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On certain families of elliptic curves in projective space*

To the memory of Fabio Bardelli

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1. Introduction

Let $\mathrm{El}(n; p_1, \ldots, p_m)$ be the family of elliptic curves of degree $n+1$ in $\mathbb{P}^n$ containing a fixed set of m distinct points $p_1, \ldots, p_m$. In modern language, $\mathrm{El}(n; p_1, \ldots, p_m)$ is a Zariski-open subset of the fibre of the evaluation map $\mathrm{ev} : M_{1,m}(\mathbb{P}^n, n+1) \to (\mathbb{P}^n)^m$, where $M_{1,m}(\mathbb{P}^n, n+1)$ is the space of regular maps of elliptic curves equipped with an ordered set of m points to a curve of degree $n+1$ in $\mathbb{P}^n$. Its general fibre, if not empty, is an irreducible variety of dimension $(n+1)^2 - m(n-1)$. The largest possible m for which the map ev is dominant is equal to $9 (n = 2, 5)$, $8 (n = 3, 4)$, $n + 3 (n \geq 6)$. In the last case we show that the general fibre is isomorphic to an open subset of a complete intersection of $n-2$ diagonal quadrics in $\mathbb{P}^{n+2}$. In particular, birationally, it is a Fano variety if $n \leq 6$, a Calabi–Yau if $n = 7$, and of general type if $n \geq 8$. The group $\mathcal{G}_n = (\mathbb{Z}/2\mathbb{Z})^{n+2}$ acts naturally in $\mathbb{P}^{n+2}$ by multiplying the projective coordinates with ± 1. The corresponding action of a subgroup of index 2 of $\mathcal{G}_n$ is induced by a certain group of Cremona transformations in $\mathbb{P}^n$ which we will describe explicitly.

There are three cases when $\mathrm{El}(n; p_1, \ldots, p_m)$ is of expected dimension 0. They are $(n, m) = (2, 9), (3, 8), (5, 9)$. It is well known that in the first two cases $\mathrm{El}(n; p_1, \ldots, p_m)$ consists of one point. Less known is the fact that the same is true in the case $(5, 9)$. D. Babbage [1] attributes this result to T. G. Room. Apparently it was proven much earlier by A. Coble [2]. We reproduce Coble's proof in the paper. This result implies the existence of a rational elliptic fibration $f : \mathbb{P}^5 - \to \mathrm{El}(5, p_1, \ldots, p_8)$ which is an analog of the well-known rational elliptic fibrations $\mathbb{P}^2 - \to \mathbb{P}^1 = \mathrm{El}(2; p_1, \ldots, p_8)$ and $\mathbb{P}^3 - \to \mathbb{P}^2 = \mathrm{El}(3; p_1, \ldots, p_7)$ defined by the pencil of plane cubics through 8 points and by the net of quadrics through 7

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points, respectively. We show that its locus of points of indeterminacy is a certain 3-fold, a Weddle variety studied intensively by Coble [4], [5], [6].

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2. Association

2.1. We start with a reminder of the classical theory of association of finite sets of points. We follow the modern exposition of this theory given in [8].

Let Z be a Gorenstein scheme of dimension 0 over a field k, $\mathcal{L}$ be an invertible sheaf on Z and $V \subset H^0(Z, \mathcal{L})$ be a linear system. The duality pairing

$$H^0(Z, \mathcal{L}) \times H^0\big(Z, \omega_Z \otimes \mathcal{L}^{-1}\big) \to H^0(Z, \omega_Z) \xrightarrow{\text{trace}} k$$

allows one to define the subspace $V^\perp \subset H^0(Z, \omega_Z \otimes \mathcal{L}^{-1})$. The pair

$$\big(V^\perp, \omega_Z \otimes \mathcal{L}^{-1}\big)$$

is called the *Gale transform* of $(V, \mathcal{L})$.

2.2. Assume that Z is realized as a closed subscheme of a connected smooth d-dimensional scheme $B \subset \mathbb{P}^n$. Let $\mathcal{I}_Z$ be its sheaf of ideals in B. We have a short exact sequence of sheaves on B

$$0 \to \mathcal{I}_Z(1) \to \mathcal{O}_B(1) \to \mathcal{O}_Z(1) \to 0,$$

which gives an exact sequence

$$H^0(B, \mathcal{O}_B(1)) \to H^0(Z, \mathcal{O}_Z(1)) \to H^1(B, \mathcal{I}_Z(1)) \to H^1(B, \mathcal{O}_B(1)).$$

Let us assume that B is embedded by the complete linear system, i.e., the restriction map $H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}) \to H^0(B, \mathcal{O}_B(1))$ is bijective. Then the image of the map $H^0(B, \mathcal{O}_B(1)) \to H^0(Z, \mathcal{O}_Z(1))$ is equal to V. If we also assume that $H^1(B, \mathcal{O}_B(1)) = 0$, then we will be able to identify $V^\perp$ with $H^1(B, \mathcal{I}_Z(1))^*$. Using Serre's duality, we get

$$V^\perp \cong \operatorname{Ext}^{d-1}(\mathcal{I}_Z(1), \omega_B). \tag{2.1}$$

2.3. In particular, taking $B = \mathbb{P}^n$, we can fix a basis in $\operatorname{Ext}^{n-1}(\mathcal{I}_Z(1), \omega_{\mathbb{P}^n})$, so that the linear system $|V^\perp|$ maps Z to a subset Z^{ass} in the projective space

$$\mathbb{P}((V^\perp)^*) \cong \mathbb{P}\big(H^1\big(\mathbb{P}^n, \mathcal{I}_Z(1)\big)\big) \cong \mathbb{P}^{m-n-2}, \tag{2.2}$$

where $m = \deg Z := \dim_k H^0(Z, \mathcal{O}_Z)$. We will always assume that $m \geq n+3$. The set Z^{ass} is called the *associated set* of Z. It depends on a choice of a basis in $H^1(\mathbb{P}^n, \mathcal{I}_Z(1))$. If Z is a reduced scheme defined by an ordered set of m distinct points $(p_1, \ldots, p_m)$ in $\mathbb{P}^n$, then the associated set Z^{ass} is an ordered set of points $(q_1, \ldots, q_m)$ in $\mathbb{P}^{m-n-2}$.

2.4. Now let B be a curve in $\mathbb{P}^n$. Then $Z \subset B$ can be identified with a positive divisor, and

$$V^{\perp} \cong \mathrm{Ext}^{d-1}(\mathcal{I}_Z(1), \omega_B) = H^0(B, \mathcal{O}_B(K_B + Z - H)), \tag{2.3}$$

where H is a hyperplane section and K_B is a canonical divisor. Thus we see that the associated scheme Z^{ass} is equal to the image of the divisor Z under the map $B \to \mathbb{P}^{m-n-2}$ given by the complete linear system $|K_B + Z - H|$.

2.5. Remark. Let $Z \subset \mathbb{P}^n$ be a reduced scheme consisting of m distinct points $(p_1, \dots, p_m)$. A choice of an order defines a basis in the space $H^0(Z, \mathcal{O}_Z(1))$. Choosing a basis in $H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(1))$ we can represent Z by the matrix A of the linear map $H^0(Z, \mathcal{O}_Z(1))^* \to H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(1))^*$ equal to the transpose of the restriction map $H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(1))^* \to H^0(Z, \mathcal{O}_Z(1))$. The columns of A are projective coordinates of the points p_i's. Let B be a matrix whose rows form a basis in the nullspace of A. Then the columns of B can be chosen as the projective coordinates of the ordered set of points $q_1, \dots, q_m$ in $\mathbb{P}^{m-n-2}$. These points represent the associated set of points Z^{ass} which is well defined up to projective equivalence. This is the original classical definition of association (see [2], [3]). Let P_n^m be the space of m ordered points in $\mathbb{P}^n$ modulo projective equivalence. This is a GIT-quotient with respect to the democratic linearization (i.e the unique linearization compatible with the action of the symmetric group S_m). The association defines an isomorphism of algebraic varieties (see [7]):

$$\mathrm{ass} : P_n^m \cong P_{m-n-2}^m.$$

As far as I know the first cohomological interpretation of the association was given by A. Tyurin [11].

3. Normal elliptic curves in $\mathbb{P}^n$ through $n+3$ points

3.1. We shall apply 2.4 to the case when B is an elliptic curve embedded in $\mathbb{P}^n$ by a complete linear system (a normal elliptic curve of degree $n+1$). Let $\mathrm{El}(n; p_1, \dots, p_m)$ be the set of normal elliptic curves in $\mathbb{P}^n$ containing $m \geq n+3$ distinct points $p_1, \dots, p_m$. As in the Introduction we consider $\mathrm{El}(n; p_1, \dots, p_m)$ as an open Zariski subset of the fibre of the evaluation map $\mathrm{ev} : M_{1,m}(\mathbb{P}^n, n+1) \to (\mathbb{P}^n)^m$ over $(p_1, \dots, p_m) \in (\mathbb{P}^n)^m$. Let Z be the reduced scheme $\{p_1, \dots, p_m\}$ and let $B \in \mathrm{El}(n; p_1, \dots, p_m)$. We fix a basis in $\mathbb{P}(H^1(\mathbb{P}^n, \mathcal{I}_Z(1)))$ to identify $Z^{\mathrm{ass}} = \{q_1, \dots, q_m\}$ with an ordered subset of $\mathbb{P}^{m-n-2}$. It follows from 2.4 that Z^{ass} lies on the image of B under the map ϕ given by the complete linear system $|Z-H|$ of degree $m-n-1$. Recall that a choice of a basis in $H^1(\mathbb{P}^n, \mathcal{I}_Z(1))$ defines a choice of a basis in $H^0(B, \mathcal{O}_B(Z-H))$; so the image of each $B \in \mathrm{El}(n; p_1, \dots, p_m)$ lies in the same space $\mathbb{P}^{m-n-2}$ and contains Z^{ass}. If $m > n+3$, the map $\phi : B \to \mathbb{P}^{m-n-2}$ is an embedding. Since the association is the duality, we obtain the following:

3.2. Theorem. *Assume $m > n+3$. Let $Z = \{p_1, \dots, p_m\}$ and $Z^{ass} = \{q_1, \dots, q_m\}$. The association defines an isomorphism of algebraic varieies*:

$$\mathrm{El}(n; p_1, \dots, p_m) \cong \mathrm{El}(m-n-2; q_1, \dots, q_m).$$

3.3. Remark. In the case $m = 2n+2$ one can speak about self-associated schemes Z, that is, Gorenstein schemes such that $Z^{ass} = Z$ after an appropriate choice of a basis in $H^1(\mathbb{P}^n, \mathcal{I}_Z(1))$. In this case $|H| = |Z - H|$, so that $Z \subset B$ is cut out in B by a quadric. The association isomorphism from Theorem 3.2 is of course the identity.

3.4. Let us consider the exceptional case $m = n+3$. In this case the map $\phi : B \to \mathbb{P}^1$ is of degree 2. It maps $p_1, \dots, p_{n+3}$ to the associated set of points $q_1, \dots, q_{n+3}$ in $\mathbb{P}^1$. Conversely, fix a set $\mathcal{Q} = \{q_1, \dots, q_{n+3}\}$ of distinct points in $\mathbb{P}^1$ and let $Z_{\mathcal{Q}}$ be the corresponding 0-dimensional scheme on $\mathbb{P}^1$. Take a degree 2 map $\phi : B \to \mathbb{P}^1$ and choose $n+3$ points $\mathcal{P} = \{p_1, \dots, p_{n+3}\} \subset B$ such that $p_i \in \phi^{-1}(q_i)$. Let $Z_{\mathcal{P}}$ be the corresponding 0-dimensional subscheme of B. Since

$$\phi_*(\phi^*(\mathcal{O}_{\mathbb{P}^1}(-1)) \otimes \mathcal{O}_{Z_{\mathcal{P}}}) = \mathcal{O}_{Z_{\mathcal{Q}}}(-1),$$

we can identify the corresponding vector spaces of sections. The associated set $Z_{\mathcal{Q}}^{\mathrm{ass}}$ lies on the image of $\mathbb{P}^1$ under the map $\mathbb{P}^1 \to \mathbb{P}^n$ given by the linear system $\left|\mathcal{O}_{\mathbb{P}^1}(Z_{\mathcal{Q}}) \otimes \mathcal{O}_{\mathbb{P}^1}(-1)\right|$. The linear system

$$\left|\mathcal{O}_B(p_1 + \ldots + p_{n+3}) \otimes \phi^*(\mathcal{O}_{\mathbb{P}^1}(-1))\right|$$

embeds B into the same $\mathbb{P}^n$ with $Z_{\mathcal{P}}^{\mathrm{ass}} = Z_{\mathcal{Q}}^{\mathrm{ass}} = Z$. Thus $(B, \mathcal{P})$ defines a point in $\mathrm{El}(n; Z)$. The double cover $\phi : B \to \mathbb{P}^1$ is defined, up to isomorphism, by a choice of the branch divisor $W \subset \mathbb{P}^1$, i.e. an unordered set of 4 distinct points in $\mathbb{P}^1$; the latter can be identified with a point in $\mathbb{P}^4 \setminus \Delta$, where Δ is the discriminant hypersurface. We will continue to identify $\mathbb{P}^k$ with the symmetric product $(\mathbb{P}^1)^{(k)}$. Let $\iota : B \to B$ be the covering involution. It is clear that the $(B, \mathcal{P})$ and $(B, \iota(\mathcal{P}))$ define the same point in $\mathrm{El}(n; Z)$. Summarizing we obtain the following:

3.5. Theorem. *Let $q_1, \dots, q_{n+3}$ be distinct points in $\mathbb{P}^1$ and let $p_1, \dots, p_{n+3}$ be the associated set of points in $\mathbb{P}^n$. The variety El$(n; p_1, \dots, p_{n+3})$ is isomorphic to the fibre of the evaluation map ev : $M_{1,n+3}(\mathbb{P}^1, 2) \to (\mathbb{P}^1)^{n+3}$ over $(q_1, \dots, q_{n+3})$. The forgetting map $M_{1,n+3}(\mathbb{P}^1, 2) \to M_1(\mathbb{P}^1, 2) \cong \mathbb{P}^4 \setminus \Delta$ restricted to the fibre is a Galois cover with the Galois group $\mathcal{G}_n$ isomorphic to $(\mathbb{Z}/2\mathbb{Z})^{n+2}$.*

3.6. Fix a set $\mathcal{P}$ of m points $p_1, \dots, p_m$ in $\mathbb{P}^1$ and consider the subvariety

$$X = \{A \in \mathbb{P}^k : A \cap \mathcal{P} \neq \emptyset\}.$$

It is clear that $X = X_1 \cup \ldots \cup X_m$, where $X_i = \{A \in \mathbb{P}^k : p_i \in A\}$. This is the image in $(\mathbb{P}^1)^{(k)}$ of a coordinate hyperplane in $(\mathbb{P}^1)^k$, and hence is a hyperplane in $\mathbb{P}^k$ of effective divisors D of degree k such that $D - p_i > 0$. Let us identify $\mathbb{P}^k$ with $|\mathcal{O}_{\mathbb{P}^1}(k)|$ and let $R = v_k(\mathbb{P}^1)$ be the Veronese curve in $\left|\mathcal{O}_{\mathbb{P}^1}(k)\right|^* = \check{\mathbb{P}}^k$. Then X_i is the hyperplane in $\left|\mathcal{O}_{\mathbb{P}^1}(k)\right|$ corresponding to the point $v_k(p_i) \in R$. Thus X is the union of hyperplanes corresponding to m points on R.

3.7. The previous discussion shows that the branch divisor of the map $\mathrm{El}(n; p_1, \dots, p_{n+3}) \to \mathbb{P}^4 \setminus \Delta$ consists of the union of open subsets of $n+3$ hyperplanes corresponding to $n+3$ points on a rational normal curve in $\mathbb{P}^4$. Since the fundamental group of the complement of N hyperplanes in a general linear position in $\mathbb{P}^k$ is isomorphic to $\mathbb{Z}^{N-1}$, the Galois $(\mathbb{Z}/2\mathbb{Z})^{n+2}$-cover $X \to \mathbb{P}^4$ branched along the union of $n+3$ hyperplanes H_i is defined uniquely up to an isomorphism. Thus $\mathrm{El}(n; p_1, \dots, p_{n+3})$ is an open subset of X. A well-known way to construct X is as follows. Let

$$l_i = \sum_{s=0}^{4} a_{is} t_s = 0, \quad i = 0, \dots, n+2$$

be linear equations defining the hyperplanes H_i. We assume that $l_i = t_i, i = 0, \dots, 4$. Consider the map

$$i : \mathbb{P}^4 \to \mathbb{P}^{n+2}, (t_0, \dots, t_4) \mapsto (l_0(t_0, \dots, t_4), \dots, l_{n+2}(t_0, \dots, t_4)).$$

Let $(x_0, \dots, x_{n+2})$ be projective coordinates in $\mathbb{P}^{n+2}$ and let $(y_0, \dots, y_{n+2})$ be projective coordinates in another copy of $\mathbb{P}^{n+2}$. Consider the map $\psi : \mathbb{P}^{n+2} \to \mathbb{P}^{n+2}$ defined by the formula $(x_0, \dots, x_{n+2}) = (y_0^2, \dots, y_{n+2}^2)$. Then X is isomorphic to the pre-image of $i(\mathbb{P}^4)$ under the map ψ. It is easy to see that X is a complete intersection of the quadrics

$$y_i^2 - \sum_{s=0}^{4} a_{is} y_s^2 = 0, \quad i = 5, \dots, n+2.$$

3.8. Corollary. $\mathrm{El}(n; p_1, \dots, p_{n+3})$ is isomorphic to an open Zariski subset of a smooth complete intersection of $n-2$ quadrics in $\mathbb{P}^{n+2}$. In particular, X is birationally of general type for $n > 7$, Calabi–Yau for $n = 7$, and Fano for $n \leq 6$.

3.9. Remark. The projection from the last point defines a map

$$\mathrm{El}(n; p_1, \dots, p_m) \to \mathrm{El}(n-1; p_1, \dots, p_{m-1}).$$

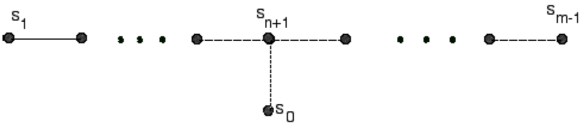
In the case $m = n+3$, this is a finite map of varieties of dimension 4. I do not know the degree of this map.

4. Cremona action

4.1. Let P_n^m be the space of m ordered points in $\mathbb{P}^n$ modulo projective equivalence (see Remark 2.5). Recall that the Cremona action on P_n^m is a homomorphism of groups:

$$\mathrm{cr}_{n,m} : W_{n,m} \to \mathrm{Bir}\big(P_n^m\big), \tag{4.1}$$

where $W_{n,m}$ is the Coxeter group corresponding to the Coxeter diagram defined by a $T_{2,n+1,m-n-1}$ graph (see [7]). Let $s_0, \dots, s_{m-1}$ be its Coxeter generators with s_0 corresponding to the vertex of the Coxeter diagram with the arm of length 2.



In the action (4.1), the element s_0 acts by means of the standard Cremona transformation

$$T : \mathbb{P}^n - \to \mathbb{P}^n, \quad (z_0, \dots, z_n) \mapsto (z_1 \cdots z_n, z_0 z_2 \cdots z_n, \dots, z_0 z_1 \cdots z_{n-1}),$$

as follows. Choose an open subset U of P_n^m representing stable orbits of sets of distinct points $\mathcal{P} = (p_1, \dots, p_m)$ such that $p_1, \dots, p_{n+1}$ span $\mathbb{P}^n$. Then one can represent any point x of U with a set $\mathcal{P} = (p_1, \dots, p_m)$ such that

$$p_1 = (1, 0, \dots, 0), \dots, p_{n+1} = (0, \dots, 0, 1). \tag{4.2}$$

Then we define

$$\mathrm{cr}_{n,m}(s_0)(x) = (p_1, \dots, p_{n+1}, T(p_{n+2}), \dots, T(p_m)).$$

The subgroup S_m of $W_{n,m}$ generated by $s_1, \dots, s_{m-1}$ acts via a permutation of factors of $(\mathbb{P}^n)^m$.

4.2. For any subset $\mathcal{P} \in (\mathbb{P}^n)^m$ representing a point $x \in U$ let $\pi_{\mathcal{P}} : X(\mathcal{P}) \to \mathbb{P}^n$ denote the blowing up of $\mathbb{P}^n$ with center at $\mathcal{P}$. The Picard group $\mathrm{Pic}(X(\mathcal{P}))$ has a natural basis $e_{\mathcal{P}} = (e_0, e_1, \dots, e_m)$, where e_0 is the class of the pre-image of a hyperplane, and $e_i, i > 0$, is the class of the exceptional divisor blown-up from the point p_i. This basis is independent of a choice of a representative of the orbit x. The group $W_{n,m}$ acts on $\mathrm{Pic}(X(\mathcal{P}))$ as follows. The subgroup S_m acts by permuting $e_i, i \neq 0$, and s_0 acts by

$$e_0 \to e_0 - e_1 - \dots - e_{n+1}, \quad e_i \to e_0 - e_1 - \dots - e_{n+1} + e_i, i = 1, \dots, n+1,$$

all e_i are invariant for $i > n+1$. We immediately check that all elements $w \in W_{n,m}$ leave the anticanonical class

$$-K_{X(\mathcal{P})} = (n+1)e_0 - (n-1)\sum_{i=1}^{n+3} e_i$$

invariant.

If $y = \mathrm{cr}_{n,m}(w)(x)$ and $\mathcal{Q}$ is a representative of y, then there is a birational map $f : X(\mathcal{P}) \to X(\mathcal{Q})$ which is an isomorphism in codimension ≥ 2 such that

$$w(e_{\mathcal{P}}) = f^*(e_{\mathcal{Q}}).$$

In particular, if $y = x$, we can choose $\mathcal{Q} = \mathcal{P}$ and obtain a pseudo-automorphism g of $X(\mathcal{P})$ (i.e. a birational automorphism which is an isomorphism in codimension ≥ 2) such that $w = g^*$.

4.3. Recall from Remark 2.5 that the association defines an isomorphism of algebraic varieties

$$\mathrm{ass}_{n,m} : P_n^m \to P_{m-n-2}^m.$$

It commutes with the Cremona action in the following sense. There is a natural isomorphism of the Coxeter groups $\tau : W_{n,m} \to W_{n,m-n-2}$ defined by the unique isomorphism of the Coxeter diagrams $T_{2,n+1,m-n-1}$ and $T_{2,m-n-1,n+1}$ which leaves the vertex corresponding to s_0 fixed. We have

$$\mathrm{cr}_{n,m}(w)(x) = \mathrm{cr}_{m-n-2,m}(\tau(w))(\mathrm{ass}_{n,m}(x)). \tag{4.3}$$

4.4. We will be interested in the special case $m = n+3$. In this case $W_{n,m} \cong W(D_{n+3})$, the Weyl group of the root system of type D_{n+3}. It is known that $W(D_{n+3}) \cong G_n \rtimes S_{n+3}$, where $G_n \cong (\mathbb{Z}/2)^{n+2}$ is generated by the element $w_1 = s_0 \circ s_{n+2}$ and their conjugates. Let $x \in P_n^{n+3}$ be represented by a point set $\mathcal{P}$ as in (4.2). We also assume that $p_{n+2} = (1, \ldots, 1)$, $p_{n+3} = (a_0, \ldots, a_n)$ with $a_i \neq 0$. Then

$$\mathrm{cr}_{n,n+3}(w_1)(x) = (p_1, \ldots, p_{n+1}, p'_{n+3}, p_{n+2}),$$

where $p'_{n+3} = (1/a_0, \ldots, 1/a_n)$. Consider the projective transformation

$$g : \mathbb{P}^n \to \mathbb{P}^n, \quad (t_0, \ldots, t_n) \mapsto (a_0t_0, a_1t_1, \ldots, a_nt_n).$$

Then $w(\mathcal{P}) = g(\mathcal{P})$, and hence $\mathrm{cr}_{n,n+3}(x) = x$. This shows that w_1 acts trivially on P_n^{n+3}. In particular, the Cremona action $\mathrm{cr}_{n,m}$ has the kernel which is a normal subgroup of $W_{n,n+3}$ containing w_1. It is easy to see that the kernel coincides with G_n. For example, this follows from (4.3) since $P_n^{n+3} \cong P_1^{n+3}$. Thus we obtain that G_n acts by pseudo-automorphisms of the blow-up $X(\mathcal{P})$.

4.5. Following [4] we show in more details how the group G_n acts on $X(\mathcal{P})$. Let V be the vector space over $\mathbb{F}_2$ consisting of subsets of $[n+3] = \{1, \ldots, n+3\}$ of even cardinality with the addition defined by $S + S' = S \cup S' \setminus (S \cap S')$. Let A be the affine space over V consisting of subsets of odd cardinality. For any $I \in A$ of cardinality $2k+1$ consider the following divisor class in $X = X(\mathcal{P})$:

$$D_I = ke_0 - (k-1)\sum_{i\in I} e_i - k\sum_{i\in \bar{I}} e_i, \tag{4.4}$$

where $\bar{I}$ denotes the complementary subset of $[n+3]$. For example, when $k = 0$, i.e., $I = \{s\}$ for some $s \in [n+3]$, we get $D_{\{s\}} = e_s$.

Lemma 1. *There is a unique isomorphism* $h : V \to G_n$, $J \mapsto w_J$ *such that* $w_{\{n+2,n+3\}} = s_0 \circ s_{n+2}$, *and*

$$w_J(D_I) = D_{J+I}, \quad \forall J \in V, \ \forall I \in A. \tag{4.5}$$

Proof. First we check this directly for $w_{\{n+2,n+3\}}$:

$$D_{I+\{n+2,n+3\}} = s_0 \circ s_{n+2}(D_I), \quad \forall I \in A.$$

Then we extend this to all subsets of cardinality 2 of V by defining, for any $\sigma \in S_{n+3}$, $w_{\sigma(\{n+2,n+3\})} = \sigma \circ w_{n+2,n+3} \circ \sigma^{-1}$ and using that

$$D_{I+\sigma(\{n+2,n+3\})} = D_{\sigma(\sigma^{-1}(I)+\{n+2,n+3\})} = \sigma \circ w_{n+2,n+3} \circ \sigma^{-1}(D_I).$$

Finally, we extend the map h to all even subsets by linearity. □

4.6. Let $J \in V$ with $\#J = 2k$. Then

$$w_J(e_s) = D_{J\cup\{s\}} = ke_0 - (k-1)\sum_{i\in J} e_i - k\sum_{i\notin J} e_i + e_s,$$

if $s \notin J$, and

$$w_J(e_s) = D_{J\setminus\{s\}} = (k-1)e_0 - (k-2)\sum_{i\in J} e_i - (k-1)\sum_{i\notin J} e_i - e_s$$

otherwise. We have

$$-K_{X(\mathscr{P})} = (n+1)e_0 - (n-1)\sum_{i=1}^{n+3} e_i = w_J\Big((n+1)e_0 - (n-1)\sum_{i=1}^{n+3} e_i\Big) =$$

$$(n+1)w_J(e_0)$$

$$-(n-1)\Big(k(n+1)e_0 - \big((k-1)(n+1)-1\big)\sum_{i\in J} e_i - \big(k(n+1)-1\big)\sum_{i\notin J} e_i\Big).$$

This gives

$$w_J(e_0) = \big(k(n-1)+1\big)e_0 - (k-1)(n-1)\sum_{i\in J} e_i - k(n-1)\sum_{i\notin J} e_i. \tag{4.6}$$

This shows that the pseudo-automorphism of $X_{\mathscr{P}}$ induced by $\mathrm{cr}_{n,m}(w_J)$ is given by the linear system of hypersurfaces of degree $kn - k + 1$ passing through the points $p_i \notin J$ with multiplicity $(k-1)(n-1)$ and through the points $p_i \in J$ with multiplicity $k(n-1)$.

4.7. Assume that $n = 2g - 1$ is odd. Applying Lemma 1 we obtain that, for any $I \in A$,

$$w_{[n+3]}(D_I) = D_{\bar{I}},$$

and hence

$$w_{[n+3]}(D_I) + D_I = D_{\bar{I}} + D_I = ge_0 - (g-1)\sum_{i=1}^{n+3} e_i \in \Big|-\frac{1}{2}K_{X(\mathscr{P})}\Big|.$$

The subgroup $G_n \subset W_{n,n+3}$ acts on $X_{\mathscr{P}}$ and leaves the half-anticanonical linear system

$$\left|-\frac{1}{2}K_{X(\mathscr{P})}\right| = \left|ge_0 - (g-1)\sum_{i=1}^{n+3} e_i\right|$$

invariant. The distinguished element $w_{[n+3]} \in G_n$ leaves 2^g divisors $D_I + D_{\bar{I}} \in \left|-\frac{1}{2}K_{X(\mathscr{P})}\right|$ invariant. Coble shows [4] that

$$\dim\left|-\frac{1}{2}K_{X(\mathscr{P})}\right| = 2^g - 1 \tag{4.7}$$

and the divisor classes $D_I + D_{\bar{I}}$ span $\left|-\frac{1}{2}K_{X(\mathscr{P})}\right|$. In particular, $w_{[n+3]}$ acts identically in the projective space $\left|-\frac{1}{2}K_{X(\mathscr{P})}\right|$.

4.8. Let $\tilde{W}_g \subset X_{\mathscr{P}}$ be the locus of fixed points of $w_{[n+3]}$. This is defined as the closure of the fixed locus of $w_{[n+3]}$ restricted to an invaraint open subset where $w_{[n+3]}$ is defined. The projection W_g of $\tilde{W}_g$ in $\mathbb{P}^n$ is a Weddle variety of dimension g. Coble proves that the linear system $\left|-\frac{1}{2}K_{X(\mathscr{P})}\right|$ maps W_g to $\mathbb{P}^{2^g-1}$ and the image is isomorphic to the Kummer variety of the Jacobian variety $\mathrm{Jac}(C)$ of a hyperelliptic curve C of genus g embedded via $|2\Theta|$ on $\mathrm{Jac}(C)$. The curve C is the hyperelliptic curve corresponding to $n+3 = 2g+2$ points on $\mathbb{P}^1$ associated to $(p_1, \ldots, p_{n+3})$.

4.9. Let $p \in W_g$ be a fixed point of $w_{[n+3]}$. Consider the set $(p_1, \ldots, p_{n+3}, p)$ and $w_{[n+3]}$ as an element of $W_{n,n+4}$. Let $\mathscr{Q} = (q_1, \ldots, q_{n+3}, q)$ be the associated set in $\mathbb{P}^2$. The projection from q defines $n+3$ points in $\mathbb{P}^1$ which are associated to $(p_1, \ldots, p_{n+3})$ (see [2], [7]). Using (4.3), we see that $\mathrm{cr}_{n,n+4}(w_{[n+3]})$ corresponds to $\mathrm{cr}_{2,n+4}(w_0)$ for some $w_0 \in W_{2,n+4}$. Moreover, we may assume that $\mathrm{cr}_{2,n+4}(w_0)$ defines an automorphism g (in dimension 2 any pseudo-automorphism is an automorphism) of the blow-up $X(\mathscr{Q})$. Since $\mathrm{cr}_{n,n+3}(w_{[n+3]})$ fixes the orbit of $(p_1, \ldots, p_{n+3})$, we see that $\mathrm{cr}_{2,n+4}(w_0)$ leaves the lines $\langle q_i \rangle$ invariant. This implies that the automorphism g is induced by a de Jonquières transformation of the projective plane (see [10, p. 150]). Its fixed set of points is a hyperelliptic curve of degree $g+2$ with g-multiple points at q and tangent lines $\langle q, q_i \rangle$. This curve is isomorphic to the curve C from 4.8. When we vary the point p in the Weddle variety W_g the subvariety of P_2^{n+4} of associated point sets $\mathscr{Q}$ consists of orbits of points sets $(q_1, \ldots, q_{n+3}, q)$ such that C admits a plane model C' of degree $g+2$ with Weierstrass points at $q_1, \ldots, q_{n+3}$ and a g-multiple point at q. The g branches at q define an effective divisor D of degree g on C such that the map $C \to C'$ is given by the linear system $|g_2^1 + D|$, where g_2^1 is the linear series of degree 2 defined by the hyperelliptic involution. In this way the Weddle variety W_g becomes birationally isomorphic to the symmetric product $C^{(g)}$ modulo the hyperelliptic involution, and hence to the Kummer variety of $\mathrm{Jac}(C)$. All of this can be found in [3, §38].

4.10. Let us see how the group G_n, in its realization as the group of pseudo-automorphisms of the blow-up $X(\mathcal{P})$, acts on the variety $\mathrm{El}(n;\, p_1, \ldots, p_n)$. For any $B \in \mathrm{El}(n;\, p_1, \ldots, p_n)$ let $\bar{B}$ be the proper inverse transform under the projection $\pi_{\mathcal{P}} : X(\mathcal{P}) \to \mathbb{P}^n$. Since $\mathcal{O}_{X(\mathcal{P})}(e_0) \cong \pi_{\mathcal{P}}^*(\mathcal{O}_{\mathbb{P}^n}(1))$, we see that $\bar{B}$ is embedded in $\mathbb{P}^n$ by the linear system $|e_0|$. It follows from (4.6) that

$$\begin{aligned} w_J(\bar{B}) \cdot e_0 &= \bar{B} \cdot w_J(e_0) \\ = \bar{B} \cdot &\Big[(k(n-1)+1)\, e_0 - (k-1)(n-1) \sum_{i \in J} e_i - k(n-1) \sum_{i \notin J} e_i \Big] \\ = (k(n-1)+1)(n+1) &- (k-1)(n-1)2k - k(n-1)(n+3-2k) = n+1. \end{aligned}$$

This shows that $B' = \pi_{\mathcal{P}}(w_J(\bar{B}))$ is embedded in $\mathbb{P}^n$ by a complete linear system. Similarly we check that

$$w_J(\bar{B}) \cdot e_i = \bar{B} \cdot w_J(e_i) = 1,$$

and hence B' passes through the points $p_1, \ldots, p_{n+3}$. Thus each w_j acts on the variety $\mathrm{El}(n, p_1, \ldots, p_{n+3})$. We would like to verify that this action corresponds to the action of the Galois group $\mathcal{G}_n$ of the cover $\mathrm{El}(n;\, p_1, \ldots, p_{n+3}) \to \mathbb{P}^4$ from Theorem 3.5. For this we consider the double covers $\phi : B \to \mathbb{P}^1$ and $\phi' : B' \to \mathbb{P}^1$ defined by the linear systems $|\mathcal{O}_B(-1) \otimes \mathcal{O}_B(p_1 + \ldots + p_{n+3})|$ and $|\mathcal{O}_{B'}(-1) \otimes \mathcal{O}_{B'}(p_1 + \ldots + p_{n+3})|$, respectively. Each maps the set $\mathcal{P} = (p_1, \ldots, p_{n+3})$ to the same set of points in $\mathbb{P}^1$, the associated set of $\mathcal{P}$. The isomorphism $w_J : B \to B'$ defines an isomorphism of the covers, and we have $\phi'(w_J(p_i)) = \phi'(p_i)$ for each $i = 1, \ldots, n+3$. Thus B and B' belong to the same fibre of the map $\mathrm{El}(n, p_1, \ldots, p_{n+3}) \to \mathbb{P}^4$.

The homomorphism $G_n \to \mathcal{G}_n$ which we have just constructed is not injective in the case when $n = 2g-1$ is odd. In fact, the linear system $\left|-\frac{1}{2}K_{X(\mathcal{P})}\right|$ cuts out a g_2^1 on each $\bar{B}$. Since $w_{[n+3]}$ acts identically on $\left|-\frac{1}{2}K_{X(\mathcal{P})}\right|$ it induces an automorphism of $\bar{B}$. Thus $B' = B$ in this case. It is clear that the fixed points of $w_{[n+3]}$ on $\bar{B}$ are the points of intersection of $\bar{B}$ and the Weddle variety W_g. In particular, we obtain

$$\#\bar{B} \cap W_g = 4 \tag{4.8}$$

unless all curves b are contained in W_g. However, by using the next lemma and induction on g we see that the latter does not happen.

4.11. Lemma. *For any $J \in V$ with $\#J = 2k$, let F_J be the set of fixed points of the involution w_J in $X_{\mathcal{P}}$. Then* $\dim F_J = k-1$. *If $k > 2$, the projection map π_J from the linear space spanned by the points $p_i,\, i \in \bar{J}$ defines a finite map of degree 2^{n+2-2k} from $\pi_{\mathcal{P}}(F_J)$ to the Weddle variety W_{k-1} in $\mathbb{P}^{2k-3}$ corresponding to the set $\pi_J(\{p_i\}_{i \in J})$.*

Proof. See [4, p. 456]. □

4.12. Theorem. *The kernel of the homomorphism $G_n \to \mathcal{G}_n$ is of order ≤ 2. It is generated by $w_{[n+3]}$ if n is odd.*

Proof. Let K be the kernel of the homomorphism $G_n \to \mathcal{G}_n$ and $B \in \mathrm{El}(n; p_1, \ldots, p_{n+3})$. Assume n is odd. Since K leaves the linear system $\left|-\frac{1}{2}K_{X(\mathcal{P})}\right|$ invariant, K leaves the g_2^1 on B cut by this linear system invariant. Thus the image of K in $\mathrm{Aut}(B)$ is equal to the image of $w_{[n+3]}$. If $w_J \in K$ acts identically on B, then B is contained in the fixed locus F_J of w_J. Hence the union of all curves from $\mathrm{El}(n; p_1, \ldots, p_{n+3})$ is contained in F_J. Let $\#J = 2k$. If $k \leq 2$, this cannot happen since $\dim F_J$ is too small. So we may assume that $k > 3$. Projecting from the subspace spanned by the points $p_i, i \in \bar{J}$, we obtain that the Weddle variety W_{k-1} contains all elliptic curves from $\mathrm{El}(2k-3; q_1, \ldots, q_{2k})$. But this contradicts (4.8).

Assume n is even. Assume B is preserved under some w_J. Since w_j sends the set of points $p_1, \ldots, p_{n+3}$ to a projectively equivalent set of points, we may assume that w_J induces a projective transformation of B. Since B is embedded in $\mathbb{P}^n$ by a linear system of odd degree, a projective automorphism of B of order 2 cannot be a translation by a 2-torsion point (when we fix a group law on B). It must fix 4 points on B. Now we finish as in the previous case. □

4.13. Remark. In the case n is even, the homomorphism $G_n \to \mathcal{G}_n$ must be injective. However I cannot prove it.

5. A rational elliptic fibration on $\mathbb{P}^5$

5.1. Now we consider the special case $n = 5, m = 9$. By Theorem 3.2

$$\mathrm{El}(5; p_1, \ldots, p_9) \cong \mathrm{El}(2; q_1, \ldots, q_9),$$

where $\mathcal{Q} = (q_1, \ldots, q_9)$ is the point set associated to $\mathcal{P} = (p_1, \ldots, p_9)$. It is well known that there is a unique nonsingular cubic through a general set of 9 points in $\mathbb{P}^2$. Thus we obtain the following:

5.2. Theorem (A. Coble). *Let $\{p_1, \ldots, p_9\}$ be a general set of 9 points in $\mathbb{P}^5$. Then there exists a unique elliptic curve of degree 6 containing these points.*

5.3. For a special set of 9 points $\mathcal{Q}$ in $\mathbb{P}^2$, the dimension of the linear system of cubics through $\mathcal{Q}$ varies from 0 to 5. However, if we assume that $\mathrm{El}(2; q_1, \ldots, q_9) \neq \emptyset$, then $\dim \mathrm{El}(2; q_1, \ldots, q_9) \leq 1$.

5.4. Proposition. *Assume that* $\dim El(2; q_1, \ldots, q_9) = 1$. *Then the linear system defining the associated set of points $\mathcal{P} = (p_1, \ldots, p_9)$ in $\mathbb{P}^5$ is equal to the restriction of the complete linear system of conics. In particular, $\mathcal{P}$ is equal to the image of $\mathcal{Q}$ under a Veronese map $v_2 : \mathbb{P}^2 \to \mathbb{P}^5$.*

Proof. The scheme $Z = \{q_1, \ldots, q_9\}$ is cut out by two cubics, and so is arithmetically Gorenstein, i.e., its homogeneous coordinate ring is Gorenstein. Now the assertion follows from Corollary 2.6 of [8]. □

5.5. Remark. The assertion is claimed by Coble in [2]. The proof uses a method which has not been justified so far (see Problem 2.3 in [8]).

5.6. Lemma (A. Coble). *Let E be an elliptic curve of degree 6 in $\mathbb{P}^5$. Then E is contained in exactly 4 Veronese surfaces.*

Proof. Take 9 points on E such that the associated set of points in $\mathbb{P}^2$ lies on a pencil of nonsingular cubics C_t. By Proposition 5.4 and Theorem 3.2, E is the image of some C_t under a Veronese map. Thus E is contained in some Veronese surface (this can also be verified by counting constants). Let V be a Veronese surface containing E. Then the linear system $|D|$ on E cut out by the complete linear system L, where L is an effective generator of $\mathrm{Pic}(V)$, satisfies $|2D| = |\mathcal{O}_E(1)|$. Also, it comes with a fixed isomorphism $\phi : S^2(H^0(\mathcal{O}_E(D))) \to H^0(\mathcal{O}_E(2D))$. Suppose two Veronese surfaces V and V' contain E and cut out the same linear system $|D|$ on E. Then the two embeddings $E \to V, E \to V'$ differ by a linear endomorphism T of $H^0(E, \mathcal{O}_E(D))$ such that $S^2(T)$ is the identity. This obviously implies that T is the identity, and hence $V = V'$. On other hand, if $|D| \neq \left|D'\right|$, we can choose an isomorphism $S^2(H^0(E, \mathcal{O}_E(D))) \to S^2(H^0(E, \mathcal{O}_E(D')))$ such that the compositions of two embeddings $E \to |\mathcal{O}_E(D)|^*$ and $E \to |\mathcal{O}_E(D')|^*$ with the corresponding Veronese maps $|\mathcal{O}_E(D)|^* \to |\mathcal{O}_E(2D)|^*$ and $\left|\mathcal{O}_E(D')\right|^* \to \left|\mathcal{O}_E(2D')\right|^*$ are the same. This proves the assertion. □

5.7. Let $E \in \mathrm{El}(5, p_1, \ldots, p_8)$. Let V be a Veronese surface containing E. We may assume that V is the image of the Veronese map $v_2 : \mathbb{P}^2 \to \mathbb{P}^5$ and $E = v_2(C)$ for some nonsingular plane cubic C. Let $q_1, \ldots, q_8$ be the points on C such that $v_2(q_i) = p_i, i = 1, \ldots, 8$. Choose the ninth point q_9 such that $\mathcal{Q} = \{q_1, \ldots, q_9\}$ is the base locus of a pencil of cubics. Then, by Proposition 5.4 the associated set of points $\mathcal{Q}^{\mathrm{ass}}$ is the image under a Veronese map. We can fix the Veronese map by requiring that $v_2(p_i) = q_i,\ i = 1, \ldots, 8$. Let $v_2(q_9) = p_9$. Then by Theorem 3.2, $\mathrm{El}(5; p_1, \ldots, p_9) \cong \mathrm{El}(2; q_1, \ldots, q_9)$ is an open subset of $\mathbb{P}^1$. We want to show that the point p_9 is on the Weddle variety W_3 corresponding to the set of points $(p_1, \ldots, p_8)$.

5.8. Lemma. *Let $(q_1, \ldots, q_9)$ be an ordered set of base points of an irreducible pencil of cubic curves. Then there exists a unique plane curve C of degree 5 with a triple point at q_9 such that the lines $\langle q_i, q_9\rangle$ are tangents to C at q_i. Conversely, given such a curve C, there exists a pencil of cubic curves with base points at $q_1, \ldots, q_9$.*

Proof. We give a modern version of the proof given by Coble in [3, §38]. Let X be the blow-up of the points $q_1, \ldots, q_9$. This has a structure of an elliptic surface together with the zero section defined by the exceptional curve E_9 blown up from q_9. Let F be the divisor class of a fibre and S be the divisor class of the proper transform of a line through the point q_9. Then $|F + S|$ defines a double cover $f : X \to \mathbb{P}^1 \times \mathbb{P}^1$ such that the pre-image of the first ruling is equal to the pencil $|F|$, and the pre-image of the second ruling is equal to the pencil $|S|$. The branch curve B of f is a curve of bi-degree $(4, 2)$. It is a hyperelliptic curve of genus 3. Its pre-image in X is equal to $2R$, where R is isomorphic to B. On each nonsingular fibre F there are two ramification points of the pencil $|S|$ restricted to F. The curve R is equal to the locus of these points when F varies in the pencil. Similarly, for each member of $|S|$ there are two ramification points of the restriction of the pencil $|F|$, and these ramification points are also on R.

Let us show that the image of R in $\mathbb{P}^2$ is the hyperelliptic curve C from the assertion of the lemma. The image $\bar{E}_9$ of E_9 is a curve of bi-degree $(1, 1)$ such that $f^{-1}(\bar{E}_9) = E_9 + E'_9$, where E'_9 is the image of E_9 under the involution of X defined by the double cover f. Since $\bar{E}_9$ is tangent everywhere to the branch curve R, we see that E_9 intersects R at 3 points. Thus C has a triple point at q_9. Let E_i be the exceptional curve blown up from q_i and let S_i be the proper inverse transform of the line $\langle q_i, q_9\rangle$. Then $E_i + S_i \in |S|$, and $f(E_i) = f(S_i)$ is a fibre of the second ruling of $\mathbb{P}^1 \times \mathbb{P}^1$ which is tangent to B at a point P such that $f^{-1}(P) = E_i \cap S_i$. This shows that $E_i \cap S_i \cap R \neq \emptyset$, and hence C is tangent to $\langle q_i, q_9\rangle$ at q_i. Thus C satisfies all the properties from the assertion of the lemma. Let us show its uniqueness. Suppose C' is another curve satisfying the same properties. Let R' be its proper inverse transform on X. We have $R \cdot R' = 25 - 9 - 16 = 0$, and $R^2 = R'^2 = 25 - 9 - 8 = 8$. This contradicts the Hodge Index Theorem.

It remains to prove the converse. We will be brief, letting the reader fill in the details. Let R be the proper inverse transform of C in X. Then the linear system $|R - S|$ defines a double cover $f : X \to \mathbb{P}^1 \times \mathbb{P}^1 \subset \mathbb{P}^3$. If the restriction of f to S is of degree 1, then its image is a plane section of $\mathbb{P}^1 \times \mathbb{P}^1$ and hence $|R - 2S| = \{S'\}$ for some curve isomorphic to S. However, $(R - 2S)^2 = 4$, $(R - 2S) \cdot K_X = 0$ imply that the arithmetical genus of S' is equal to 1. This contradiction proves that the image of S must be a fibre of a ruling on $\mathbb{P}^1 \times \mathbb{P}^1$. This implies that $|R - 2S|$ is a pencil of elliptic curves (the pre-image of another ruling). Its image in $\mathbb{P}^2$ is a pencil of cubics with base points $q_1, \dots, q_9$. □

5.9. Theorem. *Let $p_1, \dots, p_8$ be general points in $\mathbb{P}^5$. Then the locus of point $p_9 \in \mathbb{P}^5$ is such that* $\dim El(5; p_1, \dots, p_9) = 1$ *is equal to the Weddle variety W_3 associated to the points $p_1, \dots, p_8$.*

Proof. We know that if $\dim \mathrm{El}(5; p_1, \dots, p_9) = 1$, then $(p_1, \dots, p_9)$ is associated to a base point set of a pencil of cubics in $\mathbb{P}^2$. Now the assertion follows from 4.9 and the previous lemma. □

5.10. Remark. The assertion of Theorem 5.9 agrees with (4.8) and Lemma 5.6. We have 4 Veronese surfaces V_i on $E \in \mathrm{El}(5; p_1, \dots, p_8)$. In each V_i we find the ninth point $p_9 \in E$ such that there exists a pencil of cubics through $p_1, \dots, p_9$. The point p_9 must belong to W_3 and we have $E \cap W_3 = 4$.

5.11. Remark. It was shown by Coble [4, p. 489] that W_3 is of degree 19 with singular points at $p_1, \dots, p_8$ of multiplicity 9, triple lines $\langle p_i, p_j\rangle$ and the triple curve equal to the rational normal curve through the points $p_1, \dots, p_8$.

5.12. By Theorem 5.2 we have a rational elliptic fibration

$$\Phi : \mathbb{P}^5 - \to \mathrm{El}(5; p_1, \dots, p_8)$$

defined by the universal family of elliptic curves passing through the points $p_1, \dots, p_8$. Its set of points of indeterminacy is the Weddle variety W_3. If we blow up $p_1, \dots, p_9$ and then blow up the proper inverse transform $\tilde{W}_3$ of W_3 we expect to get a regular map $\tilde{\Phi} : \tilde{X} \to \mathrm{El}(5; p_1, \dots, p_8)$. Let $\pi : \tilde{X} \to \mathbb{P}^5$ be the composition of these blow-ups. For each nonsingular point $p \in W_3 \setminus \{p_1, \dots, p_8\}$ the image of the line $\pi^{-1}(p)$ under $\tilde{\Phi}$ in $\mathrm{El}(5; p_1, \dots, p_8)$ is the pencil of elliptic curves from $\mathrm{El}(5; p_1, \dots, p_8)$ passing though p. Thus there are 4 rational curves through each general point in the variety $\mathrm{El}(5; p_1, \dots, p_8)$. The restriction of $\tilde{\Phi}$ over each such curve is a rational elliptic surface corresponding to the associated set of points $\mathcal{Q}$ of $\mathcal{P} = (p_1, \dots, p_8, p)$ in $\mathbb{P}^2$. It is isomorphic to the proper inverse transform $\tilde{V}$ under π of a Veronese surface containing $(p_1, \dots, p_8, p)$. In the Cremona action $\mathrm{cr}_{2,9} : W_{2,9} \to \mathrm{Bir}(P_2^9)$ the orbit of $\mathcal{Q}$ is fixed under the subgroup G isomorphic to $H = \mathbb{Z}^8 \rtimes (\mathbb{Z}/2\mathbb{Z})$ (see [7, p. 124]). Using (4.3), we obtain that the orbit of the points set $\mathcal{P}$ is fixed under the subgroup isomorphic to H in the Cremona action $\mathrm{cr}_{5,9}$. In particular, we find that the group of pseudo-automorphisms of the blow-up $X(\mathcal{P})$ contains an infinite group G' isomorphic to H. Let V' be the proper inverse transform of the Veronese surface V in $X(\mathcal{P})$. The projection $\tilde{X} \to X(\mathcal{P})$ defines an isomorphism $\tilde{V} \to V'$. The group G leaves V' invariant and its action on V' corresponds to the action of G on $X(\mathcal{Q})$. Apparently (I have not checked it), the action of H on each $\tilde{V}$ is the restriction of the automorphism group of the generic fibre of the elliptic fibration $\tilde{\Phi}$ isomorphic to H.

5.13. Remark. The fact that $X(\mathcal{P})$ has an infinite group of pseudo-automorphisms was used by S. Mukai [9] to construct an example of a linear action of the additive group $\mathbb{G}_a^3$ in the polynomial algebra $\mathbb{C}[T_1, \dots, T_9]$ such that the algebra of invariants is not finitely generated.

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