f : X \to Y$ be a morphism of schemes. The direct image functor f_* defined on the category $\mathrm{Mod}(\mathcal{O}_X)$ restricts to the category $\mathrm{Qcoh}(X)$ to define the functor

$$f_* : \mathrm{Qcoh}(X) \to \mathrm{Qcoh}(Y)$$

(see [Hartshorne], Chap. II, Prop. 5.8). By using injective sheaves we define a functor

$$\mathbf{R}f_* : D^+_{\mathrm{qc}}(X) \to D^+_{\mathrm{qc}}(Y). \tag{2.3}$$

For any quasi-coherent sheaf $\mathcal{F}$, the sheaf $R^if_*(\mathcal{F}) = H^i(\mathbf{R}f_*(\mathcal{F}))$ is the sheaf associated with the presheaf $U \to H^i(f^{-1}(U), \mathcal{F}|f^{-1}(U))$ ([Hartshorne], Chap. 3, Prop. 8.1). It follows that $R^nf_*(\mathcal{F}) = 0, n > \dim X$.

We have the spectral sequence similar to (2.2)

$$E_2^{p,q} = R^pf_*(\mathcal{H}^q(\mathcal{F}^\bullet)) \Longrightarrow R^nf_*(\mathcal{F}^\bullet) := H^n(\mathbf{R}f_*(\mathcal{F}^\bullet)). \tag{2.4}$$

The vanishing of R^pf_* for $p > \dim X$ implies that the functor (2.3) restricts to the functor

$$\mathbf{R}f_* : D^b_{\mathrm{qc}}(X) \to D^b_{\mathrm{qc}}(Y). \tag{2.5}$$

Assume $f; X \to Y$ is a proper morphism, e.g. a projective morphism. By Grothendieck's theorem the sheaves $R^if_*(\mathcal{F})$ are coherent for any coherent sheaf $\mathcal{F}$ (([EGA III], 3.2.1), for projective morphisms see [Hartshorne], Chap. 3, Thm 8.8). This shows that in this case the functor (2.6) restricts to the functor

$$\mathbf{R}f_* : D^b(X) \to D^b(Y). \tag{2.6}$$

The spectral sequence for the composition of functors $\Gamma_Y \circ f_* = \Gamma_X$ gives the Leray spectral sequence

$$E_2^{p,q} = H^p(Y, R^qf_*(\mathcal{F}^\bullet)) \Longrightarrow H^n(X, \mathcal{F}^\bullet). \tag{2.7}$$

Example 2.1.4. Consider the left exact functor

$$\mathcal{H}om_X(\mathcal{F}, ?) : \mathrm{Mod}(X)) \to \mathrm{Mod}(\mathcal{O}_X),\ \mathcal{G} \to \mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G}),$$

from Exercise 3.1. The sheaf $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})$ is defined by

$$U \to \mathrm{Hom}_{\mathcal{O}_U}(\mathcal{F}|U, \mathcal{G}|U)$$

([Hartshorne], Chap. II, Exercise 1.15). For any quasi-cohrent (coherent) sheaves $\mathcal{F}$ and $\mathcal{G}$, the sheaf $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})$ is quasi-coherent (coherent) ([Hartshorne], Chap. II, Exercise 6.3). Thus we obtain the functor

$$\mathcal{H}om_X(\mathcal{F}, ?) : \mathrm{Coh}(X) \to \mathrm{Coh}(X).$$

Using injective sheaves as adapted objects for this functor, we can define the derived functor

$$\mathbf{R}\mathcal{H}om_X(\mathcal{F}, ?) : D^+_{\mathrm{qc}}(X) \to D^+_{\mathrm{qc}}(X)$$

and then restrict it to $D^+(X)$ to get a functor

$$\mathbf{R}\mathcal{H}om_X(\mathcal{F}, ?) : D^+(X) \to D^+(X).$$

By definition, for any quasi-coherent sheaf $\mathcal{F}$ and $\mathcal{G}^\bullet \in \mathrm{Ob}(D^+_{\mathrm{qc}}(X))$,

$$\mathcal{E}xt^i_X(\mathcal{F}, \mathcal{G}^\bullet) = \mathrm{R}^i\mathcal{H}om_X(\mathcal{F}, \mathcal{G}^\bullet) \cong \mathrm{R}^0\mathcal{H}om_X(\mathcal{F}, \mathcal{G}^\bullet[i]).$$

In particular, we have defined the sheaves $\mathcal{H}om^i_X(\mathcal{F}, \mathcal{G})$, and the definition coincides with the one given in [Hartshorne], Chap. III, §6) for $\mathcal{E}xt^i(\mathcal{F}, \mathcal{G})$. We have an isomorphism

$$\mathcal{H}om^i_X(\mathcal{F}, \mathcal{G})_x \cong \mathrm{Ext}^i_{\mathcal{O}_{X,x}}(\mathcal{F}_x, \mathcal{G}_x) \tag{2.8}$$

(see [Hartshorne], Chapter III, Prop. 6.8), where in the right-hand side the Ext is defined in commutative algebra.

To compute $\mathcal{H}om^i_X(\mathcal{F}, \mathcal{G}^\bullet)$ one uses the spectral sequence

$$E_2^{p,q} = \mathcal{H}om^p_X(\mathcal{G}, H^q(\mathcal{G}^\bullet)) \Longrightarrow \mathcal{H}om^i_X(\mathcal{F}, \mathcal{G}^\bullet).$$

In particular, taking $p = q = 0$, we obtain

$$\mathcal{H}om^0_X(\mathcal{F}, \mathcal{G}^\bullet) \cong \mathcal{H}om^0_X(\mathcal{G}, \mathcal{H}^0(\mathcal{G}^\bullet)) \cong \mathcal{H}om_X(\mathcal{F}, \mathcal{H}^0(\mathcal{G}^\bullet)),$$

hence

$$\mathcal{E}xt^i_X(\mathcal{F}, \mathcal{G}^\bullet) \cong \mathcal{H}om_X(\mathcal{F}, \mathcal{H}^i(\mathcal{G}^\bullet)).$$

For any quasi-coherent sheaf $\mathcal{G}$, the functor $\mathcal{E} \to \mathcal{H}om_X(\mathcal{E}, \mathcal{G})$ is exact on the category formed by locally free sheaves (so $\mathcal{E}$ is sort of "locally projective object" in $\mathrm{Qcoh}(X)$). Since we assume that $\mathrm{Qcoh}(X)$ has enough of locally free sheaves, we obtain that they form a set of adapted objects for the functor $\mathcal{H}om_X(?, \mathcal{G})$, and we can define the left derived functors

$$\mathbf{R}\mathcal{H}om_X(?, \mathcal{G}) : D^-(X) \to D^-(X).$$

We also have a bifunctor

$$\mathcal{H}om^\bullet(?, ?) : \mathbf{K}(\mathrm{Qcoh}(X))^{\mathrm{op}} \times \mathbf{K}(\mathrm{Qcoh}(X)) \to \mathbf{K}(\mathrm{Qcoh}(X))$$

defined by

$$\mathcal{H}om^n_X(\mathcal{F}^\bullet, \mathcal{G}^\bullet) = \prod_{i \in \mathbb{Z}} \mathcal{H}om_X(\mathcal{F}^i, \mathcal{G}^{i+n})$$

with differential $d = d_{\mathcal{F}} - (-1)^n d_{\mathcal{G}}$. One easily checks that

$$\begin{aligned} H^i(\mathcal{H}om_X(\mathcal{F}^\bullet, \mathcal{G}^\bullet))(U) &= \mathrm{Hom}_{\mathbf{K}(\mathrm{Qcoh}(U))}(\mathcal{F}^\bullet|U, \mathcal{G}^\bullet[i]|U) \\ &= \mathrm{Hom}_{\mathbf{K}(\mathrm{Qcoh}(U))}(\mathcal{F}^\bullet[-i]|U, \mathcal{G}^\bullet|U). \end{aligned} \tag{2.9}$$

Using injective complexes one extends the second partial functcor to a functor of derived categories of bounded complexes from below. One can show that, if $\mathcal{G}$ is injective, then locally free sheaves form an adapted set for the functor in the first partial functor. Thus we obtain a bi-functor

$$\mathbf{R}\mathcal{H}om(?, ?) : D^-(X)^{\mathrm{op}} \times D^+(X) \to D(X). \tag{2.10}$$

It follows from the definitions that, for any $\mathcal{F}^\bullet, \mathcal{G}^\bullet \mathrm{Ob}(D^-(X))$, the complex $\mathbf{R}\mathcal{H}om(\mathcal{F}^\bullet, \mathcal{G}^\bullet)$ is the sheaf of complexes on X given by

$$U \to \mathrm{Hom}_{D_{\mathrm{qc}}(U)}(\mathcal{F}^\bullet|U, \mathcal{G}^\bullet|U).$$

We define the *hyperext sheaves* by setting

$$\mathcal{H}om^i_X(\mathcal{F}^\bullet, \mathcal{G}^\bullet) := H^i(\mathbf{R}\mathcal{H}om_X(\mathcal{F}^\bullet, \mathcal{G}^\bullet)) = \mathcal{H}om_X(\mathcal{F}^\bullet, \mathcal{G}^\bullet[i]) \cong \mathcal{H}om_X(\mathcal{F}^\bullet[-i], \mathcal{G}^\bullet)$$

where we consider the right derived functor of the partial functor in the second argument.

We have two spectral sequences

$$E_2^{p,q} = \mathcal{H}om^p(\mathcal{F}^\bullet, \mathcal{H}^q(\mathcal{G}^\bullet)) \Longrightarrow \mathcal{H}om^{p+q}(\mathcal{F}^\bullet, \mathcal{G}^\bullet), \tag{2.11}$$
$$E_2^{p,q} = \mathcal{H}om^p(\mathcal{H}^{-q}(\mathcal{G}^\bullet), \mathcal{G}^\bullet)) \Longrightarrow \mathcal{H}om^{p+q}(\mathcal{F}^\bullet, \mathcal{G}^\bullet), \tag{2.12}$$

Suppose $\mathcal{F}^\bullet \in \mathrm{Ob}(D^-(X))$. Then the second spectral sequence implies that $\mathcal{H}om_X^i(\mathcal{F}^\bullet, \mathcal{G}^\bullet)$ are equal to zero for $i << 0$. Thus the bifunctor (2.10) takes values in $D^+(X)$. Moreover, assume that X is regular (i.e. its local rings are regular local rings in sense of commutative algebra). Since any finitely generated module over a local regular rings A admits a finite free resolution of length $\geq \dim A$, we can use (2.8) to obtain that $\mathcal{H}om_X^i(\mathcal{F}, \mathcal{G}) = 0$ for $i > \dim X$ for any coherent sheaves $\mathcal{F}, \mathcal{G}$. Using the spectral sequences this allows us to define the bi-functor

$$\mathbf{R}\mathcal{H}om_X(?, ?) : D^b(X)^{\mathrm{op}} \times D^b(X) \to D^b(X). \tag{2.13}$$

We define the *dual complex* by

$$\mathcal{F}^{\bullet\vee} := \mathbf{R}\mathcal{H}om_X(\mathcal{F}^\bullet, \mathcal{O}_X). \tag{2.14}$$

By definition,

$$\mathcal{H}^i(\mathcal{F}^{\bullet\vee}) = \mathcal{E}xt_X^i(\mathcal{F}^\bullet, \mathcal{O}_X) = \mathcal{H}om_X(\mathcal{F}^\bullet, \mathcal{O}_X[i]).$$

If $\mathcal{F}^\bullet$ consists of locally free sheaves, we apply spectral sequence (2.17) to compute $H^i(\mathcal{F}^{\bullet\vee})$. In particular, if the cohomology of $\mathcal{F}^\bullet$ are locally-free, the spectral sequence degenerates and we obtain

$$\mathcal{H}^i(\mathcal{F}^{\bullet\vee}) = \mathcal{H}om(\mathcal{H}^{-i}(\mathcal{F}^\bullet), \mathcal{O}_X). \tag{2.15}$$

For example, this applies to the case when $\mathcal{F}^\bullet$ is a complex-object made of a locally free sheaf $\mathcal{E}$. In this case, $\mathcal{F}^{\bullet\vee}$ is quas-isomorphic to the dual sheaf $\mathcal{E}^\vee = \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{E}, \mathcal{O}_X)$.

Composing the functor $\mathbf{R}\mathcal{H}om_X(\mathcal{F}^\bullet, ?)$ with the functor $\mathbf{R}\Gamma_X$ we get the functor

$$\mathbf{R}\mathrm{Hom}_X(\mathcal{F}^\bullet, ?) : D^+_{\mathrm{qc}}(X) \to D^+(\mathrm{Mod}(R)).$$

We define the *hyperext modules*

$$\mathrm{Hom}_X^i(\mathcal{F}^\bullet, \mathcal{G}^\bullet) = H^i(\mathbf{R}\mathrm{Hom}_X(\mathcal{F}^\bullet, \mathcal{G}^\bullet)).$$

It follows from (2.2.3) that

$$\mathbf{R}\mathrm{Hom}_X(\mathcal{F}^\bullet, \mathcal{G}^\bullet) = \mathrm{Hom}_{D_{\mathrm{qc}}(X)}(\mathcal{F}^\bullet, \mathcal{G}^\bullet), \tag{2.16}$$

$$\mathrm{Hom}^i(\mathcal{F}^\bullet, \mathcal{G}^\bullet) \cong \mathrm{Hom}_{D_{\mathrm{qc}}(X)}(\mathcal{F}^\bullet, \mathcal{G}^\bullet[i]).$$

The spectral sequence of composition of derived functors gives

$$E_2^{p,q} = H^p(X, \mathcal{H}om_X^q(\mathcal{F}^\bullet, \mathcal{G}^\bullet)) \Longrightarrow \mathrm{Hom}_X^n(\mathcal{F}^\bullet, \mathcal{G}^\bullet). \tag{2.17}$$

Similarly, one defines the functor

$$\mathbf{R}\mathrm{Hom}_X(?, \mathcal{G}^\bullet) : D^-(X) \to D^-(\mathrm{Mod}(R)).$$

If $\mathcal{F}^\bullet, \mathcal{G}^\bullet$ are bounded complexes, we can compute the hyperext modules by using either of these two functors.

The spectral sequence of the composition of the derived functors gives the spectral sequence

$$E_2^{p,q} = H^p(X, \mathcal{H}om_X^q(\mathcal{F}^\bullet, \mathcal{G}^\bullet)) \Longrightarrow \mathrm{Hom}_X^n(\mathcal{F}^\bullet, \mathcal{G}^\bullet). \tag{2.18}$$

If X is proper over R, it shows that the R-modules $\mathrm{Hom}_X^n(\mathcal{F}^\bullet, \mathcal{G}^\bullet)$ are finitely generated, and we get a bi-functor

$$\mathbf{R}\mathrm{Hom}_X(?, ?) : D^-(X)^{\mathrm{op}} \times D^+(X) \to D^+(\mathrm{Mod}(R)^{\mathrm{fg}}).$$

If X is regular, we get the bi-functor

$$\mathbf{R}\mathrm{Hom}_X(?, ?) : D^b(X)^{\mathrm{op}} \times D^b(X) \to D^b(\mathrm{Mod}(R)^{\mathrm{fg}}).$$

More generally, let $f : X \to Y$ be a morphism of schemes. We set

$$\mathbf{R}f_* \circ \mathcal{H}om_X(?, ?) := \mathbf{R}\mathrm{Hom}_{X/Y}(?, ?) : D^-_{\mathrm{qc}}(X)^{\mathrm{op}} \times D^+_{\mathrm{qc}}(X) \to D^+_{\mathrm{qc}}(Y).$$

We define the relative *hyperexts*

$$\mathcal{H}om^i_{X/Y}(\mathcal{F}^\bullet, \mathcal{G}^\bullet) := H^i(\mathbf{R}\mathcal{H}om_{X/Y}(\mathcal{F}^\bullet, \mathcal{G}^\bullet)).$$

They are quasi-coherent sheaves on Y. One can show that $\mathcal{H}om^i_{X/Y}(\mathcal{F}^\bullet, \mathcal{G}^\bullet)$ is the sheaf associated to the presheaf on Y given by

$$V \to \mathrm{Hom}_{D_{\mathrm{qc}}(f^{-1}(V))}(\mathcal{F}^\bullet|f^{-1}(V), \mathcal{G}[i]^\bullet|f^{-1}(V)).$$

If $f : X \to Y$ is proper, then we have a bi-functor

$$\mathbf{R}\mathrm{Hom}_{X/Y}(?, ?) : D^b(X)^{\mathrm{op}} \times D^b(X) \to D^+(Y).$$

When dealing with object complexes, the functors Hom^i and $\mathcal{H}om^i$ are usually denoted by Ext^i and $\mathcal{E}xt^i$. They are zero for $i < 0$. Let us explain the notation Ext, short for extension. Suppose we have a distinguished triangle $X^\bullet \to Y^\bullet \to Z^\bullet \to X^\bullet[1]$ in $D(\mathsf{A})$, where A is an abelian category. Applying the functor $\mathrm{Hom}_{D(\mathsf{A})}(Z^\bullet, ?)$ we get an exact sequence

$$0 \to \mathrm{Hom}_{D(\mathsf{A})}(Z^\bullet, X^\bullet) \to \mathrm{Hom}_{D(\mathcal{A})}(Z^\bullet, Y^\bullet) \to \mathrm{Hom}_{D(\mathsf{A})}(Z^\bullet, Z^\bullet)$$
$$\to \mathrm{Ext}^1_{D(\mathsf{A})}(Z^\bullet, X^\bullet) = \mathrm{Hom}_{D(\mathsf{A})}(Z^\bullet, X^\bullet[1]).$$

The image of $\mathbf{id}_{Z^\bullet}$ defines an element in $\mathrm{Ext}^1_{D(\mathcal{A}}(Z^\bullet, X^\bullet)$, called the *class of the extension*. If it is equal to zero, then there exists a morphism $Z^\bullet \to Y^\bullet$ which is the left inverse of $Y^\bullet \to Z^\bullet$ (see the proof of Corollary 1.3.8). We say that the triangle splits in this case. In particular, if $\mathrm{Ext}^1_{D(\mathsf{A})}(A, B) = 0$ for all object-complexes A, B, then all exact sequences in $\mathcal{A}$ split. The converse is also true.

Example 2.1.5. We can restrict the operation of tensor product of complexes of $\mathcal{O}_X$-modules (1.4) to define a bifunctor

$$\otimes : \mathbf{K}^-(\mathrm{Qcoh}(X)) \otimes \mathbf{K}^-(\mathrm{Qcoh}(X)) \to \mathbf{K}(\mathrm{Qcoh}(X)).$$

One checks that complexes of locally free sheaves are adapted objects for the functor $\mathcal{F}^\bullet \otimes ?$. This allows one to define the derived bi-functor

$$\overset{\mathbf{L}}{\otimes} : D^-_{\mathrm{qc}}(X) \times D^-_{\mathrm{qc}}(X) \to D^-(X).$$

Using locally free sheaves of finite rank, we get the bi-functor

$$\overset{\mathbf{L}}{\otimes} : D^-(X) \times D^-(X) \to D^-(X).$$

If additionally, X is regular, we can use finite locally free finite resolutions (see Exercise 4.10) to define the left bi-functor

$$\overset{\mathbf{L}}{\otimes} : D^b(X) \times D^b(X) \to D^b(X).$$

We set

$$\mathcal{T}or_i(\mathcal{F}^\bullet, \mathcal{G}^\bullet) = H^{-i}(\mathcal{F}^\bullet \overset{\mathbf{L}}{\otimes} \mathcal{G}^\bullet).$$

These are coherent sheaves on X. There is a spectral sequence

$$E_2^{p,q} = \mathcal{T}or_{-p}(\mathcal{H}^q(\mathcal{F}^\bullet), \mathcal{G}^\bullet) \Longrightarrow \mathcal{T}or_{-p-q}(\mathcal{F}^\bullet, \mathcal{G}^\bullet). \tag{2.19}$$

Recall that for any three modules L, M, N over a commutative ring A there are canonical isomorphisms

$$\begin{aligned} L \otimes_A (M \otimes_A N) &\cong (L \otimes_A M) \otimes_A N, \\ M \otimes_A N &\cong N \otimes_A M. \end{aligned}$$

This can be extended to the derived tensor product. We have isomorphisms in the derived category $D_{\mathrm{qc}}(X)$

$$(\mathcal{F}^\bullet \overset{\mathbf{L}}{\otimes} \mathcal{G}^\bullet) \overset{\mathbf{L}}{\otimes} \mathcal{E} \cong \mathcal{F}^\bullet \overset{\mathbf{L}}{\otimes} (\mathcal{G}^\bullet \overset{\mathbf{L}}{\otimes} \mathcal{E}), \tag{2.20}$$

$$\mathcal{F}^\bullet \overset{\mathbf{L}}{\otimes} \mathcal{G} \cong \mathcal{G}^\bullet \overset{\mathbf{L}}{\otimes} \mathcal{F} \tag{2.21}$$

One first establishes the corresponding isomorphisms in the category $\mathbf{K}(\mathrm{Qcoh}(X))$ and then extend them to the derived category.

We also have the adjoint isomorphisms of A-modules

$$\begin{aligned} \mathrm{Hom}_A(L \otimes M, N) &\cong \mathrm{Hom}_A(L, \mathrm{Hom}_A(M, N)) \\ \mathrm{Hom}_A(L, M) \otimes_A N &\cong \mathrm{Hom}_A(L, M \otimes_A N) \end{aligned}$$

This can be generalized to derived categories. We have isomorphisms

$$\mathbf{R}\mathcal{H}om_X(L^\bullet \overset{\mathbf{L}}{\otimes} M^\bullet, N^\bullet) \cong \mathbf{R}\mathcal{H}om_X(L^\bullet, \mathbf{R}\mathcal{H}om_X(M^\bullet, N^\bullet)) \tag{2.22}$$

$$\mathbf{R}\mathcal{H}om_X(L^\bullet, M^\bullet) \overset{\mathbf{L}}{\otimes} N^\bullet \cong \mathbf{R}\mathcal{H}om_X(L^\bullet, M^\bullet \overset{\mathbf{L}}{\otimes} N^\bullet) \tag{2.23}$$

$$\mathbf{R}\mathrm{Hom}_X(L^\bullet \overset{\mathbf{L}}{\otimes} M^\bullet, N^\bullet) \cong \mathbf{R}\mathrm{Hom}_X(L^\bullet, \mathbf{R}\mathrm{Hom}_X(M^\bullet, N^\bullet)) \quad (2.24)$$

$$\mathbf{R}\mathrm{Hom}_X(L^\bullet, M^\bullet) \overset{\mathbf{L}}{\otimes} N^\bullet \cong \mathbf{R}\mathrm{Hom}_X(L^\bullet, M^\bullet \overset{\mathbf{L}}{\otimes} N^\bullet) \quad (2.25)$$

In particular, (2.23) gives an isomorphism

$$\mathbf{R}\mathcal{H}om_X(\mathcal{F}, \mathcal{G}^\bullet) \cong \mathcal{F}^{\bullet\vee} \overset{\mathbf{L}}{\otimes} \mathcal{G}^\bullet. \quad (2.26)$$

Also, we obtain a natural isomorphism of complexes

$$\mathcal{F}^\bullet \cong (\mathcal{F}^{\bullet\vee})^\vee \quad (2.27)$$

Example 2.1.6. Consider the functor

$$f^* : \mathrm{Qcoh}(Y) \to \mathrm{Qcoh}(X),\ \mathcal{F} \to \mathcal{F} \otimes_{\mathcal{O}_Y} \mathcal{O}_X.$$

It is the left adjoint functor to the functor f_*. Restricting it to $\mathrm{Coh}(X)$ we get a functor $f^* : \mathrm{Coh}(Y) \to \mathrm{Coh}(X)$. Assume that each coherent sheaf admits a locally free resolution in $\mathrm{Coh}(X)$. Then we can define the derived functor

$$\mathbf{L}f^* : D^-(Y) \to D^-(X),$$

If f is of finite Tor-dimension (e.g. f is a flat morphism or Y is regular), then this defines a functor

$$\mathbf{L}f^* : D^b(Y) \to D^b(X).$$

We set

$$L_i f^* : D^b(Y) \to \mathrm{Coh}(X),\ \mathcal{F}^\bullet \to H^{-i}(\mathbf{L}f^*(\mathcal{F}^*)).$$

Moreover, if f is a flat morphism, then $\mathbf{L}f^* = f^*$. We have a spectral sequence

$$E_2^{p,q} = L_p f^*(\mathcal{H}^q(\mathcal{F}^\bullet)) \Longrightarrow L_{p+q} f^*(\mathcal{F}^\bullet). \quad (2.28)$$

Now everything is ready to state the *Grothendieck-Serre-Verdier Duality Theorem*. We will state it not in full generality.

Theorem 2.1.7. *Let $f : X \to Y$ be a proper morphism of schemes of finite type over a field k. There exists a right adjoint functor $f^! : D^b(Y) \to D^b(X)$ to the functor $\mathbf{R}f_*$ and a morphism*

$$\theta_f : \mathbf{R}f_* \circ \mathbf{R}\mathcal{H}om_X(\mathcal{F}^\bullet, f^!\mathcal{G}^\bullet) \to \mathbf{R}\mathcal{H}om_Y(\mathbf{R}f_*\mathcal{F}^*, \mathbf{R}f_*f^!\mathcal{G}^\bullet)$$

whose composition with the morphism

$$\mathrm{Tr}_f : \mathbf{R}\mathcal{H}om_Y(\mathbf{R}f_*\mathcal{F}^*, \mathbf{R}f_*f^!\mathcal{G}^\bullet) \to \mathbf{R}\mathcal{H}om_Y(\mathbf{R}f_*\mathcal{F}^\bullet, \mathcal{G}^\bullet)$$

defined by the adjunction morphism of functors $\mathbf{R}f_*f^! \to \mathbf{id}_{D^b(Y)}$ is an isomorphism

$$\mathbf{R}f_* \circ \mathbf{R}\mathcal{H}om_X(\mathcal{F}^\bullet, f^!\mathcal{G}^\bullet) \xrightarrow{\cong} \mathbf{R}\mathcal{H}om_Y(\mathbf{R}f_*\mathcal{F}^\bullet, \mathcal{G}^\bullet).$$

This isomorphism is functorial in $\mathcal{F}^\bullet, \mathcal{G}^\bullet$.

Example 2.1.8. Assume that $f : X \to Y$ is a proper smooth morphism of relative dimension r. In this case the sheaf of relative differentials $\Omega^1_{X/Y}$ is locally free of rank r. Its maximal exterior power $\omega_f = \Lambda^r(\Omega^1_{X/k})$ is the *relative canonical sheaf*. The functor $f^!$ is defined in this case by

$$f^! = (\overset{\mathbf{L}}{\otimes}\omega_f[r]) \circ \mathbf{L}f^* = (\otimes\omega_f[r]) \circ \mathbf{L}f^*.$$

Since a smooth morphism is flat, by definition, the functor $\mathbf{L}$ is just the usual f^* extended to complexes. This gives the duality isomorphism

$$\mathbf{R}f_* \circ \mathbf{R}\mathcal{H}om_X(\mathcal{F}^\bullet, f^*(\mathcal{G}^\bullet) \otimes \omega_f[r]) \longrightarrow \mathbf{R}\mathcal{H}om_Y(\mathbf{R}f_*\mathcal{F}^\bullet, \mathcal{G}^\bullet).$$

Taking the cohomology and using the definition of relative hyperext, we can rewrite it in the form

$$\mathbf{R}\mathrm{Hom}_{X/Y}(\mathcal{F}^\bullet, f^*\mathcal{G}^\bullet \otimes \omega_f[r]) \cong \mathbf{R}\mathcal{H}om(\mathbf{R}f_*\mathcal{F}^\bullet, \mathcal{G}^\bullet).$$

Taking $\mathcal{G} = \mathcal{O}_Y$, we obtain an isomorphism

$$\mathbf{R}\mathrm{Hom}_{X/Y}(\mathcal{F}^\bullet, \omega_f[r]) \cong (\mathbf{R}f_*\mathcal{F}^\bullet)^\vee. \tag{2.29}$$

Passing to cohomology, and using (2.15) we get the spectral sequence

$$E_2^{p,q} = \mathcal{H}om^p_Y(R^{-q}f_*\mathcal{F}^\bullet, \mathcal{O}_Y) \Longrightarrow \mathcal{H}om^n_{X/Y}(\mathcal{F}^\bullet, \omega_f[r])) \cong \mathrm{Ext}^{n+r}_{X/Y}((\mathcal{F}^\bullet, \omega_f)).$$

Suppose all sheaves $R^qf_*\mathcal{F}^\bullet$ are locally free on Y (e.g. $Y = \mathrm{Spec}\ k$). Then $E_2^{p,q} = 0$ if $p \neq 0$, the spectral sequence degenerates, and we get an isomorphism

$$\mathrm{Ext}^{r-q}_{X/Y}(\mathcal{F}^\bullet, \omega_f) \cong \mathcal{H}om_Y(R^qf_*\mathcal{F}^\bullet, \mathcal{O}_Y).$$

Taking $\mathcal{F}^\bullet = \mathcal{F}$, we get the relative version of Serre's Duality isomorphism

$$\mathrm{Ext}^{r-q}_{X/Y}(\mathcal{F}, \omega_f) \cong \mathcal{H}om_Y(R^qf_*\mathcal{F}, \mathcal{O}_Y). \tag{2.30}$$

Taking $Y = \mathrm{Spec}\ k$, we get the classical *Serre Duality Theorem*

$$\mathrm{Ext}^{r-q}_{X/k}(\mathcal{F}, \omega_{X/k}) \cong \mathrm{Hom}_k(H^q(X, \mathcal{F}), k).$$

If moreover, $\mathcal{F}$ is locally free, we use the spectral sequence (2.4) to get the isomorphism

$$\mathrm{Ext}^{r-q}_{X/k}(\mathcal{F}, \omega_{X/k}) \cong H^{r-q}(X, \mathcal{F}^\vee \otimes \omega_{X/k}).$$

Example 2.1.9. Let $f : X \to Y$ be a proper *Cohen-Macaulay morphism* (i.e. it is proper, flat, of finite type and its fibres are Cohen-Macaulay varieties), then we take

$$f^!(\mathcal{G}^\bullet) = f^*(\mathcal{G}^\bullet) \otimes \omega_{X/Y},$$

where $\omega_{X/Y}$ is a certain coherent sheaf, called the *relative canonical sheaf*. For example, if X is normal, and the locus of points $x \in X$ such that f is not smooth at x is a closed subset of X of codimension ≥ 2, we can take

$$\omega_f = j_*\omega_f^o,$$

where $j : X \setminus S \to X$ is the open embedding, $\omega_f^o = \omega_{f^o}$, and $f^o : X \setminus Z \to Y - f(S)$. Specializing, the duality isomorphism to the case $Y = \mathrm{Spec}\ k, \mathcal{F}^\bullet = \mathcal{F}[i], \mathcal{G}^\bullet = \mathcal{O}_Y$, we obtain the Serre Duality Isomorphism from [Hartshorne], Chap. III, §7).

Let X be a smooth projective scheme over a field k. Consider X as a morphism $f : X \to Y = \mathrm{Spec}\ k$ and take $\mathcal{G}^\bullet = \mathcal{O}_Y = k$ and $\mathcal{F}^\bullet = \mathcal{H}om_X(\mathcal{A}^\bullet, \mathcal{B}^\bullet) \cong \mathcal{A}^{\bullet\vee} \overset{\mathbf{L}}{\otimes} \mathcal{B}^\bullet$. Using the adjunction formulas (2.24), (2.14), (2.27) and the duality isomorphism (2.29), we get isomorphism

$$\begin{aligned}\mathbf{R}\mathrm{Hom}_{X/k}(\mathcal{A}^{\bullet\vee} \overset{\mathbf{L}}{\otimes} \mathcal{B}^\bullet, \omega_{X/k}[r])) &\cong \mathbf{R}\mathrm{Hom}_{X/k}(\mathcal{B}^\bullet, \mathcal{A}^\bullet \overset{\mathbf{L}}{\otimes} \omega_{X/k}[r])) \\ &\cong \mathbf{R}\mathrm{Hom}_{X/k}(\mathcal{A}^\bullet, \mathcal{B}^\bullet)^\vee\end{aligned}$$

Passing to the duals, we get an isomorphism

$$\mathbf{R}\mathrm{Hom}_{X/k}(\mathcal{A}^\bullet, \mathcal{B}^\bullet) \cong \mathbf{R}\mathrm{Hom}_{X/k}(\mathcal{B}^\bullet, \mathcal{A}^\bullet \overset{\mathbf{L}}{\otimes} \omega_{X/k}[r])^\vee. \tag{2.31}$$

Taking the cohomology we get

$$\mathrm{Ext}^i_{X/k}(\mathcal{A}^\bullet, \mathcal{B}^\bullet) \cong \mathrm{Ext}^{r-i}_{X/k}(\mathcal{B}^\bullet, \mathcal{A}^\bullet \overset{\mathbf{L}}{\otimes} \omega_{X/k})^\vee$$

Also recall that $H^0(\mathbf{R}\mathrm{Hom}_{X/k}(\mathcal{F}^\bullet, \mathcal{G}^\bullet)) = \mathrm{Hom}_{D(X)}(\mathcal{F}^\bullet, \mathcal{G}^\bullet)$. Consider the functor

$$S : D^b(X) \to D^b(X),\ \mathcal{F}^\bullet \to \mathcal{F}^\bullet \otimes \omega_{X/k}]r].$$

It is called the *Serre functor*. The duality isomorphism gives an isomorphism

$$\mathrm{Hom}_{D^b(X)}(\mathcal{B}^\bullet, \mathcal{A}^\bullet) \xrightarrow{\cong} \mathrm{Hom}_{D^b(X)}(\mathcal{A}^\bullet, S(\mathcal{B}^\bullet))^\vee \tag{2.32}$$

functorial in $\mathcal{A}^\bullet, \mathcal{B}^\bullet$.

More generally, an equivalence of k-linear categories $S : \mathsf{C} \to \mathsf{C}$ is called a *Serre functor* if there is an isomorphism of bifunctors

$$\eta : \mathsf{C}^{\mathrm{op}} \times \mathsf{C} \to \mathbf{Ab},\ (A, B) \to \mathrm{Hom}_{\mathsf{C}}(A, B),\ (A, B) \to \mathrm{Hom}_{\mathsf{C}}(B, S(A))^*. \tag{2.33}$$

Note that two Serre functors $S, S' : \mathsf{C} \to \mathsf{C}$ are isomorphic functors since, for any object A in D, we have $\mathrm{Hom}(A, A) \cong \mathrm{Hom}(A, S_1(A)^\vee \cong \mathrm{Hom}(S_1(A), S_2(A))$. The image of $\mathbf{id}_A$ defines an isomorphism $S_1(A) \to S_2(A)$, functorial in A .

Also note that a Serre functor of triangulated categories is automatically a δ-functor (see [Bondal-Kapranov], Prop. 3.3).

Example 2.1.10. Let $f : Y \hookrightarrow X$ be a closed embedding of smooth varieties over k. Let c be the codimension of $f(Y)$. Let $\omega_f = f^*(\omega_X) \otimes_{\mathcal{O}_Y} \omega_Y^\vee$. In this case

$$f^!(\mathcal{F}^\bullet) = \mathbf{L}f^*(\mathcal{F}^\bullet) \otimes \omega_f.$$

Note that $\mathbf{R}f_*(\mathcal{F}^\bullet)$ is quasi-isomorphic to the complex $f_*\mathcal{F}^\bullet = (f_*(\mathcal{F}^i))$. The duality gives an isomorphism

$$\mathrm{Hom}_Y(\mathcal{F}^\bullet, \mathbf{L}f^*(\mathcal{G}^\bullet) \otimes \omega_f[-c]) \cong \mathrm{Hom}_X(f_*\mathcal{F}^\bullet, \mathcal{G}^\bullet).$$

Definition 2.1.1. Let A be an abelian category. Its *homological dimension* is the smallest number $\mathrm{dh}(\mathsf{A})$ such that $\mathrm{Ext}^i(A,B) = 0$ $i > \mathrm{dh}(\mathsf{A})$ and all $A, B \in \mathrm{Ob}(\mathsf{A})$.

Example 2.1.11. If $\mathsf{A} = \mathrm{Mod}(R)$, then $\mathrm{dh}(\mathsf{A})$ is the homological or projective dimension $\mathrm{dh}(R)$ of the ring R as defined in any text-book on ring theory. So for any affine scheme $X = \mathrm{Spec}\ A$ the homological dimension of $\mathrm{Coh}(X)$ is equal to $\mathrm{dh}(A)$. Let X be a nonsingular projective variety of dimension r. Then $\mathrm{dh}(\mathrm{Coh}(X)) = n$. In fact, it follows from the duality that $\mathrm{Ext}^i(\mathcal{F},\mathcal{G}) = \mathrm{Ext}^{r-i}(\mathcal{G},\mathcal{F}\otimes\omega_{X/k})$. This shows that $\mathrm{dh}(X) \le r$. On the other hand, again by duality,

$$\mathrm{Ext}^r(\omega_{X/k},\mathcal{O}_X) \cong \mathrm{Ext}^0(\omega_{X/k},\omega_{X/k})^\vee \cong H^r(X,\omega_{X/k}) \cong k,$$

hence $\mathrm{dh}(\mathrm{Coh}(X))) \ge r$.

Proposition 2.1.12. *Let A be an abelian category with $\mathrm{dh}(\mathsf{A}) \le 1$. Then any complex in $D^b(\mathsf{A})$ is isomorphic to the direct sum of shifted object-complexes.*

Proof. Let $X^\bullet$ be a bounded complex in an abelian category A and let n be the smallest integer such that $H^n(X^\bullet) \ne 0$. We would like to prove that $X^\bullet$ is isomorphic o the dircect sum of shifted object-complexes. We will use induction on n. Consider the following commutative diagram of morphism of complexes.

$$\begin{array}{ccccccccccc}
X^\bullet & & \longrightarrow & X^{n-2} & \longrightarrow & X^{n-1} & \longrightarrow & X^n & \longrightarrow & X^{n+1} \\
\downarrow & & & \downarrow & & \downarrow{\scriptstyle d^{n-1}} & & \downarrow & & \downarrow \\
\tau_{\ge n}(X^\bullet) & & \longrightarrow & 0 & \longrightarrow & \mathrm{im}(d^{n-1}) & \longrightarrow & X^n & \longrightarrow & X^{n+1} \\
\uparrow & & & \uparrow & & {\scriptstyle d^{n-1}}\uparrow & & \uparrow & & \uparrow \\
\tau_{\le n}(X^\bullet) & & \longrightarrow & X^{n-2} & \longrightarrow & X^{n-1} & \longrightarrow & \ker(d^n) & \longrightarrow & 0 \\
\downarrow & & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
H^n(X^\bullet)[-n] & & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & H^n(X^\bullet) & \longrightarrow & 0
\end{array}$$

The first morphism $X^\bullet \to \tau_{\ge n}(X^\bullet)$ and the last morphism $\tau_{\le n}(X^\bullet) \to H^n(X^\bullet)[n]$ are quasi-isomorphisms. Thus this diagram defines a morphism $f : H^n(X^\bullet)[-n] \to X^\bullet$ in the derived category. It is easy to see that this morphism is a monomorphism (because it is a monomorphism on the cohomology). Consider the distinguished triangle

$$H^n(X^\bullet)[-n] \xrightarrow{f} X^\bullet \to Y^\bullet \to H^n(X^\bullet)[-n+1].$$

Here $H^i(Y^\bullet) = 0$ for $i \le n$. By induction, $Y^\bullet$ is quasi-isomorphic to the direct sum of shifted object-complexes which must be isomorphic to $H^i(Y)[-i], i \ge n+1$. We have

$$\mathrm{Ext}^1(Y^\bullet,\mathcal{H}^n(X^\bullet[-n])) = \mathrm{Hom}(Y^\bullet, H^i(X^\bullet)[1-n]$$

$$\cong \bigoplus_{i>n} \mathrm{Hom}(H^i(Y^\bullet)[-i], H^n(X^\bullet))[1-n]$$
$$\cong \bigoplus_{i>n} \mathrm{Ext}^{i+1-n}(H^i(Y^\bullet), H^n(X^\bullet)) = 0.$$

This shows that the triangle splits in the derived category. Thus $X^\bullet \cong Y^\bullet \oplus H^n(X^\bullet[-n])$, and, by induction we are done. □

Corollary 2.1.13. *Let X be a nonsingular projective curve over a field k. Then any object in $D^b(X)$ is isomorphic to the direct sum of shifted object-complexes.*

2.2 Fourier-Mukai Transform

Let C be a category with finite fibred products. Suppose we have a contravariant functor $F : \mathsf{C} \to \mathsf{D}$, and a covariant functor $G : \mathsf{C} \to \mathsf{D}$. For any morphism $f : X \to Y$ in C we set $F(f) = f^*$ and $G(f) = f_*$. Suppose also we have a morphism in D

$$K : F(X \times Y) \to G(X \times Y).$$

Let $p : X \times Y, q : X \times Y \to Y$ be the two projections. Then we can form the composition

$$\Phi_K : p_* \circ K \circ q^* : F(Y) \xrightarrow{q_*} F(X \times Y) \xrightarrow{K} G(X \times Y) \xrightarrow{p_*} G(Y).$$

This is called the *integral transform* from $F(Y)$ to $G(X)$ with *kernel* K.

Example 2.2.1. Let LC be the category whose objects are locally compact topological spaces X with a choice of a Radon measure μ_X (a linear functional μ_X on the linear space $C_c(X)$ of functions with compact support such that $\mu_X(\phi) \geq 0$ if $\phi \geq 0$). By definition, a morphism $f : (X, \mu_X) \to (Y, \mu_Y)$ is a continuous map $f : X \to Y$ such that the pull-back of μ_Y-measurable function on Y is a μ_X-measurable function on X. The space LC has direct products $(X \times Y, \mu_X \times \mu_Y)$ with the product topology. Fix a function $K(x, y)$ on $X \times Y$ such that $K(x, y)\phi(x, y)$ is measurable for any measurable function $\phi(x, y)$. Consider the functor $F : \mathrm{LC} \to \mathrm{Vect}_{\mathbb{R}}$ which assigns to each (X, μ_X) the linear space $L^2(X, \mu_X)$ of functions with μ_X-measurable square and to a morphism $f : (X, \mu_X) \to (Y, \mu_Y)$ the pull-back transform f^*. Consider the functor G with the same values on the objects and morphisms f_* defined by the formula

$$f_*(\phi(x))(y) = \int_{f^{-1}(y)} \phi(x) d\mu_X.$$

Take for $K : F(X \times Y) \to G(X \times Y)$ the operator of multiplication by the function $K(x, y)$. Then we have the integral transform $\Phi_K : F(Y) \to G(X)$ with kernel K.

For example, take $X = Y = \mathbb{R}$ with the Lebesque measure. Take $K(x, y) = \frac{1}{\sqrt{2\pi}} e^{ixy}$. This is the classical Fourier transform. It satisfies the additional property that $\Phi_K \circ \Phi_{K^{-1}} = \mathbf{id}$.

Example 2.2.2. Let $\mathsf{C} = \mathrm{AV}/\mathbb{C}$ be the category of complex algebraic varieties. A function $f : X \to \mathbb{Z}$ is called *constructible* if its level sets are constructible subset of X (see [Hartshorne], Chap. II, Exercise 3.18). Consider the functor $F : \mathsf{C} \to \mathsf{Ab}$ which assigns to X the abelian group $\mathrm{Cons}(X)$ of constructible functions on X and to a morphism $f : X \to Y$ the homomorphism $f^* : \mathrm{Cons}(Y) \to \mathrm{Cons}(X)$. Since level sets of f are closed subsets, $f^*(\phi) = \phi \circ f$ is constructible if ϕ is constructible. By a result of MacPherson (Ann. Math. v. 100) there exists a unique covariant functor

$$f_* : \mathrm{Cons} : \mathrm{AV}/\mathbb{C} \to \mathbf{Ab}, \quad (X \xrightarrow{f} Y) \to (\mathrm{Cons}(X) \xrightarrow{f^*} \mathrm{Cons}(Y))$$

that satisfies

$$f_*(\chi_W)(y) = \mathrm{Eu}(W \cap f^{-1}(y)),$$

where χ_S is the characteristic function of a subvariety S of X, and $\mathrm{Eu}(S)$ is its topological Euler-Poincarè characteristic. Let $i : Z \to X \times Y$ be a morphism of complex algebraic varieties. Define $K_Z : \mathrm{Cons}(X \times Y) \to \mathrm{Cons}(X \times Y$ to be the multiplication with the function $i_*(\chi_Z)$. This gives the integral transform

$$\Phi_Z : p_* \circ \Phi_Z \circ q^* : \mathrm{Cons}(Y) \to \mathrm{Cons}(X).$$

It is easy to see that, if $Z = \Gamma_f$ is the graph of a morphism $f : X \to Y$, then $\Phi_Z = f^*$.

Example 2.2.3. Let C be the subcategory of $\mathrm{AV}/\mathbb{C}$ formed by nonsingular projective varieties. Consider the functor $F : \mathsf{C} \to \mathsf{Ab}$ which assigns to X the cohomology group $H^*(X, \mathbb{C})$ and we take f^* to be the usual push-back. Consider $G : \mathsf{C} \to \mathsf{Ab}$ to be the covariant functor which assigns the homology group H_* with push-forward maps f_*. Take Z as above, and consider $K : H^*(X \times Y) \to H_*(X \times Y)$ defined by taking the cup-product with the fundamental class $[i_*(Z)]$ and applying the Poincarè duality. We get the Fourier transform. We can also compose it with the Poincarè duality on Y to get a transform $H^*(X) \to H^*(Y)$ (with some shift of the grading).

Note that by MacPherson's theory, there is a morphism of functors $\mathrm{Cons} \to H_*$ on the category $\mathrm{AV}/\mathbb{C}$ which assigns to a constructible function α on a nonsingular X an element $c_*(\alpha) \in H_*(X)$ such that $c_*(\chi_X)$ is the Poiincarè dual of the total Chern class of X. By Deligne's definition, the *total Chern class* of any complex algebraic variety V is equal to $c_*(\chi_V)$.

Now let us generalize. Recall the definition of a *fibred category* D over a category C. Roughly speaking it is a functor on a category C with values in a category whose objects are categories and morphisms are functors between categories. For each objects S in C, it assigns a category D_S and to each morphism $f : S' \to S$ in C, it assigns a functor $f^* : \mathsf{D}_S \to \mathsf{D}_{S'}$. We require that for any two composable morphisms $f : S' \to S$ and $g : S'' \to S'$ there is an isomorphism of functors $c_{f,g} : g^* \circ f^* \to (g \circ f)^*$. The isomorphisms $c_{f,g}$ (called *cleavages*) must satisfy some compatibility conditions:

- $c_{f,\mathbf{id}_{S'}} = c_{\mathbf{id}_S,f} = \mathbf{id}_{f^*}$,
- $c_{f,g\circ h} \circ c_{g,h}(f^*(a)) = c_{f\circ g,h} \circ h^*(c_{f,g}(a))$, for any $h : S''' \to S'', g : S'' \to S', f : S' \to S$, and $a \in \mathrm{Ob}(\mathsf{D}_S)$.

A fibred category can be viewed as a category D together with a functor $\mathsf{D} \to \mathsf{C}$ such that the objects in D mapped to an object $S \in \mathsf{C}$ form a category D_S. Its objects are denoted by (A, S), where $S \in \mathrm{Ob}(\mathsf{C})$ and $A \in \mathsf{D}_S$. A morphism $(S, A) \to (S', B)$ in D_S is a morphism $f : S' \to S$ and a morphism $B \to f^*(A)$ in $\mathsf{D}_{S'}$. The composition $(C, S'') \to (B, S') \to (A, S)$ is defined by

$$C \to g^*(B) \to g^*(f^*(A)) \xrightarrow{c_{f,g}(A)} (f \circ g)^*(A).$$

We define the *cofibred category* by reversing the arrows, this time we assign functors $f_* : \mathsf{D}_{S'} \to \mathsf{D}_S$ to each morphism $f : S' \to S$ in C. A morphism of the pair $(S, A) \to (S', B)$ is a morphism $f : S' \to S$ in C and a morphism $f_*(B) \to A$.

We leave to the reader to define a functor $F : \mathsf{D} \to \mathsf{D}'$ of (co)-fibred categories over a category C . The composition

$$(S' \xrightarrow{f} S) \to (\mathsf{D}_S \xrightarrow{f^*} \mathsf{D}_{S'}) \to (\mathsf{D}'_S \xrightarrow{F(f^*)} \mathsf{D}'_{S'})$$

must coincide with f^* in D'.

Now suppose there is a given a fibred category D over C and a cofibred category D' over C. Let $K : \mathsf{D}(X \times Y) \to \mathsf{D}'(X \times Y)$ be a functor between the categories. We define the *categorical integral transform* with kernel K as the composition

$$\Phi_K : p_* \circ K \circ q^* : \mathsf{D}(Y) \to \mathsf{D}'(X). \tag{2.34}$$

We say that Φ_K is a *Fourier transform* if Φ_K is an equivalence of categories.

Let $F_1 : \mathsf{D}^{\mathrm{op}} \to \mathsf{A}, F_2 : \mathsf{D}' \to \mathsf{A}$ be some functors with values in an abelian category. We view A as a (co)-fibred category over C with $\mathsf{A}(S) = \{F_i(S), \mathbf{id}_{F_i(S)}\}$ and require that F_i is a functor of (co)-fibred categories. By definition, for any morphism $f : S' \to S$ in C, the following diagrams are commutative

$$\begin{array}{ccc} \mathsf{D}_{S'} & \xrightarrow{F_1(S')} & \mathsf{A} \\ {\scriptstyle f^*}\uparrow & {\scriptstyle F_1(f^*)} & \uparrow \\ \mathsf{D}_S & \xrightarrow{F_1(S)} & \mathsf{A} \end{array}, \qquad \begin{array}{ccc} \mathsf{D}'_{S'} & \xrightarrow{F_2(S')} & \mathsf{A} \\ \downarrow{\scriptstyle f_*} & & \downarrow{\scriptstyle F_2(f_*)} \\ \mathsf{D}'_S & \xrightarrow{F_2(S)} & \mathsf{A} \end{array} .$$

It is also a part of definition that the isomorphisms of functors $c_{f,g}$ in D and $c'_{f,g}$ in D' define the isomorphisms

$$F_1(c_{f,g}) : F_1((f\circ g)^*) \to F_1(g^*)\circ F_1(f^*),\ F_2(c'_{f,g}) : F_2((f\circ g)_*) \to F_2(f_*)\circ F_1(g_*)$$

and $F_i(\mathbf{id}_S) = \mathbf{id}_{\mathsf{A}_i}$. The definition shows that the composition $S \to \mathsf{D}_S \xrightarrow{F_1(S)} \mathsf{A}$ defines a contravariant functor $\tilde{F}_1 : \mathsf{C} \to \mathsf{A}$ and the composition $S \to \mathsf{D}'_S \xrightarrow{F_2(S)} \mathsf{A}$ defines a functor $\tilde{F}_2 : \mathsf{C} \to \mathsf{A}$.

Choose an additive functor $\tilde{K} : \mathsf{A} \to \mathsf{A}$ such that $F_2(X \times Y) \circ K = F_1(X \times Y) \circ \tilde{K}$. Applying F_1 and F_2 to (2.34), we obtain the commutative diagram

$$\begin{array}{ccccccc}
Y & \xleftarrow{q} & X \times Y & = & X \times Y & \xrightarrow{p} & X \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
\mathsf{D}_Y & \xrightarrow{q^*} & \mathsf{D}_{X\times Y} & \xrightarrow{K} & \mathsf{D}'_{X\times Y} & \xrightarrow{p_*} & \mathsf{D}'_X \\
\downarrow{\scriptstyle F_1(Y)} & & \downarrow{\scriptstyle F_1(X\times Y)} & & \downarrow{\scriptstyle F_2(X\times Y)} & & \downarrow{\scriptstyle F_2(X)} \\
\mathsf{A} & \xrightarrow{F_1(q^*)} & \mathsf{A} & \xrightarrow{\tilde{K}} & \mathsf{A} & \xrightarrow{F_2(p_*)} & \mathsf{A}
\end{array}$$

For any $A \in \mathsf{D}_Y$, it defines the integral transform

$$\widetilde{\Phi_K} = F_2(p_*) \circ \tilde{K} \circ F_2(q^*) : \tilde{F}_1(Y) \to \tilde{F}_2(X)$$

compatible with the categorical integral transform.

Now let us specialize. Take for C the category PAV/k of projective varieties over a field k. Consider the fibred category $S \to D^b(S)$ with $f^* := \mathbf{L}f^*$ and the cofibred category $S \to D^b(S)$ with $f_* := \mathbf{R}f_*$. We choose some cleavages $\mathbf{L}f^* \circ \mathbf{L}g^* \cong \mathbf{L}(g \circ f)^*$ and $\mathbf{R}g_* \circ \mathbf{R}f_* \cong \mathbf{R}(g \circ f)_*$. For any $\mathcal{E}^\bullet \in \mathrm{Ob}(D^b(X \times Y))$ consider the functor $K_{\mathcal{E}^\bullet} : D^b(X \times Y) \to D^b(X \times Y)$ defined by $\mathcal{E}^\bullet \overset{\mathbf{L}}{\otimes} ?$. Now we can define the *integral transform* with *kernel* $\mathcal{E}^\bullet$

$$\Phi^{Y\to X}_{\mathcal{E}^\bullet} : D^b(Y) \to D^b(X) : \mathcal{G}^\bullet \to \mathbf{R}p_*(\mathcal{E}^\bullet \overset{\mathbf{L}}{\otimes} q^*(\mathcal{G}^\bullet)), \tag{2.35}$$

Note that $\mathbf{L}q^* = q^*$ since the projections $X \xleftarrow{p} X \times Y \xrightarrow{q} Y$ are flat morphisms. We define the *inverse* integral transform by

$$\Phi^{X\to Y}_{\mathcal{E}^\bullet} : D^b(X) \to D^b(Y) : \mathcal{F}^\bullet \to \mathbf{R}q_*(\mathcal{E}^\bullet \overset{\mathbf{L}}{\otimes} p^*(\mathcal{F}^\bullet)), \tag{2.36}$$

We must warn that, in general, the inverse transform is not the inverse as functors.

Note that the integral transform is a composition of exact functors, hence is an exact functor of derived categories.

Proposition 2.2.4. *Let X and Y be smooth projective varieties and $n = \dim X, m = \dim Y$. The functors*

$$\Phi^{X\to Y}_{\mathcal{E}^{\bullet\vee}\otimes q^*(\omega_{Y/k}[m])}, \ \Phi^{X\to Y}_{\mathcal{E}^{\bullet\vee}\otimes p^*(\omega_{X/k}[n])}$$

are right and left adjoints to the functor $\Phi^{Y\to X}_{\mathcal{E}^\bullet}$.

Proof. Let us show that $\Phi^{X\to Y}_{\mathcal{E}^{\bullet\vee}\otimes p^*(\omega_{X/k})[n]}$ is a left adjoint of $\Phi^{Y\to X}_{\mathcal{E}^\bullet}$. We have isomorphisms in the corresponding derived categories

$$\mathrm{Hom}_{X/k}(\mathcal{A}^\bullet, \mathbf{R}p_*(\mathcal{E}^\bullet \overset{\mathbf{L}}{\otimes} q^*(\mathcal{B}^\bullet)) \cong \mathrm{Hom}_{X\times_k Y}(p^*(\mathcal{A}^\bullet), \mathcal{E} \overset{\mathbf{L}}{\otimes} q^*(\mathcal{B}^\bullet))$$

$$\cong \mathrm{Hom}_{X\times_k Y}(\mathcal{E}^\bullet \overset{\mathbf{L}}{\otimes} q^*(\mathcal{B}^\bullet), p^*(\mathcal{A}^\bullet)\otimes \omega_{X\times_k Y}[n+m])^\vee$$

$$\cong \mathrm{Hom}_{X\times_k Y}(q^*(\mathcal{B}^\bullet \otimes \omega_{Y/k}[m]^\vee), \mathcal{E}^{\bullet\vee} \overset{\mathbf{L}}{\otimes} p^*(\mathcal{A}^\bullet \otimes \omega_{X/k}[n])^\vee$$

$$\cong \mathrm{Hom}_{Y/k}(\mathcal{B}^\bullet \otimes \omega_{Y/k}[m]^\vee, \mathbf{R}q_*(\mathcal{E}^{\bullet\vee} \overset{\mathbf{L}}{\otimes} p^*(\mathcal{A}^\bullet \otimes \omega_{X/k}[n])^\vee$$

$$\cong \mathrm{Hom}_{Y/k}(\mathcal{B}^\bullet, \mathbf{R}q_*(\mathcal{E}^{\bullet\vee} \overset{\mathbf{L}}{\otimes} p^*(\mathcal{A}^\bullet \otimes \omega_{X/k}[n])\otimes \omega_{Y/k}[m])^\vee$$

$$\cong \mathrm{Hom}_{Y/k}(\mathbf{R}q_*(\mathcal{E}^{\bullet\vee} \otimes p^*(\omega_{X/k}[n])) \overset{\mathbf{L}}{\otimes} p^*(\mathcal{A}^\bullet), \mathcal{B}^\bullet).$$

Here we used that $\mathbf{L}p^*$ is a left adjoint to $\mathbf{R}p_*$ (the first isomorphism), the Serre functor (the second and the last isomorphism), and the adjunction isomorphisms (2.24) in the rest. The obtained isomorphism shows that the functor $\Phi^{X\to Y}_{\mathcal{E}^{\bullet\vee}\otimes p^*(\omega_{X/k})[n]}$ a left adjoint of $\Phi^{Y\to X}_{\mathcal{E}^\bullet}$.

Similarly, we check that $\Phi^{X\to Y}_{\mathcal{E}^{\bullet\vee}\otimes q^*(\omega_{Y/k}[m])}$ is a right adjoint. □

Definition 2.2.1. An integral transform $\Phi_{\mathcal{E}}$ is called a *Fourier-Mukai transform* if it is an equivalence of categories.

The proof of the following fundamental theorem is omitted (see [Orlov], J. Math.Sci. 84 (1997), or Russian Math. Surveys, 58:3 (2003)).

Theorem 2.2.5. *Let X, Y be two smooth projective varieties over a field k and $F : D^b(Y) \to D^b(X)$ be a fully faithfully functor of triangulated categories which admits a left and a right adjoint. Then there exists a unique (up to isomorphism) object $\mathcal{E}^\bullet$ of $D^b(X \times_k Y)$ such that F is isomorphic to the integral transform $\Phi^{Y\to X}_{\mathcal{E}^\bullet}$.*

Since an equivalence of categories satisfies the assumptions of the theorem, any equivalence of categories between two smooth complete varieties is realized by a Fourier-Mukai transform with a unique (up to isomorphism) kernel.

Corollary 2.2.6. *Let $\mathcal{E}^\bullet \in \mathrm{Ob}(X \times_k Y)$ be the kernel of a Fourier-Mukai transform. Then*

$$\mathcal{E}^{\bullet\vee} \otimes \omega_X[\dim X] \cong \mathcal{E}^{\bullet\vee} \otimes \omega_Y[\dim Y].$$

In particular, $D^b(X) \approx D^b(Y)$ implies $\dim X = \dim Y$.

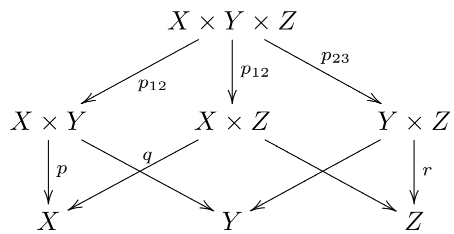
Recall that a *correspondence* over two objects X and Y is a morphism $\tau : R \to X \times Y$. An example of a correspondence is the canonical projection $\Gamma_f \to X \times Y$, where $f : X \to Y$ is the *graph* of f. One can define the *composition* $R_1 \circ R_2$ of a correspondence $\tau_1 : R_1 \to X \times Y$, and $\tau_2 : R_2 \to Y \times Z$ as the composition of the fibre product of $\tau_1 \times \mathbf{id}_Z : R_1 \times Z \to X \times Y \times Z$ and $\mathbf{id}_Y \times \tau_2 : Y \times R_2 \to X \times Y \times Z$ with the projection $p_{13} : X \times Y \times Z \to X \times Y$.

In the category of sets, if $R_1 \subset X \times Y, R_2 \subset Y \times$Z, then

$$R_1 \circ R_2 = p_{13}(p_{12}^{-1}(R_1) \cap p_{23}^{-1}(R_2)).$$

For example, an equivalence relation on a set X is a correspondence $R \subset X \times X$ that contains the diagonal, symmetric, and satisfies $R \circ R = R$.

Let $\Phi^{X\to Y}_{\mathcal{E}_1^\bullet}, \Phi^{Y\to Z}_{\mathcal{E}_1^\bullet}$ be two integral transforms. Consider the diagram



Set

$$\mathcal{E}_1^\bullet \circ \mathcal{E}_2^\bullet := \mathbf{R}p_{13}(p_{12}^*(\mathcal{E}_1^\bullet) \overset{\mathbf{L}}{\otimes} p_{23}^*(\mathcal{E}_2^\bullet)). \tag{2.37}$$

Then, one can check that

$$\Phi^{Y\to Z}_{\mathcal{E}_2^\bullet} \circ \Phi^{X\to Y}_{\mathcal{E}_1^\bullet} \cong \Phi^{X\to Z}_{\mathcal{E}_1^\bullet \circ \mathcal{E}_2^\bullet}. \tag{2.38}$$

Thus we see that the composition of integral transforms is an integral transform.

Let us consider some (easy) examples of a Fourier-Mukai transform functors.

From now on in this lecture we are in the category of smooth projective varieties over a field k. Let $f : X \to Y$ be a morphism, $i : \Gamma_f \hookrightarrow X \times Y$ be its graph. Then

$$\Phi^{Y\to X}_{i_*(\mathcal{O}_{\Gamma_f})} = \mathbf{L}f^* : D^b(Y) \to D^b(X),$$
$$\Phi^{X\to Y}_{i_*(\mathcal{O}_{\Gamma_f})} = \mathbf{R}f_* : D^b(X) \to D^b(Y).$$

Let $i_* : \Delta_X \hookrightarrow X \times_k X$ be the inclusion of the diagonal. We have

$$\Phi^{X\to X}_{i_*(\mathcal{O}_{\Delta_X}[i])}(\mathcal{F}^\bullet) = \mathcal{F}^\bullet[i].$$

$$\Phi^{X\to X}_{i_*(\mathcal{O}_{\Delta_X} \otimes \omega_{X/k}[\dim X])} = S,$$

the Serre functor.

Let $\mathcal{E}$ be an object-complex over $X \times_k Y$. Assume that $\mathcal{E}$ is flat over X. This means that for any $(x, y) \in X \times Y$, the $\mathcal{O}_{X,x}$-module $\mathcal{E}_{x,y}$ is flat. For any $x \in X$, let $j_x : p^{-1}(x) \to X \times_k Y$ be the closed embedding of the fibre $p^{-1}(x)$. Then the sheaf $\mathbf{L}j_x^*(\mathcal{E}) \cong j_x^*(\mathcal{E})$ is isomorphic to a sheaf $\mathcal{E}_x$ on Y via the second projection q. We have

$$\Phi^{X\to Y}_{\mathcal{E}}(\mathcal{O}_x) = \mathcal{E}_x. \tag{2.39}$$

Here, to simplify the notation, we identify the structure sheaf $\mathcal{O}_Z$ of a closed subscheme Z with the sheaf $i_*(\mathcal{O}_Z)$, where $i : Z \hookrightarrow X$ is the closed embedding.

Suppose $\Phi^{X\to Y}_{\mathcal{E}}$ is a Fourier-Mukai transform. The quasi-inverse functor $D^b(Y) \to D^b(X)$ must be isomorphic to $\Phi^{Y\to X}_{\mathcal{E}^\vee \otimes q^*(\omega_{Y/k}[m])}$, where $m = \dim Y$. Assume that $\mathcal{E}$ is an invertible sheaf. This implies that $\mathcal{E}^\vee$ is an object-complex defined by an invertible sheaf $\mathcal{E}^{-1}$.

Example 2.2.7. Let us take $X = Y = E$, where E is an elliptic curve over a field k. Let $x_0 \in E(k)$. Consider the invertible sheaf $\mathcal{P} = \mathcal{O}_{E\times E}(-p^{-1}(x_0) - q^{-1}(x_0) + \Delta_E)$. Let

$$P = -p^{-1}(x_0) - q^{-1}(x_0) + \Delta_E.$$

It follows from (2.39), for any closed point $x \in E$,

$$\Phi_{\mathcal{P}}^{E\to E}(\mathcal{O}_x) \cong \mathcal{O}_E(x - x_0). \tag{2.40}$$

Recall, that a choice of a point $x_0 \in E(k)$ defines an isomorphism of abelian groups

$$E \to \mathrm{Pic}^0(E),\ x \mapsto \mathcal{O}_X(x - x_0), \tag{2.41}$$

where the group law on E is defined by $x \oplus y \sim x + y - x_0$, where we use the Riemann-Roch Theorem and Serre's duality that give

$$h^0(\mathcal{O}_E(D) - h^0(\mathcal{O}_E(-D)) = \deg D, \tag{2.42}$$

giving in particular that $h^0(\mathcal{O}_E(x+y-x_0)) = 1$. Thus we see that the bijection (2.41) can be defined by the integral transform $\Phi_{\mathcal{P}}$.

Let us us see that $\Phi_{\mathcal{P}}$ is a Fourier-Mukai transform. In fact, this example appears in the paper of Mukai, where the Fourier-Mukai transform was introduced.

Before doing this let us compute explicitly some of the transforms.

We will use the following result from [Hartshorne[, Chap. III, Theorem 12.11) to compute the higher direct images of coherent sheaves. Let $f : X \to Y$ be a projective morphism of noetherian schemes and $\mathcal{F}$ be a coherent sheaf on X, flat over Y. Suppose $R^{i+1}f_*\mathcal{F}$ is locally free, then

$$R^if_*\mathcal{F}(y) := R^if_*\mathcal{F} \otimes_Y \mathrm{Spec}\ k(y) \cong H^i(X_y, \mathcal{F}_y),$$

where $X_y = f^{-1}(y)$ is the scheme-theoretical fibre of f over y and $\mathcal{F}_y = \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{O}_{X_y}$.

Let us apply this result to our morphism $q : E \times E \to E$ and use that $R^2f_*\mathcal{F} = 0$ because the relative dimension of q is equal to 1. We obtain that, for any coherent sheaf $\mathcal{F}$ on $E \times E$, flat over E, we have

$$R^1q_*(\mathcal{F})(y) \cong H^1(q^{-1}(y), \mathcal{F}|q^{-1}(y)).$$

For any $y \in E$ we have

$$\mathcal{P} \otimes p^*(\mathcal{O}_E(D)) \cong \mathcal{O}_{E\times E}(P + p^*(D)).$$

Hence

$$\mathcal{P} \otimes p^*(\mathcal{O}_E(D))|q^{-1}(y) \cong \mathcal{O}_E((P + p^*(D)) \cap q^{-1}(y)) = \mathcal{O}_E(D + y - x_0).$$

Thus we obtain

$$R^1q_*(\mathcal{P} \otimes p^*(\mathcal{O}_E(D))(x) = H^1(E, \mathcal{O}_E(D + x - x_0)) \cong H^0(E, \mathcal{O}_E(-D + x_0 - x)).$$

Applying (2.42), this gives

$$R^1q_*(\mathcal{P}\otimes p^*(\mathcal{O}_E(D))(x) = 0,\ \text{if}\ \deg D \geq 0, D + x - x_0 \not\sim 0,$$

Assume $D + x - x_0 \sim 0$. Note that this can happen only for a unique point x. Then $R^1q_*(\mathcal{P}\otimes p^*(\mathcal{O}_E(D))(x) \cong H^1(E, \mathcal{O}_E) \cong k$, and we get

$$H^1(\Phi_{\mathcal{P}}^{E\to E}(\mathcal{O}_E(D))) = \mathcal{O}_x,$$

$$H^0(\Phi_{\mathcal{P}}^{E\to E}(\mathcal{O}_E(D)))(y) = 0,\ y \neq x.$$

We have

$$H^0(\Phi_{\mathcal{P}}^{E\to E}(\mathcal{O}_E(D))(x) \cong H^0(E, q_*(\mathcal{O}_{E\times E}(P + p^*(D)))$$
$$\cong H^0(E\times E, \mathcal{O}_{E\times E}(P + p^*(D)).$$

Since $\deg D = 0$, using the intersection theory on the surface $E\times E$, we get

$$(P + q^*(D))^2 = P^2 + 2P\cdot q^*(D) = (\Delta - p^{-1}(x_0) - q^{-1}(x_0))^2 = -2 \quad (2.43)$$

(we use that $\Delta^2 = \mathrm{Eu}(E) = 0$). Since the canonical class of $E\times E$ is equal to 0, the adjunction formula gives $C^2 = 2h^1(\mathcal{O}_C) - 2$ for any irreducible curve C on $E\times E$. Since C projects surjectively to E, $h^1(\mathcal{O}_C) > 0$. Thus $C^2 \geq 0$. Suppose $H^0(E\times E, \mathcal{O}_{E\times E}(P + p^*(D))) \neq 0$, then $P + p^*(D)$ is linearly equivalent to an effective divisor, and it follows from above that its self-intersection is non-negative. This contradiction with (2.43) shows that $H^0(E\times E, \mathcal{O}_{E\times E}(P + p^*(D))) = 0$. Collecting all of this together we obtain

$$\Phi_{\mathcal{P}}^{E\to E}(\mathcal{O}_E(D))) = \mathcal{O}_x[-1] \text{ if } D \sim x_0 - x. \quad (2.44)$$

Comparing with (2.40), we find that

$$\Phi_{\mathcal{P}}^{E\to E}\circ\Phi_{\mathcal{P}}^{E\to E}(\mathcal{O}_x) = \mathcal{O}_{\ominus x}[-1], \quad (2.45)$$

where $\ominus x$ means the negative of x in the group law on E.

Now consider the distinguished triangle corresponding to the exact sequence

$$0 \to \mathcal{O}_E(-x) \to \mathcal{O}_E \to \mathcal{O}_x \to 0.$$

Applying $\Phi_{\mathcal{P}}^{E\to E}$, we obtain the distinguished triangle

$$\Phi_{\mathcal{P}}^{E\to E}(\mathcal{O}_E(-x)) \to \mathcal{O}_{x_0}[-1] \to \mathcal{O}_E(x - x_0) \to \Phi_{\mathcal{P}}^{E\to E}(\mathcal{O}_E(-x))[1].$$

It implies that

$$\Phi_{\mathcal{P}}^{E\to E}(\mathcal{O}_E(-x)) \cong \mathcal{O}_{x_0}[-1]\oplus\mathcal{O}_E(x - x_0).$$

Finally, if $\mathcal{F} = \mathcal{O}_{2x}$, then the extension

$$0 \to \mathcal{O}_x \to \mathcal{O}_{2x} \to \mathcal{O}_x \to 0$$

implies that $\Phi_{\mathcal{P}}^{E\to E}(\mathcal{O}_{2x})$ fits in the extension

$$0 \to \mathcal{O}_E(x - x_0) \to \Phi_{\mathcal{P}}^{E\to E}(\mathcal{O}_{2x}) \to \mathcal{O}_E(x - x_0) \to 0.$$

It corresponds to a non-trivial element in the space

$$\mathrm{Ext}_E^1(\mathcal{O}_E(x - x_0), \mathcal{O}_E(x - x_0)) \cong \mathrm{Ext}_E^1(\mathcal{O}_E, \mathcal{O}_E) \cong H^1(E, \mathcal{O}_E) \cong k.$$

After these concrete computations, let us start proving that $\Phi_{\mathcal{P}}^{E\to E}$ is a Fourier-Mukai transform. In fact, we will prove that

$$\Phi_{\mathcal{P}}^{E\to E} \circ \Phi_{\mathcal{P}}^{E\to E} \cong \iota^* \circ [-1], \tag{2.46}$$

where $\iota : E \to E$ is the automorphism of E defined by the negation automorphism $x \to \ominus x$, and $[-1]$ is the shift automorphism. This agrees with (2.45).

To prove (2.46), it suffices to prove that

$$\mathcal{P} \circ \mathcal{P} \cong \mathcal{O}_{\Gamma(\iota)}[-1]. \tag{2.47}$$

We will use the following well-known *seesaw principle*.

- Let $f : X \to Y$ be a smooth projective morphism of algebraic varieties over a field k admitting a section $s : Y \to X$. Assume that the function $y \to \dim_{k(y)} H^i(X_y, \mathcal{O}_{X_y})$ is constant for any $i \geq 0$. Let $\mathcal{L}$ and $\mathcal{M}$ be invertible sheaves such that $\mathcal{L}|X_y \cong \mathcal{M}|X_y$ for all closed points $y \in Y$ and $s^*(\mathcal{L}) \cong s^*(\mathcal{M})$. Then $\mathcal{L} \cong \mathcal{M}$.

To prove it consider the invertible sheaf $\mathcal{N} = \mathcal{L} \otimes \mathcal{M}^{-1}$. For any closed point $y \in \mathcal{Y}$ we have an isomorphism $\mathcal{N}|X_y \cong \mathcal{O}_{X_y}$. Since f is smooth, starting with $R^{\dim f+1} f_*(\mathcal{N}) = 0$, we obtain that the fibres $R^i f_*\mathcal{N}(y)$ are isomorphic to $H^i(X_y, \mathcal{O}_y)$ and, by assumption, their dimension is independent of y. This implies that the sheaves $(R^i f_*\mathcal{N})(y)$ are locally free of rank equal to $\dim_{k(y)} H^i(X_y, \mathcal{O}_{X_y})$. In particular, we obtain that the sheaf $\mathcal{K} = f_*(\mathcal{N})$ is an invertible sheaf on Y. Consider the homomorphism $f^* f_*\mathcal{N} \to \mathcal{N}$. Restricting to fibres, we obtain an isomorphism. Thus $\mathcal{N} \cong f^*(\mathcal{K})$. Now the second assumption on $\mathcal{L}, \mathcal{M}$ implies that $\mathcal{O}_Y \cong s^*(\mathcal{N}) \cong s^*(f^*(\mathcal{K})) \cong \mathcal{K}$. This gives $\mathcal{N} \cong \mathcal{O}_X$ and hence $\mathcal{L} \cong \mathcal{M}$.

We will apply the seesaw principle to the case when $f : X \times Y \to Y$ is the projection. Since all fibres are isomorphic, the assumption on $\dim_{k(y)} H^i(X_y, \mathcal{O}_{X_y})$ is obviously satisfied. Note that the assumption is always satisfied if $k = \mathbb{C}$.

Consider the addition map

$$\mu : E \times E \to E, (x, y) \mapsto x \oplus y.$$

We have $\mu^{-1}(z)|q^{-1}(x) = \{z \ominus x\}$, and the same is true for another projection. Restricting the divisor $\mu^{-1}(x) - \mu^{-1}(x_0)$ to fibres of p and q, and applying the seesaw principle to any of the two projections $E \times E \to E$, we easily obtain

$$\mu^{-1}(x) - \mu^{-1}(x_0) \cong p^*(\mathcal{O}_E(x - x_0)) \otimes q^*(\mathcal{O}_E(x - x_0)).$$

Now, using this, we make similar comparison and the seesaw principle to obtain an isomorphism
$$p_{12}^*(\mathcal{P}) \otimes p_{13}^*(\mathcal{P}) \cong \pi^*(\mathcal{P}),$$
where
$$\pi : E \times E \times E \to E \times E,\ (x, y, z) \mapsto (y, x + z).$$
This implies
$$\mathcal{P} \circ \mathcal{P} \cong \mathbf{R}p_{13*}(\pi^*(\mathcal{P})).$$
Applying the base-change formula for higher direct images (see [Hartshorne], Chap. III, §9) to the Cartesian diagram

$$\begin{array}{ccc} E \times E \times E & \xrightarrow{\pi} & E \times E \\ \downarrow{\scriptstyle p_{12}} & & \downarrow{\scriptstyle q} \\ E \times E & \xrightarrow{\mu} & E \end{array}$$

we find that
$$\mathbf{R}p_{13*}(\pi^*(\mathcal{P})) \cong \mu^*(\mathbf{R}q_*(\mathcal{P})).$$
By our previous computations (2.44), we have
$$\mathbf{R}q_*(\mathcal{P}) \cong \Phi_{\mathcal{P}}^{E \to E}(\mathcal{O}_E) \cong \mathcal{O}_{x_0}[-1].$$
This gives
$$\mathcal{P} \circ \mathcal{P} \cong \mu^*(\mathcal{O}_{x_0}[-1]) = \mathcal{O}_{\Gamma(\iota)}[-1].$$
This checks (2.47).

Recall that any abelian category A defines the Grothendieck group $K(\mathsf{A})$. By definition
$$K(\mathsf{A}) = \mathbb{Z}^{\mathrm{Ob}(\mathsf{A})}/H,$$
where H is the subgroup generated by elements $C - A - B = 0$, where $0 \to A \to C \to B \to 0$ is an exact sequence in A. The canonical map $\mathrm{Ob}(\mathsf{A}) \to K(\mathsf{A})$ has the following universality property: any function χ on $\mathrm{Ob}(\mathsf{A})$ satisfying $\chi(C) = \chi(A) + \chi(B)$ for any exact sequence as above, extends to a unique function on $K(A)$. We denote by $[X]$ the image of an object X of A in $K(\mathsf{A})$.

Since $\mathrm{Cp}^b(\mathsf{A})$ is an abelian category, we can define $K_0(\mathrm{Cp}^b(\mathsf{A}))$. Considering the inclusion functor $\mathsf{A} \to \mathrm{Cp}^b(\mathsf{A})$ and applying the universality property of $K(\mathsf{A})$, we obtain a homomorphism of groups $\phi : K(\mathsf{A}) \to K(\mathrm{Cp}^b(\mathsf{A}))$. Define the homomorphism $K(\mathrm{Cp}^b(\mathsf{A})) \to K(\mathsf{A})$ by sending $[X^\bullet]$ to $\sum_i (-1)^i [X^i]$. It is checked that this is well-defined and is equal to the inverse of ϕ. Notice that $[X^\bullet[1]] = -[X^\bullet]$.

Next consider the categories $\mathbf{K}^b(\mathsf{A})$ and $D^b(\mathsf{A})$. We define $K(\mathbf{K}(\mathsf{A})) = \mathbb{Z}^{\mathrm{Ob}(\mathbf{K}(\mathsf{A}))}/H$, where H is generated by $C^\bullet - A^\bullet - B^\bullet$ whenever we have a distinguished triangle $A^\bullet \to C^\bullet \to B^\bullet \to A^\bullet[1]$. Similar definition is given for $K(D^b(\mathsf{A}))$. It is immediately checked that $[X^\bullet[1]] = -[X^\bullet]$ (consider the zero map $X^\bullet \to 0$ and take the corresponding distinguished triangle, its cone is

$X^\bullet[1]$). Also an exact sequence of complexes in $\mathrm{Cp}^b(\mathsf{A})$ defines a distinguished triangle, so that this definition agrees with the definition of $K(\mathrm{Cp}^b(\mathsf{A}))$ in the sense that the universality property of K defines a unique homomorphism of groups $K(\mathsf{A}) \cong K(\mathrm{Cp}^b(\mathsf{A})) \to K(\mathbf{K}^b(\mathsf{A}))$. The exact sequences in $\mathrm{Cp}^b(\mathsf{A})$)

$$0 \to \ker(d) \to X^\bullet \to \mathrm{im}(d) \to 0,$$

$$0 \to \mathrm{im}(d)[-1] \to \ker(d) \to H \to 0$$

yield

$$[H^\bullet] = [X^\bullet].$$

This shows that the image of quasi-isomorphic complexes in $K(\mathbf{K}^b(\mathsf{A}))$ are equal. This shows that

$$K(D^b(\mathsf{A})) \cong K(\mathbf{K}^b(\mathsf{A})).$$

Also the exact sequences of object-complexes

$$0 \to \ker(d^i) \to X^i \to \mathrm{im}(d^i) \to 0,$$

$$0 \to \mathrm{im}(d^{i-1}) \to \ker(d^i) \to H^i \to 0$$

give $\sum_i (-1)^i [X^i] = \sum_i (-1)[H^i]$. Since $H^\bullet = \sum_i H^i[-i]$, we obtain

$$[X^\bullet] = [H^\bullet] = \sum_i [H^i[-i]] = \sum_i (-1)^i [H^i] = \sum_i (-1)^i [X^i]. \tag{2.48}$$

The map $\mathrm{Ob}(\mathbf{K}^b(\mathsf{A})) \to K(\mathsf{A})$ sending $X^\bullet \to \sum_i (-1)^i [X^i]$ factors through $K(\mathbf{K}^b(\mathsf{A}))$ and defines the inverse of the map $K(\mathsf{A}) \to K(\mathbf{K}^b(\mathsf{A}))$. Thus we have

$$K(D^b(\mathsf{A})) \cong K(\mathbf{K}^b(\mathsf{A})) \cong K(\mathsf{A}).$$

Note how everything agrees with finite resolution complexes $K^\bullet$. We have

$$[H^0(K^\bullet)] = [K^\bullet] = \sum_i (-1)[K^i].$$

We set

$$K_0(X) = K(\mathrm{Coh}(X)),\ K^0(X) = K(\mathrm{Qcoh}(X))$$

Assume X is a smooth projective variety. By taking finite locally free resolutions $P_{\mathcal{F}}$ of objects of $\mathrm{coh}(X)$ we can define the ring structure on $K_0(X)$ by

$$[\mathcal{F}][\mathcal{G}] = [P_{\mathcal{F}} \otimes P_{\mathcal{G}}],$$

and then, extending to any complexes, to get

$$[\mathcal{F}^\bullet] \cdot [\mathcal{G}^\bullet] = [\mathcal{F} \overset{\mathbf{L}}{\otimes} \mathcal{G}].$$

The assignment $X \to D^b(X) \to K_0(X)$ defines a fibred functor on the fibred category $X \to D^b(X)$, where we take $K_0(f^*) := f^*$ to be defined by

$[\mathcal{F}^\bullet] \to [f^*(\mathcal{F}^\bullet)]$. Also the same assignment defines a cofibred functor if we set $K_0(f_*) := f_!$ to be defined by $[\mathcal{F}] \to [\mathbf{R}f_*(\mathcal{F}^\bullet)]$. Thus we can combine it with the categorical integral transform $\Phi_{\mathcal{P}}^{Y\to X}$ by taking $\tilde{K} = \cdot[\mathcal{P}]$, to obtain the *K-theoretic integral transform*

$$[\Phi]_{[\mathcal{P}]}^{K,Y\to X} : K_0(Y) \to K_0(X),\ [\Phi]_{[\mathcal{P}]}^{K,X\to Y} : K_0(X) \to K_0(Y)$$

such that the diagram

$$\begin{array}{ccc} D^b(Y) & \xrightarrow{\Phi_{\mathcal{P}}^{Y\to X}} & D^b(X) \\ \big\downarrow [?] & & \big\downarrow [?] \\ K_0(Y) & \xrightarrow{[\Phi]_{\mathcal{P}}^{Y\to X}} & K_0(X) \end{array}$$

is commutative.

Finally we can extend Example 2.2.3 to our situation. Let $\mathrm{PAV}^{\mathrm{sm}}/\mathbb{C}$ be the category of smooth projective varieties over k. We take the functors

$$F : (\mathrm{PAV}^{\mathrm{sm}}/\mathbb{C})^{\mathrm{op}} \to \mathrm{Mod}(\mathbb{F}),\ X \to H^*(X, \mathbb{F}),$$

$$G : \mathrm{PAV}^{\mathrm{sm}}/\mathbb{C} \to \mathrm{Mod}(\mathbb{F}), X \to H^*(X, \mathbb{F}),$$

where $\mathbb{F} = \mathbb{Z}, \mathbb{Q}, \mathbb{R}$, or $\mathbb{C}$. For any morphism $f : X \to Y$ we have $F(f) = f^* : H^*(Y) \to H^*(X)$ and $G(f) := f_*$ is defined by using the Poincarè duality $H^*(X) \to H_{2\dim X-*}$ on X, then composing it with $f_* : H_{2\dim X-*}(X) \to H_{2\dim Y-*}(Y)$ and using the Poincarè duality again $H_{2\dim Y-*}(Y) \cong H^*(Y)$.

For any cohomology class $\alpha \in H^*(X\times Y)$, we define the kernel $K_\alpha : H^*(X\times Y) \to H^*(X\times Y)$ as the cup-product with α. This defines the *cohomological integral transform*

$$\Phi_\alpha^{H,Y\to X} : H^*(Y) \to H^*(X),\ \Phi_\alpha^{H,X\to Y} : H^*(X) \to H^*(Y).$$

We also take $K : H^*(X\times Y, \mathbb{F}) \to H_{2n+2m-*}(X\times Y, \mathbb{F})$ to be the Poincarè-duality map, and $q_* : H_{2n+2m-*}(X\times Y, \mathbb{F}) \to H_{2n-*}(X\times Y, \mathbb{F})$ and compose it with the Poincarè duality $H_{2n-*}(X\times Y, \mathbb{F}) \to H^*(X, \mathbb{F})$ on X to obtain a cofibred functor $f_* : H^*(X, \mathbb{F}) \to H^*(Y, \mathbb{Q})$.

Next recall the *Grothendieck-Riemann-Roch Theorem* (see [Hartshorne], Ap. A, §4).

Theorem 2.2.8. *There exists a homomorphism of abelian groups* $\mathrm{ch} : K_0(X) \to H^*(X, \mathbb{Q})$ *such that for any* $e \in K_0(X)$,

$$\mathrm{ch}(f_!(e)) \cdot \mathrm{td}(Y) = f_*(\mathrm{ch}(e) \cdot \mathrm{td}(X)).$$

Here, $\mathrm{ch}(e)$ and $\mathrm{td}(X), \mathrm{td}(Y)$ are defined as follows (see [Hartshorne], Appendix 4). For any locally-free sheaf of rank r, we write (formally) the Chern polynomial $c_t(\mathcal{E})$ in the form

$$c_t(\mathcal{E}) = 1 + \sum_{i=1}^r c_i(\mathcal{E})t^i = \prod_{i=1}^r (1 + a_i t).$$

It follows from the definition that $c_t(\mathcal{E})$ depends only on $e = [\mathcal{E}] \in K_0(X)$. We put

$$\mathrm{ch}(\mathcal{E}) = \sum_{i=1}^{r} e^{a_i},\ \mathrm{td}(\mathcal{E}) = \prod_{i=1}^{r} \frac{a_i}{1 - e^{-a_i}}$$

where

$$\frac{x}{1 - e^{-x}} = 1 + \frac{1}{2}x + \frac{1}{12}x^2 - \frac{1}{720}x^4 + \cdots .$$

(all operations are taken in the cohomology rings). We set $\mathrm{td}(X) = \mathrm{td}(\mathcal{T}_X)$, where $\mathcal{T}_X$ is the tangent sheaf of X. Since any coherent sheaf has a locally free resolution, we can extend the definition of $\mathrm{ch}(e)$ to any $e \in K_0(X)$.

For any complex $\mathcal{E}^\bullet \in D^b(X)$ we have

$$e = [\mathcal{E}^\bullet] = \sum_i (-1)^i [H^i(\mathcal{E}^\bullet)] = \sum_i (-1)^i [H^i(\mathcal{E}^i)].$$

So

$$\mathrm{ch}(\mathcal{E}^\bullet) = \mathrm{ch}([\mathcal{E}^\bullet]) = \sum_i (-1)^i \mathrm{ch}(H^i(\mathcal{E}^\bullet)) = \sum_i (-1)^i \mathrm{ch}(\mathcal{E}^i).$$

In this way we extend the Grothenidieck-Riemann-Roch formula to complexes.

Remark 2.2.9. One can extend everything to the case of nonsingular projective varieties over any field k. To do this one replaces the cohomology ring with the Chow ring $A^*(X)$ of algebraic cycles modulo rational equivalence (see [Hartshorne], Appendix A, [Fulton]).

Definition 2.2.2. The *Mukai vector* of a class $e \in K_0(X)$ is defined to be the cohomology class

$$\mathrm{Mu}(e) = \mathrm{ch}(e) \cdot \sqrt{\mathrm{td}(X)}.$$

We set

$$\mathrm{Mu}(\mathcal{E}^\bullet) := v([\mathcal{E}^\bullet]).$$

With this definition we have the following.

Theorem 2.2.10. *For any $e \in K_0(X \times Y)$ and $a \in K_0(Y)$, we have*

$$\Phi_{\mathrm{Mu}(e)}^{H, Y \to X}(\mathrm{Mu}(a)) = \mathrm{Mu}(\Phi_e^{K_0, Y \to X}(a)).$$

Proof. Consider the commutative diagram

$$\begin{array}{ccccccc}
K_0(Y) & \xrightarrow{q^*} & K_0(X \times Y) & \xrightarrow{\cdot e} & K_0(X \times Y) & \xrightarrow{p_!} & K(X) \\
\downarrow{\scriptstyle \mathrm{Mu}} & & \downarrow{\scriptstyle \mathrm{Mu}\sqrt{p^*(\mathrm{td}(X))}^{-1}} & & \downarrow{\scriptstyle \mathrm{Mu}\sqrt{q^*(\mathrm{td}(Y))}} & & \downarrow{\scriptstyle \mathrm{Mu}} \\
H^*(Y) & \xrightarrow{q^*} & H^*(X \times Y) & \xrightarrow{\cdot \mathrm{Mu}(e)} & H^*(X \times Y) & \xrightarrow{p_*} & H^*(X)
\end{array}$$

To check the commutativity of the first square, we use that

$$\mathrm{td}(X \times Y) = p^*(\mathrm{td}(X)) \cdot q^*(\mathrm{td}(Y))$$

because $\mathcal{T}_{X\times Y}) \cong p^*(\mathcal{T}_X) \oplus q^*(\mathcal{T}_Y)$ and the Todd class of the direct sum is the product, i.e. $\mathrm{td}(\mathcal{E}\oplus\mathcal{F}) = \mathrm{td}(\mathcal{E})\cdot\mathrm{td}(\mathcal{F})$. The commutativity of the second square uses the same and the multiplicativity property of ch, i.e. $\mathrm{ch}(x.y) = \mathrm{ch}(x)\mathrm{ch}(y)$. The commutativity of the third square is the GRR Theorem applied to the projection p. □

In the special case when $Y = \mathrm{Spec}\ \mathbb{C}$, the Grothendieck-Riemann-Roch Theorem gives the *Hirzebruch-Riemann-Roch formula*

$$\chi(\mathcal{F}) := \sum_{i=0}^{\dim X} (-1)^q \dim_k H^q(X,\mathcal{F}) = \int_X \mathrm{ch}([\mathcal{F}])\cdot \mathrm{td}(X),$$

where the integral means taking the projection to $H^{2\dim X}(X,\mathbb{Q})$. We stated it for object-complexes only. But, it is immediately extended to complexes by additivity

$$\chi(\mathcal{F}^\bullet) = \sum_i (-1)^i \chi(\mathcal{F}^i), \quad \chi([\mathcal{F}^\bullet]) = \sum_i (-1)^i \chi([\mathcal{F}^i]).$$

If $\mathcal{F}$ is a coherent sheaf, we take its locally free resolution $\mathcal{R}^\bullet$, and apply the formula to the complex $\mathcal{R}^\bullet$ by first computing the Chern polynomial. Note that taking $\mathcal{F} = \mathcal{O}_X$, we get

$$\chi(X,\mathcal{O}_X) = \int_X \mathrm{td}(X) = \begin{cases} 1-g & \text{if, } \dim X = 1 \\ \frac{c_X^2 + c_2(X)}{12} & \text{if } \dim X = 2 \\ \frac{c_1(X)c_2(X)}{24} & \text{if } \dim X = 3. \end{cases} \tag{2.49}$$

Example 2.2.11. Assume X is a curve of genus g. By Corollary 2.1.13, any complex is isomorphic in the derived category to the direct sum of twisted object-complexes. We know that $[\mathcal{F}^\bullet[i]] = (-1)^i[\mathcal{F}^\bullet]$. So $\chi(\mathcal{F}^\bullet[i]) = (-1)^i\chi(\mathcal{F}^\bullet)$ and $\mathrm{ch}(\mathcal{F}^\bullet[i]) = (-1)^i\mathrm{ch}(\mathcal{F}^\bullet)$, so we need only to compute $\chi(\mathcal{F})$, where $\mathcal{F}$ is either a locally free sheaf or a torsion sheaf. We have $H^*(X,\mathbb{Z}) = H^0(X,\mathbb{Z}) \oplus H^2(X,\mathbb{Z}) = \mathbb{Z}\oplus\mathbb{Z}$. Any closed point x defines its fundamental cycle $[x] \in H^2(X,\mathbb{Z})$ that we can identify with number 1. By additivity, this assigns to any divisor $D = \sum n_i x_i$ its fundamental class $[D] \in H^2(X,\mathbb{Z})$ identified with $\deg(D)$. We have

$$c_1(\mathcal{O}_X(D)) = \deg D, \quad \mathrm{ch}(\mathcal{O}_X(D)) = [X] + \deg(D),$$

$$\mathrm{td}(X) = [X] + \frac{1}{2}\deg K_X = [X] + 1 - g.$$

Any locally free sheaf $\mathcal{E}$ of rank r fits in an exact sequence of locally free sheaves

$$0 \to \mathcal{E}_1 \to \mathcal{E} \to \mathcal{E}_2 \to 0 \tag{2.50}$$

where $\mathrm{rank}(\mathcal{E}_1) = 1, \mathrm{rank}(\mathcal{E}_2) = r-1$. In fact we first tensor $\mathcal{E}$ with some $\mathcal{O}_X(D)$ with $\deg D >> 0$ to assume that $\mathcal{E}$ has a section. This defines an injective homomorphism of sheaves $\mathcal{O}_X \to \mathcal{E} \otimes \mathcal{O}_X(D)$. Then we untwist by tensoring with $\mathcal{O}_X(-D)$ to obtain an injection $\mathcal{O}_X(-D) \to \mathcal{E}$. Then we saturate this injection, by taking the largest invertible subsheaf of $\mathcal{E}$ that contains $\mathcal{O}_X(D)$ such that the quotient sheaf has no torsion, hence locally free. By using locally free resolutions, we obtain that $K_0(X)$ is generated by the classes $[\mathcal{O}_X(D)]$ of invertible sheaves.

We can do better. Consider the map

$$\det : K_0(X) \to \mathrm{Pic}(X),\ \mathcal{E} \to \det(\mathcal{E}) := \Lambda^{\mathrm{rank}\mathcal{E}}(\mathcal{E}),$$

where $\mathcal{E}$ is a locally free sheaf, and we use that $K_0(X)$ is generated by the classes of those. An exact sequence (2.50) of locally free sheaves gives $\det(\mathcal{E}) = \det(\mathcal{E}_1) \otimes \det(\mathcal{E}_2)$. This shows that the map is well-defined and is a homomorphism of abelian groups. Consider the map

$$\alpha : K_0(X) \to \mathbb{Z} \oplus \mathrm{Pic}(X),\ \mathcal{E} \to (\mathrm{rank}(\mathcal{E}), \det(\mathcal{E})).$$

It is obviously a surjective map. It is not a trivial fact that α is an isomorphism. In fact, this homomorphism is defined for varieties of any dimension, and its kernel is isomorphic to $F^2(K_0(X))$, where $F^i(K_0(X))$ is the subgroup generated by the classes of sheaves with support in a closed subscheme of codimension $\geq i$ (see [Manin, Russ. Math. Survey, 1969).

In the case when $\mathcal{E}$ is locally free, the Riemann-Roch gives

$$\chi(X, \mathcal{E}) = \deg\det(\mathcal{E}) + \mathrm{rank}(\mathcal{E})(1-g).$$

For any two complexes $\mathcal{F}^\bullet, \mathcal{G}^\bullet \in \mathrm{Ob}(D^b(X))$ define

$$\chi(\mathcal{F}^\bullet, \mathcal{G}^\bullet) = \sum_i (-1)^i \dim_k \mathrm{Ext}^i_k(\mathcal{F}^\bullet, \mathcal{G}^\bullet) = \chi(\mathcal{F}^\vee \overset{\mathbf{L}}{\otimes} \mathcal{G}^\bullet).$$

Define the Mukai pairing on $H^*(X, \mathbb{Q})$ by

$$\langle v, v' \rangle_X = \int_X \exp(c_1(X)/2) \cdot v^\vee \cdot v',$$

where, for any $v = \sum v_s \in \oplus H^{2s}(X, \mathbb{Q})$, we set $v^\vee = \sum_s (-1)^s v_s$.

Proposition 2.2.12.

$$\chi(\mathcal{F}^\bullet, \mathcal{G}^\bullet) = \langle \mathrm{Mu}(\mathcal{E}^\bullet), \mathrm{Mu}(\mathcal{G}^\bullet) \rangle.$$

Proof. By Riemann-Roch,

$$\chi(\mathcal{F}^\bullet, \mathcal{G}^\bullet) = \chi(\mathcal{F}^{\bullet\vee} \overset{\mathbf{L}}{\otimes} \mathcal{G}^\bullet) = \int_X \mathrm{ch}(\mathcal{E}^{\bullet\vee}) \cdot \mathrm{ch}(\mathcal{F}^\bullet) \cdot \mathrm{td}(X) \tag{2.51}$$

$$= \int_X (\mathrm{ch}(\mathcal{E}^{\bullet\vee}) \cdot \sqrt{\mathrm{td}(X)}) \cdot (\mathrm{ch}(\mathcal{F}^\bullet) \cdot \sqrt{\mathrm{td}(X)}).$$

If $c_t(\mathcal{F}^\bullet) = \prod(1+a_it)$, then $\mathrm{ch}(\mathcal{F}^{\bullet\vee}) = \prod(1-a_it)$. This gives $\mathrm{ch}(\mathcal{F}^\bullet) = \sum_i e^{-a_i} = \mathrm{ch}(\mathcal{E}^\bullet)^\vee$. Now it is easy to see that

$$\mathrm{td}(X)^\vee \cdot \exp(c_1(X)) = \prod \frac{-\gamma_i}{1-e^{\gamma_i}} \cdot \prod e^{\gamma_i},$$

where $c_t(\mathcal{T}_X) = \prod_i(1+\gamma_i t^i)$. Thus we can rewrite (2.51) in the form

$$\chi(\mathcal{F}^\bullet, \mathcal{G}^\bullet) = \int_X \exp(c_1(X)/2) \cdot \mathrm{Mu}(\mathcal{F}^\bullet)^\vee \cdot \mathrm{Mu}(\mathcal{G}^\bullet).$$

This proves the assertion. □

Example 2.2.13. Assume $\dim X = 1$. Let $v = v_0 + v_1 \in H^*(X)$. Then $v^\vee = v_0 - v_1$ and $\mathrm{td}(X) = [X] + \mathrm{td}(X)_1$, $\sqrt{\mathrm{td}(X)} = [X] + \frac{1}{2}\mathrm{td}_1$. Let $f = [\mathcal{F}^\bullet], g = [\mathcal{G}^\bullet]$, then

$$\chi(\mathcal{F}^\bullet, \mathcal{G}^\bullet) = \langle \mathrm{Mu}(f)^\vee, \mathrm{Mu}(g)\rangle$$

$$= \int_X ([X] + (1-g))(\mathrm{ch}_0(f) - \mathrm{ch}_1(f)(\mathrm{ch}_0(g) + \mathrm{ch}_1(g)([X] - \frac{1}{2}\mathrm{td}_1)([X] + \frac{1}{2}\mathrm{td}_1)$$

$$= \mathrm{ch}(f)_0\mathrm{ch}(g)_0(1-g) + \mathrm{ch}(f)\mathrm{ch}_1(g) - \mathrm{ch}(g)\mathrm{ch}(f))_1$$

$$= \mathrm{rank}(\mathcal{E}^\bullet)\mathrm{rank}(\mathcal{G}^\bullet)(1-g) + \mathrm{rank}(\mathcal{F}^\bullet)\deg\det(\mathcal{G}^\bullet) - \mathrm{rank}(\mathcal{G}^\bullet)\deg(\det(\mathcal{F}^\bullet)).$$

Here

$$\mathrm{rank}[\mathcal{K}^\bullet] = \sum_i (-1)^i \mathrm{rank}\mathcal{K}^i = \sum_i (-1)^i \mathrm{rank} H^i(\mathcal{K}^\bullet),$$

and the rank of a coherent sheaf is equal to the dimension of its stalk at the general point η over $k(\eta)$.

Example 2.2.14. Let $\mathcal{F}^\bullet = \mathcal{G}^\bullet = \mathcal{E}$, where $\mathcal{E}$ is an object-complex corresponding to a locally free sheaf of rank r. We get

$$\chi(\mathcal{E}, \mathcal{E}) = \chi(\mathcal{E}^\vee \otimes \mathcal{E}) = \mathrm{Mu}(\mathcal{E})^2 := \langle \mathrm{Mu}(\mathcal{E}), \mathrm{Mu}(\mathcal{E})\rangle.$$

We have $\dim H^0(\mathcal{E}^\vee \otimes \mathcal{E}) = \dim_k \mathrm{End}_k(\mathcal{E})$. A locally free sheaf (or the corresponding vector bundle) is called *simple* if $\mathrm{End}_k(\mathcal{E}) = k$. Using the deformation theory one can show that $m = \dim H^1(X, \mathcal{E}^\vee \otimes \mathcal{E})$ is equal to the dimension of the moduli space of simple vector bundles at the point corresponding to $\mathcal{E}$. If $\dim X = 1$, we get the formula

$$m = \mathrm{rank}(\mathcal{E})^2(g-1) + 1.$$

In the case $\mathrm{rank}(\mathcal{E}) = 1$, we obtain $m = g = \dim \mathrm{Pic}^d(X)$, where $d = \deg(\mathcal{E})$.

If $\dim X = 2$ and $\omega_X \cong \mathcal{O}_X$ (i.e. X is an abelian surface or a K3 surface), we can use the Serre duality, to obtain $\dim H^2(X, \mathcal{E}^\vee \otimes \mathcal{E}) = \dim \mathrm{End}(\mathcal{E})$. This gives the formula

$$m = 2 - \mathrm{Mu}(\mathcal{E})^2.$$

In particular, simple bundles which are rigid, i.e. $m = 0$, satisfy $\mathrm{Mu}(\mathcal{E}) = 2$. For example, let $\operatorname{rank} \mathcal{E} = r, c_i(\mathcal{E}) = c_i$. We have $\mathrm{ch}(\mathcal{E}) = r + c_1(\mathcal{E}) + \frac{1}{2}c_1^2 - c_2$, $\mathrm{td}(X)_1 = 0, \mathrm{td}(X)_2 = \chi(X, \mathcal{O}_X), \sqrt{\mathrm{td}(X)} = 1 + \frac{1}{2}\chi(X, \mathcal{O}_X)$. This gives

$$\mathrm{Mu}(\mathcal{E}) = (\mathrm{Mu}(\mathcal{E})_0, \mathrm{Mu}(\mathcal{E})_1, \mathrm{Mu}(\mathcal{E})_2) = (r, c_1(\mathcal{E}), \frac{1}{2}(r\chi(X, \mathcal{O}_X) + c_1^2) - c_2),$$

$$\mathrm{Mu}(\mathcal{E})^2 = (r-1)c_1^2 - 2rc_2 + r^2\chi(X, \mathcal{O}_X). \tag{2.52}$$

Assume $r = 1$. We get $\mathrm{Mu}(\mathcal{E})^2 = 0$ if X is an abelian surface, so that $m = 2$, and $\mathrm{Mu}(\mathcal{E})^2 = 2$ if X is a K3 surface, so that $m = 0$.

Remark 2.2.15. For a K3 surface X one defines the *Mukai lattice* by $\mathrm{Mu}(X) = H^0(X, \mathbb{Z}) \oplus \mathrm{Pic}(X) \oplus H^4(X, \mathbb{Z})$ with inner product

$$(a, b, c) \cdot (a', b', c')\rangle = b \cdot b' - a' \cdot c - a' \cdot c.$$

We have

$$\langle \mathrm{Mu}(e), \mathrm{Mu}(e')\rangle_X = -\mathrm{Mu}(e) \cdot \mathrm{Mu}(e').$$

2.3 Equivariant derived categories

Let C be a category and G be a *group object* in C, i.e. a Gr-object in C, where Gr is the category of groups (see section 1.1). For any $S \in \mathrm{Ob}(\mathsf{C})$, the set $G(S) := h_G(S) = \mathrm{Mor}_{\mathsf{C}}(S, G)$ is a group and, for any morphism $S' \to S$ in C, the maps $G(S) \to G(S')$ are homomorphism of groups. Assume that C has products and the functor $S \to \{e_{G(S)}\} \subset G(S)$, where $e_{G(S)}$ is the neutral element in $G(S)$, is represented by a final object e in C. By Yoneda's Lemma, there is a morphism $\mu : G \times G \to G$ (the group law) such that, for any $S \in \mathrm{Ob}(\mathsf{C})$, the map of sets $\mu(S) : G(S) \times G(S) \to G(S)$ is a group law on the set $G(S)$. There is a morphism $\epsilon : e \to G$ defined by $h_e(S) \to G(S)$ with the image equal to $e_{G(S)}$. Also there is morphism $\iota : G \to G$ with $G(S) \to G(S)$ expressing taking the inverse. The morphisms μ, ϵ, ι must satisfy certain natural commutative diagrams describing the group axioms.

An *action* of G on $X \in \mathrm{Ob}(\mathsf{C})$ is a morphism $\sigma : G \times X \to X$ such that the corresponding map of pre-sheeaves $h_G \times h_X \to h_X$ is an action in the category of presheaves. For any $S \in \mathsf{C}$ we have a group action $\sigma(S) : G(S) \times X(S) \to X(S)$ in the category of sets, where $X(S) = h_X(S)$. These actions must be functorial in S. For any $g, h \in G(S), x \in X(S)$ we write $g \cdot h = \mu(S)(g, h), g \cdot x = \sigma(S)(g, x)$.

Let (X, σ) be a pair consisting of an object X and a G-action $\sigma : G \times X \to X$. A *$G$-equivariant morphism* $(X, \sigma) \to (X', \sigma')$ is a morphism $f : X \to X'$ such that the diagram

$$\begin{array}{ccc} G \times X & \xrightarrow{\sigma} & X \\ \downarrow{\scriptstyle \mathbf{id}_G \times f} & & \downarrow{\scriptstyle f} \\ G \times X' & \xrightarrow{\sigma'} & X' \end{array}$$

is commutative. It is easy to see that the pairs (A,σ) with G-equivariant morphisms form a category which we denote by C_G. It admits a forgetting functor $\mathsf{C}_G \to \mathsf{C}$.

Suppose D is a fibred category over C. We have the functors $\sigma^* : \mathsf{D}_X \to \mathsf{D}_{G\times X}$ and $\mathrm{pr}_X^* : \mathsf{D}_X \to \mathsf{D}_{X\times G}$, where $\mathrm{pr}_X : G\times X \to X$ is the projection. A *G-linearization* on an object A in D_X is an isomorphism

$$\mathrm{pr}_X^*(A) \to \sigma^*(A)$$

satisfying the following *cocycle condition*. We have three morphisms

$$G\times G\times X \; \begin{array}{c} \xrightarrow{p_{23}} \\ \xrightarrow{\mu\times \mathbf{id}_X} \\ \xrightarrow{\mathbf{id}_G\times\sigma} \end{array} \; G\times X,$$

where p_{23} is the projection to the product of the last two factors. We require that

$$(\mu\times \mathbf{id}_X)^*(\alpha) = ((\mathbf{id}_G\times\sigma)\circ p_{12})^*(\alpha).$$

To understand this condition, let $\alpha : e \to G$ be the morphism defining the neutral elements in $G(S)$. Since e is a final object in C we may identify $e\times X$ with X. Let $g\in G(e)$ and $\sigma_g = \sigma\circ(g\times \mathbf{id}_X) : X = e\times X \to X$. For any $A\in \mathrm{Ob}(\mathsf{D}_X)$ set $g^*(A) = \sigma_g^*(A) \in \mathrm{Ob}(\mathsf{D}_X)$. The composition $\mathrm{pr}_X\circ(g\times\mathbf{id}_X) : X = e\times X \to X$ can be identified with the identity $\mathbf{id}_X$. The linearization $\alpha : \mathrm{pr}_X^*(A) \to \sigma^*(A)$ defines an isomorphism

$$\alpha_g = (g\times \mathbf{id}_X)^*(\sigma) : A \to g^*(A).$$

The cocycle condition can be expressed as follows. For any $g,h\in G(e)$

$$\alpha_{g\cdot h} = h^*(\alpha_g)\circ\alpha_h : A \to h^*(A) \to h^*(g^*(A)). \tag{2.53}$$

A *G-object* in D_X is a pair (A,α), where α is a G-linearization on A. One defines naturally the category $\mathsf{D}_{X,\sigma}$ of G-objects in D_X. A morphism $(A,\alpha)\to(B,\beta)$ is a morphism $\phi : A\to B$ in D_X such that the following diagram is commutative

$$\begin{array}{ccc} \sigma^*(A) & \xrightarrow{\sigma^*(\phi)} & \sigma^*(B) \\ \downarrow{\scriptstyle\alpha} & & \downarrow{\scriptstyle\beta} \\ q^*(A) & \xrightarrow{q^*(\phi)} & q^*(B) \end{array}$$

For any equivariant morphism $f : (X,\sigma)\to(Y,\sigma')$ and a G-object (A,α) in $\mathsf{D}_{Y,\sigma}$, the object $f^*(A)$ in D_X admits the linearization equal to $(\mathbf{id}_G\times f)^*(\alpha)$. One checks that the assignment $(X,\sigma)\to\mathsf{D}_{X,\sigma}$ defines a fibred category over C_G with functors f^* corresponding to G-equivaraint morphisms.

Example 2.3.1. Let Top be the category of topological spaces and G be a group considered as a topological group with discrete topology. Let $\sigma : G\times X\to X$ be an action of G on X in the category of topological spaces. This means that, for any $g \in G$, the map $g : X \to X, x \to g \cdot x$, is continuous. Let $\mathrm{Sh}^{\mathrm{ab}}$ be the fibred category over Top that assigns to each X the category $\mathrm{Sh}^{\mathrm{ab}}_X$ of abelian sheaves on X with the pull-back functors f^*. As above we form the category Top_G of topological spaces with an action of the group G and consider the fibred category $X \to \mathrm{Sh}^{\mathrm{ab}}_G(X)$ of G-linearized abelian sheaves, or just abelian G-sheaves. Taking for the final object e in Top a singleton $\{e\}$, we see that a G-linearization is defined by isomorphisms $\alpha_g : A \to g^*(\mathcal{A})$ satisfying the cocycle condition (2.53).

One can also express the linearization as follows. Recall the original definition of a sheaf $\mathcal{F}$ in terms of its *espace étale*. It is a topological space $\mathrm{Es}(\mathcal{F})$ which, as a set, is equal to the union $\coprod_{x \in X} \mathcal{F}_x$ of stalks. It comes with the projection $\mathrm{Es}(\mathcal{F}) \to X$ by sending $\mathcal{F}_x$ to x. We equip the set with the strongest topology such that all local sections $s : U \to \mathrm{Es}(\mathcal{F})$ defined by $s \to s_x \in \mathcal{F}_x$ are continuous maps. In this way $\mathcal{F}$ is reconstructed from $\mathrm{Es}(\mathcal{F})$ by expressing $\mathcal{F}(U)$ as the set of all continuous sections $U \to \mathrm{Es}(\mathcal{F})$. Now one can check that a G-linearization on $\mathcal{F}$ is a lift of the action of G on X to $\mathrm{Es}(\mathcal{F})$ compatible with the projection.

Recall that a morphism of ringed spaces $(X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ consists of a continuous map $f : X \to Y$ and a map of sheaves $f^\# : \mathcal{O}_Y \to f_*\mathcal{O}_X$. We consider a group G as a discrete topological space with the constant sheaf of rings $\mathbb{Z}_G$. The product $(G, \mathbb{Z}_G) \times (X, \mathcal{O}_X)$ can be identified with the disjoint sum $\coprod_{g \in G} X_g$ of copies of X together with the sheaf of ring equal to $\mathcal{O}_X$ on each component X_g. An action of G on $(X, \mathcal{O}_X)$ consists of an action $G \times X \to X$ in the category Top together with a morphism of sheaves $\sigma^\# : \mathcal{O}_X \to \sigma_*(\mathcal{O}_{G \times X})$. It is easy to see that it is defined by a collection of automorphisms $g : (X, \mathcal{O}_X) \to (X, \mathcal{O}_X)$ satisfying the cocycle condition. By definition of a morphism of ringed space, for any open subset $U \subset X$, there is an isomorphism of rings $g^*\mathcal{O}_X(U) \to \mathcal{O}_X(g^{-1}(U))$, these isomorphisms must define a G-linearization on the sheaf $\mathcal{O}_X$.

This defines a category RTop_G whose objects are ringed topological spaces together with a G-action. Now we can consider a fibred category over RTop_G by assigning to each $(X, \mathcal{O}_X)$ the category $\mathrm{Mod}_G(\mathcal{O}_X)$ of G-linearized $\mathcal{O}_X$-modules. We call its objects $G - \mathcal{O}_X$-modules.

For example, suppose G acts trivially on X and on $\mathcal{O}_X$. Then a $G - \mathcal{O}_X$-module is defined by a set of isomorphisms $\alpha_g : \mathcal{A} \to g^*(\mathcal{A}) = \mathcal{A}$ of $\mathcal{O}_X$-modules such that $\alpha_{g \cdot g'} = \alpha_g \circ \alpha_{g'}$. In other words it defines a homomorphism of groups $G \to \mathrm{Aut}_{\mathcal{O}_X}(\mathcal{A}), g \to \alpha_g$. Specialing more, we take for X a point so that $\mathcal{O}_X$ is a ring R and $\mathcal{A}$ is a R-module M. Then a $G - \mathcal{O}_X$-module becomes a representation of G in M, i.e. a homomorphism $G \to \mathrm{Aut}_R(M)$.

More generally, suppose G acts trivially on X but not necessary trivially on $\mathcal{O}_X$. We can define a new sheaf of rings $\mathcal{O}_X\#G$ whose sections on an open subset U is the skew group algebra $\mathcal{O}_X(U)\#G$. Recall that for any ring R with a group action $G \to \mathrm{Aut}(R), (g, r) \mapsto {}^g r$, one defines the skew group algebra $R\#G$. It is the free abelian group R^G of formal linear combinations $\sum_{g \in G} r_g g$ with product defined by

$$(r_g \cdot g)(r_{g'} g') = r_g {}^g r_{g'} g g'.$$

One can show that $\mathrm{Mod}_G(\mathcal{O}_X)$ is equivalent to the category $\mathrm{Mod}(\mathcal{O}_X\#G)$.

It is easy to check that the category of $G{-}\mathcal{O}_X$-modules is an abelian category. It also has enough of injective objects. Here is the proof due to A. Grothendieck.

Let M_x be a collection of $\mathcal{O}_{X,x}$-modules for all $x \in X$. It defines a sheaf by setting $\mathcal{M}(U) = \prod_{x\in U} M_x$. In other words $\mathcal{M} = \prod_{x\in X}(i_x)_*M_x$, where $i_x : x \hookrightarrow X$ is the inclusion map and M_x is the sheaf on x with value M_x. For any $\mathcal{O}_X$-module $\mathcal{F}$ a map of $\mathcal{F}$ to $\mathcal{M}$ is defined by a family of homomorphisms $\mathcal{F}_x \to M_x, x \in X$. Let $Y = X/G$ be the set of orbits of G. For each $x \in X$ let $U_x = \mathcal{O}_{X,x}\#G_x$, where G_x is the stabilizer of x. For each $y \in Y$ choose a representative $\xi(y) \in G\cdot x$ and let $U_y = U_{\xi(y)}$. Define the sheaf of algebras $\mathcal{U} = \prod_{y\in Y}(i_y)_*(U_y)$. Let $\mathcal{A}$ be the sheaf defined by a collection $(A_y)_{y\in Y}$ of U_y-modules. Now define a $G-\mathcal{O}_X$-sheaf $\tilde{\mathcal{A}}$ on X by the collection of modules $A_x = A_y$ where $x = g(\xi(y))$. The group G acts naturally on $\mathcal{A}$ via its natural action on $\mathrm{Es}(\mathcal{A})$. For any $G-\mathcal{O}_X$-sheaf $\mathcal{M}$ on X let $\overline{\mathcal{M}}$ be the sheaf of $\mathcal{U}$-modules defined by the U_y-modules $\mathcal{M}_{\xi(y)}$. Then

$$\mathrm{Hom}_{\mathcal{U}}(\overline{\mathcal{M}}, \mathcal{A}) = \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{M}, \tilde{\mathcal{A}}).$$

Indeed an element in the first set is defined by a collection of homomorphisms $\mathcal{M}_x \to A_x = gA_{\xi(y)}$ which is determined uniquely by homomorphisms $\mathcal{M}_{\xi(y)} \to A_y$.

Now it remains to take $\mathcal{A}$ to be defined by a collection of injective U_x-modules. The corresponding sheaf $\tilde{\mathcal{A}}$ is an injective G-sheaf of $\mathcal{O}_X$-modules. Since $\mathrm{Mod}(\mathcal{U})$ has enough injective modules, we can embed $\overline{\mathcal{M}}$ in an injective $\mathcal{A}$, to get an embedding of $\mathcal{M}$ in an injective $\mathcal{O}_X$-module $\tilde{\mathcal{A}}$.

Let $\mathcal{S} = \mathrm{Sch}/k$ be the category of schemes over a field k. A group object G in Sch/k is called a *group scheme* over k. Consider the fibred category $X \to \mathrm{Qcoh}(X)$. Let $\sigma : G\times X \to X$ be an action. For any field extension K/k we have an action $G_K\times X_K \to X_K$, where the subscript denotes the base extension. Now an element $g \in G(K)$ defines an automorphism $g : X_K \to X_K$ and the functor $g^* : \mathrm{Qcoh}(X_K) \to \mathrm{Qcoh}(X_K)$. A G-object in $\mathrm{Qcoh}(X)$ is a sheaf $\mathcal{F}$ such that for all K/k there is an isomorphism $\alpha_g : \mathcal{F}_K \to g^*(\mathcal{F}_K)$ satisfying the cocycle condition. These isomorphisms should be compatible with composition of extensions.

From now on we will restrict ourselves with the special case when $\mathcal{S}$ is the category AV/k of algebraic varieties over a field k and G a finite group, considered as a *constant group scheme* over k (i.e. the Yoneda functor h_G is the constant presheaf with values equal to G). In this case the action of G on X is determined by a homomorphism $G \to \mathrm{Aut}_k(X)$. A linearization on a quasi-coherent sheaf $\mathcal{F}$ is a family of isomorphisms of sheaves $\alpha_g : \mathcal{F} \to g^*(\mathcal{F})$ satisfying the cocycle condition. Let $\mathrm{Qcoh}_G(X)$ be the category of G-linearized quasi-coherent sheaves.

Example 2.3.2. Suppose $\mathcal{E}$ is a G-linearized locally free sheaf of rank r on a scheme X over a field k. Let $\mathbb{V}(\mathcal{E}) = \mathrm{Spec}\ \mathrm{Sym}^\bullet(\mathcal{E}^\vee)$ be the corresponding vector bundle. The sheaf of local sections of $\mathbb{V}(\mathcal{E})$ is isomorphic to $\mathcal{E}$. A G-linearization defines isomorphisms $\alpha_g : \mathcal{E} \to g^*(\mathcal{E})$, and hence their transposes $g^*(\mathcal{E})^\vee = g^*(\mathcal{E}^\vee) \to \mathcal{E}^\vee$. Taking the symmetric powers we obtain isomorphisms $g^*(\mathrm{Sym}^\bullet(\mathcal{E}^\vee)) \to \mathrm{Sym}^\bullet(\mathcal{E}^\vee)$ which define isomorphisms of vector bundles $\tilde{\alpha}_g : g^*(\mathbb{V}(\mathcal{E})) = X \times_X \mathbb{V}(\mathcal{E}) \to \mathbb{V}(\mathcal{E})$, the base change with respect to the morphism $g : X \to X$. For each point $x \in X$ the morphism $\tilde{\alpha}_g$ defines a $k(x)$-linear isomorphism of the fibres $\mathbb{V}(\mathcal{E})_{g^{-1}(x)} \to \mathbb{V}(\mathcal{E})_x$. Thus a G-linearization on $\mathcal{E}$ allows one to lift the action of G on X to the action of G on $\mathbb{V}(\mathcal{E})$, compatible with the natural projection morphism $\mathbb{V}(\mathcal{E}) \to X$. The converse is also true.

Example 2.3.3. Let X be an integral scheme with an action of a constant group scheme G. Let $\mathrm{Pic}(X)$ be the group of isomorphism classes of invertible sheaves on X, or, equivalently the group of Cartier divisors modulo principal divisors. Let $\mathcal{L} = \mathcal{O}_X(D)$ be the invertible sheaf defined by a Cartier divisor D on X. The group G acts naturally on the constant sheaf $k(X)$ of rational functions on X. We denote the action by $f \to {}^g f$. If f is considered as a morphism to $\mathbb{P}^1$, then ${}^g f = f \circ g^{-1}$. Via the action of G on $k(X)^*$, the group G acts on the group of Cartier divisors $\mathrm{CDiv}(X) = \Gamma(k(X)^*/\mathcal{O}_X^*)$ (by transforming the local equation $\phi = 0$ on U into the local equation ${}^g\phi = 0$ on $g^{-1}(U)$).

Suppose D is a G-invariant divisor, i.e. $g^*(D) = D$ (considered as a sections of $k(X)^*/\mathcal{O}_X^*$). Then $\mathcal{O}_X(D)(U) = \{f \in k(X)^* : fD_U \in \mathcal{O}_X(U)/\mathcal{O}_X(U)^*\}$. Thus $f \to {}^g f$ defines an isomorphism $\mathcal{O}_X(D)(U) \to \mathcal{O}_X(D)(g^{-1}(U)), f \to {}^g f$. These isomorphisms define a linearization $\alpha_g : g^*(\mathcal{O}_X(D)) \to \mathcal{O}_X(D)$. This defines a homomorphism from the group $\mathrm{Div}(X)^G$ of G-invariant Cartier divisors to the group $\mathrm{Pic}^G(X)$ of isomorphism classes G-linearized invertible sheaves. An element of the kernel corresponds to a principal G-invariant Cartier divisor $D = \mathrm{div}(f)$. The function f must satisfy $\mathrm{div}({}^g f) = \mathrm{div}(f)$ and hence satisfy ${}^g f = c_g f$, where $c_g \in \mathcal{O}_X(X)^*$ is an invertible global section of $\mathcal{O}_X$. The assignment $g \to c_g$ defines a homomorphism of groups $\chi : G \to \mathcal{O}_X(X)^*$. It is clear that $\mathcal{O}_X(D)$ belongs to the kernel of the forgetting homomorphism $r : \mathrm{Pic}^G(X) \to \mathrm{Pic}(X)$. Although as an element of $\mathrm{Pic}^G(X)$ it is trivial if and only if χ is the trivial homomorphism. In this case $D = \mathrm{div}(f)$, where f is a G-invariant section of $k(X)^*$.

Now start with any G-linearized invertible sheaf $\mathcal{L}$ defining an element of $\mathrm{Pic}^G(X)$. Choose an isomorphism $\mathcal{L} \cong \mathcal{O}_X(D)$ for some Cartier divisor D and transfer the linearization to $\mathcal{O}_X(D)$. An isomorphism $\alpha_g : g^*(\mathcal{O}_X(D)) \to \mathcal{O}_X(D)$ is defined by a rational function f_g such that $D + \mathrm{div}(f_g) = g^*(D)$. The cocycle condition implies that $f_{g' \circ g} = {}^g f_{g'} f_g$ for any $g, g' \in G$. In other words the collection $\{f_g\}_{g \in G}$ is a 1-cocycle of G with values in $k(X)^*$. If we consider $k(X)$ as a Galois extension of its field of invariants $k(X)^G$ with the Galois group G, then the famous Hilbert's Theorem 90 implies that $H^1(G, k(X)^*) = 0$. Thus we can write $f_g = {}^g a/a$ for some $a \in k(X)^*$ independent of g. Replacing D with $D' = D - \mathrm{div}(a)$, we obtain $g^*(D') = D'$ for any $g \in G$. Thus $\mathcal{L}$ is isomorphic as a G-sheaf to a sheaf $\mathcal{O}_X(D)$ corresponding to G-invariant divisor.

To sum up we obtain that $\mathrm{Pic}^G(X)$ is isomorphic to the group of G-invariant divisors modulo principal divisors. We have an exact sequence

$$0 \to \mathrm{Hom}(G, \mathcal{O}_X(X)^*) \to \mathrm{Pic}^G(X) \xrightarrow{r} \mathrm{Pic}^G(X) \to H^2(G, \mathcal{O}_X(X)^*).$$

One can prove that the last homomorphism is surjective when X is a smooth projective curve.

Let us specialize. Take X to be a smooth projective curve over an algebraically closed field k. A Cartier divisor D can be identified with a Weil divisor $D = \sum_x n_x x$. It is G-invariant if the function $x \to n_x$ is constant on G-orbits. Thus any G-invariant divisor is an integral linear combination of orbits. Let $\pi : X \to Y = X/G$ be the projection to the orbit space (which is a smooth projective curve). A G-invariant principal divisor is a linear combination of scheme-theoretical fibres of π. Let $y_1, \ldots, y_r$ be the orders with non-trivial stabilizers of orders $e_1, \ldots, e_n$. Assume $Y = \mathbb{P}^1$. Let $\mathrm{Pic}^G(X)^0$ be the subgroup of isomorphism classes of G-linearized invertible sheaves of degree 0. It is equal to the kernel of the homomorphism $\deg : \mathrm{Pic}^G(X) \to \mathbb{Z}$ defined by the degree of a divisor. The image of $\mathrm{Pic}^G(X)$ under this homomorphism is equal to a cyclic group (m). Since the canonical sheaf ω_X admits a G-linearization (the corresponding G-invariant divisor can be defined by using the Hurwitz formula $K_X = \pi^*(K_{\mathbb{P}^1}) + R$, where $R = \sum d_i y_i$ is the ramification divisor), we have $m|2g-2$.

By above any element of $\mathrm{Pic}^G(X)$ is represented by a sum $D = \sum n_y y$, where $y \in Y$ is considered as an effective divisor on X representing the orbit defined by y. For any $y \notin \{y_1, \ldots, y_n\}$, we can find a rational function f on $\mathbb{P}^1$ such that $y - e_1 y_1 = \mathrm{div}(f)$. This shows that that we may represent an element of $\mathrm{Pic}^G(X)^0$ by a linear combination $\sum n_i y_i$ with $\sum n_i = 0$. Its kernel consists of linear combinations $\sum m_i e_i y_i$, where $\sum m_i = 0$. This defines a natural isomorphism

$$\mathrm{Pic}^G(X)^0 \to A(e_1, \ldots, e_r),$$

where $A(e_1, \ldots, e_r)$ is the abelian group defined by generators $g_1, \ldots, g_r$ and relations $g_1 + \ldots + g_r = 0, e_i g_i = 0, i = 1, \ldots, r$. Using the theory of elementary divisors in the theory of abelian groups, we obtain

$$A(e_1, \ldots, e_r) \cong \mathbb{Z}/a_1\mathbb{Z} \oplus \ldots \oplus \mathbb{Z}/a_r\mathbb{Z},$$

where $a_i = c_i/c_{i-1}$, $c_0 = 1$, and

$$c_k = g.c.d.((e_{i_1} \cdots e_{i_k})_{1 \le i_1 < \cdots < i_k \le r}),\ k = 1, \ldots, r-1.$$

For example, if $g.c.d.(e_1, \ldots, e_r) = 1$ we get $\mathrm{Pic}^G(X)^0 = \{0\}$. On the other hand, let $X = \mathbb{P}^1$ and $G = \mathbb{Z}/2\mathbb{Z}$ that acts by $(t_0 : t_1) \to (-t_0, t_1)$. We assume that $\mathrm{char}(k) \neq 2$. The exceptional orbits are $0 = (1:0)$ and $\infty = (0:1)$ with $e_1 = e_2 = 2$. A field of invariant rational functions is equal to $k(x^2)$, where $x = t_1/t_0$. Thus we see that $\mathcal{O}_X$ admits two non-isomorphic linearizations corresponding to the divisors $D = 0 - \infty$ and $D = 0$. They correspond to the group of characters $\mathrm{Hom}(\mathbb{Z}/2\mathbb{Z}, k^*) \cong \mathbb{Z}/2\mathbb{Z}$. Every sheaf $\mathcal{O}_X(n)$ admits two non-isomorphic linearizations corresponding to the divisors $n{\cdot}0$ and $(n+1){\cdot}0 - \infty$.

Lemma 2.3.4. *Let G be a group scheme. The category* $\mathrm{Qcoh}_G(X)$ *is an abelian category. If G is a constant group scheme then* $\mathrm{Qcoh}_G(X)$ *has enough injective objects.*

Proof. Let $\sigma : G \times X \to X$ be the action. It is equal to the composition of the morphisms $\mathrm{pr}_X \circ (\mathbf{id}_G, \sigma) : G \times X \to G \times X \to X$. The first morphism is an automorphism and the second map is a flat morphism. Thus σ is a flat morphism. This implies that σ^*, q^* are exact functors. Let $\mathcal{K} = \ker(\mathcal{F} \to \mathcal{G})$ be the kernel of a morphism of G-bundles. Then $\sigma^*(\mathcal{K}) = \ker(\sigma^*(\mathcal{F}) \to \sigma^*(\mathcal{G}))$, $q^*(\mathcal{K}) = \ker(q^*(\mathcal{F}) \to q^*(\mathcal{G}))$ and the isomorphisms $\alpha : \sigma^*(\mathcal{F}) \to q^*(\mathcal{F}), \beta = \sigma^*(\mathcal{G}) \to q^*(\mathcal{G})$ restrict to an isomorphism $\sigma^*(\mathcal{K}) \to q^*(\mathcal{K})$ defining a linearization on $\mathcal{K}$. Similarly we prove that the cokernels exist in $\mathrm{Qcoh}_G(X)$.

The statement about sufficiently many injective objects follows from the previous example. It is easy to see that the injective sheaves of modules we used are quasi-coherent. □

Example 2.3.5. Assume $X = \mathrm{Spec}\ A$ and $G = \mathrm{Spec}\ \mathcal{O}(G)$ are affine schemes. An action of G on X is defined by a structure of a *Hopf algebra* on A. It is a group object in the dual category. In particular, it comes with a homomorphism $\mu^\# : \mathcal{O}(G) \to \mathcal{O}(G) \otimes \mathcal{O}(G)$ defining the group law on G and a homomorphism $\sigma^\# : A \to \mathcal{O}(G) \otimes A$ defining the action on X. For any ring K and $x \in X(K), g \in G(K)$, we have $\mu(g, x)$ is defined by a homomorphism $A \to \mathcal{O}(G) \otimes A \xrightarrow{g \otimes x} K \otimes K \to K$, where the last homomorphism is the multiplication. Assume G is a constant affine group scheme defined by an abstract group G. This means that $\mathcal{O}(G) = \mathbb{Z}^G$ with multiplication law of functions. An action of G on X is defined by a homomorphism of groups $\phi : G \to \mathrm{Aut}(A)$. In terms $\mu^\#$ this is defined by the homomorphism $A \to \mathcal{O}(G) \otimes A = A^G, a \to f_a : g \to {}^g a$, where $\phi(g)(a) = {}^g a$. For any A-module M let ${}^g M$ denote the A-module M with scalar multiplication defined by $a \cdot m = {}^g m$. Let ${}^g A$ be the structure of an A-algebra on A defined by the homomorphism of rings $A \to A, a \to {}^g a$A. Then ${}^g M \cong M \otimes_A {}^g A$. Geometrically, ${}^g M^\sim \cong g^*(M^\sim)$, where $g : \mathrm{Spec} A \to \mathrm{Spec} A$ defined by the homomorphism of algebras $A \to {}^g A$.

A G-linearization on M consists of a collection of isomorphisms of A-modules $\alpha_g : M \to {}^g M$. Note that, for any $g \in G, a \in A, m \in M$, we have $\alpha_g(am) = {}^g a \alpha_g(m)$. For any $h \in G$ the automorphisms α_h defines the automorphism ${}^g M \to {}^{hg} M$ which we denote by $g^*(\alpha_h)$. The collection of automorphisms α_g must satisfy the cocycle condition $\alpha_{gh} = h^*(\alpha_g) \circ \alpha_h$.

Let $A\#G$ be the skew algebra. A G-linearized A-module M defines a module over $A\#G$ by setting

$$(\sum_{g \in G} a_g g)m = \sum_{g \in G} a_g \alpha_g(m).$$

We have

$$a_g g((a_h h)m) = a_g(\alpha_g(a_h \alpha_h(m)) = a_g {}^g a_h \alpha_g(\alpha_h(m))$$

$$= a_g {}^g a_h h^*(\alpha_g)(\alpha_h(m)) = a_g {}^g a_h \alpha_{gh}(m) = ((a_g a_g)(\alpha_h h))m.$$

Conversely any $A\#G$-module M defines a G-linearized module by restricting the scalars to the subring $\{a1, a \in A\} \cong A$ of $A\#G$. In this way we get an equivalence of categories

$$\mathrm{Mod}(A\#G) \approx \mathrm{Qcoh}_G(X).$$

Remark 2.3.6. More generally assume that X is a scheme on which a constant finite group scheme G acts by automorphisms. Then there exists a coherent sheaf of algebras $\mathcal{O}_X\#G$ on X such that $\mathrm{Mod}(\mathcal{O}_X\#G)$ is equivalent to $\mathrm{Qcoh}_G(X)$. For any G-invariant open affine set U the restriction of $\mathcal{O}_X\#G$ to U is isomorphic to the sheaf of algebras associated to the $\mathcal{O}_X(U)$-algebra $\mathcal{O}_X(U)\#G$ (see D. Chan, G. Ingallis, Proc. L.M.S. 88 (2004)).

Definition 2.3.1. Let X be a smooth projective variety over a field k together with an action $\sigma : G \times X \to X$ of a constant finite group G. We define the *equivariant derived categories* by setting

$$D^G(X) = D^b(\mathrm{coh}_G(X)),\ D^G_{\mathrm{qc}}(X) = D^b(\mathrm{Qcoh}_G(X)).$$

Remark 2.3.7. By definition, objects of $D^G(X)$ are bounded complexes of coherent G-sheaves with invertible quasi-isomorphisms. It is clear that each complex defines a G-linearization on the corresponding object of $D^b(X)$ with respect to the functor $\sigma^* : D^b(X) \to D^b(G\times X)$. This defines a functor $D^G(X) \to D^b_G(X)$. If $|G|$ is invertible in k this functor is an equivalence of categories. Since taking invariants is the exact functor, one easily sees that a G-equivariant morphism of complexes $\mathcal{F}^\bullet \to \mathcal{G}^\bullet$ defines a G-invariant morphism of its components $\mathcal{F}^i \to \mathcal{G}^i$. This implies that the functor is fully faithful. A choice of a bounded resolution $\phi : \mathcal{F}^\bullet \to \mathcal{I}^\bullet$ of a G-invariant complex, allows one to transfer a G-linearization on $\mathcal{I}^\bullet$ defined by $\alpha'_g = \phi \circ \alpha_g \circ \phi^{-1} : \mathcal{I}^\bullet \to g^*\mathcal{I}^\bullet$. Since morphisms of injective complexes in the derived category are ordinary morphisms of complexes, we see that the linearized injective resolution is an object of $D^G(X)$. This shows that our functor is an equivalence of categories (see D. Ploog, Adv. Math. 216 (2007)).

Let $\mathcal{F}, \mathcal{G} \in \mathrm{Ob}(\mathrm{Qcoh}_G(X))$. The group G acts on $\mathrm{Hom}_X(\mathcal{F}, \mathcal{G})$ in the obvious way by ${}^g\phi = \beta_g^{-1} \circ g^*(\phi) \circ \alpha_g$, where $\alpha_g : \mathcal{F} \to g^*(\mathcal{F}), \beta_g : \mathcal{G} \to g^*(\mathcal{G})$ define the linearizations on $\mathcal{F}, \mathcal{G}$. It follows from the definition that

$$\mathrm{Hom}_G(\mathcal{F}, \mathcal{G}) := \mathrm{Hom}_{\mathrm{Qcoh}_G(X)}(\mathcal{F}, \mathcal{G}) \cong \mathrm{Hom}_X(\mathcal{F}, \mathcal{G})^G,$$

where for any set S on which a group G acts

$$S^G = \{s \in S : g \cdot s = s, \forall g \in G\}.$$

The structure sheaf $\mathcal{O}_X$ admits a canonical G-linearization which comes from a canonical isomorphism $\sigma^*(\mathcal{O}_X) \to \mathcal{O}_{G\times X}$ equal to the composition of the homomorphisms $\sigma^*(\mathcal{O}_X \to \sigma_*\mathcal{O}_{G\times X})$ and $\sigma^*(\sigma_*\mathcal{O}_{G\times X}) \to \mathcal{O}_{G\times X}$ and similarly for $q^*(\mathcal{O}_X) \to \mathcal{O}_{G\times X}$.

Taking in particular, $\mathcal{F} = \mathcal{O}_X$ with its canonical linearization, we see that G acts k-linearly on $\mathrm{Hom}_X(\mathcal{O}_X, \mathcal{F}) = \mathcal{F}(X)$. In particular,

$$\mathrm{Hom}_G(\mathcal{O}_X, \mathcal{F}) = \mathcal{F}(X)^G,$$

the subspace of invariant elements.

Consider the functor $\Gamma^G : \mathcal{F} \to \mathcal{F}(X)^G$ from $\mathrm{Qcoh}_G(X) \to \mathrm{Vect}(k)$. By taking injective resolutions in the category $\mathrm{Qcoh}_G(X)$ we can define the derived functor. Its values on $\mathcal{F}$ are denoted by $H^i_G(X, \mathcal{F})$ and called the *equivariant cohomology*. The functor Γ^G is equal to the composition of functors $\mathrm{Qcoh}_G(X) \to \mathrm{Vect}_G(k) \to \mathrm{Vect}(k)$, where the first functor is taking the global sections and the second functor is taking the subspace of G-invariant elements. For any G-module M (i.e. a module over the group algebra $\mathbb{Z}[G]$) we denote by $H^i(G, M)$ the cohomology group of G with values in M. It can be defined as the value of the ith left derived functor of the functor $M \to M^G$. If M arises from a module over $R[G]$ for some commutative ring R, then $M \to H^i(G, M)$ is a functor with values in $\mathrm{Mod}(A)$. The derived functors are defined by using injective objects in $\mathrm{Mod}(k[G])$ which acyclic with respect to the functor $H^0(G, ?)$. Since $\mathcal{F}(X)$ is an injective module for any injective sheaf, we can apply the spectral sequence of the composition of functors to obtain a spectral sequence

$$E_2^{p,q} = H^p(G, H^q(X, \mathcal{F}) \Longrightarrow H^n_G(X, \mathcal{F}). \tag{2.54}$$

Assume that $|G|$ is invertible in k, then the functor $V \to V^G$ in $\mathrm{Vect}_G(k)$ is exact since any G-vector space V has V^G as the direct summand in the category $\mathrm{Vect}_G(k)$ (use the averaging operator $|G|^{-1} \sum_{g \in G} g^*$). This shows that the spectral sequence of the composition of functors degenerates and we get an isomorphism

$$H^i_G(X, \mathcal{F}) \cong H^i(X, \mathcal{F})^G. \tag{2.55}$$

Similarly, we can define equivariant $\mathrm{Ext}^i_G(\mathcal{F}, \mathcal{G})$ as the derived functors of the functor $\mathcal{G} \to \mathrm{Hom}_G(\mathcal{F}, \mathcal{G})$ from $\mathrm{Qcoh}_G(X)$ to $\mathrm{Vect}(k)$ and get a spectral sequence prove that

$$E_2^{p,q} = H^p(G, \mathrm{Ext}^q(\mathcal{F}, \mathcal{G})) \Longrightarrow \mathrm{Ext}^n_G(\mathcal{F}, \mathcal{G}). \tag{2.56}$$

If $|G|$ is invertible in k, we get an isomorphism

$$\mathrm{Ext}^n_G(\mathcal{F}, \mathcal{G}) \cong \mathrm{Ext}^n(\mathcal{F}, \mathcal{G})^G.$$

Let us replace $\mathrm{Qcoh}_G(X)$ with $\mathrm{Sh}^{\mathrm{ab}}_G(X)$. This time we have a functor $\Gamma^G : \mathrm{Sh}^{\mathrm{ab}}_G(X) \to \mathsf{Ab}$ which is the composition of the functors $\mathrm{Sh}^{\mathrm{ab}}_G(X) \to \mathsf{Ab_G} \to \mathsf{Ab}$ as above. The functor $\mathcal{A} \to \mathcal{A}(X)^G$ is not exact anymore for any group $G \neq \{1\}$. There is a spectral sequence similar to (2.54).

Suppose G acts trivially on X. For any G-invariant open subset $j : U \subset X$, the sheaf $j^*\mathcal{F} = \mathcal{F}|U$ is equipped with a natural G-linearization, the pull-back $j^*(\alpha)$ of the linearization $\alpha : \sigma^*\mathcal{F} \to q^*\mathcal{F}$. Thus G acts on $H^0(U, \mathcal{F}) = \mathcal{F}(U)$ and we can take invariants $\mathcal{F}(U)^G$. The assignment $U \to \mathcal{F}(U)^G$ defines a sheaf on X which we denote by $\mathcal{F}^G$. Recall that the category $\mathrm{Qcoh}_G(X)$ is equivalent to the category $\mathrm{Mod}(\mathcal{O}_X[G])$. An injective G-sheaf $\mathcal{F}$ corresponds to an injective sheaf of modules over $\mathcal{O}_X[G]$. Its subsheaf of invariant elements is a flasque sheaf on X. In fact, for any sheaf of modules $\mathcal{G}$ and an injective sheaf of modules $\mathcal{F}$, the sheaf $\mathcal{H}om_{\mathcal{O}_X[G]}(\mathcal{F},\mathcal{G})$ is flasque (the proof is similar to the proof of Lemma 2.4 in Chapter III of [Hartshorne]). Taking $\mathcal{G} = \mathcal{O}_X$ considered as a $\mathcal{O}_X[G]$-module, we get the assertion. Thus we can construct a spectral sequence of composition of functors $\mathcal{F} \to \mathcal{F}^G \to \mathcal{F}(X)^G = \mathcal{F}(X)^G$

$$E_2^{p,q} = H^p(X, \mathcal{H}^q(G,\mathcal{F})) \Longrightarrow H^n_G(X,\mathcal{F}). \tag{2.57}$$

Again, if $|G|$ is coprime to the characteristic, we get $\mathcal{H}^q(G,\mathcal{F}) = 0, q > 0$ and obtain an isomorphism

$$H^n_G(X,\mathcal{F}) \cong H^n(X,\mathcal{F}^G).$$

All of this generalizes to objects in $D^G_{\mathrm{qc}}(X)$. The group G acts on the vector space $\mathrm{Hom}_{D_{\mathrm{qc}}(X)}(\mathcal{F}^\bullet,\mathcal{G}^\bullet)$ and we get (assuming that $|G|$ is invertible, ser Remark 2.3.7)

$$\mathrm{Hom}_G(\mathcal{F}^\bullet,\mathcal{G}^\bullet) := \mathrm{Hom}_{D^G_{\mathrm{qc}}(X)}(\mathcal{F}^\bullet,\mathcal{G}^\bullet) \cong \mathrm{Hom}_{D_{\mathrm{qc}}}(\mathcal{F}^\bullet,\mathcal{G}^\bullet)^G.$$

Since the linearization commutes with the shift, we get

$$\mathrm{Ext}^i_{D^G_{\mathrm{qc}}(X)}(\mathcal{F}^\bullet,\mathcal{G}^\bullet) := \mathrm{Hom}_G(\mathcal{F}^\bullet,\mathcal{G}^\bullet[i]) = \mathrm{Ext}^i_{D_{\mathrm{qc}}}(\mathcal{F}^\bullet,\mathcal{G}^\bullet)^G.$$

This implies that

$$\mathbf{R}\mathrm{Hom}_G(\mathcal{F}^\bullet,\mathcal{G}^\bullet) \cong \mathbf{R}\mathrm{Hom}_G(\mathcal{F}^\bullet,\mathcal{G}^\bullet)^G.$$

If $|G|$ is not invertible in k, we have to compute $\mathrm{Ext}^i_G(\mathcal{F}^\bullet,\mathcal{G}^\bullet)$ by using two spectral sequences

$$E_2^{p,q} = \mathrm{Ext}^p_G(\mathcal{F}^\bullet, H^q(\mathcal{G}^\bullet)) \Longrightarrow \mathrm{Ext}^i_G(\mathcal{F}^\bullet,\mathcal{G}^\bullet)$$

$$E_2^{p,q} = \mathrm{Ext}^p_G(H^{-q}(\mathcal{F}^\bullet), H^i(\mathcal{G}^\bullet)) \Longrightarrow \mathrm{Ext}^n_G(\mathcal{F}^\bullet, H^i(\mathcal{G}^\bullet))$$

together with spectral sequence (2.56).

Let G act on X and G' act on Y. Let $f : X \to Y$ be a ϕ-equivariant morphism with respect to a surjective homomorphism of groups $\phi : G \to G', g \to \bar{g}$, with kernel K. Let $\mathcal{F}^\bullet$ be an object of $D^G_{\mathrm{qc}}(X)$. We can equip $\mathbf{R}f_*\mathcal{F}^\bullet \in D^b_{\mathrm{qc}}(Y)$ with a G'-linearizations as follows. Consider the morphism $\sigma_Y = \sigma' \circ (\phi \times \mathbf{id}_Y) : G \times Y \to Y$, where $\sigma' : G' \times Y \to Y$ is the action of G' on Y. It describes the action of G' on Y. The base change $(G\times Y) \times_{\sigma_Y, f} X$ is isomorphic to $G \times X$. The isomorphism is defined on points by $((g,y),x) = ((g, \bar{g}^{-1}(f(x)),x) \to (g,x)$. Now we have a commutative diagram

$$\begin{array}{ccc} G \times X & \xrightarrow{\sigma} & X \\ \big\downarrow{\scriptstyle \mathbf{id}_G \times f} & & \big\downarrow{\scriptstyle f} \\ G \times Y & \xrightarrow{\sigma_Y} & Y \end{array} .$$

Since the morphism σ_Y is flat and higher direct images commute with flat base changes, we obtain

$$\mathbf{R}(\mathbf{id}_G \times f)_*(\mathrm{pr}_X^*(\mathcal{F}^\bullet)) \cong \mathrm{pr}_Y^*(\mathbf{R}f_*(\mathcal{F}^\bullet)),$$

$$\mathbf{R}(\mathbf{id}_G \times f)_*(\sigma^*(\mathcal{F}^\bullet)) \cong \sigma_Y^*(\mathbf{R}f_*(\mathcal{F}^\bullet)).$$

Let α be a G-linearization on $\mathcal{F}^\bullet$. Now $\mathbf{R}(\phi \times f)_*(\alpha)$ defines a G-linearization on $\mathbf{R}f_*(\mathcal{F}^\bullet)$. Since the group K acts trivially on Y in the action $\sigma_Y : G \times Y \to Y$, we can take the invariants $\mathbf{R}f_*(\mathcal{F}^\bullet)^K$. This comes equipped with the G'-linearization and defines the push-forward functor

$$\mathbf{R}f_*^K : D_{\mathrm{qc}}^G(X) \to D_{\mathrm{qc}}^{G'}(Y).$$

If $|G|$ is coprime with the characteristic, we get

$$H^i(\mathbf{R}f_*^K(\mathcal{F})) \cong (R^i f_*\mathcal{F})^K.$$

Note the special case, when $K = G$ and $f : X \to Y$ is an equivariant, where G acts trivially on Y.

On the other hand we can consider the left derived functors $R^i f_*^K$ of the functor $\mathcal{F}^\bullet \to (f_*\mathcal{F}^\bullet)^K$. Since the functor $f_*^K = R^0 f_*^K$ is equal to the composition of the functors f_* and the functor $\mathcal{F} \to \mathcal{F}^K$ and as noticed before the sheaf of invariants of an injective sheaf is flasque, we obtain a spectral sequence

$$E_2^{p,q} = \mathcal{H}^p(K, R^q f_*\mathcal{F}) \Longrightarrow R^n f_*^K(\mathcal{F}),$$

where $\mathcal{F} \to \mathcal{H}^p(K, R^q f_*\mathcal{F})$ are the left derived functors of the functor $\mathcal{F} \to \mathcal{F}^K$. If $|G|$ is coprime to the characteristic, this functor is exact, and we obtain an isomorphism

$$(R^n f_*\mathcal{F})^K \cong R^n f_*^K(\mathcal{F}).$$

Another spectral sequence can be obtained by considering the composition of functors $\mathcal{F}^\bullet \to (f_*\mathcal{F}^\bullet)^K \to \Gamma(Y, (f_*\mathcal{F}^\bullet)^K)$ equal to the functor $\mathcal{F}^\bullet \to \Gamma(X, \mathcal{F}^\bullet)^K$. The spectral sequence is

$$E_2^{p,q} = H^p(Y, R^q f_*^K \mathcal{F}^\bullet) \Longrightarrow H_K^n(X, \mathcal{F}^\bullet).$$

One can show that, for any coherent sheaf

$$(R^q f_*^K \mathcal{F})_y \cong H^q(K_x, \mathcal{F}_x),$$

where x is any point in the fibre $f^{-1}(y)$. In particular, when $|K|$ is coprime to the characteristic, $(R^q f_*^K \mathcal{F}) = 0, q > 0$, and we obtain an isomorphism

$$H^n(Y, (f_*\mathcal{F})^K) \cong H_K^n(X, \mathcal{F}).$$

Also, for any $\mathcal{F}^\bullet$ from $D_{\mathrm{qc}}^G(X)$, the spectral sequence

$$E_2^{p,q} = R^p f_*^K(H^q(\mathcal{F}^\bullet)) \Longrightarrow R^n f_*^K(\mathcal{F}^\bullet)$$

degenerates in this case, and we obtain

$$R^n f_*^K(\mathcal{F}^\bullet) \cong f_*^K(H^q(\mathcal{F}^\bullet)).$$

Note that we have another spectral sequence (2.2) with the same limit. When both of them degenerate, we get an isomorphism

$$H^n(X,\mathcal{F})^G \cong H^n(Y,(f_*\mathcal{F})^G).$$

For example, take $\mathcal{F} = \mathcal{O}_X$ and $f : X \to Y = X/G$, to obtain

$$H^n(X,\mathcal{O}_X)^G \cong H^n(Y,\mathcal{O}_Y).$$

2.4 The Bridgeland-King-Reid Theorem

From now on we will be considering G-actions on irreducible quasi-projective algebraic varieties, where G is a finite constant group scheme of order prime to the characteristic of the ground field k. We assume that k is algebraically closed. Now we are in a position to define an integral transform

$$\Phi_{\mathcal{P}^\bullet}^{Y\to X} : D_{\mathrm{qc}}^G(X) \to D_{\mathrm{qc}}^{G'}(Y),$$

where $\mathcal{P}^\bullet \in \mathrm{Ob}(D^{G\times G'}(X\times Y))$. It is equal to the composition of functors

$$\Phi_{\mathcal{P}^\bullet}^{Y\to X} = \mathbf{R}p_X^{G\times 1} \circ (\mathcal{P}^\bullet \overset{\mathbf{L}}{\otimes}) \circ p_Y^*.$$

Here we equip the derived tensor product $\mathcal{P}^\bullet \overset{\mathbf{L}}{\otimes} \mathrm{pr}_Y^* \mathcal{F}^\bullet$ with the canonical $G\times G'$-linearization defined by the tensor product of linearizations which we leave to the reader to define.

Similarly we define

$$\Phi_{\mathcal{P}^\bullet}^{X\to Y} : D_{\mathrm{qc}}^{G'}(Y) \to D_{\mathrm{qc}}^G(X),$$

by

$$\Phi_{\mathcal{P}^\bullet}^{X\to Y} = \mathbf{R}p_Y^{1\times G'} \circ (\mathcal{P}^\bullet \overset{\mathbf{L}}{\otimes}) \circ p_X^*.$$

We say that $\Phi_{\mathcal{P}^\bullet}^{Y\to X}$ is an *equivariant Fourier-Mukai transform* if there exists $\mathcal{Q}^\bullet \in \mathrm{Ob}(D^{G\times G'}(X\times Y))$ such that $\Phi_{\mathcal{Q}^\bullet}^{X\to Y}$ is a quasi-inverse functor of $\Phi_{\mathcal{P}^\bullet}^{Y\to X}$.

We leave the proof of the next proposition to the reader. It is an equivariant version of (2.38).

Proposition 2.4.1. *Let $(X,G),(Y,G'),(Z,G'')$ be three varieties with group actions. Let $\mathcal{P}^\bullet \in \mathrm{Ob}(D^{G\times G'}(X\times Y))$ and $\mathcal{Q}^\bullet \in \mathrm{Ob}(D^{G'\times G''}(Y\times Z))$. Then*

$$\Phi_{\mathcal{Q}^\bullet}^{Y\to Z} \circ \Phi_{\mathcal{P}^\bullet}^{X\to Y} = \Phi_{\mathcal{Q}^\bullet\circ\mathcal{P}^\bullet}^{X\to Z},$$

where $\mathcal{Q}^\bullet \circ \mathcal{P}^\bullet \in \mathrm{Ob}(D^G(X\times Z))$ is defined by

$$\mathcal{Q}^\bullet \circ \mathcal{P}^\bullet = \mathbf{R}p_{13*}^{1\times G_2\times 1}(p_{12}^*(\mathcal{P}) \overset{\mathbf{L}}{\otimes} p_{23}^*(\mathcal{Q})),$$

where the tensor product is equipped with the $(G_1 \times G_2 \times G_2)$-linearization equal to the tensor product of $G_1 \times G_2 \times G_3$ linearizations corresponding to the p_{23}-equivariant morphism $pr_{23} : X \times Y \times Z \to Y \times Z$ and the p_{12}-equivariant morphism $pr_{12} : X \times Y \times Z \to X \times Y$ (here $p_{ij} : G_1 \times G_2 \times G_3 \to G_i \times G_j$ is the projection homomorphism).

Proposition 2.4.2. *Let $\phi : G \to G'$ be a surjective homomorphism of groups with kernel $K$and $f : X \to Y$ be a ϕ-equivariant morphism. Then the functor $f^* : D^{G'}_{\mathrm{qc}}(Y) \to D^G_{\mathrm{qc}}(X)$ is a left adjoint of the functor $\mathbf{R}f^K_* : D^G_{\mathrm{qc}}(X) \to D^{G'}_{\mathrm{qc}}(Y)$.*

Proof. We know that $\mathrm{Hom}_{D^G_{\mathrm{qc}}(X)}(\mathcal{F}^\bullet, \mathcal{G}^\bullet) = \mathrm{Hom}_{D_{\mathrm{qc}}(X)}(\mathcal{F}^\bullet, \mathcal{G}^\bullet)^G$ and the functors f^* and $\mathbf{R}f_*$ are adjoint. Taking G-invariants in the isomorphism

$$\mathrm{Hom}_{D_{\mathrm{qc}}(X)}(f^*\mathcal{F}^\bullet, \mathcal{G}^\bullet) \cong \mathrm{Hom}_{D_{\mathrm{qc}}(Y)}(\mathcal{F}^\bullet, \mathbf{R}f_*\mathcal{G}^\bullet)$$

we get

$$\mathrm{Hom}_{D^G_{\mathrm{qc}}(X)}(f^*\mathcal{F}^\bullet, \mathcal{G}^\bullet) \cong \mathrm{Hom}_{D_{\mathrm{qc}}(Y)}(\mathcal{F}^\bullet, \mathbf{R}f_*\mathcal{G}^\bullet)^G$$

$$\cong \mathrm{Hom}_{D_{\mathrm{qc}}(Y)}(\mathcal{F}^\bullet, \mathbf{R}f^K_*\mathcal{G}^\bullet)^{G'} \cong \mathrm{Hom}_{D^{G'}_{\mathrm{qc}}(Y)}(\mathcal{F}^\bullet, \mathbf{R}f^K_*\mathcal{G}^\bullet).$$

This proves the assertion.

□

We leave to the reader the proof of an equivariant version of Proposition 2.2.4 that uses Proposition 2.4.2.

Proposition 2.4.3. *Assume $\Phi^{Y\to X}_{\mathcal{P}^\bullet} : D^{G'}(Y) \to D^G(X)$ is an equivariant integral transform. Assume X, Y are smooth projective varieties. Then the functors $\Phi^{X\to Y}_{\mathcal{P}^{\bullet\vee}\otimes pr_Y^*\omega_Y[\dim Y]}$ is its its right adjoint functor and $\Phi^{X\to Y}_{\mathcal{P}^{\bullet\vee}\otimes pr_X^*\omega_X[\dim X]}$ is its left-adjoint functor. Here the tensor product is equipped with the $G \times G'$-linearization of the tensor product.*

Example 2.4.4. Assume a finite group G acts freely on a smooth quasi-projective variety X of dimension n. Let $Y = X/G$ be the quotient and $f : X \to Y$ be the canonical projection. Consider f as the equivariant morphism with respect to the trivial homomorphism of groups $G \to \{1\}$. The functor $f^* : D(Y) \to D^G(X)$ is equal to the integral transform $\Phi^{Y\to X}_{\mathcal{P}^\bullet}$ with the kernel $\mathcal{P}^\bullet = \mathcal{O}_{\Gamma_f}$, where Γ_f is the graph of f. The functor $f^G_* = \mathbf{R}f^G_*$ is the integral transform $\Phi^{X\to Y}_{\mathcal{P}^\bullet}$ with the same kernel. The composition $f^G_* \circ f^*$ is the integral transform with the kernel

$$\mathcal{K} = \mathbf{R}p^G_{13}(p^*_{12}(\mathcal{O}_{\Gamma^\tau_f}) \overset{\mathbf{L}}{\otimes} p^*_{23}(\mathcal{O}_{\Gamma_f})),$$

where p_{ij} are the projection maps of $Y \times X \times Y$ and $\tau : X \times Y \to Y \times X$ is the switch. We have $p^*_{12}(\mathcal{O}_{\Gamma^\tau_f}) = \mathcal{O}_{\Gamma^\tau_f \times Y}$, and $p^*_{23}(\mathcal{O}_{\Gamma_f}) = \mathcal{O}_{Y\times\Gamma_f}$. Since G acts freely, the morphism $X \to Y$ is étale, hence Y is smooth and $Y \times \Gamma_f \hookrightarrow Y \times X \times Y$ is a regular closed embedding (i.e. locally complete intersection of codimension

n). The same is true for the embedding $\Gamma_f^\tau \times Y \hookrightarrow Y \times X \times Y$. This implies that all Tor_i vanish and

$$p_{12}^*(\mathcal{O}_{\Gamma_f^\tau}) \overset{\mathbf{L}}{\otimes} p_{23}^*(\mathcal{O}_{\Gamma_f}) \cong p_{12}^*(\mathcal{O}_{\Gamma_f^\tau}) \otimes p_{23}^*(\mathcal{O}_{\Gamma_f}) \cong \mathcal{O}_{(\Gamma_f^\tau \times Y)\cap(Y\times\Gamma_f)} \cong \mathcal{O}_X,$$

where X is embedded in $Y \times X \times Y$ by $(f, \mathbf{id}_X, f)$. Since p_{13} restricts to a finite map, we obtain

$$\mathcal{K} \cong f_*^G(\mathcal{O}_X) \cong \mathcal{O}_Y.$$

This shows that $f_*^G \circ f^* = \mathbf{id}_{\mathrm{Qcoh}_G(Y)}$.

Next we prove that $f^* \circ f_*^G = \mathbf{id}_{D^G(X)}$. We use a similar argument. Consider $X \times Y \times X$ and show that

$$p_{12}^*(\mathcal{O}_{\Gamma_f}) \cap p_{23}^*(\mathcal{O}_{\Gamma_f^\tau}) \cong \mathcal{O}_{X\times_Y X},$$

where $X \times_Y X$ is embedded in $X \times Y \times X$ by $(\mathbf{id}_X, f, \mathbf{id}_X)$. The restriction of the projection p_{13} to $X \times_Y X$ can be identified with the closed embedding $i : X \times_Y X \hookrightarrow X \times X$. Thus the composition $f^* \circ f_*^G$ is given by the integral transform $\Phi_{\mathcal{P}}^{X\to X} : D^G(X) \to D^G(X)$ with kernel isomorphic to $i_*(\mathcal{O}_{X\times_Y X})$. It is equal to the composition $p_1^G \circ p_2^*$, where p_i are the projections $X \times_Y X \to X$. Since G acts freely, we have a $G\times G$-equivariant isomorphism $(\sigma, \mathrm{pr}_X) : G\times X \to X \times_Y X$. This easily implies that $\Phi_{\mathcal{P}}^{X\to X}(\mathcal{F}^\bullet) = \sigma_*^G(\mathrm{pr}_X^*(\mathcal{F}^\bullet)) \cong \mathcal{F}^\bullet$. Here $\mathrm{pr}_X^*(\mathcal{F}^\bullet)$ is considered as a $G \times G$-sheaf on $G \times X$.

Definition 2.4.1. Let G act faithfully (i.e. with trivial kernel) on a variety X. A 0-dimensional G-invariant closed subscheme Z of X is called a G-cluster if the linear representation of G in $H^0(X, \mathcal{O}_Z)$ is isomorphic to the regular representation of G.

Let Z be a cluster which is a reduced closed subscheme. Then $h^0(\mathcal{O}_Z) = \dim k[G] = |G|$, i.e. Z consists of $|G|$ closed points. Obviously, it must be a free orbit of G, i.e. an orbit with with trivial stabilizer. Let $X//G$ be the closure of the set of points in the Hilbert scheme $\mathrm{Hilb}^{[|G|]}(X)$ representing reduced G-clusters. Since X is quasi-projective, the quotient $\pi : X \to X/G$ exists as a quasi-projective varieties X, it is obtained by gluing together the rings of invariants $\mathcal{O}_X(U_i)^G$, where $(U_i)_{i\in I}$ is a G-invariant affine open covering of X (it exists because X is quasi-projective). We assume that G acts freely on an open Zariski subset of X. This shows that free orbits are parameterized by an irreducible variety, an open subset of X/G, hence $X//G$ is an irreducible variety birationally isomorphic to X/G. In fact, one can construct a proper birational morphism $\tau : X//G \to X/G$ which is an isomorphism over the open subset of X/G parametrizing free orbits. All other points of $X//G$ represent non-reduced clusters on X. Let $q : \mathcal{Z} \to X//G$ be the restriction of the universal scheme over $\mathrm{Hilb}^{[|G|]}(X)$ to $X//G$ and $p : \mathcal{Z} \to X$ be its natural projection to X. We have a commutative diagram

$$\begin{array}{ccccc} & & \mathcal{Z} & & \\ & \stackrel{p}{\swarrow} & & \stackrel{q}{\searrow} & \\ X & & & & X/\!/G \\ & \stackrel{\pi}{\searrow} & & \stackrel{\tau}{\swarrow} & \\ & & X/G & & \end{array} \tag{2.58}$$

The fibre of q over a closed point $\xi \in X/\!/G$ representing a cluster Z is mapped isomorphically under p to Z. The fiber of p over a closed point $x \in X$ is mapped isomorphically under q to the set of points ξ in $X/\!/G$ representing clusters Z such that $x \in \mathrm{Supp}(Z)$. It is known that the action of G on X lifts to an action on the Hilbert scheme $\mathrm{Hilb}^{[|G|]}(X)$ and the universal family $\mathcal{H}^{[|G|]}(X) \to \mathrm{Hilb}^{[|G|]}(X)$ is an affine finite subscheme of the X-scheme $X \times \mathrm{Hilb}^{[|G|]}(X) \to \mathrm{Hilb}^{[|G|]}$ isomorphic to the affine spectrum of a sheaf of algebras over $\mathcal{O}_{\mathrm{Hilb}^{[|G|]}(X)}$ that admits a canonical G-linearization. Since $X/\!/G$ is a subset of fixed points of G, the restriction of $\mathcal{A}$ to $X/\!/G$ is a G-linearized sheaf of algebras over $X/\!/G$. Hence its affine spectrum $\mathcal{Z}$ admits a canonical action of G. The commutative diagram (2.58) is a commutative diagram of G-equivariant morphisms, where G acts trivially on $X/\!/G$ and on X/G.

Define the integral transform

$$\Phi = \Phi_{\mathcal{P}^\bullet}^{X/\!/G \to X} : D^b(X/\!/G) \to D^G(X)$$

with kernel $\mathcal{P}^\bullet$ equal to the object complex $i_*\mathcal{O}_{\mathcal{Z}}$, where $i : \mathcal{Z} \hookrightarrow X \times X/\!/G$ is the closed embedding. For any $\mathcal{F}^\bullet \in \mathrm{Ob}(D^b(X/\!/G))$, we have

$$\Phi(\mathcal{F}^\bullet) = \mathbf{R}p_*(q^*(\mathcal{F}^\bullet)) \in \mathrm{Ob}(D^G(X)).$$

Note that the morphism $q : \mathcal{Z} \to Y$ is known to be flat, so $i_*\mathcal{O}_{\mathcal{Z}}$ is flat over Y and the functor $\mathbf{L}q^* = q^*$ is defined.

The goal of this lecture is to prove the following theorem of Bridgeland-King-Reid.

Theorem 2.4.5. *Assume G acts on X in such a way that the canonical sheaf ω_X is locally trivial as a G-sheaf. Suppose the fibre product $X/\!/G \times_\tau X/\!/G$ has dimension $\leq \dim X + 1$. Then $\tau : X/\!/G \to X/G$ is a resolution of singularities and Φ is an equivalence of categories.*

The assumption on ω_X means that $\omega_X = p^*(\pi^*(\mathcal{L}))$ for some invertible sheaf $\mathcal{L}$ on X/G. Let $Y = X/\!/G$.

Before we start the proof, we need some facts from commutative algebra.

Recall that for any finitely generated module M over a noetherian commutative ring R its *homological dimension* or *projective dimension* $\mathrm{dh}_R(M)$ is defined to be the largest n such that $\mathrm{Ext}^n_R(M,N) \neq 0$ for some finitely generated R-module N. We have $\mathrm{dh}_R(M) \leq n$ if and only if M admits a projective resolution

$$0 \to M_n \to \ldots \to M_1 \to M_0.$$

Recall that the depth $\mathrm{depth}_R(I)$ of an ideal I is defined as the maximal length of a regular sequence in R. If R is a Cohen-Macaulay ring, then $\mathrm{depth}_R I = \dim R/I$ ([Eisenbud], Chap. 18). We have (loc. cit., Corollary 18.5)

$$\mathrm{dh}_R(M) \geq \mathrm{depth}_R(\mathrm{Ann}(M),$$

where $\mathrm{Ann}(M)$ is the annihilator ideal of M. If $(R,\mathfrak{m})$ is a regular local ring, then the equality always takes place. In particular, we have $\mathrm{dh}(R/\mathfrak{m}) = \dim R$. The next very deep result, known as the *Intersection Theorem* (see P. Roberts, Intersection theorems, in Commutative algebra (Berkeley, 1987), MSRI Publ. 15, Springer, 1989) shows that the converse is true.

Theorem 2.4.6. *Let $(R,\mathfrak{m})$ be a local algebra of dimension d. Suppose that*

$$0 \to M^{-s} \to M^{-s+1} \to \ldots \to M^0 \to 0$$

is a complex of finitely generated free R-modules with homology module $H_i(M^\bullet) = H^{-i}(M^\bullet)$ of finite length over R. Then $s \geq d$ and $s = d$ and $H^0(M^\bullet) \cong R/\mathfrak{m}$ implies that $M^\bullet$ is a free resolution of $R/\mathfrak{m}$ and R is regular.

Now we have to extend all of this to complexes of coherent sheaves. We define the *support* $\mathrm{Supp}(\mathcal{F}^\bullet)$ of a bounded complex $\mathcal{F}^\bullet$ of coherent sheaves on a scheme X as the union of supports of the homology sheaves $H_i(\mathcal{F}^\bullet) := H^{-i}(\mathcal{F}^\bullet)$. The *homological dimension* $\mathrm{dh}(\mathcal{F}^\bullet)$ of a non-zero $\mathcal{F}^\bullet$ is the smallest i such that $\mathcal{F}^\bullet$ is quasi-isomorphic to a complex of locally free sheaves of length $i+1$. For example, $\mathrm{dh}(\mathcal{F}^\bullet) = 0$ if and only if $\mathcal{F}^\bullet$ is quasi-isomorphic to $\mathcal{E}[r]$, where $\mathcal{E}$ is a locally free sheaf. The spectral sequence (2.17) gives, for any closed point $x \in X$, the spectral sequence

$$E_2^{p,q} = \mathrm{Ext}^p_X(H_q(\mathcal{F}^\bullet), \mathcal{O}_x) \Longrightarrow \mathrm{Hom}^{p+q}_{D(X)}(\mathcal{F}^\bullet, \mathcal{O}_x).$$

It shows that

$$x \in \mathrm{Supp}(\mathcal{F}^\bullet) \Leftrightarrow \mathrm{Hom}^i_{D(X)}(\mathcal{F}^\bullet, \mathcal{O}_x) \neq 0 \text{ for some } i \in \mathbb{Z}.$$

Consider the inclusion map $i : x \hookrightarrow X$ and apply the adjunction of the functors $\mathbf{L}i^*$ and $\mathbf{R}i_*$ to obtain

$$\mathbf{R}\mathrm{Hom}_x(\mathcal{F}^\bullet \overset{\mathbf{L}}{\otimes} \mathcal{O}_x) \cong \mathbf{R}\mathrm{Hom}^i_{D(X)}(\mathcal{F}^\bullet, \mathbf{L}i_*\mathcal{O}_x)$$

(here we consider $\mathcal{O}_x$ as a sky-scraper sheaf on X and also as a sheaf on $\{x\}$). Taking cohomology, and using spectral sequence (2.17), we get

$$H_i(\mathbf{F}^\bullet \overset{\mathbf{L}}{\otimes} \mathcal{O}_x)^\vee \cong \mathrm{Hom}^i_{D(X)}(\mathcal{F}^\bullet, \mathcal{O}_x). \tag{2.59}$$

It is clear that if $\mathrm{dh}(\mathcal{F}^\bullet) \leq s$, then $H_i(\mathbf{F}^\bullet \overset{\mathbf{L}}{\otimes} \mathcal{O}_x) = 0$ for all $x \in X$ unless $j \leq i \leq s+j$ for some j. One can show that the converse is also true. We skip the proof (see T. Bridgeland, A. Maciocia, JAG, 11 (2002)). Let η be a generic point of an irreducible component S_i of $\mathrm{Supp}(\mathcal{F}^\bullet)$. Take a complex of locally free sheaves $\mathcal{E}^\bullet$ of length $\leq s$ quasi-isomorphic to $\mathcal{F}^\bullet$. Restricting $\mathcal{E}^\bullet$ to $\mathcal{O}_{X,\eta}$, we obtain a non-zero complex of free $\mathcal{O}_{X,\eta}$-modules of length $m \leq s+1$ with finite cohomology modules. Applying the Intersection Theorem, we obtain that $s+1 \geq \dim \mathcal{O}_{X,\eta} = \mathrm{codim} S_i$. This implies the following.

Corollary 2.4.7.

$$\mathit{codim}(\mathit{Supp}(\mathcal{F}^\bullet)) \leq \mathrm{dh}\ \mathcal{F}^\bullet.$$

Next we need a general result (due to Bridgeland) on equivalence of derived categories.

Definition 2.4.2. A subset of objects Ω of a derived category is called a *spanning set* if $\mathrm{Hom}^i(A,X) = 0$ for all $i \in \mathbb{Z}$ and all $A \in \Omega$ implies that $X \cong 0$ and $\mathrm{Hom}^i(X,A) = 0$ for all $i \in \mathbb{Z}$ and all $A \in \Omega$ implies that $X \cong 0$.

Lemma 2.4.8. *Let $F : D \to D'$ be a functor of derived categories with a right adjoint H and a left adjoint G. Assume that for a spanning set Ω in D and any $A, B \in \Omega$, we have $\mathit{Hom}^i_D(A,B) = \mathit{Hom}^i_{D'}(F(A),F(B)), i \in \mathbb{Z}$. Then F is a fully faithful functor.*

Proof. By adjunction, we have the commutative diagram

$$\begin{array}{ccc} \mathrm{Hom}^i_D(A,B) & \longrightarrow & \mathrm{Hom}^i_D(A, H\circ F(B)) \\ \downarrow \delta & & \downarrow \beta \\ \mathrm{Hom}^i_D(G\circ F(A), B) & \longrightarrow & \mathrm{Hom}^i_{D'}(F(A),F(B)), \end{array} \tag{2.60}$$

where the bottom and the right arrows are the adjunction isomorphisms, and the left and the top arrows are defined by applying Hom^i to the adjunction morphism of functors $\mathbf{id}_D \to H\circ F$ and $G \circ F \to \mathbf{id}_D$. If $A, B \in \Omega$, the diagonal map $\mathrm{Hom}^i_D(A,B) \to \mathrm{Hom}_{D'}(F(A),F(B))$ is a bijection, so all maps in the diagram are bijective.

For any $A \in \Omega$, consider a distinguished triangle $G\circ F(A) \xrightarrow{\delta} A \to C \to G \circ F(A)[1]$. Applying the functor $\mathrm{Hom}(A, ?)$, we get the exact sequence

$$\mathrm{Hom}^{-1}(A,B) \to \mathrm{Hom}^{-1}(G\circ F(A),B) \to \mathrm{Hom}(C,B) \to \mathrm{Hom}(A,B)$$

$$\to \mathrm{Hom}(G\circ F(A),B) \to \ldots.$$

By the above, we infer that $\mathrm{Hom}^i(C,B) = 0$ for all $B \in \Omega$. Hence $C \cong 0$ and $G\circ F(A) \to A$ is an isomorphism. Now take any $B \in \mathrm{Ob}(D)$ and consider a distinguished triangle $B \to H\circ F(B) \to C \to B[1]$. Since $G \circ F(A) \to A$ is an isomorphism, the commutative diagram (2.60) implies that the homomorphism $\mathrm{Hom}^i_D(A,B) \to \mathrm{Hom}^i_D(A, H\circ F(B))$ is an isomorphism.

The long exact sequence shows that $\mathrm{Hom}^i_D(A,C) = 0$ for all $A \in \Omega$, hence $C \cong 0$, and $B \to H \circ F(B)$ is an isomorphism for any B. Thus $\mathrm{Hom}_D(A,B) \cong \mathrm{Hom}_D(A, H \circ F(B)) = \mathrm{Hom}_{D'}(F(A), F(B))$ implies that F is fully faithful. $\square$

Definition 2.4.3. A triangulated category D is called *decomposable* if there exists two full subcategories D_1 and D_2, each containing objects non-isomorphic to the zero object, such that

(i) any object X in D is isomorphic to the bi-product of an object A_1 from D_1 and an object A_2 from D_2;

(ii) $\mathrm{Hom}^i_D(A_1, A_2) = \mathrm{Hom}^i_D(A_2, A_1) = 0$ for all $i \in \mathbb{Z}$ and all $A_1 \in \mathrm{Ob}(D_1), A_2 \in \mathrm{Ob}(D_2)$.

Recall that the *biproduct* of objects A, B in an additive category is an object which is the direct sum and the direct product of A and B (i.e. corepresents the product of $h^A \times h^B$ and represents the product $h_A \times h_B$).

Note that, if $A_i \in \mathrm{Ob}(D_i)$ in the definition, then $A_i[r] \in D_i$ for any $r \in \mathbb{Z}$. In fact, obviously $\mathrm{Hom}^i_D(A_2, A_i[r]) = \mathrm{Hom}^{i+r}(A_2, A_1) = 0$ for all $i \in \mathbb{Z}$. If $A_1[r]$ is the bi-product of $A \in \mathrm{Ob}(D_1)$ and $B \in \mathrm{Ob}(D_2)$ with $B \not\cong 0$, then there is a non-zero morphism $B \to A_1[r]$. Thus B must be a zero-object, and hence $A_1[r]$ is an object of D_1.

One can restate the condition about the biproduct by saying that for any object X in D there is a distinguished triangle $A_1 \to X \to A_2 \to A_1[1]$, where $A_i \in \mathrm{Ob}(D_i)$. Since $A_1[1] \in \mathrm{Ob}(D_1)$, the morphism $A_2 \to A_1[1]$ is the zero morphism. One can prove that this implies that the triangle splits, i.e. there is a section $A_2 \to B$. Applying the functors $\mathrm{Hom}(X, ?)$ and $\mathrm{Hom}(?, X)$, we obtain that B is the bi-product of A_1 and A_2.

Example 2.4.9. Let X be a connected scheme. Let us prove that $D^b(X)$ is indecomposable. Suppose it is decomposable with subcategories D_1 and D_2 satisfying the definition. We may assume that $\mathcal{O}_X$ is an object of D_1. Since X is connected, $\mathrm{Hom}(\mathcal{O}_X, \mathcal{O}_x) \neq 0$, for all $x \in X$, hence $\mathcal{O}_X$ belongs to D_1. For any $A^\bullet \in \mathrm{Ob}(D_2), x \in X$ and any $i \in \mathbb{Z}$, we must have $\mathrm{Hom}(A^\bullet, \mathcal{O}_x[i]) = 0$. Consider the spectral sequence

$$E_2^{p,q} = \mathrm{Hom}^p(H^{-q}(A^\bullet), \mathcal{O}_x) \Longrightarrow \mathrm{Hom}^{p+q}(A^\bullet, \mathcal{O}_x)$$

and choose q minimal with $H^{-q}(A^\bullet) \neq 0$. For any point x in the support of $H^{-q}(A^\bullet)$ we have $\mathrm{Hom}(H^{-q}(A^\bullet), \mathcal{O}_x) \neq 0$. Then the term E_2^{0,q_0} survives in the limit and we get $\mathrm{Hom}^{q_0}(A^\bullet, \mathcal{O}_x) \neq 0$. This contradiction shows that all objects in D_2 are isomorphic to zero.

Example 2.4.10. Let X be a quasi-projective irreducible variety and G be a constant finite group of order prime to the characteristic acting faithfully on X. Then $D^G(X)$ is irreducible. In fact, suppose we have two full subcategories D_1 and D_2 as in the definition. If $\mathcal{F}$ is an irreducible G-sheaf , i.e it is not isomorphic to the direct sum of two non-zero sheaves, then $\mathcal{F}$ belongs to one of the categories, say D_1. Take $\mathcal{F}$ to be $\mathcal{O}_Z$, where Z is a free orbit. It is obviously an irreducible sheaf. Consider the surjection $\mathcal{O}_X \to \mathcal{O}_Z$. Recall that $H^0(\mathcal{O}_Z) \cong k[G]$ as linear representations. Let V_ρ be an irreducible $k[G]$-module corresponding to some irreducible representation of G. We consider it as a free sheaf of rank $\dim \rho$. Then we have a nontrivial G-morphism $V_\rho \to \mathcal{O}_Z$. Thus sheaves isomorphic to V_ρ, they are obviously irreducible, belong to the same category D_1. Any G-sheaf supported at a point has a section, so we can map one of V_ρ non-trivially to such a sheaf. Thus all such sheaves belong to D_1. Finally, for any G-sheaf $\mathcal{F}$ with $x \in \mathrm{Supp}(\mathcal{F})$, we have a canonical non-trivial G-homomorphism $\mathcal{F} \to i_*i^*\mathcal{F}$, where $i : x \hookrightarrow X$. Thus all non-zero G-sheaves are isomorphic to sheaves in D_1. We know that all their shifts $\mathcal{F}[i]$ belong to D_1. Suppose we have an object $(\mathcal{F}^\bullet, d)$ in D which is not isomorphic to the zero object. Then the complex $\ker(d^s)[-s]$ is mapped to $F^\bullet$ and the corresponding map on the cohomology is not-trivial. Thus this is not the zero morphism in the derived category, hence $\mathcal{F}^\bullet$ is an object of D_1. Thus D_2 consists of only zero objects. This proves the assertion.

Corollary 2.4.11. *Under assumption of Lemma 2.4.8 assume that not every object in D is isomorphic to the zero object and D' is indecomposable (i.e. not decomposable). Suppose $H(B) = 0$ implies $G(B) = 0$. Then F is an equivalence of categories.*

Proof. Consider a full subcategory D'_1 of D' that consists of objects A_1 such that $F \circ H(A_1) \cong A_1$ and a full subcategory D'_2 of objects A_2 such that $H(A_2) \cong 0$. For any A_1, A_2 as above, we have

$$Hom^i_{D'}(A_1, A_2) \cong \mathrm{Hom}^i_{D'}(F \circ H(A_1), A_2) \cong \mathrm{Hom}^i_D(H(A_1), H(A_2)) = 0,$$

$$Hom^i_{D'}(A_2, A_1) = \mathrm{Hom}^i_{D'}(A_2, F \circ H(A_1)) \cong \mathrm{Hom}^i_D(G(A_2), H(A_1)) = 0,$$

where we used the assumption $H(A_2) = 0$ implies $G(A_2) = 0$. For any object B in D', consider a distinguished triangle

$$F \circ H(B) \to B \to C \to F \circ H(B)[1] \tag{2.61}$$

Since F is fully faithful and is left adjoint to H, the canonical morphism of functors $H \circ F \to \mathbf{id}_D$ is an isomorphism. Applying H, we get $(H \circ F) \circ H(B) \cong H(B) \to H(B)$ is an isomorphism. This implies $H(C) \cong 0$ and hence $C \in \mathrm{Ob}(D'_2)$. Also $F \circ H(F \circ H(B)) \cong F \circ (H \circ F) \circ H(B) \cong F \circ H(B)$. This implies that $F \circ H(B) \in \mathrm{Ob}(D'_1)$. Since the morphism $C \to F \circ H(B)[1]$ is zero, there is a left inverse of $B \to C$ (see the proof of Corollary 1.3.8). This implies that B is isomorphic to the bi-product of $F \circ H(B)$ and C. Since B was an arbitrary object, this contradicts the assumption that D' is indecomposable. Thus D'_2 or D'_1 must consist of zero objects. If D'_1 consists of zero objects, then any object is isomorphic to an object from D'_2, hence H sends all objects to the zero objects. However, we know that $H \circ F \cong \mathbf{id}_D$, and we assumed that D contains non-zero objects. Thus D'_2 consists of zero objects. Consider again the triangle (2.61). Since $H(C) = 0$ implies $C \cong 0$, we get that $F \circ H(B) \to B$ is always an isomorphism. Thus F is an equivalence of categories. □

We will also need the following known result from the deformation theory. We skip its proof.

Lemma 2.4.12. *Let $\mathcal{Q}$ be a sheaf on $X \times Y$ flat over Y. For any closed point $y \in Y$, let $\mathcal{Q}(y) = pr_Y^*(\mathcal{O}_y) \otimes \mathcal{Q}$. Then the homomorphism*

$$pr_Y^* : Ext_Y^1(\mathcal{O}_y, \mathcal{O}_y) \to Ext_X^1(\mathcal{Q}_y, \mathcal{Q}_y)$$

is the Kodaira-Spencer map $T_yY \to Ext_X^1(\mathcal{Q}_y, \mathcal{Q}_y)$ at the point y to the scheme $\mathrm{Quot}(Y)$ *parameterizing sheaves on Y with Hilbert polynomial equal to the Hilbert polynomial of $\mathcal{Q}_y$.*

Now we are ready to start proving Theorem 2.4.5. We first start with the case when X is projective. Let $Y = X//G$ and $n = \dim X = \dim Y$.

Step 1: Since $\mathcal{Z} \to Y$ is flat and X is smooth, the sheaf $\mathcal{O}_\mathcal{Z}$ on $X \times Y$ is of finite homological dimension (i.e. has a finite locally free resolution). This implies that the complex $\mathcal{O}_\mathcal{Z}^\vee$ is of finite homological dimension. This allows one to define the integral transform $\Psi : D^G(X) \to D^b(Y)$ with kernel $\mathcal{P}^\bullet = \mathcal{O}_\mathcal{Z}^\vee \otimes \mathrm{pr}_X^*\omega_X[n] \in D^b(X \times Y)$. It is equal to the composition of three functors $G_1 \circ G_2 \circ G_3$, where $G_1 = \mathbf{R}\mathrm{pr}_{Y*}^G, G_2 = \mathcal{O}_\mathcal{Z}^\vee \otimes ?$, and $G_3 = \mathrm{pr}_X^*$. The functor Φ is the composition of three functors $F_3 \circ F_2 \circ F_1$, where $F_1 = \mathrm{pr}_Y^*, F_2 = \mathcal{O}_Z \otimes ?, F_3 = \mathbf{R}\mathrm{pr}_{X*}$. If we prove that G_i is left adjoint to F_i, then we prove that Ψ is a left adjoint to Φ. We go in all these troubles because we do not know whether Y is nonsingular. Otherwise we can apply directly Proposition 2.4.3. Clearly, G_3 is left adjoint to F_3. Applying (2.23) , we obtain that G_2 is a left adjoint of F_2. The morphism $f = \mathrm{pr}_Y : X \times Y \to Y$ is smooth and the relative canonical sheaf ω_f is isomorphic to $\mathrm{pr}_X^*\omega_X$. Applying the relative duality theorem from Example 2.1.8 and taking the invariants, we obtain that G_1 is left adjoint to F_1.

If Φ is a Fourier-Mukai transform, then Ψ must be its quasi-inverse functor. So we have to prove that Ψ is a quasi-inverse of Φ. Let $\Psi \circ \Phi = \Phi_{\mathcal{Q}^\bullet}^{Y \to Y}$, where $\mathcal{Q}^\bullet \in \mathrm{Ob}(D^b(Y \times Y))$.

Step 2: Let y be a closed point of Y and $i_y : y \times Y = Y \hookrightarrow Y \times Y$ be the closed embedding of the fibre $\mathrm{pr}_1^{-1}(y)$ identified with Y. The restriction of the second projection $p_2 : Y \times Y \to Y$ to $\mathrm{pr}_1^{-1}(y)$ is an isomorphism, hence

$$\Phi_{\mathcal{Q}^\bullet}(\mathcal{O}_y) = (\mathrm{pr}_2)_*(\mathcal{Q}^\bullet \overset{\mathbf{L}}{\otimes} \mathrm{pr}_1^*(\mathcal{O}_y)) \cong (\mathrm{pr}_2)_*(\mathbf{L}i_y^*(\mathcal{Q}^\bullet)) \cong \mathbf{L}i_y^*(\mathcal{Q}).$$

We also have $\mathcal{O}_{(y_1,y_2)} \cong i_{y_1,*}\mathcal{O}_{y_2}$. Using the adjunction of the functors $\mathbf{L}i_y^*$ and $i_{y,*}$ and of the functors Φ, Ψ, we obtain

$$\mathrm{Hom}^i_{D(Y\times Y)}(\mathcal{Q}^\bullet, \mathcal{O}_{(y_1,y_2)}) = \mathrm{Hom}^i_{D(Y\times Y)}(\mathcal{Q}^\bullet, i_{y_1,*}\mathcal{O}_{y_2}) = \mathrm{Hom}^i_{D(Y\times Y)}(\mathbf{L}i_{y_1}^*\mathcal{Q}, \mathcal{O}_{y_2})$$

$$\cong \mathrm{Hom}^i_{D(Y)}(\Psi\circ\Phi(\mathcal{O}_{y_1}), \mathcal{O}_{y_2}) \cong \mathrm{Hom}^i_{D^G(X)}(\Phi(\mathcal{O}_{y_1}), \Phi(\mathcal{O}_{y_2})) \cong \mathrm{Ext}^i_X(\mathcal{O}_{\mathcal{Z}_{y_1}}, \mathcal{O}_{\mathcal{Z}_{y_2}})^G.$$

Step 3: We show that

$$\mathrm{Hom}_{D^G(X)}(\mathcal{O}_{\mathcal{Z}_{y_1}}, \mathcal{O}_{\mathcal{Z}_{y_2}}) \cong \begin{cases} k & \text{if, } \mathcal{Z}_{y_1} = \mathcal{Z}_{y_2} \\ 0 & \text{otherwise.} \end{cases}.$$

We have

$$\mathrm{Hom}_{D^G(X)}(\mathcal{O}_{\mathcal{Z}_{y_1}}, \mathcal{O}_{\mathcal{Z}_{y_2}}) \subset \mathrm{Hom}_X(\mathcal{O}_X, \mathcal{O}_{\mathcal{Z}_{y_2}})^G = \mathrm{Hom}_G(k, k[G]) \cong k.$$

If $y_1 \neq y_2$, there are no maps, if $y_1 = y_2$ there are only constant maps. By assumption that $\omega_X \cong \mathcal{O}_X$ as a G-sheaf, we obtain $\mathcal{O}_{Z_y} \otimes \omega_X \cong \mathcal{O}_{Z_y}$(because π maps Z_y to a closed point on X/G). Applying Serre's functor, we have

$$\mathrm{Hom}^n_{D^G(X)}(\mathcal{O}_{\mathcal{Z}_{y_1}}, \mathcal{O}_{\mathcal{Z}_{y_2}}) \cong \mathrm{Hom}_{D^G(X)}(\mathcal{O}_{\mathcal{Z}_{y_2}}, \mathcal{O}_{\mathcal{Z}_{y_1}})^\vee.$$

Since any coherent sheaf $\mathcal{M}$ on $Y \times Y \setminus \Delta_Y$ has a composition series with quotient isomorphic to sheaves of the form $\mathcal{O}_{(y_1,y_2)}$, we obtain that

$$\mathrm{Hom}^i_{D(Y\times Y)}(\mathcal{Q}^\bullet, \mathcal{O}_{(y_1,y_2)}) = 0,\ i \leq 0,\ i \geq n,$$

for all $(y_1, y_2) \in Y \times Y \setminus \Delta_Y$. Hence $\mathcal{Q}^\bullet$ restricted to $Y \times Y \setminus \Delta_Y$ has homological dimension $\leq n-2$.

Step 4: Since $\mathrm{Hom}^i_{D(Y\times Y)}(\mathcal{Q}, \mathcal{O}_{(y_1,y_2)}) \cong \mathrm{Ext}^i_X(\mathcal{O}_{Z_{y_1}}, \mathcal{O}_{Z_{y_2}}) = 0$ for all $i \in \mathbb{Z}$ when Z_{y_1} is disjoint from Z_{y_2} (i.e. when $\tau(y_1) \neq \tau(y_2)$), we obtain that the support of $\mathcal{Q}^\bullet$ is contained in $Y \times_{X/G} Y$. By assumption, this fibre product is of codimension $\geq n-1$ in $Y \times Y$. Thus $\mathrm{codim}(\mathrm{Supp}(\mathcal{Q}^\bullet)) \geq n-1$. Since the homological dimension of $\mathcal{Q}^\bullet$ restricted to $Y \times Y \setminus \Delta_Y$ is less or equal than $n-2$, applying Corollary 2.4.7, we obtain that $\mathcal{Q}^\bullet$ is supported on the diagonal.

Step 5: Let $\mathcal{E}^\bullet = \Psi\circ\Phi(\mathcal{O}_y)$. It follows from Steps 1 and 3 that $\mathrm{Ext}^i_{D(Y)}(\mathcal{E}^\bullet, \mathcal{O}_{y'}) = 0$ unless $y' = y$. This implies that $\mathcal{E}^\bullet$ is supported at the point y. Thus $\mathrm{dh}(\mathcal{E}^\bullet) \geq n$. On the other hand, (2.59) and Step 3 imply that $\mathrm{dh}(\mathcal{E}^\bullet) \leq n$. Suppose we prove that $H^0(\mathcal{E}^\bullet) \cong \mathcal{O}_y$. Then the Intersection Theorem will imply that Y is nonsingular at y and $\mathcal{E}^\bullet \cong \mathcal{O}_y$.

Step 6: Let us prove that $H^0(\mathcal{E}^\bullet) \cong \mathcal{O}_y$. We have a canonical map of complexes $\mathcal{E}^\bullet = \Psi(\Phi(\mathcal{O}_Y)) \to \mathcal{O}_y$. Let $\mathsf{C}^\bullet \to \mathcal{E}^\bullet \to \mathcal{O}_y \to \mathsf{C}^\bullet[1]$ be the corresponding distinguished triangle in $D(Y)$ and let

$$\ldots \mathrm{Hom}_{D(Y)}(\mathcal{O}_y, \mathcal{O}_y) \to \mathrm{Hom}_{D(Y)}(\mathcal{E}^\bullet, \mathcal{O}_y) = \mathrm{Hom}_{D^G(X)}(\Phi(\mathcal{O}_y), \Phi(\mathcal{O}_y))$$

$$\to \mathrm{Hom}_{D(Y)}(\mathsf{C}^\bullet, \mathcal{O}_y) \to \mathrm{Ext}^1_{D(Y)}(\mathcal{O}_y, \mathcal{O}_y)) \to \mathrm{Ext}^1_{D^G(X)}(\Phi(\mathcal{O}_y), \Phi(\mathcal{O}_y)) \to \ldots$$

be the corresponding long exact sequence. By Lemma 2.4.12, the last map is the Kodaira-Spencer map to the tangent space of the Hilbert scheme at the point represented by the cluster $\mathcal{Z}_y$. Since the Hilbert scheme is a fine moduli space this map is injective. Since the first two terms in the sequence are isomorphic to k and the map is not trivial, this implies that $\mathrm{Hom}_{D(Y)}(\mathsf{C}^\bullet, \mathcal{O}_y) = 0$. The spectral sequence (2.17) implies that $H^0(\mathsf{C}^\bullet) = 0$, hence $H^0(\mathcal{E}^\bullet) \cong \mathcal{O}_y$.

Step 7: We have proved that $\mathcal{E}^\bullet \cong \mathcal{O}_y$. By adjunction,

$$\mathrm{Hom}^i_{D(Y)}(\mathcal{O}_{y_1}, \mathcal{O}_{y_2}) = \mathrm{Hom}^i_{D(Y)}(\Psi\circ\Phi(\mathcal{O}_{y_1}), \mathcal{O}_{y_2}) \cong \mathrm{Hom}^i_{D(Y)}(\Phi(\mathcal{O}_{y_1}), \Phi(\mathcal{O}_{y_2})).$$

Step 8: We prove that the sheaves $\mathcal{O}_y$ form a spanning set in $D^b(Y)$. We use the spectral sequence (2.17)

$$E_2^{p,q} = \mathrm{Ext}^p(H^{-q}(\mathcal{F}^\bullet), \mathcal{O}_y) \Longrightarrow \mathrm{Ext}^n(\mathcal{F}^\bullet, \mathcal{O}_y).$$

Since its limit is zero for al $y \in Y$, taking $p = 0$, we get $H^{-q}(\mathcal{F}^\bullet) = 0$ for all q, hence $\mathcal{F}^\bullet$ is quasi-isomorphic to 0. Applying the Serre functor, we get the other property too.

Step 9: By Step 7 and Lemma 2.4.8, Φ is fully faithful. By Example 2.4.10, $D^G(X)$ is an indecomposable category. By Lemma 2.4.2, Φ admits a left adjoint functor $G = \Psi$. Composing it with the Serre functor we get the right adjoint $H = S \circ G \circ S^{-1}$. Obviously $H(\mathcal{F}^\bullet) \cong 0$ implies $G(\mathcal{F}^\bullet) \cong 0$. Thus we can apply Corollary 2.4.11 and obtain that Φ is an equivalence of categories.

This concludes the proof in the case when X is projective.

It remains to consider the quasi-projective case. Since we used several times the Serre duality, the proof does not immediately extends to non-projective case. It follows from the proof that we have an equivalence of categories

$$\Phi_c : D_c(Y) \to D_c^G(X), \tag{2.62}$$

where the subscript indicates that we considering the derived category of complexes whose cohomology sheaves have compact support. Here we use some smooth compactification of X to be able to apply the Serre functor to objects in $D_c^b(X)$ as well as the adjunction isomorphisms. Consider the functor

$$\Upsilon = \mathbf{R}\mathrm{pr}_{Y*}^G(\omega_{\mathcal{Z}/X} \overset{\mathbf{L}}{\otimes} \mathrm{pr}_X^*(?))$$

It is a right adjoint of the functor Φ (the proof is similar to the proof that G_1 is left adjoint to F_1 in Step 1). Since Φ_c is an equivalence of categories $\Upsilon \circ \Phi(\mathcal{O}_y) \cong \mathcal{O}_y$ for any closed point $y \in Y$. This immediately implies that the complex $\mathcal{Q}^\bullet$ defining the kernel of the composition of the corresponding integral transforms is isomorphic to $i_*(\mathcal{L})$, where $\mathcal{L}$ is an invertible sheaf on Y and $i : Y \to Y \times Y$ is the diagonal morphism. Let $s : \mathcal{O}_Y \to \mathcal{L} = \Upsilon \circ \Phi(\mathcal{O}_Y)$ be a map of sheaves corresponding to the adjunction morphism of functors $\delta : \mathbf{id}_Y \to \Psi \circ \Phi$. For any surjection $\phi : \mathcal{O}_Y \to \mathcal{O}_y$ the map $\delta(\phi) : \mathcal{O}_y = \mathcal{O}_Y \otimes \mathcal{O}_y \to \mathcal{L} \otimes \mathcal{O}_y$ is an isomorphism (because of equivalence (2.62)). This easily implies that the section s is an isomorphism.

Since $\Upsilon : D^G(X) \to D^b(Y)$ is a right adjoint functor of Φ, we obtain that Φ extends to a fully faithful functor $\Phi : D^b(Y) \to D^G(X)$. By the argument in the proof of Lemma, it suffices to prove that $\Upsilon(\mathcal{F}^\bullet) \cong 0$ implies that $\mathcal{F}^\bullet \cong 0$. By adjunction, $\mathrm{Hom}^i_{D^G(X)}(\Phi(\mathcal{G}^\bullet), \mathcal{F}^\bullet) = 0$ for all i and all $\mathcal{G}^\bullet \in D^b(Y)$. Since any object in $D_c^G(X)$ is isomorphic to an object of the form $\Phi(\mathcal{G}^\bullet)$, and objects of the form $\mathcal{O}_{G\cdot x}$ is a spanning set (proven by a similar argument that $\mathcal{O}_y$ is a spanning set in $D^b(Y)$), we obtain the assertion.

Remark 2.4.13. One can also prove that, under the assumption of the Theorem, the morphism $\tau : X//G \to X/G$ is crepant, i.e. $\tau^*(\omega_{X/G}) \cong \omega_{X//G}$.

Also it follows from Step 6 that the tangent space of $X//G$ at a point y is isomorphic to the tangent space of the G-Hilbert scheme $G-\mathrm{Hilb}^{[|G|}(X)$ at the point y. Since $X//G$ is smooth this implies that $X//G$ is a connected component of $G-\mathrm{Hilb}^{[|G|}(X)$. One can prove that in the case when $\dim X \leq 3$, the scheme $G-\mathrm{Hilb}^{[|G|}(X)$ is connected (see [BKR]). This was conjectured by Nakamura.

Example 2.4.14. Let $X = \mathrm{Spec}\ R$ be an affine G-variety. Let $\rho : G \to \mathrm{GL}(V)$ be a finite-dimensional linear representation of G over k. For any G-variety T consider the morphism $T \to \mathrm{Spec}\ k$ as an equivariant morphism where G acts trivially on Spec k. Let $\mathcal{V}_{T,\rho}$ be the pull-back of V considered as a G-sheaf on Spec k. If we identify V with the fibre of the corresponding trivial vector bundle, then the group acts on its total space by $g : (t, v) \to (g \cdot t, \rho(g^{-1})(v))$.

Let $\mathcal{Z} \subset X \times Y$ be as above, i.e. the universal family over $Y = X//G$ with projections $p : \mathcal{Z} \to X$ and $q : \mathcal{Z} \to Y$. We have

$$p^*(\mathcal{V}_{X,\rho}) \cong \mathcal{V}_{\mathcal{Z},\rho} \cong q^*\mathcal{V}_{Y,\rho}.$$

Let $\Psi : D^G(X) \to D^b(Y)$ be the Fourier-Mukai transform given by the kernel $\mathcal{O}_{\mathcal{Z}}$. Then

$$\Psi(\mathcal{V}_\rho) \cong q_*^G(\mathcal{V}_{\mathcal{Z},\rho}).$$

Let $\rho^\vee$ denote the dual representation. Obviously, $\mathcal{V}_\rho^\vee \cong \mathcal{V}_{\rho^\vee}$. By adjunction

$$\mathcal{H}om_Y^G(\mathcal{V}_{Y,\rho^\vee}, q_*\mathcal{O}_{\mathcal{Z}}) \cong q_*^G\mathcal{H}om_{\mathcal{Z}}(\mathcal{V}_{\mathcal{Z},\rho^\vee}, \mathcal{O}_{\mathcal{Z}}) \cong q_*^G V_{\mathcal{Z},\rho}.$$

Let

$$\mathcal{R} := q_*\mathcal{O}_{\mathcal{Z}}, \quad \mathcal{R}_\rho := \mathcal{H}om_Y^G(\mathcal{V}_{Y,\rho^\vee}, \mathcal{R}).$$

We have

$$\Psi(\mathcal{V}_\rho) \cong \mathcal{R}_\rho$$

and

$$\mathcal{R} \cong \bigoplus_{\rho \in \mathrm{Ir}(G)} \mathcal{R}_\rho \otimes \mathcal{V}_{Y,\rho}$$

where $\mathrm{Ir}(G)$ is the set of isomorphism classes of irreducible finite-dimensional linear representations of G over k. Indeed the sum is a subsheaf of $\mathcal{E}$ and their ranks are equal to $|G| = \dim k[G]$. Applying q^* we get an isomorphism of free sheaves of rank $|G|$. Thus the sum is isomorphic to $\mathcal{R}$.

Let $X = \mathbb{C}^2$ and $G \subset \mathrm{SL}_2(\mathbb{C})$. Then X/G is an affine normal surface and the orbit of $0 \in \mathbb{C}^2$ is its unique singular (if G is not trivial) point of type $T = A_n, D_n$, or E_n (a rational double point). The resolution $\tau : X//G \to X/G$ is a minimal resolution of singularities, and $\tau^{-1}(0)$ is the union of smooth rational curves $E_1, \ldots, E_n$ with self-intersection equal to (-2). Let $\mathrm{Ir}(G)' = \{\rho_1, \ldots, \rho_m\}$ be the set of non-trivial irreducible representations of G and $a_{ij} = c_1(\mathcal{R}_{\rho_i})[E_j]$. The classical McKay correspondence asserts that $m = n$ and the matrix $C = (-a_{ij}) + 2I_n$ is the Cartan matrix of the simple root system of type T.

Let $K_G(X)$ denote the Grothendieck group of the category $\mathrm{Coh}_G(X)$ of G-linearized coherent sheaves. As we explained in Lecture 5, the Fourier-Mukai transform $\Phi^{Y\to X}_{\mathcal{P}^\bullet} : D^{G'}(Y) \to D^G(X)$ defines an isomorphism

$$\Phi^{K_G,Y\to X}_{\mathcal{P}^\bullet} : K_{G'}(Y) \to K_G(X).$$

Applying Theorem 2.4.5, we obtain

Corollary 2.4.15. *Under assumptions of Theorem 2.4.5, the Fourier-Mukai transform defines an isomorphism of abelian groups*

$$K_G(X) \cong K(X//G).$$

Example 2.4.16. Suppose $X = \mathbb{C}^n$ and $G \subset \mathrm{SL}_n(\mathbb{C})$. By Proposition 2.1.2, any bounded complex of coherent G-sheaves on X is quasi-isomorphic to a bounded complex of locally free G-sheaves on X. Since any locally free sheaf on X is free, it is isomorphic to the sheaf $\mathcal{V}_{X,\rho}$, where ρ is a linear representation of G (see Example 2.4.14). This easily implies that $K_G(X) \cong \mathrm{Rep}(G)$, where $\mathrm{Rep}(G) = K(\mathrm{Mod}(\mathbb{C}[G]))$. This is a free abelian group of rank c equal to the number of isomorphism classes of irreducible representations of G, or, equivalently, the number of conjugacy classes of elements of G. It follows from Example 2.4.14 that the generators $V_\rho, \rho \in \mathrm{Ir}(G)$ are mapped to the classes $[\mathcal{R}_\rho]$ of locally free sheaves on $Y = X//G$.

Let $K_c(X//G)$ be the subgroup of $K(X//G)$ formed by sheaves with support in $\tau^{-1}(0)$, where 0 is the orbit of $0 \in \mathbb{C}^n$. We have a natural biadditive pairing

$$K_c(X//G) \times K(X//G) \to \mathbb{Z},\ \langle [E], [F]\rangle \to p_*([E \otimes F]), \tag{2.63}$$

where we represent $[E]$ by a locally free sheaf and $p_* : K(Y) \to K(\mathrm{point})$.

Let $R = \mathbb{C}[z_1, \ldots, z_n]$ be the coordinate ring of X and Der_R be its module of derivations (the dual of Ω^1_R). Let $\alpha = \sum z_i \frac{\partial}{\partial z_i} \in \mathrm{Der}_R$ be the *Euler tangent field*. Consider the Koszul complex

$$K^\bullet : 0 \to R \to \mathrm{Der}_R \to \Lambda^2\mathrm{Der}_R \to \ldots \to \Lambda^n\mathrm{Der}_R$$

where the differentials are $\omega \to \omega \wedge \alpha$. The dual complex $K^{\bullet\vee}$ is equal to

$$K^{\bullet\vee} : 0 \to \Omega^n_R \to \Omega^{n-1}_R \to \ldots \to \Omega^1_R \to R,$$

with differentials $\delta_s : \Omega^s_R \to \Omega^{s-1}_R$ defined by $\delta_s(\omega) = \langle \omega, \alpha\rangle$. Explicitly,

$$\delta_s(dz_{j_1} \wedge \ldots \wedge dz_{j_s}) = \sum_{i=1}^{s} (-1)^{i+1} z_{j_i} dz_{j_1} \wedge \ldots \wedge \widehat{dz_{j_i}} \wedge \ldots \wedge dz_{j_s}.$$

The reader recognizes in these formulas the usual definition of the Koszul complex (see [Hartshorne], p. 245) for the set $(z_1, \ldots, z_n)$. Note that the sheaf on $\mathbb{P}^n = \mathrm{Proj}\ R$ associated with the module $\mathrm{Ker}(\delta_s), s \geq 1$, is isomorphic to the sheaf $\Omega^s_{\mathbb{P}^n}$.

We know that z_i is transformed under $g \in \mathrm{SL}_n(\mathbb{C})$ to $\sum a_i z_i$, where $(a_1, \ldots, a_n)$ is the i-th row of the matrix g^{-1}. Also $\frac{\partial}{\partial z_i}$ is transformed to $\sum b_i \frac{\partial}{\partial z_i}$, where $(b_1, \ldots, b_n)$ is the i-th column of the matrix g. This shows that α is G-invariant and the Koszul complex $K^\bullet$ is an object of $D^G(\mathrm{Mod}(R))$. It is a free resolution of the object complex $R/(z_1, \ldots, z_n) = \mathbb{C}$, i.e. $H^{-i}(K^\bullet) = 0$ and $H^0(K^\bullet) \cong \mathbb{C}$. We have

$$(\mathrm{Der}_R^1)^\sim = (\Omega_X^1)^\vee \cong \mathcal{V}_{X,\rho_0},$$

where ρ_0 is the representation of G in $\mathbb{C}^n$ defined by the inclusion $G \subset \mathrm{SL}_n(\mathbb{C})$. Let

$$\mathcal{K}^\bullet : 0 \to \mathcal{O}_X \to \mathcal{V}_{X,\rho_0} \to \ldots \to \Lambda^n(\mathcal{V}_{X,\rho_0})$$

be the complex of coherent sheaves corresponding to $\mathcal{K}^\bullet$. Obviously $\Lambda^i(\mathcal{V}_{X,\rho_0}) \cong \mathcal{V}_{X,\Lambda^i(\rho_0)}$. Since $G \subset \mathrm{SL}_n(\mathbb{C})$, the representation $\Lambda^n(\rho_0)$ is trivial, hence the last term in the complex is isomorphic to $\mathcal{O}_X$ as a G-sheaf. Applying the functor p^* we get a complex on $\mathcal{Z}$

$$p^*(\mathcal{K}^\bullet) : \mathcal{O}_\mathcal{Z} \to \mathcal{V}_{\mathcal{Z},\rho_0} \to \ldots \to \mathcal{V}_{\mathcal{Z},\Lambda^i(\rho_0)} \to \ldots \to \mathcal{O}_\mathcal{Z}.$$

Since $\mathcal{V}_{\mathcal{Z},\rho} = q^*(\mathcal{V}_Y, \rho)$, applying the projection formula, we get a complex on $Y = X//G$

$$\mathcal{S}^\bullet = q_*(p^*(\mathcal{K}^\bullet)) : \mathcal{R} \to \mathcal{V}_{Y,\rho_0} \otimes \mathcal{R} \to \ldots \to \mathcal{V}_{Y,\Lambda^i(\rho_0)} \otimes \mathcal{R} \to \ldots \to \mathcal{R}.$$

Note that the complex $\mathcal{K}^\bullet$ is exact when restricted to $\mathbb{C}^n \setminus \{0\}$. Thus $p^*(\mathcal{K}^\bullet)$ is supported on $p^{-1}(0)$. Since q_* is exact, the complex $\mathcal{S}^\bullet$ is supported on $\tau^{-1}(0)$.

Now we decompose $\mathcal{S}^\bullet$ into direct sum of complexes corresponding to an irreducible representations $\rho_i \in \mathrm{Ir}(G) = \{\rho_1, \ldots, \rho_c\}$.

$$q_*(\mathcal{K}^\bullet) \cong \bigoplus_{k=1}^c \mathcal{S}_k^\bullet,$$

where

$$\mathcal{S}_k^\bullet : \mathcal{R}_k \to \bigoplus_{i=1}^c \mathcal{R}_i^{\oplus a_{ki}^{(1)}} \to \ldots \to \bigoplus_{i=1}^c \mathcal{R}_i^{\oplus a_{ki}^{(j)}} \to \ldots \to \mathcal{R}_k, \qquad (2.64)$$

and

$$\Lambda^j(\rho_0) \otimes \rho_k = \bigoplus_{i=1}^c a_{ki}^{(j)} \rho_i, \quad j = 1, \ldots, n-1,$$

is the decomposition of the tensor product of the exterior power of ρ_0 with ρ_k into the direct sum of irreducible representations with multiplicities $a_{ki}^{(j)}$. For each $j = 1, \ldots, n-1$, we can define the Mckay graph $\Gamma(G, j)$ with vertices v_i corresponding to irreducible representations of G and $a_{ki}^{(j)}$ edges from v_k to v_i. For $n = 2$, this is the usual McKay graph equal to the extended Dynkin diagram of type ADE.

Now we have c complexes $\mathcal{S}_k^\bullet$ whose cohomology are supported in $p^{-1}(0)$. Consider the pairing (2.63).

Conjecture 2.4.17.

$$\langle [\mathcal{R}_i], [\mathcal{S}_j^{\bullet\vee}] \rangle = \delta_{ij}. \tag{2.65}$$

This is known two be true for $n = 2$ (Gonzales-Sprinberg and Verdier) and $n = 3$ (Ito and Nakajima).

2.5 Exercises

2.1 Prove the existence of the spectral sequence

$$E_2^{p,q} = \mathcal{E}xt(\mathcal{F}^\bullet, \mathcal{H}^q(\mathcal{G}^\bullet)) \Longrightarrow \mathcal{E}xt(\mathcal{F}^\bullet, \mathcal{G}^\bullet).$$

2.2 Prove the existence of the spectral sequence

$$E_2^{p,q} = \mathcal{T}or_{-p}(\mathcal{H}^q(\mathcal{F}^\bullet), \mathcal{G}^\bullet) \Longrightarrow \mathcal{T}or_{-p-q}(\mathcal{F}^\bullet, \mathcal{G}^\bullet).$$

2.3 Prove the adjunction isomorphisms in (2.24).

2.4 Prove that an abelian category has homological dimension equal to 0 if and only if all exact sequences split in A.

2.5 Prove the following *projection formula*. For any proper morphism $f : X \to Y$ of projective schemes over a field k, there is an isomorphism

$$\mathbf{R}f_*(\mathcal{F}^\bullet) \overset{\mathbf{L}}{\otimes} \mathcal{G}^\bullet \cong \mathbf{R}f_*(\mathcal{F}^\bullet \overset{\mathbf{L}}{\otimes} \mathbf{L}f^*(\mathcal{G}^\bullet)).$$

2.6 Let $i_x : x \hookrightarrow X$ be the inclusion morphism of a closed point x in a scheme X. Show that, for any complex $\mathcal{F}^\bullet$ one has $\mathcal{F}^\bullet(x) := \mathbf{L}i_x^*(\mathcal{F}^\bullet) \neq 0$ if and only if $x \in \operatorname{Supp}(\mathcal{F}^*) := \cup_i \operatorname{Supp}(H^i(\mathcal{F}^*))$.

2.7 Let $X^\bullet$ be an object of $D^b(\mathsf{A})$ and $m = \max\{i : H^i(X^\bullet) \neq 0\}$. Show that there is an epimorphism from $X^\bullet \to H^m(X^\bullet)[-m]$ in the derived category.

2.8 Let Vect_k be the category of finite-dimensional vector spaces over a field k. Show that the identity functor is a Serre functor.

2.9 Let $F : \mathsf{C} \to \mathsf{C}'$ be a functor between k-linear categories endowed with Serre functors $S : \mathsf{C} \to \mathsf{C}, S' : \mathsf{C}' \to \mathsf{C}'$.

(i) Prove that the functors $F \circ S$ and $S' \circ F$ are isomorphic.

(ii) Assume that F admits a left adjoint functor $G : \mathsf{C}' \to \mathsf{C}$. Then $S \circ G \circ S'^{-1}$ is a right adjoint of F.

2.10 Let $f : X' \to X, g : Y' \to Y$ be proper morphisms of schemes and $f \times g : X' \times Y' \to X \times Y$ be their Cartesian product. Show that there exists a morphism of functors $D^b(Y) \to D^b(X)$

$$\Phi_{\mathcal{E}^\bullet}^{Y \to X} \to \mathbf{R}f_* \circ \Phi_{\mathbf{L}(f \times g)^*(\mathcal{E}^\bullet)}^{Y' \to X'} \circ \mathbf{L}g^*.$$

2.11 Consider the category C with objects schemes and morphisms $\mathrm{Mor}_{\mathsf{C}}(X,Y)$ equal to $\mathrm{Ob}(D^b(X\times Y))$. Take for compositions the composition $\mathcal{F}^\bullet\circ\mathcal{G}^\bullet$ defined in (2.37). Check that it is indeed a category and that it admits products. Let $f_{\mathcal{E}}: X\to Y$ be a morphism in C and $f_{\mathcal{E}}^* = \Phi_{\mathcal{E}^\bullet}^{Y\to X}, f_*^{\mathcal{E}}* = \Phi_{\mathcal{E}^\bullet}^{X\to Y}$. Show that this defines a fibred and cofibred categories over C.

2.12 Let $D^b(X)^{\mathrm{tor}}$ be the full subcategory of $D^b(X)$ formed by complexes $X^\bullet$ such that $H^i(X^\bullet)$ are torsion sheaves. Suppose $\Phi_{\mathcal{E}^\bullet}^{X\to X}$ is a Fourier-Mukai transform. Does it transform $D^b(X)^{\mathrm{tor}}$ to itself?

2.13 Suppose $c_1(X)=0$. Show that the Mukai pairing is symmetric if $\dim X$ is even and alternating otherwise.

2.14 Let $\Phi_{K^\bullet}^{X\to Y}$ and $\Phi_{L^\bullet}^{X\to Y}$ be two Fourier-Mukai transforms. Prove that $\Phi^{X\to Y}_{K^\bullet\overset{\mathbf{L}}{\otimes}L^\bullet}$ is a Fourier-Mukai transform.

2.15 Let C be a small category defined by $R,X,s,t,c,e)$, where $R=\mathrm{Mor}(\mathsf{C})$ and $X=\mathrm{Ob}(\mathsf{C})$ (see section 1.1). The axioms of a category are equivalent to the following properties:

$$c\circ(c\times\mathbf{id}_R)=c\circ(\mathbf{id}_R\times c),\ s\circ\epsilon=\mathbf{id}_X=t\circ\epsilon, \tag{2.66}$$
$$c\circ(c\times\mathbf{id}_R)=\mathbf{id}_R=c\circ(\mathbf{id}_R\times\epsilon),$$
$$(R\times_{s,t}R)\times_{\mathrm{pr}_1,\mathrm{pr}_1}(R\times_{s,t}R)\cong R\times_{s,t}\times R\times_{s,t}R$$

A category $\mathcal{G}$ is called a *groupoid* if each $a\in R$ is an isomorphism. In this case there is a map $\iota: R\to R$ defined by $\iota(a)=a^{-1}$ satisfying the following conditions

$$c\circ(\mathbf{id}_R\times\iota)=e\circ s,\ c\circ(\iota\times\mathbf{id}_R)=\epsilon\circ s, \tag{2.67}$$

(i) Show that any group G defines a groupoid $\underline{G}$ with $\mathrm{Ob}(\underline{G})$ consisting of one element.

(ii) Show that, for any group object G in a category $\mathcal{S}$, the assignment $S\to\underline{G(S)}$ defines a fibred category with values in groupoids.

(iii) Show that an equivalence relation $R\subset X\times X$ on a set X defines a groupoid with ϵ equal to the diagonal map and i equal to the switch of the factors map.

2.16 A pair of objects (R,X) in a category C with fibred products is called an *groupoid* or a *pre-equivalence relation* on X if there are morphisms $R\overset{s}{\to}X, R\overset{t}{\to}X, c: R\times_X R\to R, e: X\to R, i: R\to R$ satisfying (2.66) and (2.67).

(i) Show that the Yoneda functor applied to (R,X,s,t,c,e,i) defines a fibred category in groupoids.

(ii) Show that, for any $P\in\mathrm{Ob}(\mathsf{C})$, the image of $(s(P),t(P)): h_R(P)\to h_X(P)\times h_X(P)$ is an equivalence relation on the set $h_X(P)$.

(iii) Show that a group action $\sigma: G\times X\to X$ in a category C defines a groupoid with $R=G\times X, X=X, s=\sigma, t=\mathrm{pr}_X$ and $R\times_X R\cong G\times G\times X$.

(iv) Show that for any $P \in \mathrm{Ob}(\mathsf{C})$ such that $h_X(P)$ is a singleton, the data $(h_R(P), h_X(P), h_s(P), h_t(P), h_\iota(P), h_e(P)$ defines a group structure on the set $h_R(T)$.

(iv) Let C be a topological space and $(U_i)_{i\in I}$ be its open covering. Set $X = \coprod_{i\in I} U_i$ and $R = \coprod_{i,j\in I} U_i \cap U_j$. Define two projections $s, t : R \to X$ by considering the maps $U_i \cap U_j \hookrightarrow U_i \subset X, U_i \cap U_j \hookrightarrow U_j$. Show that (R, X, s, t) generate a groupoid in Top (the *groupoid generated by the cover*).

2.17 For any groupoid $E = (R, X, s, t, c, \epsilon, \iota)$ in a category $\mathcal{S}$ and a fibred category C over $\mathcal{S}$ define the category of *descent data*. Its objects are pairs (A, α), where $A \in \mathsf{C}(X)$ and α is an isomorphism $\alpha : s^*(A) \to t^*(A)$ satisfying the following conditions

$$c^*(\alpha) = \mathrm{pr}_1^*(\alpha) \circ \mathrm{pr}_2^*(\alpha), \;\; e^*(\alpha) = \mathbf{id}_A. \tag{2.68}$$

(i) Show that in the case when (R, X) is defined by a group action, the definition of a descent datum (A, α) coincides with the definition of a G-linearized object.

(ii) Consider an example of a groupoid in Top from Exercise 6.2 (iv) and the fibred category $\mathrm{Sh}^{\mathrm{ab}}$ of abelian sheaves over Top. Show that for any sheaf $\coprod_{i\in I} \mathcal{F}_i$ over $X = \coprod_{i\in I} U_i$ equipped with a descent data (called in this case the *gluing data* there exists a unique sheaf $\mathcal{F}$ on X (up to isomorphism) such that $\mathcal{F}|U_i \cong \mathcal{F}_i, i \in I$.

2.18 Let $s, t : X \to Y$ be two morphisms in a category C. A morphism $p : Y \to Z$ is called the *co-equalizer* of the pair (s, t) if the compositions $p \circ s$ and $p \circ t$ are equal, and, for any $p' : Y \to Z'$ with this property there exists a morphism $q : Z \to Y'$ such that $p' = q \circ p$.

(i) Show that the co-equalizer of $s, t : X \to Y$ always exists in the category of sets and in the category of presheaves of sets on any category.

(ii) Let $(X, R, s, t, c, \epsilon, \iota)$ be a groupoid in a category C. Let X/R be the co-equilizer of the pair (s, t) (it may not exist). Show that there exists a canonical map of presheaves $h_X/h_R \to h_{X/R}$, where h_X/h_R is the co-equalizer of $h(s), h(t) : h_R \to h_X$ in the category presheaves of sets. Give an example showing that it is not necessary an isomorphism.

2.19 Let C be a k-linear category. An object A is called *simple* if the natural map $k \to \mathrm{End}_{\mathsf{C}}(A)$ is an isomorphism. Assume that a k-linear category is fibred over a category $\mathcal{S}$ and let G be a constant group object in $\mathcal{S}$ and $(X, \sigma) \in \mathcal{S}_G$.

(i) Show that the set of G-linearizations on an object $A \in \mathsf{C}(X, \sigma)$ is a torsor (=principal homogeneous space) over the group $\mathrm{Hom}(G, k^*)$.

(ii) Let $A \in \mathsf{C}(X,\sigma)$ be a G-invariant object (i.e. there exists an isomorphism $\mathrm{pr}_X^*(A) \cong \sigma^*(A)$). Show that one can assign to A the cohomology class $[A] \in H^2(G,k^*)$ such that a G-invariant object admits a G-linearization if and only if $[A]=0$.

2.20 Let X be a projective variety over a field k and x be its closed point. Suppose $\mathcal{F}$ is a coherent sheaf such that $\mathrm{Hom}_X(\mathcal{F},\mathcal{O}_x) \cong k$. Prove that $\mathcal{F}$ is isomorphic to the structure sheaf of a closed subscheme of X.

2.21 Suppose $\Phi_{\mathcal{K}^\bullet}^{X\to Y}$ is an equivariant Fourier-Mukai transform. Then its kernel $\mathcal{K}^\bullet$ is a simple object of the category $D^G(X \times Y)$ (see Exercise 6.5).

2.22 Let $G = \mathbb{Z}/2\mathbb{Z}$ act on $X = \mathbb{C}^2$ as $(x,y) \to (-x,-y)$. Check the last assertion from Example 2.4.14.

2.23 Let H be a normal subgroup of G. Suppose there is an equivariant Fourier-Mukai transform $\Phi : D^G(X) \to D^b(X//G)$. Show that it defines an equivariant Fourier-Mukai transform $D^{G/H}(X//H) \to D^b(X//G)$.

2.24 Suppose a non-trivial finite group of acts trivially on a quasi-projective variety X over a field of characteristic prime to the order of G . Show that the category $D^G(X)$ is always decomposable.

Lecture 3

Reconstruction Theorems

3.1 Bondal-Orlov Theorem

In this section we will prove that a smooth projective variety X with ample canonical or anti-canonical sheaf can be reconstructed from the derived category $D^b(X)$. We denote by n the dimension of X

First we see how to reconstruct closed points of X.

Definition 3.1.1. Let D be a k-linear derived category of some abelian category. Suppose D admits a Serre functor $S : \mathsf{D} \to \mathsf{D}$. An object P in D is called a *point-like object* of codimension c if

(i) $S(P) \cong P[c]$;

(ii) $\mathrm{Hom}^i(P, P) = 0, i < 0$;

(iii) $\mathrm{Hom}(P, P) := k(P)$ is a field.

Proposition 3.1.1. *Suppose X has an ample canonical or anti-canonical sheaf. Then an object $\mathcal{F}^\bullet$ in $D^b(X)$ is a point-like object if and only if $\mathcal{F}^\bullet \cong \mathcal{O}_x[r]$, where x is a closed point of X and $r \in \mathbb{Z}$.*

Proof. Let x be a closed point of X. Then $S(\mathcal{O}_x[r]) = (\mathcal{O}_x \otimes \omega_X)[r + n] \cong \mathcal{O}_x[n + r]$, thus (i) holds. Since $\mathrm{Hom}^i_X(\mathcal{F}^\bullet(r), \mathcal{G}^\bullet(r)) \cong \mathrm{Hom}^i_X(\mathcal{F}^\bullet, \mathcal{G}^\bullet)$ and $\mathrm{Hom}^i(\mathcal{O}_x, \mathcal{O}_x) = 0, i < 0$, property (ii) holds too. We have

$$\mathrm{Hom}_X(\mathcal{O}_x(r), \mathcal{O}_x[r]) \cong \mathrm{Hom}_X(\mathcal{O}_x, \mathcal{O}_x) \cong \mathrm{Hom}_{\mathcal{O}_{X,x}}(k(x), k(x)) \cong k(x).$$

This checks (iii).

Suppose $\mathcal{P}^\bullet$ is a point-like object in $D^b(X)$. Since $H^i(S(\mathcal{P}^\bullet)) = H^i(\mathcal{P}^\bullet \otimes \omega_X[n]) \cong H^{i+n}(\mathcal{P}^\bullet \otimes \omega_X)$, condition (i) implies that $H^{i+n}(\mathcal{P}^\bullet \otimes \omega_X) \cong H^{i+s}(\mathcal{P}^\bullet)$. Since the cohomology of $\mathcal{P}^\bullet$ and $\mathcal{P} \otimes \omega_X$ both vanish or not vanish, taking the largest possible j such that $H^j(\mathcal{P}^\bullet) \neq 0$ and obtain that $s = n$ and $H^i(\mathcal{P}^\bullet) \cong H^i(\mathcal{P}^\bullet \otimes \omega) \cong H^i(\mathcal{P}^\bullet) \otimes \omega_X$ for all i. Let us show that the cohomology sheaves are supported in codimension $n = \dim X$. Let $\mathcal{F}$ be a coherent sheaf on X such that $\mathcal{F}\otimes\omega_X \cong \mathcal{F}$. Tensoring with ω we get an isomorphism $\mathcal{F}\otimes\omega_X \cong \mathcal{F}\otimes\omega_X^{\otimes 2}$. Continuing in this way we obtain that $\mathcal{F}\cong\mathcal{F}\otimes\omega_X^{\otimes s}$, where $\omega_X^{\otimes s}$ is very ample if ω_X was ample. If ω_X^{-1} were ample, we use a similar argument by showing that $\mathcal{F}\cong\mathcal{F}\otimes\omega_X^{\otimes -s}$, where $\mathcal{L}=\omega_X^{\otimes -s}$ is very ample. Use $\omega_X^{\otimes \pm s}$ to find a closed embedding $i : X\hookrightarrow \mathbb{P}^N$. Since $i_*(\mathcal{F}\otimes\mathcal{L})\cong i^*(\mathcal{O}_{\mathbb{P}^N}(1))\cong i_*(\mathcal{F})(1)$, we may assume that $X\cong\mathbb{P}^N\cong \mathrm{Proj}k[T_0,\dots,T_N]$ and $\mathcal{F}=M^\sim$ for some graded module over $k[T_0,\dots,T_N]$. Let $P_M(t)$ be the Hilbert polynomial of M. Recall that $P_M(n)=\dim_k M_n, n\gg 0$. Its degree is equal to the dimension of the support of $\mathcal{F}$ ([Hartshorne], Chap. I, Theorem 7.5). Since $\mathcal{F}\cong\mathcal{F}(1)$, we have $P_M(n+1)=P_M(n)$ for n large. This is possible only if the degree of the polynomial is equal to zero. Thus $\dim\mathrm{Supp}(\mathcal{F})=0$.

Applying this to our situation we obtain that $\mathrm{Supp}(\mathcal{P}^\bullet)$ is a 0-dimensional closed subset Z. By property (iii), Z must be a single point. This follows from the following fact whose proof we leave to the reader: if $\mathrm{Supp}(\mathcal{F}^\bullet)=Z_1\coprod Z_2$ then $\mathcal{F}^\bullet\cong\mathcal{F}_1^\bullet\oplus\mathcal{F}_1^\bullet$ with $\mathrm{Supp}(\mathcal{F}_1^\bullet)=Z_1$ and $\mathrm{Supp}(\mathcal{F}_2^\bullet)=Z_2$.

Now we combine the spectral sequences (2.12) and (2.17) to obtain a spectral sequence

$$E_2^{p,q}=\bigoplus_{k-j=q}\mathrm{Ext}^p(H^j(\mathcal{P}^\bullet),H^k(\mathcal{P}^\bullet))\Longrightarrow \mathrm{Hom}^{p+q}(\mathcal{P}^\bullet,\mathcal{P}^\bullet).$$

For any two sheaves $\mathcal{F}_1$ and $\mathcal{F}_2$ supported at a closed point x there exists a nonzero homomorphism of sheaves $\mathcal{F}_1\to\mathcal{F}_2$. This follows from the fact that for any two modules M_1, M_2 over a local ring R supported at the maximal ideal $\mathfrak{m}$ there is a non-trivial homomorphism $M_1\to M_2$ (consider the filtrations on M_1 and M_2 with quotients isomorphic to $R/\mathfrak{m}$, then get a surjective homomorphism $M_1\to R/\mathfrak{m}$ and an injective homomorphism $R/\mathfrak{m}\to M_2$). Take q minimal such that $E_2^{0,q}\neq 0$. By interchanging i and j, we may assume that $q\le 0$. Then the term $E_2^{0,q}$ survives in the limit, hence $\mathrm{Hom}^q(\mathcal{P}^\bullet,\mathcal{P}^\bullet)\neq 0$. By property (ii), we obtain $q=0$. By property (iii), $\mathrm{Hom}(\mathcal{P}^\bullet,\mathcal{P}^\bullet)$ is a field. This implies that there is only one j with $H^j(\mathcal{P}^\bullet)\neq 0$. Also $\mathrm{End}(H^j(\mathcal{P}^\bullet))$ must be a field, hence $H^j(\mathcal{P}^\bullet)\cong\mathcal{O}_x$, and we are done.

□

Example 3.1.2. The condition that $\omega_X^{\pm 1}$ is ample is essential. Assume X is a smooth projective variety with $\omega_X\cong\mathcal{O}_X$. Take any closed reduced connected subvariety $i : Y\hookrightarrow X$. Then the structure sheaf $i_*\mathcal{O}_Y$ is a point-like object of codimension $\dim X$. In fact, properties (i) and (ii) are obvious. We have $\mathrm{Hom}_X(i_*\mathcal{O}_Y,i_*\mathcal{O}_Y)=\mathrm{Hom}_Y(i^*i_*\mathcal{O}_X,\mathcal{O}_Y)=\mathrm{Hom}_Y(\mathcal{O}_Y,\mathcal{O}_Y)$ is a field.

Another example is the following. Suppose we are in the situation of Theorem 2.4.5. It follows from the proof of this theorem that, for any closed point $y\in X//G$, we have $\Phi(\mathcal{O}_y)=\mathcal{O}_{Z_y}$, where Z_y is a cluster on X corresponding to y. Since an equivalence of categories sends point-like objects to point-like objects, all objects $\mathcal{O}_Z[s]$, where Z is a cluster, are point-like objects.

Definition 3.1.2. An object L of a k-linear triangulated category D is called *invertible* if for any point-like objects P in D there exists an integer s such that

(i) $\mathrm{Hom}^s(L, P) \cong k(P)$;

(ii) $\mathrm{Hom}^i(L, P) = 0,\ i \neq s$.

Proposition 3.1.3. *Let X be a smooth irreducible variety over k. Assume that all point-like objects in $D^b(X)$ are isomorphic to complexes $\mathcal{O}_x[s]$ for some $x \in X$ and $s \in Z$ (e.g. if $\omega_X^{\pm 1}$ is ample). Then an object $\mathcal{L}^\bullet$ in $D^b(X)$ is invertible if and only if $\mathcal{L}^\bullet \cong \mathcal{L}[t]$ for some invertible sheaf $\mathcal{L}$ and $t \in \mathbb{Z}$.*

Proof. Obviously $\mathcal{L}[t]$ is an invertible object in $D^b(X)$. Let $\mathcal{L}^\bullet$ be an invertible object from $D^b(X)$. Let $P = \mathcal{O}_x[s]$ be a point-like object. We have a spectral sequence

$$E_2^{p,q} = \mathrm{Hom}(H^{-q}(\mathcal{L}^\bullet), \mathcal{O}_x[p]) \Longrightarrow \mathrm{Hom}^{p+q}(\mathcal{L}^\bullet, \mathcal{O}_x).$$

Let $\mathcal{H}^{q_0} = H^{q_0}(\mathcal{L}^\bullet)$ be the nonzero cohomology sheaf with maximal possible q. The terms E_2^{0,q_0} and E_2^{1,q_0} do not change in the limit (nothing goes in and nothing goes out). Since $\mathrm{Hom}^i(\mathcal{L}^\bullet, \mathcal{O}_x)$ is allowed to be non-zero only for one i, and E_2^{0,q_0} is not zero, we obtain that $\mathrm{Hom}(\mathcal{H}^{q_0}, \mathcal{O}_x) = \mathrm{Hom}^{q_0}(\mathcal{L}^\bullet, \mathcal{O}_x)$ is a field and $\mathrm{Ext}^1(\mathcal{H}^{q_0}, \mathcal{O}_x) = 0$. The second condition gives $\mathrm{dh}(\mathcal{H}^{q_0}) = 0$, i.e. $\mathcal{H}^{q_0}$ is locally free ([Bourbaki, Commutative Algebra], Chap. 10, Prop. 4). The first condition gives that $\mathcal{H}^{q_0}$ is of rank 1. Since $\mathcal{H}^{q_0}$ is locally free, all $E_2^{p,q_0} = \mathrm{Ext}^p(\mathcal{H}^{q_0}, \mathcal{O}_x) = 0, p \neq 0$, hence $E_2^{0,q_0-1} = \mathrm{Hom}(\mathcal{H}^{q_0-1}, \mathcal{O}_x)$ survives in the limit and $\mathrm{Hom}^{q_0-1}(\mathcal{L}^\bullet, \mathcal{O}_x) \neq 0$. Since $\mathrm{Hom}^i(\mathcal{L}^\bullet, \mathcal{O}_x)$ is not zero only for one i, we get that $H^{q_0-1}(\mathcal{L}^\bullet) = 0$. Repeating the argument, we show that all cohomology sheaves vanish except $\mathcal{H}^{q_0}$. Since the latter is an invertible sheaf, we are done.

□

Now we are ready to state and prove the Bondal-Orlov Theorem.

Theorem 3.1.4. *Let X be a smooth irreducible projective variety with ample canonical or anti-canonical sheaf. If $D^b(X)$ is equivalent to $D^b(Y)$ for some smooth irreducible projective variety, then $X \cong Y$.*

Proof. The proof will consist of several steps.

Step 1: As always we assume that the equivalence is an equivalence of derived categories, i.e. it commutes with the shift functor and sends distinguished triangles to distinguished triangles. Let $F : D^b(X) \to D^b(X)$ be an equivalence of categories. Let G be its quasi-inverse functor. If S is a Serre functor in $D^b(X)$, then $S' = F \circ S \circ G$ is a Serre functor in $D^b(Y)$. We also have the Serre functor $S_Y = \otimes \omega_Y[n]$. We know that two Serre functors are isomorphic. This implies that F defines a bijection on the set of point-like objects. Since we do not assume that $\omega_Y^{\pm 1}$ is ample, we do not know whether all point-like objects in $D^b(Y)$ are really shifted $\mathcal{O}_y, y \in Y$. Suppose we have a point-like object $P^\bullet$ in $D^b(Y)$ which is not isomorphic to any object of the form $\mathcal{O}_y[i]$. Let $x_0 \in X$ such that $F(\mathcal{O}_{x_0}[a]) \cong P^\bullet$ for some $a \in \mathbb{Z}$. We know that $x_0 \neq x$ for any point $x \in X$ such that $G(\mathcal{O}_y) \cong \mathcal{O}_x[j]$ for some j. This implies that $\mathrm{Hom}^i_{D^b(Y)}(\mathcal{P}^\bullet, \mathcal{O}_y) \cong \mathrm{Hom}^j_{D^b(X)}(\mathcal{O}_{x_0}, \mathcal{O}_x) = 0$. Since the sheaves $\mathcal{O}_y$ span

$D^b(Y)$, we obtain that $P^\bullet \cong 0$. Thus all point-like objects in $D^b(Y)$ look like $\mathcal{O}_y[i]$.

Step 2:By Step 1, we can apply Proposition 3.1.3. It follows that all invertible objects in $D^b(Y)$ are isomorphic to objects of the form $\mathcal{L}[i]$, where $\mathcal{L}$ is an invertible sheaf. The functor F transforms invertible objects to invertible objects.

Step 3: Applying the shift functor we may assume that $F(\mathcal{O}_X) \cong \mathcal{L}_0$ for some invertible sheaf $\mathcal{L}_0$ on Y. But then $\mathrm{Hom}_X(\mathcal{O}_X, \mathcal{O}_x) = \mathrm{Hom}_Y(\mathcal{L}, \mathcal{O}_y[j]) = \mathrm{Ext}^j_Y(\mathcal{L}, \mathcal{O}_y)$ implies that $j = 0$. Thus $F(\mathcal{O}_x) \cong \mathcal{O}_y$ and $k(x) \cong k(y)$. This establishes a set-theoretical bijection f between closed points of X and Y.

Step 4: By composing the functor F with the functor $? \otimes \mathcal{L}_0^{-1}$, we may assume that $F(\mathcal{O}_Y) \cong \mathcal{O}_Y$ in $D^b(Y)$. Since $\mathrm{dh}(\mathcal{O}_x) = n = \mathrm{dh}(F(\mathcal{O}_x)) = \dim Y$ we obtain that $\dim X = \dim Y$. We have

$$F(\omega_X^k) = F(S_X^k(\mathcal{O}_X)[-kn] \cong S_Y^k(F(\mathcal{O}_X)[-kn] \cong S_Y^k(\mathcal{O}_Y)[-kn] \cong \omega_Y^k.$$

Since F is fully faithful, we get

$$H^0(X, \omega_X^i) \cong \mathrm{Hom}_X(\mathcal{O}_X, \omega_X^i) \cong \mathrm{Hom}_Y(F(\mathcal{O}_X), F(\omega_X^i))$$
$$\cong \mathrm{Hom}_Y(\mathcal{O}_Y, \omega_Y^i) \cong H^0(Y, \omega_Y^i),$$

for all i.

The product in the *canonical algebra*

$$A(X) = \bigoplus_{i=0}^{\infty} H^0(X, \omega_X^i)$$

can be expressed by the composition of $s_1 \in H^0(X, \omega_X^i), s_2 \in H^0(X, \omega_X^i)$,

$$s_1 \cdot s_2 = S_X^i(s_2)[-in] \circ s_1.$$

Thus we see implies that F defines an isomorphism of graded canonical rings $A(X) \to A(Y)$. If $\omega_Y^{\pm 1}$ were ample, we are done.

Step 5: It remains to show that $\omega_Y^{\pm}$ is ample. We give two proofs. The first one uses an original argument of Bondal and Orlov. For any section s of an invertible sheaf $\mathcal{L}$, its set of zeroes $(s)_0$ can be homologically described as the set of points $x \in X$ such that the composition of the homomorphisms $\mathcal{O}_X \xrightarrow{s} \mathcal{L} \to \mathcal{O}_x$ is zero. Thus we have a homological definition of a set $X_s = X \setminus (s)_0$. In our situation, we obtain that F sends $\mathcal{O}_x$ to $\mathcal{O}_{f(x)}$ and for any $x \in (s)_0$, it sends the complex $\mathcal{O}_X \xrightarrow{s} \mathcal{L} \to \mathcal{O}_x$ to a complex $\mathcal{O}_Y \xrightarrow{F(s)} F(\mathcal{L}) \to \mathcal{O}_f(y)$. Thus the bijection f sends subsets of X of the form X_s to subsets of Y of the form $Y_{F(s)}$. Since among sets X_f there are affine open subsets defining a base of topology of X (take $\mathcal{L}$ any ample invertible sheaf), we see that the sets X_s form a base of topology and our bijection $f : X \to Y$ is a homeomorphism. Now, on Y the open sets $Y_t, t \in \Gamma(Y, \omega^{\pm m})$ are the images of open affine sets $X_s, s \in \Gamma(Y, \omega^{\pm m})$ forming a base of topology of X. Hence the open sets Y_t

form a basis of topology in Y. It is known that this implies that $\omega_Y^{\pm 1}$ is ample ([EGA, II 4.5.2 and 4.5.5).

The second proof assumes that k is algebraically closed. We may assume that ω_X is very ample. By the argument from above the linear system $|K_Y|$ separates points. In fact, if all sections of ω_Y vanish at y_1, y_2, then all sections of ω_X vanish at $f^{-1}(y_1), f^{-1}(y_2)$. Let $\phi : Y \to \mathrm{Proj}\ A(Y)$ be the regular map from Y to its canonical model. Since ϕ separates points, no curves on Y are blown down to points. Applying Moishezon-Nakai criterion of ampleness, we obtain that ω_Y is ample. We could also prove this without appealing to the latter result by showing that the canonical linear system $|K_Y|$ separates tangent directions. We leave the argument to the reader. Similarly, we consider the case when ω_X^{-1} is ample.

□

Corollary 3.1.5. *Let X be a smooth projective variety with ample canonical or anti-canonical sheaf. Then any equivalence of derived categories $D^b(X) \to D^b(X)$ is a composition of f^*, where $f \in Aut(X)$, a twist by an invertible sheaf, and the shift functor. More precisely, there is an isomorphism of groups*

$$\mathrm{Auteq}(D^b(X)) \cong (\mathrm{Pic}(X) \rtimes Aut(X)) \times \mathbb{Z}.$$

Proof. This follows from the proof of the previous theorem. After we shift and twist by $\mathcal{L} = F(\mathcal{O}_X)$ we proved that $F(\mathcal{O}_X) = \mathcal{O}_X$, and then defined an automorphism of the canonical algebra $A(X)$. This defines an automorphism of X. □

Definition 3.1.3. Two smooth projective varieties are called *derived equivalent* if the exists an equivalence of their derived categories of coherent sheaves.

Using Orlov's Theorem 2.2.5, one can also prove the following.

Theorem 3.1.6. *Suppose X and Y are derived equivalent algebraic varieties. Then their canonical algebras are isomorphic. In particular, their Kodaira dimensions are equal*

Proof. Let $\Phi_{\mathcal{P}^\bullet}^{X\to Y} : D^b(X) \to D^b(Y)$ be a Fourier-Mukai transform and $\Phi_{\mathcal{Q}^\bullet}^{Y\to X}$ be its quasi-inverse with $\mathcal{P}^\bullet \circ \mathcal{Q}^\bullet \cong \mathcal{O}_Y$ in $D^b(Y)$. Use $\mathcal{A}^\bullet = p_{13}^*(\mathcal{P}^\bullet) \overset{\mathbf{L}}{\otimes} p_{24}^*(\mathcal{Q}^\bullet)$ to define the Fourie-Mukai transform $\Phi_{\mathcal{A}^\bullet} : D^b(X \times X) \to D^b(Y \times Y)$. Let $X \to X \times X$ and $Y \to Y \times Y$ denote the diagonal morphisms. For brevity of notation we identify any sheaf on X or on Y with its direct image under the diagonal morphism. Let $\mathcal{R}^\bullet = \Phi_{\mathcal{A}^\bullet}(\omega_X^s) \in D^b(Y \times Y)$. Then $\Phi_{\mathcal{R}^\bullet}^{Y\to Y}$ can be computed as the composition of functors

$$D^b(Y) \xrightarrow{\Phi_{\mathcal{Q}^\bullet}} D^b(X) \xrightarrow{\Phi_{\omega_X^s}} D^b(X) \xrightarrow{\Phi_{\mathcal{P}^\bullet}} D^b(Y)$$

We know that $\Phi_{\omega_X^s}^{X\to X} = S_X[-s\dim X]$. Since an equivalence of functors commutes with the Serre functor, we obtain that $\Phi_{\mathcal{A}^\bullet} = S_Y[-s\dim Y]$. By the uniqueness statement in Orlov's Theorem, we obtain $\mathcal{A}^\bullet \cong \omega_X^s$. Since Φ_{A} is an equivalence of categories, we obtain that an isomorphism

$$\mathrm{Hom}^i_{X\times X}(\omega_X^s, \omega_X^t) \cong \mathrm{Hom}^i_{Y\times Y}(\omega_Y^s, \omega_Y^t),$$

for all $s, t, i \in \mathbb{Z}$. Since the direct image functor under the diagonal morphism is exact, we have and isomorphism of vector spaces

$$\mathrm{Hom}^i_X(\omega_X^t, \omega_X^s) \cong \mathrm{Hom}^i_Y(\omega_Y^t, \omega_Y^s).$$

taking $s = 0$ we get an isomorphism of vector spaces $H^0(X, \omega_X^s) \cong H^0(Y, \omega_Y^s)$. As before we also show the isomorphism of the canonical rings. □

Let us see what else the derived category detects. Recall that the cohomology of a smooth projective variety over $k = \mathbb{C}$ come with the Hodge decomposition

$$H^i(X, \mathbb{C}) = \sum \bigoplus_{p+q=i} H^{p,q}(X)$$

satisfying $h^{p,q} := \dim H^{p,q} = h^{q,p}$.

A cocycle $\gamma \in H^{2p}(X, \mathbb{Z})$ represented by an integer combination of fundamental classes of closed algebraic subvarieties of codimension p is called an *algebraic cycle*. It is known that, for any algebraic cycle γ its cohomology class $[\gamma]$ belongs to $H^{2p}(X, \mathbb{Z}) \cap H^{p,p}$. It follows from the definition of Chern classes that, for any coherent sheaf $\mathcal{F}$ its Chern classes $c_i(\mathcal{F})$ are cohomology classes of algebraic cycles in $H^{2i}(X, \mathbb{Z})$. In particular, the Mukai vector $\mathrm{Mu}([\mathcal{F}])$ belongs to $\oplus_p H^{p,p} \cap H^{2p}(X, \mathbb{Q})$.

Proposition 3.1.7. *Let $\Phi^{X\to Y}_{\mathcal{P}^\bullet}$ be a Fourier-Mukai transform. Then the corresponding Fourier-Mukai transform in cohomology $\Phi^{H, X\to Y}_{\mathcal{P}^\bullet}$ defines an isomorphism*

$$\bigoplus_{p-q=i} H^{p,q}(X) \to \bigoplus_{p-q=i} H^{p,q}(Y), \ -\dim X \leq i \leq \dim X.$$

Proof. Let $\delta = \mathrm{Mu}([\mathcal{P}^\bullet]) \in H^*(X \times Y, \mathbb{Q})$. Combining the Künneth decomposition

$$H^s(X \times Y, \mathbb{Q}) \cong \bigoplus_{i+j=s} H^i(X, \mathbb{Q}) \otimes H^j(X \times Y, \mathbb{Q}),$$

with the Hodge decomposition, we obtain that

$$\delta = \sum \alpha^{p',q'} \boxtimes \beta^{r,s}, \ p' + r = q' + s.$$

Let $\gamma = \gamma^{p,q} \in H^{p,q}(X)$. It follows from the definition of the cohomology integral transform that

$$\Phi^{H, X\to Y}_{\mathcal{P}^\bullet}(\gamma) = \sum_{p=0}^{\dim X} \sum_{p',q',r,s} \int_X \gamma^{p,q} \wedge \alpha^{p',q'})\beta^{r,s} \in \bigoplus_{r,s} H^{r,s}(Y).$$

The integral is equal to zero unless $(p, q) + (p', q') = (\dim X, \dim X)$. Hence $p - q = q' - r' = r - s$. □

Remark 3.1.8. Let

$$HH_i(X) = \bigoplus_{p-q=i} H^{p,q}(X),\ HH_*(X) = \bigoplus_i HH_i(X),$$

$$HH^i(X) = \bigoplus_{p+q=i} H^p(X, \Lambda^q(\mathcal{T}_X)),\ HH^*(X) = \bigoplus_i HH^*i(X),$$

where $\mathcal{T}_X = (\Omega^1_X)^\vee$ is the tangent sheaf of X. The graded s[ace $HH^*(X)$ acquires a structure of a graded k-algebra and is called the *Hochshild cohomology algebra*. The graded vector space $HH_*(X)$ is a module over $HH^*(X)$ and is called the *Hochshild homology space* . The previous proposition asserts that the Hochshild homology is invariant with respect to the Fourier-Mukai transfrom. The same is true for the Hochshild cohomology.

We have seen already in Example 2.2.7 that the assertion of Corollary 3.1.5 is not true for elliptic curves whose canonical class is trivial. We state the following theorem due to Polishchuk and Orlov.

Theorem 3.1.9. *Two abelian varieties A and B are derived equivalent if and only if there exists an isomorphism $f : A \times \hat{A} \to B \times \hat{B}$, where the hat denotes the dual abelian variety defined as the Picard variety of linear equivalence classes of divisors algebraically equivalent to zero.*

Recall that a smooth algebraic surface X is called a $K3$-surface if $K_X = 0$ and the first Betti number $b_1(X)$ is equal to zero (this makes sense over any field). If $k = \mathbb{C}$, all K3 surfaces are diffeomorphic as compact 4-manifolds. We have $H^2(X,\mathbb{Z}) \cong \mathbb{Z}^{22}$ and the intersection form on $H^2(X,\mathbb{Z})$ defined by the cup-product has signature $(3, 19)$. It is isomorphic over $\mathbb{Z}$ to the orthogonal sum of even unimodular quadartic forms $E_8^{\perp 2} \perp U^{\perp 3}$, where E_8 is given by the negative of the Cartan matrix with Dynkin diagram of type E_8 and U is a hyperbolic plane. The Neron-Severi group $\mathrm{NS}(X)$ is a free abelian group of some rank $20 \geq \rho \geq 1$. The intersection form restricted to $\mathrm{NS}(X)$ has signature $(1, \rho - 1)$. The first Chern class $c_1 : \mathrm{NS}(X) \to H^2(X,\mathbb{Z})$ is an embedding of *quadratic lattices* (i.e. free abelian groups equipped with quadratic forms). We will identify $\mathrm{NS}(X)$ with its image. Its orthogonal complement in $H^2(X,\mathbb{Z})$ is a free abelian group $T(X)$ of rank $22 - \rho$, called the lattice of *transcendental cycles*. The Hodge decomposition

$$H^2(X,\mathbb{C}) = H^{2,0}(X) \oplus H^{1,1}(X) \oplus H^{0,2}(X)$$

has the Hodge numbers $h^{2,0} = h^{0,2} = 1, h^{1,1} = 20$. The subspace $H^{1,1}(X)$ is orthogonal to $H^{2,0}(X)$ with respect to the intersection form on $H^2(X,\mathbb{C})$. Since $\mathrm{NS}(X)$ is contained in $H^{1,1}(X)$ we see that $H^{2,0}(X) \subset T(X)_\mathbb{C} = T_X \otimes \mathbb{C}$.

An isometry $\sigma : T(X) \to T(X)$ of quadratic lattices (defined by the intersection forms) that sends $H^{2,0}(X)$ to $H^{2,0}(Y)$ under the induced map $T(X)_\mathbb{C} \to T(Y)_\mathbb{C}$ is called a *Hodge isometry*. The Global Torelli Theorem due to I. Piatetski-Shapiro and I. Shafarevich asserts that any Hodge isometry that can be extended to an isometry σ of $H^2(X,\mathbb{Z}) \to H^2(Y,\mathbb{Z})$ that sends the effective cone of X to the effective cone (or equivalently an ample class to an ample class) of Y under the induced map $H^2(X,\mathbb{R}) \to H^2(Y,\mathbb{R})$ is equal to g^*, where $g : Y \to X$ is an isomorphism of surfaces. By applying some isometry of $H^2(X,\mathbb{Z})$ identical on $T(X)$ one can assume that σ sends the effective cone to the effective cone. This implies that the surfaces X and Y are isomorphic.

The following result is due to Orlov.

Theorem 3.1.10. *Let X and Y be two complex algebraic K3 surfaces. Then $D^b(X)$ is equivalent to $D^b(Y)$ if and only if there is a Hodge isometry $H^2(X,\mathbb{Z}) \to H^2(Y,\mathbb{Z})$.*

We will prove only the 'only if' part.

Proof. Again we use Orlov's Theorem to assume that our equivalence of categories is a Fourier-Mukai transform with some kernel $\mathcal{E}^\bullet \in D^b(X \times Y)$. One can check that the Mukai vector $\mathrm{Mu}(\mathcal{E}^\bullet)$ of any $\mathcal{E}^\bullet$ belongs to $H^*(X,\mathbb{Z}) \subset H^2(X,\mathbb{Q})$. By Proposition 3.1.7, taking $i = 0$, we obtain that the Mukai transform in cohomology sends the subspace

$$H^{0,0}(X) \oplus H^{1,1}(X) \oplus H^{2,2}(X) = H^0(X,\mathbb{C}) \oplus H^{1,1}(X) \oplus H^2(X,\mathbb{C}).$$

to the similar subspace of $H^*(Y,\mathbb{C})$. It also sends the intersection of this subspace with $H^2(X,\mathbb{Z})$ equal to

$$\widetilde{\mathrm{NS}}(X) = H^0(X,\mathbb{Z}) \oplus \mathrm{NS}(X) \oplus H^4(X,\mathbb{Z}).$$

to $\widetilde{\mathrm{NS}}(Y)$. Since $K_X = K_Y = 0$, it follows from Proposition 2.2.12 that the Fourier-Mukai defines an isometry $H^*(X,\mathbb{Z}) \to H^2(Y,\mathbb{Z})$. Thus it defines an isometry of the orthogonal complements

$$\sigma : T(X) = (\widetilde{\mathrm{NS}}(X))^\perp \to T(Y) = \widetilde{\mathrm{NS}}(Y))^\perp.$$

Applying Proposition 3.1.7 with $i = 2$, we see that $\sigma_{\mathbb{C}}(H^{2,0}(X)) = H^{2,0}(Y)$. Thus σ is a Hodge isometry. □

Definition 3.1.4. Two K3 surfaces with equivalent derived categories of coherent sheaves are called *Fourier-Mukai partners*.

Proposition 3.1.11. *A K3 surface has only finitely many non-isomorphic Fourier-Mukai partners.*

Proof. Let $Y_i, i \in I$ be the set of representatives of isomorphism classes of Fourier-Mukai partners of a K3 surface X. Assume this set if infinite. It follows from the proof of Theorem 3.1.10 that an equivalence of derived categories $D^b(Y_i) \to D^b(X)$ defines an isomorphism of lattices $\widetilde{\mathrm{NS}}(Y_i) \to \widetilde{\mathrm{NS}}(X)$ and a Hodge isometry $\sigma : T(Y_i) \to T(X)$. Since the lattice $H^0(X,\mathbb{Z})+H^4(X,\mathbb{Z})$ is isomorphic to a hyperbolic plane, a theorem from the theory of integral quadratic forms implies that the lattices $NS(X)$ and $\mathrm{NS}(Y)$ are in the same genus (i.e. isomorphic over any ring of p-adic numbers $\mathbb{Z}_p$). It is known that there are only finitely many non-isomorphic forms in the same genus. Thus there are infinitely many Fourier-Mukai partners Y_i with the Neron-Severi lattices $\mathrm{NS}(Y_i)$ isomorphic to $\mathrm{NS}(X)$. Let $H = H^2(X,\mathbb{Z}), M = \mathrm{NS}(X), T = T(X)^\perp$ considered as abstract lattices. The lattice T is equipped with a Hodge structure, i.e a line in $H \otimes \mathbb{C}$. For any non-degenerate lattice N let $N^\vee = \mathrm{Hom}(N,\mathbb{Z})$ be the dual lattice. It is equipped with the symmetric bilinear form induced from the lattice $N \otimes \mathbb{Q} \cong \mathrm{Hom}(N,\mathbb{Q})$. Since the lattices M and T are non-degenerate, the natural map $(M \perp T) \to (M \perp T)^\vee, v \mapsto (v,?)$ has finite cokernel. Its order is equal to the absolute value of the discriminant of the quadratic form on $M \perp T$. We have an inclusion of lattices $N = M \perp T \subset H \subset N^\vee = (M \perp T)^\vee$. The number of intermediate lattices between N and $N^\vee$ is finite and is equal to the number of subgroups of $\mathrm{coker}(M \perp T) \to (M \perp T)^\vee)$. Thus we can find an infinite set of $Y_j, j \in J \subset I$, such that the isomorphism $\mathrm{NS}(Y_i) \perp T(Y_i) \to M \perp T$ lifts an isomorphism $H^2(Y_i,\mathbb{Z}) \to H'$, where H' lies between $M \perp T$ and $(M \perp T)^\vee$. Replacing X with one of the Y_j, we see that there are infinitely many Y_i's such that the Hodge isometry $T(Y_i) \to T(X)$ lifts to an isometry of $H^2(Y_i,\mathbb{Z}) \to H^2(X,\mathbb{Z})$. Applying the Global Torelli Theorem, we obtain that all such Y_i's are isomorphic to X. This contradiction proves the assertion. □

One can give an explicit formula for the number of Fourier-Mukai partners in terms of the structure of the Neron-Severi lattice $\mathrm{NS}(X)$.

3.2 Spherical objects

Corollary 3.1.5 on auto-equivalences of the derived category of a variety with ample canonical or anti-canonical sheaf does not extend to varieties which do not satisfy this condition. For example, in the case of a K3 surface, many new anti-equivalences arise from spherical objects.

Definition 3.2.1. An object E in a k-linear triangulated category D with a Serre functor S is called *spherical* if

(i) $S(E) \cong E[n]$, for some $n \in \mathbb{Z}$;

(ii)
$$\mathrm{Hom}_{\mathsf{D}}(E, E[i]) = \begin{cases} k & \text{if, } i = 0, n \\ 0 & \text{otherwise.} \end{cases}$$

Example 3.2.1. Assume $K_X = 0$ and $h^i(\mathcal{O}_X) = 0, 0 < i < n = \dim X$ (i.e. X is a Calabi-Yau manifold). Any invertible sheaf is a spherical object. Condition (i) holds obviously with $n = 2$. We have

$$\mathrm{Hom}(\mathcal{L}, \mathcal{L}[i]) = \mathrm{Hom}(\mathcal{O}_X, \mathcal{O}_X[i]) \cong H^i(X, \mathcal{O}_X).$$

This is not equal to zero only if $i = 0, \dim X$.

Also let R be a smooth rational curve on X such that the normal bundle $\mathcal{N}_{R/X}$ is isomorphic to $\mathcal{O}_{\mathbb{P}^1}(-1)^{\oplus n-1}$. Let $i : R \hookrightarrow X$ be the closed embedding. Let us see whether $i_*\mathcal{O}_R$ is a spherical object. We have an isomorphism

$$\mathcal{E}xt^i_X(i_*\mathcal{O}_R, i_*\mathcal{O}_R) \cong \Lambda^i \mathcal{N}_{R/X}. \tag{3.1}$$

In fact, consider the exact sequence

$$0 \to \mathcal{J}_R \to \mathcal{O}_X \to i_*\mathcal{O}_R \to 0$$

and apply $\mathcal{H}om_X(?, i_*\mathcal{O}_R)$ to get an exact sequence

$$0 \to \mathcal{H}om_X(i_*\mathcal{O}_R, i_*\mathcal{O}_R) \to \mathcal{H}om_X(\mathcal{O}_X, i_*\mathcal{O}_R) \to \mathcal{H}om_X(\mathcal{J}_R, i_*\mathcal{O}_R)$$
$$\to \mathcal{E}xt^1_X(i_*\mathcal{O}_R, i_*\mathcal{O}_R) \to 0.$$

It is easy to see that $\mathcal{H}om_X(\mathcal{J}_R, i_*\mathcal{O}_R) \cong \mathcal{N}_{R/X} \cong (\mathcal{J}_R/\mathcal{J}_R^2)^\vee$. This gives an isomorphism (3.1) for $i = 0, 1$. For $i > 1$, we consider the induced isomorphism

$$\Lambda^i \mathcal{N}_{R/X} \to \mathcal{E}xt^i_X(i_*\mathcal{O}_R, i_*\mathcal{O}_R),$$

where the target is given by the cup-product of $\mathcal{E}xt^1_X(i_*\mathcal{O}_R, i_*\mathcal{O}_R)$'s. Now we use the spectral sequence

$$E_2^{p,q} = H^p(X, \mathcal{E}xt^q_X(i_*\mathcal{O}_R, i_*\mathcal{O}_R)) \Longrightarrow \mathrm{Ext}^{p+q}(i_*\mathcal{O}_R, i_*\mathcal{O}_R)).$$

We have

$$\Lambda^q \mathcal{N}_{R/X} \cong \bigoplus \Lambda^q(\mathcal{O}_{\mathbb{P}^1}(-1)^{\oplus n-1}) \cong \mathcal{O}_{\mathbb{P}^1}(-q)^{\oplus \binom{n-1}{q}}.$$

This implies that $E_2^{p,q} \neq 0$ only if $p = q = 0$ and $p = 1, q \geq 2$. This easily implies that $i_*\mathcal{O}_R$ is a spherical objects if and only if $n = 3$.

If $n = 2$, and R is a smooth rational curve, the sheaf $\mathcal{O}_R$ is a spherical object. We use that R is a (-2)-curve, i.e. $R \cong \mathbb{P}^1$ and $R^2 = -2$. We have $k = H^0(\mathcal{O}_R, \mathcal{O}_R) \cong H^2(\mathcal{O}_R, \mathcal{O}_R)$ and playing with the exact sequence

$$0 \to \mathcal{O}_X(-R) \to \mathcal{O}_X \to \mathcal{O}_R \to 0$$

gives easily $\mathrm{Ext}^1(\mathcal{O}_R, \mathcal{O}_R) = 0$ (use that $\mathrm{Hom}(\mathcal{O}_X(-R), \mathcal{O}_R) \cong \mathrm{Hom}(\mathcal{O}_X, \mathcal{O}_R(R)) \cong H^0(R, \mathcal{O}_X(-2)) = 0$).

Another example of a spherical object on a K3 surface X is a simple rigid vector bundle (see Example 2.2.14). As we remarked earlier in Lemma 2.4.12, the space $\mathrm{Ext}^1(\mathcal{E}, \mathcal{E})$ is the tangent space to the moduli space of simple bundles at the point $[E]$.

Definition 3.2.2. A triangulated k-linear category D is called to be of *finite type* if, for any objects A and B in D,

$$\dim_k \mathrm{Hom}^\bullet_{\mathsf{D}}(A, B) < \infty \tag{3.2}$$

where

$$\mathrm{Hom}^\bullet_{\mathsf{D}}(A, B) = \bigoplus_{i \in \mathbb{Z}} \mathrm{Hom}^i_{\mathsf{D}}(A, B),$$

For example, $\mathsf{D} = D^b(X)$, where X is projective, satisfies this condition.

For any objects $E, F \in \mathrm{Ob}(\mathsf{D})$ define the objects $L_E(F)$ and $R_E(F)$ in D by means of the distinguished triangles

$$L_E(F) \to \mathrm{Hom}^\bullet_{\mathsf{D}}(E,F) \otimes E \xrightarrow{\mathrm{ev}} F \to L_E(F)[1], \tag{3.3}$$

$$F \xrightarrow{\mathrm{ev}} \mathrm{Hom}^\bullet_{\mathsf{D}}(F,E)^* \otimes E[i] \to R_E(F) \to F[1]. \tag{3.4}$$

Here the tensor product means the complex whose j-th component is equal to the direct sum of complexes $\oplus_i E[-i]^{\oplus d_i}$, where $d_i = \dim_k \mathrm{Hom}^i_{\mathsf{D}}(E,F)$. The morphism ev is defined by means of an isomorphism $\mathrm{Hom}^i_{\mathsf{D}}(E,F) \cong \mathrm{Hom}_{\mathsf{D}}(E(-i),F)$ and sending the copy of $E[-i]$ with index $\phi \in \mathrm{Hom}^i_{\mathsf{D}}(E,F)$ to F by means of ϕ. Similrly one defines the tensor product and the evaluation morphism in the second triangle.

One can show that the operations L_E and R_E define functor of triangulated categories.

Assume E is a spherical object in D. Our first observation is that

$$L_E(E) \cong E[-n],\ R_E(E) \cong E[n]. \tag{3.5}$$

Indeed we have a distinguished triangle

$$L_E(E) \to E \oplus E[-n] \to E \to L_E(E)[1].$$

It is easy to see that it is isomorphic to the direct sum of the distinguished triangles $0 \to E \to E \to 0 \xrightarrow{[1]} 0$ and $E[-n] \to E \to 0 \to E[1-n]$. A similar argument applies to R_E.

Our second observation is that

$$L_E(F) \cong F[-1],\ R_E(F) \cong F]1] \tag{3.6}$$

if $\mathrm{Hom}_{\mathsf{D}}(E, F[i]) = 0$ for all $i \in \mathbb{Z}$.

Our third observation is that there are isomorphisms of functors

$$L_{E[1]} \cong L_E,\ R_{E[1]} \cong R_E \tag{3.7}$$

From now on, let $\mathsf{D} = D^b(X)$, where X is a smooth projective n-dimensional variety. Let $i : \Delta \hookrightarrow X \times X$ be the diagonal map. Consider the natural homomorphism

$$p^*\mathcal{E}^\bullet \overset{\mathbf{L}}{\otimes} q^*\mathcal{E}^{\bullet\vee} \to \mathcal{O}_\Delta$$

defined as the composition

$$p^*\mathcal{E}^\bullet \overset{\mathbf{L}}{\otimes} q^*\mathcal{E}^{\bullet\vee} \to i_*i^*(p^*\mathcal{E}^\bullet \overset{\mathbf{L}}{\otimes} q^*\mathcal{E}^{\bullet\vee}) = i_*(\mathcal{E}^{\bullet\vee} \overset{\mathbf{L}}{\otimes} \mathcal{E}^\bullet) \xrightarrow{\mathrm{tr}} \mathcal{O}_\Delta,$$

where $\mathrm{tr} : \mathcal{E}^\bullet \overset{\mathbf{L}}{\otimes} \mathcal{E}^\bullet\vee \to \mathcal{O}_X$ is the trace map. Let

$$p^*\mathcal{E}^\bullet \overset{\mathbf{L}}{\otimes} q^*\mathcal{E}^{\bullet\vee} \to \mathcal{O}_\Delta \to \tilde{\mathcal{E}}^\bullet \to p^*\mathcal{E} \overset{\mathbf{L}}{\otimes} q^*\mathcal{E}^{\bullet\vee}[1] \tag{3.8}$$

be a distinguished triangle.

We claim that

$$L_{\mathcal{E}^\bullet} := \Phi^{X \to X}_{\tilde{\mathcal{E}}^\bullet} = \mathbf{R}q_* \circ \mathcal{E}^\bullet \overset{\mathbf{L}}{\otimes} p^*$$

To see this explicitly, we apply the integral transform to the distinguished triangle (3.8) to get the distinguished triangle

$$\mathbf{R}p_*(p^*(\mathcal{E}^\bullet) \overset{\mathbf{L}}{\otimes} q^*(\mathcal{E}^{\bullet\vee} \overset{\mathbf{L}}{\otimes} \mathcal{F}^\bullet)) \to \mathbf{R}p_*(\mathcal{O}_\Delta \overset{\mathbf{L}}{\otimes} \mathcal{F}^\bullet) \cong \mathcal{F}^\bullet \to L_{\mathcal{E}^\bullet}(\mathcal{F}^\bullet) \overset{[1]}{\to}$$

Using the projection formula, we obtain

$$\mathbf{R}p_*(p^*(\mathcal{E}^\bullet) \otimes q^*(\mathcal{E}^{\bullet\vee} \overset{\mathbf{L}}{\otimes} \mathcal{F}^\bullet)) \cong \mathcal{E}^\bullet \otimes \mathbf{R}\mathrm{Hom}(\mathcal{E}^\bullet, \mathcal{F}^\bullet),$$

where $\mathbf{R}\mathrm{Hom}^\bullet(\mathcal{E}^\bullet, \mathcal{F}^\bullet)$ is considered as a complex of free sheaves. Writing down the tensor product explicitly, we obtain a distinguished triangle

$$\bigoplus_{i \in \mathbb{Z}} \mathrm{Hom}^i(\mathcal{E}^\bullet, \mathcal{F}^\bullet) \otimes \mathcal{E}^\bullet[-i] \to \mathcal{F}^\bullet \to L_{\mathcal{E}^\bullet}(\mathcal{F}^\bullet) \overset{[1]}{\to} \tag{3.9}$$

Our first observation is that

$$L_{\mathcal{E}^\bullet}(\mathcal{E}^\bullet) \cong \mathcal{E}^\bullet[1-n] \tag{3.10}$$

Indeed, we have a distinguished triangle

$$\mathcal{E}^\bullet \oplus \mathcal{E}^\bullet[-n] \to \mathcal{E}^\bullet \to L_{\mathcal{E}^\bullet}(\mathcal{E}^\bullet) \overset{[1]}{\to}$$

We have a morphism $L_{\mathcal{E}^\bullet}(\mathcal{E}^\bullet) \to \mathcal{E}^\bullet[1] \oplus \mathcal{E}^\bullet[1-n] \to \mathcal{E}^\bullet[1-n]$. Applying the cohomology we see that it defines a quasi-isomorphism.

Our second observation is that

$$L_{\mathcal{E}^\bullet}(\mathcal{F}^\bullet) \cong \mathcal{F}^\bullet \tag{3.11}$$

if $\mathrm{Hom}(\mathcal{E}^\bullet, \mathcal{F}^\bullet[i]) = 0$ for all $i \in \mathbb{Z}$.

Our third observation is that

$$L_{\mathcal{E}^\bullet[1]} \cong L_{\mathcal{E}^\bullet} \tag{3.12}$$

Example 3.2.2. Assume $\dim X = 1$. For any closed point $x \in X$ the sheaf $\mathcal{O}_x$ is a spherical object (use the Serre duality). Suppose $\mathcal{F}^\bullet = \mathcal{F}$ is a locally free sheaf of rank r. We have $\mathrm{Hom}^i(\mathcal{O}_x, \mathcal{F}) = 0, i \neq 1$ and $\mathrm{Hom}^1(\mathcal{O}_x, \mathcal{F}) \cong \mathrm{Hom}(\mathcal{F}, \mathcal{O}_x)^\vee$. The distinguished triangle becomes

$$\mathrm{Hom}(\mathcal{F}, \mathcal{O}_x)^\vee \otimes \mathcal{O}_x[-1] \to \mathcal{F} \to L_{\mathcal{O}_x}(\mathcal{F}) \to \mathrm{Hom}(\mathcal{F}, \mathcal{O}_x)^\vee \otimes \mathcal{O}_x$$

This easily implies that $L_{\mathcal{O}_x}(\mathcal{F})$ is a locally free sheaf and, passing to duals, we get an exact sequence of sheaves

$$0 \to L_{\mathcal{O}_x}(\mathcal{F})^\vee \to \mathcal{F}^\vee \to \mathcal{F}^\vee(x) \to 0,$$

where for any sheaf $\mathcal{G}$ we denote by $\mathcal{G}(x)$ its fibre at x (not the stalk), i.e. $\mathcal{G}_x/\mathfrak{m}_{X.x}\mathcal{G}_x)$. For any non-zero linear function $\alpha : \mathcal{F}^\vee(x) \to k(x)$ the pre-image in $\mathcal{F}^\vee$ of the kernel of α is a locally free subsheaf of $\mathcal{F}^\vee$ of rank r. The data (x, α) defines the classical *elementary operation* $\mathrm{elm}_{x,\alpha}$ on locally free sheaves. Assume $r > 1$ and consider the vector bundle $\mathbb{V}(\mathcal{F})$ associated to $\mathcal{F}$, then α defines a nonzero point in the fibre $\mathbb{V}(\mathcal{F})_x$. Passing to the projective bundles $\mathbb{PV}(\mathcal{F}) = \mathrm{Proj}\ \mathrm{Sym}^\bullet(\mathcal{F})$ we consider α as a point in the fibre $\mathbb{PV}(\mathcal{F})_x$. We can blow-up this point, and then blow down the proper transform of the fibre, the result is a projective bundle isomorphic to $\mathbb{PV}(\mathrm{elm}_{x,\alpha}(\mathcal{F}))$.

Note that if $r = 1$ and $\mathcal{F} = \mathcal{O}_X(D)$ for some divisor D, then $T_{\mathcal{O}_x}(\mathcal{F}) \cong \mathcal{O}_X(D - x)$.

Assume E is a spherical object. The twist functors L_E and R_E is called the *spherical twists*.

We want to show the spherical twist is a Fourier-Mukai transform. We need the following.

Lemma 3.2.3. *Suppose $\Phi_{\mathcal{P}^\bullet}^{X\to Y}$ is a fully faithful integral transform with kernel $\mathcal{P}^\bullet$. Then it is a Fourier-Mukai transform if and only if $\dim X = \dim Y$ and*

$$\mathcal{P}^\bullet \otimes p^*\omega_X \cong \mathcal{P}^\bullet \otimes q^*\omega_Y.$$

Proof. We have seen already that these conditions are necessary. To prove the sufficiency, we use that $F = \Phi_{\mathcal{P}^\bullet}^{X\to Y}$ has the right adjoint functor H and the left adjoint functor G defined by the integral transforms with kernel $\mathcal{P}^\bullet \otimes p^*\omega_X[\dim X]$ and $\mathcal{P}^\bullet \otimes q^*\omega_Y[\dim Y]$, respectively. It follows from Corollary 2.4.11 that F is an equivalence of categories if $H(B) = 0$ implies $G(B) = 0$. Since, by assumption, $G \cong B$, the assertion follows. □

Theorem 3.2.4. *Let $\mathcal{E}^\bullet$ be a spherical object in $D^b(X)$. Then*

$$L_{\mathcal{E}^\bullet} := \Phi_{\tilde{\mathcal{E}}^\bullet}^{X\to X}$$

is a Fourier-Mukai transform.

Proof. Applying the previous lemma, it suffices to show that

$$\tilde{\mathcal{E}}^\bullet \otimes p^*\omega_X \cong \tilde{\mathcal{E}}^\bullet \otimes q^*\omega_X.$$

and that $L_{\mathcal{E}^\bullet}$ is fully faithful. The first claim is rather obvious since a distinguished triangle (3.8) defines a distinguished triangle

$$q^*(\mathcal{E}^{\bullet\vee} \otimes \omega_X) \overset{\mathbf{L}}{\otimes} p^*\mathcal{E}^\bullet \to \mathcal{O}_\Delta \otimes q^*\omega_X \to \tilde{\mathcal{E}}^\bullet \otimes q^*\omega_X \overset{[1]}{\to} \dots$$

and the same is true if we exchange the roles of the projections p and q.

We use Proposition 2.4.8, taking for the spanning set Ω the set of objects in $D^b(X)$ that consists of $\mathcal{E}^\bullet$ and the set $\mathcal{E}^{\bullet\perp}$ consisting of all objects $\mathcal{G}^\bullet$ such that $\mathrm{Hom}_{D^b(X)}(\mathcal{E}^\bullet, \mathcal{G}^\bullet[i]) = 0$ for all $i \in \mathbb{Z}$. Let us prove that it is a spanning set. Obviously, if $\mathrm{Hom}_{D^b(X)}(\mathcal{G}^\bullet,\mathcal{F}^\bullet[i])=0$ for all $\mathcal{G}^\bullet\in\Omega$ and all $i\in\mathbb{Z}$, then $\mathcal{F}^\bullet$ must be in $\mathcal{E}^\perp$. But then $\mathrm{Hom}_{D^b(X)}(\mathcal{F}^\bullet,\mathcal{F}^\bullet)=0$ implies $\mathcal{F}^\bullet\cong 0$. Now suppose $\mathrm{Hom}_{D^b(X)}(\mathcal{F}^\bullet,\mathcal{G}^\bullet[i])=0$ for all $\mathcal{G}^\bullet$ in Ω and all $i\in\mathbb{Z}$. Applying the Serre functor, we get $0=\mathrm{Hom}_{D^b(X)}(\mathcal{F}^\bullet,\mathcal{E}^\bullet[i])=\mathrm{Hom}_{D^b(X)}(\mathcal{F}^\bullet,S(\mathcal{E}^\bullet[i-n]))=\mathrm{Hom}_{D^b(X)}(\mathcal{E}^\bullet[i-n],\mathcal{F}^\bullet)$ for all i. This implies that $\mathcal{F}^\bullet\in\mathcal{E}^{\bullet\perp}$, and we finish as before.

Next we need to show that

$$L_{\mathcal{E}^\bullet}:\mathrm{Hom}(\mathcal{G}_1^\bullet,\mathcal{G}_2^\bullet[i])\to\mathrm{Hom}(L_{\mathcal{E}^\bullet}(\mathcal{G}_1^\bullet),L_{\mathcal{E}^\bullet}(\mathcal{G}_2^\bullet[i]))$$

is a bijection for all $i\in\mathbb{Z}$. This is obviously true if $\mathcal{G}_1^\bullet=\mathcal{E}^\bullet$ and $\mathcal{G}_2^\bullet\in\mathcal{E}^{\bullet\perp}$. By using observations (3.10) and (3.11), we see that this is true if $\mathcal{G}_1^\bullet=\mathcal{G}_2^\bullet$ and if $\mathcal{G}_1^\bullet,\mathcal{G}_2^\bullet\in\mathcal{E}^{\bullet\perp}$. □

Let us see how the spherical twist $L_{\mathcal{E}^\bullet}$ acts on the cohomology. Let $v\in H^*(X,\mathbb{Q})$ and $[\tilde{\mathcal{E}}^\bullet]$ be the class of $\tilde{\mathcal{E}}^\bullet$ in $K_0(X\times X)$. It follows from the definition that $[\tilde{\mathcal{E}}^{\bullet,\bullet}]=[\mathcal{O}_\Delta]-p^*([\mathcal{E}^{\bullet\vee}]\otimes q^*([\mathcal{E}^\bullet])$. Its Mukai vector in $H^*(X\times X,\mathbb{Q})$ is equal to $[\Delta]-p^*(\mathrm{Mu}(\mathcal{E}^{\bullet\vee})\cdot q^*(\mathrm{Mu}(\mathcal{E}^\bullet))$. We know that

$$L^H_{\mathcal{E}^\bullet}(v):=\Phi^{H,X\to X}_{\tilde{\mathcal{E}}^{\bullet,\bullet}}(v)=q_*(\mathrm{Mu}([\mathcal{E}^\bullet])\cdot p^*(v))$$

$$=v-\Big(\int_X v\cdot\mathrm{Mu}(\mathcal{E}^{\bullet\vee})\Big)v=v-\Big(\int_X\exp(c_1(X)/2)v\cdot\mathrm{Mu}(\mathcal{E}^\bullet)^\vee\Big)v$$

$$=v-\langle\mathrm{Mu}(\mathcal{E}^\bullet),v\rangle\mathrm{Mu}(\mathcal{E}^\bullet).$$

Applying proposition 2.2.12, we get

$$\langle\mathrm{Mu}(\mathcal{E}^\bullet),\mathrm{Mu}(\mathcal{E}^\bullet)\rangle=\chi(\mathcal{E}^\bullet,\mathcal{E}^\bullet)=\dim\mathrm{Hom}(\mathcal{E}^\bullet,\mathcal{E}^\bullet)+(-1)^n\dim\mathrm{Hom}^n(\mathcal{E}^\bullet,\mathcal{E}^\bullet)$$

$$=\begin{cases}2 & \text{if } n \text{ is even}\\ 0 & \text{otherwise.}\end{cases}$$

When n is even, we obtain that $L^H_{\mathcal{E}^\bullet}$ acts as a reflection in the vector $\mathrm{Mu}(\mathcal{E}^\bullet)$, i.e. it sends this vector to its opposite, and leaves invariant any vector orthogonal to $\mathrm{Mu}(\mathcal{E}^\bullet)$ with respect to the pairing $\langle,\rangle$. If $K_X=0$, and $\mathrm{Mu}(\mathcal{E}^\bullet)\in H^2(X\times X,\mathbb{Z})$ (e.g. if X is a K3 surface), then the pairing is a symmetric bilinear form on $H^*(X,\mathbb{Z})$ and $L^H_{\mathcal{E}^\bullet}$ is a reflection isometry of the corresponding lattice. Recall that a *reflection* with respect to a vector α in a quadratic lattice M is its isometry defined by the formula

$$r_\alpha(x)=x-\frac{2(x,\alpha)}{\alpha,\alpha}\alpha,$$

where one assumes that (v,v) divides $2(x,v)$ for any $x\in M$ (e.g. if $(\alpha,\alpha)=\pm2$).

Observe that $L^2_{\mathcal{E}^\bullet}=\mathbf{id}_{H^*}$ but $L^2_{\mathcal{E}^\bullet}(\mathcal{E}^\bullet)=\mathcal{E}^\bullet[1-n]$, so $L^2_{\mathcal{E}^\bullet}$ belongs to the kernel of the action of $\mathrm{Auteq}(D^b(X))$ on the cohomology.

Example 3.2.5. Let X be a complex algebraic K3-surface. It is known that no non-identity automorphism can act trivially on the cohomology. Thus $L^2_{\mathcal{E}^\bullet}$ cannot be induced by an automorphism of X. Consider the natural homomorphism $a : \mathrm{Aut}(X) \to \mathrm{Aut}(\mathrm{NS}(X))$. Its kernel is a finite group since it preserves the class of any ample line bundle and, its is known, that no algebraic group of positive dimension can act on X. Let W_X be the subgroup of $\mathrm{O}(\mathrm{NS}(X))$ generated by reflections in vector $[R]$, where R is a smooth rational curve on X. It is a normal subgroup of $\mathrm{O}(X)$ and we can compose a with the quotient map $\mathrm{O}(\mathrm{NS}(X)) \to \mathrm{O}(\mathrm{NS}(X))/W_X$ to obtain a homomorphism $a' : \mathrm{Aut}(X) \to \mathrm{O}(\mathrm{NS}(X))/W_X$. It follows from the Global Torelli Theorem that the cokernel of this homomorphism is a finite group. Let $\mathcal{E}^\bullet = \mathcal{O}_R$, where R as above. Then $L_{\mathcal{E}^\bullet}$ acts on $\mathrm{NS}(X)$ as the reflection with respect to the vector $[R]$ (note that $[\mathcal{O}_R] = [R]$ in $K_0(X)$ and $R^2 = -\langle \mathrm{Mu}([R],[R]\rangle = -2)$. Thus the image of the subgroup of $\mathrm{Auteq}(D^b(X)$ generated by the spherical twists $L_{\mathcal{O}_R}$ in $\mathrm{O}(\mathrm{NS}(X))$ coincides with W_X. Thus together with $\mathrm{Aut}(X)$ the cohomology spherical twists generate a finite index subgroup of $\mathrm{O}(\mathrm{NS}(X))$.

Let us see how different spherical twists compose.

Lemma 3.2.6. *Let $\Phi : D^b(X) \to D^b(X)$ be an auto-equivalence of derived categories. For any spherical object $\mathcal{E}^\bullet$ in $D^b(X)$,*

$$\Phi \circ L_{\mathcal{E}^\bullet} = L_{\Phi(\mathcal{E}^\bullet)} \circ \Phi.$$

Proof. We have a distinguished triangle

$$\mathbf{R}\mathrm{Hom}(\Phi(\mathcal{E}^\bullet), \Phi(\mathcal{F}^\bullet)) \overset{\mathbf{L}}{\otimes} \Phi(\mathcal{E}^\bullet) \to \mathcal{F}^\bullet \to L_{\Phi(\mathcal{E}^\bullet)}(\Phi(\mathcal{F}^\bullet)) \overset{[1]}{\to} \tag{3.13}$$

Since Φ is an equivalence of categories, $\mathbf{R}\mathrm{Hom}(\Phi(\mathcal{E}^\bullet), \Phi(\mathcal{F}^\bullet)) \cong \mathbf{R}\mathrm{Hom}(\mathcal{E}^\bullet, \mathcal{F}^\bullet)$, and applying Φ to the distinguished triangle defining $L_{\mathcal{E}^\bullet}$, we get a distinguished triangle

$$\mathbf{R}\mathrm{Hom}(\mathcal{E}^\bullet, \mathcal{F}^\bullet) \overset{\mathbf{L}}{\otimes} \Phi(\mathcal{E}^\bullet) \to \mathcal{F}^\bullet \to \Phi(L_{\mathcal{E}^\bullet}(\mathcal{F}^\bullet)) \overset{[1]}{\to} \tag{3.14}$$

Comparing the two triangles, we find $L_{\Phi(\mathcal{E}^\bullet)}(\Phi(\mathcal{F}^\bullet)) \cong \Phi(L_{\mathcal{E}^\bullet}(\mathcal{F}^\bullet))$. There is a subtlety here. Although we have an isomorphism for each $\mathcal{F}^\bullet$, this may not define an isomorphism of functors. Assuming that Φ is a Fourier-Mukai transform (e.g. applying Orlov's Theorem), one can show that, in fact, there is an isomorphism of functors (see [Huybrechts], p. 176). □

We apply this lemma, by taking $\Phi = L_{\mathcal{F}^\bullet}$ for some spherical object $\mathcal{F}^\bullet$. We get

$$L_{\mathcal{F}^\bullet} \circ L_{\mathcal{E}^\bullet} = L_{L_{\mathcal{F}^\bullet}(\mathcal{E}^\bullet)} \circ L_{\mathcal{F}^\bullet}. \tag{3.15}$$

This immediately implies that $L_{\mathcal{E}^\bullet}$ and $L_{\mathcal{F}^\bullet}$ commute if $\mathcal{F}^\bullet \in \mathcal{E}^{\bullet\perp}$ (for example, if $\mathcal{E}^\bullet = \mathcal{O}_R, \mathcal{F}^\bullet = \mathcal{O}_{R'}$, where R, R' are two disjoint smooth rational curves on a K3 surface).

Let $\mathcal{E}_1^\bullet, \ldots, \mathcal{E}_N^\bullet$ be a collection of spherical objects satisfying

$$a_{ij} = \oplus_p \dim \mathrm{Hom}^p(\mathcal{E}_i^\bullet, \mathcal{E}_j^\bullet) \leq 1, i \neq j. \tag{3.16}$$

Let $L_{\mathcal{E}_i^\bullet} = L_i$. Assume $a_{ij} = 1$. After shifting $\mathcal{E}_i$, using (3.12), we may assume that $\dim \mathrm{Hom}(\mathcal{E}_i^\bullet, \mathcal{E}_j^\bullet) = 1$. We have a distinguished triangle

$$\mathcal{E}_i^\bullet \to \mathcal{E}_j^\bullet \to L_i(\mathcal{E}_j^\bullet) \to \mathcal{E}_i^\bullet[1].$$

Applying L_j we get a distinguished triangle

$$L_j(\mathcal{E}_i^\bullet) \to L_j(\mathcal{E}_j^\bullet) = \mathcal{E}_j^\bullet[1-n] \to L_j(L_i(\mathcal{E}_j^\bullet)) \to L_j(\mathcal{E}_i^\bullet)[1], \tag{3.17}$$

where $\dim X = n$. Since $\dim \mathrm{Hom}(\mathcal{E}_j^\bullet, \mathcal{E}_i^\bullet) = \dim \mathrm{Hom}^n(\mathcal{E}_j^\bullet, \mathcal{E}_i^\bullet)$, we get a distinguished triangle

$$\mathcal{E}_j^\bullet[-n] \to \mathcal{E}_i^\bullet \to L_j(\mathcal{E}_i^\bullet) \to \mathcal{E}_j^\bullet[1-n].$$

After shifting, we have the triangle

$$L_j(\mathcal{E}_i^\bullet) \to \mathcal{E}_j^\bullet[1-n] \to \mathcal{E}_i^\bullet[1] \to L_j(\mathcal{E}_i^\bullet)[1].$$

Comparing it with triangle (3.17), we get $L_j(L_i(\mathcal{E}_j^\bullet)) \cong \mathcal{E}_i^\bullet[1]$. Thus $L_{L_j(L_i(\mathcal{E}_j^\bullet)} \cong L_i$. Applying (3.15) twice, we have

$$L_j \circ L_i \circ L_j = L_j \circ L_{L_i(\mathcal{E}_j^\bullet)} \circ L_i = L_{L_j(L_i(\mathcal{E}_j^\bullet))} \circ L_j \circ L_i = L_i \circ L_j \circ L_i.$$

Summing up, we obtain the relations

$$\begin{cases} L_i \circ L_j = L_j \circ L_i & \text{if } a_{ij} = 0, \\ L_i \circ L_j \circ L_i = L_j \circ L_i \circ L_j & \text{otherwise.} \end{cases} \tag{3.18}$$

Recall the definition of the *Artin-Brieskorn braid group*. Let S be a symmetric matrix of size S with integer entries $s_{ij} \geq 2$ off the diagonal and $s_{ii} = 0$. The Artin-Brieskorn braid group $B(S)$ is defined by generators $g_1, \ldots, g_N$ with defining relations

$$g_i \cdot g_j \cdot g_i \cdots = g_j \cdot g_i \cdot g_j \cdots, i \neq j,$$

where in each side there are s_{ij} factors. The quotient of $B(S)$ by the normal subgroup generated by $g_1^2, \ldots, g_N^2$ is the *Coxeter group* defined by the matrix S. The matrix S can be defined by the Coxeter-Dynkin diagram. It is a graph with vertices $v_i, i = 1, \ldots, N$, joined by $s_{ij}-2$ edges. Conversely such diagram defines a matrix S as above. For example, the Coxeter group with the Coxeter-Dynkin diagram

is the symmetric group S_{N+1}. The corresponding Artin-Brieskorn group is the classical braid group B_{N+1} introduced by E. Artin. In our case, we take $S = (s_{ij})$, where $S_{ij} = a_{ij}+2$, to obtain that the subgroup of $\mathrm{Auteq}(D^b(X)$ generated by $L_1, \ldots, L_N$ satisfies the braid relations of the Artin-Brieskorn group $B(S)$. In particular, we have a homomorphism

$$\rho : B(A) \to \mathrm{Auteq}(D^b(X)).$$

In the case when S is of type A_N, i.e. described by the Coxeter-Dynkin diagram of type A_N from above, Seidel and Thomas have proved that ρ is injective if $\dim X \geq 2$, i.e. the braid relations (3.18) are defining relations.

Example 3.2.7. Let X be a K3 surface, and $R_1, \ldots, R_N$ be a set of smooth rational curves such that $a_{ij} = R_i \cdot R_j \leq 1$ for $i \neq j$. Using the exact sequence

$$0 \to \mathcal{O}_X(-R_i) \to \mathcal{O}_X \to \mathcal{O}_{R_j} \to 0$$

and applying the functor $\mathrm{Hom}_X(\mathcal{O}_{R_i}, ?)$, we easily get

$$\begin{aligned}
\mathrm{Hom}^2(\mathcal{O}_{R_j}, \mathcal{O}_{R_i}) &\cong \mathrm{Ker}(\mathrm{Ext}^2(\mathcal{O}_{R_i}, \mathcal{O}_X(-R_j)) \to \mathrm{Ext}^2(\mathcal{O}_{R_i}, \mathcal{O}_X)) \\
&= \mathrm{Ker}(\mathrm{Hom}(\mathcal{O}_X(-R_j)), \mathcal{O}_{R_i})^\vee \to \mathrm{Hom}(\mathcal{O}_X, \mathcal{O}_{R_j})^\vee) \\
&\cong \mathrm{Ker}(H^0(X, \mathcal{O}_{R_i}(1))^\vee \to H^0(X, \mathcal{O}_{R_i})^\vee) = \mathrm{Ker}(k^2 \to k) \cong k.
\end{aligned}$$

All other Hom^i vanish. Thus the spherical objects satisfy condition (3.16) and we have an action of the corresponding Artin-Brieskorn braid group $B(A)$ on $D^b(X)$, where $A = (a_{ij})$. Note that we have a commutative diagram

$$\begin{array}{ccc}
B(A) & \longrightarrow & \mathrm{Autoeq}(D^b(X)) \\
\downarrow & & \downarrow{\scriptstyle \mathrm{Mu}} \\
C(A) & \longrightarrow & \mathrm{O}(H^*(X, \mathbb{Z}))
\end{array},$$

where $C(A)$ is the Coxeter group corresponding to $B(A)$ and $\mathrm{O}(H^*(X, \mathbb{Z}))$ is the group of isometries of the Mukai lattice (see Example 2.2.14).

The same is true if we replace a collection of (-2)-curves R_i as above with a collection of rigid simple vector bundles $\mathcal{E}_i$ with $\dim \mathrm{Ext}^1(\mathcal{E}_i, \mathcal{E}_j) \leq 1$.

3.3 Semi-orthogonal decomposition

In this section we discuss how a triangulated category could be described as a sort of span of its finitely many objects.

Let D be a triangulated category of finite type. We assume that it is equipped with a Serre functor $S : \mathsf{D} \to \mathsf{D}$.

Definition 3.3.1. A full triangulated subcategory D' of D is called *right (left) admissible* if the inclusion functor has a right (left) adjoint. The *right (left) orthogonal* $\mathsf{D}'^{\perp}$ $({}^{\perp}\mathsf{D}')$ of an admissible subcategory is the full category formed by objects B such that $\mathrm{Hom}_{\mathsf{D}}(A, B) = 0$ $(\mathrm{Hom}_{\mathsf{D}}(B, A) = 0)$ for all $A \in \mathrm{Ob}(\mathsf{D}')$. We say that D' is admissible if it is right and left admissible.

Note that $\mathsf{D}'^{\perp}$ is a triangulated subcategory as it is easy to see by using that $\mathrm{Hom}_{\mathsf{D}}(A, B[i]) = \mathrm{Hom}_{\mathsf{D}}(A[-i], B) = 0$ and applying the functor Hom to a distinguished triangle $B_1 \to B_2 \to C \to B_1[1]$ with $B_1, B_2 \in \mathrm{Ob}(\mathsf{D}'^{\perp})$.

The meaning of being right admissible is the following. For any object $C \in \mathsf{D}$ there exists a distinguished triangle $A \to C \to B \to A[1]$, where $A \in \mathrm{Ob}(\mathsf{D}'), B \in \mathrm{Ob}(\mathsf{D}'^{\perp})$. In fact we set $A = i^!(C)$ where $i^!$ is a right adjoint of the inclusion functor $i : \mathsf{D}' \to \mathsf{D}$. The identity morphism $A \to A$ defines, by the property of adjoint functors, a morphism $A \to C$ that can be extended to a distinguished triangle $A \to C \to B \to A[1]$. For any $A' \in \mathrm{Ob}(\mathsf{D}')$ we have an exact sequence

$$\mathrm{Hom}(A', A) = \mathrm{Hom}(A', i^!(C)) \xrightarrow{a} \mathrm{Hom}(A', C) \to \mathrm{Hom}(A', B) \to$$

$$\mathrm{Hom}(A', A[1]) \xrightarrow{a[1]} \mathrm{Hom}(A', C[1]),$$

where the morphisms a and $a[1]$ are isomorphisms. This gives $\mathrm{Hom}(A', B) = 0$ for all $A' \in \mathrm{Ob}(\mathsf{D}')$, hence $B \in \mathsf{D}'^{\perp}$.

Conversely, suppose each $C \in \mathrm{Ob}(\mathsf{D})$ can be realized as the midterm of a distinguished triangle $A \to C \to B \to A[1]$, where $A \in \mathrm{Ob}(\mathsf{D}')$ and $B \in \mathrm{Ob}(\mathsf{D}'^{\perp})$ for some full triangulated category D'. Then D' is right admissible. In fact, we can can set $i^!(C) = A$. In order this to be defined we have to show that the triangle is unique up to a unique isomorphism. Let $A' \to C' \to B' \to A'[1]$ be a distinguished triangle and $f : C \to C'$ be a morphism. Let us construct a unique morphism $\phi : B' \to B$ such that the following diagram is commutative

$$\begin{array}{ccccccc}
A' & \xrightarrow{g} & C' & \longrightarrow & B & \longrightarrow & A'[1] \\
\downarrow{\scriptstyle \phi} & & \downarrow{\scriptstyle f} & & \downarrow & & \downarrow{\scriptstyle \phi[1]} \\
A & \longrightarrow & C & \longrightarrow & B & \longrightarrow & A[1]
\end{array}$$

Since $\mathrm{Hom}^*(A', B) = 0$, applying $\mathrm{Hom}(A', ?)$ to the bottom triangle, we get an isomorphism $\mathrm{Hom}(A', A) \to \mathrm{Hom}(A', C)$. The morphism ϕ is the pre-image of $f \circ g$ under this isomorphism. If $C' = C$ and $f = \mathbf{id}_C$ we get the uniqueness of the isomorphism ϕ. For any $C \in \mathrm{Ob}(\mathsf{D})$ we get a distinguished triangle $0 \to i^!C \to C \to B \to i^!(C)[1].$, where $B \in \mathrm{Ob}(\mathsf{D}'^{\perp})$. Applying $\mathrm{Hom}_{\mathsf{D}}(A, ?)$ we get

$$0 \to \mathrm{Hom}_{\mathsf{D}}(A, i^!(C)) \to \mathrm{Hom}_{\mathsf{D}}(i(A), C) \to \mathrm{Hom}_{\mathsf{D}}(A, B) = 0.$$

This shows that $i^!$ is a right adjoint to i.

Our final observation is that a right (left) admissible subcategory is admissible if D admits a Serre functor.

For any set S of objects in D let $\langle S \rangle$ to be the smallest full triangulated subcategory containing S among its objects. We say that S *generates* D if $\mathsf{D} = \langle S \rangle$. We say that a subcategory D' generates D if its objects generate D.

Proposition 3.3.1. *Let D' be an admissible subcategory of D. Then D' and $\mathsf{D}'^{\perp}$ generate D.*

Proof. Any C can be included in a distinguished triangle $A \to C \to B \to A[1]$, where $A \in \mathrm{Ob}(\mathsf{D}')$ and $B \in \mathrm{Ob}(\mathsf{D}'^{\perp})$. This triangle defines the distinguished triangle $B[-1] \to A \to C \to B$. By definition of a triangulated subcategory, C must be an object of $\langle \mathsf{D}', \mathsf{D}'^{\perp} \rangle$. □

Remark 3.3.2. The notion of an admissible category is a generalization of the *torsion theory* in an abelian category. It is a pair of full subcategories T and F of an abelian category A such that $\mathrm{Hom}_{\mathsf{A}}(T, F) = 0$ for all $A \in \mathrm{Ob}(\mathsf{T})$ and $F \in \mathrm{Ob}(\mathsf{F})$ and any object X in A admits a subobject $T \in \mathrm{Ob}(\mathsf{T})$ with quotient isomorphic to an object from F. A motivating example is the category of abelian groups, where T is the subcategory of finite abelian groups, and F is the category of free abelian groups.

Definition 3.3.2. A sequence $(\mathsf{D}_1, \ldots, \mathsf{D}_m)$ of admissible subcategories is called *semi-orthogonal* if $\mathsf{D}_i \subset \mathsf{D}_j^{\perp}$ for $1 \leq i < j \leq m$. We say that $(\mathsf{D}_1, \ldots, \mathsf{D}_m)$ defines a *semi-orthogonal decomposition* of D if $\langle \mathsf{D}_1, \ldots, \mathsf{D}_m \rangle^{\perp}$ consists of zero objects, or, equivalently, $\mathsf{D} = \langle \mathsf{D}_1, \ldots, \mathsf{D}_m \rangle$.

A way to construct semi-orthogonal sequences is by using exceptional objects in the category.

Definition 3.3.3. An object E of a k-linear derived category D is called *exceptional object* if $\mathrm{Hom}^{\bullet}_{\mathsf{D}}(E, E) \cong k$. An *exceptional sequence* is a sequence $E_1, \ldots, E_s$ of exceptional objects such that $\mathrm{Hom}^{\bullet}_{\mathsf{D}}(E_j, E_i) \cong k$ if $i < j$. An exceptional sequence $(E_1, \ldots, E_m)$ is called *strong* if additionally $\mathrm{Hom}^s(E_i, E_j) = 0$ for $i < j$ and $s \neq 0$.

Example 3.3.3. Let E be an exceptional curve of the first kind (or a (-1)-curve) on a smooth projective algebraic surface X (i.e. $E \cong \mathbb{P}^1$ and $E^2 = -1$). Then $\mathcal{O}_E$, considered as a sheaf on X is an exceptional object in $D^b(X)$. This is checked in the same way as we checked that a (-2)-curve R defines a spherical object $\mathcal{O}_R$.

Example 3.3.4. Let $X = \mathbb{P}^n$ and $E_i = \mathcal{O}_{\mathbb{P}^n}(-i)$, where $i = 0, \ldots, n$. Let us check that it is an exceptional sequence. Since $\mathrm{Hom}^t(E_i, E_i) \cong \mathrm{Hom}^t(\mathcal{O}_X \mathcal{O}_X) \cong \mathrm{Hom}^t(X, \mathcal{O}_X) = 0, j \neq 0$, the sheaves E_i are exceptional. We also have

$$\mathrm{Hom}^t(E_i, E_j) \cong \mathrm{Hom}^t(\mathcal{O}_X, E_{j-i}) \cong H^t(X, \mathcal{O}_X(i-j)) = 0$$

for $t \neq 0$ (because $i - j \geq -n$).

Suppose (E, F) form an exceptional sequence. Let L_E be the left twist functor defined in (3.3). Then $(L_E(F), E)$ is an exceptional sequence. This easily follows from applying $\mathrm{Hom}(E.?)$ to the distinguished triangle (3.3). Starting from an exceptional collection $\mathcal{E} = (E_1, \ldots, E_n)$ one can replace it with the exceptional collection

$$L_i(\mathcal{E}) = (E_1, \ldots, E_{i-1}, L_{E_i}(E_{i+1}), E_i, E_{i+2}, \ldots, E_n)$$

The new collection is called the left *left mutation* of $\mathcal{E}$ at E_i. Similarly one defines the *right mutation* at E_i.

$$R_i(\mathcal{E}) = (E_1, \ldots, E_{i-1}, E_{i+1}, R_{E_{i+1}}(E_i), E_{i+2}, \ldots, E_n)$$

We omit the proof of the following proposition (see [Bondal], Izv. Math. USSR, v. 53 (1989)).

Proposition 3.3.5. *Consider R_i, L_i as transformations on the set of exceptional collections of length n. Then*

(i) $R_i = L_i^{-1}$;

(ii) $R_i \circ R_{i+1} \circ R_i = R_{i+1} \circ R_i \circ R_{i+1}$.

Note that the action braid group action described by this proposition is on the set of exceptional sequences but not on the category D, as was the case with spherical twists in $\mathsf{D} = \mathcal{D}^b(X)$. For example, one immediately checks that $R_E(E) \cong L_E(E) \cong 0$ if E is an exceptional object, so neither R_E nor L_E is an equivalence of categories.

Proposition 3.3.6. *Let $(E_1, \dots, E_m)$ be an exceptional sequence, then $\langle E_1, \dots, E_m \rangle$ is an admissible subcategory.*

Proof. We have to show that, for any $C \in \mathsf{D}$ one can find a distinguished triangle $A \to C \to B \to A[1]$, where $A \in \mathrm{Ob}(\langle E_1, \dots, E_m \rangle)$ and $B \in \mathrm{Ob}(\langle E_1, \dots, E_m \rangle^\perp)$. We use induction on m. Assume $m = 1$. We use the right twist functor R_E. Consider the distinguished triangle

$$\mathrm{Hom}_{\mathsf{D}}(E_1, C) \otimes E_1[-i] \to C \to R_{E_1}(C) \overset{[1]}{\to} .$$

The first object belongs to $\langle E_1 \rangle$. Applying $\mathrm{Hom}_{\mathsf{D}}(E_1, ?)$ we get an exact sequence

$$\to \bigoplus_i \mathrm{Hom}^i_{\mathsf{D}}(E_1, C) \otimes_k \mathrm{Hom}^j_{\mathsf{D}}(E_1, E_1[-i]) \to \mathrm{Hom}^j_{\mathsf{D}}(E_1, C) \to \mathrm{Hom}^j_{\mathsf{D}}(E_1, R_{E_1}(C)).$$

Since E_1 is exceptional, the first term is equal $\mathrm{Hom}^j_{\mathsf{D}}(E_1, C) \otimes k$. This immediately implies that $\mathrm{Hom}^j_{\mathsf{D}}(E_1, R_{E_1}(C)) = 0$ for all j.

By induction, we have a distinguished triangle $A \to C \overset{\beta}{\to} B \to A[1]$, where A is an object of $\langle E_1, \dots, E_{m-1} \rangle$ and B belongs to $\langle E_1, \dots, E_{m-1} \rangle^\perp$. In fact, we can also include in the induction the assertion that $B = R^{m-1}(C) := R_{E_{m-1}} \circ \dots \circ R_{E_1}(C)[m-1]$. Consider the morphism $\gamma = \alpha[m-1] \circ \beta$, where α is defined from the triangle defining

$$R^m(C)[-1] \to \bigoplus_i \mathrm{Hom}^i(E_m, R^{m-1}(C)) \otimes R^{m-1}[-i] \to R^{m-1}(C) \overset{\alpha}{\to} R^m(C).$$

Then it defines a distinguished triangle

$$A' \to C \overset{\gamma}{\to} R^m(C)[m] \to A'[1],$$

with $A' \in \mathrm{Ob}(\mathsf{A})$ and $R^m(C)[m] \in \mathrm{Ob}(\mathsf{A}^\perp)$.

□

Definition 3.3.4. An exceptional sequence $(E_1, \dots, E_m)$ is called *full* if $\mathsf{D} = \langle E_1, \dots, E_m \rangle$.

It follows from the definition that $\langle E_1, \dots, E_m\rangle^\perp$ consists of zero objects, if $(E_1, \dots, E_m)$ is a full exceptional sequence. If D admits a Serre functor, we obtain that ${}^\perp\langle E_1, \dots, E_m\rangle$ consists of zero objects, hence $\{E_1, \dots, E_m\}$ is a spanning set.

The following was the first non-trivial example of a full strongly exceptional sequences due to A. Beilinson.

Theorem 3.3.7. *Let $X = \mathbb{P}^n$ and $\Omega^i_{\mathbb{P}^n} = \Lambda^i(\Omega^1_{\mathbb{P}^n})$ be the sheaves of regular differential i-forms. The sequences*

$$(\mathcal{O}_{\mathbb{P}^n}, \mathcal{O}_{\mathbb{P}^n}(1), \dots, \mathcal{O}_{\mathbb{P}^n}(n)), \tag{3.19}$$

$$(\Omega^n_{\mathbb{P}^n}(n), \dots, \Omega^1_{\mathbb{P}^n}(1), \mathcal{O}_{\mathbb{P}^n}) \tag{3.20}$$

are strong full exceptional sequences.

Proof. Let $\mathbb{P}^n = \mathbb{P}(V)$ (lines in V), where V is a vector space of dimension $n+1$ over k. The standard facts about cohomology of projective spaces easily give

$$\mathrm{Hom}(\mathcal{O}_{\mathbb{P}^i}(-i), \mathcal{O}_{\mathbb{P}^j}(-j)) \cong S^{i-j}(V^*),\ \mathrm{Ext}^t(\mathcal{O}_{\mathbb{P}^i}(-i), \mathcal{O}_{\mathbb{P}^j}(-j)) = 0, t \neq 0.$$

This implies that (3.19) is a strongly exceptional sequence. To show the same for the second sequence we use the inductive description of the sheaves $\Omega^i_{\mathbb{P}^n}$ by means of short exact sequences

$$0 \to \Omega^i_{\mathbb{P}^n} \to V^*(-1) \to \Omega^{i-1}_{\mathbb{P}^n} \to 0.$$

It easily implies that

$$\mathrm{Hom}(\Omega^i_{\mathbb{P}^n}(i), \Omega^j_{\mathbb{P}^n}(j)) \cong \Lambda^{i-j}(V),\ \mathrm{Ext}^t(\Omega^i_{\mathbb{P}^n}(i), \Omega^j_{\mathbb{P}^n}(j)) = 0, t \neq 0.$$

Let $X = \mathbb{P}^n$ and Δ be the diagonal in $X \times X$, we use that $\mathcal{O}_\Delta$ admits the following locally free resolution

$$0 \to \Lambda^n(\mathcal{O}_X(-1) \boxtimes \Omega^1_X(1)) \to \dots \to \mathcal{O}_X(-1) \boxtimes \Omega^1_X(1) \to \mathcal{O}_X \boxtimes \mathcal{O}_X \to \mathcal{O}_\Delta \to 0 \tag{3.21}$$

This is the combination of two standard constructions. First we consider the Euler exact sequence

$$\mathcal{O}_X \to V \otimes \mathcal{O}_X(1) \to \mathcal{T}_X \to 0,$$

where $\mathcal{T}_X = (\Omega^1_X)^\vee$ is the tangent bundle. Twisting by $\mathcal{O}_X(-1)$, we get $H^0(X, \mathcal{T}_X(-1)) \cong V$. We also have $H^0(X, \mathcal{O}_X(1)) \cong V^\vee$. Consider the locally free sheaf $\mathcal{E}$ on $X \times X$ equal to $\mathcal{O}_X(1) \boxtimes \mathcal{T}_X(-1)$. Its dual sheaf is $\mathcal{O}_X(-1) \boxtimes \Omega^1_X(1)$. We have $H^0(X \times X, \mathcal{E}) \cong V^* \otimes V$. Let s be its section defined by $\mathbf{id}_V \in V^* \otimes V = \mathrm{Hom}(V, V)$. If $e_1, \dots, e_n$ is a basis in V and $e_{-1}, \dots, e_{-n}$ is its dual basis, then $s = \sum e_{-i} \otimes e_i$. It is easy to see that the zero scheme $Z(s)$ of s is equal to the diagonal. Now we use the Koszul resolution of the zero scheme of a section of a locally free sheaf on a regular scheme

$$0 \to \Lambda^n(\mathcal{E}^\vee) \to \ldots \to \mathcal{E}^\vee \to \mathcal{O}_{X\times X} \to \mathcal{O}_\Delta \to 0.$$

Let $\mathcal{F}$ be a coherent sheaf on $\Delta = X$ considered as a sheaf on $X \times X$. Tensoring it with the complex (3.21) we obtain a complex on $X \times X$ quasi-isomorphic to $\mathcal{F}$. Applying the derived functor $\mathbf{R}\mathrm{pr}_{2*}$ we obtain a complex X quasi-isomorphic to $\mathcal{F}$. This gives a spectral complex of the double complex

$$E_1^{p,q} = H^p(\mathbb{P}^n, \mathcal{F}(r)) \otimes \Omega_X^{-q}(-q) \Longrightarrow E^{p+q} = \begin{cases} \mathcal{F} & p+q=0 \\ 0 & p+q \neq 0 \end{cases} \tag{3.22}$$

A similar argument applied to the first projection gives a spectral sequence

$$E_1^{p,q} = H^p(\mathbb{P}^n, \Omega_X^{-q}(-q)) \otimes \mathcal{O}_X(q) \Longrightarrow E^{p+q} = \begin{cases} \mathcal{F} & p+q=0 \\ 0 & p+q \neq 0 \end{cases} \tag{3.23}$$

The spectral sequences (3.21) and (3.23) are called the *Beilinson spectral sequences*.

Let us first show that any coherent sheaf $\mathcal{F}$ orthogonal to our exceptional sequence must be zero. In fact, $0 = \mathrm{Hom}^i(\mathcal{O}_X(a), \mathcal{F}) = H^i(X, \mathcal{F}(-a)), a = 0, \ldots, n$, implies that $E_1^{p,q} = 0$ in (3.23) for all p, q, hence $\mathcal{F} = 0$. Now, for a complex $\mathcal{F}^\bullet$ we use the spectral sequence

$$E_2^{p,q} = \mathrm{Hom}^p(\mathcal{O}_X(a), H^q(\mathcal{F}^\bullet)) \Longrightarrow \mathrm{Hom}^{p+q}(\mathcal{O}(a), \mathcal{F}^\bullet)$$

If the limit is equal to zero for all $a = 0, \ldots, n$, then all $E_2^{p,q} = 0$, hence $H^q(\mathcal{F}^\bullet)$ is ortogonal to our exceptional sequence and therefore equal to zero for all q.

Using spectral sequence (3.23) we prove that the sequence (3.20) is a full strongly exceptional sequence. □

It follows from the definition that a full exceptional sequence freely generate $K_0(\mathsf{D})$. In fact, the Gram matrix G of the *Euler form*

$$\langle [E], [F] \rangle = \chi(E, F) = \sum_i (-1)^i \dim_k \mathrm{Hom}^i(F, E)$$

with respect to $([E_1], \ldots, [E_m])$ is an upper triangular unipotent matrix. Suppose also that D admits a Serre functor S. Let A be the matrix of the corresponding operator in $K_0(\mathsf{D})$ in the basis defined by the exceptional sequence. Then $\chi(E, F) = \chi(F, S(E))$ implies $A = G^{-1}G^t$.

Proposition 3.3.8. *The matrix $(-1)^d A$ is a unipotent matrix.*

Proof. Let $d = \dim X$. It is known that the group $K_0(X)$ admits a filtration

$$\{0\} \subset F^d(X) \subset \ldots F^1(X) \subset K_0(X),$$

where $F^i(X)$ is spanned by the classes of coherent sheaves with support in codimension $\geq i$. We have $F^1(X)/F^2(X) \cong \mathrm{Pic}(X)$ and $K^0(X)/F^2(X) \cong \mathrm{Pic}(X) \oplus \mathbb{Z}$. It is known that $F^i(X) \cdot F^j(X) \subset F^{i+j}(X)$ with respect to the multiplication in the ring $K_0(X)$.

The filtration in $K_0(X)$ corresponds to a filtration in $K_0(\mathcal{D}^b(X)) \cong K_0(X)$. For any two invertible sheaves $\mathcal{L}_1$ and $\mathcal{L}_2$, considered as subsheaves of the constant sheaf $k(X)$, we have $[\mathcal{L}_1] - [\mathcal{L}_1 \cap \mathcal{L}_2], [\mathcal{L}_2] - [\mathcal{L}_1 \cap \mathcal{L}_2] \in F^1(X)$, hence $[\mathcal{L}_1] - [\mathcal{L}_2] \in F^1(X)$. In particular $[\omega_X] - [\mathcal{O}_X] \in F^1(X)$. Since $[\mathcal{O}_X]$ is the unit in the ring structure on $K_0(X)$ and the shift operator in $\mathcal{D}^b(X)$ is the operator $(-1)^{\dim X}\mathbf{id}_{K_0(X)}$, we obtain that, for any $x \in F^i(X)$,

$$((-1)^d S - \mathbf{id}_{K_0(X)})(x) = ([\omega_X] - [\mathcal{O}_X] \cdot x \in F^{i+1}(X).$$

This gives $((-1)^d S - \mathbf{id}_{K_0(X)})^d = 0$, hence $((-1)^d A - 1)^d = 0$. □

Example 3.3.9. Suppose $D^b(X)$ is generated by a strongly exceptional sequence (E_1, E_2, E_3). Then the Gram matrix

$$G = \begin{pmatrix} 1 & x & y \\ 0 & 1 & z \\ 0 & 0 & 1 \end{pmatrix}$$

and

$$A = G^{-1}G^t = \begin{pmatrix} 1 - x^2 - z^2 = xyz & -x + xy^2 - yz & xy - z \\ x - yz & 1 - y^2 & -y \\ z & y & 1 \end{pmatrix}$$

The trace of $A - I_3$ is equal to $xyz - x^2 - y^2 - z^2$, so the necessary condition for $A - I_3$ to be nilpotent is that

$$xyz - x^2 - y^2 - z^2 = 0 \tag{3.24}$$

If this holds, then computing the other coefficients of the characteristic polynomial we find that all of them are equal to zero. Thus (3.24) is a necessary and sufficient condition for A to be unipotent. This shows that in the case when d is even and $D^b(X)$ is generated by a strong exceptional sequence (E_1, E_2, E_3), then the dimensions

$$x = \dim \mathrm{Hom}(E_2, E_1), y = \dim \mathrm{Hom}(E_3, E_1), z = \dim \mathrm{Hom}(E_3, E_2)$$

satisfy the equation (3.24) (called the *Markov equation*). For example, it applies to the case when $X = \mathbb{P}^2$. Taking $(E_1, E_2, E_3) = (\mathcal{O}_{\mathbb{P}^2}(2), \mathcal{O}_{\mathbb{P}^2}(1), \mathcal{O}_{\mathbb{P}^2}$, we get a solution $(x, y, z) = (3, 6, 3)$. Applying the mutations, we generate other solutions.

Note that, by Riemann-Roch, if (E_1, E_2) is a part of a strongly exceptional collection, hence

$$0 = \chi(E_2, E_1) = \langle \mathrm{Mu}([E_2]), \mathrm{Mu}([E_1]) \rangle.$$

A direct computation in the case $X = \mathbb{P}^2$ and E_i are locally free sheaves, gives

$$\dim \chi(E_2, E_1) = r_1 r_2 (1 + \frac{3}{2}(\mu_1 - \mu_2) + \frac{1}{2}(\mu_1 - \mu_2)^2 - \Delta_1 - \Delta_2),$$

where r_i is the rank of E_i, $\mu_i = \frac{k_i}{r_i}, k_i = c_1(E_i) \cdot c_1(\mathcal{O}_{\mathbb{P}^2}(1)$ is the slope of E_i and

$$\Delta_i = \frac{1}{r_i}(c_2(E_i) - \frac{(r_i - 1)}{2r_i} c_1^2(E_i))$$

is the discriminant of E_i. Since E_i are exceptional, $1 = \chi(E_i, E_i) = r_i^2 - 2\Delta_i$, hence $\Delta_i = \frac{r_i^2 - 1}{2r_i^2}$. This easily gives

$$0 = r_1^2 + r_2^2 + \delta_{12}^2 - 3\delta_{12} r_1 r_2,$$

where $\delta_{12} = k_2 r_1 - k_1 r_2$. Thus $\frac{1}{3}(r_1, r_2, \delta_{12})$ is a solution of the Markov equation (3.24). Suppose that $r_1 = r_2 = r$, solving the quadratic equation we obtain that $\delta_{12}/r^2 = (k_2 - k_1)/r = 4$ or -1.

Example 3.3.10. Starting from one of the two exceptional sequences described in Beilinson's Theorem, we may obtain new exceptional sequences via mutation. For example, we have

$$L_{\mathcal{O}_{\mathbb{P}(V)}}(\mathcal{O}_{\mathbb{P}(V)}(1)) = \ker(V^* \to O_{\mathbb{P}(V)}(1)) \cong \mathcal{T}_{\mathbb{P}(V)}(-1) \cong \Omega^{n-1}_{\mathbb{P}(V)}(n)),$$

so that the mutation of the first exceptional sequence at its first term gives the exceptional sequence

$$(\mathcal{T}_{\mathbb{P}(V)}(-1), \mathcal{O}_{\mathbb{P}(V)}(1), \ldots, \mathcal{O}_{\mathbb{P}(V)}(n)).$$

Similarly, we get $R_{\mathcal{O}_{\mathbb{P}(V)}}(\Omega^1_{\mathbb{P}(V)}(1)) \cong \mathcal{O}_{\mathbb{P}(V)}(1)$ and we get the mutated exceptional sequence

$$(\Omega^n_{\mathbb{P}(V)}(n), \ldots, \Omega^2_{\mathbb{P}(V)}(2), \mathcal{O}_{\mathbb{P}(V)}, \mathcal{O}_{\mathbb{P}(V)}(1)). \tag{3.25}$$

3.4 Tilting objects

Recall that an abelian Grothendieck category A is equivalent to the category of modules over the ring $\mathrm{End}_{\mathsf{A}}(U)$, where U is a generator of the category. In this section we will discuss a generalization of this fact to a derived category. First we consider the analogs of a generator.

Let A be an abelian category with enough injective objects. For any object $E \in \mathsf{A}$ consider the ring $R = \mathrm{End}_{\mathsf{A}}(E)^{\mathrm{op}}$ and the functor $\mathsf{A} \to \mathrm{Mod}(R)$ defined by $A \to \mathrm{Hom}_{\mathsf{A}}(E, A)$. Let $\Phi : D^+(\mathsf{A}) \to D^+(\mathrm{Mod}(R))$ be its derived functor. We say that E is a *tilting object* if its restriction to $D^b(\mathsf{A})$ defines an equivalence of triangulated categories

$$D^b(\mathsf{A}) \to D^b(\mathrm{Mod}(R)^{\mathrm{fg}})).$$

Definition 3.4.1. A tilting object in A is an object E satisfying

(i) $\mathrm{Ext}^i_{\mathsf{A}}(E,E) = 0, i \neq 0$;

(ii) for any $F^\bullet \in D^-(\mathsf{A})$, $\mathbf{R}\mathrm{Hom}_{\mathsf{A}}(E, F^\bullet) = 0$ implies $F^\bullet = 0$;

(iii) for any $F^\bullet \in D^-(\mathsf{A})$, $\mathrm{Hom}_{D^b(\mathsf{A})}(E, F^\bullet)$ is a finitely generated module over $\mathrm{End}_{\mathsf{A}}(E)$.

Theorem 3.4.1. *Let E is a tilting object in an abelian category A and $R = \mathrm{End}_{\mathsf{A}}(E)^{\mathrm{op}}$. Then the derived functor $\mathbf{R}Hom(E, ?) : D^b(\mathsf{A}) \to D^b(\mathrm{Mod}(R^{\mathrm{fg}}))$ of $Hom_{\mathsf{A}}(E, ?)$ is an an equivalence of triangulated categories.*

Proof. We know from section 1.1 that the functor $\Phi = \mathrm{Hom}(E, ?) : \mathsf{A} \to \mathrm{Mod}(R^{\mathrm{fg}})$ has the right adjoint functor $\mathrm{Mod}(R^{\mathrm{fg}}) \to \mathsf{A},\ M \to E \otimes M$. It is easy to see that its derived functor $\Psi : D^-(\mathrm{Mod}(R^{\mathrm{fg}})) \to D^-(\mathsf{A})$ is the right adjoint functor for $\mathbf{R}\mathrm{Hom}(E, ?) : D^-(\mathsf{A}) \to D^-(\mathrm{Mod}(R^{\mathrm{fg}}))$. Since E is a tilting object, we have $\Phi \circ \Psi(R) = \mathbf{R}\mathrm{Hom}(E) \cong \mathrm{Hom}(E, E) = R$. By adjunction,

$$\Psi : \mathrm{Hom}_{D^-(\mathrm{Mod}(R^{\mathrm{fg}}))}(R, M^\bullet) \xrightarrow{\sim} \mathrm{Hom}_{D^-(\mathsf{A})}(E, E \otimes M^\bullet).$$

Since R is a generator of the category $\mathrm{Mod}(R^{\mathrm{fg}})$ this easily implies that Ψ is fully faithful. Let

$$E \overset{\mathbf{L}}{\otimes} \mathbf{R}\mathrm{Hom}(E, F^\bullet) \xrightarrow{\alpha} F^\bullet \to G^\bullet \xrightarrow{[1]}$$

be the distinguished triangle corresponding to the adjunction morphism α. Applying $\mathbf{R}\mathrm{Hom}(E, ?)$, and using that Ψ is fully faithful, we obtain that $\mathbf{R}\mathrm{Hom}(E, G^\bullet) = 0$. By definition of a tilting object, we obtain $G^\bullet \cong 0$. This implies that $F^\bullet \cong \Psi(\Phi(F^\bullet))$, hence (Φ, Ψ) define an equivalence of categories $D^-(\mathsf{A}), D^-(\mathrm{Mod}(R^{\mathrm{fg}}))$. Therefore, Ψ is also a left adjoint functor to Φ, i.e. we have an isomorphism

$$\Phi : \mathrm{Hom}(E \overset{\mathbf{L}}{\otimes} M^\bullet, F^\bullet) \cong \mathrm{Hom}(M^\bullet, \mathrm{Hom}(E, F^\bullet).$$

This shows that Φ sends a bounded complex to a bounded complex and hence defines an equivalence of derived categories $D^-(\mathrm{Mod}(R^{\mathrm{fg}})) \to D^b(\mathsf{A})$. ☐

Assume that A is a k-linear category of finite type. Let E be a tilting object. The ring $R = \mathrm{End}_{\mathsf{A}}(E, E)$ is a finite-dimensional k-algebra. This implies that R considered as a left module over itself is isomorphic to the direct sum of indecomposable left ideals $R = P_1 \oplus \ldots \oplus P_n$. If $1 = e_1 + \ldots + e_n$ with $e_i \in P_i$, then e_i are *orthogonal idempotents* in R, i.e. $e_i^2 = e_i$ and $e_i e_j = 0, i \neq j$. The ideal P_i is equal to Re_i. Being a direct summand of R, it is a projective R-module. For any left R-module M, the canonical homomorphism $\mathrm{Hom}(Re_i, M) \to M,\ \phi \mapsto \phi(e_i) = e_i\phi(e_i)$ defines an isomorphism of R-modules $\mathrm{Hom}_R(P_i, M) \cong e_i M$. In particular, we have

$$\mathrm{Hom}_R(P_i, P_j) \cong e_i R e_j.$$

We assume that R is a *basic algebra* in the sense that the modules P_i are all non-isomorphic. One can always replace R by a basic algebra (by collecting idempotents from each isomorphism class) such that the categories of modules are equivalent.

The quiver Q_R assigned to any basic finite-dimensional algebra is defined as follows. We assign a vertex v_i to each idempotent e_i of R. The set of arrows between v_i and v_j is a basis of the vector space $\mathrm{Hom}_R(P_i, P_j)/K(i,j)$, where $K(i,j)$ is the subspace generated by the images of the composition maps

$$\mathrm{Hom}_R(P_i, P_t) \times \mathrm{Hom}_R(P_t, P_j) \to \mathrm{Hom}_R(P_i, P_j),\ t \neq i, j.$$

We consider the zero arrow from v_i to v_i corresponding to $e_i \in e_i R e_i$. Thus Q_R has no loops if and only if $e_i R_i e_i \cong k$. Let $k[Q]$ be the path algebra of Q_R. A two-sided ideal in $k[Q_R]$ is called *admissible* if it is generated by a linear combination of paths of length > 1 with the same source and the tail. One can show that any basic algebra R is isomorphic to the quotient of the path algebra of Q_R by some admissible ideal ([Gabriel]).

We leave the proof of the next proposition to the reader.

Proposition 3.4.2. *Let E be a tilting object in A, assume that $R = \mathrm{End}_{\mathsf{A}}(E)$ is a basic algebra and let $R = Re_1 \oplus \ldots \oplus Re_n$. Suppose Q_R does not contain loops and there are no two arrows a, b with $s(a) = t(b), t(a) = s(b)$. Then $E_i = E \overset{\mathbf{L}}{\otimes} P_i$ is an exceptional object in $D^b(\mathsf{A})$, after reindexing $(E_1, \ldots, E_n)$ is a strong exceptional sequence and $E = E_1 + \ldots + E_n$.*

Let Q be a quiver with n vertices satisfying the conditions from the previous proposition and $R = k[Q]/I$ be its path algebra with some relations. Consider the grading on R with R_i generated by the paths of length i. For example, R_0 is generated by zero arrows attached to each vertex. The graded algebra R is generated by R_0 and R_1 if and only if the arrows exist only between adjacent vertices. The vector space R_0 is the subalgebra of R isomorphic to the direct product of $n-1$ copies of the algebras $k[x]/(x(x-1))$. The vector space R_1 is a bimodule over R_0. Let $T(R_1)$ be the tensor algebra of the R_0-bimodule R_1. The algebra R is the quotient of $T(R_1)$ by some homogeneous ideal J. Assume that J is *quadratic ideal*, i.e. a two-sided ideal generated by a subspace of J_2 of $R_1 \otimes_{R_0} R_1$ (in this case a graded R_0-algebra R is called *quadratic*. Then one defines the *dual* of R as the graded algebra $B = T(R_1^*)/(J_2^{\perp})$, where $J_2^{\perp}$ is the annihilator of J_2 in the dual vector space $(R_1 \otimes_{R_0} R_1)^* = R_1^* \otimes_{R_0} R_1^*$.

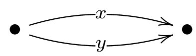
Example 3.4.3. Let $X = \mathbb{P}^1$ and $(\mathcal{E}_1^\bullet, \mathcal{E}_2^\bullet) = (\mathcal{O}_{\mathbb{P}^1}(1), \mathcal{O}_{\mathbb{P}^1})$. The algebra $R = \mathrm{End}(\mathcal{E}_1^\bullet \oplus \mathcal{E}_2^\bullet)$ is 4-dimensional with $e_1 R e_2 \cong H^0(\mathcal{O}_{\mathbb{P}^1}(1) = kx + ky$. We write R in the form of triangular matrices

$$R = \begin{bmatrix} k & 0 \\ k^2 & k \end{bmatrix},$$

where the multiplication is defined in the natural way

$$\begin{bmatrix} a & 0 \\ (x,y) & b \end{bmatrix} \cdot \begin{bmatrix} a' & 0 \\ (x',y') & b' \end{bmatrix} = \begin{bmatrix} aa' & 0 \\ (xa', yb') + (ax', by') & bb' \end{bmatrix}.$$

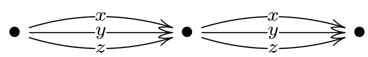
The quiver is the *Kronecker quiver*



A finitely generated R-module M is a finite-dimensional representation of the quiver with $M_1 \rightrightarrows M_2$, where $M_i = e_i M$ are vector spaces over k of dimensions n_i. the two morphisms are given by two matrices A, B of size $n_1 \times n_2$. A morphism from a representation defined by (A, B) to a representation defined by (A', B') is a pair of matrices C_1, C_2 such that $C_1 A C_2 = A', C_1 B C_2 = B'$.

Note that R is a quadratic algebra isomorphic to the non-commutative graded algebra $R_0[x, y]/(x^2, xy, yx, y^2)$, where $R_0 = k[t]/(t^2 - t)$ (note that $xe = x$ but $ex = 0$, where $e = t \mod (t^2 - t)$). Its dual algebra is isomorphic to the algebra of non-commutative polynomials in two variables over R_0.

When $X = \mathbb{P}^2$ and $(\mathcal{E}_1^\bullet, \mathcal{E}_2^\bullet, \mathcal{E}_3^\bullet) = (\mathcal{O}_{\mathbb{P}^2}(2), \mathcal{O}_{\mathbb{P}^2}(1)), \mathcal{O}_{\mathbb{P}^2})$, we get the quiver



In this case the ring R is the path algebra of the quiver with relations of type $xy = yx$, and so on.

A similar quiver with $n + 1$ vertices correspond to the exceptional sequence $(\mathcal{O}_{\mathbb{P}^n}(n), \mathcal{O}_{\mathbb{P}^n}(1)), \mathcal{O}_{\mathbb{P}^n})$. Let $\mathbb{P}^n = \mathbb{P}(V)$. For $n > 1$, the algebra R is a quadratic algebra is generated over R_0 by $R_1 = V^*$ with a basis $x_0, \ldots, x_n$ and relations $x_i x_j - x_j x_i = 0, i \neq j$.

Consider the exceptional sequence (3.20). Let the vertex v_i of the quiver corresponds to i-th term in this sequence. We know that $\mathrm{Hom}(P_i, P_j) = \Lambda^{i-j}(V)$, where $\mathbb{P}^n = \mathbb{P}(V)$. The algebra R is generated by V with the basis $e_0, \ldots, e_n$ dual to the basis $x_0, \ldots, x_n$ of V^*. Its relations are $e_i e_j + e_j e_i = 0$. The quiver Q_R coincides with the quiver for the exceptional sequence (3.19). Thus we see that R is a quadratic algebra isomorphic to the Grassmann algebra of the R_0-module R_1 (if $\mathrm{char}(k) \neq 2$) dual to the algebra corresponding to exceptional sequence (3.19).

If we take the mutated sequence (3.25) with $n = 2$, we get again the same quiver but different algebra. Its multiplication $\mathrm{Hom}(R_1, R_2) \times \mathrm{Hom}(R_2, R_3) \to \mathrm{Hom}(R_1, R_3)$ is isomorphic to the multiplication $\Lambda^2 V \times \Lambda^* V \to V$. The algebra corresponding to (3.19) is defined by the multiplication $V^* \times V^* \to S^2(V^*)$ and the algebra corresponding to (3.20) is defined by $V \times V \to \Lambda^2(V)$.

3.5 Exercises

3.1 Show that the ideal sheaf of a smooth rational curve R on a Calabi-Yau 3-fold X with normal bundle $\mathcal{O}_R(-1)^{\oplus 2}$ is a spherical object.

3.2 Show the dual, or a shift of a spherical object is a spherical object.

3.3 Show that the condition $\mathcal{E}^\bullet \times \omega_X \cong \mathcal{E}^\bullet$ can be replaced by the condition that the pairing $\mathrm{Hom}^i(\mathcal{F}^\bullet, \mathcal{E}^\bullet) \times \mathrm{Hom}^{n-i}(\mathcal{E}^\bullet, \mathcal{F}^\bullet) \to \mathrm{Hom}^n(\mathcal{E}^\bullet, \mathcal{E}^\bullet) \cong k$ is a non-dgenerate pairing for all $\mathcal{F}^\bullet$ and and all $i \in \mathbb{Z}$.

3.4 Let E be an elliptic curve and $x \neq y$ be its closed points. Show that the sheaves $\mathcal{O}_x, \mathcal{O}_y, \mathcal{O}_C$ are spherical objects satisfying (3.16) with Coxeter-Dynkin diagram of type A_3.

3.5 In the situation of the classical McKay correspondence from Example 2.4.14, $m = 2$, show that the objects $\mathcal{V}_{\rho_i} \otimes R/(z_1, z_2)R$ are transformed by the Mckay correspondence to the sheaves $\mathcal{O}_{E_i}$, where E_i is an irreducible component of $\tau^{-1}(0)$ or to $\mathcal{O}_{X//G}$, if ρ_i is trivial. Using this prove that these objects are spherical.

3.6 Let $\mathcal{E}_1^\bullet, \ldots, \mathcal{E}_N^\bullet$ be a set of spherical objects in $D^b(X)$ satisfying condition (3.16). Show that the action of $L_{\mathcal{E}_i^\bullet}$ in $H^*(X, \mathbb{Q})$ is induced from the natural homomorphism of the Artin-Brieskoen braid group to the Coxeter group.

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