DESMIC QUARTIC SURFACES IN ARBITRARY CHARACTERISTIC

IGOR DOLGACHEV AND SHIGEYUKI KONDŌ

Dedicated to Tetsuji Shioda on the occation of his 85th birthday

ABSTRACT. A desmic quartic surface is a birational model of the Kummer surface of the self-product of an elliptic curve. We recall the classical geometry of these surfaces and study their analogs in arbitrary characteristic. Moreover, we discuss the cubic line complex $\mathfrak{G}$ associated with the desmic tetrahedra introduced by G. Humbert. We prove that $\mathfrak{G}$ is a rational $\mathbb{Q}$-Fano threefold with 34 nodes and the group of projective automorphisms isomorphic to $(\mathfrak{S}_4 \times \mathfrak{S}_4) \rtimes 2$.

1. INTRODUCTION

A desmic quartic surface in characteristic zero is a member of the pencil of quartic surfaces containing three reducible members, the unions of four planes with empty intersections (desmic tetrahedra). Desmic tetrahedra were first introduced and studied by Cyparissos Stephanos [35] in 1879.[1] In 1891, George Humbert showed that an irreducible member of the pencil of quartic surfaces containing three desmic tetrahedra is birationally isomorphic to the Kummer surface of the self-product of an elliptic curve [15]. He also showed that a desmic quartic surface is isomorphic to Cremona's quartic surface, the locus of nodes of quadric surfaces that cut out the union of three conics in a smooth cubic surface whose spanning planes contain three coplanar lines on the surface. Other beautiful geometric properties of desmic quartic surfaces were later studied in two papers by Mathews [24], [25] and in Jessop's book on quartic surfaces [17].

A desmic quartic surface is singular. It contains 12 ordinary nodes lying by two on each edge of desmic tetrahedra. It also contains 16 lines, which, together with the nodes, form an abstract configuration $(12_4, 16_3)$ isomorphic to the Reye or the Hesse-Salmon configurations [7]. If the characteristic $p \neq 2$, any surface containing such a configuration is birationally isomorphic to the Kummer surface $\mathrm{Kum}(E \times E)$ of the self-product of an elliptic curve.

In this paper, we first recall the classical results about desmic surfaces and study the cubic line complex associated with the pencil first introduced by Humbert. We find its explicit equation and show that it has 34 ordinary nodes and 24 planes. We believe that the number of nodes is the maximum number of terminal singularities on a Fano threefold of degree 6 in $\mathbb{P}^5$. However, we could not prove it. We also show the rationality of the cubic line complex. The cubic complex has the group of

[1]$\delta\epsilon\sigma\mu o$s = pencil

projective symmetries isomorphic to the group $(\mathfrak{S}_4 \times \mathfrak{S}_4) \rtimes 2$, and can be viewed as the analog of the Segre cubic primal with 10 nodes and 15 planes and the group of projective symmetries isomorphic to $\mathfrak{S}_6$.

Then, we proceed to study the analogs of desmic quartic surfaces in characteristic two. One of them could be the Kummer surface of the self-product of an elliptic curve. Instead of 12 nodes, the surface has six rational double points of type A_3 in the case of an ordinary elliptic curve. We don't know the case of the supersingular elliptic curve. Another analog is Cremona's quartic surface which can be defined in any characteristic but acquires an additional rational double point if the characteristic is equal to 2. In this case, we show that the surface is birationally isomorphic to a supersingular K3 surface with Artin invariant $\sigma \leq 2$. The one-dimensional family of such surfaces contains a supersingular surface with Artin invariant one, previously studied in our paper [10], and we show how to find explicitly the configuration of $(12_4, 16_3)$ among the set of smooth rational curves on this surface.

The authors thank Ivan Cheltsov and Shigeru Mukai for a fruitful discussion about the geometry of the cubic line complex.

2. DESMIC TETRAHEDRA

Until Section 6, we assume that the ground field is an algebraically closed field of characteristic different from 2. A *tetrahedron* means either a set of four planes in $\mathbb{P}^3$ with an empty intersection or their union. We say that each plane entering a tetrahedron T is its face. The intersection line of two faces is its edge, and the intersection point of three faces is its vertex.

Let us first construct desmic tetrahedra. Fix one tetrhaedron T. Let Π be a face of T. It defines a projective involution σ_Π (a harmonic homology) with the set of fixed points equal to the union of Π and the opposite vertex v_Π. Two opposite edges ℓ, ℓ' (i.e., skew edges) define another projective involution σ_ℓ whose set of fixed points is the union of the two edges. It assigns to a point x not on the edges the unique point x' such that the pair $\{x, x'\}$ is harmonically conjugate to the intersection points $\{a, b\}$ of the edges with the unique line passing through x and intersecting the edges.

Starting from a general point P not on T, we apply the involutions defined by three pairs of opposite edges of T. Together with P we obtain four points which are the vertices of a new tetrahedron T'. Next, we apply to P four harmonic homologies defined by faces of T with its vertices as the centers. We obtain four more points that define a third tetrahedron T''. The tetrahedra T, T', T'' obtained in this way are desmic. Indeed, applying a projective transformation, we may assume that $T = V(xyzw)$ and $P = [1, 1, 1, 1]$. Then, we check that the vertices of T' are

$$[1, 1, 1, 1], \ [1, -1, -1, 1], \ [1, -1, 1, -1], \ [1, 1, -1, -1]$$

and

$$T' = V((x - y - z + w)(x - y + z - w)(x + y - z - w)(x + y + z + w)).$$

Applying, the harmonic homologies to P, we find that the vertices of T'' are

$$[-1, 1, 1, 1], \ [1, -1, 1, 1], \ [1, 1, -1, 1], \ [1, 1, 1, -1]$$

and

$$T'' = V((-x + y + z + w)(x - y + z + w)(x + y - z + w)(x + y + z - w)).$$

Next, we check that

$$-16xyzw + (x - y - z + w)(x - y + z - w)(x + y - z - w)(x + y + z + w)$$

$$+(-x + y + z + w)(x - y + z + w)(x + y - z + w)(x + y + z - w) = 0.$$

This shows that T, T', T'' are desmic tetrahedra.

The faces of T' and T'' intersect each face of T along the same four lines. This gives us 16 lines $V(x, y+z+w), V(x, -y+z+w), V(x, y-z+w), V(x, y+z-w)$, etc.

The identity

$$8(x^2 + y^2 + z^2 + w^2) = (x - y - z - w)^2 + (x - y + z + w)^2$$

$$+ (x + y - z + w)^2 + (x + y + z - w)^2 + (x - y - z + w)^2 \qquad (1)$$

$$+ (x - y + z - w)^2 + (x + y - z - w)^2 + (x + y + z + w)^2$$

shows that the three tetrahedra are self-dual with respect to the quadric $V(x^2 + y^2 + z^2 + w^2)$. Since any three faces, one from each tetrahedron, intersect along a line, any three vertices, one from each tetrahedron are on a line. This implies that each pair T_i, T_j is perspective with respect to any vertex of T_k (for the perspectivity, see [8, §2.3.1].

Proposition 2.1. *Three tetrahedra are desmic if and only if each pair is perspective from any vertex of the third one.*

Proof. By Desargues' Theorem, perspectivity from a point is equivalent to the perspectivity from a plane [8, Theorem 2.3.1]. The latter means that each face of the third tetrahedron contains the four intersection lines of the faces of the first two tetrahedra. Therefore, the tetrahedron contains the base locus of the pencil spanned by the other tetrahedra, and hence, the tetrahedron belongs to the pencil. □

Assume that $T_1 = V(xyzw), T_2 = V(l_1 l_2 l_3 l_4)$ and $T_3 = V(m_1 m_2 m_3 m_4)$ are desmic tetrahedra. We may assume that $l_1 = x + y + z + w$. The perspectivity from a vertex implies that an edge of T_1 meets T_2 and T_3 at its edges. Moreover, two opposite edges meet two opposite edges of T_2 and T_3.

Write

$$xyzw + (x + y + z + w)l_2 l_3 l_4 + m_1 m_2 m_3 m_4 = 0.$$

Then, the opposite edges $x = y = 0$ and $z = w = 0$ of $V(xyzw)$ intersect $V(x + y + z + w)$ at the points $[0, 0, 1, -1]$ and $[1, -1, 0, 0]$. This implies that we may assume that $l_2 = a_1(x + y) + a_2(z + w)$. Continuing in this way, we may assume that $l_3 = b_1(x + z) + b_2(y + w), l_4 = c_1(x + w) + c_2(y + z)$. Similar formulas hold for m_1, m_2, m_3, m_4. Now, we check that T_1, T_2, T_3 are desmic if and only if they coincide with T, T', T'' above.

So, we have proved that, up to a linear transformation, there is only one pencil containing three desmic tetrahedra. For example, the obvious desmic pencil

$$a(x^2-y^2)(z^2-w^2)+b(x^2-w^2)(y^2-z^2)+c(x^2-z^2)(w^2-y^2) = 0, \quad a+b+c = 0, \tag{2}$$

is projectively equivalent to the desmic pencil spanned by T, T'.

Note that the 12 vertices of the three tetrahdra from (2) lie by pairs on the six edges of each tetrahedron T, T', T'', and vice versa. We say that the two pencils of desmic tetrahedra are *associated* (see [23, 3.19]).

As a bonus, we obtain that the group of symmetries of a desmic pencil is isomorphic to $2^3 \rtimes \mathfrak{S}_4$.

Remark 2.2. A pencil of hypersurfaces of degree $d \geq 2$ in $\mathbb{P}^n$ with smooth base locus that contains $k \geq 3$ members equal to the reduced union of d hyperplanes with the empty intersection of any $n + 1$ of them is called a *desmic pencil*. The desmic pencils in $\mathbb{P}^2$ exist for $d \leq 5$ [36]. Among them are pencils of conics, the Hesse pencil of cubics, and a desmic pencil of quartics obtained by intersecting a desmic pencil of quartic surfaces with a general plane. A general member of the latter pencil is a smooth plane quartic curve with the property that it admits three bitangents that form the diagonals of a complete quadrilateral, whose six vertices are the points of contact of the bitangents [14].

It is not known whether desmic pencils exist in $\mathbb{P}^2$ for $d > 5$ and whether there exists a desmic pencil in $\mathbb{P}^n$, $n \geq 3$, except the desmic pencil of quartics in $\mathbb{P}^3$ [29].

3. DESMIC QUARTIC SURFACES

Recall that a *desmic quartic surface* is an irreducible member of the desmic pencil of quartics. In fact, any member of the pencil different from the tetrahedra is irreducible. To see this, we use that the base locus of the pencil is the union of 16 lines no pair of them is skew, lying by four in the faces of any of the tetrahedra. Any reducible member must contain an irreducible component of degree one or two. This easily gives a contradiction.

Recall that an *abstract configuration* of type (a_c, b_d) consists of two sets A and B of cardinalities a, b and a relation $R \subset A \times B$ such that the fibers of the projection $R \to A$ (resp. $R \to B$) have the same cardinality equal to c (resp. d). There is a natural definition of an isomorphism of abstract configurations.

In geometry, the relation R is usually the incidence relation between points and lines, lines and planes, etc.

The set of 12 singular points and 16 lines on a desmic quartic surface is an example of an abstract configuration $(12_4, 16_3)$. We call it the *desmic configuration*. Another example of an abstract configuration of the same type is the *Reye configuration*. Its set of points consists of eight vertices of a cube in $\mathbb{P}^3(\mathbb{R})$, the intersection point of the four diagonals of the cube, and three intersection points at infinity of the 12 edges. Its set of lines consists of 12 edges and 4 diagonals.

Proposition 3.1. *Let Q be a desmic quartic surface. Then Q contains 12 ordinary nodes and 16 lines that form an abstract configuration $(12_4, 16_3)$ isomorphic to the Reye configuration.*

Proof. The sixteen lines form the base locus of the pencil. For the pencil from (2) they are

$$V(x \pm y, x \pm w), \quad V(x \pm y, y \pm z), \quad V(z \pm w, x \pm w), \quad V(z \pm w, y \pm z).$$

The twelve points are

$$[0,0,0,1], \quad [0,0,1,0], \quad [0,1,0,0], \quad [1,0,0,0],$$
$$[1,1,1,1], \quad [1,1,-1,-1], \quad [1,-1,1,-1], \quad [1,-1,-1,1], \tag{3}$$
$$[1,1,1,-1], \quad [1,1,-1,1], \quad [1,-1,1,1], \quad [-1,1,1,1].$$

The singular points lie by two at the edges of any tetrahedron. Each face of one of the desmic tetrahedra contains four lines. They form a complete quadrilateral whose three diagonals are the edges lying in the face. The incidence property is verified immediately. See Figure 1 in which six black circles are nodes of Q sitting on the face.

FIGURE 1. A complete quadrilateral and three diagonals

Note that the vertices of the desmic tetrahedra from (2) are

$$[0,0,1,\pm1], \ [0,1,0,\pm1], \ [0,1,\pm1,0], \ [1,0,0,\pm1], \ [1,0,\pm1,0], \ [1,\pm1,0,0]. \tag{4}$$

They are different from the singular points of Q and occur as the nodes of a desmic quartic surface from the conjugate desmic pencil. We see that the twelve points are the vertices of the desmic tetrahedra from (2) (contrary to the assertion from [16, B.5.2.2]).

Two tetrahedra inscribed in a cube and the third tetrahedron with three vertices at infinity and one vertex at the center of the cube form a set of three tetrahedra conjugate to a set of three desmic tetrahedra. This shows that the desmic configuration is isomorphic to the Reye configuration. $\square$

Note that the Reye configuration is also isomorphic to the Hesse-Salmon plane configuration of points and lines obtained by the projection of the Reye configuration from a general point in the space (see [7, 7.3]).

Remark 3.2. As observed by Tomasz Szemberg, the set 24 points given in (3) and (4) has a peculiar property that, although the set is not a complete intersection in $\mathbb{P}^3$, its projection to $\mathbb{P}^2$ from a general point is a complete intersection of quartic

and sextic curves. Unfortunately, we do not know an explanation of this property of the set.

For an elliptic curve E, we denote by $\mathrm{Kum}(E \times E)$, called the Kummer surface, the quotient surface by the inversion of $E \times E$, and by $\widetilde{\mathrm{Kum}}(E \times E)$ the minimal resolution of singularities of $\mathrm{Kum}(E \times E)$.

Theorem 3.3. *A desmic surface is birationally isomorphic to the Kummer surface* $\mathrm{Kum}(E \times E)$ *of the self-product of an elliptic curve* E.

Proof. Choose two singular points p_1, p_2 on two different edges in a face of one of the tetrahedra. Let E_1, E_2 be the exceptional curves over these points in the minimal resolution X of the desmic quartic surface Q. The eight lines that pass through p_1 or p_2 have one common line $F_0 = \langle p_1, p_2 \rangle$. Let $F_1, \ldots, F_6$ be the remaining lines. Consider the divisor

$$H = 2(E_1 + E_2) + F_1 + \cdots + F_6 + 2F_0. \tag{5}$$

Then, we check that $H^2 = 4$ and the linear system $|H|$ defines a degree two map

$$\Phi : X \to \mathbf{F}_0 \subset \mathbb{P}^3.$$

Its branch divisor is the union of 8 lines, four from each ruling. On the other hand, the linear system $|2(E \times \{\mathrm{pt}\} + \{\mathrm{pt}\} \times E)|$ defines a double cover $\widetilde{\mathrm{Kum}}(E \times E) \to \mathbb{P}^1 \times \mathbb{P}^1$ branched along the union of 8 lines, from each ruling. Since $\mathrm{Pic}(\mathbf{F}_0)$ has no torsion, there is only one isomorphism class of such a cover. This proves the assertion. $\square$

Remark 3.4. Conversely, a desmic surface is obtained from the self-product of an elliptic curve E as follows. Let $\iota : E \to E$ be the inversion map of E and let $a_0 = 0, a_1, a_2, a_3 = a_1 + a_2$ be 2-torsion points on E. Consider the curves $E \times \{a_i\}, \{a_i\} \times E$, the diagonal Δ, and its translations Δ_i by a 2-torsion point $\{0\} \times \{a_i\}$ $(i = 1, 2, 3)$, respectively. A configuration of these curves is given in Figure 2.

FIGURE 2. Sixteen 2-torsion points and twelve elliptic curves on $E \times E$

The quotient surface of $E \times E$ by the involution $\langle \iota \times 1_E, 1_E \times \iota \rangle$ is the quadric $\mathbb{P}^1 \times \mathbb{P}^1$ on which the images of 8 elliptic curves except four curves Δ, Δ_i are 8 lines. The minimal resolution of the quotient of $E \times E$ by the inversion involution $\iota \times \iota$ is $\widetilde{\mathrm{Kum}}(E \times E)$ on which the images of sixteen 2-torsion points are sixteen disjoint (-2)-curves E_{ij} $(0 \leq i, j \leq 3)$ and the images of twelve elliptic curves $E \times \{a_i\}$, $\{a_i\} \times E$, Δ_i are twelve disjoint (-2)-curves E_i, E^i, D_i $(0 \leq i \leq 3)$. Let p_1, p_2 be the first and the second projection $E \times E \to E$ and let p_3 be the map $E \times E \to E$, $(x, y) \to (x + y, x + y)$.

Then, each p_i induces an elliptic fibration $f_i : \widetilde{\mathrm{Kum}}(E \times E) \to \mathbb{P}^1$ which has four singular fibers of type $\tilde{D}_4$ (type I_0^* in the sence of Kodaira). Let F_i be a general fiber of f_i and let

$$H = F_1 + F_2 + F_3 - \frac{1}{2} \sum_{0 \leq i,j \leq 3} E_{ij}.^{[2]} \tag{6}$$

Then, $H^2 = 4$, $H \cdot E_{ij} = 1$ and H is perpendicular to the remaining twelve (-2)-curves. Thus, the complete linear system $|H|$ gives a birational morphism from $\mathrm{Kum}(E \times E)$ to a desmic surface on which the images of E_{ij} are sixteen lines and those of twelve (-2)-curves are twelve nodes.

The Picard number of $\widetilde{\mathrm{Kum}}(E \times E)$ is at least 19. For example, consider the linear system

$$|E_{00} + E_{02} + E_{10} + E_{13} + 2(E_0 + E_{01} + E^1 + E_{11} + E_1)|$$

which defines an elliptic fibration with a singular fiber of type $\tilde{D}_8$ and sections E^2, E^3. The following curves are components of singular fibers of this fibration: E_{20}, E_2, E_{22}, E_{23} (forming a Dynkin diagram of type D_4) and D_3, E_{30}, E_3, E_{32}, E_{33} (forming a Dynkin diagram of type D_5). Thus the Picard lattice contains $U \oplus D_8 \oplus D_5 \oplus D_4$ and hence the Picard number is at least 19. If the Picard number is equal to 19, then it is well known that the transcendental lattice is isomorphic to $U(2) \oplus \langle 4 \rangle$ (see Remark 3.7), which implies that the determinant of $\mathrm{Pic}(\widetilde{\mathrm{Kum}}(E \times E))$ is equal to $\pm 2^4$. Since the above fibration has two sections E^2, E^3, $\mathrm{Pic}(\widetilde{\mathrm{Kum}}(E \times E))$ is generated by twenty-eight (-2)-curves that span an overlattice of $U \oplus D_8 \oplus D_5 \oplus D_4$ by adding E^3, that is, it is isomorphic to $U \oplus D_8 \oplus D_9$. Thus we have the following result.

Proposition 3.5. *The twenty-eight (-2)-curves forming $(12_4, 16_3)$-configuration generate a quadratic lattice isomorphic to $U \oplus D_8 \oplus D_9 (\cong U \oplus E_8 \oplus D_8 \oplus \langle -4 \rangle)$.*

Remark 3.6. In [15], Humbert showed that the Kummer surface $\mathrm{Kum}(E \times E)$ is birationally isomorphic to a desmic surface by using Weierstrass' σ-function (see also [17, Chapter 2]).

---

[2]The sum of sixteen disjoint (-2)-curves on a complex $K3$ surface X is divided by 2 in $\mathrm{Pic}(X)$ ([27]). In characteristic not equal two, this also holds replacing $H^2(X, \mathbb{F}_2)$ by $H^2_{et}(X, \mathbb{F}_2)$ ([32, Remark 3.3]).

As we noted earlier, the group of projective automorphisms of a desmic pencil is isomorphic to $2^3 \rtimes \mathfrak{S}_4$. The group acts on the base curve of the pencil. The kernel of the corresponding homomorphism is isomorphic to $2^3 \rtimes 2^2$, and the image is isomorphic to $\mathfrak{S}_3$. A desmic quartic with the group of projective automorphism larger than the group $2^3 \rtimes 2^2$ corresponds to a fixed point of the action of $\mathfrak{S}_3$ on the base. Thus, an element of order 3 corresponds to a point $(a, b, c) = (1, \zeta, \zeta^3)$ where $\zeta^3 = 1, \zeta \neq 1$. The surface is isomorphic to $\mathrm{Kum}(E_\zeta \times E_\zeta)$, where E_ζ is an elliptic curve with complex multiplication by $\mathbb{Z}[\zeta]$. Elements of order 2 give three desmic surfaces with $(a, b, c) = (0, 1, -1)$ and etc. They are isomorphic to $\mathrm{Kum}(E_{\sqrt{-1}} \times E_{\sqrt{-1}})$, where $E_{\sqrt{-1}}$ is an elliptic curve with complex multiplication by $\mathbb{Z}[\sqrt{-1}]$.

Remark 3.7. Let $T(E \times E)$ be the transcendental lattice of the abelian surface $E \times E$. As is well-known, this lattice is isomorphic to $U \oplus \langle 2 \rangle$ for a generic E. The transcendental lattice of the minimal nonsingular model $\widetilde{\mathrm{Kum}}(E \times E)$ is isomorphic to $T(E \times 2)(2)$. Hence, the Picard lattice of $\widetilde{\mathrm{Kum}}(E \times E)$ is isomorphic to the lattice $M_2 = U \oplus E_8 \oplus D_8 \oplus \langle -4 \rangle$. The special cases of curves E_ω with complex multiplication have the Picard lattices isomorphic to $U \oplus E_8 \oplus E_8 \oplus \langle -4 \rangle \oplus \langle -4 \rangle$ if $\omega = \sqrt{-1}$ and $U \oplus E_8 \oplus E_8 \oplus A_2(2)$ if $\omega = \zeta$.

The moduli space of lattice M_2 polarized K3 surfaces admits a compactification isomorphic to the quotient X_0^+ of the modular curve $X_0(2)$ by the Fricke involution. The boundary consists of the the unique cusp of the surface [6, Theorem 7.3]. It is shown in Theorem 7.6 from loc. cit. that the universal family $\mathcal{X} \to X_0(2)^+$ is isomorphic to the family of minimal nonsingular models of the Kummer surfaces $\mathrm{Kum}(E \times E')$, where $E' \to E$ is the isogeny of elliptic curves of degree 2 corresponding to the chocie of a 2-torsion point on the curve. It is explained in loc. cit. that $\mathrm{Kum}(E \times E)$ admits a rational involution such that the quotient surface is birationally isomorphic to $\mathrm{Kum}(E \times E')$. It follows that there exists a unique morphism of the base $\mathbb{P}^1$ of the desmic pencil to the modular curve $X_0(2)^+$. The pre-image of the unique cusp of $X_0(2)^+$ consists of three points corresponding to the desmic tetrahedra. We also know that the pre-image of a point on $X_0(2)^+$ corresponding to a curve $E_{\sqrt{-1}}$ consists of three points. This shows that the modular curve $X_0(2)^+$ is the quotient of the base of the desmic pencil by the group $\mathfrak{S}_3$. This implies that the base of the desmic pencil can be identified with the modular curve $X(2)$ whose three cusps correspond to the desmic polyhedra (see also [16, Corollary B.5.7]).

One can find a construction of members of the modular family of lattice M_2 polarized K3 surfaces in [6, Example 7.9]. The surfaces are birationally isomorphic to the double covers of extremal rational elliptic surfaces with a reducible fiber of type $\tilde{E}_7$ and $\tilde{A}_1$ (types III^* and I_2 in Kodaira's notation).

4. The cubic complex of lines

Let $\mathcal{N}$ be a net of quadrics in $\mathbb{P}^3$. We assume that it contains a smooth quadric. The set of lines contained in some quadric from $\mathcal{N}$ form a *line complex* $\mathfrak{G}(\mathcal{N})$, a

hypersurface in the Grassmannian $G_1(\mathbb{P}^3)$ of lines in $\mathbb{P}^3$. It is called the Montesano complex and its degree is equal to three. Recall that the latter means that $\mathfrak{G}(\mathcal{N})$ is a complete intersection of $G_1(\mathbb{P}^3)$, viewed as a quadric in the Plücker embedding in $\mathbb{P}^5$, with a cubic hypersurface.

The degree of a line complex $\mathfrak{G}$ is equal to the degree of the plane curve $\Omega(x) \cap \mathfrak{G}$, where $\Omega(x)$ is the plane in $G_1(\mathbb{P}^3)$ of lines containing a general point $x \in \mathbb{P}^3$. We easily see why the degree of $\mathfrak{G}(\mathcal{N})$ is equal to 3. The set of all quadrics from $\mathcal{N}$ that contains x is a pencil $\mathcal{N}(x)$ whose base curve is a quartic elliptic curve C containing x. The lines contained in a quadric from $\mathcal{N}(x)$ form a cone over the projection of C from x. It is a cubic cone, and $\Omega(x) \cap \mathfrak{G}(\mathcal{N})$ is the projection of C, a cubic curve.

Let us return to our desmic pencil given in (2). Take two of the desmic tetrahedra, say $V((x^2 - y^2)(z^2 - w^2))$ and $V((x^2 - z^2)(y^2 - w^2))$. The quadrics $V((x - y)(z + w)), V((x - z)(y + w)), V((x - w)(y + z))$ vanish at 8 points given in the first two rows in (3). It is immediately checked that the quadrics are linearly independent and span a net $\mathcal{N}_1$ of quadrics with 8 base points. It obviously contains a smooth quadric. The discriminant curve of $\mathcal{N}_1$ is a plane quartic curve in $\mathcal{N}_1$ with three nodes corresponding to three reducible quadrics from the net. Similarly, we find the nets $\mathcal{N}_2, \mathcal{N}_3$ with base points given by the first and the third rows, and by the second and the third rows:

$$\mathcal{N}_2 = \langle V((x - y)(z - w)), V((x - z)(y - w)), V((x - w)(y - z)) \rangle;$$
$$\mathcal{N}_3 = \langle V(x^2 - y^2), V(x^2 - z^2), V(x^2 - w^2) \rangle.$$

Thus, a desmic pencil defines three nets of quadrics $\mathcal{N}_1, \mathcal{N}_2, \mathcal{N}_3$, and hence, three cubic line complexes $\mathfrak{G}_1, \mathfrak{G}_2, \mathfrak{G}_3$.

Corollary 4.1. *The 12 lines that join a point $p \in \mathbb{P}^3$ with the 12 nodes of a desmic quartic Q are contained in a cubic cone with its vertex at p.*

Proof. If p is a base point of $\mathcal{N}_1$, the polar cubic surface with the pole at p is a cubic cone that contains all singular points of Q. So, we may assume that p is not a base point. Then, a line spanned by p and one of the base points of $\mathcal{N}_1$ is contained in a quadric from $\mathcal{N}_1$. Hence, it belongs to the cubic line complex, and, therefore, belongs to the cubic cone $K_1(p)$ of lines from $\mathfrak{G}_1$ passing through the point p. This shows that the cubic cone $K_1(p)$ contains generators spanned by p and any of the eight base points of $\mathcal{N}_1$. Let p_i be one of the remaining four nodes of Q. There are four lines joining p_i with two base points of $\mathcal{N}_1$. Thus, the pencil of quadrics from $\mathcal{N}_1$ passing through p_i contains four lines passing through p_i. It must be a cone with its vertex at p_i. So, we choose the quadric from this pencil passing through p and obtain that the line $\langle p, p_i \rangle$ belongs to the cubic line complex $\mathfrak{G}_1$. Thus, the line $\langle p, p_i \rangle$ is a generator of the cubic cone $K_1(p)$. $\square$

Since the cubic cones $K_i(x)$ of different line complexes $\mathfrak{G}_i$ intersect along 12 lines, they coincide. This gives the following:

Corollary 4.2. *The cubic complexes $\mathfrak{G}_1, \mathfrak{G}_2, \mathfrak{G}_3$ coincide.*

Let $\mathfrak{G} = \mathfrak{G}_1 = \mathfrak{G}_2 = \mathfrak{G}_3$.

Definition 4.3. The cubic line complex $\mathfrak{G}$ arising from a desmic pencil is called the *Humbert desmic line complex*.

Since $\mathfrak{G}$ contains three plane-pairs formed by faces of the desmic tetrahedra, the corollary implies another corollary.

Corollary 4.4. *Any line contained in the face of one of the desmic tetrahedra belongs to $\mathfrak{G}$.*

Remark 4.5. Since any of the twelve nodes of the desmic quartic Q is a base point of one of the nets $\mathcal{N}_i$, any line through it belongs to the complex $\mathfrak{G}$. Thus, $\mathfrak{G}$ contains twelve α-planes Λ_i of lines passing through a node and twelve β-planes Ξ_i of lines contained in some face of one of the desmic tetrahedra. Each of the 16 lines ℓ_i in Q is common to three planes Λ_i and three planes Ξ_i. This shows that the lines are singular points of $\mathfrak{G}$. The 16 singular points and 12 planes from each family of planes in the Grassmannian form two configurations $(12_4, 16_3)$. Together, they form a configuration $(16_6, 24_4)$.

Note that each edge of a desmic tetrahedron is contained in two α-planes and two β-planes. This implies that it is a singular point of $\mathfrak{G}$, and hence, the cubic line complex contains $16 + 18 = 34$ singular points. Each β-plane contains seven singular points of $\mathfrak{G}$, they form a complete quadrilateral and its three diagonals. Each α-plane contains five singular points, one edge, and two lines ℓ_i from each face containing the edge.

The group $2^3 \rtimes \mathfrak{S}_4$ of symmetries of the desmic pencil acts on $\mathfrak{G}$. This makes $\mathfrak{G}$ a Fano 3-fold similar to the Segre cubic primal that contains 10 singular points and 15 planes forming an abstract configuration $(15_4, 10_6)$ with the group of symmetries isomorphic to $\mathfrak{S}_6$.

Lemma 4.6. *Each quadric from $\mathcal{N}_i$ cuts out a desmic quartic surface Q along the union of two quartic curves intersecting at the base points of $\mathcal{N}_i$.*

Proof. A quadric $G \in \mathcal{N}_i$ intersects Q along a curve C of order 8 with double points. A general quadric from the pencil of quadrics in $\mathcal{N}_i$ passing through a general point $x \in C$ intersects C with multiplicity $2 \times 8 + 1 = 17 > 16$. This shows that the base curve of the pencil $\mathcal{N}_i(x)$ is a part of C. The residual part is another quartic curve. $\square$

The next proposition gives another proof that a desmic quartic surface is birationally isomorphic to $\mathrm{Kum}(E \times E)$.

Proposition 4.7. *Let X be a minimal resolution of singularities of Q. Then each net of quadrics $\mathcal{N}_i$ defines a base-point-free pencil $|E_i| = |E_i'|$ of elliptic curves on X, where E_i, E_i' are the proper transforms of the irreducible components of the intersection of a quadric from $\mathcal{N}_i$ with Q. We have $E_i \cdot E_j = 2$, and the linear system $|E_i + E_j|$ defines a degree two cover $X \to \mathbb{P}^1 \times \mathbb{P}^1$ with the branch curve equal to the union of 8 lines, four from each family of lines on the quadric model of $\mathbb{P}^1 \times \mathbb{P}^1$.*

This easily follows from the discussion above.

Proposition 4.8. *A desmic quartic surface contains 16 conics.*

Proof. Let Π be a plane containing one of the sixteen lines on Q. For example, we may take it to be $\ell = V(x + y, z + w)$. Then, $\Pi = V(u(x + y) + v(x + w))$. Substituting in the equation of Q, we find that Π is tangent to Q along ℓ provided that $u(b + c) + v(a + b) = 0$. The residual intersection of $Q \cap \Pi$ is a conic intersecting ℓ at two different points. $\qquad\square$

It follows from the proof of Proposition 4.8 that the pencil of cubic curves cut out by planes through any of the lines contains singular fibers of types $3\tilde{A}_5, \tilde{A}_1$.

Let us find the equation of the Humbert desmic cubic line complex $\mathfrak{G}$. The net of quadrics passing through the eight points given in rows one and three in (3) is equal to

$$t_1(xy + zw) + t_2(xz + yw) + t_3(xw + yz) = 0.$$

Let

$$(x, y, z, w) = (a_1u + b_1v, a_2u + b_2v, a_3u + b_3v, a_4u + b_4v)$$

be the parametric equation of a line in $\mathbb{P}^3$. Plugging it in the previous equation, we obtain that condition that there exist parameters (t_1, t_2, t_3) such that the quadric contains the line is

$$\det \begin{pmatrix} a_1a_2 + a_3a_4 & a_1a_3 + a_2a_4 & a_1a_4 + a_2a_3 \\ b_1b_2 + b_3b_4 & b_1b_3 + b_2b_4 & b_1b_4 + b_2b_3 \\ a_1b_2 + a_2b_1 + a_3b_4 + a_4b_3 & a_1b_3 + a_2b_4 + a_3b_1 + a_4b_2 & a_1b_4 + a_2b_3 + a_3b_2 + a_4b_1 \end{pmatrix} = 0.$$

Computing the determinant, we obtain that the equation of $\mathfrak{G}$ in Plücker coordinates $(x_1, x_2, x_3, x_4, x_5, x_6) = (p_{12}, p_{13}, p_{14}, p_{23}, p_{24}, p_{34})$ is the following:

$$p_{12}p_{34} - p_{13}p_{24} + p_{14}p_{23} = 0, \tag{7}$$

$$-p_{12}p_{13}p_{23} + p_{12}p_{14}p_{24} - p_{13}p_{14}p_{34} + p_{23}p_{24}p_{34} = 0. \tag{8}$$

Note that the equation is invariant with respect to the natural action of $\mathfrak{S}_4$ tensored with the sign representation. It is also invariant with respect to the action of 2^3 on coordinates in $\mathbb{P}^3$. Thus, we obtain that the group $2^3 \rtimes \mathfrak{S}_4$ acts on $\mathfrak{G}$. This agrees with the group of symmetry of the desmic pencil. The set $\mathrm{Sing}(\mathfrak{G}) = \mathrm{Sing}(\mathfrak{G})_1 \cup \mathrm{Sing}(\mathfrak{G})_2$ of 34 nodes of $\mathfrak{G}$, where the subset $\mathrm{Sing}(\mathfrak{G})_1$ consists of 18 points

$$\begin{array}{l} {\scriptstyle 1:}\,[1, 0, 0, 0, 0, 0],\ {\scriptstyle 2:}\,[0, 1, 0, 0, 0, 0],\ {\scriptstyle 3:}\,[0, 0, 1, 0, 0, 0], \\ {\scriptstyle 4:}\,[0, 0, 0, 1, 0, 0],\ {\scriptstyle 5:}\,[0, 0, 0, 0, 1, 0],\ {\scriptstyle 6:}\,[0, 0, 0, 0, 0, 1], \\ {\scriptstyle 7:}\,[1, 1, 0, 0, 1, 1],\ {\scriptstyle 8:}\,[1, 1, 0, 0, -1, -1],\ {\scriptstyle 9:}\,[1, -1, 0, 0, 1, -1],\ {\scriptstyle 10:}\,[1, -1, 0, 0, -1, 1], \\ {\scriptstyle 11:}\,[1, 0, 1, 1, 0, -1],\ {\scriptstyle 12:}\,[1, 0, 1, -1, 0, 1],\ {\scriptstyle 13:}\,[1, 0, -1, 1, 0, 1],\ {\scriptstyle 14:}\,[1, 0, -1, -1, 0, -1], \\ {\scriptstyle 15:}\,[0, 1, 1, 1, 1, 0],\ {\scriptstyle 16:}\,[0, 1, 1, -1, -1, 0],\ {\scriptstyle 17:}\,[0, 1, -1, 1, -1, 0],\ {\scriptstyle 18:}\,[0, 1, -1, -1, 1, 0]. \end{array} \tag{9}$$

corresponding to 18 edges which form two orbits of cardinalities 6 and 12, and $\mathrm{Sing}(\mathfrak{G})_2$ consists of 16 points

$$
\begin{array}{llll}
1\colon [1,1,1,0,0,0], & 2\colon [1,1,-1,0,0,0], & 3\colon [1,-1,1,0,0,0], & 4\colon [1,-1,-1,0,0,0] \\
5\colon [1,0,0,1,1,0], & 6\colon [1,0,0,1,-1,0], & 7\colon [1,0,0,-1,1,0], & 8\colon [1,0,0,-1,-1,0] \\
9\colon [0,1,0,1,0,1], & 10\colon [0,1,0,1,0,-1], & 11\colon [0,1,0,-1,0,1], & 12\colon [0,1,0,-1,0,-1] \\
13\colon [0,0,1,0,1,1], & 14\colon [0,0,1,0,1,-1], & 15\colon [0,0,1,0,-1,1], & 16\colon [0,0,1,0,-1,-1]
\end{array}
\tag{10}
$$

corresponding to 16 lines.

The set of 24 planes contained in $\mathfrak{G}$ consists of 12 α-planes and 12 β-planes. Recall that the Plücker equations of the α-plane of lines passing through a fixed point $[a,b,c,d] \in \mathbb{P}^3$ are

$$-cp_{12} + bp_{13} - ap_{23} = dp_{13} - cp_{14} + ap_{34} = dp_{12} - bp_{14} + ap_{24} = 0.$$

The equations of the β-plane of lines in a fixed plane $V(ax + by + cz + dw) \subset \mathbb{P}^3$ are

$$
\begin{aligned}
cp_{12} - bp_{13} + ap_{24} = dp_{13} - bp_{14} + ap_{24} = -dp_{13} + cp_{14} + ap_{34} = 0, \quad a \neq 0, \\
-cp_{12} + bp_{13} = -dp_{12} + bp_{14} = dp_{23} - cp_{24} + bp_{34} = 0, \quad a = 0.
\end{aligned}
\tag{11}
$$

For brevity of the notation, let $(p_{12}, p_{13}, p_{14}, p_{23}, p_{24}, p_{34}) = (x_1, x_2, x_3, x_4, x_5, x_6)$. Substituting the coordinates of points from (3), we obtain the equations of the twelve α-planes:

$$
\begin{array}{l}
1\colon V(x_1, x_2, x_4), 2\colon V(x_1, x_3, x_5), 3\colon V(x_2, x_3, x_6), 4\colon V(x_4, x_5, x_6), \\
5/6\colon V(x_1 + x_2 + x_4, \pm x_1 + x_3 + x_5, \pm x_2 - x_3 + x_6), 7/8\colon V(x_1 + x_2 - x_4, \pm x_1 - x_3 + x_5, \pm x_2 + x_3 + x_6), \\
9/10\colon V(x_1 - x_2 + x_4, \pm x_1 - x_3 + x_5, \pm x_2 - x_3 + x_6), 11/12\colon V(x_1 - x_2 - x_4, \pm x_1 + x_3 + x_5, \pm x_2 + x_3 + x_6).
\end{array}
$$

Substituting the equations of the faces of the three tetrahedra from (2), we obtain the equations of the twelve β-planes:

$$
\begin{array}{l}
1/2\colon V(x_1, x_2 \pm x_4, x_3 \pm x_5), 3/4\colon V(x_2, x_1 \pm x_4, x_3 \mp x_6), 5/6\colon V(x_3, x_1 \pm x_5, x_2 \pm x_6), \\
7/8\colon V(x_4, x_1 \pm x_2, x_5 \pm x_6), 9/10\colon V(x_5, x_1 \pm x_3, x_4 \mp x_6), 11/12\colon V(x_6, x_2 \pm x_3, x_4 \pm x_5).
\end{array}
$$

Changing the Plücker coordinates to the Klein coordinates

$$(x_1, x_2, x_3, y_1, y_2, y_3)$$

$$= (p_{12} + p_{34}, \; -p_{13} + p_{24}, \; p_{14} + p_{23}, \; i(p_{34} - p_{12}), \; i(p_{24} + p_{13}), \; i(p_{23} - p_{14})),$$

where $i = \sqrt{-1}$, we obtain the new equation of $\mathfrak{G}$:

$$x_1^2 + x_2^2 + x_3^2 + y_1^2 + y_2^2 + y_3^2 = x_1 x_2 x_3 + i y_1 y_2 y_3 = 0. \tag{12}$$

It is in striking resemblance with Baker's equations of the Segre cubic primal:

$$x_1 + x_2 + x_3 + y_1 + y_2 + y_3 = x_1 x_2 x_3 + y_1 y_2 y_3 = 0$$

(see [16, 3.3.2]). Note that the presence of $\sqrt{-1}$ in the equation is very important. Replacing $\sqrt{-1}$ with 1, we obtain a threefold with only 18 nodes.

We can rewrite in the new coordinates the coordinates of singular points and the equations of the 24 planes.

The set $\mathrm{Sing}(\mathfrak{G}) = \mathrm{Sing}(\mathfrak{G})_1 \cup \mathrm{Sing}(\mathfrak{G})_2$, where the subset $\mathrm{Sing}(\mathfrak{G})_1$ consists of 18 points

$$
\begin{aligned}
&1\colon [1,0,0,-i,0,0], \; 2\colon [0,1,0,0,-i,0], \; 3\colon [0,0,1,0,0,-i], \\
&\quad 4\colon [0,0,1,0,0,i], \; 5\colon [0,1,0,0,i,0], \; 6\colon [1,0,0,i,0,0], \\
&7\colon [1,0,0,0,i,0], 8\colon [0,1,0,i,0,0], 9\colon [0,1,0,-i,0,0], 10\colon [1,0,0,0,-i,0], \\
&11\colon [0,0,1,-i,0,0], 12\colon [1,0,0,0,0,-i], \; 13\colon [1,0,0,0,0,i], \; 14\colon [0,0,1,i,0,0], \\
&15\colon [0,0,1,0,i,0], 16\colon [0,1,0,0,0,i], 17\colon [0,1,0,0,0,-i], 18\colon [0,0,1,0,-i,0].
\end{aligned}
\tag{13}
$$

and $\mathrm{Sing}(\mathfrak{G})_2$ consists of 16 points

$$
[\epsilon_1, \epsilon_2, \epsilon_3, e_1, e_2, e_3], \tag{14}
$$

where $\epsilon_i^2 = -1$, $e_i^2 = 1$, and $\epsilon_1\epsilon_2\epsilon_3 + ie_1e_2e_3 = 0$.

The equations of planes become

$$
V(x_1 - \epsilon_1\sqrt{-1}y_{\sigma(1)}, x_2 - \epsilon_2\sqrt{-1}y_{\sigma(2)}, x_3 - \epsilon_3\sqrt{-1}y_{\sigma(3)}), \tag{15}
$$

where $\epsilon_i = \pm 1$, and $\epsilon_1\epsilon_2\epsilon_3 = 1$.

We can identify the equations with permutations $g \in \mathfrak{S}_4$ as follows. The vector $(\epsilon_1, \epsilon_2, \epsilon_3)$ is identified with elements from the normal subgroup 2^2 of $\mathfrak{S}_4$. For example, if we let $a = (14)(23), b = (13)(24), c = (12)(34)$, then

$$
(1,1,1) \leftrightarrow 1, \; (1,-1,-1) \leftrightarrow a, \; (-1,1,-1) \leftrightarrow b, (-1,-1,1) \leftrightarrow c.
$$

Now, we can write elements of $\mathfrak{S}_4$ as the products of elements from 2^2 and elements from $\mathfrak{S}_3$ generated by $(12), (23)$. The elements from $\mathfrak{S}_4 \setminus \mathfrak{S}_3$ are

$$
\begin{aligned}
&(14) = (23)a, \; (24) = (13)b, \; (34) = (12)c, \; (1234) = (24)a, \; (1243) = (23)b, \\
&(1342) = (23)c, \; (1324) = (34)a, \; (1423) = (34)b, \; (1432) = (24)c, \\
&(124) = (132)b, \; (142) = (123)a, \; (234) = (132)c, \\
&(243) = (123)b, \; (134) = (123)c, \; (143) = (132)a.
\end{aligned}
$$

The set of α-planes now becomes

$$
\begin{aligned}
&1\colon (12)(34), \quad 2\colon (13)(24), \quad 3\colon (14)(23), \quad 4\colon \quad 1, \quad 5\colon (142), \quad 6\colon (132), \\
&7\colon (123), \quad 8\colon \quad (124), \quad 9\colon (143), \quad 10\colon \quad (243), \quad 11\colon (234), \quad 12\colon (134).
\end{aligned}
$$

The set of β-planes becomes:

$$
\begin{aligned}
&1\colon (1342), \quad 2\colon (1243), \quad 3\colon (1432), \quad 4\colon (1234), \quad 5\colon (1423), \quad 6\colon (1324), \\
&7\colon (12), \quad 8\colon \quad (34), \quad 9\colon (24), \quad 10\colon \quad (13), \quad 11\colon (23), \quad 12\colon \quad (14).
\end{aligned}
$$

Let us now look at the incidence relation between 34 singular points and 24 planes.

Let H_1, H_2, H_3 be subgroups of $\mathfrak{S}_4$ generated by two commutating transpositions $\{(12), (34)\}$, $\{(13), (24)\}$, $\{(14), (23)\}$, respectively. Then we can identify 18 left cosets of H_1, H_2, H_3 and 18 points from $\mathrm{Sing}(\mathfrak{G})_1$, and the incidence graph of $(24_3, 18_4)$ is the same as that of 24 elements of $\mathfrak{S}_4$ and 18 cosets.

For example, the point $(i, 0, 0, 0, 0, 1)$ from $\mathrm{Sing}(\mathfrak{G})_1$ is contained in four planes $V(x_1 - iy_3, x_2 \pm iy_1, x_3 \pm iy_2)$, and $V(x_1 - iy_3, x_2 \pm iy_2, x_3 \pm iy_1)$. They correspond to the permutations $(143), (132)$ and $(1432), (13)$, respectively. The four permutations form one coset of the subgroup H_1.

The abstract configuration $(24_3, 18_4)$ defined by the incidence relation between planes and points is the configuration C_3 from [12, Figure 1.7.4] as in Figure 3.

FIGURE 3. The configuration $(24_3, 18_4)$

The incidence graph of the configuration $(24_4, 16_6)$ coincides with the incidence graph of elements of a 4×4 matrix (a_{ij}) and monomials $a_{1j_1} a_{2j_2} a_{3j_3} a_{4j_4}$ entering the expression of the determinant of the matrix. For example, let us identify (a_{ij}) with $\mathrm{Sing}(\mathfrak{G})_2$ given in (10) as follows:

$$
(a_{ij}) = \begin{pmatrix} 1 & 14 & 12 & 7 \\ 15 & 2 & 5 & 10 \\ 9 & 8 & 3 & 16 \\ 6 & 11 & 13 & 4 \end{pmatrix}.
$$

It is immediately checked that each plane contains 4 points from $\mathrm{Sing}(\mathfrak{G})_2$ and each point from $\mathrm{Sing}(\mathfrak{G})_2$ is contained in 6 planes. We say that the configuration of planes and points is of type $(24_{3+4}, 18_4 + 16_6)$ meaning that each plane contains 3 points from the set $\mathrm{Sing}(\mathfrak{G})_1$ and 4 points from the set $\mathrm{Sing}(\mathfrak{G})_2$. Also, each point from $\mathrm{Sing}(\mathfrak{G})_1$ is contained in 4 planes, and each point of the set $\mathrm{Sing}(\mathfrak{G})_2$ is contained in 6 planes.

The configuration $(24_{3+4}, 18_4 + 16_6)$ is the union of two configurations $(24_3, 18_4)$ and $(24_4, 16_6)$. In both configurations, we can identify the set of 24 elements with the set of permutations $\mathfrak{S}_4$.

Remark 4.9. We believe that the number 34 is the maximal possible number of ordinary nodes on a $\mathbb{Q}$-Fano 3-fold X of degree 6. The reason for this belief is similar to Todd's argument for believing that 45 is the maximal possible number of ordinary nodes on a 3-fold of degree 4 [37]. The proof of the latter is rather involved [4]. Here is Todd's argument. The dimension of the space of complete intersections of a quadric and a cubic in $\mathbb{P}^5$ modulo projective automorphisms is

equal to $49 - 15 = 34$. If we find an orbit represented by a sextic 3-fold with more than 34 nodes, the stabilizer subgroup of the group of projective transformations would be an infinite group. However, the group of symmetries of a set of so many points could be infinite only if the points are in a special linear position that implies that there are also non-isolated singular points.

Another argument supporting our conjecture was communicated to us by Ivan Cheltsov. According to [26], the number of nodes is bounded by $20 + h^{1,2}(X_t) - \rho(X_t)$, where X_t is a general smoothing of X. For $\mathbb{Q}$-Fano 3-folds, $h^{1,2}(X_t) = 20$ and the Picard number is equal to 1. So, the number is bounded by 39, which is rather close to 34.

The new coordinates exhibit an obvious group of projective symmetries of $\mathfrak{G}$ isomorphic to $2^4 : (\mathfrak{S}_3 \times \mathfrak{S}_3) \cong \mathfrak{S}_4 \times \mathfrak{S}_4$.

Proposition 4.10. *Assume $p \neq 2$. The group of projective automorphisms of the cubic line complex $\mathfrak{G}$ is isomorphic to the group $G := (\mathfrak{S}_4 \times \mathfrak{S}_4) \rtimes 2$. The additional generator is a projective automorphism of order 2 defined by*

$$g_0 : (x_1, x_2, x_3, y_1, y_2, y_3) \mapsto (-y_1, -y_2, -y_3, x_1, x_2, x_3).$$

The group G has two orbits $\mathrm{Sing}(\mathfrak{G})_1$ and $\mathrm{Sing}(\mathfrak{G})_2$ on the set $\mathrm{Sing}(\mathfrak{G})$ and one orbit on the set of planes in $\mathfrak{G}$.

Proof. We immediately check that g_0 is a projective automorphism of $\mathfrak{G}$ of order 4 and that the subgroup G_1 of $\mathrm{Aut}_{\mathrm{proj}}(\mathfrak{G})$ generated by g_0 and the obvious subgroup $\mathfrak{S}_4 \times \mathfrak{S}_4$ has two orbits on the set $\mathrm{Sing}(\mathfrak{G})$ and one orbit on the set of planes.

Let $G = \mathrm{Aut}_{\mathrm{proj}}(\mathfrak{G})$. Suppose $G \neq G_1$. Since G_1 acts simply transitively on the set of planes, it suffices to see that the stabilizer subgroup $H \subset G$ of a plane $\Pi \in \mathcal{P}$ is isomorphic to $\mathfrak{S}_4$. It leaves invariant the subset of 3 points from $\mathrm{Sing}(\mathfrak{G})_1$ and 4 points from $\mathrm{Sing}(\mathfrak{G})_2$. For example, we may assume that the plane is $\Pi = V(x_1 + iy_1, x_2 + iy_2, x_3 - iy_3)$ and the points are

$$[0, 0, i, 0, 0, 1], \ [0, -i, 0, 0, 1, 0], \ [-i, 0, 0, 1, 0, 0]$$

from the set $\mathrm{Sing}(\mathfrak{G})_1$ and

$$[i, i, i, -1, -1, 1], \ [i, -i, i, -1, 1, 1], \ [i, i, -i, -1, -1, -1], \ [i, -i, -i, -1, 1, -1]$$

from the set $\mathrm{Sing}(\mathfrak{G})_2$. In the coordinates y_1, y_2, y_3, the seven points form the set of vertices of a complete quadrangle. Its group of symmetries is isomorphic to $\mathfrak{S}_4$. This proves the assertion. □

Consider the projection from $P = [0, 0, 0, 0, 0, 1]$. By putting $x_1 = p_{12}, x_2 = p_{13}, x_3 = p_{14}, x_4 = p_{23}, x_5 = p_{24}, x_6 = p_{34}$ and substituting $x_1 x_6 = x_2 x_5 - x_3 x_4$ into (8), we obtain the quartic 3-fold

$$X : x_1^2 x_2 x_4 - x_1^2 x_3 x_5 + x_2^2 x_3 x_5 - x_2 x_3^2 x_4 + x_3 x_4^2 x_5 - x_2 x_4 x_5^2 = 0. \quad (16)$$

We can rewrite the equation in the form

$$x_1^2(x_2 x_4 - x_3 x_5) + (x_2 x_3 - x_4 x_5)(x_2 x_5 - x_3 x_4) = 0.$$

Let

$$Q_1 = V(x_2x_4 - x_3x_5), \quad Q_2 = V(x_2x_3 - x_4x_5), \quad Q_3 = V(x_2x_5 - x_3x_4).$$

The projection from the node $q = [1, 0, 0, 0, 0]$ of X defines a degree two map

$$\phi : \mathrm{Bl}_q(X) \to V(x_1) \cong \mathbb{P}^3$$

branched along the union $Q_1 \cup Q_2 \cup Q_3$.

The union of the quadrics $Q_2 \cup Q_3$ is equal to the hyperplane section $V(x_1) \cap X$. They intersect along a quadrangle of lines

$$V(x_1, x_2, x_4), \; V(x_1, x_3, x_5), \; V(x_1, x_2 - x_4, x_3 - x_5), \; V(x_1, x_2 + x_4, x_3 + x_5),$$

which are singular lines on X. The intersections $Q_1 \cap Q_2$ and $Q_1 \cap Q_3$ is also equal to the union of four lines.

The quartic threefold X also has 17 isolated ordinary nodes lying outside of the hyperplane $V(x_1)$.

$$
\begin{aligned}
&[1,0,0,0,0], \; [1,0,0,1,1], \; [1,0,0,-1,1], \; [1,1,0,0,1], \; [1,-1,0,0,1], \\
&[1,0,0,1,-1], \; [1,0,0,-1,-1], \; [1,1,0,0,-1], \; [1,-1,0,0,-1], \; [1,0,1,1,0] \\
&[1,0,1,-1,0], \; [1,1,1,0,0], \; [1,-1,1,0,0], \; [1,0,-1,1,0], \; [1,0,-1,-1,0] \\
&[1,1,-1,0,0], \; [1,-1,-1,0,0].
\end{aligned}
\tag{17}
$$

Note that sixteen of the nodes on $\mathfrak{G}$ are projected to the points on four singular lines of X. That explains the number $17 = 33 - 16$. Under the projection ϕ from the first node $[1, 0, 0, 0, 0]$, the remaining sixteen nodes are paired into 8 pairs, and each pair is projected by one of the singular points of $Q_1 \cap Q_3$ or $Q_1 \cap Q_3$.

Theorem 4.11. *The cubic complex $\mathfrak{G}$ is a rational 3-fold.*

Proof. It suffices to prove that the quartic 3-fold X is rational. The proof is similar to that of the Burkhardt quartic 3-fold [1, §6] (or see [16, §5.2.7]).

Consider the following three planes contained in X:

$$\Pi_1 = V(x_2, x_3), \; \Pi_2 = V(x_4, x_5), \; \Pi_3 = V(x_1 - x_2 + x_4, x_1 - x_3 + x_5).$$

Note that

$$\Pi_1 \cap \Pi_2 = [1,0,0,0,0], \; \Pi_2 \cap \Pi_3 = [1,1,1,0,0], \; \Pi_3 \cap \Pi_1 = [1,0,0,-1,-1].$$

Let $P \in X$ be a general point. Then there exists a *unique* line L meeting three planes Π_1, Π_2, Π_3 and passing through P, that is,

$$L = \langle P, \Pi_1 \rangle \cap \langle P, \Pi_2 \rangle \cap \langle P, \Pi_3 \rangle,$$

where $\langle P, \Pi_i \rangle$ is the hyperplane generated by P and Π_i. Let H be a hyperplane in $\mathbb{P}^4$. Then we have a rational map from X to H by sending P to $L \cap H$ which is obviously birational. $\square$

Remark 4.12. A general hyperplane section of the Humbert desmic cubic line complex is a K3 surface of degree 6 in $\mathbb{P}^4$. It contains two sets of twelve disjoint lines forming the configuration $(12_6, 12_6)$. The geometry of these surfaces is studied in great detail in [5].

5. CUBIC SURFACES AND DESMIC QUARTIC SURFACES

Let F be a smooth cubic surface and let $\ell_1 + \ell_2 + \ell_3$ be one of the triangles of lines on F cut out by one of the 45 tritangent planes.

Proposition 5.1. (L. Cremona) *Let K_1, K_2, K_3 be the residual conics for planes Π_1, Π_2, Π_3 from the pencils of planes through the lines ℓ_1, ℓ_2, ℓ_3, respectively. Then $K_1 + K_2 + K_3$ is cut out by a quadric $G(\Pi_1, \Pi_2, \Pi_3)$. The locus of singular points of quadrics $G(\Pi_1, \Pi_2, \Pi_3)$ is a quartic surface in $\mathbb{P}^3$.*

Proof. Choose a geometric basis $(e_0, e_1, \dots, e_6)$ in $\operatorname{Pic}(F)$ such that $[\ell_1] = e_0 - e_1 - e_2$, $[\ell_2] = e_0 - e_3 - e_4$, $[\ell_3] = e_0 - e_5 - e_6$. Then, the conics belong to $2e_0 - (e_1 + \cdots + e_6) + e_1 + e_2$, etc. This implies that the sum is equal to $6e_0 - 2(e_1 + \cdots + e_6) = -2K_F$. This proves the first assertion.

Take a general line ℓ in $\mathbb{P}^3$. For each point $x \in \ell$, there exists a unique plane $\Pi_1(x), \Pi_2(x), \Pi_3(x)$ from each pencil of planes that intersects ℓ at x. This allows us to identify the three pencils of planes with the line ℓ. Consider the map

$$\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \to |\mathcal{O}_{\mathbb{P}^3}(2)|, \quad (\Pi_1, \Pi_2, \Pi_3) \mapsto G(\Pi_1, \Pi_2, \Pi_3).$$

Obviously, the map factors through $(\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1)/\mathfrak{S}_3 \cong \mathbb{P}^3$ and defines a linear map

$$f : \mathbb{P}^3 \to |\mathcal{O}_{\mathbb{P}^3}(2)|.$$

Its image is a web of quadrics in $\mathbb{P}^3$. It is known that its Steinerian surface is a quartic surface [8, 1.1.6]. This proves the second assertion. $\square$

We call the family of quadrics from the proposition the *Cremona family of quadrics* associated with a tritangent plane.

Explicitly, fix a tritangent plane $w = 0$ intersecting F along the lines $\ell_1 = V(x, w), \ell_2 = V(y, w), \ell_3 = V(z, w)$. The equation of F can be written in the form

$$q(x, y, z, w)w + xyz = 0, \tag{18}$$

where q is a quadratic form. The pencil of planes through the line ℓ_1, ℓ_2 or ℓ_3 is given by $V(x - \alpha w), V(y - \beta w)$ or $V(z - \gamma w)$, respectively. Plugging in the equation of F, we find that the conics are $V(q + \alpha yz, x - \alpha w), V(q + \beta xz, y - \beta w)$, and $V(q + \gamma xy, z - \gamma w)$. We check that the conics lie in the quadric

$$q + \alpha yz + \beta xz + \gamma xy - (\alpha\beta z + \alpha\gamma y + \beta\gamma x)w + \alpha\beta\gamma w^2 = 0. \tag{19}$$

By homogenizing, we see that its coefficients are tri-linear functions of (α_0, α_1), $(\beta_0, \beta_1), (\gamma_0, \gamma_1)$. The locus of singular points of these quadrics is given by equations

$$\begin{aligned}
q_x + \beta z + \gamma y - \beta\gamma w &= 0, \\
q_y + \alpha z + \gamma x - \alpha\gamma w &= 0, \\
q_z + \beta x + \alpha y - \alpha\beta w &= 0, \\
q_w - \alpha\beta z - \alpha\gamma y - \beta\gamma x + 2\alpha\beta\gamma w &= 0.
\end{aligned} \tag{20}$$

The following equation of the Steinerian surface was found by Humbert:

$$G = q^2 - q_y q_z yz - q_x q_z xz - q_x q_y xy - q_x q_y q_z w + xyz q_w = 0. \tag{21}$$

It can be checked by using the equations (19) and (20).

We can find a normal form for a cubic surface (18) with a fixed tritangent plane. The only admissible transformations are $(x, y, z, w) \mapsto (\lambda x + c_1 w, \mu y + c_2 w, \gamma z + c_3 w, \delta w)$, where $\lambda \mu \gamma \delta = 1$. Using these transformations, we may assume that

$$((aw + bx + cy + dz)w + x^2 + y^2 + z^2)w + xyz = 0. \tag{22}$$

This agrees with the fact that cubic surfaces depend on four parameters. This gives the equation of Cremona's quartic surface

$$
\begin{aligned}
& - (bw + 2x)(cw + 2y)(dw + 2z)w - (bw + 2x)(cw + 2y)xy \\
& - (bw + 2x)(dw + 2z)xz - (cw + 2y)(dw + 2z)yz + (2aw + bx + cy + dz)xyz \\
& + (aw^2 + bwx + cwy + dwz + x^2 + y^2 + z^2)^2 = 0.
\end{aligned}
\tag{23}
$$

Substituting the expression for q_x, q_y, q_z, q_w in equations (20), we find the equation of the discriminant surface of the family of quadrics (19):

$$
\det \begin{pmatrix}
2\alpha_0\beta_0\gamma_0 & \alpha_0\beta_0\gamma_1 & \alpha_0\beta_1\gamma_0 & -\alpha_0\beta_1\gamma_1 + b\alpha_0\beta_0\gamma_0 \\
\alpha_0\beta_0\gamma_1 & 2\alpha_0\beta_0\gamma_0 & \alpha_1\beta_0\gamma_0 & -\alpha_1\beta_0\gamma_1 + c\alpha_0\beta_0\gamma_0 \\
\alpha_0\beta_1\gamma_0 & \alpha_1\beta_0\gamma_0 & 2\alpha_0\beta_0\gamma_0 & -\alpha_1\beta_1\gamma_0 + d\alpha_0\beta_0\gamma_0 \\
-\alpha_0\beta_1\gamma_1 + b\alpha_0\beta_0\gamma_0 & -\alpha_1\beta_0\gamma_1 + c\alpha_0\beta_0\gamma_0 & -\alpha_1\beta_1\gamma_0 + d\alpha_0\beta_0\gamma_0 & 2(a\alpha_0\beta_0\gamma_0 + \alpha_1\beta_1\gamma_1)
\end{pmatrix} = 0.
\tag{24}
$$

Here, we homogenized the affine coordinates α, β, γ in $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$. The determinantal equation is given by a multi-homogeneous form of multi-degree $(4, 4, 4)$, which is symmetric with respect to permuting the factors of $(\mathbb{P}^1)^3$. After we identify $(\mathbb{P}^1)^3/\mathfrak{S}_3$ with $\mathbb{P}^3$, we obtain a symmetric determinant whose entries are linear functions in homogenous coordinates. This is the discriminant of Cremona's web of quadrics. Its Steinerian surface is given by (23).

The following theorem is due to Cremona [3, p. 91] and Steiner [34, p. 80]. We give a proof following Humbert [15, Part IV].

Theorem 5.2. *The Steinerian quartic surface S has 12 nodes and 16 lines forming a configuration* $(12_4, 16_3)$.

Proof. Fix a tritangent plane $\Pi_0 = V(w)$ as above. Through each line ℓ_i passes 5 tritangent planes including Π_0. This gives $24 = 12 \times 2$ lines in these planes not counting ℓ_1, ℓ_2, ℓ_3. There are $45 - 12 - 1 = 32$ other tritangent planes. They are distributed in 16 pairs. Each pair contains six of the lines from the set of 24 planes and defines a reducible quadric from the Cremona family of quadrics [3, p. 91].

For example, in the standard Cremona's notation

$$a_i = e_i, \quad b_j = 2e_0 - e_1 - \cdots - e_6 + e_j, \quad c_{ij} = e_0 - e_i - e_j$$

of 27 lines on a cubic surface (see [8, 9.1.1]), take the tritangent plane T with lines $\{c_{12}, c_{34}, c_{56}\}$. Consider three tritangent planes T_1, T_2, T_3 with lines $\{c_{12}, c_{35}, c_{46}\}$, $\{c_{16}, c_{25}, c_{34}\}$, $\{c_{14}, c_{23}, c_{56}\}$, respectively. Each contains one of the lines from T. The pair of tritangent planes T_4, T_5 with lines $\{c_{13}, c_{26}, c_{45}\}$, $\{c_{15}, c_{24}, c_{36}\}$ intersect each of the planes T_1, T_2, T_3 along a line. The union of these tritangent planes is a reducible quadric that cuts out in each plane T_1, T_2, T_3 a reducible conic. Note

that the tritangent planes T, T_4, T_5 form one of the conjugate triads of tritangent planes

$$\begin{array}{ccc} c_{12} & c_{34} & c_{56} \\ c_{36} & c_{15} & c_{24} \\ c_{45} & c_{26} & c_{13}. \end{array}$$

The tritangent planes T_1, T_2, T_3 are formed by the remaining $15 - 9 = 6$ lines c_{ij}. Similarly, we find other conjugate triads with the top row c_{12}, c_{34}, c_{56}, for example

$$\begin{array}{ccc} c_{12} & c_{34} & c_{56} \\ a_1 & b_3 & c_{13} \\ b_2 & a_4 & c_{24}. \end{array}$$

The tritangent planes defined by the last two rows define a reducible quadric that cuts out line-pairs in the tritangent planes with lines $\{c_{12}, a_3, b_1\}$, $\{c_{34}, a_3, b_4\}$, $\{c_{56}, c_{14}, c_{23}\}$. Proceeding in this way, we find 16 conjugate triads of tritangent planes with the top row $\{c_{12}, c_{34}, c_{56}\}$ that give us 16 irreducible quadrics from the Cremona family.

To find 12 double points, we use (21). Let $f := qw + xyz = 0$ be the equation of F. We find

$$f^2 - f_x f_y f_z = G w^2.$$

The tangent plane to F at a point $[x_0, y_0, z_0, w_0]$ is given by equation

$$f_x(x_0, \dots, w_0) x + \cdots + f_w(x_0, \dots, w_0) w = 0.$$

This implies that the intersection points $V(f, f_x, f_y)$ are the points where the lines $V(t_0 z + t_1 w)$ are tangent to F. Each of the sixteen lines from above intersects the cubic surface at three points. The points are the intersection points with the three lines lying in each irreducible component of the plane-pair intersecting along this line. This irreducible component is a tritangent plane containing one of the lines ℓ_1, ℓ_2, ℓ_3, for example, the line $V(z, w)$. It follows that these points belong to $V(f, f_x, f_y)$, and hence, they are double points of the Steinerian quartic Q. $\qquad\square$

The following theorem is due to Humbert. We give it another proof.

Theorem 5.3. *The quartic surface Q associated with a tritangent plane of a smooth cubic surface F is isomorphic to a desmic quartic surface.*

Proof. We found the same configuration $(12_4, 16_3)$ of nodes and lines on the Steinerian quartic. Now, we follow the proof of Theorem 3.3 to show that it is isomorphic to the Kumemr surface. $\qquad\square$

Here are the immediate questions:

- A cubic surface admits 45 tritangent planes. Are the corresponding desmic quartic surfaces isomorphic?
- The construction assigns to a cubic surface and a tritangent plane (F, T) on it the isomorphism class of an elliptic curve. Is there a more direct way to see this association?

- A choice of a tritangent plane assigns a set of 12 points on the cubic surface. Is there another way to see this association?
- What are the fibers of the map from the moduli space of cubic surfaces with a marked tritangent plane to the moduli space of elliptic curves?
- What happens with the Cremona's quartic surface if the tritangent plane contains an Eckardt point?

6. CREMONA'S QUARTIC SURFACES IN CHARACTERISTIC TWO

Starting from this section, we assume that the characteristic of $\Bbbk$ is equal to 2. Let us rewrite the equation of Cremona's quartic surface from (23) in characteristic 2. The equation of the cubic surface (18) still holds. The discriminant surface of the web of quadric is given now by the pfaffian of the alternating matrix:

$$\mathrm{Pf} \begin{pmatrix} 0 & \alpha_0\beta_0\gamma_1 & \alpha_0\beta_1\gamma_0 & \alpha_0\beta_1\gamma_1+b\alpha_0\beta_0\gamma_0 \\ \alpha_0\beta_0\gamma_1 & & \alpha_1\beta_0\gamma_0 & \alpha_1\beta_0\gamma_1+c\alpha_0\beta_0\gamma_0 \\ \alpha_0\beta_1\gamma_0 & \alpha_1\beta_0\gamma_0 & 0 & \alpha_1\beta_1\gamma_0+d\alpha_0\beta_0\gamma_0 \\ \alpha_0\beta_1\gamma_1+b\alpha_0\beta_0\gamma_0 & \alpha_1\beta_0\gamma_1+c\alpha_0\beta_0\gamma_0 & \alpha_1\beta_1\gamma_0+d\alpha_0\beta_0\gamma_0 & 0 \end{pmatrix} = 0. \quad (25)$$

It is a multi-linear form of multi-degree $(2, 2, 2)$. So, it becomes a quadric in $(\mathbb{P}^1)^3/\mathfrak{S}_3 = \mathbb{P}^3$. The Humbert equation of Cremona's quartic still holds. It expresses the condition that the partial derivatives of a quadric from the web vanishes and also the quadric vanishes at the same point. From (23), we obtain the equation

$$F = bcdw^4 + bcw^2xy + bdw^2xz + cdw^2yz + (bx + cy + dz)xyz \qquad (26)$$
$$+ (aw^2 + bwx + cwy + dwz + x^2 + y^2 + z^2)^2 = 0.$$

The proof that the surface $V(F)$ contains 16 lines does not depend on the characteristic. Let us find the singularities. We have

$$F'_x = (cy+dz)(yz+bw^2), \ F'_y = (bx+dz)(xz+cw^2), \ F'_z = (bx+cy)(xy+dw^2), \qquad (27)$$

and $F'_w = 0$. One checks that, for a general parameter, no singular points lie in the plane $w = 0$. So, we may assume that $w = 1$ and find the following singular points. Four points

$$[dz_1^2, b^2, bz_1^2, bz_1], \quad b^2dz_1^3 + b^2z_1^4 + d^2z_1^4 + ab^2z_1^2 + b^3cz_1 + b^4 = 0.$$

Four points

$$[c^2, dz_2, cz_2^2, cz_2], \quad c^2dz_2^3 + c^2z_2^4 + d^2z_2^4 + ac^2z_2^2 + bc^3z_2 + c^4 = 0.$$

Four points

$$[c, b, z_3^2, z_3], \quad dz_3^3 + z_3^4 + az_3^2 + bcz_3 + b^2 + c^2 = 0.$$

There is also an additional singular point in characteristic 2:

$$P_0 = [cdz_0, bdz_0, bcz_0, bc], \quad z_0^4 = \frac{(b^5c^5d + a^2b^4c^4)}{(b^3c^3d^3 + b^4c^4 + b^4d^4 + c^4d^4)}.$$

The line connecting a point from the first group of four singular points with a point from the second group also contains one of the singular points from the third group and belongs to the surface. This defines a Reye configuration $(12_4, 16_3)$ formed by the singular points and sixteen lines. So, we may keep the name desmic surface

for Cremona's quartic $V(F)$, although no desmic pencil exists in characteristic two. Note that a minimal nonsingular model of the surface contains 16 disjoint (-2)-curves, and hence, it is a supersingular K3 surface [32].

Let us determine the type of the singular point P_0. For special parameters $(a, b, c, d) = (0, 0, 1, 1)$ for which the cubic surface is smooth, we checked directly that P_0 is a rational double point of type A_3.

Let us show that this is always true if we assume that P_0 does not lie on any of the sixteeen lines on the surface. We will show later that a K3 surface containing a $(12_4, 16_3)$-configuration is a supersingular K3 surface X with Artin invariant $\sigma \leq 2$ (see Theorem 8.2). In this case, the lattice L generated by twenty-eight (-2)-curves forming $(12_4, 16_3)$-configuration can be primitively embedded in the Picard lattice $S_\sigma = \mathrm{Pic}(X)$. Recall that $L = U \oplus D_8 \oplus D_9 \cong U \oplus D_5 \oplus D_{12}$ (Proposition 3.5). In case $\sigma = 1$, $S_\sigma = U \oplus E_8 \oplus D_{12}$ is an over lattice of the orthogonal direct sum of $L = U \oplus D_5 \oplus D_{12}$ and A_3 (E_8 is an overlattice of $D_5 \oplus A_3$). In case $\sigma = 2$, $S_\sigma = U \oplus D_8 \oplus D_{12}$ is an over lattice of the orthogonal direct sum of $L = U \oplus D_5 \oplus D_{12}$ and A_3 (D_8 is an overlattice of $D_5 \oplus A_3$ which is the intermediate one: $D_5 \oplus A_3 \subset D_8 \subset E_8$). The linear system (6) gives a quartic model which contracts twelve disjoint (-2)-curves and three (-2)-curves forming a root lattice of type A_3. Thus, the additional singularity is of type A_3.

7. Kummer surfaces $\mathrm{Kum}(E \times E)$ in characteristic two

In this section, we discuss what happens for Kummer surfaces in characteristic two, and in the next section, we will locate the Reye configuration $(12_4, 16_3)$ among 42 smooth rational curves on the supersingular K3 surfacw with Artin invariant $\sigma = 1$ studied in [10].

Let E be an ordinary elliptic curve in characteristic two and $\iota : E \to E$ the inversion map. Denote by $\{0, a\}$ the 2-torsion points of E. Let

$$E_1 = E \times \{0\}, \; E_2 = E \times \{a\}, \; F_1 = \{0\} \times E, \; F_2 = \{a\} \times E,$$

and let Δ_1 be the diagonal and Δ_2 the translation of Δ_1 by a 2-torsion $(0, a)$. The quotient surface $\mathrm{Kum}(E \times E)$ of $E \times E$ by the inversion involution $\iota \times \iota$ has four singular points of type D_4 [33]. Its minimal resolution $\widetilde{\mathrm{Kum}}(E \times E)$ has the following 22 (-2)-curves as in Figure 4:

Here, $\overline{E}_i, \overline{F}_i, \overline{\Delta}_i$ are the images of E_i, F_i, Δ_i and $\{E_i^j\}_{j=0,1,2,3}$ are the exceptional curves over a singular point of type D_4 $(i = 1, 2, 3, 4)$.

Recall that a desmic surface in characteristic $p \neq 2$ is obtained from $\widetilde{\mathrm{Kum}}(E \times E)$ by contracting twelve (-2)-curves (Remark 3.4). Let us consider its analog in characteristic two. Let p_1, p_2 be the first and second projections and let $p_3 : E \times E \to E$, $(x, y) \to (x + y, x + y)$. Each p_i induces an elliptic fibration $\pi_i : \widetilde{\mathrm{Kum}}(E \times E) \to \mathbb{P}^1$ which has two singular fibers of type $\tilde{D}_8$ (type I_4^* in the sence of Kodaira). We denote by $\mathcal{F}_i$ a general fiber of π_i.

FIGURE 4. The dual graph: ordinary case

Now, consider the divisor

$$H = \mathcal{F}_1 + \mathcal{F}_2 + \mathcal{F}_3 - \sum_{i=1}^{4}\sum_{j=1}^{3} E_i^j - 2\sum_{i=1}^{4} E_i^0.$$

It is easy to see that $H^2 = 4, H \cdot E_i^0 = 1$ ($i = 1, 2, 3, 4$) and other curves in Figure 4 are orthogonal to H. Thus, the linear system $|H|$ gives a birational morphism from $\widetilde{\mathrm{Kum}}(E \times E)$ to a quartic surface S with six rational double points of type A_3 (= the images of 18 (-2)-curves in Figure 4 except E_i^0). Let $\overline{E}_i^0$ be the images of E_i^0, all of which are lines contained in a plane, and their six intersection points are rational double points of S of type A_3.

Conversely, let

$$F_\alpha = xyz(x + y + z + w) + \alpha w^4, \ \alpha \neq 0 \in k.$$

Then, $Q_\alpha = V(F_\alpha)$ contains four lines, irreducible components of $V(xyz(x+y+z))$ in the plane $V(w)$ and has six singular points at the six intersection points of four lines:

$$[1, 0, 0, 0], \ [0, 1, 0, 0], \ [0, 0, 1, 0], \ [1, 1, 0, 0], \ [1, 0, 1, 0], \ [0, 1, 1, 0].$$

Since each singular point is formal-analytically given by $t^4 = uv$, it is a rational double point of type A_3. Thus, the four lines and six singular points form a $(6_2, 4_3)$-configuration. Contrary to the case of characteristic not equal to 2, only two (degenerate) tetrahedrons $V(xyz(x+y+z+w)), V(w^4)$ appear in the pencil $\{Q_\alpha\}_{\alpha \in \mathbb{P}^1}$.

Let X_α be the minimal resolution of singularities of Q_α. Then X_α contains twenty-eight (-2)-curves forming the dual graph in Figure 4. It follows from [32, Theorem 3.4] that X_α is birationally isomorphic to a Kummer surface, that is, there exists an abelian surface A with $X_\alpha = \widetilde{\mathrm{Kum}}(A)$. The existence of three elliptic fibrations with two singular fibers of type $\tilde{D}_8$ as π_1, π_2, π_3 implies that $A \cong E \times E$ for an ordinary elliptic curve E. We now conclude:

Theorem 7.1. *The quartic surface $Q_\alpha = V(F_\alpha)$ is birationally isomorphic to the Kummer surface $\mathrm{Kum}(E \times E)$ for an ordinary elliptic curve E.*

Remark 7.2. A pencil of planes containing one of the irreducible components of $V(xyz(x + y + z))$ induces an elliptic fibration on X_α with singular fibers of type $\tilde{E}_6$ and of type $\tilde{A}_{11}$.

Remark 7.3. The automorphism group of the Kummer surface $\mathrm{Kum}(E \times E)$ is calculated in [20].

In the case where E is supersingular, the equation of the quotient $E \times E / \langle \iota \times \iota \rangle$ is given by
$$z^2 + x^2 y^2 z + x^4 y + xy^4 = 0,$$
and it has an elliptic singularity of type $\textcircled{19}_0$ in the notation from [38] (see [18]). The dual graph of its minimal resolution is given in Figure 5. The central component has multiplicity 2 and self-intersection number -3. Other components are (-2)-curves.

$$\textcircled{19}_0 :$$

FIGURE 5. The dual graph of a minimal resolution of singularity

We do not know how to find a family of quartic surfaces that contains the Kummer surfaces of self-products of ordinary and supersingular elliptic curves.

8. SUPERSINGULAR K3 SURFACES IN CHARACTERISTIC TWO

In this section, we first show that the supersingular K3 surface X with Artin invariant 1 in characteristic two has twenty-eight (-2)-curves forming $(12_4, 16_3)$-configuration explicitly. Next we prove that a supersingular K3 surface with Artin invariant σ in characteristic two has such (-2)-curves if and only if $\sigma \leq 2$.

First, let us recall that X is obtained as the minimal resolution of an inseparable double covering $\bar{X} : w^2 = xyz(x^3 + y^3 + z^3)$ of $\mathbb{P}^2$ [10]. The covering surface $\bar{X}$ has 21 nodes over 21 points from $\mathbb{P}^2(\mathbb{F}_4) \subset \mathbb{P}^2(\Bbbk)$. Thus X contains two families of disjoint 21 (-2)-curves. One of them consists of the exceptional curves and the other the proper transforms of 21 lines on $\mathbb{P}^2(\mathbb{F}_4)$. Each member of one family meets exactly five members in other family, that is, they form a (21_5)-configuration.

Next, we recall Sylvester's duads and synthemes (see [8, 9.4.3]). First of all, we fix a set of six letters $\{1, 2, \cdots, 6\}$. We denote by ij the pair of i and j ($1 \leq i \neq j \leq 6$) which is classically called *Sylvester's duad*. Six letters $1, 2, 3, 4, 5, 6$ can be arranged in three pairs of duads, for example, 12.34.56, called *Sylvester's syntheme*. (It is understood that 12.34.56 is the same as 12.56.34 or 34.12.56). Duads

and synthemes form a symmetric (15_3)-configuration. It is possible to choose a set of five synthemes which together contain all the fifteen duads. Such a family is called a *total*. The number of possible totals is six. And every two totals have one, and only one syntheme in common between them. The following table gives the six totals in its row, and also in its columns (see [8, p. 465]) :

	T_1	T_2	T_3	T_4	T_5	T_6
T_1		14.25.36	16.24.35	13.26.45	12.34.56	15.23.46
T_2	14.25.36		15.26.34	12.35.46	16.23.45	13.24.56
T_3	16.24.35	15.26.34		14.23.56	13.25.46	12.36.45
T_4	13.26.45	12.35.46	14.23.56		15.24.36	16.25.34
T_5	12.34.56	16.23.45	13.25.46	15.24.36		14.26.35
T_6	15.23.46	13.24.56	12.36.45	16.25.34	14.26.35	

We can label the sets of 21 points, 21 lines on $\mathbb{P}^2(\mathbb{F}_4)$, six totals $T_1, \cdots, T_6$, fifteen duads, fifteen syntemes as follows (see [2, Chap. 23, §4]). We take six points on $\mathbb{P}^2(\mathbb{F}_4)$ in general position, i.e., there are no three of them are collinear and identify them with $1, \cdots, 6$. Then identify the line passing through i, j with the duad ij. The remaining three points on the line ij except i, j can be identified with three synthemes containing the duad ij. Finally, we identify the remaining six lines with the six totals. Thus, we can label 42 (-2)-curves on X, too. We remark that the incidence relation between lines and points corresponds to that between points, totals, duads and synthemes, that is, a point in $\mathbb{P}^2(\mathbb{F}_4)$ is contained in a line if and only if the corresponding point from $\{1, \cdots, 6\}$ (resp. or a syntheme) is contained in the corresponding duad (resp. or a total).

Now, we show the existence of the same type of three genus one fibrations f_1, f_2, f_3 as in Remark 3.4. In this case, they are quasi-elliptic. First, take the duad 12. The four (-2)-curves 12.35.46, 12.34.56, 12.36.45, 2 together with 12 form a singular fiber of type $\tilde{D}_4$ in which 12 is the central irreducible component. This cycle defines a genus one fibration f_1 with five singular fibers of type $\tilde{D}_4$. This is quasi-elliptic by [30, §1, Proposition]. Similarly, we define f_2 (resp. f_3) by 26 (resp. 23). In columns of Tables 1, 2, 3, we give five singular fibers of f_1, f_2, f_3, respectively.

12	13	14	15	16
12.35.46	13.25.46	4	15.23.46	6
12.34.56	13.24.56	14.23.56	5	16.23.45
12.36.45	3	14.25.36	15.24.36	16.24.35
2	13.26.45	14.26.35	15.26.34	16.25.34

TABLE 1. Singular fibers of f_1

Note that sixteen simple components of the first four singular fibers are common in f_1, f_2 and f_3. We consider these sixteen disjoint (-2)-curves and twelve disjoint (-2)-curves 12, 13, 14, 15, 26, 36, 46, 56, 23, 45, T_2, T_5 (central components of

26	36	46	56	16
2	12.36.45	12.35.46	12.34.56	1
13.26.45	3	13.25.46	13.24.56	16.23.45
14.26.35	14.25.36	4	14.23.56	16.24.35
15.26.34	15.24.36	15.23.46	5	16.25.34

TABLE 2. Singular fibers of f_2

23	45	T_2	T_5	16
2	4	12.35.46	12.34.56	1
3	5	13.24.56	13.25.46	6
14.23.56	12.36.45	14.25.36	14.26.35	16.24.35
15.23.46	13.26.45	15.26.34	15.24.36	16.25.34

TABLE 3. Singular fibers of f_3

the first four singular fibers of f_1, f_2, f_3). The incidence relation between these twenty-eight curves is the same as in Remark 3.4. The linear system $|H|$ given in (6) defines a birational morphism from X to a quartic surface such that the twelve (-2)-curves are contracted to nodes, and the images of the sixteen curves are lines. The (-2)-curves not appeared here are the following $14(= 42 - 28)$ curves:

1, 6, 16, 24, 25, 34, 35, 16.23.45, 16.24.35, 16.25.34, T_1, T_3, T_4, T_6.

By calculating their intersection numbers with H, we can see that the images of 1, 6, 16.23.45 are conics, those of 24, 25, 34, 35, T_1, T_3, T_4, T_6 are cubics and 16, 16.24.35, 16.25.34 are contracted to a rational double point of type A_3. Thus, we have the following theorem.

Theorem 8.1. *Let X be the supersingular $K3$ surface with Artin invariant $\sigma = 1$ in characteristic two. Then X contains twenty-eight (-2)-curves forming $(12_4, 16_3)$-configuration. Its quartic model has sixteen lines, twelve nodes and a rational double point of type A_3.*

Theorem 8.2. *Let X be a supersingular $K3$ surface with Artin invariant σ in characteristic two. Then X contains twenty-eight (-2)-curves forming $(12_4, 16_3)$-configuration if and only if $\sigma \leq 2$.*

Proof. Let S_σ be the Picard lattice of a supersingular $K3$ surface with Artin invariant σ in characteristic two. Recall that it depends only on σ and has signature $(1, 21)$, $S_\sigma^*/S_\sigma \cong (\mathbb{Z}/2\mathbb{Z})^{2\sigma}$, and of type I in the sense of Rudakov-Shafarevich [30]. Let L be the lattice generated by twenty-eight (-2)-curves which is isomorphic to $U \oplus E_8 \oplus D_8 \oplus \langle -4 \rangle$ (see Proposition 3.5). We need to show that L can be primitively embedded into S_σ if and only if $\sigma \leq 2$.

Assume that L can be primitively embedded in S_σ and let M be the orthogonal complement of L in S_σ which has rank 3. Then, $l(L) + l(M) \geq l(S_\sigma)$ where $l(A)$

is the number of minimal generators of the discriminant group A^*/A of an even lattice A. This implies that $3+l(M) \geq 2\sigma$, that is, $2\sigma-3 \leq l(M) \leq \text{rank}(M) = 3$ which implies $\sigma \leq 3$.

Conversely, if $\sigma = 1$, the assertion follows from the previous theorem. If $\sigma = 2$, then $S_\sigma \cong U \oplus E_8 \oplus D_8 \oplus D_4$. Obviously, $L = U \oplus E_8 \oplus D_8 \oplus \langle -4 \rangle$ can be primitively embedded in S_σ.

If $\sigma = 3$, then $S_\sigma \cong U \oplus E_8 \oplus D_4 \oplus D_4 \oplus D_4$. In this case, we use a result and the notation of [28] to show the non-existence of a primitive embedding. Since $q_{D_4} = v_+^{(2)}(2)$, $q_{D_8} = u_+^{(2)}(2)$ and $u_+^{(2)}(2) \oplus u_+^{(2)}(2) = v_+^{(2)}(2) \oplus v_+^{(2)}(2)$ ([28, Proposition 1.8.2, b]), $q_{S_\sigma} = u_+^{(2)}(2) \oplus u_+^{(2)}(2) \oplus v_+^{(2)}(2)$. On the other hand, $q_L = u_+^{(2)}(2) \oplus q_{-1}^{(2)}(2^2)$. Assume that there exists a primitive embedding of L into S_σ and let M be the orthogonal complement. Since S_σ^*/S_σ is a 2-elementary abelian group, the discriminant quadratic form of M is equal to $q = q_1^{(2)}(2^2) \oplus v_+^{(2)}(2) (= q_5^{(2)}(2^2) \oplus u_+^{(2)}(2))$ because only the unique isotropic subgroup $\mathbb{Z}/2\mathbb{Z}$ of $q_{-1}^{(2)}(2^2) \oplus q_1^{(2)}(2^2)$ (or $q_{-1}^{(2)}(2^2) \oplus q_5^{(2)}(2^2)$) can define an overlattice S_σ of $L \oplus M$ (see [28, Proposition 1.5.1]). Note that $\text{discr}(v_+^{(2)}(2)) = 12$ and $\text{discr}(q_1^{(2)}(2^2)) = 4$ ([28, Proposition 1.8.1]). Therefore, the non-existence of such M follows from [27, Theorem 1.10.1]. In fact, the condition (4) of [27, Theorem 1.10.1] does not hold in our case: $2^4 = |A_q|(= |M^*/M|) \not\equiv \pm\text{discr}(K_q) = \pm 12 \cdot 4 = \pm 3 \cdot 2^4 \ (\text{mod}(\mathbb{Z}_2^*)^2)$ because $\pm 3 \notin (\mathbb{Z}_2^*)^2$ ([31, §3.3, Remark (1)]). Thus, in case of $\sigma = 3$, L can not be primitively embedded in L_σ. $\square$

We remark that X is not a Kummer surface, but the quotient of the self-product of a cuspidal curve by a μ_2-action (see [21]). We also remark that the inseparable double cover $X \to \mathbb{P}^2$ induces an inseparable double covering $X \to \mathbb{P}^1 \times \mathbb{P}^1$. Two projections from $\mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^1$ induce quasi-elliptic fibrations f_1, f_2 with five singular fibers of type $\tilde{D}_4$.

The linear system (5) defined by the configuration of 12 nodes and 16 lines defines a degree two map φ from X to a quadric surface in $\mathbb{P}^3$. The composition with a projection $\mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^1$ gives a genus one fibration on X. If φ is separable, then a general fiber is a double cover of $\mathbb{P}^1$ ramified at four points which is impossible in characteristic two. Thus, the map φ must be inseparable, and the fibration is quasi-elliptic. This is an analog of the map from the Kummer surface to $\mathbb{P}^1 \times \mathbb{P}^1$.

We conclude that, in characteristic two, there are two types of Kummer surfaces: if we keep an elliptic curve, we get $\text{Kum}(E \times E)$, but loose $(12_4, 16_3)$-configuration, and vice versa. This situation is the same as Kummer surfaces associated with curves of genus two: if we keep genus two curves, then we loose (16_6)-configurations and vice versa ([9], [19]).

REFERENCES

[1] H. F. Baker, *A locus with* 25920 *linear selftransformations*, Cambridge Univ. Press 1946.

[2] J. Conway, N. Sloane, *Sphere packings, lattices and groups*, 3rd edition, Grundlehren der Mathematischen Wissenschaften **290**, Springer–Verlag, New York 1999.

[3] L. Cremona, *Mémoire de géométrie pure sur les surfaces du troisiéme ordre*, J. Reine Angew. Math., **68** (1868), 1–133 (Opere matematiche di Luigi Cremona, Milan, 1914, t. III, pp.1-121).

[4] A. de Jong, N. Shepherd-Barron, A. Van de Ven, *On the Burkhardt quartic*, Math. Ann., **286** (1990), 309–328.

[5] A. Degtyarev, I. Dolgachev, S. Kondō, *Humbert K3 surfaces of degree 6*, preprint.

[6] I. Dolgachev, *Mirror symmetry for lattice polarized $K3$ surfaces*, Algebraic geometry, **4**, J. Math. Sci., **81**, (1996), 2599–2630.

[7] I. Dolgachev, *Abstract configurations in algebraic geometry*, Proceedings of the Fano Conference. Università di Torino, Dipartimento di Matematica, Turin, 2004, 423–462.

[8] I. Dolgachev, *Classical Algebraic Geometry*, Cambridge Univ. Press 2012.

[9] I. Dolgachev, *$K3$ surface of Kummer type in characteristic two*, Bull. Lond. Math. Soc., **56** (2024), 1903–1919.

[10] I. Dolgachev, S. Kondō, *A supersingular K3 surfaces in characteristic 2 and the Leech Lattice*, Int. Math. Res. Notices (IMRN) **2003** (2003), 1–23.

[11] I. Dolgachev, S. Kondō, *Enriques surfaces* II, Springer–Verlag, 2025.

[12] B. Grünbaum, *Configurations of points and lines*. Grad. Stud. Math., **103** American Mathematical Society, Providence, RI, 2009.

[13] H. Hudson, *Kummer's quartic surface*, Cambridge Univ. Press 1905.

[14] G. Humbert, *Sur unde class de courbes planes et sur une surface remarkuable du quatrieme ordre*, J. Math. Pures et Appl., **6** (1890), 423–444.

[15] G. Humbert, *Sur la surface desmique du quatriéme ordre*, J. Math. Pures et Appl., **7** (1891), 353–398.

[16] B. Hunt, *The geometry of some special arithmetic quotients*, Lecture Notes in Mathematics **1637**, Springer-Verlag, Berlin, 1996.

[17] C. Jessop, *Quartic surfaces with singular points*, Cambridge Univ. Press, 1916.

[18] T. Katsura, *On Kummer surfaces in characteristic* 2, M. Nagata (ed.), Proceedings of the international symposium on algebraic geometry, 525–542, Kinokuniya Book Store, Tokyo 1978.

[19] T. Katsura, S. Kondō, *Kummer surfaces and quadratic line complexes in characteristic two*, arXiv:2301.01450.

[20] S. Kondō, S. Mukai, *The automorphism groups of Kummer surfaces in characteristic two*, arXiv:2412.18902v1.

[21] S. Kondō, S. Schröer, *Kummer surfaces associated with group schemes*, manuscripta math., **166** (2021), 323–342.

[22] E. Kummer, *Ueber die algebraische Strahlensysteme insbesondere die erste und zweiten Ordnung*, Monatsberichte Berliner Akad., **8** (1866), 1–120.

[23] E. Lord, *Symmetry and pattern in projective geometry*. Springer, London, 2013

[24] R. Mathews, *Cubic curves and desmic surfaces*, Trans. Amer. Math. Soc. **28** (1926), 502–522.

[25] R. Mathews, *Cubic curves and desmic surfaces*, II, Trans. Amer. Math. Soc. **30** (1928), 19–23.

[26] Y. Namikawa, *Smoothing Fano 3-folds,* J. Alg. Geometry **6** (1997), 307–321.

[27] V. V. Nikulin, *On Kummer surfaces*, Math. USSR-Izv., **9** (1975), 261–275.

[28] V. V. Nikulin, *Integral symmetric bilinear forms and some of their applications*, Math. USSR-Izv., **14** (1980), 103–167.

[29] J. Pereira, S. Yuzvinsky, *Completely reducible hypersurfaces in a pencil*, Adv. Math., **219** (2008), 672–688.

[30] A. N. Rudakov and I. R. Shafarevich, *Supersingular surfaces of type K3 over fields of characteristic 2,* Izv. Akad. Nauk SSSR Ser. Mat., **42** (1978), 848–869.

[31] J. P. Serre, *A course in arithmetic*, Springer-Verlag, 1973.

[32] N. Shepherd-Barron, *Some foliations on surfaces in characteristic* 2, J. Algebraic Geom., **5** (1996), 521–535.

[33] T. Shioda, *Kummer surfaces in characteristic* 2, Proc. Japan Acad., **50** (1974), 718–722.

[34] J. Steiner, *Ueber die Flächen dritten Grades*, J. Reine Angew. Math., **53** (1856), 133–141 (Gesammelte Werke, Chelsea, 1971, vol. II, pp. 649–659).

[35] C. Stephanos, *Sur les systémes desmique de trois tétraèdres*, Bull. Sci. Math. Ast., **3** (1879), 424–456.

[36] J. Stipins, *On finite k-nets in the complex projective plane*, Ph.D. thesis, University of Michigan, 2007.

[37] J. Todd, *On a quartic primal with forty-five nodes, in space of four dimensions*, Quarterly J. Math., **7** (1936), 168–174.

[38] P. Wagreich, *Elliptic singularities of surfaces*, Amer. J. Math., **92** (1970), 419–454.

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF MICHIGAN, 525 EAST UNIVERSITY AVENUE, ANN ARBOR, MI 48109-1109 USA
    Email address: idolga@umich.edu

GRADUATE SCHOOL OF MATHEMATICS, NAGOYA UNIVERSITY, NAGOYA, 464-8602 JAPAN
    Email address: kondo@math.nagoya-u.ac.jp