THE ESSENTIAL AND CREMONA DIMENSIONS OF A GROUP

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ABSTRACT. The Cremona dimension of a group G is the minimal n such that G is isomorphic to a subgroup of the Cremona group of birational transformations of an n-dimensional rational variety. In this survey article, we give many examples that give evidence to the conjecture that the Cremona dimension of a finite group over the field of complex numbers is less than or equal to the essential dimension of the group.

1. Introduction

The notion of the essential dimension of a finite group G was introduced by A. Buhler and Z. Reichstein in 1997 [6]. Since then, it has become the subject of investigation in different fields of algebra and algebraic geometry. Roughly speaking, the essential dimension $\mathrm{ed}_k(G)$ of G over a fixed field k is the smallest number of parameters needed to define a generic equation over k with the Galois group isomorphic to G. Viewing the Galois extension defined by the equation as a torsor of G, this notion can be extended to any algebraic group over k.

In a talk at the Joint Meeting of the American and Chilean Mathematical Societies in Pucòn in 2010, based on some examples in small dimension, the author conjectured that the Cremona dimension of a finite group over the field of complex numbers is greater than or equal to the essential dimension of the group. In the present note, we give a more systematic list of examples confirming this conjecture. In particular, we check the conjecture for finite p-groups, for groups generated by pseudo-reflections, and some simple or almost simple groups. I use the opportunity to survey some of the known results on the essential dimension and subgroups of the Cremona group.

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We work over an algebraically closed field k of characteristic $p \geq 0$. We employ the ATLAS notations for finite groups (see [7]). For example, if p is prime, p^n denotes $(\mathbb{Z}/p\mathbb{Z})^{\oplus n}$.

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2. The Essential and Cremona dimensions: definitions

The origin of the notion of the essential dimension $\mathrm{ed}_k(G)$ of a finite group G goes back to the classical notion of the Tschirnhaus transformation (Ehrenfried Walther von Tschirnhaus, 1659-1708). The formal definition was provided by Buhler and Reichstein in 1997 [6].

Let L/K be a finite separable extension of fields containing a fixed subfield k. The essential dimension $\mathrm{ed}_k(L/K)$ of L/K is the smallest transcendence degree of a subfield E/k of K such that L is generated by an element with the minimal polynomial $g(t) \in E[t]$.

Let G be a finite group. The essential dimension $\mathrm{ed}_k(G)$ of G is the essential dimension of the field extension $k(V)/k(V)^G$, where V is a linear faithful representation of G over k. It can be shown that $\mathrm{ed}_k(G)$ does not depend on the choice of V. It is easy to see that $\mathrm{ed}_k(G)$ is equal to the minimal dimension of a G-variety X (we assume that the action of G is *birational and faithful*) such that there exists an equivariant rational map $V \dashrightarrow X$ (a *compression*).

For example, we can take $K = k(a_1, \ldots, a_n)$, where $a_1, \ldots, a_n$ are algebraically independent over a field k of characteristic 0, and consider $L = K[t]/(f(t))$ with minimal polynomial $f(t) = t^n + a_1t^{n-1} + \cdots + a_n$. We set $d_k(n) := \mathrm{ed}_k(L/K)$. Since the Galois group of $f(t)$ is isomorphic to $\mathfrak{S}_n$ which acts linearly in $\overline{K}^n$ by permuting the roots, we see that

$$d_k(n) = \mathrm{ed}_k(\mathfrak{S}_n).$$

By Tschirnhaus's transformation (also known as Tschirnhausen or Tschirnhaus transformation) that replaces t with $t + \frac{1}{n}a_1$, we get $d_{\mathbb{C}}(n) \leq n-1$. It is easy to see that $d_{\mathbb{C}}(3) = 1$ and $d_{\mathbb{C}}(4) = 2$. Hermite proved that $d_{\mathbb{C}}(5) \leq 2$ and Klein proved that $d_{\mathbb{C}}(5) = 2$. Serre proved that $d_{\mathbb{C}}(6) = 3$ [44, 3.6] and Duncan proved that $d_{\mathbb{C}}(7) = 4$ [22]. It is known that, for $n \geq 5$,

$$n - 3 \geq \mathrm{ed}_{\mathbb{C}}(\mathfrak{S}_n) = d_k(n) \geq \left[\frac{n}{2}\right]$$

(see [6]). Moreover, the function $d_k(n)$ is not decreasing and

$$\mathrm{ed}_k(\mathfrak{S}_{n+2}) \geq \mathrm{ed}_k(\mathfrak{S}_n) + 1 \tag{2.1}$$

[6, 6.3]. It is conjectured that $\mathrm{ed}_k(\mathfrak{S}_n) = n - 3$ if $\mathrm{char}(k) = 0$ [25].

A compression X of V is obviously *G-unirational*, i.e., it is the image of a dominant rational G-map $\mathbb{P}^n \to X$. In particular, X is a unirational and, hence, a *rationally connected* algebraic G-variety. We define the *rational connected dimension* of G to be

$$\mathrm{rcd}_k(G) = \min\{\dim X : X \text{ is a rationally connected } G\text{-variety over } k\}.$$

Obviously,

$$\mathrm{rcd}_k(G) \leq \mathrm{ed}_k(G). \tag{2.2}$$

A rational variety is rationally connected. We can introduce the *Cremona dimension*

$$\mathrm{crd}_k(G) = \min\{n : \ G \text{ embeds in } \mathrm{Cr}_k(n)\}.$$

Here, $\mathrm{Cr}_k(n)$ denotes the group of birational automorphisms of $\mathbb{P}^n_k$. In algebraic terms, $\mathrm{Cr}_k(n) = \mathrm{Aut}(k(t_1, \ldots, t_n))/k)$. Obviously,

$$\mathrm{rcd}_k(G) \le \mathrm{crd}_k(G).$$

In a talk at the Joint Meeting of the American and Chilean Mathematical Societies in Pucòn in 2010, I conjectured that for $k = \mathbb{C}$,

$$\mathrm{crd}_k(G) \le \mathrm{ed}_k(G). \tag{2.3}$$

Since I have no idea how to prove this conjecture, I suggest looking for a counterexample. However, it will be hard since many examples discussed in this paper suggest that the conjecture could be true.

A finite group of essential dimension 0 is trivial. A finite group G with $\mathrm{ed}_{\mathbb{C}}(G) = 1$ is isomorphic to a cyclic group or a dihedral group $\mathfrak{D}_{4k+2}$ of order $2(2k+1)$. In fact, among all subgroups of $\mathrm{PGL}_2(\mathbb{C})$, only these groups can be isomorphically lifted to a subgroup of $\mathrm{GL}_2(\mathbb{C})$.

A group of essential dimension 2 must be isomorphic to a subgroup of $\mathrm{Cr}_k(2)$. In the case $k = \mathbb{C}$, groups of essential dimension 2 were classified by Duncan [21]. There are only a few of them among the possible subgroups of $\mathrm{Cr}_k(2)$. They are subgroups of $\mathrm{GL}_2(\mathbb{C})$, $L_2(7)$, $\mathfrak{S}_5$, or $\mathbb{G}_m^2 : H$, where $H = \mathfrak{D}_{2n}, n = 3, 4, 6$.

3. Basic properties

It is obvious that

$$\mathrm{crd}_k(H) \le \mathrm{crd}_k(G) \tag{3.1}$$

if H is a subgroup of G, and

$$\mathrm{crd}_k(G_1 \times G_2) \le \mathrm{crd}_k(G_1) + \mathrm{crd}_k(G_2). \tag{3.2}$$

The same properties are true for the essential dimension [6, Lemma 4.1].

Less known is the behavior of the Cremona and essential dimensions under extensions of groups. If $Z = \mathbb{Z}/p\mathbb{Z}$ and $Z.G$ is a central cyclic extension of G, then it is known that

$$\mathrm{ed}_k(G) = \mathrm{ed}_k(G/Z) + 1 \tag{3.3}$$

if $Z \cap [G, G] = \{1\}$ and $p \ne \mathrm{char}(k)$ [6, Theorem 5.3].

If $p = \mathrm{char}(k)$, and

$$1 \to G_1 \to G \to G_2 \to 1$$

is an extension with an abelian p-group G_1, then

$$\mathrm{ed}_k(G) \le \mathrm{ed}_k(G_1) + \mathrm{ed}_k(G_2) \tag{3.4}$$

(see [35], [45, Corollary 4.6]).

On the other hand, for any normal subgroup N of G, we, obviously, have

$$\mathrm{crd}_k(G) \geq \mathrm{rcd}_k(G/N).$$

One introduces the notion of the essential dimension $\mathrm{ed}_k(G,p)$ at a prime p (see [30]). One of the equivalent definitions is $\mathrm{ed}_k(G,p) := \mathrm{ed}_k(\mathrm{Syl}_p(G))$, where $\mathrm{Syl}_p(G)$ is any Sylow p-group of G. It follows that, for any p,

$$\mathrm{ed}_k(G) \geq \mathrm{ed}_k(G,p). \tag{3.5}$$

The computation of $\mathrm{ed}_k(G,p)$ is easier than the computation of $\mathrm{ed}_k(G)$. In fact, if $\mathrm{char}(k) \neq p$, the essential dimension of a finite p-group P is equal to the minimal dimension of a faithful linear representation of P [30, Theorem 4.1]. This implies

$$\mathrm{crd}_k(P) \leq \mathrm{ed}_k(P), \tag{3.6}$$

because P acts faithfully on the affine space of dimension $\mathrm{ed}_k(P)$.

In the case where G is an abelian group of rank n and the exponent of G is coprime to $\mathrm{char}(k)$,

$$\mathrm{ed}_k(G) = \mathrm{rank}(G) \tag{3.7}$$

[6, Theorem 5.1].

4. Fixed points of an action of an abelian group

Suppose $\mathrm{char}(k) = 0$ and G is an abelian group. It follows from (3.7) that $\mathrm{rcd}_k(G) \leq \mathrm{rank}(G)$. The following is a nice result of Kollàr and Zhuang [33] that gives a better inequality, for any abelian p-group G

$$\mathrm{rcd}_k(G) \geq \frac{p-1}{p}\mathrm{rank}(G). \tag{4.1}$$

This follows from the following proposition:

Proposition 4.1. *Assume* $\mathrm{char}(k) = 0$*. Let G be an abelian p-group of rank r acting on a smooth connected projective variety X of dimension n. Then,*

$$r \leq \nu_p(\chi(X, \mathcal{O}_X)) + \frac{p}{p-1}n.$$

Proof. First, we claim that G contains a subgroup H of index p^c such that

$$c \leq \nu_p(\chi(X, \mathcal{O}_X)) + \frac{n}{p-1}.$$

We skip the details of the proof. It is based on the equivariant theory of Chern classes. If H is a subgroup such that $[G : H] = p^c$ is the smallest orbit of the action of G, then all G-invariant algebraic cycles are of degree divisible by p^c. It follows from Hirzebruch's formula that the largest p-power dividing the denominator D of the Todd class $\mathrm{Td}_n(X)$ is equal to $[\frac{n}{p-1}]$. Thus, $\chi(X, \mathcal{O}_X) = \mathrm{Td}_n(X) = (D\mathrm{Td}_n(X))/D$ and $\nu_p(\chi(X, \mathcal{O}_X)) \geq c - [\frac{n}{p-1}]$.

This yields

$$r \leq \nu_p(\chi(X, \mathcal{O}_X)) + \frac{p}{p-1}n.$$

In fact, H is an abelian subgroup that acts on $T_x(X)$. Since we assumed that $\mathrm{char}(k)=0$, it acts faithfully on $T_x(X)$. Hence, the rank of H does not exceed n. So,

$$r \le c+n \le \nu_p(\chi(X,\mathcal{O}_X)) + \frac{n}{p-1} + n \le \nu_p(\chi(X,\mathcal{O}_X)) + \frac{p}{p-1}n.$$

□

The inequality (4.1) follows from the facts that $\chi(X,\mathcal{O}_X)=1$ for a rationally connected variety and that a birational action of a finite group on a complex algebraic variety X can be isomorphically lifted to a biregular action on a nonsingular model of X. The proof depends on the existence of an equivariant resolution of singularities (true in dimension 2 for any field).

The following example is taken from [38].

Example 4.2. Assume $k=\mathbb{C}$. Let $G=p^r$ with $\mathrm{rcd}_k(G)\le 3$. Then,

$$r \le \begin{cases} 7 & \text{if } p=2,\\ 5 & \text{if } p=3,\\ 4 & \text{if } p=5,7,11,13\\ 3 & \text{if } p\ge 17.\end{cases}$$

The inequality (4.1) improves Prokhorov's bound by one if $p\le 13$.

Example 4.3. Assume $\mathrm{char}(k)\ne 2$. We have

$$\mathrm{crd}_k(2^r) \le \left[\frac{r}{2}\right]$$

since $2^{2s}\cong \mathfrak{D}_4^s$ acts faithfully on $(\mathbb{P}^1)^s$. The inequality (4.1) shows that

$$\mathrm{crd}_k(2^r) = \mathrm{rcd}_k(2^r) < \mathrm{ed}_k(2^r) = r. \tag{4.2}$$

5. Extraspecial p-groups

A finite p-group G is called *extraspecial* if its center $Z(G)$ is cyclic and coincides with the commutator subgroup $[G,G]$. As a consquence, $Z(G)$ is of order p and $\bar{G}=G/Z(G)\cong p^{2n}$.

Let us recall some properties of extraspecial p-groups (see [28, Chapter 5, 5.5]). The group $G=p^{1+2n}$ is generated by elements $x_1,\ldots,x_n,y_1,\ldots,y_n,c$ with defining relations

$$[x_i,y_j]=1,\ [x_i,y_i]=c,\ c_p=1,\ x_i^p\in\langle c\rangle,\ y_i^p\in\langle c\rangle.$$

If p is odd, there are two isomorphism classes, one of exponent p denoted by p_+^{1+2n} and one of exponent p^2 denoted by p_-^{1+2n}. In the former case, we may take $x_1^p=c, x_i^p=1, i\ne 1, y_i^p=1$. In the latter cases $x_i^p=y_i^p=1$.

Extraspecial 2-groups have exponent 4, however, there are still two isomorphism classes. We may take $x_1^2=y_1^2=c, x_i^p=y_i^p=1, i\ne 1$, or $x_1^2=y_i^2=1$ for all i. In the former case, the group is denoted by 2_+^{1+2n}, and in the latter case it is denoted by 2_-^{1+2n}. The groups differ by the isomorphism class of the quadratic form $q : G/Z(G) \cong \mathbb{F}_2^{2n} \to Z(G) \cong \mathbb{F}_2$ defined by $x \mapsto x^2$. They are quadratic forms associated with the nondegenerate symplectic form $(x,y) \mapsto [x,y]$ (defined for all p). It is even (i.e., of Witt index 1) in the former case, and odd in the latter case.

The splitting field K of the group $G = p^{1+2}$ for linear representations over a field of characteristic 0 is equal to $\mathbb{Q}(\omega)$, where ω is a primitive p-root (resp. p^2-root) of unity if $G \cong p_+^{1+2}$ (resp. $G \cong p_+^{1+2}$). It admits p^2+p-1 non-isomorphic irreducible linear representations over K. Among them are $p-1$ faithful representations of dimension p and p^2 one-dimensional representations trivial on the center. The faithful representations $\phi : G \to \mathrm{GL}_p(K)$ defined by sending x to the cyclic permutation matrix, and sending y to the diagonal matrix $[\omega^{i(p-1)}, \ldots, \omega^i, 1]$ if $G \cong p_+^{1+2}$ and $[\omega^{i(1+(p-1)p)}, \ldots, \omega^{i(1+p)}, \omega^i]$ if $G \cong p_-^{1+2}$. Here $1 \le i \le p-1$.

The subgroup H generated by x_1, y_1 is an extraspecial group $p_\pm^{1+2}$ of order p^3. Its centralizer $C_G(H)$ is isomorphic to $p_+^{1+2(n-1)}$. By induction, we obtain that G is isomorphic to the central product of n copies of extraspecial p-groups p_+^{1+2} or the *central product* of one extraspecial group p_-^{1+2} and $n-1$ copies of the groups $p_+^{1+2(n-1)}$. In the case $p=2$, $2_+^{1+2} \cong \mathfrak{D}_8$ is the dihedral group, and we obtain that 2_+^{1+2n} is isomorphic to the central product of n copies of $\mathfrak{D}_8$. Also, $2_-^{1+2} \cong Q_8$, the quaternion group, and we obtain that 2_-^{1+2n} is isomorphic to the central product of Q_8 and $n-1$ copies of $\mathfrak{D}_8$.

An irreducible linear representation of a central product of groups G_i with the same splitting field K is isomorphic to the tensor product of irreducible representations of the groups G_i over K [28, Chapter 3, Theorem 7.1]. This shows that the group p^{1+2n} admits $p-1$ non-isomorphic irreducible linear representations of dimension p^n and p^{2n} one-dimensional representations.

Recall that $G/Z(G) \cong \mathbb{F}_p^{2n}$ has a structure of a non-degenerate symplectic space. The subspace generated by the cosets x_i and the subspace generated by the cosets of y_i form a pair of complementary maximal isotropic subspaces. It follows that the quotient of the normalizer $N(G)$ of the subgroup G in $\mathrm{SL}_{2n}(p)$ modulo G is a subgroup of $\mathrm{Sp}_{2n}(p)$. This defines an extension

$$1 \to p^{1+2n} \to H(p,n) \to H \to 1,$$

where H is a subgroup of $\mathrm{Sp}_{2n}(p)$. It is known that

$$H = \begin{cases} \mathrm{Sp}_{2n}(p) & \text{if } G = p_+^{1+2n}, p > 2, \\ p_-^{1+2(n-1)} : \mathrm{Sp}_{2n-2}(p) & \text{if } G = p_-^{1+2n}, p > 2, \\ \mathrm{GO}_{2n}^{\pm}(2) & \text{if } G = 2_\pm^{1+2n} \end{cases} \tag{5.1}$$

(see [46]).

Suppose an extraspecial p-group $G = p^{1+2n}$ acts on a complex rational variety X of dimension n. Then, the quotient $Y = X/Z(G)$ is unirational and, hence, rationally connected. The inequality (4.1) yields

$$p^n \geq \mathrm{crd}_{\mathbb{C}}(G) \geq \mathrm{rcd}_{\mathbb{C}}(p^{2n}) \geq \frac{p-1}{p}2n.$$

Following Zinovy Reichstein (a personal communication), we will prove the following:

Proposition 5.1.

$$\mathrm{crd}_{\mathbb{C}}(p^{1+2n}) \leq pn.$$

Proof. The group $G = p^{1+2n}$ is the central product of the subgroups $G_i = \langle x_i, y_i \rangle$, i.e., the quotient of the direct product $G_1 \times \cdots \times G_n$ by the subgroup $\langle (c_1, \ldots, c_n) \rangle$, where c_i generates the center of G_i.

The group G acts naturally on $X = (\mathbb{C}^p)^n / T[p]$, where $T[p]$ is the group of p-torsion points of the adjoint torus $T = (\mathbb{C}^*)^n / \mathbb{C}^*)$. The torus acts on $(\mathbb{C}^p)^n$ by multiplying diagonally each ith factor by $\alpha_i \in \mu_p$ with $\alpha_1 \cdots \alpha_n = 1$. Since the quotient of a linear action by an abelian group is known to be a rational variety, we conclude the proof. □

It follows from the result of Karpenko and Merkurjev, which we cited in Section 3, that

$$\mathrm{ed}_{\mathbb{C}}(p^{1+2n}) = p^n.$$

So, if $n > 1$, (5.1) gives a strong inequality

$$\mathrm{crd}_k(p^{1+2n}) < \mathrm{ed}_{\mathbb{C}}(p^{1+2n}).$$

Note that, in our case, one cannot apply (3.3) because the commutator subgroup of p^{1+2g} coincides with its center.

The group p_+^{1+2n} is known in the theory of abelian varieties as a subgroup $\mu_p.\mathbb{F}_p^n$ of the *Heisenberg group* $\mathcal{H}_{p,n}$, a central extension $\mathbb{C}^*.\mathbb{F}_p^{2n}$. We identify the linear space V of dimension p^n with the group algebra of the subgroup P generated by $x_1, \ldots, x_n$. Fix an isomorphism $e : Z(G) = \langle c \rangle \to \mu_p$ that sends c to a primitive p-root ω of unity, and let G act on complex valued functions $f(t), t \in P$, by

$$(x_i \cdot f)(t) = f(t + x_i), \quad (y_i \cdot f)(t) = e([y_i, t])f(t), \quad (c \cdot f)(t) = \alpha f(t).$$

The choice of an isomorphism $\mathbb{Z}/p\mathbb{Z} \to \mu_p$ defines $p - 1$ non-isomorphic representations of the extra-special group coming from the *Schrödinger representation* of $\mathcal{H}_{p,n}$, where $\mathbb{C}^*$ acts by scalar multiplication.

Example 5.2. Assume $p = 3$ and $n = 1$. The Heisenberg group 3_+^{1+3} has a linear representation in $\mathbb{C}^3$. The normalizer group N of G in $\mathrm{Sp}_2(\mathbb{C}) \cong \mathrm{SL}_2(\mathbb{C})$ is isomorphic to the extension (5.1)

$$1 \to 3_+^{1+3} \to H_{3,1} \to \mathrm{SL}_2(\mathbb{F}_3) \to 1.$$

The quotient group $H_{3,1}/Z(H_{3,1})$ is isomorphic to a subgroup G_{216} of order 216 of $\mathrm{PGL}_3(\mathbb{C})$, classically known as the *Hessian group*. The group G_{216}

leaves invariant the Hesse pencil of cubic curves. The subgroup $3^2 = G/Z(G)$ leaves invariant all members of the pencil, and the quotient group $\mathrm{SL}_2(\mathbb{F}_3)$ acts on the pencil via the natural action of $\mathrm{SL}_2(\mathbb{F}_3) \cong 2.\mathfrak{A}_4$ in $\mathbb{P}^1$. The group 3_+^{1+3} admits a faithful action on the triple cyclic cover of $\mathbb{P}^2$ branched over any member of the Hesse pencil. This is a cyclic cubic surface, so $\mathrm{crd}_{\mathbb{C}}(3_+^{1+3}) = 2$ improving (5.1) in this case.

Example 5.3. Assume $n = 2$ and $p = 3$. The group 3_+^{1+4} acts faithfully and linearly in $\mathbb{C}^9$. For any principally polarized abelian variety $(A, \mathcal{L})$ of dimension 2, the linear system $|\mathcal{L}^{\otimes 3}|$ embeds A in $\mathbb{P}(\Gamma(A, \mathcal{L}^{\otimes 3})) \cong \mathbb{P}^8$. The group of 3-torsion points $A[3] \cong 3^4$ acts on A but does not lift to an action on $\Gamma(A, \mathcal{L}^{\otimes 3})$. However, its central extension 3_+^{1+4} lifts to a linear action on this space and realizes the Schrödinger representation in $\mathbb{C}^9$. It was proved by Coble that the image of A in $\mathbb{P}^8$ is equal to the singular locus of G-invariant cubic hypersurface, called the *Coble cubic*. This remarkable fact has been extended to all $n = \dim A \geq 2$ and $p = 3$ in [1]. The group G acts faithfully on the affine cone over the Coble cubic in $\mathbb{C}^{n^3}$. Since a singular cubic hypersurface is rational, we obtain that $\mathrm{crd}_{\mathbb{C}}(3^{1+2n}) \leq 3^n - 1$. Reichstein's inequality improves substantially on this inequality.

The group $\mathrm{Sp}_{2n}(3) = H_{3,n}/3_+^{1+2n}$ acts in $V := \Gamma(A, \mathcal{L}^{\otimes 3})) \cong \mathbb{C}^{3^n}$. The corresponding linear representation is not irreducible, but decomposes into the direct sum of two linear subspaces V^+ and V^- of dimensions $\frac{1}{2}3^n(3^n + 1)$ and $\frac{1}{2}3^n(3^n - 1)$, respectively. They are eigensubspaces of the central involution of $\mathrm{Sp}_{2n}(3)$. The action of $\mathrm{Sp}_{2n}(3)$ (resp. $\mathrm{PSp}_{2n}(3)$) in V^- (resp. V^+) is the faithful linear representation of smallest dimension.

In the special case $n = 2$, the group $\mathrm{PSp}_4(3)$ is a simple group isomorphic to the index 2 subgroup $W(E_6)'$ of the Weyl group $W(E_6)$ of the root lattice of type E_6. In its projective representation in $\mathbb{P}(V^+) \cong \mathbb{P}^4$, it leaves invariant a rational quartic hypersurface. It is, classically, known as the *Bürhardt quartic*. It is distinguished from other quartic hypersurfaces in $\mathbb{P}^4$ by the property that it admits a maximal possible number of isolated singular points (equal to 45). This shows that

$$\mathrm{crd}_{\mathbb{C}}(\mathrm{PSp}_4(3)) = 3, \quad \mathrm{crd}_k(\mathrm{Sp}_4(3)) = 4. \tag{5.2}$$

(the classification of finite subgroups of $\mathrm{Cr}_C(2)$ shows that $\mathrm{crd}_{\mathbb{C}}(\mathrm{PSp}_4(3)) > 2$, and we will see later that $\mathrm{crd}_k(\mathrm{Sp}_4(3)) > 3$).

We will also see later that $\mathrm{ed}_{\mathbb{C}}(\mathrm{PSp}_4(3)) \geq 4$, and since $\mathrm{PSp}_4(3)$ acts faithfully on $V^+ \cong \mathbb{C}^5$, we obtain $\mathrm{ed}_{\mathbb{C}}(\mathrm{PSp}_4(3)) \leq 5$. In any case,

$$\mathrm{crd}_k(\mathrm{PSp}_4(3)) \leq \mathrm{ed}_{\mathbb{C}}(\mathrm{PSp}_4(3)).$$

It is known that, for any prime $p > 2$, $\mathrm{ed}_{\mathbb{C}}(\mathrm{Sp}_{2n}(p^r), p) \geq rp^{n-1}$ [32]. In particular, since we know that $\mathrm{Sp}_4(3)$ acts faithfully on $V^- \cong \mathbb{C}^4$, we obtain $4 \geq \mathrm{ed}_{\mathbb{C}}(\mathrm{Sp}_4(3)) \geq 3$. On the other hand, we will show later that $\mathrm{crd}_{\mathbb{C}}(\mathrm{Sp}_4(3)) \geq 4$. So, here we have a potential counter-example, if it turns out that $\mathrm{ed}(\mathrm{Sp}_4(3)) = 3$.

Note that the group $2 \times \mathrm{PSp}_4(2)$ acts in $V^+ \cong \mathbb{C}^5$ as a group generated by complex pseudo-reflections. It is a group No. 33 in the Shephard-Todd list of irreducible complex pseudo-reflection groups.

The group $\mathrm{Sp}_4(3)$ acts in $V^- \cong \mathbb{C}^4$ and defines a finite subgroup of $\mathrm{Aut}(\mathbb{P}^-) \cong \mathrm{PGL}_4(\mathbb{C})$. The group $3 \times \mathrm{Sp}_4(3)$ acts in $V_2(3)^-$ as a group generated by complex pseudo-reflections. It is group No. 32 in the Shephard-Todd list. It was first discovered by Maschke and, classically known, as the Maschke group.

A curious remark is that 2_+^{1+24} is a subgroup of the Monster Group $\mathbf{M}$, which occurs as the centralizer of an involution. Thus, $\mathrm{ed}_k(\mathbf{M}) \geq \mathrm{ed}_k(2_+^{1+24}) = 2^{12}$. It is known that the minimal dimension of a faithful linear representation of $\mathbf{M}$ is equal to 196383. Therefore, $2^{12} \leq \mathrm{ed}_k(\mathbf{M}) \leq 196383$.

6. Pseudo-reflection groups

Recall that a non-trivial linear transformation of finite order in a linear space V over a field K is called a *pseudo-reflection* if it is the identity on some hyperplane in V. A pseudo-reflection group is a finite group G generated by a finite set of pseudo-reflections. It is a finite reflection group if it can be generated by a finite set of pseudo-reflections of order 2. The dimension of the linear space V is called the *degree* of G and is denoted by $d(G)$. If K is algebraically closed and the order of G is not divisible by $\mathrm{char}(K)$, the algebra of invariant polynomials $K[V]^G$ is freely generated by homogeneous polynomials of degrees $2 \leq d_1 \leq \cdots \leq d_n$, where $n = \dim V$. The product $d_1 \cdots d_n$ is equal to the order of G. We will assume that G is a *tame* pseudo-reflection group, i.e., its order is prime to $\mathrm{char}(k)$.

If k is algebraically closed, which we continue to assume, a pseudo-reflection g has 1 as its eigenvalue of multiplicity $\dim V - 1$ and one simple eigenvalue. If the order of g is prime to $\mathrm{char}(k)$, g is diagonalizable and its simple eigenvalue is a primitive n-root of unity, where n is the order of g.

If $\mathrm{char}(k) = 0$, the list of isomorphism classes of irreducible (i.e., not isomorphic to the product of pseudo-reflection groups) pseudo-reflection groups coincide with the Shephard-Todd list of complex unitary reflection groups. It consists of three infinite families of groups $\mathbb{Z}/n\mathbb{Z}$, $\mathfrak{S}_n$, $G(m,p,n)$ and 34 isolated cases, which are numbered by $4, \ldots, 37$.

We already discussed the symmetric groups in Section 2. We recall that $\mathrm{ed}_{\mathbb{C}}(\mathfrak{S}_n) \geq 4$ if $n \geq 7$. The same result is known for $\mathrm{rcd}_{\mathbb{C}}(\mathfrak{S}_n)$ [39]. The groups $G(m,k,n)$ are imprimitive groups isomorphic to a semi-direct product $A(m,p,k) : \mathfrak{S}_n$, where $A(m,k,n)$ is a subgroup of $(\mathbb{Z}/m\mathbb{Z})^n$ of index p. They are generated by permutation matrices and diagonal matrices with m-th roots of unity on the diagonal, with determinant equal to m/k. Among them are Coxeter reflection groups

$$G(1,1,n) = W(A_n), \quad G(1,1,n) = W(B_n) \cong W(C_n),$$
$$G(2,2,n) = W(D_n), \quad G(m,m,2) = G_2(m).$$

Groups Nos. 4–22 act in a 2-dimensional linear space V.

So, we will discuss only the reflection groups and those for which $\dim V \geq 3$. These are the groups Nos. 23– 37.

The essential dimension of a tame pseudo-reflection group was studied by A. Duncan and Z. Reichstein [23]. They proved the following:

Proposition 6.1. *Let G be a tame irreducible pseudo-reflection group different from $\mathfrak{S}_n$. Then $\mathrm{ed}_k(G(m,m,n)) = n-1$ if $(m,n) = 1$, $\mathrm{ed}_k(W(E_6)) = 4$, and for all other groups $\mathrm{ed}_k(G) = d(G)$.*

In particular, for all pseudo-reflection groups with $\mathrm{ed}_k(G) = d(G)$, we have

$$\mathrm{crd}_k(G) \leq \mathrm{ed}_k(G).$$

Let us explain the equality

$$\mathrm{ed}_k(E_6) = 4,$$

which the authors attribute to me.

Let $W(E_n)$ be the Weyl group of a root system of type E_n defined by the Dynkin diagram of type $T_{2,3,n-3}$, and let

$$\rho_n : W(E_n) \to \mathrm{Cr}_k(2n-8) \tag{6.1}$$

be the Coble representation defined by the birational action of $W(E_n)$ on the rational quotient $\mathrm{P}_2^n := (\mathbb{P}^2)^n/\mathrm{PGL}_k(3)$ [15]. In the case $n = 6$, the Coble representation is faithful, and we get

$$\mathrm{crd}_{\mathbb{C}}(W(E_6)) \leq 4. \tag{6.2}$$

By putting six ordered points in $\mathbb{P}^2$ on a cuspidal cubic, one can define an equivariant rational map $\mathbb{A}_k^6 \dashrightarrow \mathrm{P}_2^6$ that proves that $\mathrm{ed}_k(W(E_6)) \leq 4$.

It is proven in [23] that, for any tame pseudo-reflection group

$$\mathrm{ed}_k(G,p) = |\{i : p|d_i\}|.$$

It follows from the classification that $p = 2$ or 3 divides all d_i's except in the case $G = W(E_6)$. In the latter case, 2 divides four d_i. This shows that

$$4 \geq \mathrm{ed}_k(W(E_6)) \geq \mathrm{ed}_k(W(E_6), 2) = 4 = \mathrm{crd}_k(W(E_6)).$$

Let us compare the essential and Cremona dimensions of the pseudo-reflection groups.

The imprimitive groups $G(m,m,n)$ with $\mathrm{ed}_k(G(m,m,n)) = n-1$ leave invariant the homogeneous polynomial $F = y^n + x_1 \cdots x_n$. The group acts faithfully on the rational variety $V(F) \subset \mathbb{P}^{n-1}$. This gives

$$\mathrm{crd}_k(G(m,m,n)) \leq \mathrm{ed}_k(G(m,m,n)).$$

A reflection group is a Coxeter group of one of the types

$$A_n,\ B_n,\ C_n,\ D_n,\ E_6,\ E_7,\ E_8,\ H_3,\ H_4,\ F_4,\ G_2.$$

It is isomorphic to a subgroup of the orthogonal groups and, hence, $d_1 = 2$, i.e., it admits a quadratic invariant. The converse is also true: any pseudo-reflection group G admitting a quadratic invariant is a Coxeter reflection group. Thus, G acts faithfully on an affine quadric in V, which is, obviously, rational. Therefore, for all reflection groups G with $\mathrm{ed}_k(G) = d(G)$,

$$\mathrm{crd}_k(G) < d(G) = \mathrm{ed}_k(G). \tag{6.3}$$

The Coxeter groups G of type B_n and D_n are imprimitive reflection groups. They are isomorphic to $2^n : \mathfrak{S}_n$ and $2^{n-1} : \mathfrak{S}_n$, respectively. Both of them act on $(\mathbb{P}^1)^n$ by permuting the factors and by an involution on each factor. Passing to the quotient by the diagonal action of $\mathrm{PGL}_2(k)$, we obtain that the Cremona dimensions of these groups is less than or equal to $n-3$, hence $\mathrm{crd}_k(G) \leq \mathrm{ed}_k(G) - 2$.

Using the classification of groups with $\mathrm{crd}_k(G) \leq 2$ or $\mathrm{ed}_k(G) \leq 2$, we obtain that $\mathrm{crd}_k(G) = \mathrm{ed}_k(G)$ if $d(G) = 2, 3$, unless G is a reflection group of type H_3 (No. 23) isomorphic to $2 \times \mathfrak{A}_5$. In this case, $\mathrm{ed}_k(G) = 3$, but $\mathrm{crd}_k(G) = 2$ since the group acts faithfully on $\mathbb{P}^1 \times \mathbb{P}^1$.

So, let us assume that $d(G) \geq 4$. They are groups Nos. 28–37.

Group No. 28 is of degree 4. It is the Coxeter reflection group of type F_4, isomorphic to $(\mathfrak{S}_4 \times \mathfrak{S}_4) : 2$. It was classically known as the symmetry group of the system of desmic tetrahedra. We know that $\mathrm{crd}_k(G) < \mathrm{ed}_k(G) = 4$. It is known that $\mathrm{crd}_k(G) > 2$, so we obtain $\mathrm{crd}_k(G) = 3$.

Group No. 29 is of degree 4 and order $2^5 \cdot 5!$. It is isomorphic to the group $4 \times G'$, where $G' \cong 2^4 : \mathfrak{A}_5$ is the Maschke group of projective transformations of $\mathbb{P}^3$ leaving invariant one of the six irreducible factors of a fundamental invariant of degree 24 of the group No. 31. They define six quartic surfaces in $\mathbb{P}^3$ known as the Maschke quartic surfaces with the projective symmetry group isomorphic to the Mathieu group $M_{20} \cong 2^4 : \mathfrak{S}_5$ (see [18, Remark 1], and the references therein). We know that $\mathrm{crd}_k(G) \leq \mathrm{ed}_k(G) = 4$ but I do not know whether $\mathrm{crd}_k(G) = 3$.

Group No. 30 is of degree 4. Its order is $14,400 = 120^2$. It is isomorphic to the Coxeter reflection group of type H_4. We know that $\mathrm{crd}_k(G) = 3 < \mathrm{ed}_k(G) = 4$. The group is also realized as one of the maximal subgroups of the Weyl group $W(E_8)$. It is isomorphic to $(\mathfrak{A}_5 \times \mathfrak{A}_5) : 2^2$. It is known as the group of symmetries of a regular 120-sided solid or its dual 600-sided solid in $\mathbb{R}^4$.

Group No. 31 is of degree 4 and its order is $11,520 = 2^5 \cdot 6!$. Note that the order of G is equal to the order of the Weyl group $W(D_6)$, but it is not isomorphic to this group. It is isomorphic to the group $N_{2,2} = 2_+^{1+4} : \mathfrak{S}_6$. Its extraspecial 2-subgroup acts in $\mathbb{P}^3$, leaving invariant Kummer quartic surfaces, the images of Jacobian varieties A of curves of genus 2 under the map of degree 4 given linear system $|2\Theta|$. Its general orbits are the sets of 16 nodes of Kummer surfaces. Each involution in this group has two fixed lines that give rise to the *Klein configuration* $(30_6, 60_3)$ of 30 lines and their 60 intersection points. The quotient $\mathbb{P}^3/\mathcal{H}_{2,2}$ is isomorphic to the Castelnuovo-Richmond quartic hypersurface with its Cremona-Richmond symmetri configuration (15_3) of 15 double lines intersecting by 3 at 15 points (see [17, 10.2.1]). The group G leaves invariant the Klein configuration, the set of 60 points consists of 15 tetrahedra that are permuted by $\mathfrak{S}_6$. We mentioned above that the projectivized group No. 31 contains the projectivized group No. 29 as a subgroup of index 6

We already encountered groups No. 32 and No. 33 in the previous section. These are the Burhardt and Maschke groups, which act linearly in the linear spaces $V_2(3)^{\pm}$. The Burhardt group G leaves invariant the affine cone over a rational quartic hypersurface, and hence $\mathrm{crd}_k(G) = 4 < \mathrm{ed}_k(G) = 5$.

Group No. 34 is of degree 6. Its order is $108 \cdot 9!$, and it is isomorphic to the extension $6.\mathrm{PSU}_4(3).2 = 3.\mathrm{SU}_4(3).2$ of the simple group $\mathrm{PSU}_4(3)$ (in the ATLAS notation, the latter group is $\mathrm{U}_4(3)$). It is also isomorphic to $\mathrm{O}_6^-(3)$ and a subgroup of $\mathrm{GL}_6(\mathbb{C})$ that leaves invariant a lattice of rank 6 over the Eisenstein numbers $\mathbb{Z}[\omega]$. I do not know whether $\mathrm{crd}_{\mathbb{C}}(G) < 6$. I checked that there are no G-invariant quadrics in $\mathbb{P}^5$.

Group No. 35 is the Weyl group $W(E_6)$. We discussed this group earlier.

Group No. 36 is the Weyl group $W(E_7)$. The Coble representation ρ_7 is not faithful. It is the identity on the center generated by an involution γ. We have $W(E_7)) \cong W(E_7)' \times \langle\gamma\rangle$, where $W(E_7)'$ is a simple subgroup of index 2 isomorphic to $\mathrm{Sp}_6(2)$. It is known that $\mathrm{ed}_k(\mathrm{Sp}_6(2), 2) = 6$ [32]. Thus, $\mathrm{ed}_k(\mathrm{Sp}_6(2)) \geq 6$. The minimal dimension of an irreducible linear representatrion of $\mathrm{Sp}_6(2)$ is equal to 7. Since the restriction of ρ_7 to $W(E_7)'$ is faithful, we obtain that

$$\mathrm{crd}_k(\mathrm{Sp}_6(2))) \leq 6 \leq \mathrm{ed}_k(\mathrm{Sp}_6(2)) \leq 7.$$

Finally, the group No. 38 is the Weyl group $W(E_8)$. We know that $\mathrm{crd}_k(G) < \mathrm{ed}_k(G) = 8$. The kernel of the Coble representation ρ_8 is generated by the central involution β. The group $W(E_8)/\langle\beta\rangle$ is isomorphic to $\mathrm{GO}_8^+(2)$. It contains a simple subgroup $W(E_8)'$ of index 2 isomorphic to $\mathrm{O}_8^+(2)$. It shows that

$$\mathrm{crd}_k(\mathrm{O}_8^+(2)) \leq 8.$$

It is known that $\mathrm{ed}_k(\mathrm{O}_8^+(2), 2) = 16$ [32]. Thus,

$$\mathrm{crd}_k(\mathrm{O}_8^+(2)) \leq 8 < 16 \leq \mathrm{ed}_k(\mathrm{O}_8^+(2)).$$

Note that the minimal dimension of an irreducible linear representation of $W(E_8)'$ is equal to 28.

The extraspecial 2-group 2_+^{1+8} is contained in $\mathrm{O}_8^+(2)$. Applying the Coble representation ρ_8, we obtain that, in any characteristic, $\mathrm{crd}_k(2_+^{1+8}) \leq 8$ confirming (5.1).

7. Simple and quasi-simple groups

It follows from the classification of finite subgroups of $\mathrm{Cr}_{\mathbb{C}}(2)$ that the only simple groups contained in $\mathrm{Cr}_{\mathbb{C}}(2)$ are $\mathfrak{A}_5$, $\mathfrak{A}_6$, and $L_2(7)$.

Simple groups G with $\mathrm{crd}_{\mathbb{C}}(G) = 3$ were classified by Prokhorov [40]. They are

$$\mathfrak{A}_7, \quad L_2(8), \quad W(E_6)' \cong \mathrm{PSp}_4(3) \cong \mathrm{PSU}_4(2).$$

Any non-trivial action of a simple group is faithful. The group $\mathfrak{A}_7$ admits a linear 6-dimensional representation. It acts in $\mathbb{P}^5$ leaving invariant a smooth quadric. We identify the quadric with the Grassmannian $G_1(\mathbb{P}^3)$ and use that the group of automorphisms of $G_1(\mathbb{P}^3)$ is isomorphic to a double extension of $\mathrm{Aut}(\mathbb{P}^3)$. This implies that $\mathfrak{A}_7$ acts by automorphisms of $\mathbb{P}^3$.

The group $L_2(8)$ admits a 9-dimensional irreducible representation V and leaves invariant a smooth quadric. It acts on the 10-dimensional variety $\mathrm{LG}(4,9) \subset \mathbb{P}^{15}$ of maximal isotropic subspaces in V. The projective space $\mathbb{P}^{15}$ contains two invariant subspaces of dimension 5 and 7. The intersection of these subspaces with $\mathrm{LG}(4,9)$ is a canonical curve of genus 7 with the group of automorphisms of the maximal possible order $6 \times 84 = 504 = |L_2(8)|$ and a rational Fano threefold of genus 7, respectively.

Finite simple groups with $\mathrm{ed}_{\mathbb{C}}(G) = 3$ were classified by Beauville [2]. It turns out that there is only one group $G = \mathfrak{A}_6$ and, possibly, the group $G = L_2(11)$. The inclusion of the latter group is conditional on the validity of a conjecture of Cassels and Swinnerton-Dyer: a cubic hypersurface over a field K has a rational K-point if and only if it has a K-rational 0-cycle of degree one, or, equivalently, when it has a rational point of degree prime to 3 [12]. Since $L_2(11)$ acts faithfully on the Klein cubic hypersurface in $\mathbb{P}^4$ given by the equation

$$x_0^2x_1 + x_1^2x_2 + x_2^2x_3 + x_3^2x_4 + x_4^2x_0 = 0,$$

and the latter is not rational, we obtain that our conjecture and the conjecture of Cassels and Swinnerton-Dyer are not compatible (see [24, Proposition 10.8].

Not much is known about simple groups $\mathfrak{A}_n, n \geq 8$. If $\mathrm{char}(k) = 0$, it is known that $\mathrm{ed}_k(\mathfrak{A}_n)$ is a non-decreasing function of n and

$$\mathrm{ed}_k(\mathfrak{A}_{n+4}) \geq \mathrm{ed}_k(\mathfrak{A}_n) + 2.$$

Also, it is known that

$$\mathrm{ed}_k(\mathfrak{A}_n) \geq \begin{cases} \frac{n}{2} & \text{if } n \text{ is even,} \\ \frac{n\pm 1}{2} & \text{if } n \equiv \mp 1 \mod 4 \end{cases}$$

[21]. In particular, $\mathrm{ed}_{\mathbb{C}}(\mathfrak{A}_8) \geq 4$. The character table for the group $\mathfrak{A}_8$ shows that the group admits an invariant quadric in its irreducible linear representation of dimension 6, and hence, in its faithful projective represenation in $\mathbb{P}^5$. This gives

$$\mathrm{crd}_k(\mathfrak{A}_8) = 4 \leq \mathrm{ed}_k(\mathfrak{A}_8).$$

A finite group G is called *quasi-simple* if it is perfect and the quotient of G by the center is a simple group. A quasi-simple group G with $\mathrm{rcd}_k(G) = 3$

belongs to the following list:

$$\mathrm{SL}_2(\mathbb{F}_7),\ 3.\mathfrak{A}_6,\ 2.\mathfrak{A}_6 \cong \mathrm{SL}_2(9),\ 6.\mathfrak{A}_6$$

[4]. It is not known whether the last two groups can be realized.

The group $3.\mathfrak{A}_6$ admits an irreducible 3-dimensional representation, so

$$\mathrm{crd}_{\mathbb{C}}(3.\mathfrak{A}_6) = \mathrm{ed}_{\mathbb{C}}(3.\mathfrak{A}_6) = 3.$$

The group $G = \mathrm{SL}_2(\mathbb{F}_7)$ admits an irreducible linear representation of dimension 4. It acts faithfully on the Fano varieties isomorphic to the double covers of $\mathbb{P}^3$ ramified over the surfaces given by invariants of G of degree 4 or 6. Neither of these varieties is rational. So, $\mathrm{rcd}_{\mathbb{C}}(\mathrm{SL}_2(\mathbb{F}_7)) = 3$, but I do not know whether $\mathrm{crd}_{\mathbb{C}}(\mathrm{SL}_2(\mathbb{F}_7))$ is equal to 3. We have $3 \leq \mathrm{ed}_{\mathbb{C}}(\mathrm{SL}_2(7)) \leq 4$. So, it can be a potential counterexample to my conjecture.

Note that the group $2 \times (3.\mathfrak{A}_6) \ncong 6.\mathfrak{A}_6$ (not quasi-simple in our definition) is a complex unitary reflection group of degree 3, so

$$\mathrm{crd}_{\mathbb{C}}(2 \times (3.\mathfrak{A}_6)) = \mathrm{ed}_{\mathbb{C}}(2 \times (3.\mathfrak{A}_6)) = 3.$$

It is known that $2.\mathfrak{A}_6$ admits a faithful linear representation of dimension 4, so $\mathrm{crd}_k(2.\mathfrak{A}_6) \leq 4$.

8. Positive characteristic

Not much is known about the essential dimension of finite groups of order divisible by $\mathrm{char}(k)$ (*wild groups*). In this section, we assume that this is the case.

Since $\mathrm{PGL}_{n+1}(\mathbb{F}_{p^r}) \subset \mathrm{PGL}_k(n+1)$ if $\mathrm{char}(k) = p$, we obtain that

$$\mathrm{crd}_k(\mathrm{PGL}_{n+1}(\mathbb{F}_{p^r})) \leq n.$$

In particular, $\mathrm{crd}_k(L_2(p^r)) = 1$. Also, any group p^r is contained in k, so

$$\mathrm{crd}_k(p^r) = 1.$$

The classification of finite wild subgroups of $\mathrm{PGL}_2(k)$ and $\mathrm{PGL}_3(k)$ is known (see the lists and the references in [19]). We find the following groups with $\mathrm{crd}_k(G) = 1$ besides mentioned above:

$$\mathfrak{D}_{4pk}(p \neq 2), \quad \mathfrak{D}_{4n+2}(p = 2), \quad A : \mu_s,\ A \subset k, (s,p) = 1.$$

Some new groups of Cremona dimension 2 appear as groups of automorphisms of del Pezzo surfaces in positive characteristic (see [19], [20]). For example, the group $W(E_6)'$ is the group of automorphisms of the Fermat cubic surface in characteristic 2 and the group $\mathrm{PSU}_3(2)$ is a group of automorphisms of a del Pezzo surface of degree 2 in characteristic 3.

Even for the cyclic group $\mathbb{Z}/p^r\mathbb{Z}$, we do not know either its essential dimension or its Cremona dimension. It follows from (3.4)

$$\mathrm{ed}_k(\mathbb{Z}/p^r\mathbb{Z}) \leq r.$$

It is conjectured in loc. cit. that the equality takes place.

In fact, a more general fact is true (see [45, Theorem 1.4]): for any finite p group U of order p^n, where $p = \mathrm{char}(k)$,

$$\mathrm{ed}_k(U) \le n. \tag{8.1}$$

Following Alex Duncan (a private communication), let us prove the following:

Proposition 8.1.

$$\mathrm{crd}_k(\mathbb{Z}/p^2\mathbb{Z}) = \mathrm{ed}_k(\mathbb{Z}/p^2\mathbb{Z}) = 2.^1 \tag{8.2}$$

Proof. A generator g of the group $G = \mathbb{Z}/p^2\mathbb{Z}$ acts regularly on the weighted homogeneous plane $\mathbb{P}(1,1,p-1)$ and its minimal resolution $\mathbf{F}_2$ (considered as a smooth conic bundle over $\mathbb{P}^1$) by the formula

$$g : (x,y,z) \mapsto (x+y, y, z - P(x,y)), \tag{8.3}$$

where $P(x,y) = x(x+y)\cdots(x+(p-2)y)$. We use that

$$g^p : (x,y,z) \mapsto (x,y,z-Q(x,y)),$$

where

$$Q(x,y) = \sum_{i=0}^{p-1} P(x+iy,y) \in k[x,y]_{p-2}.$$

Substituting $x = 0,1,\dots,p-1, y = 1$, we find that the values of $Q(x,1)$ are equal to $(p-1)! \mod p = -1 \mod p$. Since the polynomial $Q(x,1)$ is of degree $p-2$, we get that $Q(x,1) = -1$. This implies that g^p is of order p. □

In fact, as it was pointed out to me by Alex Duncan and Gebhard Martin, one can use the translation action on the ring $W_n(k)$ of truncated Witt vectors of length n to define a faithful action of $\mathbb{Z}/p^n\mathbb{Z}$ on $\mathbb{A}_k^n \cong \mathrm{Spec}\ W_n(k)$ proving

$$\mathrm{crd}_k(\mathbb{Z}/p^n\mathbb{Z}) \le n. \tag{8.4}$$

The translation action can be extended to a smooth equivariant compactification $\overline{W}_n(k)$ of $W_n(k)$ constructed inductively as a compactification of a line bundle over $\overline{W}_{n-1}(k)$ starting from $\overline{W}_2(k) = \mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(p))$ [26].

Let me elaborate on this and derive an explicit formula for an action of $\mathbb{Z}/27\mathbb{Z}$ on $W_3(k)$.

Recall that the ring $W_n(k)$ consits of vectors $a = (a_0, a_1, \dots, a_{n-1}) \in k^n$ with the addition $a + b := (S_0(a,b), S_1(a,b), \dots, S_{n-1}(a,b)$, where the polynomials $S_i(x,y) \in \mathbb{Z}[x_0,\dots,x_i,y_0,\dots,y_i]$ are defined by the property

$$w_n(x_0,\dots,x_n)+w_n(y_0,\dots,y_n) = w_n(S_0(x_0,y_0),\dots,S_n(x_0,\dots,x_n,y_0,\dots,y_n)),$$

where

$$w_n(X_0,\dots,X_n) = \sum_{i=0}^n p^i X_i^{p^n}.$$

[1]This corrects a mistake in [16, Theorem 8], where it is asserted that $\mathrm{crd}_k(\mathbb{Z}/p^2\mathbb{Z}) > 2$ if $p > 2$.

There are two operations V and F on $W_n(k)$ defined by

$$V(a_0,\dots,a_{n-1}) = (0,a_0,\dots,a_{n-1}),\ F(a_0,\dots,a_{n-1}) = (a_0^p,\dots,a_{n-1}^p).$$

Their composition $V\circ F$ coincides with the operation $[p]$ of multiplication by p. It follows that, for any $a\in W_n(k)$, we have $[p]^n(W_n(k)) = \{0\}$. This implies that, if $a_0\neq 0$, the homomorphism $\mathbb{Z}\to W_n(k), 1\mapsto a$, defines an injective map

$$\iota_a:\mathbb{Z}/p^n\mathbb{Z}\to W_n(k),$$

which is compatible with the multiplication by p in the source and the target. Note that, taking $a=(1,0,\dots,0)$, we obtain the natural inclusion

$$\mathbb{Z}/p^n\mathbb{Z} = W_n(\mathbb{F}_p)\subset W_n(k).$$

This defines a faithful action of $\mathbb{Z}/p^n\mathbb{Z}$ on $W_n(k)\cong \mathbf{A}_k^n$ via the translation by a, proving the inequality (8.4).

To get the explicit formula, we need to know the explicit formula for the polynomials $S_0,\dots,S_{n-1}$. Every exposition of the theory of Witt vectors includes the formulas for S_0 and S_1:

$$S_0(x_0,y_0) = x_0+y_0,\quad S_1(x_0,x_1,y_0,y_1) = x_1+y_1-\frac{(x_0+y_0)^p-x_0^p-x_1^p}{p}.$$

For example, taking $n=2, p=3, a=(1,0)$, we obtain that the action on the affine plane is defined by the formula $(x,y)\mapsto(x+1,y-x^2-x)$. After homogenizing, we get the action on $\mathbb{P}(1,1,2)$ that coincides with the action from (8.3). Homogenizing differently, we obtain the action $(x,y)\mapsto(x+t,y-x^2t-xt^2)$ on $\mathbb{P}(1,1,p)$. After we blow up the point $(0:0:1$, we obtain the minimal ruled surface $\mathbb{P}(\mathcal{O}_{\mathbb{P}^1}\oplus\mathcal{O}_{\mathbb{P}^1}(3))\cong\mathbf{F}_3$. This leads to the compactification from [26]

Let me give one more explicit example for an action of $\mathbb{Z}/p^3\mathbb{Z}$ in the case $p=3$. Other examples with larger n or larger p use very large polynomials.

The general recursive formulas for the polynomials $S_i(x,y)$ were given by Matthieu Romagny on MathOverflow. They give the following formula for S_2 in the case $p=3$:

$$\begin{aligned}
&S_2(x_0,x_1,x_2,y_0,y_1,y_2) =\\
&-x_0^8y_0-4x_0^7y_0^2-9x_0^6y_0^3-13x_0^5y_0^4-13x_0^4y_0^5-9x_0^3y_0^6\\
&-4x_0^2y_0^7-x_0y_0^8-x_0^4x_1y_0^2-x_0^4y_0^2y_1-2x_0^3x_1y_0^2-2x_0^3y_0^2y_1\\
&+x_0^2x_1^2y_0-x_0^2x_1y_0^2+2x_0^2x_1y_0y_1-x_0^2y_0^2y_1+x_0^2y_0y_1^2+x_0x_1^2y_0\\
&+2x_0x_1y_0y_1+x_0y_0y_1^2-x_1^2y_1-x_1y_1^2+x_2+y_2.
\end{aligned}$$

Substituting $(y_0,y_1,y_2)=(1,0,0)$, we obtain the following action of $\mathbb{Z}/3^3\mathbb{Z}$ on $\mathbb{A}_k^3$:

$$\begin{aligned}
&(x,y,z)\mapsto(x+1,y-x^2-x,z-x-x^8-4x^7-9x^6-13x^5\\
&-13x^4-9x^3-4x^2-x^4y-2x^3y+x^2y^2-x^2y+xy^2.
\end{aligned}$$

We could homogenize the action in two ways. We can write

$$(t, x, y, z, t) \mapsto (t, x+t, y+f_2(x,t), z+f_8(x,y)),$$

or

$$(t, x, y, z, t) \mapsto (t, x+t, y+f_3(x,t), z+f_9(x,y)),$$

where the subscript denotes the degree of a homogeneous polynomial. The first projectivization gives the action on $\mathbb{P}(1,1,p-1,p^2-1)$, the second one defines the action on $\mathbb{P}(1,1,p,p^2)$. The difference is the action on the boundary $V(t)$. In the first case, it is not trivial, and in the second case, it is the trivial action. Lifted to $\overline{W}_n(k)$, the quotient by the group action of $\mathbb{Z}/p^n\mathbb{Z}$ is a cyclic cover of $\overline{W}_n(k)$ ramified along the boundary divisor [26, Proposition 2].

A candidate for a counter-example to the inequality (2.3) is the extraspecial p-group p^{1+2n}. It follows from (3.4) that

$$\mathrm{ed}_k(p^{1+2n}) = 2.$$

Although the Heisenberg group p_+^{1+2} admits an embedding in $\mathrm{Cr}_k(2)$ via the action:

$$\phi(x_1) : (x,y,z) \mapsto (x+\alpha y, y+\beta z, z), \quad \phi(y_1) : (x,y,z) \mapsto (x+\gamma y, y+\delta z, z),$$

where $\alpha\delta - \beta\gamma = 1$ (see [14]), I do not know how to embed any group p^{1+2n} into $\mathrm{Cr}_k(2)$.

Note that the group 2_+^{1+8} acts faithfully on a del Pezzo surface of degree 1 in characteristic 2 [20], so $\mathrm{crd}_k(2_+^{1+2n}) = 2$, for $n \leq 3$.

Example 8.2. Assume $p = \mathrm{char}(k) \neq 2, 3$. Consider the Segre cubic hypersurface $\mathcal{S}(n)$ in $\mathbb{P}^n$ given by the equations in $\mathbb{P}^{n+1}$:

$$\sum_{i=0}^{n+1} x_i^3 = \sum_{i=0}^{n+1} x_i = 0$$

The symmetric group $\mathfrak{S}_{n+2}$ acts faithfully by permuting the coordinates. If $n = 2k$ is even and $\mathrm{char}(k) \neq 2$, the cubic hypersurface has $\binom{n+1}{[\frac{n}{2}]}$ nodes (it is known to be the maximal possible number of isolated singular points for a cubic hypersurface in $\mathbb{P}^{2k}$ in characteristic 0). As we mentioned earlier, a singular cubic hypersurface is rational (via the projection from a singular point). This confirms the fact that $\mathrm{crd}_k(\mathfrak{S}_{n+2}) \leq n-1$. Assume that $p|(n+1)$. The point $P = (1:1:\ldots:1)$ is an additional singular point, which is fixed under the action of $\mathfrak{S}_{n+2}$. The group acts faithfully on the tangent cone at P of dimension $n-2$. This defines a faithful action of $\mathfrak{S}_{n+2}$ on a quadric of dimension $n-2$ and shows that

$$\mathrm{crd}_k(\mathfrak{S}_{n+2}) \leq n-2 \tag{8.5}$$

if $\mathrm{char}(k)|(n+2), p \neq 2, 3$. The first possible case is $\mathfrak{S}_{10}$ and $p = 5$.

Example 8.3. As we mentioned above, it is conjectured that $\mathrm{ed}_k(\mathfrak{S}_n) \le n-3$ if $\mathrm{char}(k)=0$. However, in [25], the authors prove that $\mathrm{ed}_k(\mathfrak{S}_n) \le n-4$ for certain special n if $p = \mathrm{char}(k)|n$. Their construction uses the action of the affine group $A = \mathrm{Aff}_1(k)$ on the affine $(n-2)$-dimensional quadric Q^{n-2} given by equations

$$F_1 = \sum_{i=1}^n x_i^2 = 0, \quad F_2 = \sum_{i=1}^n x_i = 0.$$

We assume that $n \ge 5, p \ne 2$, and $p|n$. The action is

$$(x_1, \ldots, x_n) \mapsto (ax_1 + bx_n, \ldots, ax_n + bx_n).$$

We have $F_1 \mapsto aF_1 + bnx_n$ and $F_2 \mapsto a^2F_2 + 2abF_1 + b^2nx_n^2$. Since $p|n$, the quadric is A-invariant. The action is obviously faithful and commutes with $\mathfrak{S}_n$. It descends to the faithful action of $\mathfrak{S}_n$ on the orbit space of dimension $n-4$. Since $x_1 = 0$ is a cross-section of the action with the stabilizer subgroup of a general point equal to $\mathbb{G}_m$, we find that the rational quotient Q^{n-2}/A is birationally isomorphic to a quadric of dimension $n-4$ in $\mathbb{P}^{n-3}$. In particular, it is a rational variety. This gives

$$\mathrm{crd}_k(\mathfrak{S}_n) \le n-4, \ n \ge 5.$$

For example, if $n = p = 5$, $\mathrm{crd}_k(\mathfrak{S}_5) = 1$. The group is isomorphic to a maximal subgroup of $\mathrm{PGL}_2(9)$. If $n = 6, p = 3$, we get $\mathrm{crd}_k(\mathfrak{S}_6) \le 2$. In fact, S_6 is isomorphic to $\mathrm{PGL}_2(9)$, and $\mathrm{crd}_k(\mathfrak{S}_6) = 1$.

To compare this with the essential dimension, we use [25], where it is proven that, under the additional assumption that n can be written in the form $2^{m_1} + \cdots + 2^{m_r}$ for some $m_1 \ge \cdots m_r \ge 0$,

$$\mathrm{ed}_k(\mathfrak{S}_n) \le n-4.$$

This applies to the cases $(n,p) = (5,5), (6,3)$, and gives the equalities for the Cremona and essential dimensions in these cases.

If we believe that $\mathrm{crd}_k(\mathbb{Z}/p^r\mathbb{Z}) = r$, then

$$\mathrm{crd}_k(\mathfrak{S}_n) \ge \Big[\log_p n\Big], \quad \mathrm{ed}_k(\mathfrak{S}_n) \ge \Big[\log_p n\Big].$$

In positive characteristic $p > 0$, a simple algebraic group of exceptional type may contain large finite simple groups [43]. For example, if $p = 7$, the adjoint group of type E_6 contains the Mathieu group M_{22}. We will see in the next section that $\mathrm{crd}_k(E_6) = 16$. This gives $\mathrm{crd}_k(M_{22}) \le 16$. Another example is the Janko groups J_1 and J_2, which are contained in a simple group of type G_2 in characteristic $p = 2$ and $p = 11$. We will show that $\mathrm{crd}_k(G_2) = 5$. This gives $\mathrm{crd}_k(J_1) \le 5$ and $\mathrm{crd}_k(J_2) \le 5$.

The classification of wild pseudo-reflection groups is also known (see [31] and references therein). If $p \ne 2$, then among them are the reduction modulo $3, 5$ or 7 of the pseudo-reflection groups in characteristic zero with Nos. 23 = 37, except groups Nos. 25, 26, 27. Note that not all of them have the polynomial ring of invariants (for example, the group F_4 with No. 28). However, the field of invariants is always rational. One of the examples is the group $G = 3.\mathfrak{A}_7 \times 2$ of degree 3 in characteristic 5. It is known that the group $\mathfrak{A}_7$ is contained in $\mathrm{PGL}_3(k)$, so $\mathrm{crd}_k(\mathfrak{A}_7) \leq 2$. The realization of $3.\mathfrak{A}_7$ as a pseido-reflection group in characteristic $p = 5$, shows that

$$\mathrm{crd}_k(3.\mathfrak{A}_7 \times 2) = 3$$

(it is not difficult to show that $\mathrm{crd}_k(3.\mathfrak{A}_7) > 2$). Note that the reductions of the Weyl groups $W(E_6)$ (resp. $W(E_7), W(E_8)$) are wild reflection groups in characteristics 5 (resp. $3, 5, 7$). I do not know the essential dimension of wild pseudo-reflection groups, except in a few cases. Unfortunately, the results of [8] and [23] do not apply to wild pseudo-reflection groups.

9. Algebraic groups

The notion of the essential dimension can be extended to any algebraic group. It is a special case of the essential dimension of any covariant functor

$$\mathcal{F} : \mathrm{Fields}/k \to \mathrm{Sets}$$

introduced by Merkurjev [37]. Let $a \in \mathcal{F}(K)$, we say that $\mathrm{ed}_k(F) = n$ if n is the smallest integer such that, for any L/k and $a \in \mathcal{F}(L)$, there exists a subfield $K \subset L$ of transendence degree $\leq n$ over k and $a' \in \mathcal{F}(K)$ such that $\mathcal{F}(a') = a$.

For example, taking the functor $X : K \mapsto X(K)$ of points of an irreducible algebraic variety, we obtain that $\mathrm{ed}_k(X) = \dim_k(X)$.

The essential dimension of an algebraic group G is the essential dimension of the functor $K \to H^1(K, G)$, the set of G-torsors of G over K. It is equal to the minimum of the degrees of transcendence of a field extension K/k such that there exists a non-trivial torsor of G over K. Connected linear algebraic groups of essential dimension zero were classified by Grothendieck. Serre introduced such groups and referred to them as *special*. An algebraic group is special if its maximal connected semisimple subgroup is the product of groups $\mathrm{SL}_n(k)$ or $\mathrm{Sp}_n(k)$.

Let G be an algebraic group that acts rationally on an irreducible algebraic variety X. By a theorem of Rosenlicht, we can find an open subset $U \subset X$ such that the geometric quotient $U/G = Y$ exists. Suppose also that we can find U such that G acts freely on it. Then, the projection $U \to Y$ is a principal G-bundle, or, in other terms, a torsor over Y [36, Proposition 0.9]. By definition of the essential dimension,

$$\mathrm{ed}_k(G) \leq \dim Y = \dim X - \dim G.$$

In fact,

$$\mathrm{ed}_k(G) = d - \dim G,$$

where d is the smallest dimension of the image of a rational map $Y \to Z$ such that $U \to Y$ is the pull-back of a G-torsor over Z. This makes clear why $\mathrm{ed}(\mathrm{GL}_n(k)) = 0$: one should consider the linear action of the group on the linear space of matrices by left multiplication.

In the following, we assume that G is a simple algebraic group and $\mathrm{char}(k) = 0$. We have the isogenies $G^{\mathrm{sc}} \to G \to G^{\mathrm{adj}}$, where G^{sc} is simply connected and G^{adj} is adjoint. The *rank* of G is the dimension of its maximal torus.

The existence of a linear representation $\rho : G \to \mathrm{GL}(V)$ (as always assumed to be a rational representation) such that G acts almost freely on V, i.e., acts freely on an open susbet of V, gives an upper bound for $\mathrm{ed}_k(G)$.

Recall that a finite abelian subgroup of an algebraic group is called *toral* if it is contained in a maximal torus of the algebraic group. One can obtain a lower bound for $\mathrm{ed}_k(G)$ by using that, for any finite abelian non-toral subgroup A of G (i.e., not contained in a maximal torus of G),

$$\mathrm{rank}(A) \le \mathrm{ed}_k(G) \tag{9.1}$$

[42, Theorem 7.2]. An abelian toral subgroup A has rank $\le \mathrm{rank}(G)$. A non-toral abelian subgroup may have a larger rank. It is known that, for any abelian non-toral p-group A in a simply connected simple algebraic group, p divides the order of the Weil group of G [43, 1.2.2]. Any subgroup of the fundamental group of G is non-toral. Any simply connected G, not of types A_n or C_n, contains abelian non-toral subgroups. A simple algebraic group G satisfying $G^{\mathrm{adj}} = G^{\mathrm{sc}}$ must be of types G_2, F_4 or E_8. These groups contain non-toral 2-groups $2^3, 2^5, 2^9$, respectively [29]. This gives a bound

$$\mathrm{ed}_k(G_2) \ge 3, \quad \mathrm{ed}_k(F_4) \ge 5, \quad \mathrm{ed}_k(E_8) \ge 9.$$

On the other hand, the group E_6 (adjoint or simply connected) contains a group 2^5 (coming from a subgroup isomorphic to the exceptional group F_4). The group E_7^{adj} (resp. E_7 contains 2^8 (resp. 2^7). This gives

$$\mathrm{ed}_k(E_6) \ge 5, \quad \mathrm{ed}_k(E_7^{\mathrm{adj}}) \ge 8, \quad \mathrm{ed}_k(E_7^{\mathrm{sc}}) \ge 7.$$

Let us now discuss the Cremona dimension $\mathrm{crd}_k(G)$ of an affine algebraic k-group. It can be defined in a similar manner to that for a finite group G. However, my conjecture does not extend to this case.

For example, it is known that $\mathrm{ed}_k(\mathrm{PGL}_n(k)) = 2$ for $n = 2, 3, 6$ [41]. However, applying (9.2), we see that $\mathrm{PGL}_{n+1}(k)$ does not embed in $\mathrm{Cr}_{\mathbb{C}}(m)$ for any $m < n$. On the positive side, $\mathrm{ed}_k(\mathrm{PO}_{n+1}(k))) = n$, and it is known that $\mathrm{PO}_{n+1}(k)$ embeds in $\mathrm{Cr}_k(n-1)$ via a birational map from $\mathbb{P}^{n-1}$ to a quadric in $\mathbb{P}^n$.

A maximal torus T of G contains a finite abelian group p^r, where p is any prime number and $r = \dim(T)$ is the rank of G. Applying (4.1), we obtain

$$\mathrm{rank}(G) \le \mathrm{crd}_k(G). \tag{9.2}$$

Connected algebraic groups of rank n contained in $\mathrm{Cr}_k(n)$ were classified by Demazure [13]. His work extends the work of Enriques in the case $n = 2$.

A well-known theorem of Weil asserts that a birational action of an algebraic group can be regularized (see the references in [5]). This means that an algebraic subgroup G of $\mathrm{Cr}_k(n)$ acts biregularly on a rational variety X. The group of automorphisms $\mathrm{Aut}(X)$ of a complex rational variety (or even rationally connected variety) is a linear algebraic group of dimension equal to the dimension of the linear space of regular vector fields. So, if G embeds in $\mathrm{Cr}_{\mathbb{C}}(n)$, there exists a rational smooth algebraic variety X of dimension n such that $h^0(X, T_X) \geq \dim G$. Many such varieties are found among projective vector bundles. The classification of connected algebraic subgroups of $\mathrm{Cr}_{\mathbb{C}}(2)$ and $\mathrm{Cr}_{\mathbb{C}}(3)$ is known (see [4] and the references thereein).

A connected algebraic subgroup of $\mathrm{Cr}_k(n)$ is said to be of *maximal rank* if it contains a torus T of dimension n. It is known that all such tori are conjugate in $\mathrm{Cr}_k(n)$. The classification of semi-simple connected algebraic subgroups of $\mathrm{Cr}_K(n)$ that contain a maximal torus defined over a field K is known [13, §4]. Each such subgroup is is a maximal algebraic subgroup of $\mathrm{Cr}_K(n)$ and is isomorphic to the connected component of the identity of the automorphism group of the product of the projective spaces

$$(\mathbb{P}_K^{m_1})^{n_1} \times \cdots \times (\mathbb{P}_K^{m_r})^{n_r}, \quad m_1n_1 + \cdots + m_rn_r = n. \tag{9.3}$$

Example 9.1. In this example, we will compute $\mathrm{crd}_{\mathbb{C}}(G)$ of any adjoint simple algebraic linear group G over k of characteristic 0.[2]

Suppose G acts regularly on a smooth irreducible projective variety X of dimension d. If G has a fixed point $x \in X$, then G acts faithfully on the tangent space $T_x(X)$, and, since G is simple and adjoint, it acts faithfully on the projective space of dimension $d-1$. Replacing X by $\mathbb{P}^{d-1}$, we may assume that $X^G = \emptyset$. Then, G acts faithfully on each orbit in X. Replacing X by a closed orbit of smallest dimension, we may assume that X is a homogeneous space over G. After resolving singularities of X, we may assume that X is a smooth projective homogeneous space, a quotient G/P by a parabolic subgroup P. Since we want to minimize the dimension of X, we may assume that P is a maximal parabolic, so

$$\mathrm{crd}_{\mathbb{C}}(G) = M := \min_{P \text{ maximal parabolic}} \dim G/P.$$

It is known that a maximal parabolic subgroup P is determined by a simple root of its root system, in other words, a node in its Dynkin diagram. The dimension of G/P is equal to the number of positive roots, which are not supported in the complementary set of nodes.

Easy computations show that

$$M = l,\ 2l-1,\ 2l-1,\ 2l-2,\ 16,\ 27,\ 57,\ 15,\ 5,$$

if G is of type $A_l, B_l, C_l, D_l (l \geq 4), E_6, E_7, E_8, F_4, G_2$, respectively. The corresponding classical groups are $\mathrm{PGL}_{l+1}, \mathrm{PO}_{2l+1}, \mathrm{PSp}_{2l}$, and PO_{2l} (we also use that $\mathrm{PSO}_4 \cong \mathrm{PSO}_5$ and $\mathrm{PSO}_6 \cong \mathrm{PSp}_4$). The homogeneous varieties

[2] As there is no common agreement on the definitions, we assume that G is connected and admit almost simple groups as simple groups.

corresponding to classical groups are projective spaces, or quadrics in $\mathbb{P}^{2l+1}$ or $\mathbb{P}^{2l}$. The homogeneous varieties corresponding to the exceptional groups are the E_6-variety in $\mathbb{P}^{26}$ equal to the singular locus of the Cartan cubic hypersurface, the E_7-variety in $\mathbb{P}^{55}$ equal to the singular locus of the singular locus of the Cartan quartic hypersurface, the minimal adjoint orbit of the group E_8 in $\mathbb{P}^{247}$, the minimal adjoint orbit of F_4 in $\mathbb{P}^{51}$ or a cross-sections of the E_6-variety by a F_4-invariant hyperplane, the minimal adjoint orbit of G_2 in $\mathbb{P}^{13}$ of a linear section of the E_6-variety by a G_2-invariant subspace $\mathbb{P}^6$ of $\mathbb{P}^{26}$ isomorphic to a quadric (see, for example, [34, §6]).

For example, besides the products of projective general groups, $\mathrm{Cr}_{\mathbb{C}}(2)$ contains none of the other simple algebraic groups. However, $\mathrm{Cr}_{\mathbb{C}}(3)$ contains $\mathrm{PO}_5 \cong \mathrm{PSp}_4$. The group $\mathrm{Cr}_{\mathbb{C}}(4)$ contains PO_6. The first case, where a simple algebraic group G of one of the exceptional types appears, is $n = 5$ and $G \cong G_2$.

Example 9.2. We can use (3.1) to get an upper bound for the Cremona dimension of a finite simple group. For example, it is known that $L_2(13)$ is contained in a simple group G_2. This gives

$$4 \leq \mathrm{crd}_{\mathbb{C}}(L_2(13)) \leq 5.$$

The minimal dimension of an irreducible representation of $L_2(13)$ is equal to 7. Another example is the group $L_3(8)$, which is contained in a simple adjoint group E_6. We have

$$\mathrm{crd}_{\mathbb{C}}(L_3(8)) \leq 16.$$

It is known that the minimal dimension of an irreducible representation of this group is equal to 72.

10. Lattices in simple Lie groups

Let Γ be a lattice in a simple real Lie group G (in our definition, this means that G is connected and its Lie algebra is simple). Let $d(\Gamma)$ be the smallest dimension of a faithful linear representation of Γ known to be strictly larger than $\mathrm{rank}_{\mathbb{R}}(G)$ if $\mathrm{rank}_{\mathbb{R}}(G) \geq 2$. Suppose that Γ embeds in the group $\mathrm{Bir}(X)$ of birational automorphisms of a complex algebraic variety X. It is conjectured that

$$d(\Gamma) \leq \dim X - 1.$$

In particular, if $\mathrm{rank}_{\mathbb{R}}(G) \geq \dim X$, then Γ is not isomorphic to a subgroup of $\mathrm{Bir}(X)$. This is known to be true for non-cocompact lattices. Moreover, if the equality takes place, then G is isogenous to $\mathrm{SL}_{\dim X+1}(\mathbb{R})$ [10].

For example, $\Gamma = \mathrm{SL}_{n+1}(\mathbb{Z}) \subset \mathrm{SL}_{n+1}(\mathbb{R})$ embeds in $\mathrm{PGL}_{n+1}(\mathbb{C})$ for n odd and, hence, embeds in $\mathrm{Cr}_{\mathbb{C}}(n)$. For even n, it can be embedded in $\mathrm{Cr}_{\mathbb{C}}(n+1)$ as a subgroup of monomial birational transformations of $\mathbb{A}^{n+1}_{\mathbb{C}}$. However, Γ cannot be embedded in $\mathrm{Cr}_{\mathbb{C}}(n-1)$ for even or odd n.

The group $\mathrm{O}(1,n)$ of orthogonal transformations of the hyperbolic space $\mathbb{R}^{1,n}$ is of the real rank 1. It is known that for some n (e.g. $n = 9$), it contains non-cocompact lattices embedded in $\mathrm{Cr}_{\mathbb{C}}(2)$. A notorious example is the group of automorphisms of a general rational Coble surface isomorphic to such a lattice in $\mathrm{O}(1,9)$.

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